03 Rotational Motion

Introduction

Rigid body

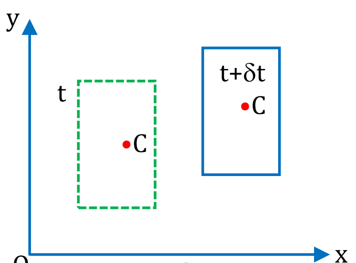
A rigid body is an assemblage of a large number of material particles, which do not change their mutual distances under any circumstance or in other words, the body is not deformed under any circumstance.

Actual material bodies are never perfectly rigid and are deformed under the action of external forces. When these deformations are small enough not to be considered during the course of motion, the body is assumed to be a rigid body. Hence, all solid objects such as stone, ball, vehicles etc are considered as rigid bodies while analysing their translational as well as rotational motion.

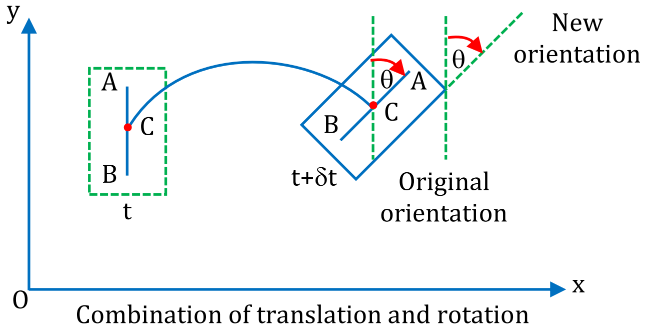
Rotational motion of a rigid body

Any kind of motion is identified by change in position or change in orientation or change in both. If a body changes its orientation during its motion it said to be in rotational motion.

In the following figures, a rectangular plate is shown moving in the x-y plane. The point C is its centre of mass. In the first case it does not change its orientation, therefore is in pure translation motion. In the second case it changes its orientation during its motion. It is a combination of translational and rotational motion.



Pure Translation



Combination of translation and rotation

Rotation i.e. change in orientation is identified by the angle through which a linear dimension or a straight line drawn on the body turns. In the figure this angle is shown by θ.

Types of motions involving rotation

Motion of body involving rotation can be classified into following three categories.

I. Rotation about a fixed axis.

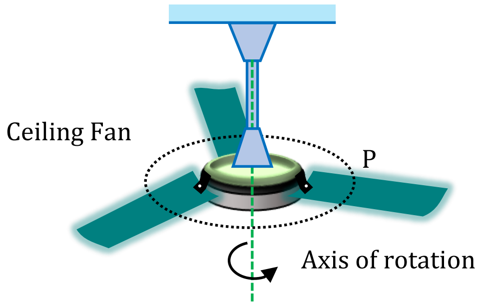
II. Rotation about an axis in translation.

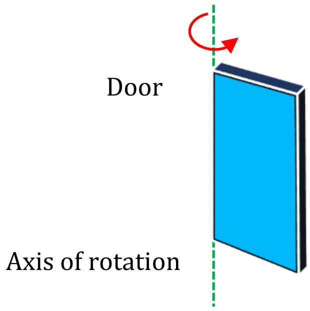
III. Rotation about an axis in rotation

Rotation about a fixed axis

Rotation of ceiling fan, opening and closing of doors and rotation of needles of a wall clock etc. come into this category.

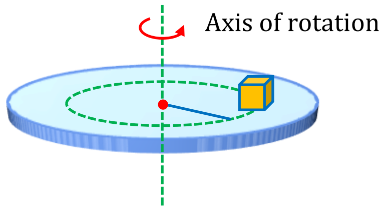
When a ceiling fan rotates, the vertical rod supporting it remains stationary and all the particles on the fan move on circular paths. Circular path of a particle P on one of its blades is shown by dotted circle. Centres of circular paths followed by every particle lie on the central line through the rod. This central line is known as the axis of rotation and is shown by a dashed line. All the particles on the axis of rotation are at rest, therefore the axis is stationary and the fan is in rotation about this fixed axis.





A door rotates about a vertical line that passes through its hinges. This vertical line is the axis of rotation. In the figure, the axis of rotation is shown by dashed line.

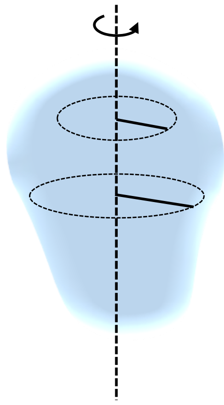
Axis of rotation



An imaginary line perpendicular to the plane of circular paths of particles of a rigid body in rotation and containing the centres of all these circular paths is known as axis of rotation.

It is not necessary that the axis of rotation should pass through the body. Consider a system shown in the figure, where a block is fixed on a rotating disc. The axis of rotation passes through the center of the disc but not through the block.

Important observations



Let us consider a rigid body of arbitrary shape rotating about a fixed axis PQ passing through the body. Two of its particles A and B are shown moving on their circular paths.

All its particles, not lying on the axis of rotation, move along circular paths with centres on the axis or rotation. All these circular paths are in parallel planes that are perpendicular to the axis of rotation.

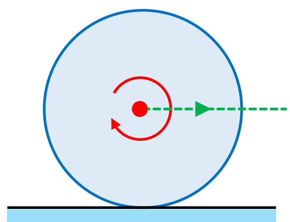
All the particles of the body undergo same angular displacement in the same time interval, therefore all of them move with the same angular velocity and angular acceleration.

Particles moving on circular paths of different radii move with different speeds and different magnitudes of linear acceleration. Furthermore, no two particles in the same plane perpendicular to the axis of rotation have same velocity and acceleration vectors.

Rotation about an axis in translation

Rotation about an axis in translation includes a broad category of motions. Rolling is an example of this kind of motion.

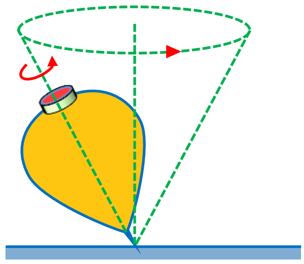
Consider the rolling of wheels of a vehicle, moving on straight levelled road. The wheel appears rotating about its stationary axle relative to a reference frame, attached with the vehicle. The rotation of the wheel as observed from this frame is rotation about a fixed axis. Relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axle, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions – rotation about the axle fixed with the vehicle and translation of the axle together with the vehicle.



Rotation about an axis in rotation.

In this kind of motion, the body rotates about an axis which in turn rotates about some other axis. Analysis of rotation about rotating axes is beyond our scope, therefore we shall keep our discussion elementary level only.

As an example consider a rotating top. The top rotates about its central axis of symmetry and this axis sweeps a cone about a vertical axis. The central axis continuously changes its orientation, therefore it is in rotational motion. This type of rotation in which the axis of rotation also rotates and sweeps out a cone is known as precession.



Another example of rotation about an axis in rotation is a swinging table-fan while running. Table-fan rotates about its shaft along which its axis of rotation passes. When running swings, its shaft rotates about a certain axis.

Note: In a rigid body angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body.

Illustration 1

In Rotational motion of a rigid body, all particles move with

(1) same linear and angular velocity

(2) same linear and different angular velocity.

(3) with different linear velocities & same angular velocities

(4) with different linear velocity & different angular velocities.

Solution: (3)

In rotational motion of the rigid body all particles cover the same angular displacements in a particular interval. So angular velocity of all the particle will be same. But linear velocity is also dependent on the distance of particles from the axis of rotation so linear velocity will be different for all particles as the distances are different for all the particles.

Kinematics of Rotational Motion

Comparison of Linear Motion and Rotational Motion

	Linear Motion		Rotational Motion
(i)	If acceleration is 0, v=constant & s = vt	(i)	If angular acceleration is 0, ω = constant & $\theta = \omega t$
(ii)	If acceleration a = constant, then (a) $s=\frac{(u+v)}{2}t$ (b) $a=\frac{v-u}{t}$ (c) $v = u + at$ (d) $s = ut + \frac{1}{2}at^2$ (e) $v^2 = u^2 + 2as$ (f) $S_{n^{th}} = u + \frac{a}{2}(2n-1)$	(ii)	If angular acceleration α = constant, then (a) $\theta=\frac{(\omega_0+\omega)}{2}t$ (b) $\alpha=\frac{\omega-\omega_0}{t}$ (c) $\omega = \omega_0 + \alpha t$ (d) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (e) $\omega^2 = \omega_0^2 + 2\alpha\theta$ (f) $\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$
(iii)	If acceleration is not constant, the above equation will not be applicable. In this case (a) $v=\frac{ds}{dt}$ (b) $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}=v\frac{dv}{ds}$	(iii)	If angular acceleration is not constant, the above equation will not be applicable. In this case (a) $\omega=\frac{d\theta}{dt}$ (b) $\alpha=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}=\omega\frac{d\omega}{d\theta}$

- In a rigid body, angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body.

Illustration 2

A wheel is rotating with angular velocity 2 rad/s. It is subjected to a uniform angular acceleration 2.0 rad/s^2.
(a) What angular velocity does the wheel acquire after 10 s?
(b) How many revolutions will it make in this time interval?

Solution:

The wheel is in uniform angular acceleration, Hence –

$\omega = \omega_0 + \alpha t$

$\omega = 2 + 2 \times 10 = 22$ rad/s

$$\theta = \frac{1}{2}(\omega_o + \omega)t$$

$$\theta = \frac{1}{2}(2+22)10 = 120\,\text{rad.}$$

In one revolution, the wheel rotates through 2π radians. Therefore the number of complete revolutions n is,

$$n = \frac{\theta}{2\pi} = \frac{120}{2\pi} \approx 19$$

Illustration 3

If the position vector $(\vec{r})$ of a point is $(\hat{i}+2\hat{j}+3\hat{k})$ m and its angular velocity $(\vec{\omega})$ is $(\hat{i}-\hat{j}+\hat{k})$ rad/s, then find the linear velocity of the particle.

Solution:

Linear velocity, $\vec{v} = \vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-5\hat{i}-2\hat{j}+3\hat{k})\,m/s$

BEGINNER'S BOX-1

1. A car has wheels of radius 0.30 m and is travelling at 36 m/s. Calculate :-
(a) the angular speed of the wheel.
(b) If the wheels describe 40 revolutions before coming to rest with a uniform acceleration.
(i) find its angular acceleration and (ii) the distance covered.

PRM168

2. A child's top is spun with angular acceleration $\alpha = 4t^3 - 3t^2 + 2t$ where t is in seconds and α is in radians per second-square. At t =0, the top has angular velocity ω_0 = 2 rad/s and a reference line on it is at an angular position θ_0 = 1 rad.

Statement I : Expression for angular velocity $\omega = (2+t^2-t^3+t^4)$ rad/s

Statement II : Expression for angular position $\theta = (1+2t-3t^2+4t^3)$ rad

(A) Only statement-I is true (B) Only statement-II is true
(C) Both of them are true (D) None of them are true

PRM169

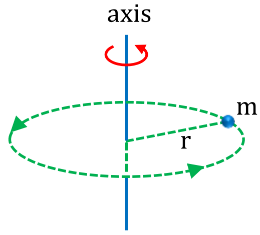
3. On account of the rotation of earth about its axis :-
(A) the linear velocity of objects at equator is greater than that at other places
(B) the angular velocity of objects at equator is more than that of objects at poles
(C) the linear velocity of objects at all places on the earth is equal, but angular velocity is different
(D) the angular velocity and linear velocity are uniform at all places

PRM170

4. A rotating rod starts from rest and acquires a rotational speed n = 600 revolutions/minute in 2 seconds with constant angular acceleration. The angular acceleration of the rod is :-
(A) 10π rad/s^2 (B) 5π rad/s^2 (C) 15π rad/s^2 (D) None of these

PRM171

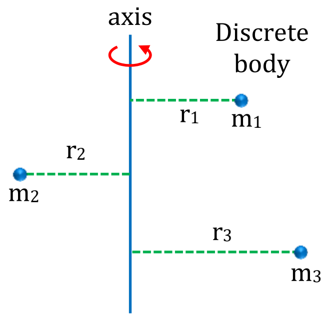
Moment of Inertia for Discrete System of Particles



- The measure of the property by virtue of which a body revolving about an axis opposes any change in state of its rotational motion is known as moment of inertia.
- The moment of inertia of a particle with respect to an axis of rotation is equal to the product of its mass and the square of its distance from the rotational axis.

  $I = mr^2$, r = perpendicular distance from the axis of rotation
- Moment of inertia of a system of particles

  $I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... = \sum mr^2$



- Moment of inertia depends on : (a) mass of the body

  (b) mass distribution of the body

  (c) position of axis of rotation
- Moment of inertia does not depend on :-

  (a) angular velocity (b) angular acceleration

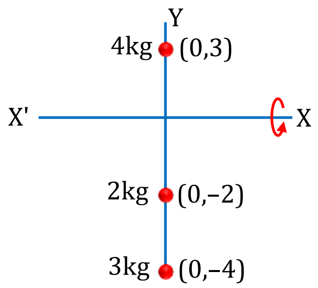
  (c) torque (d) angular momentum

  Unit : SI : $kg–m^2$, CGS : $g–cm^2$

  Dimension : $[M^1L^2T^0]$
- As the distance of mass increases from the rotational axis, the moment of inertia increases.

Illustration 4

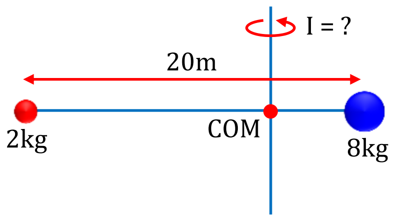
Calculate the moment of inertia about the rotational axis XX' in following figure?



Solution:

$I_{xx'} = 4 \times (3)^2 + 2 \times (2)^2 + 3 \times (4)^2 = 92$ kg–m²

Illustration 5

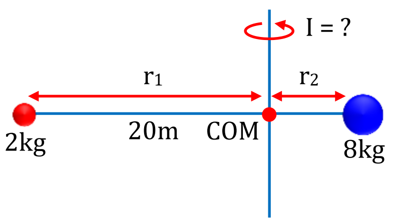


Two particles of masses 2kg and 8kg are tied at two ends of a massless rod of length 20m. Then Find MOI of system about axis passing through COM and perpendicular to the line joining particles.

Solution:

$$r_1 = \frac{m_2 r}{m_1 + m_2} = \frac{8 \times 20}{10} = 16m$$

$$r_2 = \frac{m_1 r}{m_1 + m_2} = \frac{2 \times 20}{10} = 4m$$



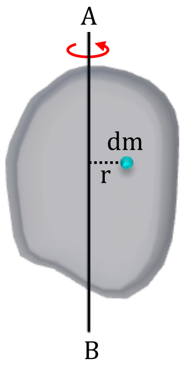
$I = m_1r_1^2 + m_2r_2^2 = 2 \times (16)^2 + 8 \times (4)^2 = 640$ kg-m²

Moment of Inertia for Simple Geometrical Objects

Moment of inertia for continuous mass distributions

Choose an appropriate element (of mass dm) on the body, at a perpendicular distance r from the axis. Then r^2dm integrated over the appropriate limits to cover the whole body gives moment of inertia.

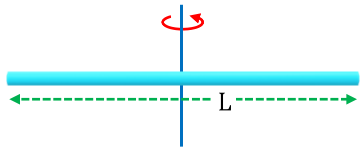
$$I_{AB} = \int r^2 dm$$



Note: Moment of inertia of two or more than two bodies can be added or subtracted only when all moment of inertia are written with respect to same axis.

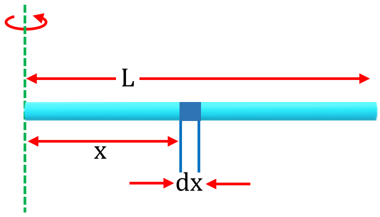
Moment of inertia of rod

Moment of inertia of rod about an axis passing through the centre of mass and perpendicular to its length = $\dfrac{ML^2}{12}$



Moment of inertia of a rod about an axis passing through its end and perpendicular to length

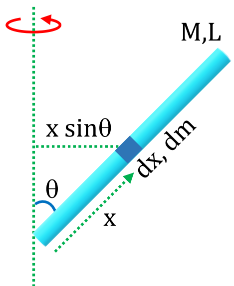
If the mass of the rod is 'M' & length of the rod is L, mass of element is dm then



$$dm = \frac{M}{L}dx$$

$$I = \int r^2 dm = \int_0^L x^2 \frac{M}{L} dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

Moment of inertia of rod about an axis inclined at some angle with axis (passing through one end)



$$dm = \frac{M}{L} \times dx$$

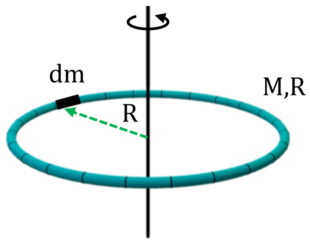
$$dI = dm \times (x \sin\theta)^2$$

$$dI = \frac{M}{L} \times dx \times x^2 \sin^2\theta$$

$$I = \int dI = \frac{M}{L}\sin^2\theta \int_0^L x^2 dx$$

$$I = \frac{M}{L}\sin^2\theta\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}\sin^2\theta$$

Moment of inertia of ring



Moment of inertia of ring about an axis perpendicular to its plane and passing through the centre = MR^2

$dI = dmR^2$

$I = \int dI = \int dmR^2$

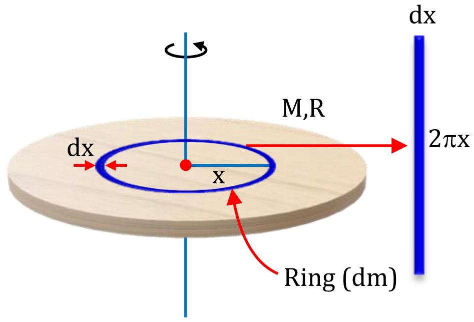
$I = R^2\int dm = R^2 \times M$

$I = MR^2$

Moment of inertia of Disc

About an axis passing through the centre and perpendicular to its plane = $\frac{1}{2}MR^2$

$dA = 2\pi x \times dx$ (Area of ring having radius as x and thickness dx)



$$dm = \frac{M}{\pi R^2} \times dA$$

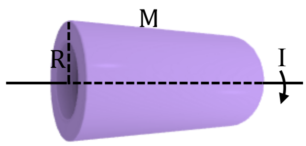
$$dm = \frac{M}{\pi R^2} \times 2\pi x\, dx = \frac{2M}{R^2} x\ dx$$

$$I = \int dI = \int_0^R \frac{2M}{R^2} \times x^3 dx$$

$$I = \frac{2m}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2}\left[\frac{R^4}{4} - 0\right] = \frac{1}{2}MR^2$$

Moment of Inertia of Hollow cylinder

Moment of Inertia of Hollow cylinder about its geometrical axis which is parallel to its length = MR^2

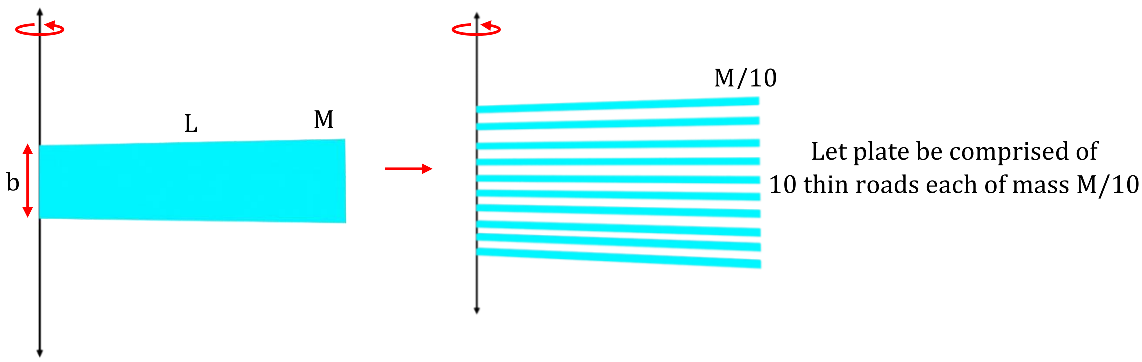


Moment of inertia of Solid cylinder

Moment of inertia of Solid cylinder about its geometrical axis, which is parallel to its length = $\frac{MR^2}{2}$

Moment of inertia of rectangular plate

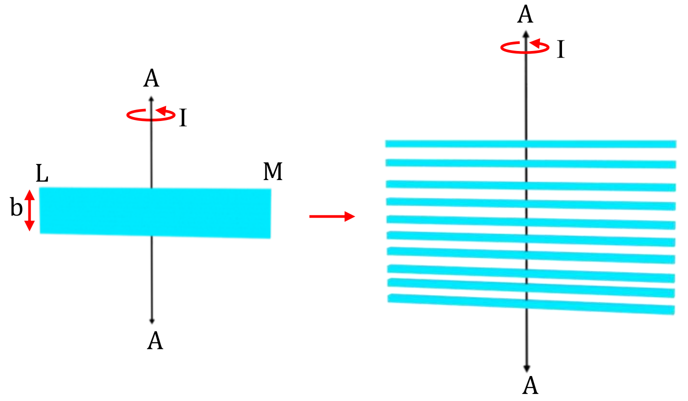
About an axis that lies in the plane of plate and parallel to shorter sides



$$I = \left[\left(\frac{M}{10}\right) \times \frac{L^2}{3}\right] \times 10 = \frac{ML^2}{3}$$

Moment of inertia of rectangular plate

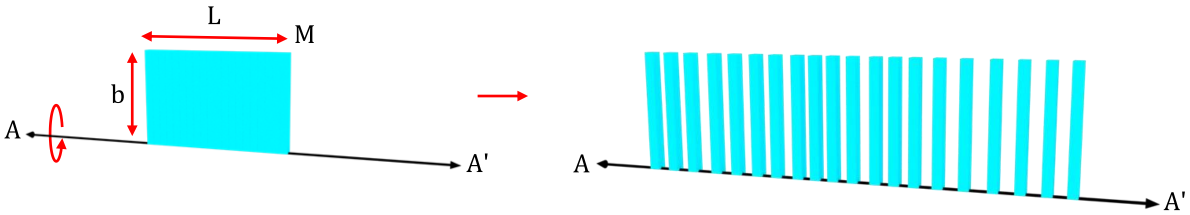
Moment of inertia of rectangular plate about the axis AA' = $\frac{ML^2}{12}$



$I_{AA'} = \frac{ML^2}{12}$

Moment of inertia of rectangular plate

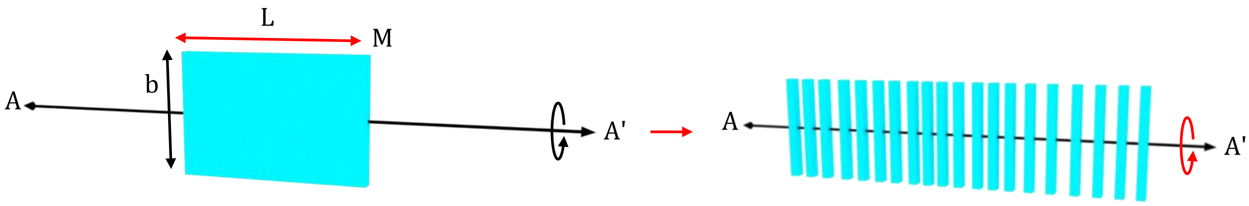
Moment of inertia of rectangular plate about the axis AA' = $\frac{Mb^2}{3}$



$I_{AA'} = \frac{Mb^2}{3}$

Moment of inertia of rectangular plate

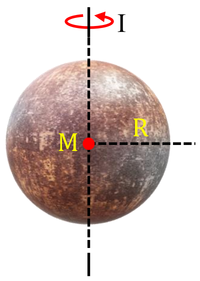
Moment of inertia of rectangular plate about the axis AA' = $\frac{Mb^2}{12}$



$I_{AA'} = \frac{Mb^2}{12}$

Moment of inertia of Solid sphere

Moment of inertia of Solid sphere about its diametric axis = $\frac{2}{5}MR^2$



Moment of inertia of Hollow sphere

Moment of inertia of Hollow sphere about its diametric axis = $\frac{2}{3}MR^2$

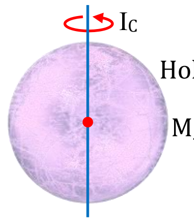


Illustration 6

MOI of Ring and MOI of disc about an axis passing through centre and perpendicular to its plane are same? What will be the ratio of there radii?

Solution:

$I_{ring} = I_{Disc} \quad MR_1^2 = \frac{MR_2^2}{2}$

$\therefore \quad \frac{R_1^2}{R_2^2} = \frac{1}{2} \Rightarrow \frac{R_1}{R_2} = \frac{1}{\sqrt{2}}$ i.e. $R_1 : R_2 = 1 : \sqrt{2}$

Illustration 7

If I_1, I_2 and I_3 are moments of inertia of solid sphere, hollow sphere and a ring of same mass and radius about geometrical axis, which of the following statement holds good?

(1) $I_1 > I_2 > I_3$ (2) $I_3 > I_2 > I_1$ (3) $I_2 > I_1 > I_3$ (4) $I_2 > I_3 > I_1$

Solution: (2)

$I_1 = \frac{2}{5}MR^2$; $I_2 = \frac{2}{3}MR^2$; $I_3 = MR^2$

Illustration 8

A solid sphere and a hollow sphere of the same mass have the same M.I. about their respective geometrical axis. The ratio of their radii will be -

Solution:

given $I_{solid\ sphere} = I_{hollow\ sphere}$

$\Rightarrow \quad \frac{2}{5}Mr_1^2 = \frac{2}{3}Mr_2^2 \Rightarrow \frac{r_1^2}{r_2^2} = \frac{5}{3} \Rightarrow \frac{r_1}{r_2} = \sqrt{5} : \sqrt{3}$

Illustration 9

What will be the MOI of disc about an axis passing through centre an perpendicular to the plane if mass of disc becomes 2 times and radius 2 times

Solution:

$I = \frac{MR^2}{2}$

$M' = 2M$ & $R' = 2R$

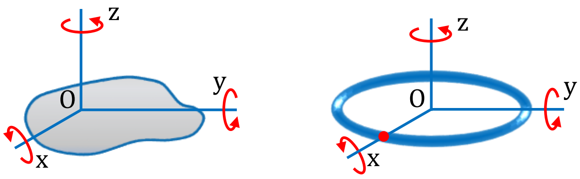
$I' = \frac{M'R'^2}{2} = \frac{(2M)\times(2R)^2}{2} = \frac{2M\times 4R^2}{2} = \frac{MR^2}{2}\times 8 = 4MR^2$

Perpendicular Axis Theorem

Theorem of perpendicular axes (applicable only for two dimensional bodies or plane lamina)

The moment of inertia (MI) of a plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about any two mutually perpendicular axes in its own plane intersecting each other at the point through which the perpendicular axis passes.

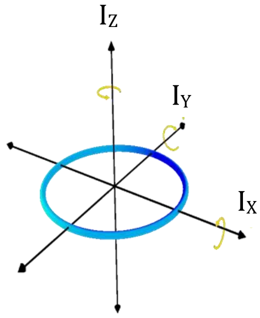
$$I_z = I_x + I_y$$



Where, I_x = MI of the body about X-axis; I_y = MI of the body about Y-axis; I_z = MI of the body about Z-axis

It is applicable only for two dimensional bodies and cannot be applied for three dimensional bodies.

MOI of Ring about an axis passing through diameter of ring



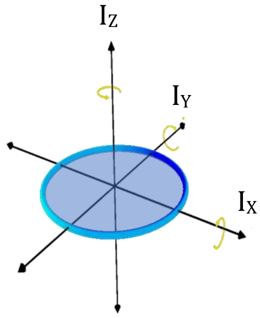
Let moment of inertia of the ring about each diameter = I_d (i.e. X and Y)

Both the diameters are perpendicular to the axis Z which is passing through the centre of ring and perpendicular to its plane,

By theorem of perpendicular axes. $I_X + I_Y = I_Z$

or $\quad I_d + I_d = I_Z \quad \Rightarrow 2I_d = MR^2 \quad \Rightarrow I_d = \frac{1}{2}MR^2$

MOI of Disc About an axis passing through diameter of disc



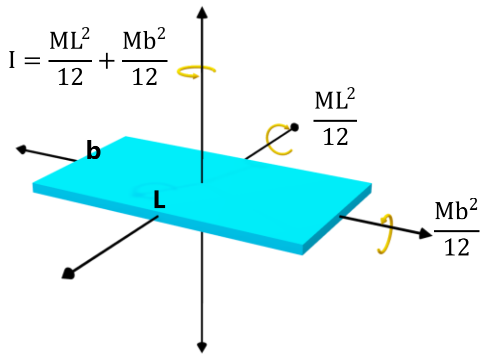
Let moment of inertia of the disc about each diameter = I_d (i.e. X and Y)

Both the diameters are perpendicular to the axis Z which is passing through the centre of disc and perpendicular to its plane,

By theorem of perpendicular axes. $I_X + I_Y = I_Z$

or $\quad I_d + I_d = I_Z \Rightarrow 2I_d = MR^2/2 \quad \Rightarrow I_d = \frac{1}{4}MR^2$

MOI of Rectangular Plate about an axis passing through centre and perpendicular to the plane of rectangular plate



Let us consider x, y & z-axis to be three mutually perpendicular axes as shown.

The moment of inertia of the rectangular lamina about the centroidal axis is $I_Z = I$.

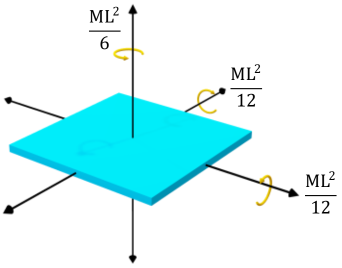
The moment of inertia about the x-axis is, $I_X = \frac{Mb^2}{12}$

The moment of inertia about the y-axis is, $I_Y = \frac{ML^2}{12}$

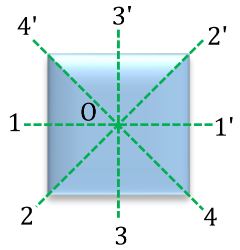
From perpendicular axes theorem, $I = I_x + I_y$

$$I = \frac{ML^2 + Mb^2}{12} = \frac{M(L^2 + b^2)}{12}$$

MOI of a square plate about an axis passing through centre and perpendicular to plane of the square plate.



MOI of a square plate about an axis passing through centre and perpendicular to plane of the square plate = $\frac{ML^2}{6}$

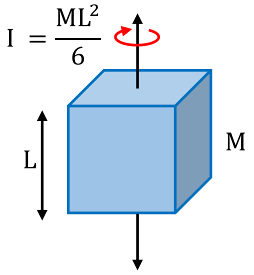


The MOI of a square plate of mass M and side "L", about the axes shown in Figure are all equal.

Thus, $I_{11'} = I_{22'} = I_{33'} = I_{44'} = \frac{ML^2}{12}$

MOI of Cube about an axis passing through centre of mass and perpendicular to its face

Since cube has similar mass distribution as square plate therefore MOI of cube about an axis passing through centre of mass and perpendicular to its face will be same as that of square plate

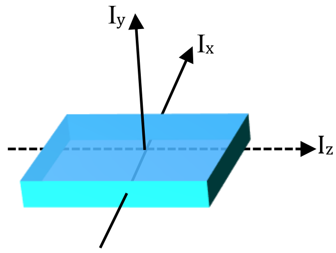


$\therefore \quad I = \frac{ML^2}{6}$

Illustration 10

A lamina lies in the x-z plane. If its moment of inertia about x-axis is 40 kgm², and about y-axis is 120 kg-m². Then find its moment of inertia about z-axis.

Solution:



$I_y = I_x + I_z$

$120 = 40 + I_z$

$I_z = 80$ kg-m²

Illustration 11

Calculate the MOI of a uniform rectangular lamina of length 3m, breadth 2m and mass 6kg about the centroidal axis perpendicular to the plane of the lamina?

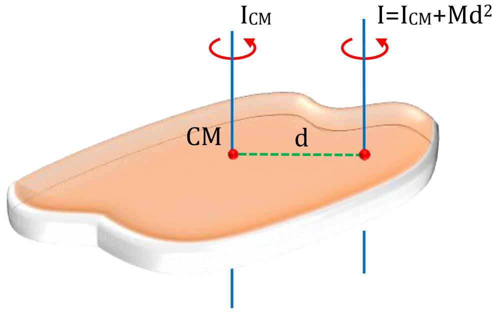
Solution:

$\because \quad L = 3\text{m and } b = 2\text{m}$

$$\therefore \quad I = \frac{M(L^2+b^2)}{12} = 6 \times \frac{\left((3)^2+(2)^2\right)}{12} = \frac{9+4}{2} = \frac{13}{2} = 6.5 \text{ kgm}^2$$

Parallel Axis Theorem

Moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis passing through the centre of mass plus product of mass of the body and the square of the distance between these two parallel axes.



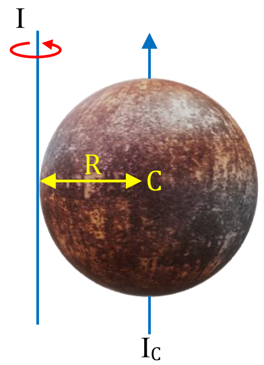
I_{CM} = Moment of inertia about the axis passing through the centre of mass.

I = Moment of inertia of rigid body about an axis which is parallel to axis passing through centre of mass and is at distance d from the axis passing through centre of mass.

Note: This Theorem is valid for all type of bodies.

Illustration 12

The moment of inertia of a sphere is 40 kg-m² about its diametric axis. Determine the moment of inertia of sphere about the tangential axis?



Solution:

Given that $\frac{2}{5}MR^2 = 40 \Rightarrow$ $MR^2 = 100$ kg-m².

By theorem of parallel axes,

$I = I_{CM} + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2 = \frac{7}{5}\times 100 = 140$ kg–m².

Illustration 13

Four rods are arranged in the form of a square. Calculate the moment of inertia of the system of rods about an axis passing through the centre and perpendicular to the plane of the square (Assume that each rod has a mass M and length L)

Solution:

From parallel axes theorem,

$I = 4I_{CM} + 4M\left(\frac{L}{2}\right)^2 = 4\left(\frac{ML^2}{12}\right) + 4\frac{ML^2}{4} = \frac{4}{3}ML^2.$

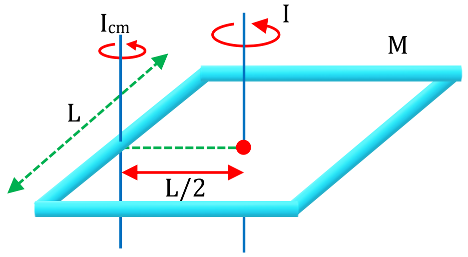


Illustration 14

The moment of inertia of a rod of mass M and length L about an axis passing through one edge and perpendicular to this length will be :

Solution:

Using parallel axes theorem $\quad I = I_{CM} + Md^2 = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$

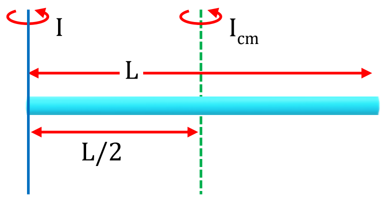
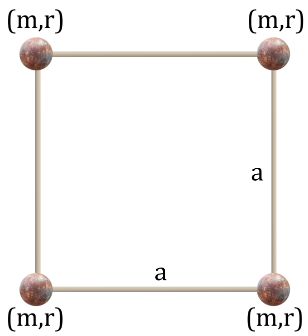
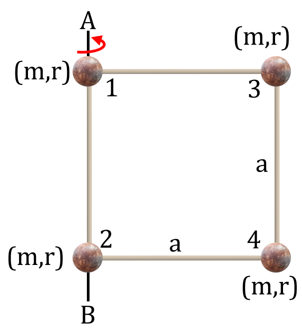


Illustration 15

Four spheres having mass m and radius r are placed with their centres on the four corners of a square of side a. Then the moment of inertia of the system about an axis along one of the sides of the square is



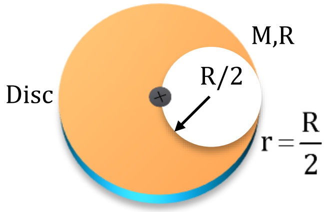
Solution:



$I_{AB} = I_{1_{AB}} + I_{2_{AB}} + I_{3_{AB}} + I_{4_{AB}} = \frac{2}{5}mr^2 + \frac{2}{5}mr^2 + \left(\frac{2}{5}mr^2 + ma^2\right) + \left(\frac{2}{5}mr^2 + ma^2\right) = \frac{8}{5}mr^2 + 2ma^2$

Illustration 16

A small disc of radius $\frac{R}{2}$ is taken out from big disc of radius R and mass M. Then find the MOI of remaining part about the given axis ?



Solution:

$I_{remaining} = I_{complete} - I_{removed}$ about same axis

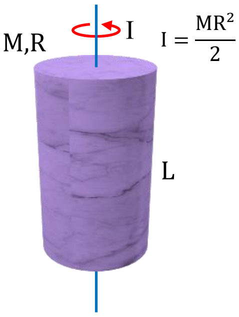
$$= \frac{MR^2}{2} - \left(\frac{mr^2}{2} + mr^2\right) \qquad \pi R^2 \rightarrow M$$

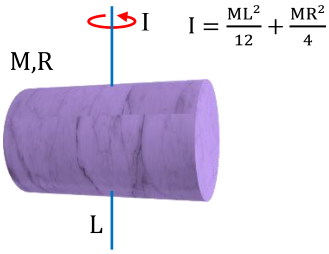
$$= \frac{MR^2}{2} - \frac{3}{2}mr^2 \qquad 1 \rightarrow \frac{M}{\pi R^2}$$

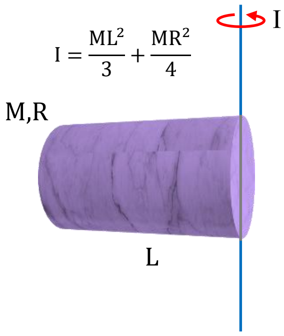
$$= \frac{MR^2}{2} - \frac{3}{2}\left(\frac{M}{4}\right)\left(\frac{R}{2}\right)^2 \qquad \pi\left(\frac{R}{2}\right)^2 \rightarrow \frac{M}{\pi R^2} \times \frac{\pi R^2}{4}$$

$$= \frac{MR^2}{2} - \frac{3MR^2}{32} = \frac{13MR^2}{32} \qquad m = \frac{M}{4}$$

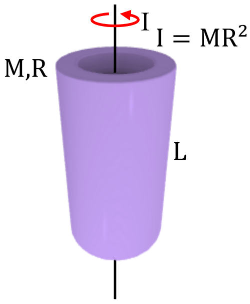
Moment of inertia of Solid Cylinder

(i)


(ii)


(iii)


Moment of inertia of Hollow Cylinder

(i)


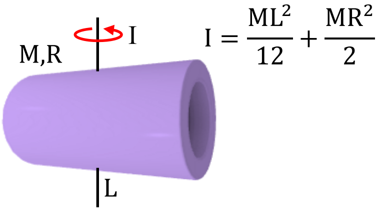
(ii)


Illustration 17

Two circular discs A and B of equal masses and thickness (t) but are made of metals with densities d_A and $d_B (d_A > d_B)$. If their moments of inertia about an axis passing through the centre and normal to the circular faces be I_A and I_B, then relation between I_A and I_B is

Solution:

Mass = volume × density ⇒ $M = (\pi R^2 t)d$ ...(i)

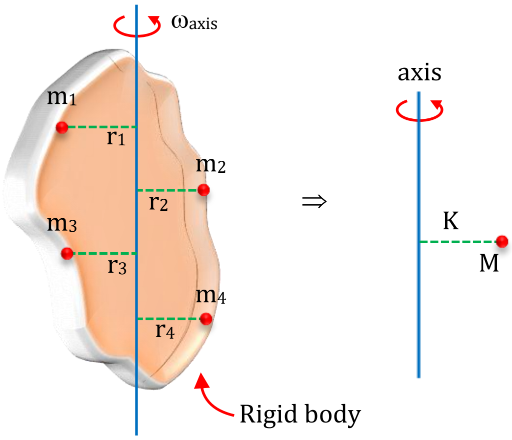
Where t is thickness

$\because \quad I = \frac{MR^2}{2}$ ...(ii)

From (i) and (ii), we get : $I = \frac{M^2}{2\pi dt} \quad \Rightarrow I \propto \frac{1}{d} \quad \Rightarrow I_A < I_B$

Radius of Gyration

The Radius of gyration of a body is the distance from axis of rotation, the square of this distance when multiplied by the mass of body then it gives the moment of inertia of the body ($I = MK^2$) about same axis of rotation.



$I = MK^2 \quad \Rightarrow I = \Sigma mr^2$

$MK^2 = \Sigma mr^2$

$$K^2 = \frac{m_1r_1^2 + m_2r_2^2 + ...+ m_nr_n^2}{M}$$

$$K = \sqrt{\frac{m_1r_1^2 + m_2r_2^2 + ...+ m_nr_n^2}{m_1 + m_2 + ...+ m_n}}$$

If $m_1 = m_2 = m_3 = m$ then, $M = mn$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + ...+ r_n^2}{n}}$$, n = total number of particles

Radius of gyration $\mathbf{K = \sqrt{\frac{I}{M}}}$

Radius of gyration depends on

(i) axis of rotation

(ii) distribution of mass of body

Radius of gyration does not depend on

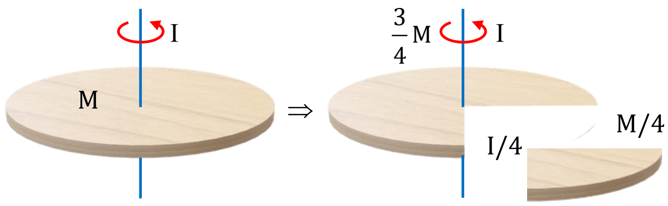
(i) Mass of the body

(ii) Angular quantities (angular displacement, angular velocity etc.)

Symmetrical separation

M = mass of disc

R = Radius of disc



If $\frac{M}{4}$ part is separated

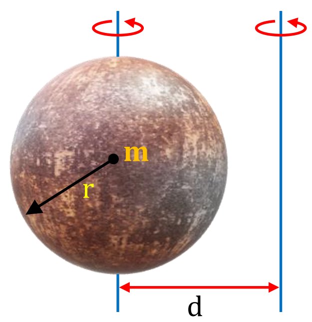
Remaining mass = $\frac{3}{4}M = M'$

Radius of gyration of remaining part

$$K' = \sqrt{\frac{\frac{3I}{4}}{\frac{3M}{4}}} = \sqrt{\frac{I}{M}} = K \Rightarrow \text{K remains unchanged}$$

Illustration 18

The radius of gyration of a solid sphere of radius r about a certain axis is r. Calculate the distance of that axis from the centre of the sphere



Solution:

From parallel axes theorem,

$\because \quad I = I_{CM} + md^2$

$\therefore \quad mr^2 = \frac{2}{5}mr^2 + md^2$

$\Rightarrow \quad d = \sqrt{\frac{3}{5}}r = \sqrt{0.6}\,r$.

BEGINNER'S BOX-2

1. Point masses of 1, 2, 3 and 10 kg are lying at the points (0, 0), (2m,0) (0, 3m) and (– 2m, – 2m) respectively in x–y plane. Find the moment of inertia of this system about y - axis. (in kg–m^2).
PRM172

2. The moment of inertia of a ring and a disc of same mass about their diameters are same. What will be the ratio of their radii?
PRM173

3. Two rings have their moments of inertia in the ratio 4 : 1 and their diameters are in the ratio 4 : 1. Find the ratio of their masses.
PRM174

4. Two bodies of masses 1 kg and 2 kg are attached to the ends of a 2 m long weightless rod. This system is rotating about an axis passing through the middle point of rod and perpendicular to length. Calculate the M.I. of system.
PRM175

5. If the moment of inertia of a disc about an axis tangential and parallel to its surface be I, then what will be the moment of inertia about an axis tangential but perpendicular to the surface?
PRM176

6. A disc is recast into a thin hollow cylinder of same radius. Which will have larger moment of inertia? Explain.
PRM177

7. Why spokes are fitted in a cycle wheel?
PRM178

8. A circular disc A of radius r is made from an iron plate of thickness t and another circular disc B of radius 4r is made from an iron plate of thickness t/4. Find the relation between the moment of inertia I_A and I_B.
PRM179

9. A square plate of side ℓ has mass per unit area μ. Find its moment of inertia about an axis passing through the centre and perpendicular to its plane.
PRM180

10. Three masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of side a. What is the moment of inertia of the system about an axis along the median passing through m_1?
PRM181

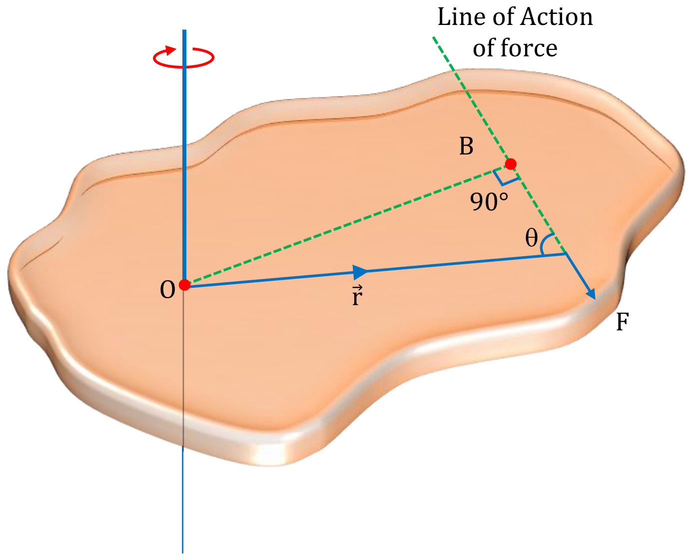
11. Two rings having the same radius and mass are placed such that their centres are at a common point and their planes are perpendicular to each other. Find the moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings. (M = mass of each ring and R = radius).
PRM182

12. Moment of inertia of a sphere about its diameter is 2/5 MR^2. What is its moment of inertia about an axis perpendicular to its two diameters and passing through their point of intersection?
PRM183

Moment of a Force-Torque

It is the physical agency which is responsible for change in state of rotation. Torque is essential for producing turning / toppling phenomena.

For producing torque the force is required & it is product of force and perpendicular distance of line of action of force (lever arm from axis).



In figure OB is the perpendicular distance of line of action from axis which is r sinθ

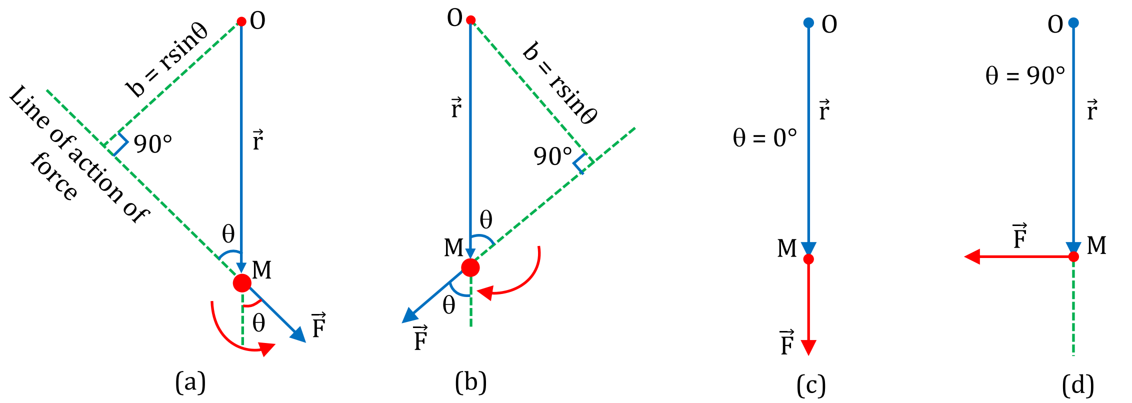
So, $\tau = Fr\sin\theta$

Vector form $\longrightarrow$ $\vec{\tau} = \vec{r} \times \vec{F}$

Where $\vec{F}$ = Force applied

$\vec{r}$ = position vector of the point of application of force w.r.t the point about which we want to determine the torque.

- It is an axial vector i.e. its direction is always perpendicular to the plane containing vector $\vec{r}$ & $\vec{F}$
- It's direction is determined by right hand thumb rule.
- Positive sign to all torques acting to turn a body anti clockwise and a minus to all torque tending to turn it clockwise.



Although force is required for producing the torque yet every force is not capable to produce torque. If line of action of force passes through the axis then torque about the axis is zero. [as shown in figure (C).]

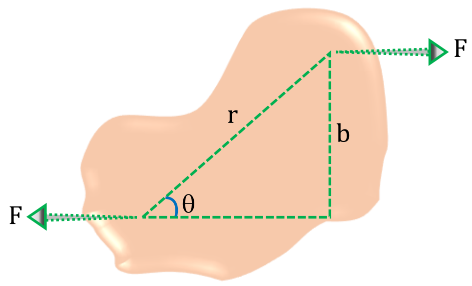
Torque is also known as moment of force.

Minimum Torque	**Maximum Torque**
When | sinθ | is min = 0	When | sinθ | is max = 1
i.e., θ = 0° or 180°	i.e., θ = 90°
$\tau_{min} = 0$	$\tau_{max} = Fr$

- **Unit of torque :** N-m (same as that of work or energy)
  Dimensions of torque : $[ML^2T^{-2}]$

Note: Unit of torque (N-m) cannot be written as joule, because joule is used specifically for work or energy.

Couple of forces



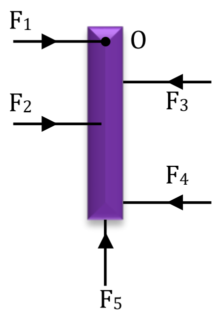
- When two forces of equal magnitude act on different points and in opposite directions with distinct lines of action these force form a couple. This couple tries to rotate body.
- Moment of couple $\tau = Fr\sin\theta = Fb$
- Rotation of a door about a hinge, rotation of grinding wheel about a pivot or unbolting a nut by a pipe-wrench can be cited as examples involving torque. These examples represent rotational effects.

$$\tau = \text{constant} \Rightarrow Fr\sin\theta = \text{constant} \Rightarrow F = \frac{\text{constant}}{r\sin\theta}$$

Longer the arm and greater the value of sinθ, lesser will be the force required for producing desired rotational effect. This fact explains why it is much easier to rotate a body about a given axis when the force is applied at maximum distance from the axis of rotation and normal to the arm.

Illustration 19

Different forces are applied on a pivoted scale as shown. Which of the following forces will produce torque?



Solution:

$F_1 \neq 0, \tau_1 = 0$

(As line of action of force passes from the pivoted point O)

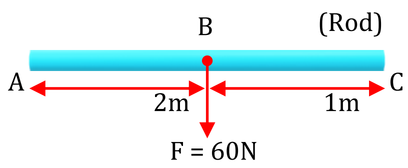
$F_2 \neq 0$ and produces τ_2 (Anticlockwise)

$F_3 \neq 0$ produces τ_3 (Clockwise)

$F_4 \neq 0$ produces τ_4 (Clockwise)

$F_5 \neq 0, \tau_3 = 0$; as line of action of force passes from the pivoted point O.

Illustration 20



(i) find $(\tau_F)_A = ?$

(ii) find $(\tau_F)_B = ?$

(iii) find $(\tau_F)_C = ?$

Solution:

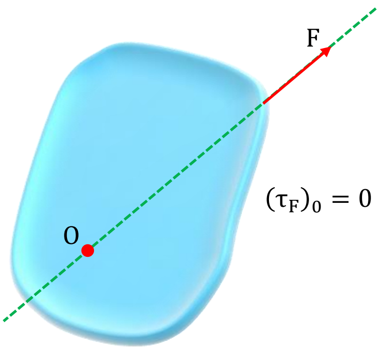
(i) $(\tau_F)_A = -60 \times 2 = -120\ \text{N}-\text{m}$ (Clockwise)

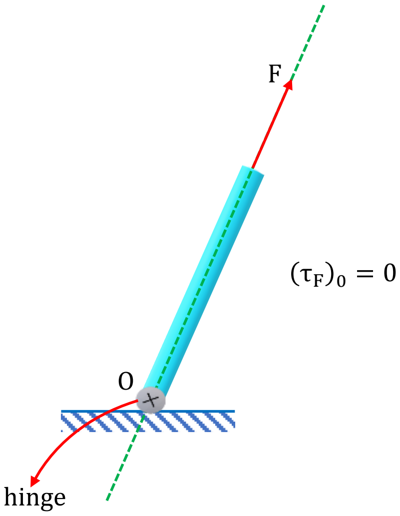
(ii) $(\tau_F)_B = 0$

(iii) $(\tau_F)_C = 60 \times 1 = +60\text{N}-\text{m}$ (Anticlockwise)

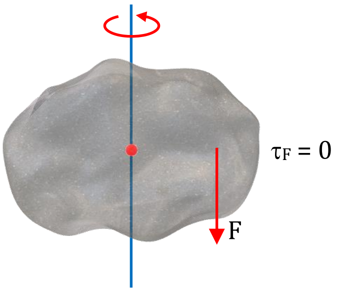
Important points

1. If line of action of a force passes through point O, then Torque of this force about point O will be zero.





2. If a force is acting parallel to the axis of rotation, then it's rotational effect about that axis will be zero.



3. If line of action of a force passes through axis of rotation, then it's rotational effect (Torque) about that axis of rotation will be zero.

Illustration 21

A force $\vec{F} = 2\hat{i} + 3\hat{j} - \hat{k}$ acts at a point (2, –3, 1). Then magnitude of torque of this force about point (0, 0, 2) will be?

Solution:

$\vec{F} = 2\hat{i} + 3\hat{j} - \hat{k}$ at point (2, –3, 1)

Torque about point (0, 0, 2)

$\vec{r} = (2\hat{i} - 3\hat{j} + \hat{k}) - 2\hat{k}$

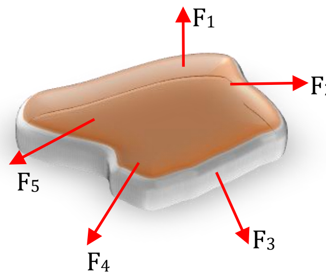
$\vec{\tau} = \vec{r} \times \vec{F} = (2\hat{i} - 3\hat{j} - \hat{k}) \times (2\hat{i} + 3\hat{j} - \hat{k}) = (6\hat{i} + 12\hat{k}) \Rightarrow |\vec{\tau}| = (6\sqrt{5})$ N-m

Equilibrium of Rigid Bodies

A system is said to be in mechanical equilibrium, if it is in translational as well as in rotational equilibrium

i.e. $\vec{F}_{net} = 0$ and $\vec{\tau}_{net} = 0$ (about every point)

If $\vec{F}_{net} = 0$, then $\vec{\tau}_{net}$ is same about every point.



Hence necessary and sufficient condition for equilibrium is $\vec{F}_{net} = 0$

i.e. $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 = 0$

$\vec{\tau}_{net} = 0$ about any one point, which we can choose as per our convenience. ($\vec{\tau}_{net}$ will automatically be zero about every point)

Rotational equilibrium

A rigid body is said to be in a state of rotational equilibrium if its angular acceleration is zero. Therefore, a body in rotational equilibrium must either be at rest or in rotation with a constant angular velocity.

If a rigid body is in rotational equilibrium under the action of several coplanar forces, the resultant torque of all the forces about any axis perpendicular to the plane containing the forces must be zero.

In the figure a body is shown under the action of several external coplanar forces F_1, F_2, ... F_i, and F_n

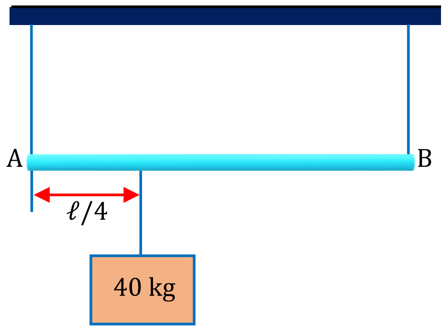
$$\sum \vec{\tau}_P = 0$$

Here P is a point in the plane of the forces about which we calculate torque of all the external forces acting on the body.

Note: A body cannot be in rotational equilibrium under the action of a single force unless the line of action passes through the axis of rotation.

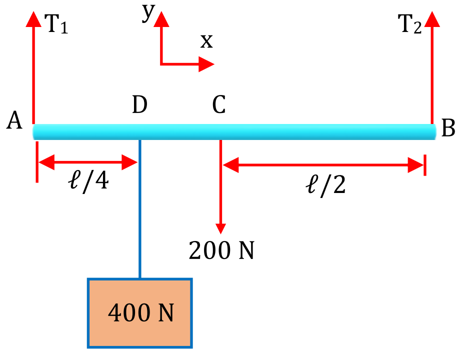
Illustration 22

A uniform rod of 20 kg is hanging in a horizontal position with the help of two threads. It also supports a 40 kg mass as shown in the figure. Find the tensions developed in each thread.



Solution:

Free body diagram of the rod is shown in the figure.



Translational equilibrium requires

$\Sigma F_y = 0 \qquad T_1 + T_2 = 400 + 200 = 600$ N ...(i)

Rotational equilibrium:

Applying the condition about A, we get T_2.

$\sum \vec{\tau}_A = \vec{0} \Rightarrow -400(\ell/4) - 200(\ell/2) + T_2\ell = 0$

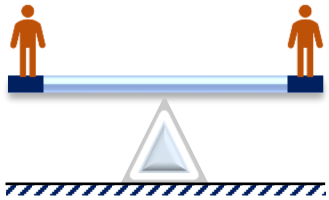
$T_2 = 200$ N

From equation (i)

$T_1 = 400$ N.

Illustration 23

Two small kids weighing 10 kg and 15kg are trying to balance a seesaw of total length 5.0 m, with the fulcrum at the centre? If one of the kids is sitting at an end, where should the other sit?



Solution:

It is clear that 10kg kid should sit at the end and the 15kg kid should sit closer to the centre. Suppose his distance from the centre is x. As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should be zero.

(a) forces are 15g downward by 15.

(b) 10g downward by 10kg kid.

(c) weight of seesaw.

(d) normal force by the fulcrum.

Taking torque about the fulcrum,

$15g.\ x = 10g\ (2.5m) \Rightarrow x = 1.7$ m.

Newton's 2nd law for rotation or Rotation about a fixed axis

Relation Between torque and Angular Acceleration

$$\therefore \tau_{net} = I\alpha$$

This equation is valid only for fixed axis.

Here I $\rightarrow$ moment of inertia of the body about the axis of rotation.

α = angular acceleration.

Illustration 24

A rod of mass M and length L is released then find out α just after releasing?

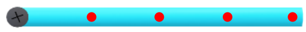
Solution:

$\tau_{hinge} = I_{hinge} \times \alpha$

$$Mg\frac{L}{2} = \frac{ML^2}{3} \cdot \alpha \Rightarrow \alpha = \frac{3g}{2L}$$

Illustration 25

A rod of mass M and length L is released then find out acceleration of marked points which are equidistant to each other?



Solution:

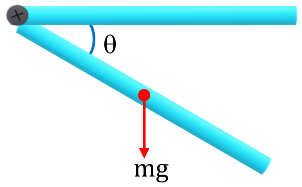
Here $\alpha_A = \alpha_B = \alpha_C = \alpha_D = \frac{3g}{2L}$ [For rigid body α is same]

$$a_D = \alpha \times \frac{L}{4} = \frac{3g}{2L} \times \frac{L}{4} = \frac{3g}{8} \quad \text{and} \quad a_A = \alpha \times \frac{L}{2} = \frac{3g}{2L} \times \frac{L}{2} = \frac{3g}{4}$$

$$a_B = \alpha \times \frac{3L}{4} = \frac{3g}{2L} \times \frac{3L}{4} = \frac{9g}{8} c \quad \text{and} \quad a_C = \alpha \times L = L \times \frac{3g}{2L} = \frac{3g}{2} c$$

Illustration 26

A rod of mass M and length L is released then α at this position will be?



Solution:

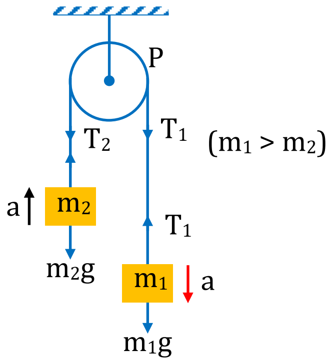
$$\tau = Mg \times \frac{L}{2}\cos\theta = \alpha \times \frac{ML^2}{3}$$

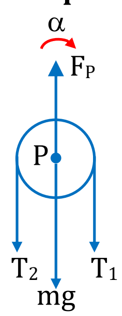
$$\alpha = \frac{Mg \times \frac{L}{2}\cos\theta}{M \times \frac{L^2}{3}} = \frac{3g\cos\theta}{2 \times L}$$

Rotating Pulley

Angular acceleration for heavy pulley

Here friction is present between the string and pulley i.e. There is no relative slipping between string and pulley.





$m_1g - T_1 = m_1a$ ...(i)

$T_2 - m_2g = m_2a$ ...(ii)

$\tau_{net/P} = I_P\alpha$

$0 + 0 + (T_1 R - T_2R) = I\alpha$ ...(iii)

$\because$ No slipping,

so $\left(\vec{a}_{string}\right)_P = (\vec{a}_{pulley})_P$

$a = R\alpha \Rightarrow \quad \alpha = \frac{a}{R}$

Substituting α in equation (iii)

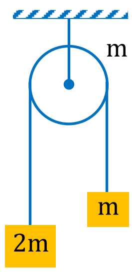
$T_1 - T_2 = \frac{Ia}{R^2}$ ...(iv)

$\therefore$ on adding eq. (i), (ii) & (iv)

$$\boxed{a = \frac{m_1g - m_2g}{m_1 + m_2 + \frac{I}{R^2}}} \Rightarrow \boxed{a = \frac{\text{Net force along string}}{\left(\text{total mass of blocks} + \frac{I}{R^2}\right)}}$$

Illustration 27

Find out the acceleration of the blocks...



Solution:

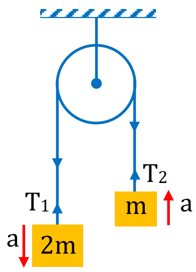
$2mg - T_1 = 2ma$ ...(i)

$T_2 - mg = ma$ ...(ii)

$T_1R - T_2R = I\alpha$;

$\alpha R = a$ ...(iii)

$\therefore \quad \alpha = \frac{a}{R}$

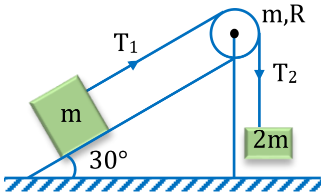


$T_1 - T_2 = \frac{I}{R^2}a$ ...(iv)

Equation (i) + (ii) + (iv) $\quad 2mg - mg = \left(2m + m + \frac{I}{R^2}\right)a \Rightarrow a = \frac{2mg - mg}{2m + m + \frac{mR^2}{2R^2}} \Rightarrow a = \frac{2g}{7}$

Illustration 28

Find out the acceleration of the blocks.



Solution:

$2mg - T_2 = 2ma$ ...(i)

$T_1 - mg\sin30° = ma$ ...(ii)

$(T_2 - T_1)\ R = \dfrac{mR^2}{2}\alpha$ ...(iii)

$\alpha R = a$ ...(iv)

$T_2 - T_1 = \dfrac{ma^2}{2}$ ...(v)

Equation (i) + (ii) + (v)

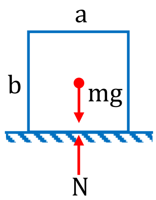
$$2mg - T_2 = 2mg - \frac{mg}{2} = ma\left(2 + 1 + \frac{1}{2}\right)$$

$$\frac{3}{2}mg = ma\left(\frac{7}{2}\right) \Rightarrow a = \frac{3g}{7}$$

Toppling

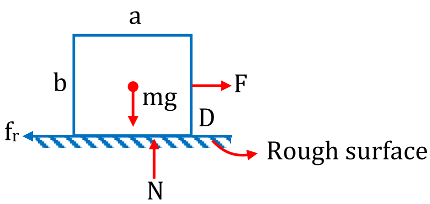
Toppling and shifting of Normal reaction

In many situations an external force is applied to a body to cause it to slide along a surface. In certain cases, the body may tip over before sliding. This is known as toppling.



(1) There is no horizontal force so pressure at bottom is uniform and Normal is collinear with mg.

(2) If a force is applied at COM, pressure is not uniform Normal Reaction shifts right so that torque of N can counter balance torque of friction.



(3) If F is continuously increased N keeps shifting towards right until it reaches the right most point D. Here we have assumed that the surface is sufficiently rough so that there is no sliding as F is increased to F_{max}.

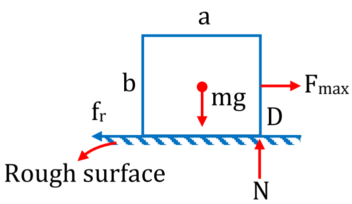
If force is increased any further, then torque of N cannot counter balance torque of friction fr and body will topple.

The value of force now is the max value for which toppling will not occur F_{max}

$F_{max} = f_r$

$N = mg$

Torque about COM



$$f_r \cdot \frac{b}{2} = N \cdot \frac{a}{2}$$

$$\Rightarrow \quad f_r = \frac{Na}{b} = mg \cdot \frac{a}{b}$$

$$\Rightarrow \quad F_{max} = mg \cdot \frac{a}{b}$$

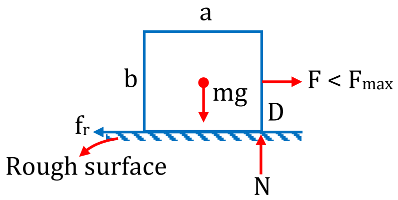
(4) If surface is not sufficiently rough and the body slides before F is increased to $F_{max} = mg\frac{a}{b}$ then body will slide before toppling. Once body starts sliding friction becomes constant and hence no toppling.

This is the case if

$F_{max} > F_{limit}$

$\Rightarrow \quad mg\frac{a}{b} > \mu mg$

$\mu < \frac{a}{b}$



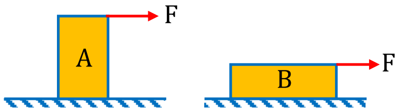
Condition for toppling

When $\mu \geq \frac{a}{b}$ in this case body will topple if $F > mg\frac{a}{b}$

But if $\mu < \frac{a}{b}$, body will not topple for any value of F, applied at COM.

Important points

(1) For shown situation (A) & (B), more changes of toppling in (A). In case of toppling, Normal Reaction must passes through end points.



If block kept on smooth horizontal surface and a force F is applied at COM in horizontal direction, for any value of force F, block can't topple.

BEGINNER'S BOX-3

1. A force $\vec{F} = \left(\hat{i} - 2\hat{j} + 5\hat{k}\right)$ N is acting at a point $\vec{r} = \left(2\hat{i} + \hat{j} + 3\hat{k}\right)$ m. Find torque about the origin.

PRM184

2. The density of a rod AB increases continuously from A to B. Is it easier to set it in rotation by clamping it at A and applying a perpendicular force at B or by clamping it at B and applying the force at A ? Explain your answer.

PRM185

3. A uniform disc of radius 20 cm and mass 2 kg can rotate about a fixed axis through the centre and perpendicular to its plane. A massless cord is round along the rim of the disc. If a uniform force of 2 newtons is applied on the cord, tangential acceleration of a point on the rim of the disc will be :-

(A) 1 m/s^2 (B) 2 m/s^2 (C) 3.2 m/s^2 (D) 1.6 m/s^2

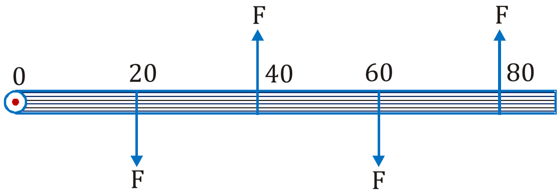
PRM186

4. A door 1.6 m wide requires a force of 1 N to be applied at the free end to open or close it. The force that is required to be applied at a point 0.4 m distant from the hinges for opening or closing the door is :-

(A) 1.2 N (B) 3.6 N (C) 2.4 N (D) 4 N

PRM187

5. Four equal and parallel forces are acting on a rod (as shown in figure at distances of 20 cm, 40 cm, 60 cm and 80 cm respectively from one end of the rod. Under the influence of these forces the rod



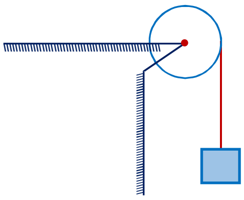
(A) is at rest
(B) experiences a torque
(C) experiences a linear motion
(D) experiences a torque and also a linear motion

PRM188

6. A weightless rod is acted upon by upward parallel forces of 2 N and 4 N at ends A and B respectively. The total length of the rod AB = 3 m. To keep the rod in equilibrium a force of 6 N should act in the following manner:
(A) downward at any point between A and B
(B) downward at the midpoint of AB
(C) downward at a point C such that AC = 1 m
(D) downward at a point D such that BD = 1 m

PRM189

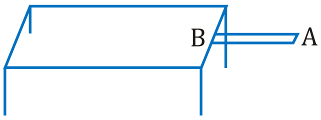
7. Figure shows a uniform disc, with mass M = 2.4 kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block of mass m = 1.2 kg hangs from a massless cord which is wrapped around the rim of the disc. The tension in the cord is :



(A) 12 N
(B) 20 N
(C) 24 N
(D) 6 N

PRM190

8. A thin uniform stick of length ℓ and mass m is held horizontally with its end B hinged on the edge of a table. Point A is suddenly released. The acceleration of the centre of mass of the stick at the time of release, is :-

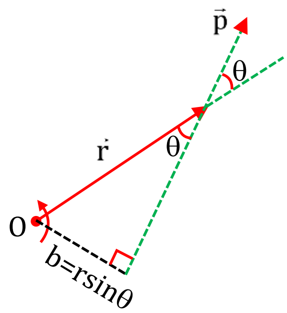


(A) $\frac{3}{4}g$
(B) $\frac{3}{7}g$
(C) $\frac{2}{7}g$
(D) $\frac{1}{7}g$

PRM191

Angular momentum of a Particle

Angular momentum (moment of linear momentum) : Angular momentum of a body about a given axis is the product of its linear momentum and perpendicular distance of line of action of linear momentum vector from the axis of rotation.



Angular momentum = Linear momentum × Perpendicular distance of line of action of momentum from the axis of rotation

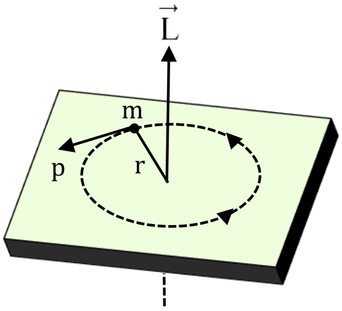
$L = mv \times r\sin\theta \Rightarrow \vec{L} = \vec{r} \times \vec{p}$

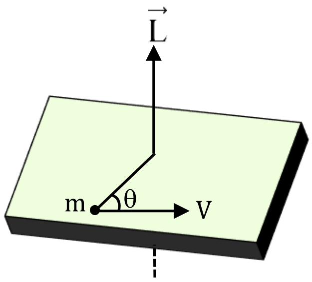
Here $\vec{L}$ = angular momentum of a moving particle about point. O

$\vec{P}$ is the linear momentum of the particle.

$\vec{r}$ is the position vector of the particle regarding the point.

- Unit : SI $\rightarrow$ J-sec or kg – m²/sec
- Dimension : $[ML^2T^{-1}]$
- Angular momentum is an axial vector



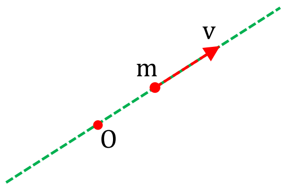


- As torque $(\vec{r}\times\vec{F})$ is defined as the 'moment of force'.
  Angular momentum is also defined as moment of linear momentum.
- In Cartesian system, Angular momentum:
  $\vec{L}=\vec{r}\times\vec{p}=m(\vec{r}\times\vec{v})$ $\quad [\because \; \vec{p}=m\vec{v}]$

$$\vec{L}=m\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ v_x & v_y & v_z\end{vmatrix}=m\left[(x\hat{i}+y\hat{j}+z\hat{k})\times(v_x\hat{i}+v_y\hat{j}+v_z\hat{k})\right]$$

i.e. $\vec{L}=m\left[\hat{i}(yv_z-zv_y)-\hat{j}(xv_z-zv_x)+\hat{k}(xv_y-yv_x)\right]$

Note: If line of motion of a particle passes through point 'O', then angular momentum about point O will be zero.



$(\vec{L})_0=0$

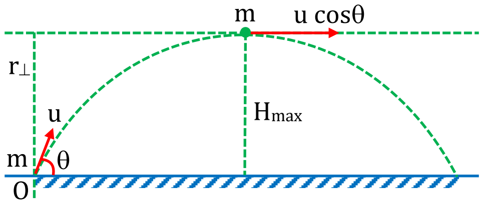
Because $r_\perp = 0$

- As the magnitude of the angular momentum is L = mvr sinθ
  (a) For θ = 0° or 180° i.e. $\vec{r}$ and $\vec{v}$ are parallel or anti-parallel ∴ sinθ = 0; L will be minimum.
  If the axis of rotation is on the line of motion the of moving particle then angular momentum is minimum and zero.
  (b) For θ = 90°, i.e., angular momentum is maximum when $\vec{r}$ and $\vec{v}$ are orthogonal
  ∴ sin θ = 1; L will be maximum.
- As $|\vec{L}|$ = mvr sinθ so if the point about which angular momentum is to be determined is not on the line of motion i.e., θ ≠ 0 or 180° then $|\vec{L}|\neq 0$ i.e., a particle in translatory motion always has an angular momentum unless the point is on the line of motion.

Illustration 29

A particle of mass m is projected with velocity 'u' at an angle θ with Horizontal. Find angular Momentum of particle about point of Projection when Particle is at its highest Point ?

Solution:



$(L)_0 = mu\cos\theta \times r_\perp$ $\quad$ $(L)_0 = mu\cos\theta \times H_{max}$

Illustration 30

A particle of mass 0.01 kg having position vector $\vec{r} = (10\hat{i} + 6\hat{j})$ metre is moving with a velocity $5\hat{i}\,m/s$. Calculate its angular momentum about the origin

Solution:

$\vec{p} = m\vec{v} = 0.01 \times 5\hat{i} = 0.05\hat{i}$ $\quad\Rightarrow\quad$ $\vec{L} = \vec{r} \times \vec{p} = (10\hat{i} + 6\hat{j}) \times 0.05\hat{i} = 0.3(-\hat{k}) = -0.3\hat{k}\,J/s$

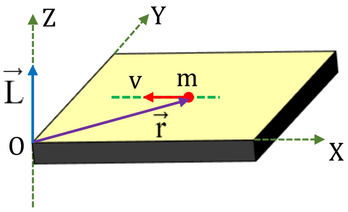
Direction of angular momentum

Direction of angular momentum due to linear motion :

Imagine your position at the point about which you want to determine the angular momentum spread the palm of your right hand along the position vector and curl the fingers in the direction of velocity of particle then the thumb of your right hand shows the direction of angular momentum **(Right hand thumb rule.)**

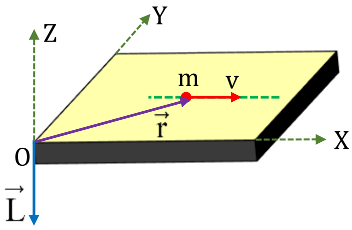
$\vec{L} = mvb(-\hat{k})$ $\quad$ here $\quad b = r\sin\theta$

A particle is moving parallel to X-axis or Y-axis or along any straight line with constant velocity then its angular momentum about the origin will remain constant.



ACW or +ve [$\hat{k}$]

or ⊥ to the plane upwards



CW or –ve [– $\hat{k}$]

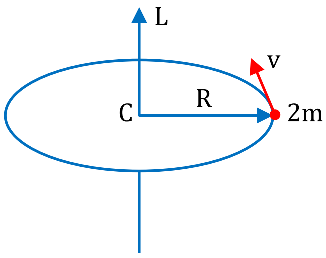
or ⊥ to the plane downwards [⊗]

Illustration 31

A particle of mass 2m is moving on a circle of radius R with angular momentum 'L'. Then find.

(i) Centripetal force on particle $\qquad$ (ii) Kinetic energy of particle

Solution:



(i) $\quad L = 2mvR$

$$\Rightarrow V = \frac{L}{2mR}$$

So, centripetal force, $F = \dfrac{2mv^2}{R} = 2m \times \dfrac{\left(\dfrac{L}{2mR}\right)^2}{R}$

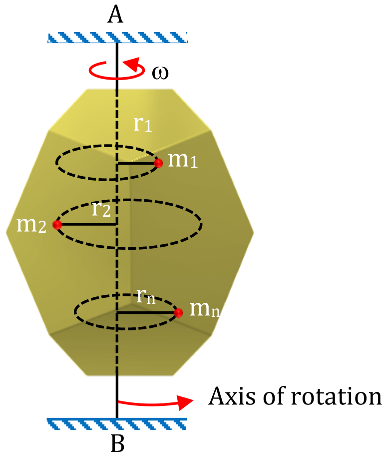
$$F = \frac{2m \times \dfrac{L^2}{4m^2R^2}}{R} \Rightarrow \frac{L^2}{2mR^2 \times R} = \frac{L^2}{2mR^3}$$

(ii) $\quad$ K.E. of particle, KE = $\dfrac{1}{2}mv^2 = \dfrac{1}{2} \times (2m) \times \left(\dfrac{L}{2mR}\right)^2 = \dfrac{1}{2} \times 2m \times \dfrac{L^2}{4m^2R^2} = \dfrac{L^2}{4mR^2} = \dfrac{L^2}{2I}$

Angular momentum of objects and Relation between Torque and Angular momentum

Angular momentum of a rigid body rotating about fixed axis

Angular momentum of a rigid body about the fixed axis AB is $L_{AB} = L_1 + L_2 + L_3 + \ldots\ldots + L_n$



$L_1 = m_1r_1\omega r_1, \quad L_2 = m_2r_2\omega r_2, \quad L_3 = m_3r_3\omega r_3,$

$L_n = m_nr_n\omega r_n$

$L_{AB} = m_1r_1\omega r_1 + m_2r_2\omega r_2 + m_3r_3\omega r_3 + \ldots\ldots + m_nr_n\omega r_n$

$$L_{AB} = \sum_{n=1}^{n=n} m_n(r_n)^2 \times \omega \Rightarrow \left[\sum_{n=1}^{n=n} m_n(r_n)^2 = I\right]$$

$L_{AB} = I_H\omega$

$L = I\omega$

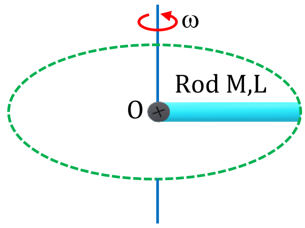
L = Angular momentum of object about axis of rotation.

I = moment of inertia of rigid body about axis of rotation

$\omega \rightarrow$ Angular velocity of the object.

Illustration 32

Find angular momentum of the rod about an axis passing through point O ?

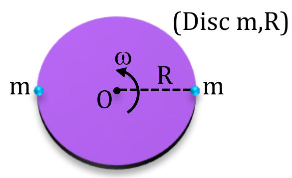


Solution:

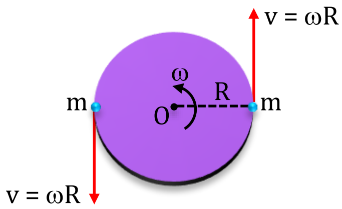
$$(L)_o = I_o\omega = \frac{ML^2}{3} \times \omega$$

Illustration 33

Two particles of mass m are placed on the disc as shown in the figure. Find the angular momentum of the system about axis of rotation.



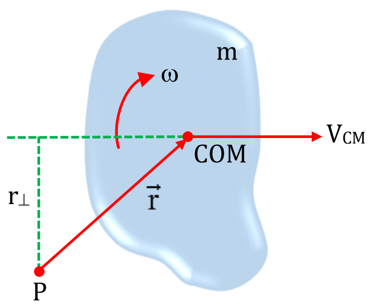
Solution:



$$L = \frac{mR^2}{2} \cdot \omega\,(\hat{K}) + [mR(\omega R)] \times 2\,(\hat{k}) = \frac{5}{2}mR^2\omega\,(\hat{k})$$

Angular momentum of combined rotation and translation

Suppose a body is rotating about an axis passing through its centre of mass with an angular velocity ω about COM and moving translationally with a linear velocity V_{CM} a. Then the angular momentum of the body about a point P outside the body in the lab frame is given by,



$$\vec{L}_P = \vec{L}_{CM} + \vec{r} \times \vec{p}_{CM}.$$

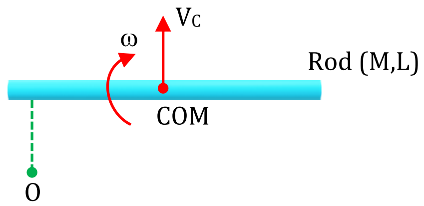
Where, r = position vector of the centre of mass with respect to point P.

Hence $\vec{L}_P = I_{CM}\vec{\omega} + \vec{r} \times m\vec{v}_{CM}$

Note: Always consider positive and negative sign while calculating these values.

Illustration 34

Find angular momentum of the Rod about point O?



Solution:

$$(\vec{L})_o = (\vec{L})_{Translation} + (\vec{L})_{Rotation}$$

$$(\vec{L})_0 = \underbrace{\left[+MV_C \times \frac{L}{2}\right]}_{ACW} + \underbrace{\left[-\frac{ML^2}{12} \times \omega\right]}_{CW} \text{ So, } (\vec{L})_0 = MV_C \times \frac{L}{2} - \frac{ML^2}{12}\omega$$

Newton's 2nd Law in Rotation

It states that net external Torque acting on a system about any point O (or axis) is equal to rate of change of angular momentum of system about any point O (or axis).

$$\Rightarrow (\vec{\tau}_{ext})_0 = \frac{d\vec{L}_0}{dt} \text{ Also, } \vec{L}_0 = I_0\vec{\omega}$$

$$\text{So, } (\vec{\tau}_{ext})_0 = \frac{d(I_0\vec{\omega})}{dt} = I_0\frac{d\vec{\omega}}{dt} = I_0\vec{\alpha}$$

Note: $\vec{\tau}_{avg} = \frac{\Delta\vec{L}}{\Delta t} = \frac{\vec{L}_f - \vec{L}_i}{t}$

Illustration 35

The diameter of a disc is 0.5 m and its mass is 16 kg. What torque will increase its angular velocity from zero to 120 rotations/ minute in 8 seconds?

Solution:

Moment of inertia of the disc $I = \frac{1}{2}MR^2 = \frac{1}{2} \times 16 \times \left[\frac{0.5}{2}\right]^2 = \frac{1}{2}kg - m^2$

Angular velocity $\omega = \frac{120 \times 2\pi}{60}$ = 4π rad/s and change in angular momentum $\Delta L = I\omega - 0 = I\omega$

$\therefore$ Angular impulse is $\tau \times t = \Delta L \Rightarrow \tau = \frac{\Delta L}{t} = \frac{1}{8} \times \frac{1}{2} \times 4\pi = \frac{\pi}{4} N - m$

BEGINNER'S BOX-4

1. If the earth is a point mass of 6×10^{24} kg revolving around the sun at a distance of 1.5×10^8 km and in time of $T = 3.14 \times 10^7$ seconds, then the angular momentum of the earth around the sun is :–

(A) 1.2×10^{18} kg-m²/s (B) 1.8×10^{20} kg-m²/s

(C) 1.5×10^{37} kg-m²/s (D) 2.7×10^{40} kg-m²/s

PRM192

2. If a particle moves along a straight line whose equation is y = mx then what will be its angular momentum about the origin ? Explain your answer.

PRM193

3. A uniform circular disc of mass 200 g and radius 4·0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. Find the kinetic energy of the disc and its angular momentum about the axis of rotation.

PRM194

Conservation of Angular Momentum (COAM)

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

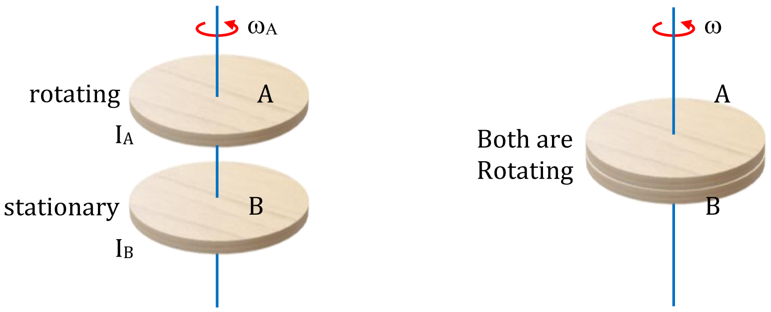
If $\vec{\tau}_{ext} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L}$ is constant $\Rightarrow \vec{L}_i = \vec{L}_f$

If the resultant external torque acting on a system is zero then the total angular momentum of the system remains constant.

If a system is isolated from its surrounding any internal interaction between its different parts cannot alter its total angular momentum.

Illustration 36

Two wheels A and B are placed coaxially. Wheel A has moment of inertia I_A and angular velocity ω_A while B is stationary. On clubbing them with a clutch they move jointly by an angular velocity 'ω' then find the M.I. of 'B'.



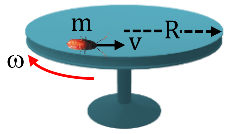
Solution:

The phenomenon occurs in the absense of external torque, so applying law of COAM, we have

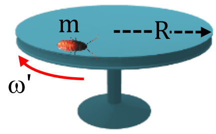
$$\because \qquad I_A\,\omega_A = (I_A + I_B)\,\omega \Rightarrow I_B = \frac{I_A\omega_A - I_A\omega}{\omega} = I_A\left(\frac{\omega_A}{\omega} - 1\right)$$

Illustration 37

A cockroach of mass 'm' is moving with velocity v in anticlockwise sense on the rim of a disc of radius R. The M.I. of the disc about the axis is 'I' and it is rotating in clockwise direction with an angular velocity 'ω'. If the cockroach stops then calculate the angular velocity of disc.



Solution:



The phenomenon occurs in the absence of external torque, so applying law of COAM, we have

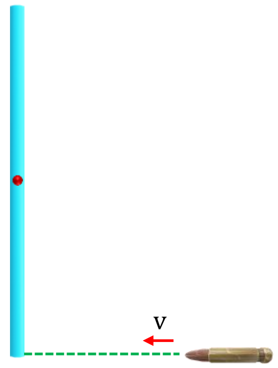
$$I\omega - mvR = \left(I + mR^2\right)\omega' \Rightarrow \omega' = \frac{I\omega - mvR}{I + mR^2}.$$

Illustration 38

Moment of inertia of uniform rod of mass M and length L about an axis through its centre and perpendicular to its length is given by $\frac{ML^2}{12}$. Now consider one such rod pivoted at its centre, free to rotate in a vertical plane. The rod is at rest in the vertical position. A bullet of mass M moving horizontally at a speed v strikes and embedded in one end of the rod. The angular velocity of the rod just after the collision will be

Solution:

Initial angular momentum of the system = Angular momentum of bullet before collision = $Mv\left(\frac{L}{2}\right)$ ...(i)



let the rod rotates with angular velocity ω.

Final angular momentum of the system = $\left(\frac{ML^2}{12}\right)\omega + M\left(\frac{L}{2}\right)^2\omega$ ...(ii)

By equation (i) and (ii) $Mv\frac{L}{2} = \left(\frac{ML^2}{12} + \frac{ML^2}{4}\right)\omega \quad \Rightarrow \quad \omega = 3v/2L$

Problems based on Conservation of Angular Momentum

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

If $\vec{\tau}_{ext} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L}$ is constant $\Rightarrow \vec{L}_i = \vec{L}_f$

If the resultant external torque acting on a system is zero then the total angular momentum of the system remains constant.

If a system is isolated from its surrounding any internal interaction between its different parts cannot alter its total angular momentum.

The principle of conservation of angular momentum governs a wide range of physical processes from subatomic to celestial world. The following examples explicate some of these applications.

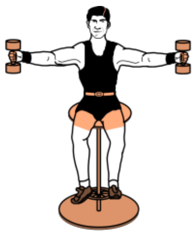
Examples Based On COAM

Spinning Ice Skater.

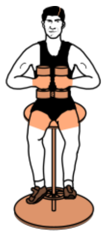
A spinning ice skater or a ballet dances can control her moment of inertia by extending or folding her arms and make use of conservation of angular momentum to perform their spins. In doing so no external force is needed and if we ignore effects of friction from the ground and the air, the angular momentum can be assumed to be conserved. When she spreads her hands or legs away from the spin axis, her moment of inertia increases therefore her angular velocity decreases and when she brings her hands or legs closer her moment of inertia decreases and consequently her angular velocity increases.

Student on rotating turntable

The student with the dumbbells and the turntable make an isolated system on which no external torque acts (if we ignore friction in the bearings of the turntable and air resistance). Initially, the student has his arms stretched. When he draws the dumbbells close to his body, his angular velocity increases due to conservation of angular momentum.



Larger moment of inertia and smaller angular velocity

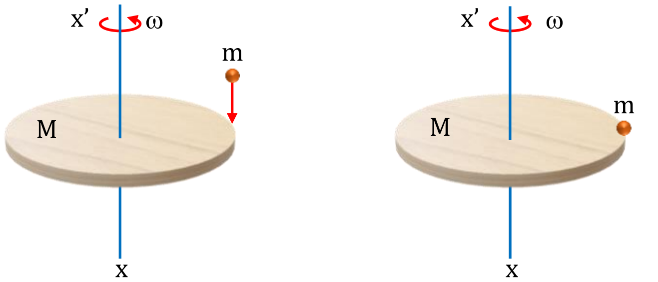


Smaller moment of inertia and larger angular velocity

- If a lady skating on ice while spinning folds her arms then her M.I. decreases and consequently 'ω' increases.

Illustration 39

A disc of mass M and radius r is rotating about its axis with angular velocity ω. Now if a mass m falls vertically on its rim and sticks to it. Then find, final angular velocity of the combined system.



Solution:

applying angular momentum conservation

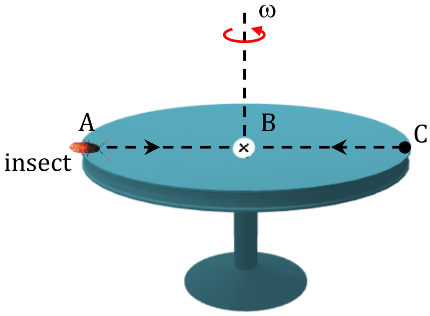
$I_1\omega_1 = I_2\omega_2$

$$\frac{MR^2}{2}\omega = \left(\frac{MR^2}{2} + mR^2\right)\omega'$$

$$\omega' = \left(\frac{M}{M+2m}\right)\omega$$

Illustration 40

If insect goes from A to C then find how ω changes?



Solution:

L =Iω=constant

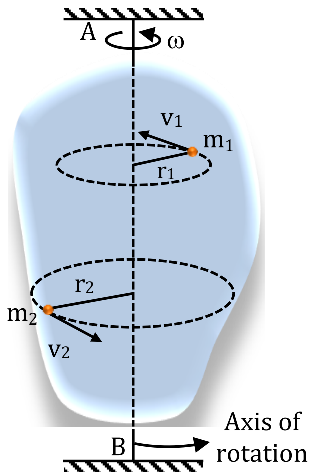
from A to B, $I\downarrow \omega\uparrow$

from B to C, $I\uparrow \omega\downarrow$

So, from A to B : first ω↑ and then ω↓

Work, Power and Energy in Rotation

Rotational kinetic energy



- The energy possessed due to rotational motion of a body is known as rotational kinetic energy.
- A rigid body is rotating about an axis with a uniform angular velocity ω. The body is assumed to be composed of particles of masses m_1, m_2 ..... The linear velocity of the particles are $v_1 = \omega r_1$, $v_2 = \omega r_2$, ... Therefore, the total kinetic energy of the rotating body is:

$$KE_r = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ... = \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + ...)\omega^2 = \frac{1}{2}I\omega^2$$

- It is a scalar quantity.

Relation between angular momentum $(\vec{L})$ and Rotational kinetic energy

$$KE = \frac{1}{2}I\omega^2$$

Multiply and divide with I on numerator and denominator.

$$KE = \frac{1}{2}\times\frac{I^2\omega^2}{I} \quad \text{i.e} \quad KE_{rotation} = \frac{L^2}{2I}$$

If $\vec{\tau}_{ext} = 0$, then $\vec{L}$ = constant

So, $KE_{rotation} \propto \frac{1}{I}$

If I increases, then $KE_{rotation}$ decreases.

Illustration 41

A person is standing on the rotating disc with his arms stretching in the first case and then be relaxed his arms in the second case. What will happen to the rotational kinetic energy of the system in the second case?

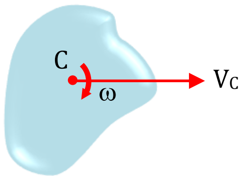
Solution:

L = constant

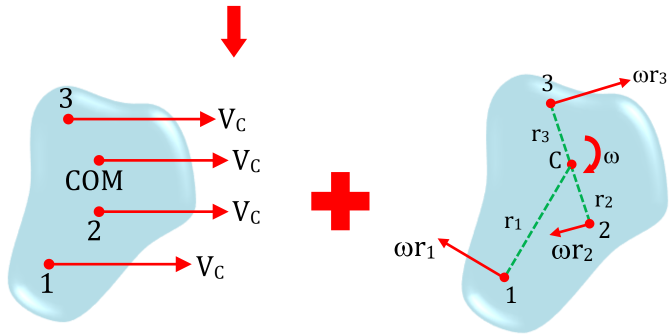
$KE = \frac{L^2}{2I} \Rightarrow KE \propto \frac{1}{I} \Rightarrow$ So, as in this case, MOI(I) decreases, KE increases.

Total kinetic energy in combined translation and rotation

If during rotation, the centre of mass of the rigid body undergoes translational motion, it will have both translational and rotational kinetic energy.



Translation + Rotation



Pure translation

Pure Rotation

$K_{translation} = \frac{1}{2} MV_C^2$ $\quad$ $K_{rotation} = \frac{1}{2} I_C\omega^2$

$K_{total} = K_{translation} + K_{rotation} \quad \Rightarrow \quad K_{total} = \frac{1}{2} MV_C^2 + \frac{1}{2} I_C\omega^2$

In pure translation, each and every particle of the body moves with same velocity as centre of mass, we can define kinetic energy of the object in case of pure translation.

$$K_{translation} = \frac{1}{2} MV_C^2 \quad ...(i)$$

In case of pure rotation the body rotate about its axes passing through its centre of mass, we define kinetic energy of the object as rotational kinetic energy.

$$K_{rotation} = \frac{1}{2} I_C\omega^2$$

Where I_C = moment of inertia of the object about an axis passing through its centre of mass.

Hence, total kinetic energy of the object

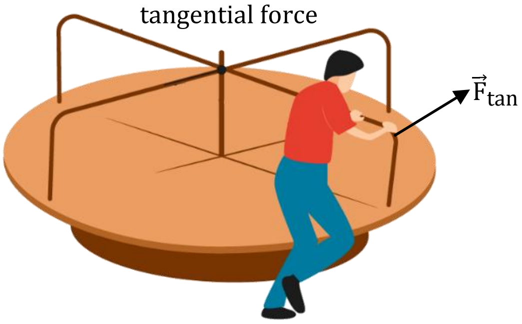
$$K_{total} = K_{translation} + K_{rotation} \quad \Rightarrow \quad K_{total} = \frac{1}{2}MV_C^2 + \frac{1}{2}I_C\omega^2$$

The first term signifies the translation kinetic energy of the rigid body which can be understood as the kinetic energy of the center of mass of the body. The second terms gives the kinetic energy of the rotation of the body about the centre of mass.

Hence, we can call the term $\frac{I_C\omega^2}{2}$ as rotational kinetic energy of the rigid body.

Work and power in rotation

Suppose a tangential force $\vec{F}_{tan}$ acts on the rim of a pivoted disc [for example, a man rotating a merry-go-round on a play ground, while simultaneously running along with it. The disc rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval dt.



The work done by the force $\vec{F}_{tan}$ while any point on the rim moves a distance ds is $dW = F_{tan}\,ds$.

If $d\theta$ the corresponding is angular displacement, then $ds = Rd\theta$ and $\therefore\ dW = F_{tan}Rd\theta$

Torque due to the force $\vec{F}_{tan}$ is $\tau = F_{tan}R$ $\qquad\therefore\ dW = \tau d\theta$ ...(i)

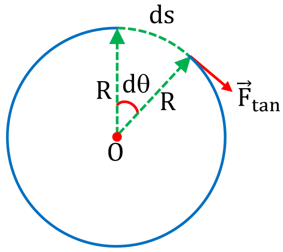
Total work W done by the torque during an angular displacement from θ_1 to θ_2 is :

$$W = \int_{\theta_1}^{\theta_2} \tau d\theta \text{ (work done by a torque)}$$

If the torque remains constant while the angle changes by a finite amount $\Delta\theta = \theta_2 - \theta_1$ then

$W = \tau(\theta_2 - \theta_1) = \tau\Delta\theta$ (work done by a constant torque)

Work done by a constant torque is the product of torque and the angular displacement.



Overhead view of merry-go-round

Let τ represent the net torque on the body so form equation $\tau = I\alpha$.

Assume the body to be rigid so that its moment of inertia I remains constant,

$$\tau d\theta = (I\alpha)d\theta = I\frac{d\omega}{dt}d\theta = I\frac{d\theta}{dt}d\omega = I\omega d\omega$$

Since τ is the net torque, the integral in equation (ii) yield the total work done on the rotating body.

$$W_{total} = \int_{\omega_1}^{\omega_2} I\omega\, d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

When a torque executes work on a rotating rigid body, the kinetic energy of the body changes by an amount equal to the work done.

Rotational power

The power associated with the work done by torque acting on a rotating body :

Divide both sides of equation $dW = \tau d\theta$ by the time interval dt during which the angular displacement $d\theta$ occurs.

$$\therefore \qquad \frac{dW}{dt} = \tau\frac{d\theta}{dt}$$

Instantaneous rotational power $P_r = \tau\omega$. In general, $P_r = \vec{\tau}\cdot\vec{\omega}$.

Work-Energy Theorem in Rotational motion

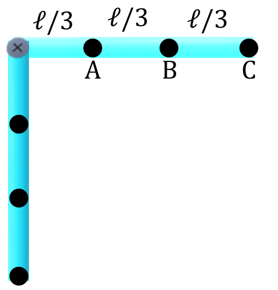
Work done by the torque = Change in kinetic energy of rotation

$$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

The change in the rotational kinetic energy of a rigid body equals the work done by external torques exerted on the body. This equation is analogous to equation corresponding to the work-energy theorem applicable for a particle.

Illustration 42

A light rod carries three equal masses A, B and C as shown in the figure. What will be the velocity of B in the vertical position of the rod, if it is released from horizontal position as shown in the figure ?



Solution:

Applying law of conservation of mechanical energy.

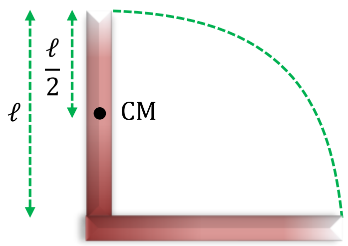
Loss in gravitational P.E. = Gain in rotational K.E.

i.e.,

$$mg\frac{\ell}{3} + mg\left(\frac{2\ell}{3}\right) + mg\ell = \frac{1}{2}\left(m\left(\frac{\ell}{3}\right)^2 + m\left(\frac{2\ell}{3}\right)^2 + m\ell^2\right)\omega^2 \Rightarrow \omega = \sqrt{\frac{36g}{14\ell}} . \Rightarrow v_B = \omega\ell_B = \frac{2\ell}{3}\sqrt{\frac{36g}{14\ell}} = \sqrt{\frac{8g\ell}{7}}$$

Illustration 43

A thin meter scale is kept vertical by placing its lower end hinged on floor. It is allowed to fall. Calculate the velocity of its upper end when it hits the floor.



Solution:

Loss in PE = gain in rotational KE

$$\frac{mg\ell}{2} = \frac{1}{2}I\omega^2$$

$$\Rightarrow \frac{mg\ell}{2} = \frac{1}{2}\frac{m\ell^2}{3} \times \frac{v^2}{\ell^2}$$

$$\Rightarrow v = \sqrt{3g\ell}$$

Illustration 44

The power output of an automobile engine is advertised to be 200 hp at 6000 rpm. What is the corresponding torque generated.

Solution:

$P = 200 \times 746\ W = 1.49 \times 10^5\ W$

& $\omega = 6000\ rev/min = 6000 \times \frac{2\pi}{60} = 628\ rad/s$

$$\tau = \frac{P}{\omega} = \frac{1.49 \times 10^5}{628} = 237.5\ N\text{-}m$$

BEGINNER'S BOX-5

1. The angular velocity of a body changes from ω_1 to ω_2 without applying external torque. Then find the ratio of initial radius of gyration to final radius of gyration.

PRM195

2. A stone is attached to a string and is revolved in a circular path with constant angular velocity ω. In this state its angular momentum is L. If the length of the string is reduced to half and again it is revolved with the same angular velocity ω then find its angular momentum.

PRM196

3. A thin uniform thin rod of mass m and length ℓ is suspended from one of its ends and is rotated at the rate of f rotations per second. Find the rotational kinetic energy of the rod.

PRM197

4. A wheel of moment of inertia 10 kg-m^2 rotates at the rate of 10 revolutions per minute. Find the work done in increasing its speed to 5 times of its initial value.

PRM198

5. A solid ball of mass 1 kg and radius 3 cm is rotating about its own axis with an angular velocity of 50 radians per second. Find its kinetic energy of rotation.

PRM199

6. When sand is poured gradually on the edge of a rotating disc, what will be the change in its angular velocity?

PRM200

7. The moment of inertia of a wheel about its axis of rotation is 3.0 MKS units. Its kinetic energy is 600 J. Find its period of rotation.

PRM201

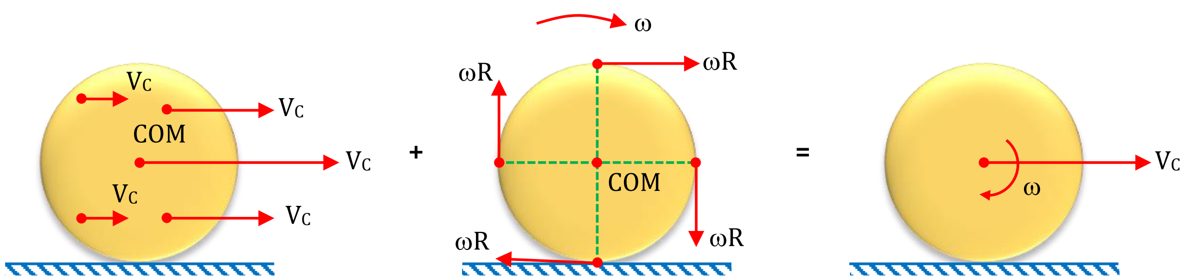
8. The moment of inertia of a wheel is 1000 kg–m^2. At a given instant, its angular velocity is 10 rad/s. After the wheel rotates through an angle of 100 radians, its angular velocity increases to 100 rad/s. Calculate the,
(a) torque applied on the wheel. (b) increase in the rotational kinetic energy.

PRM202

Introduction to Pure Rolling

Rolling Motion

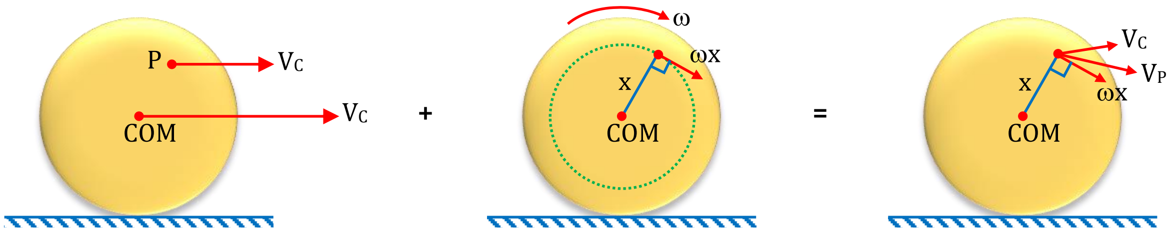
When a body performs translatory motion as well as rotatory motion combinedly then it is said to undergo rolling motion. The velocity of centre of mass represents linear motion while angular velocity about the centroid axis represents rotatory motion.



Pure translation | Pure Rotation about COM | Rolling Motion

In pure translational motion, all points move with the same linear velocity V_c. In pure rotational motion about COM, all points move with the same angular velocity about the central axis.

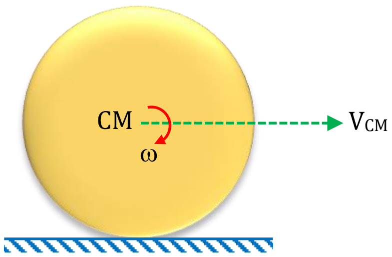
Velocity of particle on a rolling object:



Rolling motion

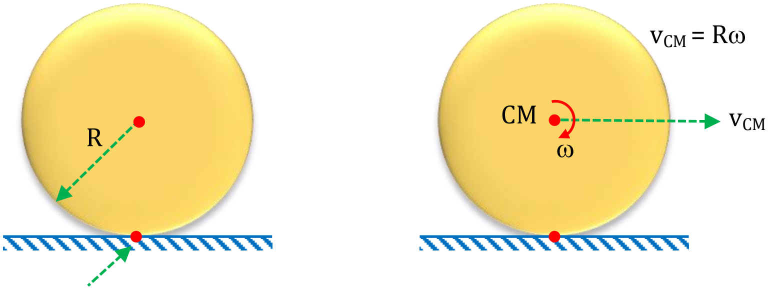
Velocity of particle on a rolling object can be seen by considering the translatory motion of the body and rotational motion of body wrt an axis passing through the centre of mass.

Rolling without slipping (Pure rolling)



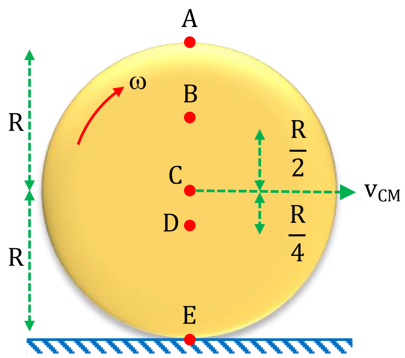
- If the relative velocity of the point of contact of the rolling body with the surface is zero then it is known as pure rolling.
- If a body is performing rolling then the velocity of any point of the body with respect to the surface is given by $\vec{v} = \vec{v}_{CM} + \vec{\omega} \times \vec{R}$

Pure rolling



Contact point

$$
\begin{aligned}
v_A &= v_{CM} + \omega R &&= v_{CM} + v_{CM} &&= 2v_{CM} \\
v_B &= v_{CM} + \frac{\omega R}{2} &&= v_{CM} + \frac{v_{CM}}{2} &&= \frac{3}{2}v_{CM} \\
v_C &= v_{CM} + 0 &&= v_{CM} && \\
v_D &= v_{CM} - \frac{\omega R}{4} &&= v_{CM} - \frac{v_{CM}}{4} &&= \frac{3}{4}v_{CM} \\
v_E &= v_{CM} - \omega R &&= v_{CM} - v_{CM} &&= 0
\end{aligned}
$$

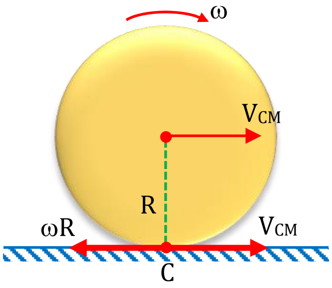


For contact point on ground

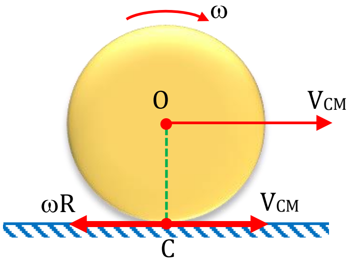
[Condition for pure rolling]

1. **Case-I** $\rightarrow$ if $V_{CM} < \omega R$

Point (will slip backward wrt ground. It is called rolling with slipping.

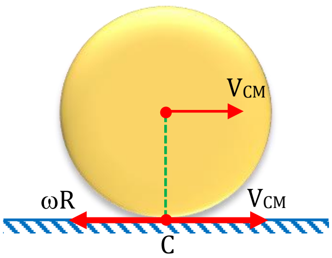


Case-II $\rightarrow$ if $V_{CM} > \omega R$



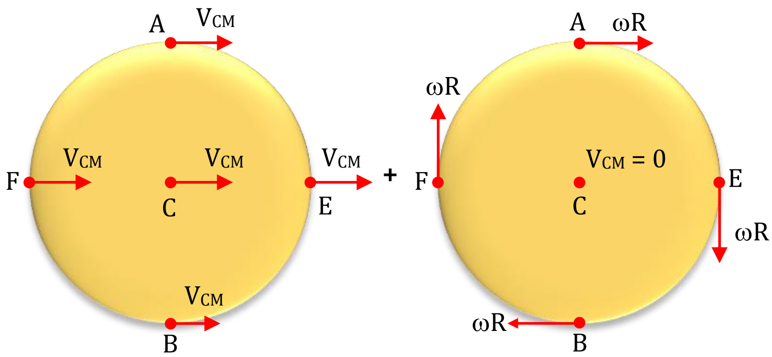
Point C will slip forward with respect to ground. It is also called rolling with slipping

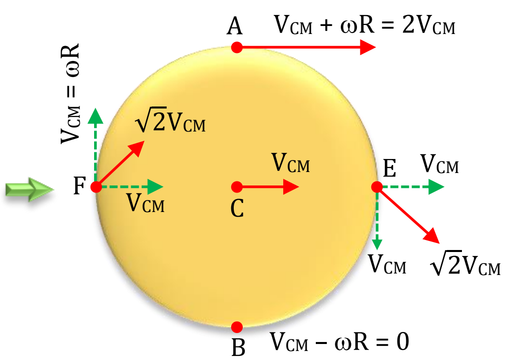
Case-III $\rightarrow$ $V_{CM} = \omega R$



Velocity of point C will be zero. So it will not slip with respect to ground. It is called pure rolling. So, for pure rolling on ground

$$V_{CM} = \omega R$$





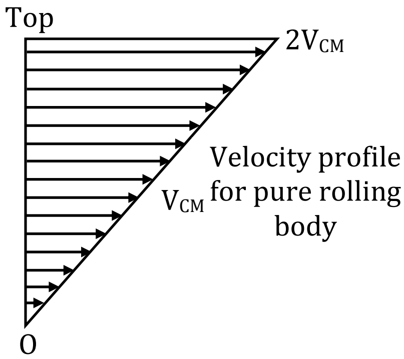
For pure rolling motion of the above mentioned body, we have

$V_A = 2v_{CM}$

$V_E = \sqrt{2}\, V_{CM}$

$V_F = \sqrt{2}\, V_{CM}$

$V_B = 0$



Value of $\frac{K^2}{R^2}$ for some objects

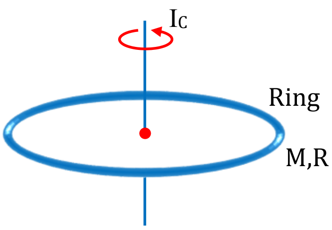
[K is radius of gyration about axis passing through COM]

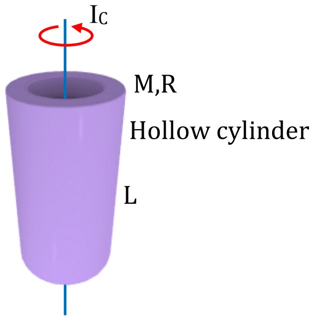
1. Ring and Hollow Cylinder

$I_C = MK^2$

$MR^2 = MK^2$

$\Rightarrow \frac{K^2}{R^2} = 1$



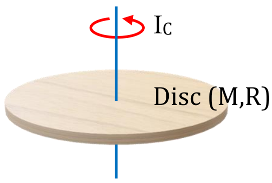


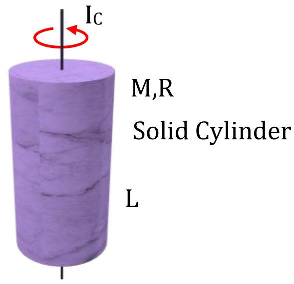
2. Disc and solid cylinder

$I_C = MK^2$

$\frac{MR^2}{2} = MK^2$

$\Rightarrow \frac{K^2}{R^2} = \frac{1}{2}$



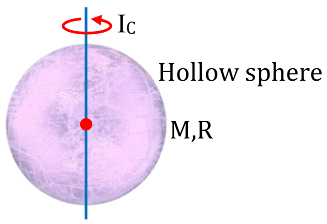


3. Hollow sphere and Solid sphere

$I_C = MK^2$

$\frac{2}{3} MR^2 = MK^2$

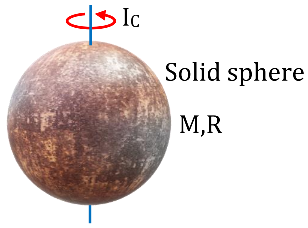
$\Rightarrow \frac{K^2}{R^2} = \frac{2}{3}$



$I_C = MK^2$

$\frac{2}{5} MR^2 = MK^2$

$\Rightarrow \frac{K^2}{R^2} = \frac{2}{5}$



So, value of $\left(\frac{K^2}{R^2}\right)$ is maximum for ring or Hollow cylinder = 1 and minimum for solid sphere = $\frac{2}{5}$ = 0.4

Total kinetic energy of a purely rolling objects

$(KE)_{total} = (KE)_{translation} + (KE)_{rotation}$

$$= \frac{1}{2}MV_C^2 + \frac{1}{2}I_C\omega^2 = \frac{1}{2}MV_C^2 + \frac{1}{2}MK^2 \times \omega^2 = \frac{1}{2}MV_C^2 + \frac{1}{2}M\left(\frac{K^2}{R^2}\right) \times (R^2\omega^2) = \frac{1}{2}MV_C^2 + \frac{1}{2}MV_C^2 \times \frac{K^2}{R^2}$$

$$\Rightarrow \quad KE_{total} = \frac{1}{2}MV_C^2\left[1 + \frac{K^2}{R^2}\right]$$

Where, $\frac{1}{2}MV_C^2 \rightarrow (KE)_{Translation}$ and $\frac{1}{2}MV_C^2 \times \left(\frac{K^2}{R^2}\right) \rightarrow (KE)_{Rotation}$

Fraction of total K.E. in form of translational K.E. (K_{Tr}) and Rotational K.E. (K_{Rot})

$$\frac{(KE)_{Tr}}{(KE)_{Total}} = \frac{\frac{1}{2}MV_C^2}{\frac{1}{2}MV_C^2\left(1+\frac{K^2}{R^2}\right)} = \frac{1}{\left(1+\frac{K^2}{R^2}\right)}$$

$$\frac{(KE)_{Rot}}{(KE)_{Total}} = \frac{\frac{1}{2}MV_C^2 \times \frac{K^2}{R^2}}{\frac{1}{2}MV_C^2\left(1+\frac{K^2}{R^2}\right)} = \frac{K^2/R^2}{\left(1+\frac{K^2}{R^2}\right)}$$

$$\frac{(KE)_{Rot}}{(KE)_{Tr}} = \frac{\frac{1}{2}MV_C^2 \times \frac{K^2}{R^2}}{\frac{1}{2}MV_C^2} = \frac{K^2}{R^2}$$

So, ratio of $(KE)_{Rot}$ to $(KE)_{Tr}$ can directly give shape of object.

Body	$\frac{K^2}{R^2}$	$\frac{KE_{Tr}}{KE_{Rot}} = \frac{1}{\left(\frac{K^2}{R^2}\right)}$	$\frac{KE_{Tr}}{KE_{Total}} = \frac{1}{\left(1+\frac{K^2}{R^2}\right)}$	$\frac{KE_{Rot}}{KE_{Total}} = \frac{K^2/R^2}{\left(1+\frac{K^2}{R^2}\right)}$
Ring	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Disc	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5}{2}$	$\frac{5}{7}$	$\frac{2}{7}$
Spherical shell	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Solid cylinder	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Hollow cylinder	1	1	$\frac{1}{2}$	$\frac{1}{2}$

Illustration 45

When a sphere of moment of inertia 'I' rolls down an inclined plane then find the percentage of rotational kinetic energy of the total energy.

Solution:

$$\frac{KE_{Rot}}{KE_{Total}} \times 100 = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2} \times 100 = \frac{\frac{1}{2} \times \frac{2}{5}MR^2 \times \frac{v^2}{R^2}}{\frac{1}{2}Mv^2\left[1+\frac{K^2}{R^2}\right]} \times 100 = \frac{\frac{2}{5}}{1+\frac{2}{5}} \times 100 = 28.6\%$$

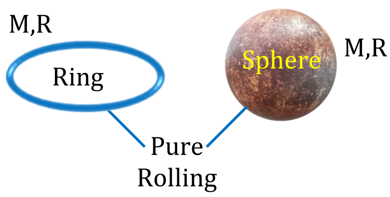
Illustration 46

A ring of mass (M, R) and sphere of mass (M, R) is performing pure rolling. Then the ratio of translation kinetic energy to total kinetic energy will be more for ?

Solution:

For ring : $\frac{KE_{Tr}}{KE_{total}} = \frac{1}{2}$

and for sphere : $\frac{KE_{Tr}}{KE_{total}} = \frac{5}{7}$



$\therefore$ sphere will have more ratio of energies.

Illustration 47

If rotational kinetic energy is 50% of total kinetic energy, the body will be

Solution:

$$\frac{KE_{Rot}}{KE_{Total}}\times 100 = 50\%$$

$$\frac{\frac{K^2}{R^2}}{1+\frac{K^2}{R^2}}\times 100 = 50 \Rightarrow \frac{\frac{K^2}{R^2}}{1+\frac{K^2}{R^2}} = \frac{1}{2} \Rightarrow \frac{K^2}{R^2} = 1 \qquad \therefore \qquad \text{Body can be ring or hollow cylinder}$$

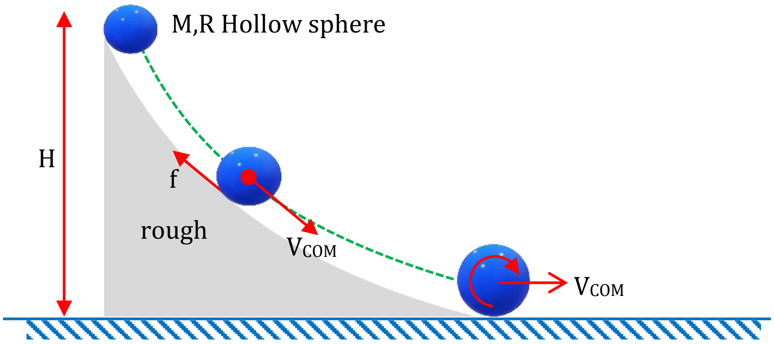
Conservation of Mechanical Energy and Rolling with Slipping

Work energy theorem application

For a body performing rolling motion over a fixed surface, work done by friction force on the body will be zero as velocity of point of application of friction always zero, during pure rolling static friction acts between object and ground. As static friction can not do any work on a system, so there will be no loss of energy due to friction in pure rolling, so we can apply conservation of mechanical energy [COME]

Illustration 48

Find velocity of COM of hollow sphere on reaching ground, if sphere maintains pure rolling throughout motion ? [Assume R<<H]. Also find direction of friction on sphere during rolling ?



Solution:

Apply conservation of mechanical energy :

$K_1 + U_1 = K_2 + U_2$

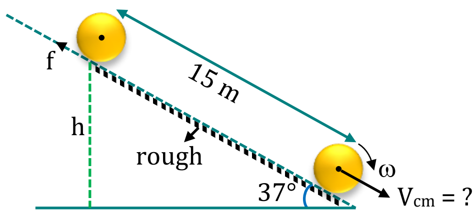
$$0 + MgH = \frac{1}{2}MV_{COM}^2\left[1+\frac{2}{3}\right]+0 \Rightarrow MgH = \frac{5}{6}MV_{COM}^2 \Rightarrow V_{COM} = \sqrt{\frac{6gH}{5}}$$

(ii) For pure rolling, $V_{COM} = \omega R$

So, as V_{COM} increases during rolling so ω must also increase to maintain pure rolling, so torque of friction should be clockwise, and friction must be upward along plane.

Illustration 49

A disc of mass M and radius R released from rest performs pure rolling motion. Find the velocity of COM of the disc ?



Solution:

h = 15 sin37°

$h = 15 \times \frac{3}{5} = 9\text{m}$

Applying conservation of mechanical energy

$W_g + W_N + W_f = KE_f - KE_i$

$$Mgh + 0 + 0 = \left(\frac{1}{2}\times I\times\omega^2 + \frac{1}{2}MV_{COM}^2\right) - (0+0) \Rightarrow Mg \times 9 + 0 + 0 = \left(\frac{1}{2}\frac{MR^2}{2}\omega^2 + \frac{1}{2}MV_{cm}^2\right) - 0$$

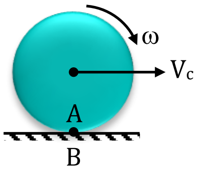
$$9Mg = \frac{1}{2}M\left(\frac{V_{cm}^2}{2} + V_{cm}^2\right) \Rightarrow 6g = \frac{V_{cm}^2}{2} \Rightarrow V_{cm} = \sqrt{12g}$$

Rolling with slipping

When a body rolls on a surface under external force, the frictional force on the body (if any) will be static in nature, less than its limiting value. But if the object rolls with slipping the nature of friction should be kinetic in nature.

- Nature of friction in the following cases assuming the body is perfectly rigid

(i) Condition for pure rolling



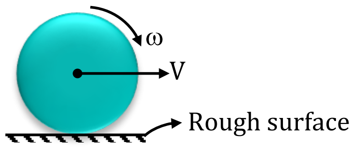
Contact point A on object do not slip relative to point B on surface. So velocity of point A must be equal to velocity of point B.

$V_A = V_B$

$V_C - \omega R = V_{surface}$

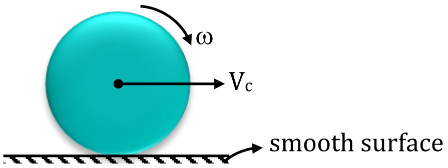
No, friction and pure rolling.

(ii) $V = \omega R$



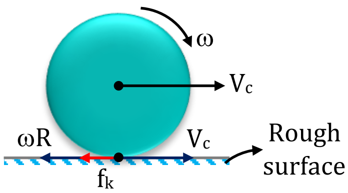
Pure rolling will continue

(iii) $V_c > \omega R$ or $V_c < \omega R$



No friction force but not pure rolling.

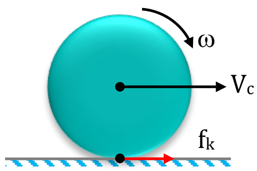
(iv) $V_c > \omega R$



If $V_c > \omega R$ contact point slips forward so kinetic friction will act in backward direction, so V_c decrease and ω increases

Ultimately, when $V_c = \omega R$, pure rolling starts

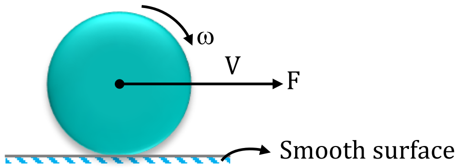
(v) If $V_c < \omega R$



If $V_c < \omega R$, contact point slips in backward direction, so kinetic friction will act in forward direction, this kinetic friction will act in forward direction, this kinetic friction will increase V and decrease ω, so after some time.

Ultimately, $V_c = \omega R$, pure rolling will starts.

(vi) $V = \omega R$ (initial)



No friction and no pure rolling

(vii) $V = \omega R$ (initial)

Static friction whose value can lie between zero and μN will act in backward direction. If co-efficient of friction is sufficiently high, then f_s compensates for increasing V due to F by increasing ω and body may continue in pure rolling with increasing V as well as ω.

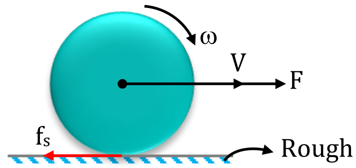
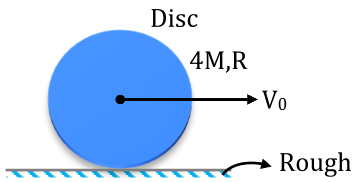


Illustration 50

A disc placed on a rough surface such that its initial angular velocity is zero. Find the velocity of COM when pure rolling starts?



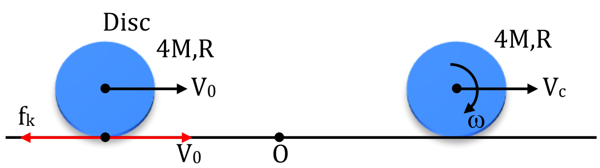
Solution:

$\omega = 0$

Applying COAM about point O.

$(\vec{L}_i)_0 = (\vec{L}_f)_0$

$$[-4MV_oR] + [0] = [-4MV_cR] + \left[-\frac{1}{2}\times 4MR^2\omega\right]$$



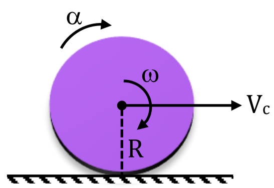
$$\Rightarrow -4MV_oR = -4MV_cR - \frac{1}{2}\times 4MV_cR$$

$$\Rightarrow -4MV_oR = -6MV_cR \Rightarrow V_c = \frac{2V_o}{3}$$

Accelerated Pure Rolling

For pure rolling $V_c = \omega R$

To maintain pure rolling in accelerated motion, if V_c increases so omega (ω) also increases. $\Rightarrow \frac{dV_c}{dt} = \frac{d}{dt}(\omega R) = \frac{R d\omega}{dt}$



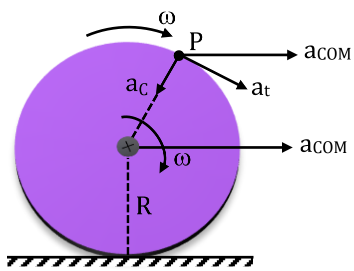
$\boxed{\Rightarrow a_{COM} = \alpha R}$ $\rightarrow$ **Condition for accelerated pure rolling**

Where, α = angular acceleration

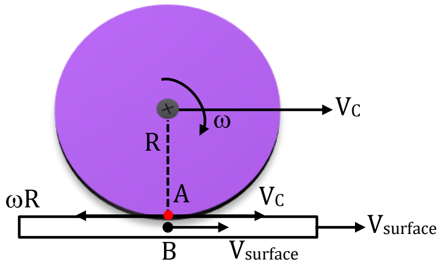
a_{COM} = acceleration of COM

- So, $(\vec{a}_{net})_P = \vec{a}_{translation} + \vec{a}_{rotation}$

  So, $(\vec{a}_{net})_P = \vec{a}_{COM} + \vec{a}_{centripetal} + \vec{a}_{tangential}$



- If surface is also moving



For pure rolling,

$V_A = V_B$

$V_C - \omega R = V_{surface}$

and $a_{T_A} = a_{T_B}$ i.e.

$\therefore a_{COM} - \alpha R = a_{plank}$

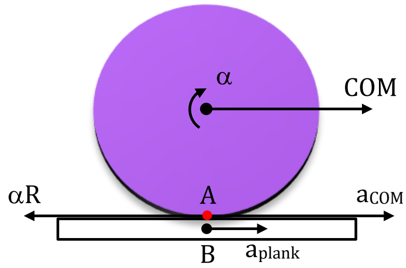
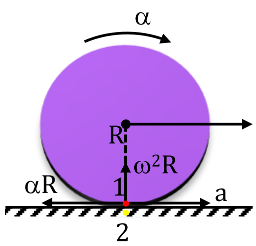


Illustration 51

If pure rolling occurs, find relation between α and a?



Solution:

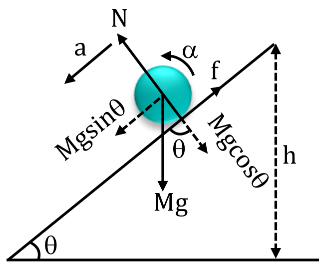
condition $a_{T_1} = a_2$

$a - R\alpha = a_2$ {∵ the surface is fixed, ∴ $a_2 = 0$}

$\Rightarrow \quad a = R\alpha$

Rolling motion on an inclined plane

A body of mass M and radius R is rolling down a plane inclined at an angle θ with the horizontal. The body rolls without slipping. The centre of mass of the body moves in a straight line. External forces acting on the body are :



- Weight Mg of the body vertically downwards through its center of mass.
- The normal reaction N of the inclined plane.
- The frictional force f acting upwards and parallel to the inclined plane.

For linear motion $\quad Mg\sin\theta - f = Ma_{CM}$; $\quad$ For angular motion $\tau = fR = I\alpha$

From the condition for pure rolling $a_{CM} = \alpha R \Rightarrow Mg\sin\theta - f = M\left(\frac{fR^2}{I}\right) = M\left(\frac{fR^2}{MK^2}\right) \Rightarrow f = \frac{Mg\sin\theta}{\left(1+\frac{R^2}{K^2}\right)}$

But $f \le \mu Mg\cos\theta \quad \Rightarrow \frac{Mg\sin\theta}{\left(1+\frac{R^2}{K^2}\right)} \le \mu Mg\cos\theta$

$\Rightarrow \mu \ge \frac{\tan\theta}{\left(1+\frac{R^2}{k^2}\right)} \quad \Rightarrow$ for pure rolling $\mu_{min} = \frac{\tan\theta}{\left(1+\frac{R^2}{K^2}\right)}$

Rolling Motion on an inclined plane

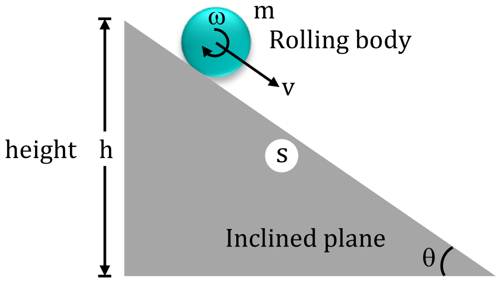
(i) Velocity at the bottom of the inclined plane

Applying Conservation of mechanical energy principle

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

$$mgh = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right) \quad ...(i)$$



$$h = s\sin\theta \quad ...(ii)$$

from (i) & (ii)

$$v_{Rolling} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}$$

When the body slides $\frac{K^2}{R^2} = 0$

$\therefore$ Velocity when the body slides $v_{sliding} = \sqrt{2gh} = \sqrt{2gs\sin\theta} \Rightarrow v_{sliding} > v_{rolling}$

(ii) Acceleration of the body

$v^2 = u^2 + 2as \quad (\because u = 0)$

$\therefore \quad v^2 = 2as$

$$\frac{2gs\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} = 2as \Rightarrow a_{rolling} = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

When body slides $\frac{K^2}{R^2} = 0$

Acceleration when the body slides $a_{sliding} = g\sin\theta \Rightarrow a_{sliding} > a_{rolling}$

(iii) Time taken by the rolling body to reach the bottom

$$s = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2s}{a}}, \quad s = \frac{h}{\sin\theta}$$

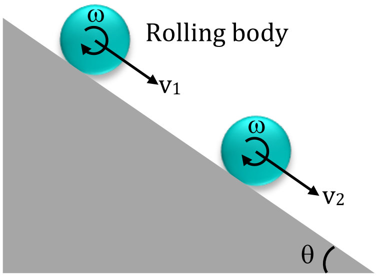
$$t_{rolling} = \sqrt{\frac{2s}{g\sin\theta}\left(1+\frac{K^2}{R^2}\right)} = \sqrt{\frac{2h}{g\sin^2\theta}\left(1+\frac{K^2}{R^2}\right)} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}$$

When the body slides, $\frac{K^2}{R^2} = 0$

Time of descent when the body slides $t_{sliding} = \sqrt{\frac{2s}{g\sin\theta}} = \sqrt{\frac{2h}{g\sin^2\theta}} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$

so $\quad t_{rolling} > t_{sliding}$

Note: If different bodies are allowed to roll down on an inclined plane then the body with



(i) least $\frac{K^2}{R^2}$ will reach first

(ii) maximum $\frac{K^2}{R^2}$ will reach last

(iii) equal $\frac{K^2}{R^2}$ will reach together.

(iv) Change in kinetic energy due to rolling $v_2 > v_1$

$$=\frac{1}{2}mv_2^2\left(1+\frac{K^2}{R^2}\right)-\frac{1}{2}mv_1^2\left(1+\frac{K^2}{R^2}\right)=\frac{1}{2}m\left(1+\frac{K^2}{R^2}\right)\left(v_2^2-v_1^2\right)$$

(v) Rolling v/s Sliding

$$v_{rolling}=\frac{v_{sliding}}{\sqrt{1+\frac{K^2}{R^2}}},\ a_{rolling}=\frac{a_{sliding}}{1+\frac{K^2}{R^2}},\ t_{rolling}=t_{sliding}\sqrt{1+\frac{K^2}{R^2}}$$

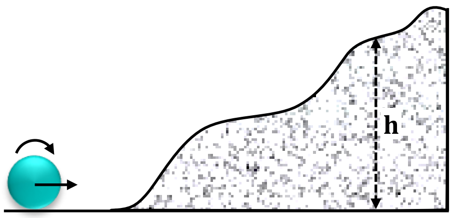
$v_{rolling} < v_{sliding}$, $a_{rolling} < a_{sliding}$, $t_{rolling} > t_{sliding}$

Comparision between formula of translatory motion and rotatory motion

Translatory Motion	**Rotatory Motion**
• $\vec{F}=\frac{d\vec{p}}{dt}$	• $\vec{\tau}=\frac{d\vec{L}}{dt}$
• $\vec{F}=m\vec{a}$	• $\vec{\tau}=I\vec{\alpha}$
• Linear momentum $(\vec{p})$, $\vec{p}=m\vec{v}$	• Angular momentum $(\vec{L})$, $\vec{L}=I\vec{\omega}$
• Translational kinetic energy, $KE=\frac{1}{2}mv^2$	• Rotational kinetic energy, $E=\frac{1}{2}I\omega^2$
• Work done $W=\vec{F}.\vec{s}$ (constant force)	• Work done $W=\vec{\tau}.\vec{\theta}$ (constant torque)
• $W=\int\vec{F}.d\vec{s}$ Variable force	• $W=\int\vec{\tau}.d\vec{\theta}$ Variable torque
• Power $P=\frac{dW}{dt}=\frac{\vec{F}.d\vec{s}}{dt}=\vec{F}.\vec{v}$	• Power $P=\frac{dW}{dt}=\frac{\vec{\tau}.d\vec{\theta}}{dt}=\vec{\tau}.\vec{\omega}$
• Work–energy theorem $W=\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2$	• Work–energy theorem $W=\frac{1}{2}I\omega_2^2-\frac{1}{2}I\omega_1^2$
• Linear impulse It is the product of large force and the small time for which it acts $\vec{F}=\frac{\Delta\vec{p}}{\Delta t}$ Impulse–momentum theorem $\Delta\vec{p}=\vec{F}\Delta t$ = Impulse	• Angular impulse It is product of large torque for small time for which it acts $\vec{\tau}=\frac{\Delta\vec{L}}{\Delta t}$ Angular Impulse–momentum theorem $\Delta\vec{L}=\vec{\tau}\Delta t$ = Angular impulse

Illustration 52

A body of mass M and radius r, rolling with velocity v on a smooth horizontal floor, rolls up a rough irregular inclined plane up to a vertical height $(3v^2/4g)$. Compute the moment of inertia of the body and comment on its shape?



Solution:

$$KE_{Total} = KE_{Tr} + KE_{Rot} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$\Rightarrow E = \frac{1}{2}Mv^2\,[1 + (I/Mr^2)]\ [\text{as } v = r\omega]$$

When it rolls up on an irregular inclined plane of height $h = (3v^2/4g)$, its KE is fully converted into PE, So, by conservation of mechanical energy

$\frac{1}{2}Mv^2\left[1+\frac{I}{Mr^2}\right] = Mg\left[\frac{3v^2}{4g}\right]$ which on simplification gives $I = (1/2)\,Mr^2$.

This result clearly indicates that the body is either a disc or a cylinder.

BEGINNER'S BOX-6

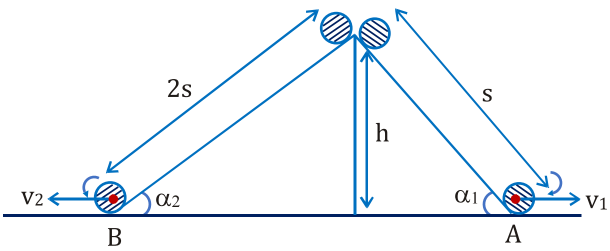
1. A hollow cylinder is rolling down an inclined plane which is inclined at an angle of 30° to the horizontal. Find its speed after travelling a distance of 10 m.

PRM203

2. A wheel is rotating about a fixed axis. Find the moment of inertia of the wheel about the axis of rotation, when its angular speed is 30 radians/s and its kinetic energy is 360 joules.

PRM204

3. Two uniform identical discs roll down on two inclined planes of length s and 2s respectively as shown in the figure. Find the ratio of velocities of the two discs at the points A and B of the inclined planes?



PRM205

4. Translational kinetic energy of a rolling hollow sphere is .... percent of its total energy.

PRM206

BEGINNER'S BOX ANSWER KEY

BEGINNER'S BOX-1

1. (a) 120 rad/s; (b) (i) $-\frac{90}{\pi}$ rad/s^2 ; (ii) 24π m

2. (A) **3.** (A) **4.** (A)

BEGINNER'S BOX-2

1. 48 kg-m^2 **2.** $1:\sqrt{2}$ **3.** 1:4 **4.** 3 Kg-m^2 **5.** $\frac{6I}{5}$

6. Moment of inertia of a hollow cylinder will be larger as compare to a disc because most of the mass of the former is located away from the axis of rotation as compared to the latter.

7. Spoke do not carry much mass. Most of the mass is located at the rim. This gives cycle wheel more moment of inertia for same mass.

When the wheel experiences any opposition in the uniform rotational motion, a steady and smooth motion can be seen in the bicycle by increasing the moment of inertia.

8. $I_B = 64\ I_A$ **9.** $\frac{\mu\ell^4}{6}$ **10.** $I = \frac{a^2}{4}(m_2 + m_3)$ **11.** $\frac{3}{2}MR^2$

12. It will be again = $\frac{2}{5}MR^2$, because the axis in question is also diameter of the sphere.

BEGINNER'S BOX-3

1. $\vec{\tau} = 11\hat{i} - 7\hat{j} - 5\hat{k}$

2. About point B moment of inertia is less than A so it is easier to rotate about point B

3. (B) **4.** (D) **5.** (B) **6.** (D)

7. (D) **8.** (A)

BEGINNER'S BOX-4

1. (D)

2. The angular momentum of particle about origin will be zero because given straight line passes through origin, so line of action of velocity (momentum) also passes through this point.

3. 4×10^{-3} J ; 8×10^{-4} J-s

BEGINNER'S BOX-5

1. $\sqrt{\frac{\omega_2}{\omega_1}}$ **2.** $\frac{L}{4}$ **3.** $\frac{2}{3}m\ell^2\pi^2 f^2$ **4.** 131.6 J

5. 0.45 J **6.** Angular velocity will decrease. **7.** $T = \frac{\pi}{10}$

8. (a) 4.95×10^4 N–m (b) 4.95×10^6 J

BEGINNER'S BOX-6

1. 7 m/s **2.** 0.8 kg–m^2 **3.** $\frac{v_1}{v_2} = 1$ **4.** 60