04 Gravitation

What is Gravitation

Gravitation is the phenomena in which objects with masses have a natural tendency to attract each other.

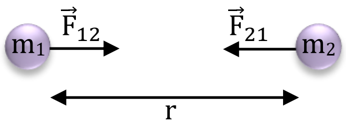
Newton's Law of Gravitation

It states that every particle in the universe attract all other particles with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

$$\left|\vec{F}_{12}\right| = \left|\vec{F}_{21}\right| = F_g$$

$F_g \propto m_1 m_2$ ...(i)

$F_g \propto \frac{1}{r^2}$ ...(ii)



By equation (i) & (ii)

$$\Rightarrow F_g \propto \frac{m_1 m_2}{r^2} \quad \boxed{F_g = \frac{Gm_1 m_2}{r^2}}$$

Universal Gravitational Constant "G"

- Universal Gravitational constant is a scalar quantity.
- **Value of G : SI :** $G = 6.67 \times 10^{-11}$ N–m²/kg² ;

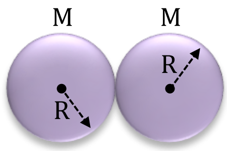
  CGS : $G = 6.67 \times 10^{-8}$ dyne–cm²/g² **Dimensions :** $[M^{-1}L^3T^{-2}]$
- Its value is same throughout the universe; G does not depend on the nature and size of the bodies; it does not depend even upon the nature of the medium between the bodies.

Note: This formula is given for point masses.

For spherical objects (planets), distance between centers is taken.

Illustration 1

Two solid spheres of same size of a certain metal are placed in contact with each other. Prove that the gravitational force acting between them is directly proportional to the fourth power of their radius.



Solution:

The mass of the spheres may be assumed to be concentrated at their centres.

So $$F = \frac{G\left[\frac{4}{3}\pi R^3\rho\right]\times\left[\frac{4}{3}\pi R^3\rho\right]}{(2R)^2} = \frac{4}{9}(G\pi^2\rho^2)R^4 \Rightarrow F \propto R^4$$

Illustration 2

Two particles of mass 1kg and 2kg are placed at a separation of 50cm. Assuming that the only forces acting on the particles are their mutual gravitation. Find the initial acceleration of heavier particle.

Solution:

$\because$ we know that $F_g = \dfrac{Gm_1m_2}{r^2}$

$\Rightarrow \quad F_g = \dfrac{6.67\times10^{-11}\times2\times1}{(0.5)^2}$

$\Rightarrow \quad F_g = 5.34 \times 10^{-10}$ N

here
$m_1 = 2$kg
$m_2 = 1$kg
$r = 50$ cm $= 0.5$ m

Acceleration of heavier particle $(a_1) = \dfrac{F_g}{m_1} \Rightarrow a_1 = \dfrac{5.34\times10^{-10}}{2} \Rightarrow a_1 = 2.67 \times 10^{-10}$ m/s^2

Important Points

(1) Gravitational force is always attractive.

(2) Gravitational forces are developed in the form of action and reaction pair. They obey Newton's third law of motion.

(3) It is the weakest of all fundamental forces in nature.
Strong Nuclear force > Electromagnetic force > Weak forces > Gravitational force

(4) To see the effect of gravitational pull practically, at least one of the bodies must be celestial.

(5) It is two body interaction i.e. gravitational force between the two particles is independent of the presence or absence of other bodies or particles.

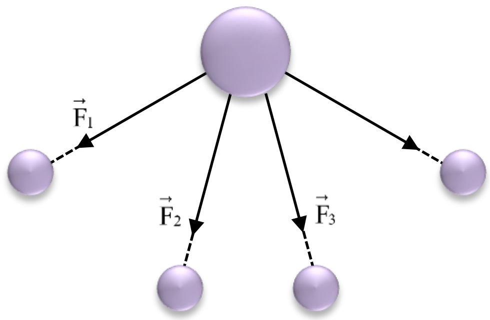
(6) Force of gravitation does not depend on medium.

(7) Force of gravitation always acts along the line joining two particles, so it is a central force.

(8) Gravitational force is conservative force, so work done by gravitational force does not depends on path.

(9) If any particle moves along a closed path under the action of gravitational force, then the work done by this force is always zero for round the trip.

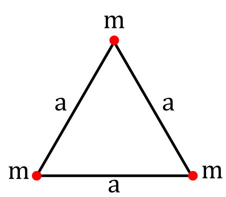
Force due to multiple particles

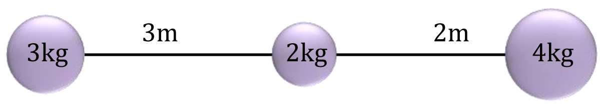


$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots\ldots$$

Illustration 3

Find net force on 2kg due to 3kg and 4kg particles.





Solution:

∵ We know that

$$F_g = \because \frac{Gm_1m_2}{r^2}$$

$$\Rightarrow \quad F_1 = \frac{6.67\times10^{-11}\times2\times3}{(3)^2} = 4.44 \times 10^{-11}\text{N}$$

Similarly $F_2 = \dfrac{6.67\times10^{-11}\times2\times4}{(2)^2} = 13.34 \times 10^{-11}\text{N}$

So Net force

$$\Rightarrow \quad F_{net} = F_2 - F_1 \quad = 13.34 \times10^{-11} - 4.44 \times 10^{-11} = 8.9 \times 10^{-11}\text{N}$$

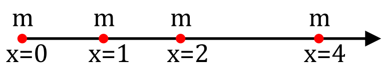
Here

F_1 = Force on 2kg due to 3kg mass

F_2 = Force on 2kg due to 4kg mass

Illustration 4

Infinite particles each of mass m are placed at positions x = 1m, x = 2m, x = 4m .... ∞. Find the magnitude of gravitational force on particle at origin (x = 0)



Solution:

∵ We know that, $F_g = \dfrac{Gm_1m_2}{r^2}$

So, the gravitational force on mass m placed at x = 0

$$F_g = \frac{Gm^2}{(1)^2} + \frac{Gm^2}{(2)^2} + \frac{Gm^2}{(4)^2} + \ldots\ldots\ldots\ldots$$

$$\Rightarrow \quad F_g = Gm^2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{4^2} + \ldots\ldots\right)$$

$$\Rightarrow \quad F_g = Gm^2\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \ldots\ldots\right)$$

$$\Rightarrow \quad F_g = Gm^2\left(\frac{1}{1-\frac{1}{4}}\right) = Gm^2\left(\frac{4}{3}\right) = \frac{4Gm^2}{3}$$

∵ from geometric progression

$$S_\infty = \frac{a}{1-r}$$

Here

a = first term

r = common ratio

$a = 1$

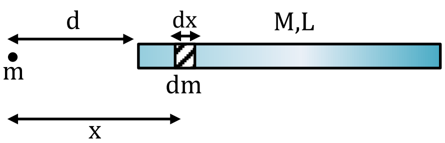
$r = \dfrac{1}{4}$

Illustration 5

A particle of mass m is placed at a distance "d" from one end of mass M and length L as shown in figure. Find the gravitational force on the particle by the rod.

Solution:

Since Rod is not a point mass.



$$dF_g = \frac{Gmdm}{x^2} \Rightarrow \int_0^F dF_g = \int_d^{d+L} \frac{Gm \times \left(\frac{M}{L}dx\right)}{x^2} = \frac{GMm}{L}\int_d^{d+L} x^{-2}dx$$

$$\Rightarrow F_g = \frac{GMm}{L}\left[\frac{x^{-1}}{-1}\right]_d^{d+L} = \frac{-GMm}{L}\left[\frac{1}{x}\right]_d^{d+L}$$

$$\Rightarrow F_g = \frac{-GMm}{L}\left[\frac{1}{d+L} - \frac{1}{d}\right] \Rightarrow F_g = \frac{-GMm}{L}\left[\frac{d-(d+L)}{d(d+L)}\right] = \frac{GMm}{d(d+L)}$$

By unitary method

$$L \longrightarrow M$$

$$L \longrightarrow \frac{M}{L}$$

$$dx \longrightarrow \frac{M}{L}dx \Rightarrow dm.$$

$$\Rightarrow dm = \frac{M}{L}dx.$$

Illustration 6

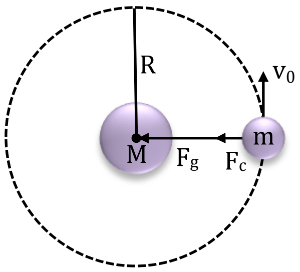
A small object of mass m rotates about a very large and heavy object of mass M with constant speed V_0. Then find the radius of its circular path R.

Solution:

$\because$ We know that, $F_g = \dfrac{GMm}{R^2}$

Here for circular motion of particle, necessary centripetal force is provided by gravitation force.

$\Rightarrow$ $F_g = F_{CP}$



R $\rightarrow$ Radius of circle.

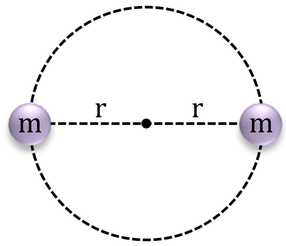
$\because$ $F_{CP} = \dfrac{mv_0^2}{R}$

$\Rightarrow$ $\dfrac{GMm}{R^2} = \dfrac{mv_0^2}{R}$

$\Rightarrow$ $R = \dfrac{GM}{v_0^2}$

☞ When mass of two objects are comparable they both rotates about a common center if both have sufficient speed.

For example, Binary star systems.



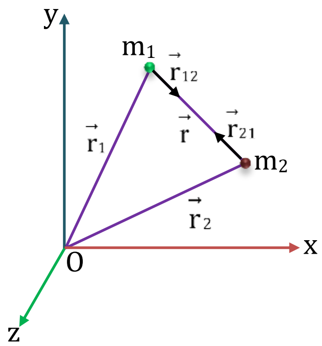
Law of gravitation in vector form

Here $\vec{F}_{12}$ = Gravitational force on m_1 by m_2.

$\vec{F}_{21}$ = Gravitational force on m_2 by m_1.

$\vec{F}_{12} = -\vec{F}_{21}$

$\Rightarrow$ $\left|\vec{F}_{12}\right| = \left|\vec{F}_{21}\right|$



$\vec{F}_{12} = -\dfrac{Gm_1m_2}{r^2}\hat{r}$

or $\vec{F}_{12} = -\dfrac{Gm_1m_2}{r^3}\vec{r}$

where $\hat{r}$ is a unit vector from m_2 to m_1 and $\vec{r} = \vec{r}_1 - \vec{r}_2$

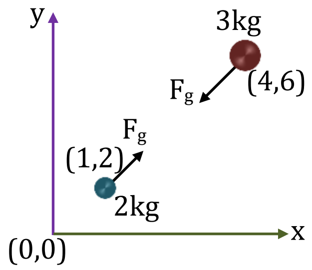
Note: The gravitational forces $\vec{F}_{12}$ and $\vec{F}_{21}$ form action-reaction pair.

Negative sign shows that this force is attractive in nature.

Illustration 7

Two particles of mass 2kg and 3kg are at points (1, 2) and (4, 6) respectively. Gravitational force on 3kg particle due to 2kg particle is?

Solution:



Force or mass 3 kg due to 2 kg in vector form is

$\vec{F} = -|Fg|\hat{r}$

$\hat{r} \rightarrow$ unit vector from 2 kg to 3 kg

$$\hat{r} = \frac{\vec{r}}{r} = \frac{(4\hat{i}+6\hat{j})-(\hat{i}+2\hat{j})}{5}$$

$$\hat{r} = \frac{3\hat{i}+4\hat{j}}{5}$$

$$F_g = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times2\times3}{(5)^2}$$

$$\vec{F} = -\frac{1.60\times10^{-11}}{5}(3\hat{i}+4\hat{j})N$$

BEGINNER'S BOX-1

1. Four identical point masses, each equal to M are placed at the four corners of a square of side a. Calculate the force of attraction on another point mass m_1 kept at the centre of the square.

PGR148

2. Three identical particles each of mass m are placed at the three corners of an equilateral triangle of side "a". Find the gravitational force exerted on one body due to the other two.

PGR149

3. Three identical point masses, each of mass 1 kg lie in the x–y plane at points (0,0) (0, 0.2m) and (0.2m, 0) respectively. The gravitational force on the mass at the origin is :–

(A) $1.67 \times 10^{-11}(\hat{i}+\hat{j})$ N (B) $3.34 \times 10^{-10}(\hat{i}+\hat{j})$ N

(C) $1.67 \times 10^{-9}(\hat{i}+\hat{j})$ N (D) $3.34 \times 10^{-10}(\hat{i}-\hat{j})$ N

PGR150

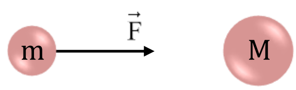
Gravitational field

- It is the region around a mass where its gravitational effects can be observed.
- Gravitational field due to a body extends till infinity.
- Strength of gravitational field decreases on moving away from a body.
  (i) At infinite gravitational field is zero. (ii) Gravitational field $\propto$ mass
- Gravitational field is conservative in nature.
  There are two ways to measure strength of gravitational field.
  (i) Gravitational field intensity. (ii) Gravitational potential

Gravitational field intensity ($\vec{I}$)

It is defined as gravitational force exerted on a unit mass placed in a gravitational field.

$$\vec{I} = \frac{\vec{F}}{m}$$



- It is a vector quantity.
- Its direction is same as that of gravitational force.
- Unit : N/kg. · Dimension $\rightarrow [M^0L^1T^{-2}]$

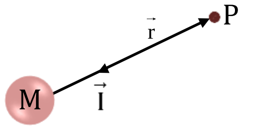
Force on a particle of mass 'm' placed in gravitational field having intensity $\vec{I}$ can be written as –

$$\vec{F} = m\vec{I}$$

Gravitational field intensity due to a point mass

Intensity due to a particle of mass M at a position $\vec{r}$ with respect to itself can be written as –

Gravitational field intensity = gravitational force exerted on unit mass.



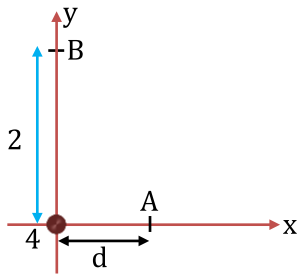
$$\vec{I}=\frac{GM}{r^2}(-\hat{r})$$

Where "M" is the mass of that particle due to which we have to find intensity.

- Negative sign indicates that the force due to gravitational field is attractive in nature.

Illustration 8

Particle of mass 4m is placed at origin. Find gravitational field intensity at point A and point B respectively.



Solution:

$\because$ we know that, $\vec{I}=\frac{GM}{r^2}(-\hat{r})$

So $\vec{I}_A=\frac{G(4m)}{d^2}(-\hat{i})$

and similarly, $\vec{I}_B=\frac{G(4m)}{(2d)^2}(-\hat{j})$

Illustration 9

Gravitational field intensity due to a particle of mass "M" at 5m distance is 10 units, then find its value at 10m.

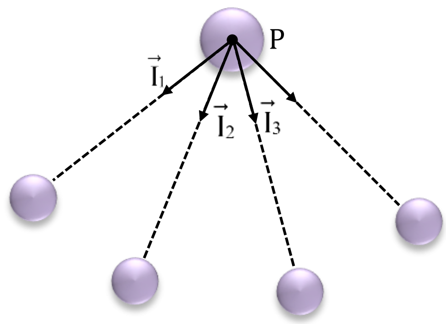
Solution:

As we know that. $I=\frac{GM}{r^2}$

Here 'M' is same

So $I\propto\frac{1}{r^2}\Rightarrow\frac{I_1}{I_2}=\frac{r_2^2}{r_1^2}\Rightarrow\frac{10}{I_2}=\frac{(10)^2}{(5)^2}\Rightarrow I_2=2.5\text{units}$

Gravitational Field intensity due to multiple particles

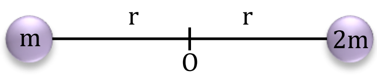


- Write gravitational field intensity due to all particles individually.
- The resultant of all the intensity vector will give net gravitational field intensity at a given point.

$$\vec{I}_{net}=\vec{I}_1+\vec{I}_2+\vec{I}_3+\ldots\ldots\ldots..$$

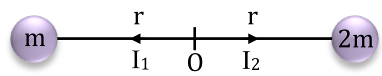
Illustration 10

Find net field intensity at mid-point + O.



Solution:

$\because$ we know that



$\vec{I}=\frac{GM}{r^2}(-\hat{r})$

$\vec{I}_1=\frac{Gm}{r^2}(-\hat{i})$

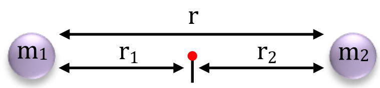
Similarly $\vec{I}_2=\frac{G(2m)}{r^2}(\hat{i})$

$\vec{I}_{net}=\vec{I}_1+\vec{I}_2 \quad\Rightarrow\quad \vec{I}_{net}=\frac{-Gm}{r^2}\hat{i}+\frac{2Gm}{r^2}\hat{i}=\left(\frac{2Gm}{r^2}-\frac{Gm}{r^2}\right)(\hat{i})\Rightarrow\vec{I}_{net}=\frac{Gm}{r^2}(\hat{i})$

Neutral point

A point in space, Where $I_{net} = 0$

- For two particle system, Neutral point exists on the line joining of the particles.



If r_1 and r_2 are distances at neutral point from m_1 and m_2 respectively then.

Intensity of both particle would be equal.

$$\Rightarrow \quad \frac{Gm_1}{r_1^2} = \frac{Gm_2}{r_2^2} \Rightarrow \frac{m_1}{r_1^2} = \frac{m_2}{r_2^2} \Rightarrow \frac{\sqrt{m_1}}{r_1} = \frac{\sqrt{m_2}}{r_2}$$

$\because \quad r = r_1 + r_2 \quad \Rightarrow \quad r_2 = r - r_1$

$$\Rightarrow \quad \frac{\sqrt{m_1}}{r_1} = \frac{\sqrt{m_2}}{(r - r_1)} \Rightarrow \sqrt{m_1}\, r - \sqrt{m_1}\, r_1 = \sqrt{m_2}\, r_1$$

$$\sqrt{m_1}\, r = (\sqrt{m_1} + \sqrt{m_2}) r_1$$

$$\Rightarrow \quad r_1 = \frac{\sqrt{m_1}\, r}{\sqrt{m_1} + \sqrt{m_2}} \quad \text{Similarly, } r_2 = \frac{\sqrt{m_2}\, r}{\sqrt{m_1} + \sqrt{m_2}}$$

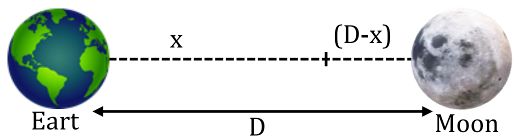
Where r is the distance between m_1 and m_2

Note: Neutral point is closer to the lighter particle.

Illustration 11

If the distance between centres of earth and moon is "D" and mass of earth is 64 times that of moon, at what distance from centre of earth gravitational field will be zero ?

Solution:



Here m_e = mass of earth

M_m = mass of moon

$\because \quad m_e = 64M_m$

As we know that at neutral point

$I_{net} = 0$

$\Rightarrow$ Intensity of earth = intensity of moon.

$$\Rightarrow \quad \frac{Gm_e}{x^2} = \frac{GM_m}{(D-x)^2} \quad \Rightarrow \quad \frac{G \times 64M_m}{x^2} = \frac{GM_m}{(D-x)^2}$$

$$\Rightarrow \quad \frac{64}{x^2} = \frac{1}{(D-x)^2} \Rightarrow \frac{8}{x} = \frac{1}{(D-x)} \quad \Rightarrow \quad x = \frac{8D}{9}$$

Here

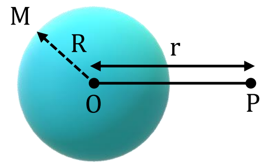
x = distance from the earth surface where gravitational field is zero.

Gravitational field Intensity due to spheres

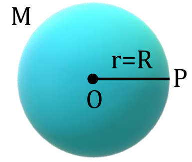
For solid sphere

Let 'M' be the mass of sphere, 'R' the radius of sphere and 'r' the distance of the point under consideration from the centre of sphere.

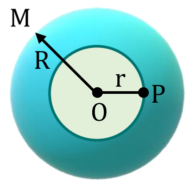
Case I : When r > R, i.e. outside the sphere then $\vec{I}_{out} = \frac{GM}{r^2}(-\hat{r})$



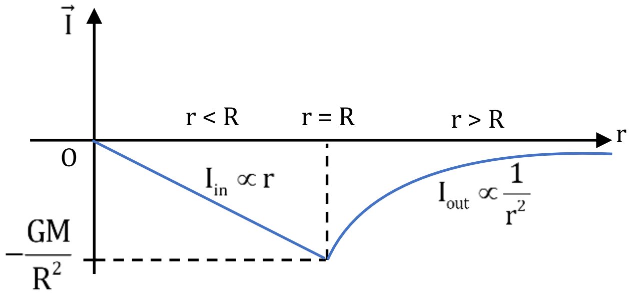
Case II : When r = R, i.e. at the surface then $\vec{I}_{surface} = \frac{GM}{R^2}(-\hat{r})$



Case III : When r < R, i.e. inside the sphere then $\vec{I}_{in} = \frac{GMr}{R^3}(-\hat{r})$



$\Rightarrow$ Graph between '$\vec{I}$' and '**r**' for a solid sphere :

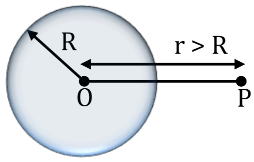


$$I_{max} = I_{surface} = \frac{GM}{R^2}$$

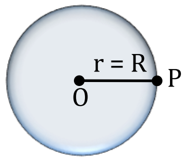
$$I_{min} = I_{centre} = \frac{GM(0)}{R^3} = 0$$

Gravitational field intensity due to a hollow sphere

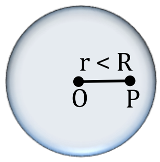
Case I : If r > R, the point is outside the shell then $\vec{I}_{out} = \frac{GM}{r^2}(-\hat{r})$



Case II : If r = R, the point is on the surface then $\vec{I}_{surface} = \frac{GM}{R^2}(-\hat{r})$



Case III : If r < R, the point is inside the shell then I = 0



$\Rightarrow$ $\vec{I}$ v/s r graph for hollow sphere :

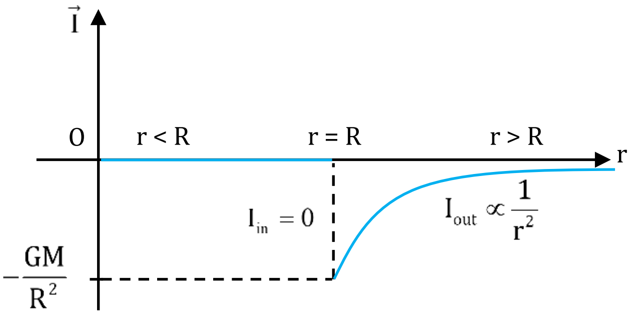
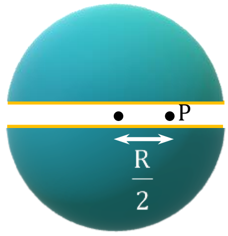


Illustration 12

Find net gravitational force on particle of mass m inside a tunnel present in a solid sphere as shown.

Mass $\rightarrow$ M

Radius $\rightarrow$ R



Tunnel

Solution:

Step A : First find I at mass 'm''.

$$I_P = \frac{GM\left(\frac{R}{2}\right)}{R^3}$$

Step B : Now calculate force. $F = mI_P = m\left[\frac{GM\left(\frac{R}{2}\right)}{R^3}\right] = \frac{GMm}{2R^2}$

Illustration 13

A spherical shell of mass M and radius 2R surrounds a smaller sphere of same mass but radius R. If both spheres have common centre (concentric), then find the intensity of gravitational field at distance r from the centre.

(i) $\frac{R}{2}$ (ii) $\frac{3R}{2}$

M,2R

Solution:

(i) At $r = \frac{R}{2}$

A we know that intensity inside a spherical shell is zero.

But intensity inside a solid sphere at point (r) here r < R.

Than, $I = \frac{GMr}{R^3}$

here, $r = \frac{R}{2} \Rightarrow I = \frac{GM\left(\frac{R}{2}\right)}{R^3} \Rightarrow I = \frac{GM}{2R^2}$

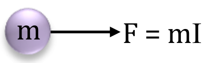
(ii) At $r = \frac{3R}{2}$

Similarly, intensity for a spherical shell inside distance r < 2R is also zero, but intensity for a solid sphere at distance $r = \frac{3R}{2}(r > R)$ will be

$$\because \quad I = \frac{GM}{r^2} \Rightarrow I = \frac{GM}{\left(\frac{3R}{2}\right)^2} \Rightarrow I = \frac{GM}{9\frac{R^2}{4}} \Rightarrow I = \frac{4GM}{9R^2}$$

Acceleration due to gravity

If a body is placed inside a gravitational field having intensity I, gravitational force applied on it is



$$F = mI$$

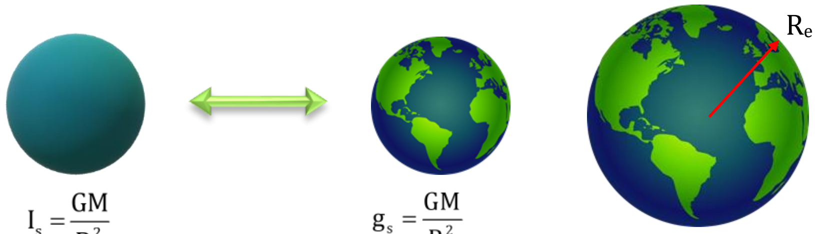
Earth also pulls every object towards its center with force

$$F = mg$$

So, by comparison, it can be deduced that – $g = |\vec{I}|$

Where, g is acceleration due to gravity I is intensity of gravitational field due to Earth

Assuming Earth to be a solid sphere



$R_e = 6400$ km

$M = 6 \times 10^{24}$ kg

$G = 6.67 \times 10^{-11}$ Nm²/kg²

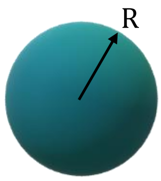
$I_s = \dfrac{GM}{R^2}$ $\qquad$ $g_s = \dfrac{GM}{R_e^2}$

On putting values of G, M and R_e in the formula,

$$g_s = 9.81 \text{ m/s}^2$$

Note: Same formula can be used to find acceleration due to gravity at surface of any planet.

In terms of density (ρ) : $g = \dfrac{GM}{R^2}$



Volume $= \dfrac{4}{3}\pi R^3$

$\because$ Mass = Density × Volume

$$M = \rho \times \frac{4}{3}\pi R^3$$

$$g = \frac{G\left(\rho \times \frac{4}{3}\pi R^3\right)}{R^2} = \frac{4}{3}\pi G R \rho$$

Illustration 14

Density of a planet is double of density of earth and radius is 1.5 times that of earth, the acceleration due to gravity on the surface of planet will be –

(1) 3/4 times that on the surface of the earth
(2) 3 times that on the surface of the earth
(3) 4/3 times that on the surface of the earth
(4) 6 times that on the surface of the earth

Solution: (2)

$\because$ We know that

$$g = \frac{4}{3}\pi G R \rho$$

$\Rightarrow$ $g \propto R\rho$

$\Rightarrow$ $\dfrac{g_p}{g_e} = \dfrac{R_p \rho_p}{R_e \rho_e}$

$\Rightarrow$ $\dfrac{g_p}{g_e} = \dfrac{1.5 R_e \times 2\rho_e}{R_e \times \rho_e} = 1.5 \times 2 \Rightarrow g_p = 3g_e$

$\because$ given

$R_p = 1.5 R_e$

$\rho_p = 2\rho_e$

Here R_p = Radius of planet

ρ_p = density of planet

% change based problem

$\because$ $g = \dfrac{GM}{R^2}$ $\qquad$ $g = \dfrac{4}{3}\pi G R \rho$

For change less than 5%

$\dfrac{\Delta g}{g} = \dfrac{\Delta M}{M} - \dfrac{2\Delta R}{R}$ $\qquad$ $\dfrac{\Delta g}{g} = \dfrac{\Delta R}{R} + \dfrac{\Delta \rho}{\rho}$

$\%\Delta g = \%\Delta M - 2\%\Delta R$ $\qquad$ $\%\Delta g = \%\Delta M + \%\Delta \rho$

Illustration 15

If the mass of the earth increased by 2% & radius of the earth decreased by 1% then find % change in g ?

Solution:

According to question

$M_e \uparrow \rightarrow 2\%$ and $R_e \rightarrow \downarrow \rightarrow 1\%$

$\because \quad \%\Delta g = \%\Delta M - 2\%\Delta R$

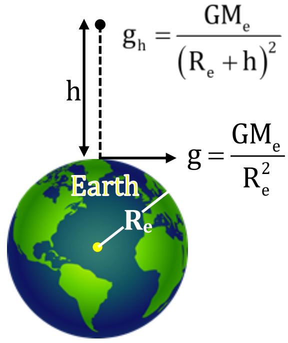
$\Rightarrow \quad \%\Delta g = 2 - 2(-1)\%$

$\Rightarrow \quad \%\Delta g = (2 + 2)\%$

$\Rightarrow \quad \%\Delta g = 4\%$

Variation in acceleration due to gravity with height

From diagram



$$\frac{g_h}{g} = \frac{\dfrac{Gm_e}{(R_e+h)^2}}{\dfrac{Gm_e}{R_e^2}} = \frac{R_e^2}{(R_e+h)^2}$$

$$\Rightarrow \quad \frac{g_h}{g} = \frac{R_e^2}{R_e^2\left(1+\dfrac{h}{R_e}\right)^2} = \frac{1}{\left(1+\dfrac{h}{R_e}\right)^2} \quad \Rightarrow \quad \frac{g_h}{g} = \left(1+\frac{h}{R_e}\right)^{-2}$$

By Binomial expansion,

$$\left(1+\frac{h}{R_e}\right)^{-2} \approx \left(1-\frac{2h}{R_e}\right) \quad [\text{if } h << R_e] \quad \Rightarrow \quad \boxed{g_h = g\left[1-\frac{2h}{R_e}\right]}$$

Note: (i) This formula is valid if h is upto 5% of earth's radius (320 km from earth's surface)

(ii) If h is greater than 5% earth's radius we use. $g_h = \dfrac{GM_e}{(R_e+h)^2}$

Illustration 16

At which height from surface of earth acceleration due to gravity becomes $\frac{1}{4}$ times of its value at earth surface.

Solution:

$g \rightarrow$ at earth surface, $g' = \dfrac{g}{4} \qquad \because \qquad g_h = \dfrac{g}{\left(1+\dfrac{h}{R_e}\right)^2}$

For more than 5% $\quad \Rightarrow \quad \dfrac{g}{4} = \dfrac{g}{\left(1+\dfrac{h}{R_e}\right)^2} \Rightarrow \dfrac{1}{2} = \dfrac{1}{\left(1+\dfrac{h}{R_e}\right)} \Rightarrow 2 = \left(1+\dfrac{h}{R_e}\right) \Rightarrow \dfrac{h}{R_e} = 1$

$\Rightarrow \quad h = R_e \Rightarrow h = 6400$ km

Illustration 17

Value of acceleration due to gravity at a height $2R_e$ from the surface of earth is (g is acceleration due to gravity at a surface of earth).

Solution:

$$\because \quad g_h = \frac{g}{\left(1+\frac{h}{R_e}\right)^2} \qquad \text{Given, } h = 2R_e$$

$$\Rightarrow \quad g_h = \frac{g}{\left(1+\frac{h}{R_e}\right)^2} = \frac{g}{(1+2)^2} \Rightarrow g_h = \frac{g}{9}$$

Variation in acceleration due to gravity with depth

At any general point inside Earth,

$$g = \frac{GM}{R^3} r \qquad R \to \text{radius of Earth}$$

here, r is distance from centre of Earth,

$$g_d = \frac{GM}{R^3}(R-d)$$

$$g_d = \frac{GM}{R^2}\left(1-\frac{d}{R}\right) \qquad \frac{GM}{R^2} = g_s \qquad (g_s = \text{gravity on surface of the earth})$$

$$g_d = g_s\left(1-\frac{d}{R}\right) \text{ For any depth below Earth Surface.}$$

Fractional decrement $\frac{(\Delta g)_d}{g} = \frac{d}{R}$ (Valid for all depths)

Note: 1. Valid for small or large depth

2. Fractional change in value of $g \to \frac{\Delta g}{g} = \frac{d}{R}$

3. At any depth, $g_d < g_s$

4. Another form that can be used, $g_d = \frac{GM}{R^3} r \quad g_d = g_s \frac{r}{R}$

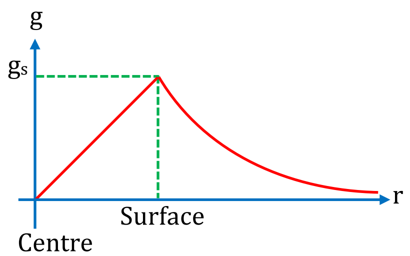
Graph :

For $r < R$

$$g = \frac{GM}{R^3} r \to g = \frac{g_s}{R} r$$

For $r \geq R$

$$g = \frac{GM}{r^2} \to g = g_s \frac{R^2}{r^2}$$



Note: Value of g is maximum at Earth's surface

Illustration 18

A body weighs 72N on surface of the earth. When it is taken to a depth equal to R/2 from the surface of earth, where R is radius of earth, it would weigh –

Solution:

$$\because \quad g_d = g_s\left(1-\frac{d}{R}\right) = g_s\left(1-\frac{R}{2\times R}\right) = g_s\left(1-\frac{1}{2}\right) = \frac{1}{2}g_s$$

$$mg_d = \frac{1}{2}mg_s \quad \Rightarrow \quad mg_d = \frac{1}{2}\times 72 = 36\text{N}$$

Illustration 19

At what depth from the earth surface acceleration due to gravity decreases by 75% of its value at the surface of earth?

Solution:

$\because \quad g_d = g\left(1-\frac{d}{R}\right)$

Here 75% decreases means 25% become

$\Rightarrow \quad \frac{25}{100}g = g\left(1-\frac{d}{R}\right) \Rightarrow \frac{1}{4} = 1-\frac{d}{R} \quad \Rightarrow \quad \frac{d}{R} = \frac{3}{4} \quad \Rightarrow \quad d = \frac{3R}{4}$

Illustration 20

At what depth from earth surface acceleration due to gravity is decreased by 2% ?

Solution:

$\because$ We know that

$\frac{\Delta g}{g} = \frac{d}{R}$

$\frac{2}{100} = \frac{d}{6400\text{km}} \Rightarrow d = 128$ km

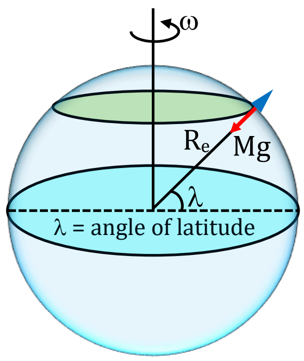
Illustration 21

Value of acceleration due to gravity at a depth $\frac{2R}{3}$ from the surface of earth is.

Solution:

$\because \quad g_d = g\left(1-\frac{d}{R}\right) = g\left(1-\frac{2R}{3\times R}\right) = g\left(1-\frac{2}{3}\right) = g\left(\frac{1}{3}\right) = \frac{g}{3}$

Variation in value of 'g' due to rotation of earth



$N = Mg - M\omega^2R_e\cos^2\lambda$

$Mg_{eff} = Mg - M\omega^2R_e\cos^2\lambda$

$g_{eff} = g - \omega^2R_e\cos^2\lambda$

Where, g : acceleration due to gravity without considering rotation of earth

g_{eff} : acceleration due to gravity considering rotation of earth

λ : Angle of latitude

For poles, $\lambda = 90°$

For equator, $\lambda = 0$

$N = Mg_{eff} = Mg - M\omega^2R_e\cos^2\lambda$

As one moves towards pole, apparent weight increases.

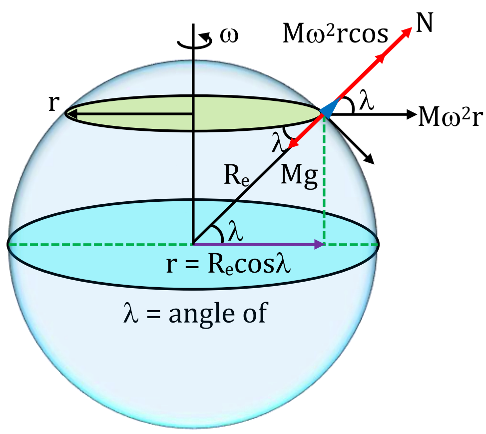


Illustration 22

What should be angular velocity of Earth so that weight of a man at equator becomes half of his present of value? (Answer in terms of g and R)

Solution:

$\because$ Weight at equator (W') $= \dfrac{W}{2}$

$\because$ $Mg' = Mg - M\omega^2R_e\cos^2(\lambda)$

at equator $\lambda = 0$

$\Rightarrow$ $Mg' = Mg - M\omega^2R_e(1)^2$ $\Rightarrow$ $\dfrac{W}{2} = W - M\omega^2R_e$ $\Rightarrow$ $M\omega^2R_e = \dfrac{W}{2} = \dfrac{Mg}{2}$

$\Rightarrow$ $\omega^2 = \dfrac{g}{2R_e}$ $\Rightarrow$ $\omega = \sqrt{\dfrac{g}{2R_e}}$

Special case

$N = Mg_{eff} = Mg - M\omega^2R_e\cos^2\lambda$

If somehow $\omega \uparrow g_{eff} \downarrow N \downarrow$

$$N = Mg_{eff} = Mg - M\omega^2R_e\cos^2\lambda = 0 \Rightarrow \underbrace{\omega = \sqrt{\frac{g}{R_e\cos^2\lambda}}}_{\text{Required } \omega}$$

For Weightlessness.

Illustration 23

At what angular speed of Earth, person at equator will feel weightlessness?

Solution:

For weightlessness required ω is

$$\omega = \sqrt{\frac{g}{R_e\cos^2\lambda}} = \frac{1}{\cos\lambda}\sqrt{\frac{g}{R_e}} \qquad (\lambda = 0°) \text{ for equator}$$

$$\Rightarrow \quad \omega = \frac{1}{\cos 0°}\sqrt{\frac{g}{R_e}} \Rightarrow \omega = \sqrt{\frac{g}{R_e}}$$

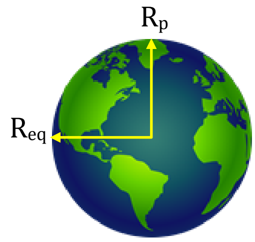
Note: $g_{eff} = g - \omega^2R_e\cos^2\lambda$

- For poles, $\lambda = 90°$ $\quad g_p = g$
- For equator, $\lambda = 0°$ $\quad g_{eq} = g - \omega^2R$
- If not mentioned in problem, assume object at equator, $\lambda = 0°$
- If the earth suddenly stops rotating about its own axis, then apparent weight of bodies or effective acceleration due to gravity will increase at all the places except poles.

Variation in value of "g" due to shape of Earth

$\Delta g = g_e - g_p \approx 0.02\ m/s^2$

Earth is not a perfect sphere.



$g_{surface} = \dfrac{GM}{R^2}$

as surface is close to center of Earth near pole as compared to equator,

$g_p > g_{eq}$

Illustration 24

Where will you get more amount of sugar in 1kg. At pole or at equator?

Solution:

$\because$ $g = \frac{Gm}{R^2}$

$\because$ $g_p > g_{eq}$ [$\because R_{eq} > R_P$]

and W = weight = mg

$\Rightarrow$ at pole at equator

$W_{pole} = mg_p$ $W_{equator} = mg_e$

$\Rightarrow$ $W_{pole} > W_{equator}$

So at poles weight is more and at equator weight is less

So at equator we will get more amount of sugar in 1kg.

BEGINNER'S BOX-2

1. The magnitudes of the gravitational field at distances r_1 and r_2 from the centre of a uniform sphere of radius R and mass M are F_1 and F_2 respectively. then :-

(A) $\frac{F_1}{F_2} = \frac{r_1}{r_2}$ if $r_1 < R$ and $r_2 < R$

(B) $\frac{F_1}{F_2} = \frac{r_2^2}{r_1^2}$ if $r_1 > R$ and $r_2 > R$

(C) $\frac{F_1}{F_2} = \frac{r_1^3}{r_2^3}$ if $r_1 < R$ and $r_2 < R$

(D) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2}$ if $r_1 < R$ and $r_2 < R$

PGR151

2. A stone dropped from a height 'h' reaches the Earth's surface in 1 s. If the same stone is taken to Moon and dropped freely from the same height then it will reach the surface of the Moon in a time (The 'g' of Moon is 1/6 times that of Earth) :-

(A) $\sqrt{6}$ seconds (B) 9 seconds (C) $\sqrt{3}$ seconds (D) 6 seconds

PGR152

3. The radius of Earth is about 6400 km and that of Mars is 3200 km. The mass of Earth is 10 times that of Mars. An object weighs 200 N on the surface of Earth. Its weight on the surface of Mars will be :–

(A) 80 N (B) 40 N (C) 20 N (D) 8 N

PGR153

4. Weight of a body decreases by 1% when it is raised to a height h above the Earth's surface. If the body is taken to a depth h in a mine, then its weight will :-

(A) decrease by 0.5% (B) decrease by 2% (C) increase by 0.5% (D) increase by 1%

PGR154

5. At which height from the earth's surface does the acceleration due to gravity decrease by 1%?

PGR155

6. Find the percentage decrement in the weight of a body when taken to a height of 16 km above the surface of earth. (radius of earth is 6400 km)

PGR156

7. What is the value of acceleration due to gravity at a height equal to half the radius of earth, from surface of earth? [take g = 10 m/s^2 on earth's surface]

PGR157

8. At which height from the earth's surface does the acceleration due to gravity decrease by 75% of its value at earth's surface?

PGR158

9. At which height above earth's surface is the value of 'g' same as in a 100 km deep mine?
PGR159

10. At what depth below the surface does the acceleration due to gravity becomes 70% of its value on the surface of earth?
PGR160

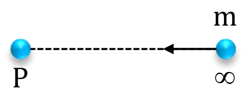
11. At what depth from earth's surface does the acceleration due to gravity becomes $\frac{1}{4}$ times that of its value at surface?
PGR161

12. If earth is assumed to be a sphere of uniform density then plot a graph between acceleration due to gravity (g) and distance from the centre of earth. **[AIPMT (Mains) 2006]**
PGR162

Gravitational potential

Gravitational potential at a point is defined as amount of work done by external agent against gravitational pull in bringing a unit mass from infinity to that point without changing its kinetic energy.

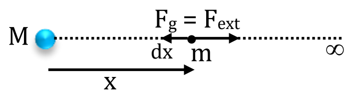
$$V_P = \frac{W_{\infty \to P}}{m}$$



Note: Motion of unit mass is considered slowly, so that no energy is spent to change the kinetic energy of system.
At infinity, Gravitational Potential is assumed to be **ZERO**.

Gravitational potential due to point mass

If a particle of mass m is at distance x from given particle of mass M



then, work done to displace mass m by distance dx towards M under equilibrium

$$dW_{ext} = \frac{GMm}{x^2}(-dx)\cos(180°)$$

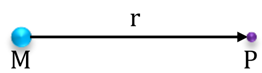
Negative sign of dx indicates that displacement is towards mass M

$$dW_{ext} = \frac{GMm}{x^2}(-dx)(-1) \Rightarrow dW_{ext} = \frac{GMm}{x^2}dx$$

$$dW_{ext} = \frac{GMm}{x^2}dx$$

total work done in bringing the particle of mass m from infinity to a distance r from mass M is-

$$W_{ext} = \int_{\infty}^{r} \frac{GMm}{x^2}(dx) = -\frac{GMm}{x}\Big|_{\infty}^{r} = \left(-\frac{GMm}{r}\right) - \left(-\frac{GMm}{\infty}\right)$$



$$W_{ext} = -\frac{GMm}{r}$$

as per definition, $V_P = \frac{W_{\infty \to P}}{m}$

$$V = \frac{W_{ext}}{m} = -\frac{GM}{r}$$

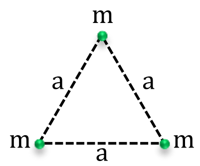
At r distance from the particle of mass M,

$$V = -\frac{GM}{r}$$

Note: Negative sign in potential indicates attractive nature of force.

Illustration 25

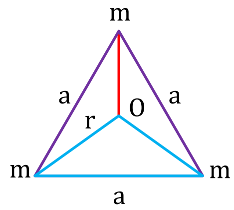
Find gravitational potential at the centroid.



Solution:

$$V = -\frac{Gm}{r}$$

$$V_{net} = -\frac{Gm}{r} - \frac{Gm}{r} - \frac{Gm}{r}$$



$\Rightarrow \quad r\cos 30 = \frac{a}{2}$

$\Rightarrow \quad r \times \frac{\sqrt{3}}{2} = \frac{a}{2} = \frac{a}{\sqrt{3}}$

$\Rightarrow \quad V_{net} = -\frac{3Gm}{a/\sqrt{3}} = -3\sqrt{3}\frac{Gm}{a}$

Relation between Potential and Intensity

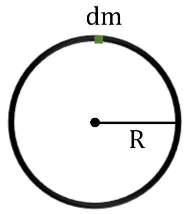
$$I = -\frac{dV}{dr}$$

Note: 1. If V is constant in a region, I = 0

2. This relation is valid for all conservative fields.

Illustration 26

Find gravitational potential at the centre of a ring of mass M and radius R.

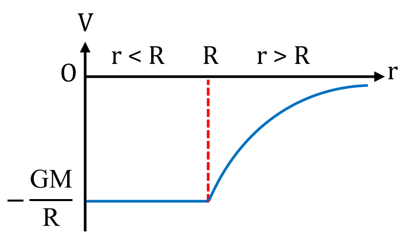


Solution:

$$V = \int -\frac{Gdm}{R} \quad \Rightarrow \quad V = -\frac{Gm}{R}$$

Standard results to remember

Spherical shell (M, R)

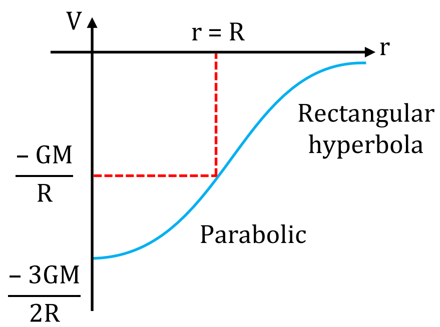


Case I r > R (outside the sphere) ; $V_{out} = -\frac{GM}{r}$

Case II r = R(on the surface) ; $V_{surface} = -\frac{GM}{R}$

Case III r < R (inside the sphere ; Potential is same every where and is equal to its value at the surface ; $V_{in} = -\frac{GM}{R}$

Solid sphere(M, R)



Case I r > R (outside the sphere) ; $V_{out} = -\frac{GM}{r}$

Case II r = R(on the surface) ; $V_{surface} = -\frac{GM}{R}$

Case III r < R (inside the sphere ; Potential is same every where and is equal to its value at the surface ; $V_{in} = -\frac{GM}{2R^3}(3R^2 - r^2)$

Illustration 27

A particle of mass M is situated at the centre of a spherical shell of same mass and radius R. Calculate the gravitational potential at a point situated at distance R/3 from the centre.

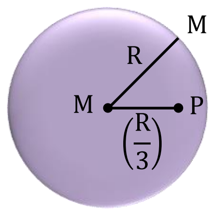
Solution:

$V_P = V_{sphere} + V_{particle}$

$$= -\left(\frac{GM}{R} + \frac{GM}{R/3}\right)$$

$$= -\frac{GM}{R}(1+3)$$

$$= -\frac{4GM}{R}$$

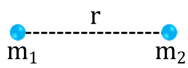


Spherical shell

Gravitational Potential Energy (GPE)

The gravitational potential energy of a particle situated at a point in some gravitational field is defined as the amount of work required to bring it from infinity to that point without changing its kinetic energy.

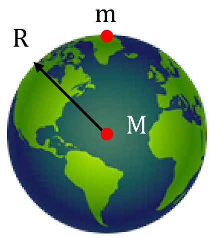
$$U = -\frac{Gm_1m_2}{r}$$



(Here negative sign shows the boundness of the two bodies)

- It is a scalar quantity.
- It's SI unit is joule and Dimensions are $[M^1L^2T^{-2}]$
- The gravitational potential energy of a particle of mass 'm' placed on the surface of earth of mass 'M' and radius 'R' is given by :

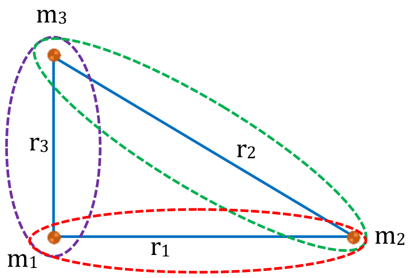
$$U = -\frac{GMm}{R}$$



Gravitational potential energy for three particle system

If there are more than two particles in a system, then the net gravitational potential energy of the whole system is the sum of gravitational potential energies of all the possible pairs in that system.

For eg. :



$$U_{system} = \left(-\frac{Gm_1m_2}{r_1}\right) + \left(-\frac{Gm_2m_3}{r_2}\right) + \left(-\frac{Gm_1m_3}{r_3}\right) = -\frac{Gm_1m_2}{r_1} - \frac{Gm_2m_3}{r_2} - \frac{Gm_1m_3}{r_3}$$

If Gravitational Potential at a point P is V_P

Then GPE of a particle of mass m placed at P is :

$U_P = m(V_P)$

Note: At infinity, $V_\infty = 0 \Rightarrow U_\infty = 0$

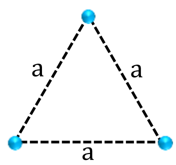
Illustration 28

If the given system of three particles, each of mass m, on the vertices of an equilateral triangle side a is to be changed to side of 2a, then find the work done on the system.

Solution:

Gravitational potential energy of an equilateral triangle of side "a"

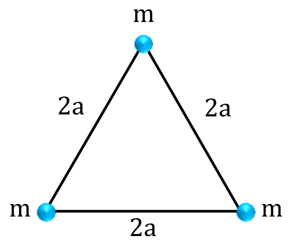
$$U_{system} = -\frac{Gmm}{a} - \frac{Gmm}{a} - \frac{Gmm}{a} = -\frac{3Gm^2}{a}$$



Now "a" is changed by "2a"

$$\text{Then } U_{system} = -\frac{Gmm}{2a} - \frac{Gmm}{2a} - \frac{Gmm}{2a}$$

$\Rightarrow \quad U_{system} = -\dfrac{3Gm^2}{2a}$



$U_i = -\dfrac{3Gm^2}{a}$ and $U_f = -\dfrac{3Gm^2}{2a}$

Workdone = ΔU

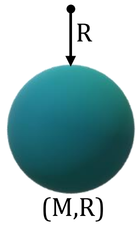
$W = U_f - U_i$

$\Rightarrow \quad W = -\dfrac{3Gm^2}{2a} - \left(-\dfrac{3Gm^2}{a}\right) = -\dfrac{3Gm^2}{a}\left[\dfrac{1}{2} - 1\right]$

$\Rightarrow \quad W = \dfrac{3Gm^2}{2a}$

Illustration 29

Find gravitational potential energy of a particle of mass "m" placed at a height R. above surface of solid sphere of mass "M" and Radius "R".



Solution:

$V = -\dfrac{GM}{r}$

$\Rightarrow \quad V = -\dfrac{GM}{(R+h)} = -\dfrac{GM}{R+R} = -\dfrac{GM}{2R}$ $\quad \because \; r = R + h$

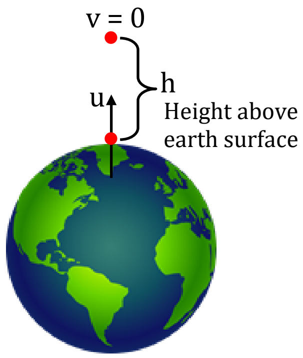
Now potential energy (U) = V(m) $\quad$ and h = R

$\Rightarrow \quad U = -\dfrac{GM}{2R}(m) \Rightarrow U = -\dfrac{GMm}{2R}$

Note: If object is thrown with velocity u from Earth Surface, maximum height achieved can be found as :

[By applying mechanical energy conservation]

$$u = \sqrt{\dfrac{2gh}{1+\dfrac{h}{R_e}}}$$



Same result for

Velocity of an object at Earth's surface when dropped from height h above Earth Surface.

Illustration 30

An object is projected from Earth's surface with speed u = $\sqrt{1.5gR_e}$, find the maximum height attained by object. (g, R_e have their usual meaning)

Solution:

$\therefore \quad u = \sqrt{\dfrac{2gh}{1+\dfrac{h}{R_e}}}$

$\sqrt{\dfrac{3}{2}gR_e} = \sqrt{\dfrac{2gh}{1+\dfrac{h}{R_e}}} \Rightarrow \dfrac{3}{2}R_e = \dfrac{2h}{1+\dfrac{h}{R_e}} \Rightarrow 3R_e + 3h = 4h$

$\Rightarrow \quad h = 3R_e$

Escape velocity (V_e)

It is the minimum velocity required for an object located at the planet's surface so that it just escapes the planet's gravitational field.

Consider a projectile of mass m, leaving the surface of a planet (or some other astronomical body or system), of radius R and mass M with escape speed v_e.

When the projectile just escapes to infinity, it has neither kinetic energy nor potential energy.

From conservation of mechanical energy $\frac{1}{2}mv_e^2+\left(-\frac{GMm}{R}\right)=0+0 \Rightarrow v_e=\sqrt{\frac{2GM}{R}}$

The escape velocity of a body from a location which is at height 'h' above the surface of planet, we can use :-

$$\boxed{v_e=\sqrt{\frac{2GM}{r}}=\sqrt{\frac{2GM}{R+h}}} \quad \{\because r = R + h\}$$

Where, r = Distance from the centre of the planet, h = Height above the surface of the planet

Escape speed depends on :

(i) Mass (M) and radius (R) of the planet

(ii) Position from where the particle is projected.

Escape speed does not depend on :

(i) Mass (m) of the body which is projected

(ii) Angle of projection.

If a body is thrown from the Earth's surface with escape speed, it goes out of earth's gravitational field and never returns back to the earth's surface.

For Earth

$$v_e=\sqrt{\frac{2GM}{R_e}}$$

Other forms, $g_s=\frac{GM}{R_e^2} \Rightarrow v_e=\sqrt{2g_s R_e}$

On putting values : v_e = 11.2 km/s, g_s = 9.81 m/s^2, R_e = 6400 km

Illustration 31

An unknown planet is of twice the size and half the mass of Earth. If escape velocity from Earth surface is V_o, what would be escape velocity of particle from the unknown planet.

Solution:

$$V_o=\sqrt{\frac{2GM}{R}} \Rightarrow V_o \propto \sqrt{\frac{M}{R}}$$

Now $\frac{V}{V_o}=\sqrt{\frac{M/2}{M}\times\frac{R}{2R}} \Rightarrow \frac{V}{V_o}=\sqrt{\frac{1}{4}} \Rightarrow \boxed{V=\frac{V_o}{2}}$

Illustration 32

If M is mass of a planet, then in order to become black hole, what should be the radius of planet?

Solution:

For a body to become a black body. Even light can not escape its gravitational pull.

$V_e \geq c$

$\sqrt{\frac{2GM}{r}} \geq c$ c = speed of light

$\frac{2GM}{c^2} \geq r \quad \Rightarrow \quad r \leq \frac{2GM}{c^2}$

Escape velocity from a point other than surface

Keep following points in mind !!

1. Total energy is zero at any point when particle start moving with escape velocity, TE=0
2. If given point is at distance r (>R) from center of Earth, $PE_{out} = -\dfrac{GM_e m}{r}$
3. If given point is at distance r (<R) from center of Earth. $PE_{in} = -\dfrac{GM(3R^2 - r^2)}{2R^3} \times m$

Illustration 33

With what velocity must a body be thrown from Earth's surface, so that it may reach at a height $4R_e$ above the Earth's surface. (R_e = 6400km, g = 9.8m/s²)

Solution:

By using conservation of mechanical energy.

$K_i + U_i = K_f + U_f$.

$$\Rightarrow \frac{1}{2}M_oV^2 - \frac{GMM_o}{R_e} = -\frac{GMM_o}{(R_e + 4R_e)}$$

$$\Rightarrow \frac{1}{2}M_oV^2 = -\frac{GMM_o}{5R_e} + \frac{GMM_o}{R_e}$$

$$\Rightarrow \frac{1}{2}M_oV^2 = \frac{4}{5}\frac{GM_oM}{R_e} \Rightarrow V^2 = \frac{8}{5}\frac{GM}{R_e}$$

$$\because \quad g = \frac{GM}{R_e^2}$$

$$\Rightarrow \quad GM = gR_e^2$$

$$\Rightarrow \quad V^2 = \frac{8}{5}\frac{gR_e^2}{R_e} = \frac{8}{5} \times 9.8 \times 6400 \times 10^3 = 10^8 \text{m/s}$$

Illustration 34

A narrow tunnel is dug along the diameter of the earth (M_e, R), and a particle of mass 'm' is placed at 'R/2' distance from the centre. Find the escape speed of the particle from that place.

Solution:

Suppose we project the particle with speed V_e. So that it just reaches infinity ($r \to \infty$)

By conservation of mechanical energy

$K_i + U_i = K_f + U_f$.

$$\frac{1}{2}mV_e^2 + m\left[-\frac{GM_e}{2R^3}\left\{3R^2 - \left(\frac{R}{2}\right)^2\right\}\right] = 0$$

$$\Rightarrow \quad V_e = \sqrt{\frac{11GM_e}{4R}}$$

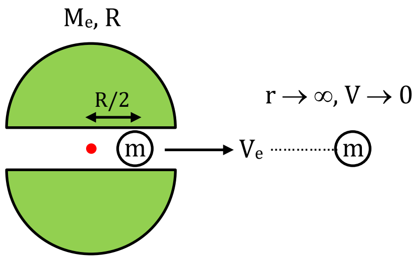


Illustration 35

Potential energy of a particle of mass 2kg is –100J at surface of planet. Find its escape velocity.

Solution:

By conservation of mechanical energy

$K_i + U_i = K_f + U_f$.

$$\Rightarrow \quad \frac{1}{2}MV_e^2 + (-100) = 0 + 0$$

$$\Rightarrow \quad \frac{1}{2} \times 2 \times V_e^2 = 100$$

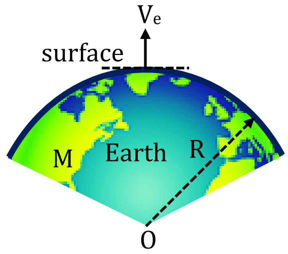
$$V_e = 10 \text{m/s}$$

at infinity

$K_f = 0$

$U_f = 0$

Escape energy-Binding Energy



Minimum energy given to a particle in the form of kinetic energy so that it can just escape the Earth's gravitational field.

Magnitude of escape energy = $\frac{GMm}{R}$

(–ve of PE on the Earth's surface)

Escape energy = Kinetic Energy corresponding to the escape velocity ⇒

$$\frac{GMm}{R} = \frac{1}{2}mv_e^2$$

Note: In the above discussion it can be any planet for that matter

KE < Escape Energy	$v_o < v_e$	Body returns to Earth surface
KE = Escape Energy	$v_o = v_e$	Body comes to rest at infinity
KE > Escape Energy	$v_o > v_e$	Body has residual velocity at infinity

Illustration 36

A body of mass m has speed $v_o(<v_e)$ at the surface of the Earth. Find minimum additional energy required so that it may escape from gravitational field of Earth. (M_e : mass of Earth, R_e : radius of Earth)

Solution:

Here $v_e < v_e$; By conservation of mechanical energy

Total energy at ∞ = 0

⇒ $E_{supplied} + KE_i + U_i = 0$

$$E_{supplied} + \frac{1}{2}mv_o^2 + \left(-\frac{GMm}{R}\right) = 0 \quad \Rightarrow \quad E_{supplied} = \left[\frac{GMm}{R} - \frac{1}{2}mv_o^2\right]$$

Note: Why do some planets have atmosphere around them, but some do not ?

If energy of a gas molecule is greater than or equal to escape energy on a planet, it will escape from that planet easily.

That's why some celestial bodies have atmosphere and others have not.

Illustration 37

What happens when you release a Helium balloon-

(i) At Earth's surface (ii) At Moon's surface

Solution:

(i) If a helium balloon is released from surface of Earth, then it "move upward" because the upward buoyant force due to surrounding air exceeds its downwards weight

(ii) When the balloon is released from surface of Moon then it will fall with "g/6" acceleration under the influence of gravitational attraction at Moon (upthrust is zero due to absence of atmosphere)

Binding energy

Total energy of a particle near Earth.

BE < 0 Particle cannot escape the gravitational field of Earth

BE ≥ 0 Particle can escape the gravitational field of Earth

BEGINNER'S BOX-3

1. The gravitational acceleration on the surface of earth is g. Find the increase in potential energy in lifting an object of mass m to a height equal to the radius of earth.

PGR163

2. In a certain region of space gravitational field is given by I = – (K/r) (Where r is the distance from a fixed point and K is constant). Taking the reference point to be at $r = r_0$ with $V = V_0$. Find the potential at a distance r.

PGR164

3. Two masses of 10^2 kg and 10^3 kg are separated by 1 m distance. Find the gravitational potential at the mid point of the line joining them.

PGR165

4. The magnitude of intensity of gravitational field at a point situated at a distance 8000 km from the centre of Earth is 6.0 N-kg. The magnitude of gravitational potential at that point in N-m/kg will be :-

(A) 6 (B) 4.8×10^7 (C) 8×10^5 (D) 4.8×10^2

PGR166

5. The gravitational field due to a certain mass distribution is $E = \frac{K}{x^3}$ in the x–direction (K is a constant). Taking the gravitational potential to be zero at infinity, its value corresponding to distance x is :–

(A) $\frac{K}{x}$ (B) $\frac{K}{2x}$ (C) $\frac{K}{x^2}$ (D) $\frac{K}{2x^2}$

PGR167

6. Two bodies of respective masses m and M are placed d distance apart. The gravitational potential (V) at the position where the gravitational field due to them is zero is :–

(A) $V = -\frac{G}{d}(m+M)$ (B) $V = -\frac{G}{d}$

(C) $V = -\frac{GM}{d}$ (D) $V = -\frac{G}{d}(\sqrt{m}+\sqrt{M})^2$

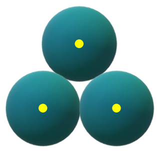
PGR168

7. A body of mass m is situated at a distance $4R_e$ above the Earth's surface, where R_e is the radius of Earth. What minimum energy should be given to the body so that it may escape ?

(A) mgR_e (B) $2mgR_e$ (C) $\frac{mgR_e}{5}$ (D) $\frac{mgR_e}{16}$

PGR169

8. Three solid spheres of mass M and radius R are placed in contact as shown in fig. find the potential energy of the system.



PGR170

Kepler's Law

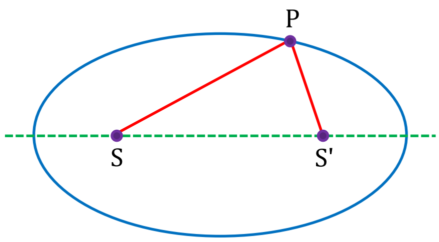
Kepler's 1st Law

Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

Ellipse : Ellipse is the set of all points such that sum of the distance from the foci (plural of focus) and any point on ellipse is constant.

General equation of ellipse

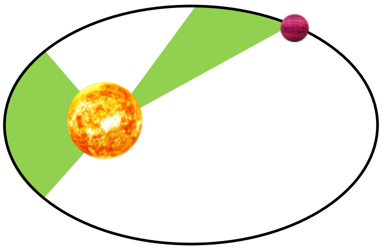
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

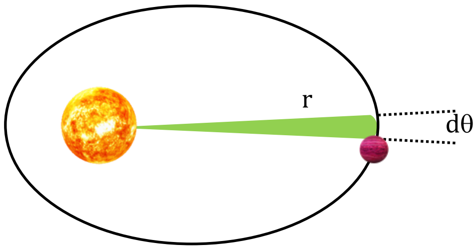


SP + S'P = constant

Kepler's 2nd law (Law of Areas)

A line joining any planet to the Sun sweeps out equal areas in equal intervals of time, i.e. the areal velocity of the planet remains constant.



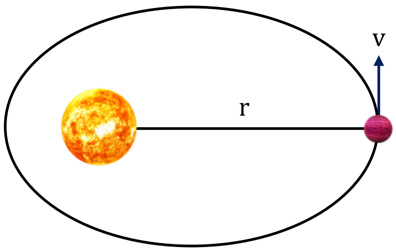


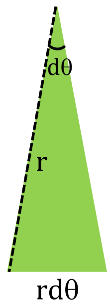
Kepler's laws of planetary motion

Area = $\frac{1}{2}$ (base) (height)

$dA = \frac{1}{2}(rd\theta)(r) = \frac{1}{2}r^2 d\theta$

So, areal velocity (area per unit time)





$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{1}{2}rv$$

$$\frac{dA}{dt} = \frac{1}{2}\omega r^2 = \frac{1}{2}rv$$

Areal velocity,

L = Iω = mr²ω = mvr

Here I = mr²(instantaneous moment of inertia of planet about sun)

Angular momentum of planet **about the Sun** L = mvr

As force of gravitation on planet by the Sun passes through Sun itself,

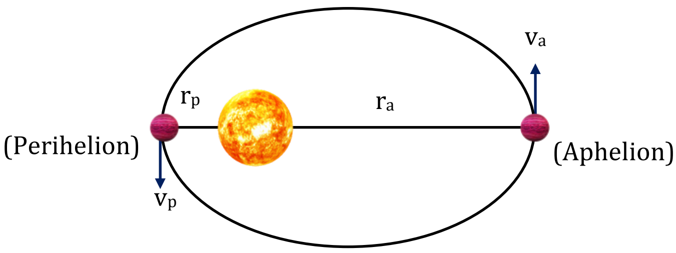
Torque of gravitational pull is zero **about the Sun**,

Angular momentum of planet about the Sun is constant

L = mvr (constant)

$$\frac{L}{m} = vr \Rightarrow \frac{L}{2m} = \frac{1}{2}vr \Rightarrow \frac{1}{2}vr = \frac{dA}{dt}$$

r × v = constant



$$v_a r_a = v_p r_p \Rightarrow \frac{v_a}{v_p} = \frac{r_p}{r_a} = \frac{r_{min}}{r_{max}}$$

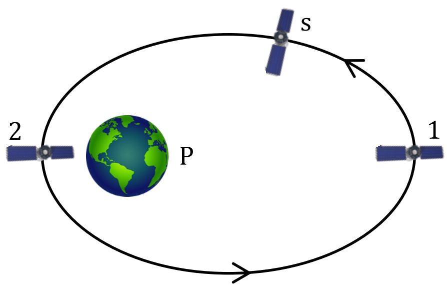
Conclusion

At minimum distance from Sun, velocity is maximum.

At maximum distance from Sun, velocity is minimum.

Illustration 38

A small satellite is in elliptical orbit around Earth as shown. If L is the magnitude of its angular momentum about the centre of the earth and K is the kinetic energy. Then compare the kinetic energies of satellite at position 1 and 2 :



Solution:

As torque of gravitational pull is zero about the sun. So angular momentum of planet about the sun is constant.

$\Rightarrow \quad L_1 = L_2 \Rightarrow mv_1r_1 = mv_2r_2 \Rightarrow v_1r_1 = v_2r_2 \Rightarrow r_1 > r_2$

$\Rightarrow \quad v_1 = \frac{v_2 r_2}{r_1}$

$\Rightarrow \quad v_2 > v_1$

$\Rightarrow \quad \frac{1}{2}mv_2^2 > \frac{1}{2}mv_1^2 \Rightarrow K_2 > K_1$

Illustration 39

A planet revolves about the sun in elliptical orbit. The areal velocity of the planet is 4.0×10^{16} m^2/s. The least distance between planet and the sun is 2×10^{12}m. Then the maximum speed of the planet in km/s is :

Solution:

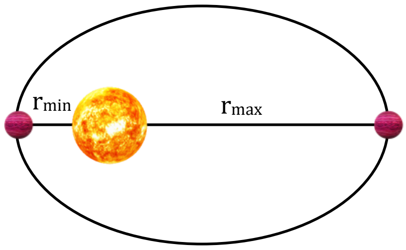
$$\because \quad \frac{dA}{dt}=\frac{r^2\omega}{2} \text{ is constant}$$

$$\because \quad \frac{dA}{dt}=\frac{r_{max}^2\omega_{min}}{2}=\frac{r_{min}^2\omega_{max}}{2} \quad \Rightarrow \quad \omega_{max}=\frac{2dA/dt}{r_{min}^2}$$

$$\Rightarrow \quad v_{max} = \omega_{max}\, r_{min} = \frac{2dA/dt}{r_{min}} = \frac{2\times4\times10^{16}}{2\times10^{12}} = 40 \text{ km/sec}$$

Kepler's 3rd law

The square of the period of revolution of any planet around the Sun is directly proportional to the cube of the semi-major axis of its elliptical orbit.

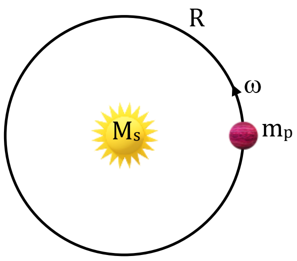


$$T^2 \propto a^3$$

a : average distance of the planet from Sun

$$a=\frac{r_{max}+r_{min}}{2}$$

3rd law can be derived assuming a circular orbit of planet around Sun Force of gravitation provides necessary centripetal acceleration to planet for circular motion



$$\frac{GM_sm_p}{R^2}=m_p(\omega^2R)$$

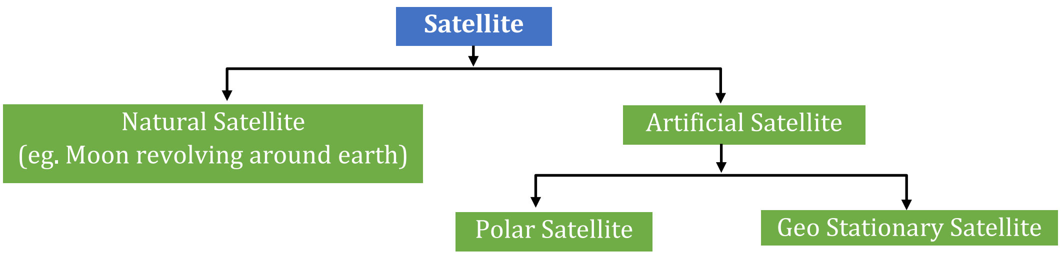
$$\frac{GM_s}{R^3}=\omega^2$$

$$\frac{GM_s}{R^3}=\left(\frac{2\pi}{T}\right)^2$$

$$\frac{GM_s}{R^3}=\frac{4\pi^2}{T^2} \Rightarrow T^2=\frac{4\pi^2}{GM_s}R^3$$

Satellite Motion

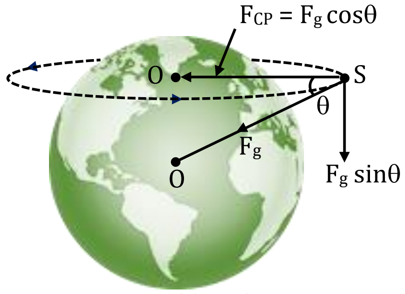
A light body revolving round a heavier planet due to gravitational attraction, is called a satellite. Moon is a natural satellite of Earth.



Essential conditions for satellite motion

- The centre of satellite's orbit should coincide with the centre of Earth.
- Plane of the orbit of satellite should pass through the centre of Earth.

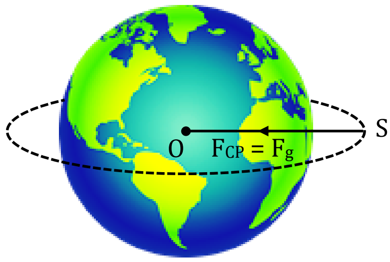
(i)



Unstable orbit

(Due to $F_g \sin\theta$, orbit will shift)

(ii)



Stable orbit

- It follows that a satellite can revolve round the earth only in those circular orbits whose centres coincide with the centre of earth. Circles drawn on globe with centres coincident with earth are known as 'great circles'. Therefore, a satellite revolves around the earth along circles concentric with great circles.

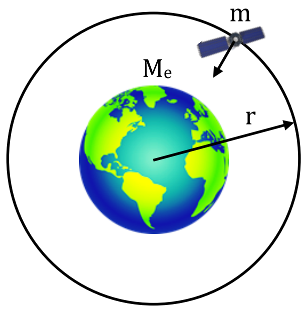
Orbital Velocity (V_o)

As centripetal acceleration is provided by Gravitational pull of Earth,

$$\frac{GM_e m}{r^2} = \frac{mv_o^2}{r}$$

$$v_o^2 = \frac{GM_e}{r}$$

$$v_o = \sqrt{\frac{GM_e}{r}}$$



For near Earth Surface Satellite

$R \approx R_e$

$$v_o = \sqrt{\frac{GM_e}{R_e}} \approx 8.1 \text{ km/s}$$

Illustration 40

Satellites A and B are orbiting around the Earth in orbits of radius ratio 1 : 4, find ratio of their orbital velocities.

Solution:

$$\Rightarrow \quad v \propto \frac{1}{\sqrt{r}}$$

$$\frac{v_1}{v_2} = \sqrt{\frac{r_2}{r_1}} \Rightarrow = \sqrt{\frac{4}{1}} \Rightarrow = \frac{2}{1}$$

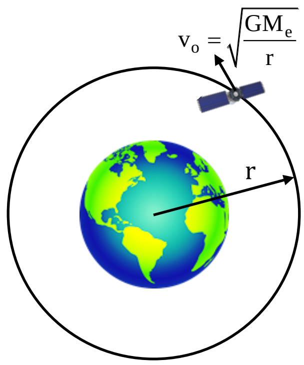
Given

$$\frac{r_1}{r_2} = \frac{1}{4}$$

Time period of orbital motion

$$T = \frac{\text{distance}}{\text{speed}} = \frac{2\pi r}{v_o}$$

$$T = \frac{2\pi r}{\sqrt{\frac{GM_e}{r}}} \Rightarrow T^2 \propto r^3$$



For Near Earth Surface satellite, $r \approx R_e$

$$T = 2\pi\sqrt{\frac{R_e^3}{GM_e}} \approx 84 \min$$

Illustration 41

The time period of a satellite in a circular orbit of radius R_o is T_o. Find time period of another satellite in a circular orbit of radius $4R_o$.

Solution:

$T_0 = 2\pi\sqrt{\frac{R_o^3}{GM_e}}$

When $R' = 4R_o \Rightarrow T' = 2\pi\sqrt{\frac{(4R_o)^3}{GM_e}} = 2\pi\sqrt{\frac{64R_o^3}{GM_e}}$

$T' = 8\left(2\pi\sqrt{\frac{R_o^3}{GM_e}}\right) \Rightarrow T' = 8T_o$

Note: In case Gravitation Force is different $F \propto \frac{1}{r^n}$

then, orbital velocity $v_o \propto r^{\frac{1-n}{2}}$

and Time period $T \propto r^{\frac{1+n}{2}}$

Satellite motion - Energy Analysis

Kinetic energy of satellite

Due to motion of satellite, $KE = \frac{1}{2}mv_o^2$

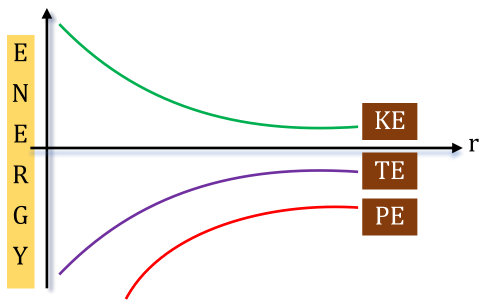
$\because \quad v_o = \sqrt{\frac{GM_e}{r}}$

$\therefore \quad KE = \frac{GM_em}{2r}$

Potential energy of satellite

Due to position of satellite

$PE = -\frac{GM_em}{r}$



Total Energy = Kinetic Energy + Potential Energy

$TE = -\frac{GM_em}{2r}$

$KE = \frac{GM_em}{2r} \qquad TE = -\frac{GM_em}{2r} \qquad PE = -\frac{GM_em}{r}$

$KE = -TE = -\frac{PE}{2}$

As total energy of a satellite is negative, it is called as bounded system

Binding energy

Minimum energy required to break a bounded system.

$\Rightarrow \quad BE + TE = 0 \quad \Rightarrow \quad BE = -TE$

$\Rightarrow \quad BE = |TE| \quad \Rightarrow \quad BE = \frac{GM_em}{2r}$

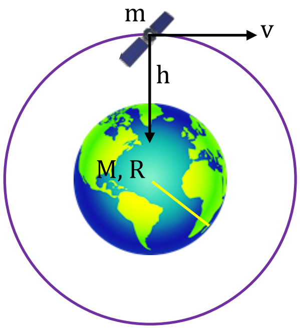
Illustration 42

A satellite of mass m, initially at rest on Earth, is launched into a circular orbit at a height equal to the radius of Earth. Energy required to do so is –

Solution:

h = R

Necessary centripetal force is provided by gravitational force.



$$\Rightarrow \quad \frac{mv^2}{2R} = \frac{GMm}{(2R)^2} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{2R}}$$

Now by conservation of mechanical energy.

$K_i + U_i = K_f + U_f$

$$K_i + -\frac{GMm}{R} = \frac{1}{2}m\left(\sqrt{\frac{GM}{2R}}\right)^2 + \left(-\frac{GMm}{2R}\right)$$

$$K_i = \frac{1}{2}m\frac{GM}{2R} + \frac{GMm}{R} - \frac{GMm}{2R}$$

$$K_i = \frac{GMm}{4R} + \frac{GMm}{2R} = \frac{3}{4}\frac{GMm}{R} \qquad \because g = \frac{GM}{R^2} \Rightarrow GM = gR^2$$

$$K_i = \frac{3}{4}mgR$$

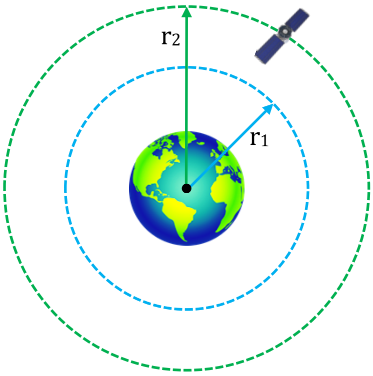
Note: $TE = -\frac{GM_e m}{2r}$ $\qquad PE = -\frac{GM_e m}{r}$ $\qquad KE = \frac{GM_e m}{2r}$

$v_o = \sqrt{\frac{GM_e}{r}}$ $\qquad T \propto R^{3/2}$

So, If radius of orbit of satellite is increased,

- KE will decrease
- TE will increase
- PE will increase
- Orbital velocity will decrease
- Time period will increase

Work done to shift orbit of satellite



$W_{ext} = \Delta TE$

$W_{ext} = TE_f - TE_i$

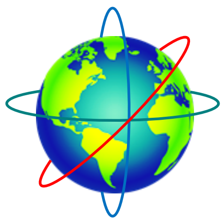
$$W_{ext} = \left(-\frac{GM_e m}{2r_2}\right) - \left(-\frac{GM_e m}{2r_1}\right) = -\frac{GM_e m}{2r_2} + \frac{GM_e m}{2r_1}$$

$$W_{ext} = \frac{GM_e m}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Essential conditions for satellite motion –

- Centre of satellite's orbit coincide with center of Earth.
- Plane of orbit of satellite is passing through center of Earth.

There are two stable orbits for a satellite



- Equatorial orbit
- Polar orbit

Any satellite is projected as near to equator as possible as-

- g is minimum near equator, lesser fuel
- Easier to set the satellite in any orbit

Polar satellite

- It rotates in Polar plane.
- Its height from Earth surface is 500 km ~ 600 km.
- Its time period is 100 min.
- Used for weather data collection

Note: Any satellite is said to be in a condition of freely falling on Earth. (as only force acting on it is force of gravity). Hence, apparent weight of any object in a satellite is always ZERO.

BEGINNER'S BOX-4

1. The mean radius of the earth's orbit around the sun is 1.5×10^{11} m. The mean radius of the orbit of mercury around the sun is 6×10^{10} m. Calculate the year of the mercury.

PGR171

2. If earth describes an orbit round the sun of double its present radius, what will be the year on earth?

PGR172

3. If the gravitational force were to vary inversely as m^{th} power of the distance, then the time period of a planet in circular orbit of radius r around the Sun will be proportional to :-

(A) $r^{-3m/2}$ (B) $r^{3m/2}$ (C) $r^{m+1/2}$ (D) $r^{(m+1)/2}$

PGR173

4. A planet is revolving around the Sun in an elliptical orbit. Its closest distance from the Sun is r_{min}. The farthest distance from the Sun is r_{max}. If the orbital angular velocity of the planet when it is nearest to the Sun is ω, then the orbital angular velocity at the point when it is at the farthest distance from the Sun is :–

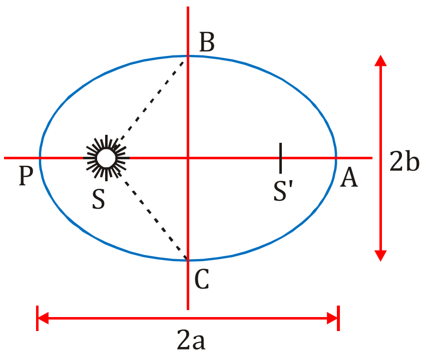
(A) $\left(\sqrt{\frac{r_{min}}{r_{max}}}\right)\omega$ (B) $\left(\sqrt{\frac{r_{max}}{r_{min}}}\right)\omega$ (C) $\left(\frac{r_{max}}{r_{min}}\right)^2\omega$ (D) $\left(\frac{r_{min}}{r_{max}}\right)^2\omega$

PGR174

5. The time period of revolution of moon around the earth is 28 days and radius of its orbit is 4×10^5 km. If $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 then find the mass of the earth.

PGR175

6. Let the speed of the planet at the perihelion P in Fig. be v_P and the Sun-planet distance SP be r_P. Relate (r_P, v_P) to the corresponding quantities at the aphelion (r_A, v_A). Will the planet take equal times to traverse BAC and CPB?



PGR176

7. A comet orbits the sun in a highly elliptical orbit. Does the comet have a constant (a) linear speed, (b) angular speed, (c) angular momentum, (d) kinetic energy, (e) potential energy (f) total energy throughout its orbit? Neglect any mass loss of the comet when it comes very close to the Sun.

PGR177

8. A satellite moves in a circular orbit around the earth. The radius of this orbit is one half that of the moon's orbit. Find the time in which the satellite completes one revolution.

PGR178

9. A small satellite revolves round a planet in an orbit just above planet's surface. Taking the mean density of planet as ρ, calculate the time period of the satellite.

PGR179

10. Two satellites A and B, having ratio of masses 3 : 1 are in circular orbits of radius r and 4r. Calculate the ratio of total mechanical energies of A to B.

PGR180

11. A satellite orbits the Earth at a height of 400 km above the surface. How much energy must be expended to rocket the satellite out of the Earth's gravitational influence? Mass of the satellite = 200 kg; mass of Earth = 6.0×10^{24} kg; radius of the Earth = 6.4×10^{6} m; G = 6.67×10^{-11} N-m^2/kg^2.

PGR181

12. An artificial satellite orbiting the earth in very thin atmosphere loses its energy gradually due to small but continuous dissipation against atmospheric resistance. Then explain why its speed increases progressively as it comes closer and closer to the earth.

PGR182

13. For a satellite moving in an orbit around the earth what is the ratio of kinetic energy to potential energy?

PGR183

14. An object weighs 10 N at the north pole of the Earth. In a geostationary satellite distant 7R from the centre of the Earth (of radius R), the true weight and the apparent weight are respectively: -

(A) 0, 0 (B) 0.2 N, 0 (C) 0.2 N, 9.8 N (D) 0.2 N, 0.2 N

PGR184

15. What is the minimum energy required to launch to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of 2R?

PGR185

BEGINNER'S BOX ANSWER KEY

BEGINNER'S BOX-1

1. Zero **2.** $\sqrt{3}\frac{Gm^2}{a^2}$ **3.** (C)

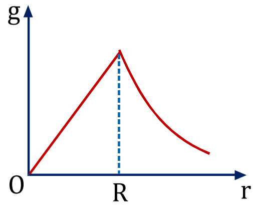
BEGINNER'S BOX-2

1. A, B **2.** (A) **3.** (A) **4.** (A)

5. 32 km **6.** 0.5% **7.** 4.44 m/s^2 **8.** h = R = 6400 km

9. 50 km **10.** 1920 km

11. $d = \frac{3}{4}R$ **12.**



BEGINNER'S BOX-3

1. $\frac{1}{2}mgR$ **2.** $V = V_0 + K\log\left(\frac{r}{r_0}\right)$ **3.** $-2200 \times 6.67 \times 10^{-11}$ J/kg

4. (B) **5.** (D) **6.** (D) **7.** (C)

8. $\boxed{PE = -\frac{3Gm^2}{2R}}$

BEGINNER'S BOX-4

1. $\left(\frac{2}{5}\right)^{3/2}$ years **2.** $2\sqrt{2}$ years **3.** (D) **4.** (D)

5. 6.47×10^{24} kg

6. angular momentum $L = m_P r_P v_P = m_P r_A v_A$

$\therefore \frac{v_P}{v_A} = \frac{r_A}{r_P}$ $\therefore$ Since $r_A > r_P$, $v_P > v_A$.

The area SBAC bounded by the ellipse and the radius vectors SB and SC is larger than SBPC in Fig. From Kepler's second law, equal areas are swept in equal time intervals. Hence the planet will take a longer time to traverse BAC than CPB.

7. All quantities vary over an orbit except angular momentum and total energy.

8. 9.7 days **9.** $\sqrt{\frac{3\pi}{G\rho}}$ **10.** $\frac{12}{1}$ **11.** 5.89×10^9 J

12. Kinetic energy increases, but potential energy decreases, and the sum decreases due to dissipation against friction.

13. $-\frac{1}{2}$ **14.** (B) **15.** $\frac{5GMm}{6R}$