01 Circular Motion

Introduction of Circular Motion

If a particle moves in a plane such that its distance from a fixed (or moving) point remains constant, then the motion is called **circular motion** with respect to that fixed or moving point.

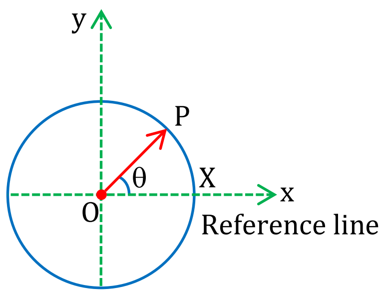
That fixed point is called the centre and the corresponding distance is called the radius of circular path.

The vector joining centre of circle and particle performing circular motion is called radius vector. It has constant magnitude and variable direction.

Angular Position, Displacement, Velocity, Frequency, Time Period

Angular position:

To decide the angular position of a point in space we need to specify (i) origin and (ii) reference line.



The angle made by the position vector w.r.t. origin, with the reference line is called angular position.

Clearly angular position depends on the choice of the origin as well as the reference line.

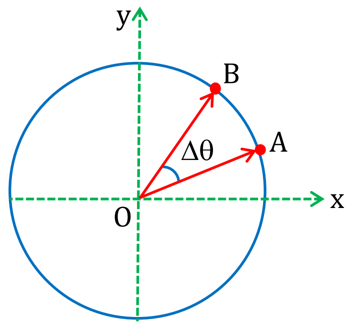
Circular motion is a two dimensional motion or motion in a plane. Suppose a particle P is moving in a circle of radius r and centre O.

The angular position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the angular position of the particle.

Unit : radian (SI)

Angular Displacement:

Angle through which the position vector of the moving particle rotates in a given time interval is called angular displacement.



Angular displacement depends on origin, but it does not depend on the reference line. As the particle moves on above circle, its angular position θ changes. Suppose the point rotates through an angle $\Delta\theta$ in time Δt, then $\Delta\theta$ is angular displacement.

Angular displacement $= \dfrac{\text{Arc}}{\text{Radius}} \Rightarrow \Delta\theta = \dfrac{s}{r}$ where s is arc length A to B

- **Unit** : radian (SI)
- It is a dimensionless quantity.
- 1 radian $= \dfrac{360^\circ}{2\pi} \Rightarrow \pi$ radian $= 180^\circ$

Important Points:

- Clockwise angular displacement is taken as negative and anticlockwise displacement as positive.
- It is an axial vector and direction of angular displacement is perpendicular to the plane along the axis of rotation and is given by right hand thumb rule.
- Small angular displacement is a vector quantity,
  But large angular displacement is not a vector quantity, because it does not follow commutative law of vector addition. $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ but $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$
- In one complete revolution angular displacement is 2π radian.
- If a body makes n revolutions, its angular displacement $\theta = 2n\pi$ radians

Frequency (f):

Number of revolutions described by particle per second is its frequency.

Unit : Revolutions per second (rps) or Revolutions per minute (rpm)

1 rps = 60 rpm

Time Period (T):

It is the time taken by particle to complete one revolution. i.e. $T = \frac{1}{f}$

Unit : hertz (Hz) or s^{-1}

Illustration 1

A particle completes one third revolution in 0.6s. Find

(i) Time period (ii) Frequency (iii) Angular displacement after 1.8s

Solution:

(i) Particle completes one third revolution in 0.6s

$\therefore$ Time taken by the particle to completes one revolution = $\frac{0.6}{1/3} = 1.8\,s$

(ii) $T = \frac{1}{f}$ $\therefore$ $f = \frac{1}{1.8}$ hertz (Hz) or s^{-1}

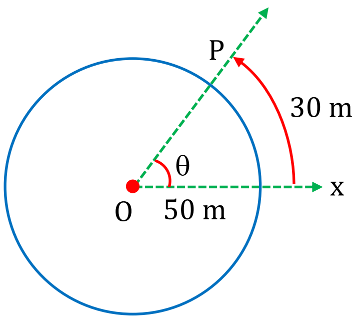
(iii) In 1.8s particle completes one revolution, so angular displacement = 2π radian

Illustration 2

A student runs 30 m around a circular track of diameter 100 m. What is his angular displacement?

Solution:

After running 30 m, the student reaches point P at an angle θ as shown in the figure.



Angular displacement,

$\Delta\theta = \theta_2 - \theta_1$

Here,

$\theta_1 = 0$ and

$\theta_2 = \theta$

Now, $\theta = \frac{\text{Arc}}{\text{Radius}}\text{rad} = \frac{30}{50}\text{rad}$ (Radius = Diameter/2)

$\Rightarrow$ Angular Displacement = $\Delta\theta$ = 0.6 rad (Anti-clockwise)

Illustration 3
A particle completes 1.5 revolutions in a circular path of radius 2cm. The angular displacement of the particle will be - (in radian)

Solution:

We have angular displacement, $\theta = \dfrac{\text{Arc}}{\text{Radius}}$

Here arc length, S = n(2πr) = 1.5 (2π × 2 × 10^{-2}) = 6π × 10^{-2}

$\therefore\ \Delta\theta = \dfrac{6\pi \times 10^{-2}}{2 \times 10^{-2}} = 3\pi$ radian

Relation between Angular Velocity and Linear Velocity

Angular velocity (ω)
It is defined as the rate of change of angular position of a moving particle with respect to time.

- **Unit :** Radian/sec
- Its dimension is $[M^0L^0T^{-1}]$

Average angular velocity (ω_{avg}):

$$\omega_{avg} = \frac{\text{Total angular displacement}}{\text{Total time taken}} = \omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2. Since angular displacement is a scalar, average angular velocity is also a scalar.

- If a body makes 'n' rotations in 't' seconds then average angular velocity in radian per second will be

$$\omega_{avg} = \frac{2\pi n}{t}$$

- If T is the period and 'f' the frequency of uniform circular motion, $\omega_{avg} = \dfrac{2\pi}{T} = 2\pi f$

Illustration 4
A particle revolving in a circular path completes first one third of the circumference in 2 s, while next one third in 1s. Calculate its average angular velocity.

Solution:

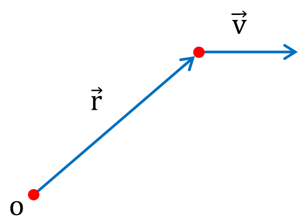
$\theta_1 = \dfrac{2\pi}{3}$ and $\theta_2 = \dfrac{2\pi}{3}$; Total time T = 2 + 1 = 3 s

$$\therefore\ \langle\omega\rangle = \frac{\theta_1 + \theta_2}{T} = \frac{\frac{2\pi}{3} + \frac{2\pi}{3}}{3} = \frac{\frac{4\pi}{3}}{3} = \frac{4\pi}{9}\ \text{rad/s}$$

Instantaneous angular velocity (ω_{ins}):

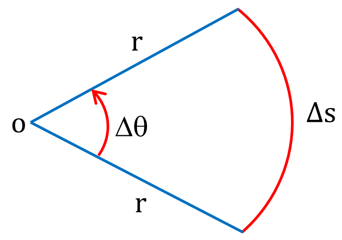
It is angular velocity at any instant of time. $\omega_{inst} = \lim_{\Delta t \to 0} \dfrac{\Delta\theta}{\Delta t} = \dfrac{d\vec{\theta}}{dt}$

Since infinitesimally small angular displacement $\overrightarrow{d\theta}$ is a vector quantity, instantaneous angular velocity $\vec{\omega}$ is also a vector, whose direction is given by right hand thumb rule.



$$\omega_{inst} = \frac{\text{component of velocity perpendicular to line joining}}{\text{length of line joining}} = \frac{V_\perp}{r}$$

Relation between linear and angular velocity



$\Delta\theta = \frac{\Delta s}{r}$

$\Delta s = r\Delta\theta$

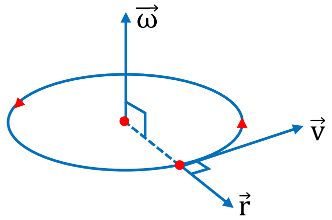
Divide both the side by total time taken

$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$

$\lim \Delta t \to 0$

$\frac{ds}{dt} = r\frac{d\theta}{dt}$

$v = \omega r$



In vector from, $\vec{v} = \vec{\omega} \times \vec{r}$

$\vec{v}$ = Linear Velocity

$\vec{\omega}$ = Angular velocity

$\vec{r}$ = radius vector or position vector

Note: For circular motion, all the three vectors $\vec{v}, \vec{\omega}$ and $\vec{r}$ are mutually perpendicular to each other.

Here, $\vec{v} \perp \vec{\omega} \perp \vec{r}$ $\quad \because \vec{v} \perp \vec{\omega} \quad \therefore \vec{v}.\vec{\omega} = 0$

$\because \vec{\omega} \perp \vec{r} \quad \therefore \vec{\omega}.\vec{r} = 0$

$\because \vec{v} \perp \vec{r} \quad \therefore \vec{v}.\vec{r} = 0$

Illustration 5

A particle is moving with constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\frac{\pi}{2}$ radian.

Solution:

Time taken to describe angle θ, $t = \frac{\theta}{\omega} = \frac{\theta R}{v} = \frac{\pi R}{2v}$

Average velocity = $\frac{\text{Total displacement}}{\text{Total time}} = \frac{\sqrt{2}R}{\pi R / 2v} = \frac{2\sqrt{2}}{\pi}v$

Instantaneous velocity = v

The ratio of average velocity to its instantaneous velocity = $\frac{2\sqrt{2}}{\pi}$

Illustration 6

An insect moves in a circular path of radius $\frac{25}{\pi}$ cm steadily and completes 10 revolutions in 50s. What is the average angular speed and the average linear speed of the motion?

Solution:

Radius of the circle = $\frac{25}{\pi}\text{cm} = \frac{1}{4\pi}\text{m}$

Time taken for 10 revolutions = 50s

So, time taken for 1 revolution = time period = $\frac{50}{10} = 5\text{s}$

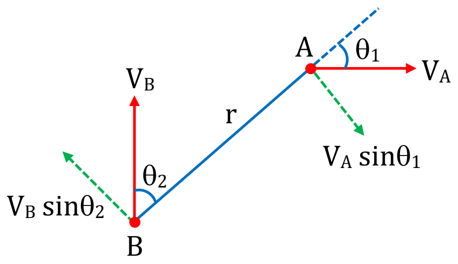
So, in 1 second, $\frac{1}{5} = 0.2$ revolution.

So, angular speed = $\frac{2\pi}{T} = 2\pi f$ = $2\pi \times 0.2 = 0.4\pi$ rad s^{-1}

And linear speed, $v = \omega r = 0.4\pi \times \frac{1}{4\pi} = 0.1 m/s = 10 cm/s$

Relative Angular Velocity

Relative angular velocity of a particle 'A' w.r.t. another moving particle B is the angular velocity of the position vector of A w.r.t. B. It means the rate at which the position vector of 'A' w.r.t. B rotates at that instant.

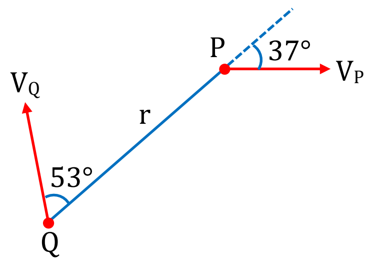


$$\omega = \frac{(V_{AB})_\perp}{r_{AB}} = \frac{\text{relative velocity of A w.r.t. B perpendicular to line AB}}{\text{separation between A and B}}$$

Here $(V_{AB})_\perp = V_A \sin\theta_1 + V_B \sin\theta_2 \therefore \omega_{AB} = \frac{V_A \sin\theta_1 + V_B \sin\theta_2}{r}$

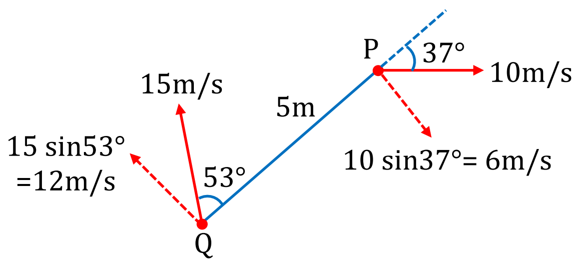
Illustration 7

Two moving particles P and Q are 5 m apart at any instant. Velocity of P is 10 m/s and that of Q is 15m/s. Calculate the angular velocity of P with respect to Q.



Solution:

Angular velocity of P with respect to Q is ; $\omega_{PQ} = \frac{(v_{PQ})_\perp}{r_{PQ}}$



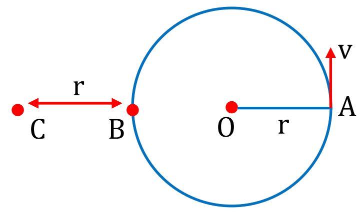
$(v_{PQ})_\perp = 6 - (-12) = 18 m/s$

$r_{PQ} = 5m$

$$\omega_{AB} = \frac{(v_{PQ})_\perp}{r_{PQ}} = \frac{18}{5} \text{radian/sec}$$

Illustration 8

A particle is moving with constant speed in a circle as shown, find the angular velocity of the particle A with respect to fixed point B and C, if angular velocity with respect to O is ω.



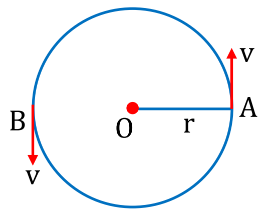
Solution:

Angular velocity of A with respect to O is ; $\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$

$$\therefore \omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{v}{2r} = \frac{\omega}{2} \text{ and } \omega_{AC} = \frac{(v_{AC})_\perp}{r_{AC}} = \frac{v}{3r} = \frac{\omega}{3}$$

Illustration 9

Particles A and B move with constant and equal speeds in a circle as shown. Find the angular velocity of the particle A with respect to B, if angular velocity of particle A w.r.t. O is ω.



Solution:

Angular velocity of A with respect to O is

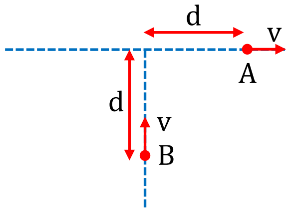
$$\omega_{AO} = \frac{(v_{AO})_{\perp}}{r_{AO}} = \frac{v}{r} = \omega \quad \Rightarrow \text{Now, } \omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}}$$

v_{AB} = 2v, since v_{AB} is perpendicular to r_{AB}

$$\therefore \ (v_{AB})_{\perp} = v_{AB} = 2v \ ; \quad r_{AB} = 2r \Rightarrow \omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}} = \frac{2v}{2r} = \omega$$

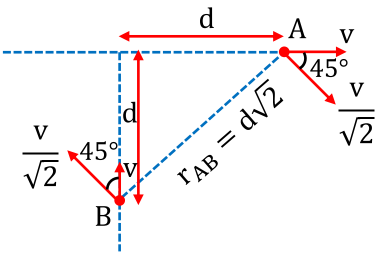
Illustration 10

Find angular velocity of A w.r.t. B at the instant shown in the figure.



Solution:

Angular velocity of A with respect to B is ; $\omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}}$



$(v_{AB})_{\perp} = v_A \cos 45° - (-v_B \cos 45°)$

$$(v_{AB})_{\perp} = \frac{v}{\sqrt{2}} + \frac{v}{\sqrt{2}} = \sqrt{2}v$$

$$r_{AB} = \sqrt{2}d$$

$$\omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}} = \frac{\sqrt{2}v}{\sqrt{2}d} = \frac{v}{d}$$

Angular Acceleration

It is defined as rate of change of angular velocity with respect to time.

Average angular acceleration ($\vec{\alpha}_{avg}$) :

Let ω_i and ω_f be the instantaneous angular velocities at times t_1 and t_2 respectively, then the average angular acceleration is defined as $\vec{\alpha}_{avg} = \frac{\text{Change in angular velocity}}{\text{Total time taken}} = \frac{\vec{\omega}_f - \vec{\omega}_i}{t_2 - t_1} = \frac{\Delta\vec{\omega}}{\Delta t}$

Instantaneous angular acceleration ($\vec{\alpha}_{inst}$) :

It is the limit of average angular acceleration as Δt approaches zero, i.e.,

$$\vec{\alpha}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{dt} = \frac{d\vec{\omega}}{dt}$$

since $\vec{\omega} = \frac{d\vec{\theta}}{dt}$, $\therefore \vec{\alpha} = \frac{d\vec{\omega}}{dt}$ {When ω is a function of t}

$\vec{\alpha} = \frac{d^2\vec{\theta}}{dt^2}$ {When θ is a function of t}

Also $\vec{\alpha} = \omega\frac{d\vec{\omega}}{d\theta}$ {When ω is a function of θ}

Important points:

- **Unit :** rad/s^2 • **Dimensions :** [$M^0L^0T^{-2}$]
- Angular acceleration is an axial vector quantity. It's direction is the direction of change in angular velocity.
- If $\alpha = 0$, then circular motion is said to be uniform.
- If $\alpha \neq 0$, then circular motion is said to be non-uniform.
- If direction of angular velocity and angular acceleration is same then magnitude of angular velocity increases.
- If direction of angular velocity and angular acceleration is opposite then magnitude of angular velocity decreases.

$$\theta \xrightarrow[\text{Differentiate w.r.t. time}]{\frac{d\theta}{dt}} \omega \xrightarrow[\text{Differentiate w.r.t. time}]{\frac{d\omega}{dt}} \alpha$$

$$\alpha \xrightarrow[\text{Integrate w.r.t. time}]{\int \alpha\, dt} \Delta\omega$$

$$\omega \xrightarrow[\text{Integrate w.r.t. time}]{\int \omega\, dt} \Delta\theta$$

Illustration 11

A particle is moving on a circular path of radius 5m and its time dependent angular position is, $\theta = \frac{t^3}{3} - \frac{t^2}{2}$ radian. Find

(i) Angular displacement in first 3s

(ii) Average angular velocity in first 3s

(iii) Angular velocity at t = 2s

(iv) Angular acceleration at t = 2s

Solution:

(i) at t = 0s, $\theta = 0$ at t = 3s $\theta = \frac{(3)^3}{3} - \frac{(3)^2}{2} = 9 - \frac{9}{2} = 4.5$ radian. Angular displacement in first 3s = 4.5 radian. (ii) $\omega_{avg} = \frac{4.5\text{rad}}{3\text{s}} = \frac{3}{2}\text{rad/s}$	(iii) $\omega_{inst} = \frac{d\theta}{dt} = \frac{3t^2}{3} - \frac{2t}{2} = (t^2 - t)$ at t = 2, $\omega_{inst} = (2)^2 - 2 = 2\text{rad/sec}$ (iv) $\alpha_{inst} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = 2t - 1$ at t = 2, $\alpha = 2(2) - 1 = 3$ rad/sec^2

Illustration 12

A disc starts from rest and gains an angular acceleration given by $\alpha = 3t - t^2$ (where t is in seconds) upon the application of a torque. Calculate its (i) angular velocity after 2 sec and (ii) angular displacement after 2 sec.

Solution:

(i) $\alpha = \frac{d\omega}{dt} = 3t - t^2$

$\Rightarrow \int_0^{\omega} d\omega = \int_0^{t} (3t - t^2)dt$

$\Rightarrow \omega = \frac{3t^2}{2} - \frac{t^3}{3}$

$\Rightarrow$ at t = 2 s, $\omega = \frac{10}{3}$ rad/s

(ii) $\omega = \frac{d\theta}{dt}$

$\Rightarrow \int_0^{\theta} d\theta = \int_0^{t} \left(\frac{3t^2}{2} - \frac{t^3}{3}\right)dt$

$\Rightarrow \theta = \left[\frac{t^3}{2} - \frac{t^4}{12}\right]_0^2$

$\Rightarrow \theta = \left[\frac{8}{2} - \frac{16}{12}\right] = \frac{8}{3}$ rad

Equation of circular motion

Translatory / Linear Motion	Rotational Motion
Initial velocity (u)	Initial angular velocity (ω_0)
Final velocity (v)	Final angular velocity (ω)
Displacement (s)	Angular displacement (θ)
Acceleration (a)	Angular Acceleration (α)
If a = constant, then $v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$ $s_{n^{th}} = u + \frac{a}{2}(2n-1)$ $s = \left(\frac{u+v}{2}\right)t$	If α = constant, then $\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha\theta$ $\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$ $\theta = \left(\frac{\omega_0+\omega}{2}\right)t$

Illustration 13

A fan is rotating with angular velocity 100 rev/sec. Then it is switched off. It takes 5 minutes to stop. (a) Find the total number of revolutions made before it stops. (Assume uniform angular retardation) (b) Find the value of angular retardation (c) Find the average angular velocity during this interval.

Solution:

(a) $\theta = \left(\frac{\omega_0+\omega}{2}\right)t = \left(\frac{100+0}{2}\right)\times 5\times 60 = 15000$ revolutions

(b) $\omega = \omega_0 + \alpha t \Rightarrow 0 = 100 - \alpha(5\times 60) \Rightarrow \alpha = -\frac{1}{3}$ rev/sec^2 $\Rightarrow$ Angular retardation = $\frac{1}{3}$ rev/sec^2

(c) $\omega_{avg} = \frac{\text{Total Angle of Rotation}}{\text{Total time taken}} = \frac{15000}{5\times 60} = 50$ rev./sec.

Illustration 14

A particle performs circular motion with initial angular velocity 10 rad/s and constant angular acceleration of 2 rad/s^2. Then find number of rotations in 5th sec.

Solution:

Given, ω_0 = 10 rad/s, α = 2rad/s^2

$$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1) = 10+\frac{2}{2}(2\times5-1) = 19 \text{ rad}$$

$$\text{Number of rotations} = N = \frac{\theta}{2\pi} = \frac{19}{2\pi} \approx 3$$

Illustration 15

A grind stone starts from rest and has a constant-angular acceleration of 4.0 rad/sec^2. The angular displacement and angular velocity, after 4 sec. will respectively be -

Solution:

Angular displacement after 4 sec is,

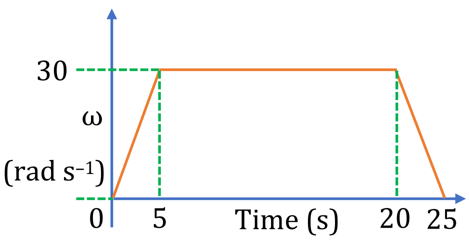
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2 = \frac{1}{2} \times 4 \times 4^2 = 32 \text{ rad}$$

Angular velocity after 4 sec.

$\omega = \omega_0 + \alpha t = 0 + 4 \times 4 = 16$ rad/sec

Illustration 16

The figure shows the angular velocity versus time graph of a flywheel. The angle, in radians through which the flywheel turns during 25s is



Solution:

$$\omega = \frac{d\theta}{dt} \Rightarrow \int d\theta = \int \omega dt \quad \{\text{Area under the } \omega - t \text{ curve gives } \Delta\theta\}$$

$$\Rightarrow \theta = \frac{1}{2}\{15+25\}\times 30 \Rightarrow \theta = 600 \text{ rad}$$

Acceleration in Circular Motion

Tangential and Radial acceleration

Velocity $\vec{v} = \vec{\omega}\times\vec{r}$

Differentiate both the side w.r.t time

$$\frac{d\vec{v}}{dt} = \frac{d(\vec{\omega}\times\vec{r})}{dt} = \frac{d(\vec{\omega})}{dt}\times\vec{r} + \vec{\omega}\times\frac{d(\vec{r})}{dt}$$

$$\vec{a} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v}$$

$\vec{a}_T = \vec{\alpha}\times\vec{r}$ is tangential acceleration ; $\vec{a}_{cp} = \vec{\omega}\times\vec{v}$ is centripetal acceleration

$\vec{a} = \vec{a}_T + \vec{a}_c$ ($\vec{a}_T$ and $\vec{a}_c$ are the two component of net linear acceleration)

$\vec{a}_T \perp \vec{a}_c$

$$|\vec{a}| = \sqrt{a_T^2 + a_c^2}$$

Tangential Acceleration:

Its magnitude is the rate of change of speed of the particle.

$a_T = \frac{d|\vec{v}|}{dt}$ = Rate of change of speed

As $\vec{a}_T$ is along the direction of motion (in the direction of $\vec{v}$ or opposite to $\vec{v}$) so $\vec{a}_T$ is responsible for change in speed of the particle.

- If a particle is moving on a circular path with constant speed then tangential acceleration(a_T) is zero.

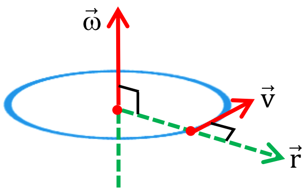
Relation Between Linear and angular Acceleration

In vector form $\vec{a}_T = \vec{\alpha} \times \vec{r}$

Magnitude of tangential acceleration in case of circular motion :

$a_T = \alpha\, r \sin 90° = \alpha r$ ($\vec{\alpha}$ is axial, $\vec{r}$ is radial so that $\vec{\alpha} \perp \vec{r}$)

Centripetal Acceleration:

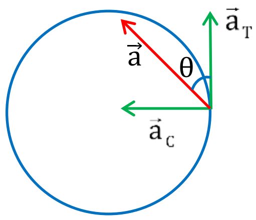


The velocity of the particle changes while moving on the curved path, this change in velocity is brought by a force known as centripetal force and the acceleration so produced in the body is known as centripetal acceleration.

$\vec{a}_C = \vec{\omega} \times \vec{v}$

- Centripetal acceleration is always perpendicular to the velocity at each point, therefore it is responsible for change in direction of velocity.
- In terms of magnitude $a_c = \omega v = \omega^2 r = \frac{v^2}{r}$

Net acceleration:



$\vec{a} = \vec{a}_T + \vec{a}_C$

$$|\vec{a}| = \sqrt{a_T^2 + a_C^{\,2}} = \sqrt{(\alpha r)^2 + \left(\frac{v^2}{r}\right)^2}$$

$\tan\theta = \frac{a_C}{a_T}$ {θ is the angle made by $\vec{a}$ with $\vec{a}_T$}

Important Points:

(i) Differentiation of speed gives tangential acceleration.

(ii) Differentiation of velocity $(\vec{v})$ gives total acceleration.

(iii) $\left|\frac{d\vec{v}}{dt}\right|$ & $\frac{d|\vec{v}|}{dt}$ are not same physical quantity. $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of rate of change of velocity, i.e. magnitude of total acceleration and $\frac{d|\vec{v}|}{dt}$ is a rate of change of speed, i.e. tangential acceleration.

Illustration 17

A particle travels in a circle of radius 20 cm at a speed that uniformly increases. If the speed changes from 5.0 m/s to 6.0 m/s in 2.0 s, find the angular acceleration.

Solution:

Since speed increases uniformly, average tangential acceleration is equal to instantaneous tangential acceleration.

$$a_T = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{6.0 - 5.0}{2.0} \ m/s^2 = 0.5 \ m/s^2$$

$$\alpha = \frac{a_T}{r} = \frac{0.5 m/s^2}{20 cm} = 2.5 rad/s^2$$

Illustration 18

A body of mass 2kg lying on a smooth surface is attached to a string 3m long and then whirled round in a horizontal circle making 60 revolutions per minute. The centripetal acceleration will be -

Solution:

m = 2kg, r = 3m

ω = 60rev./minute = $\frac{60 \times 2\pi}{60}$ rad/sec. = 2π rad/sec.

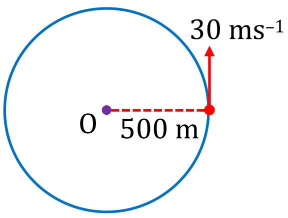
Because the angle described during 1 revolution is 2π radian.

$v = r\omega = 2\pi \times 3$ m/s $= 6\pi$ m/s

Now $a_c = \omega^2 r = (2\pi)^2 \times 3 = 12\pi^2 = 118.4 \ m/s^2$

Illustration 19

A particle is revolving in a circular path of radius 500 m at a speed 30 ms^{-1}. It is increasing its speed at the rate of 2ms^{-2}. What is its acceleration?



Solution:

r = 500 m, u = 30 ms^{-1}

a_T = 2ms^{-2}

$$a_C = \frac{v^2}{r} = \frac{30^2}{500} = \frac{9}{5} ms^{-2}$$

Total acceleration, $a = \sqrt{a_T^2 + a_C^2} = \sqrt{2^2 + \frac{9^2}{5^2}} \Rightarrow a = \sqrt{\frac{181}{25}} ms^{-2}$

Illustration 20

A particle is performing circular motion of radius 1 m. Its speed is v = ($2t^2$) m/s. What will be the magnitude of its acceleration at t = 1s?

Solution:

Tangential acceleration = $a_T = \frac{dv}{dt}$ = 4t, at t = 1s, a_T = 4 m/s^2

Centripetal acceleration $a_c = \frac{v^2}{r} = \frac{4t^4}{1}$ = $4t^4$, at t = 1 s, a_c = 4 m/s^2

Net acceleration (a) = $\sqrt{a_T^2 + a_c^2}$ = $\sqrt{4^2 + 4^2}$ = $4\sqrt{2}$ m/s^2.

BEGINNER'S BOX-1

1. If angular velocity of a particle depends on the angle rotated θ as $\omega = \theta^2 + 2\theta$, then its angular acceleration α at $\theta = 1$ rad is :

(A) 8 rad/s^2 (B) 10 rad/s^2 (C) 12 rad/s^2 (D) None of these

PCM101

2. The second's hand of a watch has 6 cm length. The speed of its tip and magnitude of difference in velocities of its at any two perpendicular positions will be respectively :

(A) 2π & 0 mm/s (B) $2\sqrt{2}\,\pi$ & 4.44 mm/s

(C) $2\sqrt{2}\,\pi$ & 2π mm/s (D) 2π & $2\sqrt{2}\,\pi$ mm/s

PCM102

3. A particle is moving on a circular path of radius 6 m. Its linear speed is v = 2t, here t is time in second and v is in m/s. Calculate its centripetal acceleration at t = 3 s.

PCM103

4. Two particles move in concentric circles of radii r_1 and r_2 such that they maintain a straight line through the centre. Find the ratio of their angular velocities.

PCM104

5. If the radii of circular paths of two particles are in the ratio of 1 : 2, then in order to have same centripetal acceleration, their speeds should be in the ratio of :

(A) 1 : 4 (B) 4 : 1 (C) $1:\sqrt{2}$ (D) $\sqrt{2}:1$

PCM105

6. A stone tied to the end of a 80 cm long string is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, the magnitude of its acceleration is :

(A) 20 m/s^2 (B) 12 m/s^2 (C) 9.9 m/s^2 (D) 8 m/s^2

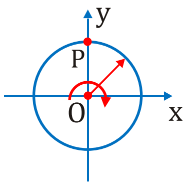
PCM106

7. For a body in a circular motion with a constant angular velocity, the magnitude of the average acceleration over a period of half a revolution is.... times the magnitude of its instantaneous acceleration.

(A) $\frac{2}{\pi}$ (B) $\frac{\pi}{2}$ (C) π (D) 2

PCM107

8. A ring rotates about z axis as shown in figure. The plane of rotation is xy. At a certain instant the acceleration of a particle P (shown in figure) on the ring is $\left(6\hat{i} - 8\hat{j}\right)$ m / s^2. Find the angular acceleration of the ring and its angular velocity at that instant. Radius of the ring is 2 m.



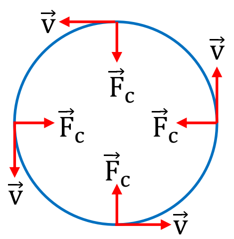
PCM108

Dynamics of Circular Motion

Centripetal, Centrifugal and Tangential Force

Centripetal Force:

The force that is necessary to keep an object moving in a curved path and that is directed inward towards the centre of rotation.



- $\left|\vec{F}_c\right| = m|\vec{a}_c| = m\omega^2 r = m\frac{v^2}{r}$
- It is a vector quantity
- In vector form $\vec{F}_c = -\frac{mv^2}{r}.\,\hat{r} = -\frac{mv^2}{r^2}\vec{r} = -m\omega^2 r\,\hat{r} = -m\omega^2\vec{r} = -m(\vec{v}\times\vec{\omega})$

  –ve sign indicates direction only.

  $\vec{F}_c = m(\vec{\omega}\times\vec{v})$

Centrifugal Force

- The apparent force that is felt by an object moving in a curved path that acts radially away from the center of rotation.
- Its magnitude is equal to centripetal force.
- $\left|\vec{F}_c\right| = m\omega^2 r = m\dfrac{v^2}{r}$
- Its direction is radially outwards.

Note: Centripetal force and centrifugal force are really the exact same force, just in opposite directions because they're experienced from different frames of reference.

Tangential Force

- Tangential force is the force acting on a body in a circular motion in the tangential direction of a curved path.
- A tangential force is the follow up of a tangential acceleration which is always at right angle to the radius which originate from the axis of rotation.
- $\vec{F}_T = m\vec{a}_T$

Horizontal Circular Motion

Uniform Circular Motion (U.C.M.)

When a particle moves on a circular path with constant speed, then its motion is called uniform circular motion.

In uniform circular motion a resultant non-zero force acts on the particle. The acceleration is due to the change in direction of the velocity vector. In uniform circular motion, tangential acceleration (a_T) is zero.

The acceleration of the particle is towards the centre and its magnitude is $\dfrac{v^2}{r}$. Here, v is the speed of the particle and r the radius of the circle.

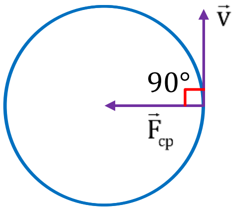
The direction of the resultant force F is therefore, towards the centre and its magnitude is $F = \dfrac{mv^2}{r} = mr\omega^2$ $(\text{as } v = r\omega)$

Here, ω is the angular speed of the particle. This force F is called the centripetal force. Thus, a centripetal force of magnitude $\dfrac{mv^2}{r}$ is needed to keep the particle moving in a circle with constant speed. This force is provided by some external agent such as friction, magnetic force, coulomb force, gravitational force, tension. etc.

In this motion :

- Speed = constant
- |velocity| = constant
- Velocity ≠ constant (because its direction continuously changes)
- K.E. = $\dfrac{1}{2}$ mv^2 = constant
- $\vec{\omega}$ = constant (because magnitude and direction, both are constant)
- $\vec{a}_T = 0 \left[\because a_T = \dfrac{d|\vec{v}|}{dt} = \dfrac{d(\text{constant})}{dt} = 0\right]$

- $\alpha = 0$ [From $a_T = \alpha r$]
- $|\vec{a}| = |\vec{a}_{cp}| = \omega v = \omega^2 r = v^2/r$ = constant
- $\vec{a} = \vec{a}_{cp} \neq$ constant (because the direction of a_{cp} is towards the centre of circle which changes as the particle revolves)
- Uniform circular motion is usually executed in horizontal plane.



Total work done = $\int \vec{F}_{net}.d\vec{s}$ = $\int \vec{F}_{cp}.d\vec{s}$ $\left[\because \vec{F}_{net} = \vec{F}_{cp}\right]$

= $\int (F_{cp})(ds)\cos 90° = 0$

- Power = $\dfrac{\text{Work}}{\text{time}} = \dfrac{0}{t} = 0$

  or Power = $\vec{F}_{net} \cdot \vec{v} = \vec{F}_{cp} \cdot \vec{v}$ = $F_{cp}v\cos 90°$ = 0
- Concept (For U.C.M.); $\vec{F}_{net} = m\vec{a}_{cp}$

Illustration 21

A body of mass 0.1 kg is moving on a circular path of diameter 1m at the rate of 10 revolutions per 31.4 seconds. The centripetal force acting on the body is -

Solution:

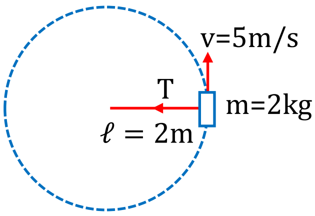
$F = \dfrac{mv^2}{r} = mr\omega^2$; Here m = 0.10kg, r = 0.5m and

$\omega = \dfrac{2\pi n}{t} = \dfrac{2 \times 3.14 \times 10}{31.4}$ = 2rad/s

F = 0.10 × 0.5 × $(2)^2$ = 0.2 N

Illustration 22

A block of mass 2kg is tied to a string of length 2m, the other end of which is fixed. The block is moved on a smooth horizontal table with constant speed 5 m/s. Find the tension in the string.



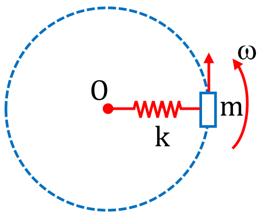
Solution:

Here centripetal force is provided by tension.

$T = \dfrac{mv^2}{r} = \dfrac{2 \times 5^2}{2} = 25N$

Illustration 23

A block of mass m is tied to a spring of spring constant k, natural length ℓ, and the other end of spring fixed at O. If the block moves in a circular path on a smooth horizontal surface with constant angular velocity ω, find tension in the spring



Solution:

Assume extension in the spring is x.

Here centripetal force is provided by spring force.

Centripetal force, kx = $m\omega^2(\ell + x)$ $\Rightarrow x = \dfrac{m\omega^2\ell}{k - m\omega^2}$

therefore, Tension = $kx = \dfrac{km\omega^2\ell}{k - m\omega^2}$

Illustration 24

A certain string which is 1m long will break, if the load on it is more than 0.5kg. A mass of 0.05kg is attached to one end of it and the particle is whirled round a horizontal circle by holding the free end of the string by one hand. The greatest number of revolutions per minute possible without breaking the string will be-

Solution:

m = 0.05kg, r = 1m

The maximum tension the string can withstand

= 0.5 kg wt.= 0.5 × 9.8 N = 4.9 N

Hence the centripetal force required to produce the maximum tension in the string is 4.9N

i.e.$mr\omega^2 = 4.9 \Rightarrow \omega^2 = \frac{4.9}{mr} = \frac{4.9}{0.05 \times 1} = 98 \Rightarrow \omega = \sqrt{98}$

$\Rightarrow 2\pi f = \sqrt{98} \Rightarrow f = \frac{\sqrt{98}}{2\pi}$ = 1.576 rev/sec = 94.5 rev/min

Illustration 25

A small object placed on a rotating horizontal turn table just slips when it is placed at a distance 4 cm from the axis of rotation. If the angular velocity of the turn-table is doubled, the object slips when its distance from the axis of rotation is

Solution:

The object will slip if required centripetal force ≥ force of friction

$mr\omega^2 \geq \mu mg$ or $r\omega^2 \geq \mu g$

$r\omega^2 \geq$ constant

or $\left(\frac{r_1}{r_2}\right) = \left(\frac{\omega_2}{\omega_1}\right)^2$

$\frac{4cm}{r_2} = \left(\frac{2\omega}{\omega}\right)^2 \quad \therefore r_2 = 1$ cm

Banking of Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which provides the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by the following three ways :

- By friction only.
- By banking of roads only.
- By friction and banking of roads both.

In real life the necessary centripetal force is provided by friction and banking of roads both.

Case-1; Circular turning on roads by friction only :

Suppose a car of mass m is moving with a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force will be provided to the car by the force of friction f acting towards centre of the circular path

$f = \frac{mv^2}{r}$

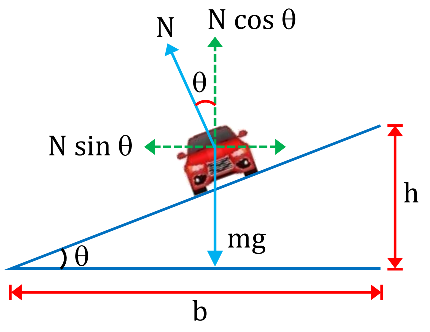
$f_L = \mu N = \mu mg$

Therefore, for a safe turn without skidding

$\frac{mv^2}{r} \leq f_L \Rightarrow \frac{mv^2}{r} \leq \mu mg \Rightarrow v \leq \sqrt{\mu rg}$

Case-2; Circular turning on roads by banking of roads only :

Friction is not always reliable at turns particularly when high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn in the sense that the outer part of the road is some what lifted compared to the inner part.



$N\sin\theta = \dfrac{mv^2}{r}$ and $N\cos\theta = mg$

$\tan\theta = \dfrac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta}$ $\qquad \because \tan\theta = \dfrac{h}{b}$

Note: $\tan\theta = \dfrac{v^2}{rg} = \dfrac{h}{b}$

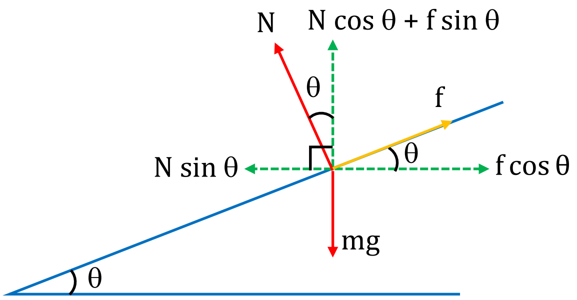
Case-3; Circular turning on roads by friction and banking of road both :

- If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle.
- Of these the first force, i.e., weight (mg) is fixed both in magnitude and direction.
- The direction of second force, i.e., normal reaction N is perpendicular to road.
- The direction of the third force, i.e., friction f can be either inwards or outwards, while its magnitude can be varied up to a maximum limit ($f_L = \mu N$).
- So, direction and the magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\dfrac{mv^2}{r}$ towards the Centre.

(a) If $v < \sqrt{rg\tan\theta}$, then friction acts outwards.

In this case, $N\cos\theta + f\sin\theta = mg$ ...(i)

and $N\sin\theta - f\cos\theta = \dfrac{mv^2}{r}$ ...(ii)



For minimum speed f = μN

So, by dividing equation (i) by equation (ii)

$$\frac{N\cos\theta + \mu N\sin\theta}{N\sin\theta - \mu N\cos\theta} = \frac{mg}{mv_{min}^2 / r}$$

Therefore $\qquad v_{min} = \sqrt{rg\left(\dfrac{\tan\theta - \mu}{1 + \mu\tan\theta}\right)}$

If we assume $\mu = \tan\phi$, then

$$v_{min} = \sqrt{rg\left(\frac{\tan\theta - \tan\phi}{1 + \tan\phi\tan\theta}\right)} = \sqrt{rg\tan(\theta - \phi)}$$

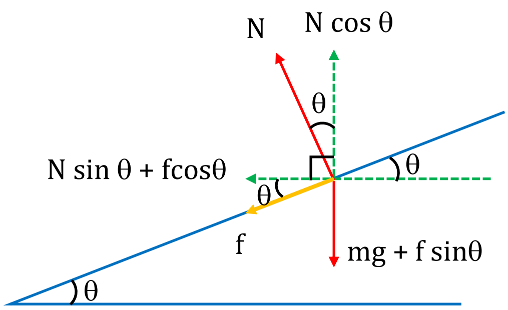
(b) If speed of the vehicle is high then friction acts inwards.

In this case for maximum speed,

$N\cos\theta - \mu N\sin\theta = mg$

and $N\sin\theta + \mu N\cos\theta = \dfrac{mv_{max}^2}{r}$

Which gives $v_{max} = \sqrt{rg\left(\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)}$



If we assume $\mu = \tan\phi$, then

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \tan\phi}{1 - \tan\phi\tan\theta}\right)} = \sqrt{rg\tan(\theta+\phi)}$$

Hence for successful turning on a rough banked road, velocity of vehicle must satisfy following relation.

$$\sqrt{rg\tan(\theta-\phi)} \le v \le \sqrt{rg\tan(\theta+\phi)}$$

where θ = banking angle and $\phi = \tan^{-1}(\mu)$

Note: The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle of banking for an aircraft, i.e., the angle through which it should tilt while negotiating a curve, to avoid deviation from the circular path.

The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle at which a cyclist should lean inward, when rounding a corner. In this case, θ is the angle which the cyclist must make with the vertical.

Illustration 26

An unbanked curve has a radius of 100m. The maximum speed at which a car can make a turn if the coefficient of static friction is 0.8, is (acceleration due to gravity = 10 m/s²).

Solution:

Here centripetal force is provided by friction so,

$$\frac{mv^2}{r} \le \mu mg$$

$$v_{max} = \sqrt{\mu rg} = \sqrt{0.8\times100\times10} \simeq 28 m/s$$

Illustration 27

A circular road of radius 40 m has the angle of banking equal to 37°. At what speed should a vehicle go on this road so that the friction is not used?

Solution:

Radius (r) = 40 m ; Angle of banking = θ = 37°

$$\therefore \tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta} \Rightarrow v = \sqrt{40\times10\times\tan37^\circ} = 10\sqrt{3} m/s$$

Illustration 28

A circular road of radius 0.5 km has a banking angle of 37°. What will be the maximum safe speed of a car whose mass is 500 kg, and the coefficient of friction between the tyre and the road is 0.5.

Solution:

Given; Radius of the circular road (r) = 0.5 km = 500 m ; Mass of the car (m) = 500 kg

Angle of banking = 37° ; Coefficient of friction between the tyre and the road = 0.5

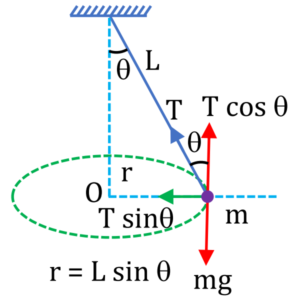
$$v_{max} = \sqrt{rg\left(\frac{\tan\theta+\mu}{1-\mu\tan\theta}\right)} = \sqrt{500\times10\left(\frac{\tan37^\circ+0.5}{1-0.5\times\tan37^\circ}\right)} = \sqrt{500\times10\left(\frac{\frac{3}{4}+0.5}{1-0.5\times\frac{3}{4}}\right)} = \sqrt{5000\frac{\left(\frac{5}{4}\right)}{\left(\frac{2.5}{4}\right)}} = 100 m/s$$

Conical Pendulum, Rotor and Death Well

Conical Pendulum

If a small particle of mass m tied to a string is whirled along a horizontal circle, as shown in figure then the arrangement is called conical pendulum. In case of conical pendulum, the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force. Thus,
The forces acting on the bob are (a) Tension (b) weight

The horizontal component T sinθ of the tension T provides the centripetal force and the vertical component T cosθ balances the weight of bob



$$T\sin\theta = \frac{mv^2}{r} \quad \text{...(i)}$$

$$T\cos\theta = mg \quad \text{...(ii)}$$

From equations (i) and (ii)

$$\tan\theta = \frac{v^2}{rg} \quad \text{...(iii)}$$

$$\Rightarrow v = \sqrt{rg\tan\theta}$$

$\therefore$ Angular speed $\omega = \dfrac{v}{r} = \sqrt{\dfrac{g\tan\theta}{r}}$

So, the time period of pendulum is $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{r}{g\tan\theta}} = 2\pi\sqrt{\dfrac{L\cos\theta}{g}}$

Death Well or Rotor

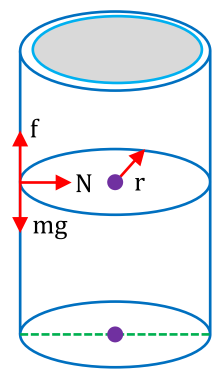
In case of 'death well' a person drives a motorcycle on the vertical surface of a large wooden well while in case of a rotor a person hangs resting against the wall without any support from the bottom at a certain angular speed of rotor. In death well, walls are at rest and person revolves while in case of rotor person is at rest w.r.t wall and the wall rotates.

In both cases, friction balances the weight of person while reaction provides the centripetal force for circular motion, i.e.

By the free body diagram

$$f_r = mg \quad \text{...(i)}$$

$$N = \frac{mv^2}{r} = mr\omega^2 \quad \text{...(ii)}$$



Death well

Balancing the vertical forces

$$f_r = mg \quad \text{or} \quad \mu N = mg$$

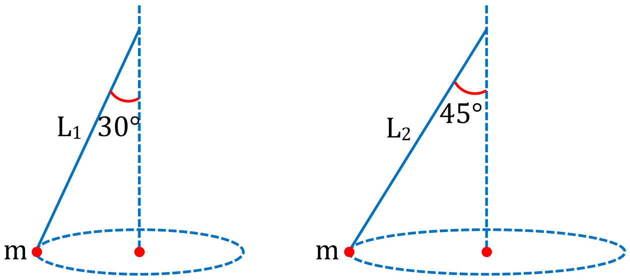
$$\mu\frac{mv^2}{r} = mg$$

$$v^2 = \frac{gr}{\mu}$$

$$v_{min} = \sqrt{\frac{gr}{\mu}}$$

Illustration 29

Two particles tied to different strings are whirled in a horizontal circle as shown in figure. The ratio of lengths of the strings so that they complete their circular path with equal time period is :



(a) (b)

Solution:

Since $T = 2\pi\sqrt{\dfrac{L\cos\theta}{g}}$

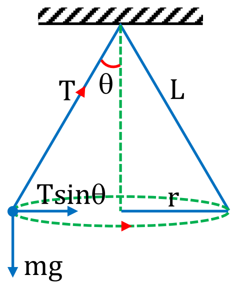
$\therefore \quad T_1 = T_2 \quad \Rightarrow \quad L_1 \cos\theta_1 = L_2 \cos\theta_2$

$\therefore \quad \dfrac{L_1}{L_2} = \dfrac{\cos\theta_2}{\cos\theta_1} = \dfrac{\cos 45°}{\cos 30°} = \dfrac{\sqrt{2}}{\sqrt{3}}$

Illustration 30

A string breaks under a load of 50 kg. A mass of 1 kg is attached to one end of the string 10 m long and is rotated in horizontal circle. Calculate the greatest number of revolutions that the mass can make in one second without breaking the string.

Solution:



$\omega = 2\pi f,\ T_{max} = 500$ N, $r = L\sin\theta$

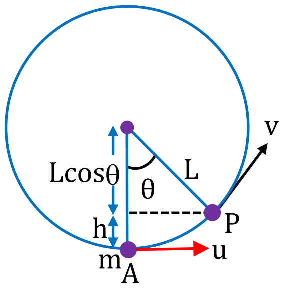
$T\sin\theta = m\omega^2 r$

$\Rightarrow \quad T = m\omega^2 L$

$\Rightarrow \quad T_{max} = m\,\omega_{max}^2 L \quad \Rightarrow \quad T_{max} = m(2\pi f)^2 L$

$f = \dfrac{1}{2\pi}\sqrt{\dfrac{T_{max}}{mL}} = \dfrac{1}{2\pi}\sqrt{\dfrac{500}{1\times 10}} = \dfrac{\sqrt{50}}{2\pi}$ revolution per second.

Vertical Circular Motion



Suppose a particle of mass m is attached to a light inextensible string of length L. The particle is moving in a vertical circle of radius L about a fixed-point O. It is imparted a velocity u in the horizontal direction at lowest point A. Let v be its velocity at point P of the circle as shown in the figure.

Velocity at a point P

Then apply mechanical energy conservation between point A and P(considering potential energy zero at point A)

Total (PE + KE) at A = Total (PE + KE) at P

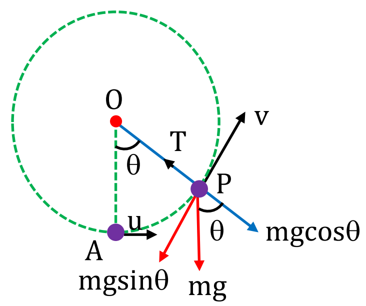
$\Rightarrow \quad 0 + \dfrac{1}{2}mu^2 = mgh + \dfrac{1}{2}mv^2$

$\Rightarrow \quad \dfrac{1}{2}mu^2 = mg(L - L\cos\theta) + \dfrac{1}{2}mv^2 \qquad$ as $h = (L - L\cos\theta)$

[Where L is length of the string]

$\Rightarrow \quad v = \sqrt{u^2 - 2gL(1-\cos\theta)}$

Tension at a point P



At point P required centripetal force = $\frac{mv^2}{L}$

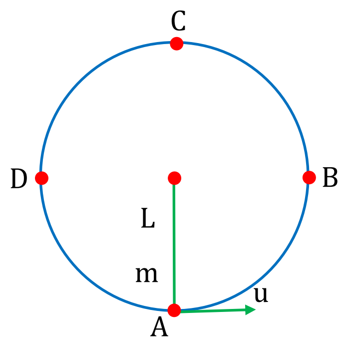
Net force towards the centre = T – mg cosθ

This net force provides required centripetal force.

$$\therefore \qquad T - mg\cos\theta = \frac{mv^2}{L}$$

$$\Rightarrow \qquad T = mg\cos\theta + \frac{mv^2}{L}$$

Special Case



At point A : (Bottom point)

$T_A = \frac{mv_A^2}{L} + mg \qquad T_A = \frac{mu^2}{L} + mg \qquad$ (Here θ = 0°)

At point B :

$v_B = \sqrt{u^2 - 2gL}$

$T_B = \frac{m{v_B}^2}{L} \qquad T_B = \frac{mu^2}{L} - 2mg \qquad$ (Here θ = 90°)

At point C : (Top point)

$v_C = \sqrt{u^2 - 4gL}$

$T_C = \frac{mv_C^2}{L} - mg \qquad T_C = \frac{mu^2}{L} - 5mg \qquad$ (Here θ = 180°)

From above equation we can see, $T_{bottom} - T_{top} = T_A - T_C = 6$ mg, this difference in tension remain same.

Illustration 31

A 4kg balls swings in a vertical circle at the end of a cord 1m long. The maximum speed at which it can swing if the cord can sustain maximum tension of 183.2N will be -

Solution:

Maximum tension $T = \frac{mv^2}{r} + mg \qquad$ (Tension will be maximum at lowest point)

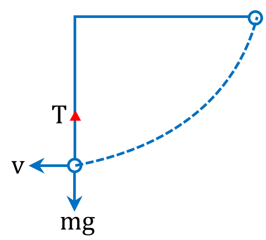
$\therefore \frac{mv^2}{r} = T - mg$ or $\frac{4v^2}{1} = 183.2 - 4 \times 9.8 \Rightarrow v = 6$m/s

Illustration 32

The string of a pendulum is horizontal. The mass of the bob is m. Now the string is released. The tension in the string in the lowest position is -

Solution:

The situation is shown in fig. Let v be the velocity of the bob at the lowest position. In this position the P.E. of bob is converted into K.E. hence -



$mgL = \frac{1}{2}mv^2 \Rightarrow v^2 = 2gL \qquad$ ...(i)

If T be the tension in the string,

then $\quad T - mg = \frac{mv^2}{\ell} \qquad$ ...(ii)

From (i) & (ii) $\qquad$ T = 3mg

Illustration 33

A body weighing 0.4 kg is whirled in a vertical circle with a string making 2 revolutions per second. If the radius of the circle is 1.2 m. Find the tension (a) at the top of the circle, (b) at the bottom of the circle. Given: g = 10 ms^{-2} and π = 3.14.

Solution:

$m = 0.4$ kg ; $T = \frac{1}{2}$ sec, $r = 1.2$ m

$\omega = \frac{2\pi}{1/2} = 4\pi$ rad s^{-1} = 12.56 rad s^{-1}.

(a) At the top of the circle, $T = \frac{mv^2}{r} - mg = mr\omega^2 - mg = m\left(r\omega^2 - g\right)$

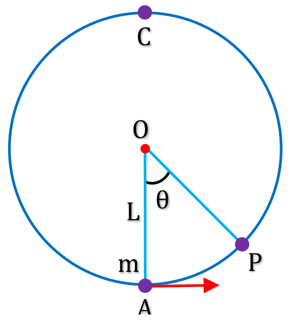
$= 0.4\ (1.2 \times 12.56 \times 12.56 - 9.8)$ N = 72 N

(b) At the lowest point, $T = m\left(r\omega^2 + g\right) = 80N$

Condition for Looping the Loop

For String

What should be the minimum velocity of particle connected with string of length L at point A so that it just performs vertical circular motion about a fixed point. The point mass will complete the circle only and only if tension is never zero (except momentarily. If at all) if tension becomes zero at any point, string will go slack and subsequently, the only force acting on the body is gravity. Hence its subsequent motion will be similar to that of a projectile.



Tension at point P, $T - mg\cos\theta = \frac{mv^2}{L}$

From equation, it is evident that tension decreases with increase in θ because cos θ is a decreasing function and v decreases with height. Hence tension is minimum at the top most point. i.e. $T_{min} = T_{topmost}$.

However if tension is momentarily zero at highest point the body would still be able to complete the circle. Hence condition for completing the circle (or looping the loop) is : -

$T_{min} \geq 0$ or $T_{top} \geq 0$.

$T_{top} + mg = \frac{mv_{top}^2}{L}$

$T_{top} = \frac{mv_{top}^2}{L} - mg$

For looping the loop, $T_{top} \geq 0$

$\frac{mv_{top}^2}{L} - mg \geq 0$

$\frac{mv_{top}^2}{L} \geq mg \quad \Rightarrow \quad v_{top} \geq \sqrt{gL}$ ...(i)

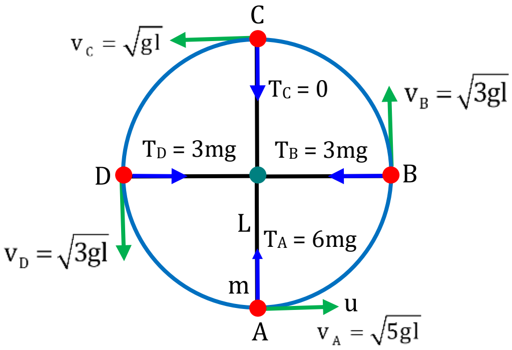
If speed at the lowest point is u, then from conservation of mechanical energy between lowest point and top most point.

$\frac{1}{2}mu^2 = \frac{1}{2}m\,v_{top}^2 + mg \cdot 2L$ ...(ii)

using equation (i) and (ii) we get $u \geq \sqrt{5gL}$

i.e., for looping the loop, velocity at lowest point must be $\geq \sqrt{5gL}$.

If velocity at lowest point is just enough for looping the loop, value of various quantities at point A, B, C and D are :



		A	B,D	C
1	Velocity	$\sqrt{5gL}$	$\sqrt{3gL}$	$\sqrt{gL}$
2	Tension	6mg	3mg	0
3	Potential Energy	0	mgL	2mgL
4	Radial acceleration	5g	3g	g
5	Tangential acceleration	0	g	0

Condition for looping for Massless Rod

In case of light rod tension at top most point can never be zero so velocity will become zero.

$\therefore$ For completing the loop $v_L \geq \sqrt{4gR}$

Illustration 34

A stone weighing 1kg is whirled in a vertical circle at the end of a rope of length 0.5m. Find the velocity of a stone and tension in string (a) at lowest position (b) midway when the string is horizontal (c) at topmost position to just complete the circle.

Solution:

Lower most point; $v_L = \sqrt{5gr} = \sqrt{5 \times 9.8 \times 0.5} = 4.95\ \text{m/s}$

$T_L = 6mg = 6 \times 1 \times 9.8 = 58.8\text{N}$

When string is horizontal; $v_M = \sqrt{3gr} = \sqrt{3 \times 9.8 \times 0.5} = 3.83\ \text{m/s}$

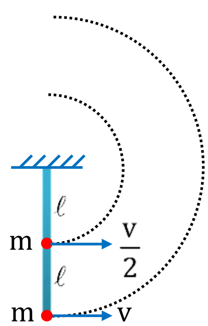
$T_M = 3mg = 3 \times 1 \times 9.8 = 29.4\text{N}$

Top most point; $v_T = \sqrt{gr} = \sqrt{9.8 \times 0.5} = 2.21\ \text{m/s}$

$T_T = 0\text{N}$

Illustration 35

Two-point masses, each m is connected to a light rod of length 2ℓ and it is free to rotate in vertical plane as shown. Calculate the minimum horizontal velocity given to lower mass so that it completes the circular motion in vertical plane.



Solution:

For rod to just complete the loop velocity at the top most point is zero.

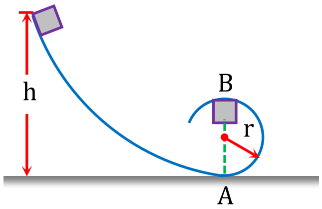
Particle have same angular velocity but different linear velocity.

Loss of KE = gain of PE

$$\frac{1}{2}mv^2 + \frac{1}{2}m\left(\frac{v}{2}\right)^2 = mg(2\ell) + mg(4\ell)$$

$$\Rightarrow\ v = 4\sqrt{\frac{3g\ell}{5}}$$

Illustration 36



A block is released from the top of a smooth vertical track, which ends in a circle of radius r as shown.

(i) Find the minimum value of h so that the block completes the circle.

(ii) If h = 3r, find normal reaction when the block is at the points A and B.

Solution:

(i) For completing the circle, velocity at lowest point of circle (say A) is $\sqrt{5gr}$

from energy conservation

$$mgh = \frac{1}{2}m\left(\sqrt{5gr}\right)^2 \Rightarrow h = \frac{5r}{2}$$

(ii) h = 3r

From energy conservation, velocity at point A and B are :

$$mg\cdot 3r = \frac{1}{2}mv_A^2 \Rightarrow v_A = \sqrt{6gr}$$

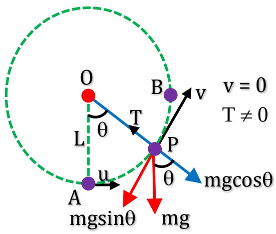
$$mg\cdot 3r = \frac{1}{2}mv_B^2 + mg\cdot 2r \Rightarrow v_B = \sqrt{2gr}$$

Therefore, normal reaction at A and B is: -

$$N_A - mg = \frac{mv_A^2}{r} \Rightarrow N_A = 7mg$$

$$N_B + mg = \frac{mv_B^2}{r} \Rightarrow N_B = mg$$

Condition of Oscillation $\left(0 < u \le \sqrt{2gL}\right)$



The particle will oscillate if velocity of the particle becomes zero but tension in the string is not zero.

(In lower half circle (A to B))

Here, $T - mg\cos\theta = \dfrac{mv_A^2}{L}$

$T = \dfrac{mv_A^2}{L} + mg\cos\theta$

In the lower part of circle, when velocity become zero and tension is non-zero, means when v = 0, but $T \neq 0$

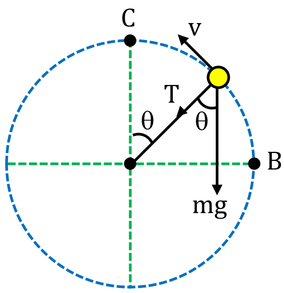
So, to make the particle oscillate in lower half cycle, maximum possible velocity at A can be. given by

$\dfrac{1}{2}mv_A^2 + 0 = mgL + 0$ (by COME between A and B)

$v_A = \sqrt{2gL}$ ...(i)

Thus, for $0 < u \le \sqrt{2gL}$, particle oscillates in lower half of the circle ($0^0 < \theta \le 90°$)

Condition of Leaving the Circle : $(\sqrt{2gL} < u < \sqrt{5gL})$

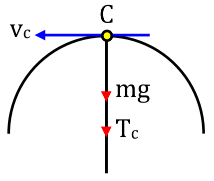


In upper half cycle (B to C)

Here, $T + mg\cos\theta = \dfrac{mv^2}{L}$

$T = \left(\dfrac{mv^2}{L} - mg\cos\theta\right)$ ...(ii)

In this part of circle tension force can be zero without having zero velocity mean when T = 0, v ≠ 0 form equation (ii) it is clear that tension decreases if velocity decreases. So, to complete the loop, tension force should not be zero, in between B to C. Tension will be minimum at C i.e., $T_c \geq 0$ is the required condition.



At Top $T_c + mg = \dfrac{mv_c^2}{L}$

if $T_c = 0$

Then $mg = \dfrac{mv_c^2}{L}$

$v_c^2 = gL \Rightarrow v_c = \sqrt{gL}$

By COME (Between A and C)

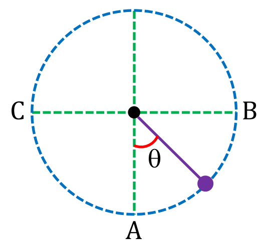
$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_c^2 + mg(2L)$$

$v_A^2 = v_c^2 + 4gL \Rightarrow \quad v_A^2 = 5gL \quad \Rightarrow v_A = \sqrt{5gL}$

Therefore, if $\sqrt{2gL} < u < \sqrt{5gL}$, the particle leaves the circle.

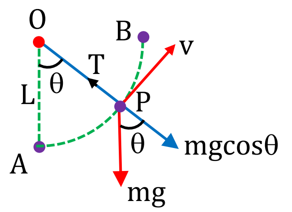
Note: After leaving the circle, the particle will follow a parabolic path.

Illustration 37



Find minimum speed at A so that the ball can reach at point B as shown in figure. Also discuss the motion of particle when T = 0, v = 0 simultaneously at θ = 90°.

Solution:



From energy conservation ,

$\frac{1}{2}mv_A^2 + 0 = 0 + mgL$ (for minimum speed at A, $v_B = 0$)

$v_{min} = \sqrt{2gL}$

at the position B, v = 0 and T = 0

$T - mg\cos\theta = \dfrac{mv_B^2}{L}$ ...(i)

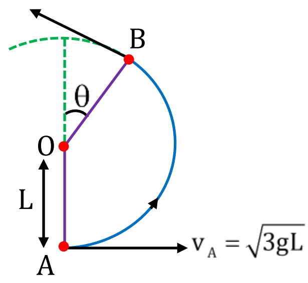
(putting $v_B = 0$ and θ = 90°, in equation ...(ii)

ball will return back, motion is oscillatory

Illustration 38

A particle in projected with velocity $\sqrt{3gL}$ at point A (lowest point of the circle) in the vertical plane. Find the maximum height about horizontal level of point A if the string slacks at the point B as shown.

Solution:



As tension at B ; T = 0

$\therefore\ mg\cos\theta = \dfrac{mv_B^2}{L}$

$\therefore\ v_B = \sqrt{gL\cos\theta}$ ...(1)

Now by equation of energy between A and B.

$$0 + \frac{1}{2}m \times 3gL = \frac{1}{2}mv_B^2 + mgL(1 + \cos\theta)$$

Put $v_B = \sqrt{gL\cos\theta}$

$\therefore \quad \cos\theta = \frac{1}{3}$

$\therefore$ Height attended by particle after the point B where the string slacks is ;

$$h' = \frac{v_B^2 \sin^2\theta}{2g} = \frac{gL\cos\theta(1-\cos^2\theta)}{2g} = \frac{4L}{27}$$

$\therefore$ Maximum height about point A is given by ;

$$H_{max} = L + L\cos\theta + h'$$

$$= L + \frac{L}{3} + \frac{4L}{27} = \frac{40L}{27}$$

BEGINNER'S BOX-2

1. A particle of mass m_1 is fastened to one end of a string and another one of mass m_2 to the middle point; the other end of the string being fastened to a fixed point on a smooth horizontal table. The particles are then projected, so that the two portions of the string are always in the same straight line and describe horizontal circles. Find the ratio of the tensions in the two parts of the string.

PCM109

2. A road is 8 m wide. Its average radius of curvature is 40 m. The outer edge is above the lower edge by a distance of 1.28 m. Find the velocity of vehicle for which the road is most suited ? ($g = 10\ m/s^2$)

PCM110

3. A stone of mass 1 kg tied to a light string of length $\ell = 10$ m is whirling in a circular path in the vertical plane. If the ratio of the maximum to minimum tensions in the string is 3, find the speeds of the stone at the lowest and highest points.

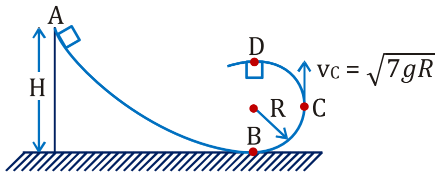
PCM111

4. Calculate the following for the situation shown :-

(a) Speed at D

(b) Normal reaction at D

(c) Height H



PCM112

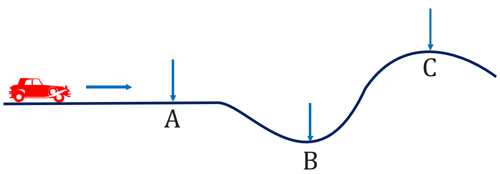
5. A car is moving along a hilly road as shown (side view). The coefficient of static friction between the tyres and the pavement is constant and the car maintains a steady speed. If at one of the points shown the driver applies brakes as hard as possible without making the tyres slip, the magnitude of the frictional force immediately after the brakes are applied will be maximum if the car was at :-

(A) point A

(B) point B

(C) point C

(D) friction force same for positions A, B and C



PCM113

6. A stone weighing 0.5 kg tied to a rope of length 0.5 m revolves along a circular path in a vertical plane. The tension of the rope at the bottom point of the circle is 45 newton. To what height will the stone rise if the rope breaks at the moment when the velocity is directed upwards? ($g=10\ m/s^2$)

PCM114

BEGINNER'S BOX ANSWER KEY

BEGINNER'S BOX-1

1.	(C)	**2.**	(D)	**3.**	6 m/s^2 towards the centre.
4.	1 : 1	**5.**	(C)	**6.**	(C)
7.	(A)	**8.**	$-3\hat{k}$ rad/s^2, $-2\hat{k}$ rad/s		

BEGINNER'S BOX-2

1.	$\frac{2m_1}{m_2+2m_1}$	**2.**	8 m/s	**3.**	$v_{lowest} = 20\sqrt{2}$ m/s ; $v_{highest} = 20$ m/s
4.	(a) $\sqrt{5gR}$	(b) 4mg	(c) $\frac{9}{2}R$		
5.	(B)	**6.**	1.5 m		