01 Basic Mathematics Used in Physics and Vectors

Basic Mathematics Used in Physics

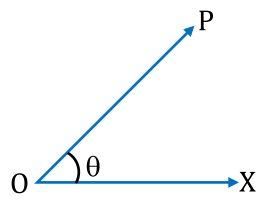
Trigonometry

Introduction to Angle

Consider a revolving line OP. Suppose that it revolves in anticlockwise direction starting from its initial position OX.

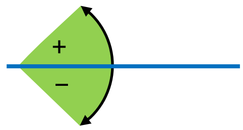
The angle is defined as the amount of revolution that the revolving line makes with its initial position.

From fig. the angle covered by the revolving line OP is $\theta = \angle POX$



The angle is taken positive if it is traced by the revolving line in anticlockwise direction.

The angle is taken negative if it is covered in clockwise direction.



$1° = 60'$ (minute) $1' = 60''$ (second)

1 right angle = 90° (degrees) also 1 right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle subtended at the centre of a circle by an arc of the circle, whose length is equal to the radius of the circle.

$1 \text{ rad} = \frac{180°}{\pi} \approx 57.3°$

Units of Angle

Practical units : degrees (°) ; 1° = 60'(minute) ; 1' = 60"(second)

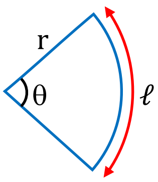
To convert an angle from degree to radian multiply it by $\frac{\pi}{180°}$

To convert an angle from radian to degree multiply it by $\frac{180°}{\pi}$

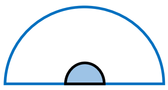
Relation between Angle and Arc

$$\text{Angle } (\theta) = \frac{\text{Arc}}{\text{Radius}} = \frac{\ell}{r}$$

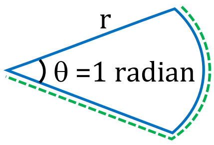
SI UNIT → Radian



Radian



π rad = 180°



$2\pi = 360°$	$\pi = 180°$	$\frac{\pi}{2} = 90°$
$\frac{\pi}{3} = 60°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{6} = 30°$
$\frac{2\pi}{3} = 120°$	$\frac{3\pi}{4} = 135°$	$\frac{5\pi}{6} = 150°$

Illustration 1:

Convert the given angles in desired units.

(i) 5° to minutes (ii) 6' to second (iii) 120" to minutes

Solution:

(i) 1° = 60' so, 5° × 60' = 300' (ii) 1' = 60" so, 6' × 60" = 360" (iii) 60" = 1' so, $\frac{120"}{60"} = 2'$

Illustration 2:

Convert the given angles in desired units.

1. Convert 45° to radians 2. Convert $\frac{5\pi}{6}$ rad to degree

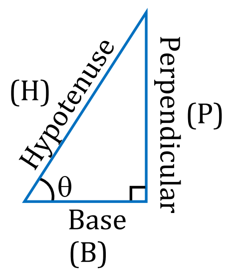
Solution:

1. $45° \times \frac{\pi}{180°} = \frac{\pi}{4}$ radians 2. $\frac{5\pi}{6} \times \frac{180°}{\pi} = 150°$

Pythagoras Theorem and Trigonometric Ratio

Pythagoras Theorem

$P^2 + B^2 = H^2$



Pythagorean Triplets

3, 4, 5 ($3^2 + 4^2 = 5^2$)

6, 8, 10 ($6^2 + 8^2 = 10^2$)

7, 24, 25 ($7^2 + 24^2 = 25^2$)

12, 16, 20 ($12^2 + 16^2 = 20^2$)

Remember for fast calculations in Physics!!

Trigonometric Ratios (or T ratios)

$\sin\theta = \frac{P}{H}$ $\quad \text{cosec}\theta = \frac{H}{P}$ $\quad \cos\theta = \frac{B}{H}$ $\quad \sec\theta = \frac{H}{B}$ $\quad \tan\theta = \frac{P}{B}$ $\quad \cot\theta = \frac{B}{P}$

It can be easily proved that : $\text{cosec}\theta = \frac{1}{\sin\theta}$ $\quad \sec\theta = \frac{1}{\cos\theta}$ $\quad \cot\theta = \frac{1}{\tan\theta}$

Trigonometric Identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

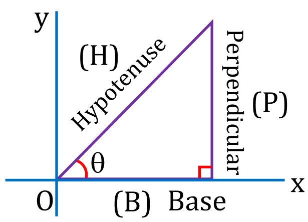
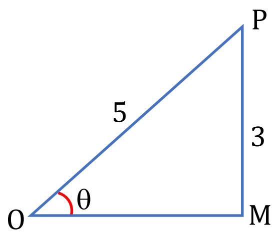


Illustration 3:

Given sinθ = 3/5. Find all the other T-ratios, if θ lies in the first quadrant.

Solution:



In ΔOMP, $\sin\theta = \frac{3}{5}$ so, MP = 3 and OP = 5

$\because$ $OM = \sqrt{(5)^2 - (3)^2} = \sqrt{25-9} = \sqrt{16} = 4$

Now, $\cos\theta = \frac{OM}{OP} = \frac{4}{5}$ $\quad \tan\theta = \frac{MP}{OM} = \frac{3}{4}$

$\cot\theta = \frac{OM}{MP} = \frac{4}{3}$ $\quad \sec\theta = \frac{OP}{OM} = \frac{5}{4}$ $\quad \text{cosec}\theta = \frac{OP}{MP} = \frac{5}{3}$

Quadrant Theory and Trigonometric Formulae

Quadrants & ASTC Rule :

In first quadrant, all trigonometric ratios are positive.

In second quadrant, only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive.

In third quadrant, only $\tan\theta$ and $\cot\theta$ are positive.

In fourth quadrant, only $\cos\theta$ and $\sec\theta$ are positive

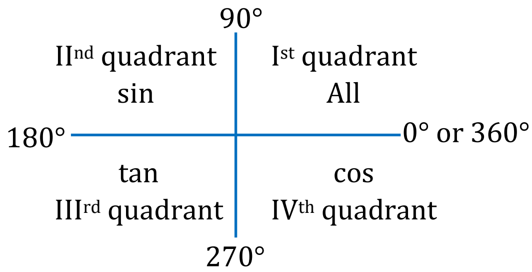
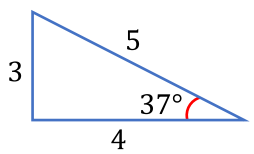
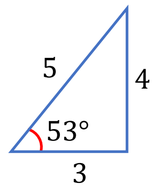


Table to Remember

Angle(θ)	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ (not defined)

T-Ratios of Special Angles





$\sin\theta = \frac{3}{5}; \cos\theta = \frac{4}{5}; \tan\theta = \frac{3}{4}$

$\sin\theta = \frac{4}{5}; \cos\theta = \frac{3}{5}; \tan\theta = \frac{4}{3}$

T-ratios of angles greater than 90°

STEP-1	Decide sign according to quadrant.	
STEP-2	$\alpha = A \pm \theta$	A : Integral multiple of 90° θ : Acute angle
STEP-3	If A is odd multiple of 90°	$\sin\theta \rightleftharpoons \cos\theta$; $\operatorname{cosec}\theta \rightleftharpoons \sec\theta$; $\tan\theta \rightleftharpoons \cot\theta$
	If A is Even multiple of 90°	No Change

Trigonometrical Ratios of General Angles (Reduction Formulae)

(i) Trigonometric function of an angle $(2n\pi + \theta)$ where n=0, 1, 2, 3,.... will be remain same.

$\sin(2n\pi + \theta) = \sin\theta$ $\quad$ $\cos(2n\pi + \theta) = \cos\theta$ $\quad$ $\tan(2n\pi + \theta) = \tan\theta$

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will remain same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

$\sin(\pi - \theta) = +\sin\theta$ $\quad$ $\cos(\pi - \theta) = -\cos\theta$ $\quad$ $\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi + \theta) = -\sin\theta$ $\quad$ $\cos(\pi + \theta) = -\cos\theta$ $\quad$ $\tan(\pi + \theta) = +\tan\theta$

$\sin(2\pi - \theta) = -\sin\theta$ $\quad$ $\cos(2\pi - \theta) = +\cos\theta$ $\quad$ $\tan(2\pi - \theta) = -\tan\theta$

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos\theta$ $\quad$ $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ $\quad$ $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos\theta$ $\quad$ $\cos\left(\frac{\pi}{2} - \theta\right) = +\sin\theta$ $\quad$ $\tan\left(\frac{\pi}{2} - \theta\right) = +\cot\theta$

(iv) Trigonometric function of an angle $-\theta$ (negative angles)

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = +\cos\theta$ $\tan(-\theta) = -\tan\theta$

BEGINNER'S BOX-1

1. Find the values of :

(i) tan (–30°) (ii) sin 120° (iii) sin 135° (iv) cos 150° (v) sin 270° (vi) cos 270° **PBM149**

2. If $\sec\theta = \frac{5}{3}$ and $0 < \theta < \frac{\pi}{2}$. Find all the other T-ratios. **PBM150**

Sum property for sine function

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

Sum property for cosine function

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

Sum property for tan function

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Double angle property

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$ < $\cos 2A = 2\cos^2 A - 1$; $\cos 2A = 1 - 2\sin^2 A$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Illustration 4:

Write the function in terms of acute angle θ

(1) $\cos(270° - \theta) = ?$ (2) $\cos(90° + \theta) = ?$ (3) $\cos\left(\frac{\pi}{2} - \theta\right) = ?$ (4) $\sin\left(\frac{\pi}{2} - \theta\right) = ?$

Solution:

(1) $\cos(270° - \theta) = \cos(\frac{3\pi}{2} - \theta) = -\sin\theta$

(2) $\cos(90° + \theta) = \cos(\frac{\pi}{2} + \theta) = -\sin\theta$

(3) $\cos(\frac{\pi}{2} - \theta) = +\sin\theta$

(4) $\sin(\frac{\pi}{2} - \theta) = +\cos\theta$

Illustration 5:

Evaluate :

(a) sin120° (b) tan150° (c) cos330°

Solution:

(a) $\sin 120° = \sin(90° + 30°) = \sin(\frac{\pi}{2} + 30°) = +\cos 30° = \frac{\sqrt{3}}{2}$

(b) $\tan 150° = \tan(90° + 60°) = \tan(\frac{\pi}{2} + 60°) = -\cot 60° = -\frac{1}{\sqrt{3}}$

(c) $\cos 330° = \cos(360° - 30°) = \cos(2\pi - 30°) = +\cos 30° = \frac{\sqrt{3}}{2}$

Illustration 6:

Evaluate :

(a) cos(–30°) (b) sin(–45°)

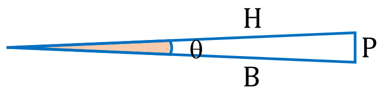
Solution:

(a) $\cos(-30°) = \cos(30°) = \frac{\sqrt{3}}{2}$

(b) $\sin(-45°) = -\sin(45°) = -\frac{1}{\sqrt{2}}$

Range of Trigonometric Functions :

Small angle approximation



Here θ must be in radians

$\sin\theta \approx \theta$

$\tan\theta \approx \theta$

$\cos\theta \approx 1$

Illustration 7:

Find :

1. $\sin 2^\circ$ 2. $\tan 1^\circ$ 3. $\sin \pi^\circ$

Solution:

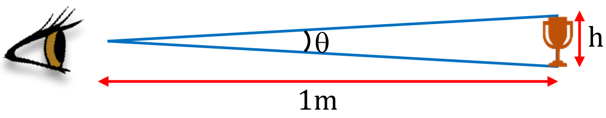
1. $\sin 2^\circ = 2^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{90}$ 2. $\tan 1^\circ = 1^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{180}$ 3. $\sin \pi^\circ = \pi \times \frac{\pi}{180}$

Illustration 8:

A normal human eye can see an object making an angle 1.8° at the eye. What is the minimum height of object which can be seen by an eye from 1 m distance.

Solution:

θ is very small

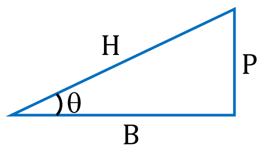


$\therefore \quad \tan\theta \approx \theta$

$\theta = \frac{1.8^\circ \times \pi}{180^\circ} = \frac{\pi}{100}\text{rad}$

$\frac{h}{1} = \frac{\pi}{100} \quad \therefore \quad h = 0.031 \text{ m}$

Range of Trigonometric Functions



$\sin\theta = \frac{P}{H} \longrightarrow -1 \le \sin\theta \le 1$

$\cos\theta = \frac{B}{H} \longrightarrow -1 \le \cos\theta \le 1$

$\tan\theta = \frac{P}{B} \longrightarrow -\infty < \tan\theta < +\infty$

Illustration 9:

Find the maximum value of y = (sinx) (cosx)

Solution:

$y = \sin x\cos x = \frac{1}{2}(\sin 2x) = \frac{1}{2}$ $\{\sin 2x = 2\sin x\cos x\}$

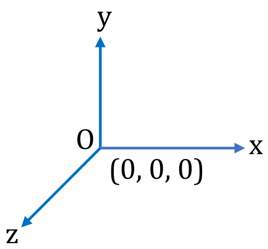
Important result : Range of function : "asinθ + bcosθ" $\Rightarrow -\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

Co-ordinate Geometry

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If a point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

- **Origin:** This is any fixed point which is convenient to you. All measurements are taken w.r.t. this fixed point.

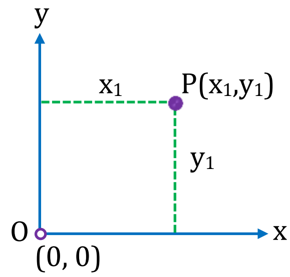
- **Axis or Axes:** Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.



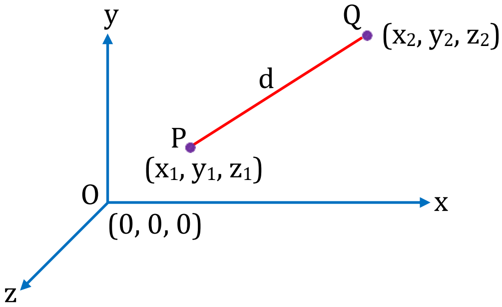
Position of a point

x_1 = Abscissa : Distance of point from y axis

y_1 = Ordinate : Distance of point from x axis



Distance Formula in space



In space : $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Illustration 10:

Find distance between two points A (1, 2, 5) and B (3, 4, 6).

Solution:

$A(x_1\ y_1\ z_1) = A(1, 2, 5)$, $B(x_2\ y_2\ z_2) = B\ (3, 4, 6)$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(3-1)^2 + (4-2)^2 + (6-5)^2} = \sqrt{4+4+1} = 3$ unit

Illustration 11:

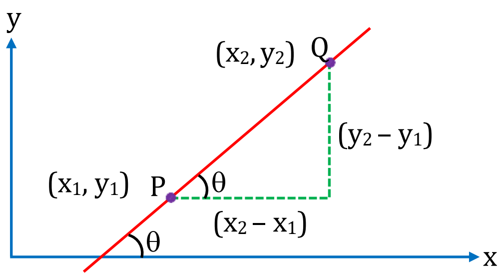
Find possible values of a if distance between the points (–9 cm, a cm) and (3 cm, 3 cm) is 13 cm.

Solution:

$13 = \sqrt{(3+9)^2 + (3-a)^2} \Rightarrow 169 = 144 + (3 - a)^2 \Rightarrow \pm 5 = (3 - a) \Rightarrow a = -2\text{cm}$ or $a = 8\text{cm}$

Slope of Line joining Two points

$\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$



Angle θ is measured from x axis.

Illustration 12:

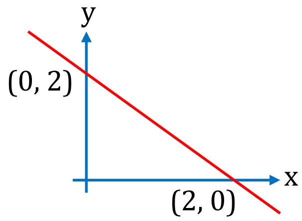
Find slope of a line passing through points A(2, 4) and B(3, 8)

Solution:

$A(x_1, y_1) = A(2, 4)$, $B(x_2, y_2) = B(3, 8)$ so $\tan\theta = \dfrac{8-4}{3-2} = \dfrac{4}{1}$ so, slope = 4

Illustration 13:

Calculate slope of the shown line and its angle with x axis.



Solution:

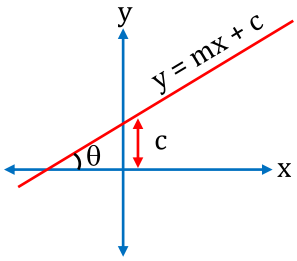
$\tan\theta = \dfrac{2-0}{0-2} = -1 \Rightarrow$ So, slope $= -1$ angle with x-axis 135°.

Equation of Straight Line

Genera equation of straight line

$y = mx + c$

Slope or gradient = $\tan\theta$ Intercept on y-axis



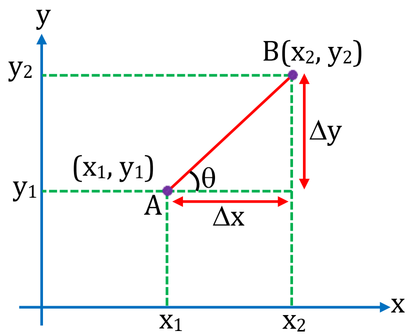
Slope of A Line

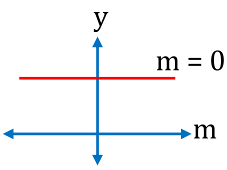
The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by

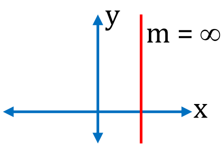
$m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \tan\theta$ [If both axes have identical scales]

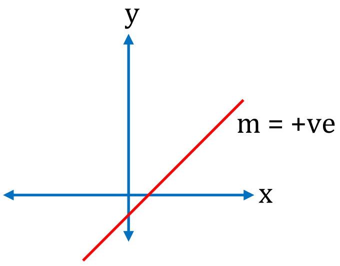
Here θ is the angle made by line with positive x-axis.

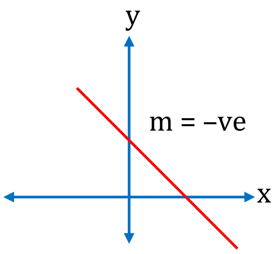
Slope of a line is a quantitative measure of inclination.



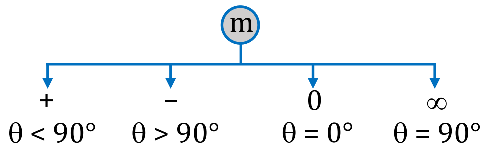




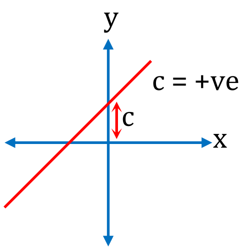


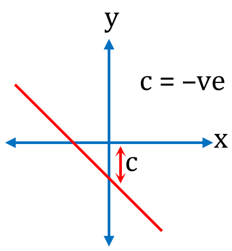


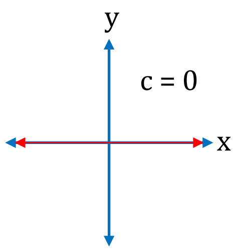
Slope of a Straight line



Intercept of A Line







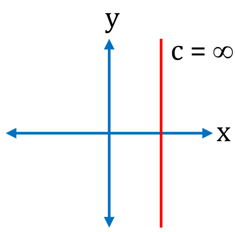
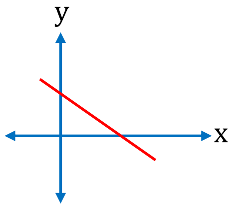


Illustration 14:

Find slope and intercept of a line 4y + 3x = 8, also draw the line.

Solution:



y = mx + c (general equation) ...(i)

4y + 3x = 8 (given equation)

4y = – 3x + 8

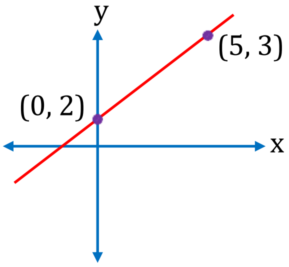
$y = -\frac{3}{4}x + 2$ ...(ii)

comparing equation (i) & (ii)

$m = \text{slope} = -\frac{3}{4}$; c = 2

Illustration 15:

Write equation of the line drawn



Solution:

$\text{slope} = \tan\theta = \frac{3-2}{5-0} = \frac{1}{5}$

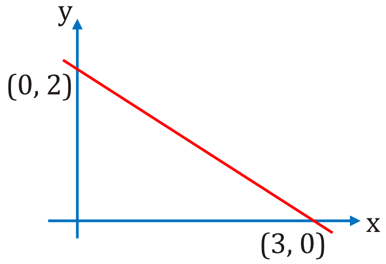
c = 2 $\therefore$ equation of straight line $y = \frac{1}{5}x + 2$

BEGINNER'S BOX-2

1. Distance between two points (8, – 4) and (0, a) is 10. All the values are in the same unit of length. Find the positive value of a. **PBM151**

2. Calculate the distance between two points (0, –1, 1) and (3, 3, 13). **PBM152**

3. Calculate slope of shown line **PBM153**



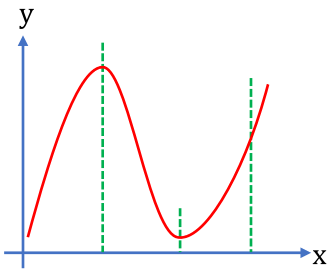
Differentiation

Constant : A quantity, whose value remains unchanged during mathematical operations, is called a constant quantity. The integers, fractions like π, e etc are all constants.

Variable : A quantity, which can take different values, is called a variable quantity. A variable is usually represented as x, y, z, etc.

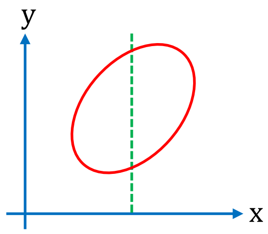
Function : A quantity y is called a function of a variable x, if corresponding to any given value of x, there exists a single definite value of y. The phrase 'y is function of x' is represented as y = f (x)

(A)



It is a function

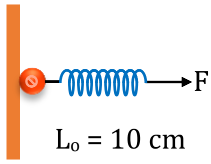
(B)



It is not a function

Finite change (Δx)

If change in a quantity is comparable to its initial value, change is said to be finite. For example, if length of a spring is 10 cm, then a change of 2 cm in length is considered as finite change in length.



L_o = 10 cm

ΔL = 2 cm

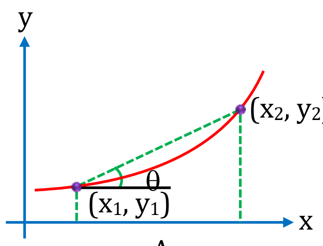
Infinitesimal change

If change in a quantity is not comparable to its initial value, change is said to be infinitesimal.

- ➢ A very small change in x is called as dx
- ➢ A very small change in z is called as dz
- ➢ A very small change in t is called as dt

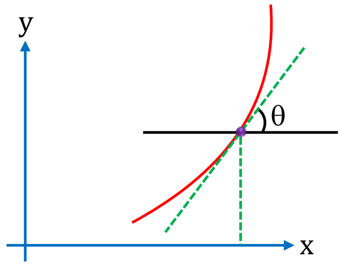
Slope of A Curve

Average slope of curve (Between two different points)



$$m_{avg} = \frac{\Delta y}{\Delta x} = \tan\theta$$

Instantaneous slope of curve (At a Single point)

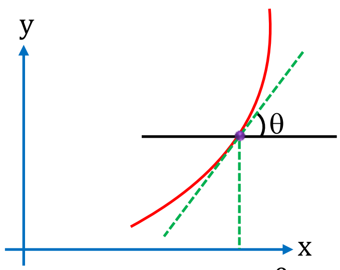


$$m_{instantaneous} = \tan\theta$$

Instantaneous slope of curve

(At a Single point)

To find the slope we require the tool called DIFFERENTIATION



$$m_{instantaneous} = \tan\theta$$

Definition of Differentiation/Derivative

At a point :

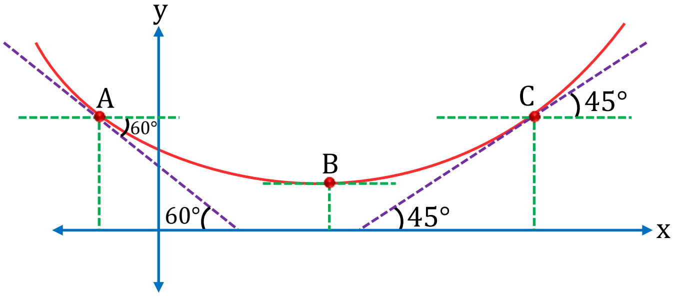
$\frac{dy}{dx}$ = "instantaneous rate of change of y w.r.t. x"

If y is a function of x : y = f(x)

Then derivative of y "w.r.t "x is given by : $y' = f'(x) = \frac{dy}{dx}$

Illustration 16:

Find slope at A, B & C



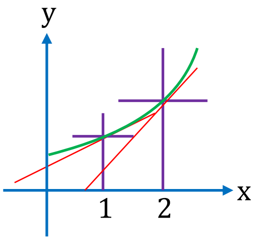
Solution:

at A, slope = tanθ = tan (120^0) = tan(90° + 30°) = –cot(30°) = $-\sqrt{3}$

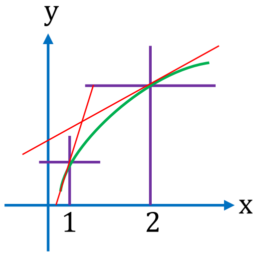
at B, slope = tanθ = tan(0°) = 0

at C, slope = tanθ = tan(45°) = 1

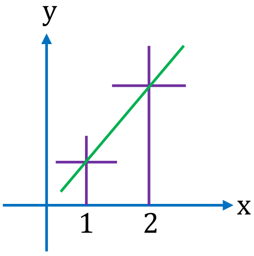
Increasing functions



|Slope| → 1 < 2
|Slope| is increasing.

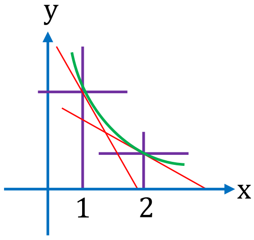


|Slope| → 1 > 2
|Slope| is decreasing.

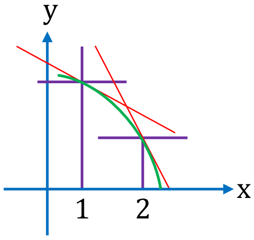


|Slope| → 1 = 2
|Slope| is constant.

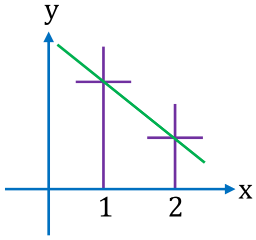
Decreasing functions



|Slope| → 1 > 2
|Slope| is decreasing.



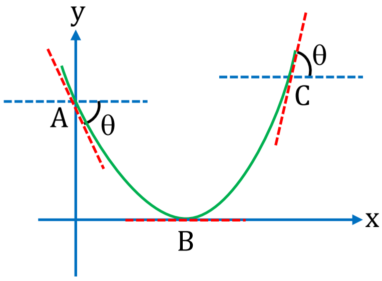
|Slope| → 1 < 2
|Slope| is increasing.



|Slope| → 1 = 2
|Slope| is constant.

Illustration 17:

Comment on slope and its magnitude from A to B & B to C



Solution:

A → B slope increasing

B → C slope increasing

A → B magnitude decreasing

B → C magnitude increasing

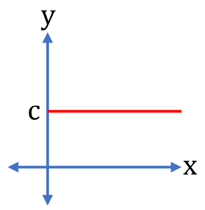
Differentiation of Standard Functions

Physical meaning of $\frac{dy}{dx}$

- The ratio of small change in the function y and the variable x is called the average rate of change of y w.r.t. x.

  For example, the velocity of a body changes by a small amount Δv in small time Δt, then average acceleration of the body, $a_{av} = \frac{\Delta v}{\Delta t}$
- When $\Delta x \to 0$ The limiting value of $\frac{\Delta y}{\Delta x}$ is $\lim\limits_{\Delta t \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

Derivative of Constant Functions



y = f (x) = constant, then it's derivative is ZERO.

$$\frac{dy}{dx} = \frac{dc}{dx} = 0$$

Derivative of Algebraic Functions

y = f (x) = x^n, then it's derivative is :

$$\frac{dy}{dx} = \frac{dx^n}{dx} = nx^{(n-1)}$$

Illustration 18:

Find the derivative of given functions w.r.t x

(1) x^4 (2) x^{-2}

Solution:

(1) $\frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3$ (2) $\frac{d}{dx}(x^{-2}) = -2x^{(-2-1)} = -2x^{-3}$

Formulae for Differentiation of Trigonometric Functions

y = sinx $\longrightarrow$ $\frac{d(\sin x)}{dx} = \cos x$

y = cosx $\longrightarrow$ $\frac{d(\cos x)}{dx} = -\sin x$

y = tanx $\longrightarrow$ $\frac{d(\tan x)}{dx} = \sec^2 x$

y = cosecx $\longrightarrow$ $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$

y = secx $\longrightarrow$ $\frac{d(\sec x)}{dx} = \sec x \tan x$

y = cotx $\longrightarrow$ $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

Formulae for Differentiation of Exponential Functions

y = e^x $\longrightarrow$ $\frac{d(e^x)}{dx} = e^x$

Formulae for Differentiation of Logarithmic Functions

y = $\log_e x$ or lnx $\longrightarrow$ $\frac{d(\ln x)}{dx} = \frac{1}{x}$

Rules of Differentiation – Basic

Constant Multiple Rule

If y = cf(x) = cU, then : $\frac{dy}{dx}=\frac{d(cU)}{dx}=c\frac{dU}{dx}$

Illustration 19:

Find the derivative of given functions w.r.t x

(1) 3(sinx) (2) 4π(tanx) (3) 20(lnx) (4) 0.6(e^x)

Solution:

(1) $\frac{d}{dx}(3(\sin x))=3\frac{d}{dx}(\sin x)=3\cos x$

(2) $\frac{d}{dx}(4\pi(\tan x))=4\pi\frac{d}{dx}(\tan x)=4\pi\sec^2 x$

(3) $\frac{d}{dx}(20(\ell n x))=20\frac{d}{dx}(\ell nx)=\frac{20}{x}$

(4) $\frac{d}{dx}\left(0.6(e^x)\right)=0.6\frac{d}{dx}(e^x)=0.6e^x$

Addition/Subtraction Rule

If U and V are functions of x and y is sum of functions U and V :

$$y = U \pm V,\ \frac{d(U\pm V)}{dx}=\frac{dU}{dx}\pm\frac{dV}{dx}$$

Illustration 20:

Find value of $\frac{dy}{dx}$; (1) $y = x^2 + 2x$ (2) $y = x^3 - x^{-2} + 1$

Solution:

(1) $\frac{d}{dx}(x^2+2x)=\frac{d}{dx}(x^2)+\frac{d}{dx}(2x)=2x+2$

(2) $\frac{d}{dx}(x^3-x^{-2}+1)=\frac{d}{dx}(x^3)-\frac{d}{dx}(x^{-2})+\frac{d}{dx}(1)=3x^2+2x^{-3}+0$

Illustration 21:

Find value of $\frac{dx}{dt}$ if : $x = 4t^2 + 3t + 2$

Solution:

$\frac{d}{dt}(4t^2+3t+2)=\frac{d}{dx}(4t^2)+\frac{d}{dx}(3t)+\frac{d}{dx}(2)=8t+3$

Illustration 22:

Find value of $\frac{dy}{dx}$; (1) y = 3sinx + cosx (2) $y = 2\tan x - x^3$

Solution:

(1) $\frac{d}{dx}(3\sin x+\cos x)=\frac{d}{dx}(3\sin x)+\frac{d}{dx}(\cos x)=3\cos x-\sin x$

(2) $\frac{d}{dx}(2\tan x-x^3)=\frac{d}{dx}(2\tan x)-\frac{d}{dx}x^3=2\sec^2 x-3x^2$

Rules of Differentiation - Product Rule and Quotient Rule

Product Rule

If we need to differentiate the product of two functions, then we apply product rule.

$$\frac{d(UV)}{dx}=V\frac{dU}{dx}+U\frac{dV}{dx}$$

Illustration 23:

Find value of $\frac{dy}{dx}$; (1) y = x lnx (2) y = e^xcosx

Solution:

(1) $\frac{d}{dx}(x\ell nx)=\ell nx\frac{d}{dx}(x)+x\frac{d}{dx}(\ell nx)=\ell nx(1)+x\left(\frac{1}{x}\right)=\ell nx+1$

(2) $\frac{d}{dx}(e^x\cos x)=\cos x\frac{d}{dx}(e^x)+e^x\frac{d}{dx}\cos x=\cos xe^x+e^x(-\sin x)$

Quotient Rule

If we need to differentiate a function which is the ratio of two functions, then we apply Quotient Rule.

$$\frac{d\left(\frac{U}{V}\right)}{dx}=\frac{V\frac{dU}{dx}-U\frac{dV}{dx}}{V^2}$$

Illustration 24:

Find derivative of $y=\frac{4x}{x-7}$ w.r.t. x

Solution:

$$\frac{d\left(\frac{4x}{x-7}\right)}{dx}=\frac{(x-7)\frac{d}{dx}(4x)-4x\frac{d}{dx}(x-7)}{(x-7)^2}=\frac{(x-7)(4)-4x(1)}{(x-7)^2}=-\frac{28}{(x-7)^2}$$

Illustration 25:

Find value of $\frac{dy}{dx}$; (1) y = x^2cosx (2) y = e^xsinx

Solution:

(1) $\frac{d}{dx}(x^2\cos x)=\cos x\frac{d}{dx}(x^2)+x^2\frac{d}{dx}(\cos x)=(2x)(\cos x)-x^2\sin x$

(2) $\frac{d}{dx}(e^x\sin x)=\sin x\frac{d}{dx}e^x+e^x\frac{d}{dx}(\sin x)=e^x\sin x+e^x\cos x$

Rules of Differentiation - Chain Rule

Chain Rule: The Chain Rule tells us how to find the derivative of a composite function.

If y = f(g(x)) then $\frac{dy}{dx}$ will be given by : $\frac{dy}{dx}=\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$

Another way to represent the Chain Rule is :

If y = f(U) then $\frac{dy}{dx}$ will be given by : $\frac{dy}{dx}=\frac{df(U)}{dU}\times\frac{dU}{dx}$

Illustration 26:

Find value of $\frac{dy}{dx}$; (1) y = cos(2x + 3) (2) y = sin(x^2 + x^3)

Solution:

(1) Let u = 2x + 3 $\therefore$ y = cos(u)

$\frac{du}{dx}=\frac{d}{dx}(2x+3)=2$ $\frac{dy}{du}=\frac{d}{du}(\cos(u))=-\sin u$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-(\sin u)(2)=-2\sin(2x+3)$

(2) Let $u = x^2 + x^3$ $\therefore$ $y = \sin(u)$

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + x^3) = 2x + 3x^2 \qquad \frac{dy}{du} = \frac{d}{du}(\sin(u)) = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u(2x + 3x^2) = \cos(x^2 + x^3)(2x + 3x^2)$$

Illustration 27:

Find derivative of y w.r.t. x if : $y = \ln(x^3 + 4)$

Solution::

Let $u = x^3 + 4$ $\therefore$ $y = \ln(u)$

$$\frac{du}{dx} = \frac{d}{dx}(x^3 + 4) = 3x^2 \qquad \frac{dy}{du} = \frac{d}{du}(\ln(u)) = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{1}{u}\right)(3x^2) = \frac{3x^2}{x^3 + 4}$$

Illustration 28:

Find value of $\frac{dy}{dt}$ $y = A\sin(\omega t + \phi)$

Solution:

Let $u = \omega t + \phi$ $\therefore$ $y = A\sin u$

$$\frac{du}{dx} = \frac{d}{dx}(\omega t + \phi) = \omega \qquad \frac{dy}{du} = \frac{d}{du}(A\sin u) = A\cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (A\cos u)(\omega) = A\omega\cos(\omega t + \phi)$$

Power Chain Rule

If $y = [f(x)]^n$ then $\frac{dy}{dx}$ will be given by :

$$\frac{dy}{dx} = n[f(x)]^{(n-1)} \frac{d[f(x)]}{dx}$$

$$\frac{dy}{dx} = n[f(x)]^{(n-1)} f'(x)$$

If $y = U^n$ then $\frac{dy}{dx}$ will be given by :

$$\frac{dy}{dx} = nU^{(n-1)} \frac{dU}{dx}$$

Illustration 29:

Find the derivative of $y = (x^2 + 3)^6$ w.r.t. x

Solution:

Let $u = x^2 + 3$ $\therefore$ $y = u^6$

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 3) = 2x \qquad \frac{dy}{du} = \frac{d}{du}(u^6) = 6u^5$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (6u^5)(2x) = 6(x^2 + 3)^5(2x)$$

Application of Derivatives

Instantaneous rate of change of a quantity "w.r.t." another quantity

$$\vec{v}_{inst}=\frac{d\vec{x}}{dt};\ \vec{a}_{inst}=\frac{d\vec{v}}{dt}=\frac{d^2x}{dt^2};\ \vec{F}_{inst}=\frac{d\vec{p}}{dt}$$

Illustration 30:

If height of magical tree depends on time as $h = 3t^2 + 5t + 2$ m. Find out :-

(1) Rate of change of height at t = 3 sec.

(2) Rate of change of height from t = 0 to t = 3 sec.

Solution:

(1) $\frac{dh}{dt}=\frac{d}{dt}(3t^2+5t+2)=6t+5$; $\frac{dh}{dt}$ = rate of change of height = 6t + 5

$\therefore$ rate of change of height at t = 3 = 6(3) + 5 = 23 m

(2) Height at t = 3, h = $3(3^2) + 5(3) + 2 = 44$ m

height at t = 0, h = $3(0^2) + 5(0) + 2 = 2$ m

$$\frac{\Delta h}{\Delta t}=\frac{h_f-h_i}{t_f-t_i}=\frac{44-2}{3-0}=14\text{m}$$

Illustration 31:

If position of particle is given by $x = (3t^2 + 4t - 1)$ m. Find its initial velocity and initial acceleration.

Solution:

$$\frac{dx}{dt}=\frac{d}{dt}(3t^2+4t-1)=6t+4\Rightarrow \text{Velocity}=\frac{dx}{dt}_{(t=0)}=6(0)+4=4\text{m/s}\Rightarrow a=\frac{d^2x}{dt^2}=6$$

$\Rightarrow$ Acceleration = 6m/s^2 (constant)

Illustration 32:

The area A of a circle is related to its radius by the equation $A = \pi r^2$. How fast is the area changing with respect to the radius when the radius is 10 m?

Solution:

$A = \pi r^2$

$$\frac{dA}{dr}=\frac{d}{dr}(\pi r^2)=\pi(2r)=\pi(2)(10)=20\pi\text{ m}$$

Illustration 33:

If side of a cube is changing by a rate of 4 m/s find rate of change of its volume w.r.t. time when side length is 2m.

Solution:

$$V=a^3;\ \frac{dV}{dt}=\frac{d}{dt}(a^3)=\frac{3a^2da}{dt}=3(2)^2(4)=48\text{ m}^3/\text{s}$$

Illustration 34:

The area of a block of ink is growing such that after t second its area is given by $A = (3t^2 + 7)$ cm^2. Calculate the rate of increase of area at = 5second.

Solution:

$$\frac{dA}{dt}=\frac{d}{dt}(3t^2+7)=6t\ ;\ \text{at } t=5,\ \frac{dA}{dt}=6(5)=30\text{ cm}^2/\text{s}$$

BEGINNER'S BOX-3

1. Find $\frac{dy}{dx}$ for the following :

(i) $y = x^{7/2}$ (ii) $y = x^{-3}$ (iii) $y = x$ (iv) $y = x^5 + x^3 + 4x^{1/2} + 7$

(v) $y = 5x^4 + 6x^{3/2} + 9x$ (vi) $y = ax^2 + bx + c$ (vii) $y = 3x^5 - 3x - \frac{1}{x}$

PBM154

2. Given $s = t^2 + 5t + 3$, find $\frac{ds}{dt}$. **PBM155**

3. If $s = ut + \frac{1}{2}at^2$, where u and a are constants. Obtain the value of $\frac{ds}{dt}$. **PBM156**

4. The area of a blot of ink is growing such that after t seconds, its area is given by $A = (3t^2+7)\ cm^2$. Calculate the rate of increase of area at t=5 second. **PBM157**

5. The area of a circle is given by $A = \pi r^2$, where r is the radius. Calculate the rate of increase of area w.r.t. radius. **PBM158**

6. Find the derivatives of the following with respect to x :

(i) $(x - 1)(2x + 5)$ (ii) $\frac{1}{2x+1}$ (iii) $\frac{3x+4}{4x+5}$ (iv) $\frac{x^2}{x^3+1}$

PBM159

Application of Derivatives

Slope of a curve at a given point

$m_{inst} = \tan\theta = \frac{dy}{dx}$

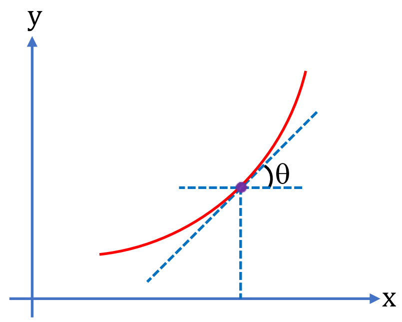


Illustration 35:

Find the slope of the tangent of a curve $y = x^2 + 2x + 4$ at $x = 0$ and $x = -1$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 2x + 4) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(4) = 2x + 2$$

Slope of the tangent at $x = 0$, $\frac{dy}{dx} = 2(0) + 2 = 2$

Slope of the tangent at $x = -1$, $\frac{dy}{dx} = 2(-1) + 2 = 0$

Concept of Maxima and Minima

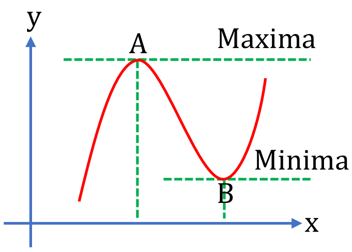
Double Differentiation

$$y = f(x) \xrightarrow{\text{Differentiation}} \frac{dy}{dx} \xrightarrow{\text{Differentiation}} \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$$

Function First Derivative Second Derivative

Maxima and Minima

Suppose a quantity y depends on another quantity x in a manner shown in the figure.



First derivative

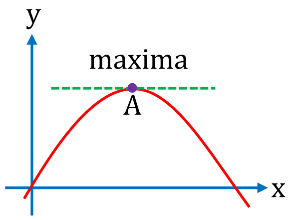
$\frac{dy}{dx}$ = rate of change of y w.r.t. x = slope

Second derivative

$\frac{d^2y}{dx^2}$ = rate of change of slope

$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\text{slope})$

Maxima

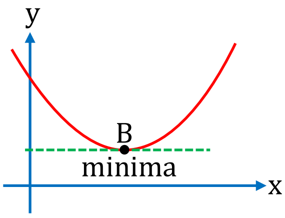


Condition for maxima are :

$\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}<0$

in figure at point A (maxima)

Minima



Condition for minima are :

$\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$

in figure at point B (minima)

Summary

How to check Maxima & Minima of a function Y

Step-1 : Calculate $\frac{dy}{dx}$ and put it equal to zero.

Step-2 : Find value of x from above equation.

Step-3 : Find $\frac{d^2y}{dx^2}$ to check for maxima and minima

- If $\frac{d^2y}{dx^2}>0$ "then minima" • If $\frac{d^2y}{dx^2}<0$ then maxima • If $\frac{d^2y}{dx^2}=0$ then neither maxima, nor minima

Illustration 36:

Find maximum or minimum value for given equation $y = x^2 - 4x + 8$

Solution:

Step-1 : $\frac{dy}{dx}=2x-4$; Step-2 : $2x - 4 = 0 \Rightarrow x = 2$; Step-3 : $\frac{d^2y}{dx^2}=2$

$\frac{d^2y}{dx^2}>0$; So, minima at x = 2 ; $\therefore$ minimum value of given equation at x = 2; $y = (2)^2 - 4(2) + 8 = 4$

Integration

In integral calculus, the differential coefficient of a function is given. We are required to find the function. Integration is basically used for summation . Σ is used for summation of discrete values, while $\int$ sign is used for continuous function.

Reverse process of differentiation

If $F'(x)=f(x)$, then $\int f(x)dx=F(x)+c$

Here, c is called constant of integration or arbitrary constant

$$\int f(x)dx = F(x) + c$$

Integration symbol Integrand Variable of Integration Constant of Integration

Types of Integration

1. Indefinite Integration

$$\int f'(x)dx = f(x)+c$$

2. Definite Integration

$$\int_a^b f'(x)dx = [f(x)]_a^b = [f(b)-f(a)]$$

Formulae of Integration for Algebraic function

$$\int x^n dx = \frac{x^{n+1}}{n+1}+c \qquad \text{(Provided } n \neq -1)$$

$$\int \frac{1}{x}dx = \ln x + c \;;\; \int k dx = kx + c \;;\; \int e^x dx = e^x + c \;:\; \int \sin x\, dx = -\cos x + c \;;\; \int \cos x\, dx = \sin x + c$$

Illustration 37:

Evaluate: (i) $\int x^6 dx$ (ii) $\int x^3 dx$ (iii) $\int \frac{1}{\sqrt{x}}dx$

Solution:

(i) $\int x^6 dx = \frac{x^{6+1}}{6+1}+c = \left(\frac{x^7}{7}\right)+c$ (ii) $\int x^3 dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4}+c$ (iii) $\int \frac{1}{x^{\frac{1}{2}}}dx = \int x^{-\frac{1}{2}}dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x} + c$

Rules For Integration

Rule 1 $\int (u \pm v)dx \Rightarrow \int u dx \pm \int v dx$

Rule 2 $\int cf(x)dx \Rightarrow c\int f(x)dx$

Illustration 38:

Find the value of $\int (4x^2 - 6x + 2)dx$.

Solution:

$$I = \int (4x^2 - 6x + 2)dx = \int 4x^2 dx - \int 6x dx + \int 2dx = 4\left[\frac{x^{2+1}}{2+1}\right] - 6\left[\frac{x^{1+1}}{1+1}\right] + 2\left[\frac{x^{0+1}}{0+1}\right] = \frac{4}{3}x^3 - 3x^2 + 2x + c$$

Illustration 39:

Find the value of $\int (\cos\theta - \sin\theta + 3)d\theta$.

Solution:

$$I = \int (\cos\theta - \sin\theta + 3)d\theta = \int \cos\theta d\theta - \int \sin\theta d\theta + \int 3d\theta = \sin\theta + \cos\theta + 3\theta + c$$

Illustration 40:

Find the value of $\int (e^x + x^e + e^e)dx$.

Solution:

$$I = \int (e^x + x^e + e^e)\,dx = \int e^x dx + \int x^e dx + \int e^e dx = e^x + \frac{x^{e+1}}{e+1} + e^e x + c$$

Linear Substitution in Integration Algebraic function

$$\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)} + c \quad ; \quad \int \frac{1}{ax+b}dx = \frac{1}{a}\ln(ax+b) + c \quad ; \quad \int \sin(ax+b)dx = -\frac{\cos(ax+b)}{a} + c$$

$$\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a} + c \quad ; \quad \int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$$

Illustration 41:

Find the value of (i) $\int \cos(2x+4)dx$ (ii) $\int \frac{1}{4t-2}dt$

Solution:

(i) $I = \int \cos(2x+4)dx = \frac{\sin(2x+4)}{2} + c$ (ii) $I = \int \frac{1}{4t-2}dt = \frac{\ln(4t-2)}{4} + c$

BEGINNER'S BOX-4

1. Evaluate the following integrals :

(i) $\int x^{15} dx$ (ii) $\int x^{-3/2} dx$ (iii) $\int (3x^{-7} + x^{-1})dx$

(iv) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$ (v) $\int \left(x + \frac{1}{x}\right)dx$ (vi) $\int \left(\frac{a}{x^2} + \frac{b}{x}\right)dx$ (a and b are constant)

PBM160

Definite Integration :

When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

If $\frac{d}{dx}(f(x)) = f'(x)$, then

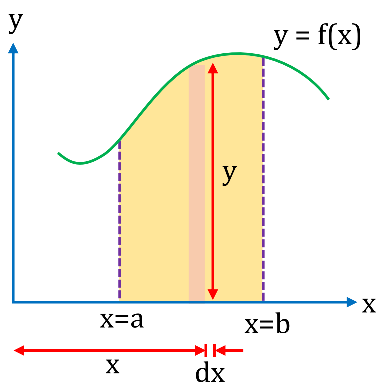
$\int f'(x)dx$ is called indefinite integral and $\int_a^b f'(x)dx$ is called definite integral

Here, a and b are called lower and upper limits of the variable x.

After carrying out integration, the result is evaluated between upper and lower limits as explained below :

$$\int_a^b f'(x)dx = |f(x)|_a^b = [f(b) - f(a)]$$

Area under a curve and definite integration



Area of small shown darkly shaded element = ydx = f(x) dx

If we sum up all areas between x=a and x= b then

$$\int_a^b f(x)dx = \text{shaded area between curve and x-axis.}$$

$$\int_a^b f(x)dx = [F(x)]_a^b = [F(b) - F(a)]$$

Illustration 42:

Find the value of $\int_3^4 6x\,dx$

Solution:

$$I=\int_3^4 6x\,dx=6\int_3^4 x\,dx=6\left[\frac{x^2}{2}\right]_3^4=6\left[\frac{(4)^2}{2}-\frac{(3)^2}{2}\right]=6\left[\frac{7}{2}\right]=21$$

Illustration 43:

Find the value of $\int_1^2 (10x^2-4x+4)dx$

Solution:

$$I=\int_1^2 (10x^2-4x+4)dx=\int_1^2 10x^2dx-\int_1^2 4x\,dx+\int_1^2 4\,dx=10\left[\frac{(x^3)}{3}\right]_1^2-4\left[\frac{x^2}{2}\right]_1^2+4[x]_1^2$$

$$I=10\left[\frac{2^3}{3}-\frac{1}{3}\right]-4\left[\frac{2^2}{2}-\frac{1}{2}\right]+4[2-1]=\frac{128}{6}$$

Illustration 44:

Find the value of $\int_0^{\frac{\pi}{3}} \cos x\,dx$

Solution:

$$I=\int_0^{\frac{\pi}{3}} \cos x\,dx=[+\sin x]_0^{\frac{\pi}{3}}=\left(\sin\frac{\pi}{3}-\sin 0\right)=\frac{\sqrt{3}}{2}$$

Illustration 45:

Find the value of $\int_\infty^0 e^{-2t}dt$

Solution:

$$I=\int_\infty^0 e^{-2t}dt=\left[\frac{e^{-2t}}{-2}\right]_\infty^0=-\frac{1}{2}\left[e^{-2(0)}-e^{-2(\infty)}\right]=-\frac{1}{2}$$

BEGINNER'S BOX-5

1. Evaluate the following integrals

(i) $\int_R^\infty \frac{GMm}{x^2}dx$ **(ii)** $\int_{r_1}^{r_2} -k\frac{q_1q_2}{x^2}dx$ **(iii)** $\int_u^v Mv\,dv$ **(iv)** $\int_0^\infty x^{-1/2}dx$

(v) $\int_0^{\pi/2} \sin x\,dx$ **(vi)** $\int_0^{\pi/2} \cos x\,dx$ **(vii)** $\int_{-\pi/2}^{\pi/2} \cos x\,dx$

PBM161

Applications of Integration – Analytical

There are many applications of integration such as :

(a) Displacement/Change in position $\Delta x = x_2 - x_1$

(b) Change in velocity $\Delta v = v_2 - v_1$

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt}$$

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} vdt \qquad \int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} adt$$

(Change in position) $x_2 - x_1 = \int_{t_1}^{t_2} vdt$ (Change in velocity) $v_2 - v_1 = \int_{t_1}^{t_2} adt$

Illustration 46:

Initial position of a particle is x = 20 m and its velocity is $v = (2t^2 - 4t)$ m/s. Find position of the particle at t = 3 sec.

Solution:

$$v = \frac{dx}{dt} \Rightarrow \int_{20}^{x} dx = \int_{t=0}^{t=3} vdt \Rightarrow x - 20 = \int\left(2t^2 - 4t\right)dt$$

$$x - 20 = \left[\frac{2t^3}{3} - 2t^2\right]_0^3$$ so, $x - 20 = 18 - 18$ finally x = 20m

Illustration 47:

Find change in momentum from t = 1s to t = 2s if a force $F = 4t^2 - 6$ N acts on a particle. (Use $\Delta p = \int_{t_1}^{t_2} Fdt$)

Solution:

$$\Delta P = \int_{t_1}^{b_2} Fdt = \int_1^2\left(4t^2 - 6\right)dt = \int_1^2\left(4t^2\right)dt - \int_1^2 6dt = \left[\frac{4t^3}{3}\right]_1^2 - [6t]_1^2 = \left[\frac{32}{3} - \frac{4}{3}\right] - [12 - 6] = \frac{28}{3} - 6 = \frac{10}{3}\text{ Ns}$$

Area Under The Curve

Area of shaded element small shown darkly = ydx = f(fx) dx

If we sum all areas between x = a and x = b then

$$\int_a^b f(x)dx = F(b) - F(a)$$

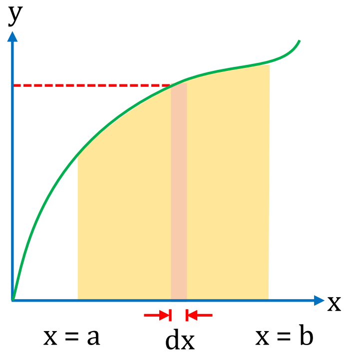
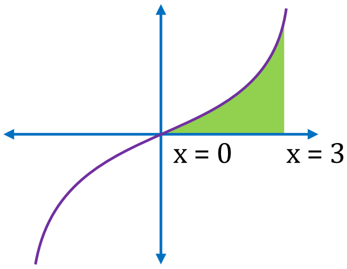


Illustration 48:

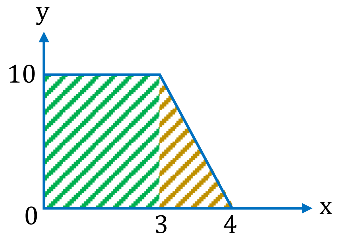
Find area between the curve $y = x^3$ and x axis from x = 0 to x = 3.



Solution:

Area under the curve = $\int ydx = \int_{x=0}^{x=3} x^3 dx = \left[\frac{x^4}{4}\right]_0^3 = \frac{81}{4}$ unit

Illustration 49:



Find the values of (a) $\int_0^3 y\,dx$ (b) $\int_3^4 y\,dx$

Solution:

(a) $\int_0^3 y\,dx$ = Area under the curve from x = 0 to x = 3= 10 × 3 = 30 unit

(b) $\int_3^4 y\,dx$ = Area under the wave from x = 3 to x = 4 $= \frac{1}{2}\times 1\times 10 = 5$ unit

Illustration 50:

Find area under the curve $y = 2x^2 - 4x + 6$, from x = 2 to x = 4.

Solution:

Area under the curve = $\int_{x_1}^{x_2} y\,dx = \int_2^4 \left(2x^2 - 4x + 6\right)dx = \int_2^4 2x^2 dx - \int_2^4 4x\,dx + \int_2^4 6\,dx$

$$= \left[\frac{2x^3}{3}\right]_2^4 - \left[2x^2\right]_2^4 + \left[6x\right]_2^4 = \left(\frac{112}{3}\right) - (24) + 12 = \frac{76}{3} \text{ unit}$$

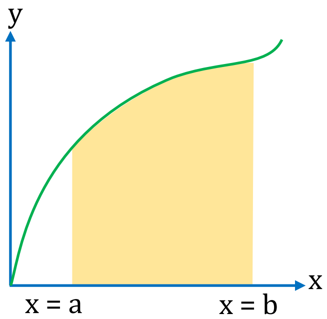
Average Value of Function

Average value of a function y = f(x), over an interval $a \le x \le b$ is given by $\langle y \rangle = \frac{\int_a^b y\,dx}{\int_a^b dx} = \frac{\int_a^b y\,dx}{b-a}$

Suppose there is a function y = f(x) Then average value of y = f(x) from x_1 = a to x_2 = b is

$$\langle y \rangle = \frac{\int_{x_1}^{x_2} f(x)dx}{\int_{x_1}^{x_2} dx} \qquad \langle y \rangle = \frac{\int_a^b f(x)dx}{\int_a^b dx}$$

Average Value of Function



In case of Graphical section : - The average value of y = f(x) from x_1 = a to x_2 = b is

$$\langle y \rangle = \frac{\text{area under } y - x \text{ curve}}{\text{range of } x\ (b-a)}$$

Illustration 51:

What is the average value of the function x^3 on the interval [0, 4]?

Solution:

Average value = $\frac{\int_0^4 x^3 dx}{4-0} = \frac{\left[\frac{x^4}{4}\right]_0^4}{4} = \frac{256}{16} = 16$ unit

Illustration 52:

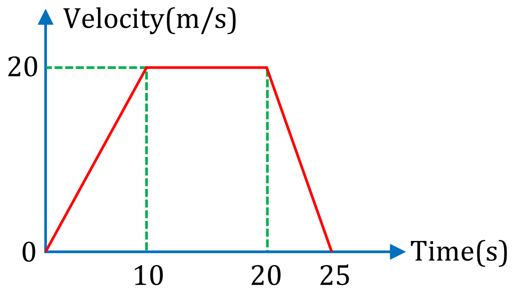
Find average value of a function I = I_0cost in interval of time [0, $\frac{\pi}{2}$]

Solution:

$$<I>=\frac{\int_0^{\pi/2}(I_0\cos t)dt}{\frac{\pi}{2}-0}=\frac{I_0\int_0^{\pi/2}\cos t dt}{\frac{\pi}{2}}=\frac{I_0}{\frac{\pi}{2}}=\frac{2I_0}{\pi}\text{ unit}$$

Illustration 53:

The velocity-time graph of a car moving along a straight road is shown in figure. The average velocity of the car in first 25 seconds is –



Solution:

$$\text{Average velocity} = \frac{\int_0^{25} vdt}{25-0}=\frac{\text{Area of v-t graph from t=0 to t = 25 s}}{25}=\frac{1}{25}\left[\left(\frac{25+10}{2}\right)(20)\right]=14m/s$$

Illustration 54:

A particle is moving with velocity, v = $(3t^2 + 4t^3 + 4)$m/s. Find <v> for interval 0 to 2 sec.

Solution:

$$<v>=\frac{\int_0^2 vdt}{2-0}=\frac{\int_0^2\left(3t^2+4t^3+4\right)}{2-0}=\frac{\left(\frac{3t^3}{3}+\frac{4t^4}{4}+4t\right)_0^2}{2}=\frac{8+16+8}{2}=16\text{ unit}$$

Illustration 55:

What is the average value of the function f(t) = cosπt in the interval [0, 1]?

Solution:

$$\langle f(t)\rangle=\frac{\int_0^1 f(t)dt}{1-0}=\frac{\int_0^1(\cos\pi t)}{1}=\frac{\left[\sin(\pi t)\right]_0^1}{\pi}=\frac{\left[\sin\pi-\sin(0)\right]}{\pi}=0$$

Quadratic Equation

An algebraic equation of second order (Highest power of the variable is equal to 2) is called a quadratic equation.

$ax^2 + bx + c = 0$ is the general Quadratic Equation.

where $a \neq 0$

Roots of Quadratic Equation

The general solution of the quadratic equation or it's roots are:

$$x = \frac{-b \pm \sqrt{D}}{2a} \quad \begin{cases} x_1 = \dfrac{-b + \sqrt{D}}{2a} \\ x_2 = \dfrac{-b - \sqrt{D}}{2a} \end{cases}$$ Where $D = b^2 - 4ac$

Condition for Real and Imaginary Roots

For Real Roots	For Imaginary Roots
$D \geq 0$	$D < 0$
$b^2 - 4ac \geq 0$	$b^2 - 4ac < 0$

Sum and Product of Roots

Sum of the roots	Product of the roots
$x_1 + x_2 = -\frac{b}{a}$	$x_1 x_2 = \frac{c}{a}$

Illustration 56:

Solve the equation to $x^2 + 3x - 18 = 0$

Solution:

$$x^2 + 3x - 18 = 0,\ x = \frac{-3 \pm \sqrt{9 - 4(1)(-18)}}{2(1)} = \frac{-3 \pm \sqrt{81}}{2} \text{ So } x_1 = \frac{-3+9}{2} = 3\ ;\ x_2 = \frac{-3-9}{2} = -6$$

BEGINNER'S BOX-6

1. Solve for x : (i) $10x^2 - 27x + 5 = 0$ (ii) $pqx^2 - (p^2 + q^2)\,x + pq = 0$ **PBM162**
2. In quadratic equation $ax^2 + bx + c = 0$, if discriminant is $D = b^2 - 4ac$, then roots of the quadratic equation are : (choose the correct alternative)
   (1) Real and distinct, if $D > 0$
   (2) Real and equal (i.e., repeated roots), if $D = 0$.
   (3) Non-real (i.e. imaginary), if $D < 0$
   (4) All of the above are correct
   PBM163

Binomial Theorem

An algebraic expression containing two terms is called a binomial expression.

For example : $(a + b)$, $(a + b)^3$, $(2x - 3y)^{-2}$ etc.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\times1}x^2 + \frac{n(n-1)(n-2)}{3\times2\times1}x^2 + \ldots\ldots$$

Binomial Approximation

If x is very small, compared to 1, then terms containing higher powers of x can be neglected so $(1+x)^n \approx 1 + nx$.

If $|x| << 1$ $\xrightarrow[\text{Approximation}]{\text{Binomial}}$
- $(1+x)^n \approx 1 + nx$
- $(1-x)^n \approx 1 - nx$
- $(1+x)^{-n} \approx 1 - nx$
- $(1-x)^{-n} \approx 1 + nx$

Illustration 57:

Find the value of (i) $(1.01)^4$ (ii) $(0.997)^2$ (iii) $\sqrt{0.99}$

Solution:

(i) $(1.01)^4 = (1+0.01)^4 \approx 1 + 4(0.01) \approx 1.04$ (ii) $(0.997)^2 = (1 - 0.003)^2 \approx 1 - 2(0.003) \approx 0.994$

(iii) $\sqrt{0.99} = (1 - 0.01)^{1/2} \approx 1 - \frac{1}{2}(0.01) \approx 1 - 0.005 \approx 0.995$

Progression

Arithmetic Progression (AP)

General form: a, (a + d), (a + 2d), ..., [a + (n – 1) d]

Here First Term : a

Common Difference : d

n_{th} Term : $\left[a+(n-1)d\right]$

Sum of first n terms: $S_n = \frac{n}{2}\left[1^{st}\text{ term} + n^{th}\text{ term}\right] \Rightarrow S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

Illustration 58:

Find the fifth term of given Arithmetic Progression: 5, 7, 9, . . .

Solution:

a = 5, d = 7 – 5 = 9 – 7 = 2

Fifth term = (a + 4d) = (5 + 4 (2)) = 13

Illustration 59:

Find the sum of given series: 4 + 8 + 12 + ... + 64

Solution:

First term = 4 ; Last term = 64 ; n^{th} term = a + (n – 1) d $\Rightarrow$ 64 = 4 + (n – 1) 4 $\Rightarrow$ n = 16

$S_n = \frac{16}{2}\left[4+64\right] = 544$

Illustration 60:

Find the sum of first 20 natural numbers:

Solution:

First 20 natural numbers = 1, 2, 3, ... 20 ; a = 1, d = 1 ; $S_n = \frac{20}{2}\left[2(1)+(19)(1)\right] = 210$

Geometric Progression (GP)

General form : a, ar, ar^2, ... $ar^{(n-1)}$

First Term : a

Common Ratio : r

n^{th} Term : $ar^{(n-1)}$

Sum of first n terms : $S_n = \frac{a\left(1-r^n\right)}{(1-r)}$

Sum of all the terms of an infinite GP : $S_\infty = \frac{a}{(1-r)}$; Only when |r| < 1

Illustration 61:

Find the sixth term of 1, 2, 4, ...

Solution:

Sixth term = ar^5

a = 1, r = $\frac{2}{1} = \frac{4}{2} = 2$

$\therefore$ Sixth term = $1(2)^5 = 32$

Illustration 62:

Find sum of all the terms of an infinite GP : $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.........\infty$

Solution:

$a = 1, r = \frac{1/2}{1}=\frac{1/4}{1/2}=\frac{1}{2}$; $S_n = \frac{a}{1-r}=\frac{1}{1-1/2}=2$

Formulae to Remember

Sum of first n natural numbers:

$1 + 2 + 3 + 4 + 5 + ... + n = \frac{n(n+1)}{2}$

Sum of squares of first n natural numbers:

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 +......+n^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of cubes of first n natural numbers :

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 +......+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

BEGINNER'S BOX-7

1. Find sum of first 50 natural numbers. **PBM164**
2. Find $1^2 + 2^2 +................+ 10^2$. **PBM165**
3. Find $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+........\infty$. **PBM166**
4. Find $F_{net} = GMm\left[\frac{1}{r^2}+\frac{1}{2r^2}+\frac{1}{4r^2}+..............\text{up to }\infty\right]$. **PBM167**

Logarithm

The exponent or power to which a base must be raised to yield a given number.

Expressed mathematically, x is the logarithm of n to the base b

$x = \log_b n$ (if $b^x = n$, exponential form)

If base is 10 it is called **Standard log** $\Rightarrow \log_{10} n$

If base is e then it is called **Natural log** $\Rightarrow \log_e n$

$e \approx 2.71$

Common Formulae of Logarithm

Product Formula $\Rightarrow$ log (mn) = log m + log n

Quotient Formula $\Rightarrow \log\left(\frac{m}{n}\right)=\log m-\log n$

Power Formula $\Rightarrow$ log (m^n) = n log m

Standard Values of Logarithm

Base Changing Formula $\Rightarrow \log_e m = 2.303 \log_{10} m$

For any Base $\Rightarrow \log_b 1 = 0$

For Base a $\Rightarrow \log_a a = 1$

Standard Values to remember

Base 10	Base e
log 2 = 0.301	ln 2 = 0.693
log 3 = 0.477	ln 3 = 1.09
log 5 = 0.699	ln 5= 1.6

Illustration 63:

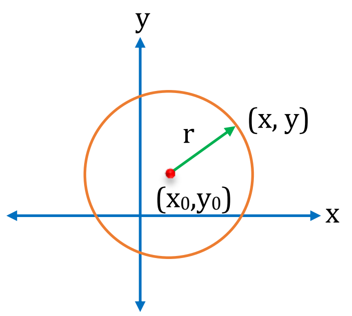
Find the value of : (i) $\ln e^5$ (ii) $\ln e^{2/3}$

Solution:

(i) $\ell ne^5 = 5\ell ne = 5\log_e e = 5$ (ii) $\ell ne^{2/3} = \frac{2}{3}\ell n_e e = \frac{2}{3}$

Some standard graphs and their equations

Circle



Assume that (x, y) are the coordinates of a point on the circle shown, the centre is at (x_0, y_0) and the radius is r.

Equation of circle = $(x - x_0)^2 + (y - y_0)^2 = r^2$

Equation of Circle with center at Origin

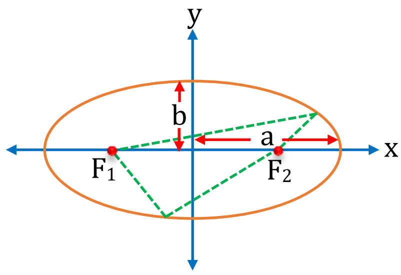
$(x - 0)^2 + (y - 0)^2 = r^2$

$x^2 + y^2 = r^2$

Ellipse

An ellipse is the locus of points in a plane, the sum of whose distances from two fixed points is a constant value.

The two fixed points are called the **foci** of the ellipse.



In this diagram: -

a = semi major axis ; b = semi minor axis

F_1 and F_2 = foci of the ellipse.

Equation of Ellipse with center at Origin

The equation of ellipse is written in terms of it's

semi-major axis and semi-minor axis as: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Illustration 64:

What is the radius of the circle given by: $x^2 + y^2 = 49$

Solution:

$x^2 + y^2 = 49$ (given equation) ...(1)

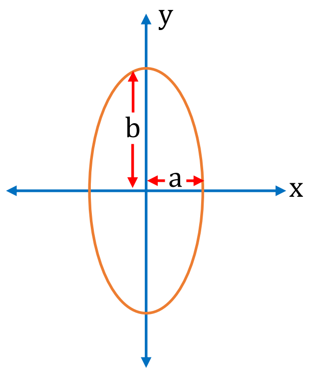
$x^2 + y^2 = r^2$ (general equation) ...(2)

Comparing both equations : $r^2 = 49$; Radius of the circle = r = 7

Illustration 65:

Find length of semi-major axis and semi-minor axis for ellipse

$\frac{x^2}{16} + \frac{y^2}{36} = 1$



Solution:

$\frac{x^2}{16} + \frac{y^2}{36} = 1$ (given equation) ...(1)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (general equation) ...(2)

Comparing both equations :

a = 4 (semi minor axis) b = 6 (semi major axis)

Illustration 66:

If the length of semi-major axis is 5 and semi-minor axis is 3 then write the equation of ellipse centered at origin.

Solution:

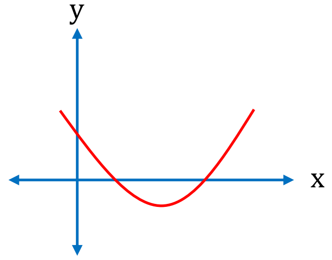
$a = 5\ ;\ b = 3\ ;\ \frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ so, $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Parabola

The equation of parabola is given by:

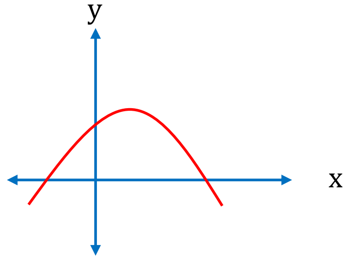
$$\mathbf{y = ax^2 + bx + c}$$

If a > 0



Upward Opening Parabola

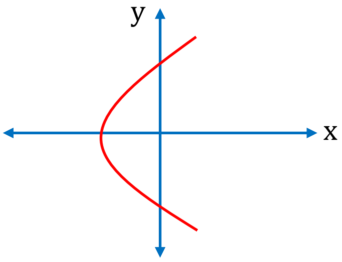
If a < 0



Downward Opening Parabola

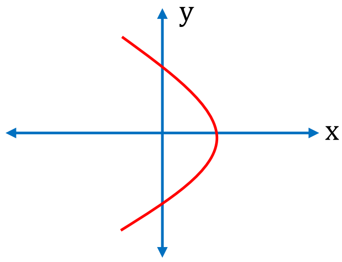
$$\mathbf{x = ay^2 + by + c}$$

If a > 0



Rightward Opening Parabola

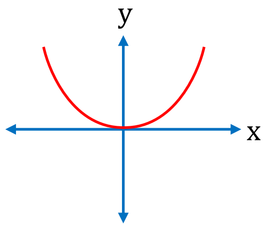
If a < 0



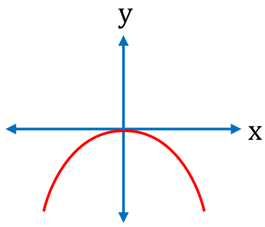
Leftward Opening Parabola

Some standard Parabola

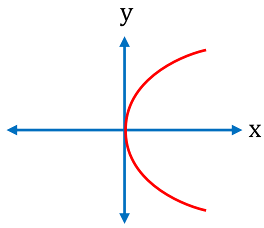
$y = x^2$



$y = -x^2$



$x = y^2$



$x = -y^2$

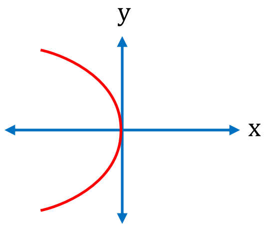


Illustration 67:

If $x = 9t^2$ and $y = 3t$ represents the coordinate of a particle, then its path will be?

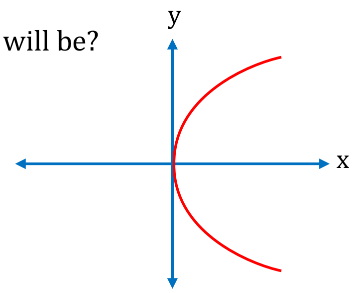
Solution:

$x = 9t^2$ ...(1)

$y = 3t$ ...(2)

From equation (2) $t = \frac{y}{3}$ ...(3)

Now, from equation (1) and (3) $x = 9\frac{y^2}{9} = y^2$



Rectangular Hyperbola

The equation of Rectangular Hyperbola is given by:

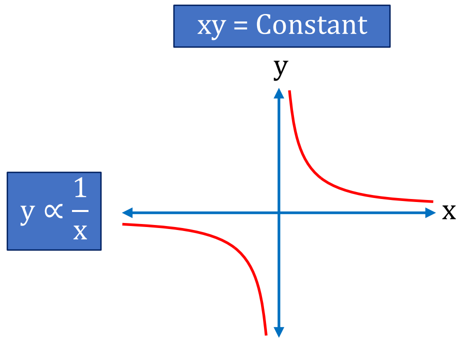


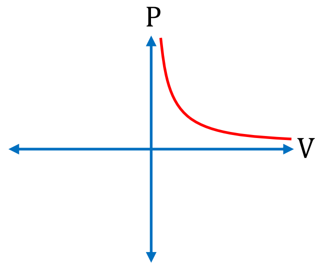
Illustration 68:

Draw graph between pressure and volume for an ideal gas at constant temperature (PV = Constant)

Solution:

PV = nRT

PV = constant

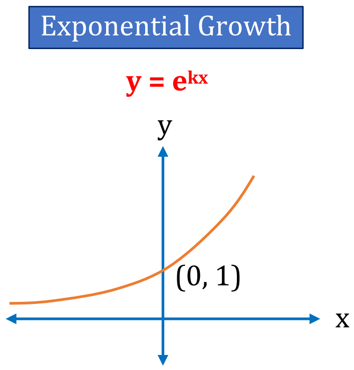


Exponential Graphs

There are two types of exponential graphs:

(i) Exponential Growth

(ii) Exponential Decay



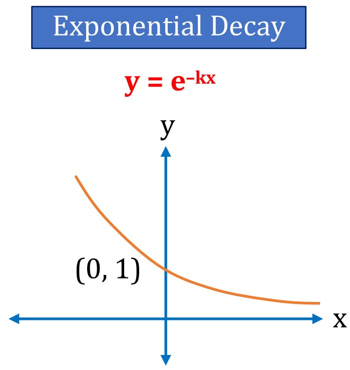


Illustration 69:

A particle moves along path $y = 9x^2 - 2x + 4$, then its path will be?

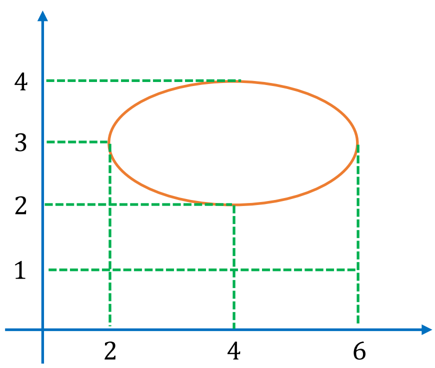
Solution:

$y = ax^2 - bx + c$

$\therefore$ path will be parabola

Illustration 70:

Calculate the area enclosed by shown ellipse



Solution:

Shaded area = Area of ellipse = π ab

Here $a = 6 - 4 = 2$ and $b = 4 - 3 = 1$

$\Rightarrow$ Area = $\pi \times 2 \times 1 = 2\pi$ units