

MODULUS OF COMPLEX NUMBERS

ARGUMENT OF COMPLEX NUMBERS

MAXIMA AND MINIMA

CUBIC ROOT OF UNITY

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MODULUS OF COMPLEX NUMBERS

1). If α, β are two non zero complex numbers and β is uni modular then

the value of $\left| \frac{\alpha - \beta}{1 - \alpha\beta} \right|$ is

1) 0

2) 1

3) 2

4) $|\alpha|$

Key (2)

Sol
$$\left| \frac{\alpha - \beta}{1 - \bar{\alpha}\beta} \right| = \frac{|\alpha - \beta|}{\left| 1 - \alpha \cdot \frac{1}{\bar{\beta}} \right|} = \frac{|\alpha - \beta|}{\left| \frac{\bar{\beta} - \alpha}{\bar{\beta}} \right|} = |\bar{\beta}| = 1$$

2). If $Z = \begin{vmatrix} 1 & 1-2i & 3+5i \\ 1+2i & -5 & 10i \\ 3-5i & -10i & 11 \end{vmatrix}$ then

1) Z is purely imaginary

2) Z is purely real

3) $Z = 0$

4) $\arg Z = \frac{-\pi}{4}$

Key (2)

Sol
$$\bar{z} = \begin{vmatrix} 1 & 1+2i & 3-5i \\ 1-2i & -5 & -10i \\ 3+5i & -10i & 11 \end{vmatrix} = z$$

$\therefore z$ is purely real.

3). Number of complex numbers z such that $|Z|=1$ and $\left| \frac{z}{\bar{z}} + \frac{\bar{z}}{z} \right| = 1$ and $\arg Z \in (0, 2\pi)$

1) 4

2) 6

3) 8

4) more than 8

Key (3)

Sol $z = e^{i\theta}, \left| \frac{z^2 + \bar{z}^{-2}}{2\bar{z}} \right| = 1 \Rightarrow |\cos 2x + i \sin 2x + \cos 2x - i \sin 2x| = 1$

$$\Rightarrow \cos 2x = \pm \frac{1}{2} \Rightarrow x \text{ has eight values in } (0, 2\pi)$$

4). If the ratio $\frac{1-z}{1+\bar{z}}$ is purely imaginary, then

1) $0 < |z| < 1$ 2) $|z| = 1$ 3) $|z| > 1$ 4) bounds for $|z|$ can not be decided

Key (2)

Sol $0 = \frac{1-z}{1+z} + \frac{1-\bar{z}}{1+\bar{z}} = \frac{(1-z)(1+\bar{z}) + (1-\bar{z})(1+z)}{(1+z)(1+\bar{z})} = \frac{2(1-|z|)^2}{|1+z|^2} \Rightarrow |z| = 1$

5). If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$

then $|z_1 + z_2 + z_3|$ is

1) equal to 1 2) less than 1 3) greater than 3 4) equal to 3

Key (1)

Sol $\because |z_1| = |z_2| = |z_3| = 1$ we get $z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = 1$

$$\Rightarrow \bar{z}_1 = \frac{1}{z_1}, \bar{z}_2 = \frac{1}{z_2}, \bar{z}_3 = \frac{1}{z_3}$$

$$= |\bar{z}_1 + \bar{z}_2 + \bar{z}_3|$$

$$= |z_1 + z_2 + z_3|$$

7). If $|z_1| = 2, |z_2| = 3, |z_3| = 4$ and $|z_1 + z_2 + z_3| = 5$ then $|4z_2z_3 + 9z_3z_1 + 16z_1z_2| =$

1) 20 2) 24 3) 48 4) 120

Key (4)

Sol $|4z_2z_3 + 9z_3z_1 + 16z_1z_2|$
 $= |z_1\bar{z}_1z_2z_3 + \bar{z}_2z_2z_3z_1 + \bar{z}_3z_3z_1z_2|$

$$= \|z_1\| \|z_2\| \|z_3\| \|z_1 + z_2 + z_3\| = 120$$

8). Let z_r ($1 \leq r \leq 4$) be complex numbers such that $|z_r| = \sqrt{r+1}$ and

$$|30z_1 + 20z_2 + 15z_3 + 12z_4| = k |z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2|$$

Then the value of k equals

- 1) $|z_1z_2z_3|$ 2) $|z_2z_3z_4|$ 3) $|z_4z_1z_2|$ 4) None of these

Key (3)

Sol We have $\left| \frac{z_1}{2} + \frac{z_2}{3} + \frac{z_1}{4} + \frac{z_4}{5} \right| = \frac{k}{60} |z_1z_2z_3z_4| \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} \right|$

Now, $z_1\bar{z}_1 = 2, z_2\bar{z}_2 = 3, z_3\bar{z}_3 = 4$ and $z_4\bar{z}_4 = 5$

So, $k = \frac{60}{|z_1z_2z_3z_4|} = \frac{60}{\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5}} = \sqrt{30} = |z_4z_1z_2|$

Note for objective take $z_1 = \sqrt{2}; z_2 = \sqrt{3}; z_3 = 2; z_4 = \sqrt{5}$

9). If $\log_{\tan 30^\circ} \left(\frac{2|z|^2 + 2|z| - 3}{|z| + 1} \right) < -2$ then

- 1) $|z| < \frac{3}{2}$ 2) $|z| > \frac{3}{2}$ 3) $|z| > 2$ 4) $|z| < 2$

Key (3)

Sol $\log_{\tan 30^\circ} \left(\frac{2|z|^2 + 2|z| - 3}{|z| + 1} \right) < -2$

$$\Rightarrow \frac{2|z|^2 + 2|z| - 3}{|z| + 1} > 3$$

$$\Rightarrow ((|z| - 2)(2|z| + 3)) > 0$$

$$\Rightarrow |z| > 2$$

10). If z_1 is a root of the equation $a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 3$, where $|a_i| < 2$ for $i = 0, 1, \dots, n$. Then

- 1) $|z_1| > \frac{1}{3}$ 2) $|z_1| < \frac{1}{4}$ 3) $|z_1| > \frac{1}{4}$ 4) $|z| < \frac{1}{3}$

Key (1)

Sol $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n = 3$

$$|3| = |a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n|$$

$$3 \leq |a_0| |z|^n + |a_1| |z|^{n-1} + \dots + |a_{n-1}| |z| + |a_n|$$

$$3 < 2(|z|^n + |z|^{n-1} + \dots + |z| + 1)$$

$$\frac{3}{2} < 1 + |z| + |z|^2 + \dots + |z|^n$$

$$\frac{1 - |z|^{n+1}}{1 - |z|} > \frac{3}{2}$$

$$2 - 2|z|^{n+1} < 3|z| - 1$$

$$3|z| - 1 > 0$$

$$|z| > \frac{1}{3}$$

12). $\theta \in [0, 2\pi]$ and z_1, z_2, z_3 are three complex numbers such that they are collinear and $(1 + |\sin \theta|)z_1 + (|\cos \theta| - 1)z_2 - \sqrt{2} z_3 = 0$. If at least one of the complex numbers z_1, z_2, z_3 is non-zero then number of possible values of θ is

- 1) Infinite 2) 0 3) 2 4) 4

Key (4)

Sol If z_1, z_2, z_3 are collinear and $az_1 + bz_2 + cz_3 = 0$ then $a+b+c=0$. Hence

$$1 + |\sin \theta| + |\cos \theta| - 1 - \sqrt{2} = 0 \Rightarrow |\sin \theta| + |\cos \theta| = \sqrt{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

- 14). If Z_1, Z_2 are two complex numbers such that $|Z_1|=1, |Z_2|=1$ then the maximum value of $|Z_1+Z_2|+|Z_1-Z_2|$ is
- 1) 2 2) $2\sqrt{2}$ 3) 4 4) none of these

Key (2)

Sol

$$\begin{aligned} Z_1 &= \cos \alpha + i \sin \alpha, & Z_2 &= \cos \beta + i \sin \beta \\ |Z_1+Z_2|+|Z_1-Z_2| &= \sqrt{2+2\cos(\alpha-\beta)} + \sqrt{2-2\cos(\alpha-\beta)} \\ \text{let } \alpha-\beta &= \theta \\ 2\cos\frac{\theta}{2}+2\sin\frac{\theta}{2} &= 2\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \end{aligned}$$

- 17). If z_1, z_2 are complex numbers such that $z_1^3-3z_1z_2^2=2$ and $3z_1^2z_2-z_2^3=11$ then $|z_1^2+z_2^2| =$

Key (5)

Sol $z_1^3-3z_1z_2^2+3iz_1^2z_2-iz_2^3=2+11i \Rightarrow (z_1+iz_2)^3=2+11i$

Similarly, $(z_1-iz_2)^3=2-11i$

$$|z_1^2+z_2^2| = |(z_1+iz_2)(z_1-iz_2)| = |(2+11i)^{1/3}(2-11i)^{1/3}| = 5$$

- 19). If the complex number Z satisfying $Z+|Z|=2+8i$ then value of $|Z| =$
- 1) 8 2) 17 3) 15 4) 24

Key (2)

Sol $z = a+ib$

$$\begin{aligned} \Rightarrow a+ib+\sqrt{a^2+b^2} &= 2+8i \\ \Rightarrow b=8, a+\sqrt{a^2+64} &= 2 \\ a^2+64 &= a^2-4a+4 \\ \Rightarrow -4a &= 60 \Rightarrow a = -15 \\ \therefore |z| &= \sqrt{a^2+b^2} = \sqrt{225+64} = \sqrt{289} = 17 \end{aligned}$$

ARGUMENT OF COMPLEX NUMBERS

- 32). The greatest positive argument of z satisfying $|Z-4| = \operatorname{Re}(Z)$

1. $\frac{\pi}{3}$ 2. $\frac{2\pi}{3}$ 3. $\frac{\pi}{2}$ 4. $\frac{\pi}{4}$

Key (4)

Sol $|x+iy-4| = x \sqrt{(X-4)^2 + y^2 + x^2}$
 $y^2 - 8x + 16 = 0$

z lies on the parabola with vertex (2,0) focus (4,0) and tangents from (0,0) a point on the directrix in x always include 90°

$\therefore$ greatest $\arg(z)$ is $45^\circ = \frac{\pi}{4}$

33). If Z is a complex number such that $-\frac{\pi}{2} \leq \arg Z \leq \frac{\pi}{2}$ then which of the following in equality is true.

- 1) $|z - \bar{z}| \leq |z| |\arg z - \arg \bar{z}|$ 2) $|z - \bar{z}| \leq |\arg z - \arg \bar{z}|$
 3) $|z - \bar{z}| > |z| |\arg z - \arg \bar{z}|$ 4) $|z - \bar{z}| = 1$

Key (1)

Sol $z = re^{i\theta}, \bar{z} = re^{-i\theta}$

$$\left| \frac{z}{r} - \frac{\bar{z}}{r} \right| = |2i \sin \theta| \leq |\arg z - \arg \bar{z}|$$

$$\Rightarrow |z - \bar{z}| \leq |z| |\arg z - \arg \bar{z}|$$

34). If $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$ then Modulus and argument of $(1 + \cos 2\alpha) + i \sin 2\alpha$ is

- 1) $-2 \sin \alpha, \frac{\pi}{6}$ 2) $-2 \cos \alpha, \alpha - \pi$ 3) $-2 \sin \alpha, \alpha - \pi$ 4) None of these

Key (2)

Sol Let $z = (1 + \cos 2\alpha) + i \sin 2\alpha$

$$= 2 \cos^2 \alpha + 2i \sin \alpha \cos \alpha$$

$$= 2 \cos \alpha [\cos \alpha + i \sin \alpha]$$

$$= -2 \cos \alpha [-\cos \alpha - i \sin \alpha]$$

$$= -2 \cos \alpha [\cos(\alpha - \pi) + i \sin(\alpha - \pi)] \quad \left[\because \frac{\pi}{2} < \alpha < \frac{3\pi}{2} \right]$$

$$\therefore |z| = -2 \cos \alpha \text{ and } \arg z = \alpha - \pi$$

36). Let a complex number z , with minimum argument is lying on the curve $|z + 4i| = 2$ then

$$\left| \arg(z) + \sum_{r=0}^{17} \left(\frac{z}{2\sqrt{3}} \right)^r \right| =$$

1) $5 - \frac{\pi}{3}$

2) $3 - \frac{\pi}{3}$

3) $\frac{\pi}{3}$

4) $\frac{2\pi}{3}$

Key (4)

Sol Clearly $z = -\sqrt{3} - 3i \Rightarrow \frac{z}{2\sqrt{3}} = -\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

37). If $z = (1 + \sqrt{3}i)^{10} + (1 - \sqrt{3}i)^{10}$, then $\arg z$ is

1) $\frac{\pi}{2}$

2) π 4) $\frac{\pi}{4}$

(D)

none of these

Key (2)

Sol $z = a + \bar{a}$

= always real

$$\Rightarrow \arg z = 0 \text{ or } \pi.$$

38). If $z = \cos \alpha + i \sin \alpha$, $0 < \alpha < \pi/6$ then the argument of $\frac{z^4 - 1}{z^3 + 1}$ is

1) $\frac{\pi}{2} + \frac{\alpha}{2}$

2) $\frac{\pi}{2} - \frac{\alpha}{2}$

3) $\frac{3\alpha}{2}$

4) $2\alpha - \frac{\pi}{2}$

Key (1)

Sol $\arg\left(\frac{z^4 - 1}{z^3 + 1}\right) = \arg(z^4 - 1) - \arg(z^3 + 1) = \left(2\alpha - \frac{\pi}{2} + \pi\right) - \frac{3\alpha}{2} = \frac{\pi + \alpha}{2}$

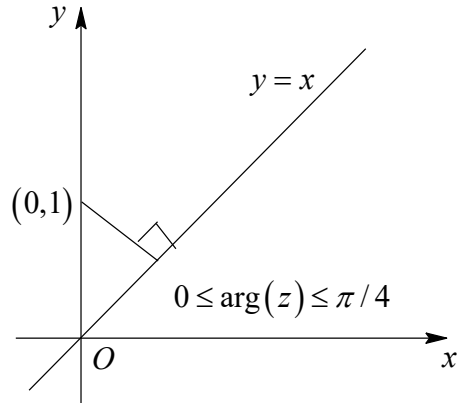
39). If $0 \leq \arg(z) \leq \pi/4$, then the least value of $\sqrt{2}|z - i|$ is

Key (1)

Sol $0 \leq \arg(z) \leq \pi/4$ represents the region of the complex plane lying in the first quadrant and bounded by the x -axis and the line $y = x$.

The least value of $|z-i|$ is the length of perpendicular from $(0,1)$ to $y=x$, which is $\frac{1}{\sqrt{2}}$.

Thus, least value of $\sqrt{2}|z-i|=1$.



MAXIMA AND MINIMA

45). If $|z_1|=15$ and $|z_2-3-4i|=5$, then

1) $|z_1 - z_2|_{\min} = 5$

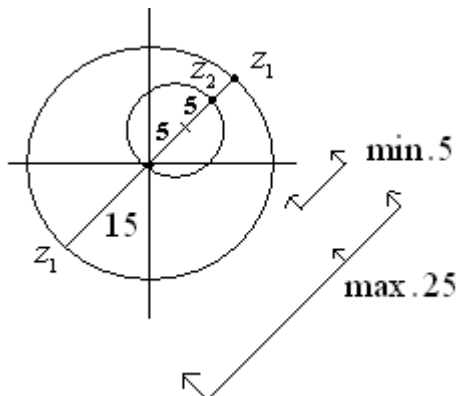
2) $|z_1 - z_2|_{\min} = 10$

3) $|z_1 - z_2|_{\max} = 20$

4) $|z_1 - z_2|_{\max} = 25$

Key: (1, 4)

Sol:



Min. value of $|z_1 - z_2| = 5$

Max. value of $|z_1 - z_2| = 25$

46). For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$,

the minimum value of $|z_1 - z_2|$ is

1) 0

2) 2

3) 7

4) 17

Key: (2)

Sol: $|z_1| = 12 \Rightarrow z_1$ lies on circle with centre c_1 at origin and radius 12

$|z_2 - 3 - 4i| = 5 \Rightarrow z_2$ lies on the circle with centre $c_2 (3 + 4i)$ and radius 5.

$\therefore |z_1 - z_2|$ will be minimum.

If z_1 and z_2 lies on the line joining c_1 and c_2 i.e on the line $z = 3 + 4i$

Minimum value of $|z_1 - z_2| = AB$

$$= c_1 B - c_1 A$$

$$= 12 - 10 = 2$$

47). If $|z_1 - 1| < 1, |z_2 - 2| < 2$ and $|z_3 - 3| < 3$, then $|z_1 + z_2 + z_3|$

1) is less than 6

2) is greater than 6

3) is less than 12

4) lies between 6 and 12

Key: (3)

Sol: We have $|z_1 + z_2 + z_3 - 6 + 6| \leq |z_1 + z_2 + z_3 - 6| + 6$

$$= |(z_1 - 1) + (z_2 - 2) + (z_3 - 3)| + 6$$

$$\leq |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$$

$$< 1 + 2 + 3 + 6 = 12.$$

48). If $\left|z - \frac{4}{z}\right| = 2$, then the greatest value of $|z|$ is

1) $1 + \sqrt{2}$

2) $2 + \sqrt{2}$

3) $\sqrt{3} + 1$

4) $\sqrt{5} + 1$

Key: (4)

Sol: We have $|z| = \left| z - \frac{4}{z} + \frac{4}{z} \right| \leq \left| z - \frac{4}{z} \right| + \frac{4}{|z|}$

$$\Rightarrow |z| \leq 2 + \frac{4}{|z|}$$

$$\Rightarrow |z|^2 \leq 2|z| + 4 \Rightarrow (|z| - 1)^2 \leq 5$$

$$\Rightarrow |z| - 1 \leq \sqrt{5} \Rightarrow |z| \leq \sqrt{5} + 1$$

Also, for $z = \sqrt{5} + 1$,

$$\left| z - \frac{4}{z} \right| = 2.$$

Therefore, the greatest value of $|z|$ is $\sqrt{5} + 1$ which is attained when $z = \sqrt{5} + 1$.

49). If $|Z + 2 - i| = 5$ then maximum value of $|3Z + 9 - 7i| =$ _____

1) 20

2) 15

3) 5

4) 16

Key: (1,)

Sol: $|3Z + 9 - 7i| = |3Z + 6 - 3i + 3 - 4i|$

$$\leq |3(z + 2 - i)| = |3 - 4i|$$

$$\leq 3|z + 2 - i| + \sqrt{3^2 + 4^2}$$

$$3(5) + 5 = 20$$

50) If Z is a non-zero complex number such that $\left| Z^3 + \frac{1}{Z^3} \right| \leq 2$, then the

maximum value of $\left| Z + \frac{1}{Z} \right|$ is

1) 4

2) $\frac{1}{2}$

3) $2\sqrt{2}$

4) 2

Key: (4)

Sol: $\left(Z + \frac{1}{Z} \right)^3 = Z^3 + \frac{1}{Z^3} + 3\left(Z + \frac{1}{Z} \right)$

$$\left| Z + \frac{1}{Z} \right|^3 \leq \left| Z^3 + \frac{1}{Z^3} \right| + 3\left| Z + \frac{1}{Z} \right|$$

$$\left|Z + \frac{1}{Z}\right|^3 \leq 2 + 3\left|Z + \frac{1}{Z}\right|$$

$$\left|Z + \frac{1}{Z}\right|^3 - 3\left|Z + \frac{1}{Z}\right| - 2 \leq 0$$

$$a^3 - 3a - 2 \leq 0 \text{ where } a = \left|Z + \frac{1}{Z}\right| \geq 0.$$

$$(a-2)(a^2 + 2a + 1) \leq 0$$

$$(a-2)(a+1)^2 \leq 0$$

$$a \leq 2.$$

51). If $|z - 3 + 2i| \leq 4$, then the absolute difference between the maximum and minimum values of $|z|$ is

1) $2\sqrt{11}$

2) $3\sqrt{11}$

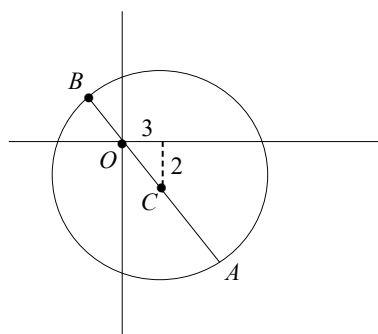
3) $2\sqrt{13}$

4) $4 + \sqrt{13}$

Key: (4)

Sol:

$C = 3 - 2i$ be the center of the circle $|z - 3 + 2i| = 4$. Join the origin to C and let it meet the circle in A and B (see figure).



Least value of $|z| = OB$

$$= CB - OC$$

$$= 4 - \sqrt{3^2 + 2^2}$$

$$= 4 - \sqrt{13}$$

Maximum value of $|z| = OA = 4 + \sqrt{13}$.

The absolute difference between the maximum and minimum values of $|z|$ is $2\sqrt{13}$.

53). If $|z - 25i| \leq 15$ then $|\text{maximum arg } z - \text{minimum arg } z|$ equals

- 1) $2 \cos^{-1} \frac{3}{5}$ 2) $2 \cos^{-1} \frac{4}{5}$ 3) $\frac{\pi}{2} + \cos^{-1} \frac{3}{5}$ 4) $\sin^{-1} \frac{3}{5} - \cos^{-1} \frac{3}{5}$

Key: (2)

Sol:

If $|z - 25i| \leq 15$ then z lies either in the interior and or on the boundary of the circle with centre at $C(0, 25)$ and radius equal to 15.

The least argument is for point A and greatest argument is for point B

from right $\triangle OAC$, $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{C}{c}$

$$\frac{\pi}{2} - \theta = \cos^{-1}\left(\frac{4}{5}\right)$$

Now for $|z - 25i| \leq 15$

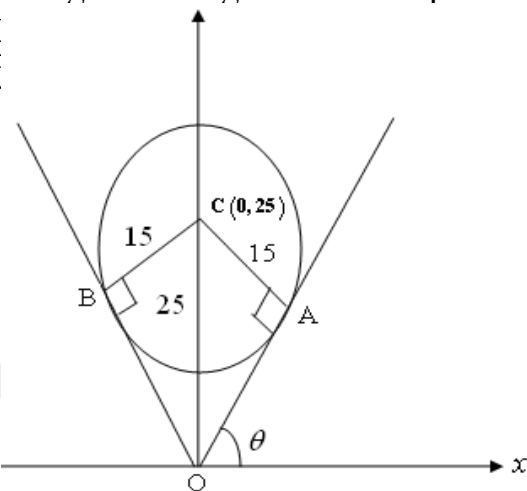
$|\text{Maximum arg } z - \text{Minimum arg } z|$

$$= \angle BOA$$

$$= \angle BOX - \angle AOX = \frac{\pi}{2} + \frac{\pi}{2} - \theta - \theta$$

$$= \pi - 2\theta$$

$$= 2 \cos^{-1} \frac{4}{5}$$



54). If Z_1, Z_2 are two complex numbers such that $|Z_1| = 1, |Z_2| = 1$ then

the maximum value of $|Z_1 + Z_2| + |Z_1 - Z_2|$ is

- 1) 2 2) $2\sqrt{2}$ 3) 4 4) none of these

Key: (2)

Sol:

$$Z_1 = \cos \alpha + i \sin \alpha, \quad Z_2 = \cos \beta + i \sin \beta$$

$$|Z_1 + Z_2| + |Z_1 - Z_2| = \sqrt{2 + 2 \cos(\alpha - \beta)} + \sqrt{2 - 2 \cos(\alpha - \beta)}$$

let $\alpha - \beta = \theta$

$$2 \cos \frac{\theta}{2} + 2 \sin \frac{\theta}{2} = 2\sqrt{2} \sin \left(\frac{\pi}{4} + \frac{\theta}{2} \right)$$

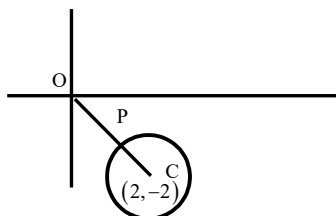
55). If z is a complex number having least absolute value and $|z - 2 + 2i| = 1$ then $z =$

- 1) $\left(2 - \frac{1}{\sqrt{2}}\right)(1-i)$ 2) $\left(2 - \frac{1}{\sqrt{2}}\right)(1+i)$
- 3) $\left(2 + \frac{1}{\sqrt{2}}\right)(1-i)$ 4) $\left(2 + \frac{1}{\sqrt{2}}\right)(1+i)$

Key: (1)

Sol:

$$OP = OC - CP$$



$$= 2\sqrt{2} - 1$$

$$\therefore (0,0) (2,-2)$$

$$2\sqrt{2} - 1 : 1$$

$$\frac{2(2\sqrt{2} - 1)}{2\sqrt{2}}, \frac{-2(2\sqrt{2} - 1)}{2\sqrt{2}}$$

$$= \left(\left(2 - \frac{1}{\sqrt{2}}\right), -\left(2 - \frac{1}{\sqrt{2}}\right) \right)$$

CUBIC ROOT OF UNITY

58). If $2w + 1 = \sqrt{3}i$, then the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 - w^2 & w^2 \\ 1 & w^2 & w^4 \end{vmatrix} \text{ Is}$$

1) $3w$

2) $3w(w-1)$

3) $3w^2$

4) $3w(1-w)$

Key (2)

Sol $w = \frac{-1}{2} + \frac{i\sqrt{3}}{2}, w^3 = 1, \text{ and } \Delta = 3w - 3w^2$

59). If w is complex cube root of 1, then

$$\cos\left(\left((1-w)(1-w^2) + (2-w)(2-w^2) + \dots + (10-w)(10-w^2)\right)\frac{\pi}{900}\right) =$$

- 1) -1 2) 0 3) 1 4) $\frac{\sqrt{3}}{2}$

Key (2)

Sol $\sum_{r=1}^{10} (r-w)(r-w^2) = 450$ since $w + w^2 = -1, w^3 = 1$

60). If a, b, c are distinct integers and $\omega \neq 1$ is a cube root of unity then minimum value of

$$x = \left|a + b\omega + c\omega^2\right| + \left|a + b\omega^2 + c\omega\right| \text{ is}$$

- 1) $2\sqrt{3}$ 2) 3 3) $4\sqrt{2}$ 4) 2

Key (1)

Sol Let $z = a + b\omega + c\omega^2$

$$\text{Then } |z|^2 = z\bar{z} = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

$$= a^2 + b^2 + c^2 - bc - ca - ab$$

$$= \frac{1}{2}[(b-c)^2 + (c-a)^2 + (a-b)^2]$$

$$x = |z| + |\bar{z}| = 2|z|$$

$$= \sqrt{2}[(b-c)^2 + (c-a)^2 + (a-b)^2]^{1/2}.$$

61). If a, b, c are integers not all equal and ω is a cube root of unity ($\omega \neq 1$) then the minimum value of $|a + b\omega + c\omega^2|$ is

- 1) 2 2) $a+b+c$ 3) 1 4) 0

Key (3)

Sol

$$\begin{aligned}
 \text{Consider } |a+b\omega+c\omega^2|^2 &= (a+b\omega+c\omega^2)(\overline{a+b\omega+c\omega^2}) \\
 &= (a+b\omega+c\omega^2)(a+b\bar{\omega}+c\bar{\omega}^2) \\
 &= (a+b\omega+c\omega^2)(a+b\omega^2+c\omega) \\
 &= a^2+b^2+c^2-ab-bc-ca \\
 &= \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}
 \end{aligned}$$

When $a=b=1, c=2$, it gives minimum value (Since a, b, c not all equal)

$$\therefore \text{Minimum value of } (a+b\omega+c\omega^2) = \sqrt{\frac{1}{2}(0+1+1)} = \sqrt{1} = 1$$

62). If $\alpha = -1 + i\sqrt{3}$ and n is a positive integer which is not a multiple of 3, then $\alpha^{2n} + 2^n \alpha^n + 2^{2n} =$

- 1) 1 2) -1 3) 0 4) α^2

Key (3)

Sol

$$\begin{aligned}
 \text{We have } \alpha^{2n} + 2^n \alpha^n + 2^{2n} &= 2^{2n} \left[\left(\frac{\alpha}{2} \right)^{2n} + \left(\frac{\alpha}{2} \right)^n + 1 \right] \\
 &= 2^{2n} (w^{2n} + w^n + 1) \left[\because \frac{\alpha}{2} = \frac{-1 + i\sqrt{3}}{2} \text{ and } \left(\frac{\alpha}{2} \right)^3 = 1 \right] \\
 &= 2^{2n} (0) = 0. \text{ (since 3 does not divide } n \text{).}
 \end{aligned}$$

63). If $\omega \neq 1$ be a cube root of unity and $(1+\omega)^7 = l + m\omega$, then the value of $l + m =$

- 1) 0 2) 1 3) 2 4) -1

Key (3)

Sol

$$\because \omega \text{ is cube root of unity} \quad \therefore 1 + \omega + \omega^2 = 0$$

$$\Rightarrow 1 + \omega = -\omega^2$$

$$\text{Now if } (1 + \omega)^7 = l + m\omega$$

$$\Rightarrow (-\omega^2)^7 = l + m\omega \quad \Rightarrow -\omega^{12} \cdot \omega^2 = l + m\omega$$

$$\Rightarrow -(\omega^3)^4 \omega^2 = l + m\omega$$

$$\Rightarrow -\omega^2 = l + m\omega$$

$$\Rightarrow 1 + \omega = l + m\omega$$

can comparison $l = 1, m = 1$

66). If $x^2 + x + 1 = 0$ then the value of $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \dots + \left(x^{27} + \frac{1}{x^{27}}\right)^2$ is

1) 27

2) 72

3) 45

4) 54

Key (4)

Sol $x^2 + x + 1 = 0 \Rightarrow x = w \text{ or } w^2$

$$\text{Let } x = w \text{ then } x + \frac{1}{x} = w + \frac{1}{w} = w + w^2 = -1$$

$$x^2 + \frac{1}{x^2} = w^2 + \frac{1}{w^2} = w^2 + w = -1$$

$$x^3 + \frac{1}{x^3} = w^3 + \frac{1}{w^3} = 1 + 1 = 2$$

$$x^4 + \frac{1}{x^4} = w^4 + \frac{1}{w^4} = w + \frac{1}{w} = -1 \text{ etc.}$$

$$\therefore \left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \dots + \left(x^{27} + \frac{1}{x^{27}}\right)^2$$

$$= 18 + 9(2)^2 = 54$$

67). If ω is a non-real cube root of unity and $f(x) = \sum_{k=0}^8 A_k x^k$.

If $f(x) + f(\omega x) + f(\omega^2 x) = n(A_0 + A_n x^n + A_{2n} x^{2n})$ then 'n' must be equal to

Key (3)

Sol $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots + A_8 x^8$

$$f(\omega x) = A_0 + A_1 \omega x + A_2 \omega^2 x^2 + A_3 x^3 + \dots + A_8 \omega^2 x^8$$

$$f(\omega^2 x) = A_0 + A_1 \omega^2 x + A_2 \omega x + A_3 x^3 + \dots$$

$$\overline{f(x) + f(\omega x) + f(\omega^2 x)} = 3(A_0 + A_3 x^3 + A_6 x^6)$$

$$\therefore n = 3$$

70). If $x + \frac{1}{x} = 1$; and $p = x^{4000} + \frac{1}{x^{4000}}$ and q be the digit at unit place in the number 2^{2^n} , $n \in N$ and $n > 1$, then the value of $p + q =$

Key (5)

Sol

$$x + \frac{1}{x} = 1$$

$$x^2 - x + 1 = 0$$

$$\therefore x = -\omega, -\omega^2$$

$$\text{Now for } x = -\omega, p = \omega^{4000} + \frac{1}{\omega^{4000}} = \omega + \frac{1}{\omega} = -1$$

Similarly for $x = -\omega^2$ also $p = -1$

For $n > 1, 2^n = 4k$

$$2^{2^n} = 2^{4k} = (16)^k \text{ is a number with last digit } 6$$

$$\Rightarrow q = 6 + 1 = 7$$

$$\therefore p + q = -1 + 7 = 6$$

71). The number of common roots of the equations $x^5 - x^3 + x^2 - 1 = 0$ and $x^4 - 1 = 0$ is

Key (2)

Sol $x^5 - x^3 + x^2 - 1 = 0$

$$\Rightarrow (x^3 + 1)(x^2 - 1) = 0$$

$$\Rightarrow (x + 1)^2 (x - 1)(x^2 - x + 1) = 0$$

Roots are $-1, -1, 1, -\omega, -\omega^2$

$$x^4 - 1 = 0$$

$$\Rightarrow (x^2 + 1)(x^2 - 1) = 0$$

Roots are $1, -1, i, -i$

$\therefore$ Common roots are 1 and -1 .

72). Sum of common roots of the equation $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is

1) -1

2) 1

3) 0

4) 1

Key (1)

Sol $(z + 1)(z^2 + z + 1)$

$$\Rightarrow z = -1, \omega, \omega^2$$

$$\text{Let } f(z) = z^{1985} + z^{100} + 1$$

$$f(-1) \neq 0, f(\omega) = f(\omega^2) = 0$$

76). The value of $1 + \sum_{k=0}^{12} \left\{ \cos \frac{(2k+1)\pi}{13} + i \sin \frac{(2k+1)\pi}{13} \right\}$ is

- 1) 0 2) -1 3) 1 4) i

Key (3)

Sol Sum of the roots = 0

$$\sum_{k=0}^{12} \left\{ \cos \frac{(2k+1)\pi}{13} + i \sin \frac{(2k+1)\pi}{13} \right\} = 0$$

77). $\sum_{p=1}^{32} (3p+2) \left\{ \sum_{r=1}^{10} \left(\sin \frac{2r\pi}{11} - i \cos \frac{2r\pi}{11} \right) \right\}^p =$

- 1) $48i$ 2) $16(1-i)$ 3) $48(1-i)$ 4) $16(i-1)$

Key (3)

Sol $\sin \frac{2r\pi}{11} - i \cos \frac{2r\pi}{11} = -ie^{\frac{2r\pi}{11}i}$ Hence

$$\sum_{r=1}^{10} \left(\sin \frac{2r\pi}{11} - i \cos \frac{2r\pi}{11} \right) = -i \frac{e^{\frac{2\pi}{11}i} \left(\left(e^{\frac{2\pi i}{11}} \right)^{10} - 1 \right)}{e^{\frac{2\pi i}{11}} - 1}$$

$$= \frac{-i \left(1 - e^{\frac{2\pi i}{11}} \right)}{e^{\frac{2\pi i}{11}} - 1} = i$$

$$\therefore \sum_{p=1}^{32} (3p+2) \left\{ \sum_{r=1}^{10} \left(\sin \frac{2r\pi}{11} - i \cos \frac{2r\pi}{11} \right) \right\}^p = \sum_{p=1}^{32} (3p+2) i^p$$

$$= 3 \sum_{p=1}^{32} p i^p + 2 \sum_{p=1}^{32} i^p$$

$$= 3A + 2B \rightarrow (1)$$

$$A = i + 2i^2 + 3i^3 + \dots + 32i^{32} = \frac{i(1-i^{32})}{(1-i)^2} \frac{32i}{-1-i} = \frac{-32i}{1-i} = 16(1-i)$$

$$B = \sum_{p=1}^{32} i^p = \frac{i(1-i^{32})}{1-i} = 0 \quad \text{Ans: } 48(1-i)$$

78). If $Z_K = \cos \frac{K\pi}{10} + i \sin \frac{K\pi}{10}$ then $z_1 z_2 z_3 z_4$ is equal to

1) -1

2) 1

3) -2

4) 2

Key (1)

Sol Let $z_K = w^K$ where $w = \cos \frac{\pi}{10} + i \sin \frac{\pi}{10}$

$$\therefore z_1 z_2 z_3 z_4 = w.w^2.w^3.w^4$$

$$= w^{10}$$

$$= \cos \frac{10\pi}{10} + i \sin \frac{10\pi}{10} \quad (\because \text{Demoviere's theorem})$$

$$= \cos \pi + i \sin \pi$$

$$= -1$$

79). $\sin \frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} + \frac{1}{3} \sin \frac{3\pi}{3} + \dots \infty =$

1) 0

2) $\frac{\pi}{2}$

3) $\frac{\pi}{4}$

4) $\frac{\pi}{3}$

Key (4)

Sol

$$\text{Let } S = \sin \frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} + \frac{1}{3} \sin \frac{3\pi}{3} + \dots \infty$$

$$C = \cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} + \frac{1}{3} \cos \frac{3\pi}{3} + \dots \infty$$

$$C + iS = \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) + \frac{1}{2} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) + \frac{1}{3} \left(\cos \frac{3\pi}{3} + i \sin \frac{3\pi}{3} \right) + \dots \infty$$

$$\begin{aligned}
&= e^{\frac{i\pi}{3}} + \frac{1}{2} \left(e^{\frac{i\pi}{3}} \right)^2 + \frac{1}{3} \left(e^{\frac{i\pi}{3}} \right)^3 + \dots \infty \\
&= -\log \left(1 - e^{\frac{i\pi}{3}} \right) \\
&= -\log \left(1 - \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right) \\
&= -\log \left(\left(1 - \frac{1}{2} \right) - i \frac{\sqrt{3}}{2} \right) \\
&= -\log \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \\
&= - \left(\log \sqrt{\frac{1}{4} + \frac{3}{4}} + i \operatorname{Arg} \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \right) \quad (\because \log z = \log |z| + i \operatorname{Arg} z) \\
&= - \left(\log 1 + i \tan^{-1} (-\sqrt{3}) \right) \\
&= i \tan^{-1} \sqrt{3} \quad \Rightarrow S = \frac{\pi}{3}
\end{aligned}$$

80). Let $z = e^{i\theta}$ then the value of $\sum_{m=1}^{15} \operatorname{Im}(z^{2m-1})$, when $\theta = 2^\circ$ is

1) $\frac{1}{\sin 2^\circ}$ 2) $\frac{1}{3 \sin 2^\circ}$ 3) $\frac{1}{2 \sin 2^\circ}$ 4) $\frac{1}{4 \sin 2^\circ}$

Key (4)

Sol $\operatorname{Im}(Z^{2m-1}) = \sin(2m-1)\theta$

$$\sum_{m=1}^{15} \sin(2m-1)\theta = \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin 29\theta$$

$$= \frac{\sin \left(\frac{15 \cdot 2\theta}{2} \right)}{\sin \frac{2\theta}{2}} \sin \left(\frac{\theta + 29\theta}{2} \right) = \frac{\sin 15\theta}{\sin \theta} \cdot \sin 15\theta$$

$$= \frac{1}{4 \sin 2^\circ} \quad (\text{when } \theta = 2^\circ)$$

82). Roots of the equation $(z+1)^5 = (z-1)^5$ are

$$1) \pm i \tan\left(\frac{\pi}{5}\right), \pm i \tan\left(\frac{2\pi}{5}\right) \quad 2) \pm i \cot\left(\frac{\pi}{5}\right), \pm i \cot\left(\frac{2\pi}{5}\right)$$

$$3) \pm i \cot\left(\frac{\pi}{5}\right), \pm i \tan\left(\frac{2\pi}{5}\right) \quad 4) \pm i \tan\left(\frac{\pi}{5}\right), \pm i \cot\left(\frac{2\pi}{5}\right)$$

Key (2)

Sol For $z \neq 1$, the given equation can be written as

$$\left(\frac{z+1}{z-1}\right)^5 = 1 = e^{2k\pi i}, k \in I$$

$$\Rightarrow \frac{z+1}{z-1} = e^{2k\pi i/5}, \text{ where } k = -2, -1, 1, 2$$

[For $k=0$, we get $1=-1$]

If we denote this value of z by z_k , then

$$\begin{aligned} z_k &= \frac{e^{2k\pi i/5} + 1}{e^{2k\pi i/5} - 1} = \frac{e^{k\pi i/5} + e^{-k\pi i/5}}{e^{k\pi i/5} - e^{-k\pi i/5}} \\ &= -i \cot\left(\frac{k\pi}{5}\right), k = -2, -1, 1, 2. \end{aligned}$$

$\therefore$ Roots are

$$\pm i \cot\left(\frac{\pi}{5}\right), \pm i \cot\left(\frac{2\pi}{5}\right).$$

83). If z_1, z_2 are two complex numbers satisfying the equation $\left|\frac{z_1+z_2}{z_1-z_2}\right|=1$, then $\frac{z_1}{z_2}$ is a number which is

1) Positive real

(2) Negative real

3) Zero

4) Lying on imaginary axis

Key (4)

Sol $\left|\frac{z_1+z_2}{z_1-z_2}\right|=1 \Rightarrow \frac{z_1+z_2}{z_1-z_2} = \cos \alpha + i \sin \alpha$

where α is the argument of $\frac{z_1+z_2}{z_1-z_2}$. Applying componendo and dividendo, we get

$$\frac{z_1}{z_2} = \frac{1 + \cos \alpha + i \sin \alpha}{-1 + \cos \alpha + i \sin \alpha}$$

$$= \frac{2 \cos\left(\frac{\alpha}{2}\right) \left[\cos\left(\frac{\alpha}{2}\right) + i \sin\left(\frac{\alpha}{2}\right) \right]}{2i \sin\left(\frac{\alpha}{2}\right) \left[\cos\left(\frac{\alpha}{2}\right) + i \sin\left(\frac{\alpha}{2}\right) \right]} = -i \cot\left(\frac{\alpha}{2}\right)$$

Purely imaginary in nature

n^{th} ROOTS OF UNITY

84). If $1, w, w^2, \dots, w^{n-1}$ are n^{th} roots of 1 then value of $\frac{1}{2-w} + \frac{1}{2-w^2} + \dots + \frac{1}{2-w^{n-1}}$

equals to

1) $\frac{1}{2^n - 1}$ 2) $\frac{n(2^n - 1)}{2^n + 1}$ 3) $\frac{(n-2)2^{n-1} + 1}{2^n - 1}$ 4) $\frac{n2^{n-1}}{2^n - 1}$

Key (3)

Sol

$$\frac{z^n - 1}{z - 1} = (z - w)(z - w^2) \dots (z - w^{n-1})$$

Take 'log' both sides and diff w.r. to z both sides and put $z = 2$

85). If $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$ be the n^{th} roots of unity, then value of $\sum_{i=0}^{n-1} \frac{\alpha_i}{(3 - \alpha_i)}$ is equal to

1) $\frac{n}{3^n - 1}$ 2) $\frac{n-1}{3^n - 1}$ 3) $\frac{n+1}{3^n - 1}$ 4) $\frac{n+2}{3^n - 1}$

Key (1)

Sol Let $P = \sum_{i=0}^{n-1} \frac{\alpha_i}{3 - \alpha_i} = - \sum_{i=0}^{n-1} \frac{(3 - \alpha_i) - 3}{(3 - \alpha_i)} = 3 \sum_{i=0}^{n-1} \frac{1}{3 - \alpha_i} - \sum_{i=0}^{n-1} 1 \dots (i)$

$$Z^n - 1 = \prod_{i=0}^{n-1} (Z - \alpha_i) \log(Z^n - 1) = \sum_{i=0}^{n-1} \ln(Z - \alpha_i)$$

Diff. both sides w.r.t Z

$$\frac{nZ^{n-1}}{Z^n - 1} = \sum_{i=0}^{n-1} \frac{1}{Z - \alpha_i} \text{ Put } Z = 3$$

$$\Rightarrow \frac{n3^{n-1}}{3^n - 1} = \sum_{i=0}^{n-1} \frac{1}{3 - \alpha_i}$$

$$P = \frac{3n3^{n-1}}{3^n - 1} - n = \frac{n3^n}{3^n - 1} - n = \frac{n}{3^n - 1}$$

86). If $1, \alpha_1, \alpha_2, \dots, \alpha_{99}$ are roots of $Z^{100} = 1$ then $\sum_{1 \leq i < j \leq 99} \alpha_i \alpha_j$ equal to

1) 0

2) 1

3) 99

4) 100

Key (2)

Sol

$$z^{100} - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_{99}) = 0$$

$$\text{Sum of roots} = 1 + \alpha_1 + \alpha_2 + \dots + \alpha_{99} = 0$$

$$\text{Sum of roots taken two at a time} = 0$$

$$\Rightarrow \sum_{1 \leq i < j \leq 99} \alpha_i \alpha_j + \alpha_1 + \alpha_2 + \dots + \alpha_{99} = 0$$

$$\Rightarrow \sum \alpha_i \alpha_j = 1$$

87). $\sum_{j=1}^{n-1} \frac{1}{1 - e^{\frac{2\pi i j}{n}}} = \frac{n-1}{k}$, find k. ($i = \sqrt{-1}$)

Key (2)

Sol

$$\text{Let } e^{\frac{2\pi i}{n}} = \alpha \text{ then } \sum_{j=1}^{n-1} \frac{1}{1 - e^j} = \frac{1}{1 - \alpha} + \frac{1}{1 - \alpha^2} + \dots + \frac{1}{1 - \alpha^{n-1}}$$

Where α is a n th root of unity ($\alpha, \alpha^2, \alpha^3, \dots, \alpha^{n-1}$) are the roots of

$$\frac{x^n - 1}{x - 1} = (x - \alpha)(x - \alpha^2) \dots (x - \alpha^{n-1})$$

Taking log on both side

$$\log \frac{x^n - 1}{x - 1} = \log(x - \alpha) + \log(x - \alpha^2) + \dots + \log(x - \alpha^{n-1})$$

Diff w.r.t. x and use $\lim_{x \rightarrow 1}$

$$\Rightarrow \frac{n-1}{2} = \frac{1}{1 - \alpha} + \frac{1}{1 - \alpha^2} + \dots + \frac{1}{1 - \alpha^{n-1}}$$

88). If α is non real root of $x^7 = 1$, then $1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + \dots + 13\alpha^6$ is equal to

1) 0

2) $\frac{14}{1 - \alpha}$

3) $\frac{14}{\alpha - 1}$

4) none of these

Key (3)

Sol Let $A = 1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + \dots + 11\alpha^5 + 13\alpha^6$
 $\alpha A = \alpha + 3\alpha^2 + 5\alpha^3 + 7\alpha^4 + \dots + 11\alpha^6 + 13\alpha^7$
 $(1 - \alpha) A = 1 + 2\alpha + 2\alpha^2 + 2\alpha^3 + \dots + 2\alpha^6 - 13\alpha^7$
 $= -12 + 2[\alpha + \alpha^2 + \dots + \alpha^6] = -14$
 $A = -\frac{14}{1-\alpha}$

89). If $\beta \neq 1$ be any n^{th} root of unity then $1 + 3\beta + 5\beta^2 + \dots + (2n-1)\beta^{n-1}$ terms equals

1) $\frac{2n}{1-\beta}$ 2) $\frac{-2n}{1-\beta}$ 3) $-\frac{2n}{(1-\beta)^2}$ 4) $\frac{2n}{(1-\beta)^2}$

Key (2)

Sol $\beta = (1)^{1/n} \Rightarrow \beta^n = 1$ (1)

If S be the sum of the given series which is arithmetic geometric series, then

$$S = 1 + 3\beta + 5\beta^2 + \dots + (2n-1)\beta^{n-1}$$

$$\beta S = \beta + 3\beta^2 + \dots + (2n-3)\beta^{n-1} + (2n-1)\beta^n$$

$$S(1-\beta) = 1 + [2\beta + 2\beta^2 + \dots + (n-1)\text{terms}] - (2n-1)$$

$$= 2 + 2\beta + 2\beta^2 + \dots + n \text{ terms} - 2n$$

$$= 2 \frac{(1-\beta^n)}{1-\beta} - 2n = 0 - 2n, \quad \therefore S = \frac{-2n}{1-\beta}$$

90). If α is the n^{th} root of 1 their $1 + 2\alpha + 3\alpha^2 + \dots$ to n terms equal to

1) $\frac{-n}{(1-\alpha)^2}$ 2) $\frac{-n}{1-\alpha}$ 3) $\frac{-2n}{1-\alpha}$ 4) $\frac{-2n}{(1-\alpha)^2}$

Key (2)

Sol $s = 1 + 2\alpha + 3\alpha^2 + \dots + n\alpha^{n-1}$

$$\alpha s = \alpha + 2\alpha^2 + 3\alpha^3 + \dots + n\alpha^n$$

$$(1-\alpha)s = 1 + (\alpha + \alpha^2 + \dots + \alpha^{n-1}) - n\alpha^n$$

$$= 1 + \frac{\alpha(1-\alpha^{n-1})}{1-\alpha} - n\alpha^n$$

$$s = \frac{1}{1-\alpha} + \frac{\alpha-1}{(1-\alpha)^2} - \frac{n}{1-\alpha} = \frac{-n}{1-\alpha}$$

91). If α is non real fifth root of 1 then $\log_2 \left| 1 + \alpha + \alpha^2 + \alpha^3 - \frac{1}{\alpha} \right|$

1) 0

2) $\frac{1}{2}$

3) 1

4) 2

Key (3)

Sol $\log_2 \left| (\alpha + \alpha^2 + \alpha^3 + 1) - \alpha^4 \right|$

$$\log_2 \left| -2\alpha^4 \right| = \log_2^2 = 1$$

92). If α is any of 7th roots of unity, then $\alpha = \frac{\text{cis } 2K\pi}{7}$ (K= 0 to 6) and $\sum_{i=1}^6 \alpha^i = -1$ ($\alpha \neq 1$) and

$\alpha^7 = 1$ Answer the following

$$\sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7} = \sin \frac{4\pi}{7} \sin \frac{5\pi}{7} \sin \frac{6\pi}{7} =$$

1) $\frac{\sqrt{7}}{16}$

2) $\frac{\sqrt{7}}{8}$

3) $\frac{\sqrt{7}}{32}$

4) $\frac{\sqrt{7}}{64}$

Key (2)

Sol Let $\alpha = \text{cis } \frac{2\pi}{7}$ then $a = \alpha + \alpha^2 + \alpha^4, b = \alpha^3 + \alpha^5 + \alpha^6$

$$\Rightarrow a + b = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = -1$$

$$\& ab = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6) = 2$$

$$\therefore \text{Equation whose roots are a,b is } x^2 - x(-1) + 2 = 0$$

$$f(x) + f(\alpha x) + f(\alpha^2 x) + f(\alpha^3 x) + \dots + f(\alpha^6 x)$$

93). If ω is a tenth root of unity then $\log_2 \left| 1 + \omega + \omega^2 + \omega^3 + \dots + \omega^8 - \omega^{-1} \right|$ is equal to

Key (1)

Sol $\omega^{10} = 1$

$$1 + \omega + \omega^2 + \dots + \omega^9 = 0 \text{ and } |\omega| = 1$$

$$\log_2 \left| \frac{-2}{\omega} \right| = \log_2 \left| \frac{-2}{\omega} \right| = \log_2 (2) = 1.$$

94). If ω is a tenth root of unity then $\log_2 \left| 1 + \omega + \omega^2 + \omega^3 + \dots + \omega^8 - \omega^{-1} \right|$ is equal to

Key (1)

Sol $\omega^{10} = 1$

$$1 + \omega + \omega^2 + \dots + \omega^9 = 0 \text{ and } |\omega| = 1$$

$$\log_2 \left| \frac{-2}{\omega} \right| = \log_2 \left| \frac{-2}{\omega} \right| = \log_2 (2) = 1.$$

95). If α is non real fifth root of 1 then $\log_2 \left| 1 + \alpha + \alpha^2 + \alpha^3 - \frac{1}{\alpha} \right|$

- 1) 0 2) $\frac{1}{2}$ 3) 1 4) 2

Key (3)

Sol $\log_2 \left| (\alpha + \alpha^2 + \alpha^3 + 1) - \alpha^4 \right|$

$$\log_2 \left| -2\alpha^4 \right| = \log_2^2 = 1$$

96). If α is the non real fifth root of unity, then

- 1) $\left| 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 \right| = 0$ 2) $\left| 1 + \alpha + \alpha^2 + \alpha^3 \right| = 1$
 3) $\left| 1 + \alpha + \alpha^2 \right| = 2 \cos \frac{\pi}{5}$ 4) $\left| 1 + \alpha \right| = 2 \cos \frac{\pi}{10}$

Key (1,2,3)

Sol

We have $\alpha = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$

and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

$\therefore |1 + \alpha + \alpha^2 + \alpha^3| = |-\alpha^4| = |\alpha^4| = 1$

Also,

$$|1 + \alpha + \alpha^2| = |-\alpha^3 (1 + \alpha)| = |1 + \alpha| \quad (i)$$

$$= \left| 1 + \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} \right|$$

$$= \left| 2 \cos \frac{\pi}{5} \left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right) \right|$$

$$= 2 \cos \frac{\pi}{5}$$

Again, from (i), $|1 + \alpha| = |1 + \alpha + \alpha^2| = 2 \cos \frac{\pi}{5}$