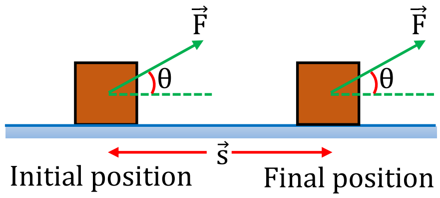
05 Work, Energy and Power

Work

Whenever a force acts on a body, displaces its point of application of force in any direction except perpendicular to it, work is said to be done by the force.

If a constant force $\vec{F}$ is applied on a body and displaces it through displacement $\vec{s}$ then work done by $\vec{F}$ on the body is given by scalar product of $\vec{F}$ and $\vec{s}$.



Mathematically,

$W = \vec{F}.\vec{s} = Fs\cos\theta$

Where θ is angle between $\vec{F}$ and $\vec{s}$.

Units and Dimensions

- **Unit**

  SI Unit of work : Joule (J)

  Joule : One joule of work is said to be done when a force of one newton displaces a body by one meter in the direction of force.

  1 joule = 1 newton × 1 meter = 1 kg-m²/s² ($1\ \text{joule} = 1\ \text{newton} \times 1\ \text{meter} = 1\ \text{kg-m}^2/\text{s}^2$)

  Erg : One erg of work is said to be done when a force of one dyne displaces a body by one centimeter in the direction of force.

  $1\ \text{erg} = 1\ \text{dyne} \times 1\ \text{cm} = 1\ \text{g-cm}^2/\text{s}^2$

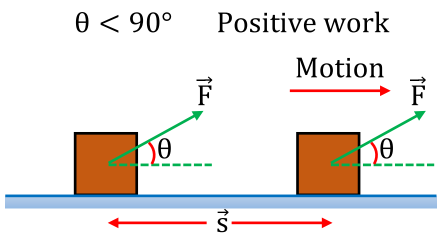
  Other Units :
  (a) 1 joule = 10^7 ergs
  (b) 1 erg = 10^{-7} joules
  (c) 1 eV = 1.6×10^{-19} joules
  (d) 1 joule = 6.25×10^{18} eV
  (e) 1 MeV = 1.6×10^{-13} J
  (f) 1 J = 6.25×10^{12} MeV
  (g) 1 kilowatt-hour (kWh) = 3.6×10^6 joules

- **Dimensions**

  [Work] = [Force] [Displacement]= $[MLT^{-2}][L] = [ML^2T^{-2}]$

Nature of Work Done

Work is **scalar quantity**, but it can be positive, negative or zero.



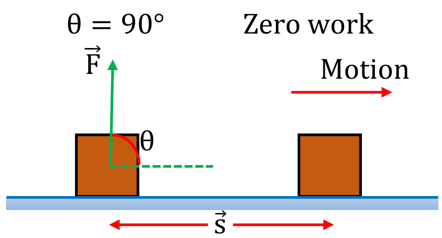
(a) Positive work

$W = \vec{F}.\vec{s} = Fs\cos\theta$

If the angle θ is acute ($\theta < 90°$) then the work done is said to be positive. Positive work signifies that the external force favours the motion of the body.

Example :-

- When a body falls freely under the action of gravity ($\theta = 0°$), the work done by gravity is positive.
- When a spring is stretched, stretching force and the displacement both are in the same direction. So, work done by stretching force is positive.



(b) Zero work

$W = \vec{F}.\vec{s} = Fs\cos\theta$

W will be zero if F = 0 or s = 0 or $\theta = 90°$

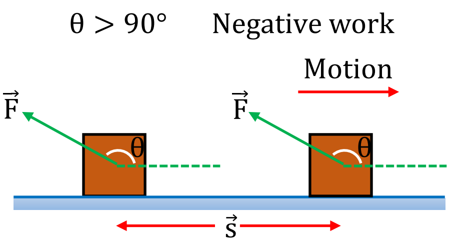
Example:-

- Body moving with uniform velocity. (F = 0)
- Net force on the particle is zero. (F = 0)
- We push the wall and it remains at rest. (s = 0)
- A pendulum is oscillating, work done by tension. ($\theta = 90°$)
- Electron is moving round the nucleus. ($\theta = 90°$)
- Satellite is moving around the earth. ($\theta = 90°$)
- Coolie **"Sahayak"** is carrying baggage on a horizontal platform (W.D. by gravitational force is zero)

(c) Negative work

$W = \vec{F}.\vec{s} = Fs\cos\theta$

If the angle θ is obtuse ($\theta > 90°$) then the work is said to be negative. It signifies that the external force opposes the motion of the body.



Example: -

- Work done by frictional force is negative when it opposes the motion.
- Work done by braking force on the car is negative.

Dependency on Frame of Reference

A force does not depend on frame of reference and assumes same value in all frame of references, but displacement depends on frame of reference and may assume different values relative to different reference frames. Therefore, work of a force depends on choice of reference frame. For example, consider a man with a suitcase standing in a lift that is moving up. In the reference frame attached to the lift, the man applies a force equal to the weight of the bag but the displacement of the bag is zero, therefore work due to this force on the bag is zero. However, in a reference frame attached to the ground the bag has displacement equal to that of the lift and the force applied by the man does a non-zero work.

Illustration 1:

If displacement of a particle is $\vec{s} = (\hat{i} + 4\hat{j} + \hat{k})$m due to a force $\vec{F} = (2\hat{i} + 3\hat{j} + \hat{k})$N, then find work done by this force?

Solution:

$W = \vec{F}\cdot\vec{s} = (\hat{i} + 4\hat{j} + \hat{k}).(2\hat{i} + 3\hat{j} + \hat{k}) = 15\text{ J}$

Illustration 2:

A force $\vec{F} = (4\hat{i} - 3\hat{j} - 7\hat{k})$ N displaces a particle from point A (0,1,4) to point B (2, 3, 2), then find work done by this force?

Solution:

$\vec{s} = (2-0)\hat{i} + (3-1)\hat{j} + (2-4)\hat{k} = 2\hat{i} + 2\hat{j} - 2\hat{k}$

$W = \vec{F}\cdot\vec{s} = (4\hat{i} - 3\hat{j} - 7\hat{k}).(2\hat{i} + 2\hat{j} - 2\hat{k}) = 16\text{ J}$

Illustration 3:

A force $\vec{F} = (\hat{i} + 2\hat{j} - 15\hat{k})$ N displaces a particle from origin to point $P(\alpha, \alpha, 1)$. Then find values of α for which force and displacement are orthogonal to each other?

Solution:

$\vec{s} = (\alpha - 0)\hat{i} + (\alpha - 0)\hat{j} + (1 - 0)\hat{k} = \alpha\hat{i} + \alpha\hat{j} + 1\hat{k}$

$W = \vec{F}\cdot\vec{s} = (\hat{i} + 2\hat{j} - 15\hat{k}).(\alpha\hat{i} + \alpha\hat{j} + 1\hat{k}) = \alpha + 2\alpha - 15 = 3\alpha - 15$

$\because W = 0 \quad \therefore 3\alpha - 15 = 0 \quad \Rightarrow \alpha = 5$

Golden Key Points

- Work is defined for an interval i.e. there is no term like instantaneous work similar to instantaneous velocity.
- For a particular displacement, work is independent of time.
- When several forces act, work done by a force, for a particular displacement, is independent of other forces.
- As displacement depends on frame of reference so work done by a force is frame dependent. It means that work done by a force can be different in different frame of reference.
- Work is done by an energy source or agent that applies the force.
- When θ is acute work done is positive.
- When θ is obtuse work done is negative.
- When θ = 0°, force does maximum positive work.
- When θ = 180°, force does maximum negative work.

Work done by a constant force

Work done by a constant force $\vec{F}$ acting on a body and displacing it by displacement $\vec{s}$, is given as

$W = \vec{F}\cdot\vec{s} = Fs\cos\theta = F(s\cos\theta) = (F\cos\theta)s$

Since $F\cos\theta$ is component of $\vec{F}$ along $\vec{s}$ and $s\cos\theta$ is component of $\vec{s}$ along $\vec{F}$.

Hence, we can rewrite above expression of work in a special form if force $\vec{F}$ is constant.

W = Force × component of displacement along the force

= displacement × component of force along the displacement.

Illustration 4:

A body of mass 20 kg is at rest. A force of 5 N is applied on it. Calculate the work done in the first second.

Solution:

Displacement in the direction of force is $d = \frac{1}{2}at^2 = \frac{1}{2}\frac{F}{m}t^2$ $\quad (\because s = ut + \frac{1}{2}at^2)$

So, work done is $W = \frac{1}{2}\frac{F^2}{m}t^2 = \frac{1}{2}\times\frac{5^2}{20}\times 1^2 = \frac{5}{8}$ J.

Illustration 5:

Calculate the work done to move a body of mass 10 kg along a smooth inclined plane (θ = 30°) with constant velocity through a distance of 10 m.

Solution:

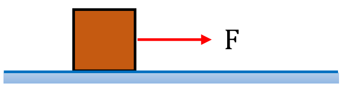
Here the motion is not accelerated, the resultant force parallel to the plane must be zero. So

$F - Mg\sin 30° = 0 \Rightarrow F = Mg\sin 30°$ & d = 10 m

$W = Fd\cos\theta = (Mg\sin 30°)\, d\cos 0° = 10 \times 10 \times \frac{1}{2} \times 10 \times 1 = 500$ J.

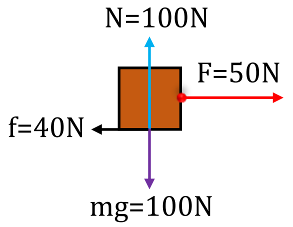
Illustration 6:

A 10 kg block placed on a rough horizontal floor is being pulled by a constant force of 50 N. Coefficient of kinetic friction between the block and the floor is 0.4. Find the work done by kinetic friction acting on the block over a displacement of 5 m.



Solution:

F.B.D. of the block is shown here

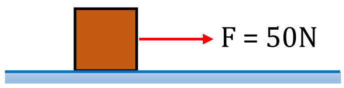


Work done by the force of kinetic friction

$W_f = \vec{f} \cdot \vec{s} = -fs = -40 \times 5 = -200$ J.

Illustration 7:

A block (mass 20 kg) kept on a horizontal floor is being pulled by a horizontal force of 50N and displaces it by 3m. Find work done by normal reaction force by floor on the block.



Solution:

Since $\vec{N}$ and $\vec{s}$ are always perpendicular. Hence, $W_N = 0$.

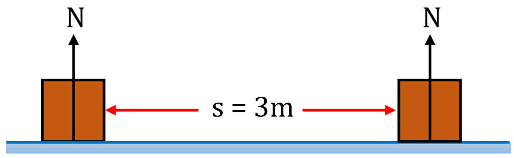
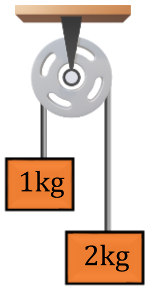


Illustration 8:

Find work done by tension force on 2kg block during time interval of 3 sec after released.



Solution:

Tension $T = \dfrac{2 \times 1 \times 2}{1+2} g = \dfrac{40}{3}$ N

Acceleration of 2kg block, $a = \dfrac{2-1}{2+1} g = \dfrac{g}{3}$

Displacement of 2kg block in 3sec, $s = \dfrac{1}{2} a\left(3^2\right) = \dfrac{1}{2} \times \dfrac{g}{3} \times 9 = 15\text{m}$

$W_T = T.S = T \times S \times \cos 180° = -T \times S$

Work done by tension force $W_T = -\dfrac{40}{3} \times 15 = -200\text{J}$

Work Done by Variable Force

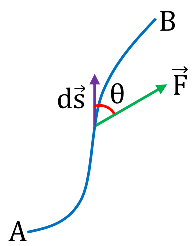
Variable Force means of force with changing direction or magnitude or both. Now, suppose a variable force displaces a body from position P_1 to position P_2. To calculate the work done by the force, the path from P_1 to P_2 can be divided into infinitesimal elements; each element is so small that during the displacement of the body through it, the force can be supposed to be constant. If $d\vec{r}$ be the small displacement of point of application and $\vec{F}$ be the force acting on the body during the displacement, '$d\vec{r}$', the work done by the force is $dW = \vec{F} \cdot d\vec{r}$

Total work done in displacing the body from P_1 to P_2 is given by $\int dW = \int_{P_1}^{P_2} \vec{F}.d\vec{r} \Rightarrow W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$

If $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of the points P_1 and P_2 respectively then the total work done $W = \int_{r_1}^{r_2} \vec{F}.d\vec{r}$

Cartesian Form

When magnitude and direction of the force varies with position, work done by the force for infinitesimal displacement $d\vec{s}$ is $dW = \vec{F}.d\vec{s}$



The total work done for the displacement from position A to B is

$W_{AB} = \int_A^B \vec{F}.d\vec{s} = \int_A^B (F\cos\theta)ds$

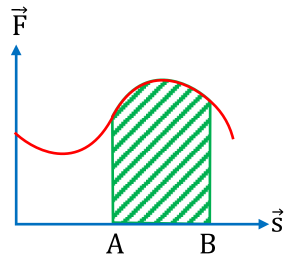
In terms of rectangular components

$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, $d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$W_{AB} = \int_A^B (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k}) = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

Graphical Meaning of Work

As we have seen, $W_{AB} = \int_A^B \vec{F}.d\vec{s}$



Since, above expression in right hand side represents **Area Under Curve (AUC)** of $\vec{F} - \vec{s}$ between positions A and B. Thus, if $\vec{F} - \vec{s}$ graph is given then we can find out the work done on the body between two particular positions by calculating AUC between that two positions.

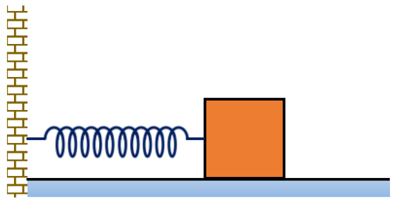
Here, area of the shaded portion in the above $\vec{F} - \vec{s}$ graph represents work done between A and B.

Note: - To calculate the work done by graphical method, for the sake of simplicity, here we have assumed the direction of force and displacement as same, but if they are not in same direction, the graph must be plotted between F cosθ and r.

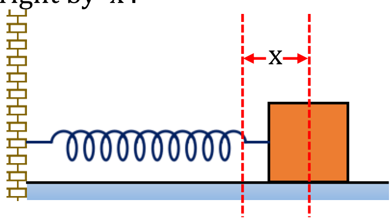
Work done by Spring Force

Here we have a block attached to one end of a spring (force constant-k) and other end is fixed.

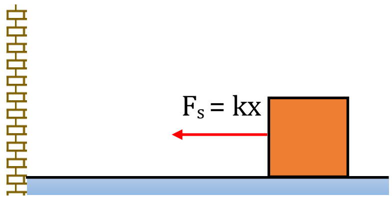
Initially the spring is in natural length as shown here.



Now, the block is displaced towards right by 'x'.

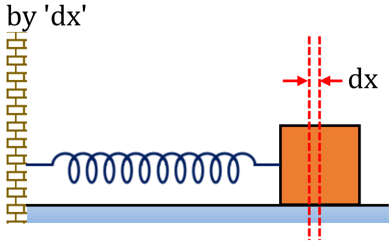


Spring force acting on the block at this instant will be



F.B.D. of the block at this instant will be

Now, if the block is further displaced by 'dx'



As we have already learnt that we can assume $F_s = kx$ to be constant for displacement 'dx'.

Now, work done by spring force for displacement 'dx'

$dW = \vec{F}_s \cdot d\vec{x} = -kxdx \qquad (\theta = 180°)$

Total work done $W_S = \int dW = \int_{x_i}^{x_f} -kxdx = \frac{1}{2}k\left(x_i^2 - x_f^2\right)$

Where x_f and x_i represent final and initial deformation of spring.

Note: Here, formula derived for elongation of spring but also valid for compression of spring.

Illustration 9:

A position dependent force F = 7 – 2x + 3x² acts on a small body of mass 2 kg and displaces it from x = 0 to x = 5 m. Calculate the work done in joules.

Solution:

$$W = \int_{x_1}^{x_2} F dx = \int_0^5 (7 - 2x + 3x^2) dx = \left[7x - \frac{2x^2}{2} + \frac{3x^3}{3}\right]_0^5 = 135\,J$$

Illustration 10:

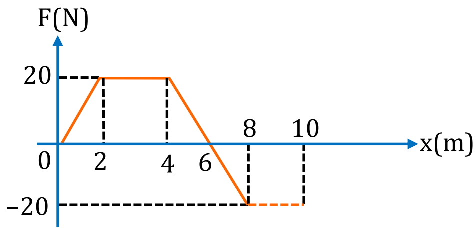
Find WD by force, $F = 3x^2 - 8x + 4N$ from x =2 m to x = 3 m.

Solution:

$$W = \int_2^3 F dx = \int_2^3 \left(3x^2 - 8x + 4\right) dx = \left[x^3 - 4x^2 + 4x\right]_2^3. = (27-8) - 4(9-4) + 4(3-2) = 19 - 20 + 4 = 3J$$

Illustration 11:

A graph shown below is plotted between force F acting on a block, kept on a horizontal floor v/s position (x) of the particle. Find work done on the particle when block reaches to x = 10 from x = 0.



Solution:

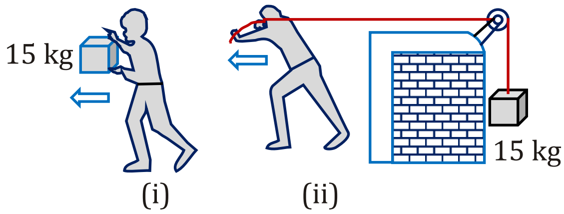
Work done = Area under F-x curve

$$W = \frac{1}{2}\times 2\times 20 + 20\times 2 + \frac{1}{2}\times 20\times 2 - \frac{1}{2}\times 2\times 20 - 2\times 20 = 20\ J$$

BEGINNER'S BOX-1

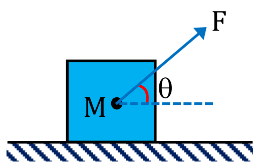
1. A body moves a distance of 10 m along a straight line under the action of a force of 5 N. If the work done is 25 joules, find the angle which the force makes with the direction of motion of the body. **PWP130**

2. In Fig. (i), the man walks 2 m carrying a mass of 15 kg on his hand. In Fig. (ii), he walks the same distance pulling the rope behind him. The rope goes over a pulley, and a mass of 15 kg hangs at its other end. In which case is the work done by gravity is greater?



PWP131

3. Calculate the work done in pulling an object with a constant force F as shown in figure, so that the block moves with a constant velocity through a distance 'd'. (given that the ground is rough with coefficient of friction μ)



PWP132

4. A particle moves along the x-axis from $x = x_1$ to $x = x_2$ under the influence of a force given by F = 2x. Find the work done in the process. **PWP133**

5. A force F = (10 + 0.5x) N acts on a particle in x direction, where x is in metres. Find the work done by this force during a displacement from x = 0 to x =2. **PWP134**

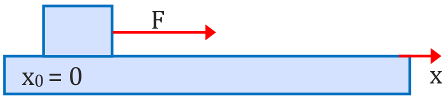
6. A particle of mass 0.5 kg travels in a straight line with velocity $v = ax^{3/2}$ where $a = 5\ m^{-1/2}s^{-1}$. What is the work done by the net force during its displacement from x = 0 to x = 2 m? **PWP135**

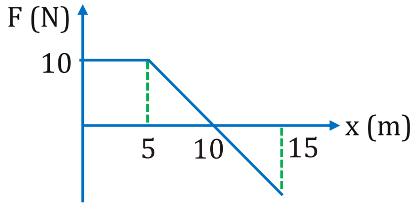
7. A particle moves from position $\vec{r}_1 = \left(3\hat{i}+2\hat{j}-6\hat{k}\right)$ m to position $\vec{r}_2 = \left(14\hat{i}+13\hat{j}+9\hat{k}\right)$ m under the action of force $\left(4\hat{i}+\hat{j}+3\hat{k}\right)$ N. Find the work done. **PWP136**

8. $\vec{F} = K(x\hat{i}+y\hat{j})$ is working on a particle which moves from (0, 0) to (a, 0) along x axis and (a, 0) to (a, a) along y axis. Find the work done by the force. **PWP137**

9. Three forces $3(\hat{i}+3\hat{j}+\hat{k}), \frac{5}{7}\left(-2\hat{i}+9\hat{k}\right)$ and $11\left(2\hat{i}+\hat{j}+6\hat{k}\right)$ are acting on a particle. Calculate the work done in displacing the particle from point (4, −1, 1) to point (11, 6, 8). **PWP138**

10. A horizontal force F is used to pull a box placed on a floor. Variation in the force with position coordinate x measured along the floor is shown in the graph.





(a) Calculate the work done by the force in moving the box from x = 0 to x = 10 m.
(b) Calculate the work done by the force in moving the box from x = 10 m to x = 15 m.
(c) Calculate the work done by the force in moving the box from x = 0 to x = 15 m.

PWP139

Energy

Energy is defined as the internal capacity to do work. When we say that a body has energy it means that it can do work.

Different forms of energy

Mechanical energy, electrical energy, optical (light) energy, acoustical (sound), molecular, atomic and nuclear energy etc., are various forms of energy.

These forms of energy can change from one form to the other.

Mass energy relation

According to Einstein's mass-energy equivalence principle, mass and energy are inter convertible i.e. they can be changed into each other.

Equivalent energy corresponding to mass m is $E = mc^2$

where, m : mass of the particle

c : speed of light

• Energy is a scalar quantity • Dimensions : $[M^1L^2T^{-2}]$ • SI Unit : Joule

Other units : 1 erg = 10^{-7} J, 1kwh = 36×10^5 J, 1 eV = 1.6×10^{-19} J, 1 cal = 4.2 J

In mechanics we are only concerned with the mechanical energy, which is of two types.

(a) kinetic energy (b) Potential energy

Kinetic Energy

Kinetic energy is the internal capacity of doing work of an object by virtue of its motion. (or)

K.E. of a body can be calculated by the amount of work done in stopping the moving body or by the amount of work done in imparting the present velocity to the body from the state of rest.

If a particle of mass m is moving with velocity 'v' much less than the velocity of light, then the kinetic energy K.E. is given by K.E. = $\frac{1}{2}mv^2$

Relation Between Kinetic Energy & Momentum

$K.E. = \frac{1}{2}mv^2$; Momentum p = mv Hence, $K = \frac{p^2}{2m}$ or $p = \sqrt{2mk}$

Thus, $K \propto p^2 \Rightarrow \frac{K_1}{K_2} = \frac{p_1^2}{p_2^2}$

For small changes (< 5%) ; $\frac{\Delta K}{K} = 2\frac{\Delta p}{p}$

Illustration 12:

Kinetic energy of a particle is increased by 300%. Find the percentage increase in its momentum.

Solution:

Kinetic energy $E = \frac{1}{2}mv^2$; Momentum p = mv

When E is increased by 300%, E' = E + 3E = 4E = $4\left(\frac{1}{2}mv^2\right) = 2mv^2$

If v' is the new velocity of the body, then $\frac{1}{2}mv^2 = 2mv^2 \Rightarrow v' = 2v$

Hence, percentage change in momentum = $\frac{2mv - mv}{mv} \times 100 = 100\%$.

Illustration 13:

If momentum is increased by 3%, find percentage change in KE.

Solution:

For small change (< 5%) ; $\frac{\Delta K}{K} = 2\frac{\Delta p}{p} = 2 \times \frac{3}{100} = \frac{6}{100}$

Percentage change in KE ; $\frac{\Delta K}{K} \times 100 = \frac{6}{100} \times 100 = 6\%$

Illustration 14:

If two particles of mass 4 kg and 7 kg have same momentum, then find ratio of their KE.

Solution:

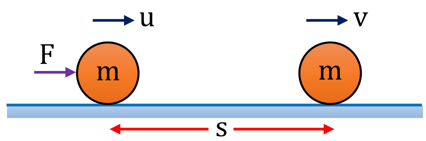
Ratio of Kinetic Energies

$$\frac{KE_1}{KE_2} = \left(\frac{\frac{P_1^2}{2M_1}}{\frac{P_2^2}{2M_2}}\right) = \frac{M_2}{M_1} = \frac{7}{4} \qquad (\because P_1 = P_2)$$

Golden Key Points

- As mass m and v^2 are always positive so K.E. can never be negative.
- The kinetic energy depends on the frame of reference as v depends on the frame of reference.
- The expression KE = $\frac{1}{2}mv^2$ holds even when the force applied varies in magnitude or in direction or in both.

Work Energy Theorem



It states that work done by all the forces (internal and external) on a particle equals to change in its kinetic energy.

$$W_{all} = \Delta KE$$

$$W_{all} = K_f - K_i = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Proof

(i) For constant force:

$$v^2 = u^2 + 2as \Rightarrow v^2 - u^2 = 2\left(\frac{F}{m}\right)s \Rightarrow s = \frac{\left(v^2 - u^2\right)m}{2F} \Rightarrow s = \frac{\Delta KE}{F} \Rightarrow Fs = \Delta KE \Rightarrow W = \Delta KE$$

(ii) For variable force :

$$W = \int Fdx = \int madx = \int m\left(\frac{vdv}{dx}\right)dx = \int_u^v mvdv = m\left[\frac{v^2}{2}\right]_u^v = \frac{m}{2}\left[v^2 - u^2\right] \Rightarrow W = \Delta KE$$

How to apply Work – Energy Theorem

The work-kinetic energy theorem is deduced here for a single body moving relative to an inertial frame, therefore it is recommended at present, to use it for a single body in inertial frame. To use work-kinetic energy theorem the following steps should be followed:

- Identify the initial and final positions as position 1 and 2 and write expressions for kinetic energies, whether known or unknown.
- Draw the free body diagram of the body at any intermediate stage between positions 1 and 2. The forces shown will help in deciding their work. Calculate work by each force and add them to obtain the total work done $W_{1\to 2}$ by all the forces.
- Use the work obtained in step 2 and kinetic energies obtained in step 1 in the equation.

Illustration 15:

A force applied on a particle of mass 2 kg changes its velocity from $\left(2\hat{i} + 6\hat{j}\right)$m/s to $\left(3\hat{i} + 4\hat{j}\right)$m/s. Find WD in the process.

Solution:

$$v_i = \sqrt{(2)^2 + (6)^2} = \sqrt{40}\ m/s\ ;\ v_f = \sqrt{(3)^2 + (4)^2} = 5\,m/s$$

$$W = \Delta KE = \frac{1}{2}m\left(v_f^2 - v_i^2\right) = \frac{1}{2}2\left(5^2 - (\sqrt{40})^2\right) = -15J$$

Golden Key Points

- If K.E. of the body decreases then work done is negative & vice-versa.
- In the above discussion, we have assumed that the work done by the force is effective only in changing the kinetic energy of the body. It should however be remembered that the work done on a body may also be stored in the system in the form of potential energy.

Problems based on Work Energy Theorem

Illustration 16:

A bullet weighing 10 g is fired with a velocity 800 m/s. its velocity decreases to 100 m/s after passing through a 1m thick mud wall. Find the average resistance offered by the mud wall.

Solution:

Work done by the average resistance (offered by the wall) = change in K.E. of the bullet.

$Fs=\frac{1}{2}mv^2-\frac{1}{2}mu^2 \quad \Rightarrow \quad F=\frac{m(v^2-u^2)}{2s}=\frac{0.01(100^2-800^2)}{2\times 1}=-3150\text{ N}$

$\Rightarrow$ Resistance offered = 3150 N.

Illustration 17:

In a ballistics demonstration, a police officer fires a bullet of mass 50.0 g with a speed of 200 m/s on soft plywood of thickness 2.00 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet?

Solution:

Initial kinetic energy, $K_i=\frac{1}{2}\times\frac{50}{1000}\times 200\times 200=1000\text{ J}$

Final kinetic energy, $K_f=\frac{10}{100}\times 1000=100\text{ J}$

If v_f is emergent speed of the bullet, then $\frac{1}{2}\times\frac{50}{1000}\times v_f^2=100\Rightarrow v_f^2=4000\Rightarrow v_f=63.2\text{ m/s}$

Note that the speed is reduced by approximately 68% and not 90%.

Illustration 18:

A particle of mass m moves with velocity $v=a\sqrt{x}$ where a is a constant. Find the total work done by all the forces during a displacement from x = 0 to x = d.

Solution:

Work done by all forces = W = $\Delta KE=\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2$

Here $v_1=a\sqrt{0}=0, v_2=a\sqrt{d}$ So, $W=\frac{1}{2}ma^2d-0=\frac{1}{2}ma^2d$

BEGINNER'S BOX-2

1. If the linear momentum of a body is increased by 50%, then by what percentage will its kinetic energy increase? **PWP140**
2. The kinetic energy of a body is numerically equal to thrice the momentum of the body. Find the velocity of the body. **PWP141**
3. A body of mass 10 kg is released from the top of a tower which acquires a velocity of 10 m/s after falling through the distance of 20 m. Calculate the work done by the drag force of the air on the body? (take g = 10 m/s²) **[AIPMT(Mains) 2008]** **PWP142**
4. The displacement x of a body of mass 1 kg on smooth horizontal surface as a function of time t is given by $x=\frac{t^3}{3}$ (where x is in metres and t is in seconds). Find the work done by the external agent for the first one second. **PWP143**

Conservative Force, Non Conservative Force and Central Force

Conservative Force

A force is said to be conservative if the work done by the force is independent of the path and depends only on initial and final positions.

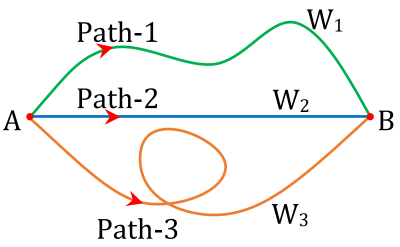
It does not depend on the nature of the path followed between the initial and the final positions.

Examples of Conservative Force

All central forces are conservative like Gravitational, Electrostatic, Elastic force, restoring force due to spring, Intermolecular force etc.

A body is taken from A to B by the help of a conservative force along three different paths as shown above and W_1, W_2 and W_3 are work done by the force along these paths respectively.

Then, $W_1 = W_2 = W_3$



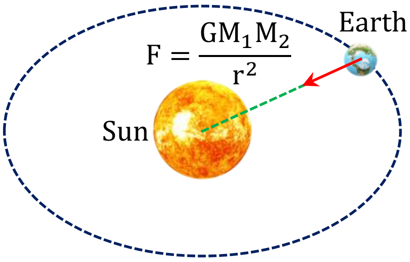
Central Force

A force whose line of action always passes through a fixed point (which is known as centre of force) and whose magnitude depends only on the distance from this point is known as **central force**.

$\vec{F} = F(r)\hat{r}$

All forces following inverse square law are called central forces.

$\vec{F} = \frac{k}{r^2}\hat{r}$ is a central force such as Gravitational force and Coulomb force.



- All central forces are conservative forces.
- Central forces are functions of position only.

Non-Conservative Force

A force is said to be non–conservative if work done by the force in moving a body depends upon the path between the initial and final positions.

Work done in a closed path is not zero in a non–conservative force field.

Frictional forces are non–conservative forces. This is because the work done against friction depends on the length of the path along which the body is moved. It does not depend on the initial and final positions. The work done by frictional force in a round trip is not zero.

Examples of non-conservative force

All velocity–dependent forces such as air resistance, viscous force are non–conservative forces.

Difference Between Conservative forces & Non-conservative Forces

Conservative Forces	Non-Conservative Forces
• Work done does not depend upon path. • Work done in a round trip is zero e.g. gravitational force • Central forces, spring forces etc. are conservative forces • When a conservative force acts within a system, the kinetic energy and potential energy can change. However, their sum, the mechanical energy of the system, does not change. • Work done is completely recoverable.	• Work done depends upon path. • Work done in a round trip is not zero e.g. friction. • Forces which are velocity-dependent in nature e.g. dragging force, viscous force, etc. • Work done against a non-conservative force may be dissipated as heat energy. • Work done is not completely recoverable.

Potential Energy

The energy possessed by a body by virtue of its position or configuration in a conservative force field.

Potential energy is a relative quantity.

Potential energy is defined only for conservative force field.

Potential energy of a body at any position in a conservative force field is defined as the external work done against the action of conservative force in order to shift it from a certain reference point (PE = 0) to the present position.

Potential energy of a body in a conservative force field is equal to the work done by the conservative force in moving the body from its present position to reference position.

At a certain reference position, the potential energy of the body is assumed to be zero.

Relationship between conservative force field and potential energy :

$$\vec{F} = -\nabla U = -\text{grad}(U) = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

If force varies with only one dimension (say along x-axis) then $F = -\frac{dU}{dx} \Rightarrow dU = -Fdx$

$$\Rightarrow \quad \int_{U_1}^{U_2} dU = -\int_{x_1}^{x_2} Fdx \Rightarrow \quad \boxed{\Delta U = -W_C}$$

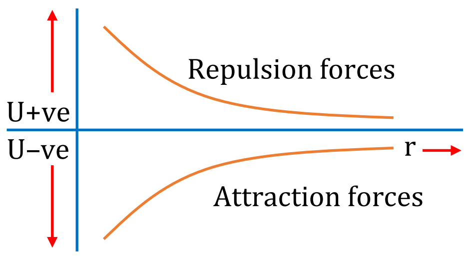
Potential energy may be positive or negative or even zero

(i) Potential energy is positive, if force field is repulsive in nature

(ii) Potential energy is negative, if force field is attractive in nature

If r $\uparrow$ (separation between body and force centre), U $\uparrow$, force field is attractive or vice–versa.

If r $\uparrow$, U $\downarrow$, force field is repulsive in nature.



PE $\uparrow$ increases when positive work is done by the external force (against the conservative force)

PE $\downarrow$ decreases when positive work is done by the conservative force

Illustration 19:

The potential energy of a particle of mass 1 kg free to move along the x-axis is given by $U(x) = \left(\frac{x^2}{2} - x\right)$ joules. If total mechanical energy of the particle is 2 J, then find its maximum speed.

Solution:

Potential energy $U = \left(\frac{x^2}{2} - x\right)$

For minimum U, $\frac{dU}{dx} = \frac{2x}{2} - 1 = 0$ and $\frac{d^2U}{dx^2} = 1 = \text{positive}$

So, at x = 1, U is minimum. Hence $U_{min} = -\frac{1}{2}J$.

Total mechanical energy = Max KE + Min PE $\Rightarrow$ Max KE = $\frac{1}{2}mv_{max}^2 = 2 - \left(-\frac{1}{2}\right) = \frac{5}{2} \Rightarrow v_{max}\sqrt{\frac{2}{1} \times \frac{5}{2}} = \sqrt{5}\ ms^{-1}$.

Illustration 20:

The potential energy for a conservative force system is given by U = ax^2 – bx, where a and b are constants. Find out the (a) expression for force (b) potential energy at equilibrium.

Solution:

(a) For conservative force F = $-\frac{dU}{dx} = -(2ax - b) = -2ax + b$ (b) U = $a\left(\frac{b}{2a}\right)^2 - b\left(\frac{b}{2a}\right) = \frac{b^2}{4a} - \frac{b^2}{2a} = -\frac{b^2}{4a}$.

BEGINNER'S BOX-3

1. Match the following items –

(A) Conservative force
(B) Non conservative force

(p) Mechanical energy = constant
(q) Kinetic energy = constant
(r) Work done in a closed path = 0
(s) $F = -\frac{dU}{dx}$
(t) Potential energy is not defined **PWP144**

2. The kinetic energy of a particle increases continuously with time. Then select correct alternative.
(A) the magnitude of its linear momentum is increasing continuously.
(B) its height above the ground must continuously decrease.
(C) the work done by all forces acting on the particle must be positive.
(D) the resultant force on the particle must be parallel to the velocity at all times.
PWP145

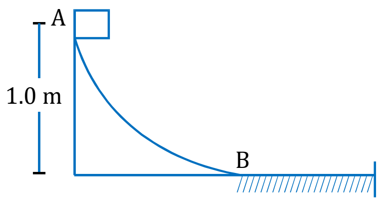
3. A particle moves in one dimensional field with total mechanical energy E. If its potential energy is U(x), then
(A) particle has zero speed where U(x) = 0
(B) particle has zero acceleration where U(x) = E
(C) particle has zero velocity where $\frac{dU(x)}{dx} = 0$
(D) particle has zero acceleration where $\frac{dU(x)}{dx} = 0$ **PWP146**

4. Match the following items –

	Column-I		Column-II
A	Work done by all the forces	p	Change in mechanical energy
B	Work done by internal conservative forces	q	Change in kinetic energy
C	Work done by external forces	R	Negative of changes in potential energy

PWP147

5. A block weighing 10 N travels down a smooth curved track AB joined to a rough horizontal surface. The rough surface has a friction coefficient of 0.20 with the block. If the block starts slipping on the track from a point 1.0 m above the horizontal surface, then it would move a distance S on the rough surface. Calculate the value of S [g = 10 m/s²]



PWP148

Golden Key Points

- In presence of conservative forces mechanical energy remains conserved.
- Work done along a closed path is zero by a conservative force i.e. $\oint \vec{F}.d\vec{r} = 0$
- Potential energy depends on the frame of reference but change in potential energy is independent of reference frame.
- Potential energy should be considered to be a property of the entire system, rather than assigning it to any specific particle.
- Potential energy is a function of position and does not depend on the path.
- Whenever work is done by the conservative forces the potential energy decreases and whenever work is done against the conservative forces, potential energy increases.

Gravitational and Spring Potential Energy

Gravitational Potential Energy

As we have seen, when a body is taken against Earth's Gravitational Force then work done by the weight on the body, $W_g = -mgh$

Now, since Earth's Gravitational Force is a conservative force hence this work done must be stored in the body in form of Gravitational Potential Energy (ΔU_g).

From the concept of potential energy

$$\Delta U = -W_{\text{Conservative force}} \Rightarrow \Delta U_g = -W_g \Rightarrow U_f - U_i = -(-mgh) \Rightarrow U_f - U_i = mgh$$

Where $U_f \to$ Gravitational Potential Energy at height h

$U_i \to$ Gravitational Potential Energy on ground

Here we can assume $U_i = 0$ for our convenience as potential energy is a relative quantity.

Thus, we have following relation $U_f = mgh$

Note: - For regularly shaped uniform bodies, the potential energy change can be calculated by considering their mass to be centred at the geometrical centre point.

For Example: For a uniform vertical rod of length L ; GPE = $mg\frac{L}{2}$

Spring Potential Energy

Work done by spring force $W_s = \frac{1}{2}k\left(x_i^2 - x_f^2\right)$

As we can see work done by spring force depends only on initial and final position. Thus, it is also a conservative force and hence Spring Potential Energy (U_s) can be defined as

$$U_s = -W_s = \frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

Illustration 21:

A body is dropped from height 8 m. After striking the surface it rises to 6 m, what is the fractional loss in kinetic energy during impact? Assuming the frictional resistance to be negligible.

Solution:

Fractional loss in kinetic energy $= \frac{\text{loss in kinetic energy}}{\text{initial kinetic energy}} = \frac{\text{loss in potential energy}}{\text{initial potential energy}} = \frac{mg(8-6)}{mg(8)} = \frac{2}{8} = \frac{1}{4}$

Illustration 22:

A chain of mass m and length L is held on a frictionless table in such a way that $\frac{1}{n}$th part is hanging below the edge of table. Calculate the work done to pull the hanging part of the chain.

Solution:

Required work done = change in potential energy of chain

Now, let Potential energy (U) = 0 at table level

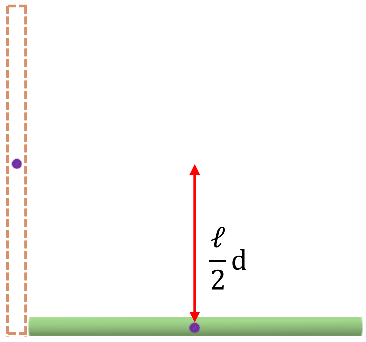
So, potential energies of chain initially and finally are respectively

$$U_i = -mg\left(\frac{L}{2n}\right) = -\left(\frac{M}{L}\right)\frac{L}{n}g\left(\frac{L}{2n}\right) = -\frac{MgL}{2n^2}, U_f = 0$$

$\therefore$ Required work done = $U_f - U_i = \frac{MgL}{2n^2}$

Illustration 23:

A rod of mass 'm' and length 'ℓ' is placed in a horizontal position. Find the work done by the external force against gravity to alter its configuration from horizontal position to vertical position.



Solution:

Displacement between the positions of center of mass of rod is $\frac{\ell}{2}$

(in a direction parallel to the force of gravity)

W = – change in PE of the rod = $mg\frac{\ell}{2}$.

Types of Equilibrium

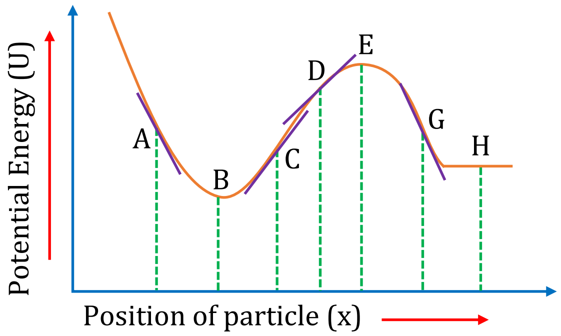
Equilibrium

At equilibrium net force is zero. If net force acting on a body is zero then the body is said to be in equilibrium and positions where body achieves equilibrium are called equilibrium position.

At equilibrium $F = -\frac{dU}{dx} = 0$

Potential Energy Curve

It is a curve which shows the change in potential energy with the position of a particle.



Stable Equilibrium

After a particle is slightly displaced from its equilibrium position if it tends to come back towards equilibrium due to restoring force then it is said to be in stable equilibrium.

Restoring Force is a force which acts to bring a body to its equilibrium position.

At point **A** : Slope $\frac{dU}{dx}$ is negative so F is positive

At point **C** : Slope is positive so F is negative

At point **B** : It is the point of stable equilibrium

At point **B** : $U = U_{min} \Rightarrow \frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} = \text{positive}$

Unstable Equilibrium

After a particle is slightly displaced from its equilibrium position, if it tends to move away from equilibrium position then it is said to be in unstable equilibrium.

At point **D** : slope $\frac{dU}{dx}$ is positive so F is negative ; At point **G** : slope $\frac{dU}{dx}$ is negative so F is positive

At point **E** : it is the point of unstable equilibrium ; At point E $U = U_{max} \Rightarrow \frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ = negative

Neutral equilibrium

After a particle is slightly displaced from its equilibrium position if no force acts on it then the equilibrium is said to be neutral equilibrium.

Point **H** corresponds to neutral equilibrium $\Rightarrow$ U = constant; $\frac{dU}{dx}=0, \frac{d^2U}{dx^2}=0$.

Illustration 24:

A particle moves in a potential region given by $U = 2x^2 - 2x + 100$ J. Value of x at which particle will be in equilibrium.

Solution:

$F=-\frac{dU}{dx}$; For equilibrium $\frac{dU}{dx}=0$; $4x - 2 = 0 \Rightarrow x = \frac{2}{4} = 0.5$

Illustration 25:

If $F = 2x^2 - 3x - 2$ then find out the position of stable equilibrium.

Solution:

$F = 2x^2 - 3x - 2 = 0$

For equilibrium, $F = 0$

$2x^2 - 3x - 2 = 0 \Rightarrow 2x^2 - 4x + x - 2 = 0$

$2x(x-2) + 1(x-2) = 0$; so, $x = 2$ and $x = -\frac{1}{2}$

$\frac{dF}{dx}=4x-3$; $\frac{dF}{dx}<0$ for stable equilibrium

Hence $x = -\frac{1}{2}$ is the position for stable equilibrium.

Law of Conservation of Mechanical Energy

Statement:- Energy can neither be created nor be destroyed, it can only be converted from one form to another.

The total potential energy of a system and the total kinetic energy of all the constituent bodies together is known as the mechanical energy of the system. If E, K and U respectively denote the total mechanical energy, total kinetic energy and the total potential energy of a system in any configuration then we have ;

$E = K + U$

Consider a system on which no external force acts and all the internal forces are conservative. If we apply work-kinetic energy $\left(W_{1\to2}=K_2-K_1\right)$ theorem, the work will be the work $W_{1\to2}$ done by internal conservative forces, negative of which equals the change in potential energy. Rearranging the kinetic energy and potential energy terms, we have, $E = K_1 + U_1 = K_2 + U_2$

The above equation takes the following form, $E = K + U$ = constant

$\Delta E=0 \Rightarrow \Delta K+\Delta U=0$

Above equations, express the principle of conservation of mechanical energy.

If there is no net work done by any external force or any internal non-conservative force, then the total mechanical energy of a system is conserved.

The principle of conservation of mechanical energy is developed from the work-kinetic energy principle for systems where change in configuration takes place under internal conservative forces only. Therefore, in physical situations, where external forces or non-conservative internal forces are involved, the use of work-kinetic energy principle should be preferred.

In systems, where external forces or internal nonconservative forces do work, the net work done by these forces becomes equal to the change in the mechanical energy of the system.

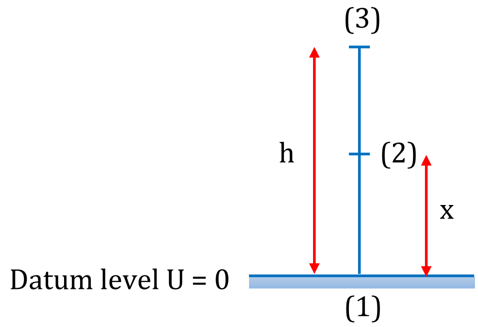
Application to gravitational field

Let a ball of mass m be dropped from position (3) (as shown in figure)

At point 1:

PE = 0 ; KE = $\frac{1}{2}mv^2$ $\quad \because v = \sqrt{2gh}$

so KE = mgh



At point 2:

PE = mgx ; KE = $\frac{1}{2}mv^2$ $\quad \because v = \sqrt{2g(h-x)}$

So KE = mg(h – x)

At point 3:

PE = mgh ; KE = 0

So during the motion of the ball at any position

$ME_{(1)} = ME_{(2)} = ME_{(3)}$ and PE = mgx ; KE = mg(h – x) (at any point x above ground)

Illustration 26:

A body is dropped from a height h. When loss in its potential energy is U then its velocity is v. The mass of the body is –

Solution:

$$U = \frac{1}{2}mv^2 \Rightarrow m = \frac{2U}{v^2}$$

Illustration 27:

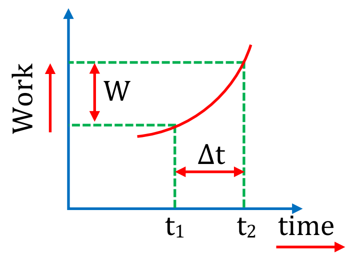
A block of mass 4 kg is moving on a frictionless horizontal surface with velocity 2m/s and comes to rest after pressing a spring. If the force constant of the spring is 100 N/m then the compression in the spring will be –

Solution:

By COME, $\frac{1}{2}mv^2 = \frac{1}{2}kx^2 \Rightarrow 4 \times (2)^2 = 100 \times x^2 \Rightarrow \frac{4\times4}{100} = x^2 \Rightarrow \frac{16}{100} = x^2 \Rightarrow x = 0.4$ m

Power

Average Power



When we purchase a car or jeep we are interested in the horsepower of its engine. We know that, usually an engine with large horsepower is most effective in accelerating the automobile.

In many cases, it is useful to know not just the total amount of work being done, but also how fast the work is being done. We can define power as the rate at which work is being done.

$$\text{Average power} = \frac{\text{Total Work done}}{\text{Time taken to do work}} = \frac{\text{Total change in kinetic energy}}{\text{Total change in time}}$$

If W is the amount of work done in a time interval Δt as shown in the below graph, then $P = \frac{W}{\Delta t} = \frac{W}{t_2 - t_1}$

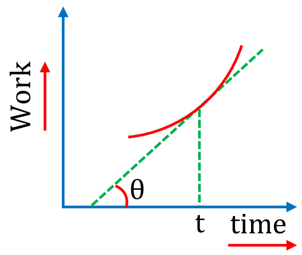
Instantaneous Power

Instantaneous power is the scalar product of force and velocity at a particular instant.

Instantaneous power $P = \dfrac{dW}{dt} = \dfrac{\vec{F}.d\vec{r}}{dt} = \vec{F}.\vec{v}$

where v is the instantaneous velocity of the particle. Here dot product is used because only that component of force will contribute to power which is acting in the direction of instantaneous velocity.

Also, Instantaneous power = slope of work-time curve = tanθ



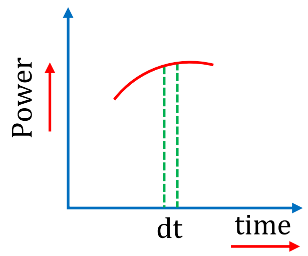
For a system of varying mass $F = \dfrac{d}{dt}(mv) = m\dfrac{dv}{dt} + v\dfrac{dm}{dt}$

If v = constant then F = $v\dfrac{dm}{dt}$ then $P = \vec{F}.\vec{v} = v^2\dfrac{dm}{dt}$

- Power is a scalar quantity with dimensions $M^1L^2T^{-3}$
- **SI unit : -** J/s or watt
- 1 horsepower = 746 watts = 550 ft–lb/s
  (for motors and engines, power is usually measured in horsepower)
- The definition of power is applicable to all types of energies also like mechanical, electrical, thermal.

Graphical Analysis

Area under power–time graph gives the work done $W = \int P dt$.



Efficiency

Machines are designed to convert energy into useful work, however because of frictional effects and other dissipative forces work performed by the machine is always less than the energy supplied to the machine.

Thus, we define efficiency of a machine which denote how much a machine is effective in converting energy into useful work.

Efficiency of a machine is given by $\eta = \dfrac{\text{work done}}{\text{energy input}}$

Illustration 28:

A small motor is used to power a lift that raises a load of bricks weighing 800N to a height of 20m in 40s. What is the minimum power motor needed?

Solution:

Assuming that the bricks are lifted without acceleration, the upwards force equal to the force of gravity 800N. Speed of the bricks is = $\dfrac{20m}{40s}$ = 0.5 m/s.

Now, power, P = Fv = (800N)(0.5 m/s) = 400 W

If there are no energy losses, e.g. to frictional forces, the motor must have a power output of 400W.

Illustration 29:

A truck of mass 10,000 kg moves up an inclined plane rising 1 in 50 with a speed of 36 km/h. Find the power of the engine (g = 10 m/s^2).

Solution:

Force against which work is done F = mg sin θ = 10,000 × 10 × $\dfrac{1}{50}$ = 2000N

Speed v = $\dfrac{36 \times 5}{18}$ = 10 m/s so P = 2000 × 10 = 20 kW.

Problems based on Power

Illustration 30:

A pump is used to deliver water at a certain rate from a given pipe. By what amount should velocity of water, Force on the water and power of motor be increased to obtain n times water from the same pipe in the same time?

Solution:

Amount of water flowing per unit time $\frac{dm}{dt} = Av\rho$

v = velocity of flow, A is area of cross–section, ρ = density of liquid

To get n times water in the same time, $\left(\frac{dm}{dt}\right)' = n\frac{dm}{dt} \Rightarrow Av'\rho = nAv\rho \Rightarrow v' = nv$

$$F = \frac{vdm}{dt} \Rightarrow F' = v'\frac{dm'}{dt} = n^2v\frac{dm}{dt} = n^2F$$

To gets n times water, force must be increased n^2 times.

$\because\ P = v^2\frac{dm}{dt}$ So $\frac{P'}{P} = \frac{v'^2(dm/dt)'}{v^2(dm/dt)} = \frac{n^2v^2n\ \ dm/dt}{v^2\ \ dm/dt} = n^3 \Rightarrow P' = n^3P$

Thus to get n times water, the power must be increased n^3 times.

Illustration 31:

A pump can take out 7200 kg of water per hour from a 100 m deep well. Calculate the power of the pump, assuming that its efficiency is 50%. (g = 10 m/s^2)

Solution:

Output power = $\frac{mgh}{t} = \frac{7200\times10\times100}{3600} = 2000\ W$ Efficiency $\eta = \frac{\text{output power}}{\text{input power}}$

Input power = $\frac{\text{output power}}{\eta} = \frac{2000\times100}{50} = 4\,kW$

Illustration 32:

An engine pumps water of density ρ, through a hose pipe. Water leaves the hose pipe of cross-sectional area A with a velocity v. Find the

(i) rate at which kinetic energy is imparted to water

(ii) power of the engine.

Solution:

(i) Rate of change of kinetic energy $= \frac{dE_k}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}v^2\frac{dm}{dt} = \frac{1}{2}v^2\frac{d}{dt}(\rho Ax) = \frac{1}{2}\rho Av^2\frac{dx}{dt} = \frac{1}{2}\rho Av^3$

(ii) Power = Fv = $\left(v\frac{dm}{dt}\right)v = v^2\frac{dm}{dt} = v^2(\rho Av) = \rho Av^3$

BEGINNER'S BOX-4

1. Two springs have their respective force constants K_1 and K_2. Both are stretched till their elastic potential energies are equal. If the stretching forces are F_1 and F_2 find $\frac{F_1}{F_2}$.

PWP149

2. A long spring is stretched by 1 cm. If work done in this process is W, then find out the work done in further stretching it by 1 cm.

PWP150

3. A 20 newtons stone falls accidentally from a height of 4 m on to a spring of stiffness constant 1960 N/m. Write down the equation to find out the maximum compression (x_m) of the spring.

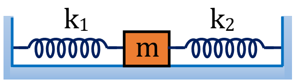
PWP151

4. A block of mass m moving with speed v compresses a spring through distance x before its speed is halved. What is the value of spring constant?

(A) $\frac{3mv^2}{4x^2}$ (B) $\frac{mv^2}{4x^2}$ (C) $\frac{mv^2}{2x^2}$ (D) $\frac{2mv^2}{x^2}$

PWP152

5. A block of mass m is attached to two springs of spring constants k_1 and k_2 as shown in figure. The block is displaced by x towards right and released. The velocity of the block when it is at x/2 will be :



(A) $\sqrt{\frac{(k_1+k_2)x^2}{2m}}$ (B) $\sqrt{\frac{3}{4}\frac{(k_1+k_2)x^2}{m}}$

(C) $\sqrt{\frac{(k_1+k_2)x^2}{m}}$ (D) $\sqrt{\frac{(k_1+k_2)x^2}{4m}}$

PWP153

6. A pump on the ground floor of a building can pump up water to fill a tank of volume 30 m^3 in 15 min. If the tank is 40 m above the ground, and the efficiency of the pump is 30%, how much electric power is consumed by the pump?

PWP154

7. An engine pumps up 100 kg of water through a height of 10 m in 5 s. If the efficiency of the engine is 60%, find out the power of the engine (g = 10 m/s^2).

PWP155

BEGINNER'S BOX ANSWER KEY

BEGINNER'S BOX-1							
1.	60°	**2.**	In second case work has to be done against gravity.				
3.	$\frac{\mu Mgd\cos\theta}{\cos\theta+\mu\sin\theta}$	**4.**	$x_2^2 - x_1^2$	**5.**	21 J	**6.**	50 J
7.	100 J	**8.**	Ka^2	**9.**	833 units	**10.**	(a) 75 J (b) –25 J (c) 50 J
BEGINNER'S BOX-2							
1.	125%	**2.**	6 units	**3.**	–1500 J	**4.**	0.5 J
BEGINNER'S BOX-3							
1.	(A) – p, r, s, (B) – t			**2.**	(A, C)	**3.**	D
4.	(A) – q, (B) – r, (C) – p			**5.**	5 m		
BEGINNER'S BOX-4							
1.	$\sqrt{\frac{K_1}{K_2}}$	**2.**	3W	**3.**	$49x_m^2 - x_m - 4 = 0$		
4.	A	**5.**	B	**6.**	43.6 kW	**7.**	3.33 kW