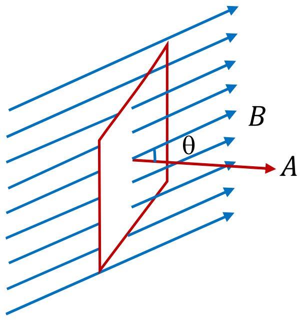
02 EMI and AC

ELECTROMAGNETIC INDUCTION

Magnetic Flux

Faraday's great insight lay in discovering a simple mathematical relation to explain the series of experiments he carried out on electromagnetic induction. However, before we state and appreciate his laws, we must get familiar with the notion of magnetic flux, Φ_B. Magnetic flux is defined in the same way as electric flux is defined in the chapter Electrostatics.



A plane of surface area ***A*** placed in a uniform magnetic field ***B***.

Magnetic flux through a plane of area A placed in a uniform magnetic field B can be written as $\phi_B = B.A = BA\cos\theta$

where θ is angle between B and A. The notion of the area as a vector has been discussed earlier in electrostatics. Above Equation can be extended to curved surfaces and nonuniform fields.

$$\phi = \int \vec{B}.d\vec{A}$$

Faraday's laws of electromagnetic induction

Faraday stated experimental observations in the form of laws called Faraday's laws of electromagnetic induction. The law is stated below.

The magnitude of the induced emf in a circuit is equal to the time rate of change of magnetic flux through the circuit.

Mathematically, the induced emf is given by

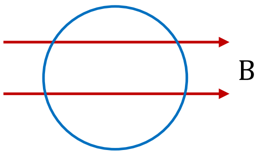
$$E = -\frac{d\phi}{dt}$$

(–) sign indicates that the emf will be induced in such a way that it will oppose the change of flux (Len'z law).

SI unit of magnetic flux = Weber.

Illustration 1:

A coil is placed in a constant magnetic field. The magnetic field is parallel to the plane of the coil as shown in figure. Find the emf induced in the coil.



Solution:

ϕ = 0 (always) since area is perpendicular to magnetic field.

∴ emf = 0

Illustration 2:

Find the emf induced in the coil shown in figure. The magnetic field is perpendicular to the plane of the coil and is constant.

Solution:

$\phi = BA$ (always)

$= \text{const.}$

$\therefore$ emf = 0

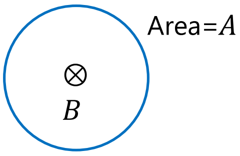
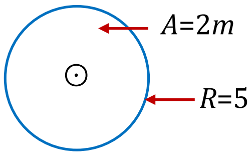


Illustration 3:

Figure shows a coil placed in decreasing magnetic field applied perpendicular to the plane of coil. The magnetic field is decreasing at a rate of $10T/s$. Find out current in magnitude and direction.



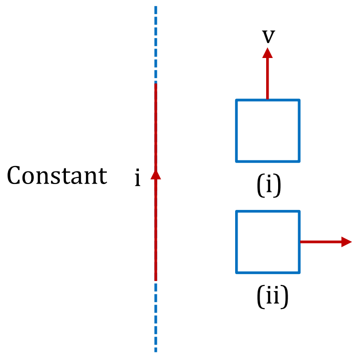
Solution:

$$\phi = B.A \qquad \Rightarrow \quad emf = A \cdot \frac{dB}{dt} = 2 \times 10 = 20\ V$$

$\therefore$ $i = 20/5 = 4$ amp. From Lenz's law direction of current will be anticlockwise.

Illustration 4:

Figure shows a long current carrying wire and two rectangular loops moving with velocity v. Find the direction of current in each loop.

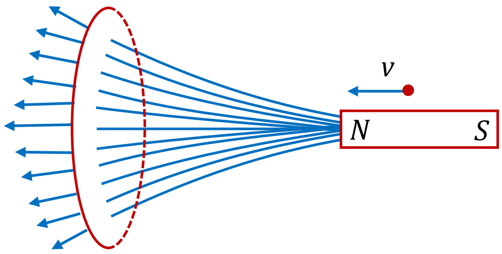


Solution:

In loop (i) no emf will be induced because there is no flux change.

In loop (ii) emf will be induced because the coil is moving in a region of decreasing magnetic field which is inward in direction. Therefore, to oppose the flux decrease in inward direction, current will be induced such that its magnetic field will be inwards. For this direction of current should be clockwise.

Lenz's Law (Conservation of Energy Principle)



According to this law, emf will be induced in such a way that it will oppose the cause which has produced it. Figure shows a magnet approaching a ring with its north pole towards the ring.

We know that magnetic field lines come out of the north pole and magnetic field intensity decreases as we move away from magnet. So, the magnetic flux (here towards left) will increase with the approach of magnet. This is the cause of flux change. To oppose it, induced magnetic field will be towards right. For this the current must be anticlockwise as seen by the magnet.

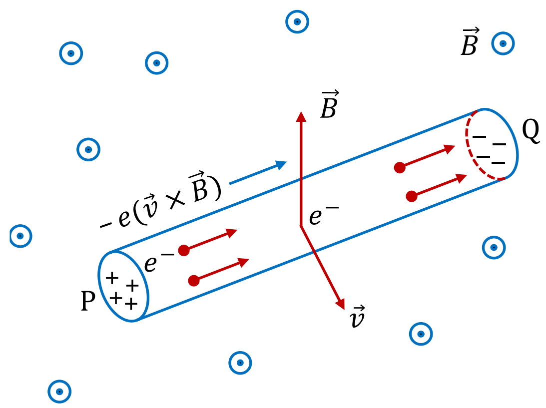
If we consider the approach of North pole to be the cause of flux change, the Lenz's law suggests that the side of the coil towards the magnet will behave as North pole and will repel the magnet. We know that a current carrying coil will behave like North pole if it flows anticlockwise. Thus, as seen by the magnet, the current will be anticlockwise.

If we consider the approach of magnet as the cause of the flux change, Lenz's law suggests that a force opposite to the motion of magnet will act on the magnet, whatever be the mechanism.

Lenz's law tells that if the coil is set free, it will move away from the magnet, because in doing so it will oppose the 'approach' of magnet. If the magnet is given some initial velocity towards the coil and is released, it will slow down. It can be explained as following.

The current induced in the coil will produce heat. From the energy conservation, if heat is produced there must be an equal decrease of energy in some other form, here it is the kinetic energy of the moving magnet. Thus, the magnet must slow down. So, we can justify that the **Lenz's law is conservation of energy principle**.

Motional emf from Lorentz force



A conductor PQ is placed in a uniform magnetic field B, directed outwards, perpendicular to the plane of the paper. PQ is moved with a velocity v, the free electrons of PQ also move with the same velocity. The electrons experience a magnetic Lorentz force,

$$F = e\vec{v}\times\vec{B}$$

According to right hand rule, this force acts in the direction PQ and hence the free electrons will move towards Q. Negative charge accumulates at Q and positive charge at P. An electric field E is setup in the conductor from P to Q. Force exerted by electric field on the free electrons is,

$$\vec{F}_e = e\vec{E}$$

The accumulation of charges at the two ends continues till these two forces balance each other.

So $\quad \vec{F}_m = -\vec{F}_e \Rightarrow e\left(\vec{v}\times\vec{B}\right) = -e\vec{E} \Rightarrow \vec{E} = -\left(\vec{v}\times\vec{B}\right)$

The potential difference between the ends P and Q is

$$V = \vec{E}.\vec{l} = \left(\vec{v}\times\vec{B}\right).\vec{l}$$

It is the magnetic force on the moving free electrons that maintains the potential difference and produces the emf $E = B\ell v$.

For $\quad \vec{B}\perp\vec{v}\perp\vec{\ell}$

As this emf is produced due to the motion of a conductor, it is called a motional emf. The concept of motional emf for a conductor can be generalized for any shape moving in any magnetic field uniform or not. For an element $d\vec{l}$ of conductor the contribution due to the emf of the magnitude

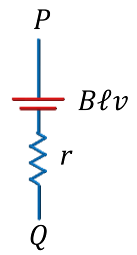
$$de = \left(\vec{v}\times\vec{B}\right).d\vec{l}$$

Therefore total EMF across any arbitrary shape wire is given by

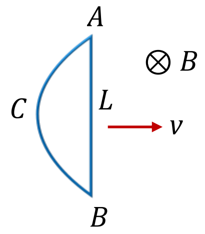
$$E = \int\left(\vec{v}\times\vec{B}\right).d\vec{l}$$

Motional emf across an arbitrary shape wire moving translationally in uniform magnetic field:

From previous section it can be said that a moving rod is electrically equivalent to the following diagram.

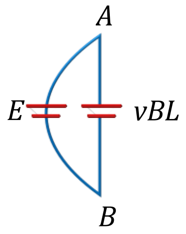


Now consider a situation as shown by the Figure in which a closed coil *ABCA* is moving in a uniform magnetic field ***B*** with velocity ***v*****.** The flux passing through the coil is constant and therefore the induced emf is zero.



Now consider rod *AB*, which is a part of the coil. Emf induced in the rod = $B\,L\,v$

Suppose the emf induced in part *ACB* is *E*, as shown.



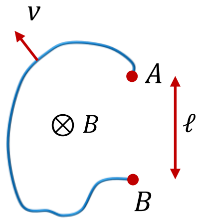
Since the emf in the coil is zero, Emf (in *ACB*) + Emf (in *BA*) = 0

or $\quad -E + vBL = 0 \quad$ or $\quad E = vBL$

Thus, emf induced in any path joining *A* and *B* is same, provided the magnetic field is uniform. Also, the equivalent emf between *A* and *B* is *BLv* (here the two emf's are in parallel)

Illustration 5:

Figure shows an irregular shaped wire AB moving with velocity v, as shown.



Find the emf induced in the wire.

Solution:

The same emf will be induced in the straight imaginary wire joining A and B, which is $Bv\ell \sin\theta$

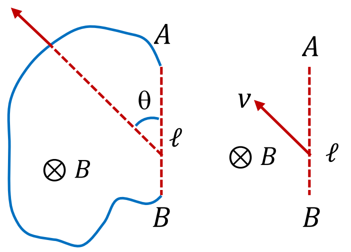
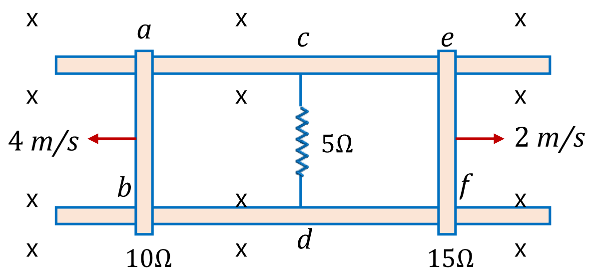


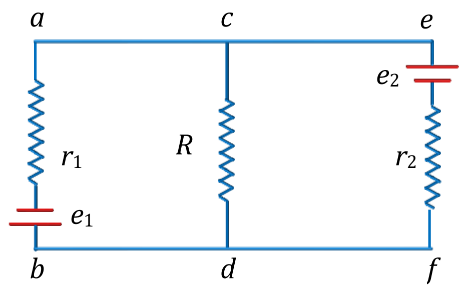
Illustration 6:

Two parallel rails with negligible resistances are 10 cm apart. They are connected by a 5 Ω resistor. The circuit also contains two metal rods having resistance of 10 Ω and 15 Ω along the rails (fig). The rods are pulled away from the resistor at constant speeds 4 m/s and 2 m/s respectively. A uniform magnetic field of magnitude 0.01 T is applied perpendicular to the plane of the rails. Determine the current in the 5 Ω resistor.



Solution:

As two conductors are moving in uniform magnetic field, motional emf will be induced across them. The rod ab will act as a source of emf $e_1 = Bv\ell = (0.01)(4)(0.1) = 4\times10^{-3}V$



And it's internal resistance r_1 = 10 Ω

Similarly, rod ef will also act as source of emf,

$$e_2 = (0.01)(2)(0.1) = 2\times10^{-3}V$$

with internal resistance r_2 = 15 Ω

From right hand rule: $V_b > V_a$ and $V_e > V_f$

Also R = 5 Ω,

$$E_{eq} = \frac{e_1r_2 - e_2r_1}{r_1 + r_2} = \frac{60\times10^{-3} - 20\times10^{-3}}{15+10} = \frac{40}{25}\times10^{-3} = 1.6\times10^{-3}\,volt$$

$$r_{eq} = \frac{15\times10}{15+10} = 6\Omega$$

and $$I = \frac{E_{eq}}{r_{eq}+R} = \frac{1.6\times10^{-3}}{6+5} = \frac{1.6}{11}\times10^{-3} = \frac{8}{55}\times10^{-3}\,Amp$$ from d to c

Illustration 7:

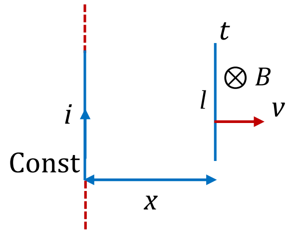
A rod of length *l* is kept parallel to a long wire carrying constant current *i*. It is moving away from the wire with a velocity *v*. Find the emf induced in the wire when its distance from the long wire is *x*.

Solution:

$$E = Blv = \frac{\mu_0 ilv}{2\pi x}$$

OR

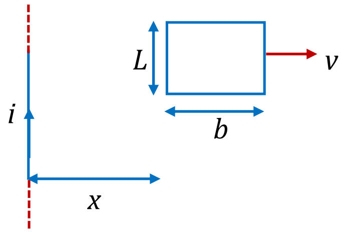
Emf is equal to the rate with which magnetic field lines are cut. In dt time the area swept by the rod is $l\,v\,dt$. The magnetic field lines cut in dt time = $Blvdt = \frac{\mu_0 ilvdt}{2\pi x}$.



$\therefore$ The rate with which magnetic field lines are cut = $\frac{\mu_0 ilv}{2\pi x}$

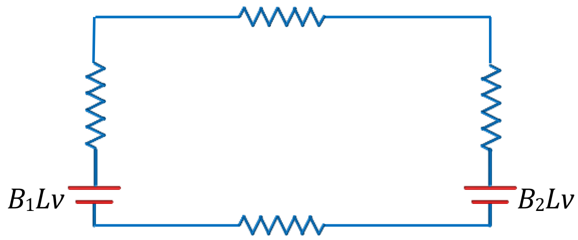
Illustration 8:

A rectangular loop, as shown in the figure, moves away from an infinitely long wire carrying a current *i*. Find the emf induced in the rectangular loop.



Solution:

$$E = B_1Lv - B_2Lv = \frac{\mu_0 i}{2\pi x}Lv - \frac{\mu_0 i}{2\pi(x+b)}Lv = \frac{\mu_0 iLbv}{2\pi x(x+b)}$$



Alter:

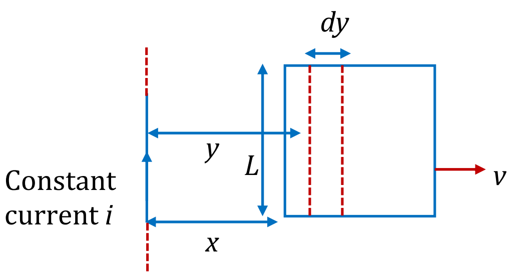
Consider a small segment of width *dy* at a distance *y* from the wire.

Let flux through the segment be $d\phi$.

$\therefore \quad d\phi = \frac{\mu_0 i}{2\pi y}Ldy$

$\therefore \quad \phi = \frac{\mu_0 iL}{2\pi}\int_x^{x+b}\frac{dy}{y} = \frac{\mu_0 iL}{2\pi}\left(\ell n(x+b) - \ell nx\right)$

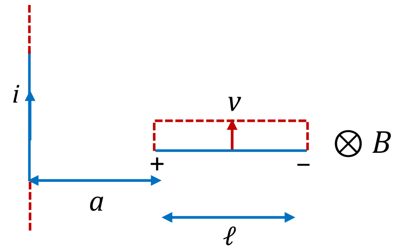
Now $\quad \frac{d\phi}{dt} = \frac{\mu_0 iL}{2\pi}\left[\frac{1}{x+b}\frac{dx}{dt} - \frac{1}{x}\frac{dx}{dt}\right]v = \frac{-\mu_0 ibLv}{2\pi x(x+b)}$



$\therefore \quad$ Induced emf = $\frac{\mu_0 ibLv}{2\pi x(x+b)}$

Illustration 9:

A rod of length l is placed perpendicular to a long wire carrying current i. The rod is moved parallel to the wire with a velocity v. Find the emf induced in the rod, if its nearest end is at distance *'a'* from the wire.



Solution:

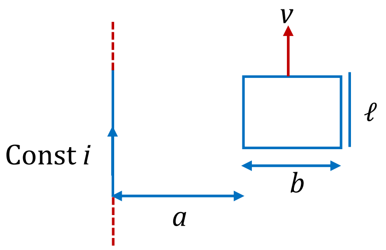
Consider a segment of the rod of length dx, at a distance x from the wire. Emf induced in the segment

$$dE = \frac{\mu_0 i}{2\pi x} dx.v$$

$$\therefore \qquad E = \int_a^{a+\ell} \frac{\mu_0 i v dx}{2\pi x} = \frac{\mu_0 i v}{2\pi} \ell n\left(\frac{\ell + a}{a}\right)$$

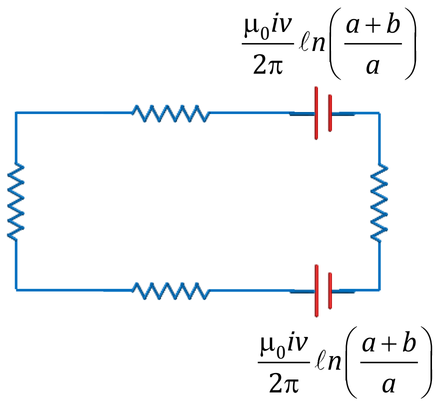
Illustration 10:

A rectangular loop is moving parallel to a long wire carrying current i with velocity v. Find the emf induced in the loop, if its nearest end is at a distance *'a'* from the wire. Draw equivalent electrical diagram.



Solution:

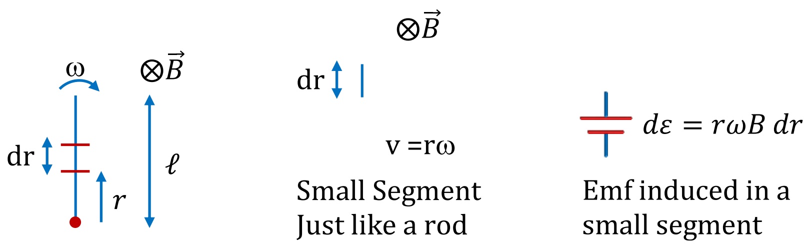
emf = 0



Induced emf due to rotation

Rotation of the rod

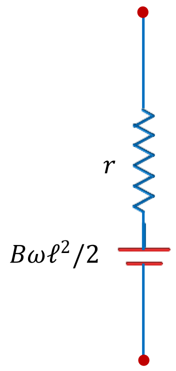
Consider a conducting rod of length l rotating in a uniform magnetic field.



Emf induced in a small segment of length dr of the rod = $v\,B\,dr = r\omega\,B\,dr$

$\therefore$ emf induced in the rod = $\omega B\int_0^\ell r dr = \frac{1}{2}B\omega\ell^2$

Equivalent of this rod is as following (figure):



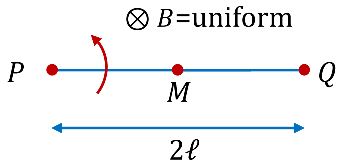
or $\varepsilon = \frac{d\Phi}{dt}$

$$\varepsilon = \frac{d\Phi}{dt} = \frac{\text{Flux through the area swept by the rod in time dt}}{dt}$$

$$= \frac{B\frac{1}{2}\ell^2\omega dt}{dt} = \frac{1}{2}B\omega\ell^2$$

Illustration 11:

A rod PQ of length 2ℓ is rotating about one end P in a uniform magnetic field B which is perpendicular to the plane of rotation of the rod. Point M is the mid point of the rod. Find the induced emf between M and Q if that between P and Q = $100V$.



Solution:

$E_{MQ} + E_{PM} = E_{PQ}$

$$E_{PM} = \frac{Bw\ell^2}{2}$$

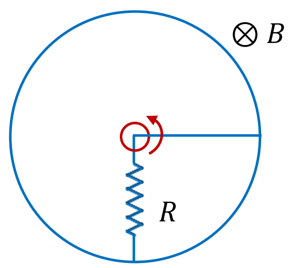
$$E_{PQ} = \frac{1}{2}B\omega(2\ell)^2 = 100$$

$$B\omega\ell^2 = 50$$

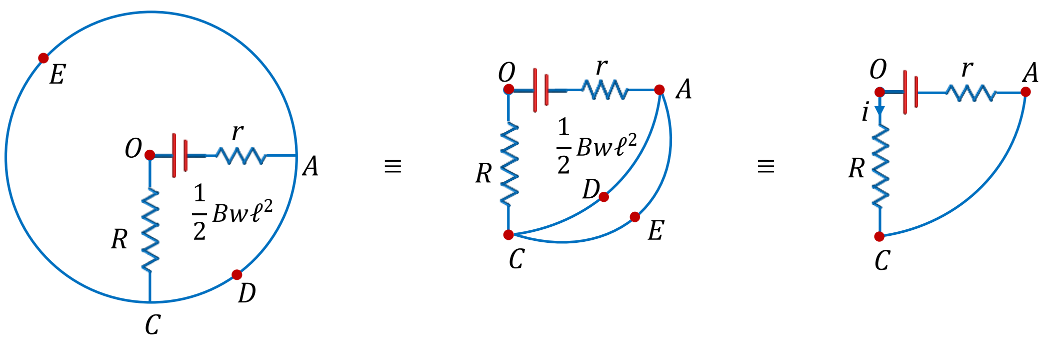
$$E_{MQ} = E_{PQ} - E_{PM} = 100 - \frac{50}{2} = 75V$$

Illustration 12:

A rod of length ℓ and resistance r rotates about one end as shown in figure. Its other end touches a conducting ring of negligible resistance. A resistance R is connected between centre and periphery. Draw the electrical equivalence and find the current in the resistance R. There is a uniform magnetic field B directed as shown.



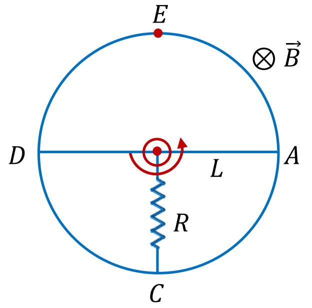
Solution:



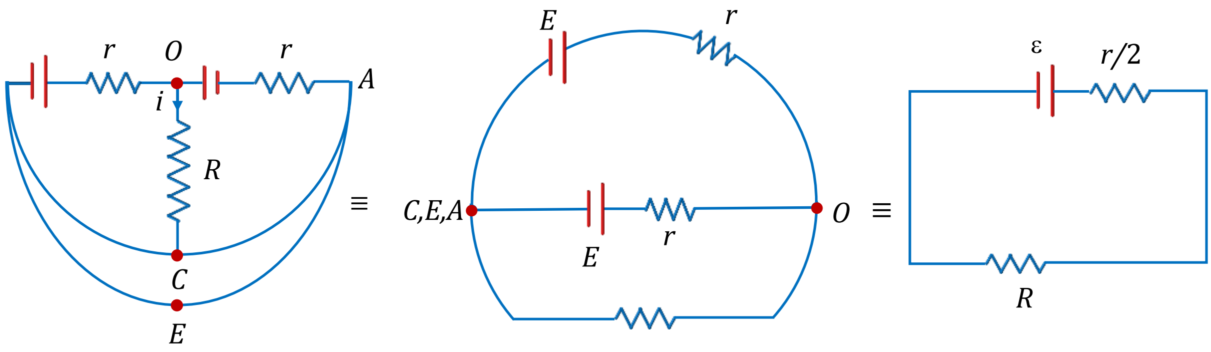
$$\text{current } i = \frac{\frac{1}{2}B\omega\ell^2}{R+r}$$

Illustration 13:

Solve the above question if the length of rod is $2L$ and resistance $2r$ and it is rotating about its centre. Both ends of the rod now touch the conducting ring.

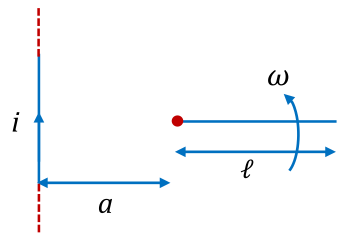


Solution:



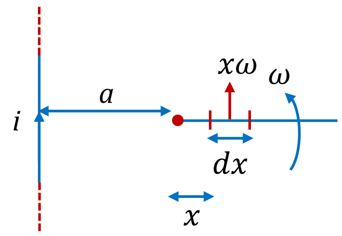
$$i=\frac{\varepsilon}{R+\frac{r}{2}}=\frac{\frac{1}{2}B\omega L^2}{R+\frac{r}{2}}$$

Illustration 14:



A rod of length l is rotating with an angular speed ω about its one end which is at a distance 'a' from an infinitely long wire carrying current i. Find the emf induced in the rod at the instant shown in the figure.

Solution:



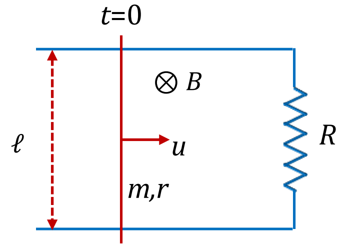
Consider a small segment of rod of length dx, at a distance x from one end of the rod. Emf induced in the segment

$$dE=\frac{\mu_0 i}{2\pi(x+a)}(x\omega)dx$$

$$\therefore \qquad E=\int_0^{\ell}\frac{\mu_0 i}{2\pi(x+a)}(x\omega)dx=\frac{\mu_0 i\omega}{2\pi}\left[\ell-a.\ell n\left(\frac{\ell+a}{a}\right)\right]$$

Velocity, force and power analysis for a conductor moving a magnetic field:

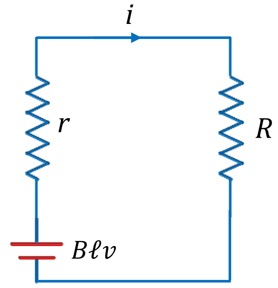
Illustration 15:



A rod of mass m and resistance r is placed on fixed, resistance less, smooth conducting rails (closed by a resistance R) and it is projected with an initial velocity u. Find its velocity as a function of time.

Solution:

Let at an instant the velocity of the rod be v. The emf induced in the rod will be $B\ell v$. The electrically equivalent circuit is shown in the following diagram.



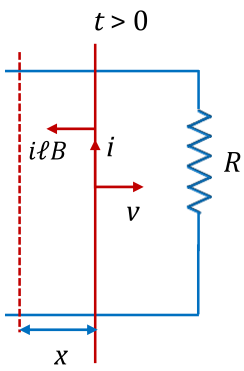
$\therefore$ Current in the circuit,

$$i = \frac{B\ell v}{R+r}$$

At time t magnetic force acting on the rod is $F = i\ell B$, opposite to the motion of the rod.

$$i\ell B = -m\frac{dv}{dt} \qquad ...(1)$$

$$i = \frac{B\ell v}{R+r} \qquad ...(2)$$



Now solving these two equations

$$\frac{B^2\ell^2 v}{R+r} = -m.\frac{dv}{dt}$$

$$-\frac{B^2\ell^2}{(R+r)m}.dt = \frac{dv}{v}$$

Let $$\frac{B^2\ell^2}{(R+r)m} = k$$

$$-K.dt = \frac{dv}{v}$$

$$\int_u^v \frac{dv}{v} = \int_0^t -K.dt$$

ln $$\left(\frac{v}{u}\right) = -Kt$$

$v = ue^{-Kt}$

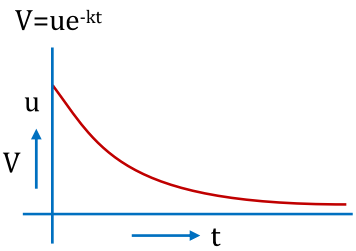


Illustration 16:

In the above question find the force required to move the rod with constant velocity v, and also find the power delivered by the external agent.

Solution:

The force needed to keep the velocity constant,

$$F_{ext} = i\ell B = \frac{B^2\ell^2 v}{R+r}$$

Power due to external force = $\dfrac{B^2\ell^2v^2}{R+r} = \dfrac{\varepsilon^2}{R+r} = i^2(R+r)$

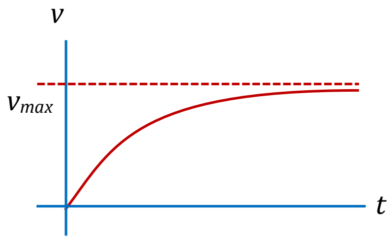
Note:

The power delivered by the external agent is converted into joule heating in the circuit. That means magnetic field helps in converting the mechanical energy into joule heating.

Illustration 17:

In the above question if a constant force F is applied on the rod. Find the velocity of the rod as a function of time, assuming it started with zero initial velocity.

Solution:



$$m\frac{dv}{dt} = F - i\ell B \qquad ...(1)$$

$$i = \frac{B\ell v}{R+r} \qquad ...(2)$$

$$m\frac{dv}{dt} = F - \frac{B^2\ell^2 v}{R+r}$$

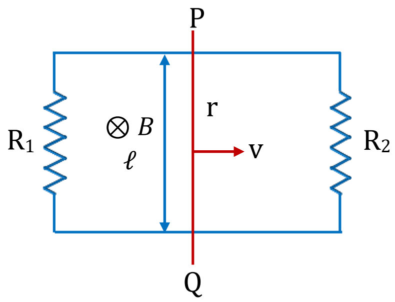
Let $\quad K = \dfrac{B^2\ell^2}{R+r} \quad \Rightarrow \quad \displaystyle\int_0^v \frac{dv}{F-Kv} = \int_0^t \frac{dt}{m}$

$$-\frac{\left[\ell n(F-Kv)\right]_0^v}{K} = \frac{t}{m} \quad \Rightarrow \quad \ell n\left(\frac{F-Kv}{F}\right) = -\frac{Kt}{m}$$

$$F - Kv = Fe^{-kt/m} \quad \Rightarrow \quad v = \frac{F}{K}\left(1 - e^{-kt/m}\right)$$

Illustration 18:

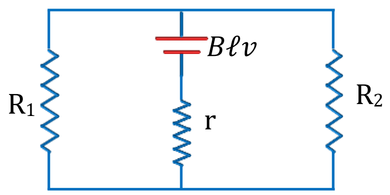
A rod PQ of mass m and resistance r is moving on two fixed, resistance less, smooth conducting rails (closed on both sides by resistances R_1 and R_2). Find the current in the rod at the instant its velocity is v.



Solution:

$$i = \frac{B\ell v}{r + \dfrac{R_1R_2}{R_1+R_2}}$$

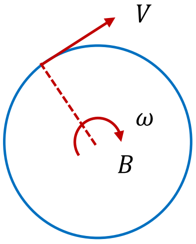
This circuit is equivalent to the following diagram.



EMF Induced due to rotation of a coil

Illustration 19:

A ring rotates with angular velocity ω about an axis perpendicular to the plane of the ring passing through the center of the ring. A constant magnetic field B exists parallel to the axis. Find the emf induced in the ring.

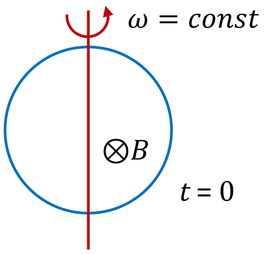


Solution:

Flux passing through the ring $\phi = B.A$ is a constant here, therefore emf induced in the coil is zero. Every point of this ring is at the same potential, by symmetry.

Illustration 20:

A ring rotates with angular velocity ω about an axis in the plane of the ring and which passes through the center of the ring. A constant magnetic field B exists perpendicular to the plane of the ring. Find the emf induced in the ring as a function of time.



Solution:

At any time, t, $\phi = BA\cos\theta = BA\cos\omega t$

Now induced emf in the loop

$$e = \frac{-d\phi}{dt} = BA\omega\sin\omega t$$

If there are N turns

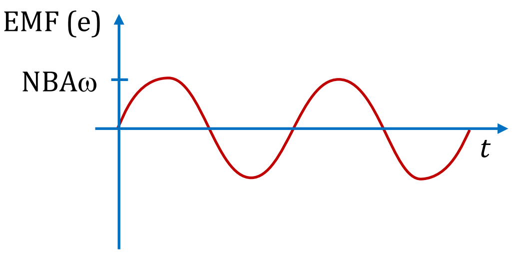
$$\text{emf} = BA\omega N\sin\omega t$$

$BA\omega N$ is the amplitude of the emf

$$e = e_m \sin\omega t$$

$$i = \frac{e}{R} = \frac{e_m}{R}\sin\omega t = i_m \sin\omega t$$

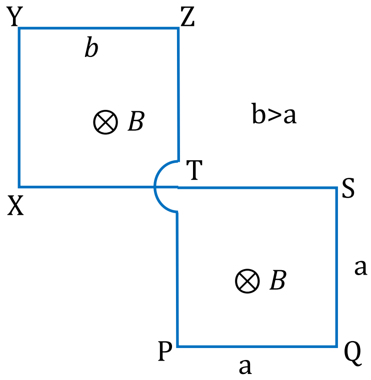
$$i_m = \frac{e_m}{R}$$



The rotating coil thus produces a sinusoidally varying current or alternating current. This is also the principle used in generator.

Illustration 21:

Figure shows a wire frame *PQSTXYZ* placed in a time varying magnetic field given as $B = \beta t$, where β is a positive constant. Resistance per unit length of the wire is λ. Find the current induced in the wire and draw its electrical equivalent diagram.



Solution:

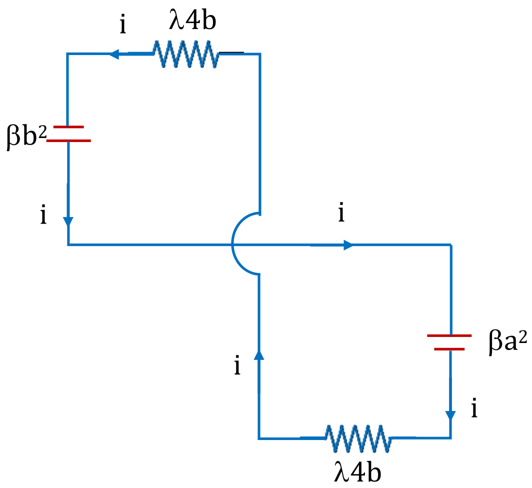
Induced emf in part *PQST* = βa^2 (in anticlockwise direction, from Lenz's Law)

Similarly Induced emf in part *TXYZ* = βb^2

(in anticlockwise direction, from Lenz's Law)

Total resistance of the part *PQST* = $\lambda 4a$.

Total resistance of the part *TXYZ* = $\lambda 4b$. The equivalent circuit is as shown in the following diagram.



Writing *KVL* along the current flow

$$\beta b^2 - \beta a^2 - \lambda\, 4ai - \lambda\, 4bi = 0$$

$$i = \frac{\beta}{4\lambda}(b-a)$$

Emf induced in a rotating disc:

Consider a disc of radius r rotating in a magnetic field B.

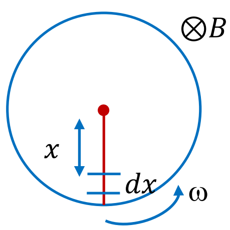
Consider an element dx at a distance x form the centre. This element is moving with speed $v = \omega x$.

$\therefore$ Induced emf across dx

$$= B(dx)\,v = Bdx\omega x = B\omega x dx$$

$\therefore$ Emf between the centre and the edge of disc.

$$= \int_0^r B\omega x dx = \frac{B\omega r^2}{2}$$



Induced Electric Field

Consider a conducting loop placed at rest in a magnetic field $\vec{B}$. Suppose, the field is constant till $t = 0$ and then changes with time. An induced current starts in the loop at $t = 0$.

The free electrons were at rest till $t = 0$ (we are not interested in the random motion of the electrons). The magnetic field cannot exert force on electrons at rest. Thus, the magnetic force cannot start the induced current. The electrons can be forced to move only by an electric field and hence we conclude that an electric field appears at $t = 0$. This electric field is produced by the changing magnetic field and not by any charged particles according to the Coulomb's law or the Gauss's law. The electric field produced by the changing magnetic field is non electrostatic and nonconservative in nature. We cannot define a potential corresponding to this field. We call it induced electric field. The lines of induced electric field are closed curves. There are no starting and terminating points of the field lines.

If $\vec{E}$ be the induced electric field, the force on a charge q placed in field is $q\vec{E}$. The work done per unit charge as the charge moves through $d\vec{l}$ is $\vec{E}.d\vec{l}$.

The emf developed in the loop is, therefore,

$$\varepsilon = \oint \vec{E}.d\vec{l}$$

Using Faraday's law of induction,

$$\varepsilon = -\frac{d\phi}{dt} \qquad \text{or} \qquad \oint \vec{E}.d\vec{l} = -\frac{d\phi}{dt}$$

The presence of a conducting loop is not necessary to have an induced electric field. As long as $\vec{B}$ keeps changing, the induced electric field is present. If a loop is there, the free electrons start drifting and consequently an induced current results.

Calculating the Induced Field

The induced electric field is peculiar in another way: It is nonconservative. Recall that a force is conservative if it does no net work on a particle moving around a closed path. "Uphill" are balanced by "downhills." We can associate a potential energy with a conservative force, hence we have gravitational potential energy for the conservative gravitational force and electric potential energy for the conservative electric force of charges (Coulomb electric field).

But a charge moving around a closed path in the induced electric field is always being pushed in the same direction by the electric force $F = qE$. There's never any negative work to balance the positive work, so the net work done in going around a closed path is not zero. Because it is nonconservative, we cannot associate an electric potential with an induced electric field. Only the Coulomb field of charges has an electric potential.

However, we can associate the induced field with the emf of Faraday's law. The emf was defined as the work required per unit charge to separate the charge. That is,

$$\varepsilon = \frac{W}{q}$$

In batteries this work, a familiar source of emf, is done by chemical forces. But the emf that appears in Faraday's law arises when work is done by the force of an induced electric field.

If a charge q moves through a small displacement $d\vec{s}$, the small amount of work done by the electric field is $dW = \vec{F}.d\vec{s} = q\vec{E}.d\vec{s}$. The emf of Faraday's law is an emf around a closed curve through which the magnetic flux ϕ_m is changing. The work done by the induced electric field as charge q moves around a closed curve is

$$W_{\text{closed curve}} = q\oint \vec{E}.d\vec{s}$$

Where the integration symbol with the circle is the same as the one we used in Ampere's law to indicate an integral around a closed curve. If we use this work in equation, we find that the emf around a closed loop is

$$\varepsilon = \frac{W_{\text{closed curve}}}{q} = \oint \vec{E}.d\vec{s}$$

If we restrict ourselves to situations such as figure where the loop is perpendicular to the magnetic field and only the field is changing, we can write Faraday's law as $\varepsilon = |d\phi_m / dt| = A|dB/dt|$. Consequently

$$\oint \vec{E}.d\vec{s} = A\left|\frac{dB}{dt}\right|$$

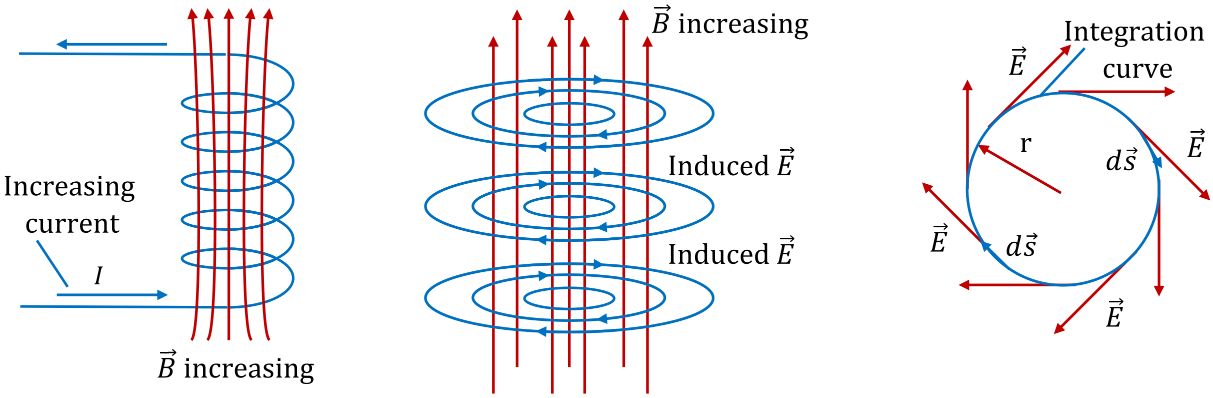
Above equation is an alternative of statement of Faraday's law that relates the induced electric field to the changing magnetic field.

The solenoid in figure (a) provides a good example of the connection between $\vec{E}$ and $\vec{B}$. If there were a conduction loop inside the solenoid, we could use Lenz's law to determine that the direction of the induced current would be clockwise. But Faraday's law, in the form of above equation, tells us that an induced electric field is present whether there is a conducting loop or not. The electric field is induced simply due to the fact that $\vec{B}$ is changing.

(a) The current through the solenoid is increasing.

(b) The induced electric field circulates around the magnetic field lines.

(c) Top view into the solenoid $\vec{B}$ is coming out of the page.



In figure, the induced electric field circulates around the changing magnetic field inside a solenoid.

The shape and direction of the induced electric field have to be such that it could drive a current around a conducting loop, if one were present and it has to be consistent with the cylindrical symmetry of the solenoid. The only possible choice, shown in Figure is an electric field that circulates clockwise around the magnetic field lines.

Note:

Circular electric field lines violate the rule that electric field lines have to start and stop on charges. However, that rule applied only to Coulomb fields created by source charges. An induced electric field is a non-Coulomb field created not by source charges but by a changing magnetic field. Without source charges, induced electric field lines must form closed loops.

To use Faraday's law, choose a clockwise circle of radius *r* as the closed curve for evaluating the integral. Figure (c) shows that the electric field vectors are everywhere along the tangent to the curve, so the line integral of $\vec{E}$ is

$$\oint \vec{E}.d\vec{s} = El = 2\pi rE$$

Where $l = 2\pi r$ is the length of the closed curve. This is exactly like the integrals we did for Amperes law.

If we stay inside the solenoid ($r < R$), the flux passes through area $A = \pi r^2$ and above equation becomes

$$\oint \vec{E}.d\vec{s} = 2\pi rE = A\left|\frac{dB}{dt}\right| = \pi r^2\left|\frac{dB}{dt}\right|$$

Illustration 22:

A 4 *cm* diameter solenoid is wound with 2000 turns per meter. The current through the solenoid oscillates at 60 *Hz* with an amplitude of 2 *A*. What is the maximum strength of the induced electric field inside the solenoid?

Solution:

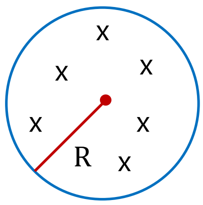
You learned in that the magnetic field strength inside a solenoid with *n* turns per meter is $B = \mu_0 nI$. In this case, the current through the solenoid is $I = I_0 \sin \omega t$, where $I_0 = 2$ *A* is the peak current and $\omega = 2\pi(60Hz) = 377$ *rad/s*. Thus, the induced electric field strength at radius *r* is

$$E = \frac{r}{2}\left|\frac{dB}{dt}\right| = \frac{r}{2}\frac{d}{dt}\left(\mu_0 nI_0 \sin \omega t\right) = \frac{1}{2}\mu_0 nr\omega I_0 \cos \omega t$$

The field strength is maximum at maximum radius ($r = R$) and at the instant when $\cos \omega t = 1$. That is,

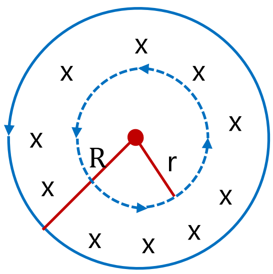
$$E_{\max} = \frac{1}{2}\mu_0 nR\omega I_0 = 0.019V/m$$

Illustration 23:



The magnetic field at all points within the cylindrical region whose cross-section is indicated in the figure start increasing at a constant rate $\alpha \dfrac{\text{tesla}}{\text{second}}$. Find the magnitude of electric field as a function of *r*, the distance from the geometric centre of the region.

Solution:



For $r \le R$:

$\because \quad E\ell = A\left|\dfrac{dB}{dt}\right|$

$\therefore \quad E(2\pi r) = (\pi r^2)\alpha$

$\Rightarrow \quad E = \dfrac{r\alpha}{2} \quad \Rightarrow \quad E \propto r$

E-r graph is straight line passing through origin.

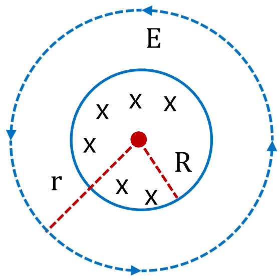
At $\quad r = R,\ E = \dfrac{R\alpha}{2}$

For $\quad r \geq R$

$\because \quad E\ell = A\left|\dfrac{dB}{dt}\right|$

$\therefore \quad E(2\pi r) = (\pi R^2)\alpha$

$\Rightarrow \quad E = \dfrac{\alpha R^2}{2r} \Rightarrow E \propto \dfrac{1}{r}$



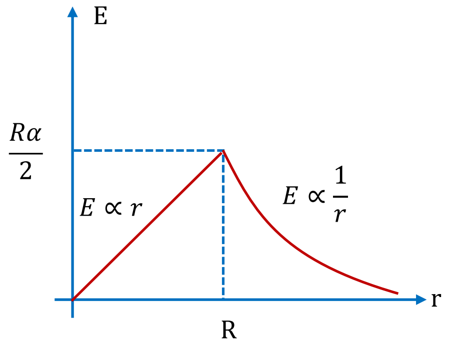
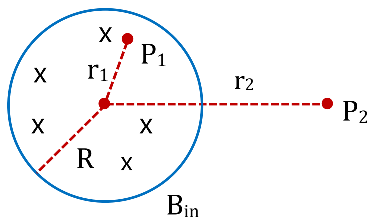


Illustration 24:

For the situation described in figure the magnetic field changes with time according to, $B = (2t^3 - 4t^2 + 0.8)T$ and $r_2 = 2R = 5cm$

(a) Calculate the force on an electron located at P_2 at $t = 2s$

(b) What are the magnitude and direction of the electric field at P_1 when $t = 3$ s and $r_1 = 0.02m$?



Solution:

$$E\ell = A\left|\frac{dB}{dt}\right| \Rightarrow E = \frac{\pi R^2}{2\pi r_2}\frac{d}{dt}(2t^3 - 4t^2 + 0.8) = \frac{R^2}{2r_2}(6t^2 - 8t)$$

(a) Force on electron at P_2 is $F = eE$

At $\quad t = 2s,\ F = \dfrac{1.6\times10^{-19}\times(2.5\times10^{-2})^2}{2\times5\times10^{-2}}\times\left[6(2)^2 - 8(2)\right]$

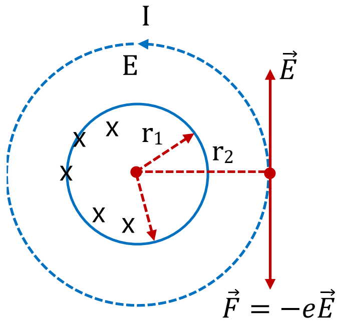
$$= \frac{1.6}{4}\times2.5\times10^{-21}\times(24-16) = 8\times10^{-21}N \text{ at } t = 2s$$

$\dfrac{dB}{dt}$ is positive so it is increasing.

$\therefore$ Direction of induced current and E are as shown in figure and hence force on electron having charge-e is right perpendicular to r_2 downwards.

(b) For $r_1 = 0.02m$ and at $t = 3s$,

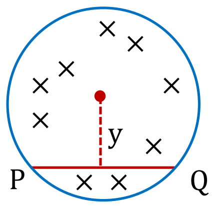
$$E = \frac{\pi r_1^2}{2\pi r_1}\left(6t^2 - 8t\right) = \frac{0.02}{2}\times\left[6(3)^2 - 8(3)\right] = 0.3V/m$$



At $t = 3\sec$, $\frac{dB}{dt}$ is positive, so B is increasing and hence direction of E is same as in case (a) and it is left perpendicular to r_1 upward.

Illustration 25:

For the situation described in figure the magnetic field changes with time according to $B = kt$. Find the value of induced electric field across PQ.



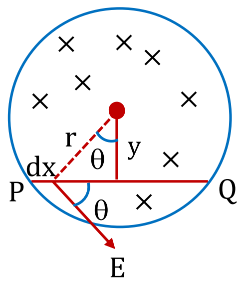
Solution:

M-1: $dE = \vec{E}.d\vec{x}$

$$dE = \frac{k}{2}(r.\cos\theta).dx$$

$$\Rightarrow \quad \int dE = \int \frac{ky}{2}dx = \frac{ky}{2}\times \ell$$

$$E_{PQ} = \frac{ky\ell}{2}$$



M-2: $\phi_{OPQ} = B\times\frac{1}{2}(y\ell)$

$$E_{OPQ} = \frac{d\phi}{dt} = \frac{ky\ell}{2}$$

$$E_{OP} = +E_{PQ} + E_{QO} = \frac{ky\ell}{2}$$

$$E_{PQ} = \frac{ky\ell}{2} \qquad (\because E_{OP} = E_{QO} = 0)$$

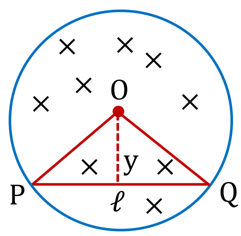
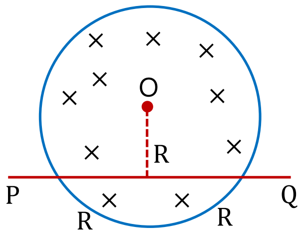


Illustration 26:

For the situation described in figure the magnetic field changes with time according to $B = kt$. Find the value of induced electric field across PQ.



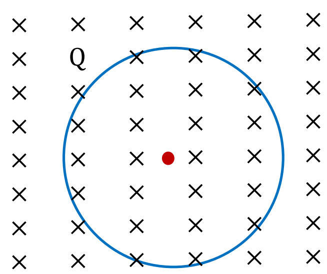
Solution:

$$E = \frac{d\phi}{dt} \quad \Rightarrow \quad E = \frac{dB}{dt} \times A$$

$$E_{OP} + E_{PQ} + E_{QO} = k \times \frac{\pi R^2}{4}$$

Illustration 27:

A non-conducting ring of mass m and radius R is placed horizontally in region where magnetic field varies such that $B = kt$. Find angular acceleration α.



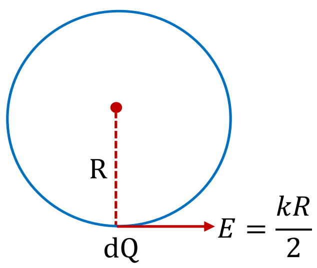
Solution:

$$E\ell = A\left|\frac{dB}{dt}\right| = E(2\pi R) = \pi R^2 (K) \Rightarrow E = \frac{KR}{2}$$

$$\int d\tau = \int ErdQ$$

$$\tau = \text{Torque} = EQR = I\alpha$$

$$\alpha = \frac{EQR}{mR^2} \quad \Rightarrow \quad \boxed{\alpha = \frac{kQ}{2m}}$$



Self-Inductance:

In the absence of magnetic materials, the B-field at any point due to a single circuit of given shape is proportional to the current in it, so the current is measured by it magnetic effect. It follows that the total flux ϕ linking a circuit due to its own current I is proportional to I. We define the self inductance L of a circuit as the ratio of ϕ to I so that

$$\boxed{\phi = LI \quad \text{(definition of L)}}$$

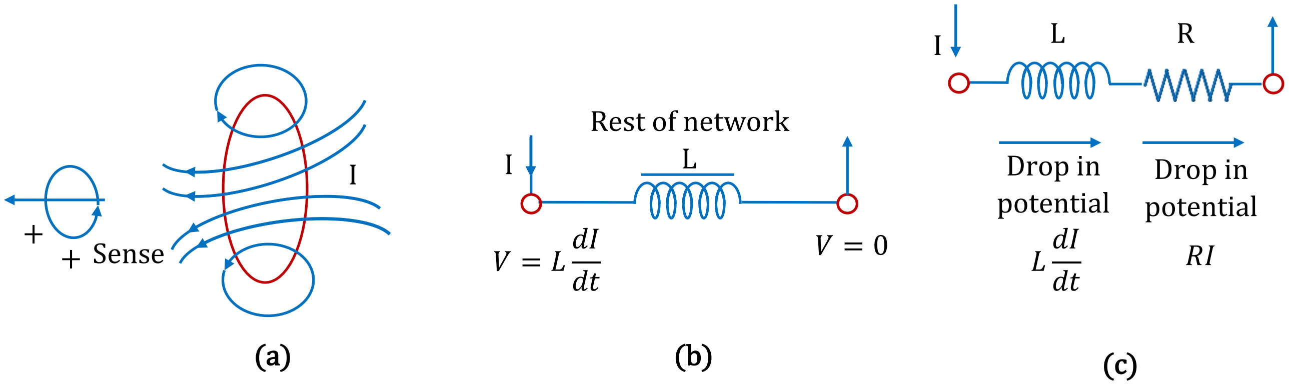


Figure: (a) Self-flux of a circuit: increase of flux due to an increase in I causes an electromotance in the negative sense and this opposes the growth of I; (b) self-inductance as a circuit element; (c) the equivalent circuit of a self-inductor with resistance.

And L is thus a constant for a circuit of given shape and size. The self-flux of a circuit defines the positive direction for ϕ so that L is essentially positive. Its SI unit, the WbA^{-1}, is called the henry, symbol H.

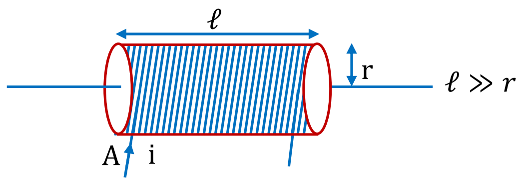
Self-induction when the current in a rigid circuit changes, the induced electromotance is

$$\boxed{\varepsilon = -LdI/dt}$$

The negative sign indicating that when I increases, ε is negative and is thus in a direction opposing the growth of I (fig. a). When a device is specially constructed so that above equation applies, it is known as an inductor and has the symbol as in Fig. b in a circuit diagram, crossed by an arrow if variable and with parallel lines adjacent to it if it has a ferromagnetic core.

In a network with varying currents, an inductor acts as an electromotance – LdI/dt with an internal resistance which cannot be avoided. Like a cell, therefore, it can be represented by an equivalent network consisting of a pure inductance L in series with its resistance R as in Fig. c. The pure inductance has no resistance and thus has a potential difference across its ends equal to the electromotance: once the direction of I has been chosen arbitrarily in any problem the potential drop across an inductor L is LdI/dt just as that across R is RI (fig. c).

Self inductance of solenoid:



Let the volume of the solenoid be V, the number of turns per unit length be n.

Let a current i be flowing in the solenoid.

Magnetic field in the solenoid is given as $B = \mu_0 ni$.

The magnetic flux through one turn of solenoid $\phi = \mu_0\, niA$.

The total magnetic flux through the solenoid $= N\phi = N\mu_0 ni\,A = \mu_0 n^2 i\,A\,\ell$

$$\therefore \quad L = \mu_0\, n^2\, \ell A = \mu_0\, n^2\, V \quad \Rightarrow \quad \phi = \mu_0\, n\, i\, \pi r^2\, (n\ell) \quad \Rightarrow \quad L = \frac{\phi}{i} = \mu_0\, n^2\, \pi r^2 \ell$$

Inductance per unit volume $= \mu_0 n^2$.

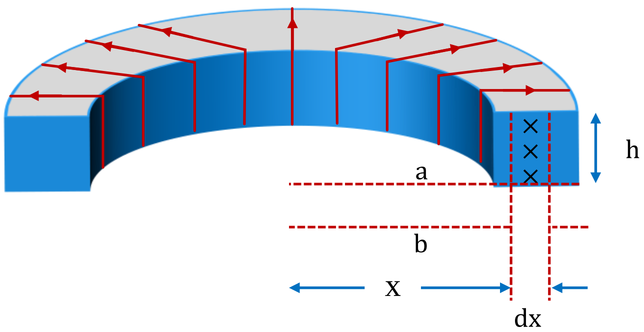
Self inductance is the physical property of the loop due to which it opposes the change in current that means it tries to keep the current constant. Current cannot change suddenly in the inductor.

Self-Inductance of Toroid:

$$L=\frac{N\phi}{I}=\frac{N}{I}\int\vec{B}.d\vec{A}$$

$$\int\vec{B}.d\vec{A}=\int B.dA=\int_a^b\frac{\mu_0 NI}{2\pi x}hdx=\frac{\mu_0 NIh}{2\pi}\ell n\left(\frac{b}{a}\right)$$

$$L=\frac{\mu_0 N^2}{2\pi}h\,\ell n\left(\frac{b}{a}\right)$$



Let, $b-a=d$

$$\ell n\left(\frac{b}{a}\right)=\ell n\left(1+\frac{b-a}{a}\right)$$

If, $b-a<<a$

$b-a<<b$

Then, $\ell n\left(1+\frac{b-a}{a}\right)\simeq\left(\frac{b-a}{a}\right)$

$$L=\frac{\mu_0 N^2}{2\pi}\frac{h(b-a)}{a}=\frac{\mu_0 N^2 A}{\ell}=\mu_0 n^2\ell A$$

$$\frac{L}{\ell}=\mu_0 n^2 A$$

which is Inductance per unit length of infinitely long solenoid.

Note:

If there is a resistance in the inductor (resistance of the coil of inductor) then :

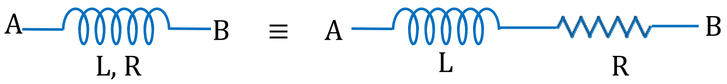
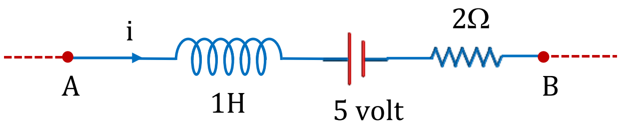


Illustration 28:

A B is a part of circuit. Find the potential difference $V_A - V_B$ if



(i) current i = 2A and is constant.

(ii) current i = 2A and is increasing at the rate of 1 amp/sec.

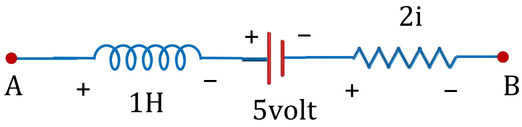
(iii) current i = 2A and is decreasing at the rate 1 amp/sec.

Solution:

$$L\frac{di}{dt}=1\frac{di}{dt}$$

Writing *KVL* from *A* to *B*

$$V_A-1\frac{di}{dt}-5-2i=V_B$$



(i) Put $i = 2, \frac{di}{dt} = 0$

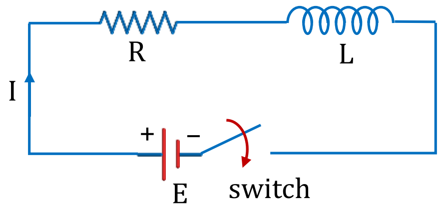
$V_A - 5 - 4 = V_B \qquad \therefore \quad V_A - V_B = 9 volt$

(ii) Put $i = 2, \frac{di}{dt} = 1; V_A - 1 - 5 - 4 = V_B$ or $V_A - V_B = 10 volt$

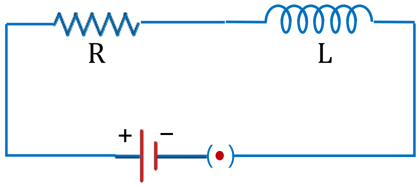
(iii) Put $i = 2, \frac{di}{dt} = -1; V_A + 1 - 5 - 2 \times 2 = V_B$ or $V_A - V_B = 8 Volt$

R-L DC circuit:

Current Growth



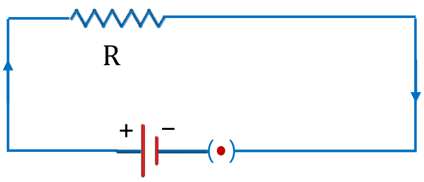
(i) Emf equation $E = IR + L\frac{dl}{dt}$



(ii) Current at any instant

When key is closed the current in circuit increases exponentially with respect to time. The current in circuit at any instant 't' is given by $I = I_0\left[1 - e^{\frac{-t}{\lambda}}\right]$

$$T = 0 \text{ (just after the closing of key)} \Rightarrow I = 0$$
$$T = \infty \text{ (sometime after closing of key)} \Rightarrow I \to I_0$$



(iii) Just after the closing of the key inductance behaves like open circuit and current in circuit is zero. Open circuit, $t = 0$, $I = 0$, inductor provide infinite resistance.

(iv) Sometime after closing of the key inductance behaves like simple connecting wire (short circuit) and current in circuit is constant.

Short circuit, $t \to \infty$, $I \to I_0$. Inductor provide zero resistance $I_0 = \frac{E}{R}$

(Final, steady, maximum or peak value of current) or ultimate current.

Note:

Peak value of current in circuit does not depends on self inductance of coil.

(v) Time constant of circuit (λ)

$$\lambda = \frac{L}{R_{\sec}}$$

It is a time in which current increases up to 63% or 0.63 times of peak current value.

(vi) Rate of growth of current at any instant:

$$\left[\frac{dI}{dt}\right] = \frac{E}{L}\left(e^{-t/\lambda}\right)$$

$$\Rightarrow \quad t = 0 \quad \Rightarrow \quad \left[\frac{dI}{dt}\right]_{\max} = \frac{E}{L}$$

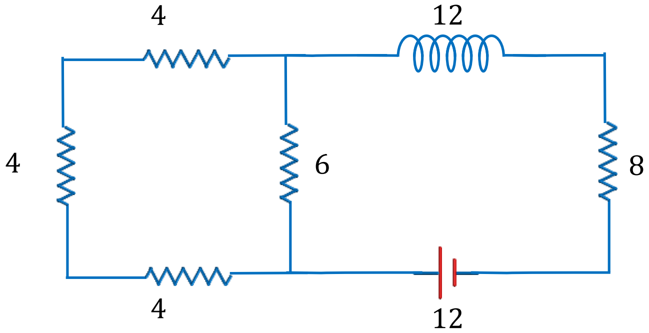
$$t = \infty \quad \Rightarrow \quad \left[\frac{dI}{dt}\right] \to 0$$

Note:

Maximum or initial value of rate of growth of current does not depends upon resistance of coil.

Illustration 29:

For the circuit shown, which of the following statement(s) is (are) correct?



(A) Its time constants is 2 second.

(B) In steady state, current through inductance will be 1A.

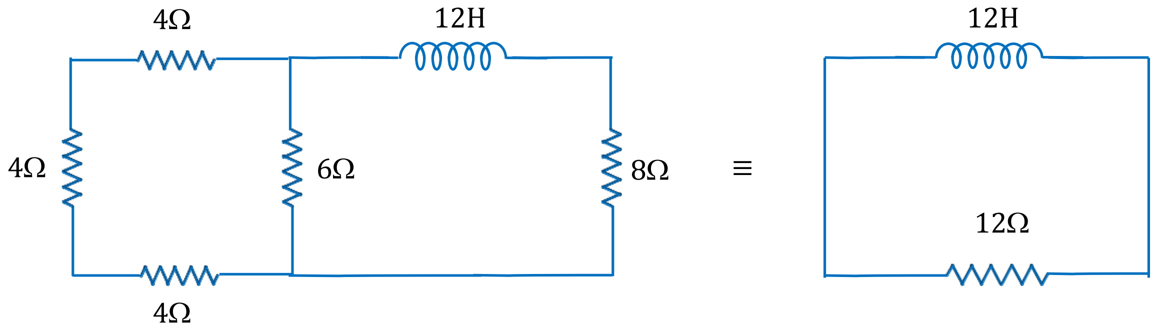
(C) In steady state, current through 4Ω resistance will be 2/3A.

(D) In steady state, current through 8Ω resistance will be zero.

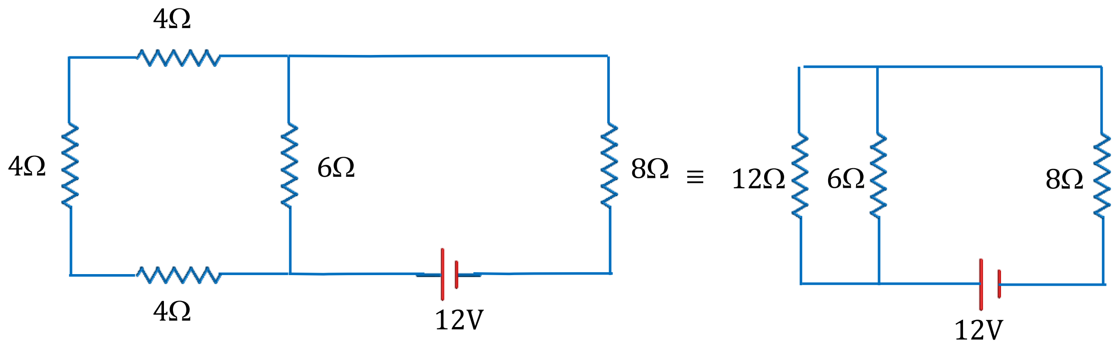
Ans. (B)

Solution:

Time constant $\tau = \frac{L}{R} = \frac{12}{12} = 1s$



In steady state

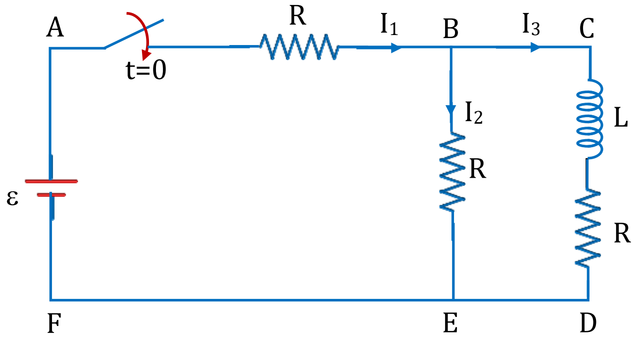


Current through 4Ω resistance = $\frac{6}{6+12}\times 1=\frac{1}{3}A$

Illustration 30:

In the following circuit, the switch is close at $t = 0$. Find the currents i_1, i_2, i_3 and $\frac{di_3}{dt}$ at $t = 0$ and at $t = \infty$.

Initially all current are zero.



Solution:

At $t = 0$

i_3 is zero, since current cannot suddenly change due to the inductor.

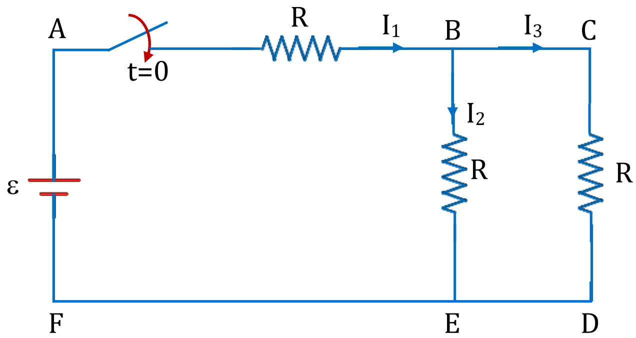
$\therefore \quad i_1 = i_2$ (From KCL)

Applying KVL in the part ABEF we get, $i_1 = i_2 = \frac{\varepsilon}{2R}$

Also $\quad \frac{di}{dt} = \frac{\varepsilon}{L}$

At $\quad t = \infty$

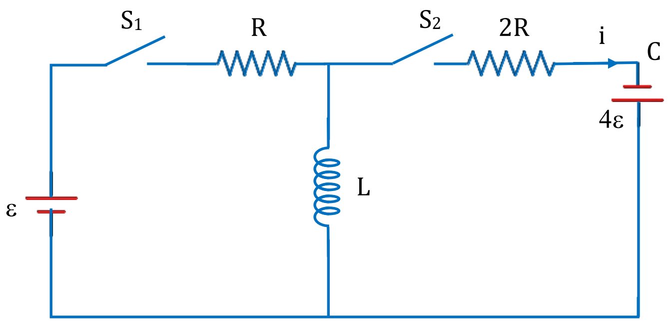
i_3 will become constant and hence potential difference across the inductor will be zero. It is just like a simple wire and the circuit can be solved assuming it to be like shown in the following diagram.



$$i_2 = i_3 = \frac{\varepsilon}{3R}, i_1 = \frac{2\varepsilon}{3R}, \text{ also } \frac{di_3}{dt} = 0$$

Illustration 31:

In the circuit shown in figure S_1 remains closed for a long time and S_2 remains open. Now S_2 is closed and S_1 is opened. Find out the di/dt just after that moment.



Solution:

Before S_2 is closed and S_1 is closed current in the left part of the circuit $= \frac{\varepsilon}{R}$. Now when S_2 is closed S_1 is opened, current through the inductor can not change suddenly, current $\frac{\varepsilon}{R}$ will continue to move in the inductor.

Applying KVL in loop l.

$$L\frac{di}{dt}+\frac{\varepsilon}{R}(2R)+4\varepsilon=0 \quad \Rightarrow \quad \frac{di}{dt}=-\frac{6\varepsilon}{L}$$

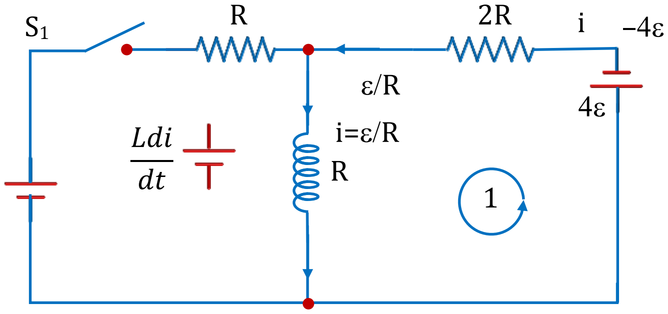
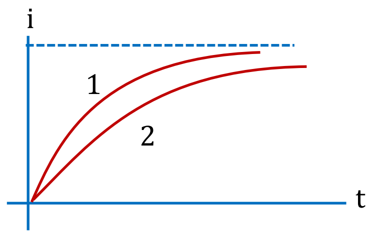


Illustration 32:

Which of the two curves shown has less time constant.



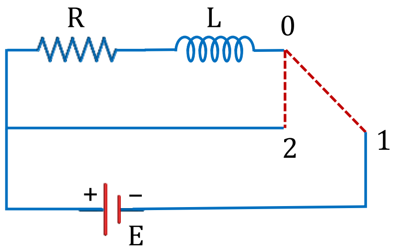
Ans. Curve 1

Solution:

$$i=i_0\left(1-e^{-t/\lambda}\right) \Rightarrow \frac{di}{dt}=\frac{i_0}{\lambda}=slope$$

Current Decay:

(i) **Emf equation** $IR + L\dfrac{dI}{dt} = 0$



(ii) **Current at any instant**

Once current acquires its final max steady value, if suddenly switch is put off then current start decreasing exponentially wrt to time. At switch put off condition $t = 0, I = I_0$, source emf E is cut off from circuit $I = I_0\left(e^{-t/\lambda}\right)$

Just after opening of key $\quad t = 0 \quad \Rightarrow \quad I = I_0 = \dfrac{E}{R}$

Some time after opening of key $\quad t \to \infty \quad \Rightarrow \quad I \to 0$

(iii) **Time constant (λ)**

It is a time in which current decreases up to 37% or 0.37 times of peak current value.

(iv) **Rate of decay of current at any instant**

$$\left[-\frac{dI}{dt}\right] = \left[\frac{E}{L}\right]e^{-t/\lambda}$$

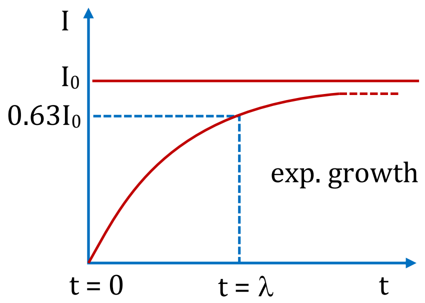
$$t = 0, \left[-\frac{dI}{dt}\right]_{\text{max.}} = \frac{E}{L}$$

$$t \to \infty \Rightarrow \left[-\frac{dI}{dt}\right] \to 0$$

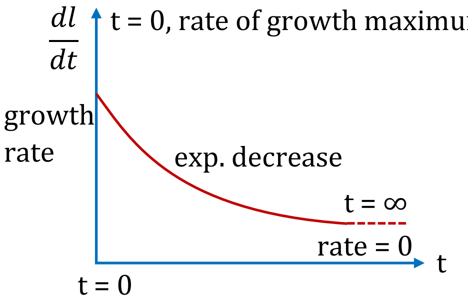
Graph for R-L circuit:

Current Growth:

(a)

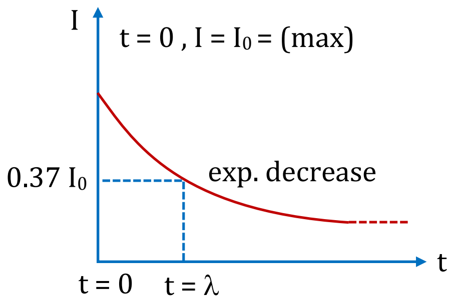


(b)



Current decay:

(a)



(b)

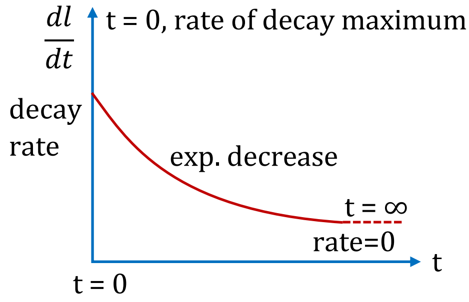
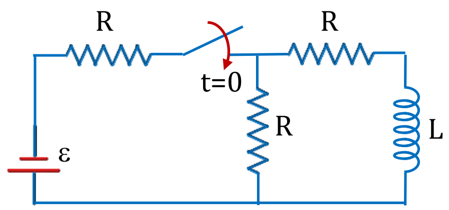


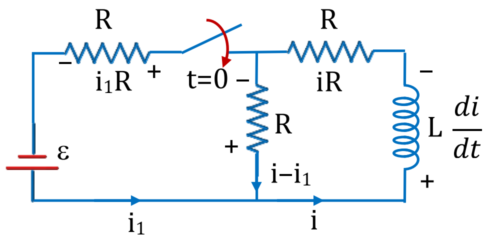
Illustration 33:

In the following circuit the switch is closed at $t = 0$. Initially there is no current in inductor. Find out current in the inductor coil as a function of time.



Solution:

At any time t



$$-\varepsilon+i_1R-(i-i_1)R=0$$

$$-\varepsilon+2i_1R-iR=0$$

$$i_1=\frac{iR+\varepsilon}{2R}$$

Now, $$-\varepsilon+i_1R+iR+L.\frac{di}{dt}=0$$

$$-\varepsilon+\left(\frac{iR+\varepsilon}{2}\right)+iR+L.\frac{di}{dt}=0$$

$$-\frac{\varepsilon}{2}+\frac{3iR}{2}=-L.\frac{di}{dt}$$

$$\left(\frac{-\varepsilon+3iR}{2}\right)dt=-L.di$$

$$\frac{-dt}{2L}=\frac{di}{-\varepsilon+3iR}$$

$$-\int_0^t\frac{dt}{2L}=\int_0^i\frac{di}{-\varepsilon+3iR}$$

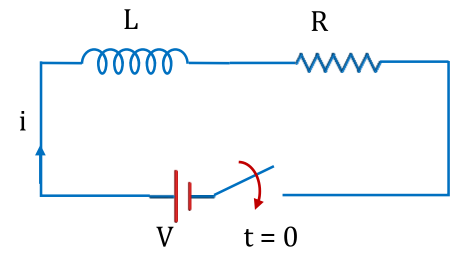
$$-\frac{t}{2L}=\frac{1}{3R}\ln\left(\frac{-\varepsilon+3iR}{-\varepsilon}\right)$$

$$-\ln\left(\frac{-\varepsilon+3iR}{-\varepsilon}\right)=\frac{3Rt}{2L}$$

$$i=+\frac{\varepsilon}{3R}\left(1-e^{-\frac{3Rt}{2L}}\right)$$

Illustration 34:

In the following circuit the switch is closed at $t = 0$. Initially there is no current in inductor. Find charge flown at time t when current flowing in the circuit is i.



Solution:

$$V - L\frac{di}{dt} - iR = 0$$

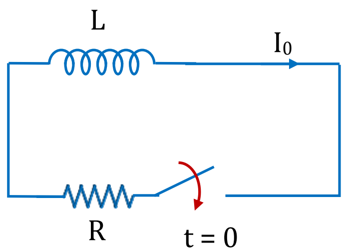
$$\int Vdt - \int_0^i Ldi = \int idtR$$

$$Vt - Li = QR$$

$$Q = \frac{Vt - Li}{R} \qquad Q = \text{charge flown}$$

Illustration 35:

Prove that decay energy loss in resistor from $t = 0$ to ∞ is equal to $\frac{1}{2}Li_0^2$.



Solution:

$$dH = \int i^2 rdt$$

$$\Rightarrow \qquad dH = \int \left(i_o e^{\frac{-t}{L/R}} \right)^2 Rdt$$

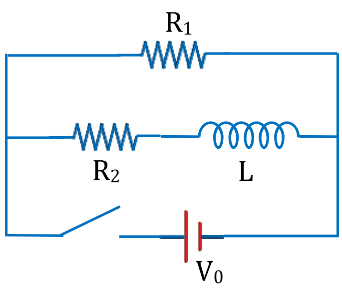
$$H\Big|_0^H = \int i_0^2 R \left(e^{\frac{-tR}{L}} \right)^2 dt$$

$$H - 0 = i_0^2 R \int e^{\frac{-2tR}{L}} dt$$

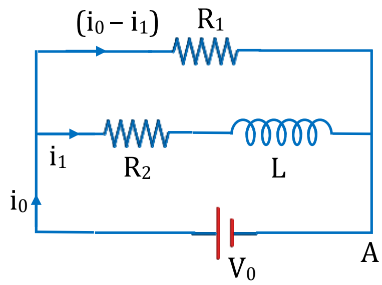
$$H - 0 = i_0^2 R \int e^{\frac{-2tR}{L}} dt = i_0^2 R e^{\frac{-2tR}{L}} \left(\frac{-L}{2R} \right)_0^\infty = \frac{1}{2}Li_0^2$$

Illustration 36:

Switch was open for long time and now it is closed at $t = 0$. Find current from R_t as a function of time.



Solution:



$$V_A + V_0 - i_1R_2 - \frac{Ldi_1}{dt} = V_A$$

$$V_0 - i_1R_2 = \frac{Ldi_1}{dt}$$

$$\frac{-dt}{L} = \frac{di}{(i_1R_2 - V_0)} \quad \Rightarrow \quad \frac{-t}{L}\Big|_0^t = \Big|_0^{i_1} \frac{\ell n(i_1R_2 - V_0)}{R_2}$$

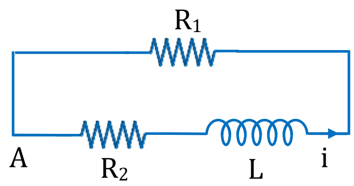
$$\frac{-t}{L} = \ell n\left(\frac{i_1R_2 - V_0}{-V_0}\right) \times \frac{1}{R_2} \quad \Rightarrow \quad \frac{-tR_2}{L} = \ell n\left(\frac{i_1R_2 - V_0}{-V_0}\right)$$

$$\frac{-tR_2}{L} = \ell n\left(\frac{V_0 - i_1R_2}{V_0}\right)$$

$$\frac{V_0 - i_1R_2}{V_0} = e^{\frac{-tR_2}{L}} \quad \Rightarrow \quad V_0 - i_1R_2 = V_0 e^{\frac{-tR_2}{L}}$$

$$i_1R_2 = V_0\left(1 - e^{-tR_2/L}\right) \quad \Rightarrow \quad i_1 = \frac{V_0}{R_2}\left(1 - e^{-tR_2/L}\right)$$

$$i_1 = \frac{V_0}{R_2}\left(1 - e^{-tR_2/L}\right) \quad \Rightarrow \quad i_0 = \frac{V_0}{R_2}$$



$$V_A - iR_2 - \frac{Ldi}{dt} - iR_1 = V_A$$

$$\frac{Ldi}{dt} = -i(R_1 + R_2) \quad \Rightarrow \quad \frac{di}{-i(R_1 + R_2)} = \frac{dt}{L}$$

$$-\frac{di}{i} = \left(\frac{R_1 + R_2}{L}\right)dt \quad \Rightarrow \quad \ell ni\Big|_{i_0}^{i} = \left|\left(\frac{R_1 + R_2}{L}\right)\right| t \Big|_0^t$$

$$-\ell n\frac{i}{i_0} = \left(\frac{R_1 + R_2}{L}\right)t \quad \Rightarrow \quad i = i_0 e^{\frac{-t}{L}(R_1 + R_2)}$$

$$i_0 = \frac{V_0}{R_2}$$

Energy stored in inductor

The energy of a capacitor is stored in the electric field between its plates. Similarly, an inductor has the capability of storing energy in its magnetic field. An increasing current in an inductor causes an emf between its terminals.

Power P = The work done per unit time $= \frac{dW}{dt} = -ei = -\left[L\frac{di}{dt}\right]i = -Li\frac{di}{dt}$

Here i = instantaneous current and L=inductance of the coil

From $dW = -dU$ (energy stored) so $= \frac{dW}{dt} = -\frac{dU}{dt}$ $\quad \therefore \quad \frac{dU}{dt} = Li\frac{di}{dt} \Rightarrow dU = Li\,di$

The total energy U supplied while the current increases from zero to final value I is.

$$U = L\int_0^I i\,di = \frac{1}{2}L\left(i^2\right)_0^I \quad \therefore\ U = \frac{1}{2}LI^2$$

the energy stored in the magnetic field of an inductor when a current I is $=\frac{1}{2}LI^2$.

The source of this energy is the external source of emf that supplies the current.

After the current has reached its final steady state value I, $\frac{di}{dt}=0$ and no more energy is input to the inductor.

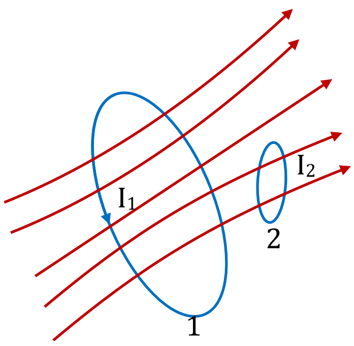
When the current decreases from I to zero, the inductor acts as a source that supplies a total amount of energy $\frac{1}{2}Li^2$ to the external circuit. If we interrupt the circuit suddenly by opening a switch the current decreases very rapidly, the induced emf is very large and the energy may be dissipated.

Magnetic energy density = $\frac{1}{2}\frac{B^2}{\mu_0}$

Mutual inductance

Consider two arbitrary conducting loops 1 and 2. Suppose that I_1 is the instantaneous current flowing around loop 1. This current generates a magnetic field B_1 which links the second circuit, giving rise to a magnetic flux ϕ_2 through that circuit. If the current I_1 doubles, then the magnetic field B_1 doubles in strength at all points in space, so the magnetic flux ϕ_2 through the second circuit also doubles. Furthermore, it is obvious that the flux through the second circuit is zero whenever the current flowing around the first circuit is zero. It follows that the flux ϕ_2 through the second circuit is directly proportional to the current I_1 flowing around the first circuit. Hence, we can write $\phi_2 = M_{21}I_1$ where the constant of proportionality M_{21} is called the mutual inductance of circuit 2 with respect to circuit 1. Similarly, the flux ϕ_2 through the first circuit due to the instantaneous current I_2 flowing around the second circuit is directly proportional to that current, so we can write $\phi_1 = M_{12}I_2$ where M_{12} is the mutual inductance of circuit 1 with respect to circuit 2. It can be shown that $\boldsymbol{M_{21} = M_{12}}$ **(Reciprocity Theorem)**. Note that M is a purely geometric quantity, depending only on the size, number of turns, relative position, and relative orientation of the two circuits. The S.I. unit of mutual inductance is called Henry (H). One Henry is equivalent to a volt-second per ampere.

Suppose that the current flowing around circuit 1 changes by an amount ΔI_1 in a small time interval Δt. The flux linking circuit 2 changes by an amount $\Delta\phi_2 = M\Delta I_1$ in the same time interval. According to Faraday's law, an emf $\varepsilon_2 = -\frac{\Delta\phi_2}{\Delta t}$ is generated around the second circuit due to the changing magnetic flux linking that circuit. Since, $\Delta\phi_2 = M\Delta I_1$, this emf can also be written $\varepsilon_2 = -M\frac{\Delta I_1}{\Delta t}$.



Thus, the emf generated around the second circuit due to the current flowing around the first circuit is directly proportional to the rate at which that current changes. Likewise, if the current I_2 flowing around the second circuit changes by an amount ΔI_2 in a time interval Δt then the emf generated around the first circuit is $\varepsilon_1 = -M\dfrac{\Delta I_2}{\Delta t}$ Note that there is no direct physical connection(coupling) between the two circuits: the coupling is due entirely to the magnetic field generated by the currents flowing around the circuits.

Note:

(1) $M \le \sqrt{L_1 L_2}$

(2) For two coils in series if mutual inductance is considered then

$L_{eg} = L_1 + L_2 \pm 2M$

Illustration 37:

Two insulated wires are wound on the same hollow cylinder, so as to form two solenoids sharing a common air-filled core. Let I be the length of the core, A the cross-sectional area of the core, N_1 the number of times the first wire is wound around the core, and N_2 the number of turns the second wire is wound around the core. Find the mutual inductance of the two solenoids, neglecting the end effects.

Solution:

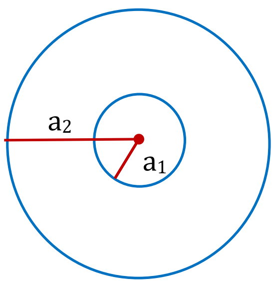
If a current I_1 flows around the first wire then a uniform axial magnetic field of strength $B_1 = \dfrac{\mu_0 N_1 I_1}{\ell}$ is generated in the core. The magnetic field in the region outside the core is of negligible magnitude. The flux linking a single turn of the second wire is $B_1 A$. Thus, the flux linking all N_2 turns of the second wire is

$$\phi_2 = N_2 B_1 A = \frac{\mu_0 N_1 N_2 I_1 A}{\ell} = M I_1 \qquad \therefore \quad M = \frac{\mu_0 N_1 N_2 A}{\ell}$$

As described previously, M is a geometric quantity depending on the dimensions of the core and the manner in which the two wires are wound around the core, but not on the actual currents flowing through the wires.

Illustration 38:

Find the mutual inductance of two concentric coils of radii a_1 and a_2 ($a_1 << a_2$) if the planes of coils are same.



Solution:

Let a current i flow in coil of radius a_2.

Magnetic field at the centre of coil = $\dfrac{\mu_0 i}{2a}$

Flux linkage = Mi

$$Mi = \frac{\mu_0 i}{2a}\pi a_1^2 \qquad \text{or} \qquad M = \frac{\mu_0 \pi a_1^2}{2a_2}$$

Illustration 39:

Solve the above question, if the planes of coil are perpendicular.

Solution:

Let a current i flow in the coil of radius a_1. The magnetic field at the centre of this coil will now be parallel to the plane of smaller coil and hence no flux will pass through it, hence $M = 0$.

Illustration 40:

Solve the above problem if the planes of coils make θ angle with each other.

Solution:

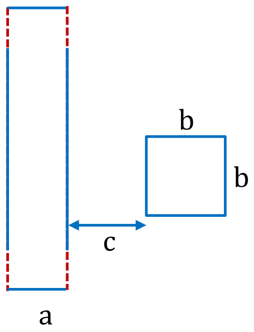
If i current flows in the larger coil, magnetic field produced at the centre will be perpendicular to the plane of larger coil.

Now the area vector of smaller coil which is perpendicular to the plane of smaller coil will make an angle θ with the magnetic field.

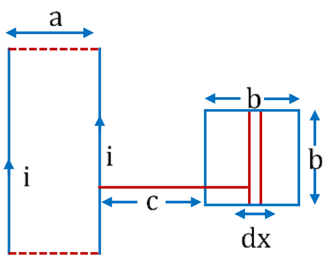
Thus flux = $\vec{B}.\vec{A} = \dfrac{\mu_0 i}{2a_2}.\pi a_1^2.\cos\theta$ or $M = \dfrac{\mu_0 \pi a_1^2.\cos\theta}{2a_2}$

Illustration 41:

Find the mutual inductance between two rectangular loops, shown in the figure.



Solution:



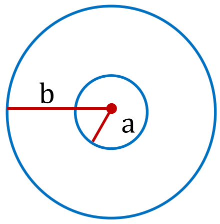
Let current i flow in the loop having ∞-ly long sides. Consider a segment of width dx at a distance x as shown flux through the region

$$d\phi = \left[\frac{\mu_0 i}{2\pi x} - \frac{\mu_0 i}{2\pi(x+a)}\right]bdx$$

$$\Rightarrow \qquad \phi = \int_c^{c+b}\left[\frac{\mu_0 i}{2\pi x} - \frac{\mu_0 i}{2\pi(x+a)}\right]bdx = \frac{\mu_0 ib}{2\pi}\left[\ell n\frac{c+b}{c} - \ell n\frac{a+b+c}{a+c}\right] = Mi$$

Illustration 42:

Figure shows two concentric coplanar coils with radii a and b ($a << b$). A current $i = 2t$ flows in the smaller loop. Neglecting self inductance of larger loop.



(a) Find the mutual inductance of the two coils.

(b) Find the emf induced in the larger coil.

(c) If the resistance of the larger loop is R find the current in it as a function of time.

Solution:

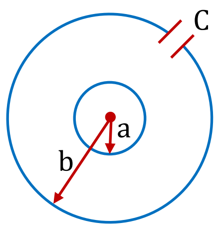
(a) To find mutual inductance, it does not matter in which coil we consider current and in which flux is calculated (Reciprocity theorem) Let current i be flowing in the larger coil. Magnetic field at the centre = $\frac{\mu_0 i}{2b}$.

Flux through the smaller coil = $\frac{\mu_0 i}{2b}\pi a^2$ $\qquad \therefore \quad M = \frac{\mu_0}{2b}\pi a^2$

(b) $$|\text{emf induced in larger coil}| = M\left[\left(\frac{di}{dt}\right)\text{in smaller coil}\right] = \frac{\mu_0}{2b}\pi a^2(2) = \frac{\mu_0 \pi a^2}{b}$$

(c) Current in the larger coil = $\frac{\mu_0 \pi a^2}{bR}$.

Illustration 43:



If the current in the inner loop changes according to $i = 2t^2$ then, find the current in the capacitor as a function of time. ($a << b$)

Solution:

$$M = \frac{\mu_0}{2b}\pi a^2$$

$$|\text{emf induced in larger coil}| = M\left[\left(\frac{di}{dt}\right)\text{in smaller coil}\right]$$

$$\Rightarrow \qquad e = \frac{\mu_0}{2b}\pi a^2(4t) = \frac{2\mu_0 \pi a^2 t}{b}$$

Applying KVL :

$$+e - \frac{q}{C} - iR = 0$$

$$\frac{2\mu_0 \pi a^2 t}{b} - \frac{q}{C} - iR = 0$$

Differentiate wrt time :-

$$\frac{2\mu_0 \pi a^2}{b} - \frac{i}{C} - \frac{di}{dt}R = 0$$

On solving it

$$i = \frac{2\mu_0 \pi a^2 C}{b}\left[1 - e^{-t/RC}\right]$$

Coupled Coils:

The coils of figure have a mutual inductance of magnitude M. If Q and R are connected and a current I passed through both coils in series then, considered as one circuit, the total flux linkage is $L_A I + L_B I + 2MI$ and the

$$L_1 = L_A + L_B + 2M$$

If, on the other hand, Q and S are connected and a current passed from P to R the total flux linkage is now $L_A I + L_B I + 2MI$ (the self-fluxes must be positive and the mutual fluxes oppose them) so that

$$L_2 = L_A + L_B - 2M$$

The same effect could be obtained by retaining the connections but rotating one coil through 180°–a method which can therefore be used to obtain a self inductance varying smoothly from L_1 to L_2. Alternatively, because $M = \frac{1}{4}(L_1 - L_2)$ from above equation a mutual inductance may be obtained from measurements only of self-inductance, which are often simpler.

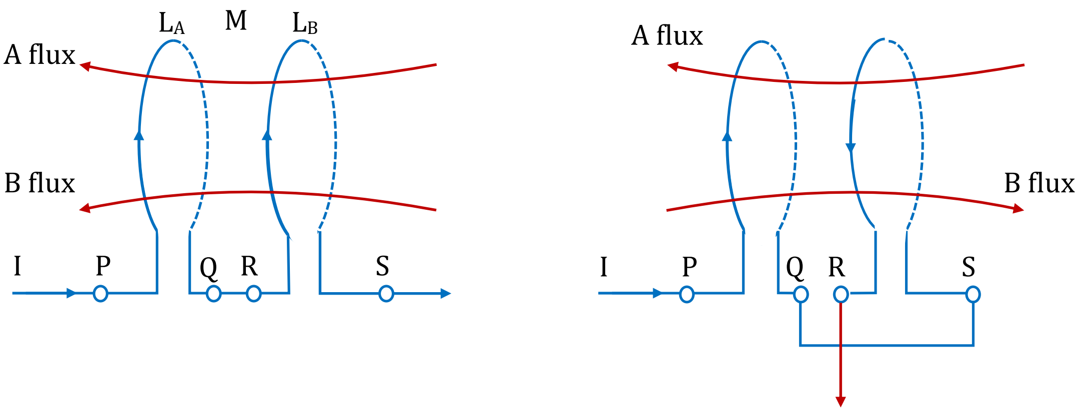


Figure : Two coupled coils.

Magnetic energy due to mutual inductance

Consider now two circuits with self inductances L_1 and L_2 and a mutual inductance M_{21}. We do not now assume that $M_{21} = M_{12}$.

First insert an electromotance (e.g. a cell) in circuit 1 so that the current in it grows as above to its final value I_1, keeping any induced currents in 2 at zero by extra electromotances (these do no work since $I_2 = 0$). The energy stored, as above, is $\frac{1}{2}L_1 I_1^2$.

Now insert an electromotance in circuit 2 to make the current in it grow to its final value I_2. When it has done so, naturally an energy $\frac{1}{2}L_2 I_2^2$ is stored in 2, but while I_2 is changing an extra electromotance $-M_{21}di_2/dt$ is induced in circuit 1, i_2 being the instantaneous value. We insert in 1 an adjustable electromotance of value $+M_{21}di_2/dt$, thus keeping I_1 constant (and not upsetting the $\frac{1}{2}L_1 I_1^2$ already stored). This time, however, the extra electromotance passes the current I_1 and thus puts energy into the system at the rate $I_1 M_{21} di_2/dt$. The total energy put in by this extra electromotance is therefore $I_1 M_{21} di_2$ integrated from 0 to I_2, or $M_{21} I_1 I_2$.

Hence for a pair of circuits

$$U_M = \frac{1}{2}L_1 I_1^2 + M_{21} I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

And because the same situation can be reached by allowing I_2 to grow first and finding that U_M is then $\frac{1}{2}L_1I_1^2 + M_{21}I_1I_2 + \frac{1}{2}L_2I_2^2$, it follows that $M_{12} = M_{21}$

There is only one mutual inductance M between two circuits and the magnetic energy is

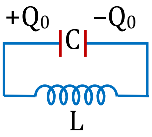
Two coupled circuits: $\boxed{U_M = \frac{1}{2}L_1I_1^2 + M_{21}I_1I_2 + \frac{1}{2}L_2I_2^2}$

This can also be written as

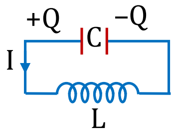
$$U_M = \frac{1}{2}I_1\left(L_1I_1 + MI_2\right) + \frac{1}{2}I_2\left(L_2I_2 + MI_1\right) = \frac{1}{2}I_1\phi_1 + \frac{1}{2}I_2\phi_2$$

Where ϕ_1 and ϕ_2 are total magnetic fluxes linking each circuit.

LC oscillations:



At $t = 0$



At $t = t$

When capacitor C is completely charged up to Q_0 and connected to an inductor L at $t = 0$ then at $t = t$

$$L\frac{dI}{dt} - \frac{Q}{C} = 0, \qquad -L\frac{d^2Q}{dt^2} - \frac{Q}{C} = 0 \quad Q = -LC\frac{d^2Q}{dt^2}$$

$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$ therefore, charge Q oscillates with $Q = Q_0 \cos \omega t$

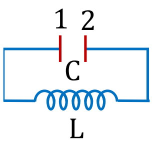
Hence initial phase of oscillation is $\frac{\pi}{2}$ and angular frequency $\omega = \frac{1}{\sqrt{LC}}$

One can prove that the energy in the system remains conserved.

Therefore $\frac{Q_0^2}{2C} + 0 = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{1}{2}LI_0^2 + 0$

Illustration 44:

Consider a $L - C$ oscillation circuit. Circuit elements has zero resistance. Initially at $t = 0$ all the energy is stored in the form of electric field and plate-1 is having positive charge:



At time $t = t_1$ plate-2 attains half of the maximum $+ve$ change for the first time. Value of t_1 is :

(A) $\frac{2\pi}{3}\sqrt{LC}$

(B) $\frac{\pi}{3}\sqrt{LC}$

(C) $\frac{4\pi}{3}\sqrt{LC}$

(D) $\pi\sqrt{LC}$

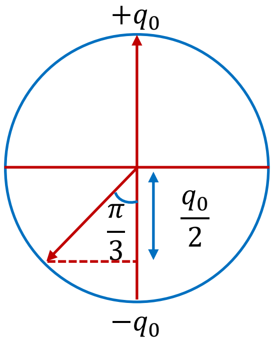
Solution:

$q_1 = q_0 \sin(\omega t + \pi/2)$

At $\quad t = t_1, \quad q_1 = -\dfrac{q_0}{2}$

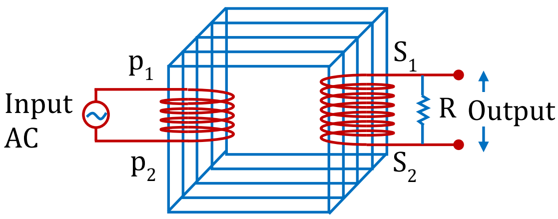
$$t_1 = \frac{\pi - \frac{\pi}{3}}{\omega}$$

$$= \frac{2\pi}{3\omega} = \frac{2\pi}{3}\sqrt{LC}$$



Transformers

One of the great advantages of ac over dc for electric-power distribution is that it is much easier to step voltage levels up and down with ac than with dc. The necessary conversion is accomplished by a static device called transformer using the principle of mutual induction.



Laminated core

The figure shows an idealised transformer which consists of two coils or windings, electrically insulated from each other but wound on the same core. The winding to which power is supplied is called primary, the winding from which power is delivered is called the secondary.

The ac source causes an alternating current in the primary which sets up an alternating flux in the core and this induces an emf in each winding of secondary in accordance with Faraday's law. For ideal transformer we assume that primary has negligible resistance and all the flux in core links both primary and secondary. The primary winding has N_1 turns and secondary has N_2 turns. When the magnetic flux changes because of changing currents in the two coils, the resulting induced emf are

$$E_1 = -N_1 \frac{d\phi_B}{dt} \text{ and } E_2 = -N_2 \frac{d\phi_B}{dt}$$

The flux per turn B is same in both primary and the secondary so that the emf per turn is same in each. The ratio of secondary emf E_2 to the primary emf E_1 is therefore equal at any instant to the ratio of secondary to primary turns.

$$\frac{E_2}{E_l} = \frac{N_2}{N_1}$$

If the windings have zero resistance, the induced emf 1 and 2 are equal to the terminal voltage across the primary and the secondary respectively, hence

$$\frac{V_2}{V_l} = \frac{N_2}{N_1}$$

If $N_2 > N_1$ then $V_2 > V_1$ and we have step up transformer, if $N_2 < N_1$ then $V_2 < V_1$ and we have a step down transformer.

If the transformer is assumed to be 100% efficient (no energy losses) the power input is equal to the power output i.e.

$$I_2V_2 = I_1V_1$$

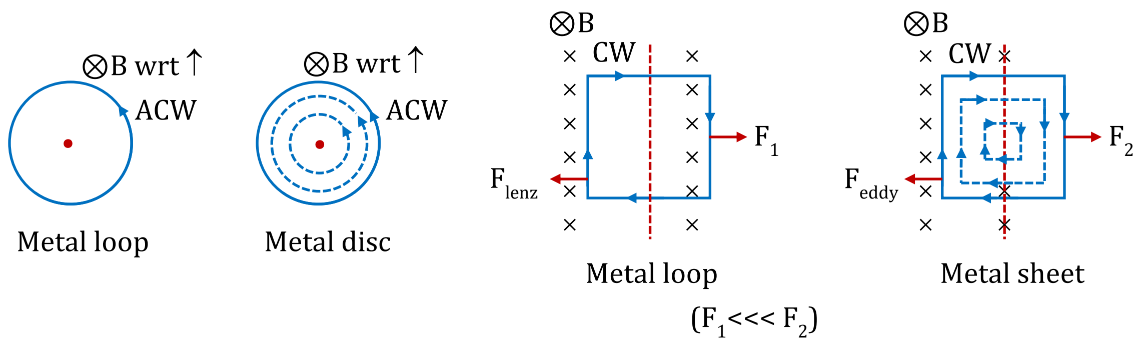
$$\therefore \quad \frac{I_2}{I_1} = \frac{N_1}{N_2}$$

All the currents and voltages derived above have same frequency as that of source. The equations obtained above apply to ideal transformers, although some energy is lost but well designed transformers have efficiency more than 95%, this is a good approximation. The causes of energy losses and their rectification is given below

	Cause		Rectification
1.	Due to poor design and air gaps in the core, all the flux due to primary does not pass through the secondary.	**1.**	By winding the primary and secondary coil one over the other.
2.	Resistance of windings causes I^2R loss.	**2.**	In high current, low voltage, these are minimised by using thick wire.
3.	The alternating magnetic flux induces eddy currents in the core and causes heating.	**3.**	By using laminated core it can be reduced.
4.	Alternating magnetisating of core causes hystersis loss.	**4.**	It is kept minimum by using a magnetic material having low hystersis loss. (e.g. soft iron)

Eddy current :

Eddy currents (or Focalt's currents) [Experimental verification by Focalt]



Def. It is a group of induced currents which are produced, when metal bodies placed in time varying magnetic field or they moves in external magnetic field in such a way that flux through them changes with respect to time.

ALTERNATING CURRENT

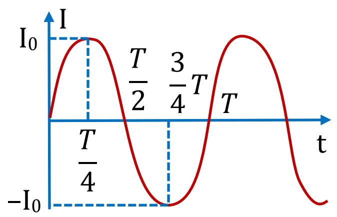
Alternating Current and Voltage

Voltage or current is said to be alternating if it is changed continuously in magnitude and periodically in direction with time. It can be represented by a sine curve or cosine curve

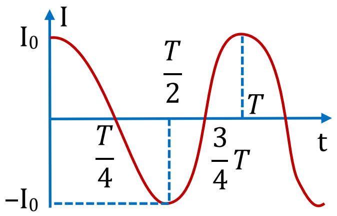
$I = I_0 \sin \omega t$ or $I = I_0 \cos \omega t$

where I = Instantaneous value of current at time t, I_0 = Amplitude or peak value

ω = Angular frequency $\omega = \dfrac{2\pi}{T} = 2\pi f$ T = time period f = frequency



I as a sine function of t



I as a cosine function of t

Amplitude of AC:

The maximum value of current in either direction is called peak value or the amplitude of current. It is represented by I_0. Peak to peak value = $2I_0$.

Periodic Time:

The time taken by alternating current to complete one cycle of variation is called periodic time or time period of the current.

Frequency:

The number of cycles completed by an alternating current in one second is called the frequency of the current.

Unit : cycle/s ; (Hz)

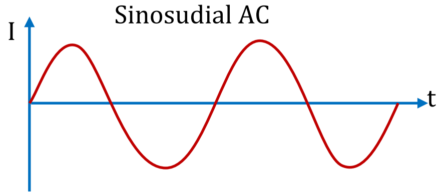
In India : $f = 50\ Hz$, supply voltage = 220 volt

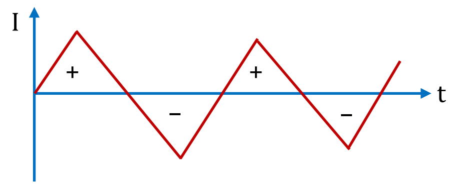
In USA : $f = 60\ Hz$, supply voltage = 110 volt

Condition required for current/ voltage to be Alternating:

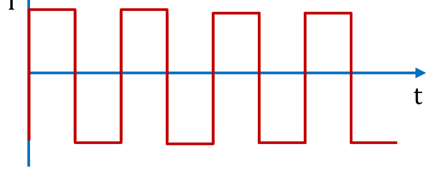
- Amplitude is constant.
- Alternate half cycle is positive and half negative.

The alternating current continuously varies in magnitude and periodically reverses its direction.

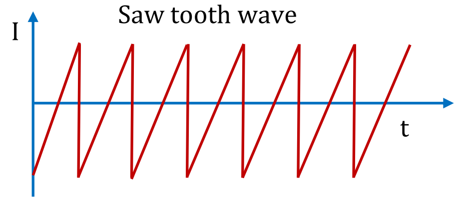


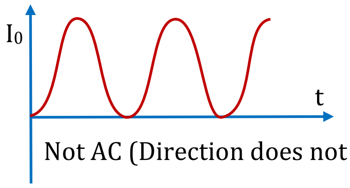


Square wave AC

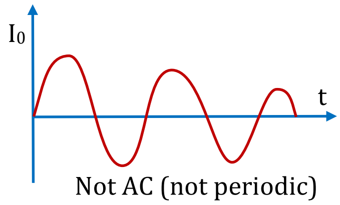


Saw tooth wave

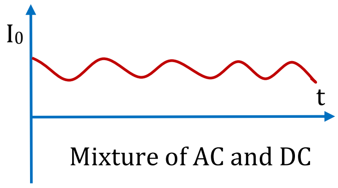




Not AC (Direction does not change)



Not AC (not periodic)



Mixture of AC and DC

Average Value or Mean Value

The mean value of *AC* over any half cycle (either positive or negative) is that value of *DC* which would send same amount of charge through a circuit as is sent by the *AC* through same circuit in the same time.

Average value of current for half cycle $< I > = \dfrac{\int_0^{T/2} I dt}{\int_0^{T/2} dt}$

Average value of $I = I_0 \sin \omega t$ over the positive half cycle :

$$I_{av} = \frac{\int_0^{\frac{T}{2}} I_0 \sin \omega t dt}{\int_0^{\frac{T}{2}} dt} = \frac{2I_0}{\omega T}[-\cos \omega t]_0^{\frac{T}{2}} = \frac{2I_0}{\pi} 1$$

$< \sin \theta > = < \sin 2\theta > = 0$
$< \cos \theta > = < \cos 2\theta > = 0$
$< \sin \theta \cos \theta > = 0$
$< \sin^2 \theta > = < \cos^2 \theta > = \frac{1}{2}$

- For symmetric *AC*, average value over full cycle = 0,

Average value of sinusoidal *AC*

Full cycle	(+ve) half cycle	(–ve) half cycle
0	$\frac{2I_0}{\pi}$	$-\frac{2I_0}{\pi}$

As the average value of AC over a complete cycle is zero, it is always defined over a half cycle which must be either positive or negative.

Maximum Value

- $I = a \sin\theta \Rightarrow I_{Max.} = a$
- $I = a + b \sin \theta \Rightarrow I_{Max.} = a + b$ (if a and $b > 0$)
- $I = a \sin\theta + b \cos \theta \Rightarrow I_{Max.} = \sqrt{a^2 + b^2}$
- $I = a \sin^2 \theta \Rightarrow I_{Max.} = a$ ($a > 0$)

Root Mean Square (rms) Value

It is value of *DC* which would produce same heat in given resistance in given time as is done by the alternating current when passed through the same resistance for the same time.

$I_{rms} = \sqrt{\dfrac{\int_0^T I^2 dt}{\int_0^T dt}}$ *rms* value = Virtual value = Apparent value

rms **value of** $I = I_0 \sin \omega t$ **:**

$$I_{rms} = \sqrt{\frac{\int_0^T (I_0 \sin\omega t)^2 dt}{\int_0^T dt}} = \sqrt{\frac{I_0^2}{T}\int_0^T \sin^2\omega t\, dt}$$

$$= I_0\sqrt{\frac{1}{T}\int_0^T \left[\frac{1-\cos 2\omega t}{2}\right]dt} = I_0\sqrt{\frac{1}{T}\left[\frac{t}{2}-\frac{\sin 2\omega t}{2\times 2\omega}\right]_0^T} = \frac{I_0}{\sqrt{2}}$$

- If nothing is mentioned then values printed in *a.c.* circuit on electrical appliances, any given or unknown values, reading of *AC* meters are assumed to be *RMS*.

Current	Average	Peak	RMS	Angular frequency
$I_1 = I_0 \sin \omega t$	0	I_0	$\frac{I_0}{\sqrt{2}}$	ω
$I_2 = I_0 \sin \omega t \cos \omega t = \frac{I_0}{2}\sin 2\omega t$	0	$\frac{I_0}{2}$	$\frac{I_0}{2\sqrt{2}}$	2ω
$I_3 = I_0 \sin\omega t + I_0 \cos\omega t$	0	$\sqrt{2}\, I_0$	I_0	ω

- For above varieties of current $rms = \frac{\text{Peak value}}{\sqrt{2}}$

Illustration 45:

If $I = 2\sqrt{t}$ ampere then calculate average and rms values over $t = 2$ to $4s$.

Solution:

$$\langle I\rangle = \frac{\int_2^4 2\sqrt{t}.dt}{\int_2^4 dt} = \frac{4}{3}\frac{(t^{\frac{3}{2}})_2^4}{(t)_2^4} = \frac{2}{3}\left[8-2\sqrt{2}\right] \quad \text{and} \quad I_{rms} = \sqrt{\frac{\int_2^4 (2\sqrt{t})^2 dt}{\int_2^4 dt}} = \sqrt{\frac{\int_2^4 4t dt}{2}} = \sqrt{2\left[\frac{t^2}{2}\right]_2^4} = 2\sqrt{3}$$

Illustration 46:

Find the time required for 50Hz alternating current to change its value from zero to rms value.

Solution:

$$\because \quad I = I_0 \sin \omega t \qquad \therefore \frac{I_0}{\sqrt{2}} = I_0 \sin\omega t \Rightarrow \sin\omega t = \frac{1}{\sqrt{2}} \Rightarrow \omega t = \frac{\pi}{4}$$

$$\Rightarrow \quad \left(\frac{2\pi}{T}\right)t = \frac{\pi}{4} \qquad \Rightarrow t = \frac{T}{8} = \frac{1}{8\times 50} = 2.5\ ms$$

Illustration 47:

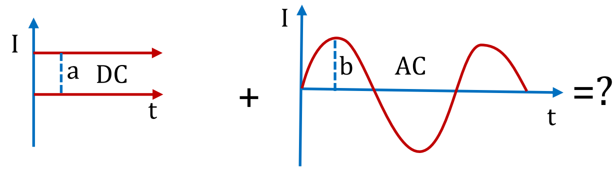
If $E = 20 \sin (100\pi\, t)$ volt then calculate value of E at $t = \frac{1}{600}$ *s.*

Solution:

At $\quad t = \frac{1}{600}$ *s*, $E = 20 \text{ Sin}\left[100\pi\times\frac{1}{600}\right] = 20 \sin\left[\frac{\pi}{6}\right] = 20 \times \frac{1}{2} = 10V$

Illustration 48:

If a direct current of value a ampere is superimposed on an alternating current $I = b \sin \omega t$ flowing through a wire, what is the effective value of the resulting current in the circuit?



Solution:

As current at any instant in the circuit will be,

$$I = I_{DC} + I_{AC} = a + b \sin\omega t$$

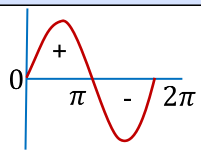
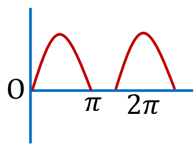
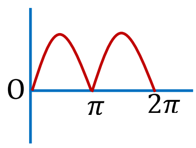
$$\therefore \quad I_{eff} = \sqrt{\frac{1}{T}\int_0^T I^2 dt} = \sqrt{\frac{1}{T}\int_0^T (a + b\sin\omega t)^2 dt} = \sqrt{\frac{1}{T}\int_0^T (a^2 + 2ab\sin\omega t + b^2\sin^2\omega t)dt}$$

But as $\frac{1}{T}\int_0^T \sin\omega t dt = 0$

and $\frac{1}{T}\int_0^T \sin^2\omega t dt = \frac{1}{2}$

$$\therefore \quad I_{eff} = \sqrt{a^2 + \frac{1}{2}b^2}$$

Some Important wave forms and their RMS and Average Value:

Nature of wave form	Wave–form	RMS Value	Average or mean Value
Sinusoidal		$\frac{I_0}{\sqrt{2}} = 0.707\ I_0$	$\frac{2I_0}{\pi} = 0.637\ I_0$
Half wave rectified		$\frac{I_0}{2} = 0.5\ I_0$	$\frac{I_0}{\pi} = 0.318\ I_0$
Full wave rectified		$\frac{I_0}{\sqrt{2}} = 0.707\ I_0$	$\frac{2I_0}{\pi} = 0.637\ I_0$

Phase and phase difference

(a) Phase :

$I = I_0 \sin(\omega t \pm \phi)$

Initial phase = ϕ (it does not change with time)

Instantaneous phase = $\omega t \pm \phi$ (it changes with time)

- Phase decides both value and sign. ◎ **Unit:** radian

(b) Phase difference :

Voltage $V = V_0 \sin(\omega t + \phi_1)$ Current $I = I_0 \sin(\omega t + \phi_2)$

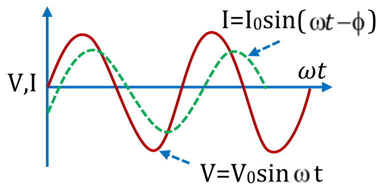
- Phase difference of I w.r.t. V $\phi = \phi_2 - \phi_1$
- Phase difference of V w.r.t. I $\phi = \phi_1 - \phi_2$

Lagging and leading Concept:

(a) V leads I or I lags $V \rightarrow$ It means, V reach maximum before I

Let if $V = V_0 \sin \omega t$ then $I = I_0 \sin(\omega t - \phi)$

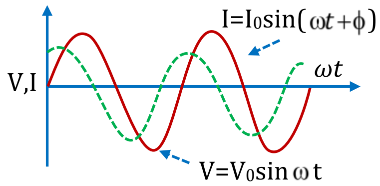
and if $V = V_0 \sin(\omega t + \phi)$ then $I = I_0 \sin \omega t$



(b) V lags I or I leads $V \rightarrow$ It means V reach maximum after I

Let if $V = V_0 \sin \omega t$ then $I = I_0 \sin(\omega t + \phi)$

and if $V = V_0 \sin(\omega t - \phi)$ then $I = I_0 \sin \omega t$



Phasor and Phasor diagram

A diagram representing alternating current and voltage (of same frequency) as vectors (phasor) with the phase angle between them is called phasor diagram.

Let $V = V_0 \sin \omega t$ and $I = I_0 \sin(\omega t + \phi)$

In figure (a) two arrows represents phasors. The length of phasors represents the maximum value of quantity. The projection of a phasor on y-axis represents the instantaneous value of quantity

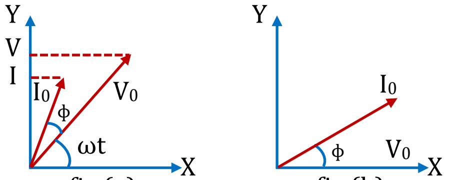


fig (a) fig (b)

Illustration 49:

The Equation of current in AC circuit is $I = 4\sin\left[100\pi t + \frac{\pi}{3}\right]$ A. Calculate.

(i) RMS Value (ii) Peak Value (iii) Frequency (iv) Initial phase (v) Current at $t = 0$

Solution:

(i) $I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}A$

(ii) Peak value $I_0 = 4A$

(iii) $\because$ $\omega = 100\,\pi\ rad/s$

$\therefore$ Frequency $f = \frac{100\pi}{2\pi} = 50\ Hz$

(iv) Initial phase $= \frac{\pi}{3}$

(v) At $t = 0$, $I = 4\sin\left[100\pi \times 0 + \frac{\pi}{3}\right] = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}\ A$

Illustration 50:

If $I = I_0 \sin \omega t$, $E = E_0 \cos\left[\omega t + \frac{\pi}{3}\right]$. Calculate phase difference between E and I.

Solution:

$I = I_0 \sin \omega t$ and $E = E_0 \sin\left[\frac{\pi}{2} + \omega t + \frac{\pi}{3}\right]$

$\therefore$ Phase difference = $\frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

Illustration 51:

If $E = 500 \sin (100\ \pi t)$ volt then calculate time taken to reach from zero to maximum.

Solution:

$\because$ $\omega = 100\ \pi$ $\Rightarrow T = \frac{2\pi}{100\pi} = \frac{1}{50} s$, time taken to reach from zero to maximum $= \frac{T}{4} = \frac{1}{200} s$

Illustration 52:

Show that average heat produced during a cycle of AC is same as produced by DC with $I = I_{rms}$.

Solution:

For AC, $I = I_0 \sin \omega t$, the instantaneous value of heat produced (per second) in a resistance R is,

$H = I^2 R = I_0^2 \sin^2 \omega t \times R$ the average value of heat produced during a cycle is :

$$H_{av} = \frac{\int_0^T H dt}{\int_0^T dt} = \frac{\int_0^T (I_0^2 \sin^2 \omega t \times R) dt}{\int_0^T dt} = \frac{1}{2} I_0^2 R \qquad \left[\because \int_0^T I_0^2 \sin^2 \omega t dt = \frac{1}{2} I_0^2 T\right]$$

$$\Rightarrow \quad H_{av} = \left(\frac{I_0}{\sqrt{2}}\right)^2 R = I_{rms}^2 R \qquad \text{...(i)}$$

However, in case of DC, $H_{DC} = I^2 R$ ...(ii)

$\because$ $I = I_{rms}$ so from equation (i) and (ii) $H_{DC} = H_{av}$

AC produces same heating effects as DC of value $I = I_{rms}$. This is also why AC instruments which are based on heating effect of current give rms value.

Different types of AC circuits:

In order to study the behaviour of $A.C.$ circuits we classify them into two categories :

(a) Simple circuits containing only one basic element i.e. resistor (R) or inductor (L) or capacitor (C) only.

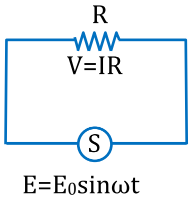
(b) Complicated circuit containing any two of the three circuit elements R, L and C or all of three elements.

AC Circuit Containing Pure Resistance:

Let at any instant the current in the circuit = I.

Potential difference across the resistance = IR.

with the help of Kirchoff's circuital law $E - IR = 0$

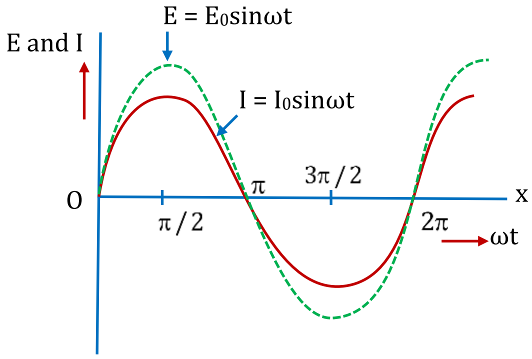


$\Rightarrow$ $E_0 \sin \omega t = IR$

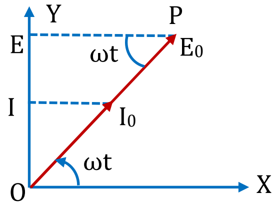
$$\Rightarrow \quad I = \frac{E_0}{R} \sin \omega t = I_0 \sin \omega t$$

($I_0 = \frac{E_0}{R}$ = peak or maximum value of current)

Alternating current developed in a pure resistance is also of sinusoidal nature. In an *a.c.* circuits containing pure resistance, the voltage and current are in the same phase. The vector or phasor diagram which represents the phase relationship between alternating current and alternating *e.m.f.* as shown in figure.



In the *a.c.* circuit having R only, as current and voltage are in the same phase, hence in fig. both phasors E_0 and I_0 are in the same direction, making an angle ωt with OX. Their projections on Y-axis represent the instantaneous values of alternating current and voltage.



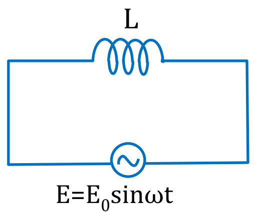
i.e. $I = I_0 \sin \omega t$ and $E = E_0 \sin \omega t$

Since $I_0 = \dfrac{E_0}{R}$, Hence $\dfrac{I_0}{\sqrt{2}} = \dfrac{E_0}{R\sqrt{2}}$

$\Rightarrow \quad I_{rms} = \dfrac{E_{rms}}{R}$

AC circuit containing pure inductance:

A circuit containing a pure inductance L (having zero ohmic resistance) connected with a source of alternating emf.



Let the alternating e.m.f., $E = E_0 \sin \omega t$

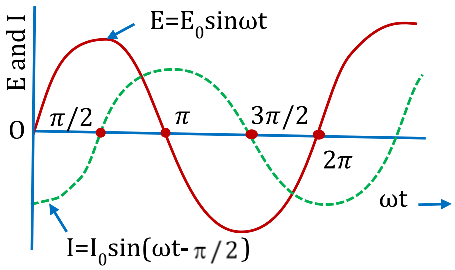
When *a.c.* flows through the circuit, emf induced across inductance, $E = -L\dfrac{dI}{dt}$

Negative sign indicates that induced *emf* acts in opposite direction to that of applied *emf*.

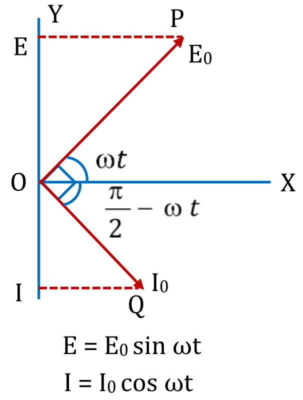
Because there is no other circuit element present in the circuit other than inductance so with the help of Kirchhoff's circuital law $E + \left(-L\dfrac{dI}{dt}\right) = 0 \Rightarrow E = L\dfrac{dI}{dt}$ so we get $I = \dfrac{E_0}{\omega L}\sin\left(\omega t - \dfrac{\pi}{2}\right)$

Maximum current, $I_0 = \dfrac{E_0}{\omega L} \times 1 = \dfrac{E_0}{\omega L}$, hence, $I = I_0 \sin\left(\omega t - \dfrac{\pi}{2}\right)$

In a pure inductive circuit current always lags behind the emf by $\dfrac{\pi}{2}$.



Or alternating *emf* leads the *a. c.* by a phase angle of $\frac{\pi}{2}$.



Expression $I_0 = \frac{E_0}{\omega L}$ resembles the expression $\frac{E}{I} = R$.

This non-resistive opposition to the flow of *A.C.* in a circuit is called the inductive reactance (X_L) of the circuit.

$X_L = \omega L = 2\,\pi f L$ where f = frequency of *A.C.*

Unit of X_L : ohm

(ωL) = Unit of L × Unit of ω = henry × $\sec^{-1}$

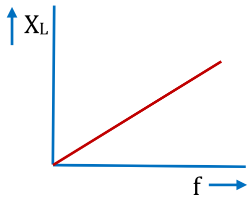
$$= \frac{\text{Volt}}{\text{Ampere/sec}} \times \sec^{-1} = \frac{\text{Volt}}{\text{Ampere}} = \text{ohm}$$

Inductive reactance $X_L \propto f$

Higher the frequency of *A.C.* higher is the inductive reactance offered by an inductor in an *A.C.* circuit.

For *d.c.* circuit, $f = 0$

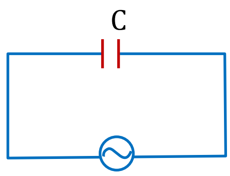
$\therefore \quad X_L = \omega L = 2\,\pi f L = 0$



Hence, inductor offers no opposition to the flow of *d.c.* whereas a resistive path to *a.c.*

AC circuit containing pure capacitance

A circuit containing an ideal capacitor of capacitance C connected with a source of alternating *emf* as shown in fig. The alternating *e.m.f.* in the circuit $E = E_0 \sin \omega t$

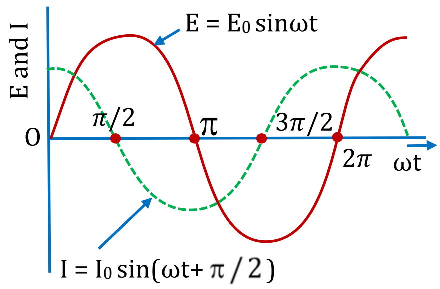


$E = E_0 \sin \omega t$

When alternating *e.m.f.* is applied across the capacitor a similarly varying alternating current flows in the circuit.

The two plates of the capacitor become alternately positively and negatively charged and the magnitude of the charge on the plates of the capacitor varies sinusoidally with time. Also, the electric field between the plates of the capacitor varies sinusoidally with time. Let, at any instant charge on the capacitor = q.

Instantaneous potential difference across the capacitor,



$$E = \frac{q}{C}$$

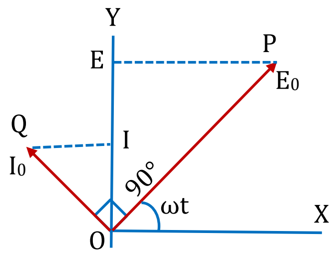
$\Rightarrow \quad q = C\,E$

$\Rightarrow \quad q = CE_0 \sin \omega t$

The instantaneous value of current

$$I = \frac{dq}{dt} = \frac{d}{dt}(CE_0 \sin \omega t) = CE_0 \omega \cos \omega t$$

$$\Rightarrow \quad I = \frac{E_0}{(1/\omega C)} \sin\left(\omega t + \frac{\pi}{2}\right) = I_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$



where $I_0 = \omega C V_0$

In a pure capacitive circuit, the current always leads the *e.m.f.* by a phase angle of $\pi/2$. The alternating *emf* lags behinds the alternating current by a phase angle of $\pi/2$.

Important Points

$\frac{E}{I}$ is the resistance R when both E and I are in phase, in present case they differ in phase by $\frac{\pi}{2}$, hence $\frac{1}{\omega C}$ is not the resistance of the capacitor, the capacitor offer opposition to the flow of *A.C.* This non-resistive opposition to the flow of *A.C.* in a pure capacitive circuit is known as capacitive reactance $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$.

Unit of X_C : ohm

Capacitive reactance X_C is inversely proportional to frequency of *A.C.* X_C decreases as the frequency increases.

This is because with an increase in frequency, the capacitor charges and discharges rapidly following the flow of current.

For *d.c.* circuit $f = 0$

$\therefore \quad X_C = \frac{1}{2\pi f C} = \infty$, but has a very small value for *a.c.*

This shows that capacitor blocks the flow of *d.c.* but provides an easy path for *a.c.*

Individual Components (R or L or C)

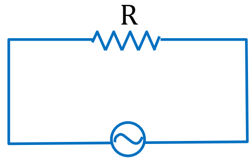
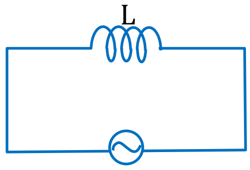
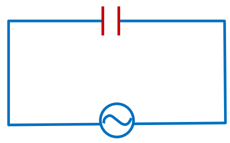
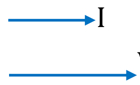
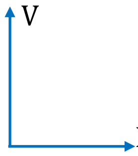
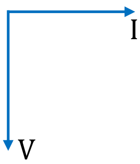
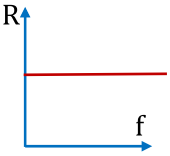
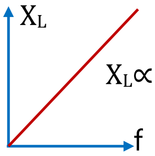
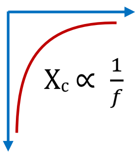
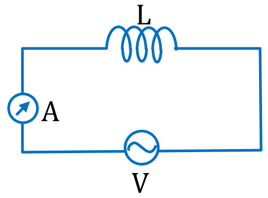
Term	R	L	C
Circuit			
Supply Voltage	$V = V_0 \sin \omega t$	$V = V_0 \sin \omega t$	$V = V_0 \sin \omega t$
Current	$I = I_0 \sin \omega t$	$I = I_0 \sin (\omega t - \frac{\pi}{2})$	$I = I_0 \sin (\omega t + \frac{\pi}{2})$
Peak Current	$I_0 = \frac{V_0}{R}$	$I_0 = \frac{V_0}{\omega L}$	$I_0 = \frac{V_0}{1/\omega C} = V_0 \omega C$
Impedance (Ω)	$\frac{V_0}{I_0} = R$	$\frac{V_0}{I_0} = \omega L = X_L$	$\frac{V_0}{I_0} = \frac{1}{\omega C} = X_C$
$\boldsymbol{Z = \frac{V_0}{I_0} = \frac{V_{rms}}{I_{rms}}}$	R = Resistance	X_L=Inductive reactance	X_C=Capacitive reactance
Phase difference	zero (in same phase)	$+\frac{\pi}{2}$ (V leads I)	$-\frac{\pi}{2}$ (V lags I)
Phasor diagram			
Variation of *Z* with *f*	 R does not depend on f		
$\boldsymbol{G, S_L, S_C}$ **(mho, seiman)**	$G = 1/R$ = conductance	$S_L = 1/X_L$ Inductive susceptance	$S_C = 1/X_C$ Capacitive susceptance
Behaviour of device in *D.C.* and *A.C*	Same in *A C* and *D C*	*L* passes *DC* easily (because $X_L = 0$) while gives a high impedance for the *A.C.* of high frequency ($X_L \propto f$)	*C* - blocks *DC* (because $X_C = \infty$) while provides an easy path for the *A.C.* of high frequency $\left[X_C \propto \frac{1}{f}\right]$
Ohm's law	$V_R = IR$	$V_L = IX_L$	$V_C = IX_C$

Illustration 53:

In given circuit applied voltage $V = 50 \sin 100\pi t$ volt and ammeter reading is $2A$ then calculate value of L.



Solution:

$V_{rms} = I_{rms} X_L$

$\because$ Reading of ammeter = I_{rms}

$$X_L = \frac{V_{rms}}{I_{rms}} = \frac{V_0}{\sqrt{2}I_{rms}} = \frac{50\sqrt{2}}{\sqrt{2}\times 2} = 25\ \Omega$$

$$\Rightarrow \quad L = \frac{X_L}{\omega} = \frac{25}{100\pi} = \frac{1}{4\pi}\ H$$

Illustration 54:

A 50 W, 100 V lamp is to be connected to an AC main of 200 V, 50 Hz. What capacitance is essential to be put in series with the lamp ?

Solution:

$\because$ Resistance of the lamp $R = \frac{V^2}{P} = \frac{(100)^2}{50} = 200\ \Omega$ and the maximum current $I = \frac{V}{R} = \frac{100}{200} = \frac{1}{2} A$

$\therefore$ When the lamp is put in series with a capacitance and run at 200 V AC, from $V = IZ$

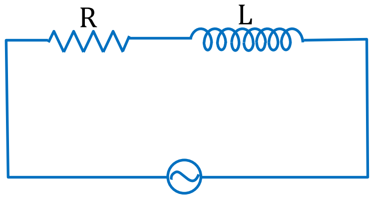
$$Z = \frac{V}{I} = \frac{200}{\frac{1}{2}} = 400\Omega \quad \text{Now as in case of } C\text{–}R \text{ circuit } Z = \sqrt{R^2 + \frac{1}{(\omega C)^2}}$$

$$\Rightarrow \quad R^2 + \frac{1}{(\omega C)^2} = (400)^2 \quad \Rightarrow \frac{1}{(\omega C)^2} = 16\times 10^4 - (200)^2 = 12\times 10^4 \quad \Rightarrow \frac{1}{\omega C} = \sqrt{12}\times 10^2$$

$$\Rightarrow \quad C = \frac{1}{100\pi\times\sqrt{12}\times 10^2} F = \frac{100}{\pi\sqrt{12}} \mu F = 9.2\mu F$$

Resistance and Inductance in Series (R-L Circuit)

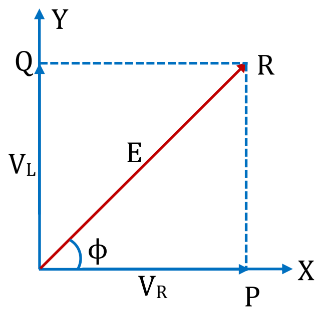
A circuit containing a series combination of a resistance R and an inductance L, connected with a source of alternating *e.m.f.* E as shown in figure.



Phasor Diagram for L-R Circuit

Let in a *L-R* series circuit, applied alternating emf is $E = E_0 \sin \omega t$. As *R* and *L* are joined in series, hence current flowing through both will be same at each instant. Let *I* be the current in the circuit at any instant and V_L and V_R the potential differences across *L* and *R* respectively at that instant.

Then $V_L = IX_L$ and $V_R = IR$



Now, V_R is in phase with the current while V_L leads the current by .

So V_R and V_L are mutually perpendicular (Note : $E \neq V_R + V_L$)

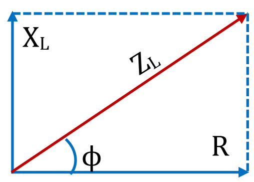
The vector *OP* represents V_R (which is in phase with *I*),

while *OQ* represents V_L (which leads *I* by 90°).

The resultant of V_R and V_L = the magnitude of vector OR $E = \sqrt{V_R^2 + V_L^2}$

Thus $E^2 = V_R^2 + V_L^2 = I^2 (R^2 + X_L^2)$

$$\Rightarrow \quad I = \frac{E}{\sqrt{R^2 + X_L^2}}$$



The phasor diagram shown in fig. also shows that in *L-R* circuit the applied emf *E* leads the current *I* or conversely the current *I* lags behind the *e.m.f. E.* by a phase angle ϕ $\tan\phi = \frac{V_L}{V_R} = \frac{IX_L}{IR} = \frac{X_L}{R} = \frac{\omega L}{R}$

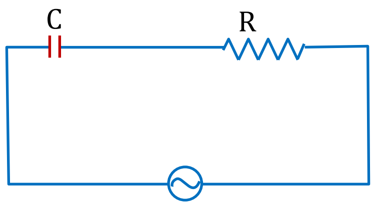
$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

Inductive Impedance Z_L:

In *L-R* circuit the maximum value of current $I_0 = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}}$. Here $\sqrt{R^2 + \omega^2 L^2}$ represents the effective opposition offered by *L-R* circuit to the flow of *a.c.* through it. It is known as impedance of *L-R* circuit and is represented by Z_L. $Z_L = \sqrt{R^2 + \omega^2 L^2} = \sqrt{R^2 + (2\pi f L)^2}$. The reciprocal of impedance is called admittance $Y_L = \frac{1}{Z_L} = \frac{1}{\sqrt{R^2 + \omega^2 L^2}}$.

Resistance and Capacitor in Series (*R-C* Circuit)

A circuit containing a series combination of a resistance *R* and a capacitor *C*, connected with a source of *e.m.f.* of peak value E_0 as shown in fig.

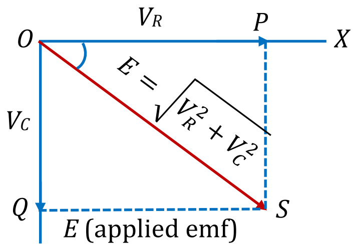


Phasor Diagram for R-C Circuit

Current through both the resistance and capacitor will be same at every instant and the instantaneous potential differences across C and R are

$V_C = IX_C$ and $V_R = IR$

where X_C = capacitive reactance and I = instantaneous current.



Now, V_R is in phase with I, while V_C lags behind I by 90°.

The phasor diagram is shown in fig.

The vector OP represents V_R (which is in phase with I) and the vector OQ represents V_C (which lags behind I by $\frac{\pi}{2}$).

The vector OS represents the resultant of V_R and V_C = the applied *e.m.f. E.*

Hence $V_R^2 + V_C^2 = E^2$ $\qquad \Rightarrow E = \sqrt{V_R^2 + V_C^2}$

$\Rightarrow \qquad E^2 = I^2 (R^2 + X_C^2) \qquad \Rightarrow I = \dfrac{E}{\sqrt{R^2 + X_C^2}}$

The term $\sqrt{(R^2 + X_C^2)}$ represents the effective resistance of the R-C circuit and called the capacitive impedance Z_C of the circuit.

Hence, in C-R circuit, $Z_C = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}$

Capacitive Impedance Z_C :

In R-C circuit the term $\sqrt{R^2 + X_C^2}$ effective opposition offered by R-C circuit to the flow of *a.c.* through it.

It is known as impedance of R-C circuit and is represented by Z_C.

The phasor diagram also shows that in R-C circuit the applied *e.m.f.* lags behind the current I

(or the current I leads the *emf E*) by a phase angle ϕ given by

$$\tan\phi = \frac{V_C}{V_R} = \frac{X_C}{R} = \frac{1/\omega C}{R} = \frac{1}{\omega CR}, \quad \tan\ \phi = \frac{X_C}{R} = \frac{1}{\omega CR}$$

$$\Rightarrow \qquad \phi = \tan^{-1}\left(\frac{1}{\omega CR}\right)$$

Combination of Components (*R-L* or *R-C* or *L-C*)

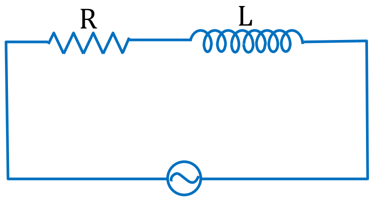
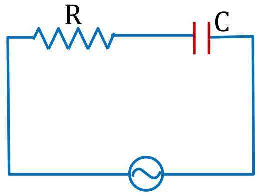
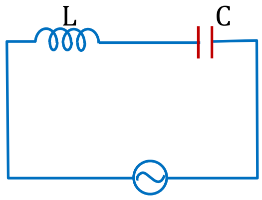
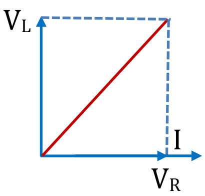
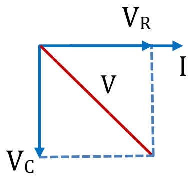
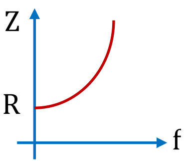
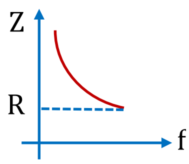
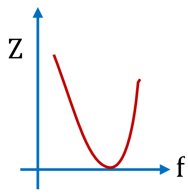
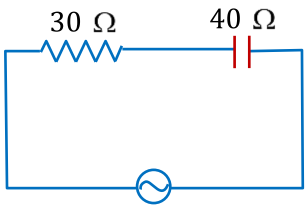
Term	*R-L*	*R-C*	*L-C*
Circuit	 *I* is same in R & L	 *I* is same in R & C	 *I* is same in L & C
Phasor diagram	 $V^2 = V_R^2 + V_L^2$	 $V^2 = V_R^2 + V_C^2$	 $V = V_L - V_C$ ($V_L > V_C$) $V = V_C - V_L$ ($V_C > V_L$)
Phase difference between *V* and *I*	*V* leads *I* ($\phi = 0$ to $\frac{\pi}{2}$)	*V* lags *I*($\phi = -\frac{\pi}{2}$ to 0)	*V* lags *I*($\phi = -\frac{\pi}{2}$, if $X_C > X_L$) *V* leads *I* ($\phi = \frac{\pi}{2}$, if $X_L > X_C$)
Variation of *Z*	$Z = \sqrt{R^2 + (X_L)^2}$ as $f\uparrow, Z\uparrow$	$Z = \sqrt{R^2 + (X_C)^2}$ as $f\uparrow, Z\downarrow$	$Z = \lvert X_L - X_C \rvert$ as $f\uparrow$, Z first $\downarrow$ then$\uparrow$
with *f*			
At very low *f*	$Z \simeq R$ ($X_L \to 0$)	$Z \simeq X_C$	$Z \simeq X_C$
At very high *f*	$Z \simeq X_L$	$Z \simeq R$ ($X_C \to 0$)	$Z \simeq X_L$

Illustration 55:

Calculate the impedance of the circuit shown in the figure.



Solution:

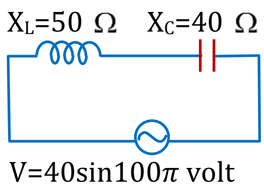
$$Z = \sqrt{R^2 + (X_c)^2}$$

$$= \sqrt{(30)^2 + (40)^2}$$

$$= \sqrt{2500} = 50\ \Omega$$

Illustration 56:

If X_L = 50 Ω and X_C = 40 Ω Calculate effective value of current in given circuit.



Solution:

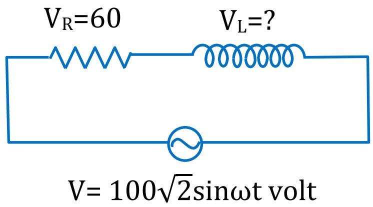
$Z = X_L - X_C = 10\ \Omega$

$I_0 = \dfrac{V_0}{Z} = \dfrac{40}{10} = 4A$

$\Rightarrow \quad I_{rms} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}\ A$

Illustration 57:

In given circuit calculate, voltage across inductor.



Solution:

$\because \quad V^2 = V_R^2 + V_L^2 \qquad \therefore \quad V_L^2 = V^2 - V_R^2$

$V_L = \sqrt{V^2 - V_R^2} = \sqrt{(100)^2 - (60)^2} = \sqrt{6400} = 80\ V$

Illustration 58:

When 10V, dc is applied across a coil current through it is 2.5 A. if 10V, 50 Hz $A.C.$ is applied current reduces to 2 A. Calculate reactance of the coil.

Solution:

For 10 V $D.C.$ $\because$ $V = IR$

$\therefore$ Resistance of coil, $R = \dfrac{10}{2.5} = 4\Omega$. For 10 V $A.C.$: $V = IZ$

$\Rightarrow \quad Z = \dfrac{V}{I} = \dfrac{10}{2} = 5\Omega$

$\because \quad Z = \sqrt{R^2 + X_L^2} = 5$

$\Rightarrow \quad R^2 + X_L^2 = 25$

$\Rightarrow \quad X_L^2 = 5^2 - 4^2$

$\Rightarrow \quad X_L = 3\ \Omega$

Illustration 59:

When an alternating voltage of 220 V is applied across a device X, a current of 0.5 A flows through the circuit and is in phase with the applied voltage. When the same voltage is applied across another device Y, the same current again flows through the circuit but it leads the applied voltage by $\pi/2$ radians.

(a) Name the devices X and Y.

(b) Calculate the current flowing in the circuit when same voltage is applied across the series combination of X and Y.

Solution:

(a) X is resistor and Y is a capacitor

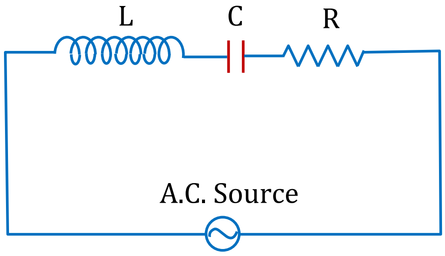
(b) Since the current in the two devices is the same (0.5A at 220 *volt*)

When R and C are in series across the same voltage then

$$R = X_C = \frac{220}{0.5} = 440\ \Omega \quad \Rightarrow I_{rms} = \frac{V_{rms}}{\sqrt{R^2 + X_C^2}} = \frac{220}{\sqrt{(440)^2 + (440)^2}} = \frac{220}{440\sqrt{2}} = 0.35A$$

Inductance, Capacitance and Resistance in series (*l-C-R* series circuit)

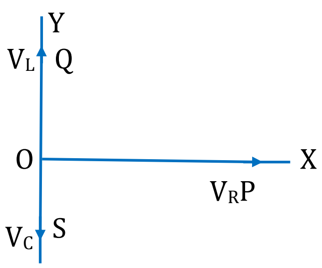
A circuit containing a series combination of an resistance R, a coil of inductance L and a capacitor of capacitance C, connected with a source of alternating *e.m.f.* of peak value of E_0, as shown in fig.



$E = E_0 \sin\omega t$

Phasor Diagram for Series L-C-R Circuit

Let in series LCR circuit applied alternating *emf* is $E = E_0 \sin \omega t$. As L, C and R are joined in series, therefore, current at any instant through the three elements has the same amplitude and phase.



However, voltage across each element bears a different phase relationship with the current.

Let at any instant of time t the current in the circuit is I.

Let at this time t the potential differences across L, C, and R

$V_L = IX_L$, $V_C = IX_C$ and $V_R = IR$

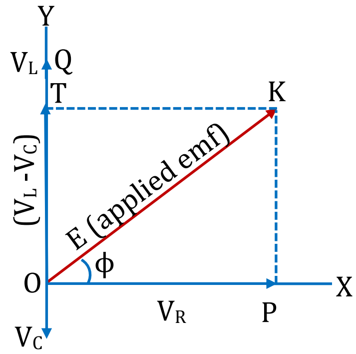
Now, V_R is in phase with current I but V_L leads I by 90°.

While V_C lags behind I by 90°.

The vector OP represents V_R (which is in phase with I) the vector OQ represent V_L (which leads I by 90°) and the vector OS represents V_C (which lags behind I by 90°) V_L and V_C are opposite to each other.

If $V_L > V_C$ (as shown in figure) then their resultant will be $(V_L - V_C)$ which is represented by OT.

Finally, the vector OK represents the resultant of V_R and $(V_L - V_C)$, that is, the resultant of all the three = applied *e.m.f.*



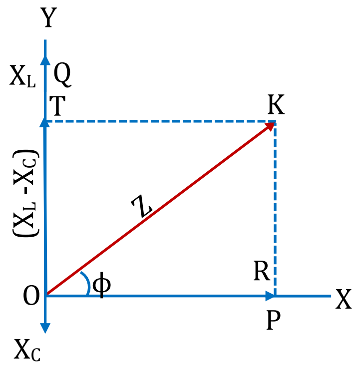
Thus $\quad E = \sqrt{V_R^2 + (V_L - V_C)^2}$

$$= I\sqrt{R^2 + (X_L - X_C)^2}$$

$$\Rightarrow \qquad I = \frac{E}{\sqrt{R^2 + (X_L - X_C)^2}}$$

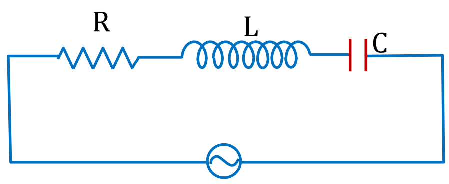
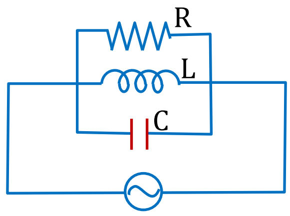
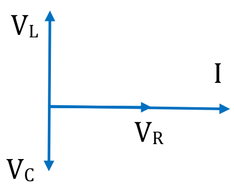
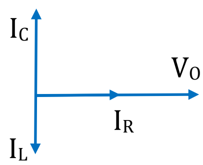
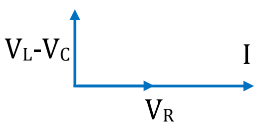
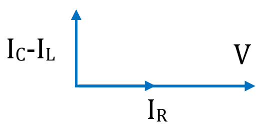
Impedance $Z = \sqrt{R^2 + (X_L - X_C)^2}$

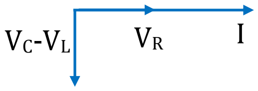
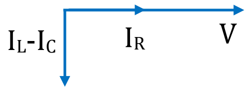
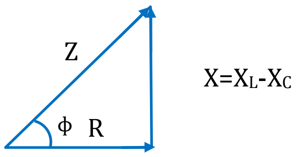
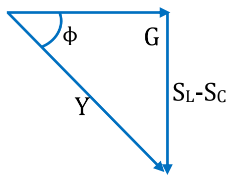
$$= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$



The phasor diagram also shown that in LCR circuit the applied *e.m.f.* leads the current I by a phase angle ϕ $\tan\phi = \dfrac{X_L - X_C}{R}$

Series LCR and Parallel LCR Combination

	Series L-C-R Circuit		Parallel L-C-R Circuit
1.	**Circuit diagram** 		V same for R, L & C  V same for R,L and C
2.	**Phasor diagram** 		
(i)	If $V_L > V_C$ then 	**(i)**	if $I_C > I_L$ then 

	Series *L-C-R* Circuit		Parallel *L-C-R* Circuit
(ii)	If $V_C > V_L$ then 	**(ii)**	if $I_L > I_C$ then 
(iii)	$V = \sqrt{V_R^2 + (V_L - V_C)^2}$ Impedance $Z = \sqrt{R^2 + (X_L - X_C)^2}$ $\tan\phi = \dfrac{X_L - X_C}{R} = \dfrac{V_L - V_C}{V_R}$	**(iii)**	$I = \sqrt{I_R^2 + (I_L - I_C)^2}$ Admittance $Y = \sqrt{G^2 + (S_L - S_C)^2}$ $\tan\phi = \dfrac{S_L - S_C}{G} = \dfrac{I_L - I_C}{I_R}$
(iv)	Impedance triangle 	**(iv)**	Admittance triangle 

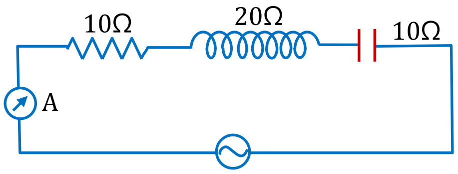
Golden Key Points

	Series		Parallel
(a)	if $X_L > X_C$ then V leads I, ϕ (positive) circuit nature inductive	**(a)**	if $S_L > S_C$ ($X_L < X_C$) then V leads I,ϕ (positive) circuit nature inductive
(b)	if $X_C > X_L$ then V lags I,ϕ (negative) circuit nature capacitive	**(b)**	if $S_C > S_L$ ($X_C < X_L$) then V lags I, ϕ (negative) circuit nature capacitive

- In A.C. circuit voltage for L or C may be greater than source voltage or current but it happens only when circuit contains L and C both and on R it is never greater than source voltage or current.
- In parallel A.C. circuit phase difference between I_L and I_C is π.

Illustration 60:

Find out reading of *AC* ammeter and also calculate the potential difference across, resistance and capacitor.



E = 100 sin100 π t volt

Solution:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = 10\sqrt{2}\,\Omega \;\Rightarrow\; I_0 = \frac{V_0}{Z} = \frac{100}{10\sqrt{2}} = \frac{10}{\sqrt{2}}A$$

$\because$ Ammeter reads *RMS* value, so its reading = $\dfrac{10}{\sqrt{2}\sqrt{2}} = 5A$

So, $V_R = 5 \times 10 = 50\ V$ and $V_C = 5 \times 10 = 50\ V$

Illustration 61:

In LCR circuit with an AC source $R = 300\ \Omega$, $C = 20\ \mu F$, $L = 1.0\ H$, $E_{rms} = 50V$ and $f = 50/\pi\ Hz$. Find RMS current in the circuit.

Solution:

$$I_{rms} = \frac{E_{rms}}{Z} = \frac{E_{rms}}{\sqrt{R^2 + \left[\omega L - \frac{1}{\omega C}\right]^2}} = \frac{50}{\sqrt{300^2 + \left[2\pi \times \frac{50}{\pi} \times 1 - \frac{1}{20 \times 10^{-6} \times 2\pi \times \frac{50}{\pi}}\right]^2}}$$

$$\Rightarrow \quad I_{rms} = \frac{50}{\sqrt{(300)^2 + \left[100 - \frac{10^3}{2}\right]^2}} = \frac{50}{100\sqrt{9+16}} = \frac{1}{10} = 0.1A$$

Resonance

A circuit is said to be resonant when the natural frequency of circuit is equal to frequency of the applied voltage. For resonance both L and C must be present in circuit.

There are two types of resonance : (i) Series Resonance (ii) Parallel Resonance

Series Resonance

(a) At Resonance

(i) $X_L = X_C$ (ii) $V_L = V_C$

(iii) $f = 0$ (V and I in same phase) (iv) $Z_{min} = R$ (impedance minimum)

(v) $I_{max} = \frac{V}{R}$ (current maximum)

(b) Resonance frequency

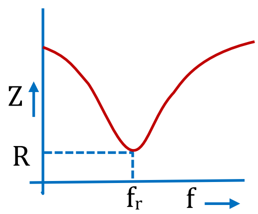
$$\because X_L = X_C \quad \Rightarrow \omega_r L = \frac{1}{\omega_r C} \quad \Rightarrow \omega_r^2 = \frac{1}{LC} \quad \Rightarrow \omega_r = \frac{1}{\sqrt{LC}} \quad \Rightarrow f_r = \frac{1}{2\pi\sqrt{LC}}$$

(c) Variation of Z with f

(i) If $f < f_r$ then $X_L < X_C$ circuit nature is capacitive, ϕ (negative)

(ii) At $f = f_r$ then $X_L = X_C$ circuit nature is Resistive, ϕ = zero

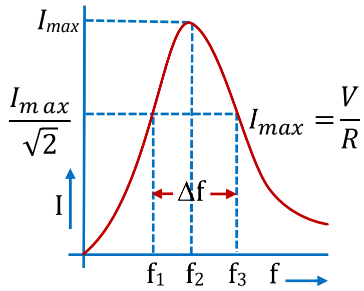
(iii) If $f > f_r$ then $X_L > X_C$ circuit nature is inductive, ϕ (positive)



Variation of *I* with *f* as f increase, Z first decreases then increases.

(d) As f increase, I first increase then decreases

At resonance impedance of the series resonant circuit is minimum so it is called 'acceptor circuit' as it most readily accepts that current out of many currents whose frequency is equal to its natural frequency. In radio or TV tuning we receive the desired station by making the frequency of the circuit equal to that of the desired station.



Half power frequencies

The frequencies at which, power become half of its maximum value called half power frequencies

Band width = $\Delta f = f_2 - f_1$

Quality factor Q: Q-factor of AC circuit basically gives an idea about stored energy & lost energy.

$$Q = 2\pi \frac{\text{maximum energy stored per cycle}}{\text{maximum energy loss per cycle}}$$

(i) It represents the sharpness of resonance.

(ii) It is unit less and dimension less quantity.

(iii) $Q = \frac{(X_L)_r}{R} = \frac{(X_C)_r}{R} = \frac{2\pi f_r L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{f_r}{\Delta f} = \frac{f_r}{\text{band width}}$

Magnification

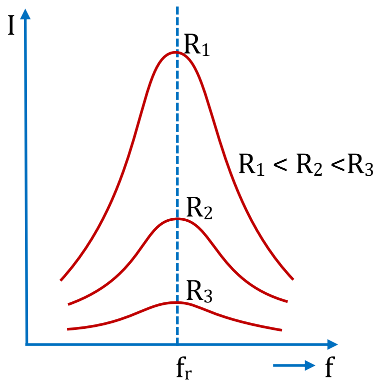
At resonance V_L or $V_C = QE$ (where E = supplied voltage)

So, at resonance, Magnification factor = Q-factor

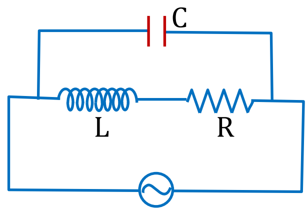
Sharpness

Sharpness ∝ Quality factor ∝ Magnification factor

R decrease $\Rightarrow Q$ increases $\Rightarrow$ Sharpness increases



Parallel Resonance



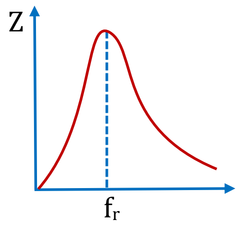
(a) At resonance

(i) $S_L = S_C$ (ii) $I_L = I_C$

(iii) $\phi = 0$ (iv) $Z_{max} = R$ (impedance maximum)

(v) $I_{min} = \frac{V}{R}$ (current minimum)

(b) Resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

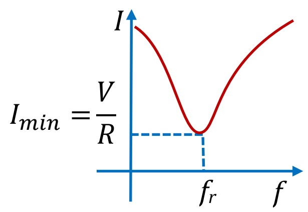


(c) Variation of Z with *f* as f increases, z first increases then decreases

◎ If $f < f_r$ then $S_L > S_C$, f (positive), circuit nature is inductive

◎ If $f > f_r$ then $S_C > S_L$, f (negative), circuit nature is capacitive.

(d) Variation of I with f as f increases, I first decreases then increases

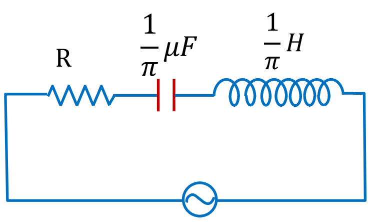


Note:

For this circuit $f_r = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}} \Rightarrow Z_{\max} = \dfrac{L}{RC}$ $\Rightarrow$ For resonance $\dfrac{1}{LC} > \dfrac{R^2}{L^2}$

Illustration 62:

For what frequency the voltage across the resistance R will be maximum.

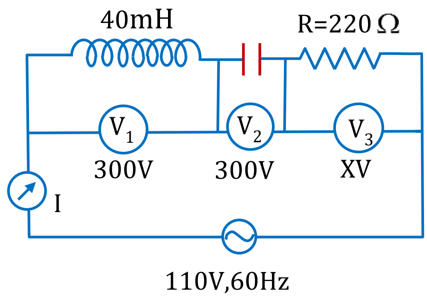


Solution:

It happens at resonance

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{\frac{1}{\pi}\times 10^{-6}\times\frac{1}{\pi}}} = 500$$

Illustration 63:



A capacitor, a resistor and a 40 mH inductor are connected in series to an AC source of frequency 60Hz, calculate the capacitance of the capacitor, if the current is in phase with the voltage. Also calculate the value of X and I.

Solution:

At resonance,

$$\omega L = \frac{1}{\omega C}, C = \frac{1}{\omega^2 L} = \frac{1}{4\pi^2 f^2 L} = \frac{1}{4\pi^2 \times (60)^2 \times 40 \times 10^{-3}} = 176\mu F$$

$$V = V_R \Rightarrow X = 110\ V \quad \text{and} \quad I = \frac{V}{R} = \frac{110}{220} = 0.5A$$

Illustration 64:

A coil, a capacitor and an $A.C.$ source of rms voltage 24 V are connected in series. By varying the frequency of the source, a maximum rms current 6 A is observed. If this coil is connected to a battery of emf 12 V, and internal resistance 4Ω, then calculate the current through the coil.

Solution:

At resonance, current is maximum. $I = \frac{V}{R} \Rightarrow$ Resistance of coil $R = \frac{V}{I} = \frac{24}{6} = 4\Omega$

When coil is connected to battery, suppose I current flow through it then

$$I = \frac{E}{R+r} = \frac{12}{4+4} = 1.5A$$

Illustration 65:

Radio receiver receives a message at 300m band, If the available inductance is 1 mH, then calculate required capacitance

Solution:

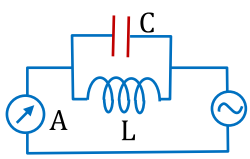
Radio receives EM waves (Velocity of EM waves $c = 3 \times 10^8\ m/s$)

$$\therefore \quad c = f\lambda \Rightarrow f = \frac{3\times 10^8}{300} = 10^6\ Hz$$

Now $$f = \frac{1}{2\pi\sqrt{LC}} = 1\times 10^6 \Rightarrow C = \frac{1}{4\pi^2 L \times 10^{12}} = 25pF$$

Illustration 66:

In a $L–C$ circuit parallel combination of inductance of 0.01 H and a capacitor of 1 μF is connected to a variable frequency alternating current source as shown in figure. Draw a rough sketch of the current variation as the frequency is changed from 1kHz to 3kHz.

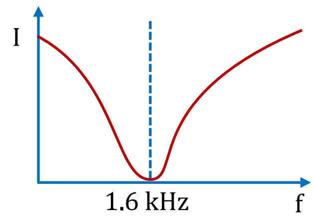


Solution:

L and C are connected in parallel to the AC source,

So, resonance frequency, $f=\dfrac{1}{2\pi\sqrt{LC}}=\dfrac{1}{2\pi\sqrt{0.01\times10^{-6}}}=\dfrac{10^4}{2\pi}\simeq 1.6kHz$

In case of parallel resonance, current in L–C circuit at resonance is zero, so the I-f curve will be as shown in figure.



Power in AC Circuit

The average power dissipation in LCR ac circuit:

Let $V = V_0 \sin \omega t$ and $I = I_0 \sin(\omega t - \phi)$

Instantaneous power, $P = (V_0 \sin \omega t)(I_0 \sin(\omega t - \phi)) = V_0I_0 \sin \omega t (\sin \omega t \cos \phi - \sin \phi \cos \omega t)$

Average power $<P>$ = $\dfrac{1}{T}\int_0^T (V_0I_0\sin^2\omega t\cos\phi - V_0I_0\sin\omega t\cos\omega t\sin\phi)dt$

$$=V_0I_0\left[\frac{1}{T}\int_0^T \sin^2\omega t\cos\phi dt-\frac{1}{T}\int_0^T\sin\omega t\cos\omega t\sin\phi dt\right]=V_0I_0\left[\frac{1}{2}\cos\phi-0\times\sin\phi\right]$$

$$\Rightarrow \quad <P>=\frac{V_0I_0\cos\phi}{2}=V_{rms}I_{rms}\cos\phi$$

Instantaneous power	Average power/actual power/Virtual power/ dissipated power/power loss	apparent Power/rms	Peak power Power
$P = VI$	$P = V_{rms} I_{rms} \cos \phi$	$P = V_{rms} I_{rms}$	$P = V_0 I_0$

- $I_{rms} \cos\phi$ is known as active part of current or wattfull current or workfull current. It is in phase with voltage.
- $I_{rms} \sin\phi$ is known as inactive part of current or wattless current or workless current. It is in quadrature (90°) with voltage.

Power Factor:

Average power $\overline{P}=E_{rms}I_{rms}\cos\phi = rms \text{ power}\times\cos\phi \Rightarrow P_{rms}\times\cos\phi$

Power factor ($\cos \phi$) = $\dfrac{\text{Average power}}{rms \text{ power}}$ and $\cos\phi = \dfrac{R}{Z}$

Power factor : (i) is leading if I leads V

(ii) is lagging if I lags V

Golden Key points:

- $P_{av} \le P_{rms}$.
- Power factor varies from 0 to 1
-

Pure/Ideal	ϕ	V	Power factor = $\cos\phi$	Average power
R	0	V, I same Phase	1 (maximum)	V_{rms}. I_{rms}
L	$+\frac{\pi}{2}$	V leads I	0	0
C	$-\frac{\pi}{2}$	V lags I	0	0
Choke coil	$+\frac{\pi}{2}$	V leads I	0	0

- At resonance power factor is maximum ($\phi = 0$ so $\cos\phi = 1$) and $P_{av} = V_{rms} I_{rms}$

Illustration 67:

A voltage of 10 V and frequency 10^3 Hz is applied to $\frac{1}{\pi}$ μF capacitor in series with a resistor of 500Ω. Find the power factor of the circuit and the power dissipated.

Solution:

$\because$ $$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 10^3 \times \frac{10^{-6}}{\pi}} = 500\Omega$$

$\therefore$ $$Z = \sqrt{R^2 + X_C^2} = \sqrt{(500)^2 + (500)^2} = 500\sqrt{2}\Omega$$

Power factor $\cos\phi = \frac{R}{Z} = \frac{500}{500\sqrt{2}} = \frac{1}{\sqrt{2}}$,

Power dissipated = $V_{rms} I_{rms} \cos\phi = \frac{V_{rms}^2}{Z}\cos\phi = \frac{(10)^2}{500\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{10} W$

Illustration 68:

If V = 100 sin 100t volt and I = 100 sin (100t + $\frac{\pi}{3}$) mA for an $A.C.$ circuit then find out

(a) phase difference between V and I
(b) total impedance, reactance, resistance
(c) power factor and power dissipated
(d) components contains by circuits

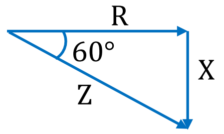
Solution:

(a) Phase difference $\phi = -\frac{\pi}{3}$ (I leads V)

(b) Total impedance $Z = \frac{V_0}{I_0} = \frac{100}{100 \times 10^{-3}} = 1k\Omega$

Now resistance $R = Z\cos 60° = 1000 \times \frac{1}{2} = 500\Omega$

Reactance $X = Z\sin 60° = 1000 \times \frac{\sqrt{3}}{2} = \frac{500}{\sqrt{3}}\Omega$



(c) ϕ = – 60° $\Rightarrow$ Power factor = cosϕ = cos (–60°) = 0.5 (leading)

Power dissipated $P = V_{rms} I_{rms} \cos\phi = \frac{100}{\sqrt{2}} \times \frac{0.1}{\sqrt{2}} \times \frac{1}{2} = 2.5W$

(d) Circuit must contains R as $\phi \neq \frac{\pi}{2}$ and as ϕ is negative

So, C must be there, (L may exist but $X_C > X_L$).

Illustration 69:

If power factor of a R-L series circuit is $\frac{1}{2}$ when applied voltage is V = 100 sin 100πt volt and resistance of circuit is 200Ω then calculate the inductance of the circuit.

Solution:

$$\cos\phi = \frac{R}{Z} \Rightarrow \frac{1}{2} = \frac{R}{Z} \Rightarrow Z = 2R \Rightarrow \sqrt{R^2 + X_L^2} = 2R \Rightarrow X_L = \sqrt{3}R$$

$$\omega L = \sqrt{3}R \Rightarrow L = \frac{\sqrt{3}R}{\omega} = \frac{\sqrt{3}\times 200}{100\pi} = \frac{2\sqrt{3}}{\pi}H$$

Illustration 70:

A circuit consisting of an inductance and a resistance joined to a 200 volt supply (A.C.). It draws a current of 10 ampere. If the power used in the circuit is 1500 watt. Calculate the wattless current.

Solution:

Apparent power = 200 × 10 = 2000 W

$\therefore$ Power factor $\cos\phi = \frac{\text{True power}}{\text{Apparent power}} = \frac{1500}{2000} = \frac{3}{4}$

Wattless current = $I_{rms}\sin\phi$ = 10 $\sqrt{1-\left(\frac{3}{4}\right)^2} = \frac{10\sqrt{7}}{4}A$

Illustration 71:

A coil has a power factor of 0.866 at 60 Hz. What will be power factor at 180 Hz.

Solution:

Given that $\cos\phi = 0.866$, $\omega = 2\pi f = 2\pi \times 60 = 120\pi$ rad/s,

$\omega' = 2\pi f' = 2\pi \times 180 = 360\pi$ rad/s

Now, $\cos\phi = R/Z \quad \Rightarrow R = Z\cos\phi = 0.866\,Z$

But $Z = \sqrt{R^2 + (\omega L)^2} \Rightarrow \omega L = \sqrt{Z^2 - R^2} = \sqrt{Z^2 - (0.866Z)^2} = 0.5Z$

$\therefore \quad L = \frac{0.5Z}{\omega} = \frac{0.5Z}{120\pi}$

When the frequency is changed to $\omega' = 2\pi \times 180 = 3 \times 120\pi = 300$ rad/s,

then inductive reactance $\omega' L = 3\,\omega L = 3 \times 0.5\,Z = 1.5\,Z$

$\therefore$ New impedance, $Z' = \sqrt{\left[R' + (\omega' L)^2\right]} = \sqrt{(0.866Z^2) + (1.5Z)^2} = Z\sqrt{\left[(0.0866)^2 + (1.5)^2\right]}$

$= 1.732\,Z$

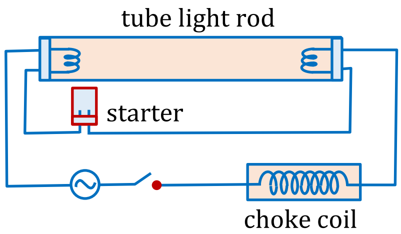
$\therefore$ New power factor $= \frac{R}{Z'} = \frac{0.0866Z}{1.732Z} = 0.5$

Choke Coil

In a direct current circuit, current is reduced with the help of a resistance. Hence there is a loss of electrical energy $I^2 R$ per sec in the form of heat in the resistance. But in an AC circuit the current can be reduced by choke coil which involves very small amount of loss of energy. Choke coil is a copper coil wound over a soft iron laminated core. This coil is put in series with the circuit in which current is to be reduced. It also known as ballast.

Circuit with a choke coil is a series L-R circuit. If resistance of choke coil = r (very small)

The current in the circuit $I=\dfrac{E}{Z}$ with $Z=\sqrt{(R+r)^2+(\omega L)^2}$ so due to large inductance L of the coil, the current in the circuit is decreased appreciably. However, due to small resistance of the coil r.



The power loss in the choke $P_{av}=V_{rms}I_{rms}\cos\phi\to 0$

$$\because \qquad \cos\phi=\frac{r}{Z}=\frac{r}{\sqrt{r^2+\omega^2L^2}}\approx\frac{r}{\omega L}\to 0$$

Illustration 72:

A choke coil and a resistance are connected in series in an *a.c* circuit and a potential of 130 volt is applied to the circuit. If the potential across the resistance is 50 V. What would be the potential difference across the choke coil.

Solution:

$$V=\sqrt{V_R^2+V_L^2}\Rightarrow V_L=\sqrt{V^2-V_R^2}=\sqrt{(130)^2-(50)^2}=120V$$

Illustration 73:

An electric lamp which runs at 80V DC consumes 10 A current. The lamp is connected to 100 V – 50 Hz ac source compute the inductance of the choke required.

Solution:

Resistance of lamp $R=\dfrac{V}{I}=\dfrac{80}{10}=8\Omega$

Let z be the impedance which would maintain a current of 10 A through the Lamp when it is run on 100 Volt *a.c.* then. $Z=\dfrac{V}{I}=\dfrac{100}{10}=10\Omega$ but $Z=\sqrt{R^2+(\omega L)^2}$

$\Rightarrow \qquad (\omega L)^2 = Z^2 - R^2 = (10)^2 - (8)^2 = 36$

$\Rightarrow \qquad \omega L = 6 \Rightarrow L=\dfrac{6}{\omega}=\dfrac{6}{2\pi\times 50}=0.02H$

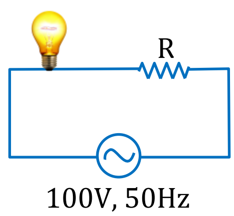
Illustration 74:

Calculate the resistance or inductance required to operate a lamp (60V, 10W) from a source of (100 V, 50 Hz).

Solution:

(a) Maximum voltage across lamp = 60V

$\because \qquad V_{Lamp} + V_R = 100 \qquad \therefore \qquad V_R = 40V$



Now current through Lamp is $=\dfrac{\text{Wattage}}{\text{voltage}}=\dfrac{10}{60}=\dfrac{1}{6}A$

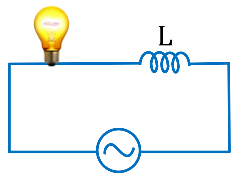
But $V_R = IR \quad \Rightarrow 40=\dfrac{1}{6}(R)\Rightarrow R=240\Omega$

(b) Now in this case $(V_{Lamp})^2 + (V_L)^2 = (V)^2$

$(60)^2 + (V_L)^2 = (100)^2 \Rightarrow V_L = 80\ V$

Also, $V_L = IX_L = \frac{1}{6}\ X_L$

So, $X_L = 80 \times 6 = 480\ \Omega = L\ (2\pi f) \Rightarrow L = 1.5\ H$



A capacitor of suitable capacitance replace a choke coil in an AC circuit, the average power consumed in a capacitor is also zero. Hence, like a choke coil, a capacitor can reduce current in AC circuit without power dissipation.

Cost of capacitor is much more than the cost of inductance of same reactance that's why choke coil is used.

Illustration 75:

A choke coil of resistance R and inductance L is connected in series with a capacitor C and complete combination is connected to *a.c.* voltage, Circuit resonates when angular frequency of supply is $\omega = \omega_0$.

(a) Find out relation between ω_0, L and C.

(b) What is phase difference between V and I at resonance, is it changes when resistance of choke coil is zero.

Solution:

(a) At resonance condition $X_L = X_C \Rightarrow \omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$

(b) $\because\ \cos\phi = \frac{R}{Z} = \frac{R}{R} = 1\ \therefore$ No, it is always zero.

Damped Harmonic Oscillator

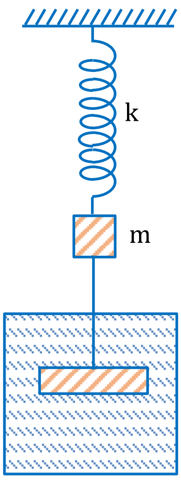
In the previous section we have considered several examples of oscillatory systems. In each case the amplitude of the oscillation does not depend on time and the system, once set to oscillate, will continue to do so forever. Physical systems are never as ideal as that. Even in the most controlled case there will be dissipative elements.

In each of these cases the amplitudes of the oscillation will gradually decrease and the motion will be 'damped'.

Most of the time we are interested in the motion of an object through air or other viscous fluids. The motion of the object is then subject to a resistive force called aerodynamic or viscous drag. Experimentally the resistive force is found to be proportional to the velocity of the body for low relative speed with respect to the medium.

A laboratory model for a damped spring-mass system is shown in figure. Where a vertical spring has at its end a mass m which is connected to massless piston which moves through a liquid. For low speeds it is reasonable to write the frictional force as

$$F_{f_f} = -\alpha v \quad (\alpha \text{ in } kg/s)$$



where α is a positive constant known as damping constant. The equation of motion of a harmonic oscillator becomes

$$m = \frac{d^2x}{dt^2} = -kx - \alpha\frac{dx}{dt}$$

which can be rewritten as

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

where $2\gamma = \alpha/m > 0$ and ω_0^2, as before is equal to $k/m.\gamma$ is known as damping coefficient

Damped Oscillations

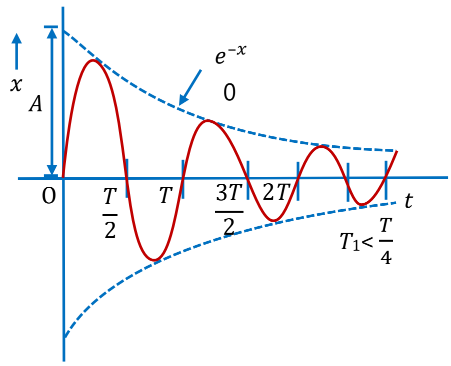
If the dissipative forces are small, i.e. if $\gamma^2 < \omega_0^2$, the dominant force acting on the mass is the conservative restoring force. The system therefore shows oscillations, even though the amplitudes of oscillation continuously decrease with time.

we get $x = Ae^{-\gamma t}\sin(\Omega t + \phi)$

where the new angular frequency of oscillation Ω is given by

$\Omega^2 = \omega_0^2 - \gamma^2$

i.e the frequency shift to a lower value due to damping. The amplitude of oscillation is no longer constant but decreases with time in an exponential fashion. The variation of x with time is shown in figure.



It may be seen that sin $(\Omega t + \phi)$ is a periodic function with period $2\pi/\Omega$. when this is multiplied with $e^{-\gamma t}$ the zeroes of the product would still be at the same place where sin $(\Omega t + \phi)$ becomes zero. The maxima however does not fall midway between the two minima because of the exponential $e^{-\gamma t}$ but occurs at a slightly earlier time.

Power Loss in a Damped Oscillation-Q-factor:

Because of the work done against the nonconservative forces, the energy stored in an oscillator continuously decreases. The fraction of energy lost in a period with respect to its average value for that period is a measure of the quality of the oscillator. The less the value of this fraction is, the more is the ability of the oscillator to sustain periodic motion. Quantitatively, the quality of an oscillator is measured through a *Q*-factor defined as

$$Q = 2\pi \times \frac{\text{Average energy stored in one period}}{\text{average loss of energy in that period}}$$

If the damping is light i.e. $\gamma << \omega_0$, The instantaneous energy of the oscillator is

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2\omega_0^2 e^{-2\gamma t}$$

where we have replaced k by $m\omega_0^2$. At $t = 0$ the energy of the oscillator is $E_0 = mA^2\omega_0^2/2$ so that

$$E(t) = E_0 e^{-2\gamma t}$$

$$P(t) = \frac{dE}{dt} = -2\gamma E_0 e^{-2\gamma t} = 2\gamma E(t)$$

Energy lost in one period can be approximately written as

$$< p(t) > \frac{2\pi}{\Omega} = 2\gamma < E(t) > \frac{2\pi}{\Omega}$$

where the angular bracket < ..... > denotes the time average value during a period at time t. The quality or *Q*-factor for an oscillator is

$$2\pi \frac{< E(t) >}{2\gamma < E(t) > \frac{2\pi}{\Omega}} = \frac{\Omega}{2\pi} \cong \frac{\omega_0}{2\gamma}$$

It may be noted that Q is a dimensionless quantity, greater than unity by our assumption that. $\gamma << \omega_0$

Time to reach extreme:

The natural logarithm of two successive maxima is called logarithmic decrement and is denoted by λ. From Equation it follows that

$$\lambda = \frac{2\pi\gamma}{\Omega} = \frac{\pi}{Q}$$

Overdamped Motion:

In equation if $\gamma > \omega_0$ we do not get oscillatory solution.

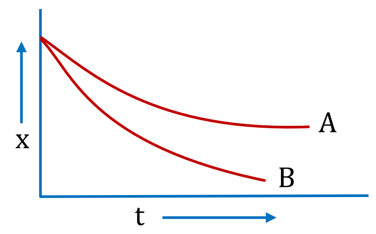
Critical Damping ($\gamma = \omega_0$):

The solution for the displacement is therefore

$$x = (A + Bt)e^{-\gamma t}$$

$$x' = -\frac{mg}{k}(1+\gamma t)e^{-\gamma t}$$

It may be seen that once again there are no oscillation. As $t \to \infty$, $x \to 0$ so that the particle takes an infinite time before reaching the equilibrium position. The difference with the overdamped case is that the approach to the equilibrium is faster. The variation of displacement with time is shown in figure (curve B.)



Forced Oscillations and Resonance

The oscillations in a system can be indefinitely maintained by supplying energy continuously. In mechanical system this can be done by subjecting it to an external force which itself has harmonic time dependence.

An interesting phenomenon occurs if the frequency of the external source is equal or nearly equal to the natural frequency of the system. The amplitude of oscillations is found to increase many folds in such cases. This is called resonance.

Forced Damped Oscillations:

In the presence of resistive forces proportional to velocity, equation becomes

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Let us therefore try a solution of the type

$$x = A\cos\omega t + B\sin\omega t + Ce^{-\lambda t}$$

The steady - state displacement is then given by

$$x = C\cos(\omega t + \delta)$$

with $$C = \frac{F_0}{m}\sqrt{\frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\gamma^2}}$$

and $$\tan\delta = \frac{2\omega\gamma}{\omega^2 - \omega_0^2}$$