02 Collisions and Centre of Mass

Introduction of Centre of Mass

Centre of mass is a point where we can concentrate whole mass of the body and it behave in similar manner as a point object behave under same circumstances.

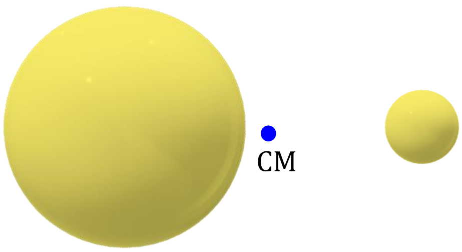
OR

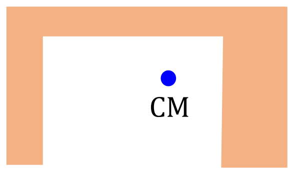
The point in a system at which whole mass of the system may be assumed to be concentrated for all translational effects of system is called Centre of Mass (CM).

Properties of Centre of Mass

(i) For symmetrical bodies having uniform distribution of mass, it coincides with centre of symmetry or geometrical centre.

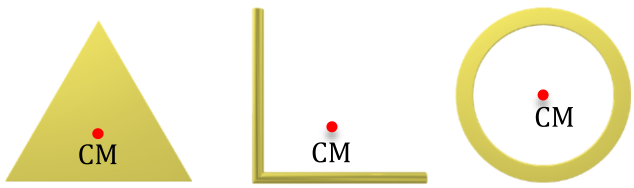
(ii) For a given shape it depends on the distribution of mass within the body and is closer to massive part.





Irregular mass distribution

(iii) There may or may not be any mass present physically at centre of mass and it may be within or outside the body.



Centre of Mass of Discrete mass System

Assume there are n-discrete particles with position vector $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$ respectively.

We can say that $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

From definition of COM, Position vector of centre of mass of all n-particles

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n}{m_1 + m_2 + m_3 + ... + m_n}$$

$$\vec{r}_{cm} = \frac{m_1(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+m_2(x_2\hat{i}+y_2\hat{j}+z_2\hat{k})+m_3\left(x_3\hat{i}+y_3\hat{j}+z_3\hat{k}\right)+..........+m_n\left(x_n\hat{i}+y_n\hat{j}+z_n\hat{k}\right)}{m_1+m_2+m_3+......................+m_n}$$

$$\vec{r}_{cm} = \frac{(m_1x_1+m_2x_2+........+m_nx_n)\hat{i}}{m_1+m_2+m_3+.......+m_n}+\frac{(m_1y_1+m_2y_2.......+m_ny_n)\hat{j}}{m_1+m_2+m_3+.......+m_n}+\frac{(m_1z_1+m_2z_2+........+m_nz_n)\hat{k}}{m_1+m_2+m_3+.......+m_n}$$

$$\vec{r}_{cm} = x_{cm}\hat{i}+y_{cm}\hat{j}+z_{cm}\hat{k}$$

Where;

$$x_{cm} = \frac{m_1x_1+m_2x_2+.......+m_nx_n}{m_l+m_2+.......+m_n}\ ;\ y_{cm} = \frac{m_1y_1+m_2y_2+.......+m_ny_n}{m_l+m_2+.......+m_n}\ ;$$

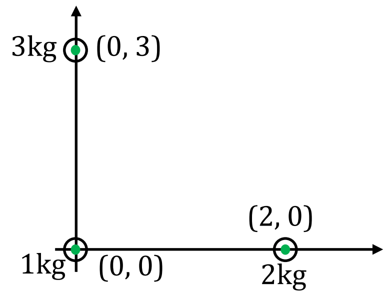
$$z_{cm} = \frac{m_1z_1+m_2z_2+.......+m_nz_n}{m_l+m_2+.......+m_n}$$

Note : If COM is at origin, $\vec{r}_{cm}=0$

$$m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3+.....=\vec{0}$$

Illustration 1

Find position of COM?



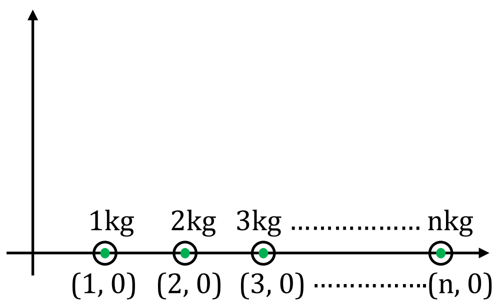
Solution:

$$x_{cm} = \frac{1(0)+2(2)+3(0)}{6}=\frac{2}{3} \text{ and } y_{cm} = \frac{1(0)+2(0)+3(3)}{6}=\frac{3}{2}$$

So position of COM $\left(\frac{2}{3},\frac{3}{2}\right)$.

Illustration 2

Find position of COM?



Solution:

$$x_{cm} = \frac{1(1)+2(2)+3(3).........n(n)}{1+2+3+.........}$$

$$=\frac{1^2+2^2+3^2+........+n^2}{1+2+3+.........+n} = \frac{\left[\frac{n(n+1)(2n+1)}{6}\right]}{\left[\frac{n(n+1)}{2}\right]}=\frac{2n+1}{3}$$

Illustration 3

The position vector of three particles of masses m_1 = 1 kg, m_2 = 2 kg and m_3 = 3 kg are $\vec{r}_1=(\hat{i}+4\hat{j}+\hat{k})$m, $\vec{r}_2=(\hat{i}+\hat{j}+\hat{k})$m and $\vec{r}_3=(2\hat{i}-\hat{j}-2\hat{k})$m respectively. Find the position vector of their centre of mass?

Solution:

The position vector of COM of the three particles will be given by

$$\vec{r}_{COM}=\frac{m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3}{m_1+m_2+m_3}$$

Substituting the values, we get

$$\vec{r}_{COM}=\frac{(1)(\hat{i}+4\hat{j}+\hat{k})+(2)(\hat{i}+\hat{j}+\hat{k})+(3)(2\hat{i}-\hat{j}-2\hat{k})}{1+2+3}=\frac{1}{2}(3\hat{i}+\hat{j}-\hat{k})\text{m}$$

Centre of Mass of Two Particle System

Consider two particles of masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively. Let their centre of mass have position vector.

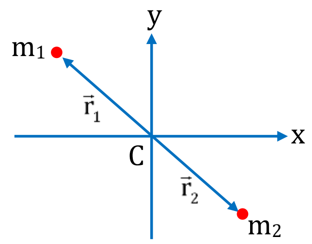
From definition, we have

$$\vec{r}_{CM} = \frac{\Sigma m_i \vec{r}_i}{M} \Rightarrow \vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

From the result obtained above we have, $x_{CM} = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$ and $y_{CM} = \dfrac{m_1y_1 + m_2y_2}{m_1 + m_2}$

If we assume origin to be at the centre of mass, then the vector $\vec{r}_{CM}$ vanishes and we have

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$



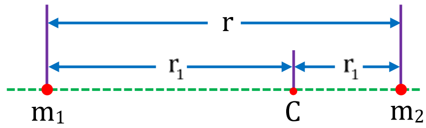
Since neither of the masses m_1 and m_2 can be negative, to satisfy the above equation, vectors $\vec{r}_1$ and $\vec{r}_2$ must have opposite signs. It is geometrically possible only when the centre of mass lies between the two particles on the line joining them as shown in the figure.

If we substitute magnitudes r_1 and r_2 of vectors $\vec{r}_1$ and $\vec{r}_2$ in the above equation, we have; $m_1r_1 = m_2r_2$

$$\Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1} \qquad \text{...(i)}$$

We conclude that the centre of mass of the two particles system lies between the two particles on the line joining them which divides the distance between them in the inverse ratio of their respective masses.

Consider two particles of masses m_1 and m_2 at a distance r from each other. Let the distances of these particles from the centre of mass be r_1 and r_2.



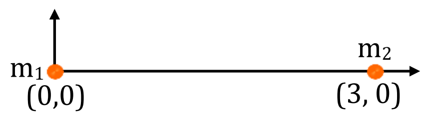
$r = r_1 + r_2$ ...(ii)

using equation (i) and (ii) we can write

$$r_1 = \frac{m_2 r}{m_1 + m_2} \text{ and } r_2 = \frac{m_1 r}{m_1 + m_2}$$

Illustration 4

Two particles of mass 1 kg and 2 kg are located at x = 0 and x = 3 m. Find the position of their centre of mass?

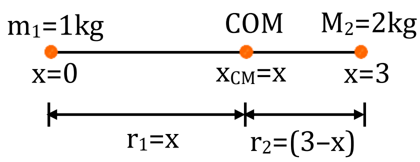


Solution:

Since, both the particles lies on x-axis, the COM will also lie on x-axis. Let the COM is located at x_{CM} = x, then

Distance of COM from the particle of mass 1 kg is $r_1 = x$

and distance of COM from the particle of mass 2 kg is $r_2 = (3 - x)$



Using $\dfrac{r_1}{r_2} = \dfrac{m_2}{m_1} \Rightarrow \dfrac{x}{3-x} = \dfrac{2}{1} \Rightarrow x = 2$ m

Thus, the COM of the two particles is located at x = 2 m.

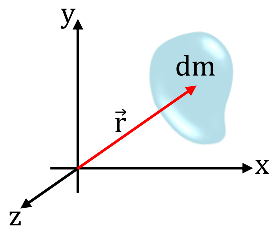
Centre of Mass of Continuous Mass Distribution

If a system has continuous distribution of mass, treating the mass element dm at position $\vec{r}$ as a point mass and replacing summation by integration:

$$\vec{r}_{cm} = \frac{\int \vec{r}dm}{m} \quad ; \quad \text{where } m = \int dm$$

$$x_{cm} = \frac{\int xdm}{\int dm} = \frac{\int xdm}{m} \quad \Rightarrow \quad y_{cm} = \frac{\int ydm}{\int dm} = \frac{\int ydm}{m}$$

$$z_{cm} = \frac{\int zdm}{\int dm} = \frac{\int zdm}{m}$$



Mass Densities

Sometimes mass of an object is not uniform in that case we can conveniently express masses in terms of mass density.

On the basis of mass distribution density are of 3 type-

(i) Linear density (λ) = $\frac{m}{\ell}$ $\Rightarrow m = \lambda\ell$ (ii) Area density (σ) = $\frac{m}{A}$ $\Rightarrow m = \sigma A$

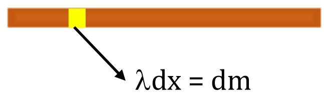
(iii) Volume density (ρ) = $\frac{m}{V}$ $\Rightarrow m = \rho V$

Types of Mass Densities

(a) Linear Mass Density (For 1-D Objects)

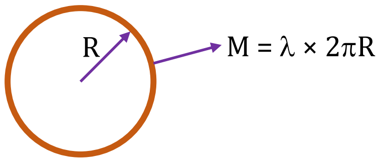
The linear mass density of the rod, $\lambda = \frac{m}{L}$

If we take an element of mass 'dm' having length 'dx' from the rod shown below, then we can write mass of the element $dm = \lambda dx$



Total mass of the rod $m = \int dm = \int \lambda dx$

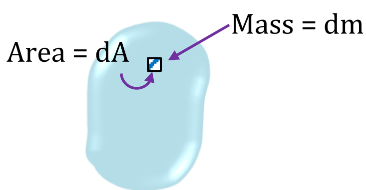
Example:- If linear mass density of ring is λ. Then mass of ring is $\lambda \times 2\pi R$



(b) Areal Mass Density (For 2-D Objects)

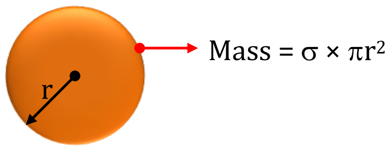
The areal mass density of the object, $\sigma = \frac{m}{A}$.

If we take an element of mass 'dm' having area 'dA', then we can write mass of the element $dm = \sigma dA$



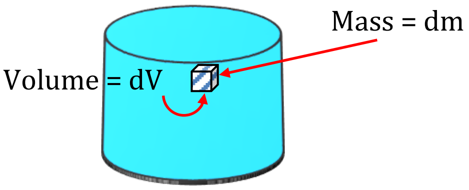
Total mass of the object $m = \int dm = \int \sigma dA$

Example:- If areal mass density of disc is σ then mass of disc is $\sigma \times \pi r^2$



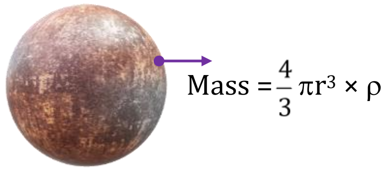
(c) Volumetric mass density (For 3-D Objects)

The volumetric mass density of the object, $\rho = \frac{m}{V}$.



If we take an element of mass 'dm' having volume 'dV' from the cylinder shown below, then we can write mass of the element $dm = \rho dV$

Total mass of the object $m = \int dm = \int \rho dV$



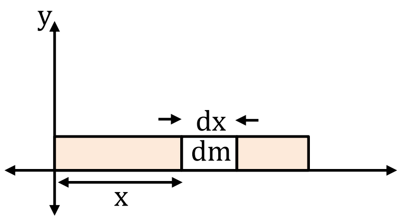
Example:- If volumetric mass density of solid sphere is ρ then its mass is $\frac{4}{3}\pi r^3 \times \rho$

Special Cases of Linear Mass Density

$$x_{cm} = \frac{\int x dm}{\int dm} = \frac{\int x(\lambda dx)}{\int \lambda dx} \qquad (dm = \lambda dx)$$

Case-I : λ = constant.

$$x_{cm} = \frac{\int_0^L x dx}{\int_0^L dx} = \frac{\frac{L^2}{2}}{L} = \frac{L}{2}$$



Case-II : $\lambda = \lambda_0 x$

$$x_{cm} = \frac{\int_0^L x(\lambda_0 x)dx}{\int_0^L \lambda_0 x dx} = \frac{\frac{L^3}{3}}{\frac{L^2}{2}} = \frac{2L}{3}$$

Case-III : $\lambda = \frac{\lambda_0 x^2}{L^2}$

$$x_{cm} = \frac{\int x\left(\frac{\lambda_0 x^2}{L^2}\right)dx}{\int \frac{\lambda_0 x^2}{L^2}dx} = \frac{\int_0^L x^3 dx}{\int_0^L x^2 (dx)} = \frac{\left[\frac{x^4}{4}\right]_0^L}{\left[\frac{x^3}{3}\right]_0^L} = \frac{3L}{4}$$

Case-IV : $\lambda = ax + b$

$$x_{cm} = \frac{\int_0^L (ax^2 + bx)dx}{\int_0^L (ax + b)dx} = \frac{\left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^L}{\left[\frac{ax^2}{2} + bx\right]_0^L} = \frac{\left[\frac{2ax^3 + 3bx^2}{6}\right]_0^L}{\left[\frac{ax^2 + 2bx}{2}\right]_0^L} = \frac{2aL^2 + 3bL}{3aL + 6b}$$

Illustration 5

The mass per unit length of a non-uniform rod of length L is given by $\mu = \lambda x^2$, where λ is a constant and x is distance from one end of the rod. The distance of the centre of mass of rod from this end is -

Solution:

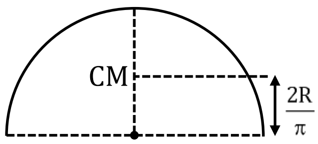
$$X_{CM} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^L \lambda x^3 dx}{\int_0^L \lambda x^2 dx} = \frac{3L}{4}$$

Uniform Symmetric Bodies

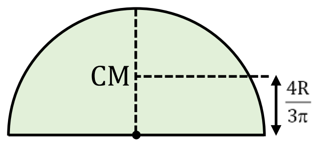
For symmetrical bodies having homogeneous distribution of mass, CM coincides with the centre of symmetry or the geometrical centre.

Centre of mass of some uniform symmetric bodies are given below: -

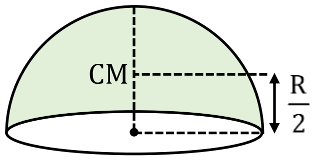
(i) Semi-circular ring of radius R



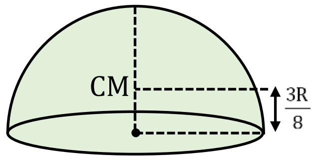
(ii) Semi-circular disc



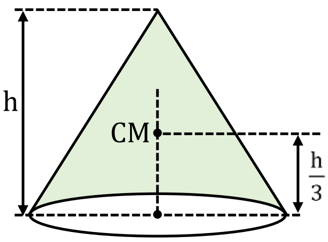
(iii) Hemispherical shell



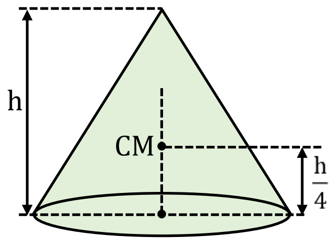
(iv) Solid hemisphere



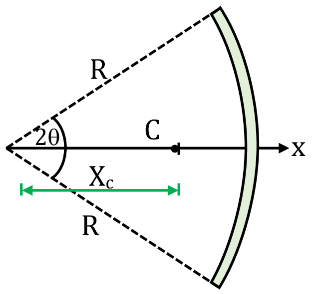
(v) Hollow cone



(vi) Solid cone

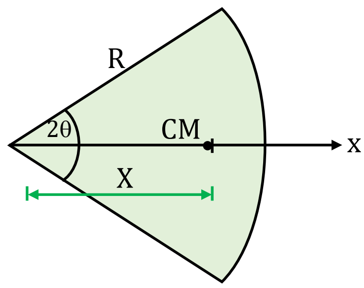


(vii) Circular arc



$$x_{CM} = \frac{R\sin\theta}{\theta}$$

(viii) Sector of a circular plate



$$x_{CM} = \frac{2R\sin\theta}{3\theta}$$

Note : Here θ is in radians.

Illustration 6

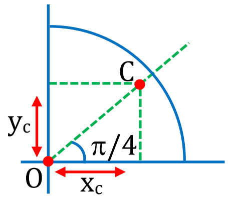
Find coordinates of center of mass of a quarter ring of radius r placed in the first quadrant of a Cartesian coordinate system, with centre at origin.

Solution:

Making use of the result of circular arc, distance OC of the center of mass from the center is

$$OC = \frac{r\sin(\pi/4)}{\pi/4} = \frac{2\sqrt{2}r}{\pi}$$

Coordinates of the center of mass (x_c, y_c) are $\left(OC\cos\frac{\pi}{4}, OC\sin\frac{\pi}{4}\right) \equiv \left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)$



Centre of Mass of composite bodies

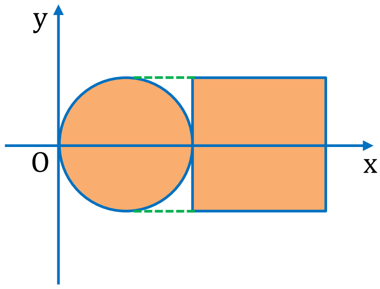
Composite Bodies are those bodies which consist of two or more than two individual body (continuous). In order to find the centre of mass, the component bodies are assumed to be particles of masses equal to the corresponding bodies located at their respective centres of masses. Then we use the equation to find the coordinates of the centre of mass of the composite body.

Here as we can see we have a composite body made of a disc and a square body placed side by side as shown in the figure.

Now, we have to calculate centre of mass of the composite body.

Co-ordinates of centre of mass of composite body are

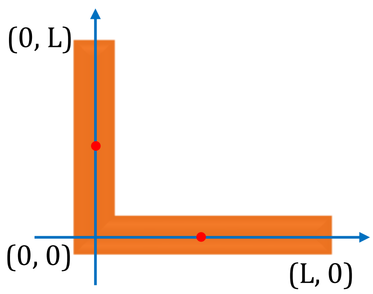
$$x_{CM} = \frac{m_d x_d + m_s x_s}{m_d + m_s} \text{ and } y_{CM} = \frac{m_d y_d + m_s y_s}{m_d + m_s}$$



Where m_d and m_s are masses of disc and square and x_d and x_s are x-coordinates of centre of mass of disc and square respectively and y_d and y_s are y-coordinates of centre of mass of disc and square respectively

Illustration 7

Find position of COM.



Solution:

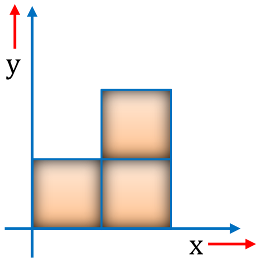
Centre of mass of rods are $\left(\frac{L}{2}, 0\right), \left(0, \frac{L}{2}\right)$

For the system $x_{cm} = \frac{m\left(\frac{L}{2}\right) + m(0)}{2m} = \frac{L}{4}$ & $y_{cm} = \frac{m(0) + m\left(\frac{L}{2}\right)}{2m} = \frac{L}{4}$

So, position of COM $\left(\frac{L}{4}, \frac{L}{4}\right)$.

Illustration 8

Find position of COM for given combination (each plate is square of side a and mass m).



Solution:

Centre of mass of square plate are $\left(\frac{a}{2},\frac{a}{2}\right),\left(\frac{3a}{2},\frac{a}{2}\right),\left(\frac{3a}{2},\frac{3a}{2}\right)$

For the system $x_{cm}=\dfrac{m\left(\frac{a}{2}\right)+m\left(\frac{3a}{2}\right)+m\left(\frac{3a}{2}\right)}{3m}=\dfrac{7a}{6}$ and $y_{cm}=\dfrac{m\left(\frac{a}{2}\right)+m\left(\frac{a}{2}\right)+m\left(\frac{3a}{2}\right)}{3m}=\dfrac{5a}{6}$

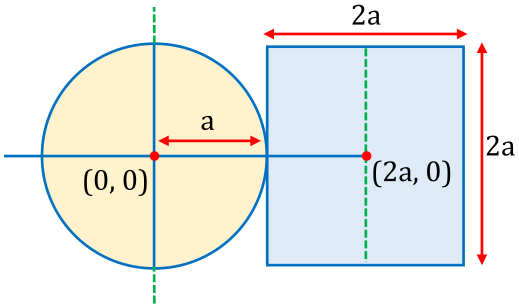
So, position of COM of system will be $\left(\frac{7a}{6},\frac{5a}{6}\right)$.

Illustration 9

Find out position of COM from given reference point.

(1) If their masses are equal

(2) If their density are equal



Solution:

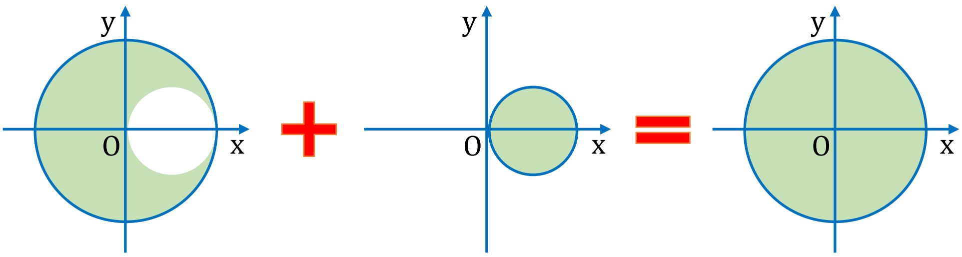
(1) $x_{cm}=\dfrac{m(0)+m(2a)}{2m}=a$

So, position of COM (a, 0)

(2) $x_{cm}=\dfrac{(\sigma\pi a^2)\times 0+(\sigma\times 4a^2)\times 2a}{\sigma\times\pi a^2+\sigma\times 4a^2}=\left(\dfrac{8a}{\pi+4}\right)$ $(m = \sigma A)$

Centre of Mass of Truncated Bodies

To find the centre of mass of truncated bodies or bodies with cavities we can make use of superposition principle that is, if we restore the removed portion in the same place we obtain the original body. The idea is illustrated in the following figure.



The removed portion is added to the truncated body keeping their location unchanged relative to the coordinate frame.

If a portion of a body is taken out, the remaining portion may be considered as,

[Original mass (M) – mass of the removed part (m)] = {original mass (M)} + {– mass of the removed part (m)}

The formula changes to : $x_{cm}=\dfrac{Mx-mx'}{M-m}$; $y_{cm}=\dfrac{My-my'}{M-m}$; $z_{cm}=\dfrac{Mz-mz'}{M-m}$

Where x', y' and z' represent the coordinates of the centre of mass of the removed part.

Illustration 10

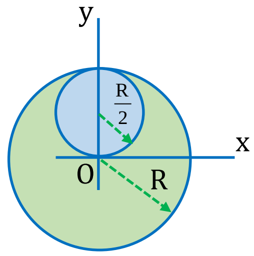
A disc of radius R is cut off from a uniform thin sheet of metal. A circular hole of radius $\frac{R}{2}$ is now cut out from the disc, with the hole being tangent to the rim of the disc. Find the distance of the centre of mass from the centre of the original disc.

Solution:

By symmetry, the CM lies along the +y–axis in figure, so $x_{CM} = 0$. With the origin at the centre of the original circle whose mass is assumed to be m.

Mass of original uncut disc $m_1 = m$ & Location of CM original uncut disc = (0,0)

Mass of disc removed : $m_2 = \frac{m}{4}$; Location of CM disc removed = $\left(0, \frac{R}{2}\right)$



Thus $y_{CM} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2} = \frac{m(0) - \frac{m}{4}\frac{R}{2}}{m - \frac{m}{4}} = -\frac{R}{6}$

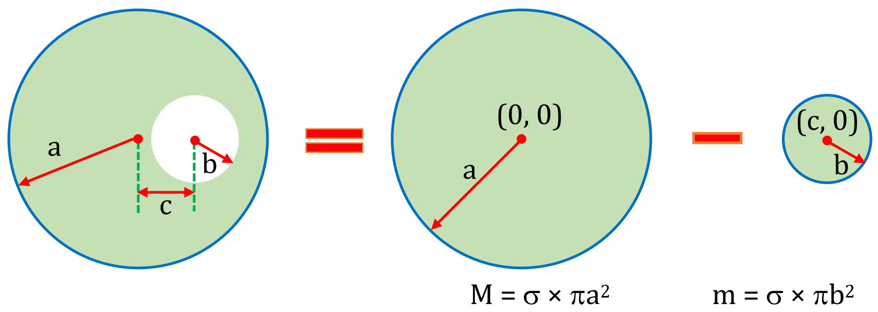
So, the centre of mass is at the point $\left(0, -\frac{R}{6}\right)$.

Thus, the distance of the centre of mass from the centre of the original disc is R/6.

Illustration 11

Find the centre of mass of a uniform disc of radius 'a' from which a circular section of radius 'b' has been removed. The centre of the hole is at a distance c from the centre of the disc.

Solution:

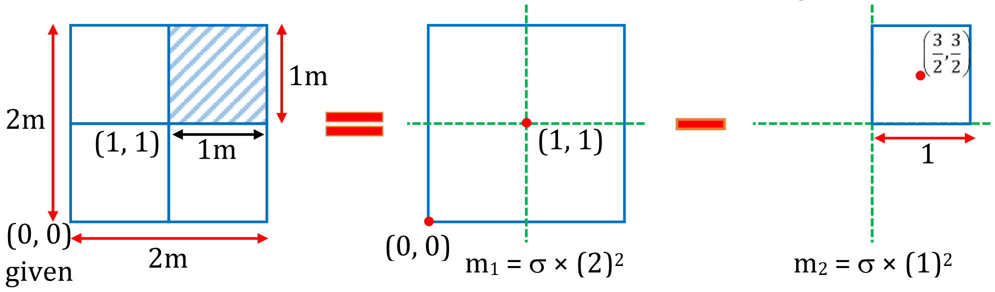


$$x_{cm} = \frac{(\sigma \times \pi a^2) \times 0 - (\sigma \times \pi b^2) \times c}{\sigma \pi a^2 - \sigma \pi b^2} = \frac{-b^2c}{a^2 - b^2}$$

So, coordinate of COM of remaining portion is $\left(\frac{-b^2c}{a^2 - b^2}, 0\right)$

Illustration 12

From given plate if shaded position is removed then find out COM of remaining.



Solution:

$$x_{cm} = \frac{\left[\sigma\times(2)^2\right]\times 1-\left[\sigma(1)^2\right]\times\frac{3}{2}}{(\sigma\times 2^2)-(\sigma\times(1)^2)} = \frac{4-\frac{3}{2}}{4-1} = \frac{5}{6}\text{ m}$$

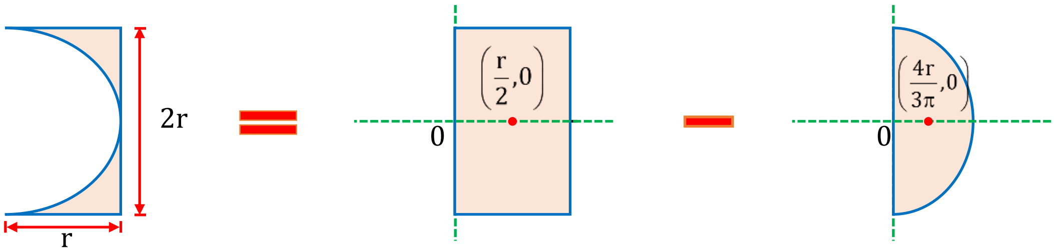
Similarly, $y_{cm} = \frac{5}{6}$ m

So, position of COM $\left[\frac{5}{6},\frac{5}{6}\right]$

Illustration 13

A semi-circular disc of radius r is removed from a rectangular plate of side 'r' and as '2r' as shown in figure. Then find out position of COM of remaining part.

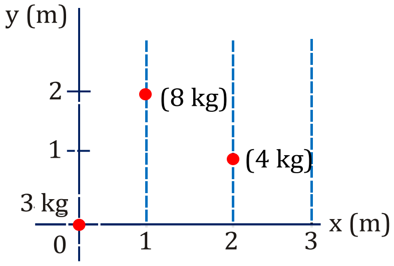
Solution:



$$x_{cm} = \frac{\sigma\left(2r^2\right)\left(\frac{r}{2}\right)-\sigma\left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right)}{\sigma\left(2r^2\right)-\sigma\left(\frac{\pi r^2}{2}\right)} = \frac{r-\frac{2r}{3}}{\frac{4-\pi}{2}} = \frac{2r}{3[4-\pi]}$$

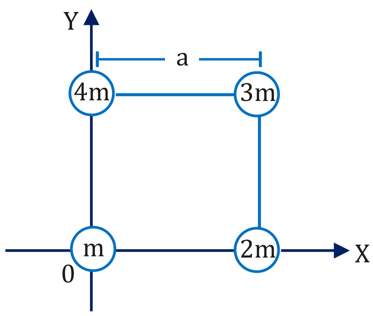
BEGINNER'S BOX-1

1. What are the co–ordinates of the centre of mass of the three particles system shown in figure?



PSC100

2. Four particles of masses m, 2m, 3m, 4m are placed at the corners of a square of side 'a' as shown in fig. Find out the co-ordinates of centre of mass.



PSC101

3. A rigid body consists of a 3 kg mass connected to a 2 kg mass by a massless rod. The 3 kg mass is located at $\vec{r}_1 = \left(2\hat{i}+5\hat{j}\right)$m and the 2 kg mass at $\vec{r}_2 = \left(4\hat{i}+2\hat{j}\right)$m. Find the position and coordinates of the centre of mass.

PSC102

4. Fig. shows a uniform square plate from which one or more of the four identical squares at the corners will be removed.

(a) Where is the COM of the plate originally.

(b) Where is the COM after square 1 is removed.

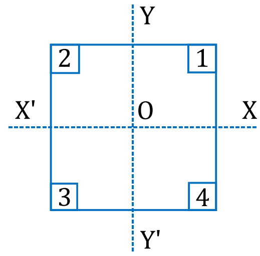
(c) Where is the COM after squares 1 and 2 removed.

(d) Where is the COM after squares 1 and 3 are removed.

(e) Where is the COM after squares 1, 2 and 3 are removed.

(f) Where is the COM after all the four squares are removed.

Give your answers in terms of quadrants and axis.



PSC103

5. Find the centre of mass of a uniform disc of radius 'a' from which a circular section of radius 'b' has been removed. The centre of the hole is at a distance c from the centre of the disc.

PSC104

Motion of Centre of Mass

Velocity of COM

For a system of particles, position of centre of mass is given by

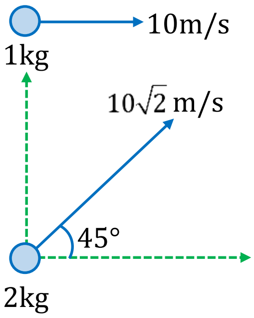
$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...}{m_1 + m_2 + m_3 + ...} \qquad \text{So, } \frac{d}{dt}\left(\vec{R}_{CM}\right) = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + ...}{m_1 + m_2 + m_3 + ...}$$

velocity of centre of mass ;

$$\vec{v}_{CM} = \frac{d\vec{R}_{CM}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ...}{m_1 + m_2 + ...}$$

Illustration 14

For a given system of particles find out velocity of COM.



Solution:

$$\vec{v}_{cm} = \frac{1(10\ \hat{i}) + 2(10\hat{i}+10\hat{j})}{3} = \frac{10\hat{i}+20\hat{i}+20\hat{j}}{3} = \frac{30\hat{i}+20\hat{j}}{3} = \frac{30\hat{i}}{3} + \frac{20\hat{j}}{3} = \left(10\hat{i} + \frac{20}{3}\hat{j}\right) m/s$$

Acceleration of COM

For a system of particles,

Acceleration of centre of mass $\vec{a}_{CM} = \dfrac{d}{dt}\left(\vec{v}_{CM}\right) = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2 + ...}{m_1 + m_2 + ...}$

Illustration 15

Two particles of masses 2 kg and 4 kg are approaching towards each other with accelerations of 1 m/s^2 and 2 m/s^2 respectively, on a smooth horizontal surface. Find the acceleration of centre of mass of the system.

Solution:

The acceleration of centre of mass of the system

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1+m_2} \Rightarrow a_{cm} = \frac{|m_1\vec{a}_1 + m_2\vec{a}_2|}{m_1+m_2}$$

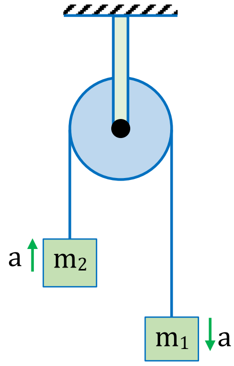
Since $\vec{a}_1$ and $\vec{a}_2$ are anti–parallel

$$\text{So, } a_{cm} = \frac{|m_1a_1 - m_2a_2|}{m_1+m_2} = \frac{|(2)(1)-(4)(2)|}{2+4} = 1\ m/s^2$$

Since $m_2a_2 > m_1a_1$ so the direction of acceleration of centre of mass is along in the direction of a_2.

Illustration 16

For a given system find out $\vec{a}_{cm}$



Solution:

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1+m_2} = \frac{(m_1-m_2)a}{(m_1+m_2)} \qquad \left(\because a = \frac{(m_1-m_2)g}{m_1+m_2}\right)$$

$$\text{So } \vec{a}_{cm} = \left(\frac{m_1-m_2}{m_1+m_2}\right)g\left(\frac{m_1-m_2}{m_1+m_2}\right) = \left(\frac{m_1-m_2}{m_1+m_2}\right)^2 g$$

Linear Momentum

We can write $M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + ... = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... [\because \vec{p} = m\vec{v}]$

$$M\vec{v}_{CM} = \vec{p}_{CM} \qquad [\because \Sigma\vec{p}_i = \vec{p}_{CM}]$$

Linear momentum of a system of particles is equal to the product of mass of the system with velocity of its centre of mass. From Newton's second law $\vec{F}_{ext.} = \frac{d(M\vec{v}_{CM})}{dt}$

Illustration 17

Two bodies of masses 10 kg and 2 kg are moving with velocities $(2\hat{i} - 7\hat{j} + 3\hat{k})$ m/s and $(-10\hat{i} + 35\hat{j} - 3\hat{k})$ m/s respectively. Find the momentum of their centre of mass.

Solution:

$$\vec{P}_{CM} = M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 = 10(2\hat{i} - 7\hat{j} + 3\hat{k}) + 2(-10\hat{i} + 35\hat{j} - 3\hat{k}) = 24\hat{k}$$

Illustration 18

Two Particles of mass 1 kg and 3kg are having acceleration $(\hat{i} - 2\hat{j} + 5\hat{k})$ m/s^2 and $\left(-2\hat{i} + \frac{4}{3}\hat{j} - \hat{k}\right)$ m/s^2.

Find out $\vec{a}_{CM}$.

Solution:

$$\vec{a}_{cm} = \frac{1(\hat{i} - 2\hat{j} + 5\hat{k}) + 3(-2\hat{i} + \frac{4}{3}\hat{j} - \hat{k})}{4} = \frac{\hat{i} - 2\hat{j} + 5\hat{k} - 6\hat{i} + 4\hat{j} - 3\hat{k}}{4} = \left(\frac{-5}{4}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) m/s^2$$

Effect of External Force on Centre of Mass

From Newton's second law $\vec{F}_{ext.} = \frac{d(M\vec{v}_{CM})}{dt}$

$M\,\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...\quad \Rightarrow \quad \vec{F}_{Net} = M\,\vec{a}_{CM}$

If $F_{ext} = 0 \Rightarrow a_{CM} = 0$

Illustration 19

Two forces $\vec{F}_1 = (4\hat{i} - 2\hat{j} + 3\hat{k})N$ and $\vec{F}_2 = (7\hat{i} - 8\hat{j} + 5\hat{k})N$ are acting on two particles of mass 10kg and 5kg respectively then find out $\vec{a}_{CM}$.

Solution:

$$\vec{a}_{CM} = \frac{\vec{F}_{net}}{m_{total}} \Rightarrow \vec{a}_{CM} = \frac{(4\hat{i} - 2\hat{j} + 3\hat{k}) + (7\hat{i} - 8\hat{j} + 5\hat{k})}{15} = \frac{(11\hat{i} - 10\hat{j} + 8\hat{k})}{15} m/s^2$$

Special Case

If $\vec{F}_{ext} = 0$ and initial velocity of Centre of Mass $\vec{u}_{CM} = 0$

Hence, $\vec{a}_{CM} = 0 \quad \Rightarrow \vec{v}_{CM} = \vec{u}_{CM} = 0$

Where $\vec{v}_{CM}$ = Final velocity of Centre of Mass

Thus, Centre of Mass will not move and change in position of centre of mass will be zero.

$\therefore \qquad \Delta\vec{r}_{CM} = 0$

$\Rightarrow \qquad (\vec{r}_{CM})_{Final} - (\vec{r}_{CM})_{Initial} = 0$

$$\Rightarrow \qquad \frac{\Sigma m_i \vec{r}_{i_F}}{\Sigma m_i} - \frac{\Sigma m_i \vec{r}_{i_I}}{\Sigma m_i} = 0 \Rightarrow \frac{\Sigma m_i (\vec{r}_{i_F} - \vec{r}_{i_I})}{\Sigma m_i} = 0 \Rightarrow \frac{\Sigma m_i (\Delta\vec{r})_i}{\Sigma m_i} = 0$$

$\Rightarrow \qquad \Sigma m_i (\Delta\vec{r})_i = 0$

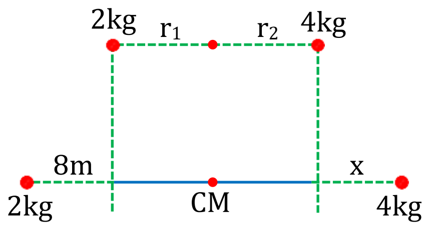
$\Rightarrow \qquad m_1\Delta r_1 + m_2\,\Delta r_2 + ... m_i\Delta r_i = 0$

Where $(\Delta\vec{r})_i$ = change in position of i^{th} particle

Illustration 20

Two particles of mass 2kg and 4kg are present at a fixed distance from each other as shown. If 2 kg mass is displaced towards left by 8m than find out by what distance 4kg mass should be displaced. So, COM of system remains at its initial position.

Solution:



COM of system remains at its initial position,

$m_1r_1 = m_2r_2$

Initial $\quad 2r_1 = 4r_2 \qquad$ ...(i)

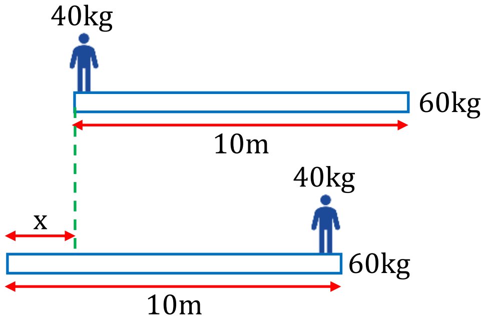
Final $\quad 2(r_1 + 8) = 4(r_2 + x)$

$\qquad 2r_1 + 16 = 4r_2 + 4x$

$\qquad x = 4m$

Illustration 21

A person of mass 40 kg is standing on a plank of length 10m. If person moves on the other side of plank find distance moved by the plank.



Solution:

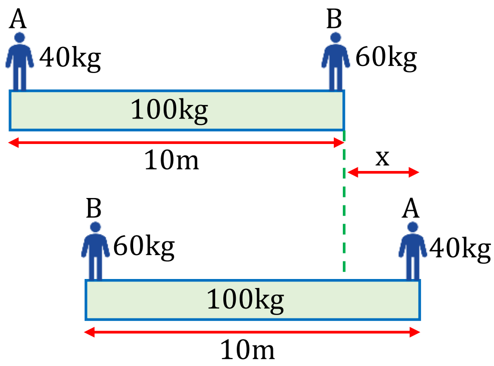
Let plank moves distance 'x'.

$40(10-x)+60(-x)=0 \Rightarrow 40(10-x)=60(x)$

$\Rightarrow 40-4x=6x \Rightarrow x=4\text{m}$

Illustration 22

Two persons A & B are standing on a plank as shown in figure. If A & B exchange their position find displacement of the plank.



Solution:

As there is no external force acting on the system so COM of system does not change its position. Let plank moves 'x' distance.

$40(10+x)+100(x)+60(x-10)=0$

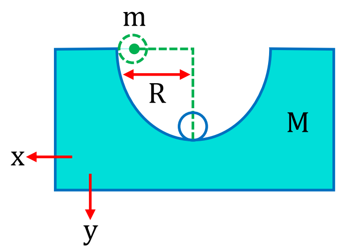
$\Rightarrow 400+40x+100x+60x-600=0$

$\Rightarrow -200+200x=0$

$\Rightarrow x=1\text{m}$

Illustration 23

A sphere of mass m is rolling down on a wedge of mass M. When sphere reaches at lowest point then find distance moved by wedge in horizontal direction.



Solution:

Let distance moved by wedge is 'x'

$m(R-x)-Mx=0$

$\Rightarrow mR-mx-Mx=0 \Rightarrow mR-x(M+m)=0 \Rightarrow x=\dfrac{mR}{M+m}$

Conservation of Linear Momentum of System and Impulse

As we have studied that

$$\vec{F}_{ext}=\frac{d(M\vec{v}_{CM})}{dt}=\frac{d(\vec{p}_{System})}{dt}$$

Now, if $\vec{F}_{ext}=0$ Then, $\vec{p}_{System}=\text{constant}$

It means that total linear momentum of a system of particles remains conserved in a time interval in which impulse of external forces is zero.

Total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

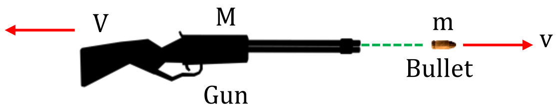
$$\vec{p}_{final}=\vec{p}_{initial}$$

The above statement is known as the principle of conservation of momentum.

Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

No external force $\Rightarrow$ Stationary mass relative to an inertial frame remains at rest.

Example : Firing a Bullet from a Gun :



If the bullet and gun is the system, then the force exerted by trigger will be internal.

Now, as initially both bullet and gun are at rest so $\vec{p}_G + \vec{p}_B = 0$. From this it is evident that:

$\vec{p}_G = -\vec{p}_B$, i.e., if bullet acquires forward momentum, the gun will acquire equal and opposite (backward) momentum.

From (i) $m\vec{v} + M\vec{V} = \vec{0}$, i.e, $\vec{V} = -\frac{m}{M}\vec{v}$ i.e, if the bullet moves forward, the gun 'recoils' or 'kicks backwards'.

Heavier the gun lesser will be the recoil velocity V.

Kinetic energy $K = \frac{p^2}{2m}$ and $|\vec{p}_B| = |\vec{p}_G| = p$. Kinetic energy of gun $K_G = \frac{p^2}{2M}$,

Kinetic energy of bullet $K_B = \frac{p^2}{2m}$ $\therefore \frac{K_G}{K_B} = \frac{m}{M} < 1 \left(\because M >> m\right)$. Thus, kinetic energy of gun is lesser than that of bullet i.e., kinetic energy of bullet and gun will not be equal. Initial kinetic energy of the system is zero as both are at rest. Final kinetic energy of the system is greater than zero.

So, here kinetic energy of the system is not constant but increases. However, energy is always conserved. Here chemical energy of gun powder is converted into KE.

Example : Block-Bullet System :

(a) When bullet remains embedded in the block

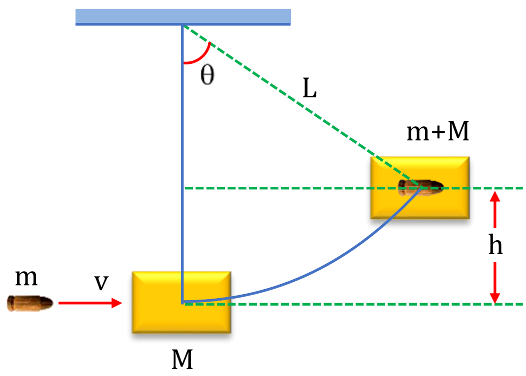
Conserving momentum of bullet and block ; mv + 0= (M+m) V

Velocity of block $V = \frac{mv}{M+m}$ ...(i)

By conservation of mechanical energy

$\frac{1}{2}(M+m)V^2 = (M+m)gh \Rightarrow V = \sqrt{2gh}$ ...(ii)

Maximum height gained by block $h = \frac{V^2}{2g} = \frac{m^2v^2}{2g(M+m)^2}$



(b) If bullet emerges out of the block

Conserving momentum $mv + 0 = mv_1 + Mv_2$

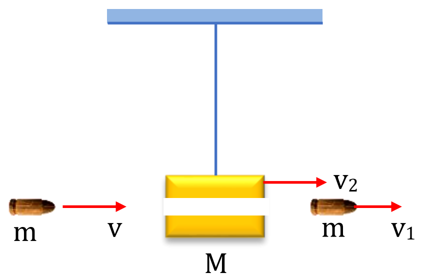
$m(v - v_1) = Mv_2$ ...(i)

Conserving energy

$\frac{1}{2}Mv_2^2 = Mgh \Rightarrow v_2 = \sqrt{2gh}$ ...(ii)

From eq^n. (i) & eq^n. (ii)

$m(v - v_1) = M\sqrt{2gh} \Rightarrow h = \frac{m^2(v-v_1)^2}{2gM^2}$



Important Points

- For an isolated system, initial momentum of the system is equal to the final momentum of the system. If the system consists of n bodies having momenta

  $\vec{p}_1, \vec{p}_2, \vec{p}_3, \dots\dots\vec{p}_n$, then $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots\dots + \vec{p}_n$ = constant

- As linear momentum depends on frame of reference, observers in different frames would find different values of linear momenta of a given system but each would agree that his own value of linear momentum does not change with time. But the system should be isolated and closed, i.e., law of conservation of linear momentum is independent of frame of reference though linear momentum depends on the frame of reference.
- Conservation of linear momentum is equivalent to Newton's III law of motion for a system of two particles. In the absence of external force from law of conservation of linear momentum,

  $\Rightarrow \vec{p}_1 + \vec{p}_2 =$ constant i.e. $m_1\vec{v}_1 + m_2\vec{v}_2$ = constant

  Differentiating the above expression with respect to time $m_1\dfrac{d\vec{v}_1}{dt} + m_2\dfrac{d\vec{v}_2}{dt} = \vec{0}$ [as m is constant]

  $\Rightarrow m_1\vec{a}_1 + m_2\vec{a}_2 = \vec{0}$ $\left[\because \dfrac{d\vec{v}_1}{dt} = \vec{a}_1\right]$ $\Rightarrow \vec{F}_1 + \vec{F}_2 = \vec{0}$ $\left[\because \vec{F} = m\vec{a}\right]$ $\Rightarrow \vec{F}_1 = -\vec{F}_2$

- This law is universal, i.e., it applies to macroscopic as well as microscopic systems.
- Total linear momentum of the system in centroidal frame is zero.

  It implies $\Sigma m_n\vec{v}_n = \vec{0}$ or $m_1\vec{v}_1 + m_2\vec{v}_2 + \dots\dots\dots + m_n\vec{v}_n = \vec{0}$.

Illustration 24

A bomb of mass 12 kg explodes into two pieces of 4 kg and 8 kg. The velocity of 4 kg piece is 24 m/s. The kinetic energy of 8 kg piece is

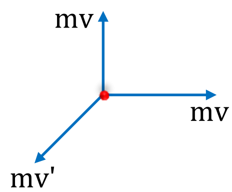
Solution:

$4 \times 24 = 8 \times v \Rightarrow v = 12$ m/s $\therefore$ $KE = \dfrac{1}{2} \times 8 \times 12 \times 12 = 576J$

Illustration 25

A body of mass 3m at rest explodes into three-identical pieces. Two of the pieces move with a speed v each in mutually perpendicular directions. The total kinetic energy released is: -

Solution:



$\therefore$ $mv' = \sqrt{2}mv \Rightarrow v' = \sqrt{2}v$

K.E. $= \left(\dfrac{1}{2}mv^2\right) \times 2 + \dfrac{1}{2}m \times \left(\sqrt{2}v\right)^2 = 2mv^2$

Illustration 26

A bomb initially at rest explodes by itself into three equal mass fragments. The velocities of two fragments are $\left(2\hat{i} + 3\hat{j}\right)$ m/s and $\left(\hat{i} - 2\hat{j}\right)$ m/s. The velocity of the third fragment is (in m/s): -

Solution:

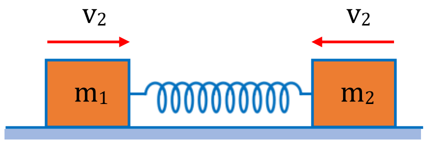
Apply momentum conservation,

$m\left(2\hat{i} + 3\hat{j}\right) + m\left(\hat{i} - 2\hat{j}\right) + m\vec{v} = 0$

$\therefore$ $\vec{v} = (-3\hat{i} - \hat{j})$

Spring Block System

Consider two blocks, resting on a frictionless surface and connected by a massless spring as shown in figure. If the spring is stretched (or compressed) and then released from rest,



Then $\vec{F}_{ext} = 0$ so $\vec{p}_s = \vec{p}_1 + \vec{p}_2$ = constant

However, initially both the blocks were at rest so, $\vec{p}_1 + \vec{p}_2 = \vec{0}$

It is clear that :

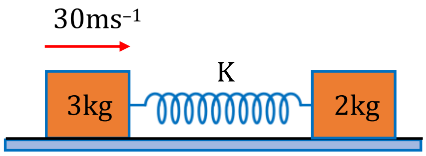
- $\vec{p}_2 = -\vec{p}_1$, i.e., at any instant the two blocks will have momentum equal in magnitude but opposite in direction (though they have different values of momentum at different positions).
- As momentum, $\vec{p} = m\vec{v}, m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = -\left(\frac{m_1}{m_2}\right)\vec{v}_1$

  The two blocks always move in opposite directions with lighter block moving faster.
- Kinetic energy $KE = \frac{p^2}{2m}$ and $|\vec{p}_1| = |\vec{p}_2|, \frac{KE_1}{KE_2} = \frac{m_2}{m_1}$ or the kinetic energy of two blocks will not be equal but in the inverse ratio of their masses and so lighter block will have greater kinetic energy.
- Initially kinetic energy of the blocks is zero (as both are at rest) but after some time kinetic energy of the blocks is not zero (as both are in motion). So, kinetic energy is not constant but changes. Here during the motion of the blocks KE is converted into elastic potential energy of the spring and vice-versa but total mechanical energy of the system remain constant.

  Kinetic energy + Potential energy = Mechanical Energy = Constant

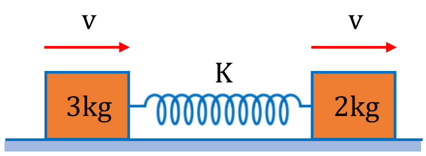
Illustration 27

Find maximum compression in the spring.



Solution:

Note : At the state of max. compression or max. elongation, velocity of connecting blocks must be same.



Apply COLM,

3 × 30 = 3v + 2v ⇒ v = 18 m/s

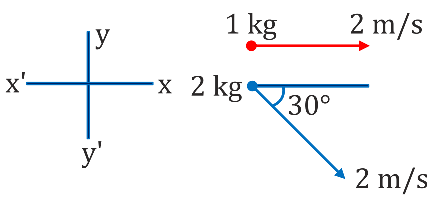
Apply COME,

$$\frac{1}{2}\times 3\times(30)^2 = \frac{1}{2}\times 3\times 18^2 + \frac{1}{2}\times 2\times 18^2 + \frac{1}{2}Kx^2$$

$\Rightarrow 2700 = 18^2 \times 5 + Kx^2 \Rightarrow 1080 = Kx^2 \Rightarrow x = \sqrt{\frac{1080}{K}}$

BEGINNER'S BOX-2

1. The velocity of centre of mass of the system as shown in the figure is :-



(A) $\left(\frac{2-2\sqrt{3}}{3}\right)\hat{i} - \frac{1}{3}\hat{j}$ (B) $\left(\frac{2+2\sqrt{3}}{3}\right)\hat{i} - \frac{2}{3}\hat{j}$ (C) $4\hat{i}$ (D) None of these

PSC105

2. Two bodies of masses 10 kg and 2 kg are moving with velocities $\left(2\hat{i}-7\hat{j}+3\hat{k}\right)$ m/s and $\left(-10\hat{i}+35\hat{j}-3\hat{k}\right)$ m/s respectively. Find the velocity of their centre of mass.

PSC106

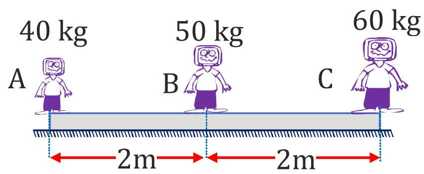
3. Two blocks of masses 5 kg and 2 kg placed on a frictionless surface are connected by a spring. An external kick gives a velocity of 14 m/s to the heavier block in the direction of the lighter one. Calculate the velocity gained by the centre of mass.

PSC107

4. Three particles of masses 1 kg, 2 kg and 3 kg are subjected to forces $\left(3\hat{i}-2\hat{j}+2\hat{k}\right)$N, $\left(-\hat{i}+2\hat{j}-\hat{k}\right)$N and $\left(\hat{i}+\hat{j}+\hat{k}\right)$N respectively. Find the magnitude of the acceleration of the CM of the system.

PSC108

5. Three men A, B & C of masses 40 kg, 50 kg and 60 kg are standing on a plank of mass 90 kg, which is kept on a smooth horizontal plane. If A & C exchange their positions then mass B will shift



(A) 1/3 m towards left (B) 1/3 m towards right
(C) will not move w.r.t. ground (D) 5/3 m towards left

PSC109

6. Consider a system having two masses m_1 and m_2 in which first mass is pushed towards the centre of mass by a distance a. The distance by which the second mass should be moved to keep the centre of mass at same position is :-

(A) $\frac{m_1}{m_2}a$ (B) $\frac{m_1}{(m_1+m_2)}a$ (C) $\frac{m_2}{m_1}a$ (D) $\left(\frac{m_2}{m_1+m_2}\right)a$

PSC110

7. Two particles A and B initially at rest, move towards each other under their mutual force of attraction. At the instant when the speed of A is v and the speed of B is 2v, the speed of the centre of mass of the system is:–
(A) 3v (B) v (C) 1.5v (D) zero

PSC111

8. An isolated particle of mass m is moving in a horizontal plane (x–y), along the x–axis, at a certain height above the ground. It suddenly explodes into two fragments of masses $\frac{m}{4}$ and $\frac{3m}{4}$. An instant later, the smaller fragment is at y = +15 cm. The larger fragment at this instant is at :–
(A) y = –5 cm (B) y = +20 cm (C) y = +5 cm (D) y = –20 cm

PSC112

9. In which of the following cases, the centre of mass of a rod may be at its centre?
(1) The linear mass density decreases continuously from left to right.
(2) The linear mass density increases continuously from left to right.
(3) The linear mass density decreases from left to right upto the centre and then increases.
(4) The linear mass density increases from left to right upto the centre and then decreases.
(A) 1, 2 (B) 3, 4 (C) 2, 4 (D) 1, 4

PSC113

Collision

Collision or Impact is the interaction between two bodies during very small duration in which they exert relatively large forces on each other. Interaction forces during an impact are created either due to direct contact or strong repulsive force fields or some connecting links.

The duration of the interaction is short enough to permit us only to consider the states of motion just before and after the event and not during the impact. Duration of an impact ranges from 10^{-23} s for impacts between elementary particles to millions of years for impacts between galaxies. The impacts we observe in our everyday life such as that between two balls last from 10^{-3} s to few seconds.

Example: -

- **Direct collision: -** When a rubber ball strikes a floor, it remains in contact with the floor for very short time in which it changes its velocity. This is an example of collision where physical contact takes place between the colliding bodies.
- **Indirect collision: -** When an α-particle passes by the nucleus of a gold atom in Rutherford's experiment, it gets deflected in a very short time. Deflection means a change in the direction of motion- a change in velocity. In this process, the particles do not touch each other.

  Effects of collision, the momentum and kinetic energy of the interacting bodies change.

  Interacting forces during collision are so large as compared to other external forces acting on either of the bodies that the effects of later can be neglected. Hence, forces involved in a collision are action–reaction pairs, i.e., the internal forces of the system and thus the **total momentum remains conserved in any type of collision**.

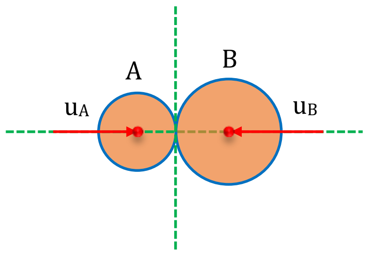
Types of Collision

(a) On the basis of direction:-

On the basis of direction there are two types of collision.

(i) Head-on Collision

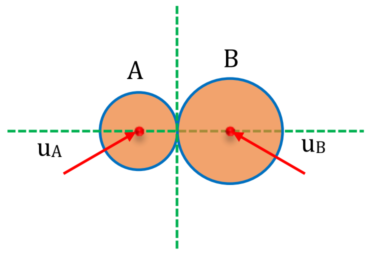
A Collision in which the particles move along the same straight line before and after the collision is defined as Head-on Collision or One-dimensional Collision.



Head-on Collision

(ii) Oblique Collision

A Collision in which the particles move along the same plane at different angles before and after the collision is defined as Oblique Collision or Two-dimensional Collision.



Oblique Collision

(b) On the basis of kinetic energy:-

On the basis of kinetic energy there are three types of collision.

(i) Elastic Collision

A Collision is said to be elastic if the total kinetic energy before and after the collision remains same.

(ii) Inelastic Collision

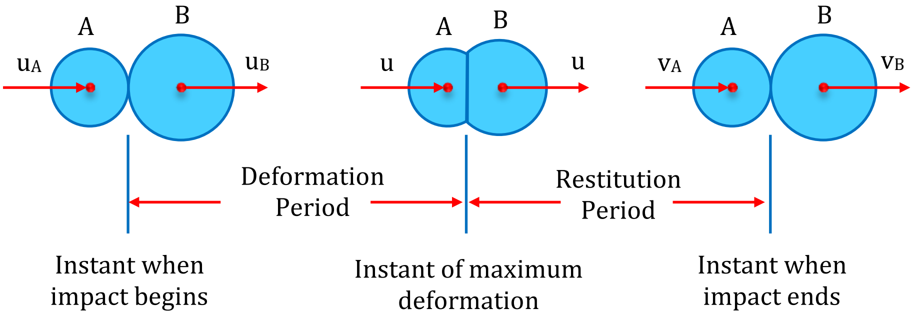
A Collision is said to be Inelastic if the total kinetic energy before and after the collision does not remain same.

(iii) Perfectly Inelastic Collision

In this type of collision loss in energy is maximum.

Process of Collision and Coefficient of Restitution

To understand mechanism of a collision, let us consider two balls A and B of masses m_A and m_B moving with velocities u_A and u_B in the same direction as shown. Velocity u_A is larger than u_B so that the ball A hits the ball B. During the impact, both the bodies push each other and first they get deformed till the deformation reaches a maximum value and then they try to regain their original shapes due to elastic behaviour of the materials the balls formed of.



Deformation period: - The time interval during which deformation takes place.

During the period of deformation, due to push applied by the balls on each other, speed of ball A decreases and that of ball B increases and at the end of the deformation period, when the deformation is maximum both the balls move with the same velocity say it is u.

Thereafter, the balls will either move together with this velocity or follow the period of restitution.

Restitution period: - The time interval in which the bodies try to regain their original shapes.

During the period of restitution due to push applied by the balls on each other, speed of the ball A decreases further and that of ball B increases further till they separate from each other.

Let us denote the velocities of the balls A and B after the impact by v_A and v_B respectively.

Coefficient of Restitution

Usually the force D applied by the bodies A and B on each other during the period of deformation differs from the force R applied by the bodies on each other during the period of restitution. Therefore, it is not necessary that the magnitude of impulse $\int D dt$ due to deformation equals to that of impulse $\int R dt$ due to restitution.

The ratio of magnitudes of impulse of restitution to that of deformation is called the coefficient of restitution and is denoted by e.

$$e = \frac{\text{impulse of recovery}}{\text{impulse of deformation}} = \frac{\int R dt}{\int D dt}$$

$$e = \frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along line of impact}} = \frac{|\vec{v}_B - \vec{v}_A|}{|\vec{u}_A - \vec{u}_B|}$$

Coefficient of restitution depends on various factors as elastic properties of materials forming the bodies, velocities of the contact points before impact, state of rotation of the bodies and temperature of the bodies. In general, its value ranges from zero to one but in collisions where additional kinetic energy is generated, its value may exceed one.

Depending on the values of coefficient of restitution, two particular cases are of special interest.

Perfectly Plastic or Inelastic Impact For these impacts e = 0, and bodies undergoing impact move with same velocity after the impact.

Perfectly Elastic Impact For these impacts e = 1.

Strategy to solve problems of head-on impact:

Write the momentum conservation equation;

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad ...(A)$$

Write the equation involving coefficient of restitution

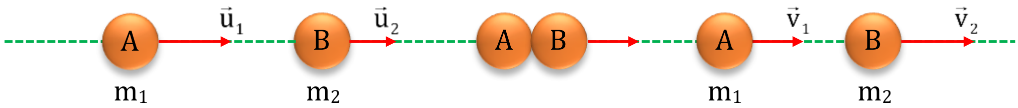
$$v_B - v_A = e(u_A - u_B) \quad ...(B)$$

Head on Collision

If velocity vectors of the colliding bodies are directed along the line of impact, the impact is called a direct or Head-on impact.

Head on Elastic Collision

The head on elastic collision is one in which the colliding bodies move along the same straight-line path before and after the collision.



Before collision During collision After collision

Assuming initial direction of motion to be positive and $u_1 > u_2$ (so that collision may take place) and applying law of conservation of linear momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \Rightarrow m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad ...(i)$$

For elastic collision, kinetic energy before collision must be equal to kinetic energy after collision, i.e.,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \Rightarrow m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad ...(ii)$$

Dividing equation (ii) by (i)

$$u_1 + v_1 = v_2 + u_2 \Rightarrow (u_1 - u_2) = (v_2 - v_1) \quad ...(iii)$$

In 1–D elastic collision 'velocity of approach' before collision is equal to the 'velocity of separation after collision, no matter what the masses of the colliding particles are.

This law is called **Newton's law for elastic collision**.

Thus, value of coefficient of restitution for elastic collision; $e = \dfrac{v_2 - v_1}{u_1 - u_2} = 1$

If we multiply equation (iii) by m_2 and subtract it from (i)

$$(m_1 - m_2)\vec{u}_1 + 2m_2\vec{u}_2 = (m_1 + m_2)\vec{v}_1 \Rightarrow \vec{v}_1 = \frac{m_1 - m_2}{m_1 + m_2}\vec{u}_1 + \frac{2m_2}{m_1 + m_2}\vec{u}_2 \quad ...(iv)$$

Similarly, multiplying equation (iii) by m_1 and adding it to equation (i)

$$2m_1\vec{u}_1 + (m_2 - m_1)\vec{u}_2 = (m_2 + m_1)\vec{v}_2 \Rightarrow \vec{v}_2 = \frac{m_2 - m_1}{m_1 + m_2}\vec{u}_2 + \frac{2m_1\vec{u}_1}{m_1 + m_2} \quad ...(v)$$

Special cases of head on elastic collision

Case-1: If the two bodies are of equal masses : $m_1 = m_2 = m$ then $v_1 = u_2$ and $v_2 = u_1$

Thus, if two bodies of equal masses undergo elastic collision in one dimension, then the bodies exchange their velocities after the collision.

Case-2: If the two bodies are of equal masses and second body is at rest.

$m_1 = m_2$ and initial velocity of second body $u_2 = 0$, $v_1 = 0$, $v_2 = u_1$

When body A collides against body B of equal mass at rest, then body A comes to rest and body B moves on with the velocity of body A. In this case transfer of energy is hundred percent

e.g. Billiard's Ball, Nuclear moderation.

Case-3: If the mass of one body is negligible as compared to the other.

If $m_1 >> m_2$ and $u_2 = 0$ then $v_1 = u_1$, $v_2 = 2u_1$

When a heavy body A collides against a light body B at rest, then body A should keep on moving with same velocity whereas body B moves with velocity double that of A.

If $m_2 >> m_1$ and $u_2 = 0$ then $v_2 = 0$, $v_1 = -u_1$

When a light body A collides against a heavy body B at rest, the body A starts moving with same speed just in opposite direction while the body B practically remains at rest.

Head on Inelastic Collision

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \quad \text{...(i)}$$

By definition of coefficient of restitution

$$\vec{v}_2 - \vec{v}_1 = e(\vec{u}_1 - \vec{u}_2) \quad \text{...(ii)}$$

$$\vec{v}_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)\vec{u}_1 + \left(\frac{(1+e)m_2}{m_1 + m_2}\right)\vec{u}_2 \quad \text{and} \quad \vec{v}_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)\vec{u}_2 + \left(\frac{(1+e)m_1}{m_1 + m_2}\right)\vec{u}_1$$

Can also be expressed $\vec{v}_1 = \dfrac{\vec{p}_i + m_2 e(\vec{u}_2 - \vec{u}_1)}{m_1 + m_2}$ and $\vec{v}_2 = \dfrac{\vec{p}_i + m_1 e(\vec{u}_1 - \vec{u}_2)}{m_1 + m_2}$ $(\vec{p}_i = m_1\vec{u}_1 + m_2\vec{u}_2)$

Loss in Kinetic Energy of particles = $\dfrac{1}{2}\dfrac{m_1 m_2}{(m_1 + m_2)}(1 - e^2)(\vec{u}_1 - \vec{u}_2)^2$

Value of coefficient of restitution for Inelastic collision is $0 < e < 1$

Head on Perfectly Inelastic Collision

$$m_1\vec{u}_1 + m_2\vec{u}_2 = (m_1 + m_2)\vec{v}$$

Where final velocity $\vec{v} = \dfrac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$

Loss in Kinetic Energy of particles = $\dfrac{1}{2}\dfrac{m_1 m_2}{(m_1 + m_2)}(\vec{u}_1 - \vec{u}_2)^2$

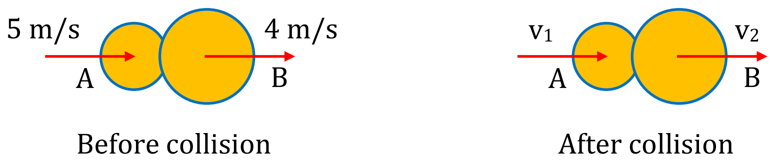
Value of coefficient of restitution for Perfectly Inelastic collision $e = 0$.

Illustration 28

A ball of mass 2 kg moving with a speed of 5 m/s collides directly with another ball of mass 3 kg moving in the same direction with a speed of 4 m/s. The coefficient of restitution is 2/3. Find the velocities after collision.

Solution:

Denoting the first ball by A and the second ball by B, velocities immediately before and after the impact are shown in the figure.



By COLM : $2(5) + 3(4) = 2v_1 + 3v_2 \Rightarrow 2v_1 + 3v_2 = 22$ ...(i)

By definition of e : $e = \dfrac{v_2 - v_1}{u_1 - u_2} \Rightarrow \dfrac{2}{3} = \dfrac{v_2 - v_1}{5 - 4} \Rightarrow 3v_2 - 3v_1 = 2$ ...(ii)

by solving equations (i) and (ii), we have $v_1 = 4$ m/s and $v_2 = 4.67$ m/s

Illustration 29

A body of 2 kg mass having velocity 3 m/s collides with a body of 1 kg mass moving with a velocity of 4m/s in the opposite direction. After collision both bodies stick together and move with a common velocity. Find the velocity in m/s.

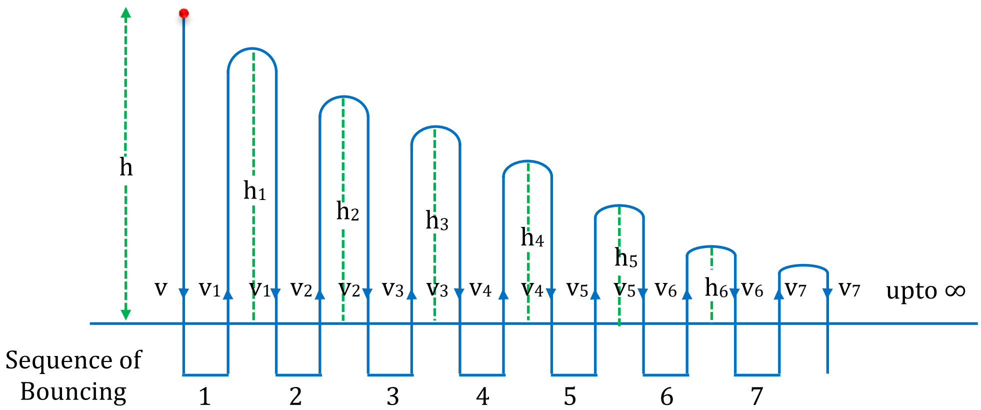
Solution:

$$v_{system} = \frac{m_1v_1 - m_2v_2}{m_1 + m_2} = \frac{2\times3-1\times4}{3} = \frac{2}{3}\,m/s$$

Bouncing of Ball

Let a ball fall from a height (h) and let it touch the ground with a velocity v taking time (t) to reach the ground. Let v_1, v_2, v_3....... be the velocities immediately after first, second, third......collisions with the ground.

Velocity immediately After the nth Rebound



By Newton's formula $(\vec{v}_2 - \vec{v}_1) = e(\vec{u}_1 - \vec{u}_2)$

Here $\vec{v}_2 = 0$, $\vec{u}_1 = v$, $\vec{u}_2 = 0$ (surface at rest)

$v_1 = ev$ (opposite direction)

Similarly, $v_2 = ev_1$

$v_2 = e(ev)$

$v_2 = e^2v$

Similarly, $v_3 = e^3v$, and $v_4 = e^4v$

Thus, $v_n = e^n v$

$v_n = e^n\sqrt{2gh}$ $\left[\because v = \sqrt{2gh}\right]$

Height Attained by the Ball After the nth Rebound

$v_1 = ev$

$\Rightarrow \quad \sqrt{2gh_1} = e\sqrt{2gh} \quad \Rightarrow \quad h_1 = e^2h$

Similarly, $v_2 = e^2v$

$\Rightarrow \quad \sqrt{2gh_2} = e^2\sqrt{2gh} \quad \Rightarrow \quad h_2 = e^4h$

Thus, $h_n = e^{2n}h$

Time Taken in n^{th} Rebound

$$h_1 = e^2 h$$

$$\Rightarrow \quad \frac{1}{2}gt_1^2 = e^2\frac{1}{2}gt^2 \quad \Rightarrow \quad t_1^2 = e^2t^2$$

Here t_1 is time of ascent or descent after first collision

$$\Rightarrow \quad t_1 = et \quad \Rightarrow \quad t_1 = e\sqrt{\frac{2h}{g}}$$

Similarly, $h_2 = e^4 h$

$$\Rightarrow \quad \frac{1}{2}gt_2^2 = e^4\left(\frac{1}{2}gt^2\right) \quad \Rightarrow \quad t_2^2 = e^4t^2 \quad \Rightarrow t_2 = e^2t \quad \Rightarrow t_2 = e^2\sqrt{\frac{2h}{g}}$$

Similarly, $t_n = e^n\sqrt{\frac{2h}{g}} \Rightarrow t_n = e^n t$

Total time taken in bouncing before the ball stops

$T = t + 2t_1 + 2t_2 + \ldots\ldots\ldots\ldots$

$T = t + 2et + 2e^2t + 2e^3t + \ldots\ldots\ldots\ldots$

$T = t + 2t\,(e + e^2 + e^3 + \ldots\ldots\ldots\ldots)$

$$T = t + 2t\left(\frac{e}{1-e}\right) = t\left(\frac{1+e}{1-e}\right) = \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)$$

Distance Covered by The Ball Before It Stops

$s = h + 2h_1 + 2h_2 + \ldots\ldots\ldots\ldots + \infty$

$s = h + 2e^2h + 2e^4h + 2e^6h + \ldots\ldots\ldots\ldots$

$s = h + 2e^2h(1 + e^2 + e^4 + e^6 + \ldots\ldots\ldots\ldots)$

$$s = h + 2e^2h\left(\frac{1}{1-e^2}\right) = h\left[1 + \frac{2e^2}{1-e^2}\right] = h\left(\frac{1+e^2}{1-e^2}\right)$$

Average Speed

$$v_{av.} = \frac{\text{Total distance}}{\text{Total time}} = \frac{h\left(\frac{1+e^2}{1-e^2}\right)}{\sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)} \quad \Rightarrow \quad v_{av.} = \sqrt{\frac{gh}{2}}\left[\frac{1+e^2}{(1+e)^2}\right]$$

Average Velocity

$$v_{av.} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{h}{\sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)} \quad \Rightarrow \quad v_{av.} = \sqrt{\frac{gh}{2}}\left(\frac{1-e}{1+e}\right)$$

Illustration 30

A particle falls from a height 'h' upon a fixed horizontal plane and rebounds. If e = 0.2 is the coefficient of restitution. Find the total distance travelled before rebounding has stopped.

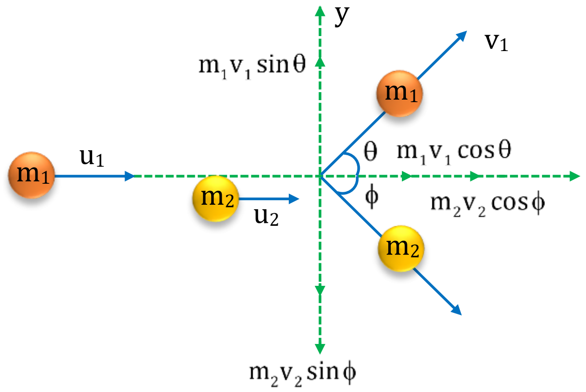
Solution:

$S = h + 2h_1 + 2h_2 + \ldots\ldots\ldots\ldots$

$$S = h\left(\frac{1+e^2}{1-e^2}\right) = h\left[\frac{1+\frac{1}{25}}{1-\frac{1}{25}}\right] = \frac{13}{12}h$$

Oblique Collision

A collision in which the particles move in the same plane at different angles before and after the collision is called Oblique Collision.



By COLM along x-axis, $m_1u_1 + m_2u_2 = m_1v_1\cos\theta + m_2v_2\cos\phi$

By COLM along y-axis, $0 + 0 = m_1v_1\sin\theta - m_2v_2\sin\phi$

If collision is elastic then,

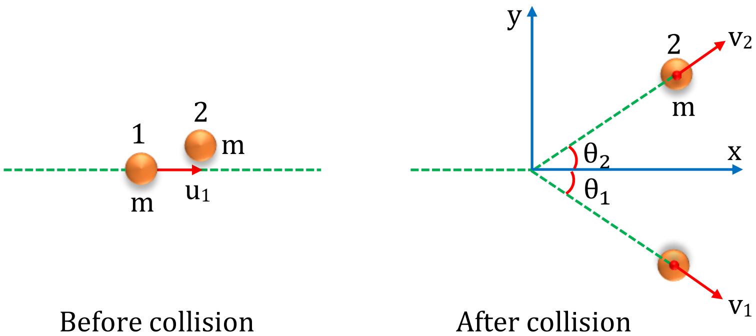
By conservation of kinetic energy,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Illustration 31

A body strikes obliquely with another identical stationary rest body elastically. Prove that they will move perpendicular to each other after collision.

Solution:



Before collision After collision

Conservation of linear momentum in x–direction gives

$mu_1 = mv_1\cos\theta_1 + mv_2\cos\theta_2 \Rightarrow u_1 = v_1\cos\theta_1 + v_2\cos\theta_2$ ...(i)

Conservation of linear momentum in y–direction gives

$0 = mv_2\sin\theta_2 - mv_1\sin\theta_1$

$\Rightarrow 0 = v_2\sin\theta_2 - v_1\sin\theta_1$ ...(ii)

Conservation of kinetic energy

$\frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \Rightarrow u_1^2 = v_1^2 + v_2^2$ ...(iii)

By, $(i)^2 + (ii)^2$

$$\Rightarrow u_1^2 + 0 = v_1^2\cos^2\theta_1 + v_2^2\cos^2\theta_2 + 2v_1v_2\cos\theta_1\cos\theta_2 + v_1^2\sin^2\theta_1 + v_2^2\sin^2\theta_2 - 2v_1v_2\sin\theta_1\sin\theta_2$$

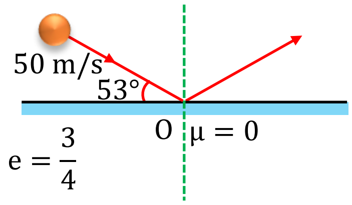
$$\Rightarrow u_1^2 = v_1^2\left(\cos^2\theta_1 + \sin^2\theta_1\right) + v_2^2\left(\cos^2\theta_2 + \sin^2\theta_2\right) + 2v_1v_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)$$

$$\Rightarrow u_1^2 = v_1^2 + v_2^2 + 2v_1v_2\cos(\theta_1+\theta_2) \qquad \left\{\because u_1^2 = v_1^2 + v_2^2\right\}$$

$$\Rightarrow \cos(\theta_1+\theta_2) = 0 \Rightarrow \theta_1 + \theta_2 = 90°$$

Illustration 32

Find maximum height, time of flight & range from O of particle after collision.



Solution:

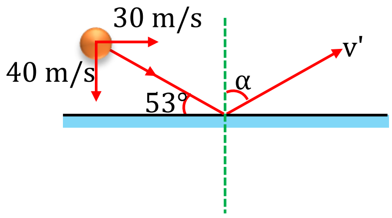
$$\frac{v'\cos\alpha}{40} = e = \frac{3}{4}$$

$$\therefore v'\cos\alpha = 30$$

$$v'\sin\alpha = 30$$

$$\alpha = 45° \ \& \ v' = 30\sqrt{2}$$

$$T = \frac{2v'\cos\alpha}{g} = 6\,\text{sec}$$



Range 6 × 30 = 180

$$H = \frac{v_y^2}{2g} = \frac{v'^2\cos^2\alpha}{2g} = 45\,\text{m}$$

BEGINNER'S BOX-3

1. A body of mass 2 kg makes an elastic collision with another body at rest and continues to move in the original direction with one fourth of its original speed. Find the mass of the second body.
PSC114

2. A particle of mass m moving with a velocity v makes a head on elastic collision with another particle of same mass initially at rest. Find the velocity of the first particle after the collision.
PSC115

3. A particle of mass m moving with velocity v strikes a stationary particle of mass 2m and sticks to it. Find the speed of the system.
PSC116

4. Two putty balls of equal masses moving in mutually perpendicular directions with equal speed, stick together after collision. If the balls were initially moving with a velocity of $45\sqrt{2}$ m/s each, find the velocity of the combined mass after collision.
PSC117

5. A body of 2 kg mass having velocity 3 m/s collides with a body of 1 kg mass moving with a velocity of 4m/s in the opposite direction. After collision both bodies stick together and move with a common velocity. Find the velocity in m/s.
PSC118

6. A ball of mass 1kg is dropped from 20 m height. Find (i) velocity of ball after second collision (ii) maximum height attained by the ball after second collision (iii) average speed for whole interval (If e = 0.5) ($g = 10\ m/s^2$)

PSC119

7. A ball is thrown vertically upward from ground with speed 40 m/s. It collides with ground after returning. Find the total distance travelled and time taken during its bouncing. (e = 0.5) ($g = 10\ m/s^2$)

PSC120

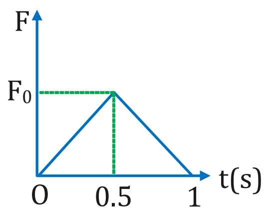
8. A particle falls from a height 'h' upon a fixed horizontal plane and rebounds. If e = 0.2 is the coefficient of restitution. Find the total distance travelled before rebounding has stopped.

PSC121

9. Two balls of equal masses undergo a head-on collision with speeds 6 m/s moving in opposite direction. If the coefficient of restitution is $\frac{1}{3}$, find the speed of each ball after impact in m/s.

PSC122

10. A body of mass 1 kg moving with velocity 1 m/s makes an elastic one dimensional collision with an identical stationary body. They are in contact for a brief period 1 s. Their force of interaction increases from zero to F_0 linearly in 0.5 s and decreases linearly to zero in a further 0.5 s as shown in figure. Find the magnitude of force F_0 in newtons.



PSC123

11. An object A of mass 1 kg is projected vertically upward with a speed of 20 m/s. At the same moment another object B of mass 3 kg, which is initially above the object A, is dropped from a height h = 20 m. The two point like objects collide and stick to each other. Find the kinetic energy of the combined mass just after the collision.

PSC124

12. A particle of mass 2 kg moving with a velocity $5\hat{i}$ m/s collides head–on with another particle of mass 3 kg moving with a velocity $-2\hat{i}$ m/s. After the collision the first particle has speed of 1.6 m/s in negative x direction. Find the :

(a) velocity of the centre of mass after the collision

(b) velocity of the second particle after the collision

(c) coefficient of restitution.

PSC125

BEGINNER'S BOX ANSWER KEY

BEGINNER'S BOX-1

1. 1·1m, 1·3 m
2. $\left(\frac{a}{2},\frac{7}{10}a\right)$
3. $\left(\frac{14}{5}\hat{i}+\frac{19}{5}\hat{j}\right)$ m, (2.8, 3.8)
4. (a) at O; (b) III quadrant; (c) on OY' axis; (d) at O; (e) iv quadrant; (f) at O
5. $x=\frac{-cb^2}{(a^2-b^2)}$

BEGINNER'S BOX-2

1.	(B)	**2.**	$2\hat{k}$ m/s	**3.**	10 m/s	**4.**	$\frac{\sqrt{14}}{6}$ m/s^2		
5.	(B)	**6.**	(A)	**7.**	(D)	**8.**	(A)	**9.**	(B)

BEGINNER'S BOX-3

1.	1.2 kg	**2.**	0	**3.**	$\frac{v}{3}$	**4.**	45 m/s

5. m/s, towards initial direction of velocity of 2 kg mass
6. (i) 5 m/s; (ii) $\frac{5}{4}$m ; (iii) $\frac{50}{9}$m/s
7. 213.33 m, 16 s

8.	$\frac{13}{12}h$	**9.**	2 m/s	**10.**	2N	**11.**	50 J

12. (a) $0.8\hat{i}$ m/s; (b) $2.4\hat{i}$ m/s; (c) $\frac{4}{7}$