

## DEFINITE MIXED QUESTIONS

1. The value of  $\int_{\pi}^{3\pi} \frac{\sin^{-1}(\sin x)}{\sin^4 x + \cos^4 x} dx$  equals

1) 0

2)  $\pi$

3)  $\frac{\pi}{2}$

4)  $2\pi$

Key(1)

**Sol** Period =  $2\pi \Rightarrow I = \int_0^{\pi} \frac{\sin^{-1}(\sin x)}{\sin^4 x + \cos^4 x} dx \Rightarrow I = -I, \Rightarrow I = 0$

2. The value of the integral  $\int_{-\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\cos x + \sin x}{1 + e^{x - \frac{\pi}{4}}} dx$  is

Key(0)

**Sol**

$$I = \int_{-\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\cos x + \sin x}{1 + e^{x - \frac{\pi}{4}}} dx$$

$$\Rightarrow 2I = \int_{-\frac{3\pi}{4}}^{\frac{5\pi}{4}} \sin x + \cos x \, dx \Rightarrow I = 0$$

3.  $\int_{\frac{\pi}{2008}}^{\frac{1003\pi}{2008}} \frac{\sqrt[7]{\sin^8 x} + \sqrt[7]{\cos^8 x}}{\sqrt[9]{\sec^5 x} + \sqrt[9]{\cos^5 x}} \log \left( \frac{3 + 4 \tan x}{3 + 4 \cot x} \right) dx = \underline{\hspace{2cm}}$

1) 0

2)  $\frac{\pi}{4}$

3)  $\frac{1003\pi}{2008}$

4)  $\frac{\pi}{2008}$

Key(1)

**Sol** use  $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

4.  $\int_0^{\infty} \frac{\cos x + x \sin x}{x^2 + \cos^2 x} dx = k \frac{\pi}{6}$ ; the value of k is

Key(3)

**Sol**  $\int \frac{\cos x + x \sin x}{x^2 + \cos^2 x} dx = \int \frac{d}{dx} \left( -\tan^{-1} \frac{\cos x}{x} \right) dx$

$$\therefore \frac{k\pi}{6} = -\lim_{x \rightarrow \infty} \tan^{-1} \frac{\cos x}{x} + \lim_{x \rightarrow 0^+} \tan^{-1} \frac{\cos x}{x} = 0 + \frac{\pi}{2}$$

$$\therefore k = 3$$

5.  $\int_0^{\pi/2} \frac{8+7\cos x}{(7+8\cos x)^2} dx = \underline{\hspace{2cm}}$

1)  $\frac{2}{7}$

2)  $\frac{-2}{7}$

3)  $\frac{-1}{7}$

4)  $\frac{1}{7}$

**Key(4)**

**Sol** Multiply and divide by  $\operatorname{cosec}^2 x$  and put denominator =  $t$

6.  $\int_0^{\frac{\pi}{4}} \frac{x^2 dx}{(x \sin x + \cos x)^2} =$

1)  $\frac{4+\pi}{4-\pi}$

2)  $\frac{4+\pi}{2(4-\pi)}$

3)  $\frac{4-\pi}{4+\pi}$

4)  $2 \left[ \frac{4-\pi}{4+\pi} \right]$

**Key(3)**

**Sol** Given =  $\frac{x \cos x}{(x \sin x + \cos x)^2} \times \frac{x}{\cos x}$  and using by parts

7. If  $\int_{1/2}^2 \frac{1}{x} \operatorname{cosec}^{101} \left( x - \frac{1}{x} \right) dx = k$  then the value of k is

1) 1

2) 1/2

3) 0

4) 1/101

**Key(3)**

**Sol**  $\therefore \int_{\frac{1}{2}}^2 \frac{1}{x} \operatorname{cosec}^{101} \left( x - \frac{1}{x} \right) dx = k$

Put  $x = \frac{1}{t} \quad dx = -\frac{1}{t^2} dt$

$$\Rightarrow \int_2^{1/2} t \operatorname{cosec}^{101} \left( \frac{1}{t} - t \right) \times \frac{-1}{t^2} dt = k$$

$$\Rightarrow -k = k$$

$$2k = 0 \Rightarrow k = 0$$

8. The value of the definite  $\int_0^{\pi/2} (\sin x + \cos x) \cdot \sqrt{\frac{e^x}{\sin x}} dx$  equals

- 1)  $2\sqrt{e^{\pi/2}}$       2)  $\sqrt{e^{\pi/2}}$       3)  $2\sqrt{e^{\pi/2}} \cdot \cos 1$       4)  $\frac{1}{2}e^{\pi/4}$

**Key(1)**

**Sol** 
$$I = \int_0^{\pi/2} e^x (\sin x + \cos x) \frac{1}{\sqrt{e^x \sin x}} dx$$

Put  $e^x \sin x = t^2 \Rightarrow e^x (\sin x + \cos x) dx = 2t dt$

$$\therefore I = 2 \int_0^{e^{\pi/4}} dt = 2t \Big|_0^{e^{\pi/4}} = 2e^{\pi/4} = 2\sqrt{e^{\pi/2}}$$

9. Let  $\int_0^x \left( \frac{bt \cos 4t - a \sin 4t}{t^2} \right) dt = \frac{a \sin 4x}{x}$  then a and b are given by

- 1)  $a = \frac{1}{4}, b = 1$       2)  $a = 2, b = 2$       3)  $a = -1, b = 4$       4)  $a = 2, b = 4$

**Key(1)**

**Sol**

Differ. both sides w.r.t 'x'

$$\frac{bx \cos 4x - a \sin 4x}{x^2} = \frac{a(4x \cos 4x - \sin 4x)}{x^2}$$

On comparing  $b = 4a$

$$\therefore a = \frac{1}{4}, b = 1$$

10.  $\int_0^{\pi/4} \frac{dx}{x \tan x + 1} = \dots\dots\dots$

- 1)  $\frac{1}{4 + \pi}$       2)  $\frac{2}{4 + \pi}$       3)  $\frac{3}{4 + \pi}$       4)  $\frac{4}{4 + \pi}$

**Key(4)**

$$\text{Sol} \quad I = \int \frac{\cos^2 x}{x \sin x + \cos x} dx = \int \underbrace{\frac{x \cos x}{x \sin x + \cos x}}_{\text{II}} \cdot \underbrace{\frac{\cos x}{x}}_{\text{I}} dx$$

Using by parts rule

$$\begin{aligned} I &= \frac{\cos x}{x} \cdot \frac{-1}{x \sin x + \cos x} + \int \frac{-x \sin x - \cos x}{x^2} \cdot \frac{1}{x \sin x + \cos x} dx \\ &= \frac{-\cos x}{x \sin x + \cos x} + \frac{1}{x} + C \end{aligned}$$

$$\therefore \int_0^{\pi/4} \frac{\cos^2 x}{x \sin x + \cos x} dx = \frac{-1}{\pi/4} + \frac{4}{\pi} = \frac{4}{\pi} - \frac{16}{4 + \pi} = \frac{4}{4 + \pi}$$

$$11. \int_0^{\pi/4} (\pi x - 4x^2) \ln(1 + \tan x) dx = \underline{\hspace{2cm}}$$

$$1) \frac{\pi^3}{192}$$

$$2) \frac{\pi^3}{192} \ln 2$$

$$3) \frac{\pi^2}{96}$$

$$4) \frac{\pi^2}{96} \ln 2$$

Key(2)

$$\begin{aligned} \text{Sol} \quad I &= \int_0^{\pi/4} 4x \left( \frac{\pi}{4} - x \right) \ln(1 + \tan x) dx = \int_0^{\pi/4} 4 \left( \frac{\pi}{4} - x \right) x \ln \left( 1 + \tan \left( \frac{\pi}{4} - x \right) \right) dx \\ &= 4 \ln 2 \int_0^{\pi/4} x \left( \frac{\pi}{4} - x \right) dx - I \\ \Rightarrow I &= 2 \ln 2 \int_0^{\pi/4} x \left( \frac{\pi}{4} - x \right) dx \\ &= \frac{\pi^3}{192} \ln 2 \end{aligned}$$

$$12. \int_0^{2\pi} x \ln \left( \frac{3 + \cos x}{3 - \cos x} \right) dx = \underline{\hspace{2cm}}$$

$$1) \frac{\pi}{2} \ln 3$$

$$2) \frac{\pi}{6} \ln 3$$

$$3) \frac{\pi}{12} \ln 3$$

$$4) 0$$

Key(4)

**Sol**

$$I = \int_0^{2\pi} (2\pi - x) \ln \left( \frac{3 + \cos x}{3 - \cos x} \right) dx \Rightarrow 2I = 2\pi \int_0^{2\pi} \ln \left( \frac{3 + \cos x}{3 - \cos x} \right) dx$$

$$\Rightarrow I = 2\pi \int_0^{\pi} \ln \left( \frac{3 + \cos x}{3 - \cos x} \right) dx = 2\pi \int_0^{\pi} \ln \left( \frac{3 + \cos x}{3 - \cos x} \right) dx = -I \Rightarrow I = 0$$

13. If  $I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$ , then the integral part of I is

**Key(9)**

**Sol**

$$I = 2 \int_0^{\pi} \frac{2x \sin x}{1 + \cos^2 x} = 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} = 4\pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$$

$$= 4\pi \left( -\tan^{-1}(\cos x) \right)_0^{\pi/2} = 4\pi \left( 0 + \frac{\pi}{4} \right) = \pi^2 \therefore [I] = 9$$

14. If  $\int_0^1 \frac{\sin x}{1+x} dx = K$  then the value of  $\int_{4\pi-2}^{4\pi} \frac{\sin\left(\frac{x}{2}\right)}{4\pi+2-x} dx$  equals

- 1)  $-k$                       2)  $\frac{k}{2}$                       3)  $\frac{-k}{2}$                       4)  $\frac{k-1}{2}$

**Key(1)**

**Sol**

$$4\pi - x = 2t \Rightarrow \int_1^0 \frac{\sin t}{1+t} dt = -k$$

15. Let  $u = \int_0^{\pi/4} \frac{\cos^2 x}{1 + \sin 2x} dx$ ,  $v = \int_0^{\pi/8} \frac{1}{(1 + \tan 2x)^2} dx$  then the value of  $\frac{u}{v}$  equals

- 1) 1                      2)  $\frac{1}{2}$                       3) 2                      4) 4

**Key(3)**

**Sol**

$$u = \int_0^{\pi/4} \left( \frac{\cos x}{\sin x + \cos x} \right)^2 dx = \int_0^{\pi/4} \left( \frac{1}{1 + \tan x} \right)^2 dx, \quad \text{put } x = 2t = 2 \int_0^{\pi/8} \left( \frac{1}{1 + \tan 2t} \right)^2 dt = 2.v \Rightarrow \frac{u}{v} = 2$$

7. If  $\int_{1/2}^2 \frac{1}{x} \operatorname{cosec}^{101} \left( x - \frac{1}{x} \right) dx = k$  then the value of k is

- 1) 1                      2) 1/2                      3) 0                      4) 1/101

Key(3)

**Sol**  $\therefore \int_{\frac{1}{2}}^2 \frac{1}{2} \operatorname{cosec}^{101} \left( x - \frac{1}{x} \right) dx = k$

Put  $x = \frac{1}{t} \quad dx = -\frac{1}{t^2} dt$

$$\Rightarrow \int_2^{1/2} t \operatorname{cosec}^{101} \left( \frac{1}{t} - t \right) \times \frac{-1}{t^2} dt = k$$

$$\Rightarrow -k = k$$

$$2k = 0 \Rightarrow k = 0$$

16. The value of  $\int_0^{3\pi/2} \cos^4 3x \cdot \sin^2 6x dx$  equals

1)  $\frac{15\pi}{64}$

2) 0

3)  $\frac{5\pi}{64}$

4)  $\frac{5\pi}{32}$

Key(1)

**Sol**

$$I = \int_0^{3\pi/2} 4 \cdot \sin^2(3x) \cdot \cos^6(3x) dx = \frac{4}{3} \int_0^{9\pi/2} \sin^2 x \cdot \cos^6 x dx = \frac{4}{3} \left[ \int_0^{4\pi} \sin \cdot \cos^6 x dx + \int_{4\pi}^{9\pi/2} \sin^2 x \cdot \cos^6 x dx \right]$$

$$= \frac{4}{3} (9) \cdot \int_0^{\pi/2} \sin^2 x \cdot \cos^6 x dx = \frac{4}{3} \times 9 \times \frac{(1)(5.3.1)}{8.6.4.2} \cdot \frac{\pi}{2} = \frac{15\pi}{64}$$

17. The value of  $\lim_{t \rightarrow 0} \frac{1}{t} \int_0^\pi \tan(t \sin x) dx$  equals

1) 0

2) 1

3) 2

4) -1

Key(3)

**Sol**  $G.E = \int_0^\pi \lim_{t \rightarrow 0} \frac{\tan(t \sin x)}{t} \sin x dx = \int_0^\pi \sin x dx = 2$

18.  $\int_0^{\frac{\pi}{2}} \sec\left(x - \frac{\pi}{6}\right) \sec\left(x - \frac{\pi}{3}\right) dx =$

1)  $\ln \sqrt{3}$

2)  $\ln 3$

3)  $2 \ln 3$

4)  $3 \ln 3$

Key(3)

Sol 
$$I = 2 \int_0^{\pi/2} \frac{\sin \frac{\pi}{6} dx}{\cos\left(x - \frac{\pi}{6}\right) \cos\left(x - \frac{\pi}{3}\right)}$$

$$= 2 \int_0^{\pi/2} \frac{\sin\left[\left(x - \frac{\pi}{6}\right) - \left(x - \frac{\pi}{3}\right)\right] dx}{\cos\left(x - \frac{\pi}{6}\right) \cos\left(x - \frac{\pi}{3}\right)}$$

$$= 2 \int_0^{\pi/2} \left[ \tan\left(x - \frac{\pi}{6}\right) - \tan\left(x - \frac{\pi}{3}\right) \right] dx = 2 \log 3$$

19.  $\int_0^{\pi/2} \frac{\sin x}{1 + \cos x + \sin x} dx =$

1)  $\frac{\pi}{4}$

2)  $\frac{\pi}{4} + \log \sqrt{2}$

3)  $\frac{\pi}{4} - \log \sqrt{2}$

4)  $\frac{\pi}{4} - \log 2$

Key(3)

Sol 
$$I = \int_0^{\pi/2} \frac{\sin \frac{x}{2}}{\sin \frac{x}{2} + \cos \frac{x}{2}} dx = 2 \int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x} dx$$

Let  $\sin x = A(\sin x + \cos x) + B(\cos x - \sin x)$

$A - B = 1$  and  $A + B = 0$ ,  $A = \frac{1}{2}, B = -\frac{1}{2}$ .

$$I = 2 \times \frac{1}{2} \times \frac{\pi}{4} - 2 \times \frac{1}{2} \left[ \log(\sin x + \cos x) \right]_0^{\pi/4} = \frac{\pi}{4} - \log \sqrt{2}$$

20.  $\int_0^a \ln(\cot a + \tan x) dx$ , where  $a \in \left(0, \frac{\pi}{2}\right)$  is

1)  $a \ln(\sin a)$

2)  $-a \ln(\sin a)$

3)  $-a \ln(\cos a)$

4) none of these

Key(2)

Sol

$$I = \int_0^a \ln \frac{\cos(a-x)}{\sin a \cdot \cos x} dx = \int_0^a \ln \frac{\cos x}{\sin a \cos(a-x)} dx$$

$$\text{adding } 2I = \int_0^a \ln \frac{1}{\sin^2 a} dx = \int_0^a -2 \ln \sin a dx = -2a \ln \sin a$$

21. If  $f(x) = \int_0^x (\cos^4 t + \sin^4 t) dt$ , then  $f(x + \pi)$  equal to

1)  $f(x) + f(\pi)$       2)  $f(x) + 2f(\pi)$       3)  $f(x) + f\left(\frac{\pi}{2}\right)$       4)  $f(x) + 2f\left(\frac{\pi}{2}\right)$

Key(1,4)

Sol 
$$f(x + \pi) = \int_0^{x+\pi} (\cos^4 t + \sin^4 t) dt = \int_0^x (\cos^4 t + \sin^4 t) dt + \int_x^{x+\pi} (\sin^4 t + \cos^4 t) dt$$

$$= f(x) + \int_x^{x+\pi} (\sin^4 t + \cos^4 t) dt$$

Put  $t = x + y = f(x) + \int_0^\pi (\sin^4 y + \cos^4 y) dy = f(x) + f(\pi)$

22.  $I_1 = \int_0^\pi \frac{\sin 884x \cos 1122x}{\sin x} dx, I_2 = \int_0^1 \frac{x^{238} (x^{1768} - 1)}{(x^2 - 1)} dx$  then find the value of  $\frac{I_1}{I_2}$ .

Key(2)

Sol 
$$I_1 = \int_0^\pi \frac{\sin 884x \cos 1122x}{\sin x} dx = 2 \int_0^{\pi/2} \frac{\sin 884x \cos 1122x}{\sin x} dx$$

$$= \int_0^{\pi/2} \frac{\sin 2006x - \sin 238x}{\sin x} dx$$

$$= \int_0^{\pi/2} \frac{\sin 2006x}{\sin x} dx - \int_0^{\pi/2} \frac{\sin 238x}{\sin x} dx$$

Use  $\frac{\sin 2kx}{\sin x} = (2 \cos x + \cos 3x + \dots + \cos(2k-1)x)$

$$I_1 = 2 \left[ 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2005} \right] - \frac{1}{2} \left( 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{237} \right)$$

$$I_2 = \int_0^1 x^{238} \left( \frac{1 - x^{1768}}{1 - x^2} \right) dx = \int_0^1 x^{238} (1 + x^2 + x^4 + \dots + x^{1766}) dx$$

$$= \int_0^1 (x^{238} + x^{240} + \dots + x^{2004}) dx$$

$$= \frac{1}{239} + \frac{1}{241} + \dots + \frac{1}{2005}$$

$$\frac{I_1}{I_2} = 2$$

23. If  $I_1 = \int_0^{\pi} \sqrt{1 - \sin 2x} \, dx$  and  $I_2 = \int_0^{\pi} \sqrt{1 + \sin 2x} \, dx$  then the value of  $I_1 \cdot I_2$  is equal to

Key(8)

Sol We have  $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \Rightarrow I_1 = I_2$

$$\begin{aligned} \text{Also } I_1 &= \int_0^{\pi} |\sin x - \cos x| \, dx = \int_0^{\pi/4} -\sin x - \cos x \, dx + \int_{\pi/4}^{\pi} \sin x - \cos x \, dx \\ &= \cos x + \sin x \Big|_0^{\pi/4} + -\cos x - \sin x \Big|_{\pi/4}^{\pi} \\ &= \sqrt{2} - 1 + 1 + \sqrt{2} = 2\sqrt{2} \end{aligned}$$

$$\therefore I_1 I_2 = I_1^2 = 8$$

24. Let F be a positive function and

$$\text{Let } A = \int_{\frac{\pi}{2}-a^2-a-1}^{a^2+a+1} \sin^2 x \, F(\sin^2 x \cos^2 x) \, dx, \quad B = \int_{\frac{\pi}{2}-a^2-a-1}^{a^2+a+1} F(\sin^2 x \cos^2 x) \, dx \quad ; \quad \text{where}$$

$$a \in \mathbb{R}, \text{ then } \frac{B}{A} =$$

Key(2)

$$\text{Sol } A = \int_{\pi/2-a^2-a-1}^{a^2+a+1} \sin^2 x \, F(\sin^2 x \cos^2 x) \, dx$$

$$A = \int_{\pi/2-a^2-a-1}^{a^2+a+1} \cos^2 x \, F(\sin^2 x \cos^2 x) \, dx$$

$$2A = B \Rightarrow B/A = 2$$

25. Given  $\int_0^{\pi/2} \ln \sin x \, dx = \frac{\pi}{2} \ln \frac{1}{2}$  and  $\int_0^{\pi/2} \left( \frac{x}{\sin x} \right)^2 \, dx = \frac{k \cdot \pi}{2} \ln 2$  then  $k = \dots\dots\dots$

Key(2)

Sol

$$\begin{aligned}
 I &= \int_0^{\pi/2} x^2 \operatorname{cosec}^2 x dx = \left[ -x^2 \cot x \right]_0^{\pi/2} + \int_0^{\pi/2} 2x \cot x dx \\
 &= 0 + \lim_{x \rightarrow 0^+} \frac{x^2}{\tan x} + 2 \left[ x \ln \sin x \right]_0^{\pi/2} - 2 \int_0^{\pi/2} \ln \sin x dx \\
 &= 0 - 2 \lim_{x \rightarrow 0^+} \frac{x \ln \sin x}{1/x} - 2 \frac{\pi}{2} \ln \frac{1}{2} = \pi \ln 2 \\
 &\left( \because \lim_{x \rightarrow 0^+} \frac{\ln \sin x}{1/x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin x} \cdot \cos x}{-\frac{1}{x^2}} = 0 \right)
 \end{aligned}$$

26.  $\int_0^{\pi/2} \frac{\sin^2 9x}{\sin x} dx =$

1)  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{9}$

2)  $\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{18}$

3)  $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{19}$

4)  $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{17}$

**Key(4)**

**Sol** Observe that

$$\sin x + \sin 3x + \sin 5x + \dots + \sin 17x = \frac{\sin^2 9x}{\sin x}$$

$$\begin{aligned}
 \therefore \int_0^{\pi/2} \frac{\sin^2 9x}{\sin x} dx &= - \left( \cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \dots + \frac{\cos 17x}{17} \right) \Bigg|_0^{\pi/2} \\
 &= 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{17}
 \end{aligned}$$

27. If  $f(x) = \begin{vmatrix} \sin x + \sin 2x + \sin 3x & \sin 2x & \sin 3x \\ 3 + 4 \sin x & 3 & 4 \sin x \\ 1 + \sin x & \sin x & 1 \end{vmatrix}$  then the value of  $\int_0^{\pi/2} f(x) dx$  is

1) 3

2) 2/3

3) 1/3

4) 0

**Key: 3**

**Sol:**  $C_1 \rightarrow C_1 - (C_2 + C_3) \Rightarrow f(x) = \sin 3x$

28.  $\int_1^{16} \tan^{-1} \sqrt{\sqrt{x}-1} dx =$

$$1) \frac{8\pi}{3} + \sqrt{3} \quad 2) \frac{8\pi}{3} - \sqrt{3} \quad 3) \frac{16\pi}{3} - 2\sqrt{3} \quad 4) \frac{16\pi}{3} + 2\sqrt{3}$$

Key(3)

**Sol**  $\sqrt{x} - 1 = t^2 \Rightarrow x = (t^2 + 1)^2$

$$I = \int_0^{\sqrt{3}} \tan^{-1} t \, d(t^2 + 1)^2 = \tan^{-1} t (t^2 + 1)^2 \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} (t^2 + 1) dt$$

$$= \frac{16\pi}{3} - \left( \frac{t^3}{3} + t \right) \Big|_0^{\sqrt{3}} = \frac{16\pi}{3} - 2\sqrt{3}$$

29.  $\int_0^1 \cot^{-1}(1 - x + x^2) dx = \underline{\hspace{2cm}}$

$$1) \frac{\pi}{2} - \log 2 \quad 2) \frac{\pi}{2} + \log 2 \quad 3) \pi - \log 2 \quad 4) \pi + \log 2$$

Key(1)

**Sol**  $\int_0^1 \tan^{-1} \left( \frac{1}{1-x(1-x)} \right) dx = \int_0^1 \tan^{-1} \left( \frac{x+(1-x)}{1-x(1-x)} \right) dx$

$$\int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1} (1-x) \, dx = 2 \int_0^1 \tan^{-1} x \, dx$$

30.  $\int_{-2\pi}^{5\pi} \cot^{-1}(\tan x) dx = \underline{\hspace{2cm}}$

$$1) \frac{7\pi}{2} \quad 2) \frac{2\pi}{7} \quad 3) \frac{2\pi^2}{7} \quad 4) \frac{7\pi^2}{2}$$

Key(4)

**Sol**  $I = 7 \int_0^{\pi} \cot^{-1} \left( \cot \left( \frac{\pi}{2} - x \right) \right) dx \quad [\because \text{period is } \pi]$

$$\left[ \cot^{-1}(\cot x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi + x & \pi/2 < x < \pi \end{cases} \right]$$

31.  $\int_{-\frac{\pi}{2}}^{2\pi} \sin^{-1}(\sin x) dx =$

1)  $\frac{\pi^2}{8}$

2)  $-\frac{\pi^2}{8}$

3)  $\frac{\pi^2}{4}$

4)  $-\pi$

**Key(2)**

**Sol** 
$$\int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x) dx = \int_{-\pi/2}^{\pi/2} x dx + \int_{\pi/2}^{3\pi/2} (\pi - x) dx + \int_{3\pi/2}^{2\pi} (x - 2\pi) dx = \frac{-\pi^2}{8}$$

32. The value of the integral  $\int_{-1}^3 \left( \tan^{-1} \frac{x}{x^2+1} + \tan^{-1} \frac{x^2+1}{x} \right) dx$  is equal to

1)  $\pi$

2)  $2\pi$

3)  $4\pi$

4)  $6\pi$

**Key(1)**

**Sol** 
$$I = \int_{-1}^1 \left( \tan^{-1} \frac{x}{x^2+1} + \tan^{-1} \frac{x^2+1}{x} \right) dx = \int_1^3 \frac{\pi}{2} dx = (3-1) \frac{\pi}{2} = \pi$$

33.  $\int_0^1 \tan^{-1}(1-x+x^2) dx = \underline{\hspace{2cm}}$

1)  $\ln 2$

2)  $\frac{\pi}{4}$

3)  $\frac{\pi}{2}$

4)  $\frac{\pi}{2} - \ln 2$

**Key(1)**

**Sol** 
$$\begin{aligned} \int_0^1 \tan^{-1}(1-x+x^2) dx &= \int_0^1 \left( \frac{\pi}{2} - \tan^{-1} \left( \frac{x+(1-x)}{1-x(1-x)} \right) \right) dx \\ &= \frac{\pi}{2} - \int_0^1 \left( \tan^{-1}(1-x) + \tan^{-1} x \right) dx \\ &\Rightarrow \frac{\pi}{2} - 2 \int_0^1 \tan^{-1}(x) dx = \ln 2 \end{aligned}$$

34.  $\int_0^1 \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}} dx =$

1)  $\frac{\sqrt{2}-1}{3}$

2)  $2 \left( \frac{\sqrt{2}-1}{3} \right)$

3)  $\frac{2(2\sqrt{2}-1)}{3}$

4)  $\frac{2\sqrt{2}-1}{3}$

**Key(3)**

**Sol** 
$$\int_0^1 \frac{\sqrt{1+x}(\sqrt{1+x}+\sqrt{x})}{\sqrt{x}+\sqrt{1+x}} dx$$

$$= \int_0^1 \sqrt{1+x} \, dx = \frac{2}{3} \left( (1+x)^{3/2} \right)_0^1 = \frac{2}{3} (2\sqrt{2} - 1)$$

35.  $\int_0^1 \frac{x^6 - x^3}{(2x^3 + 1)^3} dx$  is equal to

- 1) 0                                      2)  $-\frac{1}{6}$   
 3)  $-\frac{1}{12}$                                   4)  $-\frac{1}{36}$

Key(4)

Sol

$$\int_0^1 \frac{\left(1 - \frac{1}{x^3}\right)}{\left(2x + \frac{1}{x^2}\right)^3} dx \quad \text{and proceed}$$

36.  $\int_0^2 \frac{2x^3 - 6x^2 + 9x - 5}{x^2 - 2x + 5} dx =$

- 1) 0                                      2)  $\frac{3}{2}$                                       3) 3                                      4) 4

Key(1)

Sol

$$I = \int_0^2 \left( (2x-2) + \frac{5-5x}{x^2-2x+5} \right) dx$$

$$= \left( x^2 - 2x - \frac{5}{2} \log(x^2 - 2x + 5) \right)_0^2 = 0$$

37.  $\int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})} =$

- 1)  $\frac{1}{2\sqrt{11}} \log_e \left( \frac{\sqrt{11}+1}{\sqrt{11}-1} \right)$     2)  $\frac{1}{2\sqrt{11}} \log_e \left( \frac{\sqrt{11}+1}{\sqrt{11}} \right)$   
 3)  $\frac{1}{\sqrt{11}} \log_e \left( \frac{\sqrt{11}-1}{\sqrt{11}} \right)$     4)  $\frac{1}{\sqrt{11}} \log_e \left( \frac{\sqrt{11}-1}{\sqrt{11}+1} \right)$

Key(1)

**Sol**

$$\begin{aligned}
 I &= \int_0^1 \frac{dx}{(5+2(1-x)-2(1-x)^2)(1+e^{2-4(1-x)})} \\
 &= \int_0^1 \frac{e^{2-4x}}{(5+2x-2x^2)(e^{2-4x}+1)} \\
 2I &= \int_0^1 \frac{1+e^{2-4x}}{(5+2x-2x^2)(1+e^{2-4x})} dx \\
 &= \int_0^1 \frac{dx}{5+2x-2x^2} = \frac{1}{2\sqrt{11}} \log \left( \frac{\sqrt{11}+1}{\sqrt{11}-1} \right)
 \end{aligned}$$

38. If  $\int_2^3 \frac{x^2}{\sqrt{x^4 - x^2 + 1}} dx = J$  then  $\int_2^3 \frac{x dx}{\sqrt{1 + \left(x - \frac{1}{x}\right)^2}} - \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{dx}{x^3 \sqrt{x^2 + \frac{1}{x^2} - 1}}$  is equal to .....

1) - J

2) 0

3) J

4) 2J

**Key(4)****Sol**

$$\begin{aligned}
 I_1 &= \int_2^3 \frac{x}{\sqrt{1 + \left(x - \frac{1}{x}\right)^2}} dx \text{ and } I_2 = \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{dx}{x^3 \sqrt{x^2 + \frac{1}{x^2} - 1}} \quad x = \frac{1}{t} \quad I_2 \\
 \therefore I_2 &= \int_2^3 \frac{1}{\frac{1}{t^3} \sqrt{\frac{1}{t^2} + t^2 - 1}} \cdot -\frac{1}{t^2} dt = \int_2^3 \frac{-t^2}{\sqrt{t^4 - t^2 + 1}} dt = -I_1 \\
 \text{and } I_1 &= J \quad \therefore I_1 - I_2 = 2J
 \end{aligned}$$

39.  $\int_0^1 \frac{1}{(x^2 + 3x + 2)^2} dx =$

1)  $\frac{2}{3} + 2 \log \frac{4}{3}$

2)  $\frac{4}{3} + \log \frac{2}{3}$

3)  $\frac{1}{3} + \log \frac{3}{2}$

4)  $\frac{2}{3} - 2 \log \frac{4}{3}$

**Key(3)****Sol**

$$\frac{1}{(x^2+3x+2)^2} = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} - 2\left(\frac{1}{x+1} - \frac{1}{x+2}\right)$$

40.  $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx$

1)  $\frac{3}{16}$

2)  $\frac{5}{8}$

3)  $\frac{7}{16}$

4)  $\frac{3}{32}$

Key (4)

**Sol** using the substitution  $2x = 3 \tan \theta$

$$I = \frac{3}{16} \int_0^{\frac{\pi}{3}} \sin \theta \cdot \tan^2 \theta d\theta$$

41.  $\int_0^1 \frac{x^4 (1-x)^4}{1+x^2} dx = \underline{\hspace{2cm}}$

1) 0

2)  $\pi - \frac{3}{7}$

3)  $\frac{22}{7} - \pi$

4)  $\pi + \frac{22}{7}$

Key (3)

**Sol**

$$I = \int_0^1 \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{1+x^2} dx$$

$$= \int_0^1 \left( x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$$

$$= \frac{22}{7} - \pi$$

42. Let  $I_1 = \int_0^\infty \frac{x^2 \sqrt{x}}{1+x^6} dx$ ,  $I_2 = \int_0^\infty \frac{x \sqrt{x}}{1+x^6} dx$  then

1)  $I_1 = 2I_2$

2)  $I_2 = 2I_1$

3)  $I_1 = I_2^2$

4)  $I_1 = I_2$

Key (4)

**Sol** Putting  $x = \frac{1}{y}$  in  $I_1$ , gives  $I_2$ .

43. If  $u = \int_0^{\infty} \frac{dx}{x^4 + 7x^2 + 1}$  and  $v = \int_0^{\infty} \frac{x^2 dx}{x^4 + 7x^2 + 1}$  then

1)  $u > v$

2)  $u < v$

3)  $u = v$

4)  $u = \frac{\pi}{6}$

Key(3,4)

Sol

$$u = \int_0^{\infty} \frac{1}{x^4 + 7x^2 + 1} dx$$

Put  $x = \frac{1}{t}$ ,  $dx = \frac{-1}{t^2} = dt$

$$u = \int_{\infty}^0 \frac{1}{\frac{1}{t^4} + \frac{7}{t^2} + 1} \left( \frac{-dt}{t^2} \right) = \int_0^{\infty} \frac{t^2 dt}{t^4 + 7t^2 + 1} = v$$

$$\Rightarrow u = v$$

$$\text{So, } u + v = 2y = \int_0^{\infty} \frac{1+x^2}{x^4 + 7x^2 + 1} dx$$

$$\Rightarrow 2y = \int_0^{\infty} \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 9} = \int_{-\pi/2}^{\pi/2} \frac{dt}{t^2 + 9} = \frac{\pi}{3}$$

44. The value of  $\int_{1/e}^{\tan x} \frac{t dt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$  is

1)  $\frac{1}{2 + \tan^2 x}$

2) 1

3)  $\frac{\pi}{4}$

4)  $\frac{2}{\pi} \int_{-1}^1 \frac{dt}{1+t^2}$

Key(2,4)

Sol

$$\text{Let } I = \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$$

$$\text{Put } t = \frac{1}{z}$$

$$\therefore dt = -\frac{1}{z^2} dz$$

$$\therefore I = \int_e^{\tan x} \frac{-\frac{1}{z^2} dz}{\frac{1}{z} \left(1 + \frac{1}{z^2}\right)} = \int_{\tan x}^e \frac{z dz}{(z^2 + 1)}$$

$$\begin{aligned}
&= \int_{\tan x}^e \frac{t \, dt}{(1+t^2)} \\
\therefore \int_{1/e}^{\tan x} \frac{t \, dt}{(1+t^2)} + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)} \\
&= \int_{1/e}^{\tan x} \frac{t}{(1+t^2)} dt + \int_{\tan x}^e \frac{t}{(1+t^2)} dt \\
&= \int_{1/e}^e \frac{t \, dt}{(1+t^2)} = \frac{1}{2} \left[ \ln(1+t^2) \right]_{1/e}^e \\
&= \frac{1}{2} \left\{ \ln(1+e^2) - \ln\left(1+\frac{1}{e^2}\right) \right\} \\
&= \frac{1}{2} (\ln e^2) = 1.
\end{aligned}$$

Also,  $\frac{2}{\pi} \int_{-1}^1 \frac{dt}{1+t^2} = \frac{4}{\pi} \int_0^1 \frac{dt}{1+t^2} = \frac{4}{\pi} \cdot \tan^{-1} 1 = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1.$

45.  $I_1 = \int_0^1 1 - x^{50} \, dx$ ,  $I_2 = \int_0^1 1 - x^{50} \, dx$  then  $\left[ \frac{I_1}{I_2} \right] = \underline{\hspace{2cm}}$

**Key(1)**

**Sol**

(Here  $[\cdot]$  denotes G.I.F))

$$\begin{aligned}
I_2 &= \int_0^1 1 - x^{50} \, dx \\
&= I_1 - \int_0^1 x^{50} \, dx \\
&= I_1 - \int_0^1 x^{50} \cdot x^{49} \, dx
\end{aligned}$$

Using by parts we get

$$I_1 = I_2 \left( 1 + \frac{1}{5050} \right) = \frac{5051}{5050} I_2$$

$$\left[ \frac{I_1}{I_2} \right] = 1$$

46.  $\int_0^{\infty} \frac{x^2}{x^2+1} \cdot \frac{x^2}{x^2+4} \cdot \frac{x^2}{x^2+9} dx = \frac{\pi}{n+2!}$  then  $n = \underline{\hspace{2cm}}$

Key(3)

Sol Use partial fractions and integrate to get  $I = \frac{\pi}{120} \therefore n = 3$

47. If  $\int_0^{\infty} \frac{x^2}{1+x^2} dx = \frac{k}{15}$ , then  $k = \underline{\hspace{2cm}}$

Key(2)

Sol

Put  $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta}{\sec^7 \theta} \cdot \sec^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^2 \theta \cdot \cos^3 \theta d\theta = \frac{2}{15}$$

48. Find the value of  $\frac{29 \int_0^1 (1-x^4)^7 dx}{4 \int_0^1 (1-x^4)^6 dx}$

Key(7)

Sol

$$\begin{aligned} 1 &= \int_0^1 (1-x^4)^7 - 1 dx \\ &= \left[ x(1-x^4)^7 \right]_0^1 + 7 \times 4 \int_0^1 x(1-x^4)^6 x^3 dx \\ &= -28 \int_0^1 (1-x^4)^6 dx + 28 \int_0^1 (1-x^4)^6 dx = -28I + 28 \int_0^1 (1-x^4)^6 dx \\ &29I - 28 \int_0^1 (1-x^4)^6 dx \\ &\frac{29 \int_0^1 (1-x^4)^7 dx}{4 \int_0^1 (1-x^4)^6 dx} = 7 \end{aligned}$$

49. Let  $I_1 = \int_0^1 (1-x^{50})^{100} dx$ ;  $I_2 = \int_0^1 (1-x^{50})^{101} dx$  Then  $\frac{I_2}{I_1} = \underline{\hspace{2cm}}$

Key(1)

**Sol**                      1)  $\frac{5051}{5050}$                       2)  $\frac{5050}{5051}$                       3) 1                      4) 2

50. If  $f(2-x) = f(2+x)$  and  $f(4-x) = f(4+x)$  for all  $x$  and  $\int_0^2 f(x) dx = 5$  then

$$\int_0^{50} f(x) dx =$$

1) 0                      2) 125                      3) 25                      4) 50

**Key(2)**

**Sol**                      Replace  $x$  with  $2-x$

$$f(x) = f(4-x) = f(4+x)$$

$f(x)$  is periodic with period is

$$\begin{aligned} \int_0^{50} f(x) dx &= \int_0^{48} f(x) dx + \int_{48}^{50} f(x) dx \\ &= 24 \int_0^2 f(x) dx + \int_0^2 f(x) dx = 25 \int_0^2 f(x) dx = 25(5) = 125 \end{aligned}$$

51. Let  $f(x)$  be a differentiable function on  $\mathbb{R}$  such that  $f(6+x) = f(6-x)$ ,

$f(4+x) = f(4-x)$  and  $f(7) = 1$  and  $\int_k^{k+4} f(x) dx = 5$  then the value of

$\int_3^7 x f'(x) dx$  equals

1) 1                      2) -1                      3) 2                      4) -2

**Key(2)**

F is periodic with period = 4,

$$GI = \left( x \cdot f(x) \right)_3^7 - \int_3^7 f(x) dx = (7)(f(7)) - (3)(f(3)) - 5 = (7)(1) - 3(1) - 5 = -1$$

52. If  $f(x) = \begin{cases} e^{\cos x} \sin x, & \text{for } |x| \leq 2 \\ 2, & \text{otherwise} \end{cases}$  then  $\int_{-2}^3 f(x) dx =$

1) 0

2) 1

3) 2

4) 3

**Key(3)****Sol** Solution we have,

$$\int_{-2}^3 f(x) dx = \int_{-2}^2 e^{\cos x} \sin x dx + \int_{-2}^3 2 dx$$

$$= 0 + 2(3-2) \left[ \begin{array}{l} \therefore g(x) = e^{\cos x} \sin x \text{ is an odd function} \\ \therefore \int_{-2}^2 g(x) dx = 0 \end{array} \right]$$

53. If  $\int_0^1 \frac{e^t dt}{t+1} = a$  then  $\int_{b-1}^b \frac{e^{-t} dt}{t-b-1} =$

1)

2)  $-ae^{-b}$ 3)  $-be^{-a}$ 4)  $ae^b$ **Key(2)****Sol**

$$\therefore a = \int_0^1 \frac{e^t}{t+1} dt$$

$$\therefore \int_{b-1}^b \frac{e^t}{t-b-1} = e^{-b} \int_0^1 \frac{e^y}{y+1} dy$$

Put  $t = b - y$   
 $dt = -dy$

$$= -e^{-b} . a$$

54. The natural number  $n(\leq 5)$  for which  $I_n = \int_0^1 e^x (x-1)^n dx = 16-6e$  is

1) 2

2) 3

3) 4

4) 5

**Key(2)**

**Sol** We have  $I_0 = \int_0^1 e^x dx = e-1$  and for  $n \geq 1$

$$I_n = \left[ e^x \cdot (x-1)^n \right]_0^1 - n \int_0^1 e^x (x-1)^{n-1} dx$$

$$I_2 = -1 - 2I_1 = -1 - 2(2-e) = 2e-5$$

$$I_3 = 1 - 3I_2 = 16 - 6e$$

55. If  $f(\pi) = 2$  and  $\int_0^{\pi} (f(x) + f''(x)) \sin x \, dx = 5$  then  $f(0)$  is equal to

1) 7

2) 3

3) 5

4) 1

Key(2)

Sol

$$\int_0^{\pi} f(x) \sin x \, dx + \int_0^{\pi} f''(x) \sin x \, dx =$$

I      II                  II      I

Use by parts

$$\Rightarrow f(\pi) + f(0) = 5$$

$$\therefore f(0) = 3$$

56.  $\int_{-4}^{-5} e^{(x+5)^2} \, dx + 3 \int_{\frac{1}{3}}^{\frac{2}{3}} e^{9\left(\frac{2}{3}-x\right)^2} \, dx = \underline{\hspace{2cm}}$

1) 0

2)  $-\frac{1}{2}$ 3)  $-\frac{3}{2}$ 4)  $\sqrt{e}$ 

Key(1)

Sol

In  $I_1$  put  $x+5=t$  and In  $I_2$  put  $2-3x=z$

$$\int_1^0 e^{t^2} \, dt - \int_1^0 e^{z^2} \, dz = 0$$

57. If  $\frac{dy}{dx} = y + \int_0^1 y \, dx$  and  $y(0) = 1$  then  $y(2) =$

1)  $\frac{e^2 - e + 1}{3 - e}$

2)  $\frac{2e^2 + e - 1}{e - 3}$

3)  $\frac{e - 1 - 2e^2}{e - 3}$

4)  $\frac{2e^2 - e + 1}{e - 3}$

Key(3)

Sol

$$\text{Let } \int_0^1 y \, dx = \lambda \rightarrow (1)$$

$$\frac{dy}{dx} = y + \lambda$$

$$\int \frac{dy}{y+\lambda} = \int dx$$

$$\log_e(y+\lambda) = x+c$$

$$\text{When } x=0 \log_e 1 + \lambda = c$$

$$\log_e y + \lambda = x + \log_e 1 + \lambda$$

$$\frac{y+\lambda}{1+\lambda} = e^x$$

$$y = (1+\lambda)e^x - \lambda$$

$$\text{From (1) } \int_0^1 ((1+\lambda)e^x - \lambda) dx = \lambda$$

$$= ((1+\lambda)e^x - \lambda x)_0^1 = \lambda$$

$$= (1+\lambda)e - \lambda - 1 - \lambda = \lambda$$

$$e = 3\lambda + 1 - e\lambda$$

$$\frac{e-1}{3-e} = \lambda$$

$$\therefore y(x) = \left(1 + \frac{e-1}{3-e}\right)e^x - \frac{e-1}{3-e} \Rightarrow y(2) = \frac{e^2(2)}{3-e} - \frac{e-1}{3-e}$$

$$= \frac{2e^2 - e + 1}{3-e}$$

58. If  $\int_0^{100} f(x) dx = a$ , then  $\sum_{r=1}^{100} \left( \int_0^1 f(r-1+x) dx \right) =$

1) 100 a

2) a

3) 0

4) 10 a

Key(2)

**Sol** 
$$\sum_{r=1}^{100} \left( \int_0^1 f(r-1+x) dx \right) = \int_0^1 f(x) dx + \int_0^1 f(1+x) dx + \int_0^1 f(2+x) dx + \dots + \int_0^1 f(99+x) dx$$

$$= \int_0^1 f(x) dx + \int_1^2 f(x) dx + \dots + \int_{99}^{100} f(x) dx$$

{using shifting property}

$$= \int_0^{100} f(x) dx = a$$

59.  $\lim_{x \rightarrow 0} x \frac{\int_0^x e^{t^2} dt}{1 - e^{x^2}} =$

1) 0

2) 1

3) -1

4) -5/2

Key(3)

Sol Required limit =  $\lim_{x \rightarrow 0} \frac{x^2}{1 - e^{x^2}}$

60.  $\lim_{x \rightarrow \infty} (1+x) \int_0^x e^{t^2 - x^2} dt =$

1) 1

2)  $\frac{1}{2}$

3)  $\frac{1}{3}$

4) 4

Key(2)

Sol  $L = \lim_{x \rightarrow \infty} \frac{\int_0^x e^{t^2} dt}{\frac{e^{x^2}}{1+x}} = \lim_{x \rightarrow \infty} \frac{e^{x^2}}{e^{x^2} \cdot \frac{2x}{1+x} - \frac{e^{x^2}}{(1+x)^2}}$

$$= \lim_{x \rightarrow \infty} \frac{1}{\frac{2x}{1+x} - \frac{1}{(1+x)^2}} = \frac{1}{2}$$

61.  $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_{\frac{\pi^2}{16}}^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}}$  equals

(1)  $\frac{8}{\pi} f(2)$

(2)  $\frac{2}{\pi} f(2)$

(3)  $\frac{2}{\pi} f\left(\frac{1}{2}\right)$

(4)  $4f(2)$

Key(1)

Sol Required limits is of the form  $\frac{0}{0}$ , so it is equal to

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{2 \sec x \sec x \tan x f(\sec^2 x)}{2x} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sec^2 x \tan x f(\sec^2 x)}{x} = \frac{8}{\pi} f(2)$$

62.  $f: \mathbb{R} \rightarrow \mathbb{R}$  is periodic with period 4 and  $f(x) = \begin{cases} 9-x^2, & 0 < x \leq 2 \\ 3x-1, & 2 < x \leq 4 \end{cases}$ . Then  $\int_{-2}^8 f(x) dx =$
- 1)  $76\frac{1}{3}$                       2)  $78\frac{2}{3}$                       3)  $80\frac{2}{3}$                       4)  $78\frac{3}{4}$

**Key(2)**

**Sol**  $I = 2 \int_0^2 f(x) dx + 3 \int_2^4 f(x) dx$

63. Let  $a, b$  and  $c$  be positive constants. The value of ' $a$ ' in terms of ' $c$ ' if the value of

integral  $\int_0^1 (acx^{b+1} + a^3bx^{3b+5}) dx$  is independent of ' $b$ ' equals

- 1)  $\sqrt{\frac{3c}{2}}$                       2)  $\sqrt{\frac{2c}{3}}$                       3)  $\sqrt{\frac{c}{3}}$                       4)  $\sqrt{\frac{3}{2c}}$

**Key(1)**

**Sol** If the integral is independent of  $b$ , then  $I'(b) = 0$

$$\begin{aligned} \Rightarrow I'(b) &= \frac{-ac}{(b+2)^2} + D\left(\frac{a^3}{3} \cdot \left(\frac{b+2-2}{b+2}\right)\right) = \frac{-ac}{(b+2)^2} + \frac{a^3}{3} \left(\frac{2}{(b+2)^2}\right) \\ &= \frac{1}{(b+2)^2} \left(\frac{2a^3}{3} - ac\right) = \frac{2a^3}{3} = ac \Rightarrow a = \sqrt{\frac{3c}{2}} \end{aligned}$$

64. Let  $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x)) dx$  and let  $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x)) dx$

where ' $f$ ' is a continuous function and ' $z$ ' is any real number, then  $\frac{I_1}{I_2} =$

- 1)  $\frac{1}{2}$                       2) 1                      3)  $\frac{2}{3}$                       4) NONE

**Key(1)**

$$I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x)) dx$$

**Sol**

$$\begin{aligned}
 I_2 &= \int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x)) dx \\
 2I_1 &= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)f((3-x)x) dx \\
 2I_1 &= \int_{\sec^2 z}^{2-\tan^2 z} 3f(x(3-x)) dx \\
 &= 3 \int_{\sec^2 z}^{2-\tan^2 z} ((x)(3-x)) dx \\
 2I_1 &= 3I_2 \\
 \Rightarrow \frac{I_1}{I_2} &= \frac{3}{2}
 \end{aligned}$$

65.  $\lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n} =$

1) e

2)  $\frac{1}{e}$

3)  $e-5$

4)  $e-5$

Key(2)

Sol

$$\begin{aligned}
 \therefore \text{Let } y &= \lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n} = \lim_{n \rightarrow \infty} \left( \frac{n!}{n^n} \right)^{1/n} \\
 \Rightarrow \log y &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \log \left[ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \log \frac{r}{n} = \int_0^1 \log x dx
 \end{aligned}$$

66. If  $\lim_{n \rightarrow \infty} \sum_{r=1}^{4n} \frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r}+4\sqrt{n})^2} = \lambda$  then  $10\lambda =$  \_\_\_\_\_

Key(1)

**Sol** 
$$\lambda = \lim_{n \rightarrow \infty} \sum_{r=1}^{4n} \frac{1}{\sqrt{\frac{r}{n}} \left( 3 \sqrt{\frac{r}{n}} + 4 \right)^2} = \int_0^1 \frac{dx}{\sqrt{x} (3\sqrt{x} + 4)^2} = \frac{1}{10}$$

$$\therefore 10\lambda = 1$$

67. The value of  $\lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{{}^nC_r}{n^r (r+3)} = e - x$ , then x is \_\_\_\_\_

**Key(2)**

**Sol** 
$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{{}^nC_r}{n^r (r+3)} &= \lim_{n \rightarrow \infty} \left( \sum_{r=0}^n \frac{{}^nC_r}{n^r} \int_0^1 x^{r+2} dx \right) \\ &= \int_0^1 \left( x^2 \lim_{n \rightarrow \infty} \sum_{r=0}^n {}^nC_r \left( \frac{x}{n} \right)^r \right) dx = \int_0^1 \left( x^2 \lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n \right) dx \\ &= \int_0^1 x^2 e^x dx = e - 2 \end{aligned}$$

68. If  $\lim_{n \rightarrow \infty} \frac{\sum_{r=1}^n \sqrt{r} \sum_{r=1}^n \frac{1}{\sqrt{r}}}{\sum_{r=1}^n r} = \frac{k}{3}$  then find k.

**Key(8)**

**Sol** 
$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sum_{r=1}^n \sqrt{r} \sum_{r=1}^n \frac{1}{\sqrt{r}}}{\frac{n(n+1)}{2}} &= \lim_{n \rightarrow \infty} \frac{\sum_{r=1}^n \sqrt{\frac{r}{n}} \sum_{r=1}^n \frac{1}{\sqrt{\frac{r}{n}}}}{\frac{n^2}{2} \left( 1 + \frac{1}{n} \right)} \\ &= 2 \times \int_0^1 \sqrt{x} dx \int_0^1 \frac{1}{\sqrt{x}} dx \\ &= 2 \times \left[ \frac{x^{3/2}}{\frac{3}{2}} \right]_0^1 \left[ \frac{x^{1/2}}{\frac{1}{2}} \right]_0^1 \\ &= 2 \times \frac{2}{3} \times 2 = \frac{8}{3} \\ \therefore k &= 8 \end{aligned}$$

69.  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^{1/a} \left\{ n^{a-\frac{1}{a}+k} n^{a-\frac{1}{a}} \right\}}{n^{a+1}}$  is equal to

**Key(1)**

**Sol**

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^{\frac{1}{a}} \left\{ n^{\frac{a-1}{a}} + k^{\frac{a-1}{a}} \right\}}{n^{a+1}} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \cdot \left\{ \left( \frac{k}{n} \right)^{1/a} + \left( \frac{k}{n} \right)^a \right\} \\
&= \int_0^1 (x^{1/a} + x^a) dx \\
&= \left[ \frac{x^{(1/a)+1}}{\frac{1}{a}+1} + \frac{x^{a+1}}{a+1} \right]_0^1 \\
&= \frac{a}{a+1} + \frac{1}{a+1} = 1
\end{aligned}$$

70. If the value of the definite integral  $\int_0^1 207_{C_7} x^{200} (1-x)^7 dx$  is equal to  $\frac{k}{208}$  then k is

**Key(1)**

**Sol**  $I = \int_0^1 207_{C_7} x^{200} (1-x)^7 dx$  integrate by parts

$$= 207_{C_7} \cdot \frac{7}{201} \int_0^1 (1-x)^6 x^{201} dx$$

Integrate six more times

$$I = 207_{C_7} \frac{7!}{201 \cdot 202 \cdot 203 \cdot 204 \cdot \dots \cdot 207} \int_0^1 x^{207} dx = \frac{1}{208} = \frac{k}{208} \Rightarrow k = 1.$$

71. If  $\lim_{x \rightarrow 0} \frac{t \int_0^x \frac{t^2 dt}{(x - \sin x) \sqrt{a+t}}}{x - \sin x} = 1$  then the value of a is .....

**Key(4)**

**Sol**

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{t \int_0^x \frac{t^2 dt}{\sqrt{a+t}}}{x - \sin x} = \lim_{x \rightarrow 0} \frac{t \frac{x^2}{\sqrt{a+x}(1-\cos x)}}{x - \sin x} = \lim_{x \rightarrow 0} \frac{t \frac{x^2}{\sqrt{a+x} \left( 2 \sin^2 \frac{x}{2} \right)}}{x - \sin x} \\
&= \frac{2}{\sqrt{a}} = 1 \Rightarrow a = 4
\end{aligned}$$

72. Let  $I_n = \int_0^1 (1-x^a)^n dx$  and if  $\frac{I_n}{I_{n+1}} = 1 + \frac{\lambda}{a}$  then  $\lambda$  equals

- 1)  $n$                       2)  $n+1$                       3)  $\frac{1}{n}$                       4)  $\frac{1}{n+1}$

Key(4)

**Sol** 
$$I_{n+1} = \int_0^1 (1-x^a)^{n+1} \cdot 1 \cdot dx = \left\{ (1-x^a)^{n+1} \cdot x \right\}_0^1 - \int_0^1 (n+1) \cdot (1-x^a)^n \cdot (-ax^{a-1}) \cdot x \, dx$$

$$= (n+1)a \int_0^1 (1-x^a)^n \cdot (x^a - 1 + 1) \, dx = (n+1)a I_n - (n+1)a I_{n+1} \Rightarrow \frac{I_n}{I_{n+1}} = 1 + \frac{1}{(n+1)a} \Rightarrow \lambda = \frac{1}{n+1}$$

73. Let 'f' be a one-one continuous function such that  $f(1)=4$ ,  $f(2)=6$ , given that  $\int_1^2 f(x) \, dx = 20$  then the value of  $\int_4^6 f^{-1}(x) \, dx$  equals

- 1) 6                      2) -8                      3) -10                      4) -12

Key(4)

**Sol** 
$$\int_{f(1)}^{f(2)} f^{-1} \, dx \quad f^{-1}(x) = t \Rightarrow x = f(t) \Rightarrow dx = f'(f) \, dt$$

$$= \int_1^2 t \cdot f'(t) \, dt = (t \cdot f(t))_1^2 - \int_1^2 f(t) \, dt = 2f(2) - 1f(1) - 20$$

$$= 2(6) - 1(4) - 20 = -12$$

74. The value of  $I = \int_0^1 \ln(\cos x) \, dx$  has the same value as that of the definite integral given below

- 1)  $\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \, dx$                       2)  $\int_{\cos 1}^1 \frac{\ln x}{\sqrt{1-x^2}} \, dx$                       3)  $\frac{1}{4} \int_{\cos^2 1}^1 \frac{\ln x}{\sqrt{x-x^2}} \, dx$                       4)  $\int_{\cos 2}^1 \frac{1}{\sqrt{1-x^2}} \, dx$

Key(3)

**Sol**  $\cos x = \sqrt{t} \Rightarrow I = \int_1^{\cos^2 1} \ln(\sqrt{t}) \cdot \frac{1}{(-\sin x)} \cdot \frac{1}{2\sqrt{t}} \, dt = \frac{1}{4} \int_{\cos^2 1}^1 \frac{\ln t}{\sqrt{t}\sqrt{1-t}} \, dt = \frac{1}{4} \int_{\cos^2 1}^1 \frac{\ln x}{\sqrt{x-x^2}} \, dx$

75. Let  $f(x)$  be a continuous function such that  $\int_0^1 f(x)(2x - f(x))dx = \frac{1}{3}$  then the value of  $f(3)$  equals

1) 1

2) 2

3) 3

4) 6

Key(3)

**Sol**  $G.I \Rightarrow \int_0^1 (x^2 - (f(x) - x)^2)dx = \frac{1}{3} \Rightarrow \int_0^1 (f(x) - x)^2 dx = 0$

$$\Rightarrow f(x) - x = 0 \Rightarrow f(x) = x$$

$$\Rightarrow f(3) = 3$$

76.  $\int_{\sqrt{\frac{3a^2+b^2}{4}}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}} dx = \underline{\hspace{2cm}}$

1)  $\frac{\pi}{12}$

2)  $\frac{\pi}{24}$

3)  $\frac{\pi}{6}$

4)  $\frac{\pi}{4}$

Key(1)

**Sol** Put  $x^2 = a^2 \cos^2 t + b^2 \sin^2 t$

$$\therefore x dx = \frac{b^2 - a^2}{2} \cdot \sin 2t dt$$

$$I = \frac{\pi}{12}$$

77. The value of  $\int_0^1 \left( \prod_{r=1}^n (x+r) \right) \left( \sum_{k=1}^n \frac{1}{x+k} \right) dx = \underline{\hspace{2cm}}$

1)  $n!$

2)  $(n+1)!$

3)  $n.n!$

4)  $n(n+1)!$

Key(3)

**Sol** Put  $(x+1)(x+2)(x+3)\dots(x+n) = t$

Take log and differential

$$\left( \sum_{k=1}^n \frac{1}{x+k} \right) dx = \frac{1}{t} dt \therefore I = \int_{n!}^{(n+1)!} \frac{1}{t} dt = n!n$$

78. If  $f(x)$  be a continuous function for all real values of 'x' and satisfies

$$\int_3^x f(t) dt = \frac{x^2}{2} + \int_x^8 t^2 f(t) dt, \text{ then } \int_{-\frac{1}{2}}^1 f(x) dx = \underline{\hspace{2cm}}$$

1)  $\log\left(\frac{5}{8}\right)$

2)  $\frac{1}{2}\log\left(\frac{8}{5}\right)$

3)  $2\log\left(\frac{8}{5}\right)$

4)  $\frac{1}{2}\log\left(\frac{5}{8}\right)$

Key(2)

Sol

differentiate w.r.t 'x'

$$f(x) = x - x^2 f(x)$$

$$f(x) = \frac{x}{1+x^2}$$

79. If  $\alpha, \beta$  ( $\beta > \alpha$ ) are the roots of  $g(x) = ax^2 + bx + c = 0$  ( $a \neq 0$ ) and  $f(x)$  is an even function, then

$$\int_{\alpha}^{\beta} \frac{e^{f\left(\frac{g(x)}{x-\alpha}\right)}}{e^{f\left(\frac{g(x)}{x-\alpha}\right)} + e^{f\left(\frac{g(x)}{x-\beta}\right)}} dx =$$

1)  $\left|\frac{b}{a}\right|$

2)  $\left|\frac{b}{2a}\right|$

3)  $\frac{\sqrt{b^2 - 4ac}}{|2a|}$

4)  $\sqrt{b^2 - 4ac}$

Key(3)

Sol  $g(x) = a(x - \alpha)(x - \beta)$

$$\text{Use } \int_a^b f(x) dx = \int_a^b f(a + b - x) dx$$

80. If  $f(x)$  is a function satisfying  $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$  for all non-zero  $x$ , then

$$\int_{\sin \theta}^{\operatorname{cosec} \theta} f(x) dx \text{ equals to :}$$

1)  $\sin \theta + \operatorname{cosec} \theta$

2)  $\sin^2 \theta$

3)  $\operatorname{cosec}^2 \theta$

4) 0

Key(4)

**Sol**  $f\left(\frac{1}{x}\right) + x^2 f(x) = 0 \Rightarrow f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right) \Rightarrow I = \int_{\sin \theta}^{\operatorname{cosec} \theta} f(x) dx = \int_{\sin \theta}^{\operatorname{cosec} \theta} -\frac{1}{x^2} f\left(\frac{1}{x}\right) dx$

Put  $\frac{1}{x} = t \Rightarrow -\frac{1}{x^2} dx = dt \Rightarrow I = \int_{\operatorname{cosec} \theta}^{\sin \theta} f(t) dt$

$\Rightarrow I = - \int_{\sin \theta}^{\operatorname{cosec} \theta} f(t) dt = -I \Rightarrow 2I = 0 \Rightarrow I = 0$

**81.** Let  $f: [1, 3] \rightarrow R$  be a function defined by  $f(x) = \int_1^x t(t^2 - 3t + 2) dt$ . Then the range of  $f(x)$  is

- 1)  $[0, 2]$                       2)  $\left[-\frac{1}{4}, 4\right]$                       3)  $\left[-\frac{1}{4}, 2\right]$                       4)  $\left[-\frac{1}{2}, 1\right]$

**Key(3)**

**Sol**  $f'(x) = 0 \Rightarrow x = 2$  (interior point)

$f(1) = 0 \quad f(2) = -\frac{1}{4} \quad f(3) = 2$

**82.** If  $3f\left(\frac{1}{x}\right) + 2f(x) = \frac{1}{x} - 5 \quad \forall x \neq 0$  Then  $\int_1^2 f(x) dx = \underline{\hspace{2cm}}$

- A)  $\frac{3 \ln 2 + 4}{5}$                       B)  $\frac{3 \ln 2 - 4}{5}$                       C)  $\frac{3 \ln 2 - 8}{5}$                       D)  $\frac{3 \ln 2 + 8}{5}$

**Key(3)**

**Sol** put  $\frac{1}{x}$  for  $x$

$3f\left(\frac{1}{x}\right) + 2f(x) = x - 5$

Solving with given relation we get

$f(x) = \frac{1}{5} \left( \frac{3}{x} - 2x - 5 \right)$

Now  $\int_1^2 f(x) dx = \frac{1}{5} (3 \ln 2 - 8)$

83. If  $f(x) = ax^2 + bx + c$ ,  $\int_0^4 f(x) dx = 1$ ,  $\int_0^4 xf(x) dx = 2$ ,  $\int_0^4 x^2 f(x) dx = 3$  Then in the interval  $(0,4)$ ;  $f(x)=0$  has

- 1) No root                      2) exactly one root  
3) at most one root    4) at least one root

**Key(4)**

**Sol**  $\int_0^4 x^2 - 2x + 1 f(x) dx = 0$

$$\Rightarrow \int_0^4 x - 1^2 f(x) dx = 0$$

$\Rightarrow f(x)$  Changes its sign in  $(0, 4)$

$\therefore f(x)$  has at least one root in  $(0, 4)$

84. If  $f(x) = x + \int_0^1 x + t f(t) dt$  Then  $\int_0^1 f(x) dx = \underline{\hspace{2cm}}$

- 1)  $-4$                       2)  $4$                       3)  $-7$                       4)  $7$

**Key(3)**

**Sol**  $f(x) = x \left( 1 + \int_0^1 f(t) dt \right) + \left( \int_0^1 t f(t) dt \right)$   
 $= Ax + B$

Where  $A = 1 + \int_0^1 f(t) dt$

$$\& B = \int_0^1 t f(t) dt$$

Solving for A & B we get

$$A = -6, B = -4$$