01 Modern Physics

Photon Theory of Light

We have learnt that light has wave character as well as particle character. Depending on the situation, one of the two characters dominates. When light is passed through a double slit, it shows interference. This observation can only be understood in terms of wave theory which was discussed in detail in an earlier chapter. There are some phenomena which can only be understood in terms of the particle theory of light. When light of sufficiently low wavelength falls on a metal surface, electrons are ejected. This phenomenon is called the photoelectric effect and can be understood only in terms of the particle nature of light.

The particles of light have several properties in common with the material particles and several other properties which are different from the material particles. The particles of light are called photons. We list some of the important properties of photons.

1. A photon always travels at a speed

   $c = 299,792,458 ms^{-1} \approx 3.0 \times 10^{8} ms^{-1}$ in vacuum.

   This is true for any frame of reference used to observe the photon.
2. The mass of a photon is not defined in the sense of Newtonian mechanics. We shall ignore this concept. We simply state that the rest mass of a photon is zero.
3. Each photon has a definite energy and a definite linear momentum.
4. Let E and p be the energy and linear momentum of a photon of light, and ν and λ be the frequency and wavelength of the same light when it behaves as a wave. Then,

   And $\quad E = h\nu = hc/\lambda$

   $\quad p = h/\lambda = E/c$

   where h is a universal constant known as the Planck constant and has a value $= 4.136 \times 10^{-15}$ eVs.

   (h = 6.6 × 10^{-34} J.S)

   Thus, all photons of light of a particular wavelength λ have the same energy $E = hc/\lambda$ and the same momentum $p = h/\lambda$.
5. A photon may collide with a material particle. The total energy and the total momentum remain conserved in such a collision. The photon may get absorbed and/or a new photon may be emitted. Thus, the number of photons may not be conserved.
6. If the intensity of light of a given wavelength is increased, there is an increase in the number of photons crossing a given area in a given time. The energy of each photon remains the same.

Illustration 1:

Consider a parallel beam of light of wavelength 600 nm and intensity 100 $W\,m^{-2}$.

(a) Find the energy and linear momentum of each photon.

(b) How many photons cross 1 cm^{2} area perpendicular to the beam in one second?

Solution:

(a) The energy of each photon

$$E = hc/\lambda = \frac{\left(4.14\times10^{-15}eVs\right)\times\left(3\times10^{8}ms^{-1}\right)}{600\times10^{-9}m} = 2.07eV$$

The linear momentum is

$$p = \frac{E}{c} = \frac{2.07eV}{3\times10^{8}ms^{-1}} = 0.69\times10^{-8}eVsm^{-1}$$

(b) The energy crossing $1cm^2$ in one second

$$= \left(100Wm^{-2}\right)\times\left(1cm^{2}\right)\times\left(1s\right) = 1.0\times10^{-2}J$$

The number of photons making up this amount of energy is

$$n = \frac{1.0\times10^{-2}J}{2.07eV} = \frac{1.0\times10^{-2}}{2.07\times1.6\times10^{-19}} = 3.0\times10^{16}$$

For a given wavelength λ, the energy of light is an integer times hc/λ. Thus, the energy of light can be varied only in quantum (steps) of $\frac{hc}{\lambda}$. The photon theory is, therefore, also called the quantum theory of light.

Illustration 2:

A small potassium foil is placed (perpendicular to the direction of incidence of light) at a distance r (= 0.5 m) from a point light source whose output power P_0 is 1.0W. Assuming wave nature of light how long would it take for the foil to soak up enough energy (= 1.8 eV) from the beam to eject an electron? Assume that the ejected photoelectron collected its energy from a circular area of the foil whose radius equals the radius of a potassium atom (1.3 × 10^{-10} m).

Solution:

If the source radiates uniformly in all directions, the intensity I of the light at a distance r is given by

$$I = \frac{P_0}{4\pi r^2} = \frac{1.0W}{4\pi(0.5m)^2} = 0.32W/m^2$$

The target area A is $\pi\left(1.3\times10^{-10}m\right)^2$ or $5.3\times10^{-20}m^2$, so that the rate at which energy falls on the target is given by

$$\pi\left(1.3\times10^{-10}m\right)^2 \text{ or } 5.3\times10^{-20}m^2 = 1.7\times10^{-20}J/s$$

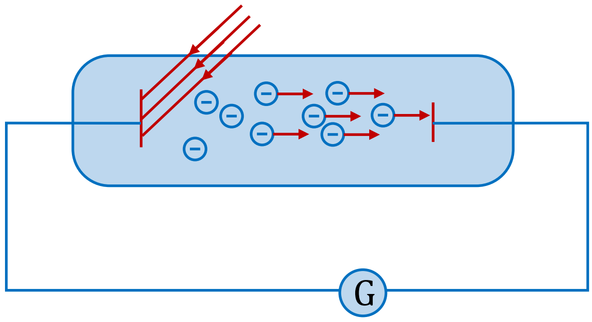
If all this incoming energy is absorbed, the time required to accumulate enough energy for the electron to escape is

$$t = \left(\frac{1.8eV}{1.7\times10^{-20}J/s}\right)\left(\frac{1.6\times10^{-19}J}{1eV}\right) = 17\ s$$

Our selection of a radius for the effective target area was some-what arbitrary, but no matter what reasonable area we choose, we should still calculate a "soak-up time" within the range of easy measurement. However, no time delay has ever been observed under any circumstances, the early experiments setting an upper limit of about 10^{-9} s for such delays.

Photoelectric Effect

When electromagnetic radiations of suitable wavelength are incident on a metallic surface then electrons are emitted, this phenomenon is called photo electric effect.



Photoelectron: The electron emitted in photoelectric effect is called photoelectron.

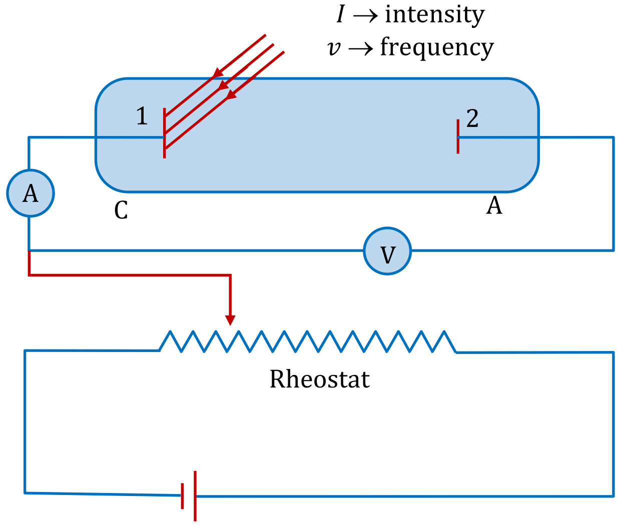
Photoelectric current: If current passes through the circuit in photoelectric effect then the current is called photoelectric current.

Work function: The minimum energy required to make an electron free from the metal is called work function. It is constant for a metal and denoted by ϕ or W. It is the minimum for Cesium. It is relatively less for alkali metals.

Work functions of some photosensitive metals

Metal	Work function (eV)	Metal	Work function (eV)
Cesium	1.9	Calcium	3.2
Potassium	2.2	Copper	4.5
Sodium	2.3	Silver	4.7
Lithium	2.5	Platinum	5.6

To produce photo electric effect only metal and light is necessary but for observing it, the circuit is completed. Figure shows an arrangement used to study the photoelectric effect.



Here the plate (1) is called emitter or cathode and other plate (2) is called collector or anode.

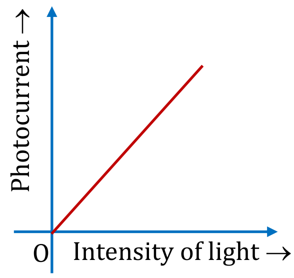
Saturation current: When all the photo electrons emitted by cathode reach the anode then current flowing in the circuit at that instant is known as saturation current, this is the maximum value of photoelectric current.

Stopping potential: Minimum magnitude of negative potential of anode with respect to cathode for which current is zero is called stopping potential. This is also known as cut off voltage. This voltage is independent of intensity.

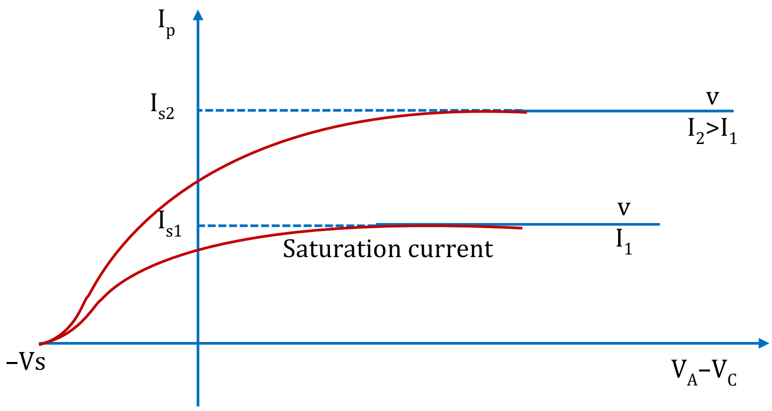
Retarding potential: Negative potential of anode with respect to cathode which is less than stopping potential is called retarding potential.

Observations made by Einstein:

A graph between intensity of light and photoelectric current is found to be a straight line as shown in figure. Photoelectric current is directly proportional to the intensity of incident radiation. In this experiment the frequency and retarding potential are kept constant.



A graph between photoelectric current and potential difference between cathode and anode is found as shown in figure.



In case of saturation current,

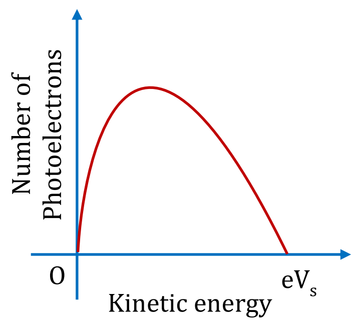
Rate of emission of photoelectrons = rate of flow of photoelectrons,

Here, $v_s \rightarrow$ stopping potential and it is a positive quantity

Electrons emitted from surface of metal have different energies.

Maximum kinetic energy of photoelectron on the cathode = eV_s

$$KE_{\max} = eV_s$$



Whenever photoelectric effect takes place, electrons are ejected out with kinetic energies ranging from

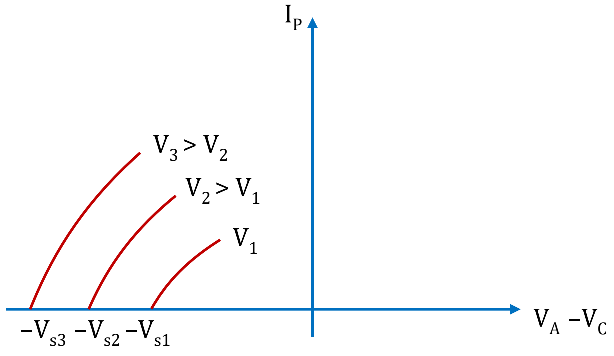
$$0 \text{ to } K.E_{\max} \qquad \text{i.e., } 0 \le KE_e \le eV_s$$

The energy distribution of photoelectron is shown in figure.

If intensity is increased (keeping the frequency constant) then saturation current is increased by same factor by which intensity increases. Stopping potential is same, so maximum value of kinetic energy is not effected.

If light of different frequencies is used then obtained plots are shown in figure.

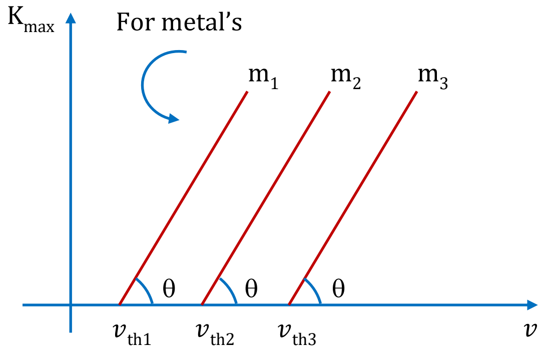
It is clear from graph, as ν increases, stopping potential increases, it means maximum value of kinetic energy increases.



Graphs between maximum kinetic energy of electrons ejected from different metals and frequency of light used are found to be straight lines of same slope as shown in figure below.

Graph between K_{max} and ν.

m_1, m_2, m_3 : Three different metals



It is clear from graph that there is a minimum frequency of electromagnetic radiation which can produce photoelectric effect, which is called **threshold frequency**.

ν_{th} = Threshold frequency

For photoelectric effect

$\nu \geq \nu_{th}$

For no photoelectric effect

$\nu < \nu_{th}$

Minimum frequency for photoelectric effect = ν_{th}

$\nu_{min} = \nu_{th}$

Threshold wavelength (λ_{th}) → The maximum wavelength of radiation which can produce photoelectric effect.

$\lambda \leq \lambda_{th}$ for photo electric effect

Maximum wavelength for photoelectric effect

$\lambda_{max} = \lambda_{th}$

Now writing equation of straight line from graph.

We have $K_{max} = A\nu + B$

When $\nu = \nu_{th}$, $K_{max} = 0$ and $B = -A\nu_{th}$

Hence $[K_{max} = A(\nu - \nu_{th})]$ and $A = \tan\theta = 6.63 \times 10^{-34}\ J\text{-}s$ (from experimental data)

Later on '*A*' was found to be '*h*'.

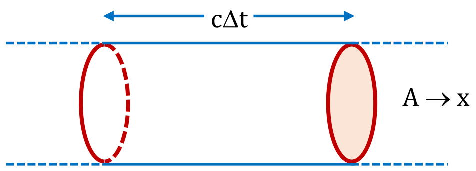
It is also observed that photoelectric effect is an instantaneous process. When light falls on surface electrons start ejecting without taking any time.

Three major features of the photoelectric effect cannot be explained in terms of the classical wave theory of light.

Intensity: The energy crossing per unit area per unit time perpendicular to the direction of propagation is called the intensity of a wave.

Consider a cylindrical volume with area of cross-section A and length $c\,\Delta t$ along the X-axis. The energy contained in this cylinder crosses the area A in time Δt as the wave propagates at speed c. The energy contained.

$$U = u_{av}(c\Delta t)A$$



The intensity is

$$I = \frac{U}{A\Delta t} = u_{av}\,c$$

In the terms of maximum electric field,

$$I = \frac{1}{2}\varepsilon_0\,E_0^2 c$$

If we consider light as a wave then the intensity depends upon electric field.

If we take work function

$$W = I\,.\,A\,.\,t, \qquad \text{then} \qquad t = \frac{W}{IA}$$

So, for photoelectric effect there should be time lag because the metal has work function.

But it is observed that photoelectric effect is an instantaneous process.

Hence, light is not of wave nature.

The Intensity Problem : Wave theory requires that the oscillating electric field vector $\boldsymbol{E}$ of the light wave increases in amplitude as the intensity of the light beam is increased. Since the force applied to the electron is $e\boldsymbol{E}$, this suggests that the kinetic energy of the photoelectrons should also increase as the light beam is made more intense. However, observation shows that maximum kinetic energy is independent of the light intensity.

The Frequency Problem : According to the wave theory, the photoelectric effect should occur for any frequency of the light, provided only that the light is intense enough to supply the energy needed to eject the photoelectrons. However, observations shows that there exists for each surface a characteristic cut off frequency ν_{th}, for frequencies less than ν_{th}, the photoelectric effect does not occur, no matter how intense is light beam.

The Time Delay Problem : If the energy acquired by a photoelectron is absorbed directly from the wave incident on the metal plate, the "effective target area" for an electron in the metal is limited and probably not much more than that of a circle of diameter roughly equal to that of an atom. In the classical theory, the light energy is uniformly distributed over the wave front. Thus, if the light is feeble enough, there should be a measurable time lag, between the impinging of the light on the surface and the ejection of the photoelectron. During this interval the electron should be absorbing energy from the beam until it had accumulated enough to escape. However, no detectable time lag has ever been measured.

Now, quantum theory solves these problems in providing the correct interpretation of the photoelectric effect.

Planck's Quantum Theory

The light energy from any source is always an integral multiple of a smaller energy value called quantum of light. Hence energy, $Q = NE$,

where $E = h\nu$ and N (number of photons) = 1, 2, 3, ...

Here energy is quantized. $h\nu$ is the quantum of energy, it is a packet of energy called as **photon**.

$$E = h\nu = \frac{hc}{\lambda} \quad \text{and} \quad hc = 12400\ eV\,Å$$

Einstein's Photon Theory

In 1905 Einstein made a remarkable assumption about the nature of light that, under some circumstances, it behaves as if its energy is concentrated into localized bundles, later called photons. The energy E of a single photon is given by

$$E = h\nu,$$

If we apply Einstein's photon concept to the photoelectric effect, we can write

$$h\nu = W + K_{max} \text{ (energy conservation)}$$

Equation says that a single photon carries an energy $h\nu$ into the surface where it is absorbed by a single electron. Part of this energy W (called the work function of the emitting surface) is used in causing the electron to escape from the metal surface. The excess energy $(h\nu - W)$ becomes the electron's kinetic energy; if the electron does not lose energy by internal collisions as it escapes from the metal, it will still have this much kinetic energy after it emerges. Thus, K_{max} represents the maximum kinetic energy that the photoelectron can have outside the surface. There is complete agreement of the photon theory with experiment.

Now, $IA = Nh\nu$

$\Rightarrow$ $N = \dfrac{IA}{h\nu}$ = number of photons incident per unit time on an area 'A' when light of intensity 'I' is incident normally.

If we double the light intensity, we double the number of photons and thus double the photoelectric current; we do not change the energy of the individual photons or the nature of the individual photoelectric processes.

The second objection (the frequency problem) is met if K_{max} equals zero, we have

$$h\nu_{th} = W,$$

Which asserts that the photon has just enough energy to eject the photoelectrons and none extra to appear as kinetic energy. If ν is reduced below ν_{th}, $h\nu$ will be smaller than W and the individual photons, no matter how many of them there are (that is, no matter how intense the illumination), will not have enough energy to eject photoelectrons.

The third objection (the time delay problem) follows from the photon theory because the required energy is supplied in a concentrated bundle. It is not spread uniformly over the beam cross section as in the wave theory.

Hence Einstein's equation for photoelectric effect is given by

$$h\nu = h\nu_{th} + K_{max} \ ; \quad K_{max} = \frac{hc}{\lambda} - \frac{hc}{\lambda_{th}}$$

Illustration 3:

In an experiment on photo electric emission, following observations were made;

(i) Wavelength of the incident light = 1.98×10^{-7} m; (ii) Stopping potential = 2.5 *volt*.

Find (a) Kinetic energy of photoelectrons with maximum speed, (b) Work function and (c) Threshold frequency.

Solution:

(a) Since $V_s = 2.5\ V$, $K_{max} = eV_s$ so, $K_{max} = 2.5\ eV$

(b) Energy of incident photon

$$E = \frac{12400}{1980} eV = 6.26\ eV$$

$$W = E - K_{max} = 3.76\ eV$$

(c) $h\nu_{th} = W = 3.76 \times 1.6 \times 10^{-19}\ J$

$$\therefore \quad \nu_{th} = \frac{3.76 \times 1.6 \times 10^{-19}}{6.6 \times 10^{-34}} \approx 9.1 \times 10^{14}\ Hz$$

Illustration 4:

A beam of light consists of four wavelength 4000 Å, 4800 Å, 6000 Å and 7000 Å, each of intensity 1.5×10^{-3} Wm^{-2}. The beam falls normally on an area 10^{-4} m^2 of a clean metallic surface of work function 1.9 eV. Assuming no loss of light energy (i.e. each capable photon emits one electron) calculate the number of photoelectrons liberated per second.

Solution:

$$E_1 = \frac{12400}{4000} = 3.1\ eV, \quad E_2 = \frac{12400}{4800} = 2.58\ eV, \quad E_3 = \frac{12400}{6000} = 2.06\ eV \quad \text{and} \quad E_4 = \frac{12400}{7000} = 1.77\ eV$$

Therefore, light of wavelengths 4000 Å, 4800 Å and 6000 Å can only emit photoelectrons.

$\therefore$ Number of photoelectrons emitted per second = Number of photons incident per second)

$$= \frac{I_1 A_1}{E_1} + \frac{I_2 A_2}{E_2} + \frac{I_3 A_3}{E_3} = IA\left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{1}{E_3}\right)$$

$$= \frac{(1.5 \times 10^{-3})(10^{-4})}{1.6 \times 10^{-19}}\left(\frac{1}{3.1} + \frac{1}{2.58} + \frac{1}{2.06}\right) = 1.12 \times 10^{12}$$

Illustration 5:

A metallic surface is irradiated with monochromatic light of variable wavelength. Above a wavelength of 5000 Å, no photoelectrons are emitted from the surface. With an unknown wavelength, stopping potential is 3V. Find the unknown wavelength.

Solution:

Using equation of photoelectric effect

$$K_{max} = E - W \ (K_{max} = eV_s)$$

$$\therefore \quad 3\ eV = \frac{12400}{\lambda} - \frac{12400}{5000} = -2.48\ eV \quad \text{or} \quad \lambda = 2262\ \text{Å}$$

Illustration 6:

Illuminating the surface of a certain metal alternately with light of wavelengths λ_1 = 0.35 μm and λ_2 = 0.54 μm, it was found that the corresponding maximum velocities of photo electrons have a ratio η = 2. Find the work function of that metal.

Solution:

Using equation for two wavelengths

$$\frac{1}{2}mv_1^2 = \frac{hc}{\lambda_1} - W \quad \text{...(i)}$$

$$\frac{1}{2}mv_2^2 = \frac{hc}{\lambda_2} - W \quad \text{...(ii)}$$

Dividing equation (i) with equation (ii), with $v_1 = 2v_2$, we have

$$4 = \frac{\frac{hc}{\lambda_1} - W}{\frac{hc}{\lambda_2} - W}$$

$$3W = 4\left(\frac{hc}{\lambda_2}\right) - \left(\frac{hc}{\lambda_1}\right) = \frac{4 \times 12400}{5400} - \frac{12400}{3500} = 5.64\ eV$$

$$W = \frac{5.64}{3}\ eV = 1.88\ eV$$

Illustration 7:

A photocell is operating in saturation mode with a photocurrent 4.8 μA when a monochromatic radiation of wavelength 3000 $\mathring{A}$ and power 1 mW is incident. When another monochromatic radiation of wavelength 1650 Å and power 5 mW is incident, it is observed that maximum velocity of photoelectron increases to two times. Assuming efficiency of photoelectron generation per incident to be same for both the cases, calculate,

(a) threshold wavelength for the cell

(b) efficiency of photoelectron generation.
[(Number of photoelectrons emitted per incident photon) × 100]

(c) saturation current in second case

Solution:

(a) $K_1 = \frac{12400}{3000} - W = 4.13 - W$ ...(i)

$K_2 = \frac{12400}{1650} - W = 7.51 - W$ ...(ii)

Since $v_2 = 2v_1$

So, $K_2 = 4K_1$ ...(iii)

Solving above equations, we get

$W = 3\ eV$

$\therefore$ Threshold wavelength $\lambda_0 = \frac{12400}{3} \approx 4133\ \mathring{A}$

(b) Energy of a photon in first case = $\frac{12400}{3000}$ = 4.13 eV

or $E_1 = 6.6 \times 10^{-19}\,J$

Rate of incident photons (number of photons per second)

$$\frac{P_1}{E_1} = \frac{10^{-3}}{6.6\times10^{-19}} = 1.5 \times 10^{15} \text{ per second}$$

Number of electrons ejected = $\frac{4.8\times10^{-6}}{1.6\times10^{-19}}$ per second = 3.0×10^{13} per second

∴ Efficiency of photoelectron generation

$$(\eta) = \frac{3.0\times10^{13}}{1.5\times10^{15}} \times 100 = 2\%$$

(c) Energy of photon in second case

$$E_2 = \frac{12400}{1650} = 7.51\, eV = 12 \times 10^{-19}\, J$$

Therefore, number of photons incident per second

$$n_2 = \frac{P_2}{E_2} = \frac{5.0\times10^{-3}}{12\times10^{-19}} = 4.17 \times 10^{15} \text{ per second}$$

Number of electrons emitted per second = $\frac{2}{100} \times 4.7 \times 10^{15}$

$= 9.4 \times 10^{13}$ per second

∴ Saturation current in second case $i = (9.4 \times 10^{13})\,(1.6 \times 10^{-19})\, amp = 15\,\mu A$

Illustration 8:

Light described at a place by the equation $E = (100\ V/m)\ [\sin(5 \times 10^{15}\ s^{-1})\ t + \sin(8 \times 10^{15}\ s^{-1})t]$ falls on a metal surface having work function 2.0 eV. Calculate the maximum kinetic energy of the photoelectrons.

Solution:

The light contains two different frequencies. The one with larger frequency will cause photoelectrons with largest kinetic energy. This larger frequency is

$$\nu = \frac{\omega}{2\pi} = \frac{8\times10^{15}\,s^{-1}}{2\pi}$$

The maximum kinetic energy of the photoelectrons is

$$K_{max} = h\nu - W$$

$$= (4.14 \times 10^{-15}\ eV\text{-}s) \times \left(\frac{8\times10^{15}}{2\pi}s^{-1}\right) - 2.0\ eV$$

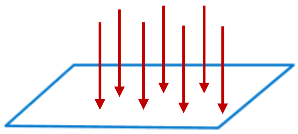
$$= 5.27\ eV - 2.0\ eV = 3.27\ eV$$

Force due to radiation (Photon)

Each photon has a definite energy and a definite linear momentum. All photons of light of a particular wavelength λ have the same energy $E = hc/\lambda$ and the same magnitude of momentum $p = h/\lambda$.

When light of intensity I falls on a surface, it exerts force on that surface. Assume absorption and reflection coefficient of surface be 'a' and 'r' and assuming no transmission.

Assume light beam falls on surface of surface area 'A' perpendicularly as shown in figure.



For calculating the force exerted by beam on surface, we consider following cases.

Case : (I)

$a = 1, \quad r = 0$

Initial momentum of the photon = $\dfrac{h}{\lambda}$

Final momentum of photon = 0

Change in momentum of photon = $\dfrac{h}{\lambda}$ (upward)

$$\Delta P = \frac{h}{\lambda}$$

Energy incident per unit time = IA

Number of photons incident per unit time = $\dfrac{IA}{h\nu} = \dfrac{IA\lambda}{hc}$

$\therefore$ Total change in momentum per unit time = $n\,\Delta P$

$$= \frac{IA\lambda}{hc} \times \frac{h}{\lambda} = \frac{IA}{c} \text{ (upward)}$$

Force on photons = total change in momentum per unit time = $\dfrac{IA}{c}$ (upward)

$\therefore$ Force on plate due to photons(F) = $\dfrac{IA}{c}$ (downward)

$$\text{Pressure} = \frac{F}{A} = \frac{IA}{cA} = \frac{I}{c}$$

Case : (II)

When $r = 1, a = 0$

Initial momentum of the photon = $\dfrac{h}{\lambda}$ (downward)

Final momentum of photon = $\dfrac{h}{\lambda}$ (upward)

Change in momentum = $\dfrac{h}{\lambda} + \dfrac{h}{\lambda} = \dfrac{2h}{\lambda}$

$\therefore$ Energy incident per unit time = IA

Number of photons incident per unit time = $\dfrac{IA\lambda}{hc}$

$\therefore$ Total change in momentum per unit time = $n\,.\,\Delta P = \dfrac{IA\lambda}{hc}\,.\,\dfrac{2h}{\lambda} = \dfrac{2IA}{C}$

Force = total change in momentum per unit time

$F = \dfrac{2IA}{c}$ (upward on photons and downward on the plate)

$$\text{Pressure, } P = \frac{F}{A} = \frac{2IA}{cA} = \frac{2I}{c}$$

Case : (III)

When $o < r < 1$

$a + r = 1$

Change in momentum of photon when it is reflected = $\frac{2h}{\lambda}$ (upward)

Change in momentum of photon when it is absorbed = $\frac{h}{\lambda}$ (upward)

Number of photons incident per unit time = $\frac{IA\lambda}{hc}$

Number of photons reflected per unit time = $\frac{IA\lambda}{hc}\, r$

Number of photon absorbed per unit time = $\frac{IA\lambda}{hc}\,(1 - r)$

Force due to absorbed photon $(F_a) = \frac{IA\lambda}{hc}(1 - r) \cdot \frac{h}{\lambda} = \frac{IA}{c}(1 - r)$ (downward)

Force due to reflected photon $(F_r) = \frac{IA\lambda}{hc} . r\, \frac{2h}{\lambda} = \frac{2IAr}{c}$ (downward)

Total force = $F_a + F_r$ (downward)

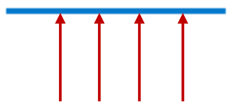
$$= \frac{IA}{c}(1 - r) + \frac{2IAr}{c} = \frac{IA}{c}(1 + r)$$

Now pressure,

$$P = \frac{IA}{c}(1 + r) \times \frac{1}{A} = \frac{I}{c}(1 + r)$$

Illustration 9:

A plate of mass 10 gm is in equilibrium in air due to the force exerted by light beam on plate. Calculate power of beam. Assume plate is perfectly absorbing.



Solution:

Since plate is in air, so gravitational force will act on this

$F_{\text{gravitational}} = mg$ (downward)

$= 10 \times 10^{-3} \times 10$

$= 10^{-1}\ N$

For equilibrium force exerted by light beam should be equal to $F_{\text{gravitational}}$

$F_{\text{photon}} = F_{\text{gravitational}}$

Let power of light beam be P

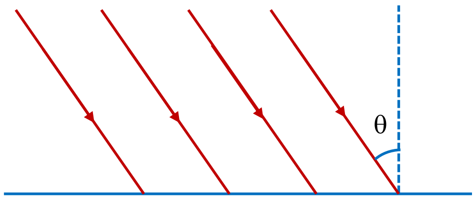
$\therefore \quad F_{\text{photon}} = \frac{P}{c}$

$\therefore \quad \frac{P}{c} = 10^{-1} \quad ; \quad P = 3.0 \times 10^{8} \times 10^{-1}$

$P = 3 \times 10^{7}\ W$

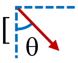
Illustration 10:

Calculate force exerted by light beam if light is incident on surface at an angle θ as shown in figure. Consider all cases.

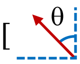


Solution:

Case I: $a = 1, \quad r = 0$

Initial momentum of photon (in downward direction at an angle θ with vertical) = $\frac{h}{\lambda}$ []

Final momentum of photon = 0

Change in momentum (in upward direction at an angle θ with vertical) = $\frac{h}{\lambda}$ []

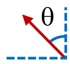
Energy incident per unit time = $IA\cos\theta$

Intensity = power per unit normal area

$I = \frac{P}{A\cos\theta}$; $P = IA\cos\theta$

Number of photons incident per unit time = $\frac{IA\cos\theta}{hc}.\ \lambda$

Total change in momentum per unit time (in upward direction at an angle θ with vertical)

$= \frac{IA\cos\theta.\lambda}{hc} \cdot \frac{h}{\lambda} = \frac{IA\cos\theta}{c}$ []

Force (F) = total change in momentum per unit time

$F = \frac{IA\cos\theta}{c}$

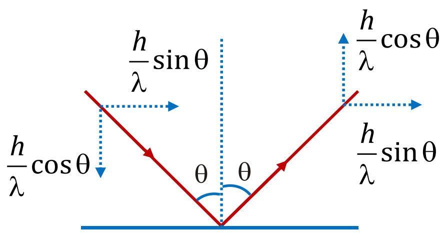
Pressure = normal force per unit Area

Pressure = $\frac{F\cos\theta}{A}$; $P = \frac{IA\cos^2\theta}{cA} = \frac{I}{c}\cos^2\theta$

Case II: When $r = 1$, $a = 0$

∴ Change in momentum of one photon

$= \frac{2h}{\lambda}\cos\theta$ (upward)



Number of photons incident per unit time

$= \frac{\text{Energy incident per unit time}}{h\nu}$

$= \frac{IA\cos\theta\cdot\lambda}{hc}$

$\therefore$ Total change in momentum per unit time

$$= \frac{IA\cos\theta\cdot\lambda}{hc} \times \frac{2h}{\lambda}\cos\theta = \frac{2IA\cos^2\theta}{c} \text{ (upward)}$$

$\therefore$ Force on the plate $= \dfrac{2IA\cos^2\theta}{c}$ (downward)

Pressure $= \dfrac{2IA\cos^2\theta}{cA}$; $P = \dfrac{2I\cos^2\theta}{c}$

Case III: $0 < r < 1$, $a + r = 1$

Change in momentum of photon when it is reflected $= \dfrac{2h}{\lambda}\cos\theta$ (downward)

Change in momentum of photon when it is absorbed $= \dfrac{h}{\lambda}$ (in the opposite direction of incident beam)

Energy incident per unit time $= IA\cos\theta$

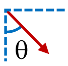
Number of photons incident per unit time $= \dfrac{IA\cos\theta\cdot\lambda}{hc}$

Number of reflected photon (n_r) $= \dfrac{IA\cos\theta.\lambda r}{hc}$

Number of absorbed photon (n_a) $= \dfrac{IA\cos\theta.\lambda}{hc}(1 - r)$

Force on plate due to absorbed photons $F_a = n_a \,.\, \Delta P_a$

$$= \frac{IA\cos\theta.\lambda}{hc}(1 - r)\frac{h}{\lambda}$$

$= \dfrac{IA\cos\theta}{c}(1 - r)$ (At an angle θ with vertical )

Force on plate due to reflected photons

$F_r = n_r\,\Delta P_r$

$$= \frac{IA\cos\theta\cdot\lambda}{hc} \times \frac{2h}{\lambda}\cos\theta \text{ (vertically downward)} = \frac{IA\cos^2\theta}{c}\cdot 2r$$

Now resultant force is given by $F_R = \sqrt{F_r^2 + F_a^2 + 2F_aF_r\cos\theta}$

$$= \frac{IA\cos\theta}{c}\sqrt{(1-r)^2 + (2r)^2\cos^2\theta + 4r(r-1)\cos^2\theta}$$

And, pressure

$$P = \frac{F_a\cos\theta + F_r}{A} = \frac{IA\cos\theta(1-r)\cos\theta}{cA} + \frac{IA\cos^2\theta\cdot 2r}{cA}$$

$$= \frac{I\cos^2\theta}{c}(1 - r) + \frac{I\cos^2\theta}{c}2r = \frac{I\cos^2\theta}{c}(1 + r)$$

De-Broglie Wavelength of Matter Wave

A photon of frequency ν and wavelength λ has energy.

$$E = h\nu = \frac{hc}{\lambda}$$

By Einstein's energy mass relation, $E = mc^2$ the equivalent mass m of the photon is given by,

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2} = \frac{h}{\lambda c} \quad \text{...(i)}$$

or $$\lambda = \frac{h}{mc} \text{ or } \lambda = \frac{h}{p} \quad \text{...(ii)}$$

Here p is the momentum of photon. By analogy de-Broglie suggested that a particle of mass m moving with speed v behaves in some ways like waves of wavelength λ (called de-Broglie wavelength and the wave is called matter wave) given by,

$$\lambda = \frac{h}{mv} = \frac{h}{p} \quad \text{...(iii)}$$

Where p is the momentum of the particle. Momentum is related to the kinetic energy by the equation,

$$p = \sqrt{2Km}$$

and a charge q when accelerated by a potential difference V gains a kinetic energy $K = qV$. Combining all these relations equation (iii), can be written as,

$$\lambda = \frac{h}{mv} = \frac{h}{p} = \frac{h}{\sqrt{2Km}} = \frac{h}{\sqrt{2qVm}} \text{ (de-Broglie wavelength)} \quad \text{...(iv)}$$

De-Broglie Wavelength for an Electron:

If an electron (charge = e) is accelerated by a potential of V volts, it acquires a kinetic energy,

$$K = eV$$

Substituting the values of h, m and q in equation (iv), we get a simple formula for calculating de-Broglie wavelength of an electron.

$$\lambda(in \quad \mathring{A}) = \sqrt{\frac{150}{V(in \quad volts)}}$$

De-Broglie Wavelength of a Gas Molecule

Let us consider a gas molecule at absolute temperature T. Kinetic energy of gas molecule is given by

$$K.E. = \frac{3}{2}kT \quad ; \quad k = \text{Boltzmann constant}$$

$$\therefore \quad \lambda_{\text{gas molecule}} = \frac{h}{\sqrt{3mkT}}$$

Illustration 11:

An electron is accelerated by a potential difference of 50 volt. Find the de-Broglie wavelength associated with it.

Solution:

For an electron, De-Broglie wavelength is given by,

$$\lambda = \sqrt{\frac{150}{V}} = \sqrt{\frac{150}{50}} = \sqrt{3} = 1.73\ \mathring{A}$$

Illustration 12:

Find the ratio of De-Broglie wavelength of molecules of hydrogen and helium which are at temperatures $27°C$ and $127°C$ respectively.

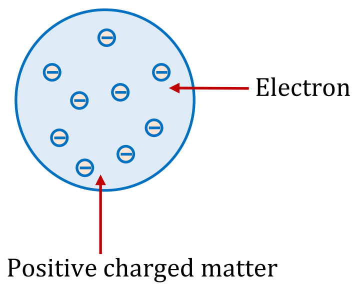
Solution:

De-Broglie wavelength is given by

$$\therefore \quad \frac{\lambda_{H_2}}{\lambda_{He}} = \sqrt{\frac{m_{He}T_{He}}{m_{H_2}T_{H_2}}} = \sqrt{\frac{4}{2}\cdot\frac{(127+273)}{(27+273)}} = \sqrt{\frac{8}{3}}$$

Thomson's Atomic Model:

J.J. Thomson suggested that atoms are just positively charge lumps of matter with electrons embedded in them like raisins in a fruit cake. Thomson's model called the 'plum pudding' model is illustrated in figure.



Thomson played an important role in discovering the electron, through gas discharge tube by discovering cathode rays. His idea was taken seriously.

But the real atom turned out to be quite different.

Rutherford's Nuclear Atom:

Rutherford suggested that; " all the positive charge and nearly all the mass were concentrated in a very small volume of nucleus at the centre of the atom. The electrons were supposed to move in circular orbits round the nucleus (like planets round the sun). The electrostatic attraction between the two opposite charges being the required centripetal force for such motion.

Hence, $\frac{mv^2}{r} = \frac{kZe^2}{r^2}$

and total energy = potential energy + kinetic energy = $\frac{-kZe^2}{2r}$

Rutherford's model of the atom, although strongly supported by evidence for the nucleus, is inconsistent with classical physics. This model suffers from two defects.

Regarding Stability of Atom : An electron moving in a circular orbit round a nucleus is accelerating and according to electromagnetic theory it should therefore, emit radiation continuously and thereby lose energy. If total energy decreases then radius increases as given by above formula. If this happened the radius of the orbit would decrease and the electron would spiral into the nucleus in a fraction of second. But atoms do not collapse. In 1913 an effort was made by Neil Bohr to overcome this paradox.

Regarding Explanation of Line Spectrum : In Rutherford's model, due to continuously changing radii of the circular orbits of electrons, the frequency of revolution of the electrons must be changing. As a result, electrons will radiate electromagnetic waves of all frequencies, i.e., the spectrum of these waves will be 'continuous' in nature. But experimentally the atomic spectra are not continuous. Instead they are line spectra.

The Bohr's Atomic Model:

In 1913, Prof. Neil Bohr removed the difficulties of Rutherford's atomic model by the application of Planck's quantum theory. For this he proposed the following postulates

(1) An electron moves only in certain circular orbits, called stationary orbits. In stationary orbits electron does not emit radiation, contrary to the predictions of classical electromagnetic theory.

(2) According to Bohr, there is a definite energy associated with each stable orbit and an atom radiates energy only when it makes a transition from one of these orbits to another. If the energy of electron in the higher orbit be E_2 and that in the lower orbit be E_1, then the frequency ν of the radiated waves is given by

$$h\nu = E_2 - E_1 \qquad \text{or} \qquad \nu = \frac{E_2 - E_1}{h} \qquad \text{...(i)}$$

(3) Bohr found that the magnitude of the electron's angular momentum is quantized, and this magnitude for the electron must be integral multiple of $\frac{h}{2\pi}$. The magnitude of the angular momentum is $L = mvr$ for a particle with mass m moving with speed v in a circle of radius r. So, according to Bohr's postulate,

$(n = 1, 2, 3....)$

Each value of n corresponds to a permitted value of the orbit radius, which we will denote by r_n. The value of n for each orbit is called **principal quantum number** for the orbit. Thus,

$$mv_nr_n = mvr = \frac{nh}{2\pi} \qquad \text{...(ii)}$$

According to Newton's second law a radially inward centripetal force of magnitude $F = \frac{mv^2}{r_n}$ is needed by the electron which is being provided by the electrical attraction between the positive proton and the negative electron.

Thus, $$\frac{mv_n^2}{r_n} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n^2} \qquad \text{...(iii)}$$

Solving equations (ii) and (iii), we get

$$r_n = \frac{\varepsilon_0 n^2 h^2}{\pi m e^2} \qquad \text{...(iv)}$$

and $$v_n = \frac{e^2}{2\varepsilon_0 nh} \qquad \text{...(v)}$$

The smallest orbit radius corresponds to $n = 1$. We'll denote this minimum radius, called the **Bohr radius** as a_0. Thus,

$$a_0 = \frac{\varepsilon_0 h^2}{\pi m e^2}$$

Substituting values of ε_0, h, p, m and e, we get

$$a_0 = 0.529 \times 10^{-10}\ m = 0.529\ \mathring{A} \qquad \text{...(vi)}$$

Eq. (iv), in terms of a_0 can be written as,

$$r_n = n^2 a_0 \quad \text{or} \quad r_n \propto n^2 \qquad \text{...(vii)}$$

Similarly, substituting values of e, ε_0 and h with $n = 1$ in equation (v), we get

$$v_1 = 2.19 \times 10^6\ m/s \qquad \text{...(viii)}$$

This is the greatest possible speed of the electron in the hydrogen atom. Which is approximately equal to $c/137$ where c is the speed of light in vacuum.

Equation (v), in terms of v_1 can be written as,

$$v_n = \frac{v_1}{n} \quad \text{or} \quad v_n \propto \frac{1}{n}$$

Energy Levels:

Kinetic and potential energies K_n and U_n in n^th orbit are given by,

$$K_n = s\, m v_n^2 = \frac{me^4}{8\varepsilon_0^2 n^2 h^2}$$

and $$U_n = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n} = -\frac{me^4}{4\varepsilon_0^2 n^2 h^2}$$ (assuming infinity as a zero potential energy level)

The total energy E_n is the sum of the kinetic and potential energies.

So, $$E_n = K_n + U_n = -\frac{me^4}{8\varepsilon_0^2 n^2 h^2}$$

Substituting values of m, e, ε_0 and h with $n = 1$, we get the least energy of the atom in first orbit, which is $-13.6\ eV$. Hence,

$$E_1 = -13.6\ eV \qquad \text{...(x)}$$

and $$E_n = \frac{E_1}{n^2} = -\frac{13.6}{n^2}\ eV \qquad \text{...(xi)}$$

Substituting $n = 2, 3, 4, \ldots$, etc., we get energies of atom in different orbits.

$$E_2 = -3.40\ eV,\ E_3 = -1.51\ eV,\ \ldots\ E_\infty = 0$$

Hydrogen Like Atoms

The Bohr model of hydrogen can be extended to hydrogen like atoms, i.e., one electron atoms, the nuclear charge is $+ze$, where z is the atomic number, equal to the number of protons in the nucleus.

The effect in the previous analysis is to replace e^2 everywhere by ze^2. Thus, the equations for, r_n, v_n and E_n are altered as under:

$$r_n = \frac{\varepsilon_0 n^2 h^2}{nmze^2} = a_0 \frac{n^2}{z} \quad \text{or} \quad r_n \propto \frac{n^2}{z} \qquad \text{...(i)}$$

where $a_0 = 0.529\ \mathring{A}$ (radius of first orbit of H)

$$v_n = \frac{ze^2}{2\varepsilon_0 nh} = \frac{z}{n} v_1 \quad \text{or} \quad v_n \propto \frac{z}{n} \qquad \text{...(ii)}$$

where $v_1 = 2.19 \times 10^6\ m/s$ (speed of electron in first orbit of H)

$$E_n = -\frac{mz^2 e^4}{8\varepsilon_0^2 n^2 h^2} = \frac{z^2}{n^2} E_1 \quad \text{or} \quad E_n \propto \frac{z^2}{n^2} \qquad \text{...(iii)}$$

where $E_1 = -13.60\ eV$ (energy of atom in first orbit of H)

Definitions Valid for Single Electron System

(1) **Ground state:** Lowest energy state of any atom or ion is called ground state of the atom.

Ground state energy of H atom = –13.6 eV
Ground state energy of He^+ Ion = –54.4 eV
Ground state energy of Li^{++} Ion = –122.4 eV

(2) **Excited State:** State of atom other than the ground state are called its excited states.

$n = 2$	first excited state
$n = 3$	second excited state
$n = 4$	third excited state
$n = n_0 + 1$	n_0^{th} excited state

(3) **Ionisation energy (I.E.):** Minimum energy required to move an electron from ground state to $n = \infty$ is called ionisation energy of the atom or ion.

Ionisation energy of H atom = 13.6 eV
Ionisation energy of He^+ Ion = 54.4 eV
Ionisation energy of Li^{++} Ion = 122.4 eV

(4) **Ionisation potential (I.P.):** Potential difference through which a free electron must be accelerated from rest such that its kinetic energy becomes equal to ionisation energy of the atom is called ionisation potential of the atom.

I.P. of H atom = 13.6 V
I.P. of He^+ Ion = 54.4 V

(5) **Excitation energy:** Energy required to move an electron from ground state of the atom to any other excited state of the atom is called excitation energy of that state.

Energy in ground state of H atom = –13.6 eV
Energy in first excited state of H-atom = –3.4 eV
I^{st} excitation energy = 10.2 eV.

(6) **Excitation Potential:** Potential difference through which an electron must be accelerated from rest so that its kinetic energy becomes equal to excitation energy of any state is called excitation potential of that state.

I^{st} excitation energy = 10.2 eV.
I^{st} excitation potential = 10.2 V.

(7) **Binding energy or Separation energy:** Energy required to move an electron from any state to $n = \infty$ is called binding energy of that state. or energy released during formation of an H-like atom/ion from $n = \infty$ to some particular n is called binding energy of that state.

Binding energy of ground state of H-atom = 13.6 eV

Illustration 13:

First excitation potential of a hypothetical hydrogen like atom is 15 volt. Find third excitation potential of the atom.

Solution:

Let energy of ground state = E_0

$E_0 = -13.6\,Z^2\,eV$ and $E_n = \dfrac{E_0}{n^2}$

$n = 2,\ E_2 = \dfrac{E_0}{4}$

Given, $\dfrac{E_0}{4} - E_0 = 15$; $-\dfrac{3E_0}{4} = 15$

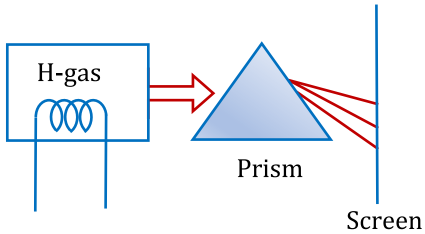
For, $n = 4,\quad E_4 = \dfrac{E_0}{16}$

Third excitation energy = $\frac{E_0}{16} - E_0 = -\frac{15}{16}E_0 = -\frac{15}{16}\cdot\left(\frac{-4\times 15}{3}\right) = \frac{75}{4}\ eV$

$\therefore$ Third excitation potential is $\frac{75}{4}$ V.

Emission Spectrum of Hydrogen Atom

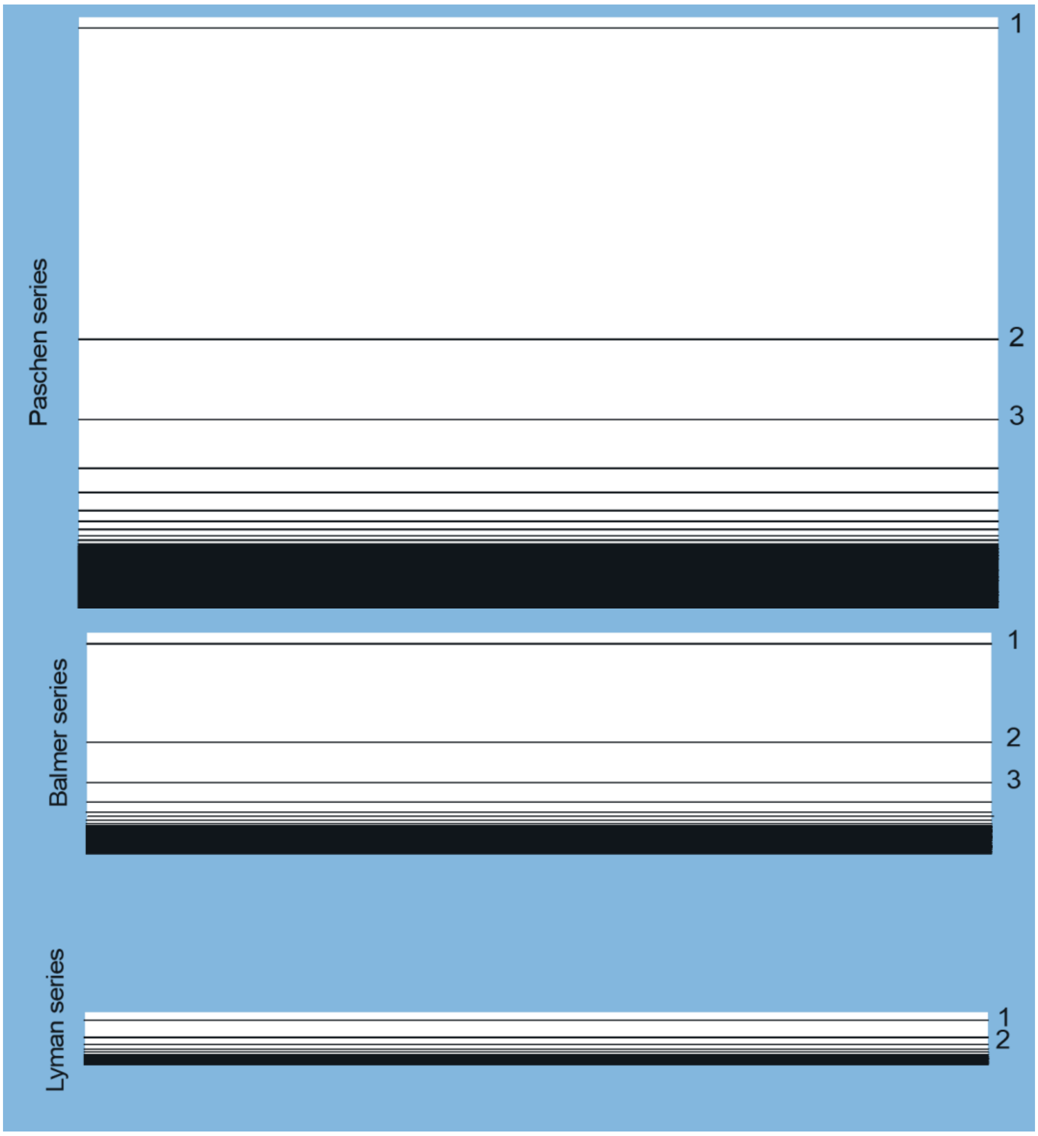
Under normal conditions the single electron in hydrogen atom stays in ground state (n = 1). It is excited to some higher energy state when it acquires some energy from external source. But it hardly stays there for more than 10^{-8} second.



A photon corresponding to a particular spectrum line is emitted when an atom makes a transition from a state in an excited level to a state in a lower excited level or the ground level.

Let n_i be the initial and n_f the final energy state, then depending on the final energy state following series are observed in the emission spectrum of hydrogen atom.

On Screen:

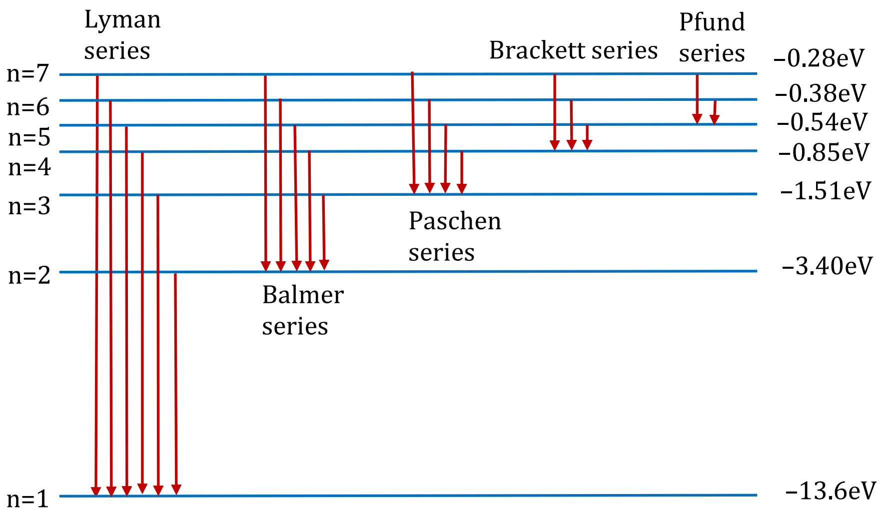


A photograph of spectral lines of the Lyman, Balmer, Paschen series of atomic hydrogen.
1, 2, 3..... represents the I, II and III line of Lyman, Balmer, Paschen series.
The hydrogen spectrum (some selected lines):

Name of series	Number of Line	Quantum Number			
		n_i (Lower State)	n_f (Upper State)	Wavelength (nm)	Energy
Lyman	I	1	2	121.6	10.2 eV
	II	1	3	102.6	12.09 eV
	III	1	4	97	12.78 eV
	series limit	1	∞ (series limit)	91.2	13.6 eV
Balmer	I	2	3	656.3	1.89 eV
	II	2	4	486.1	2.55 eV
	III	2	5	434.1	2.86 eV
	series limit	2	∞ (series limit)	364.6	3.41 eV
Paschen	I	3	4	1875.1	0.66 eV
	II	3	5	1281.8	0.97 eV
	III	3	6	1093.8	1.13 eV
	series limit	3	∞ (series limit)	822	1.51 eV

Series Limit:

Line of any group having maximum energy of photon and minimum wavelength of that group is called series limit.



For the Lyman series $n_f = 1$, for Balmer series $n_f = 2$ and so on.

Wavelength of Photon Emitted in De-Excitation:

According to Bohr when an atom makes a transition from higher energy level to a lower level it emits a photon with energy equal to the energy difference between the initial and final levels. If E_i is the initial energy of the atom before such a transition, E_f is its final energy after the transition, and the photon's energy is $h\nu = \frac{hc}{\lambda}$, then conservation of energy gives,

$$h\nu = \frac{hc}{\lambda} = E_i - E_f \text{ (energy of emitted photon)} \qquad ...(i)$$

By 1913, the spectrum of hydrogen had been studied intensively. The visible line with longest wavelength, or lowest frequency is called H_α, the next line is called H_β and so on.

In 1885, Johann Balmer, a Swiss teacher found a formula that gives the wave lengths of these lines. This is now called the Balmer series. The Balmer's formula is,

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \quad ...(ii)$$

Here, $n = 3, 4, 5$ ...., etc.

R = Rydberg constant = $1.097 \times 10^7\ m^{-1}$

and λ is the wavelength of light/photon emitted during transition,

For $n = 3$, we obtain the wavelength of H_α line.

Similarly, for $n = 4$, we obtain the wavelength of H_β line. For $n = \infty$, the smallest wavelength (= 3646 $\mathring{A}$) of this series is obtained. Using the relation, $E = \frac{hc}{\lambda}$ we can find the photon energies corresponding to the wavelength of the Balmer series.

$$E = \frac{hc}{\lambda} = hcR\left(\frac{1}{2^2} - \frac{1}{n^2}\right) = \frac{Rhc}{2^2} - \frac{Rhc}{n^2}$$

This formula suggests that,

$$E_n = -\frac{Rhc}{n^2}\ ,\ n = 1, 2, 3..... \quad ...(iii)$$

The wavelengths corresponding to other spectral series (Lyman, Paschen, (etc.)) can be represented by formula similar to Balmer's formula.

Lyman Series : $\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right)$, $n = 2, 3, 4.....$

Paschen Series : $\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right)$, $n = 4, 5, 6.....$

Brackett Series : $\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right)$, $n = 5, 6, 7.....$

Pfund Series : $\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n^2}\right)$, $n = 6, 7, 8$

The Lyman series is in the ultraviolet and the Paschen, Brackett and Pfund series are in the infrared region.

Illustration 14:

Calculate (a) the wavelength and (b) the frequency of the H_β line of the Balmer series for hydrogen.

Solution:

(a) H_β line of Balmer series corresponds to the transition from $n = 4$ *to* $n = 2$ level. The corresponding wavelength for H_β line is,

$$\frac{1}{\lambda} = (1.097 \times 10^7)\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = 0.2056 \times 10^7$$

$\therefore\quad \lambda = 4.9 \times 10^{-7}\ m$

(b) $\nu = \frac{c}{\lambda} = \frac{3.0 \times 10^8}{4.9 \times 10^{-7}} = 6.12 \times 10^{14}\ Hz$

Illustration 15:

Find the largest and shortest wavelengths in the Lyman series for hydrogen. In what region of the electromagnetic spectrum does each series lie?

Solution:

The transition equation for Lyman series is given by,

$$\frac{1}{\lambda}=R\left[\frac{1}{(1)^2}-\frac{1}{n^2}\right] \quad ; \ n = 2, 3, ...$$

For largest wavelength,

$n = 2$

$$\frac{1}{\lambda_{max}}=1.097\times10^7\left(\frac{1}{1}-\frac{1}{4}\right) = 0.823 \times 10^7$$

$\therefore \qquad \lambda_{max} = 1.2154 \times 10^{-7}\ m = 1215\ \mathring{A}$

The shortest wavelength corresponds to $n = \infty$

$$\therefore \qquad \frac{1}{\lambda_{max}}=1.097\times10^7\left(\frac{1}{1}-\frac{1}{\infty}\right)$$

or $\qquad \lambda_{min} = 0.911 \times 10^{-7}\ m = 911\ \mathring{A}$

Both of these wavelengths lie in ultraviolet (UV) region of electromagnetic spectrum.

Illustration 16:

How many different wavelengths may be observed in the spectrum from a hydrogen sample if the atoms are excited to states with principal quantum number n ?

Solution:

From the nth state, the atom may go to $(n - 1)^{th}$ state, ..., 2^{nd} state or 1^{st} state. So, there are $(n - 1)$ possible transitions starting from the nth state. The atoms reaching $(n - 1)^{th}$ state may make $(n - 2)$ different transitions. Similarly, for other lower states. The total number of possible transitions is

$$(n - 1) + (n - 2) + (n - 3) + ...2 + 1 = \frac{n(n-1)}{2} \text{ (Remember)}$$

Illustration 17:

(a) Find the wavelength of the radiation required to excite the electron in Li^{++} from the first to the third Bohr orbit.

(b) How many spectral lines are observed in the emission spectrum of the above excited system?

Solution:

(a) The energy in the first orbit = $E_1 = Z^2 E_0$ where $E_0 = -13.6\ eV$ is the energy of a hydrogen atom in ground state thus for Li^{++},

$$E_1 = 9E_0 = 9 \times (-13.6\ eV) = -122.4\ eV$$

The energy in the third orbit is $E_3 = \frac{E_1}{n^2}=\frac{E_1}{9} = -13.6\ eV$

Thus, $\quad E_3 - E_1 = 8 \times 13.6\ eV = 108.8\ eV$

Energy required to excite Li^{++} from the first orbit to the third orbit is given by

$$E_3 - E_1 = 8 \times 13.6\ eV = 108.8\ eV$$

The wavelength of radiation required to excite Li^{++} from the first orbit to the third orbit is given by

$$\frac{hc}{\lambda} = E_3 - E_1 \qquad \text{or} \qquad \lambda = \frac{hc}{E_3 - E_1} = \frac{1240\ eV - nm}{108.8\ eV} \approx 11.4\ nm$$

(b) The spectral lines emitted are due to the transitions $n = 3 \rightarrow n = 2$, $n = 3 \rightarrow n = 1$ and $n = 2 \rightarrow n = 1$. Thus, there will be three spectral lines in the spectrum.

Illustration 18:

Find the kinetic energy potential energy and total energy in first and second orbit of hydrogen atom if potential energy in first orbit is taken to be zero.

Solution:

$E_1 = -13.60\ eV$, $K_1 = -E_1 = 13.60\ eV$, $U_1 = 2E_1 = -27.20\ eV$

$E_2 = \frac{E_1}{(2)^2} = -3.40\ eV$, $K_2 = 3.40\ eV$, $U_2 = -6.80\ eV$

Now $U_1 = 0$, i.e., potential energy has been increased by 27.20 eV while kinetic energy will remain unchanged. So, values of kinetic energy, potential energy and total energy in first orbit are 13.60 eV, 0, 13.60 respectively and for second orbit these values are 3.40 eV, 20.40 eV and 23.80 eV.

Illustration 19:

The energy levels of a hypothetical one electron atom are shown in the figure.

(a) Find the ionization potential of this atom.

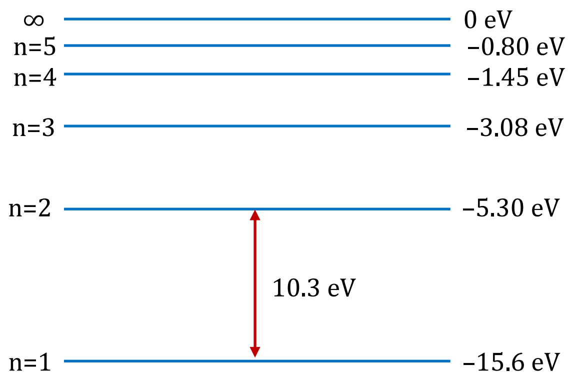
(b) Find the short wavelength limit of the series terminating at $n = 2$.

(c) Find the excitation potential for the state $n = 3$.

(d) Find wave number of the photon emitted for the transition $n = 3$ to $n = 1$.

(e) What is the minimum energy that an electron will have after interacting with this atom in the ground state if the initial kinetic energy of the electron is

(i) 6 eV (ii) 11 eV



Solution:

(a) Ionization potential = 15.6 V

(b) $\lambda_{min} = \frac{12400}{5.3} = 2340\ Å$

(c) $\Delta E_{31} = -3.08 - (-15.6) = 12.52\ eV$

Therefore, excitation potential for state $n = 3$ is 12.52 volt.

(d) $\frac{1}{\lambda_{31}} = \frac{\Delta E_{31}}{12400}$, $Å^{-1} = \frac{12.52}{12400} Å^{-1} \approx 1.01 \times 10^7\ m^{-1}$

(e) (i) $E_2 - E_1 = 10.3\ eV > 6\ eV$.

Hence, electron cannot excite the atoms. So, $K_{min} = 6\ eV$.

(ii) $E_2 - E_1 = 10.3\ eV < 11\ eV$.

Hence, electron can excite the atoms. So, $K_{min} = (11 - 10.3) = 0.7\ eV$.

Illustration 20:

A particle known as μ-meson, has a charge equal to that of an electron and mass 208 times the mass of the electron. It moves in a circular orbit around a nucleus of charge $+3e$. Take the mass of the nucleus to be infinite. Assuming that the Bohr's model is applicable to this system,

(a) derive an expression for the radius of the n^{th} Bohr orbit,

(b) find the value of n for which the radius of the orbit is approximately the same as that of the first Bohr orbit for a hydrogen atom and

(c) find the wavelength of the radiation emitted when the μ–meson jumps from the third orbit to the first orbit.

Solution:

(a) We have, $\dfrac{mv^2}{r} = \dfrac{Ze^2}{4\pi\varepsilon_0 r^2}$

or, $v^2 r = \dfrac{Ze^2}{4\pi\varepsilon_0 m}$ ...(i)

The quantization rule is $vr = \dfrac{nh}{2\pi m}$

The radius is $r = \dfrac{(vr)^2}{v^2 r} = \dfrac{4\pi\varepsilon_0 m}{Ze^2}$

$= \dfrac{n^2 h^2 \varepsilon_0}{Z\pi m e^2}$ ...(ii)

For the given system, $Z = 3$ and $m = 208\ m_e$.

Thus, $r_\mu = \dfrac{n^2 h^2 \varepsilon_0}{624\pi m_e e^2}$

(b) From (ii), the radius of the first Bohr orbit for the hydrogen atom is

$r_h = \dfrac{h^2 \varepsilon_0}{\pi m_e e^2}$

For $r_\mu = r_h$, $\dfrac{n^2 h^2 \varepsilon_0}{624\pi m_e e^2} = \dfrac{h^2 \varepsilon_0}{\pi m_e e^2}$

or, $n^2 = 624$

or, $n \approx 25$

(c) From (i), the kinetic energy of the atom is $\dfrac{mv^2}{2} = \dfrac{Ze^2}{8\pi\varepsilon_0 r}$

and the potential energy is $-\dfrac{Ze^2}{4\pi\varepsilon_0 r}$

The total energy is $E_n = \dfrac{Ze^2}{8\pi\varepsilon_0 r}$

Using (ii),

$$E_n = -\frac{Z^2 \pi m e^4}{8\pi\varepsilon_0^2 n^2 h^2} = -\frac{9 \times 208 m_e\, e^4}{8\varepsilon_0^2 n^2 h^2} = \frac{1872}{n^2}\left(-\frac{m_e e^4}{8\varepsilon_0^2\ h^2}\right)$$

But $\left(-\dfrac{m_e e^4}{8\varepsilon_0^2\ h^2}\right)$ is the ground state energy of hydrogen atom and hence is equal to – 13.6 eV.

From (iii),

$$E_n = -\frac{1872}{n^2} \times 13.6\ eV = \frac{-25459.2\ eV}{n^2}$$

Thus, E_1 = – 25459.2 eV and $E_3 = \frac{E_1}{9}1$ = – 2828.8 eV. The energy difference is $E_3 - E_1$ = 22630.4eV.

The wavelength emitted is

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\ eV - nm}{22630.4\ eV} = 55\ pm.$$

Illustration 21:

A gas of hydrogen like atoms can absorb radiations of 68 eV. Consequently, the atoms emit radiations of only three different wavelength. All the wavelengths are equal or smaller than that of the absorbed photon.

(a) Determine the initial state of the gas atoms.

(b) Identify the gas atoms.

(c) Find the minimum wavelength of the emitted radiations.

(d) Find the ionization energy and the respective wavelength for the gas atoms.

Solution:

(a) $\dfrac{n(n-1)}{2} = 3$

$\therefore$ $n = 3$

i.e., after excitation atom jumps to second excited state.

Hence, $n_f = 3$. So, n_i can be 1 or 2

If $n_i = 1$ then energy emitted is either equal to, greater than or less than the energy absorbed.

Hence the emitted wavelength is either equal to, less than or greater than the absorbed wavelength.

Hence, $n_i \neq 1$.

If $n_i = 2$, then $E_e \geq E_a$. Hence, $\lambda_e \leq \lambda_0$

(b) $E_3 - E_2 = 68\ eV$

$\therefore$ $(13.6)\ (Z^2)\left(\frac{1}{4} - \frac{1}{9}\right) = 68$

$\therefore$ $Z = 6$

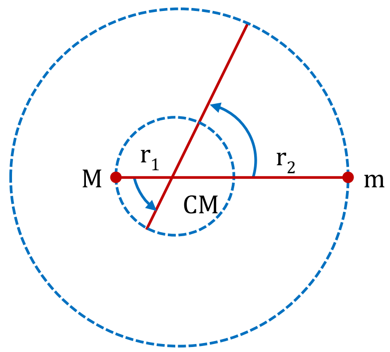
(c) $\lambda_{min} = \dfrac{12400}{E_3 - E_1} = \dfrac{12400}{(13.6)\ (6)^2\left(1 - \frac{1}{9}\right)} = \dfrac{12400}{435.2} = 28.49$

(d) Ionization energy = $(13.6)\ (6)^2 = 489.6\ eV$

$\lambda = \dfrac{12400}{489.6} = 25.33\ \mathring{A}$

Effect of Nucleus Motion on Energy of Atom:

Let both the nucleus of mass M, charge Ze and electron of mass m, and charge e revolve about their centre of mass (CM) with same angular velocity (ω) but different linear speeds. Let r_1 and r_2 be the distance of CM from nucleus and electron. Their angular velocity should be same then only their separation will remain unchanged in an energy level.



Let r be the distance between the nucleus and the electron. Then

$$Mr_1 = mr_2$$

$$r_1 + r_2 = r$$

$$\therefore \quad r_1 = \frac{mr}{M+m} \quad \text{and} \quad r_2 = \frac{Mr}{M+m}$$

Centripetal force to the electron is provided by the electrostatic force.

So, $$mr_2\omega^2 = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2}$$

or $$m\left(\frac{Mr}{M+m}\right)\omega^2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Ze^2}{r^2}$$

or $$\left(\frac{Mm}{M+m}\right)r^3\omega^2 = \frac{Ze^2}{4\pi\varepsilon_0}$$

or $$\mu r^3\omega^2 = \frac{e^2}{4\pi\varepsilon_0}$$

where $\dfrac{Mm}{M+m} = \mu$

Moment of inertia of atom about CM,

$$I = Mr_1^2 + mr_2^2 = \left(\frac{Mm}{M+m}\right)r^2 = \mu r^2$$

According to Bohr's theory,

$$\frac{nh}{2\pi} = I\omega$$

or $$\mu r^2\omega = \frac{nh}{2\pi}$$

Solving above equations for r, we get

$$r = \frac{\varepsilon_0 n^2h^2}{\pi\mu e^2 Z} \quad \text{and} \quad r = (0.529\,\text{Å})\,\frac{n^2}{Z}\cdot\frac{m}{\mu}$$

Further electrical potential energy of the system,

$$U = \frac{-Ze^2}{4\pi\varepsilon_0 r}\ ;\quad U = \frac{-Z^2e^4\mu}{4\varepsilon_0^2n^2h^2}$$

and kinetic energy, $K = \frac{1}{2}I\omega^2 = \frac{1}{2}\mu r^2\omega^2$ and $K = \frac{1}{2}\mu v^2$

v-speed of electron with respect to nucleus ($v = r\omega$)

here, $\omega^2 = \dfrac{Ze^2}{4\pi\varepsilon_0\mu r^3}$

$\therefore$ $K = \dfrac{Ze^2}{8\pi\varepsilon_0 r} = \dfrac{Z^2e^4\mu}{8\pi\varepsilon_0^2 n^2h^2}$

$\therefore$ Total energy of the system

$E_n = K + U$

$E_n = -\dfrac{\mu e^4}{8\varepsilon_0{}^2 n^2h^2}$

This expression can also be written as $E_n = -(13.6\ eV)\dfrac{Z^2}{n^2}\cdot\left(\dfrac{\mu}{m}\right)$.

The expression for E_n without considering the motion of proton is $E_n = -\dfrac{me^4}{8\varepsilon_0{}^2n^2h^2}$, i.e., m is replaced by μ while considering the motion of nucleus.

Illustration 22:

A positronium 'atom' is a system that consists of a positron and an electron that orbit each other. Compare the wavelength of the spectral lines of positronium with those of ordinary hydrogen.

Solution:

Here the two particles have the same mass m, so the reduced mass is

$$\mu = \frac{mm}{m+m} = \frac{m^2}{2m} = \frac{m}{2}$$

where m is the electron mass. We know that

$E_n \propto m$

$\therefore$ $\dfrac{E'_n}{E_n} = \dfrac{\mu}{m} = \dfrac{1}{2}$; energy of each level is halved.

$\therefore$ Their difference will also be halved.

Hence, $\lambda'_n = 2\lambda_n$

Atomic Collision

In such collisions assume that the loss in the kinetic energy of system is possible only if it can excite or ionise.

Illustration 23:

What will be the type of collision, if $K = 14eV$, $20.4\ eV$, $22\ eV$, $24.18\ eV$ (elastic/inelastic/perfectly inelastic)

Head on Collison

Solution:

Loss in energy (ΔE) during the collision will be used to excite the atom or electron from one level to another.

According to quantum Mechanics, for hydrogen atom.

$\Delta E = \{0, 10.2\ eV, 12.09\ eV, ..., 13.6\ eV\}$

According to Newtonian mechanics

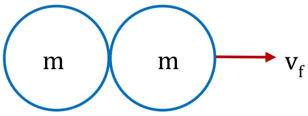
Minimum loss = 0. (elastic collision)

For maximum loss collision will be perfectly inelastic if neutron collides perfectly inelastically.

Then, Applying momentum conservation

$m\text{v}_0 = 2m\text{v}_f$

$v_f = \dfrac{v_0}{2}$



Final $K.E. = \dfrac{1}{2} \times 2m \times \dfrac{\text{v}_0^2}{4} = \dfrac{\frac{1}{2}m\text{v}_0^2}{2} = \dfrac{K}{2}$

Maximum loss = $\dfrac{K}{2}$

According to classical mechanics (ΔE) = $[0, \dfrac{K}{2}]$

(a) If $K = 14\ eV$,

According to quantum mechanics

$(\Delta E) = \{0, 10.2 eV, 12.09\ eV\}$

According to classical mechanics

$\Delta E = [0, 7\ eV]$

loss = 0

Hence it is elastic collision speed of particle changes.

(b) If $K = 20.4\ eV$

According to classical mechanics

loss = $[0, 10.2\ eV]$

According to quantum mechanics

loss = $\{0, 10.2eV, 12.09\ eV, ...\}$

loss = 0 elastic collision.

loss = $10.2eV$ perfectly inelastic collision

(c) If $K = 22\ eV$

Classical mechanics $\Delta E = [0, 11]$

Quantum mechanics $\Delta E = \{0, 10.2eV, 12.09eV, ...\}$

loss = 0 elastic collision

loss = $10.2\ eV$ inelastic collision

(d) If $K = 24.18\ eV$

According to classical mechanics, $\Delta E = [0, 12.09eV]$

According to quantum mechanics, $\Delta E = \{0, 10.2eV, 12.09eV, ... 13.6eV\}$

loss = 0 elastic collision

loss = $10.2\ eV$ inelastic collision

loss = $12.09\ eV$ perfectly inelastic collision

Illustration 24:

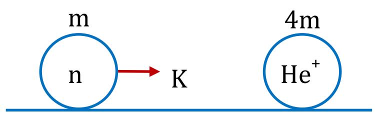
A He^+ ion is at rest and is in ground state. A neutron with initial kinetic energy K collides head on with the He^+ ion. Find minimum value of K so that there can be an inelastic collision between these two particle.

Solution:

Here the loss during the collision can only be used to excite the atoms or electrons.

So, according to quantum mechanics

$$\text{loss} = \{0, 40.8eV, 48.3eV, ..., 54.4eV\} \qquad ...(1)$$



$$E_n = -\frac{Z^2}{n^2}\ 13.6\ eV$$

Now according to Newtonian mechanics

Minimum loss = 0

Maximum loss will be for perfectly inelastic collision.

Let v_0 be the initial speed of neutron and v_f be the final common speed.

So, by momentum conservation $mv_0 = mv_f + 4mv_f$; $v_f = \dfrac{v_0}{5}$

Where m = Mass of Neutron

$\therefore$ Mass of He^+ ion = $4m$

So, final kinetic energy of system

$$K.E. = \frac{1}{2}m v_f^2 + \frac{1}{2}4m v_f^2 = \frac{1}{2}.(5m).\frac{v_0^2}{25} = \frac{1}{5}.(\frac{1}{2}mv_0^2) = \frac{K}{5}$$

Maximum loss = $K - \dfrac{K}{5} = \dfrac{4K}{5}$

So, loss will be $\left[0, \dfrac{4K}{5}\right]$ ...(2)

For inelastic collision there should be at least one common value other than zero in set (1) and (2)

$\therefore$ $\dfrac{4K}{5} > 40.8\ eV$

$K > 51\ eV$

Minimum value of $K = 51\ eV$.

Illustration 25:

A moving hydrogen atom makes a head on collision with a stationary hydrogen atom. Before collision both atoms are in ground state and after collision they move together. What is the minimum value of the kinetic energy of the moving hydrogen atom, such that one of the atoms reaches one of the excitation state.

Solution:

Let K be the kinetic energy of the moving hydrogen atom and K', the kinetic energy of combined mass after collision.

From conservation of linear momentum,

$p = p'$ or $\sqrt{2Km} = \sqrt{2K'(2m)}$

or $K = 2K'$ ...(i)

From conservation of energy,

$$K = K' + \Delta E \qquad \text{...(ii)}$$

Solving equations (i) and (ii), we get

$$\Delta E = \frac{K}{2}$$

Now minimum value of ΔE for hydrogen atom is 10.2 eV.

or $\quad \Delta E \geq 10.2\ eV$

$\therefore \quad \frac{K}{2} \geq 10.2$

$\therefore \quad K \geq 20.4\ eV$

Therefore, the minimum kinetic energy of moving hydrogen is 20.4 eV

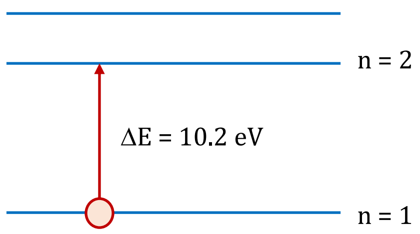


Illustration 26:

A neutron moving with speed v makes a head-on collision with a hydrogen atom in ground state kept at rest. Find the minimum kinetic energy of the neutron for which inelastic (completely or partially) collision may take place. The mass of neutron = mass of hydrogen = 1.67×10^{-27} kg.

Solution:

Suppose the neutron and the hydrogen atom move at speed v_1 and v_2 after the collision. The collision will be inelastic if a part of the kinetic energy is used to excite the atom. Suppose an energy ΔE is used in this way. Using conservation of linear momentum and energy.

$$mv = mv_1 + mv_2 \qquad \text{...(i)}$$

and $\quad \frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \Delta E \qquad \text{...(ii)}$

From (i), $v^2 = v_1^2 + v_2^2 + 2v_1v_2$,

From (ii), $v^2 = v_1^2 + v_2^2 + \frac{2\Delta E}{m}$

Thus, $\quad 2v_1v_2 = \frac{2\Delta E}{m}$

Hence, $(v_1 - v_2)^2 - 4v_1v_2 = v^2 - \frac{4\Delta E}{m}$

As $v_1 - v_2$ must be real, $v^2 - \frac{4\Delta E}{m} \geq 0$

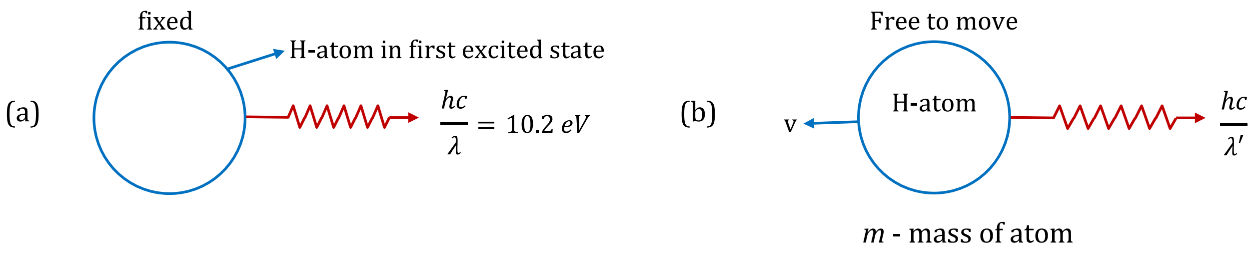
or, $\quad \frac{1}{2}mv^2 > 2\Delta E.$

The minimum energy that can be absorbed by the hydrogen atom in ground state to go in an excited state is 10.2 eV. Thus, the minimum kinetic energy of the neutron needed for an inelastic collision is

$$\frac{1}{2}mv_{\min}^2 = 2\times 10.2 \ \ eV = 20.4 \ \ eV$$

Calculation of Recoil Speed of Atom on Emission of a Photon

Momentum of photon = $mc = \dfrac{h}{\lambda}$



According to momentum conservation

$$mv = \frac{h}{\lambda'} \qquad \text{...(i)}$$

According to energy conservation

$$\frac{1}{2}mv^2 + \frac{hc}{\lambda'} = 10.2 \ eV$$

Since mass of atom is very large than photon

Hence $\frac{1}{2}mv^2$ can be neglected

$$\frac{hc}{\lambda'} = 10.2 \ eV \quad ; \quad \frac{h}{\lambda} = \frac{10.2}{c} \ eV$$

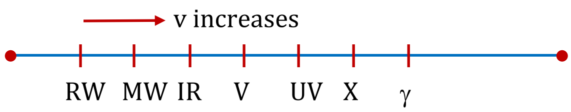
$$mv = \frac{10.2}{c} eV \quad ; \quad v = \frac{10.2}{cm}$$

Recoil speed of atom = $\dfrac{10.2}{cm}$

X-Rays

It was discovered by **ROENTGEN**. The wavelength of x-rays is found between 0.1 $Å$ to 10 $Å$. These rays are invisible to eye. They are electromagnetic waves and have speed $c = 3 \times 10^8 \ m/s$ in vacuum.

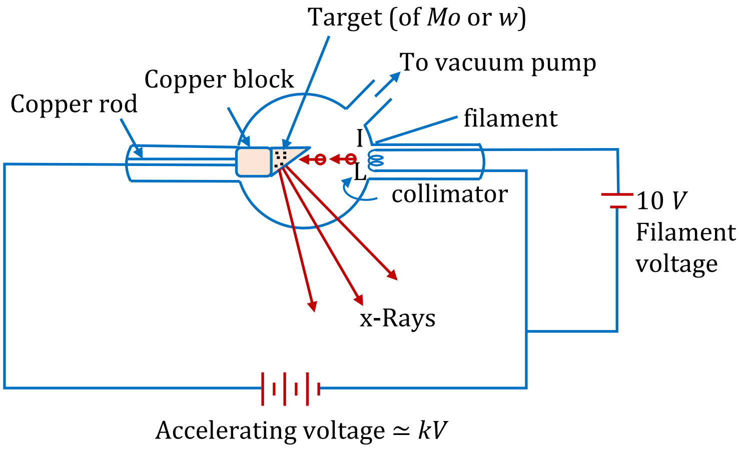
Its photons have energy around 1000 times more than the visible light.



When fast moving electrons having energy of order of several KeV strike the metallic target then x-rays are produced.

Production of x-Rays by Coolidge Tube:

The melting point, specific heat capacity and atomic number of target should be high. When voltage is applied across the filament then filament on being heated emits electrons from it. Now for giving the beam shape of electrons, collimator is used. Now when electron strikes the target then x-rays are produced.



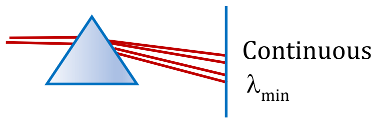
When electrons strike with the target, some part of energy is lost and converted into heat. Since, target should not melt or it can absorb heat so that the melting point, specific heat of target should be high.

Here copper rod is attached, so that heat produced can go behind and it can absorb heat and target does not get heated very high.

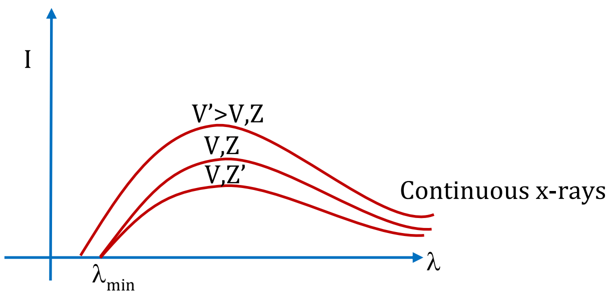
For more energetic electron, accelerating voltage is increased.

For more number of photons voltage across filament is increased.

The x-ray was analysed by mostly taking their spectrum.



Variation of Intensity of x-Rays with λ is Plotted as shown in figure:



The minimum wavelength corresponds to the maximum energy of the x-rays which in turn is equal to the maximum kinetic energy eV of the striking electrons thus

$$eV = h\nu_{max} = \frac{hc}{\lambda_{min}} \; ; \; \lambda_{min} = \frac{hc}{eV} = \frac{12400}{V(\text{involts})} \text{Å}$$

We see that cut off wavelength λ_{min} depends only on accelerating voltage applied between target and filament. It does not depend upon material of target, it is same for two different metals (Z and Z').

Illustration 27:

An X-ray tube operates at 20 kV. A particular electron loses 5% of its kinetic energy to emit an X-ray photon at the first collision. Find the wavelength corresponding to this photon.

Solution:

Kinetic energy acquired by the electron is $K = eV = 20 \times 10^3\ eV$.

The energy of the photon = 0.05 × 20 = $10^3\ eV = 10^3\ eV$.

Thus, $\frac{h\nu}{\lambda} = 10^3 eV$

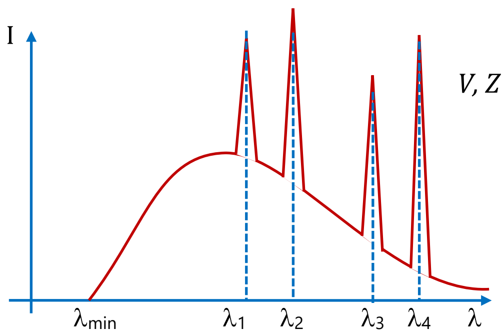
$$\Rightarrow \qquad \lambda = \frac{(4.14\times10^{-15} eV-s)\times(3\times10^8 m/s)}{10^3 eV}$$

$$= \frac{1242\ \ eV-nm}{10^3 eV} = 1.24\ \ nm$$

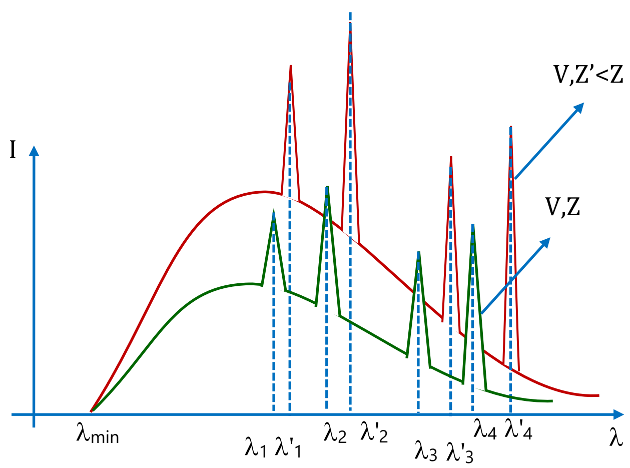
Characteristic x-Rays

The sharp peaks obtained in graph are known as characteristic x-rays because they are characteristic of target material.

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots\ldots$ = characteristic wavelength of material having atomic number Z are called characteristic x-rays and the spectrum obtained is called characteristic spectrum. If target of atomic number Z' is used then peaks are shifted.



Characteristic x-ray emission occurs when an energetic electron collides with target and remove an inner shell electron from atom, the vacancy created in the shell is filled when an electron from higher level drops into it. Suppose vacancy created in innermost K-shell is filled by an electron dropping from next higher level L-shell then K_α characteristic x-ray is obtained. If vacancy in K-shell is filled by an electron from M-shell, K_β line is produced and so on similarly, L_α, L_β, ... M_α, M_β lines are produced.



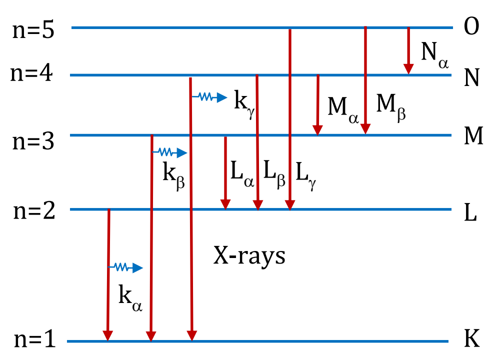
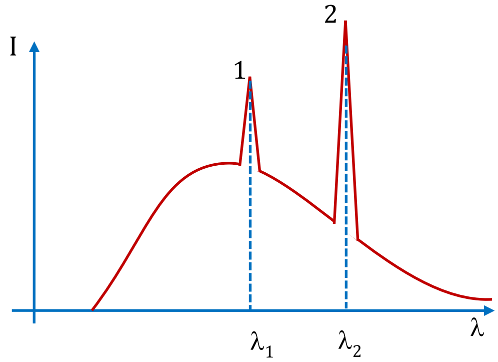


Illustration 28:

Find which is K_α and K_β.



Solution:

$$\Delta E = \frac{hc}{\lambda}, \lambda = \frac{hc}{\Delta E}$$

Since energy difference of K_α is less than K_β

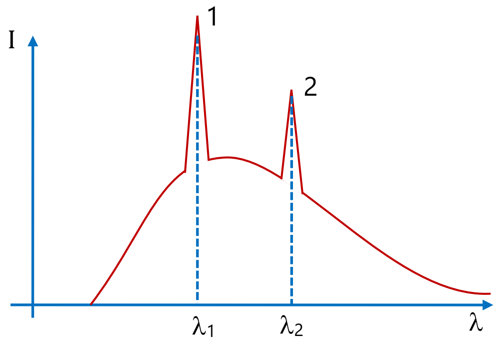
$$\Delta E_{k\alpha} < \Delta E_{k\beta}$$

$$\lambda_{k\beta} < \lambda_{k\alpha}$$

1 is K_β and 2 is K_α.

Illustration 29:

Find which is K_α and L_α.



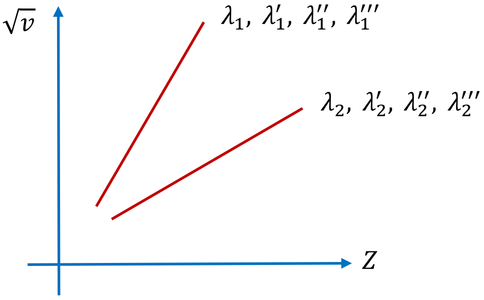
Solution:

$\therefore \quad \Delta E_{K\alpha} > \Delta E_{L\alpha}$

1 is K_α and 2 is L_α.

Moseley's Law

Moseley measured the frequencies of characteristic *x*-rays for a large number of elements and plotted the square root of frequency against position number in periodic table. He discovered that plot is very closed to a straight line not passing through origin.



Z_1	I_1	I_2
Z_2	I_1'	I_2'
Z_3	I_1''	I_2''
Z_4	I_1'''	I_2'''

Wavelength of characteristic wavelengths

Moseley's observations can be mathematically expressed as

$$\sqrt{\nu} = a(Z - b)$$

a and *b* are positive constants for one type of *x*-rays and for all elements (independent of *Z*).

Moseley's Law can be derived on the basis of Bohr's theory of atom, frequency of *x*-rays is given by

$$\sqrt{\nu} = \sqrt{cR\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)} \cdot (Z - b)$$

By using the formula $\dfrac{1}{\lambda} = R\,z^2\left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right)$ with modification for multi electron system.

$b \rightarrow$ known as screening constant or shielding effect, and $(Z - b)$ is effective nuclear charge.

For K_α line

$n_1 = 1, \quad n_2 = 2$

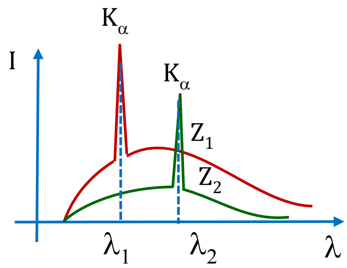
$\therefore \quad \sqrt{\nu} = \sqrt{\dfrac{3Rc}{4}}\,(Z - b)$

$\sqrt{\nu} = a(Z - b)$

Here $a = \sqrt{\dfrac{3Rc}{4}}$, [$b = 1$ for K_α lines]

Illustration 30:

Find in Z_1 and Z_2 which one is greater.



Solution:

$\because \quad \sqrt{\nu} \equiv \sqrt{cR\left(\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right)} \cdot (Z - b)$

If Z is greater then ν will be greater, λ will be less

$\therefore \quad \lambda_1 < \lambda_2$

$\therefore \quad Z_1 > Z_2$

Illustration 31:

A cobalt target is bombarded with electrons and the wavelength of its characteristic spectrum are measured.
A second, fainter, characteristic spectrum is also found because of an impurity in the target. The wavelength of the K_α lines are 178.9 pm (cobalt) and 143.5 pm (impurity). What is the impurity?

Solution:

Using Moseley's law and putting c/λ for ν (and assuming b = 1), we obtain

$$\sqrt{\frac{c}{\lambda_{c_0}}} = aZ_{c_0} - a$$

and $$\sqrt{\frac{c}{\lambda_x}} = aZ_x - a$$

Dividing yields

$$\sqrt{\frac{\lambda_{c_0}}{\lambda_x}} = \frac{Z_x - 1}{Z_{c_0} - 1}$$

Substituting gives us

$$\sqrt{\frac{178.9\ pm}{143.5\ pm}} = \frac{Z_x - 1}{27 - 1}.$$

Solving for the unknown, we find Z_x = 30.0; the impurity is zinc.

Illustration 32:

Find the constants a and b in Moseley's equation $\sqrt{\nu} = a(Z - b)$ from the following data.

Element	Z	Wavelength of K_α X-ray
Mo	42	71 pm
Co	27	178.5 pm

Solution:

Moseley's equation is

$$\sqrt{\nu} = a(Z - b)$$

Thus, $$\sqrt{\frac{c}{\lambda_1}} = a(Z_1 - b) \quad \text{...(i)}$$

and $$\sqrt{\frac{c}{\lambda_2}} = a(Z_2 - b) \quad \text{...(ii)}$$

From (i) and (ii),

$$\sqrt{c}\left(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}}\right) = a\ (Z_1 - Z_2)$$

or, $$a = \frac{\sqrt{c}}{(Z_1 - Z_2)}\left(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}}\right)$$

$$= \frac{(3\times10^8 m/s)^{1/2}}{42-27}\left[\frac{1}{(71\times10^{-12}m)^{1/2}} - \frac{1}{(178.5\times10^{-12}m)^{1/2}}\right]$$

$$= 5.0 \times 10^7\ (Hz)^{1/2}$$

Dividing (i) by (ii),

$$\sqrt{\frac{\lambda_2}{\lambda_1}} = \frac{Z_1 - b}{Z_2 - b}$$

or, $$\sqrt{\frac{178.5}{71}} = \frac{42 - b}{27 - b}$$

or, $b = 1.37$

Illustration 33:

Find the momentum of a 12.0 MeV photon.

Solution:

$$p = \frac{E}{c} = 12\ MeV/c$$

Illustration 34:

Monochromatic light of wavelength 3000 $Å$ is incident normally on a surface of area 4 cm^2. If the intensity of the light is $15 \times 10^{-2}\ W/m^2$, determine the rate at which photons strike the surface.

Solution:

Rate at which photons strike the surface = $\frac{IA}{hc/\lambda} = \frac{6\times10^{-5}\ J/s}{6.63\times10^{-19}\ J/photon} = 9.05 \times 10^{13}$ photon/s.

Illustration 35:

The kinetic energies of photoelectrons range from zero to $4.0 \times 10^{-19}\ J$ when light of wavelength 3000 Å falls on a surface. What is the stopping potential for this light ?

Solution:

$$K_{max} = 4.0 \times 10^{-19}\ J \times \frac{1eV}{1.6\times10^{-19}\ J} = 2.5\ eV.$$

Then, from $eV_s = K_{max}$, $V_s = 2.5\ V$.

Illustration 36:

What is the threshold wavelength for the material in above problem?

Solution:

$$2.5\ eV = \frac{12.4\times10^3 eV.Å}{3000\ Å} - \frac{12.4\times10^3 eV.Å}{\lambda_{th}}$$

Solving, $\lambda_{th} = 7590\ Å$.

Illustration 37:

Find the de Broglie wavelength of a 0.01 kg pellet having a velocity of 10 m/s.

Solution:

$$\lambda = h/p = \frac{6.63\times10^{-34}\ J.s}{0.01\ kg\times10\ m/s} = 6.63 \times 10^{-23}\ Å.$$

Illustration 38:

Determine the accelerating potential necessary to give an electron a de Broglie wavelength of 1 $\mathring{A}$, which is the size of the interatomic spacing of atoms in a crystal.

Solution:

$$V = \frac{h^2}{2m_0 e\lambda^2} = 151\ V.$$

Illustration 39:

Determine the wavelength of the second line of the Paschen series for hydrogen.

Solution:

$$\frac{1}{\lambda} = (1.097 \times 10^{-3}\ \mathring{A}^{-1})\left(\frac{1}{3^2} - \frac{1}{5^2}\right)$$

or $\lambda = 12{,}820\ \mathring{A}$.

Illustration 40:

How many different photons can be emitted by hydrogen atoms that undergo transitions to the ground state from the $n = 5$ state ?

Solution:

No of possible transition from $n = 5$ are ${}^5C_2 = 10$

10 photons.

Illustration 41:

An electron rotates in a circle around a nucleus with positive charge Ze. How is the electrons' velocity related to the radius of its orbit ?

Solution:

The force on the electron due to the nuclear provides the required centripetal force

$$\frac{1}{4\pi\varepsilon_0}\frac{Ze.\ e}{r^2} = \frac{mv^2}{r}$$

$$\Rightarrow \quad v = \sqrt{\frac{Ze^2}{4\pi\varepsilon_0 .rm}}$$

Illustration 42:

(i) Calculate the first three energy levels for positronium.

(ii) Find the wavelength of the H_a line ($3 \to 2$ transition) of positronium.

Solution:

In positronium electron and positron revolve around their centre of mass

$$\frac{1}{4p\varepsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r/2} \qquad ...(1)$$

$$\frac{n\ \ h}{2\pi} = 2 \times mv_{k/2} \qquad ...(2)$$

From (1) and (2)

$$V =. \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0} \times 2p = \frac{e^2}{4\varepsilon_0 nh}$$

$$TE = -\frac{1}{2}mv^2 \times 2 = -m.\frac{e^4}{16\varepsilon_0{}^2 n^2 h^2}$$

$$= -6.8\ \frac{1}{n^2} eV$$

(i) $E_1 = -6.8\ ev$

$$E_2 = -6.8 \times \frac{1}{2^2} eV = -1.70\ eV$$

$$E_3 = -6.8 \times \frac{1}{3^2} eV = -0.76\ eV$$

(ii) $\Delta E\ (3 \rightarrow 2) = E_3 - E_2 = -0.76 - (-1.70)\ eV = 0.94\ eV$

The corresponding wave length

$$\lambda = \frac{1.24 \times 10^4}{0.94} \mathring{A} = 1313\ \mathring{A}$$

(i) $-6.8\ eV,\ -1.7\ eV, -0.76\ eV$

(ii) $1313\ \mathring{A}$.

Illustration 43:

A H-atom in ground state is moving with initial kinetic energy K. It collides head on with a He^+ ion in ground state kept at rest but free to move. Find minimum value of K so that both the particles can excite to their first excited state.

Solution:

Energy available for excitation = $\frac{4k}{5}$

Total energy required for excitation

= 10.2 ev + 40.8 eV

= 51.0 ev

$\therefore \quad \frac{4k}{5} = 51 \quad \Rightarrow \quad k = 63.75\ eV$

Illustration 44:

A TV tube operates with a 20 kV accelerating potential. What are the maximum–energy X–rays from the TV set ?

Solution:

The electrons in the TV tube have an energy of 20 keV, and if these electrons are brought to rest by a collision in which one X–ray photon is emitted, the photon energy is 20 keV.

Illustration 45:

In the Moseley relation, $\sqrt{v} = a(Z-b)$ which will have the greater value for the constant a for K_α or K_β transition?

Solution:

A is larger for the K_β transitions than for the K_α transitions.

Nuclear Physics

It is the branch of physics which deals with the study of nucleus.

Nucleus:

(a) **Discoverer :** Rutherford

(b) **Constituents :** neutrons (n) and protons (p) [collectively known as nucleons]

1. **Neutron :** It is a neutral particle. It was discovered by J. Chadwick (in 1932).
Mass of neutron, $m_n = 1.6749286 \times 10^{-27}\ kg$.

2. **Proton :** It has a charge equal to +e. It was discovered by Goldstein.
Mass of proton, $m_p = 1.6726231 \times 10^{-27}\ kg$

$m_p \gtrsim m_n$

(c) **Representation :**

${}_zX^A$ or ${}^A_ZX$

Where, $X \Rightarrow$ Symbol of the atom

$Z \Rightarrow$ Atomic number = Number of protons

$A \Rightarrow$ Atomic mass number = Total number of nucleons.
= Number of protons + Number of neutrons.

Atomic Mass Number:

It is the nearest integer value of mass represented in a.m.u. (Atomic mass unit).

1 a.m.u. = $\frac{1}{12}$ [Mass of one atom of ${}_6C^{12}$ atom at rest and in ground state]

$1.6603 \times 10^{-27}\ kg : 931.478\ MeV/c^2$

Mass of proton (m_p) = Mass of neutron (m_n) = 1 a.m.u.

Some Definitions:

(1) **Isotopes:**
The nuclei having the same number of protons but different number of neutrons are called isotopes.

(2) **Isotones:**
Nuclei with the same neutron number N but different atomic number Z are called isotones.

(3) **Isobars:**
The nuclei with the same mass number but different atomic number are called isobars.

(d) **Size** of nucleus: Order of $10^{-15}\ m$ (fermi)
Radius of nucleus; $R = R_0A^{1/3}$
where $R_0 = 1.1 \times 10^{-15}\ m$ (which is an empirical constant)
A = Atomic mass number of atoms.

(e) **Density :**

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} \cong \frac{Am_p}{\frac{4}{3}\pi R^3} = \frac{Am_p}{\frac{4}{3}\pi(R_0A^{1/3})^3} = \frac{3m_p}{4\pi\ R_0^3}$$

$$= \frac{3\times1.67\times10^{-27}}{4\times3.14\times(1.1\times10^{-15})^3} = 3 \times 10^{17}\ kg/m^3$$

Nuclei of almost all atoms have almost same density as nuclear density is independent of the mass number (A) and atomic number (Z).

Illustration 46:

Calculate the radius of ^{70}Ge.

Solution:

We have, $R = R_0 A^{1/3} = (1.1\,fm)\,(70)^{1/3}$

$= (1.1\,fm)\,(4.12) = 4.53\,fm.$

Illustration 47:

Calculate the electric potential energy of interaction due to the electric repulsion between two nuclei of ^{12}C when they 'touch' each other at the surface.

Solution:

The radius of a ^{12}C nucleus is

$R = R_0 A^{1/3}$

$= (1.1\,fm)\,(12)^{1/3} = 2.52\,fm.$

The separation between the centres of the nuclei is $2R = 5.04\,fm$. The potential energy of the pair is

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = (9 \times 10^9\,N\text{–}m^2/C^2)\,\frac{(6\times1.6\times10^{-19}C)^2}{5.04\times10^{-15}m}$$

$= 1.64 \times 10^{-12}\,J = 10.2\,MeV.$

Mass Defect

It has been observed that there is a difference between expected mass and actual mass of a nucleus.

$M_{\text{expected}} = Z\,m_p + (A - Z)\,mn$

$M_{\text{observed}} = M_{\text{atom}} - Zm_e$

It is found that $M_{\text{observed}} < M_{\text{expected}}$

Hence, mass defect is defined as Mass defect = $M_{\text{expected}} - M_{\text{observed}}$

$\Delta m = [Zm_p + (A - Z)\,mn] - [M_{\text{atom}} - Zm_e]$

Binding Energy

It is the minimum energy required to break the nucleus into its constituent particles.

or

Amount of energy released during the formation of nucleus by its constituent particles and bringing them from infinite separation.

Binding Energy $(B.E.) = \Delta mc^2$

$BE = \Delta m\,(in\ amu) \times 931.5\,MeV/amu$

$= \Delta m \times 931.5\,MeV$

Note:

If binding energy per nucleon is more for a nucleus then it is more stable.

For example, if $\left(\frac{B.E_1}{A_1}\right) > \left(\frac{B.E_2}{A_2}\right)$ then nucleus 1 would be more stable.

Illustration 48:

Following data is available about 3 nuclei P, Q and R. Arrange them in decreasing order of stability:

	P	Q	R
Atomic mass number (A)	10	5	6
Binding Energy (MeV)	100	60	66

Solution:

$$\left(\frac{B.E}{A}\right)_P = \frac{100}{10} = 10 \qquad \Rightarrow \left(\frac{BE}{A}\right)_Q = \frac{60}{5} = 12$$

$$\left(\frac{B.E.}{A}\right)_R = \frac{66}{6} = 11 \qquad \therefore \text{ Stability order is } Q > R > P.$$

Illustration 49:

The three stable isotopes of neon: ${}^{20}_{10}Ne, {}^{21}_{10}Ne$ and ${}^{22}_{10}Ne$ have respective abundances of 90.51% 0.27% and 9.22%. The atomic masses of three isotopes are 19.99 u. 20.99 u, 22 u, respectively. Obtain the average atomic mass of neon.

Solution:

$$m = \frac{90.51 \times 19.99 + 0.27 \times 20.99 + 9.22 \times 22}{100} = 20.18\ u$$

Illustration 50:

A nuclear reaction is given as $A + B \rightarrow C + D$. Binding energies of A, B, C and D are given as B_1, B_2, B_3 and B_4. Find the energy released in the reaction.

Solution:

$(B_3 + B_4) - (B_1 + B_2)$

Illustration 51:

Calculate the binding energy of an alpha particle from the following data:

Mass of ${}^{1}_{1}H$ atom = 1.007826 u

Mass of ${}^{4}_{2}He$ neutron = 1.008665 u

Mass of atom = 4.00260 u

Take 1 u = 931 MeV/c^2.

Solution:

The alpha particle contains 2 protons and 2 neutrons. The binding energy is

$B = (2 \times 1.007826\ u + 2 \times 1.008665\ u - 4.00260\ u)c^2$

$= (0.03038\ u)c^2 = 0.03038 \times 931\ MeV = 28.3\ MeV.$

Illustration 52:

Find the binding energy of ${}^{56}_{26}Fe$. Atomic mass of ${}^{56}Fe$ is 55.9349 u and that of ${}^{1}H$ is 1.00783 u. Mass of neutron = 1.00867 u.

Solution:

The number of protons in ${}^{56}_{26}Fe$ = 26 and the number of neutrons = 56 – 26 = 30.

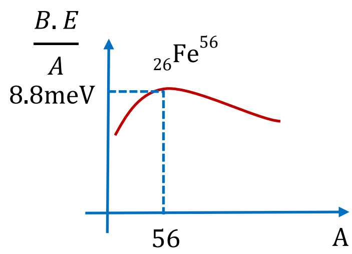
The binding energy of ${}^{56}_{26}Fe$ is

$= [26 \times 1.00783\,u + 30 \times 1.00867\,u - 55.9349\,u]\,c^2 = (0.52878\,u)c^2$

$= (0.52878\,u)\,(931\,MeV/u) = 492\,MeV.$

Variation of Binding Energy Per Nucleon with Mass Number

The binding energy per nucleon first increases on an average and reaches a maximum of about 8.8 MeV for A 50 $\rightarrow$ 80. For still heavier nuclei, the binding energy per nucleon slowly decreases as A increases. Binding energy per nucleon is maximum for ${}_{26}Fe^{56}$, which is equal to 8.8 MeV. Binding energy per nucleon is more for medium nuclei than for heavy nuclei. Hence, medium nuclei are highly stable.



* The heavier nuclei being unstable have tendency to split into medium nuclei. This process is called **Fission**.
* The Lighter nuclei being unstable have tendency to fuse into a medium nucleus. This process is called **Fusion**.

Radioactivity

It was discovered by Henry Becquerel.

Spontaneous emission of radiations (α, β, γ) from unstable nucleus is called **radioactivity**. Substances which shows radioactivity are known as **radioactive substance**.

Radioactivity was studied in detail by Rutherford.

In radioactive decay, an unstable nucleus emits α particle or β particle. After emission of α or β the remaining nucleus may emit γ-particle, and converts into more stable nucleus.

α-Particle

It is a doubly charged helium nucleus. It contains two protons and two neutrons.

Mass of α-particle = Mass of ${}_2He^4$ atom – $2m_e \approx 4\,m_p$

Charge of α-particle = + 2 e

β-Particle

(a) **β^- (electron):**

Mass = m_e ; Charge = $-e$

(b) **β^+ (positron):**

Mass = m_e ; Charge = $+e$

Positron is an antiparticle of electron.

Antiparticle

A particle is called antiparticle of other if on collision both can annihilate (destroy completely) and converts into energy.

For example:

(i) Electron ($-e$, m_e) and positron ($+e$, m_e) are anti particles.

(ii) Neutrino (ν) and antineutrino ($\bar{\nu}$) are antiparticles.

γ-Particle

They are energetic photons of energy of the order of *MeV* and having rest mass zero.

Radioactive Decay (Displacement Law)

α-Decay

Nuclei with mass number greater than 210 undergo α-decay.

$$_ZX^A \rightarrow {}_{Z-2}Y^{A-4} + {}_2He^4 + Q$$

Q Value

It is defined as energy released during the decay process.

Q value = rest mass energy of reactants – rest mass energy of products.

This energy is available in the form of increase in K.E. of the products.

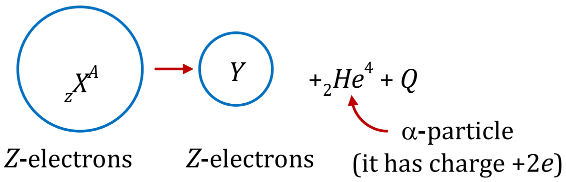
Let, M_x = mass of atom $_ZX^A$

M_y = mass of atom $_{Z-2}Y^{A-4}$

M_{He} = mass of atom $_2He^4$

$$Q \text{ value} = [(M_x - Zm_e) - \{(M_y - (Z-2)\,me) + (M_{He} - 2m_e)\}]c^2$$

$$= [M_x - M_y - M_{He}]\,c^2$$



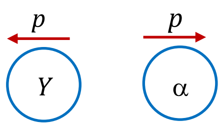
Considering actual number of electrons in α-decay

$$Q \text{ value} = [M_x - (M_y + 2m_e) - (M_{He} - 2m_e)]c^2$$

$$= [M_x - M_y - M_{He}]\,c^2$$

Calculation of Kinetic Energy of Final Products:

As atom X was initially at rest and no external forces are acting, so final momentum also has to be zero. Hence both Y and α-particle will have same momentum in magnitude but in opposite direction.

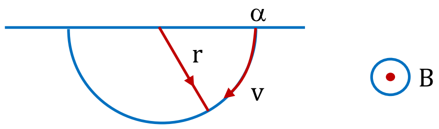


$p_\alpha^2 = p_Y^2$; $2m_\alpha T_\alpha = 2m_Y T_Y$ (Here we are representing T for kinetic energy)

$Q = T_y + T_\alpha$; $m_\alpha T_\alpha = m_Y T_Y$

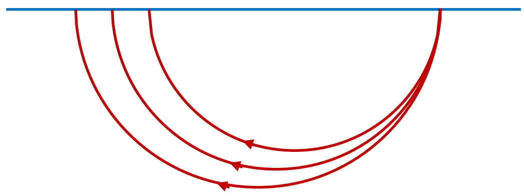
$$T_\alpha = \frac{m_Y}{m_\alpha + m_Y}Q \; ; \; T_Y = \frac{m_\alpha}{m_\alpha + m_Y}Q \quad \Rightarrow \quad T_\alpha = \frac{A-4}{A}\,Q \; ; \; T_Y = \frac{4}{A}\,Q$$

From the above calculation, one can see that all the α-particles emitted should have same kinetic energy. Hence, if they are passed through a region of uniform magnetic field having direction perpendicular to velocity, they should move in a circle of same radius.



$$r = \frac{mv}{qB} = \frac{mv}{2eB} = \frac{\sqrt{2Km}}{2eB}$$

Experimental Observation:



Experimentally it has been observed that all the α-particles do not move in the circle of same radius, but they move in `circles having different radii.

This shows that they have different kinetic energies. But it is also observed that they follow circular paths of some fixed values of radius i.e. yet the energy of emitted α-particles is not same but it is quantized. The reason behind this is that all the daughter nuclei produced are not in their ground state but some of the daughter nuclei may be produced in their excited states and they emits photon to acquire their ground state.

X ⟶ Y* (excited state) + α + Q

↓

Y + photon (γ particle)

The only difference between Y and Y^* is that Y^* is in excited state and Y is in ground state.

Let, the energy of emitted γ-particles be E

$\therefore \quad Q = T_\alpha + T_Y + E$

where $Q = [M_x - M_y - M_{He}]\, c^2$

$T_\alpha + T_Y = Q - E$

$$T_\alpha = \frac{m_Y}{m_\alpha + m_Y}(Q-E) \quad ; \quad T_Y = \frac{m_\alpha}{m_\alpha + m_Y}(Q-E)$$

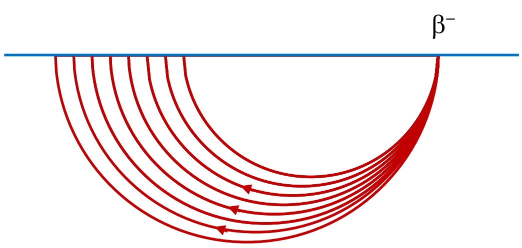
β^- Decay

$${}_zX^A \rightarrow {}_{z+1}Y^A + {}_{-1}e^0 + Q$$

${}_{-1}e^0$ can also be written as ${}_{-1}\beta^0$.

Here also one can see that by momentum and energy conservation, we will get

$$T_e = \frac{m_Y}{m_e + m_Y}Q\ ; \ T_Y = \frac{m_e}{m_e + m_Y}Q$$



as $m_e << m_Y$, we can consider that all the energy is taken away by the electron.

From the above results, we will find that all the β-particles emitted will have same energy and hence they have same radius if passed through a region of perpendicular magnetic field. But, experimental observations were completely different.

On passing through a region of uniform magnetic field perpendicular to the velocity, it was observed that β-particles take circular paths of different radius having a continuous spectrum.
To explain this, Pauling has introduced the extra particles called neutrino and antineutrino (antiparticle of neutrino).

$\bar{\nu} \rightarrow$ antineutrino, $\nu \rightarrow$ neutrino

Properties of Antineutrino ($\bar{\nu}$) and Neutrino (ν)

(1) They have rest mass equal to zero or, at most, the mass equivalent of a few electron-volts.
speed = c (or nearly equal to c)
Energy, $E = mc^2$

(2) They are charge less (neutral)

(3) They have spin quantum number, $s = \pm \frac{1}{2}$

Considering the emission of antineutrino, the equation of β^--decay can be written as

$${}_zX^A \rightarrow {}_{z+1}Y^A + {}_{-1}e^0 + Q$$

(4) They are not electromagnetic in nature as is the photon, the neutrino can pass unimpeded through vast amounts of matter.

Production of antineutrino along with the electron helps to explain the continuous spectrum because the energy is distributed randomly between electron and $\bar{\nu}$. It also helps to explain the spin quantum number balance (p, n and $\pm e$ each has spin quantum number $\pm 1/2$).
During β^--decay, inside the nucleus a neutron is converted to a proton with emission of an electron and antineutrino.

$$n \rightarrow p + {}_{-1}e^0 + \bar{\nu}$$

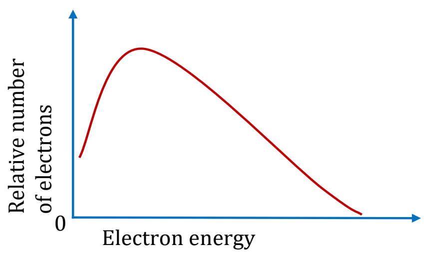
Let, M_x = mass of atom ${}_ZX^A$
M_y = mass of atom ${}_{Z+1}Y^A$
m_e = mass of electron
Q value = $[(M_X - Zm_e) - \{(M_Y - (z+1)\,m_e) + m_e\}]\,c^2 = [M_X - M_Y]\,c^2$

Considering actual number of electrons.

Q value = $[M_X - \{(M_Y - m_e) + m_e\}]\,c^2 = [M_X - M_Y]\,c^2$

Energy Spectrum of β-Particles:

The figure below shows the energy spectrum of the electrons emitted in the beta decay.



β⁺ Decay

$${}_ZX^A \rightarrow {}_{Z-1}Y^A + {}_{+1}e^0 + \nu + Q$$

In β^+ decay, inside a nucleus a proton is converted into a neutron, positron and neutrino.

$$p \rightarrow n + {}_{+1}e^0 + \nu$$

As mass increases during conversion of proton to a neutron, hence it requires energy for β^+ decay to take place,

$\therefore$ β^+ decay is rare process. It can take place in the nucleus where a proton can take energy from the nucleus itself.

$$Q \text{ value} = [(M_X - Zm_e) - \{(M_Y - (Z-1)\,m_e) + m_e\}]\,c^2 = [M_X - M_Y - 2m_e]\,c^2$$

Considering actual number of electrons.

$$Q \text{ value} = [M_X - \{(M_Y + m_e) + m_e\}]\,c^2 = [M_X - M_Y - 2m_e]\,c^2$$

Illustration 53:

Calculate the Q-value in the following decays:

(a) ${}^{19}O \rightarrow {}^{19}F + e^- + \bar{\nu}$ (b) ${}^{25}Al \rightarrow {}^{25}Mg + e^+ + \nu$

The atomic masses needed are as follows:

${}^{19}O$	${}^{19}F$	${}^{25}Al$	${}^{25}Mg$
19.003576 u	18.998403 u	24.990432 u	24.985839 u

Solution:

(a) The Q-value of β^--decay is

$$Q = [m({}^{19}O) - m({}^{19}F)]c^2$$

$$= [19.003576\,u - 18.998403\,u]\,(931\,MeV/u) = 4.816\,MeV$$

(b) The Q-value of β^+ -decay is

$$Q = [m({}^{25}Al) - m({}^{25}Mg) - 2m_e]c^2$$

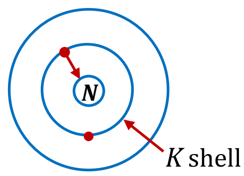
$$= \left[24.99032u - 24.985839u - 2\times 0.511\frac{MeV}{c^2}\right]c^2$$

$$= (0.004593\,u)\,(931\,MeV/u) - 1.022\,MeV$$

$$= 4.276\,MeV - 1.022\,MeV = 3.254\,MeV.$$

K Capture

It is a rare process which is found only in few nucleus. In this process the nucleus captures one of the atomic electrons from the K shell. A proton in the nucleus combines with this electron and converts itself into a neutron. A neutrino is also emitted in the process and is emitted from the nucleus.



Electron capture is competitive with positron emission. It occurs more often than positron emission in heavy nuclides because electrons are relatively closer to nucleus which allows more interaction.

$$p + {}_{-1}e^0 \rightarrow n + \nu$$

If X and Y are atoms then reaction is written as:

$${}_ZX^A \rightarrow {}_{Z-1}Y^A + \nu + Q + \text{characteristic } x\text{-rays of } Y.$$

If X and Y are taken as nucleus, then reaction is written as:

$${}_ZX^A + {}_{-1}e^0 \rightarrow {}_{Z-1}Y^A + \nu$$

γ Decay

Like an atom a nucleus can also exist in states whose energies are higher than that of its ground state. Excited nuclei return to their ground states by emitting photons whose energies correspond to the energy differences between the various initial and final states in the transitions involved. The photons emitted by nuclei have energy up to several *MeV*, and are traditionally called gamma rays.

$${}_{12}Mg^{27} \to {}_{13}^{\ *}Al^{27} + {}_{-1}e^{0}$$

$${}_{13}^{\ *}Al^{27} \to {}_{13}Al^{27} + \gamma$$

Al^* represents aluminium nucleus in its excited state.

When γ-rays are passed through a slab their intensity decreases exponentially with slab thickness x. $I = I_0e^{-\mu x}$ where μ is absorption coefficient. It depends on the slab.

Note:

(1) Nuclei having atomic numbers from $Z = 84$ to 112 shows radioactivity.

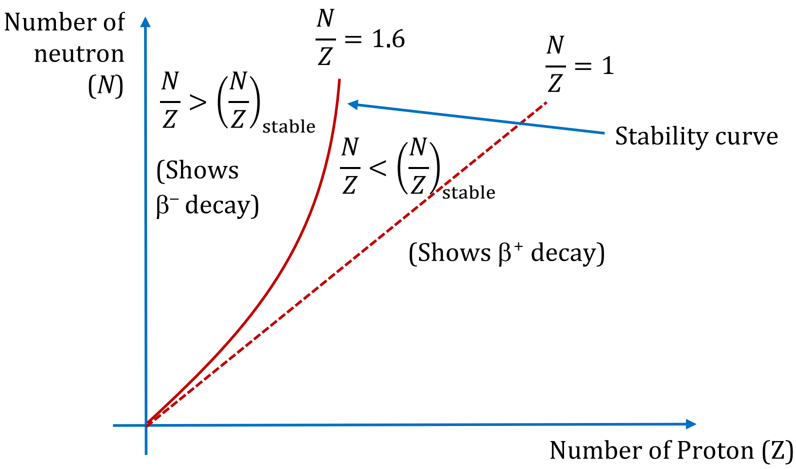
(2) Nuclei having $Z = 1$ to 83 are stable (only few exceptions are there)

(3) Whenever a neutron is produced, a neutrino is also produced.

(4) Whenever a neutron is converted into a proton, a antineutrino is produced.

Nuclear Stability

Figure shows a plot of neutron number N versus proton number Z for the nuclides found in nature. The solid line in the figure represents the stable nuclides. For light stable nuclides, the neutron number is equal to the proton number so that ratio N/Z is equal to 1. The ratio N/Z increases for the heavier nuclides and becomes about 1.6 for the heaviest stable nuclides. The points (Z, N) for stable nuclides fall in a rather well-defined narrow region.



There are nuclides to the left of the stability belt as well as to the right of it. The nuclides to the left of the stability region have excess neutrons, whereas, those to the right of the stability belt have excess protons. These nuclides are unstable and decay with time according to the laws of radioactive disintegration. Nuclides with excess neutrons (lying above stability belt) show β^- decay while nuclides with excess protons (lying below stability belt) show β^+ decay and K-capture.

Nuclear Force:

(i) Nuclear forces are basically attractive and are responsible for keeping the nucleons bound in a nucleus in spite of repulsion between the positively charge protons.

(ii) It is strongest force with in nuclear dimensions (F_n; 100 F_e)

(iii) It is short range force (acts only inside the nucleus)

(iv) It acts only between neutron-neutron, neutron-proton and proton-proton i.e. between nucleons.

(v) It does not depend on the nature of nucleons.

(vi) An important property of nuclear force is that it is not a central force. The force between a pair of nucleons is not solely determined by the distance between the nucleons. For example, the nuclear force depends on the directions of the spins of the nucleons. The force is stronger if the spins of the nucleons are parallel (i.e., both nucleons have m_s = + 1/2 or – 1/2) and is weaker if the spins are antiparallel (i.e., one nucleon has m_s = + 1/2 and the other has m_s = – 1/2). Here m_s is spin quantum number.

Radioactive decay: Statistical Law:

(Given by Rutherford and Soddy)

Rate of radioactive decay $\propto N$

$\Rightarrow \quad \frac{dN}{dt} \propto N$

$$\frac{dN}{dt} = -\lambda N$$

Where N = number of active nuclei

Where λ = decay constant of the radioactive substance.

Decay constant is different for different radioactive substances, but it does not depend on amount of substance and time.

SI unit of λ *is* s^{-1}

If $\lambda_1 > \lambda_2$ then first substance is more radioactive (less stable) than the second one.

For the case, if A decays to B with decay constant λ

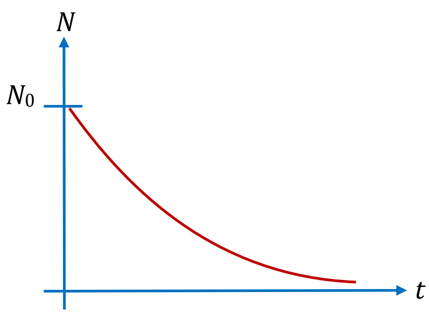
$A \xrightarrow{\lambda} B$

$t = 0$	N_0	0	where N_0 = number of active nuclei of A at $t = 0$
$t = t$	N	N'	where N = number of active nuclei of A at $t = t$

Rate of radioactive decay of $A = -\frac{dN}{dt} = \lambda N$

$$-\int_{N_0}^{N} \frac{dN}{N} = \int_0^t \lambda dt$$

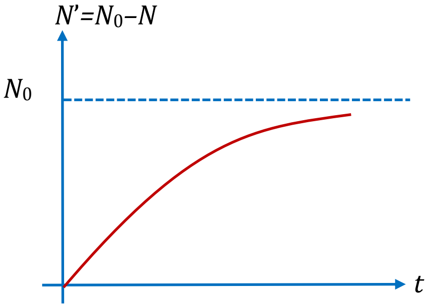
$\Rightarrow \quad \boxed{N = N_0 e^{-\lambda} t}$ (It is exponential decay)



Number of nuclei decayed (i.e. the number of nuclei of B formed)

$N' = N_0 - N = N_0 - N_0e^{-\lambda t}$

$N' = N_0(1 - e^{-\lambda t})$



Half Life ($T_{1/2}$)

It is the time in which number of active nuclei becomes half.

$N = N_0\, e^{-\lambda t}$

After one half life,

$$N = \frac{N_0}{2}$$

$$\frac{N_0}{2} = N_0\, e^{-\lambda t} \quad \Rightarrow t = \frac{\ln 2}{\lambda} \Rightarrow \frac{0.693}{\lambda} = t_{1/2}$$

$$\boxed{t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}}$$ **(to be remembered)**

Number of nuclei present after n half lives i.e. after a time $t = n\, t_{1/2}$

$$N = N_0e^{-\lambda t} = N_0\, e^{-\lambda n t1/2} = N_0\; e^{-\lambda n \frac{\ln 2}{\lambda}}$$

$$= N_0\, e^{\ln 2^{(-n)}} = N_0\,(2)^{-n} = N_0(1/2)^{n} = \frac{N_0}{2^n}$$

$\{n = \dfrac{t}{t_{1/2}}$. It may be a fraction, need not to be an integer$\}$

or $$N_0 \xrightarrow[\text{half life}]{\text{after 1st}} \frac{N_0}{2} \xrightarrow{2} N_0\left(\frac{1}{2}\right)^2 \xrightarrow{3} N_0\left(\frac{1}{2}\right)^3 \ldots\ldots\ldots\ldots \xrightarrow{n} N_0\left(\frac{1}{2}\right)^n$$

Illustration 54:

A radioactive sample has 6.0×10^{18} active nuclei at a certain instant. How many of these nuclei will still be in the same active state after two half-lives?

Solution:

In one half-life the number of active nuclei reduces to half the original number. Thus, in two half lives the number is reduced to $\left(\frac{1}{2}\right)\times\left(\frac{1}{2}\right)$ of the original number. The number of remaining active nuclei is, therefore,

$6.0 \times 10^{18} \times \left(\frac{1}{2}\right)\times\left(\frac{1}{2}\right) = 1.5 \times 10^{18}$.

Illustration 55:

The number of ^{238}U atoms in an ancient rock equals the number of ^{206}Pb atoms. The half-life of decay of ^{238}U is $4.5 \times 10^9\ y$. Estimate the age of the rock assuming that all the ^{206}Pb atoms are formed from the decay of ^{238}U.

Solution:

Since the number of ^{206}Pb atoms equals the number of ^{238}U atoms, half of the original ^{238}U atoms have decayed. It takes one half-life to decay half of the active nuclei. Thus, the sample is $4.5 \times 10^9\ y$ old.

Activity:

Activity is defined as rate of radioactive decay of nuclei.

It is denoted by A or R

$$A = \lambda N$$

If a radioactive substance changes only due to decay then

$$A = -\frac{dN}{dt}$$

As in that case,

$$N = N_0\, e^{-\lambda t}$$

$$A = \lambda N = \lambda N_0 e^{-\lambda t}$$

$$A = A_0\, e^{-\lambda t}$$

SI Unit of activity: becquerel (Bq) which is same as 1 dps (disintegration per second)

The popular unit of activity is curie which is defined as

1 curie = 3.7×10^{10} dps (which is activity of 1 gm Radium)

Illustration 56:

The decay constant for the radioactive nuclide ^{64}Cu *is* $1.516 \times 10^{-5}\ s^{-1}$. Find the activity of a sample containing 1 μg *of* ^{64}Cu. Atomic weight of copper = 63.5 $g/mole$. Neglect the mass difference between the given radioisotope and normal copper.

Solution:

63.5 g of copper has 6×10^{23} atoms. Thus, the number of atoms in 1 μg *of Cu is*

$$N = \frac{6 \times 10^{23} \times 1\mu g}{63.5 g} = 9.45 \times 10^{15}$$

The activity = λN

$$= (1.516 \times 10^{-5}\ s^{-1}) \times (9.45 \times 10^{15}) = 1.43 \times 10^{11} \text{ disintegrations/s}$$

$$= \frac{1.43 \times 10^{11}}{3.7 \times 10^{10}}\ Ci = 3.86\ Ci.$$

Illustration 57:

The half-life of a radioactive nuclide is 20 hours. What fraction of original activity will remain after 40 hours?

Solution:

40 hours means 2 half lives.

Thus, $A = \dfrac{A_0}{2^2} = \dfrac{A_0}{4}$ or $\dfrac{A}{A_0} = \dfrac{1}{4}$

So, one fourth of the original activity will remain after 40 hours.

Specific Activity:
The activity per unit mass is called specific activity.

Average Life:

$$T_{avg} = \frac{\text{Sum of ages of all the nuclei}}{N_0} = \frac{\int_0^{\infty} \lambda N_0 e^{-\lambda t} dt.t}{N_0} = \frac{1}{\lambda}$$

Illustration 58:
The half-life of ^{198}Au is 2.7 days.
Calculate (a) the decay constant, (b) the average-life and (c) the activity of 1.00 mg of ^{198}Au. Take atomic weight of ^{198}Au to be 198 g/mol.

Solution:

(a) The half-life and the decay constant are related as

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$$

or $$\lambda = \frac{0.693}{t_{1/2}} = \frac{0.693}{2.7 days} = \frac{0.693}{2.7 \times 24 \times 3600 s} = 2.9 \times 10^{-6}\ s^{-1}.$$

(b) The average-life is $t_{av} = \frac{1}{\lambda} = 3.9$ days.

(c) The activity is $A = \lambda N$. Now, 198 g of ^{198}Au has 6×10^{23} atoms.
The number of atoms in 1.00 mg of ^{198}Au is

$$N = 6 \times 10^{23} \times \frac{1.0\ mg}{198\ g} = 3.03 \times 10^{18}$$

Thus, $A = \lambda N = (2.9 \times 10^{-6}\ s^{-1})(3.03 \times 10^{18})$

$$= 8.8 \times 10^{12} \text{ disintegrations}/s = \frac{8.8 \times 10^{12}}{3.7 \times 10^{10}} Ci = 240\ Ci$$

Illustration 59:
Suppose, the daughter nucleus in a nuclear decay is itself radioactive. Let λ_p and λ_d be the decay constants of the parent and the daughter nuclei. Also, let N_p and N_d be the number of parent and daughter nuclei at time t. Find the condition for which the number of daughter nuclei becomes constant.

Solution:

The number of parent nuclei decaying in a short time interval t to $t + dt$ is $\lambda_p N_p dt$. This is also the number of daughter nuclei decaying during the same time interval is $\lambda_d N_d dt$. The number of the daughter nuclei will be constant if

$$\lambda_p N_p dt = \lambda_d N_d dt$$

or $$\lambda_p N_p = \lambda_d N_d$$

Illustration 60:
A radioactive nucleus can decay by two different processes. The half-life for the first process is t_1 and that for the second process is t_2. Show that the effective half-life t of the nucleus is given by $\frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2}$.

Solution:

The decay constant for the first process is $\lambda_1 = \frac{\ln 2}{t_1}$ and for the second process it is $\lambda_2 = \frac{\ln 2}{t_1}$. The probability that an active nucleus decays by the first process in a time interval dt is $\lambda_1 dt$. Similarly, the probability that it decays by the second process is $\lambda_2 dt$. The probability that it either decays by the first process or by the second process is $\lambda_1 dt + \lambda_2 dt$. If the effective decay constant is λ, this probability is also equal to λdt. Thus.

$$\lambda dt = \lambda_1 dt + \lambda_2 dt$$

or $$\lambda = \lambda_1 + \lambda_2$$

or $$\frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2}$$

Illustration 61:

A factory produces a radioactive substance A at a constant rate R which decays with a decay constant λ to form a stable substance. Find

(i) the number of nuclei of A and

(ii) Number of nuclei of B, at any time t assuming the production of A starts at $t = 0$.

(iii) Also find out the maximum number of nuclei of 'A' present at any time during its formation.

Solution:

(i) Factory $\xrightarrow[\text{const. rate}]{R} A \xrightarrow[\text{decay}]{\lambda} B$

Let N be the number of nuclei of A at any time t

$$\therefore \quad \frac{dN}{dt} = R - \lambda N \ ; \ \int_0^N \frac{dN}{R - \lambda N} = \int_0^t dt$$

On solving we will get $N = R/\lambda(1 - e^{-\lambda t})$

(ii) Number of nuclei of B at any time t, $N_B = R\,t - N_A = Rt - R/\lambda(1 - e^{-\lambda t}) = R/\lambda\,(\lambda\,t - 1 + e^{-\lambda t})$.

(iii) Maximum number of nuclei of 'A' present at any time during its formation $= R/\lambda$.

Alternate Solution of (b) Part without use of Answer of Part (a)

If $\lambda_1 > \lambda_2$ that means A will decay very fast to 'B' and B will then decay slowly. We can say that practically N_1 vanishes in very short time and B has initial number of atoms as N_0.

Now $N_2 = N_0\,e^{-\lambda_2 t}$ and $N_1 = N_0\,e^{-\lambda_1 t}$

If $\lambda_1 << \lambda_2$ then B is highly unstable and it will soon decay into C.

So, it's rate of formation $\approx$ its rate of decay.

$$\therefore \quad \lambda_1 N_1 \approx \lambda_2 N_2 \quad \Rightarrow N_2 = \frac{\lambda_1 N_1}{\lambda_2} = \frac{\lambda_1 N_0}{\lambda_2}\,(e^{-\lambda_1 t})$$

Nuclear Fission

In nuclear fission heavy nuclei of A, above 200, break up into two or more fragments of comparable masses. The most attractive bid, from a practical point of view, to achieve energy from nuclear fission is to use $_{92}U^{236}$ as the fission material. The technique is to hit a uranium sample by slow-moving neutrons (kinetic energy ≈ 0.04 eV, also called thermal neutrons). A $_{92}U^{235}$ nucleus has large probability of absorbing a slow neutron and forming $_{92}U^{236}$ nucleus. This nucleus then fissions into two or more parts. A variety of combinations of the middle-weight nuclei may be formed due to the fission. For example, one may have

$$_{92}U^{235} + {_0n^1} \rightarrow {_{92}U^{236}} \rightarrow X + Y + 2\,{_0n^1} \quad \text{or} \quad {_{92}U^{235}} + {_0n^1} \rightarrow {_{92}U^{236}} \rightarrow X' + Y' + 3\,{_0n^1}$$

and a number of other combinations.

* On an average 2.5 neutrons are emitted in each fission event.
* Mass lost per reaction $\approx$ 0.2 a.m.u.
* In nuclear fission the total $B.E.$ increases and excess energy is released.

* In each fission event, about 200 MeV of energy is released a large part of which appears in the form of kinetic energies of the two fragments. Neutrons take away about 5 MeV.

  e.g. ${}^{235}_{92}U + {}_{o}n^{1} \rightarrow {}^{236}_{92}U \rightarrow {}^{141}_{56}Ba + {}^{92}_{36}Kr + 3\,{}_{o}n^{1}$ + energy

  Q value = $[(M_U - 92m_e + m_n) - \{(M_{Ba} - 56\,m_e) + (M_{Kr} - 36m_e) + 3m_n\}]c^2$

  $= [(M_U + m_n) - (M_{Ba} + M_{Kr} + 3m_n)]c^2$

* A very important and interesting feature of neutron-induced fission is the chain reaction.

Nuclear Reactor:

Nuclear reactors utilize energy released in nuclear fission reaction to produce power. Some nuclear reactor are research reactors. Their primary aim is to provide a facility for research on different aspects of nuclear science and technology. Some reactors are used to produce power.

Important Components of a Nuclear Reactor:

(i) **Moderators :** The average energy of neutrons liberated in fission of a ${}_{92}U^{235}$ is 2 MeV. These neutrons unless slowed down will escape from the reactor without interacting with uranium nuclei. Fast neutrons need to be slowed down for them to be able to get absorbed by Uranium.

When neutrons are made to strike a light nuclei like that of a hydrogen it loses almost all of it $K.E.$ In reactor light nuclei called moderators are used. Commonly used moderators are water, heavy water (D_2O) and graphite 'Apsara' reactor in BARC user H_2O.

RAPP uses D_2O as moderator.

(ii) **Multiplication Factor (K) :** The ratio of number of fissions produced by given generation of neutrons to the number of fissions of the preceding generation

If $K = 1$, the operation of reactor is said to be critical.

For steady generation of power K must be equal to 1.

If $K > 1$, the reaction rate and reactor power increase exponentially if K is not brought down the reactor will become super critical and may explode.

(iii) **Control Rods :** The reaction rate is controlled through control-rods made out of neutron absorbing material such as cadmium.

(iv) **Safety Rods :** These rods are provided in reactors in addition to control rods These, when required, can be inserted into the reactor and K can be reduced.

Nuclear Fusion (Thermo Nuclear Reaction)

(a) Some unstable light nuclei of A below 20, fuse together, the $B.E.$ per nucleon increases and hence the excess energy is released. The easiest thermonuclear reaction that can be handled on earth is the fusion of two deuterons (D–D reaction) or fusion of a deuteron with a triton (D–T reaction).

${}_1H^2 + {}_1H^2 \rightarrow {}_2He^3 + {}_0n^1 + 3.3\ MeV\ (D\text{–}D)$

Q value = $[2(M_D - m_e) - \{(M_{He3} - 2m_e) + m_n\}]c^2 = [2M_D - (M_{He3} + m_n)]c^2$

${}_1H^2 + {}_1H^2 \rightarrow {}_1H^3 + {}_1H^1 + 4.0\ MeV\ (D\text{–}D)$

$Q\ value = [2(M_D - m_e) - \{(M_T - m_e) + (M_H - m_e)\}]c^2$

$= [2M_D - (M_T + M_H)]c^2$

${}_1H^2 + {}_1H^3 \rightarrow {}_2He^4 + n + 17.6\ MeV\ (D - T)$

Q value = $[\{(M_D - m_e) + (M_T - m_e)\} - \{(M_{He4} - 2m_e) + m_n\}]c^2$

$= [(M_D + M_T) - (M_{He4} + m_n)]c^2$

Note:

In case of fission and fusion, $\Delta m = \Delta m_{\text{atom}} = \Delta m_{\text{nucleus}}$

(b) These reactions take place at ultra high temperature ($\cong 10^7$ to 10^9). At high pressure it can take place at low temperature also. For these reactions to take place nuclei should be brought upto 1 fermi distance which requires very high kinetic energy.

(c) Energy released in fusion exceeds the energy liberated in the fission of heavy nuclei.

Fusion Reactions in Sun:

The fusion reaction in sun is multi-step process which involves conversion of hydrogen in helium.

The below set of reactions is called as $p-p$ cycle of nuclear fusion in stars

$$_1H^1 + {}_1H^1 \rightarrow {}_1H^2 + {}_{+1}e^0 + \nu + 0.42\ MeV \quad ...(i)$$

$$_{+1}e^0 + {}_{-1}e^0 \rightarrow \gamma + \gamma + 1.02\ MeV \quad ...(ii)$$

$$_1H^2 + {}_1H^1 \rightarrow {}_2He^3 + \gamma + 5.49\ MeV \quad ...(iii)$$

$$_2He^3 + {}_2He^3 \rightarrow {}_2He^4 + 2{}_1H^1 + 12.86\ MeV \quad ...(iv)$$

2[(i) + (ii) + (iii)] + (iv)

$$6{}_1H^1 + 2{}_{-1}e^0 \rightarrow {}_2He^4 + 2{}_1H^1 + 6\gamma + 2\nu$$

$$4{}_1H^1 + 2{}_{-1}e^0 \rightarrow {}_2He^4 + 2\nu + 6\gamma + 26.7\ MeV$$

Illustration 62:

Calculate the energy released when three alpha particles combine to form a ^{12}C nucleus. The atomic mass of ^4_2He is 4.002603 u.

Solution:

The mass of a ^{12}C atom is exactly 12 u.

The energy released in the reaction $3\left({}^4_2He\right) \rightarrow {}^{12}_6C$ is

$$[3\ m({}^4_2He) - m({}^{12}_6C)]\ c^2 = [3 \times 4.002603\ u - 12\ u]\ (931\ MeV/u) = 7.27\ MeV.$$

Illustration 63:

Consider two deuterons moving towards each other with equal speeds in a deuteron gas. What should be their kinetic energies (when they are widely separated) so that the closest separation between them becomes 2fm? Assume that the nuclear force is not effective for separations greater than 2 fm. At what temperature will the deuterons have this kinetic energy on an average?

Solution:

As the deuterons move, the Coulomb repulsion will slow them down. The loss in kinetic energy will be equal to the gain in Coulomb potential energy. At the closest separation, the kinetic energy is zero and the potential energy is $\frac{e^2}{4\pi\varepsilon_0 r}$. If the initial kinetic energy of each deuteron is K and the closest separation is 2fm, we shall have

$$2K = \frac{e^2}{4\pi\varepsilon_0(2\ fm)} = \frac{(1.6\times10^{-19}C)^2\times(9\times10^9 N-m^2/C^2)}{2\times10^{-15}m}$$

or, $K = 5.7 \times 10^{-14}\ J$

If the temperature of the gas is T, the average kinetic energy of random motion of each nucleus will be 1.5 kT. The temperature needed for the deuterons to have the average kinetic energy of $5.7 \times 10^{-14}\ J$ will be given by

$$1.5\ kT = 5.7 \times 10^{-14}\ J$$

where k = Boltzmann constant

or, $T = \frac{5.7\times10^{-14}J}{1.5\times1.38\times10^{-23}J/K} = 2.8 \times 10^9\ K.$