Maxima and Minima

SOLUTIONS

EXERCISE - O

1. **Ans. (A)**

$f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$

$f'(x) = -\sin x - \sin 2x + \sin 3x = 2\cos 2x \sin x - 2\sin x \cos x$

$= 2\sin x(\cos 2x - \cos x)$

$f'(x) = 2\sin x\left(-2\sin\left(\frac{3x}{2}\right)\sin\frac{x}{2}\right)$

$f'(x) = 0$ (for maxima + minima)

$\sin x = 0, \sin\frac{3x}{2} = 0, \sin\left(\frac{x}{2}\right) = 0 \Rightarrow x = n\pi, \frac{3x}{2} = n\pi, \frac{x}{2} = n\pi$

$\Rightarrow x = n\pi, x = \frac{2n\pi}{3}, x = 2n\pi \Rightarrow 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi, \ldots\ldots$

$f(0) = 1 + \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$ (maximum)

$f\left(\frac{2\pi}{3}\right) = \frac{-1}{2} - \frac{1}{4} - \frac{1}{3} = \frac{-13}{12}$ (minimum)

$f(\pi) = -1 + \frac{1}{2} + \frac{1}{3} = \frac{-1}{6}$

$f\left(\frac{4\pi}{3}\right) = \frac{-1}{2} - \frac{1}{4} - \frac{1}{3} = \frac{-13}{12}$ (minimum)

$f(2\pi) = f(0) = \frac{7}{6}$

Difference $= \frac{7}{6} - \left(\frac{-13}{12}\right) = \frac{27}{12} = \frac{9}{4}$

2. **Ans. (B)**

$f(x) = 2x + (x - 100)^2$

$f'(x) = 2 + 2(x - 100) = 0$

$\Rightarrow x = 99$

$f'(99^+) > 0$

minima

$f'(99^-) < 0$

So, number are 99, 1

3. **Ans. (B)**

Since $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$.

$f''(x) = \lambda x$

$f'(x) = \lambda\frac{x^2}{2} + c$ $\quad\{f'(-1) = 0\}$

$\Rightarrow \frac{\lambda}{2} + c = 0 \Rightarrow \lambda = -2c$ ...(i)

again, Integrating both sides we get

$f(x) = \lambda\frac{x^3}{6} + cx + d$ ...(ii)

$f(2) = \lambda\left(\frac{8}{6}\right) + 2c + d = 18$

and $\quad f(1) = \frac{\lambda}{6} + c + d = -1 \quad$ ...(iii)

using (i), (ii) and (iii) we get

$f(x) = \frac{1}{4}(19x^3 - 57x + 34)$

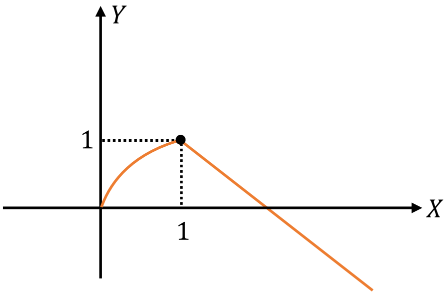
$f'(x) = \frac{1}{4}(57x^2 - 57) = \frac{57}{4}(x-1)(x+1),$

using number line rule $f(x)$ is increasing for $[1, 2\sqrt{5}]$

and $\quad f(x)$ has local maximum at $x = -1$ and $f(x)$ has local minimum at $x = 1$

also, $\quad f(0) = \frac{34}{4}$

4. Ans. (A)



$$f(x) = \begin{cases} \sin\frac{\pi x}{2} & , 0 \le x < 1 \\ 3 - 2x & , x \ge 1 \end{cases}$$

5. Ans. (B)

$f(x) = \frac{a}{x} + bx$

$(1, 6)$ satisfy $6 = a + b \quad$ ...(1)

and $f'(x) = \frac{-a}{x^2} + b = 0$ at $(1, 6)$

$\Rightarrow a = b \quad$ ...(2)

from (1) & (2) we get $a = 3, b = 3$.

6. Ans. (B)

$\frac{dy}{dx} = \begin{cases} 12x^2 - 2x + 2 & \text{for } 2 < x \le 3 \\ 12x^2 + 2x - 2 & \text{for } 0 \le x < 2 \end{cases}$ not derivable at $x = 2$ & $\frac{dy}{dx} = 0$

at $x = 1/3$, Decreasing in $(0, 1/3)$ & Increasing for $\left(\frac{1}{3}, 2\right) \cup (2, 3)$

$\Rightarrow$ Minima occurs at $x = 1/3$ So $f\left(\frac{1}{3}\right) = -11/27$.

7. Ans. (B)

$x - y = 2$

$x + y + xy$ is to be minimized.

Putting $y = x - 2$.

we get, $f(x) = 2x - 2 + x(x-2)$

$f'(x) = 2 + 2x - 2 = 0$

$f'(x) = 2x \Rightarrow x = 0$ is a point of minima

Thus minimum value is at $x = 0$ & $y = -2$.

Hence (B) is correct.

8. Ans. (C)

$f'(x)=3x^2+2ax+b$

$$\left.\begin{array}{l} f'(3)=27+6a+b=0 \\ f'(-1)=3-2a+b=0 \end{array}\right\} \begin{array}{l} a=-3 \\ b=-9 \end{array};c\in R$$

9. Ans. (B)

$f(1)=1(0-2)=-2$

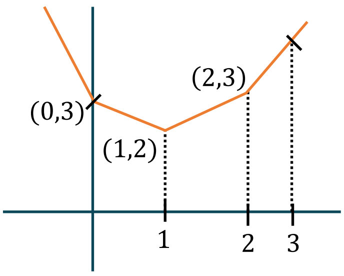
$f(e^2)=e^2(2-2)=0$

$f'(x)=\ell n\,x-2+x.\dfrac{1}{x}=0 \Rightarrow \ell n\,x=1 \;\Rightarrow\; x=e$

$\therefore f(e)=e(1-2)=-e$

Difference = e

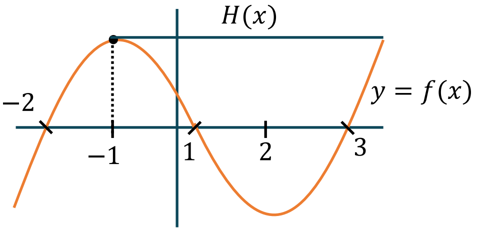
10. Ans. (A)



$\therefore$ minima at $x=1$

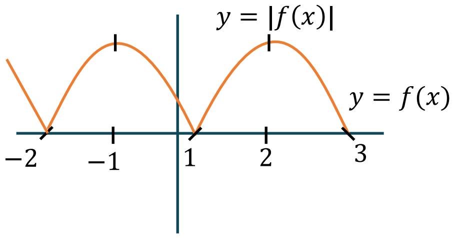
11. Ans. (A,B,D)

(A)



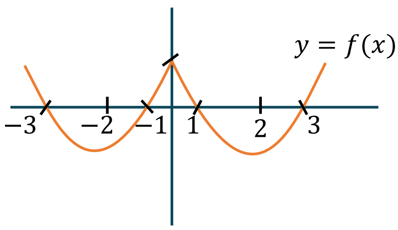
$\therefore$ local minima as well as maxima (in constant portion)

(B)



5 pts. Of extrema

(C), (D)

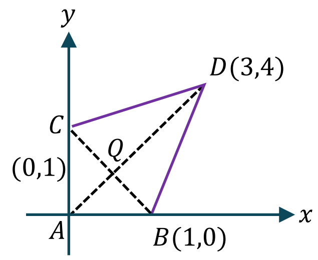


3 pts. Of extrema

12. **Ans. (A,C,D)**

$f(x,y)$ represent sum of distances of $P(x,y)$ from four pts $A(0,0)$, $B(1,0)$, $C(0,1)$ and $D(3,4)$

$f(x,y)$ is minimum when P is at pt. of intersection of diagonals

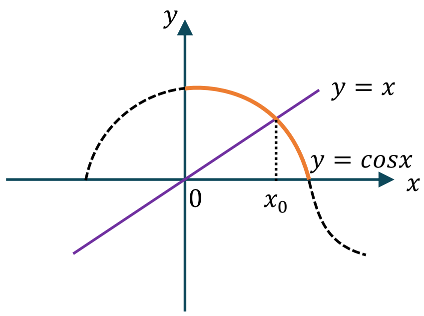


$$\left.\begin{cases} x+y=1 \\ y=\frac{4}{3}x \end{cases}\right\} \Rightarrow x=\frac{3}{7}\,;\; y=\frac{4}{7}$$

$$f\left(\frac{3}{7},\frac{4}{7}\right)=5+\sqrt{2}$$

13. **Ans. (B,D)**

$$f'(x) = \frac{\sec^2 x(\cos x+x)(\cos x-x)}{(1+x\tan x)^2}$$



The only factor in $f'(x)$ which changes sign is $\cos x - x$.

Let us consider graph of $y = \cos x$ and $y = x$

It is clear from figure that for $x \in (0, x_0)$, $\cos x - x > 0$ and for

$x \in \left(x_0, \frac{\pi}{2}\right)$

$\cos x - x < 0, \Rightarrow f'(x)$ has maxima at x_0

14. **Ans. (B,D)**

we have

$x = t^5 - 5t^3 - 20t + 7$

and, $y = 4t^3 - 3t^2 - 18t + 3$

$\therefore \frac{dx}{dt} = 5t^4 - 15t^2 - 20$ and $\frac{dy}{dt} = 12t^2 - 6t - 18$

$$\Rightarrow \frac{dx}{dt} = 5(t^2-4)(t^2+1) \text{ and } \frac{dy}{dt} = 6(2t-3)(t+1) \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6(2t-3)(t+1)}{5(t^2-4)(t^2+1)}$$

Now, $\frac{dy}{dx} = 0 \Rightarrow t = \frac{3}{2}, -1$

clearly, these values satisfy $-2 < t < 2$.

$$\text{Now, } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \cdot \frac{dt}{dx}$$

$$= \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} = \frac{(5t^4-15t^2-20)(24t-6)-(12t^2-6t-18)(20t^3-30t)}{(5t^4-15t^2-20)^3}$$

When $t = -1$, we have

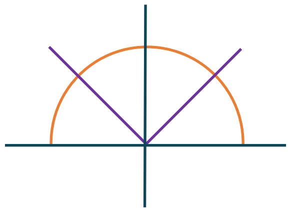
$$\frac{d^2y}{dx^2} = \frac{-30\times-30}{(-30)^3} < 0, x = 31 \text{ and } y = 14$$

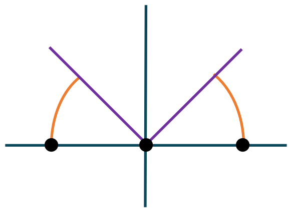
when $t = \frac{3}{2}$, we have

$\frac{d^2y}{dx^2} > 0$, $x = \frac{-1033}{32}$ and $y = \frac{-69}{4}$

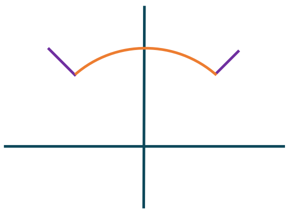
Hence, the function $y = f(x)$ attains a local maximum at (31, 14) and a local minimum at (– 1033/32, – 69/4)

15. Ans. (B,C,D)





$f(x) \rightarrow$ 3 local minima



$g(x) \rightarrow$ 3 local maxima

16. Ans. (A)

17. Ans. (B)

18. Ans. (A)

Solution For Q.16 to Q.18

$$g(x) = \frac{x^2+3x+1}{x^2+x+1} = 1 + \frac{2x}{x^2+x+1}$$

$$= 1 + \frac{2}{x + \frac{1}{x} + 1}$$

If $x > 0 \rightarrow x + \frac{1}{x} \geq 2$

$\therefore\ 1 < g(x) \leq \frac{5}{3}$

If $x < 0 \rightarrow x + \frac{1}{x} \leq -2$

$\therefore -1 \leq g(x) < 1$

If $x = 0 \rightarrow g(0) = 1$

$\therefore\ g(x) \in \left[-1, \frac{5}{3}\right] \rightarrow A = -1,\ B = \frac{5}{3}$

$f(x) = 1 + a^2x - x^3$

$f'(x) = a^2 - 3x^2 = 0 \rightarrow x = \pm\frac{|a|}{\sqrt{3}}$ $(\because a > 0)$

$f''(x) = -6x\ \&\ x = -\frac{a}{\sqrt{3}}, f''(x) > 0$

19. Ans. (A,D)

$L(x) = f(g(x)) = (x^3 - x^2 + x)^2\ e^{x^3 - x^2 + x}$

$\Rightarrow L'(x) = e^{x^3 - x^2 + x} \cdot (x^3 - x^2 + x)\ (3x^2 - 2x + 1)\ (x^3 - x^2 + x + 2)$

$$\underset{}{+}\ \underset{\alpha}{|}\ -\ \underset{0}{|}\ +$$

Local minimum value $f(g(0)) = 0$

20. Ans. (C.D)

$\frac{d}{dx}(h(f(x))) = e^x x(x+2) = 0$

$$\Rightarrow\ +\ \underset{-2}{|}\ -\ \underset{0}{|}\ +$$

$\therefore\ h(f(0)) = 2$

21. Ans. (0)

22. Ans. (-1)

23. Ans. (6)

Solution for Q.21 to Q.23)

$f(x) = x^3(x-2)^2(x-1)$

$f'(x) = x^3(x-2)^2 + (x-2)^2(x-1)3x^2 + 2x^3(x-1)\ (x-2)$

$f'(x) = (x-2)x^2[x(x-2) + (x-2)\ 3(x-1) + 2x(x-1)]$

$f'(x) = (x-2)x^2[x^2 - 2x + 3(x^2 - 3x + 2) + 2x^2 - 2x]$

$= (x-2)x^2[6x^2 - 13x + 6]$

$= (x-2)x^2(2x-3)\ (3x-2)$

$$-\ \underset{0}{|}\ -\ \underset{\frac{2}{3}}{|}\ +\ \underset{\frac{3}{2}}{|}\ -\ \underset{2}{|}\ +$$

minima at $x = \frac{2}{3}, 2$

maxima at $x = \frac{3}{2}$

minimum value of $= f\left(\frac{2}{3}\right) = -\frac{128}{729}$

24. Ans. (A→P;B→S;C→Q;D→R)

(A) $f'(x) = 2x - \frac{2}{x^3} = \frac{2(x^2-1)(x^2+1)}{x^3}$

No point of local maxima

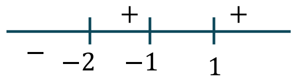
(B) Given expression = $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

$= \left(\frac{\pi}{2}\right)^3 - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right).\frac{\pi}{2} = \frac{3\pi}{2}(\sin^{-1}x)^2 - \frac{3\pi^2}{4}\sin^{-1}x + \frac{\pi^3}{8}$

This is quadratic in $\sin^{-1}x$. Therefore it will give maximum value when

$\sin^{-1}x = -\frac{\pi}{2} \Rightarrow x = -1$

(C) $f'(x) = 12(x+2)(x+1)(x-1)$



sing of $f'(x)$

$\Rightarrow a = -2, b = -1$

(D) $\frac{a^3+b^3}{48} = \frac{a^3+(8-a)^3}{48} = \frac{8}{48}(3a^2 - 24a + 64)$

Minimum $\frac{a^3+b^3}{48}$ is $\frac{8}{48}\frac{(4.3.64-24^2)}{4.3} = \frac{8}{3}$

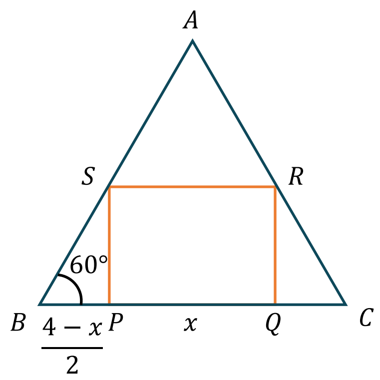
25. Ans. (A→R;B→S;C→Q;D→P)

(A) Let $PQ = x$

Then $BP = \frac{4-x}{2}$

$\therefore PS = \frac{4-x}{2}\tan 60° = \frac{\sqrt{3}(4-x)}{2}$

$\therefore$ area A of rectangle = $\frac{\sqrt{3}}{2}(4-x)x$



$\frac{dA}{dx} = \frac{\sqrt{3}}{2}(4-2x) = 0 \Rightarrow x = 2$

$\frac{d^2A}{dx^2} = -\sqrt{3} < 0$ $\quad\therefore A$ is maximum, when $x = 2$.

$\therefore$ Maximum area = $\frac{\sqrt{3}}{2}.2.2 = 2\sqrt{3}$

Square of maximum area = 12

(B) Dimensions be $x, 2x, h$

$72 = x.2x.h$

$36 = x^2h$ ...(1)

$S = 4x^2 + 6xh$

$S = 4x^2 + 6\frac{36}{x}$

EXERCISE - S

1. **Ans. (177)**

$f'(x) = 3x^2 - 24x + 36$

$= 3(x^2 - 8x + 12) = 3(x-2)(x-6)$

$f'(x) = 0 \Rightarrow x = 2, 6$

$f''(x) = 3(2x - 8)$

$f''(2) < 0$ local maxima

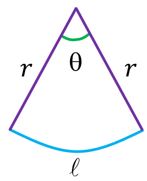
$f(2) = 8 - 48 + 72 + 17 = 49$

$f(1) = 42,$ so maximum $x = 10$

$f(10) = 177$ maxi. i.e. $(f(x)) = 177$

2. **Ans. (25)**

Given $2r + \ell = 20$



Area $= \dfrac{\pi r^2}{360°} \cdot \theta = \dfrac{1}{2} r^2 \theta$

$A = \dfrac{r^2}{2} \cdot \dfrac{\ell}{r} = \dfrac{1}{2} \ell r$

$A = \dfrac{r(20-2r)}{2} = (10r - r^2)$

$\dfrac{dA}{dr} = (10 - 2r) = 0 \Rightarrow r = 5 \Rightarrow \ell = 20 - 10 = 10$

Area $= \dfrac{1}{2}(10 \times 5) = 25$

3. **Ans. (80)**

Let A be $(h, 0) \Rightarrow D$ is $(h, 4h)$

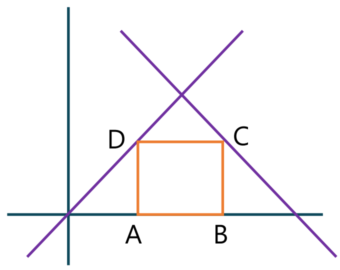
$\Rightarrow C\left(\dfrac{6-4h}{5}, 4h\right) \Rightarrow B\left(\dfrac{6-4h}{5}, 0\right)$

Area $f(h) = 4h\left(\dfrac{6-4h}{5} - h\right) = \dfrac{12}{5}(2h - 3h^2)$

Maximum $f(h) \Rightarrow f'(h) = 0$

$f'(h) = \dfrac{12}{5}(2 - 6h) = 0 \Rightarrow h = \dfrac{1}{3}$

$\therefore$ Area $= 4\left(\dfrac{1}{3}\right)\left(\dfrac{6-4\left(\frac{1}{3}\right)}{5} - \dfrac{1}{3}\right) = \dfrac{4}{3}\left(\dfrac{14-5}{5\times 3}\right) = 0.8$



4. **Ans. (4)**

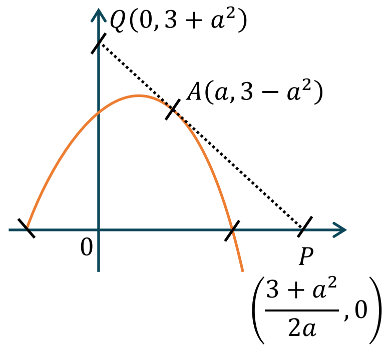
$y = 3 - x^2 = y' = -2x\,|_{x=a}$

$\Rightarrow$ slope of tangent $= -2a$

$\therefore$ eq. of tangent at A

$\therefore y - (3 - a^2) = -2a(x - a)$

$\therefore P = \left(\dfrac{3+a^2}{2a}, 0\right); Q(0, 3+a^2)$



Ar $(\Delta OP) = \dfrac{1}{2}\left(\dfrac{3+a^2}{2a}\right)(3+a^2) = A$

$\dfrac{dA}{da} = 0 \Rightarrow a = 1$ or $a = 3$ (reject)

$\therefore A_{\min} = \dfrac{1}{2}\left(\dfrac{3+1}{2}\right)(3+1) = 4$

5. Ans. (15)

Let x tree be added then

$P(x) = (x + 50)\,(800 - 10x)$

now $P'(x) = 0 \Rightarrow x = 15$

6. Ans. (74)

(i) St. $\uparrow$: $[\underset{a}{0}, \underset{b}{2}]\,;\,[\underset{b}{4}, \underset{d}{6}]\,;\,[\underset{c}{8}, \underset{f}{9})$

St. $\downarrow$: $[\underset{p}{2}, \underset{q}{4}]\,;\,[\underset{r}{6}, \underset{s}{8}]$

(ii) Local minima at $x_1 = 4$ & $x_2 = 8$

(iii) $f''(x) > 0$ in $(\underset{\ell}{3}, \underset{m}{6})\,;\,(\underset{n}{6}, \underset{t}{9})$

(iv) Inflection pt. at $x = 3 \rightarrow k$

(v) Number of critical pts. is $4 \rightarrow w$

$\therefore\ (a + b + c + d + e) + (p + q + r + s) + (l + m + n) + (x_1 + x_2) + (k + w)$

$= (0 + 2 + 4 + 6 + 8) + (2 + 4 + 6 + 8) + (3 + 6 + 6) + (4 + 8) + (3 + 4)$

$= 74$

7. Ans. (6)

$$\lim_{x\to 0}\frac{P(x)}{x^3} - 2 = 4 \Rightarrow \lim_{x\to 0}\frac{P(x)}{x^3} = 6$$

Let $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

$$\therefore\ \underset{x\to 0}{\ell t}\ \frac{\mathrm{ax^5 + bx^4 + cx^3 + dx^2 + ex + f}}{x^3} = 6$$

$\Rightarrow d = e = f = 0\,;\, c = 6$ (for limit to exist)

$\therefore\ P(x) = ax^5 + bx^4 + 6x^3$

$$P'(-1) = 0; P'(1) = 0 \Rightarrow \left.\begin{array}{l} a = -\dfrac{18}{5} \\ b = 0 \end{array}\right\}$$

$$\therefore\ P(x) = -\frac{18}{5}x^5 + 6x^3$$

$P'(x) = -18x^4 + 18x^2 = -18(x^4 - x^2)$

$P''(x) = -18(4x^3 - 2x) = 0$

$\Rightarrow 2x(2x^2 - 1) = 0$

$$\Rightarrow x = 0\,;\,\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$$

$= \{x : x^2 + 6 \le 5x\} \Rightarrow x^2 - 5x + 6 \le 0$

$\Rightarrow x \in [2, 3]$

$\therefore\ P'(2) = -18(16 - 4) = -18 \times 12 = M$

$P'(3) = -18(81 - 9) = -18 \times 72 = m$

$$\therefore\ \frac{m}{M} = \frac{72}{12} = 6$$

8. **Ans. (0)**

$y = \dfrac{6}{x^2+3} \Rightarrow y' = 6\cdot\left(\dfrac{-2x}{(x^2+3)^2}\right)$ (slope function)

$y'' = -12\left[\dfrac{(x^2+3)^2 - x\cdot 2(x^2+3)\cdot 2x}{(x^2+3)^4}\right] = 0$

$\Rightarrow (x^2+3)(x^2+3-4x^2) = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1$

At $x = 1$: $y = \dfrac{6}{4} = \dfrac{3}{2}$ At $x = -1$: $y = \dfrac{6}{4} = \dfrac{3}{2}$

$y'(1) = \dfrac{-12(1)}{(1^2+3)^2} = -\dfrac{3}{4}$ $y'(-1) = \dfrac{3}{4}$

$T : y - \dfrac{3}{2} = \dfrac{-3}{4}(x-1)$ $T : y - \dfrac{3}{2} = \dfrac{3}{4}(x+1)$

$\Rightarrow 3x + 4y - 9 = 0$ $\Rightarrow 3x - 4y + 9 = 0$

9. **Ans. (360)**

$\pi r + \pi r + 2\ell = 440$

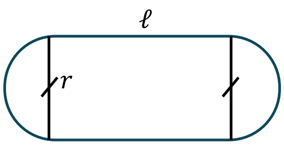
$A = 2r\,.\,\ell$

$\Rightarrow A = r(440 - 2\pi r) \Rightarrow A = 440r - 2\pi r^2$

$\dfrac{dA}{dr} = 0 \Rightarrow 440 - 4\pi r = 0 \Rightarrow r = \dfrac{440}{4\times 22}\times 7 \Rightarrow r = 35$

$\Rightarrow 2r = 70$ and $\ell = 110$

Perimeter $= 2(2r + l) = 360$



10. **Ans. (320)**

$f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$

$D = 36(a-7)^2 - 4\times 3\times 3(a^2-9)$

$= 36(a^2 + 49 - 14a - a^2 + 9) = 36(58 - 14a)$

$f'(x) = 0$

at $x = \dfrac{-6(a-7) \pm \sqrt{36(58-14a)}}{6}$

$\alpha = 7 - a - \sqrt{58-14a}\cdot\beta = 7 - a + \sqrt{58-14a}$

Now $\alpha > 0$ & $D > 0$

$a^2 - 9 > 0$ & $a < \dfrac{58}{14}$

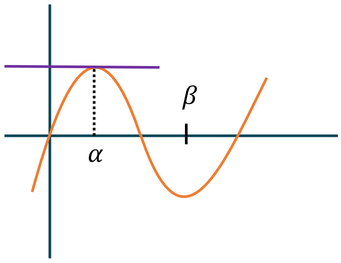
$a \in (-\infty, -3) \cup \left(3, \dfrac{58}{14}\right)$

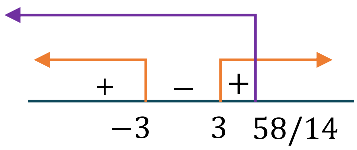
So, $a_2 = -3$

$a_3 = 3$

$a_4 = \dfrac{58}{14}$

$a_2 + 11a_3 + 70a_4 = 320$





EXERCISE - JEE (Main) PYQ

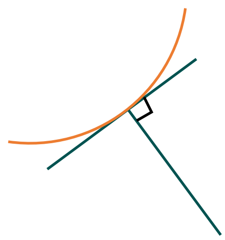
1. Ans. (3)

$y - x^{\frac{3}{2}} = 7, (x \geq 0)$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$

$\left(\frac{3}{2}\sqrt{x}\right)\left(\frac{7-y}{\frac{1}{2}-x}\right) = -1$

$\left(\frac{3}{2}\sqrt{x}\right)\left(\frac{-x^{3/2}}{\frac{1}{2}-x}\right) = -1$



$\frac{3}{2}x^2 = \frac{1}{2} - x$

$y = 7 + x^{3/2} = 7 + \left(\frac{1}{3}\right)^{3/2}$

$3x^2 = 1 - 2x$

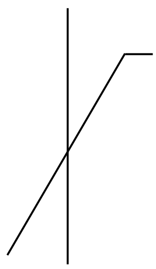
$3x^2 + 2x - 1 = 0$

$3x^2 + 3x - x - 1 = 0$

$(x+1)\,(3x-1) = 0$

$\therefore\ x = -1$ (rejected)

$x = \frac{1}{3}$



$\ell_{AB} = \sqrt{\left(\frac{1}{2}-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3} = \sqrt{\frac{1}{36}+\frac{1}{27}}$

$= \sqrt{\frac{3+4}{9\times 12}}$

$= \sqrt{\frac{7}{108}} = \frac{1}{6}\sqrt{\frac{7}{3}}$

Option (3)

2. Ans. (2)

Let a is first term and d is common difference

then, $a + 5d = 2$ (given) ...(1)

$f(d) = (2-5d)(2-2d)(2-d)$

$f'(d) = 0 \Rightarrow d = \frac{2}{3}, \frac{8}{5}$

$f''(d) < 0$ at $d = 8/5$

$\Rightarrow d = \frac{8}{5}$

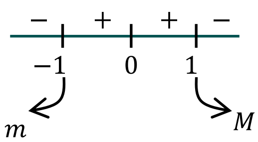
3. Ans. (2)

$\lim\limits_{x\to 0} \frac{f(x)}{x^3} = 2$

Let $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

For limit to exit $d = e = f = 0$; $c = 2$

$\therefore\ f(x) = ax^5 + bx^4 + 2x^3$



$f'(x) = 5ax^4 + 4bx^3 + 6x^2$

$$\left.\begin{array}{l} f'(1)=0 \Rightarrow 5a+4b+6=0 \\ f'(-1)=0 \Rightarrow 5a-4b+6=0 \end{array}\right\} \begin{array}{l} a=-6/5 \\ b=0 \end{array}$$

$\therefore f(x) = \left(-\frac{6}{5}\right)x^5 + 2x^3 \rightarrow$ odd f^n.

$f(x) = -6x^4 + 6x^2 = -6x^2(x-1)(x+1)$

4. **Ans. (3)**

Let $f''(x) = a(x-1)$

$\Rightarrow f'(x) = \frac{ax^2}{2} - ax + b$

$\therefore f(x)$ is cubic

$\therefore f''(x)$ will be linear

$f'(-1)=0 \Rightarrow \frac{a}{2}+a+b=0 \Rightarrow \boxed{b=\frac{-3a}{2}}$

$\Rightarrow f(x) = \frac{ax^3}{6} - \frac{ax^2}{2} - \frac{3a}{2} + c$

$f(1) = -6 \Rightarrow \frac{a}{6} - \frac{a}{2} - \frac{3a}{2} + c = -6$

$\Rightarrow \boxed{-11a+6c=-36}$

Similarly, $f(-1) = 10$

$\Rightarrow \boxed{5a+6c=60}$

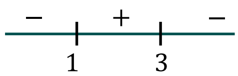
Solving, we get : $a = 6$; $c = 5$

$\therefore f(x) = x^3 - 3x^2 - 9x + 5$

$f'(x) = 3(x^2 - 2x - 3)$

$= 3(x-3)(x+1)$

minima at $x = 3$



5. **Ans. (3)**

Let thickness of ice be 'h'.

Vol. of ice = $v = \frac{4\pi}{3}\left((10+h)^3 - 10^3\right)$

$\frac{dv}{dt} = \frac{4\pi}{3}\left(3(10+h)^2\right).\frac{dh}{dt}$

given $\frac{dv}{dt} = 50\ cm^3/min$ and $h = 5\ cm$

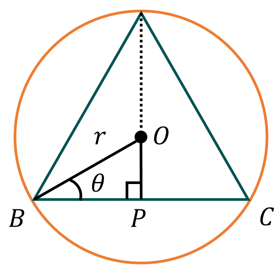
$\Rightarrow 50 = \frac{4\pi}{3}\left(3(10+5)^2\right)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{50}{4\pi \times 15^2} = \frac{1}{18\pi} cm/\text{min}$

6. **Ans. (3)**

$h = r\sin\theta + r$

base = $BC = 2r\cos\theta$

$\theta \in \left[0, \frac{\pi}{2}\right)$

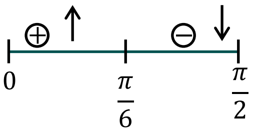


Area of $\Delta ABC = \frac{1}{2}(BC).h$

$\Delta = \frac{1}{2}(2r\cos\theta).(r\sin\theta + r) = r^2\,(\cos\theta).(1 + \sin\theta)$

$\frac{d\Delta}{d\theta} = r^2\left[\cos^2\theta - \sin\theta - \sin^2\theta\right] = r^2[1 - \sin\theta - 2\sin^2\theta]$

$= \underbrace{r^2[1+\sin\theta]}_{positive}[1-2\sin\theta] = 0 \Rightarrow \theta = \frac{\pi}{6}$



$\Rightarrow \Delta$ is maximum where $\theta = \frac{\pi}{6}$

$\Delta_{max.} = \frac{3\sqrt{3}}{4}r^2$ = area of equilateral Δ with $BC = \sqrt{3}r$.

7. **Ans. (3)**

$f(x) = x^3 - 3x^2 - \frac{3}{2}f''(2)x + f''(1)$ ...(i)

$f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$ ...(ii)

$f''(x) = 6x - 6$ ...(iii)

Now is 3rd equation

$f''(2) = 12 - 6 = 6$

$f''(11 = 0)$

Use (ii)

$f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$

$f'(x) = 3x^2 - 6x - \frac{3}{2}\times 6$

$f'(x) = 3x^2 - 6x - 9$

$f'(x) = 0$

$3x^2 - 6x - 9 = 0$

$\Rightarrow x = -1\ \&\ 3$

Use (iii)

$f''(x) = 6x - 6$

$f''(-1) = -12 < 0$ maxima

$f''(3) = 12 > 0$ minima.

Use (i)

$f(x) = x^3 - 3x^2 - \frac{3}{2}f''(2)x + f''(1)$

$f(x) = x^3 - 3x^2 - \frac{3}{2}\times 6 \times x + 0$

$f(x) = x^3 - 3x^2 - 9x$

$f(3) = 27 - 27 - 9 \times 3 = -27$

8. Ans. (1)

$$f(x)=\begin{cases} x^2-4x-2, & \forall x\in\left(-1,\dfrac{3-\sqrt{17}}{2}\right) \\ -x^2+2x+2, & \forall x\in\left(\dfrac{3-\sqrt{17}}{2},2\right) \end{cases}$$

$f'(x)$ when $x\in\left(-1,\dfrac{3-\sqrt{17}}{2}\right)$

$f'(x) = 2x-4=0 \Rightarrow x=2$

$f'(x)=2(x-2) \Rightarrow f'(x)$ is always $\downarrow$

$f(2)=2$

$f(-1)=3$

$$f\left(\frac{3-\sqrt{17}}{2}\right)=\frac{\sqrt{17}-3}{2}$$

$f'(x)$ when $x\in\left(\dfrac{3-\sqrt{17}}{2},2\right)$

$f'(x)=-2x+2$

$f'(x)=-2(x-1)$

$f'(x)=0$ when $x=1$

$f(1)=3$

absolute minimum value = $\dfrac{\sqrt{17}-3}{2}$

absolute maximum value = 3

Sum = $\dfrac{\sqrt{17}-3}{2}+3=\dfrac{\sqrt{17}+3}{2}$

9. Ans. (1)

$y(x)=ax^3+bx^2+cx+5$ is passing through (–2,0) then

$8a-4b+2c=5$ ...(1)

$y'(x)=3ax^2+2bx+c$ touches x-axis at (–2,0)

$12a-4b+c=0$ ...(2)

again, for $x=0, y'(x)=3$

$c=3$ ...(3)

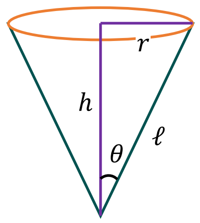
Solving eq. (1), (2) & (3) $a=-\dfrac{1}{2}$, $b=-\dfrac{3}{4}$

$$y'(x)=-\frac{3}{2}x^2-\frac{3}{2}x+3$$

$y(x)$ has local maxima at $x=1$

$y(1)=\dfrac{27}{4}$

10. Ans. (5)



$$\tan\theta=\frac{3}{4}=\frac{r}{h}$$

$$\frac{dV}{dt}=6$$

$$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi h^3\tan^2\theta=\frac{9\pi}{48}h^3=\frac{3\pi}{16}h^3$$

$$\Rightarrow\frac{dV}{dt}=\frac{3\pi}{16}.3h^2.\frac{dh}{dt}=6\Rightarrow\left(\frac{dh}{dt}\right)_{h=4}=\frac{2}{3\pi}m/hr$$

Now, S = $\pi r\ell$ = $\frac{15}{16}\pi h^2$

$$\Rightarrow\frac{dS}{dt}=\frac{15\pi}{16}.2h\frac{dh}{dt}\Rightarrow\left(\frac{dS}{dt}\right)_{h=4}=5m^2/hr$$

11. Ans. (1)

$$f'(x)=8x^3-36x+8=4(2x^3-9x+2)$$

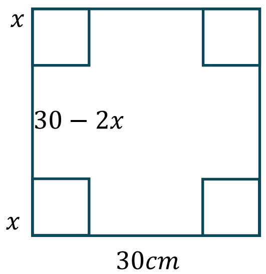
$$f'(x)=0$$

$$\therefore x=\frac{\sqrt{6}-2}{2}$$

Now $f(x)=\left(x^2-2x-\frac{9}{2}\right)\left(2x^2+4x-1\right)+24x+7.5$

$$\therefore f\left(\frac{\sqrt{6}-2}{2}\right)=M=12\sqrt{6}-\frac{33}{2}$$

12. Ans. (3)



Volume (V) = x $(30-2x)^2$

$$\frac{dV}{dx}=(30-2x)(30-6x)=0$$

x = 5 cm

Surface area = 4 × 5 × 20 + $(20)^2$ = 800 cm^2

13. Ans. (1)

$$f(x)=x-\sin 2x+\frac{1}{3}\sin 3x$$

$$f'(x)=1-2\cos 2x+\cos 3x=0$$

$$x=\frac{5\pi}{6},\frac{\pi}{6}$$

$$\therefore f''(x)=4\sin 2x-3\sin 3x$$

$f''\left(\frac{5\pi}{6}\right)<0\Rightarrow\left(\frac{5\pi}{6}\right)$ is point of maxima

$$f\left(\frac{5\pi}{6}\right)=\frac{5\pi}{6}+\frac{\sqrt{3}}{2}+\frac{1}{3}$$

14. Ans. (4)

$f'(x)=x^2+2b+ax$

$g'(x)=x^2+a+2bx$

$(2b-a)-x(2b-a)=0$

$\therefore x=1$ is the common root

Put $x = 1$ in $f'(x)=0$ or $g'(x)=0$

$1 + 2b + a = 0$

$7 + 2b + a = 6$

EXERCISE - JEE (Advanced) PYQ

1. Ans. (D)

$f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x) = 2x\left(e^{x^2}-e^{-x^2}\right) \geq 0 \forall x \in [0,1]$

Clearly for $0 \leq x \leq 1 : 1 \geq x \geq x^2$

$f(x) \geq g(x) \geq h(x)$

$f(1) = g(1) = h(1) = e+\frac{1}{e}$ and $f(1)$ is the greatest

$\therefore a = b = c = e+\frac{1}{e} \Rightarrow a = b = c$

2. Ans. (1)

$f(x) = \ell n\,(g(x))$

$g(x) = e^{f(x)}$

$g'(x) = e^{f(x)}.f'(x)$

$g'(x) = 0 \Rightarrow f'(x) = 0$ as $e^{f(x)} \neq 0$

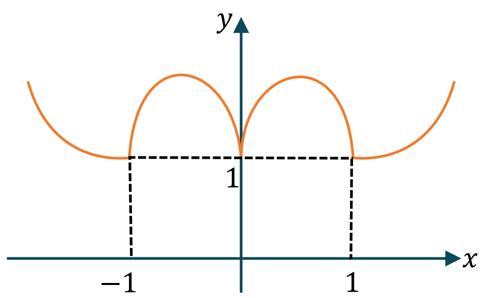
$\Rightarrow 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4 = 0$

So there is only one point of local maxima.

3. Ans. (5)

$f(x) = |x| + |(x+1)(x-1)|$

$$\Rightarrow f(x) = \begin{cases} x^2-x-1 & x<-1 \\ -x^2-x+1 & -1 \leq x < 0 \\ -x^2+x+1 & 0 \leq x < 1 \\ x^2+x-1 & x \geq 1 \end{cases}$$



$\therefore f$ has 5 points where it attains either a local maximum or local minimum.

4. Ans. (9)

Let $P'(x) = k(x-1)(x-3)$

$= k(x^2-4x+3)$

$\Rightarrow P(x) = k\left(\frac{x^3}{3}-2x^2+3x\right)+c$

$\because P(1)=6$

$\Rightarrow \quad \dfrac{4k}{3}+c=6$ ...(1)

$P(3) = 2$

$\Rightarrow c = 2$ ...(2)

by (i) and (ii)

$k = 3$

$\therefore\ P'(x) = 3(x-1)(x-3)$

$\Rightarrow P'(0) = 9$

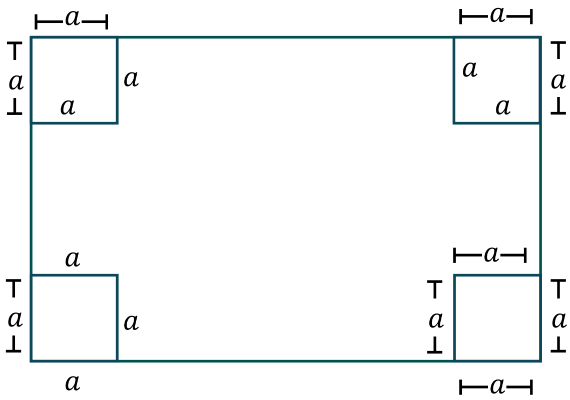
5. **Ans. (A,C)**

Let $\ell = 8x, b = 15x$

$\therefore$ Volume = $(8x-2a)(15x-2a)(a) = 4a^3 - 46a^2x + 120\ ax^2$

$\dfrac{dV}{da} = 6a^2 - 46ax + 60x^2$

$\left(\dfrac{dV}{da}\right)_{a=5} = 0 \Rightarrow \ \therefore\ \ x = 3 \text{ and } \dfrac{5}{6}$



$4a^2 = 100$

$\Rightarrow a = 5$

$\dfrac{d^2V}{da^2} = 6a - 23x$

$\left(\dfrac{d^2V}{da^2}\right)_{at\ a=5\ \&\ x=3} < 0,$

So, at $x = 3$ given maxima

$\left(\dfrac{d^2V}{da^2}\right)_{at\ a=5\ \&\ x=\frac{5}{6}} > 0$

So, at $x = \frac{5}{6}$ given minima.

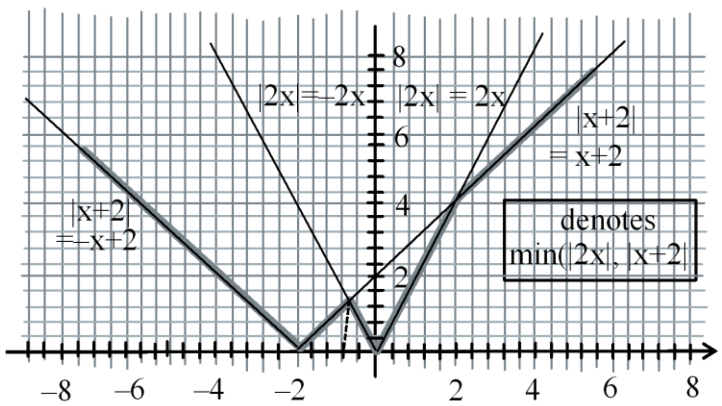
So by $x = 3$ (for max volume)

$8x = 24, 15x = 45$

6. **Ans. (A,B)**

$As, \dfrac{f(x)+g(x)-|f(x)-g(x)|}{2} = \text{Min}\big(f(x), g(x)\big)$

$\Rightarrow \quad \dfrac{2|x|+|x+2|-|x+2|x||}{2} = \text{Min}\big(|2x|, |x+2|\big)$



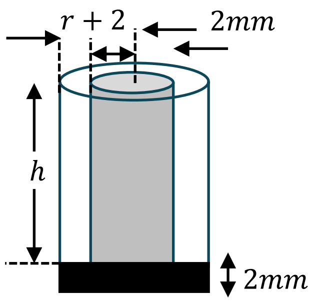
According to the figure shown, points of local minima/maxima are $x = -2, \dfrac{-2}{3}, 0$

7. Ans. (4)

Let the inner radius of cylindrical container be r then radius of outer cylinder is $(r+2)$.

Now, $V = \pi r^2 h$ where h is height of cylinder

Now, Let volume of total material used be T



$$\therefore\ T = \pi\left((r+2)^2 - r^2\right).\frac{V}{\pi r^2} + \pi(r+2)^2.2$$

$$T = V\left(\frac{r+2}{r}\right)^2 + 2\pi(r+2)^2 - V$$

Now, $\dfrac{dT}{dr} = 2V\left(\dfrac{r+2}{r}\right)\times\left(-\dfrac{2}{r^2}\right) + 4\pi(r+2)$

Now, At $r = 10mm$ $\dfrac{dT}{dr} = 0$

$$\therefore\ 0 = (r+2).4\left(\pi - \frac{V}{r^3}\right)$$

$$\Rightarrow\ \frac{V}{\pi} = 1000\ \Rightarrow \frac{V}{250\pi} = 4$$

8. Ans. (C)

$$f(x) = 4\alpha x^2 + \frac{1}{x};\ x > 0$$

$$f'(x) = 8\alpha x - \frac{1}{x^2} = \frac{8\alpha x^3 - 1}{x^2}$$

$f(x)$ attains its minimum at $x = \left(\dfrac{1}{8\alpha}\right)^{1/3}$

$$f\left(\left(\frac{1}{8\alpha}\right)^{1/3}\right) = 1\ \Rightarrow\ 4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} = 1$$

$$\Rightarrow\ 3\alpha^{1/3} = 1\ \Rightarrow \alpha = \frac{1}{27}$$

9. Ans. (A,D)

Using L' *Hôpital's* Rule

$$\lim_{x\to 2}\frac{f'(x)g(x) + f(x)g'(x)}{f''(x)g'(x) + f'(x)g''(x)} = 1$$

$$\Rightarrow\ \frac{f(2)g'(2)}{f''(2)g'(2)} = 1\ \Rightarrow\ f''(2) = f(2) > 0$$

option (D) is right and option (C) is wrong

also $f'(2) = 0$ and $f''(2) > 0$

$\therefore\ x = 2$ is local minima.

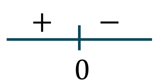
10. Ans. (B,D)

Expansion of determinant

$f(x) = \cos 2x + \cos 4x$

$f'(x) = -2\sin 2x - 4\sin 4x = -2\sin 2x(1 + 4\cos 2x)$

$\therefore$ Maxima at $x = 0$



$f'(x) = 0 \Rightarrow x = \dfrac{n\pi}{2}, \cos 2x = -\dfrac{1}{4}$

$\Rightarrow$ more than three solutions

11. **Ans. (A,B,D)**

$f(x)$ can't be constant throughout the domain. Hence we can find $x \in (r, s)$ such the $f(x)$ is one-one option (A) is true.

Option (B) : $|f'(x_0)| = \left|\dfrac{f(0)-f(-4)}{4}\right| \le 1(LMVT)$

Option (C) : $f(x) = \sin\left(\left(\sqrt{85}\right)x\right)$ satisfies given condition

but $\lim\limits_{x\to\infty} \sin\left(\sqrt{85}\right)$ D.N.E.

$\Rightarrow$ Incorrect

Option (D) : $g(x) = f^2(x) + \left(f'(x)\right)^2$

$|f'(x_1) \le 1$ (by LMVT)

$|f(x_1)| \le 2$ (given)

$\Rightarrow g(x_1) \le 5\ \exists x_1 \in (-4, 0)$

Similarly $g(x_2) \le 5\ \exists x_2 \in (0, 4)$

$g(0) = 85 \Rightarrow g(x)$ has maxima in (x_1, x_2) say at α.

$g'(\alpha) = 0\ \&\ g(\alpha) \ge 85$

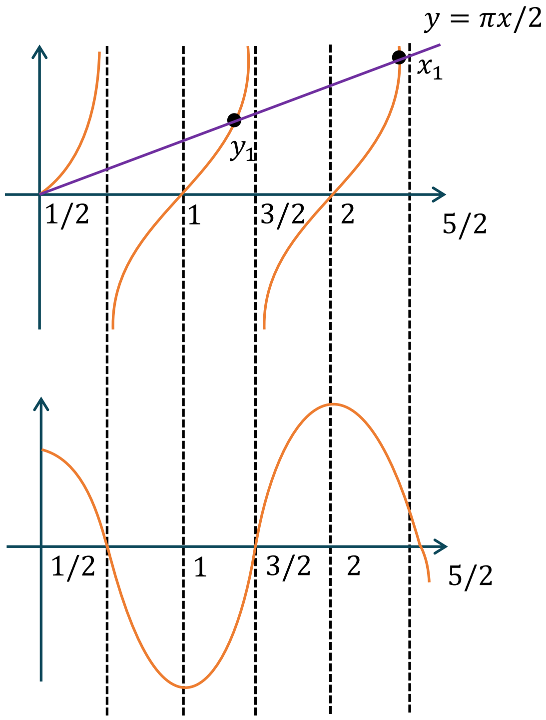
$2f'(\alpha)(f(\alpha) + f''(\alpha)) = 0$

If $f'(\alpha) = 0 \Rightarrow g(\alpha) = f^2(\alpha) \ge 85$ Not possible

$f(\alpha) + f''(\alpha) = 0\ \ \exists \alpha \in (x_1, x_2) \in (-4, 4)$

option (D) correct.

12. **Ans. (A,C,D)**



$$f(x) = \frac{\sin \pi x}{x^2}$$

$$\Rightarrow\ f'(x) = \frac{\pi x^2 \cos \pi x - 2x \sin \pi x}{x^4}$$

$$= \frac{2\cos \pi x\left(\dfrac{\pi x}{2} - \tan \pi x\right)}{x^3}$$

$$f'(x) = 0\ \Rightarrow\ \cos \pi x = 0\ \ or\ \ \frac{\pi x}{2} = \tan \pi x$$

$$\Rightarrow\ \pi x = (2n+1)\frac{\pi}{2}\ \ or\ \ \frac{\pi x}{2} = \tan \pi x$$

$$x = \frac{(2n+1)}{2}, n \in I$$

from graph, we can see that $\forall x = \dfrac{2n+1}{2}$

$\Rightarrow f'(x)$ doesn't change sign so these points are neither local maxima nor local minimum.

Similarly $\forall x : \dfrac{\pi x}{2} = \tan \pi x$,

Where $y_n \in (2n - 1, 2n - 1/2)\ \forall n = 1, 2, 3$ ....

and $x_n \in (2n, 2n + 1/2\ )\ \forall n = 1, 2, 3$ ....

$x_{n+1} - y_{n+1} > 1$ and $y_{n+1} - x_n > 1 \Rightarrow x_{n+1} - x_n > 2$.

13. Ans. (C)

Perimeter = 2(2α + 2cos 2α)

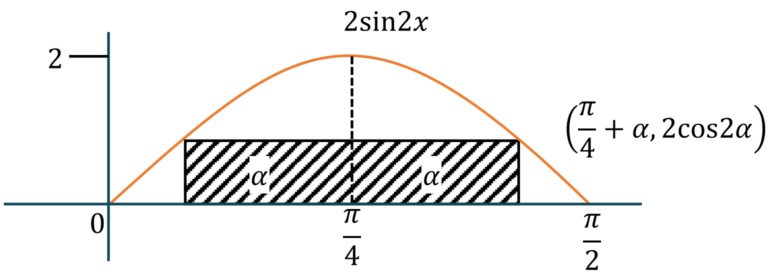
$P = 4(\alpha + \cos 2\alpha)$

$$\frac{dP}{d\alpha} = 4(1-2\sin 2\alpha) = 0$$

$$\sin 2\alpha = \frac{1}{2}$$

$$2\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{d^2P}{d\alpha^2} = -4\cos 2\alpha$$



for maximum $\alpha = \frac{\pi}{12}$

Area = (2α) (2 cos 2α)

$$= \frac{\pi}{6} \times 2 \times \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$$

14. Ans. (0.50)

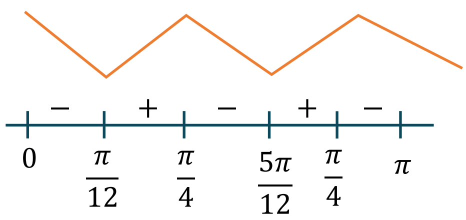
$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$

$f(\theta) = \sin^2 2\theta - \sin 2\theta + 2$

$f(\theta) = 2(\sin 2\theta)(2\cos 2\theta) - 2\cos 2\theta$

$= 2\cos 2\theta\,(2\sin 2\theta - 1)$

critical points



so, minimum at $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$$\lambda_1 + \lambda_2 = \frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$$

15. Ans. (57)

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j\, dt$$

$$\Rightarrow f_1'(x) = \prod_{j=1}^{21} (x-j)^j$$

Therefore $m_1 = 6, n_1 = 5$

$2m_1 + 3n_1 + m_1 n_1$

= 2(6) + 3(5) + 30

= 12 + 15 + 30 = 57

16. Ans. (6)

$$f_2'(x) = (98)(50)(x-1)^{49} - (600)(49)(x-1)^{48}$$

$= (49)(100)(x-1)^{48}(x-1-6)$

$m_2 = 1, n_2 = 0$

$6m_2 + 4n_2 + 8m_2 n_2$

6(1) + 4(0) + 8(0) = 6

17. Ans. (A,B,C)

$$\alpha = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \ldots + \alpha = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

$$\therefore g(x) = 2^{\frac{x}{3}} + 2^{\left(\frac{1}{3}\right)(1-x)}$$

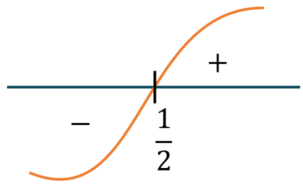
$$\therefore g(x) = 2^{\frac{x}{3}} + \frac{2^{1/3}}{2^{x/3}}$$

where $g(0) = 1 + 2^{\frac{1}{3}}$ & $g(1) = 1 + 2^{\frac{1}{3}}$

$$\therefore g'(x) = \frac{1}{3}\left(2^{x/3} - \frac{2^{1/3}}{2^{x/3}}\right) = 0$$

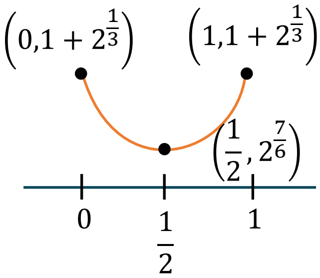
$\Rightarrow 2^{\frac{2x}{3}} = 2^{\frac{1}{3}} \Rightarrow x = \frac{1}{2}$ = critical point

graph of $g'(x)$ =



& $g\left(\frac{1}{2}\right) = 2^{\frac{7}{6}}$

$\therefore$ graph of $g(x)$ in [0, 1]



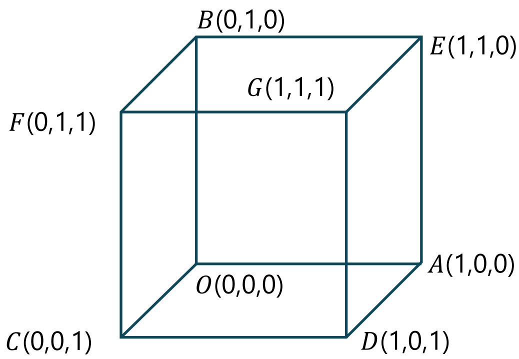
18. Ans. (A)

$DR'S$ of OG = 1, 1, 1

$DR'S$ of AF = −1, 1, 1

$DR'S$ of CE = 1, 1, −1

$DR'S$ of BD = 1, −1, 1



Equation of $OG \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

Equation of $AB \Rightarrow \frac{x-1}{1} = \frac{y}{-1} = \frac{z}{0}$

Normal to both the line's

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \hat{i} + \hat{j} - 2\hat{k}$$

$\overrightarrow{OA} = \hat{i}$

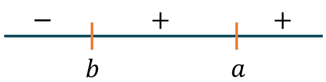
$$S.D. = \frac{\left|\hat{i}.(\hat{i}+\hat{j}-2\hat{k})\right|}{\left|\hat{i}+\hat{j}-2\hat{k}\right|} = \frac{1}{\sqrt{6}}$$

JEE (Main) Practice Paper

SECTION–A

1. Ans. (1)

$f'(x) = (x-a)^{2m}(x-b)^{2n+1}$



$b \rightarrow$ minima

$a \rightarrow$ neither maximum non minima

2. **Ans. (2)**

$C = aV + \frac{b}{V}$

$$\left.\begin{aligned}75 &= 30.a + \frac{b}{30}\\ 65 &= 40.a + \frac{b}{40}\end{aligned}\right\} \quad a = \frac{1}{2},\ b = 1800$$

For C_{min} (Most economical cost) = $C' = 0$

$\Rightarrow C' = a - \frac{b}{V^2} = 0 \quad \Rightarrow V^2 = \frac{b}{a} = 3600$

$\Rightarrow V = 60\ km/h$

3. **Ans. (1)**

Let $f(x) = 2sinx + tanx - 3x$

$f'(x)2cosx + \sec^2 x - 3$

$= \frac{(\cos x - 1)^2\ (2\cos x + 1)}{\cos^2 x} > 0$

$f(x)$ is M.I.

$x > 0$

$f(x) > f(0)$

$2\ sinx\ +\ tanx > 3x$

$3x < 2\ sinx + tan\ x$

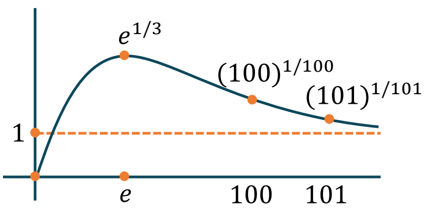
$\Rightarrow \frac{3x}{2\sin x + \tan x} < 1$ for $x \in \left(0, \frac{\pi}{2}\right)$ and $\underset{x\to 0^+}{Lt} \frac{3x}{3\sin x + \tan x} = 1$

$\Rightarrow \underset{x\to 0^+}{Lt}\left[\frac{3x}{3\sin x + \tan x}\right] = 0$

4. **Ans. (2)**

Assume $f(x) = x^{\frac{1}{x}}$ and let us examine monotonic nature of $f(x)$

$f'(x) = x^{\frac{1}{x}} . \left(\frac{1 - \ell nx}{x^2}\right)$



$f'(x) > 0 \quad \Rightarrow x \in (0, e)$

and $f'(x) < 0 \Rightarrow x \in (e, \infty)$

Hence $f(x)$ is M.D. for $x \geq e$

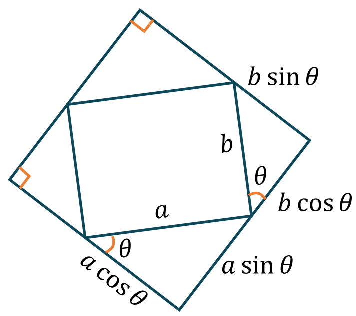
and since $100 < 101$

$\Rightarrow f(100) > f(101)$

$\Rightarrow (100)^{1/100} > (101)^{1/101}$

5. **Ans. (2)**

Area = ab + $\left(\frac{1}{2}a^2 \sin\theta \ \ \cos\theta + \frac{1}{2}b^2 \sin\theta\cos\theta\right).2$



$= ab + \frac{(a^2 + b^2)}{2}\ \ sin\ 2\theta$

Maximum area is $ab + \frac{(a^2 + b^2)}{2}$

6. Ans. (1)

At $x = 0, f(0) = 0 \rightarrow f(x)_{min}$ as for then x, values of $f(x)$ will be positive

$$f(x)=\frac{1}{\frac{1}{(\tan x)^n}+\frac{1}{(\tan x)^{n-1}}+---+\frac{1}{\tan x}+1+\tan x+--+(\tan x)^{n-1}+(\tan x)^n}$$

$$\Rightarrow f(x)=\frac{1}{\underbrace{\left(\underbrace{(\tan x)^n+\frac{1}{(\tan x)^n}}_{\geq 2}\right)+\left(\underbrace{(\tan x)^{n-1}+\frac{1}{(\tan x)^{n-1}}}_{\geq 2}\right)+---+\left[\underbrace{\tan x+\frac{1}{\tan x}}_{\geq 2}\right]+1}_{n \text{ terms, each with least value } = 2}}$$

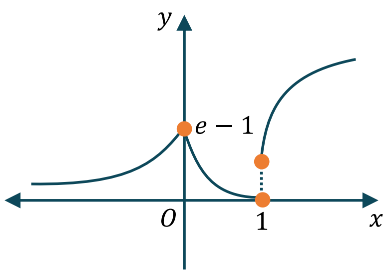
$\therefore f(x)_{\max}=\frac{1}{2n+1}$ at $x = \frac{\pi}{4}$ (each term 2)

$\therefore$ $Range:\left[0,\frac{1}{2n+1}\right]$

for $n\in N,\frac{1}{2n+1}$ is a fraction (Ans. A)

7. Ans. (1)

Graph of the functions



Hence x = 0 is point of maxima and x = 1 is neither maxima nor minima.

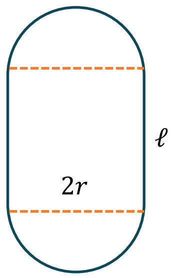
8. Ans. (3)

$2\ell + 2\pi r = 440$

$A = \ell\, 2r = -2\pi r^2 + 440r$

$\frac{dA}{dr} = -4\pi r + 440 = 0$

at r =



Figure

9. Ans. (1)

$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{C-R_1}$

$R = R_1(C - R_1)/C$

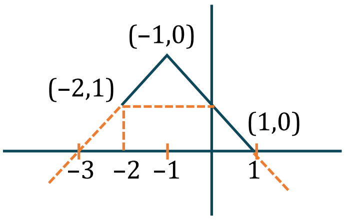
$\frac{dR}{dR_1}=\frac{C-2R_1}{C}$

$\frac{dR}{dR_1} = 0$ at $R_1 = \frac{C}{2}$

$R_2 = \frac{C}{2}$ So, $R_1 = R_2$

10. Ans. (3)

$f(x) = 2 - |x + 1|$



Figure

From figure it is clear that greatest, least values are respectively 2, 0

11. Ans. (3)

$x = 4\cos\theta, y = 3\sin\theta$

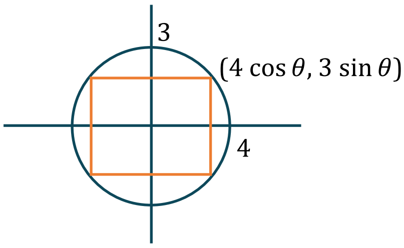
Let A be area.

$A = 4(4\cos\theta)(3\sin\theta)$

$= 24\sin 2\theta$

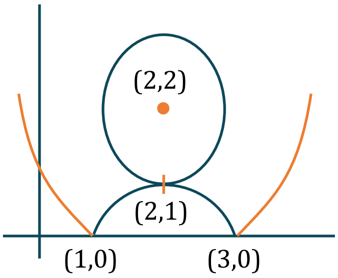
A is maximum when $2\theta = \frac{\pi}{2}$.

$\Rightarrow$ Dimensions are $\frac{2\,.\,4}{\sqrt{2}}, \frac{2.3}{\sqrt{2}}$



Figure

12. Ans. (3)



13. Ans. (3)

Girth = Circumference of solid

$\therefore\ \ell + 2\pi r = 3$

$v = \pi r^2\ell = \pi r^2(3 - 2\pi r)$

$v = \pi(3r^2 - 2\pi r^3)$

$\frac{dv}{dr} = 6r - 6\pi r^2 = 0 \Rightarrow r = 0 \text{ or } r = \frac{1}{\pi}$

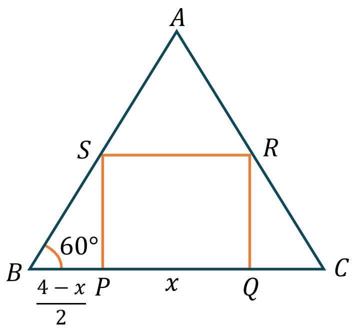
$\therefore\ v_{max} = \pi\left(\frac{3}{\pi^2} - \frac{2\pi}{\pi^3}\right) = \frac{1}{\pi}cm^3$

14. Ans. (4)

Let $PQ = x$ Then $BP = \frac{4-x}{2}$

$\therefore\ PS = \frac{4-x}{2}\tan 60° = \frac{\sqrt{3}\ (4-x)}{2}$

$\therefore$ area A of rectangle $= \frac{\sqrt{3}}{2}\ (4-x)x$



$\frac{dA}{dx} = \frac{\sqrt{3}}{2}(4 - 2x) = 0 \qquad \Rightarrow x = 2$

$\frac{d^2A}{dx^2} = -\sqrt{3} < 0 \therefore A$ is maximum, when $x = 2$.

$\therefore$ Maximum area $= \frac{\sqrt{3}}{2} 2.2 = 2\sqrt{3}$.

Square of maximum area = 12

15. **Ans. (1)**

$\frac{x^2+x+2}{x^2+5x+6} < 0 \Rightarrow x \in (-3, -2)$

For maximum or minimum of the function, put $f'(x) = 0$

$\Rightarrow a^2 - 3x^2 = 0 \Rightarrow x = -\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}$

If $a > 0$, then point of minima is $x = -\frac{a}{\sqrt{3}} \Rightarrow -3 < -\frac{a}{\sqrt{3}} < -2$ or $2\sqrt{3} < a < 3\sqrt{3}$

if $a < 0$, then point of minima is $x = \frac{a}{\sqrt{3}}$

$\Rightarrow -3 < \frac{a}{\sqrt{3}} < -2 \Rightarrow -3\sqrt{3} < a < -2\sqrt{3}$

16. **Ans. (4)**

$f'(x) = \sin x \, \cos x \, (3 \sin x + 2\lambda)$

$f'(x) = 0$

$\Rightarrow \sin x = 0 \quad$ or $\quad \cos x = 0 \quad$ or $\quad \sin x = \frac{-2\lambda}{3}$

$\Rightarrow x = 0 \qquad$ or $\quad \sin x = \frac{-2\lambda}{3} \quad$ (as $\cos x = 0$ is not possible).

If $\lambda = 0$ then $f'(x) \geq 0 \Rightarrow$ no extrema,

hence $\lambda \neq 0$

$\Rightarrow -1 < \frac{-2\lambda}{3} < 0 \quad$ or $\quad 0 < \frac{-2\lambda}{3} < 1$

$\Rightarrow 0 < \lambda < \frac{3}{2} \qquad$ or $\quad -\frac{3}{2} < \lambda < 0$

17. **Ans. (2)**

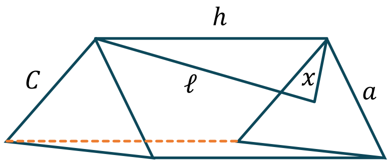
$\ell^2 = h^2 + x^2$

Area of base (triangle) is $\frac{\sqrt{3}}{4}a^2$

$3x = \frac{\sqrt{3}}{2}a$

Volume $V = \frac{\sqrt{3}}{4} ha^2$

$= h\frac{\sqrt{3}}{4}.4.3.x^2$



Figure

$= 3\sqrt{3}\, h(\ell^2 - h^2)$

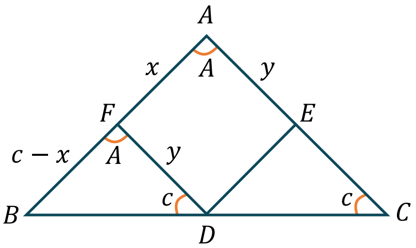
$\dfrac{dV}{dh} = 3\sqrt{3}\ (\ell^2 - 3h^2)$

V is maximum when h = $\dfrac{\ell}{\sqrt{3}}$.

18. **Ans. (1)**

From similar triangles $\Delta ABC, \Delta FBD$

$\dfrac{c-x}{y} = \dfrac{c}{b}$



Area of $AFDE = xy \sin A$

$= \dfrac{b}{c}(c - x) \sin A$

It is maximum when $x = \dfrac{c}{2}$

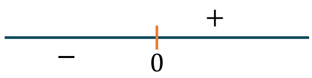
$\therefore$ Maximum area = $\dfrac{bc}{4}$ sin A

statement -1 is true

statement-2 is obvious.

19. **Ans. (3)**

$f'(x) = \dfrac{2x.\ \ell n2.2^{x^2}(2^{x^2}+1-\sqrt{2})\ (2^{x^2}+1+\sqrt{2})}{(2^{x^2}+1)^2}$



$2^{x^2} \geq 1$

$2^{x^2} + 1 - \sqrt{2} \geq 2 - \sqrt{2} > 0$

At $x = 0$, $f(x)$ is least.

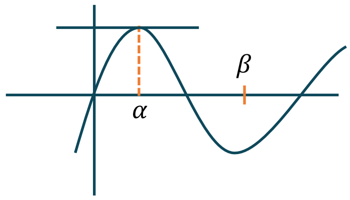
Least value = $f(0) = 1$

20. **Ans. (1)**

$f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$

$D = 36(a-7)^2 - 4\times3\times3(a^2-9) = 36(a^2+49-14a-a^2+9)$

$= 36(58-14a)$



$f'(x) = 0$

at $x = \dfrac{-6(a-7)\pm\sqrt{36(58-14a)}}{6}$

$\alpha = 7-a-\sqrt{58-14a} \cdot \beta = 7-a+\sqrt{58-14a}$

Now $\alpha > 0$ & $D > 0$

$a^2 - 9 > 0$ & $a < \dfrac{58}{14}$

$a \in (-\infty,-3)\cup\left(3,\dfrac{58}{14}\right)$

So, $a_2 = -3$

$a_3 = 3$

$a_4 = \frac{58}{14}$

$a_2 + 11a_3 + 70a_4 = 320$

SECTION-B

1. Ans. (36)

Let $y = \sqrt{-3+4x-x^2}$

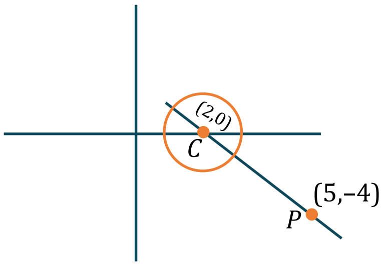
$x^2 + y^2 - 4x + 3 = 0$

$(x-2)^2 + y^2 = 1$, center $C = (2, 0)$

Consider point $P(5, -4)$

$CP = \sqrt{9+16} = 5$

Maximum value of $\left(\sqrt{-3+4x-x^2}+4\right)^2 + (x-5)^2$ is $(5 + 1)^2 = 36$.



Figure

2. Ans. (4)

Let $f(x) = x^4 + 4x^3 - 8x^2 + k$

$f'(x) = 4x^3 + 12x^2 - 16x$

$= 4x(x^2+3x-4) = 4x(x+4)(x-1)$

$\Rightarrow f'(x) = 0 \Rightarrow x = -4, 0, 1$

$f''(x) = 12x^2 + 24x - 16 = 4(3x^2+6x-4)$

$f''(-4) = 20 > 0$

$f''(0) = -16 < 0$

$f''(1) = 20 > 0$

$\Rightarrow x = -4$ and $x = 1$ are points of local minima whereas

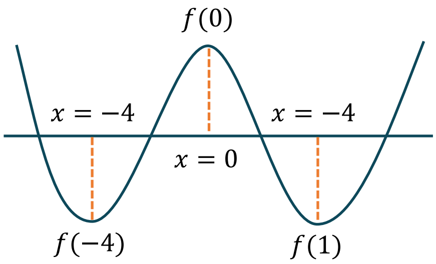
$x = 0$ is point of local maxima

for $f(x) = 0$ to have 4 real roots

$f(-4) < 0 \quad \Rightarrow k < 128$

$f(0) > 0 \quad \Rightarrow k > 0$

$f(1) < 0 \quad \Rightarrow k < 3 \quad \Rightarrow k \in (0, 3)$



Figure

3. Ans.(2)

$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2$

$f''(x) = 12x - 18a$

For maximum and minimum,

$6x^2 - 18ax + 12a^2 = 0 \Rightarrow x^2 - 3ax + 2a^2 = 0$

$x = a$ or $x = 2a$, at $x = a$ maximum and at $x = 2a$ minimum

$\because \; p^2 = q$,

$\therefore \; a^2 = 2a \Rightarrow a = 2$ or $a = 0$

But $a > 0$, therefore $a = 2$.

4. Ans. (10)

Let $u = 2^x + 2^{-x}$

$u^3 = 8^x + 8^{-x} + 3(2^x)(2^{-x})(2^x + 2^{-x}) \Rightarrow u^3 - 3u = 8^x + 8^{-x}$

also, $4^x + 4^{-x} = u^2 - 2 \Rightarrow f(x) = u^3 - 3u - 4(u^2 - 2) = u^3 - 4u^2 - 3u + 8$

Let $g(u) = u^3 - 4u^2 - 3u + 8$; $u \geq 2$

$g'(u) = 3u^2 - 8u - 3 = (3u+1)(u-3)$

putting $g'(u) = 0$; we get $u = 3$

$g''(u) = 6u - 8 \Rightarrow g''(3) = 1 > 0$

$\Rightarrow u = 3$ is point of minima $\Rightarrow g(3) = 27 - 36 - 9 + 8 = -10$

$\Rightarrow$ minimum $f(x) = -10$

5. Ans. (1)

$y = 1 - x^2$

Consider point $P(x_0, 1 - x_0^2)$

$0 < x_0 \leq 1$

equation of tangent at P is

$y - (1 - x_0^2) = -2x_0(x - x_0)$

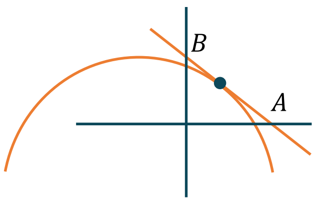
intersection with x-axis at

$\Rightarrow A \equiv \left(x_0 + \dfrac{(1 - x_0^2)}{2x_0}, 0\right)$

intersection with y-axis at

$B(0, 2x_0^2 + (1 - x_0^2))$

area of ΔOAB $\Delta = \dfrac{1}{2}\dfrac{(x_0^2+1)^2}{2x_0} = \dfrac{1}{4}\dfrac{(x_0^2+1)^2}{x_0}$



Figure

$\dfrac{dA}{dx_0} = \dfrac{(x_0^2+1)}{4x_0^2}[3x_0^2 - 1] = 0 \Rightarrow x_0 = \dfrac{1}{\sqrt{3}}$

$\dfrac{dA}{dx_0}$ changes sign from $+ve$ to $-ve$

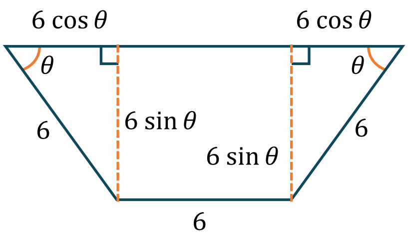
at $x_0 = \dfrac{1}{\sqrt{3}}$ So point of minimum $\Rightarrow A_{min} = \dfrac{4\sqrt{3}}{9}$

6. Ans. (81)

$Area = 36(1 + \cos\theta)\sin\theta = 36.4.\cos^3(\theta/2).\sin(\theta/2)$

Maximum occurs when

$\tan\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1}{3}} \Rightarrow \theta = \dfrac{\pi}{3}$



Figure

7. Ans. (45)

$f(x)=x^3y=x^3(60-x)$

$f'(x)=4x^2(45-x)$

$f(x)$ is maximum at $x = 45$

8. Ans. (5)

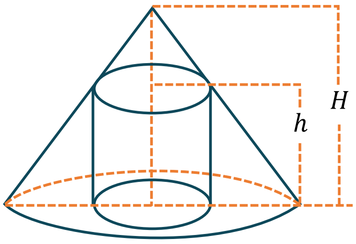
$\frac{H}{R}=\frac{H-h}{r}$

$S=2\pi rh$

$=2\pi H\left(r-\frac{r^2}{R}\right)$

$\frac{dS}{dr}=2\pi H\left(1-\frac{2r}{R}\right)$

Maximum at $r=\frac{R}{2}$



Figure

9. Ans. (3)

$x + y = k$ constant

Area of triangle be A.

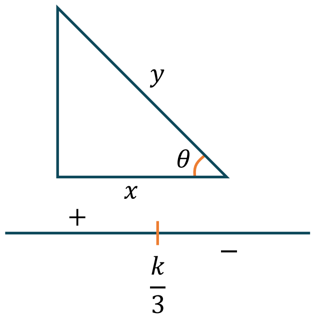
$A=\frac{1}{2}\times\sqrt{y^2-x^2}$

$A=\frac{1}{2}\times\sqrt{k}\sqrt{k-2x}$

$\frac{dA}{dx}=\frac{\sqrt{k}}{2}\frac{(k-3x)}{\sqrt{k-2x}}$

Area is maximum when $x=\frac{k}{3}$.

$\Rightarrow y=\frac{2k}{3}\Rightarrow \cos\theta=\frac{\frac{k}{3}}{\frac{2k}{3}}=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$.



sign of $\frac{dA}{dx}$

10. Ans. (3)

$\frac{a^3+b^3}{48}=\frac{a^3+(8-a)^3}{48}=\frac{8}{48}\left(3a^2-24a+64\right)$

Minimum $\frac{a^3+b^3}{48}$ is $\frac{8}{48}\frac{(4.3.64-24^2)}{4.3}=\frac{8}{3}$

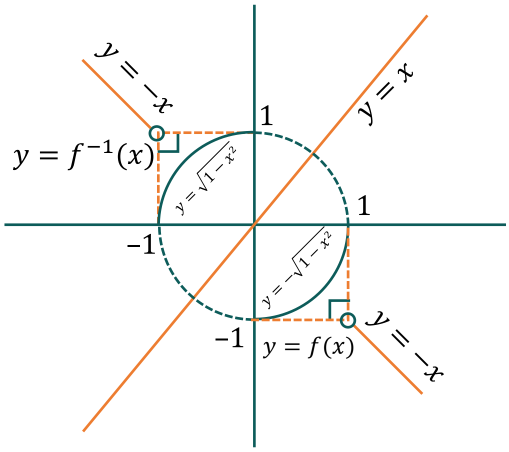
JEE (Advanced) Practice Paper

1. Ans. (A,C)

From graph $f^{-1}(x)=\begin{cases}-x & ; \quad x<-1\\ \sqrt{1-x^2} & ; \quad -1\le x\le 0\end{cases}$

Maximum of $f(x)$ exist at $x = 1$

Minimum of $f^{-1}(x)$ exist at $x = -1$



Figure

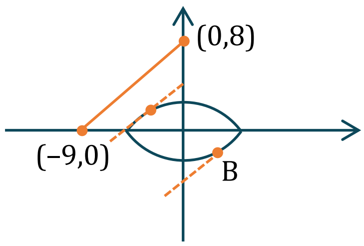
2. **Ans. (A, D)**

$4x^2+9y^2=1 \quad , \quad 9y-8x=72$

$\frac{x^2}{\frac{1}{4}}+\frac{y^2}{\frac{1}{9}}=1 \quad , \quad \frac{x}{-9}+\frac{y}{8}=1$

$\therefore$ closest & earthest points on ellipse lie on tangent parallel to $9y = 8x + 72$

$\therefore m_T=\frac{8}{9}$



$T: y=m_T x \pm \sqrt{a^2m^2+b^2}$

$y=\frac{8}{9}x\pm\sqrt{\frac{1}{4}\cdot\frac{64}{9^2}+\frac{1}{9}}$

$y=\frac{8}{9}x\pm\frac{5}{9}\rightarrow 9y=8x\pm 5$

Tangent to ellipse on (x_1,y_1) is

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1\rightarrow 4xx_1+9yy_1=1$

$\therefore T: \frac{-8}{5}x+\frac{9}{5}y=1, \frac{8}{5}x-\frac{9}{5}y=1$

$\therefore x_1=\pm\frac{2}{5}, y_1=\mp\frac{1}{5}$

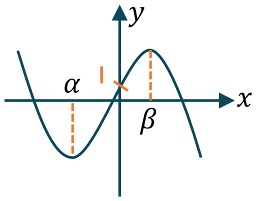
So, $A\left(\frac{-2}{5},\frac{1}{5}\right)$ & $B\left(\frac{2}{5},\frac{-1}{5}\right)$

$A(P,q)$ & $B(h,k)$

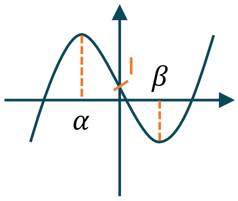
$P+q=\frac{-1}{5}, h+k=\frac{1}{5}$

3. **Ans. (B, C, D)**

Let $\alpha < 0$ & $\beta > 0$; $f(0) = 1$



$(f(\alpha) < 0, f(\beta) > 0)$
(three distinct real roots with one positive root)



$(f(\alpha) > 0, f(\beta) < 0)$
(three distinct real roots with one negative root)

4. **Ans. (A,B,C)**

$f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{2}\,\dfrac{1}{x}$, $x > 0$

$= \dfrac{-(x-1)^2}{2x(1+x^2)} \leq 0 \;\; \forall \;\; x > 0.$

$f(x)$ is decreasing $\forall$ $x > 0$.

On $\left[\dfrac{1}{\sqrt{3}}, \;\; \sqrt{3}\right]$, greatest value is

$f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6} - \dfrac{1}{2}\,\ell n\left(\dfrac{1}{\sqrt{3}}\right)$ and least value is

$f(\sqrt{3}) = \dfrac{\pi}{3} - \dfrac{1}{2}\ell n.\sqrt{3}$

5. **Ans. (B,D)**

$x = r\cos\theta, y = r\sin\theta$

$\Rightarrow \dfrac{4}{r^2} = 5(1+\cos 2\theta) + 3\sin 2\theta + 1 - \cos 2\theta$

$= 6 + 4\cos 2\theta + 3\sin 2\theta$, which has maximum 11 and minimum 1

$\therefore$ OP has minimum $\dfrac{2}{\sqrt{11}}$ and maximum 2.

6. **Ans. (A,B,C)**

$V = k\sqrt{\left(\dfrac{\lambda}{a}\right)+\left(\dfrac{a}{\lambda}\right)}$

V will be minimum when $\dfrac{\lambda}{a} + \dfrac{a}{\lambda}$ will be minimum

$A.M. \geq G.M.$

$\dfrac{\frac{\lambda}{a}+\frac{a}{\lambda}}{2} \geq \sqrt{\dfrac{\lambda}{a} \times \dfrac{a}{\lambda}}$

$\dfrac{\lambda}{a}+\dfrac{a}{\lambda} \geq 2 \Rightarrow$ minimum of $\dfrac{\lambda}{a}+\dfrac{a}{\lambda} = 2$

$V_{min} = K\sqrt{2}$ which is independent of a.

7. Ans. (A)

$P'(x)=k(x-1)(x-2)(x-3)=k(x^3-6x^2+11x-6)$

$P(x)=k\left(\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x\right)$

$P(-1)=k\left(\frac{1}{4}+2+\frac{11}{2}+6\right)$

$55=\frac{k}{4}(1+8+22+24)$

$k=4$

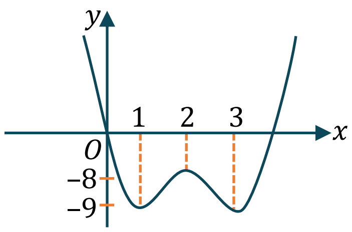
$\therefore\ P(x)=x^4-8x^3+22x^2-24x$

Points of extremum are

(1, −9), (2, −8), (3, −9)

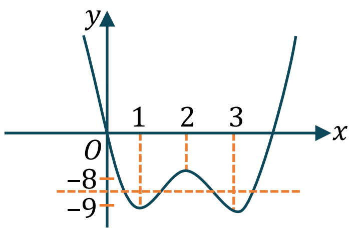
Area = 1

8. Ans. (D)



$\Rightarrow$ negative integral values which $f(x)$ can have −9,−8, −7, −6, −5, −4, −3, −2, −1 i.e. 9 values.

9. Ans. (B)



$\Rightarrow 8<\mu<9$

10. Ans. (C)

$\sin x$ is concave downward in $(0, \pi)$

and $\sin x$ is concave upward in $(\pi, 2\pi)$

11. Ans. (D)

e^x, 2^x, $\tan^{-1}x$ (If $x \in R^-$) is concave upward and ℓnx is concave downward

12. Ans. (A)

$f(x)$ concave downward $(f''(x)<0)$

$f(x)$ increasing $\quad (\therefore f'(x) \leq 0)$

Let $g(x)=f^{-1}(x)=x$

$f'(g(x))=x$

$f'(g(x)).g'x=1$

$g'(x) = \dfrac{1}{f'(g(x))} > 0$

$g''(x) = -\dfrac{1}{(f'(g(x))^2} \times f''(x).g'(x)$

$g''(x) > 0$

$g(x) = f^{-1}(x)$ concave upward

13. Ans. (191)

Let No. of children of john & anglina = y

$\therefore\ x + (x + 1) + y = 24$

$y = 23 - 2x$

Number of fights

$F = x(x+1) + x(23-2x) + (x+1)(23-2x)$

$F = -3x^2 + 45x + 23$

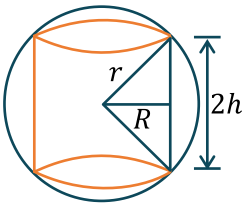
$\dfrac{df}{dx} = 0 \Rightarrow -6x + 45 = 0 \Rightarrow x = 7.5$

But 'x' wil be integral.

check $x = 6$ or $x = 7$

$F = 191$

14. Ans. (9)



Figure

$R^2 + r^2 = h^2$

$R^2 = h^2 - r^2$

volume of cylinder,

$V = \pi R^2(2h) = \pi(2h)(\sqrt{r^2 - h^2})^2$

$\dfrac{dV}{dh} = 2\pi(r^2 - h^2) + 2\pi h(-2h) = 0$

$\Rightarrow r^2 = 3h^2 \Rightarrow h = \dfrac{r}{\sqrt{3}}$

$\dfrac{d^2V}{dh^2} < 0$ at $h = \dfrac{r}{\sqrt{3}}$ $\quad \Rightarrow V_{max} = 2\pi \dfrac{r}{\sqrt{3}}\left(r^2 - \dfrac{r^2}{3}\right) = \dfrac{4\pi r^3}{3\sqrt{3}}$

15. Ans. (2)

$A = 2rh + \pi r^2 + \pi rh$

$V = \dfrac{\pi r^2 h}{2} \Rightarrow h = \dfrac{2V}{\pi r^2}$

$$A=\frac{4V}{\pi r}+\pi r^2+\frac{2V}{r}$$

For A_{min} = $\frac{dA}{dr}=0$

$$\Rightarrow \frac{dA}{dr}=\frac{-4V}{\pi r^2}+2\pi r-\frac{2V}{r^2}=0$$

$$\Rightarrow \frac{V}{\pi r^3}=\frac{\pi}{\pi+2}=\frac{h}{2r}$$

16. Ans. (40 mph)

Fuel charges per hour = $k\nu^2 \Rightarrow 48 = k.\,16^2$

$\Rightarrow$ Fuel charges per hour = $\frac{3}{16}\,\nu^2$

Charges per hour = $\frac{3}{16}\nu^2 + 300$

Expenses of journey = $(\frac{3}{16}\nu^2 + 300)\frac{s}{\nu}$

where ν = speed s = distance

Maximum occures when $\frac{3\nu}{16}=\frac{300}{\nu}$

($\because$ $ax + \frac{b}{x}$, $a, b, > 0, x > 0$, has minimum when $ax = \frac{b}{x}$)

$\nu^2 = 16.100$

$\nu = 40$

17. Ans. (32)

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6$$

$$\lim_{x\to 0}\left(1+\frac{f(x)}{x^3}\right)=1 \Rightarrow a_0=a_1=a_2=a_3=0$$

$$\lim_{x\to 0}\left(1+\frac{f(x)}{x^3}\right)^{1/x}=e^2$$

$$\lim_{x\to 0}e^{(a_4+a_5x+a_6x^2)} = e^2 \Rightarrow a_4 = 2$$

$$f(x)=2x^4+a_5x^5+a_6x^6$$

$$f'(x)=x^3\left(8+5a_5x+6a_6x^2\right)$$

$$f'(1)=0, f'(2)=0$$

$$a_5=-\frac{12}{5}\ ,\ a_6=\frac{2}{3}$$

$$f(x) = 2x^4-\frac{12}{5}x^5+\frac{2}{3}x^6$$

18. Ans. (5)

Let $f(x) = \dfrac{\sin x}{x}$

$f'(x) = \dfrac{x\cos x - \sin x}{x^2} = \dfrac{\cos x(x - \tan x)}{x^2} < 0 \ \forall \ x \in \left(0, \dfrac{\pi}{2}\right); (\therefore \tan x > x)$

$f''(x) = \dfrac{-x^2 \sin x - 2x\cos x + 2\sin x}{x^3}$

Let $g(x) = -x^2 sinx - 2xcosx + 2sin\, x$

$g'(x) = -x^2 cosx < 0 \ \forall \ x \in (0, \pi/2)$

for $x > 0$, we have $g(x) < g(0)$ i.e. $g(x) < 0$

$\therefore f'(x) < 0$ and $f''(x) < 0 \ \forall \ x \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow f\left(\dfrac{A+B+C}{3}\right) > \left(\dfrac{f(A)+f(B)+f(C)}{3}\right)$

$$\Rightarrow \frac{\sin\left(\dfrac{A+B+C}{3}\right)}{\dfrac{A+B+C}{3}} \geq \left(\frac{\dfrac{\sin A}{A} + \dfrac{\sin B}{B} + \dfrac{\sin C}{C}}{3}\right)$$

$$\Rightarrow \frac{\sin\ A}{A} + \frac{\sin\ B}{B} + \frac{\sin\ C}{C} \leq \frac{9\sqrt{3}}{2\pi}$$