

EVEN AND ODD FUNCTIONS

1. If $I = \int_{-\pi/6}^{\pi/6} \frac{\pi + 4x^5}{1 - \sin\left(|x| + \frac{\pi}{6}\right)} dx$, then I equals to

- 1) 4π 2) $2\pi + \frac{1}{\sqrt{3}}$ 3) $2\pi - \sqrt{3}$ 4) $4\pi + \sqrt{3} - \frac{1}{\sqrt{3}}$

Key (1)

Sol As $\frac{4x^5}{1 - \sin\left(|x| + \frac{\pi}{6}\right)}$ is an odd function, and $\frac{\pi}{1 - \sin\left(|x| + \frac{\pi}{6}\right)}$ is an even

function, we get

$$I = 2\pi \int_0^{\pi/6} \frac{dx}{1 - \sin\left(x + \frac{\pi}{6}\right)}$$

$$\text{Put } x + \frac{\pi}{6} = t$$

$$= 2\pi \int_{\pi/6}^{\pi/3} \frac{dt}{1 - \sin t} = 4\pi$$

2. $\int_{-2\pi}^{2\pi} \frac{\sin^6 x}{(\sin^6 x + \cos^6 x)(1 + e^{-x})} dx =$

- 1) 2π 2) π 3) $\pi/2$ 4) 4π

Key (2)

Sol $\therefore \int_{-a}^a f(x) dx = \int_0^a (f(x) + f(-x)) dx$

3. $\int_{-2}^2 \frac{x^4 \tan^{-1} x + \sin x - 3x^2}{3 - |x|} dx$

- 1) $48 + 54\ln 3$ 2) $48 - 54\ln 3$ 3) $36 + 28\ln 2$ 4) 0

Key (2)

Sol $I = 3 \int_{-2}^2 \frac{x^2}{|x| - 3} dx = 6 \int_0^2 \frac{x^2}{x^2 - 3} dx$

4. $\int_{-\pi/4}^{\pi/4} \frac{e^x \cdot \sec^2 x dx}{e^{2x} - 1}$ is equal to

1) 0

2) 2

3) e

4) 2e

Key (1)

Sol : Let $I = \int_{-\pi/4}^{\pi/4} \frac{e^x \sec^2 x dx}{e^{2x} - 1}$

If $f(x) = \frac{e^x \sec^2 x}{e^{2x} - 1}$

$\therefore f(-x) = \frac{e^x \sec^2 x}{e^{2x} - 1}$

$$= \frac{e^x \sec^2 x}{1 - e^{2x}}$$

$$= -\frac{e^x \sec^2 x}{e^{2x} - 1}$$

$$= -f(x)$$

$\therefore I = 0$ ($f(x)$ is odd function)

5. $\int_{\ln \lambda}^{\ln \left(\frac{1}{\lambda}\right)} \frac{f\left(\frac{x^2}{3}\right)(f(x) + f(-x))}{g(3x^2)(g(x) - g(-x))} dx =$ where $\lambda > 1$.

1) 0

2) 1

3) λ

4) $\frac{1}{\lambda}$

Key (1)

Sol odd function

6. $\int_{-\pi/2}^{\pi/2} \frac{\cos 6x + 6 \cos 4x + 15 \cos 2x + 10}{\cos 5x + 5 \cos 3x + 10 \cos x} dx =$ _____

Key (4)

Sol
$$\begin{aligned} & \cos 6x + 6 \cos 4x + 15 \cos 2x + 10 \\ &= \cos 6x + \cos 4x + 5 \cos 4x + \cos 2x + 10 \cdot 1 + \cos 2x \\ &= 2 \cos x \cos 5x + 5 \cos 3x + 10 \cos x \end{aligned}$$

$$\therefore I = \int_{-\pi/2}^{\pi/2} 2 \cos x dx = 4$$

8. $\int_{-\pi/2}^{+\pi/2} \frac{\cos x}{1+e^x} dx = \underline{\hspace{2cm}}$

Key (1)

Sol

$$\begin{aligned} \int_{-\pi/2}^{+\pi/2} \frac{\cos x}{1+e^x} dx &= \int_0^{\pi/2} \left(\frac{\cos x}{1+e^x} + \frac{\cos(-x)}{1+e^{-x}} \right) dx \\ &= \int_0^{\pi/2} \cos x \left(\frac{1+e^x}{1+e^x} \right) dx \\ &= (\sin x)_0^{\pi/2} = 1 \end{aligned}$$

The value of $\int_{-\pi/4}^{\pi/4} [(x^9 - 3x^5 + 7x^3 - x + 1) / \cos^2 x] dx$ is _____

Key (2)

Sol

$$\begin{aligned} f(x) &= \frac{x^9 - 3x^5 + 7x^3 - x}{\cos^2 x} + \sec^2 x \\ &= \sec^2 x (x^9 - 3x^5 + 7x^3 - x) + \sec^2 x \\ \Rightarrow \int_{-\pi/4}^{\pi/4} f(x) dx &= \int_{-\pi/4}^{\pi/4} \sec^2 x dx \\ &= 2 \int_0^{\pi/4} \sec^2 x dx \\ &= 2 \tan x \Big|_0^{\pi/4} = 2 \end{aligned}$$

17. The function $F(x) = \int_0^x \log(t + \sqrt{1+t^2}) dt$ is

- 1) Periodic function 2) an even function 3) an odd function 4) even or odd

Key (2)

Sol Solution we know that if $f(t)$ is an odd function, then

$$F(x) = \int_0^x f(t) dt \text{ is an even function}$$

Since $\log(t + \sqrt{1+t^2})$ is odd,

$\therefore$ then given integral is even

$$18. \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx =$$

$$1) \frac{\pi^2}{4}$$

$$2) \pi^2$$

$$3) 0$$

$$4) \frac{\pi}{2}$$

Key (2)

Sol Given $= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x} dx$

$$= 4 \int_{-\pi}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx +$$

$$\because \frac{2x}{1+\cos^2 x} \text{ is odd and } \frac{x \sin x}{1+\cos^2 x} \text{ is even}$$

$$\text{Apply } \int_0^{2\pi} f(x) dx = 2 \int_0^{\pi} f(x) dx \text{ if } f(2a-x) = f(x) \text{ and}$$

Put $\cos x = t$

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$10. \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5) dx = \underline{\hspace{2cm}}$$

$$1) 0$$

$$2) 2$$

$$3) -2$$

$$4) 63$$

Key (1)

Sol Use $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$13. \text{ The value of definite integral } \int_2^3 \left[\sqrt{2x - \sqrt{5(4x-5)}} + \sqrt{2x + \sqrt{5(4x-5)}} \right] dx \text{ is equal to}$$

$$1) 4\sqrt{3} - \frac{2\sqrt{2}}{3} \quad 2) 4\sqrt{2} \quad 3) 4\sqrt{3} - \frac{4}{3} \quad 4) \frac{\sqrt{10}}{2} + \frac{7\sqrt{7} - 5\sqrt{5}}{3\sqrt{2}}$$

Key (4)

Sol $5(4x-5) = t^2 \Rightarrow 20x-25 = t^2 \Rightarrow x = \frac{t^2+25}{20}$

Also $20 dx = 2t dt$ or $dx = \frac{t}{10} dt$

$$\begin{aligned} \therefore I &= \int_{\sqrt{15}}^{\sqrt{35}} \left(\sqrt{\frac{t^2+25}{10}} - t + \sqrt{\frac{t^2+25}{10}} + t \right) \frac{t}{10} dt = \int_{\sqrt{15}}^{\sqrt{35}} \left(\frac{|t-5|}{\sqrt{10}} + \frac{|t+5|}{\sqrt{10}} \right) \frac{t}{10} dt \\ &= \int_{\sqrt{15}}^5 \left(\frac{-t+5}{\sqrt{10}} + \frac{t+5}{\sqrt{10}} \right) \frac{t}{10} dt + \int_5^{\sqrt{35}} \left(\frac{t-5}{\sqrt{10}} + \frac{t+5}{\sqrt{10}} \right) \frac{t}{10} dt = \frac{1}{\sqrt{10}} \left[\frac{t^2}{2} \right]_{\sqrt{15}}^5 + \frac{1}{5\sqrt{10}} \int_5^{\sqrt{35}} t^2 dt \\ &= \frac{1}{\sqrt{10}} \left[\frac{25}{2} - \frac{15}{2} \right] + \frac{1}{15\sqrt{10}} \left[(\sqrt{35})^2 - 125 \right] = \frac{\sqrt{10}}{2} + \frac{7\sqrt{7} - 5\sqrt{5}}{3\sqrt{2}} \end{aligned}$$

15. The value of $\int_3^6 \left(\sqrt{x + \sqrt{12x-36}} + \sqrt{x - \sqrt{12x-36}} \right) dx$ is equal to

$$1) 6\sqrt{3} \quad 2) 4\sqrt{3} \quad 3) 12\sqrt{3} \quad 4) 2\sqrt{3}$$

Key (1)

Sol $\int_0^3 \left(\sqrt{(x+3) + 2\sqrt{3}\sqrt{x}} + \sqrt{(x+3) - 2\sqrt{3}\sqrt{x}} \right) dx$

$$\int_0^3 \left((\sqrt{x} + \sqrt{3}) + (\sqrt{3} - \sqrt{x}) \right) dx = \int_0^3 2\sqrt{3} dx = 6\sqrt{3}$$

16. Value of $\int_1^5 \left(\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} \right) dx$ is

$$1) \frac{8}{3} \quad 2) \frac{16}{3} \quad 3) \frac{32}{3} \quad 4) \frac{34}{3}$$

Key (4)

Sol $\because \sqrt{x + (2\sqrt{x-1})} = \sqrt{(\sqrt{x-1})^2 + 1^2 + 2\sqrt{x-1}}$

$$= \sqrt{(\sqrt{x-1} + 1)^2} = \sqrt{x-1} + 1$$

And $\sqrt{x - 2\sqrt{x-1}} = \sqrt{(\sqrt{x-1})^2 + 1^2 - 2\sqrt{x-1}}$

$$= \left| \sqrt{(x-1)} - 1 \right|$$

Then
$$\begin{aligned} & \int_1^5 \sqrt{x+2}\sqrt{(x-1)} + \sqrt{x-2}\sqrt{(x-1)} dx \\ &= \int_1^5 (\sqrt{(x-1)} + 1) + \int_1^5 |\sqrt{(x-1)} - 1| dx \\ &= \int_1^5 (\sqrt{(x-1)} + 1) dx + \int_1^2 (1 - \sqrt{(x-1)}) dx + \int_2^5 (\sqrt{(x-1)} - 1) dx \\ &= \int_0^4 (\sqrt{x} + 1) dx + \int_0^1 (1 - \sqrt{x}) dx + \int_1^4 (\sqrt{x} - 1) dx \\ &= \left[\frac{2}{3} x^{3/2} + x \right]_0^4 + \left[x - \frac{2}{3} x^{3/2} \right]_0^1 + \left[\frac{2}{3} x^{3/2} - x \right]_1^4 \\ &= \left(\frac{16}{3} + 4 \right) + \left(1 - \frac{2}{3} \right) + \left(\frac{16}{3} - 4 \right) + \left(\frac{2}{3} - 1 \right) = \frac{32}{3} \end{aligned}$$

If $I = \int_{-1}^1 (1+x)^{1/2} (1-x)^{3/2} dx$, then the value of $\sec^3(I/2)$ is _____

Key (8)

Sol

Using property $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$I = \int_{-1}^1 (1-x)^{1/2} (1+x)^{3/2} dx$$

$$\Rightarrow 2I = \int_{-1}^1 (1+x)^{1/2} (1-x)^{1/2} [(1-x) + (1+x)] dx$$

$$\Rightarrow 2I = 2 \int_{-1}^1 \sqrt{1-x^2} dx$$

$$\Rightarrow I = 2 \int_0^1 \sqrt{1-x^2} dx \quad (x = \sin \theta) \Rightarrow dx = \cos \theta d\theta$$

$$\Rightarrow I = 2 \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{\pi}{2}$$

2. Let $f: R \rightarrow R$ defined as $f(x) = x + \sin x$ then value of $\int_0^{2\pi} f^{-1}(x) dx =$

1) $2\pi^2$

2) $2\pi^2 - 2$

3) $2\pi^2 + 2$

4) π^2

Key (1)

Sol

$$\int_0^{2\pi} f^{-1}(x) dx$$

Put $x = f(t) = t + \sin t$ when $x = 0 \Rightarrow t = 0$

$$dx = f'(t) dt \quad x = 2\pi \Rightarrow t = 2\pi$$

$$\int_0^{2\pi} f^{-1}(x) dx = \int_0^{2\pi} t f'(t) dt$$

$$= \int_0^{2\pi} t(1 + \cos t) dt = \int_0^{2\pi} t dt + \int_0^{2\pi} t \cos t dt$$

$$= \left(\frac{t^2}{2} \right)_0^{2\pi} + (t \sin t)_0^{2\pi} - \int_0^{2\pi} \sin t dt$$

$$= 2\pi^2 + 0 + (\cos t)_0^{2\pi}$$

$$= 2\pi^2$$