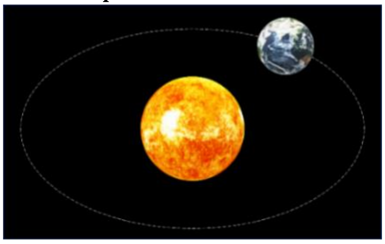
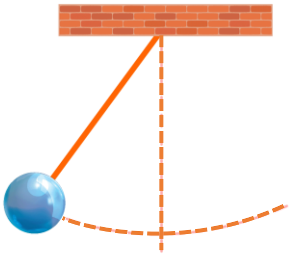
02 Oscillations

Introduction

(a) **Periodic Motion:** Any motion which repeats itself after regular interval of time is called periodic motion or harmonic motion.
The constant interval of time after which the motion is repeated is called time period.

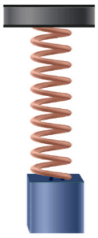
Ex : **(i)** Motion of planets around the sun. **(ii)** Motion of the pendulum.





(b) **Oscillatory Motion:** The motion of a body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after certain interval of time.
The fixed point about which the body oscillates is called mean position or equilibrium position.

Ex : **(i)** Vibration of the wire of 'Sitar'. **(ii)** Oscillation of the mass suspended from spring.



(c) **Harmonic Functions:** The trigonometric function of constant amplitude and single frequency is defined as harmonic function (among all the trigonometrical functions only sin and cos functions are taken as harmonic function in basic form).

$$\left.\begin{array}{l} y = A\sin\theta = A\sin\omega t \\ y = A\cos\theta = A\cos\omega t \end{array}\right\} \text{Harmonic Functions}$$

Golden Key Points

- Any motion that repeats itself after regular internal of time is called periodic motion.
- The motion of a body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after certain interval of time.
- Among all the trigonometrical functions only sin and cos functions are taken as harmonic function in basic form.

Simple harmonic motion (SHM) and its equation

Some basic terms

Mean Position: The point at which the restoring force on the particle is zero and potential energy is minimum, is known as its mean position.

Restoring Force:

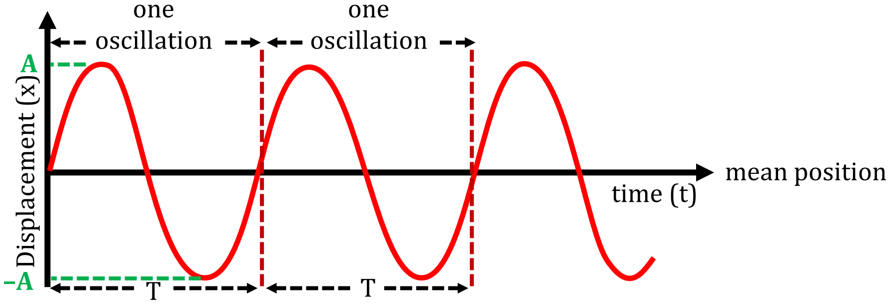
- The force acting on the particle which tends to bring the particle towards its mean position, is known as restoring force.
- This force is always directed towards the mean position.
- Restoring force always acts in a direction opposite to that of particle displacement. This displacement is measured from the mean position.
- It is given by F = –kx and has dimensions [MLT^{-2}].
- Direction of displacement and restoring force change at mean position.

Amplitude : The maximum displacement of particle from mean position is defined as amplitude.

Time period (T) : The minimum time after which the particle keeps on repeating its motion is known as time period. It is given by $T=\frac{2\pi}{\omega}$, $T=\frac{1}{n}$ where ω is angular frequency and n is frequency.

One Oscillation or Vibration

When a particle goes to one side from mean position and returns back and then it goes to other side and again returns back to mean position, then this process is known as one oscillation.



Frequency (n or f)

The number of oscillations per second is defined as frequency.

It is given by $n=\frac{1}{T}, n=\frac{\omega}{2\pi}$.

SI UNIT : Hertz (Hz), 1 Hertz = 1 cycle per second (cycle is a number not a dimensional quantity).

Dimension : $\left[M^0L^0T^{-1}\right]$

Angular frequency (ω)

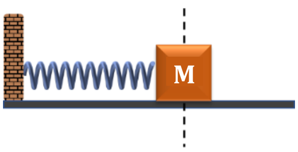
The rate of change of phase angle of a particle with respect to time is define as its angular frequency.

SI UNIT : radian/second **Dimension :** $\left[M^0L^0T^{-1}\right]$

S.H.M. are of two types

(a) Linear S.H.M.

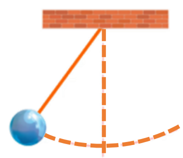
When a particle moves to and fro about a fixed point (called equilibrium position) along a straight line then its motion is called linear simple harmonic motion.



Example : Motion of a mass connected to spring.

(b) Angular S.H.M.

When a system oscillates angularly with respect to a fixed axis then its motion is called angular simple harmonic motion.



Example :- Motion of a bob of simple pendulum.

Necessary Conditions to execute S.H.M.

- Motion of particle should be oscillatory.
- Total mechanical energy of particle should be conserved (Kinetic energy + Potential energy = constant)

- Extreme position should be well defined.
- In linear S.H.M the restoring force (or acceleration) acting on the particle should always be proportional to the displacement of the particle and directed towards the equilibrium position.

  $\therefore$ $F \propto -x$ or $a \propto -x$

  Negative sign shows that direction of force and acceleration is towards equilibrium position and x is displacement of particle from equilibrium position.
- In angular S.H.M. the restoring torque (or angular acceleration) acting on the particle should always be proportional to the angular displacement of the particle and directed towards the equilibrium position

  $\therefore$ $\tau \propto -\theta$ or $\alpha \propto -\theta$

Differential equation of SHM and Graphs

Linear S.H.M.	Angular S.H.M.
$F \propto -x \Rightarrow F = -kx$ Where k is the restoring force constant $a = -\frac{k}{m}x \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ It is known as differential equation of linear S.H.M. $x = A\sin\omega t \Rightarrow a = -\omega^2 x$ where ω is the angular frequency. $\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega}$ where T is time period and n is frequency. $T = 2\pi\sqrt{\frac{m}{k}}$ $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ This concept is valid for all types of linear S.H.M.	$\tau \propto -\theta \Rightarrow \tau = -C\theta$ Where C is the restoring torque constant $\alpha = -\frac{C}{I}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$ It is known as differential equation of angular S.H.M. $\theta = \theta_0 \sin\omega t \Rightarrow \alpha = -\omega^2\theta$ $\omega^2 = \frac{C}{I} \Rightarrow \omega = \sqrt{\frac{C}{I}} = \frac{2\pi}{T} = 2\pi n$ $T = 2\pi\sqrt{\frac{I}{C}}$ $n = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$ This concept is valid for all types of angular S.H.M.

Illustration 1

Which of the following functions represent SHM :

(i) sin 2ωt (ii) sin ωt + 2cos ωt (iii) sin ωt + cos 2ωt

Solution:

(i) As $y = \sin 2\omega t$

$$\Rightarrow v = \frac{dy}{dt} = 2\omega\cos 2\omega t,\ \text{Acceleration} = \frac{d^2y}{dt^2} = -4\omega^2\sin 2\omega t = -4\omega^2 y$$

So, y = sin 2ωt represents S.H.M.

(ii) $y = \sin\omega t + 2\cos\omega t$

$$\Rightarrow v = \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin\omega t,$$

$$\text{Acceleration} = \frac{dv}{dt} = -\omega^2\sin\omega t - 2\omega^2\cos\omega t = -\omega^2(\sin\omega t + 2\cos\omega t) = -\omega^2 y$$

$\therefore$ The given function represents SHM.

(iii) $y = \sin\omega t + \cos 2\omega t$

$$\Rightarrow \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin 2\omega t, \frac{d^2y}{dt^2} = -\omega^2\sin\omega t - 4\omega^2\cos 2\omega t$$

$$\frac{d^2y}{dt^2} = -\omega^2(\sin\omega t + 4\cos 2\omega t) \Rightarrow \frac{d^2y}{dt^2} \neq (-y)$$ (S.H.M. is not possible).

Illustration 2

A particle in SHM according to equation $16\frac{d^2x}{dt^2}+25x=0$. Find the nature of motion and its time period.

Solution:

$$16\frac{d^2x}{dt^2}+25x=0 \Rightarrow \frac{d^2x}{dt^2}+\left(\frac{25}{16}\right)x=0$$

Motion is linear SHM.

We get $\omega^2=\frac{25}{16} \Rightarrow \omega=\frac{5}{4} \Rightarrow \frac{2\pi}{T}=\frac{5}{4} \Rightarrow T=\frac{8\pi}{5}s$

BEGINNER'S BOX-1

1. Which of the following functions of time represent (a) SHM and (b) Periodic but not SHM (c) Non-periodic motion?

(i) $\sin \omega t + \cos \omega t$ (ii) $\sin \omega t + \cos 2\omega t + \sin 4\omega t$ (iii) $e^{-\omega t}$

(iv) $\log(\omega t)$ (v) $\sin \omega t - \cos \omega t$ (vi) $\sin^3 \omega t$

POS168

2. The equation of motion of a particle executing simple harmonic motion is $a+16\pi^2x=0$. In this equation, a is the linear acceleration in m/s^2 of the particle at a displacement x in metre. Find the time period.

POS169

3. Displacement of a particle executing SHM is represented by $Y=0.08\sin\left(3\pi t+\frac{\pi}{4}\right)$ metre. Then calculate: -

(a) Time period (b) Initial phase

(c) Displacement from mean position at $t=\frac{7}{36}$sec.

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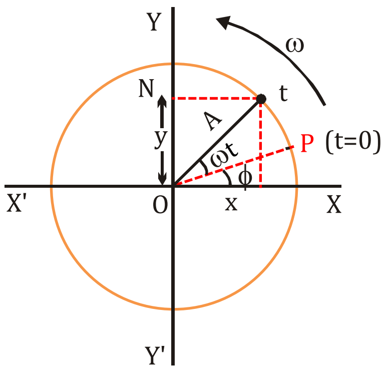
Explaining SHM with Phasor Diagram

Geometrical meaning of S.H.M.

If a particle is moving with uniform speed along the circumference of a circle then the straight line motion of the foot of perpendicular drawn from the particle on the diameter of the circle is called S.H.M.

Description of S.H.M. based on circular motion.



(a) Draw a circle, having radius equal to amplitude (A).

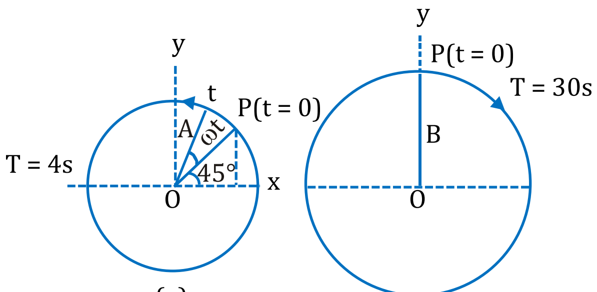
(b) Suppose a particle is moving with uniform speed with angular frequency ω along the circumference of the circle.

(c) Shadow (foot of the perpendicular from particle position) of particle performs S.H.M. on vertical and horizontal diameter of circle.

(d) Position of particle's shadow can be represented on diameter at t = 0 or any instant and position of particle performing circular motion can be determined by direction of velocity.

(e) By joining centre of circle to particle's position, angle θ is determined from horizontal or vertical diameter. After time t radius vector will turn ωt. so θ = ωt.

Ex. : Depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated on the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.



(a) (b)

Sol. (a) At t = 0, OP makes an angle of 45° = π/4 rad with the (positive direction of) x-axis. After time t, it covers an $\frac{2\pi}{T}t$ in the anticlockwise sense, and makes an angle of $\frac{2\pi}{T}t+\frac{\pi}{4}$ with the x-axis.

The projection of OP on the x-axis at time t is given by,

$x(t)=A\cos\left(\frac{2\pi}{T}t+\frac{\pi}{4}\right)$ For T = 4 s, $x(t)=A\cos\left(\frac{2\pi}{4}t+\frac{\pi}{4}\right)$

which is a SHM of amplitude A, period 4 s, and an initial phase = $\frac{\pi}{4}$.

(b) In this case at t = 0, OP makes an angle of 90° = $\frac{\pi}{2}$ with the x-axis. After a time t, it t covers an angle of $\frac{2\pi}{T}t$ sense and makes an angle of $\frac{\pi}{2}-\frac{2\pi}{T}t$ with the x-axis. The projection of OP on the x-axis at time t is given by

$x(t)=B\cos\left(\frac{\pi}{2}-\frac{2\pi}{T}t\right)=B\sin\left(\frac{2\pi}{T}t\right)$ For T = 30 s, $x(t)=B\sin\left(\frac{\pi}{15}t\right)$

Writing this as $x(t)=B\cos\left(\frac{\pi}{15}t-\frac{\pi}{2}\right)$. We find that this represents a SHM of amplitude B, period 30 s, and an initial phase of $\frac{\pi}{2}$.

Illustration 3

Which of the following expressions represent simple harmonic motion?

(1) $x=A\sin(\omega t+\phi)$ (2) $x=B\cos(\omega t+\phi)$ (3) $x=A\tan(\omega t+\phi)$ (4) $x=A\sin\omega t.\cos\omega t$

Solution:

(1) $x=A\sin(\omega t+\phi)$ (2) $x=B\cos(\omega t+\phi)$

(4) $x = A\sin\omega t\cos\omega t\times\frac{2}{2} \Rightarrow x=\frac{A}{2}\sin 2\omega t$

Illustration 4

The displacement of a particle in S.H.M. is indicated by equation y = 10 sin(20t + π/3) where y is in metres. The value of time period of vibration will be (in seconds) :

Solution:

On comparing $y = 10\sin(20t+\frac{\pi}{3})$ with $y = A\sin(\omega t+\phi)$,

we get $\omega = 20 \Rightarrow T=\frac{2\pi}{\omega}=\frac{2\pi}{20}=\frac{\pi}{10}$ sec.

Illustration 5
The equation of a simple harmonic motion is x = 0.34cos (3000t + 0.74). Where x and t are in mm and sec. respectively. The frequency of the motion is (in Hertz):

Solution:

On comparing, x = 0.34 cos (3000t + 0.74) with x = A cos (ωt + kx)

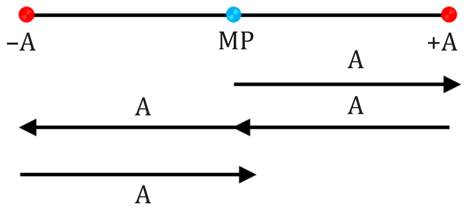
We get, ω = 3000   Now, ω = 2πn ⇒ 3000 = 2πn ⇒ $n = \frac{3000}{2\pi} \Rightarrow n = \frac{1500}{\pi}$

Illustration 6
The distance covered by a particle executing SHM, in one time period is equal to :

(1) Four times the amplitude   (2) Two times the amplitude

(3) One times the amplitude   (4) Eight times the amplitude

Solution: (1)

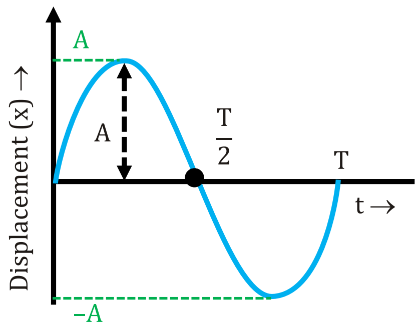


The distance covered by a particle executing SHM in one time period is 4 A.

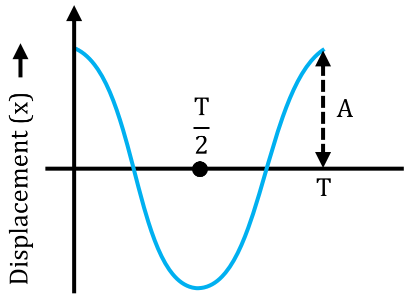
Displacement in SHM
(i) The displacement of a particle executing linear S.H.M. at any instant is defined as the distance of the particle from the mean position at that instant.

(ii) It can be given by relation x = Asinωt or x = Acosωt.
The first relation is valid when the time is measured from the mean position and the second relation is valid when the time is measured from the extreme position of the particle executing S.H.M. along a straight-line path.

(iii) Maximum value of displacement is = ± A



X = A Sinωt



X = A cosωt

Illustration 7
A particle executes S.H.M. from extreme position and covers a distance equal to half of its amplitude in 1s. Determine the time period of motion.

Solution:

For particle starting S.H.M. from extreme position

$$y = A\cos\omega t \Rightarrow \frac{A}{2} = A\cos(\omega \times 1) \Rightarrow \cos\omega = \cos\frac{\pi}{3} \Rightarrow \omega = \frac{\pi}{3} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi \times 3}{\pi} = 6s$$

Illustration 8
An object performs S.H.M. of amplitude 5 cm and time period 4 s. If timing is started when the object is at the centre of the oscillation i.e., x = 0 then calculate –

(i) Frequency of oscillation   (ii) The displacement at 0.5 sec.

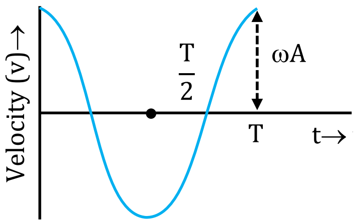
Solution:

(i) Frequency $f=\frac{1}{T}=\frac{1}{4}=0.25\text{Hz}$

(ii) The displacement equation of object $x=A\sin\omega t$

So, at $t=0.5\,s$ $x=5\sin(2\pi\times0.25\times0.5)=5\sin\frac{\pi}{4}=\frac{5}{\sqrt{2}}\text{ cm}$

Velocity in S.H.M.



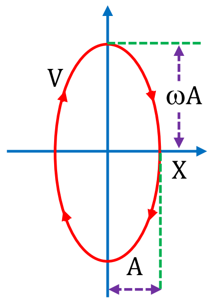
(i) It is defined as the time rate of change of the displacement of the particle at a given instant.

(ii) Velocity in S.H.M. is given by

$v=\frac{dx}{dt}=\frac{d}{dt}(A\sin\omega t)\Rightarrow \quad v=A\omega\cos\omega t$

$v=\pm A\omega\sqrt{1-\sin^2\omega t}\Rightarrow v=\pm A\omega\sqrt{1-\frac{x^2}{A^2}}\left[\because x=A\sin\omega t\right]$

$v=\pm\omega\sqrt{\left(A^2-x^2\right)}$



Squaring both the sides $v^2=\omega^2\left(A^2-x^2\right)\Rightarrow\frac{v^2}{\omega^2}=A^2-x^2$

$\frac{v^2}{\omega^2A^2}=1-\frac{x^2}{A^2}\quad\Rightarrow\frac{x^2}{A^2}+\frac{v^2}{A^2\omega^2}=1$

This is equation of ellipse. So, curve between displacement and velocity of particle executing S.H.M. is ellipse. If particle oscillates with unit angular frequency $(\omega=1)$ then curve between V and x will be circle.

Note: (i) The direction of velocity of a particle in S.H.M. is either towards or away from the mean position.

(ii) At mean position (x = 0), velocity is maximum (= Aω) and at extreme position (x = ±A), the velocity of particle executing S.H.M. is zero.

Illustration 9

Amplitude of a harmonic oscillator is A, when velocity of particle is half of maximum velocity, then determine position of particle.

Solution:

$v=\omega\sqrt{A^2-x^2}\Rightarrow\frac{A\omega}{2}=\omega\sqrt{A^2-x^2}\Rightarrow A^2=4\left[A^2-x^2\right]\Rightarrow x^2=\frac{4A^2-A^2}{4}\Rightarrow x=\pm\frac{\sqrt{3}A}{2}$

Illustration 10

A particle performing SHM is found at its equilibrium position at t = 1 sec and it is found to have a speed of 0.25 m/s at t = 2 sec. If the period of oscillation is 8 sec. Calculate the amplitude of oscillations.

Solution:

x = Asin(ωt + ϕ), at t = 1 sec. particle at mean position $0=A\sin\left(\frac{2\pi}{8}\times1+\phi\right)\Rightarrow\phi=-\frac{\pi}{4}$

at t = 2 sec. velocity of particle is 0.25 m/s $\Rightarrow$ $0.25=A\omega\cos\left(\frac{\pi}{4}\times2-\frac{\pi}{4}\right)\Rightarrow 0.25=\frac{A\omega}{\sqrt{2}}\Rightarrow A=\frac{\sqrt{2}}{\pi}$

Illustration 11

A particle is executing the motion $x=A\cos(\omega t-\theta)$. The maximum velocity of the particle is

Solution:

Velocity of a particle executing SHM is given by $v=\omega\sqrt{A^2-x^2}$ ...(1)

For maximum velocity we need x = 0, So, by putting x = 0 in equation (1) we have, $V_{max}=A\omega$

Illustration 12

The maximum velocity of a simple harmonic motion represented by $y = 3\sin\left(100t + \frac{\pi}{6}\right)$ is given by

Solution:

$v_{max} = A\omega = 3 \times 100 = 300$

Illustration 13

Velocity at mean position of a particle executing S.H.M. is v, they velocity of the particle at a distance equal to half of the amplitude.

Solution:

Velocity in mean position $v = A\omega$, velocity at a distance of half amplitude.

$$v' = \omega\sqrt{A^2 - x^2} = \omega\sqrt{A^2 - \frac{A^2}{4}} = \sqrt{\frac{3}{2}}A\omega = \sqrt{\frac{3}{2}}v$$

Illustration 14

The instantaneous displacement of a simple pendulum oscillator is given by $x = A\cos\left(\omega t + \frac{\pi}{4}\right)$. Its speed will be maximum at time ?

Solution:

$$x = A\cos\left(\omega t + \frac{\pi}{4}\right) \text{ and } v = \frac{dx}{dt} = -A\omega\sin\left(\omega t + \frac{\pi}{4}\right)$$

For maximum speed, $\sin\left(\omega + \frac{\pi}{4}\right) = 1 \Rightarrow \omega t + \frac{\pi}{4} = \frac{\pi}{2}$ or $\omega t = \frac{\pi}{2} - \frac{\pi}{4} \Rightarrow t = \frac{\pi}{4\omega}$

Acceleration in S.H.M.

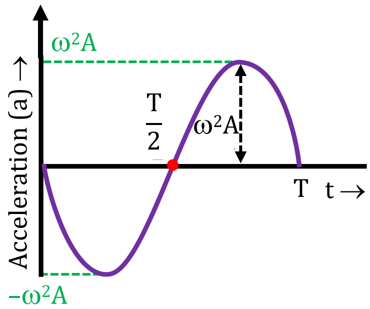
(i) It is defined as the time rate of change of the velocity of the particle at given instant.

(ii) Acceleration in S.H.M. is given by

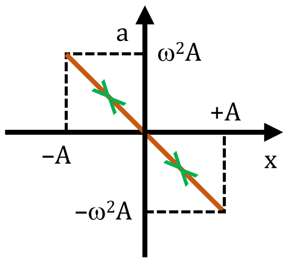
$$a = \frac{dv}{dt} = \frac{d}{dt}(A\omega\cos\omega t)$$

$$a = -\omega^2 A\sin\omega t$$

$$a = -\omega^2 x$$

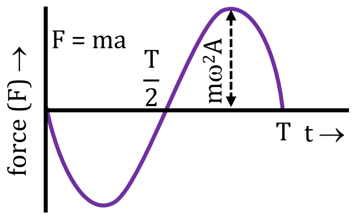


(iii) The graph between acceleration and displacement is a straight line as shown in figure.

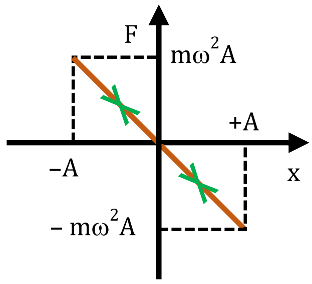


(iv) Relation between force(F) and time(t)

$\Rightarrow F = ma = -m\omega^2 A\sin\omega t$



(v) Relation between force(F) and displacement(x) $\Rightarrow F = -m\omega^2 x$



Note: (i) The acceleration of a particle executing S.H.M. is always directed towards the mean position.

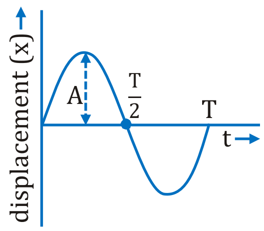
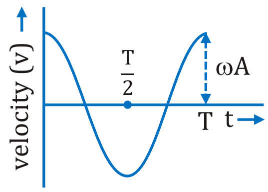
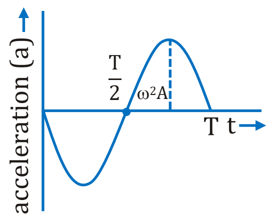
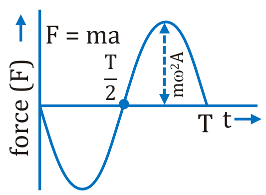
(ii) The acceleration of the particle executing S.H.M. is maximum at extreme position ($\pm\omega^2A$) and minimum at mean position (= zero).

(iii) Maximum value of acceleration is $\pm\omega^2A$.

(iv) Maximum value of force is $\pm m\omega^2A$.

Graphical Representation

Graphical study of displacement, velocity, acceleration and force in S.H.M.

S. No.	Graph	In form of t	In form of x	Maximum value
1.	Displacement 	$x = A\sin\omega t$	$x = x$	$x = +A$
2.	Velocity 	$v = A\omega\cos\omega t$	$v = \pm\omega\sqrt{A^2 - x^2}$	$v = \pm\omega A$
3.	Acceleration 	$a = -\omega^2A\sin\omega t$	$a = -\omega^2x$	$a = \pm\,\omega^2A$
4.	Force (F = ma) 	$F = -\,m\omega^2A\sin\omega t$	$F = -\,m\omega^2x$	$F = \pm\,m\omega^2A$

Golden Key Points

- The direction of displacement is always away from the mean position whether the particle is moving from or coming towards the mean position.
- In linear S.H.M., the length of S.H.M. path = 2A.
- In S.H.M., the total work done and displacement in one complete oscillation is zero but total travelled length is 4A.
- In S.H.M., the velocity and acceleration varies simple harmonically with the same frequency as displacement.
- Velocity is always ahead of displacement by phase angle $\frac{\pi}{2}$ radian.
- Acceleration is ahead of displacement by phase angle π radian i.e., opposite to displacement.
- Acceleration leads the velocity by phase angle $\frac{\pi}{2}$ radian.

Illustration 15

An object performs S.H.M. of amplitude 5 cm and time period 4 s. If timing is started when the object is at the centre of the oscillation i.e. x = 0 then calculate

(i) Frequency of oscillation
(ii) The displacement at 0.5 sec.
(iii) The maximum acceleration of the object.
(iv) The velocity at a displacement of 3 cm.

Solution:

(i) Frequency $f = \frac{1}{T} = \frac{1}{4} = 0.25$ Hz

(ii) The displacement equation of object $x = A\sin\omega t$

so at t = 0.5 s $\quad x = 5\sin(2\pi \times 0.25 \times 0.5) = 5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}}$ cm

(iii) Maximum acceleration $a_{max} = \omega^2 A = (0.5\,\pi)^2 \times 5 = 12.3$ cm/s²

(iv) Velocity at x = 3 cm is $v = \pm\omega\sqrt{A^2 - x^2} = \pm 0.5\pi\sqrt{5^2 - 3^2} = \pm\ 6.28$ cm / sec

Illustration 16

A particle executing S.H.M. having amplitude 0.01 m and frequency 60 Hz. Determine maximum acceleration of particle.

Solution:

Maximum acceleration $a_{max.} = \omega^2 A = 4\pi^2 n^2 A = 4\pi^2 (60)^2 \times (0.01) = 144\,\pi^2$ m/sec

Illustration 17

Which one of the following statements is true for the speed v and the acceleration a of a particle executing simple harmonic motion?

(1) When v is maximum, a is maximum
(2) Value of a is zero, whatever may be the value of v
(3) When v is zero, a is zero
(4) When v is maximum, a is zero

Solution: (4)

In S.H.M. $v = \omega\sqrt{a^2 - x^2}$ and $a = -\omega^2 x$ when $x = a$

$\Rightarrow v_{min} = 0$ and $a_{max} = -\omega^2 a$, and at y = 0 $\Rightarrow V_{max} = A\omega$, $a_{min} = 0$

Illustration 18

In simple harmonic motion, the ratio of acceleration of the particle to its displacement at any time is a measure of

(1) Spring constant (2) Angular frequency (3) (Angular frequency)^2 (4) Restoring force

Solution: (3)

$a = -\omega^2 x \Rightarrow \left|\frac{a}{x}\right| = \omega^2$

BEGINNER'S BOX-2

1. A particle executing simple harmonic motion completes 1200 oscillations per minute and passes through the mean position with a velocity of $3.14 ms^{-1}$. Determine the maximum displacement of the particle from its mean position. Also obtain the displacement equation of the particle if its displacement be zero at the instant t = 0.

POS171

2. A particle oscillates along the x–axis according to equation $x = 0.05 \sin\left(5t - \frac{\pi}{6}\right)$ where x is in metre and t is in second. Find its velocity at t = 0 second.

POS172

3. A particle is executing SHM given by $x = A \sin(\pi t + \phi)$. The initial displacement of particle is 1 cm and its initial velocity is π cm/sec. Find the amplitude of motion and initial phase of the particle.

POS173

4. A body executing S.H.M. has its velocity 10 cm/sec and 7 cm/sec when its displacement from the mean position are 3 cm and 4 cm respectively. Calculate the length of the path.

POS174

5. A particle is executing S.H.M. of time period 4s. What is the time taken by it to move from the

(a) Mean position to half of the amplitude.

(b) Extreme position to half of the amplitude.

POS175

6. A particle undergoes simple harmonic motion having time period T. Find the time taken to complete $\frac{3}{8}$ oscillation.

POS176

7. The displacement of a particle executing simple harmonic motion is given by $y = 10\sin(6t + \frac{\pi}{3})$. Here y is in metre and t is in second. Find initial displacement & velocity of the particle.

POS177

8. For a particle executing S.H.M. In which part of a complete oscillation

(a) Acceleration supports the velocity. (b) Acceleration opposes the velocity.

POS178

9. A particle is in linear simple harmonic motion between two points A and B, 10 cm apart. Take the direction from A to B as the positive direction and given the signs of velocity, acceleration and force on the particle when it is

(a) at the end A. (b) at the end B.

(c) at the mid-point of AB going towards A. (d) at 2 cm away from B going towards A.

(e) at 3 cm away from A going towards B, and (f) at 4 cm away from A going towards A.

POS179

10. The piston in the cylinder head of a locomotive has a stroke (twice of the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min., what is its maximum speed?

POS180

Energy in SHM – Potential & Kinetic Energies

Potential Energy in S.H.M.

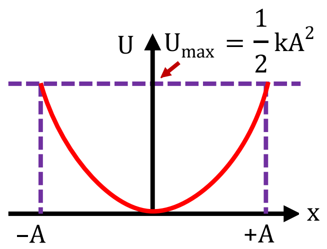
(i) In terms of displacement

The potential energy is related to force by the relation

$F=-\frac{dU}{dx} \Rightarrow \int dU=-\int F dx$

For S.H.M. F = – kx

So, $\int dU=-\int(-kx)dx=\int kx\,dx \Rightarrow U=\frac{1}{2}kx^2+C$



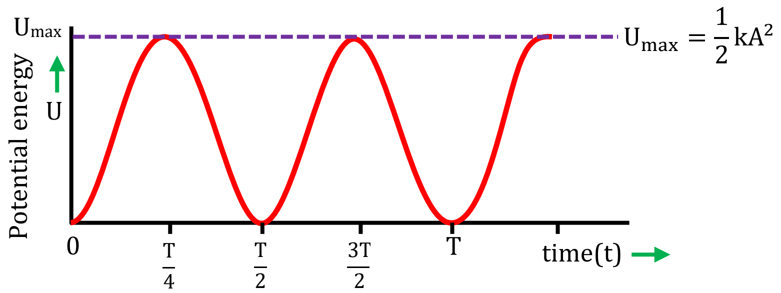
At x = 0, U = $U_0 \Rightarrow C = U_0$ So, $U=\frac{1}{2}kx^2+U_0$

Where the potential energy at equilibrium position = U_0 when $U_0 = 0$ then $u=\frac{1}{2}kx^2$

(ii) In terms of time

Since, $x = A\sin(\omega t+\phi)$, $U=\frac{1}{2}kA^2\sin^2(\omega t+\phi)$

If initial phase (ϕ) is zero then $U=\frac{1}{2}kA^2\sin^2\omega t=\frac{1}{2}m\omega^2A^2\sin^2\omega t$



Note: (i) In S.H.M. the potential energy is a parabolic function of displacement; the potential energy is minimum at the mean position (x = 0) and maximum at extreme position (x = ±A)

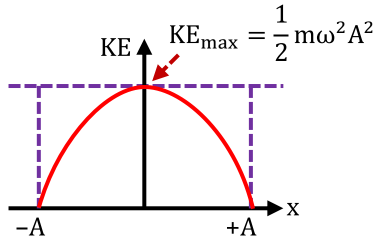
(ii) The potential energy is the periodic function of time.

It is minimum at $t=0,\frac{T}{2},T,\frac{3T}{2}$.... and maximum at $t=\frac{T}{4},\frac{3T}{4},\frac{5T}{4}$.......

Kinetic Energy (K)

(i) In terms of displacement

If mass of the particle executing S.H.M. is m and Its velocity is v then kinetic energy at any instant.



K.E. = $\frac{1}{2}mv^2=\frac{1}{2}m\omega^2(A^2-x^2)=\frac{1}{2}k(A^2-x^2)$

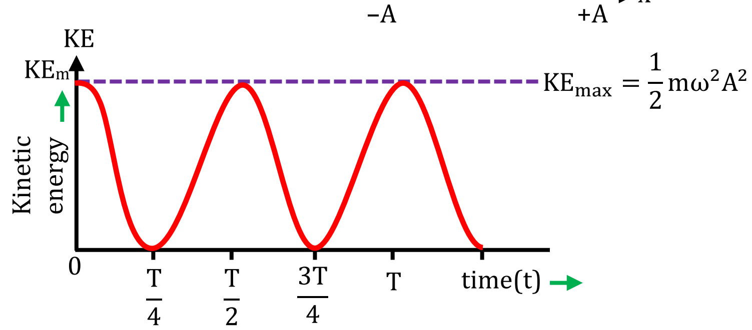
(ii) In terms of time

$v = A\omega\cos(\omega t+\phi)$

$\therefore$ K.E. = $\frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi)$

If initial phase ϕ is zero,

K.E. = $\frac{1}{2}m\omega^2A^2\cos^2\omega t$



Note: (i) In S.H.M. the kinetic energy is a inverted parabolic function of displacement. The kinetic energy is maximum $\left(\frac{1}{2}kA^2\right)$ at mean position (x = 0) and minimum (zero) at extreme position $(x = \pm A)$.

(ii) The kinetic energy is the periodic functions of time. It is maximum at $t = 0, \frac{T}{2}, T, \frac{3T}{2}$............. and minimum at ... $\frac{T}{4}, T, \frac{3T}{4}, \frac{5T}{4}$.

Illustration 19

A particle starts oscillating simple harmonically from its equilibrium position with time period T. Determine ratio of K.E. and P.E. of the particle at time t = $\frac{T}{12}$.

Solution:

at t = $\frac{T}{12}$ x = $A\sin\frac{2\pi}{T}\times\frac{T}{12} = A\sin\frac{\pi}{6} = \frac{A}{2}$

so K.E. = $\frac{1}{2}k(A^2 - x^2) = \frac{3}{4}\times\frac{1}{2}kA^2$ and P.E. = $\frac{1}{2}kx^2 = \frac{1}{4}\times\frac{1}{2}kA^2$ $\therefore$ Ratio $\frac{K.E.}{P.E.} = \frac{3}{1}$

Illustration 20

A harmonic oscillator of force constant 4×10^6 Nm^{-1} and amplitude 0.01 m has total energy 240 J. What is maximum kinetic energy and minimum potential energy?

Solution:

k = 4×10^6 N/m, A = 0.01 m, T.E. = 240 J, As $\omega^2 = \frac{k}{m}$

Maximum kinetic energy = $\frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2 = \frac{1}{2}\times 4\times 10^6\times(0.01)^2$ = 200 J

Minimum potential energy = Total energy – Maximum kinetic energy = 40 J

Illustration 21

A particle executes simple harmonic motion along a straight line with an amplitude A. The potential energy is maximum when the displacement is

Solution:

P.E. $=\frac{1}{2}m\omega^2x^2$, it is clear P.E. will be maximum when x will be maximum i.e., at $x = \pm A$

Illustration 22

A particle is vibrating in a simple harmonic motion with an amplitude of 4 cm. At what displacement from the equilibrium position, its energy is half potential and half kinetic.

Solution:

Let x be the point where K.E. = P.E. $\Rightarrow \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}m\omega^2x^2 \Rightarrow 2x^2 = A^2 \Rightarrow x = 2\sqrt{2}$cm

Illustration 23

The maximum K.E. of a oscillating spring is 5 joules and its amplitude 10 cm. The force constant of the spring is :

Solution:

Maximum K.E. = $\frac{1}{2}kA^2 \Rightarrow 5 = \frac{1}{2}\times k\times\left(\frac{10}{100}\right)^2 \Rightarrow 5\times 2 = k\times\frac{1}{100} \Rightarrow k = 1000\ N/m$

Total Energy in SHM and Graphs

Total energy in S.H.M. is given by :-

E = potential energy + kinetic energy = U + K.E.

(i) w.r.t. position : $E = \frac{1}{2}kx^2 + \frac{1}{2}k(A^2 - x^2)$ $\left[U = \frac{1}{2}kx^2, K.E. = \frac{1}{2}k\left(A^2 - x^2\right)\right]$

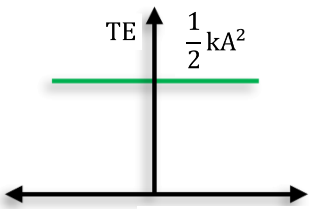
$\Rightarrow E = \frac{1}{2}kA^2 = \text{constant}$

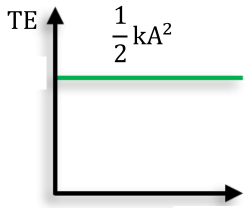
(ii) w.r.t. time

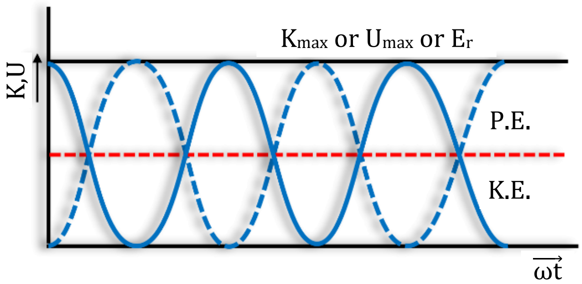
$E = \frac{1}{2}m\omega^2A^2\sin^2\omega t + \frac{1}{2}m\omega^2A^2\cos^2\omega t$

$= \frac{1}{2}m\omega^2A^2\left(\sin^2\omega t + \cos^2\omega t\right)$

$= \frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2 = \text{constant}$







Note: (i) Total energy of a particle in S.H.M. is same at all instant and at all displacement.

(ii) Total energy depends upon mass, amplitude and frequency of vibration of the particle executing S.H.M.

Average energy in S.H.M.

The time average of P.E. and K.E. over one cycle is

(a) $<KE>_t = <\frac{1}{2}m\omega^2A^2\cos^2\omega t> = \frac{1}{2}m\omega^2A^2 <\cos^2\omega t> = \frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right) = \frac{1}{4}m\omega^2A^2 = \frac{1}{4}kA^2$

(b) $<PE>_t = <\frac{1}{2}m\omega^2A^2\sin^2\omega t> = \frac{1}{2}m\omega^2A^2 <\sin^2\omega t> = \frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right) = \frac{1}{4}m\omega^2A^2 = \frac{1}{4}kA^2 + U_0$

(c) $<TE>_t = <\frac{1}{2}m\omega^2A^2 + U_0> = \frac{1}{2}m\omega^2A^2 + U_0 = \frac{1}{2}kA^2 + U_0$

Illustration 24

In case of simple harmonic motion –

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude.

(b) At what displacement the kinetic and potential energies are equal.

Solution:

In S.H.M. $\Rightarrow$ K. E. $= \frac{1}{2}k(A^2 - x^2)$, P. E. $= \frac{1}{2}kx^2$, T.E. $= \frac{1}{2}kA^2$

(1) $f_{K.E.} = \frac{K.E.}{T.E.} = \frac{A^2 - x^2}{A^2}$ and $f_{P.E.} = \frac{P.E.}{T.E} = \frac{x^2}{A^2}$

at $x = \frac{A}{2}$ $f_{K.E.} = \frac{A^2 - A^2/4}{A^2} = \frac{3}{4}$ and $f_{P.E.} = \frac{A^2/4}{A^2} = \frac{1}{4}$

(2) K. E. = P. E. $\Rightarrow \frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kx^2 \Rightarrow 2x^2 = A^2 \Rightarrow x = \pm\frac{A}{\sqrt{2}}$

Illustration 25

The potential energy of a particle executing S.H.M. is 2.5 J, when its displacement is half of the amplitude, then determine total energy of particle.

Solution:

P.E. = $\frac{1}{2}kx^2 \Rightarrow \frac{1}{2}k\frac{A^2}{4} = 2.5 \Rightarrow$ Total energy $= \frac{1}{2}kA^2 = 2.5 \times 4 = 10$ J

Illustration 26

The potential energy of a particle oscillating on x-axis is $U = 20 + (x - 2)^2$. Here U is in joules and x in meters. Total mechanical energy of the particle is 36 J.

(1) State whether the motion of the particle is simple harmonic or not.

(2) Find the mean position.

(3) Find the maximum kinetic energy of the particle.

Solution:

(1) $F = -\frac{dU}{dx} = -2(x - 2)$ By assuming $x - 2 = X$, we have $F = -2X$

Since, $F \propto -X$ The motion of the particle is simple harmonic

(2) The mean position of the particle is $X = 0 \Rightarrow x - 2 = 0$, which gives $x = 2$m

(3) Maximum kinetic energy of the particle is, $K_{max} = E - U_{min} = 36 - 20 = 16$ J

BEGINNER'S BOX-3

1. A point particle of mass 0.1 Kg is executing SHM with amplitude of 0.1m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} Joule. Obtain the equation of motion of this particle if the initial phase of oscillation is 45°.

POS181

2. In case of SHM what fraction of total energy is kinetic and what fraction is potential, when displacement is one fourth of amplitude.

POS182

3. A particle execute S.H.M. with frequency f. Find frequency with which its kinetic energy oscillates?

POS183

4. A particle of mass 10g is placed in potential field given by $V = (50x^2 + 100)$ erg/g. What will be frequency of oscillation of particle?

POS184

Oscillations of Spring Block System

Spring Block System

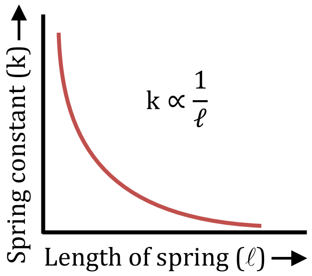
(i) When spring is given small displacement by stretching or compressing it, then restoring elastic force is developed in it which obeys Hook's law.

$F \propto -x \quad \Rightarrow F = -kx \quad$ (Here k is spring constant)

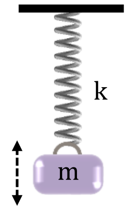
(ii) Spring is assumed massless, so restoring elastic force in spring is assumed same everywhere.

(iii) Spring constant (k) depends on length (ℓ), radius and material of wire used in spring.

For spring, $k\ell$ = constant



Spring Pendulum



(i) When a small mass is suspended from a mass-less spring then this arrangement is known as spring pendulum. For small linear displacement the motion of spring pendulum is simple harmonic.

(ii) For a spring pendulum

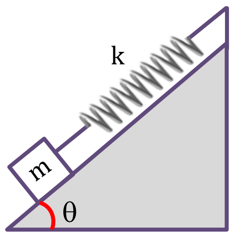
$F = -kx \Rightarrow m\dfrac{d^2x}{dt^2} = -kx\ [\because F = ma = m\dfrac{d^2x}{dt^2}]$

$\Rightarrow \dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$

$\because \dfrac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega^2 = \dfrac{k}{m}$

This is standard equation of linear S.H.M.

Time period; $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}}$. Frequency $n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$

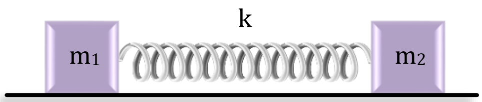


(iii) Time period of a spring pendulum is independent of acceleration due to gravity. This is why a clock based on oscillation of spring pendulum will keep proper time everywhere on a hill or moon or in a satellite or different places of earth.

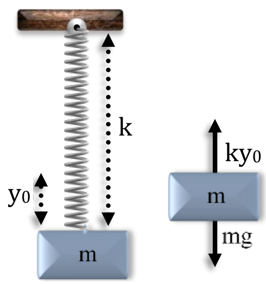
(iv) If a spring pendulum oscillates in a vertical plane is made to oscillate on a horizontal surface or on an inclined plane then time period will remain unchanged.

(v) By increasing the mass, time period of spring pendulum increases $(T \propto \sqrt{m})$, but by increasing the force constant of spring (k), Its time period decreases $\left[T \propto \dfrac{1}{\sqrt{k}}\right]$ whereas frequency increases $(n \propto \sqrt{k})$

(vi) If two masses m_1 and m_2 are connected by a spring and made to oscillate then time period $T = 2\pi\sqrt{\dfrac{\mu}{k}}$



Here, $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ = reduced mass

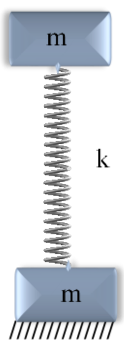


(vii) If the stretch in a vertically loaded spring is y_0 then for equilibrium of mass m.

$ky_0 = mg$

So, time period $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{y_0}{g}}$

But remember time period of spring pendulum is independent of acceleration due to gravity.



(viii) If two particles are attached with spring in which only one is oscillating then the

Time period $= 2\pi\sqrt{\dfrac{\text{mass of oscillating particle}}{\text{force constant}}} = 2\pi\sqrt{\dfrac{m_1}{k}}$

Illustration 27

A body of mass m attached to a spring which is oscillating with time period 4 seconds. If the mass of the body is increased by 4 kg, its timer period increases by 2 sec. Determine value of initial mass m.

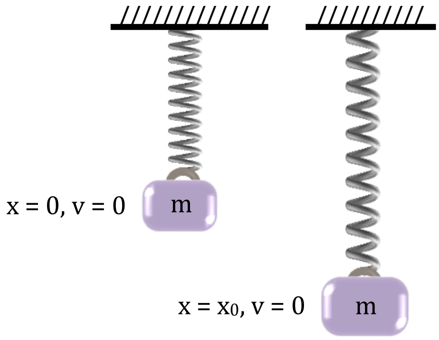
Solution:

In Ist case : $T=2\pi\sqrt{\frac{m}{k}} \Rightarrow 4=2\pi\sqrt{\frac{m}{k}}$ ...(i) and in IInd case: $6=2\pi\sqrt{\frac{m+4}{k}}$ ...(ii)

Divide (i) by (ii) $\frac{4}{6}=\sqrt{\frac{m}{m+4}} \Rightarrow \frac{16}{36}=\frac{m}{m+4} \Rightarrow m = 3.2\text{ kg}$

Illustration 28

A block of mass m is attached from a spring of spring constant k and dropped from its natural length. Find the amplitude of S.H.M.



Solution:

Let amplitude of S.H.M. be A then by work energy theorem

W = ΔK.E.

$mgx_0-\frac{1}{2}kx_0^2=0 \Rightarrow x_0=\frac{2mg}{k}$

So, amplitude $A=\frac{mg}{k}$

Illustration 29

Two springs with spring constants K_1 = 1500 N/m and K_2 = 3000 N/m are stretched by the same force. The ratio of potential energy stored in spring will be

Solution:

$U=\frac{F^2}{2K} \Rightarrow U \propto \frac{1}{K} \Rightarrow \frac{U_1}{U_2}=\frac{K_2}{K_1}=2$

Illustration 30

The time period of a mass suspended from a spring is T. If the spring is cut into four equal parts and the same mass is suspended from one of the parts, then the new time period will be

Solution:

By cutting spring in four equal parts force constant (K) of each parts becomes four times $\left(\because k \propto \frac{1}{\ell}\right)$ so by using $T=2\pi\sqrt{\frac{m}{k}}$; time period will be half i.e. $T'=T/2$

Illustration 31

If a body of mass 0.98 kg is made to oscillate on a spring of force constant 4.84 N/m, the angular frequency of the body is

Solution:

$\omega=\sqrt{k/m}=\sqrt{\frac{4.84}{0.98}}=2.22\text{ rad/sec}$

Illustration 32

A spring executes SHM with mass of 10kg attached to it. The force constant of spring is 10N/m. If at any instant its velocity is 40cm/sec, the displacement will be (where amplitude is 0.5m)

Solution:

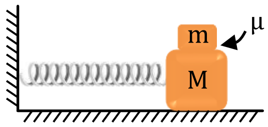
$\omega=\sqrt{\left(\frac{k}{m}\right)}=\sqrt{\left(\frac{10}{10}\right)}=1$ rad Now, $u=\omega\sqrt{A^2-y^2}$ by solving, we get $y=0.3\,\text{m}$

Illustration 33

A block of mass m is on a horizontal slab of mass M which is moving horizontally and executing S.H.M. The coefficient of static friction between block and slab is μ. If block is not separated from slab then determine angular frequency of oscillation.

Solution:

If block is not separated from slab then restoring force due to S.H.M. should be less than frictional force between slab and block.

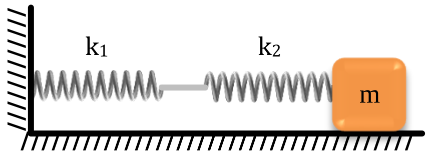


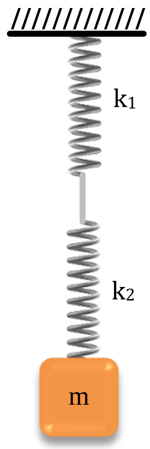
$$F_{restoring} \le F_{friction} \Rightarrow m\, a_{max.} \le \mu mg \Rightarrow a_{max.} \le \mu g \Rightarrow \omega^2 A \le \mu g \Rightarrow \omega \le \sqrt{\frac{\mu g}{A}}$$

Series and Parallel Combinations of Springs

Series combination of springs

In series combination same restoring force exerts in all springs but extension will be different.





Total displacement $x = x_1 + x_2$

Force acting on both springs $F = -k_1x_1 = -k_2x_2$

$$\because x_1 = -\frac{F}{k_1} \text{ and } x_2 = -\frac{F}{k_2}$$

$$\therefore \quad x = -\left[\frac{F}{k_1} + \frac{F}{k_2}\right] \qquad \text{...(i)}$$

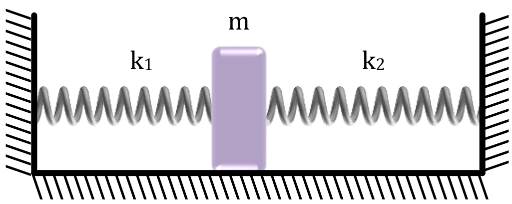
If equivalent force constant is k_s then $F = -k_sx$

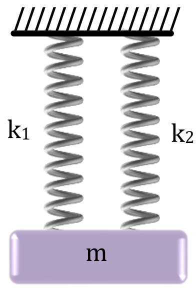
so by equation (i) $-\frac{F}{k_s} = -\frac{F}{k_1} - \frac{F}{k_2}$

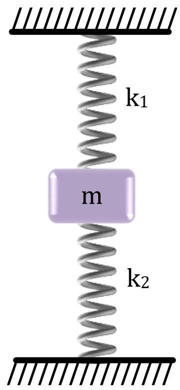
$$\Rightarrow \frac{\mathbf{1}}{\mathbf{k_s}} = \frac{\mathbf{1}}{\mathbf{k_1}} + \frac{\mathbf{1}}{\mathbf{k_2}} \Rightarrow k_s = \frac{k_1 k_2}{k_1 + k_2}$$

Time period $T = 2\pi\sqrt{\frac{m}{k_s}}$ Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k_s}{m}}$, Angular frequency $\omega = \sqrt{\frac{k_s}{m}}$

Parallel Combination of springs







In parallel combination displacement on each spring is same but restoring force is different.

Force acting on the system $F = F_1 + F_2 \Rightarrow F = -k_1x - k_2x$ ...(i)

If equivalent force constant is k_P then, $F = -k_Px$, so by equation (i) $-k_Px = -k_1x - k_2x \Rightarrow \mathbf{k_P = k_1 + k_2}$

Time period $T = 2\pi\sqrt{\frac{m}{k_P}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$; Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k_P}{m}}$; Angular frequency $\omega = \sqrt{\frac{k_1 + k_2}{m}}$

Illustration 34
Infinite springs with force constants k, 2k, 4k, 8k, ... respectively are connected in series. Calculate the effective force constant of the spring.

Solution:

$$\frac{1}{k_{eff}}=\frac{1}{k}+\frac{1}{2k}+\frac{1}{4k}+\frac{1}{8k}+.............\infty \quad \text{(For infinite G.P. } S_\infty=\frac{a}{1-r} \text{ where a = First term, r = common ratio)}$$

$$\frac{1}{k_{eff}}=\frac{1}{k}\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+..........\right]=\frac{1}{k}\left[\frac{1}{1-\frac{1}{2}}\right]=\frac{2}{k} \qquad \text{so } k_{eff}=\frac{k}{2}$$

Illustration 35
Frequency of oscillation of a body is 6 Hz when force F_1 is applied and 8 Hz when F_2 is applied. If both forces F_1 & F_2 are applied together then find out the frequency of oscillation.

Solution:

According to question $F_1 = -K_1x$ & $F_2 = -K_2x$

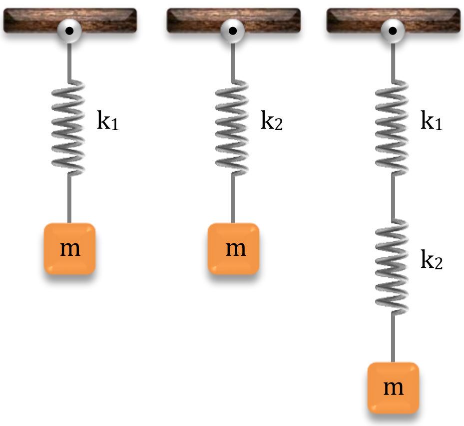
So $\quad n_1=\frac{1}{2\pi}\sqrt{\frac{K_1}{m}}=6\text{Hz}\ ;\ n_2=\frac{1}{2\pi}\sqrt{\frac{K_2}{m}}=8\text{ Hz}$

Now $\quad F = F_1 + F_2 = -(K_1+K_2)x$ Therefore $n=\frac{1}{2\pi}\sqrt{\frac{K_1+K_2}{m}}$

$$\Rightarrow \quad n=\frac{1}{2\pi}\sqrt{\frac{4\pi^2n_1^2m+4\pi^2n_2^2m}{m}}=\sqrt{n_1^2+n_2^2}=\sqrt{8^2+6^2}=10\text{ Hz}$$

Illustration 36
Periodic time of oscillation T_1 is obtained when a mass is suspended from a spring. If another spring is used with same mass then periodic time of oscillation is T_2. Now if this mass is suspended from series combination of above springs then calculate the time period.

Solution:



$$T_1=2\pi\sqrt{\frac{m}{k_1}} \Rightarrow T_1^2=4\pi^2\frac{m}{k_1} \Rightarrow k_1=\frac{4\pi^2m}{T_1^2} \text{ and}$$

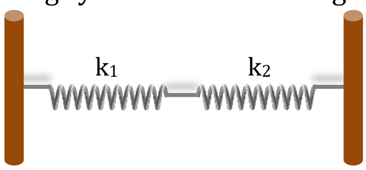
$$T_2=2\pi\sqrt{\frac{m}{k_2}} \Rightarrow T_2^2=4\pi^2\frac{m}{k_2} \Rightarrow k_2=\frac{4\pi^2m}{T_2^2}$$

$$K_{eq.}=\frac{4\pi^2m}{T_{eq.}^2}$$

In series combination –

$$\frac{1}{K_{eq.}}=\frac{1}{K_1}+\frac{1}{K_2} \Rightarrow \frac{T_{eq.}^2}{4\pi^2m}=\frac{T_1^2}{4\pi^2m}+\frac{T_2^2}{4\pi^2m} \Rightarrow T_{eq.}=\sqrt{T_1^2+T_2^2}$$

Illustration 37
The effective spring constant of two spring system as shown in figure will be



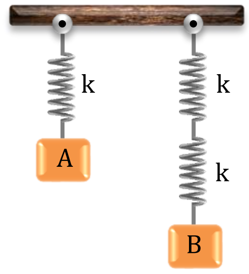
Solution:

When external force is applied, one spring gets extended and another one gets contracted by the same distance hence force due to two springs act in same direction.

i.e. $F = F_1 + F_2 \Rightarrow -kx = -k_1x - k_2x \Rightarrow k = k_1 + k_2$

Illustration 38

The springs shown are identical. When A = 4kg, the elongation of spring is 1 cm. If B = 6 kg, the elongation produced by it is



Solution:

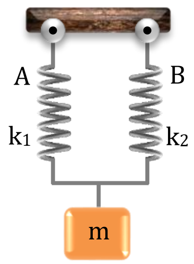
$F = kx \Rightarrow mg = kx \Rightarrow m \propto kx$

Hence $\dfrac{m_1}{m_2} = \dfrac{k_1}{k_2} \times \dfrac{x_1}{x_2} \Rightarrow \dfrac{4}{6} = \dfrac{k}{k/2} \times \dfrac{1}{x_2}$

$\Rightarrow x_2 = 3\text{cm}.$

Illustration 39

A mass m is suspended by means of two coiled spring which have the same length in unstretched condition as in figure. Their force constant are k_1 and k_2 respectively. When set into vertical vibrations, the period will be



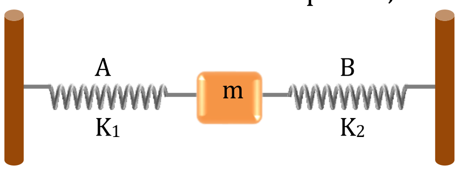
Solution:

Given spring system has parallel combination, so

$k_{eq} = k_1 + k_2$ and time period $T = 2\pi\sqrt{\dfrac{m}{(k_1 + k_2)}}$

Illustration 40

In arrangement given in figure, if the block of mass m is displaced, the frequency is given by



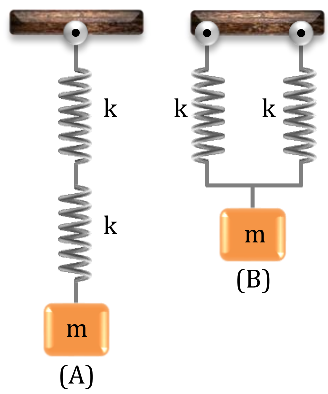
Solution:

With respect to the block the springs are connected in parallel combination.

$\therefore$ Combined stiffness $k = k_1 + k_2$ and $n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1 + k_2}{m}}$

Illustration 41

Two identical spring of constant K are connected in series and parallel as shown in figure. A mass m is suspended from them. The ratio of their frequencies of vertical oscillations will be



Solution:

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow \frac{n_S}{n_P} = \sqrt{\frac{k_S}{k_P}} \Rightarrow \frac{n_s}{n_p} = \sqrt{\frac{\left(\frac{k}{2}\right)}{2k}} = \frac{1}{2}$$

Illustration 42

One body is suspended from a spring of length *l*, spring constant k and has time period T. Now if spring is divided in two equal parts which are joined in parallel and the same body is suspended from this arrangement then determine new time period.

Solution:

Spring constant in parallel combination k' = 2k + 2k = 4k

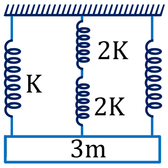
$$\therefore \quad T' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{4k}} = 2\pi\sqrt{\frac{m}{k}} \times \frac{1}{\sqrt{4}} = \frac{T}{\sqrt{4}} = \frac{T}{2}$$

BEGINNER'S BOX-4

1. A man with a wrist watch on his hand falls from the top of a tower. Does the watch give correct time during the free fall?

POS185

2. Calculate the time period of following system.



POS186

3. A body of mass 'm' is suspended from a vertical spring of length 'L'. It execute S.H.M. with a time period T. Find the time period if

(a) Length of spring is made half. (b) Mass of body is made half.

POS187

4. When a mass of 1 kg is suspended from a vertical spring, its length increases by 0.98 m. If this mass is pulled downwards and then released, what will be periodic time of vibration of spring? ($g = 9.8\ m/s^2$).

POS188

5. A block of mass 'm' moving with velocity (v) collides perfectly inelastically with another identical block attached to spring of force constant K. What will be the amplitude of resulting SHM?

POS189

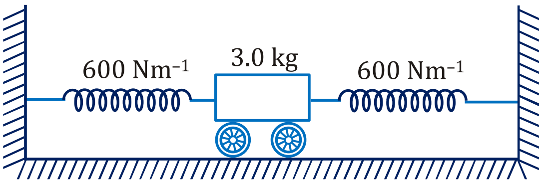
6. A spring having a spring constant 1200 Nm^{-1} is mounted on a horizontal table. A mass of 3kg is attached to the free end of the spring. The mass is then pulled to a distance of 2.0 cm and released. Determine Maximum acceleration of mass.

POS190

7. A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?

POS191

8. A trolley of mass 3.0 kg, as shown in Fig., is connected to two springs, each of spring has spring constant 600 Nm^{-1}. If the trolley is displaced from its equilibrium position by 5.0 cm and released, what is (a) the period of ensuing oscillations, and (b) the maximum speed of the trolley? How much energy is dissipated as heat by the time the trolley comes to rest due to damping forces?

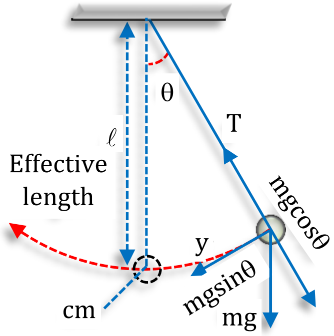


POS192

Simple Pendulum

If a heavy point mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum

Expression for time period



Method-I (Force Method)

For small angular displacement, $\sin\theta \approx \theta$, so that

$F = -mg\sin\theta = -mg\theta$

$$= -\left(\frac{mg}{\ell}\right)y = -ky \qquad \Rightarrow \qquad k = \frac{mg}{\ell}$$

because $y = \ell\theta$, Thus, the time period of the simple pendulum is

$$t = 2\pi\sqrt{\frac{m}{k}} \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{\ell}{g}}$$

Method-II (Torque Method)

Bob of pendulum moves along the arc of circle in vertical plane. Here motion involved is angular and oscillatory where restoring torque is provided by gravitational force.

$\tau = -(mg)(\ell\sin\theta)$ (Negative sign shows opposite direction of τ and angular displacement)

$\tau = -mg\ell(\theta)$ (If angular displacement is small, then $\sin\theta \simeq \theta$)

$I\alpha = -mg\ell(\theta)$

$\alpha = \frac{mg\ell}{m\ell}(\theta)$ (where I = moment of inertia) $\Rightarrow I = m\ell^2$

Now $\alpha = -\frac{mg\ell}{m\ell^2}\theta \Rightarrow \alpha = -\frac{g}{\ell}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$

It is differential equation of angular SHM of simple pendulum.

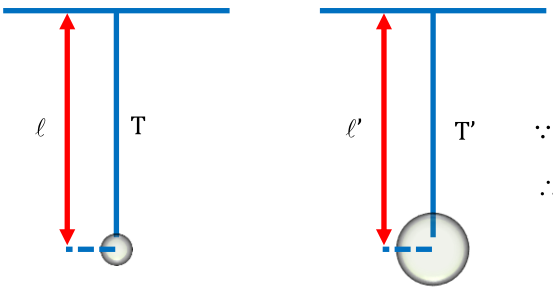
Comparing with standard differential equation of angular SHM $\left(\frac{d^2\theta}{dt^2} + \omega^2\theta = 0\right)$

Therefore $\quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$

Note: Simple pendulum is the example of SHM but only when its angular displacement is very small.

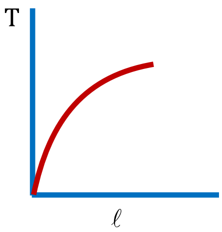
Golden Key Points

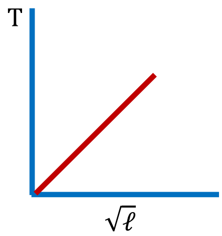
1. The time period of simple pendulum is independent from mass of the bob but it depends on size of bob (position of centre of mass). So in simple pendulum when a solid iron bob is replaced by light aluminium bob of same radius then time period remains unchanged.
2. Time period of simple pendulum is directly proportional to square root of length.



$\because \ell' > \ell$

$\therefore T' > T$





When a person sitting on an oscillating swing comes in standing position then centre of mass raises upwards and length decreases, so time period decreases and frequency increases means swing oscillates faster.

3. When a hollow spherical bob of simple pendulum is completely filled with water and a small hole is made in bottom of it, then as water drain out, at first its time period increases, after that it decrease and when sphere becomes empty then finally it becomes as before (T).

4. If simple pendulum is shifted to poles, equator or hilly areas, then its time period may be different $\left(T \propto \frac{1}{\sqrt{g}}\right)$

5. If a clock based on oscillation of simple pendulum is shifted from earth to moon then it becomes slow because its time period increases and becomes $\sqrt{6}$ times compare to earth. $\frac{g_M}{g_E} = \frac{1}{6} \Rightarrow T_M = \sqrt{6}T_E$

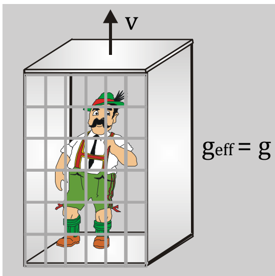
Cases of Effective Acceleration in Simple Pendulum

Periodic time of simple pendulum in reference system

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$$

where, $g_{eff.}$ = effective gravity acceleration in reference system or total downward acceleration.

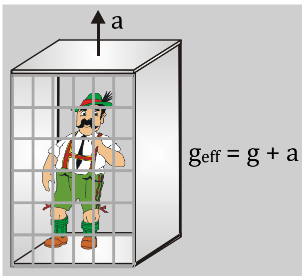
(a) If reference system is lift



(i) If velocity of lift (v) = constant

acceleration a = 0 and $g_{eff.} = g$

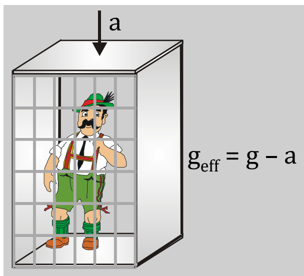
$$\therefore \quad T = 2\pi\sqrt{\frac{\ell}{g}}$$



(ii) If lift is moving upwards with acceleration 'a'

$g_{eff.} = g + a$

$$T = 2\pi\sqrt{\frac{\ell}{g+a}} \quad \Rightarrow \quad \text{T decreases.}$$



(iii) If lift is moving downwards with acceleration 'a'

$g_{eff.} = g - a$

$$\therefore \quad T = 2\pi\sqrt{\frac{\ell}{g-a}} \quad \Rightarrow \quad \text{T increases.}$$

(iv) If lift falls downwards freely

$g_{eff.} = g - g = 0 \quad \Rightarrow \quad T = \infty$

simple pendulum will not oscillate.

Note: If simple pendulum is shifted to the centre of earth, freely falling lift, in artificial satellite then it will not oscillate and its time period is infinite ($\because$ $g_{eff.} = 0$).

(b) A simple pendulum is mounted on a moving truck

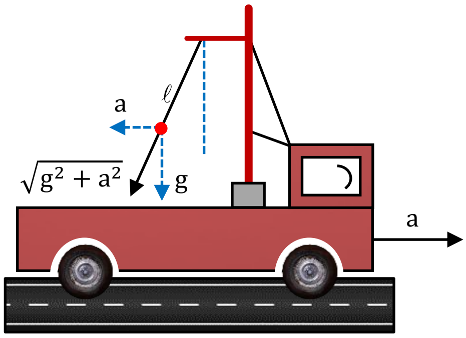
(i) If truck is moving with constant velocity, no pseudo force acts on the pendulum and time period remains same $T = 2\pi\sqrt{\frac{\ell}{g}}$

(ii) If truck accelerates forward with acceleration a then a pseudo force acts in opposite direction.

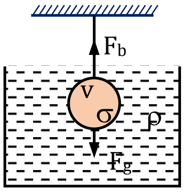
So effective acceleration,

$g_{eff.} = \sqrt{g^2 + a^2}$ and $T' = 2\pi\sqrt{\frac{\ell}{g_{eff.}}}$

Time period $T' = 2\pi\sqrt{\frac{\ell}{\sqrt{g^2+a^2}}} \Rightarrow$ T' decreases.



(c) If a simple pendulum of density σ is made to oscillate in a liquid of density ρ then its time period will increase as compare to that of air and is given by

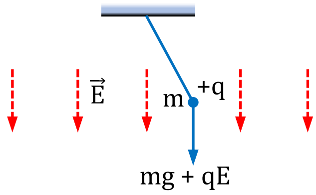


$F_{net} = F_g - F_b$

$$\frac{mg_{net}}{m} = \frac{mg}{m} - \frac{V\rho g}{m},\ g_{net} = g - \frac{V\rho g}{V\sigma} = g\left(1 - \frac{\rho}{\sigma}\right),\ T = 2\pi\sqrt{\frac{\ell}{\left[1-\frac{\rho}{\sigma}\right]g}}$$

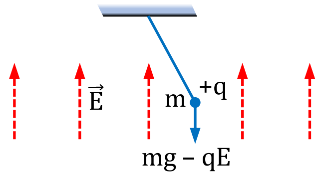
(d) If the bob of simple pendulum has positive charge q and pendulum is placed in uniform electric field which is in downward direction then time period decreases

$$T = 2\pi\sqrt{\frac{\ell}{g + \frac{qE}{m}}}$$



(e) If the bob of simple pendulum has positive charge q and is made oscillate in uniform electric field acting in upward direction then time period increases

$$T = 2\pi\sqrt{\frac{\ell}{g - \frac{qE}{m}}}$$



(f) $T = 2\pi\sqrt{\frac{\ell}{g}}$ is valid when length of simple pendulum (ℓ) is negligible as compare to radius of earth (ℓ<< R) but if ℓ is comparable to radius of earth

then time period $T = 2\pi\sqrt{\frac{R}{\left[1+\frac{R}{\ell}\right]g}} = 2\pi\sqrt{\frac{\ell}{g\left[1+\frac{\ell}{R}\right]}} = 2\pi\sqrt{\frac{1}{\left[\frac{1}{\ell}+\frac{1}{R}\right]g}}$

The time period of oscillation of simple pendulum of infinite length.

$T = 2\pi\sqrt{\frac{R}{g}} \simeq 84.6$ minute ≈ 1.5 hour it is maximum time period.

(g) Second's pendulum

If the time period of a simple pendulum is 2 second then it is called second's pendulum. Second's pendulum takes one second to go from one extreme position to other extreme position.

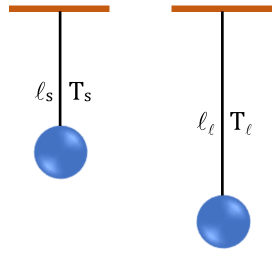
For second's pendulum, time period $T = 2 = 2\pi\sqrt{\frac{\ell}{g}}$

At the surface of earth $g = 9.8\ m/s^2 \approx \pi^2\ m/s^2$,

So, length of second pendulum at the surface of earth $\ell \approx 1$ metre

(h) When a long and short pendulum start oscillation simultaneously then both will be in same phase in minimum time when short pendulum complete one more oscillation compare to long pendulum.

After starting, if long pendulum completes N oscillation to come in same phase in minimum time then short will complete (N +1) oscillation.



$$t = NT_\ell = (N+1)T_s$$

$$N\left(2\pi\sqrt{\frac{\ell_\ell}{g}}\right) = (N+1)\left(2\pi\sqrt{\frac{\ell_s}{g}}\right)$$

$$\boxed{N\sqrt{\ell_\ell} = (N+1)\sqrt{\ell_s}}$$

Illustration 43

The period of a simple pendulum measured inside a stationary lift is found to be T. If the lift starts accelerating upwards with acceleration of g/3 then the time period of the pendulum is ?

Solution:

For stationary lift $T_1 = 2\pi\sqrt{\frac{l}{g}}$

For ascending lift with acceleration, a, $T_2 = 2\pi\sqrt{\frac{l}{g+a}} \Rightarrow \frac{T_1}{T_2} = \sqrt{\frac{g+a}{g}} \Rightarrow \sqrt{\frac{g+\frac{g}{3}}{g}} = \sqrt{\frac{4}{3}} \Rightarrow T_2 = \frac{\sqrt{3}}{2}T$

Illustration 44

A simple pendulum is vibrating in an evacuated chamber, it will oscillate with

(1) Increasing amplitude (2) Constant amplitude

(3) Decreasing amplitude (4) First (3) then (1)

Solution: (2)

As it is clear that in vacuum, the bob will not experience any frictional force. Hence, there shall be no dissipation therefore, it will oscillate with constant amplitude.

Illustration 45

The length of a second's pendulum is ?

Solution:

$$T = 2\pi\sqrt{l/g} \Rightarrow l = \frac{gT^2}{4\pi^2} = \frac{9.8\times 4}{4\times\pi^2} = 99\,\text{cm}$$

Illustration 46

The time period of a simple pendulum when it is made to oscillate on the surface of moon

(1) Increases (2) Decreases (3) Remains unchanged (4) Becomes infinite

Solution: (1)

At the surface of moon, g decreases hence time period increases $\left(T \propto \frac{1}{\sqrt{g}}\right)$

Illustration 47

If the length of the simple pendulum is increased by 44%, then what is the change in time period of pendulum

Solution:

$T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow \frac{T_2}{T_1} = \sqrt{\frac{l_2}{l_1}} = \sqrt{\frac{144}{100}} = \frac{12}{10} \Rightarrow T_2 = 1.2\ T_1$. Hence % increase $= \frac{T_2 - T_1}{T_1}\times 100 = 20\%$

Illustration 48

The ratio of frequencies of two pendulums are 2 : 3, then their length are in ratio ?

Solution:

Frequency $n \propto \frac{1}{\sqrt{l}} \Rightarrow \frac{n_1}{n_2} = \sqrt{\frac{l_2}{l_1}} \Rightarrow \frac{l_1}{l_2} = \frac{n_2^2}{n_1^2} = \frac{3^2}{2^2} = \frac{9}{4}$

Illustration 49

A simple pendulum hanging from the ceiling of a stationary lift has a time period T_1. When the lift moves downward with constant velocity, the time period is T_2, then

(1) T_2 is infinity (2) $T_2 > T_1$ (3) $T_2 < T_1$ (4) $T_2 = T$

Solution: (4)

$T \propto \frac{1}{\sqrt{g}}$ and g is same in both cases so time period remain same.

Illustration 50

If the length of a pendulum is made 9 times and mass of the bob is made 4 times then the value of time period becomes

Solution:

$T = 2\pi\sqrt{\frac{\ell}{g}} \Rightarrow T \propto \sqrt{\ell}$, hence if ℓ made 9 times T becomes 3 times.

Also, time period of simple pendulum does not depend on the mass of the bob.

Illustration 51

A simple pendulum is taken from the equator to the pole. Its period

(1) Decreases (2) Increases (3) Unchanged (4) Decreases and then increases

Solution: (1)

As we go from equator to pole the value of g increases. Therefore, time period of simple pendulum $\left(T \propto \frac{1}{\sqrt{g}}\right)$ decreases. $\left(\because T \propto \frac{1}{\sqrt{g}}\right)$

BEGINNER'S BOX-5

1. The angle made by the string of a simple pendulum with the vertical depends upon time as $\theta = \frac{\pi}{90}$ sin πt. Find the length of the pendulum if $g = \pi^2$ ms^{-1}

POS193

2. The bob of a simple pendulum is a hollow sphere filled with water. Explain how will the period of oscillation change if the water begins to drain out of the hollow sphere through a small hole in the bottom?

POS194

3. A girl is swinging on the swing in the sitting position. What shall be effect on the frequency of oscillation if she stands up ?

POS195

4. Why a pendulum clock does not work during free fall or in an artificial satellite?

POS196

5. Find the time period and frequency of a simple pendulum of 1·000 m at a location where g = 9·800 m/s^2.

POS197

6. What is the length of a simple pendulum whose time period of oscillation for small amplitudes equals 2·0 seconds

POS198

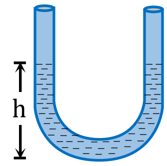
7. What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity?

POS199

Oscillations of General Bodies

Various Formulae of S.H.M.

(1) **S.H.M. of a liquid in U tube:** If a liquid of density ρ contained in a vertical U tube performs S.H.M. in its two limbs. Then time period

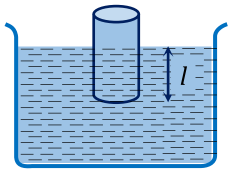


$$T = 2\pi\sqrt{\frac{L}{2g}} = 2\pi\sqrt{\frac{h}{g}}$$

where L = Total length of liquid column,

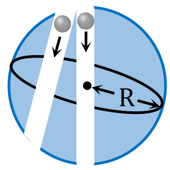
h = Height of undisturbed liquid in each limb (L = 2h)

(2) **S.H.M. of a floating cylinder :** If 'l' is the length of cylinder dipping in liquid



then Time period $T = 2\pi\sqrt{\frac{l}{g}}$

(3) **S.H.M. of a body in a tunnel dug along any chord of earth**



$$T = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ minutes}$$

Superposition of SHM

If particle is acted upon by two separate forces each of which can produce a SHM, the resultant motion of the particle is a combination of two SHM's.

Example: - $x_1 = A_1 \sin\omega t$, $x_2 = A_2 \sin(\omega t + \delta)$

$[x = x_1 + x_2]$

Amplitude of Resultant SHM, $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$

Illustration 52

If two S.H.M. are represented by equations $y_1 = 10\sin\left[3\pi t + \frac{\pi}{4}\right]$ and $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ then find the ratio of their amplitudes and phase difference in between them.

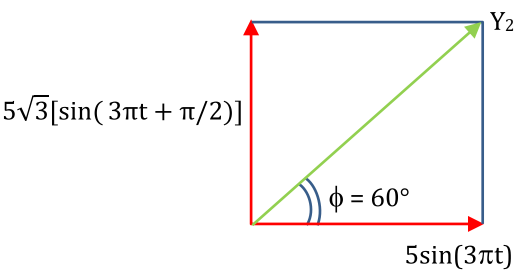
Solution:

As $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ ...(i)

So, if $5 = A\cos\phi$ and $5\sqrt{3} = A\sin\phi$

Then $A = \sqrt{5^2 + (5\sqrt{3})^2} = 10$

and $\tan\phi = \frac{5\sqrt{3}}{5} = \sqrt{3}$ so $\phi = \frac{\pi}{3}$



$y_2 = 10\sin(3\pi t + \phi)$

the above equation (i) becomes

$y_2 = A\cos\phi\sin(3\pi t) + A\sin\phi\cos(3\pi t)$

$\Rightarrow y_2 = A\sin(3\pi t + \phi)$

But $y_2 = 10\sin\left[3\pi t + \left(\frac{\pi}{3}\right)\right]$

So, $\frac{A_1}{A_2} = \frac{10}{10} \Rightarrow A_1 : A_2 = 1 : 1,$

Phase difference $= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$ rad.

BEGINNER'S BOX — ANSWER KEY

BEGINNER'S BOX-1

1. (i) – (a), (ii) – (b), (iii) – (c), (iv) – (c), (v) – (a), (vi) – (b)

2. 0.5 s **3.** (a) 0.67 s, (b) $\frac{\pi}{4}$ rad, (c) 0.04 metre

BEGINNER'S BOX-2

1. 0.025m, y = 0.025 sin (40πt) m **2.** $\frac{\sqrt{3}}{8}$ ms^{-1} **3.** $\sqrt{2}$ cm and $\frac{\pi}{4}$ rad

4. 9.52 cm **5.** (a) $\frac{1}{3}$ s, (b) $\frac{2}{3}$ s **6.** $\frac{5T}{12}$

7. $5\sqrt{3}$ m, 30 m/s

8. (a) When particle moves from extreme to mean position
(b) When particle moves from mean to extreme position

9. (a) 0, +, +; (b) 0, –, –; (c) –, 0, 0; (d) –, –, – ; (e) +, +, +; (f) –, +, +

10. 100 m/min

BEGINNER'S BOX-3

1. $Y = 0.1\sin\left[4t + \frac{\pi}{4}\right]$ **2.** $\frac{15}{16}$ and $\frac{1}{16}$ **3.** 2f **4.** $\frac{5}{\pi}$ s^{-1}

BEGINNER'S BOX-4

1. watch will give correct time because it depends on spring action and does not depend on gravity.

2. $2\pi\sqrt{\frac{m}{K}}$ **3.** (a) $T' = \frac{T}{\sqrt{2}}$, (b) $T' = \frac{T}{\sqrt{2}}$ **4.** 2s **5.** $A = v\sqrt{\frac{m}{2K}}$

6. 8 m/s **7.** 218.7 N **8.** (a) 0.314 s, (b) 1 m/s, 1.5 J

BEGINNER'S BOX-5

1. 1 m

2. The time period will increase at first, then decrease until the sphere is empty acquire its initial value.

3. increase

4. In both cases effective acceleration due to gravity becomes zero. In absence of 'g_{eff}' there is no restoring force and pendulum does not oscillate.

5. 2 s, 0.5 Hz

6. 0·99 m

7. zero