01 Electrostatics and Gravitation

ELECTROSTATICS

Introduction

"Electrostatic" pertains to electric charges at rest or to fields or phenomena produced by stationary charge(s).

Electric Charge

Charge of a material body or particle is the property (acquired or natural) due to which it produces and experiences electrical and magnetic effects. Some of naturally occurring charged particles are electrons, protons, α-particles etc.

Charge is a derived physical quantity and is measured in Coulomb in S.I. unit. In practice we use $mC(10^{-3}C)$, $\mu C\ (10^{-6}C)$, $nC(10^{-9}C)$ etc.

C.G.S. unit of charge = electrostatic unit = esu.

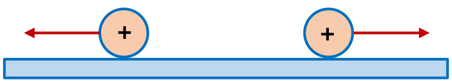
1 coulomb = 3×10^9 esu of charge

Dimensional formula of charge = $[M^0L^0T^1I^1]$

Properties of Charge

(i) **Charge is a scalar quantity:** It adds algebraically and represents excess or deficiency of electrons.

(ii) **Charge is of two types: (i) Positive charge and (ii) Negative charge** Charging a body implies transfer of charge (electrons) from one body to another. Positively charged body means loss of electrons i.e. deficiency of electrons. Negatively charged body means excess of electrons. This also shows that **mass of a negatively charged body > mass of a positively charged identical body.** Charges with the same electrical sign repel each other, and charges with opposite electrical sign attract each other.



(iii) **Charge is conserved:** In an isolated system, total charge (sum of positive and negative) remains constant whatever change takes place in that system.

(iv) **Charge is quantized:** Charge on any body always exists in integral multiples of a fundamental unit of electric charge. This unit is equal to the magnitude of charge on electron ($1e = 1.6 \times 10^{-19}$ coulomb). So, charge on anybody is $Q = \pm ne$, where n is an integer and e is the charge of the electron. **Millikan's oil drop** experiment proved the quantization of charge or atomicity of charge

(v) **Comparison of charge and mass :** We are familiar with role of mass in gravitation, and we have just studied some features of electric charge. We can compare the two as shown below

Table - Charge v/s Mass

Charge	Mass
(1) Electric charge can be positive, negative or zero.	(1) Mass of a body is a positive quantity.
(2) Charge carried by a body does not depend upon velocity of the body.	(2) Mass of a body increases with its velocity as $m=\dfrac{m_0}{\sqrt{1-v^2/c^2}}$ where c is velocity of light in vacuum, m is the mass of the body moving with velocity v and m_0 is rest mass of the body.
(3) Charge is quantized.	(3) The quantization of mass is yet to be established.
(4) Electric charge is always conserved.	(4) Mass is not conserved as it can be changed into energy and vice-versa.
(5) Force between charges can be attractive or repulsive, accordingly as charges are unlike or like charges.	(5) The gravitational force between two masses is always attractive.

(vi) Charge is always associated with mass, i.e., charge cannot exist without mass though mass can exist without charge. The particle such as photon or neutrino which have no (rest) mass can never have a charge.

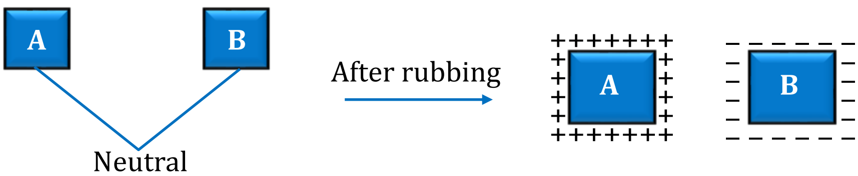
(vii) **Charge is relativistically invariant:** This means that charge is independent of frame of reference i.e., charge on a body does not change whatever be its speed. This property is worth mentioning as in contrast to charge, the mass of a body depends on its speed and increases with increase in speed.

(viii) A charge at rest produces only electric field around itself, a charge having uniform motion produces electric as well as magnetic field around itself while a charge having accelerated motion emits electromagnetic radiations.

Charging of a Body

A body can be charged by means of (a) friction, (b) conduction, (c) induction

(a) **Charging by Friction:** When a neutral body is rubbed against other neutral body then some electrons are transferred from one body to other. The body which can hold electrons tightly, draws some electrons and the body which can not hold electrons tightly, loses some electrons. The body which draws electrons becomes negatively charged and the body which loses electrons becomes positively charged.



For example: Suppose a glass rod is rubbed with a silk cloth. As the silk can hold electrons more tightly and a glass rod can hold electrons less tightly (due to their chemical properties), some electrons will leave the glass rod and get transferred to the silk. So, in the glass rod their will be deficiency of electrons, therefore it will become positively charged. And in the silk, there will be some extra electrons, so it will become negatively charged.

Different materials have different affinities for electrons. By rubbing a variety of materials against each other and testing their resulting interaction with objects of known charge, the tested materials can be ordered according to their affinity for electrons. Such an ordering of substances is known as a **triboelectric series**. One such ordering for several materials is shown in the table at the right. Materials shown highest on the table tend to have a greater affinity for electrons than those below it. Subsequently, when any two materials in the table are rubbed together, the one that is higher can be expected to pull electrons from the material that is lower. As such, the materials highest on the table will have the greatest tendency to acquire the negative charge. Those below it would become positively charged.

Triboelectric Series
Celluloid
Sulfur
Rubber
Copper, Brass
Amber
Wood
Cotton
Human Skin
Silk
Cat Fur
Wool
Glass
Rabbit Fur
Asbestos

(b) Charging by conduction (flow): There are three types of materials in nature:

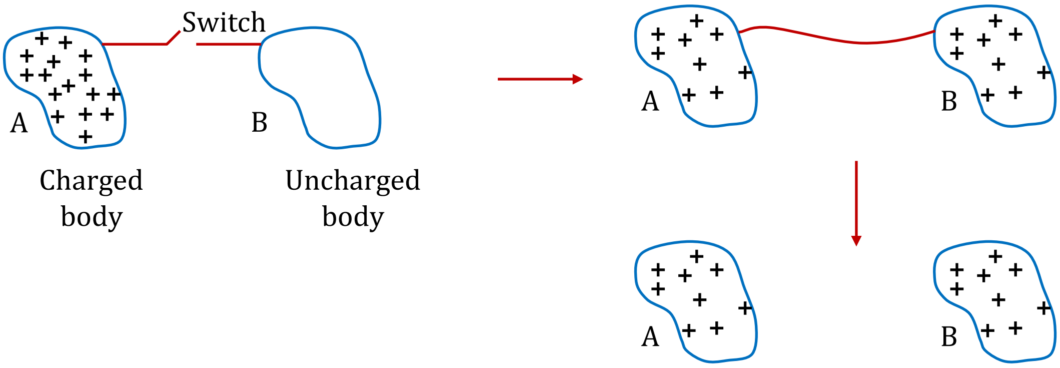
(i) Conductor: Conductors are the material in which the outer most electrons are very loosely bound, so they are free to move (flow). So, in a conductor, there are large number of free electrons.

Ex. Metals like Cu, Ag, Fe, Al

(ii) Insulator or Dielectric or Non-conductor: Non-conductors are the materials in which outer most electrons are very tightly bound, so that they cannot move (flow). Hence in a non-conductor there are no free electrons. Ex. plastic, rubber, wood etc.

(iii) Semi-conductor: Semiconductors are the materials which have free electrons but very less in number.

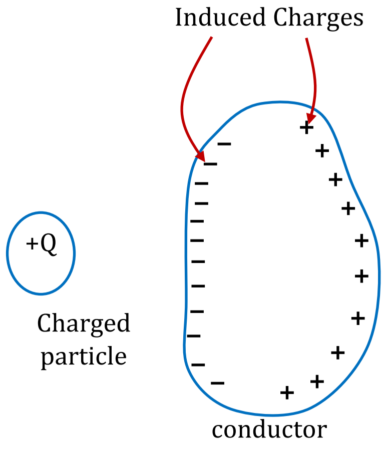
Now lets see how the charging is done by conduction. In this method, we take a charged conductor *'A'* and an uncharged conductor *'B'*. When both are connected, some charge will flow from the charged body to the uncharged body. If both the conductors are identical & kept at large distance and connected to each other, then charge will be divided equally in both the conductors otherwise they will flow till their electric potential becomes same. Its detailed study will be done in last section of this chapter.



(c) **Charging by Induction:** To understand this, let's have introduction to induction.

We have studied that there are lot of free electrons in the conductors. When a charged particle $+Q$ is brought near a neutral conductor, due to attraction of $+Q$ charge, many electrons ($-ve$ charges) come closer and accumulate on the closer surface.

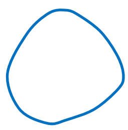
On the other hand, a positive charge (deficiency of electrons) appears on the other surface. The flow of charge continues till the resultant force on free electrons of the conductor becomes zero. This phenomena is called induction and charges produced are called induced charges.



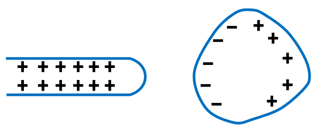
A body can be charged by induction in the following two ways :

Method I:

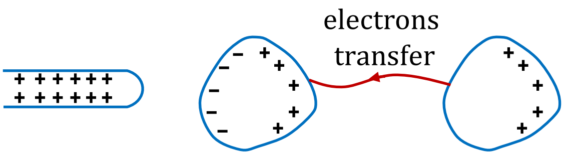
Step 1 : Take an isolated neutral conductor.



Step 2 : Bring a charged rod near it. Due to the charged rod, charges will induce on the conductor.



Step 3 : Connect another neutral conductor with it. Due to attraction of the rod, some free electrons will move from the right conductor to the left conductor and due to deficiency of electrons positive charges will appear on right conductor. On the left conductor, there will be excess of electrons due to transfer from right conductor.

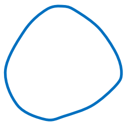


Step 4 : Now disconnect the connecting wire and remove the rod.

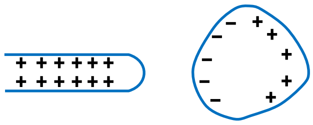
The first conductor will be negatively charged and the second conductor will be positively charged.

Method II:

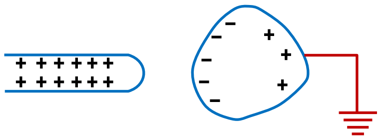
Step 1 : Take an isolated neutral conductor.



Step 2 : Bring a charged rod near it. Due to the charged rod, charges will induce on the conductor.



Step 3 : Connect the conductor to the earth (this process is called grounding or earthing). Due to attraction of the rod, some free electrons will move from earth to the conductor, so in the conductor there will be excess of electrons due to transfer from the earth, so net charge on conductor will be negative.



Step 4 : Now disconnect the connecting wire. Conductor becomes negatively charged.

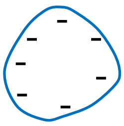
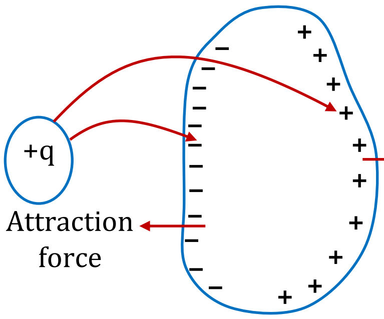


Illustration 1:

If a charged body is placed near a neutral conductor, will it attract the conductor or repel it?

Solution:

If a charged body (+*ve*) is placed left side near a neutral conductor, (–*ve*) charge will induce at left surface and (+*ve*) charge will induce at right surface. Due to positively charged body –*ve* induced charge will feel attraction and the +*ve* induced charge will feel repulsion. But as the –*ve* induced charge is nearer, so the attractive force will be greater than the repulsive force.



So, the net force on the conductor due to positively charged body will be attractive. Similarly, we can prove for negatively charged body also.

From the above example we can conclude that. "A charged body can attract a neutral body."

If there is attraction between two bodies then one of them may be neutral. But if there is repulsion between two bodies, both must be charged (similarly charged). So **"repulsion is the sure test of electrification".**

Illustration 2:

Five styrofoam balls *A, B, C, D* and *E* are used in an experiment. Several experiments are performed on the balls and the following observations are made :

(i) Ball *A* repels *C* and attracts *B*.

(ii) Ball *D* attracts *B* and has no effect on *E*.

(iii) A negatively charged rod attracts both *A* and *E*.

For your information, an electrically neutral styrofoam ball is very sensitive to charge induction and gets attracted considerably, if placed nearby a charged body. What are the charges, if any, on each ball ?

	A	*B*	*C*	*D*	*E*
(A)	+	–	+	0	+
(B)	+	–	+	+	0
(C)	+	–	+	0	0
(D)	–	+	–	–	0

Ans. (C)

Solution:

From (i), as *A* repels *C*, so both *A* and *C* must be charged similarly. Either both are +*ve* or both are –*ve*. As *A* also attract *B*, so charge on *B* should be opposite of *A* or *B* may be uncharged conductor.

From (ii) as *D* has no effect on *E*, so both *D* and *E* should be uncharged and as *B* attracts uncharged *D*, so *B* must be charged and *D* must be an uncharged conductor.

From (iii), a –vely charged rod attracts the charged ball *A*, so *A* must be +*ve* and from exp. (i) *C* must also be +*ve* and *B* must be –*ve*.

Illustration 3:

Charge conservation is always valid. Is it also true for mass?

Solution:

No, mass conservation is not always. In some nuclear reactions, some mass is lost and it is converted into energy.

Illustration 4:

If a glass rod is rubbed with silk, it acquires a positive charge because:

(A) protons are added to it

(B) protons are removed from it

(C) electrons are added to it

(D) electrons are removed from it.

Ans. (D)

Coulomb's Law (Inverse Square Law) and coulombic force:

On the basis of experiments Coulomb established the following law known as Coulomb's law:

The magnitude of electrostatic force between two-point charges is directly proportional to the product of charges and inversely proportional to the square of the distance between them.

i.e. $F \propto q_1 q_2$ and $F \propto \frac{1}{r^2}$

$\Rightarrow$ $F \propto \frac{q_1 q_2}{r^2}$ $\Rightarrow$ $F = \frac{k q_1 q_2}{r^2}$

Important points regarding Coulomb's law:

(i) **point charges:** By the word "point charges" in Coulomb's law we mean, the bodies having very small size in compare to distance between them.

(ii) The constant of proportionality k in SI units in vacuum is expressed as $\frac{1}{4\pi\varepsilon_0}$ and in any other medium expressed as $\frac{1}{4\pi\varepsilon}$. If charges are dipped in an infinitely extended medium then electrostatic force on one charge is $\frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1 q_2}{r^2}$ where ε_0 and ε are called permittivity of vacuum and absolute permittivity of the medium respectively. The ratio $\varepsilon/\varepsilon_0 = \varepsilon_r$ is called relative permittivity of the medium, which is a dimensionless quantity.

(iii) The value of relative permittivity ε_r is constant for a medium and can have values between 1 to ∞. For vacuum, by definition it is equal to 1. For air it is nearly equal to 1 and may be taken to be equal to 1 for calculations. For metals, the value of ε_r is ∞ and for water is 81. The material in which more charge can induce ε_r will be higher.

(iv) The value of $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9\ Nm^2\ C^{-2}$ and $\varepsilon_0 = 8.855 \times 10^{-12}\ C^2/Nm^2$.

Dimensional formula of ε is $[M^{-1}\ L^{-3}\ T^4\ A^2]$

(v) The force acting on one point charge due to the other point charge is always along the line joining these two charges. It is equal in magnitude and opposite in direction on two charges, irrespective of the medium in which they lie.

(vi) The force is conservative in nature i.e., work done by electrostatic force in moving a point charge along a closed loop of any shape is zero.

(vii) Since the force is a central force, in the absence of any other external force, angular momentum of one particle w.r.t. the other particle (in two particle system) is conserved.

(viii) In vector form formula can be given as below.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1 q_2}{|\vec{r}|^3}\vec{r} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1 q_2}{|\vec{r}|^2}\hat{r}$$

(q_1 and q_2 are to be substituted with sign)

Here, $\vec{r}$ is position vector of the test charge (on which force is to be calculated) with respect to the source charge (due to which force is to be calculated).

Illustration 5:

Find out the electrostatic force between two-point charges placed in air (each of +1 C) if they are separated by 1m.

Solution:

$$Fe = \frac{kq_1q_2}{r^2} = \frac{9\times10^9\times1\times1}{1^2} = 9\times10^9\ N$$

Note:

From the above result, we can say that 1 C charge is too large to realize. In nature, charge is usually of the order of μC.

Illustration 6:

A particle of mass m carrying charge q_1 is revolving around a fixed charge $-q_2$ in a circular path of radius r. Calculate the period of revolution and its speed also.

Solution:

$$\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} = mr\omega^2 = \frac{4\pi^2 mr}{T^2}$$

$$T^2 = \frac{(4\pi\varepsilon_0)r^2(4\pi^2 mr)}{q_1q_2} \quad \text{or} \quad T = 4\pi r\sqrt{\frac{\pi\varepsilon_0 mr}{q_1q_2}}$$

and also we can say that

$$\frac{q_1q_2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{q_1q_2}{4\pi\varepsilon_0 mr}}$$

Illustration 7:

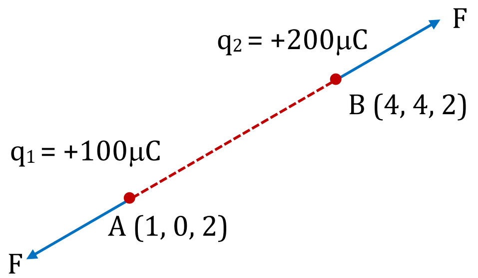
A point charge q_A = + 100 μC is placed at point A (1, 0, 2) m and another point charge q_B = +200μC is placed at point B (4, 4, 2) m. Find :

(i) Magnitude of electrostatic interaction force acting between them

(ii) Find $\vec{F}_A$ (force on A due to B) and $\vec{F}_B$ (force on B due to A) in vector form

Solution:

(i)



Value of F: $$|\vec{F}| = \frac{kq_Aq_B}{r^2} = \frac{(9\times10^9)\ (100\times10^{-6})\ (200\ \times10^{-6})}{\left(\sqrt{(4-1)^2 + (4-0)^2 + (2-2)^2}\right)^2} = 7.2\ N$$

(ii) Force on B, $\vec{F}_B = \frac{kq_Aq_B}{|\vec{r}|^3}\vec{r} = \frac{(9\times10^9)(100\times10^{-6})(200\times10^{-6})}{\left(\sqrt{(4-1)^2+(4-0)^2+(2-2)^2}\right)^3}\left[(4-1)\hat{i}+(4-0)\hat{j}+(2-2)\hat{k}\right]$

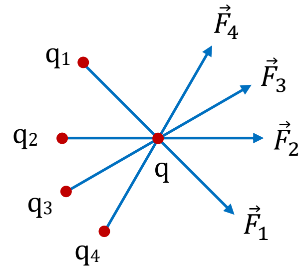
$$= 7.2\left(\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}\right)N$$

Similarly, $\vec{F}_A = 7.2N\left(-\frac{3}{5}\hat{i}-\frac{4}{5}\hat{j}\right)N$

Action ($\vec{F}_A$) and Reaction ($\vec{F}_B$) are equal but in opposite direction.

Principle of Superposition

The electrostatic force is a two body interaction i.e., electrical force between two point charges is independent of presence or absence of other charges and so the principle of superposition is valid i.e., force on charged particle due to number of point charges is the resultant of forces due to individual point charges. Therefore, force on a point test charge due to many charges is given by.



$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$$

Illustration 8:

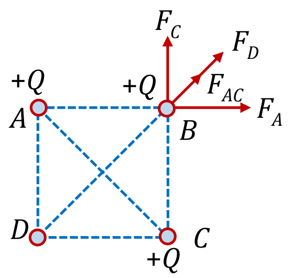
Equal charges Q are placed at the four corners A, B, C, D of a square of length a. The magnitude of the force on the charge at B will be

(A) $\frac{3Q^2}{4\pi\varepsilon_0 a^2}$ (B) $\frac{4Q^2}{4\pi\varepsilon_0 a^2}$ (C) $\left(\frac{1+2\sqrt{2}}{2}\right)\frac{Q^2}{4\pi\varepsilon_0 a^2}$ (D) $\left(2+\frac{1}{\sqrt{2}}\right)\frac{Q^2}{4\pi\varepsilon_0 a^2}$

Ans. (C)

Solution:

After following the guidelines mentioned above



$$F_{net} = F_{AC} + F_D = \sqrt{F_A^2 + F_C^2} + F_D$$

Since $F_A = F_C = \frac{kQ^2}{a^2}$ and $F_D = \frac{kQ^2}{(a\sqrt{2})^2}$

$$F_{net} = \frac{\sqrt{2}kQ^2}{a^2} + \frac{kQ^2}{2a^2} = \frac{kQ^2}{a^2}\left(\sqrt{2}+\frac{1}{2}\right) = \frac{Q^2}{4\pi\varepsilon_0 a^2}\left(\frac{1+2\sqrt{2}}{2}\right)$$

Illustration 9:

Two equal charges are separated by a distance d. A third charge placed on a perpendicular bisector at x distance, will experience maximum coulomb force when

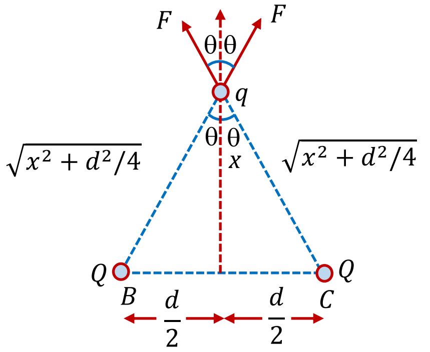
(A) $x = \frac{d}{\sqrt{2}}$ (B) $x = \frac{d}{2}$ (C) $x = \frac{d}{2\sqrt{2}}$ (D) $x = \frac{d}{2\sqrt{3}}$

Ans. (C)

Solution:

Suppose third charge is similar to Q and it is q

So net force on it $F_{net} = 2F\cos\theta$



Where $F = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Qq}{\left(x^2 + \dfrac{d^2}{4}\right)}$ and $\cos\theta = \dfrac{x}{\sqrt{x^2 + \dfrac{d^2}{4}}}$

$$\therefore \qquad F_{net} = 2 \times \frac{1}{4\pi\varepsilon_0} \cdot \frac{Qq}{\left(x^2 + \frac{d^2}{4}\right)} \times \frac{x}{\left(x^2 + \frac{d^2}{4}\right)^{1/2}} = \frac{2Qqx}{4\pi\varepsilon_0 \left(x^2 + \frac{d^2}{4}\right)^{3/2}}$$

For F_{net} to be maximum $\dfrac{dF_{net}}{dx} = 0$ i.e. $\dfrac{d}{dx}\left[\dfrac{2Qqx}{4\pi\varepsilon_0\left(x^2 + \dfrac{d^2}{4}\right)^{3/2}}\right] = 0$

or $\qquad \left[\left(x^2 + \dfrac{d^2}{4}\right)^{-3/2} - 3x^2\left(x^2 + \dfrac{d^2}{4}\right)^{-5/2}\right] = 0$ i.e. $\qquad x = \pm\dfrac{d}{2\sqrt{2}}$

Illustration 10:

ABC is a right angle triangle in which *AB* = 3 *cm*, *BC* = 4 *cm* and $\angle ABC = \dfrac{\pi}{2}$. The three charges +15, +12 and – 20 e.s.u. are placed respectively on *A*, *B* and *C*. The force acting on *B* is

(A) 125 dynes (B) 35 dynes (C) 25 dynes (D) Zero

Ans. (C)

Solution:

Net force on *B*, $F_{net} = \sqrt{F_A^2 + F_C^2}$

$$F_A = \frac{15 \times 12}{(3)^2} = 20\ dyne$$

$$F_C = \frac{12 \times 20}{(4)^2} = 15\ dyne$$

$$F_{net} = 25\ dyne$$

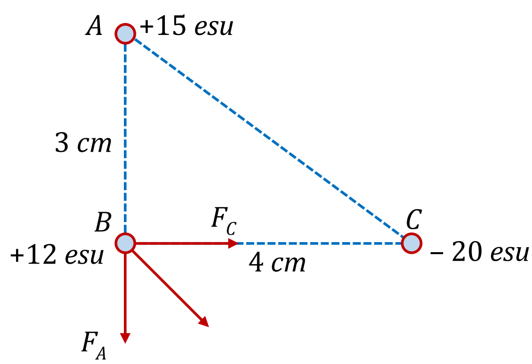
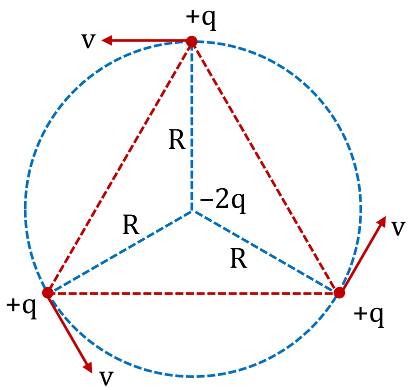
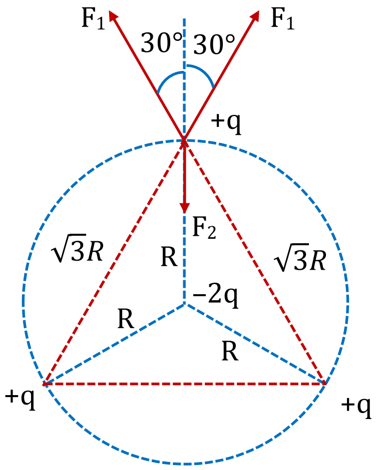


Illustration 11:

Three equal point charges of charge $+q$ each are moving along a circle of radius R and a point charge $-2q$ is also placed at the centre of circle (as shown in figure). If charges are revolving with constant and same speed in the circle then calculate speed of charges



Solution:



$$F_2 - 2F_1 \cos 30° = \frac{mv^2}{R}$$

$$\Rightarrow \quad \frac{kq\ (2q)}{R^2} - \frac{2(kq^2)}{(\sqrt{3}R)^2} \cos 30 = \frac{mv^2}{R}$$

$$\Rightarrow \quad v = \sqrt{\frac{kq^2}{Rm}\left[2 - \frac{1}{\sqrt{3}}\right]}$$

Illustration 12:

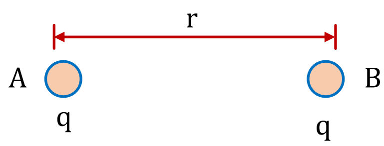
Two equally charged identical small metallic spheres A and B repel each other with a force 2×10^{-5} N when placed in air (neglect gravitational attraction). Another identical uncharged sphere C is touched to B and then placed at the mid point of line joining A and B. What is the net electrostatic force on C?

Solution:

Let, initially the charge on each sphere be q and separation between their centres be r. Then according to given problem:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q \times q}{r^2} = 2 \times 10^{-5}\ N$$

When sphere C touches B, the charge of B i.e. q will distribute equally on B and C as sphere are identical



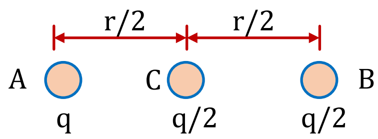
$$q_B = q_C = \frac{q}{2}$$

So sphere C will experience a force

$$F_{CA} = \frac{1}{4\pi\varepsilon_0}\frac{q(q/2)}{(r/2)^2} = 2F \text{ along } \overrightarrow{AB} \text{ due to charge on } A.$$

and, $$F_{CB} = \frac{1}{4\pi\varepsilon_0}\frac{(q/2)(q/2)}{(r/2)^2} = F$$

along $\overrightarrow{BA}$ due to charge on B.



So, the net force F_C on C due to charges on A and B,

$$F_C = F_{CA} - F_{CB} = 2F - F = 2 \times 10^{-5}\ N \text{ along } \overrightarrow{AB}.$$

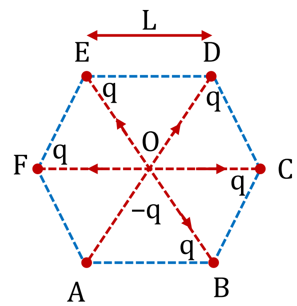
Illustration 13:

Five point charges, each of value q are placed on five vertices of a regular hexagon of side L. What is the magnitude of the force on a point charge of value $-q$ coulomb placed at the centre of the hexagon?

Solution:

Method-I:

If there had been a sixth charge $+q$ at the remaining vertex of hexagon, force due to all the six charges on $-q$ at O would have been zero (as the forces due to individual charges will balance each other), i.e., $\overrightarrow{F_R} = 0$



Now if $\vec{f}$ is the force due to sixth charge and $\vec{F}$ due to remaining five charges.

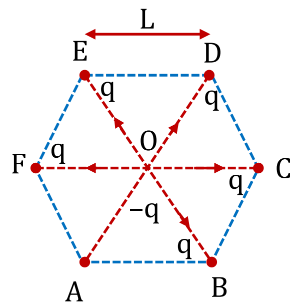
From $\vec{F} + \vec{f} = 0$ i.e. $\vec{F} = -\vec{f}$

or, $$|F| = |f| = \frac{1}{4\pi\varepsilon_0}\frac{q \times q}{L^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{L^2}$$

$$\vec{F}_{Net} = \vec{F}_{OD} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{L^2} \text{ along } OD$$

Method-II:

In the diagram, we can see that force due to charge A and D are opposite to each other



$$\vec{F}_{OF} + \vec{F}_{OC} = \vec{0} \quad \text{...(i)}$$

Similarly $$\vec{F}_{OB} + \vec{F}_{OE} = \vec{0} \quad \text{...(ii)}$$

So $$\vec{F}_{OF} + \vec{F}_{OB} + \vec{F}_{OC} + \vec{F}_{OD} + \vec{F}_{OE} = \vec{F}_{Net}$$

Using (i) and (ii) $\vec{F}_{Net} = \vec{F}_{OD} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2}{L^2}$ along OD.

Electrostatic Equilibrium

The point where the resultant force on a charged particle becomes zero is called equilibrium position.

Stable Equilibrium:

A charge is initially in equilibrium position and is displaced by a small distance. If the charge tries to return back to the same equilibrium position then this equilibrium is called position of stable equilibrium.

Unstable Equilibrium:

If charge is displaced by a small distance from its equilibrium position and the charge has no tendency to return to the same equilibrium position. Instead it goes away from the equilibrium position.

Neutral Equilibrium:

If charge is displaced by a small distance and it is still in equilibrium condition then it is called neutral equilibrium.

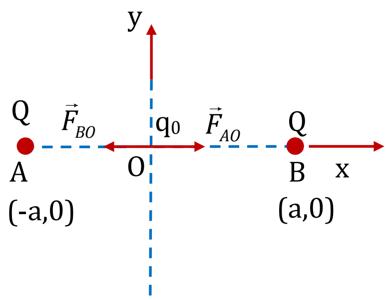
Illustration 14:

Two equal positive point charges *'Q'* are fixed at points $B(a, 0)$ and $A(-a, 0)$. Another test charge q_0 is also placed at $O(0, 0)$. Show that the equilibrium at *'O'* is

(i) Stable for displacement along *X*-axis. (ii) Unstable for displacement along *Y*-axis.

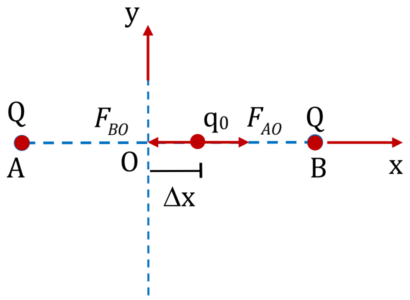
Solution:

(i)



Initially $\vec{F}_{AO} + \vec{F}_{BO} = 0 \Rightarrow |\vec{F}_{AO}| = |\vec{F}_{BO}| = \dfrac{kQq_0}{a^2}$

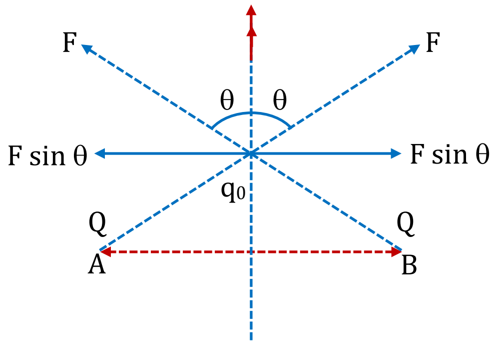
When charge is slightly shifted towards + x axis by a small distance Δx, then.



$|\vec{F}_{AO}| < |\vec{F}_{BO}|$

Therefore, the particle will move towards origin (its original position). Hence, the equilibrium is stable.

(ii) When charge is shifted along y axis:



After resolving components, net force will be along y axis So, the particle will not return to its original position & it is unstable equilibrium. Finally, the charge will move to infinity.

Illustration 15:

Two-point charges of charge q_1 and q_2 (both of same sign) and each of mass m are placed such that gravitational attraction between them balances the electrostatic repulsion. Are they in stable equilibrium? If not, then what is the nature of equilibrium?

Solution:

In given example: $\dfrac{kq_1q_2}{r^2} = \dfrac{Gm^2}{r^2}$

We can see that irrespective of distance between them charges will remain in equilibrium. If now distance is increased or decreased then there is no effect in their equilibrium. Therefore, it is a neutral equilibrium.

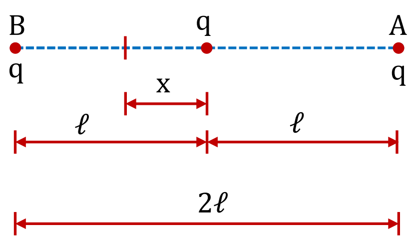
Illustration 16:

A particle of mass m and charge q is located midway between two fixed charged particles each having a charge q and a distance 2ℓ apart. Prove that the motion of the particle will be SHM if it is displaced slightly along the line connecting them and released. Also find its time period.

Solution:

Let the charge q at the mid-point is displaced slightly to the left. The force on the displaced charge q due to charge q at A,

$$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(\ell + x)^2}$$



The force on the displaced charge q due to charge at B,

$$F_2 = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(\ell - x)^2}$$

Net restoring force on the displaced charge q.

$$F = F_2 - F_1 \text{ or } F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(\ell - x)^2} - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(\ell + x)^2}$$

or $$F = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{(\ell - x)^2} - \frac{1}{(\ell + x)^2}\right] = \frac{q^2}{4\pi\varepsilon_0}\frac{4\ell x}{(\ell^2 - x^2)^2}$$

Since $\ell >> x$,

$$\therefore \quad F = \frac{q^2 \ell x}{\pi\varepsilon_0 \ell^4}$$

$$\Rightarrow \quad F = \frac{q^2 x}{\pi\varepsilon_0 \ell^3}$$

Hence, we see that $F \propto x$ and it is opposite to the direction of displacement. Therefore, the motion is SHM.

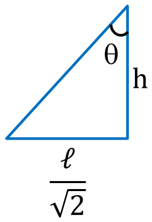
$$T = 2\pi\sqrt{\frac{m}{k}}, \text{ (here } k = \frac{q^2}{\pi\varepsilon_0 \ell^3}\text{)}$$

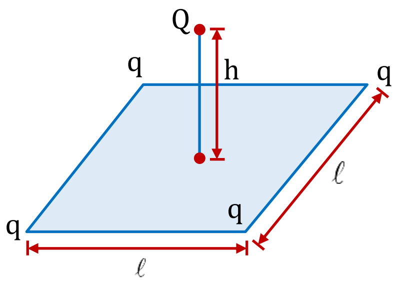
$$T = 2\pi\sqrt{\frac{m\pi\varepsilon_0 \ell^3}{q^2}}$$

Illustration 17:

Find out mass of the charge Q, so that it remains in equilibrium for the given configuration.

Solution:





$$\Rightarrow \quad 4F\cos\theta = mg$$

$$\Rightarrow \quad 4 \times \frac{kQq}{\left(\frac{\ell^2}{2} + h^2\right)^{3/2}} h = mg \qquad \therefore m = \frac{4kQqh}{g\left(\frac{\ell^2}{2} + h^2\right)^{3/2}}$$

Electric Field

Electric field is the region in space in which a charge particle experiences electric force. Electrostatic field is the region around charged particle or charged body in which if another charge is placed, it experiences electrostatic force.

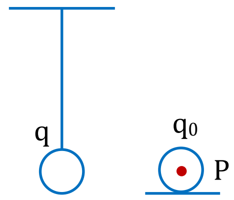
Electric field intensity $\vec{E}$

Electric field intensity at a point is equal to the electrostatic force experienced by a unit positive point charge both in magnitude and direction.

If a test charge q_0 is placed at a point in an electric field and experiences a force $\vec{F}$ due to some charges (called source charges), the electric field intensity at that point due to source charges is given by $\vec{E} = \frac{\vec{F}}{q_0}$.

If the $\vec{E}$ is to be determined practically then the test charge q_0 should be small otherwise it will affect the charge distribution on the source which is producing the electric field and hence modify the quantity which is measured.

Illustration 18:



A positively charged ball hangs from a long silk thread. We wish to measure E at a point P in the same horizontal plane as that of the hanging charge. To do so, we put a positive test charge q_0 at the point and measure F/q_0. Will F/q_0 be less than, equal to, or greater than E at the point in question?

Solution:

When we try to measure the electric field at point P then after placing the test charge at P, it repels the source charge (suspended charge) and the measured value of electric field $E_{\text{measured}} = \frac{F}{q_0}$ will be less than the actual value E_{act}, that we wanted to measure.

Physical significance of electric field:

You may wonder why the notion of electric field has been introduced here at all. After all, for any system of charges, the measurable quantity is the force on a charge which can be directly determined using Coulomb's law and the superposition principle [Eq. (1.5)]. Why then introduce this intermediate quantity called the electric field? For electrostatics, the concept of electric field is convenient, but not necessary. Electric field is an elegant way of characterising the electrical environment of a system of charges. Electric field at a point in the space around a system of charges tells you the force a unit positive test charge would experience if placed at that point (without disturbing the system). Electric field is a characteristic of the system of charges and is independent of the test charge that you place at a point to determine the field. The term field in physics generally refers to a quantity that is defined at every point in space and may vary from point to point.

Electric field is a vector field, since force is a vector quantity. The true physical significance of the concept of electric field, however, emerges only when we go beyond electrostatics and deal with time-dependent electromagnetic phenomena. Suppose we consider the force between two distant charges q_1, q_2 in accelerated motion. Now the greatest speed with which a signal or information can go from one point to another is, the speed of light. Thus, the effect of any motion of q_1 on q_2 cannot arise instantaneously. There will be some time delay between the effect (force on q_2) and the cause (motion of q_1). It is precisely here that the notion of electric field (strictly, electromagnetic field) is natural and very useful. The field picture is this: the accelerated motion of charge q_1 produces electromagnetic waves, which then propagate with the speed c, reach q_2 and cause a force on q_2. The notion of field elegantly accounts for the time delay. Thus, even though electric and magnetic fields can be detected only by their effects (forces) on charges, they are regarded as physical entities, not merely mathematical constructs. They have an independent dynamic of their own, i.e., they evolve according to laws of their own. They can also transport energy. Thus, a source of time-dependent electromagnetic fields, turned on for a short interval of time and then switched off, leaves behind propagating electromagnetic fields transporting energy. The concept of field was first introduced by Faraday and is now among the central concepts in physics.

Properties of electric field intensity $\vec{E}$

(i) It is a vector quantity. Its direction is the same as the force experienced by positive charge.

(ii) Direction of electric field due to positive charge is always away from it while due to negative charge, always towards it.

(iii) Its S.I. unit is Newton/Coulomb.

(iv) Its dimensional formula is $[MLT^{-3}A^{-1}]$

(v) Electric force on a charge q placed in a region of electric field at a point where the electric field intensity is $\vec{E}$ is given by $\vec{F} = q\vec{E}$.

Electric force on point charge is in the same direction of electric field on positive charge and in opposite direction on a negative charge.

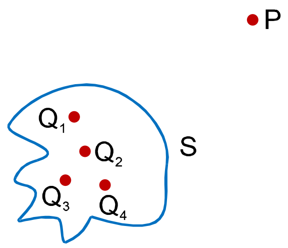
(vi) It obeys the superposition principle, that is, the field intensity at a point due to a system of charges is vector sum of the field intensities due to individual point charges.

i.e. $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + ...$

(vii) It is produced by source charges. The electric field will be a fixed value at a point unless we change the distribution of source charges.

Illustration 19:

Electrostatic force experienced by $-3\ \mu C$ charge placed at point P due to a system S of fixed point charges as shown in figure is $\vec{F} = (21\hat{i} + 9\hat{j})\,\mu N$.



(i) Find out electric field intensity at point P due to S.

(ii) If now, $2\ \mu C$ charge is placed and $-3\ \mu C$ is removed at point P then force experienced by it will be.

Solution:

(i) $\vec{F} = q\vec{E} \Rightarrow (21\hat{i} + 9\hat{j})\mu N = -3\mu C\left(\vec{E}\right)$

$\Rightarrow \quad \vec{E} = -7\hat{i} - 3\hat{j}\,\dfrac{N}{C}$

(ii) Since the source charges are not disturbed the electric field intensity at P will remain same.

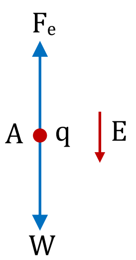
$$\vec{F}' = +2\left(\vec{E}\right) = 2\left(-7\hat{i} - 3\hat{j}\right) = \left(-14\hat{i} - 6\hat{j}\right)\mu N$$

Illustration 20:

Calculate the electric field intensity which would be just sufficient to balance the weight of a particle of charge $-10\ mC$ and mass $10\ mg$. (*Take* $g = 10\ ms^{2}$)

Solution:

As force on a charge q in an electric field $\vec{E}$ is $\vec{F}_e = q\vec{E}$



So, according to given problem:

$|\vec{F}_e| = |\vec{W}|$

Where W = weight of particle

i.e., $|q|E = mg$

i.e., $E = \dfrac{mg}{|q|} = 10\ N/C.$, in downward direction.

Illustration 21:

Find out electric field intensity at point A (0, 1m, 2m) due to a point charge $-20\mu C$ situated at point $B(\sqrt{2}\,m, 0, 1m)$.

Solution:

$\Rightarrow \quad \vec{r}$ = Position vector of A – Position vector of B = $\left(-\sqrt{2}\hat{i}+\hat{j}+\hat{k}\right)$

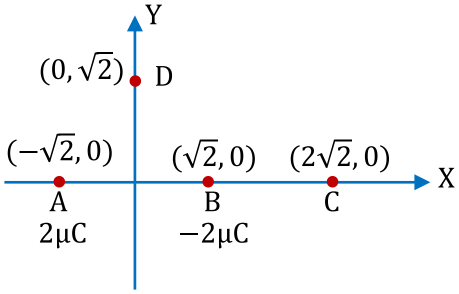
and $\quad |\vec{r}| = \sqrt{(\sqrt{2})^2+(1)^2+(1)^2} = 2$

$$\vec{E} = \frac{KQ}{|\vec{r}|^3}\vec{r} = \frac{KQ}{|\vec{r}|^2}\hat{r}$$

$$\vec{E} = \frac{9\times10^9\times(-20\times10^{-6})}{8}\left(-\sqrt{2}\hat{i}+\hat{j}+\hat{k}\right) = -22.5\times10^3\left(-\sqrt{2}\hat{i}+\hat{j}+\hat{k}\right)\ N/C.$$

Illustration 22:

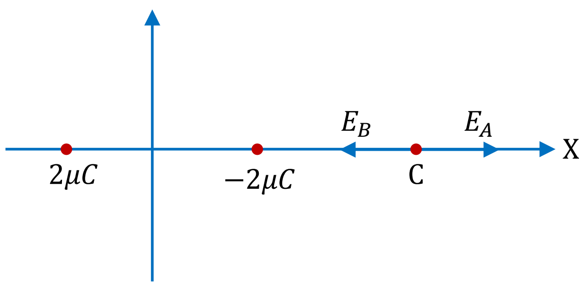
Two point charges $2\mu C$ and $-2\mu C$ are placed at points A and B as shown in figure. Find out electric field intensity at points C and D. [All the distances are measured in meter].



Solution:

Electric field at point C :

Electric field due to positive charge is away from it while due to negative charge, it is towards the charge. It is clear that $E_B > E_A$.



$\therefore \quad E_{Net} = (E_B - E_A)$ towards negative X-axis

$$= \frac{k(2\mu C)}{(\sqrt{2})^2} - \frac{k(2\mu C)}{(3\sqrt{2})^2} \text{ towards negative } x\text{-axis}$$

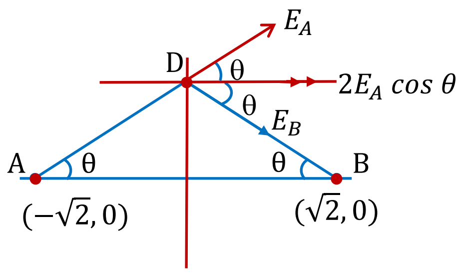
$$= 8000\ (-\hat{i})\ N/C$$

Electric field at point D.

Since magnitude of charges are same and also $AD = BD$.

So, $\quad E_A = E_B$

Vertical components of $\vec{E}_A$ and $\vec{E}_B$ cancel each other while horizontal components are in the same direction.

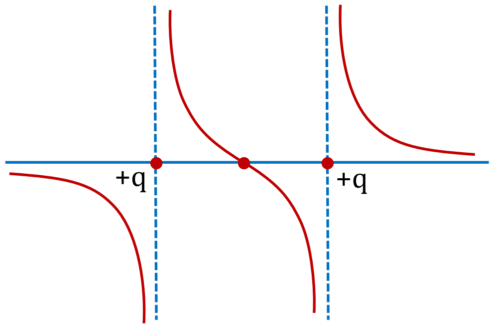


So, $\quad E_{net} = 2E_A\cos\theta = \dfrac{2.k(2\mu C)}{2^2}\cos45°$

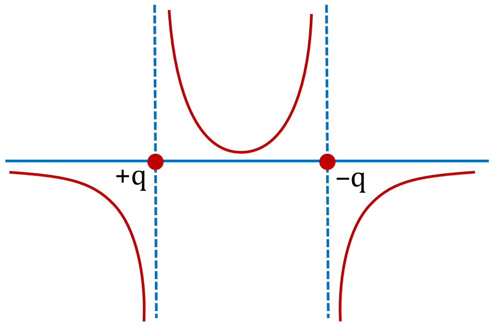
$$= \frac{k\times10^{-6}}{\sqrt{2}} = \frac{9000}{\sqrt{2}}\hat{i}\ N/C.$$

Variation of Electric Field Intensity on the line joining two point charges:

Variation of electric field for charges $+q$ and $+q$ separated by a distance r.



Variation of electric field for charges $+q$ and $-q$ separated by a distance r.



Variation of electric field for charges $+q$ and $+4q$ separated by a distance r.

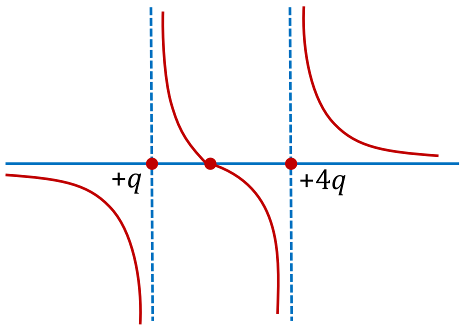
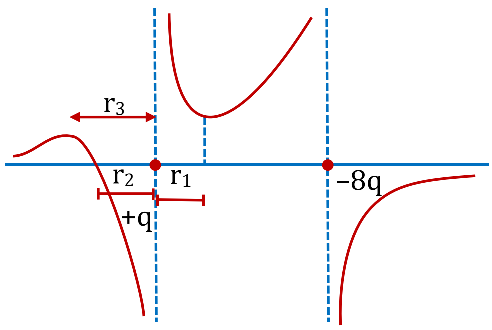


Illustration 23:

Two-point charges $-8q$ and q are kept at a distance of r. Variation of electric field is as shown in figure. Find value of r_1, r_2 and r_3.



Solution:

Electric field at $r_2 = 0$

So $$\frac{k8q}{(r+r_2)^2} = \frac{kq}{r_2^2}$$

$$\left(\frac{r+r_2}{r_2}\right)^2 = 8 \qquad \frac{r+r_2}{r_2} = -2\sqrt{2}$$

$$r_2 = \frac{r}{-2\sqrt{2}-1}$$

$\therefore$ $r_2 = \dfrac{r}{2\sqrt{2}+1}$ to the left of charge $+q$

Now for r_1 and r_3:

$$E_1 = \frac{kq}{x^2} \quad \text{and} \quad E_2 = \frac{8kq}{(r+x)^2}$$

$$E_{net} = \frac{8kq}{(r+x)^2} - \frac{kq}{x^2}$$

Now $\dfrac{dE_{net}}{dx} = 0$ to get position where E is maximum or minimum.

$$x = r, \frac{-r}{3}, \qquad \text{So} \quad r_1 = r$$

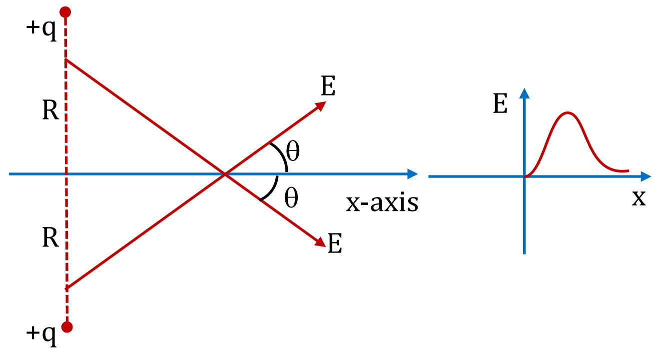
and $r_2 = \dfrac{r}{3}$ to the left of charge $+q$

Variation of Electric Field Intensity on perpendicular bisector of line joining two point charges:

$$E_{net} = 2E\cos\theta$$

$$= \frac{2kq}{x^2+r^2} \times \frac{x}{\sqrt{x^2+R^2}} = \frac{2kqx}{(x^2+R^2)^{\frac{3}{2}}}$$

$$\boxed{E_{net} = \frac{2kqx}{(x^2+R^2)^{\frac{3}{2}}}}$$

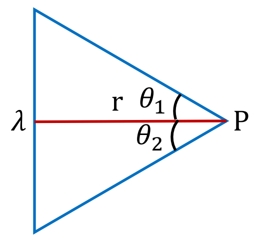


For two-point charges (same charges) on perpendicular bisector Electric Field Intensity is max at distance $\dfrac{R}{\sqrt{2}}$.

Electric field due to uniformly charged wire:

(i) **Line charge of finite length:** Derivation of expression for intensity of electric field at a point due to line charge of finite size of uniform linear charge density λ. The perpendicular distance of the point from the line charge is r and lines joining ends of line charge distribution make angle θ_1 and θ_2 with the perpendicular line.

Consider a small element dx on line charge distribution at distance x from point A (see fig.). The charge of this element will be $dq = \lambda dx$. Due to this charge (dq), the intensity of electric field at the point P is dE.



Then, $dE = \dfrac{k(dq)}{r^2+x^2} = \dfrac{k(\lambda dx)}{r^2+x^2}$

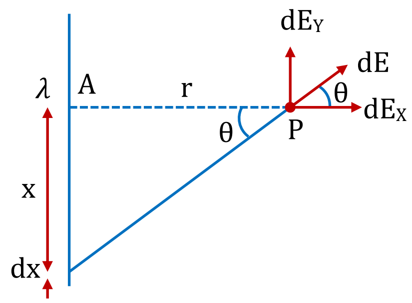
There will be two components of this field: dE_x and dE_y

$$E_x = \int dE_x = \int dE\cos\theta = \int \frac{k\lambda dx}{r^2+x^2}.\cos\theta$$

Assuming, $x = r\tan\theta \Rightarrow dx = r\sec^2\theta.\ d\theta$

so $$E_x = \int_{-\theta_2}^{+\theta_1} \frac{k\ \ \lambda r\sec^2\theta\cdot\cos\theta\cdot d\theta}{r^2+r^2\tan^2\theta} = \frac{k\lambda}{r}\int_{-\theta_2}^{+\theta_1}\cos\theta.d\theta$$

$$E_x = \frac{k\lambda}{r}\ [\sin\theta_1 + \sin\theta_2] \qquad ...(1)$$



Similarly y-component.

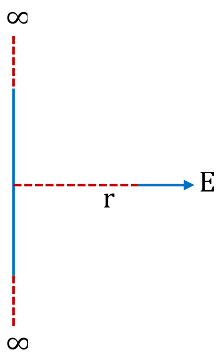
$$E_y = \frac{k\lambda}{r}\int_{-\theta_2}^{+\theta_1}\sin\theta.d\theta$$

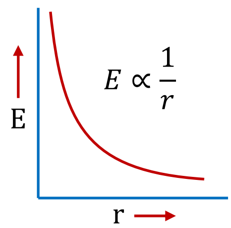
$$E_y = \frac{k\lambda}{r}\ [\cos\theta_2 - \cos\theta_1] \qquad ...(2)$$

Net electric field at the point: $E_{net} = \sqrt{E_x^2+E_y^2}$

(ii) We can derive a result for infinitely long line charge: In above eq. (1) and (2), if we put $\theta_1 = \theta_2 = 90°$, we can get required result.

$$E_{net} = E_x = \frac{2k\lambda}{r}$$





(iii) For Semi- infinite wire:

$\theta_1 = 90°$

and $\theta_2 = 0°$

so, $E_x = \dfrac{k\lambda}{r}$

$E_y = \dfrac{k\lambda}{r}$

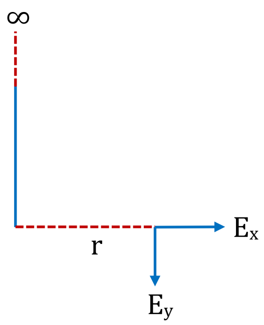
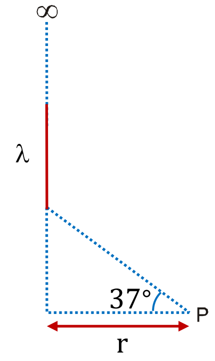
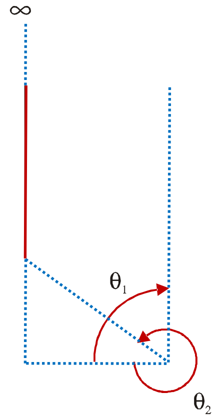


Illustration 24:

Figure shows a long wire having uniform charge density λ as shown in figure. Calculate electric field intensity at point *P*.



Solution:



$\theta_1 = 90°$

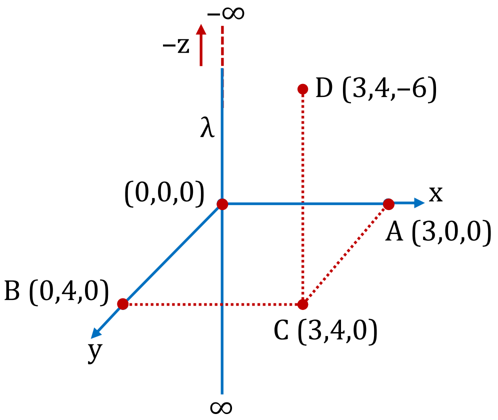
and $\theta_2 = 360° - 37°$

So $E_x = E_r = \dfrac{2K(P\cos\theta)}{r^3}$ $[\sin\theta_1 + \sin\theta_2]$

$E_y = \dfrac{k\lambda}{r}$ $[\cos\theta_2 - \cos\theta_1]$

Illustration 25:

Find electric field at point *A, B, C, D* due to infinitely long uniformly charged wire with linear charge density λ and kept along *z*-axis (as shown in figure). Assume that all the parameters are in S.I. units.



Solution:

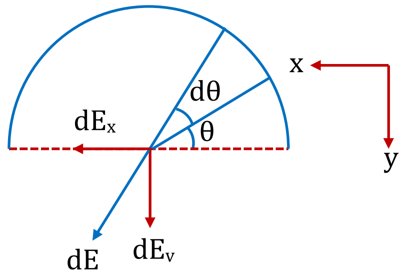
$E_A = \dfrac{2k\lambda}{3}(\hat{i})$; $E_B = \dfrac{2k\lambda}{4}(\hat{j})$

$E_C = \dfrac{2k\lambda}{5}\ \hat{OC} = \dfrac{2k\lambda}{5}\left(\dfrac{3\hat{i}+4\hat{j}}{5}\right)$; $E_D = \dfrac{2k\lambda}{5}\left(\dfrac{3\hat{i}+4\hat{j}}{5}\right) \Rightarrow E_D = E_C$

Electric field due to a uniformly charged ring and arc.

Derivation of electric field intensity at the centre of a uniformly charged semi-circular ring of radius *R* and linear charge density λ.

The arc is the collection of large number of point charges. Consider a part of ring as an element of length $Rd\theta$ which subtends an angle $d\theta$ at centre of ring and it lies between θ and $\theta + d\theta$



$$\overrightarrow{dE} = dE_x\hat{i} + dE_y\,\hat{j}$$

$$E_x = \int dE_x = 0 \text{ (due to symmetry)}$$

and $$E_y = \int dE_y = \int_0^{\pi} dE\sin\theta = \frac{k\lambda}{R}\int_0^{\pi}\sin\theta.d\theta = \frac{2k\lambda}{R}$$

Illustration 26:

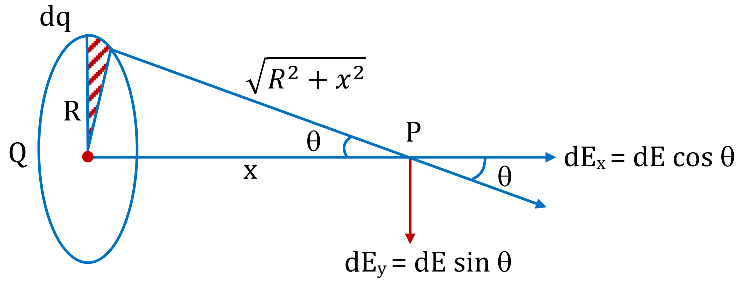
Find out electric field intensity at the centre of uniformly charged quarter ring of radius R and linear charge density λ.

Solution:

Refer to the previous question $\overrightarrow{dE} = dE_x\hat{i} + dE_y\,\hat{j}$

$\therefore$ On solving $E_{net} = \dfrac{k\lambda}{R}(\hat{i}+\hat{j})$

Derivation of electric field intensity at a point on the axis at a distance *x* from centre of uniformly charged ring of radius *R* and total charge *Q*.



Consider an element of charge dq. Due to this element, the electric field at the point on axis, which is at a distance x from the centre of the ring is dE.

There are two components of this electric field dE_x and dE_y.

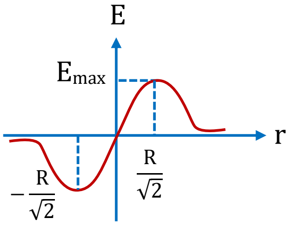
The y-component of electric field due to all the elements will be cancelled out to each other. So net electric field intensity at the point will be only due to X-component of each element.

$$E_{net} = \int dE_x = \int dE\cos\theta$$

$$= \int_0^Q \frac{k(dq)}{R^2+x^2}\times\frac{x}{\sqrt{R^2+x^2}} = \frac{k\ x}{(R^2+x^2)^{3/2}}\int_0^Q dq$$

$$E_{net} = \frac{kQx}{[R^2+x^2]^{3/2}}$$

Graph for variation of E with r.



E will be max when $\frac{dE}{dx} = 0$,

that is at $x = \frac{R}{\sqrt{2}}$ and $E_{max} = \frac{2kQ}{3\sqrt{3}\,R^2}$

Case (i) : If $x >> R$, $E = \frac{kQ}{x^2}$. Hence the ring will act like a point charge.

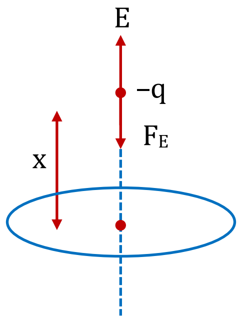
Case (ii) : If $x << R$, $E = \frac{kQ\,x}{R^3}$

Illustration 27:

Positive charge Q is distributed uniformly over a circular ring of radius a. A point particle having a mass m and a negative charge $-q$, is placed on its axis at a distance x from the centre. Find the force on the particle. Assuming $x << a$, find the time period of oscillation of the particle if it is released from there.
(Neglect gravity)

Solution:

When the negative charge is shifted at a distance x from the centre of the ring along its axis then force acting on the point charge due to the ring:



$$F_E = qE \text{ (towards centre)} = q\left[\frac{kQx}{(a^2+x^2)^{3/2}}\right]$$

If $a >> x$ then $a^2 + x^2 \cong a^2$

$\therefore$ $F_E = \frac{1}{4\pi\varepsilon_0}\frac{Qqx}{a^3}$ (Towards centre)

Since, restoring force $F_E \propto x$, therefore motion of charge the particle will be S.H.M.

Time period of SHM

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\left(\frac{Qq}{4\pi\varepsilon_0 a^3}\right)}} = \left[\frac{16\pi^3\varepsilon_0 m a^3}{Qq}\right]^{1/2}$$

Illustration 28:

Calculate electric field intensity at a point on the axis which is at distance x from the centre of half ring, having total charge Q distributed uniformly on it. The radius of half ring is R.

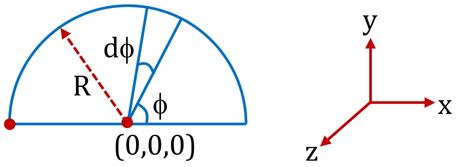
Solution:

Consider an element of small angle $d\phi$ at an angle ϕ as shown.

Coordinates of element: $(R\cos\phi, R\sin\phi, 0)$

Coordinates of point: $(0, 0, x)$

Now electric field due to element:



$$\overrightarrow{dE} = \frac{k(\lambda R d\phi)\cdot[-R\cos\phi\hat{i} - R\sin\phi\hat{j} + x\hat{k}]}{(R^2\cos^2\phi + R^2\sin^2\phi + x^2)^{3/2}}$$

$$E_x = \int dE_x = -\int_0^{\pi}\frac{k\lambda R^2\cos\phi d\phi}{(R^2+x^2)^{3/2}} = 0$$

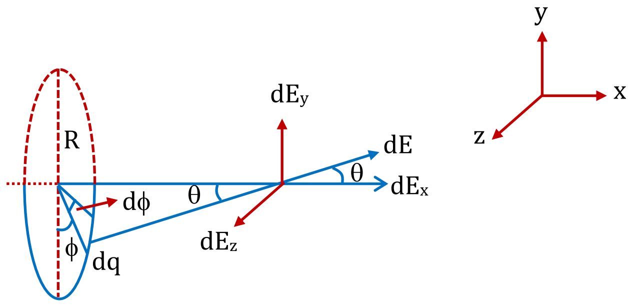
$$E_y = \int dE_y = -\int_0^{\pi}\frac{k\lambda R^2\sin\phi d\phi}{(R^2+x^2)^{3/2}} = \frac{2k\lambda R^2}{(R^2+x^2)^{3/2}} = \frac{2kQR}{\pi(R^2+x^2)^{3/2}}1$$

$$E_z = \int dE_z = \int_0^{\pi}\frac{k\lambda Rx d\phi}{(R^2+x^2)^{3/2}} = \frac{kQx}{(R^2+x^2)^{3/2}}$$

$$E_{net} = \sqrt{E_x^2+E_y^2+E_z^2} = \frac{kQ}{(R^2+x^2)^{3/2}}\sqrt{\frac{4R^2}{\pi^2}+x^2}$$

Alternate solution:

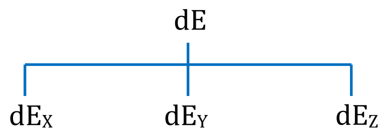
Consider an element of charge dq at an angle ϕ on circumference of half ring. Due to this element electric field at the point on axis, which is at a distance x from the centre of half ring is dE. This electric field can be resolved into three component.



$$E_Z = \int_{-\pi/2}^{\pi/2} dE\sin\theta\sin\phi = 0$$

$$E_X = \int_{-\pi/2}^{\pi/2} dE\cos\theta = \frac{kQx}{[R^2+x^2]^{3/2}}$$

$$E_Y = \int_{-\pi/2}^{\pi/2} dE\sin\theta\cos\phi = \int\frac{kdq}{R^2+x^2}\ \sin\theta.\cos\phi$$



$\therefore \quad dq = \lambda R d\phi$ and $\sin\theta = \dfrac{R}{\sqrt{R^2+x^2}}$

So $\quad E_Y = \dfrac{2k\lambda R\sin\theta}{R^2+x^2}$

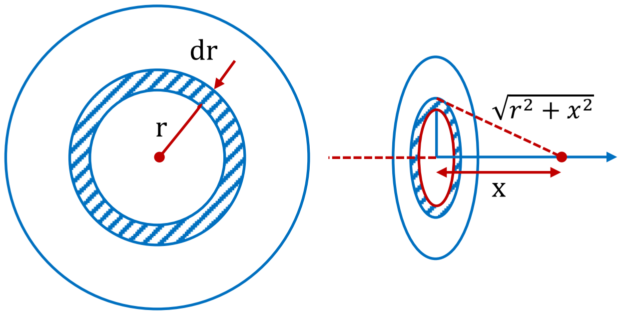
$\Rightarrow \quad E_{\text{net}} = \sqrt{E_X^2+E_Y^2}$

Electric field due to a uniformly charged disc.

Now we are going to derive the expression of electric field intensity at a point P which is situated at a distance x on the axis of uniformly charged disc of radius R and surface charge density σ. Also, derive results for:

(i) $x >> R$ (ii) $x << R$

The disc can be considered to be a collection of large number of concentric rings. Consider an element of the shape of rings of radius r and of width dr. Electric field due to this ring at P is



$$dE=\frac{k.\sigma 2\pi r.dr.x}{(r^2+x^2)^{3/2}}$$

Put, $r^2 + x^2 = y^2$

$2rdr = 2ydy$

$$\therefore \quad dE = \frac{k.\sigma 2\pi y.dy.x}{y^3} = 2k\sigma\pi.x\ \frac{ydy}{y^3}$$

Electric field at P due to all rings is along the axis:

$$\therefore \quad E=\int dE \Rightarrow E = 2k\sigma\pi x\int_x^{\sqrt{R^2+x^2}}\frac{1}{y^2}dy = 2k\rho\pi x.\left[-\frac{1}{y}\right]_x^{\sqrt{R^2+x^2}}$$

$$= 2K\sigma\pi x\left[+\frac{1}{x}-\frac{1}{\sqrt{R^2+x^2}}\right] = 2k\sigma\pi\left[1-\frac{x}{\sqrt{R^2+x^2}}\right]$$

$$=\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{\sqrt{R^2+x^2}}\right] \text{ along the axis}$$

Cases : (i) If $x >> R$

$$E=\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{x\sqrt{\frac{R^2}{x^2}+1}}\right] = \frac{\sigma}{2\varepsilon_0}\left[1-\left(1+\frac{R^2}{x^2}\right)^{-1/2}\right]$$

$$\frac{\sigma}{2\varepsilon_0}\ [1-1+\frac{1}{2}\frac{R^2}{x^2} + \text{higher order terms}] = \frac{\sigma}{4\varepsilon_0}\frac{R^2}{x^2} = \frac{\sigma\pi R^2}{4\pi\varepsilon_0 x^2} = \frac{Q}{4\pi\varepsilon_0 x^2}$$

i.e., behaviour of the disc is like a point charge.

(ii) If $x << R$

$$E=\frac{\sigma}{2\varepsilon_0}\ [1-0] = \frac{\sigma}{2\varepsilon_0}$$

i.e., behaviour of the disc is like infinite sheet.

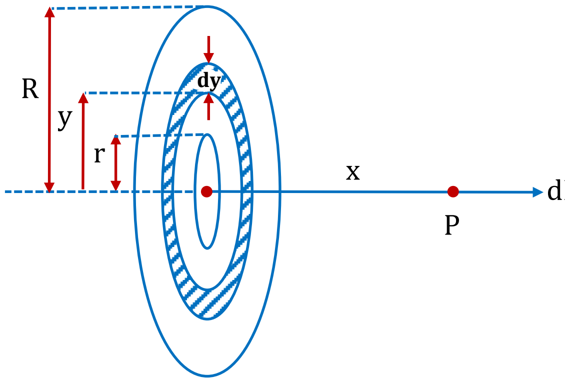
$E_{net} = \dfrac{\sigma}{2\varepsilon_o}$ towards normal direction

Illustration 29:

Calculate electric field at a point on axis, which at a distance x from centre of uniformly charged disc having surface charge density σ and R which also contains a concentric hole of radius r.

Solution:

Consider a ring of radius y ($r < y < R$) and width dy concentric with disc and in the plane of the disc. Due to this ring, the electric field at the point P



$$dE = \frac{k(dq)x}{[x^2+y^2]^{3/2}}$$

$$E_{net} = \int_r^R \frac{kx.\sigma(2\pi y)dy}{[x^2+y^2]^{3/2}} \qquad [\because dq = \sigma 2\pi y dy]$$

Put $x^2 + y^2 = t$

$2y.\, dy = dt$

$$E_{net} = \frac{2\pi\sigma kx}{2}\int_{x^2+r^2}^{x^2+R^2}\frac{dt}{t^{3/2}} = \frac{\sigma x}{2\varepsilon_0}\left[\frac{1}{\sqrt{x^2+r^2}} - \frac{1}{\sqrt{x^2+R^2}}\right] \quad \text{away from centre}$$

Alternate method:

We can also use superposition principle to solve this problem.

(i) Assume a disc without hole of radius R having surface charge density $+\sigma$.

(ii) Also assume a concentric disc of radius r in the same plane of first disc having charge density $-\sigma$.

Now using derived formula in last example the net electric field at the centre is :

$$\vec{E}_{net} = \vec{E}_R + \vec{E}_r = \frac{\sigma x}{2\varepsilon_0}\left[\frac{1}{\sqrt{r^2+x^2}} - \frac{1}{\sqrt{R^2+x^2}}\right] \quad \text{away from centre.}$$

Electric field due to uniformly charged infinite sheet

$$E_{net} = \frac{\sigma}{2\varepsilon_o} \quad \text{towards normal direction,}$$

This field is applicable whenever $x << R$, means distance of the point of observation is very small in compared to the size of the charge sheet.

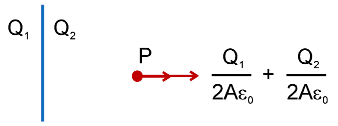
Illustration 30:

If an isolated infinite sheet contains charge Q_1 on its one surface and charge Q_2 on its other surface then prove that electric field intensity at a point in front of sheet will be $\frac{Q}{2A\varepsilon_0}$, where $Q = Q_1 + Q_2$.

Solution:

Electric field at point P :

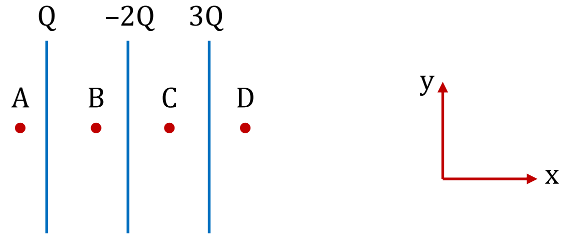
$$\vec{E} = \vec{E}_{Q_1} + \vec{E}_{Q_2} = \frac{Q_1}{2A\varepsilon_0}\hat{n} + \frac{Q_2}{2A\varepsilon_0}\hat{n} = \frac{Q_1+Q_2}{2A\varepsilon_0}\hat{n} = \frac{Q}{2A\varepsilon_0}\hat{n}$$



[This shows that the resultant field due to a sheet depends only on the total charge of the sheet and not on the distribution of charge on individual surfaces].

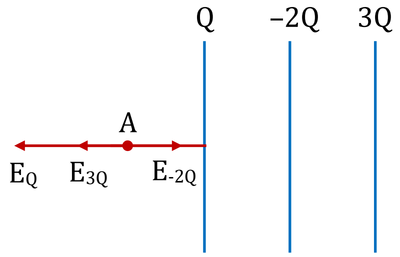
Illustration 31:

Three large conducting parallel sheets are placed at a finite distance from each other as shown in figure. Find out electric field intensity at points *A*, *B*, *C* and *D*.



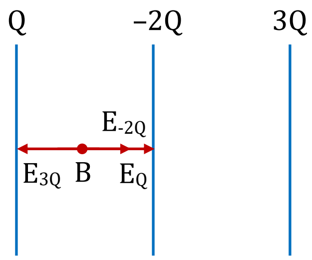
Solution:

For point A:



$$\vec{E}_{net} = \vec{E}_Q + \vec{E}_{3Q} + \vec{E}_{-2Q} = -\frac{Q}{2A\varepsilon_0}\hat{i} - \frac{3Q}{2A\varepsilon_0}\hat{i} + \frac{2Q}{2A\varepsilon_0}\hat{i} = -\frac{Q}{A\varepsilon_0}\hat{i}$$

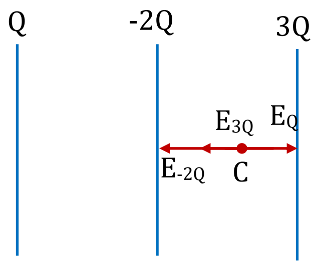
For point B:



$$\vec{E}_{net} = \vec{E}_{3Q} + \vec{E}_{-2Q} + \vec{E}_Q$$

$$= -\frac{3Q}{2A\varepsilon_0}\hat{i} + \frac{2Q}{2A\varepsilon_0}\hat{i} + \frac{Q}{2A\varepsilon_0}\hat{i} = \vec{0}$$

For point C:



$$\vec{E}_{net} = \vec{E}_Q + \vec{E}_{3Q} + \vec{E}_{-2Q}$$

$$= +\frac{Q}{2A\varepsilon_0}\hat{i} - \frac{3Q}{2A\varepsilon_0}\hat{i} - \frac{2Q}{2A\varepsilon_0}\hat{i} = -\frac{2Q}{A\varepsilon_0}\hat{i}$$

For point D:

$$\vec{E}_{net} = \vec{E}_Q + \vec{E}_{3Q} + \vec{E}_{-2Q}$$

$$= +\frac{Q}{2A\varepsilon_0}\hat{i} + \frac{3Q}{2A\varepsilon_0}\hat{i} - \frac{2Q}{2A\varepsilon_0}\hat{i}$$

$$= \frac{2Q}{A\varepsilon_0}\hat{i}$$

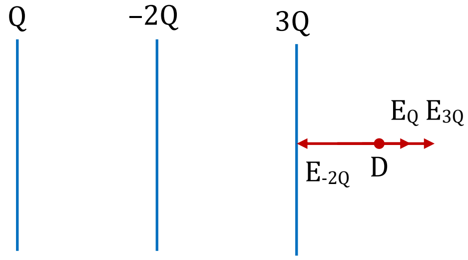


Illustration 32:

Determine and draw the graph of electric field due to infinitely large non-conducting sheet of thickness t and uniform volume charge density ρ as a function of distance x from its symmetry plane.

(a) For $x \le \frac{t}{2}$ (b) For $x > \frac{t}{2}$

Solution:

(a) We can consider two sheets of thickness $\left(\frac{t}{2} - x\right)$ and $\left(\frac{t}{2} + x\right)$

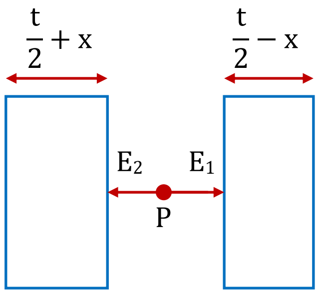
Where the point P lies inside the sheet.

Now, net electric field at point P:

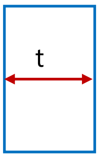
$$E = E_1 - E_2 = \frac{Q_1}{2A\varepsilon_0} - \frac{Q_2}{2A\varepsilon_0}$$

[Q_1 : charge of left sheet; Q_2 : charge of right sheet.]

$$= \frac{A\rho\left(\frac{t}{2} + x\right) - \rho A\left(\frac{t}{2} - x\right)}{2A\varepsilon_0} = \frac{\rho x}{\varepsilon_0}$$



(b) For point which lies outside the sheet we can consider a complete sheet of thickness t



$\longrightarrow \frac{Q}{2A\varepsilon_0}$

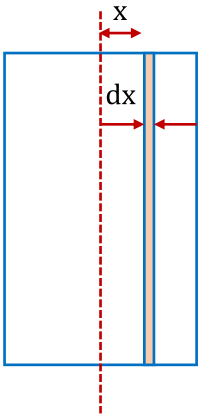
$$E = \frac{\sigma t A}{2A\varepsilon_0} = \frac{\sigma t}{2\varepsilon_0}$$

Alternate : We can assume thick sheet to be made of large number of uniformly charged thin sheets. Consider an elementary thin sheet of width dx at a distance x from symmetry plane.

Charge in sheet = $\rho A dx$ (A : assumed area of sheet)

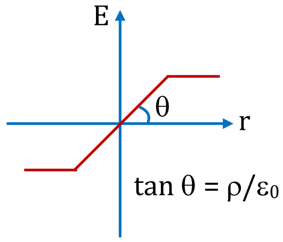
Surface charge density, $\sigma = \frac{\rho A dx}{A}$

So, electric field intensity due to elementary sheet : $dE = \frac{\rho dx}{2\varepsilon_0}$



(a) When $x < \frac{t}{2}$

$$\Rightarrow \quad E_{Net} = \int_{-t/2}^{x} \frac{\rho dx}{2\varepsilon_0} - \int_{x}^{t/2} \frac{\rho dx}{2\varepsilon_0} = \frac{\rho x}{\varepsilon_0}$$



(b) When $x > \frac{t}{2}$

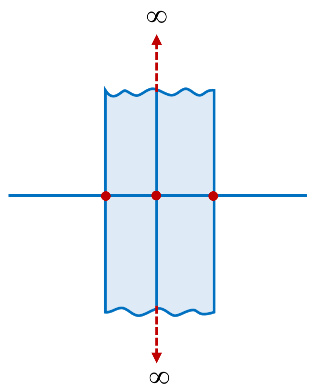
$$\Rightarrow \quad E_{Net} = \int_{-t/2}^{t/2} \frac{\rho dx}{2\varepsilon_0} = \frac{\rho t}{2\varepsilon_0}$$

Illustration 33:

Thin infinite sheet of width w contains uniform charge distribution σ. Find out electric field intensity at following points :

(a) A point which lies in the same plane at a distance d from one of its edge.

(b) A point which is on the symmetry plane of sheet at a perpendicular distance d from it.



Solution:

(a) Consider a thin strip of width dx. Linear charge density of strip:

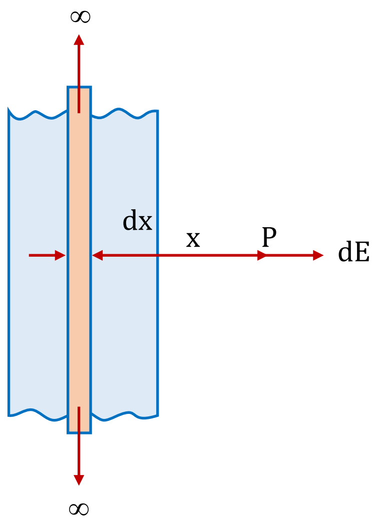
$$\lambda = \sigma dx$$

So, electric field due to this strip at point P

$$dE = \frac{2k\sigma dx}{x}$$

$$E_{net} = \int_{d}^{d+w} \frac{\sigma}{2\pi\varepsilon_0} \frac{dx}{x}$$

$$= \frac{\sigma}{2\pi\varepsilon_0} \ell n\left(\frac{d+w}{d}\right)$$



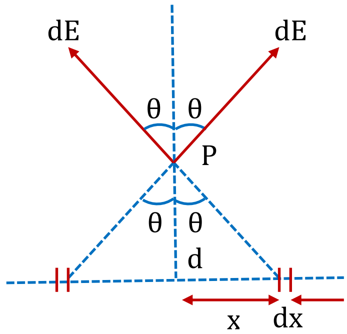
(b) Consider a thin strip of width dx. Linear charge density of strip:

$$\lambda = \sigma dx$$

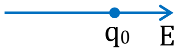
$$\therefore \quad E_p = \int 2dE\cos\theta$$

or $$E_p = 2. \int_0^{w/2} \frac{\sigma dx}{2\pi\varepsilon_0\sqrt{d^2+x^2}} . \frac{d}{\sqrt{d^2+x^2}}$$

$$= \frac{\sigma d}{\pi\varepsilon_0}\int_0^{w/2} \frac{dx}{d^2+x^2} = \frac{\sigma}{\pi\varepsilon_0}\tan^{-1}\frac{w}{2d}$$



Motion of Charged particle under electric field:



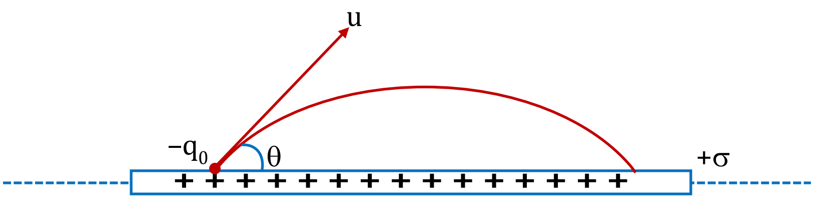
$$\boxed{\vec{F} = q_0\vec{E}}$$

If q_0 is $+ve$ then $\vec{F}$ in the direction of $\vec{E}$.

If q_0 is $-ve$ then $\vec{F}$ in opposite to $\vec{E}$.

Illustration 34:

An infinitely large plate of surface charge density $+\sigma$ is lying in horizontal xy-plane. A particle having charge $-q_o$ and mass m is projected from the plate with velocity u making an angle θ with sheet. Find :

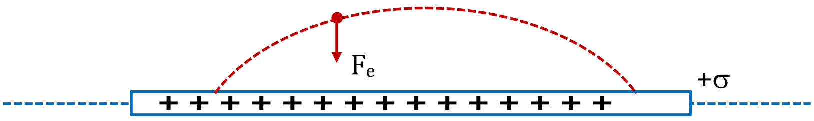


(i) The time taken by the particle to return on the plate.

(ii) Maximum height achieved by the particle.

(iii) At what distance will it strike the plate (Neglect gravitational force on the particle)

Solution:



Electric force acting on the particle $F_e = q_oE : F_e = (q_o)\left(\dfrac{\sigma}{2\varepsilon_o}\right)$ downward

So, acceleration of the particle : $a = \dfrac{F_e}{m} = \dfrac{q_o\sigma}{2\varepsilon_o m}$ = Uniform

This acceleration will act like 'g' (Acceleration due to gravity).

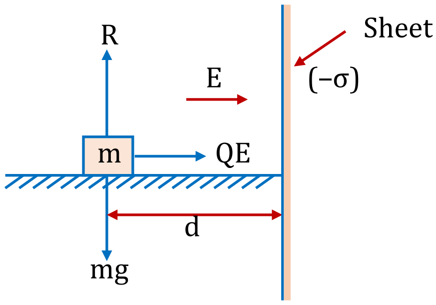
So, the particle will perform projectile motion.

(i) $T = \dfrac{2u\sin\theta}{g_{eff}} = \dfrac{2u\sin\theta}{\left(\dfrac{q_o\sigma}{2\varepsilon_o m}\right)}$ (ii) $H = \dfrac{u^2\sin^2\theta}{2g_{eff}} = \dfrac{u^2\sin^2\theta}{2\left(\dfrac{q_o\sigma}{2\varepsilon_o m}\right)}$ (iii) $R = \dfrac{u^2\sin 2\theta}{g_{eff}} = \dfrac{u^2\sin 2\theta}{\left(\dfrac{q_o\sigma}{2\varepsilon_o m}\right)}$

Illustration 35:

A block having mass m and charge Q is resting on a frictionless plane at a distance d from fixed large non-conducting infinite sheet of uniform charge density $-\sigma$ as shown in Figure.

Assuming that collision of the block with the sheet is perfectly elastic, find the time period of oscillatory motion of the block. Is it SHM?



Solution:

The situation is shown in Figure. Electric force produced by sheet will accelerate the block towards the sheet producing an acceleration. Acceleration will be uniform because electric field E due to the sheet is uniform.

$$a = \frac{F}{m} = \frac{QE}{m}, \text{ where } E = \sigma/2\varepsilon_0$$

As initially the block is at rest and acceleration is constant, from second equation of motion, time taken by the block to reach the wall

$$d = \frac{1}{2} at^2$$

i.e., $$t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2md}{QE}} = \sqrt{\frac{4md\varepsilon_0}{Q\sigma}}$$

As collision with the wall is perfectly elastic, the block will rebound with same speed and as now its motion is opposite to the acceleration, it will come to rest after traveling same distance d in same time t.

After stopping, it will again be accelerated towards the wall and so the block will execute oscillatory motion with 'span' d and time period.

$$T = 2t = 2\sqrt{\frac{2md}{QE}}$$

$$= 2\sqrt{\frac{4md\varepsilon_0}{Q\sigma}}$$

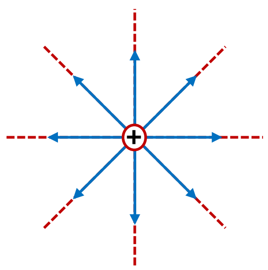
However, as the restoring force $F = QE$ is constant and not proportional to displacement x, the motion is not simple harmonic.

Electric field lines or electric lines of force:

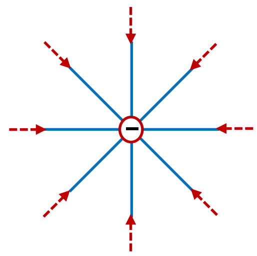
The line of force in an electric field is an imaginary line, the tangent to which at any point on it represents the direction of electric field at the given point and strength of field is proportional to the field lines density (number of field lines per unit cross-sectional area).

Properties:

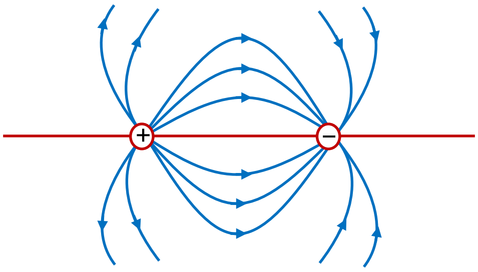
(i) Line of force originates out from a positive charge and terminates on a negative charge. If there is only one positive charge, then lines start from positive charge and terminates at ∞. If there is only one negative charge, then lines start from ∞ and terminates at negative charge.



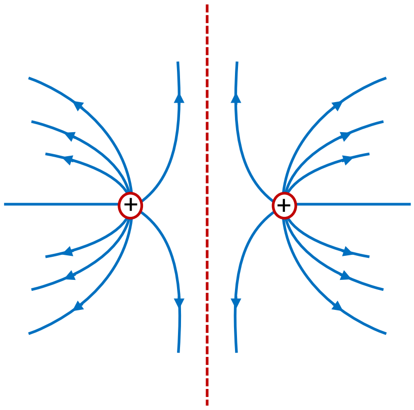
ELOF of Isolated positive charge



ELOF of Isolated negative charge

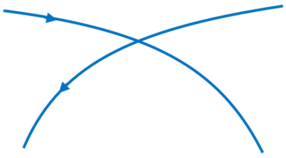


ELOF due to equal positive and negative charges



ELOF due to two equal positive charges

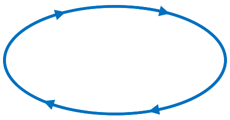
(ii) Two lines of force never intersect each other because there cannot be two directions of $\vec{E}$ at a single Point



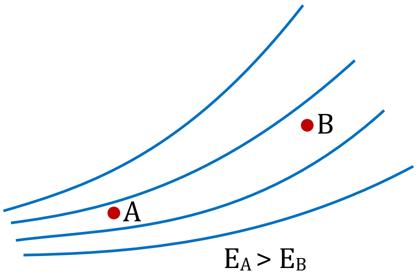
Impossible

(iii) Electric lines of force produced by static charges do not form closed loop.

If lines of force make a closed loop, then work done to move *a* $+q$ charge along the loop will be non-zero. So, it will not be conservative field. So, these type of lines of force are not possible in electrostatics.



(iv) The Number of lines per unit area (line density) represents the magnitude of electric field.



If lines are dense $\Rightarrow$ E will be more

If Lines are rare $\Rightarrow$ E will be less

If $E = 0$, no line of force will be found there

(v) Number of lines originating (terminating) at a charge is proportional to the magnitude of charge

(vi) Electric lines of force end or start perpendicularly on the surface of a conductor.

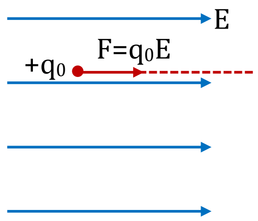
(vii) Electric lines of force never enter into conductors.

Illustration 36:

If a charge is released in electric field, will it follow lines of force?

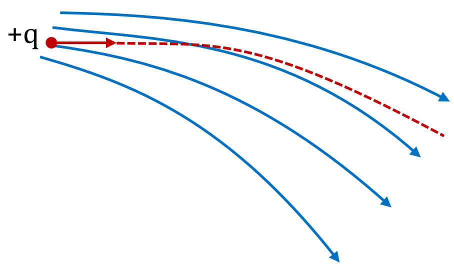
Solution:

Case I : If lines of force are parallel (in uniform electric field) :



In this type of field, if a charge is released, force on it will be q_0E and its direction will be along. So the charge will move in a straight line, along the lines of force.

Case II: If lines of force are curved (in non-uniform electric field) :



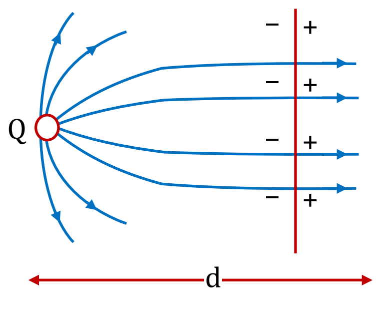
The charge will not follow lines of force.

Illustration 37:

A charge + Q is fixed at a distance d in front of an infinite metal plate. Draw the lines of force indicating the directions clearly.

Solution:

There will be induced charge on two surfaces of conducting plate, so $ELOF$ will start from $+Q$ charge and terminate at conductor and then will again start from other surface of conductor.



Electric Flux

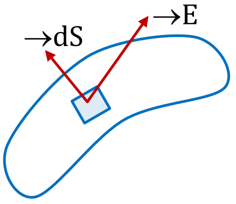
Consider some surface in an electric field $\vec{E}$. Let us select a small area element $\overrightarrow{dS}$ on this surface. The electric flux of the field over the area element is given by $d\phi = \vec{E}.\overrightarrow{dS}$

Direction of $\overrightarrow{dS}$ is normal to the surface. It is along $\hat{n}$

or $\quad d\phi = EdS \cos\theta$

or $\quad d\phi = (E \cos\theta)\, dS$

or $\quad d\phi = E_n\, dS$



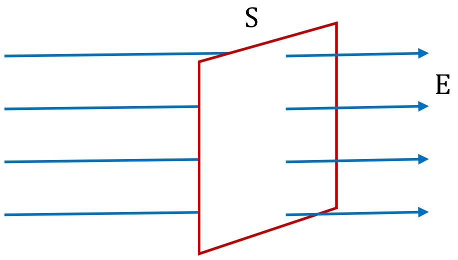
where E_n is the component of electric field in the direction of $\overrightarrow{dS}$.

The electric flux over the whole area is given by $\phi = \int_S \vec{E}.\overrightarrow{dS} = \int_S E_n dS$.

If the electric field is uniform over that area then $\phi = \vec{E}\cdot\vec{S}$.

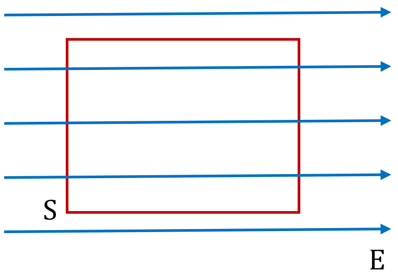
Special Cases:

Case I: If the electric field is normal to the surface, then angle of electric field $\vec{E}$ with normal will be zero.



So $\quad \phi = ES \cos 0° \quad$ or $\quad \phi = ES$

Case II: If electric field is parallel of the surface (grazing), then angle made by $\vec{E}$ with normal = 90°.



So $\quad \phi = ES \cos 90° = 0$

Physical Meaning:

The electric flux through a surface inside an electric field represents the total number of electric lines of force crossing the surface. It is a property of electric field.

Unit:

(i) The SI unit of electric flux is $Nm^2\, C^{-1}$ (Gauss) or $J\, m\, C^{-1}$.

(ii) Electric flux is a scalar quantity. It can be positive, negative or zero.

Illustration 38:

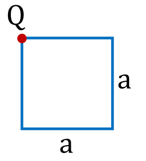
The electric field in a region is given by $\vec{E} = \frac{3}{5}E_0\hat{i} + \frac{4}{5}E_0\hat{j}$ with $E_0 = 2.0 \times 10^3\, N/C$. Find the flux of this field through a rectangular surface of area $0.2m^2$ parallel to the Y–Z plane.

Solution:

$$\phi_E = \vec{E}\cdot\vec{S} = \left(\frac{3}{5}E_0\hat{i} + \frac{4}{5}E_0\hat{j}\right).\left(0.2\hat{i}\right) = 240\frac{N-m^2}{C}$$

Illustration 39:

A point charge Q is placed at the corner of a square of side a, then find the flux through the square.

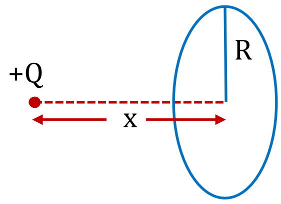


Solution:

The electric field due to Q at any point of the square will be along the plane of square and the electric field lines are perpendicular to square ; so $\phi = 0$. In other words we can say that no line is crossing the square so, flux = 0.

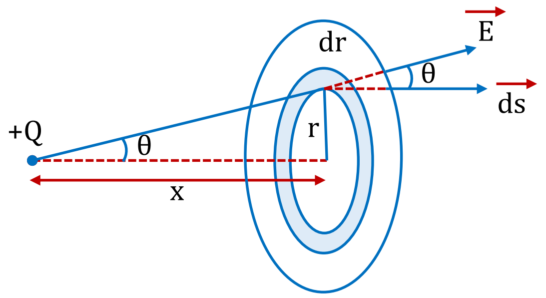
Illustration 40:

Find the electric flux due to a point charge 'Q' through the circular region of radius R if the charge is placed on the axis of ring at a distance x.



Solution:

We can divide the circular region into small rings.

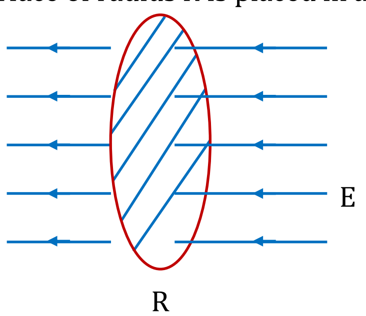


Let's take a ring of radius r and width dr. Flux through this small element $d\phi = E\, ds \cos\theta$

$$\therefore \qquad \phi_{net} = \int E\, ds \cos\theta = \int_{r=o}^{r=R} \frac{kQ}{(x^2+r^2)}(2\pi r\, dr)\left(\frac{x}{\sqrt{x^2+r^2}}\right) = \frac{Q}{2\varepsilon_0}\left[1-\frac{x}{\sqrt{x^2+R^2}}\right]$$

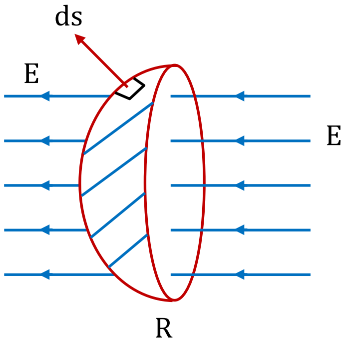
Case III: Curved surface in uniform electric field.

(i) Suppose a circular surface of radius R is placed in a uniform electric field as shown.



Flux passing through the surface $\phi = E\,(\pi R^2)$

(ii) Now suppose, a hemispherical surface, is placed in the electric field. Flux through hemispherical surface:



$\phi = \int Eds \cos\theta$

$\phi = E\int ds \cos\theta$

where, $\int ds \cos\theta$ is projection of the spherical surface Area on base.

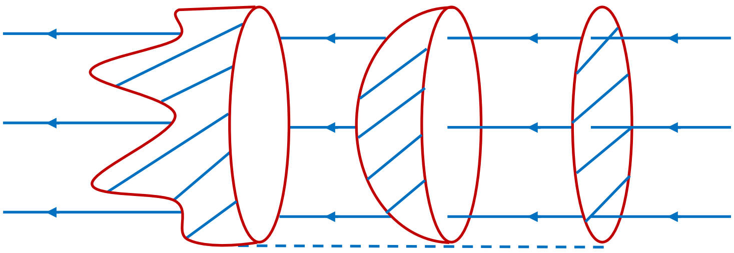
and $\int ds\ cos\theta = \pi R^2$

So, $\phi = E(\pi R^2)$ = same answer as in previous case

So, we can conclude that.

Note:

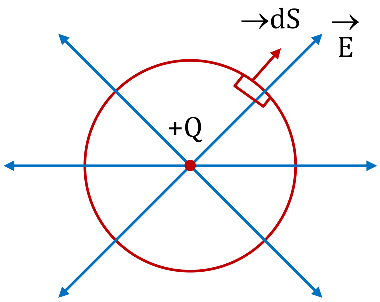
If the number of electric field lines passing through two surfaces are same, then flux passing through these surfaces will also be same, irrespective of the shape of surface.



$\phi_1 = \phi_2 = \phi_3 = E(\pi R^2)$

Case IV: Flux through a closed surface :

Suppose there is a spherical surface and a charge 'q' is placed at centre.



∴ Flux through the spherical surface

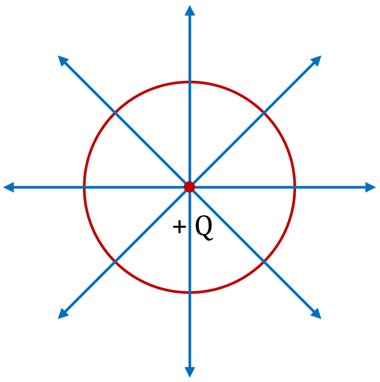
$$\phi = \int \vec{E}.\vec{ds} = \int E\,ds \quad (\text{as } \vec{E} \text{ is along } \vec{ds} \text{ (normal)})$$

$$\therefore \quad \phi = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R^2}\int ds \qquad \text{where, } \int ds = 4\pi R^2$$

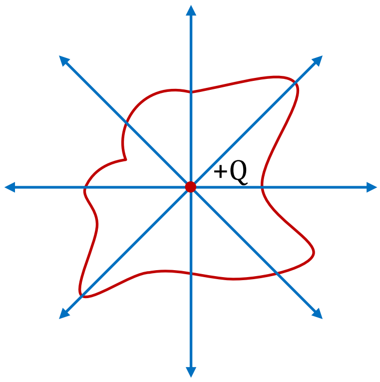
$$\phi = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\right)\left(4\pi R^2\right) \Rightarrow \phi = \frac{Q}{\varepsilon_o}$$

Now if the charge Q is enclosed by any other closed surface, still same no of lines of force will pass through the surface.

So, here also flux will be $\phi = \frac{Q}{\varepsilon_o}$. That's what Gauss Theorem is.



$\phi = \frac{Q}{\varepsilon_o}$



$\phi = \frac{Q}{\varepsilon_o}$

Gauss's Law in Electrostatics or Gauss's theorem

This law was stated by a mathematician Karl F Gauss. This law gives the relation between the electric field at a point on a closed surface and the net charge enclosed by that surface. This surface is called Gaussian surface. It is a closed hypothetical surface. Its validity is shown by experiments. It is used to determine the electric field due to some symmetric charge distributions.

Statement and Details:

Gauss's law is stated as given below:

The surface integral of the electric field intensity over any closed hypothetical surface (called Gaussian surface) in free space is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed within the surface. Here, ε_0 is the permittivity of free space.

If S is the Gaussian surface and $\sum_{i=1}^{n} q_i$ is the total charge enclosed by the Gaussian surface, then according to Gauss's law,

$$\phi = \oint \vec{E} \cdot \overrightarrow{dS} = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i$$

The circle on the sign of integration indicates that the integration is to be carried out over the closed surface.

Note:

(i) Flux through Gaussian surface is independent of its shape.

(ii) Flux through Gaussian surface depends only on total charge present inside Gaussian surface.

(iii) Flux through Gaussian surface is independent of position of charges inside Gaussian surface.

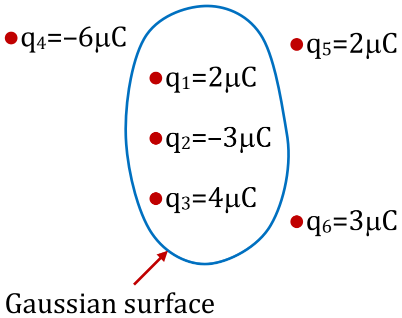
(iv) Electric field intensity at the Gaussian surface is due to all the charges present inside as well as outside the Gaussian surface.

(v) In a closed surface incoming flux is taken negative, while outgoing flux is taken positive, because $\hat{n}$ is taken positive in outward direction.

(vi) In a Gaussian surface, $\phi = 0$ does not imply $E = 0$ at every point of the surface but $E = 0$ at every point implies $\phi = 0$.

Illustration 41:

Find out flux through the given Gaussian surface.



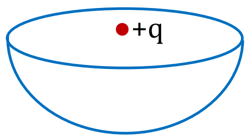
Solution:

$$\phi = \frac{Q_{in}}{\varepsilon_0} = \frac{2\mu C - 3\mu C + 4\mu C}{\varepsilon_0} = \frac{3\times 10^{-6}}{\varepsilon_0}\ Nm^2/C$$

Flux through open surfaces using Gauss's Theorem:

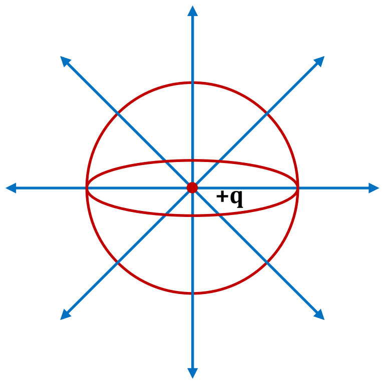
Illustration 42:

A point charge $+q$ is placed at the centre of curvature of a hemisphere. Find flux through the hemispherical surface.



Solution:

Lets put an upper half hemisphere. Now, flux passing through the entire sphere $= \frac{q}{\varepsilon_0}$.

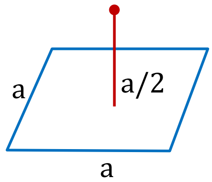


As the charge q is symmetrical to the upper half and lower half hemispheres, so half-half flux will emit from both the surfaces.

Flux emitting from lower half surface $= \frac{q}{2\varepsilon_0}$

Flux emitting from upper half surface $= \frac{q}{2\varepsilon_0}$

Illustration 43:
A charge Q is placed at a distance $a/2$ above the centre of a horizontal, square surface of edge a as shown in figure. Find the flux of the electric field through the square surface.



Solution:
We can consider imaginary faces of cube such that the charge lies at the centre of the cube. Due to symmetry, we can say that flux through the given area (which is one face of cube), $\phi = \dfrac{Q}{6\varepsilon_0}$

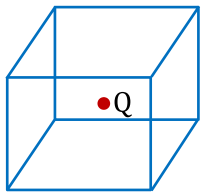
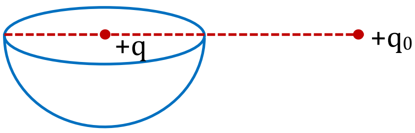


Illustration 44:
Find flux through the hemispherical surface.



Solution:
(i) Flux through the hemispherical surface due to $+q = \dfrac{q}{2\varepsilon_0}$ (we have seen in previous examples)

(ii) Flux through the hemispherical surface due to $+q_0$ is 0, because due to $+q_0$, field lines entering the surface = field lines coming out of the surface.

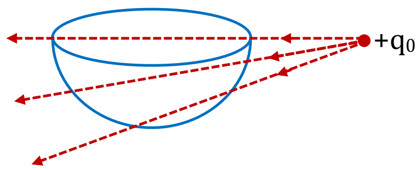
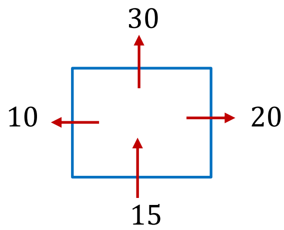


Illustration 45:
Flux (in S.I. units) coming out and entering a closed surface is shown in the figure. Find charge enclosed by the closed surface.



Solution:
Net flux through the closed surface

$= +20 + 30 + 10 - 15 = 45\ N.m^2/c$

From Gauss`s theorem : $\phi_{net} = \dfrac{q_{in}}{\varepsilon_0}$ or $45 = \dfrac{q_{in}}{\varepsilon_0}$ $\therefore\ q_{in} = (45)\varepsilon_0$

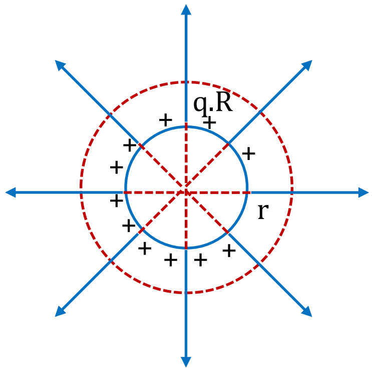
Finding electric field from Gauss's Theorem:

From Gauss`s theorem, we can say $\oint \vec{E}.\vec{ds} = \phi_{net} = \dfrac{q_{in}}{\varepsilon_0}$

Finding E due to a spherical shell:

Electric field outside the Sphere:

Since, electric field due to a shell will be radially outwards. So, lets choose a spherical Gaussian surface Applying Gauss's theorem for this spherical Gaussian surface,



$$\oint \vec{E}.\vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_o} = \frac{q}{\varepsilon_o}$$

As $\vec{E}$ is normal to the surface

$$\oint \vec{E}.\vec{ds} = \oint E ds \cos 0° = E \oint ds$$

(because value of E is constant at the surface)

$= E\ (4\pi r^2)$

$(\therefore \oint ds$ total area of the spherical surface $= 4\ \pi r^2)$

$\Rightarrow \qquad E\ (4\pi r^2) = \dfrac{q_{in}}{\varepsilon_0}$

$\therefore \qquad E_{out} = \dfrac{q}{4\pi\varepsilon_o r^2}$

Electric field inside the sphere:

$$\oint \vec{E}\,\vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_0} = 0$$

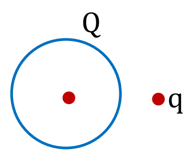
$$\oint \vec{E}.\vec{ds} = \oint E ds \cos 0° = E \oint ds = E\ (4\pi r^2)$$

$\Rightarrow \qquad E\ (4\pi r^2) = 0$

$\therefore \qquad E_{in} = 0$

Illustration 46:

Figure shows a uniformly charged sphere of radius R and total charge Q. A point charge q is situated outside the sphere at a distance r from centre of sphere. Find out the following :



(i) Force acting on the point charge q due to the sphere.

(ii) Force acting on the sphere due to the point charge.

Solution:

(i) Electric field at the position of point charge $\vec{E} = \dfrac{kQ}{r^2}\hat{r}$

So, $\vec{F} = \dfrac{kqQ}{r^2}\hat{r} \qquad \Rightarrow \ |\vec{F}| = \dfrac{kqQ}{r^2}$

(ii) Since we know that every action has equal and opposite reaction so

$$\vec{F}_{sphere} = \frac{kqQ}{r^2}$$

Illustration 47:

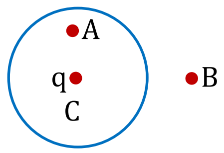


Figure shows a uniformly charged thin sphere of total charge Q and radius R. A point charge q is also situated at the centre of the sphere. Find out the following:

(i) Force on charge q.

(ii) Electric field intensity at A.

(iii) Electric field intensity at B.

Solution:

(i) Electric field at the centre of the uniformly charged hollow sphere = 0

So force on charge q = 0

(ii) Electric field at A

$$\vec{E}_A = \vec{E}_{Sphere} + \vec{E}_q = 0 + \frac{kq}{r^2} \; ; r = CA$$

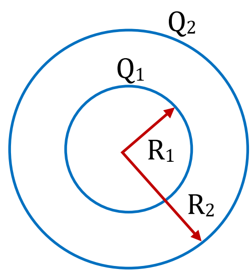
E due to sphere = 0, because point lies inside the charged hollow sphere.

(iii) Electric field $\vec{E}_B$ at point $B = \vec{E}_{Sphere} + \vec{E}_q = \frac{kQ}{r^2}.\hat{r} + \frac{kq}{r^2}.\hat{r} = \frac{k(Q+q)}{r^2}.\hat{r}$; $r = CB$

Note:

Here, we can also assume that the total charge of sphere is concentrated at the centre, for calculation of electric field at B.

Illustration 48:



Two concentric uniformly charged spherical shells of radius R_1 and R_2 ($R_2 > R_1$) have total charges Q_1 and Q_2 respectively. Derive an expression of electric field as a function of r for following positions.

(i) $r < R_1$

(ii) $R_1 \le r < R_2$

(iii) $r \ge R_2$

Solution:

(i) For $r < R_1$, therefore, point lies inside both the spheres

$$E_{net} = E_{Inner} + E_{outer} = 0 + 0$$

(ii) For $R_1 \le r < R_2$, point lies outside inner sphere but inside outer sphere:

$\therefore$ $E_{net} = E_{inner} + E_{outer}$

$$\frac{kQ_1}{r^2}\hat{r} + 0 = \frac{kQ_1}{r^2}\hat{r}$$

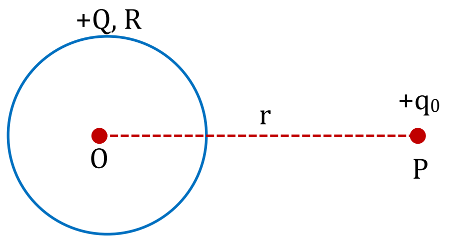
(iii) For $r \ge R_2$

Point lies outside inner as well as outer sphere.

Therefore, $E_{Net} = E_{inner} + E_{outer} = \frac{kQ_1}{r^2}\hat{r} + \frac{kQ_2}{r^2}\hat{r} = \frac{k(Q_1+Q_2)}{r^2}\hat{r}$

Illustration 49:

A spherical shell having charge $+Q$ (uniformly distributed) and a point charge $+q_0$ are placed as shown. Find the force between shell and the point charge ($r >> R$).



Solution:

Force on the point charge $+q_0$ due to the shell

$$= q_0 \vec{E} = (q_0)\left(\frac{kQ}{r^2}\right)\hat{r} = \frac{kQq_0}{r^2}\hat{r}$$

where $\hat{r}$, is unit vector along OP.

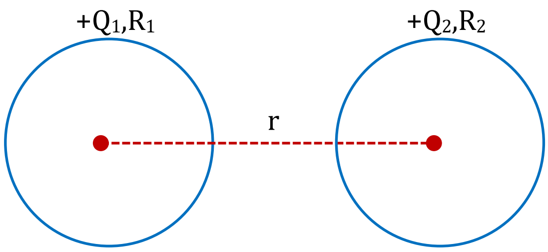
From action - reaction principle, force on the shell due to the point charge will also be :

$$F_{\text{shell}} = \frac{kQq_0}{r^2}(-\hat{r})$$

Conclusion : To find the force on a hollow sphere due to outside charges, we can replace the sphere by a point charge kept at centre.

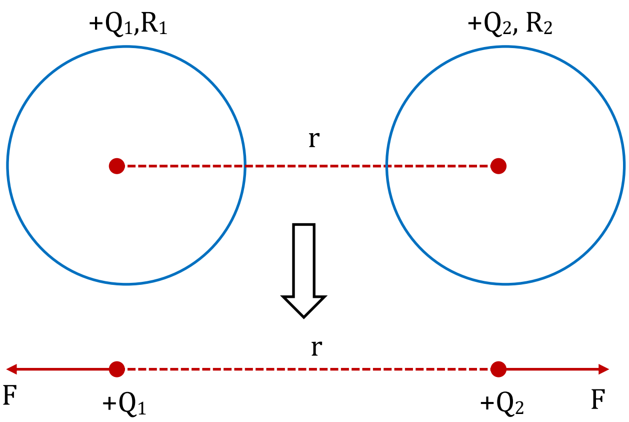
Illustration 50:

Find force acting between two shells of radius R_1 and R_2 which have uniformly distributed charges Q_1 and Q_2 respectively and distance between their centers is r.



Solution:

The shells can be replaced by point charges kept at centre, so force between them



$$F = \frac{kQ_1Q_2}{r^2}$$

Electric field due to uniformly charged solid sphere:

For a solid sphere having uniformly distributed charge Q and radius R.

Electric field outside the sphere:

Direction of electric field is radially outwards, so we will choose a spherical Gaussian surface Applying Gauss's theorem

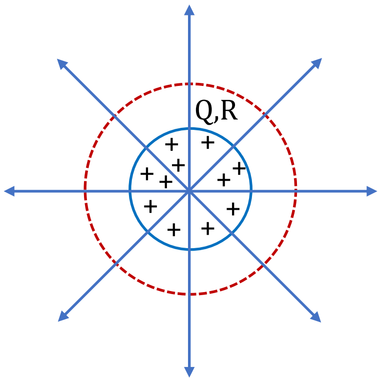
$$\int \vec{E}\,\vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_o} = \frac{Q}{\varepsilon_o}$$

$$\int \vec{E}.\vec{ds} = \int E ds\cos 0^\circ = E\int ds$$

$$= E\,(4\pi r^2)$$

$$E\,(4\pi r^2) = \frac{Q}{\varepsilon_o}$$

or $$E_{out} = \frac{Q}{4\pi\varepsilon_o r^2}$$



Electric field inside a solid sphere :

For this choose a spherical Gaussian surface inside the solid sphere Applying Gauss's theorem for this surface

$$\int \vec{E}\,\vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_o} = \frac{\dfrac{Q}{\frac{4}{3}\pi R^3}\times\frac{4}{3}\pi r^3}{\varepsilon_o} = \frac{Qr^3}{\varepsilon_o R^3}$$

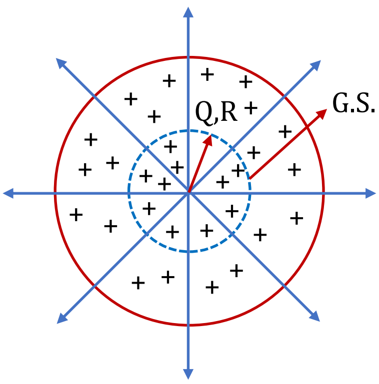
As $$\int \vec{E}.\vec{ds} = \int E ds\cos 0^\circ = E\int ds = E\,(4\pi r^2)$$

$\Rightarrow$ $$E(4\pi r^2) = \frac{Qr^3}{\varepsilon_o R^3}$$

$$E = \frac{Qr}{4\pi\varepsilon_o R^3}$$

$\therefore$ $$E_{in} = \frac{kQ}{R^3}\,r$$

and $$\vec{E} = \frac{\rho\vec{r}}{3\varepsilon_0}$$ (ρ = Volume charge density)



Q = Total charges contained by solid sphere
R = Radius of sphere

Illustration 51:

A solid non conducting sphere of radius R and uniform volume charge density ρ has its centre at origin. Find out electric field intensity in vector form at following positions :

(i) $(R/2, 0, 0)$ (ii) $\left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, 0\right)$ (iii) $(R, R, 0)$

Solution:

(i) At $(R/2, 0, 0)$: Distance of point from centre $= \sqrt{(R/2)^2 + 0^2 + 0^2} = R/2 < R$, so point lies inside the sphere, so $\vec{E} = \frac{\rho\vec{r}}{3\varepsilon_0} = \frac{\rho}{3\varepsilon_0}\left[\frac{R}{2}\hat{i}\right]$

(ii) At $\left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, 0\right)$ distance of point from centre $= \sqrt{(R/\sqrt{2})^2 + (R/\sqrt{2})^2 + 0^2} = R = R$, so point lies at the surface of sphere, therefore

$$\vec{E} = \frac{kQ}{R^3}\vec{r}$$

$$= \frac{k\frac{4}{3}\pi R^3\rho}{R^3}\left[\frac{R}{\sqrt{2}}\hat{i} + \frac{R}{\sqrt{2}}\hat{j}\right]$$

$$= \frac{\rho}{3\varepsilon_0}\left[\frac{R}{\sqrt{2}}\hat{i} + \frac{R}{\sqrt{2}}\hat{j}\right]$$

(iii) The point is outside the sphere

So, $\vec{E} = \frac{kQ}{r^3}\vec{r}$

$$= \frac{k\frac{4}{3}\pi R^3\rho}{(\sqrt{2}R)^3}\left[R\hat{i} + R\hat{j}\right]$$

$$= \frac{\rho}{6\sqrt{2}\varepsilon_0}\left[R\hat{i} + R\hat{j}\right]$$

Illustration 52:

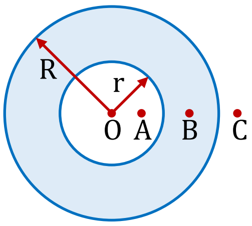
A Uniformly charged solid non-conducting sphere of uniform volume charge density ρ and radius R is having a concentric spherical cavity of radius r. Find out electric field intensity at following points, as shown in the figure :

(i) Point A

(ii) Point B

(iii) Point C

(iv) Centre of the sphere



Solution:

Method-I :

(i) For point A : We can consider the solid part of sphere to be made of large number of spherical shells which have uniformly distributed charge on its surface. Now, since point A lies inside all spherical shells so electric field intensity due to all shells will be zero.

$$\vec{E_A} = 0$$

(ii) For point B : All the spherical shells for which point B lies inside will make electric field zero at point B. So electric field will be due to charge present from radius r to OB.

So, $$\overrightarrow{E_B} = \frac{k\frac{4}{3}\pi(OB^3 - r^3)\rho}{OB^3}\overrightarrow{OB} = \frac{\rho}{3\varepsilon_0}\frac{[OB^3 - r^3]}{OB^3}\overrightarrow{OB}$$

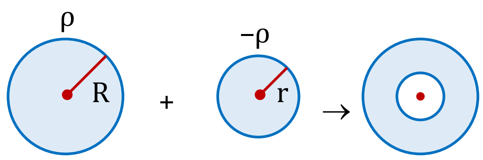
(iii) For point C, similarly we can say that for all the shell points C lies outside the shell

So, $$\overrightarrow{E_C} = \frac{k[\frac{4}{3}\pi(R^3 - r^3)]}{[OC]^3}\overrightarrow{OC} = \frac{\rho}{3\varepsilon_0}\frac{R^3 - r^3}{[OC]^3}\overrightarrow{OC}$$

Method-II :

We can consider that the spherical cavity is filled with charge density ρ and also –ρ, thereby making net charge density zero after combining. We can consider two concentric solid spheres: One of radius R and charge density ρ and other of radius r and charge density –ρ. Applying superposition principle :

(i) $$\overrightarrow{E_A} = \overrightarrow{E_\rho} + \overrightarrow{E_{-\rho}} = \frac{\rho(\overrightarrow{OA})}{3\varepsilon_0} + \frac{[-\rho(\overrightarrow{OA})]}{3\varepsilon_0} = 0$$



(ii) $$\overrightarrow{E_B} = E_\rho + E_{-\rho} = \frac{\rho(\overrightarrow{OB})}{3\varepsilon_0} + \frac{k\left[\frac{4}{3}\pi r^3(-\rho)\right]}{(OB)^3}\overrightarrow{OB}$$

$$= \left[\frac{\rho}{3\varepsilon_0} - \frac{r^3\rho}{3\varepsilon_0(OB)^3}\right]\overrightarrow{OB} = \frac{\rho}{3\varepsilon_0}\left[1 - \frac{r^3}{OB^3}\right]\overrightarrow{OB}$$

(iii) $$E_C = E_\rho + E_{-\rho} = \frac{k\left(\frac{4}{3}\pi R^3\rho\right)}{OC^3}\overrightarrow{OC} + \frac{k\left(\frac{4}{3}\pi r^3(-\rho)\right)}{OC^3}\overrightarrow{OC} = \frac{\rho}{3\varepsilon_0(OC)^3}[R^3 - r^3]\overrightarrow{OC}$$

(iv) $$\overrightarrow{E_O} = \overrightarrow{E_\rho} + \overrightarrow{E_{-\rho}} = 0 + 0 = 0$$

Illustration 53:

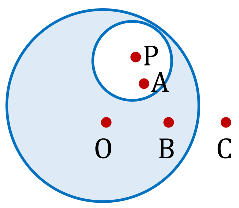
In above question, if cavity is not concentric and centered at point P then repeat all the steps.

Solution:

Again assume ρ and –ρ in the cavity, (similar to the previous example) :

(i) $$\overrightarrow{E_A} = \overrightarrow{E_\rho} + \overrightarrow{E_{-\rho}} = \frac{\rho[\overrightarrow{OA}]}{3\varepsilon_0} + \frac{(-\rho)\overrightarrow{PA}}{3\varepsilon_0}$$

$$\frac{\rho}{3\varepsilon_0}[\overrightarrow{OA} - \overrightarrow{PA}] = \frac{\rho}{3\varepsilon_0}[\overrightarrow{OP}]$$



Note:

Here, we can see that the electric field intensity at point A is independent of position of point A inside the cavity. Also, the electric field is along the line joining the centres of the sphere and the spherical cavity.

(ii) $\vec{E_B} = \vec{E_\rho} + \vec{E_{-\rho}} = \dfrac{\rho(\overrightarrow{OB})}{3\varepsilon_0} + \dfrac{k[\frac{4}{3}\pi r^3(-\rho)]}{[PB]^3}\overrightarrow{PB}$

(iii) $\vec{E_C} = \vec{E_\rho} + \vec{E_{-\rho}} = \dfrac{k[\frac{4}{3}\pi R^3\rho]}{[OC]^3}\overrightarrow{OC} + \dfrac{k[\frac{4}{3}\pi r^3(-\rho)]}{[PC]^3}\overrightarrow{PC}$

(iv) $\vec{E_O} = \vec{E_\rho} + \vec{E_{-\rho}} = 0 + \dfrac{k[\frac{4}{3}\pi r^3(-\rho)]}{[PO]^3}\overrightarrow{PO}$

Illustration 54:

A non-conducting solid sphere has volume charge density that varies as $\rho = \rho_0 r$, where ρ_0 is a constant and r is distance from centre. Find out electric field intensities at following positions.

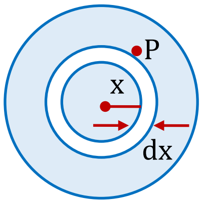
(i) $r < R$ (ii) $r \geq R$

Solution:

Method I :

(i) For $r < R$:

The sphere can be considered to be made of large number of spherical shells. Each shell has uniform charge density on its surface. So the previous results of the spherical shell can be used. Consider a shell of radius x and thickness dx as an element. Charge on shell $dq = (4\pi x^2 dx)\rho_0 x$.



$\therefore$ Electric field intensity at point P due to shell, $dE = \dfrac{kdq}{x^2}$

Since all the shell will have electric field in same direction

$$\therefore \quad E = \int_0^R dE = \int_0^r dE + \int_r^R dE$$

Due to shells which lie between region $r < x \leq R$, electric field at point P will be zero.

$$\therefore \quad |\vec{E}| = \int_0^r \frac{kdq}{r^2} + 0 = \int_0^r \frac{k.4\pi x^2 dx \rho_0 x}{r^2}$$

$$= \frac{4\pi k\rho_0}{r^2}\left[\frac{x^4}{4}\right]_0^r = \frac{\rho_0 r^2}{4\varepsilon_0}$$

(ii) For $r \geq R$, $\vec{E} = \displaystyle\int_0^R dE = \int_0^R \frac{k.4\pi x^2 dx \rho_0 x}{r^2} = \frac{\rho_0 R^4}{4\varepsilon_0 r^2}\hat{r}$

Method II:

(i) The sphere can be considered to be made of large number of spherical shells. Each shell has uniform charge density on its surface. So, the previous results of the spherical shell can be used. we can say that all the shells for which point lies inside will make electric field zero at that point,

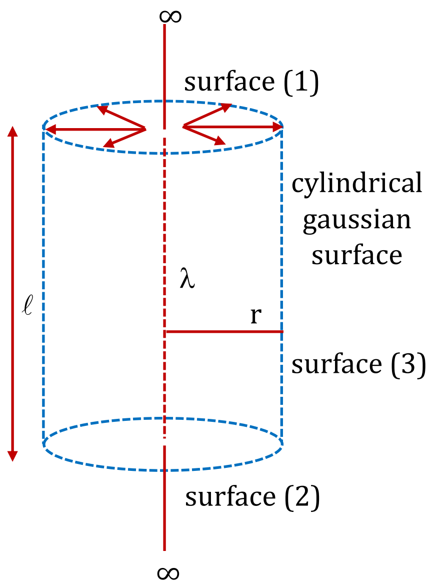
$$\text{So} \quad \vec{E}_{(r<R)} = \frac{k\int_0^r (4\pi x^2 dx)\rho_0 x}{r^2} = \frac{\rho_0 r^2}{4\varepsilon_0}\hat{r}$$

(ii) Similarly, for $r \geq R$, all the shells will contribute in electric field. Therefore :

$$\vec{E}_{(r<R)} = \frac{k\int_0^R (4\pi x^2 dx)\rho_0 x}{r^2} = \frac{\rho_0 R^4}{4\varepsilon_0 r^2}\hat{r}$$

Electric field due to infinite line charge (having uniformly distributed charge of charge density λ) :

Electric field due to infinitely wire is radial so we will choose cylindrical Gaussian surface as shown is figure:



ϕ_{net} → $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 \neq 0$

$$\phi_3 = \int \vec{E}.\vec{ds} = \frac{q_{in}}{\varepsilon_o}$$

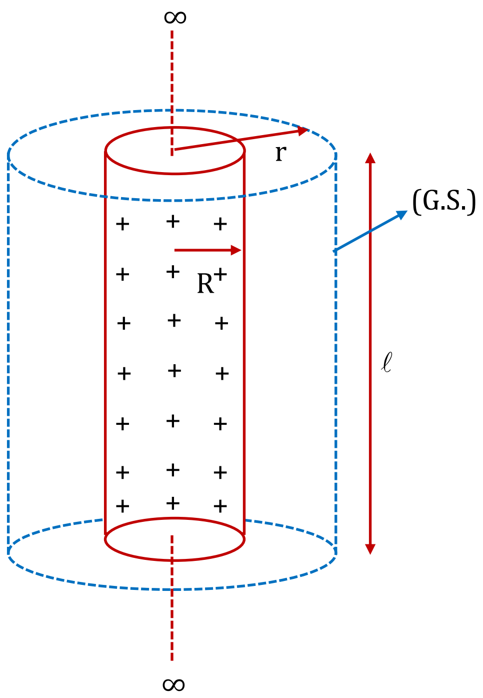
$$\int E\, ds = \frac{\lambda \ell}{\varepsilon_o}$$

$$E\int ds = \frac{\lambda \ell}{\varepsilon_o}$$

$$E\,(2\pi r\ell) = \frac{\lambda \ell}{\varepsilon_o}$$

$$\therefore \quad E = \frac{\lambda}{2\pi\varepsilon_o r} = \frac{2k\lambda}{r}$$

Electric field due to infinitely long charged tube (having uniform surface charge density σ and radius R):



(i) ***E* outside the tube :** Lets choose a cylindrical Gaussian surface of length ℓ :

$$\therefore \quad \phi_{net} = \frac{q_{in}}{\varepsilon_o} = \frac{\sigma 2\pi R\ell}{\varepsilon_o}$$

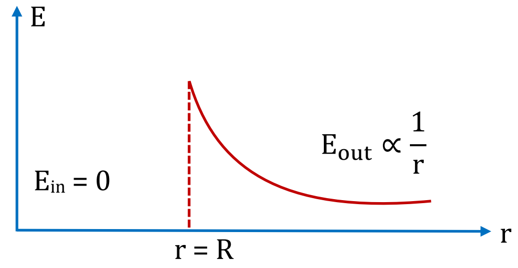
$$\Rightarrow \quad E_{out} \times 2\pi r\ell = \frac{\sigma 2\pi R\ell}{\varepsilon_o}$$

$$\therefore \quad E = \frac{\sigma R}{r\,\varepsilon_0}$$

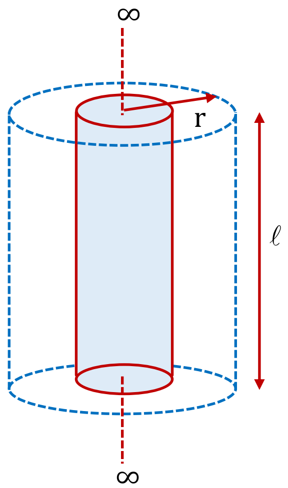
(ii) E inside the tube :

Lets choose a cylindrical Gaussian surface inside the tube.

$$\phi_{net} = \frac{q_{in}}{\varepsilon_o} = 0, \qquad \text{So} \qquad E_{in} = 0$$



***E* due to infinitely long solid cylinder of radius *R* (having uniformly distributed charge in volume (volume charge density ρ)):**



(i) *E* at outside point:

Lets choose a cylindrical Gaussian surface.

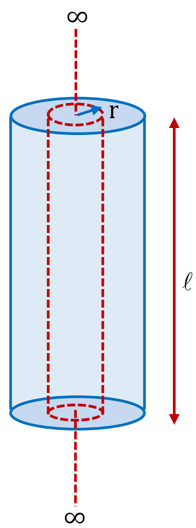
Applying Gauss's theorem :

$$E \times 2\pi r\ell = \frac{q_{in}}{\varepsilon_o} = \frac{\rho \times \pi R^2 \ell}{\varepsilon_o}$$

$$E_{out} = \frac{\rho R^2}{2r\,\varepsilon_0}$$

(ii) *E* at inside point:

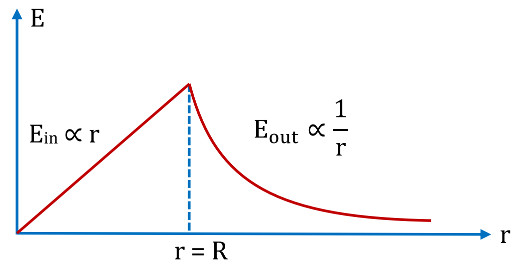
Lets choose a cylindrical Gaussian surface inside the solid cylinder. Applying Gauss's theorem



$$E \times 2\pi r\ell = \frac{q_{in}}{\varepsilon_o}$$

$$= \frac{\rho \times \pi r^2 \ell}{\varepsilon_o}$$

$$\Rightarrow \qquad E_{in} = \frac{\rho r}{2\varepsilon_0}$$



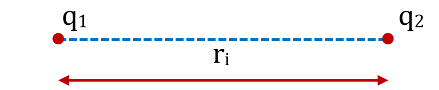
Electrostatic/Electric potential energy:

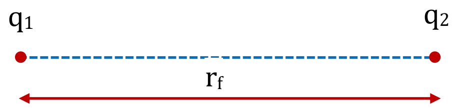
Since Electrostatic force is a conservative force therefore whenever system will be choose such that electrostatic force behave as internal force. We can find an expression of change in potential energy associated with electrostatic forces by following the definition of change in potential energy, which is "Change in potential energy is equal to the negative of work done by internal conservative forces i.e.,

$$\Delta U = -W_{\text{int, cons.}}$$

Electrostatic potential energy of system of two point charges:

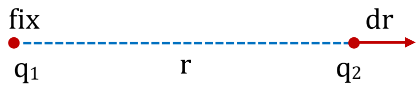
Consider two point charges as shown in the figure. Initial distance between them is r_i and final distance is r_f.





To calculate change in potential energy between these two situation, we have to calculate work by electrostatic force in doing so.

To calculate work done by electrostatic force, consider that q_1 is fixed and q_2 is at r distance from q_1 during the process & further moves distance dr as shown in figure.



Then, work done by electrostatic force in this process is

$$dW = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}dr$$

$$\therefore \qquad \Delta U = U_f - U_i = -\int dW = -\frac{1}{4\pi\varepsilon_0}\int_{r_i}^{r_f}\frac{q_1 q_2}{r^2}dr$$

or $$U_f - U_i = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_f} - \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_i}$$

Therefore, electric potential energy of two point charges can be defined as

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} + U_0$$

where r is the distance between the charges & U_0 is a constant. It is customary to choose $U_0 = 0$

$$\therefore \qquad U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

Few important points regarding above expression of potential energy:

(i) The choice $U_0 = 0$ is equivalent to saying that the potential energy of two charged particle is zero only when they are infinitely far apart. This makes sense because two charged particles cease interacting only when they are infinitely far apart.

(ii) we derived the equation for two like charges, but it is equally valid for two opposite charges. The potential energy of two like charges is positive and of two opposite charges is negative.

(iii) Because the electric field outside a sphere of charge is the same as that of a point charge at the centre, therefore the above equation is also the electric potential energy of two charged spheres. Distance r is the distance between their centres.

Electric potential energy of system having more than two charges

For a system having more then two charged particle, potential energy (electric) is equal to then algebraic summation of potential energy of each pair of charge present in the system.

Therefore, $U = \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r}, \quad i \neq j$

Few points regarding above expression:

(i) In this expression, electrostatic potential energy of system is taken as zero, when distance between each two pair of charge of system is zero.

(ii) For the proof of above expression, superposition of electric force is used (try to prove above expression by yourself).

(iii) Value of charges (q_i and q_j) should be kept with sign.

Illustration 55:

A particle of mass 40 mg and carrying a charge 5×10^{-9} C is moving directly towards a fixed positive point charge of magnitude 10^{-8} C. When it is at a distance of 10 cm from the fixed point charge it has speed of 50 cm/s. At what distance from the fixed point charge will the particle come momentarily to rest? Is the acceleration constant during the motion?

Solution:

If the particle comes to rest momentarily at a distance r from the fixed charge, then from conservation of energy, we have?

$$\frac{1}{2}mu^2 + \frac{1}{4\pi\varepsilon_0}\frac{Qq}{a} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r}$$

Substituting the given data, we get: $\frac{1}{2} \times 40 \times 10^{-6} \times \frac{1}{2} \times \frac{1}{2} = 9 \times 10^{9} \times 5 \times 10^{-9} \times 10^{-8}\left[\frac{1}{r} - 10\right]$

or, $\quad \frac{1}{r} - 10 = \frac{5 \times 10^{-6}}{9 \times 5 \times 10^{-8}} = \frac{100}{9}$

$\Rightarrow \quad \frac{1}{r} = \frac{190}{9} \qquad \Rightarrow \qquad r = \frac{9}{190}\,m$

or, $\quad$ i.e., $\quad r = 4.7 \times 10^{-2}\,m$.

As here, $F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$

So $\quad acc. = \frac{F}{m} \propto \frac{1}{r^2}$

i.e., Acceleration is not constant during the motion.

Illustration 56:

A proton moves from a large distance with a speed u m/s directly towards a free proton originally at rest. Find the distance of closest approach for the two protons in terms of mass of proton m and its charge e.

Solution:

As here the particle at rest is free to move, when one particle approaches the other, due to electrostatic repulsion other will also start moving and so the velocity of first particle will decrease while of other will increase and at closest approach, both will move with same velocity. So, if v is the common velocity of each particle at closest approach, then by 'conservation of momentum' of the two proton system.

$$mu = mv + mv$$

i.e., $\quad v = \frac{1}{2}u$

And by conservation of energy,

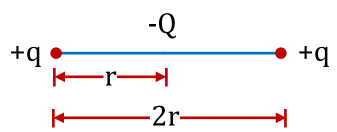
$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

$$\Rightarrow \quad \frac{1}{2}mu^2 - m\left(\frac{u}{2}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \qquad [as\ v = \frac{u}{2}]$$

$$\Rightarrow \quad \frac{1}{4}mu^2 = \frac{e^2}{4\pi\varepsilon_0 r} \qquad \Rightarrow \qquad r = \frac{e^2}{\pi m\varepsilon_0 u^2}$$

Illustration 57:

Figure shows an arrangement of three point charges. The total potential energy of this arrangement is zero. Calculate the ratio $\frac{q}{Q}$.



Solution:

$$U_{sys} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-qQ}{r} + \frac{(+q)(+q)}{2r} + \frac{Q(-q)}{r}\right] = 0$$

$$-Q + \frac{q}{2} - Q = 0 \quad \text{or} \quad 2Q = \frac{q}{2} \quad \text{or} \quad \frac{q}{Q} = \frac{4}{1}$$

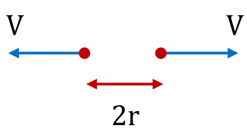
Illustration 58:

Two point charges, each of mass m and charge q are released when they are at a distance r from each other. What is the speed of each charged particle when they are at a distance $2r$?

Solution:

According to momentum conservation, both the charge particles will move with same speed. Now applying energy conservation:

$$0 + 0 + \frac{kq^2}{r} = 2\frac{1}{2}mv^2 + \frac{kq^2}{2r}$$



$$\Rightarrow \quad v = \sqrt{\frac{kq^2}{2rm}}$$

Illustration 59:

Two charged particles each having equal charges 2×10^{-5} *C* are brought from infinity to within a separation of 10 *cm*. Calculate the increase in potential energy during the process and the work required for this purpose.

Solution:

$\Delta U = U_f - U_i = U_f - 0 = U_f$

We have to simply calculate the electrostatic potential energy of the given system of charges

$\therefore \quad \Delta U = U_f = \dfrac{kq_1q_2}{r} = \dfrac{9\times10^{9}\times2\times10^{-5}\times2\times10^{-5}\times100}{10} J = 36\, J$

$\therefore$ Work required = 36 *J* = equal to change in potential energy of system

Illustration 60:

Three equal charges *q* each are placed at the corners of an equilateral triangle of side a.

(i) Find out potential energy of charge system.

(ii) Calculate work required to decrease the side of triangle to $a/2$.

(iii) If the charges are released from the shown position and each of them has same mass *m* then find the speed of each particle when they lie on triangle of side $2a$.

Solution:

(i) $U = U_{12} + U_{13} + U_{23} = \dfrac{kq^2}{a} + \dfrac{kq^2}{a} + \dfrac{kq^2}{a} = \dfrac{3kq^2}{a}$

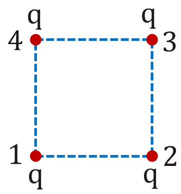
(ii) Work required to decrease the sides, $W = U_f - U_i = \dfrac{3kq^2}{a/2} - \dfrac{3kq^2}{a} = \dfrac{3kq^2}{a}$ *Joules*

(iii) Work done by electrostatic forces = Change is kinetic energy of particles.

$U_i - U_f = K_f - K_i \Rightarrow \dfrac{3kq^2}{a} - \dfrac{3kq^2}{2a} = 3(\dfrac{1}{2}mv^2) - 0 \Rightarrow \quad v = \sqrt{\dfrac{kq^2}{am}}$

Illustration 61:

Four identical point charges *q* each are placed at four corners of a square of side *a*. Find out potential energy of the charge system.



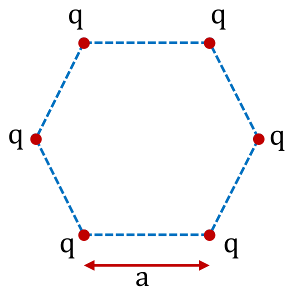
Solution:

$U = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}$

$= \dfrac{kq^2}{a} + \dfrac{kq^2}{a\sqrt{2}} + \dfrac{kq^2}{a} + \dfrac{kq^2}{a} + \dfrac{kq^2}{a\sqrt{2}} + \dfrac{kq^2}{a} = \left[\dfrac{4kq^2}{a} + \dfrac{2kq^2}{a\sqrt{2}}\right] = \dfrac{2kq^2}{a}\left[2 + \dfrac{1}{\sqrt{2}}\right]$

Illustration 62:

Six equal point charges q each are placed at six corners of a hexagon of side a. Find out potential energy of charge system.



Solution:

$$U_{Net} = \frac{U_1+U_2+U_3+U_4+U_5+U_6}{2}$$

Due to symmetry

$$U_1 = U_2 = U_3 = U_4 = U_5 = U_6$$

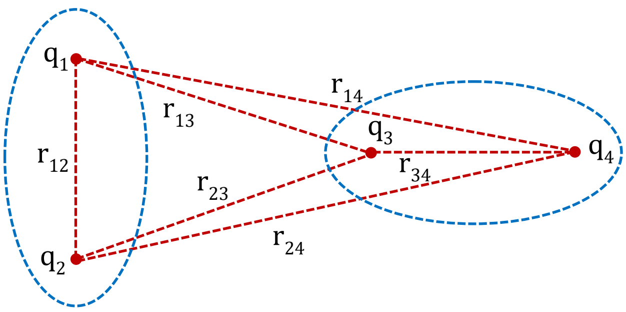
So $\quad U_{net} = 3U_1 = \dfrac{3kq^2}{a}\left[2+\dfrac{2}{\sqrt{3}}+\dfrac{1}{2}\right]$

Self and interaction electrostatic potential energy:

Self-energy is the total electric potential energy developed due to interaction of all the charges of the same system with each other.

Interaction energy of one system with other system is energy developed due to interaction of charge particles of one system with charge particles of other system.

For example consider two system A and B as shown in figure.



System A = $q_1 + q_2$ System B = $q_3 + q_4$

Self energy of system

$$A = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}}$$

Self energy of system

$$B = \frac{1}{4\pi\varepsilon_0}\frac{q_3q_4}{r_{34}}$$

Interaction energy of system A with system B

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1q_4}{r_{14}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2q_3}{r_{23}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2q_4}{r_{24}}$$

Note:

Here we are ignoring self energy of point charges, which are not defined.

The Electrostatic Potential

The potential energy per unit charge U/q_0 is independent of the value of q_0 and has a value at every point in an electric field. This quantity U/q_0 is called the electric potential (or simply the potential) V. Thus, the electric potential at any point in an electric field is:

$$V = \frac{U}{q_0}$$

Since potential energy is a scalar quantity so the electric potential also is a scalar quantity. If the test charge is moved between two positions A and B in an electric filed, the charge-field system experiences a change in potential energy. Potential is defined as the change in potential energy of the system when a test charge is moved between the points divided by the test charge q_0.

$$\Delta V = \frac{\Delta U}{q_0} = -\int_A^B \vec{E}.d\vec{\ell}$$

Other physical interpretation of electric potential:

In electrostatic field, the electric potential (due to some source charges) at a point P is defined as the work done by external agent in taking a unit point positive charge from a reference point (generally taken at infinity) to that point P without changing its kinetic energy.

Mathematical representation:

If $(W_{\infty \to P})_{ext}$ is the work required in moving a point charge q from infinity to a point P, the electric potential of the point P is

$$V_p = \left.\frac{\left(W_{\infty \to p}\right)_{ext}}{q}\right]_{\Delta K = 0} = \frac{\left(-W_{elc}\right)_{\infty \to p}}{q}$$

Note:

(i) $(W_{\infty \to P})_{ext}$ can also be called as the work done by external agent against the electric force on a unit positive charge due to the source charge.

(ii) Write both W and q with proper sign.

Properties :

(i) Potential is a scalar quantity, its value may be positive, negative or zero.

(ii) S.I. Unit of potential is *volt* = $\frac{\text{joule}}{\text{coulomb}}$ and its dimensional formula is $[M^1L^2T^{-3}I^{-1}]$.

(iii) Electric potential at a point is also equal to the negative of the work done by the electric field in taking the point charge from reference point (i.e. infinity) to that point.

(iv) Electric potential due to a positive charge is always positive and due to negative charge it is always negative except at infinity. (Taking $V_\infty = 0$).

(v) Potential decreases in the direction of electric field.

(vi) $V = V_1 + V_2 + V_3 + ...$

Potential due to a point charge:

Derivation of expression for potential due to point charge Q, at a point which is at a distance r from the point charge.

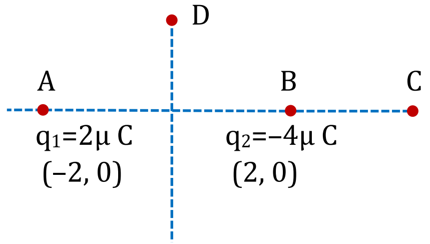
From definition of potential

$$V = \frac{W_{ext(\infty \to p)}}{q_o} = \frac{-\int_{\infty}^{r} (q_0\vec{E})\cdot d\vec{r}}{q_o} = -\int_{\infty}^{r} \vec{E}\cdot\overrightarrow{dr}$$

$$\Rightarrow \qquad V = -\int_{\infty}^{r} \frac{kQ}{r^2}(-dr)\cos 180^\circ = \frac{kQ}{r}$$

Illustration 63:

Two-point charges $2\mu C$ and – $4\mu C$ are situated at points (–2*m*, 0*m*) and (2 *m*, 0 *m*) respectively. Find out potential at point $C(4\,m, 0\,m)$ and $D(0\,m, \sqrt{5}\ m)$.



Solution:

Potential at point *C*

$$V_C = V_{q_1} + V_{q_2} = \frac{k(2\mu C)}{6} + \frac{k(-4\mu C)}{2} = \frac{9\times10^9\times2\times10^{-6}}{6} - \frac{9\times10^9\times4\times10^{-6}}{2} = -15000\ V.$$

Similarly, $V_D = V_{q_1} + V_{q_2} = \frac{k(2\mu C)}{\sqrt{(\sqrt{5})^2+2^2}} + \frac{k(-4\mu C)}{\sqrt{(\sqrt{5})^2+2^2}} = \frac{k(2\mu C)}{3} + \frac{k(-4\mu C)}{3} = -\ 6000\ V.$

Illustration 64:

When charge 10 μC is shifted from infinity to a point in an electric field, it is found that work done by electrostatic forces is $-10\ \mu J$. If the charge is doubled and taken again from infinity to the same point without accelerating it, then find the amount of work done by electric field and against electric field.

Solution:

$$(W_{ext})_{\infty \to p} = (-W_{elec})_{\infty \to p} = (W_{elec})_{p \to \infty} = 10\ \mu J$$

because $\Delta KE = 0$

$$\therefore \qquad V_p = \frac{(W_{ext})_{\infty \to p}}{10\mu C} = \frac{10\mu J}{10\mu C} = 1V$$

So, if now the charge is doubled and taken from infinity then

$$1 = \frac{(W_{ext})_{\infty \to p}}{20\mu C}$$

or $\qquad (W_{ext})_{\infty \to P} = 20\ \mu J$

$\Rightarrow \qquad (W_{elec})_{\infty \to P} = -20\ \mu J$

Illustration 65:

A charge $3\mu C$ is released from rest from a point P where electric potential is 20 V. What will be its kinetic energy when it reaches infinity?

Solution:

$W_{elec} = \Delta K = K_f - 0$

$\therefore$ $(W_{elec})_{P\to\infty} = qV_P = 60\ \mu J$

So, $K_f = 60\ \mu J$

Illustration 66:

$1\mu C$ charge is shifted from A to B and it is found that work done by an external force is $40\mu J$ in doing so against electrostatic forces, then find potential difference $V_A - V_B$.

Solution:

$(W_{AB})_{ext} = q(V_B - V_A)$

$\Rightarrow$ $40\ \mu J = 1\mu C\ (V_B - V_A)$

$\Rightarrow$ $V_A - V_B = -\ 40\ V$

Relation Between Electric Field and Potential:

(1) In an electric field rate of change of potential with distance is known as **potential gradient.**

(2) Potential gradient is a vector quantity and it's direction is opposite to that of electric field.

(3) Potential gradient relates with component of electric field according to the following relation $E_r = -\frac{dV}{dr}$. This relation gives another unit of electric field is $\frac{\text{volt}}{\text{meter}}$.

(4) In the above relation negative sign indicates that in the direction of electric field potential decreases.

(5) Negative of the slope of the V-r graph denotes intensity of electric field *i.e.* $\tan\theta = \frac{V}{r} = -E$

(6) In space around a charge distribution we can also write $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$

where $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$ and $E_z = -\frac{\partial V}{\partial z}$

(7) Potential difference between any two points in an electric field can be determined by knowing the boundary conditions

$$dV = -\int_{r_1}^{r_2} \vec{E}.\vec{dr} = -\int_{r_1}^{r_2} E.dr\cos\theta$$

Illustration 67:

Find expression of electric field, if expression of electric potential depends on x, y and z as

$$V = \frac{V_0}{a^3}x^2y^3z$$

Solution:

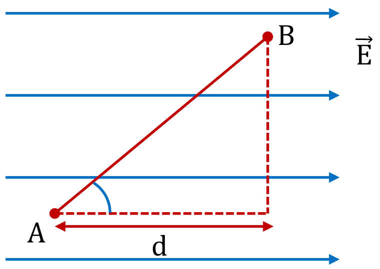
$$E_x = -\frac{\partial V}{\partial x} = -\frac{2V_0}{a^3}xy^3z$$

$$E_y = -\frac{\partial V}{\partial y} = -\frac{3V_0}{a^3}x^2y^2z$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{V_0}{a^3}x^2y^3$$

So, $\vec{E} = -\frac{V_0}{a^3}\left[2xy^3z\,\hat{i} + 3x^2y^2z\,\hat{j} + x^2y^3\,\hat{k}\right]$

Potential difference in a uniform electric field:



$V_B - V_A = -\,\vec{E}\cdot\overrightarrow{AB}$

$\Rightarrow$ $V_B - V_A = -\,|E|\,|AB|\cos\theta$

$= -\,|E|\,d$

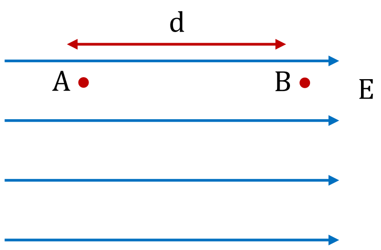
$= -\,Ed$

where d = effective distance between A and B along electric field.

or we can also say that $E = -\frac{\Delta V}{\Delta d}$

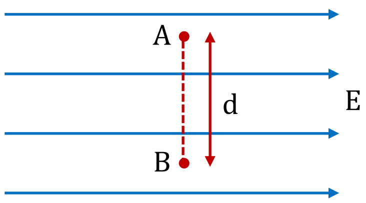
Special Cases :

Case-1. Line AB is parallel to electric field.



$\therefore$ $V_B - V_A = -Ed$

Case-2. Line AB is perpendicular to electric field.



$\therefore$ $V_B - V_A = 0$ $\Rightarrow$ $V_A = V_B$

Note:

In the direction of electric field potential always decreases.

Illustration 68:

A uniform electric field is present in the positive x-direction. If the intensity of the field is $5N/C$ then find the potential difference ($V_B - V_A$) between two points A (0m, 2 m) and B (5 m, 3 m).

Solution:

$$V_B - V_A = -\vec{E}\,.\,\overrightarrow{AB} = -(5\hat{i}).(5\hat{i}+\hat{j}) = -25V.$$

The electric field intensity in uniform electric field, $E = -\dfrac{\Delta V}{\Delta d}$

Where ΔV = Potential difference between two points.

Δd = Effective distance between the two points.

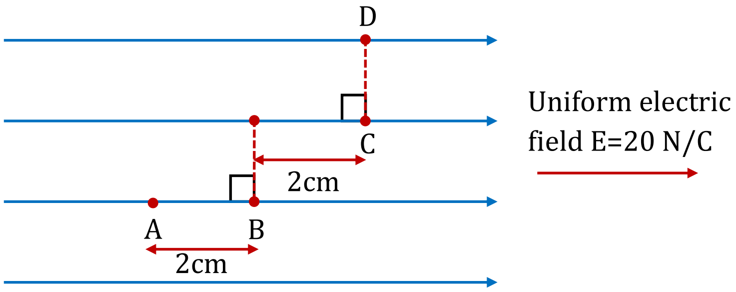
(Projection of the displacement along the direction of electric field.)

Illustration 69:

Find out following:

(i) $V_A - V_B$ (ii) $V_B - V_C$ (iii) $V_C - V_A$ (iv) $V_D - V_C$

(v) $V_A - V_D$ (vi) Arrange the order of potential for points A, B, C and D.



Solution:

(i) $|\Delta V_{AB}| = Ed = 20 \times 2 \times 10^{-2} = 0.4$ so, $V_A - V_B = 0.4\ V$

because **in the direction of electric field potential always decreases.**

(ii) $|\Delta V_{BC}| = Ed = 20 \times 2 \times 10^{-2} = 0.4$ so, $V_B - V_C = 0.4\ V$

(iii) $|\Delta V_{CA}| = Ed = 20 \times 4 \times 10^{-2} = 0.8$ so, $V_C - V_A = -0.8\ V$

because **in the direction of electric field potential always decreases.**

(iv) $|\Delta V_{DC}| = Ed = 20 \times 0 = 0$ so, $V_D - V_C = 0$

because the effective distance between D and C is zero.

(v) $|\Delta V_{AD}| = Ed = 20 \times 4 \times 10^{-2} = 0.8$ so, $V_A - V_D = 0.8\ V$

because **in the direction of electric field potential always decreases.**

(vi) The order of potential is : $V_A > V_B > V_C = V_D$.

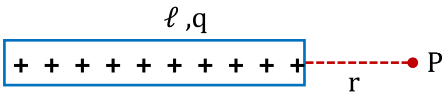
Potential due to continuous charges:

Finding potential due to continuous charges

If formula of E is tough, then we take a small element and integrate	If formula of E is easy then, we use
$V = \int dV$	$V = -\int_{r\to\infty}^{r=r} \vec{E}.d\vec{r}$ (i.e. for sphere, plate, infinite wire etc.)

Illustration 70:

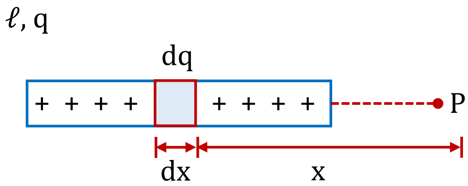
A rod of length ℓ is uniformly charged with charge q. Calculate potential at point P.



Solution:

Take a small element of length dx, at a distance x from point P.

Potential due to this small element, $dV = \dfrac{k(dq)}{x}$



$\therefore$ Total potential $\Rightarrow$ $V = \displaystyle\int_{x=r}^{x=r+\ell} \frac{kdq}{x}$

Where $dq = \dfrac{q}{\ell}dx$

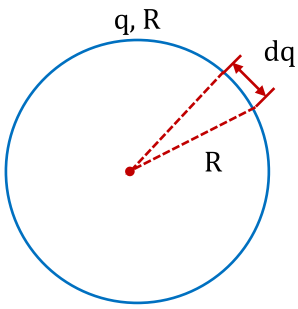
$$\Rightarrow \quad V = \int_{x=r}^{x=r+\ell} \frac{k\left(\frac{q}{\ell}dx\right)}{x} = \frac{kq}{\ell}\log_e\left(\frac{\ell+r}{r}\right)$$

Potential due to a ring:

(i) Potential at the centre of uniformly charged ring:

Potential due to the small element dq

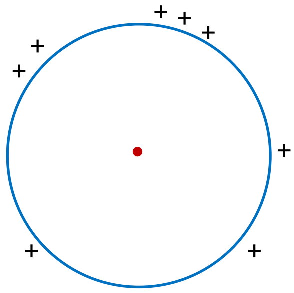
$$dV = \frac{kdq}{R}$$



$\therefore$ Net potential : $V = \displaystyle\int \frac{kdq}{R}$

$\therefore$ $V = \dfrac{k}{R}\displaystyle\int dq = \frac{kq}{R}$

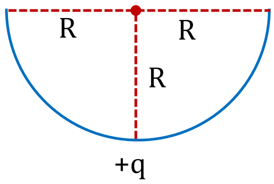
(ii) For non-uniformly charged ring potential at the center is



$$V = \frac{kq}{R}$$

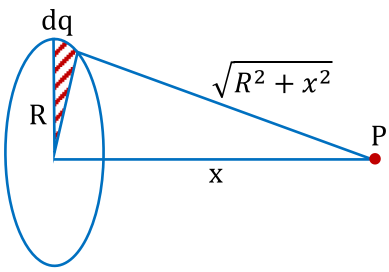
where q = total charge on the ring.

(iii) Potential due to half ring at center is:



$$V = \frac{kq}{R}$$

(iv) Potential at the axis of a ring: Now we calculate potential at a point on the axis which is at a distance x from centre of uniformly charged (total charge Q) ring of radius R.



Consider an element of charge dq on the ring. Potential at point P due to charge dq will be

$$dV = \frac{k(dq)}{\sqrt{R^2 + x^2}}$$

$\therefore$ Net potential at point P due to all such element will be :

$$V = \int dV = \frac{kQ}{\sqrt{R^2 + x^2}}$$

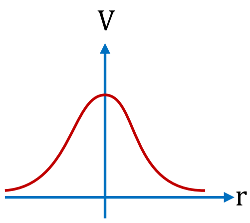
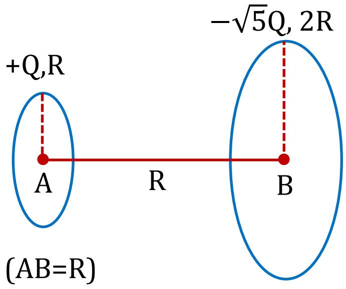


Illustration 71:

Figure shows two rings having charges Q and $-\sqrt{5}\,Q$. Find Potential difference between A and B i.e. $(V_A - V_B)$.



Solution:

$$V_A = \frac{kQ}{R} + \frac{k\left(-\sqrt{5}\,Q\right)}{\sqrt{(2R)^2 + (R)^2}} \text{ and } V_B = \frac{k\left(-\sqrt{5}\,Q\right)}{2\,R} + \frac{k(Q)}{\sqrt{(R)^2 + (R)^2}}$$

From above, we can easily find $V_A - V_B$.

Illustration 72:

A point charge q_0 is placed at the centre of uniformly charged ring of total charge Q and radius R. If the point charge is slightly displaced with negligible force along axis of the ring then find out its speed when it reaches a large distance.

Solution:

Only electric force is acting on q_0

$\therefore \quad W_{\text{elec}} = \Delta K = \frac{1}{2}mv^2 - 0$

$\Rightarrow \quad$ Now $(W_{\text{elec}})_{c\to\infty} = q_0\,V_c = q_0\,\frac{KQ}{R}$.

By energy conservation

$\therefore \quad \frac{Kq_0Q}{R} = \frac{1}{2}mv^2$

$\Rightarrow \quad V = \sqrt{\frac{2Kq_0Q}{mR}}$

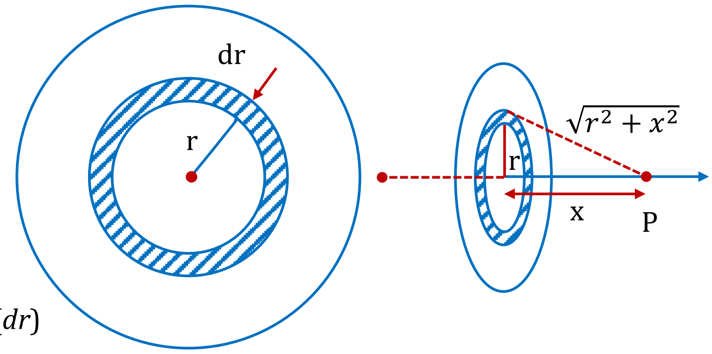
Potential due to uniformly charged disc:

A disc of radius 'R' has surface charge density (charge/area) = σ. We have to find potential at its axis, at point 'P' which is at a distance x from the centre.

For this, we can divide the disc into thin rings and let's consider a thin ring of radius r and thickness dr. Suppose charge on the small ring element = dq. Potential due to this ring at point 'P' is:

$$dV = \frac{Kdq}{\sqrt{r^2 + x^2}}$$

So, net potential : $V = \int \frac{Kdq}{\sqrt{r^2 + x^2}}$



Here, σ = Charge/area = $\frac{dq}{d(area)}$

So, $dq = \sigma \times d\,(\text{area}) = \sigma\,(2\pi r\,dr)$

Here, d (area) = Area of the small ring element

= (Length of ring) × (Width of the ring) = $(2\pi r)\,.\,(dr)$

So, $V = \int_{r=0}^{r=R} \frac{K\sigma(2\pi r\,dr)}{\sqrt{r^2 + x^2}}$

To integrate it, let, $r^2 + x^2 = y^2$

$2r\,dr = 2y\,dy$. Substituting we will get :

$$V = \int_{y=x}^{y=R^2+x^2} \frac{1}{4\pi\varepsilon_0} \frac{\sigma(2\pi)y\,dy}{y}$$

$$\Rightarrow \quad V = \frac{\sigma}{2\varepsilon_0}[y]_{y=x}^{y=R^2+x^2}$$

$$\Rightarrow \quad V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+x^2} - x\right)$$

If a test charge q_0 is placed at point P, then potential energy of this charge q_0 due to the disc = $U = q_0 V$

$$\Rightarrow \quad U = q_0\left[\frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+x^2} - x\right)\right]$$

Graph of V v/s x :

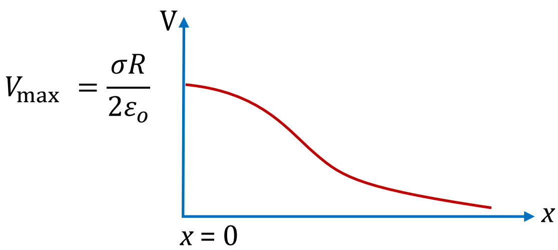
$$V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+x^2} - x\right)$$

at $x = 0$, $V = \frac{\sigma R}{2\varepsilon_0}$ to check whether V will increase with x or decrease, lets multiply and divide by conjugate.

$$V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+x^2} - x\right) \times \frac{\left(\sqrt{R^2+x^2} + x\right)}{\left(\sqrt{R^2+x^2} + x\right)}$$

$$\Rightarrow \quad V = \frac{\sigma R^2}{2\varepsilon_0}\left(\frac{1}{\left(\sqrt{R^2+x^2} + x\right)}\right)$$

Now, we can say that as $x\uparrow \Rightarrow V\downarrow$ so curve will be



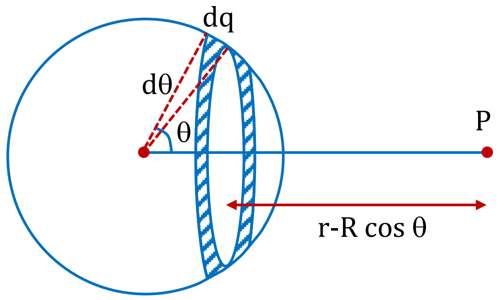
Potential Due to Uniformly Charged Spherical shell:

Derivation of expression for potential due to uniformly charged hollow sphere of radius *R* and total charge *Q*, at a point which is at a distance *r* from centre for the following situation

(i) $r > R$ (ii) $r < R$

Assume a ring of width $Rd\theta$ at angle θ from X axis (as shown in figure). Potential due to the ring at the point P will be

$$dV = \frac{K(dq)}{\sqrt{(r - R\cos\theta)^2 + (R\sin\theta)^2}}$$



Where $dq = 2\pi R \sin\theta\,(Rd\theta)\sigma$ where $Q = 4\pi R^2\sigma$

Then, net potential $V = \int dV = \frac{KQ}{2}\int_0^{\pi} \frac{\sin\theta . d\theta}{\sqrt{(r - R\cos\theta)^2 + (R\sin\theta)^2}}$

Solving this equation, we find $V = \frac{KQ}{r}$ (for $r > R$)

and $V = \frac{KQ}{R}$ for ($r < R$)

Alternate Method : As the formula of E is easy, we use $V = -\int_{r \to \infty}^{r = r} \vec{E} \cdot d\vec{r}$

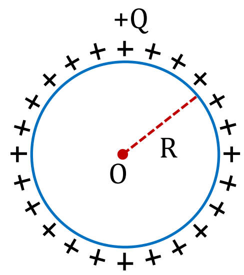
(i) At outside point ($r \geq R$) : $V_{\text{out}} = -\int_{r \to \infty}^{r = r} \left(\frac{KQ}{r^2}\right) dr$

$$\Rightarrow \quad V_{\text{out}} = \frac{KQ}{r} = \frac{KQ}{(\text{Distance from centre})}$$

For outside point, the hollow sphere acts like a point charge.

(ii) Potential at the centre of the sphere ($r = 0$) :

As all the charges are at a distance R from the centre,

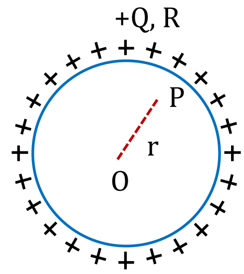


So, $V_{\text{centre}} = \frac{KQ}{R}$

$$= \frac{KQ}{(\text{Radius of the sphere})}$$

(iii) Potential at inside point ($r < R$) : Suppose we want to find potential at point P, inside the sphere.

$\therefore$ Potential difference between Point P and O :



$V_P - V_O = -\int_O^P \vec{E}_{in} \cdot d\vec{r}$ Where, $E_{in} = 0$

So $V_P - V_O = 0 \quad \Rightarrow V_P = V_O = \frac{KQ}{R}$

$$\Rightarrow V_{in} = \frac{KQ}{R} = \frac{KQ}{(\text{Radius of the sphere})}$$

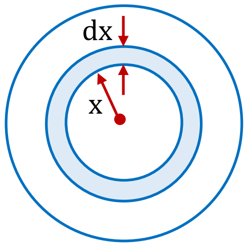
Potential Due to Uniformly Charged Solid Sphere :

Derivation of expression for potential due to uniformly charged solid sphere of radius R and total charge Q (distributed in volume), at a point which is at a distance r from centre for the following situations.

(i) $r \geq R$

(ii) $r \leq R$

Consider an elementary shell of radius x and width dx



(i) For $r \geq R$: $V = \int_0^R \frac{K \cdot 4\pi x^2 dx \rho}{r} = \frac{KQ}{r}$

(ii) For $r \leq R$: $V = \int_0^r \frac{K \cdot 4\pi x^2 dx \rho}{r} + \int_r^R \frac{K 4\pi x^2 dx \rho}{x}$

$$= \frac{KQ}{2R^3}(3R^2 - r^2) \Rightarrow \left(\rho = \frac{Q}{\frac{4}{3}\pi R^3}\right)$$

From definition of potential

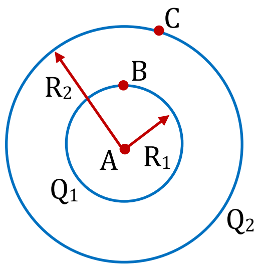
(i) For $r \geq R$: $V = -\int_\infty^r \frac{KQ}{r^2}\hat{r} \cdot dr = \frac{KQ}{r}$

(ii) For $r \leq R$: $V = \int_\infty^R \frac{KQ}{r^2}.dr - \int_R^r \frac{KQr}{R^3}dr$

$$V = \frac{KQ}{R} - \frac{KQ}{2R^3}[r^2 - R^2] = \frac{KQ}{2R^3}[2R^2 - r^2 + R^2] = \frac{KQ}{2R^3}(3R^2 - r^2)$$

Illustration 73:

Two concentric spherical shells of radius R_1 and R_2 ($R_2 > R_1$) are having uniformly distributed charges Q_1 and Q_2 respectively. Find out potential



(i) at point A

(ii) at surface of smaller shell (i.e. at point B)

(iii) at surface of larger shell (i.e. at point C)

(iv) at $r \leq R_1$

(v) at $R_1 \leq r \leq R_2$

(vi) at $r \geq R_2$

Solution:

Using the results of hollow sphere

(i) $V_A = \frac{KQ_1}{R_1} + \frac{KQ_2}{R_2}$ (ii) $V_B = \frac{KQ_1}{R_1} + \frac{KQ_2}{R_2}$ (iii) $V_C = \frac{KQ_1}{R_2} + \frac{KQ_2}{R_2}$

(iv) For $r \leq R_1$, $V = \frac{KQ_1}{R_1} + \frac{KQ_2}{R_2}$ (v) For $R_1 \leq r \leq R_2$, $V = \frac{KQ_1}{r} + \frac{KQ_2}{R_2}$ (vi) For $r \geq R_2$, $V = \frac{KQ_1}{r} + \frac{KQ_2}{r}$

Illustration 74:

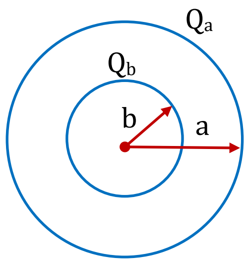
Two hollow concentric non-conducting spheres of radius the a and b ($a > b$) contain charges Q_a and Q_b respectively. Prove that potential difference between the two spheres is independent of charge on outer sphere. If outer sphere is given an extra charge, is there any change in potential difference?

Solution:

$$V_{\text{inner sphere}} = \frac{KQ_b}{b} + \frac{KQ_a}{a}$$

$$V_{\text{outer sphere}} = \frac{KQ_b}{a} + \frac{KQ_a}{a}$$

$$V_{\text{inner sphere}} - V_{\text{outer sphere}} = \frac{KQ_b}{b} - \frac{KQ_b}{a}$$

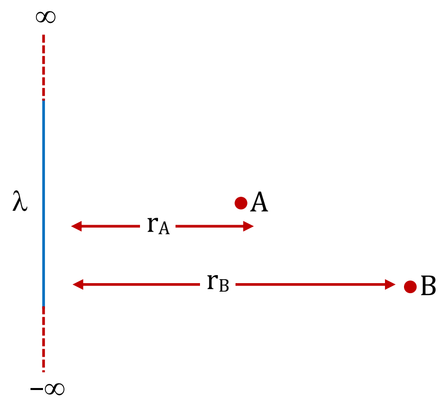


$$\therefore \quad \Delta V = KQ_b\left[\frac{1}{b} - \frac{1}{a}\right]$$

Which is independent of charge on outer sphere. If outer sphere is given any extra charge, then there will be no change in potential difference.

Potential difference due to infinitely long wire:

Derivation of expression for potential difference between two points, which have perpendicular distance r_A and r_B from infinitely long line charge of uniform linear charge density λ.



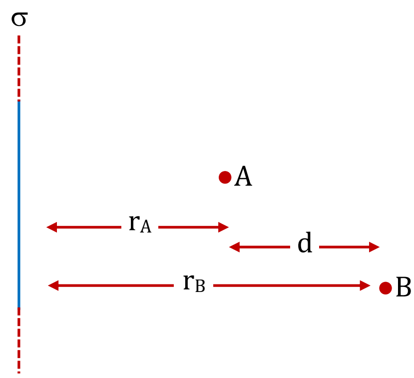
From definition of potential difference :

$$V_{AB} = V_B - V_A = -\int_{r_A}^{r_B} \vec{E}\cdot\overrightarrow{dr} = -\int_{r_A}^{r_B} \frac{2K\lambda}{r}\hat{r}\cdot\overrightarrow{dr}$$

$$\therefore \quad V_{AB} = -2K\lambda\, \ell n\left(\frac{r_B}{r_A}\right)$$

Potential difference due to infinitely long thin sheet:

Derivation of expression for potential difference between two points, having separation d in the direction perpendicularly to a very large uniformly charged thin sheet of uniform surface charge density σ.



Let the points A and B have perpendicular distance r_A and r_B respectively then from definition of potential difference.

$$V_{AB} = V_B - V_A = -\int_{r_A}^{r_B} \vec{E}\cdot\overrightarrow{dr} = -\int_{r_A}^{r_B} \frac{\sigma}{2\varepsilon_0}\hat{r}\cdot\overrightarrow{dr}$$

$$\Rightarrow \quad V_{AB} = -\frac{\sigma}{2\varepsilon_o}(r_B - r_A) = -\frac{\sigma d}{2\varepsilon_o}$$

Equipotential Surface:

Definition:

If potential of a surface (imaginary or physically existing) is same throughout, then such surface is known as an equipotential surface.

Properties of equipotential surfaces:

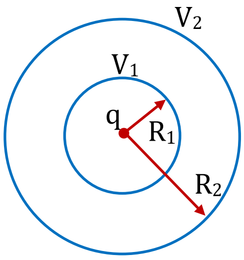
(i) When a charge is shifted from one point to another point on an equipotential surface, then work done against electrostatic forces is zero.

(ii) Electric field is always perpendicular to equipotential surfaces.

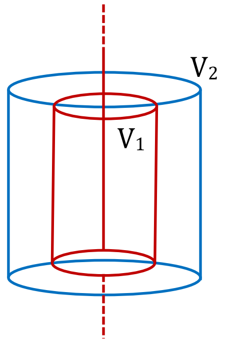
(iii) Two equipotential surfaces do not cross each other.

Equipotential surfaces for different type of charged bodies:

(i) **Point charge:** Equipotential surfaces are concentric and spherical as shown in figure. In figure, we can see that sphere of radius R_1 has potential V_1 throughout its surface and similarly for other concentric sphere potential is same.

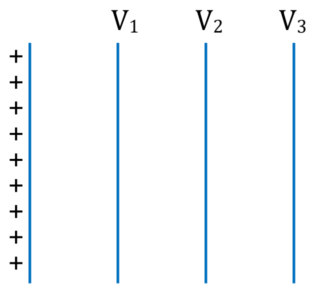


(ii) **Line charge :** Equipotential surfaces have curved surfaces as that of coaxial cylinders of different radii.



(iii) **Uniformly charged large conducting / non conducting sheets:**

Equipotential surfaces are parallel planes.

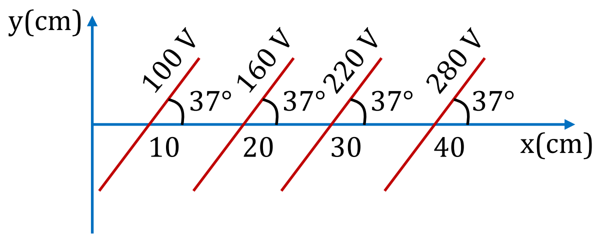


Note:

In uniform electric field equi-potential surfaces are always parallel planes.

Illustration 75:

Some equipotential surfaces are shown in figure. What can you say about the magnitude and the direction of the electric field ?



Solution:

Here, we can say that the electric field will be perpendicular to equipotential surfaces.

Also, $|\vec{E}| = \dfrac{\Delta V}{\Delta d}$

Where, ΔV = potential difference between two equipotential surfaces.

Δd = perpendicular distance between two equipotential surfaces.

So $|\vec{E}| = \dfrac{60}{(10\sin 37°)\times 10^{-2}} = 1000\ V/m$

Now there are two perpendicular directions: either direction 1 or direction 2 as shown in figure, but since we know that in the direction of electric field, electric potential decreases, so the correct direction is direction 2.

Hence $E = 1000\ V/m$, making an angle $127°$ with the x-axis.

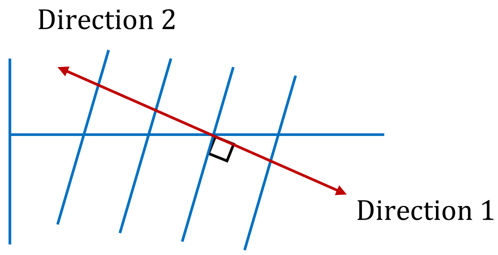
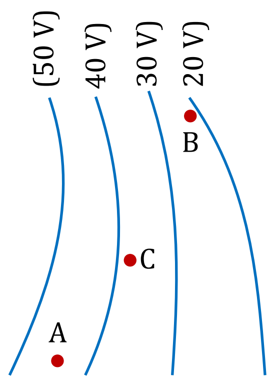


Illustration 76:

Figure shows some equipotential surfaces produce by some charges. At which point, the value of electric field is greatest?



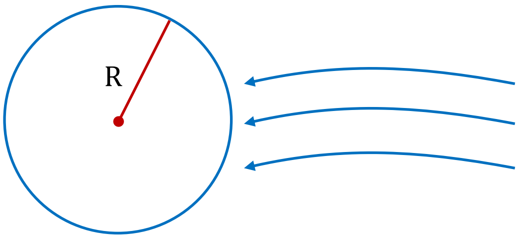
Solution:

E is larger where equipotential surfaces are closer. $ELOF$ are $\perp$ to equipotential surfaces. In the figure, we can see that for point B, they are closer so E at point B is maximum.

Derivation of self-energy for continuous charge system:

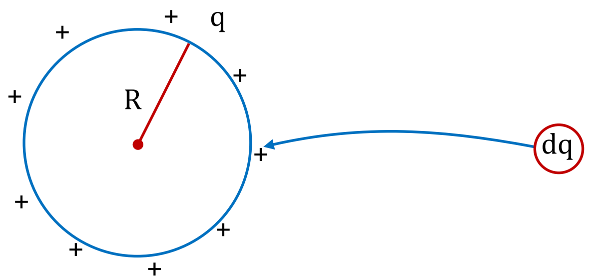
(i) Self Energy (*P.E.*) of a uniformly Charged spherical shell:

For this, lets use method 1 : Take an uncharged shell. Now bring charges one by one from infinity to the surface for the shell. The work required in this process will be stored as potential Energy.



Uncharged shell

Suppose, we have given charge q to the sphere and now we are giving extra charge dq to it. Work required to bring dq charge from infinity to the shell is



$$dW = (dq)\,(V_f - V_i) \quad \Rightarrow \quad dW = (dq)\left(\frac{kq}{R} - 0\right) = \frac{kq}{R}\,dq$$

$\Rightarrow$ Total work required to give charge Q is $W = \int_{q=o}^{q=Q} \frac{Kq}{R}\,dq = \frac{KQ^2}{2R}$

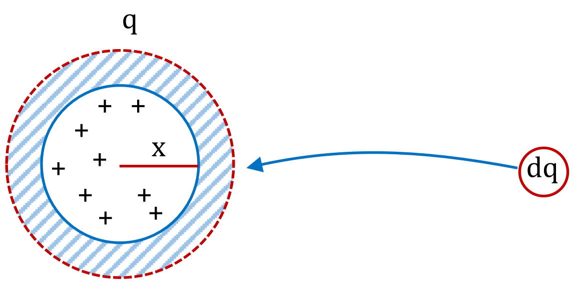
This work will be stored in the form of *P.E.* (self energy)

So, *P.E.* of a charged spherical shell : $U = \frac{kQ^2}{2R}$

(ii) Self energy of uniformly charged solid sphere :

In this case we have to assemble a solid charged sphere. So, we bring the charges one-by-one from infinity to the sphere so that the size of the sphere increases.

Suppose we have given charge q to the sphere, and its radius becomes 'x'. Now we are giving extra charge dq to it, which will increase its radius by 'dx'.



$\therefore$ Work required to bring charge dq from infinity to the sphere

$$= dq\,(V_f - V_i) = (dq)\left(\frac{kq}{x} - o\right) = \frac{kqdq}{x}$$

∴ Total work required to give charge Q to the sphere

$$W = \int \frac{kqdq}{x}, \text{ where } q = \rho\left(\frac{4}{3}\pi x^3\right)$$

and $dq = \rho\,(4\,\pi x^2\,dx)$ $\Rightarrow$ $W = \int_{x=o}^{x=R} k\frac{\rho\left(\frac{4}{3}\pi x^3\right)\rho(4\pi x^2 dx)}{x}$

Solving, well get : $W = \frac{3}{5}\frac{kQ^2}{R} = U_{\text{self}}$ (for a solid sphere)

Illustration 77:

A spherical shell of radius R with uniform charge q is expanded to a radius $2R$. Find the work performed by the electric forces and external agent against electric forces in this process.

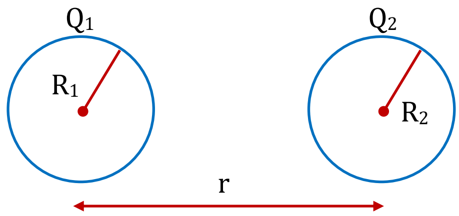
Solution:

$$W_{ext} = U_f - U_i = \frac{q^2}{16\pi\varepsilon_0 R} - \frac{q^2}{8\pi\varepsilon_0 R} = -\frac{q^2}{16\pi\varepsilon_0 R}$$

$$W_{elec} = U_i - U_f = \frac{q^2}{8\pi\varepsilon_0 R} - \frac{q^2}{16\pi\varepsilon_0 R} = \frac{q^2}{16\pi\varepsilon_0 R}$$

Illustration 78:

Two non-conducting hollow uniformly charged spheres of radii R_1 and R_2 with charge Q_1 and Q_2 respectively are placed at a distance r. Find out total energy of the system.

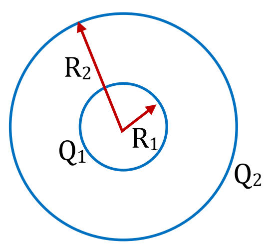


Solution:

$$U_{\text{total}} = U_{\text{self}} + U_{\text{Interaction}} = \frac{Q_1^2}{8\pi\varepsilon_0 R} + \frac{Q_2^2}{8\pi\varepsilon_0 R_2} + \frac{Q_1 Q_2}{4\pi\varepsilon_0 r}$$

Illustration 79:

Two concentric spherical shells of radius R_1 and R_2 ($R_2 > R_1$) are having uniformly distributed charges Q_1 and Q_2 respectively. Find out total energy of the system.



Solution:

$$U_{\text{total}} = U_{\text{self 1}} + U_{\text{self 2}} + U_{\text{Interaction}} = \frac{Q_1^2}{8\pi\varepsilon_0 R_1} + \frac{Q_2^2}{8\pi\varepsilon_0 R_2} + \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_2}$$

Energy density:

Energy density is defined as energy stored in unit volume in any electric field. Its mathematical formula is given as following :

Energy density = $\frac{1}{2}\varepsilon E^2$

where E = electric field intensity at that point

$\varepsilon = \varepsilon_0\varepsilon_r$ = electric permittivity of medium

Illustration 80:

Find out energy stored in an imaginary cubical volume of side a in front of a infinitely large non-conducting sheet of uniform charge density σ.

Solution:

Energy stored :

$U = \int \frac{1}{2}\varepsilon_0 E^2 dV$; where dV is small volume

$\therefore \quad U = \frac{1}{2}\varepsilon_0 E^2 \int dV$

$\because \quad E$ is constant

$\therefore \quad U = \frac{1}{2}\varepsilon_0 \frac{\sigma^2}{4\varepsilon_0^2} . a^3 = \frac{\sigma^2 a^3}{8\varepsilon_0}$

Illustration 81:

Find out energy stored in the electric field of uniformly charged thin spherical shell of total charge Q and radius R.

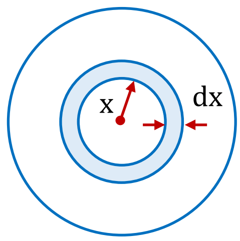
Solution:

We know that electric field inside the shell is zero, so the energy is stored only outside the shell, which can be calculated by using energy density formula.

$$U_{self} = \int_{x=R}^{x\to\infty} \frac{1}{2}\varepsilon_0 E^2 dV$$

Consider an elementary shell of thickness dx and radius x ($x > R$).

Volume of the shell = $(4\pi x^2 dx) = dV$



$$U = \int_R^\infty \frac{1}{2}\varepsilon_0 \left[\frac{KQ}{x^2}\right]^2 . 4\pi x^2\, dx = \frac{1}{2}\varepsilon_0\, K^2Q^2 4\pi \int_R^\infty \frac{1}{x^2}dx$$

$$= \frac{4\pi\varepsilon_0}{2}\frac{Q^2}{(4\pi\varepsilon_0)^2} . \left(\frac{1}{R}\right) = \frac{Q^2}{8\pi\varepsilon_0 R} = \frac{KQ^2}{2R}$$

Illustration 82:

Find out energy stored inside a solid non-conducting sphere of total charge Q and radius R. Assume charge is uniformly distributed in its volume.

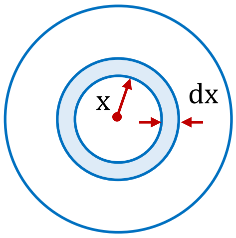
Solution:

We can consider solid sphere to be made of large number of concentric spherical shells. Also, electric field intensity at the location of any particular shell is constant.

$$\therefore \qquad U_{\text{inside}} = \int_0^R \frac{1}{2}\varepsilon_0 E^2 dV$$

Consider an elementary shell of thickness dx and radius x.

Volume of the shell = $(4\pi x^2 dx)$



$$U_{\text{inside}} = \int_0^R \frac{1}{2}\varepsilon_0 \left[\frac{KQx}{R^3}\right]^2 .4\pi x^2\, dx = \frac{1}{2}\varepsilon_0 \frac{K^2Q^2 4\pi}{R^6}\int_0^R x^4 dx$$

$$= \frac{4\pi\varepsilon_0}{2R^6}\,\frac{Q^2}{(4\pi\varepsilon_0)^2}\cdot\frac{R^5}{5} = \frac{Q^2}{40\pi\varepsilon_0 R} = \frac{KQ^2}{10R}$$

Electric Dipole

If two point charges, equal in magnitude *'q'* and opposite in sign separated by a distance *'a'* such that the distance of field point $r >> a$, the system is called a dipole. The electric dipole moment is defined as a vector quantity having magnitude $p = (q \times a)$ and direction from negative charge to positive charge.

Note:

In chemistry, the direction of dipole moment is assumed to be from positive to negative charge.

The C.G.S unit of electric dipole moment is **debye** which is defined as the dipole moment of two equal and opposite point charges each having charge 10^{-10} Franklin and separation of 1 Å, i.e.,

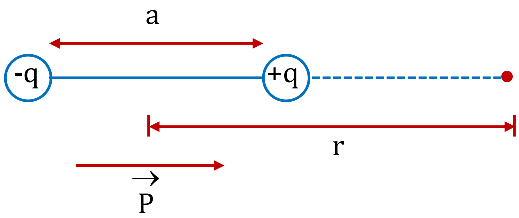
$$1 \text{ debye } (D) = 10^{-10} \times 10^{-8} = 10^{-18}\ Fr \times cm$$

or $$1\ D = 10^{-18} \times \frac{C}{3\times 10^9} \times 10^{-2}\ m = 3.3 \times 10^{-30}\ C \times m$$

S.I. Unit is coulomb × metre = $C \,.\, m$

Electric Field Intensity Due to Dipole:

(i) At the axial point:



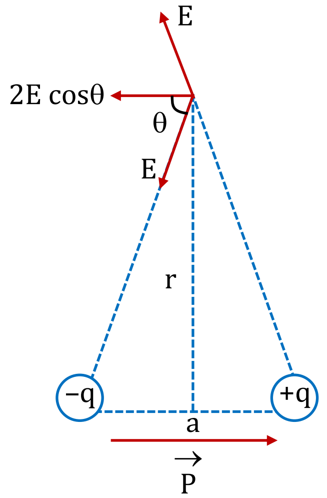
$$\vec{E} = \frac{Kq}{\left(r-\frac{a}{2}\right)^2} - \frac{Kq}{\left(r+\frac{a}{2}\right)^2} \text{ (along the } \vec{P}) = \frac{Kq(2ra)}{\left(r^2-\frac{a^2}{4}\right)^2}\hat{P}$$

If $r >> a$ then, $\vec{E} = \dfrac{Kq\,2ra}{r^4}\hat{P} = \dfrac{2K\vec{P}}{r^3}$

As the direction of electric field at axial position is along the dipole moment ($\vec{P}$)

So, $$\vec{E}_{axial} = \frac{2K\vec{P}}{r^3}$$

(ii) Electric field at perpendicular Bisector (Equatorial Position)



$E_{net} = 2\ E \cos\theta$ (along $-\hat{P}$)

$$\vec{E}_{net} = 2\left(\frac{kq}{\left(\sqrt{r^2+\left(\frac{a}{2}\right)^2}\right)^2}\right)\frac{\frac{a}{2}}{\sqrt{r^2+\left(\frac{a}{2}\right)^2}}(-\hat{P})$$

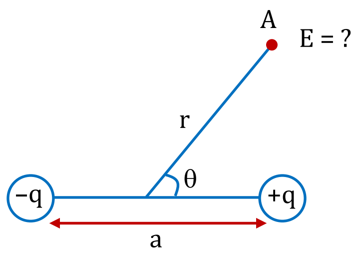
$$= \frac{Kqa}{\left(r^2+\left(\frac{a}{2}\right)^2\right)^{3/2}}(-\hat{P})$$

If $r >> a$ then $\vec{E}_{net} = \frac{KP}{r^3}(-\hat{P})$

As the direction of $\vec{E}$ at equatorial position is opposite of $\vec{P}$ so we can write in vector form:

$$\vec{E}_{eqt} = -\frac{K\vec{P}}{r^3}$$

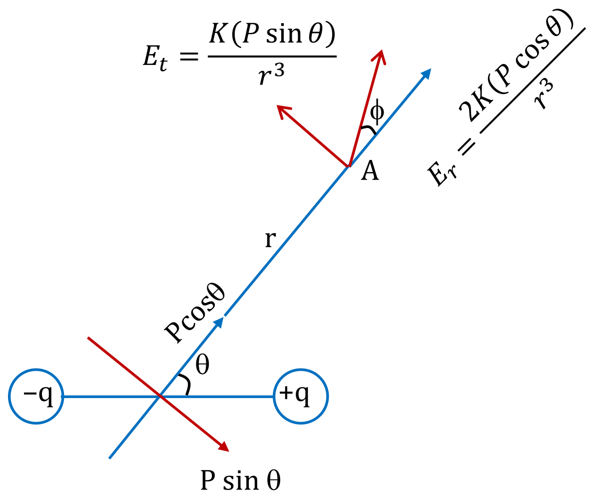
(iii) Electric field at general point (r, θ):



For this, let's resolve the dipole moment $\vec{P}$ into components.

One component is along radial line (= $P\cos\theta$) and other component is $\perp_r$ to the radial line (= $P\sin\theta$)

From the given figure



$$E_{net} = \sqrt{E_r^2 + E_t^2} = \sqrt{\left(\frac{2KP\cos\theta}{r^3}\right)^2 + \left(\frac{KP\sin\theta}{r^3}\right)^2}$$

$$= \frac{KP}{r^3}\sqrt{1+3\cos^2\theta}$$

$$\tan\phi = \frac{E_t}{E_r} = \frac{\frac{KP\sin\theta}{r^3}}{\frac{2KP\cos\theta}{r^3}} = \frac{\tan\theta}{2}$$

$$E_{net} = \frac{KP}{r^3}\sqrt{1+3\cos^2\theta} \text{ and } \tan\phi = \frac{\tan\theta}{2}$$

Illustration 83:

The electric field due to a short dipole at a distance r, on the axial line, from its mid point is the same as that of electric field at a distance r', on the equatorial line, from its mid-point. Determine the ratio $\frac{r}{r'}$.

Solution:

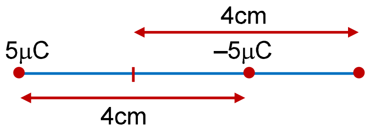
$$\frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r'^3} \quad \text{or} \quad \frac{2}{r^3} = \frac{1}{r'^3} \quad \text{or} \quad \frac{r^3}{r'^3} = 2 \quad \text{or} \quad \frac{r}{r'} = 2^{1/3}$$

Illustration 84:

Two charges, each of 5 μC but opposite in sign, are placed 4 cm apart. Calculate the electric field intensity of a point that is at a distance 4 cm from the mid point on the axial line of the dipole.

Solution:

We cannot use formula of short dipole here because distance of the point is comparable to the distance between the two point charges.



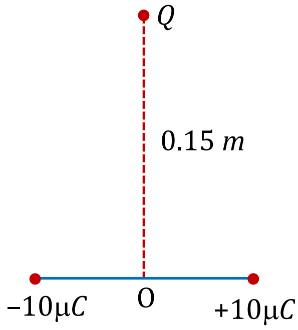
$q = 5 \times 10^{-6}\ C$, $a = 4 \times 10^{-2}\ m$, $r = 4 \times 10^{-2}\ m$

$$E_{res} = E_+ + E_- = \frac{K(5\mu C)}{(2cm)^2} - \frac{K(5\mu C)}{(6\ cm)^2} = \frac{144}{144\times10^{-8}}\ NC^{-1} = 10^8\ NC^{-1}$$

Illustration 85:

Two charges $\pm10\mu C$ are placed $5 \times 10^{-3}\ m$ apart as shown in figure. Determine the electric field at a point Q which is 0.15 m away from O, on the equatorial line.

Solution:



In the given problem, $r >> a$

$$\therefore \quad E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}$$

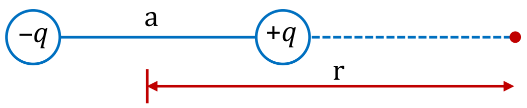
or $$E = 9 \times 10^9 \times \frac{10\times10^{-6}\times5\times10^{-3}}{0.15\times0.15\times0.15}\ NC^{-1}$$

$$= 1.33 \times 10^5\ NC^{-1}$$

Electric Potential due to a small dipole:

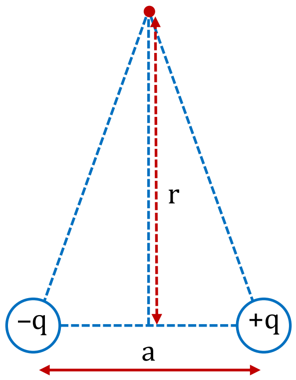
(i) Potential at axial position:

$$V = \frac{Kq}{\left(r-\frac{a}{2}\right)} + \frac{K(-q)}{\left(r+\frac{a}{2}\right)} \quad \Rightarrow \quad V = \frac{Kqa}{\left(r^2-\left(\frac{a}{2}\right)^2\right)}$$



If $r >> a$ then $V = \dfrac{Kqa}{r^2}$; where, $qa = p$

$$\boxed{\therefore V_{axial} = \frac{KP}{r^2}}$$



(ii) Potential at equatorial position:

$$V = \frac{Kq}{\sqrt{r^2 + \left(\frac{a}{2}\right)^2}} + \frac{K(-q)}{\sqrt{r^2 + \left(\frac{a}{2}\right)^2}} = 0$$

$$\boxed{V_{eqt} = 0}$$

(iii) Potential at general point (r, θ) : Lets resolve the dipole moment $\vec{P}$ into components : $P\cos\theta$ along radial line and $P\sin\theta \perp_r$ to the radial line.

For the $P\cos\theta$ component, the point A is an axial point,

So, potential at A due to $P\cos\theta = \dfrac{K(P\cos\theta)}{r^2}$

And for $P\sin\theta$ component, the point A is an equatorial point,

So potential at A due to $P\sin\theta = 0$

$$V_{net} = \frac{K(P\cos\theta)}{r^2}$$

$$\therefore \quad V = \frac{K\left(\vec{P}.\vec{r}\right)}{r^3}$$

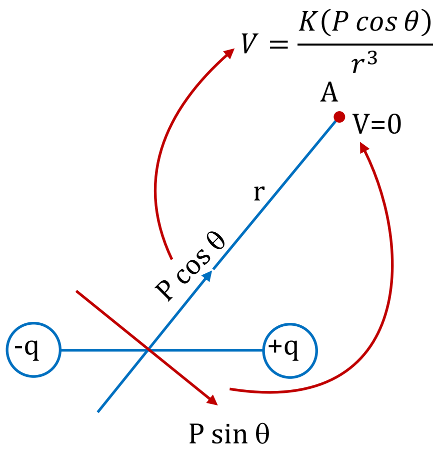
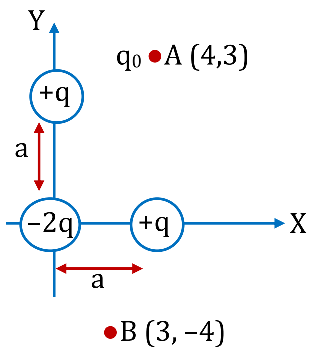


Illustration 86:

(i) Find potential at point A and B due to the small charge - system fixed near origin. (Distance between the charges is negligible).

(ii) Find work done to bring a test charge q_0 from point A to point B, slowly. All parameters are in S.I. units.



Solution:

(i) Dipole moment of the system is

$$\vec{P} = (qa)\,\hat{i} + (qa)\,\hat{j}$$

Potential at point A due to the dipole

$$V_A = K\frac{(\vec{P}\cdot\vec{r})}{r^3} = \frac{K[(qa)\hat{i}+(qa)\hat{j}]\cdot(4\hat{i}+3\hat{j})}{5^3} = \frac{(7)k(qa)}{125}$$

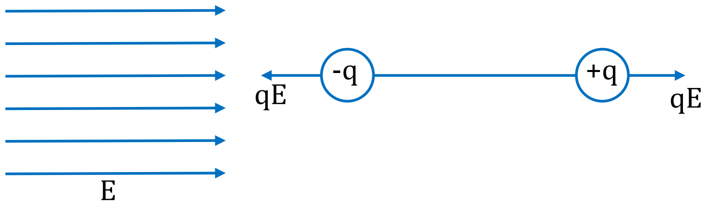
$$\Rightarrow \quad V_B = \frac{K\left[(qa)\hat{i}+(qa)\hat{j}\right]\cdot\left(3\hat{i}-4\hat{j}\right)}{(5)^3} = \frac{-K(qa)}{125}$$

(ii) $$W_{A\to B} = U_B - U_A = q_0\,(V_B - V_A) = q_0\left[-\frac{K(qa)}{125}-\left(\frac{K(qa)(7)}{125}\right)\right]$$

$$\Rightarrow \quad W_{A\to B} = \frac{-Kqq_0a}{125}(8)$$

Dipole in uniform electric field:

(i) Dipole is placed along electric field :



In this case, $F_{net} = 0$, $\tau_{net} = 0$, so it is an equilibrium state. And it is a stable equilibrium position.

(ii) If the dipole is placed at angle θ from $\vec{E}$:

In this case, $F_{net} = 0$ but

Net torque, $\tau = (qE\sin\theta)\ (a)$

Here $\quad qa = P$

$\Rightarrow \quad \tau = PE\sin\theta$

In vector form : $\vec{\tau} = \vec{P}\times\vec{E}$

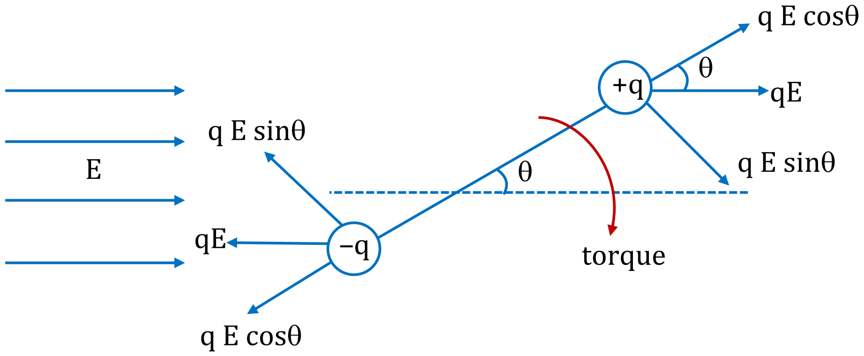


Illustration 87:

A dipole is formed by two point charge $-q$ and $+q$, each of mass m, and both the point charges are connected by a rod of length ℓ and mass m. This dipole is placed in uniform electric field $\vec{E}$. If the dipole is disturbed by a small angle θ from stable equilibrium position, prove that its motion will be almost SHM. Also find its time period.

Solution:

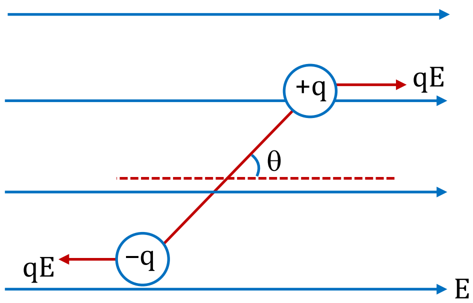
If the dipole is disturbed by θ angle, $\tau_{net} = -PE\sin\theta$ (Here *-ve* sign indicates that direction of torque is opposite to θ)

If θ is very small, $\sin\theta \approx \theta$

$\therefore \quad \tau_{net} = -(PE)\theta$

$\tau_{net} \propto (-\theta)$ so motion will be almost SHM and $C = PE$ (where, $P = q\ell$)

$\because \quad T = 2\pi\sqrt{\dfrac{I}{C}}$



$$\therefore \quad T = 2\pi\sqrt{\dfrac{\dfrac{m\ell}{12} + 2m\left(\dfrac{\ell}{2}\right)^2}{P.E}} = 2\pi\sqrt{\dfrac{\dfrac{m\ell}{12} + \dfrac{m\ell^2}{2}}{q\ell E}}$$

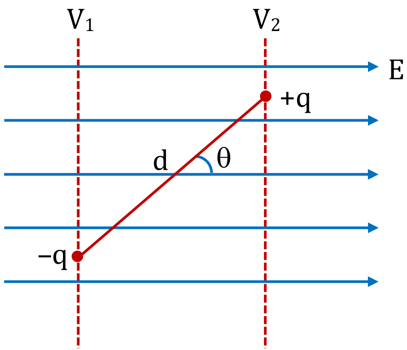
$$= 2\pi\sqrt{\dfrac{7m\ell^2}{12q\ell E}} = 2\pi\sqrt{\dfrac{7m\ell}{12qE}}$$

$$T = \pi\sqrt{\dfrac{7m\ell}{3qE}}$$

Interaction Energy of a Dipole in Electric Field:

Figure shows a dipole placed at an angle θ with the direction of electric field. In this situation we consider that the two charges of dipoles are located on two equipotential surfaces having potentials V_1 and V_2 which can be related as

$$V_1 - V_2 = Ed\cos\theta$$



The interaction energy of the dipole with electric field can be given as

$$U = -qV_1 + qV_2$$

$$\Rightarrow \quad U = q(V_2 - V_1)$$

$$\Rightarrow \quad U = -qEd\cos\theta$$

$$\Rightarrow \quad U = -pE\cos\theta$$

In vector notations we can write interaction energy of dipole in electric field from above equation as

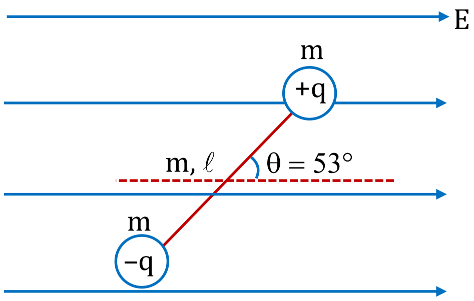
$$U = -pE\cos\theta$$

(i) At θ = 0, there is minimum of *P.E.* so it is a stable equilibrium position.

(ii) At θ = 180°, there is maxima of *P.E.* so it is a position of unstable equilibrium.

Illustration 88:

Two point masses of mass m and equal and opposite charge of magnitude q are attached on the corners of a non-conducting uniform rod of mass m and the system is released from rest in uniform electric field E as shown in figure from $\theta = 53°$



(i) Find angular acceleration of the rod just after releasing.

(ii) What will be angular velocity of the rod when it passes through stable equilibrium.

(iii) Find work required to rotate the system by 180°.

Solution:

(i) $\tau_{net} = PE \sin 53° = I\alpha$

$$\therefore \quad \alpha = \frac{(q\ell)\ E\left(\frac{4}{5}\right)}{\frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2} = \frac{48qE}{35\ m\ell}$$

(ii) From energy conservation : $K_i + U_i = K_f + U_f$

$$\therefore \quad 0 + (-PE\cos 53°) = \frac{1}{2}I\omega^2 + (-PE\cos 0°)$$

where $I = \frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{7m\ell^2}{12}$

$$\therefore \quad \frac{1}{2}I\omega^2 = PE\,(1 - 3/5) = \frac{2}{5}PE$$

$$\therefore \quad \frac{1}{2} \times \frac{7m\ell^2}{12} \times \omega^2 = \frac{2}{5}\ q\,\ell\,E$$

$$\text{or} \quad \omega = \sqrt{\frac{48qE}{35\ m\ell}}$$

(iii) $W_{ext} = U_f - U_i$

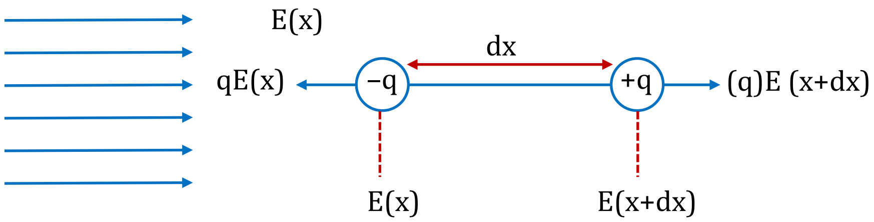
$$\therefore \quad W_{ext} = (-PE\cos(180° + 53°)) - (-PE\cos 53°)$$

$$\text{or} \quad W_{ext} = (q\,\ell)E\left(\frac{3}{5}\right) + (q\,\ell)E\left(\frac{3}{5}\right)$$

$$\Rightarrow \quad W_{ext} = \left(\frac{6}{5}\right) q\,\ell\,E$$

Dipole in non-uniform electric field:

If the dipole is placed along $\vec{E}$, (shown in figure)



Then, Net force on the dipole : $F_{net} = qE(x + dx) - qE(x)$

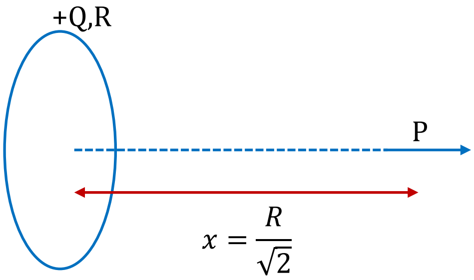
$$F_{net} = q\,\frac{E(x+dx)-E(x)}{dx}\,(dx)\,;$$

Here $q(dx) = P$

$$\therefore \quad F_{net} = P\left(\frac{dE}{dx}\right)$$

Illustration 89:

A short dipole is placed on the axis of a uniformly charged ring (total charge $-Q$, radius R) at a distance $\frac{R}{\sqrt{2}}$ from centre of ring as shown in figure. Find the Force on the dipole due to the ring.



Solution:

$$\because \quad F = P\left(\frac{dE}{dx}\right)$$

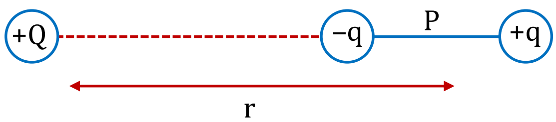
$$\therefore \quad F = P\frac{d}{dx}\left(\frac{KQx}{(R^2+x^2)^{3/2}}\right)\ ; \ \left(\text{at } x = \frac{R}{\sqrt{2}}\right)$$

Solving we get,

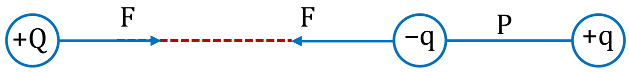
$$F = 0$$

Illustration 90:

A short dipole of dipole moment P is placed near a point charge Q as shown in figure. Find force on the dipole due to the point charge



Solution:



Force on the point charge due to the dipole $F = (Q)\ E_{\text{dipole}}$

$$F = (Q)\left(\frac{2KP}{r^3}\right) \text{ (towards right)}$$

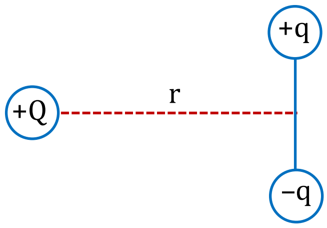
From action reaction concept, force on the dipole due to point charge will be equal to the force on charge due to dipole

$$F = \frac{2KPQ}{r^3} \text{ (towards left)}$$

is force on dipole due to point Charge.

Illustration 91:

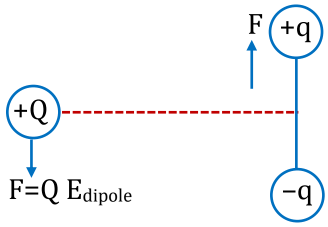
A short dipole of dipole moment P is placed near a point charge Q as shown in figure. Find force on the dipole due to the point charge.



Solution:

Force on the point charge due to dipole $F = (Q)\ (E_{\text{dipole}})$

$$F = (Q)\left(\frac{KP}{r^3}\right)(\downarrow)$$

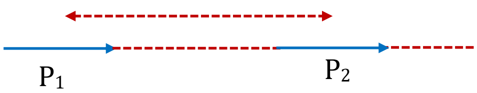


So, force on the dipole due to the point charge will also be

$F = \left(\frac{KPQ}{r^3}\right)(\uparrow)$ (but in opposite direction) as shown.

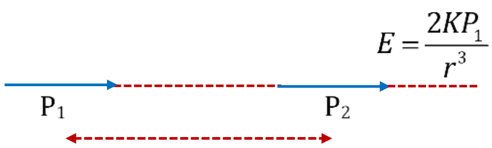
Illustration 92:

Find force on short dipole P_2 due to short dipole P_1 if they are placed at a distance r a part as shown in figure.



Solution:

Force on P_2 due to P_1



$$F_2 = (P_2)\left(\frac{dE_1}{dr}\right)$$

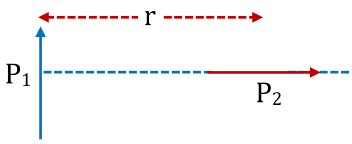
$\therefore \qquad F_2 = (P_2)\left(\dfrac{d}{dr}\left(\dfrac{2\ kP_1}{r^3}\right)\right)$

or $\qquad F_2 = -\dfrac{6KP_1P_2}{r^4}$

Here – sign indicates that this force will be attractive (opposite to r).

Illustration 93:

Find force on short dipole P_2 due to short dipole P_1 if they are placed a distance r apart as shown in figure:

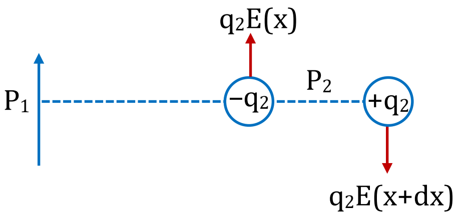


Solution:

$F_{net} = q_2\, E\,(x + dx) - q_2\, E\,(x)$

$F_{net} = q_2 \left(\dfrac{E(x+dx)-E(x)}{dx}\right) dx$

or $\qquad F_{net} = (P_2)\left(\dfrac{dE}{dx}\right)$

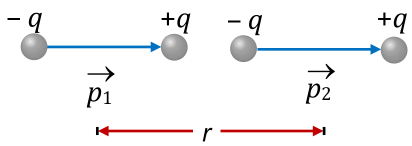
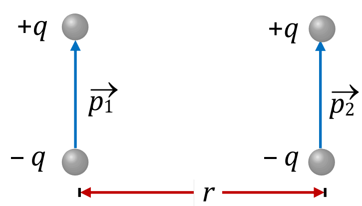
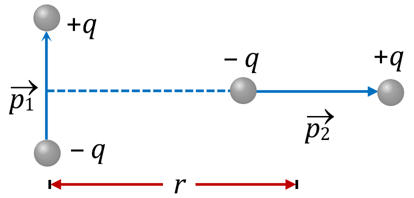


(Usually this formula is valid when the dipole is placed along $\vec{E}$. However, in this case also, we are getting the same formula)

$\therefore \qquad F_{net} = (P_2)\left(\dfrac{d}{dr}\left(\dfrac{KP_1}{r^3}\right)\right)$

$\Rightarrow \qquad F_{net} = \dfrac{3KP_1P_2}{r^4}$ (in magnitude) and (direction upwards)

Table: Dipole-dipole interaction

Relative position of dipole	Force	Potential energy
	$\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{6p_1p_2}{r^4}$ (attractive)	$\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{2p_1p_2}{r^3}$
	$\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{3p_1p_2}{r^4}$ (repulsive)	$\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{p_1p_2}{r^3}$
	$\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{3p_1p_2}{r^4}$ (perpendicular to r)	0

Conductor and its properties:

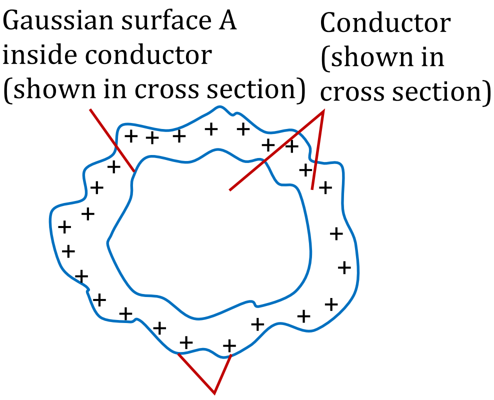
A conductor is a material in which the electrons at the outer periphery of an atom have no great affinity for any particular individual atom. They are not bound or tied to individual atoms. These so-called conduction electrons are essentially free to move readily and quickly in response to electric fields.

We described how a piece of paper can be polarized by nearby charges. The polarization is the paper's response to an externally applied electric field. The separation of charge in the paper produces an electric field of its own. The net electric field at any point-whether inside or outside the paper is the sum of the applied fields and the field due to the separated charges in the paper.

How much charge separation occurs depends on both the strength of the applied field and properties of the atoms and molecules that make up the paper. Some materials are more easily polarized than others. The most easily polarized materials are conductors because they contain highly mobile charges that can move freely through the entire volume of the material.

In this article we restrict our attention to a conductor in which the mobile charges are at rest in equilibrium, a situation called electrostatic equilibrium.

1. When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface, not in the interior of the material. (By excess we mean charges other that the ions and free electrons that make up the neutral conductor.) In an electrostatic situation the electric field $\vec{E}$ at every point in the interior of a conducting material is zero. If $\vec{E}$ were not zero, the excess charges would move. Suppose we construct a Gaussian surface inside the conductor, such as surface A in Figure. Because $\vec{E}$ =0 everywhere on this surface, Gauss's law requires that the net charge inside the surface is zero.

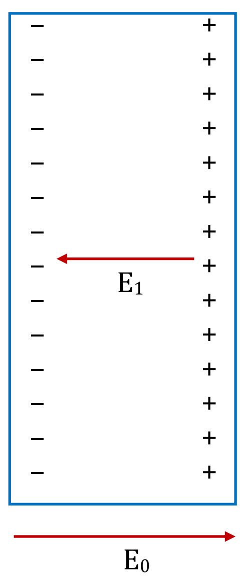


Now imagine the surface encloses a region so small region that we may consider it as a point P. Then the charge at that point must be zero. We can do this anywhere inside the conductor, so there can be no excess charge at any point within a solid conductor any excess charge must reside on the conductor's surface.

2. ***E* = 0 inside a conductor :** Because if there were any field, those free charges would move, and it wouldn't be electrostatics any more. When you put a conductor into an external electric field $\vec{E}_0$.

Due to force of electric field, field will drive any free positive charges to the right, and negative ones to the left. Note that only the negative charges-electrons move but when they shift the right side is left with a net positive charge. When they come to the edge of the material, the charges pile up: plus on the right side, minus on the left.

Now, these induced charges produce a field of their own, $\vec{E}_1$, which is in the opposite direction to $\vec{E}_0$. It means that the field of the induced charges tends to cancel off the original field. Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero. Outside the conductor the field is not zero, for here $\vec{E}_0$ and $\vec{E}_1$ do not cancel. The whole process is practically instantaneous, it takes place with speed of light.



3. Surface of a conductor (in static equilibrium) is always an equipotential surface irrespective of the charge on the surface or of nearby charges.

A conductor is an equipotential. If *a* and *b* are any two points within (or at the surface of) a given conductor.

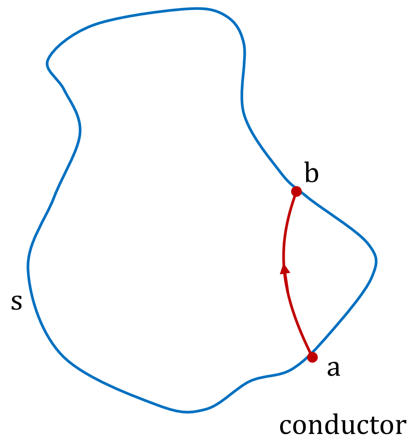
$$V_b - V_a = -\int_a^b \vec{E}.d\vec{\ell} = 0 \text{ and hence } V_a = V_b$$

Concept: When a solid conductor in equilibrium carries a net charge, the charge resides on the conductor's outer surface, the electric field just outside the conductor is perpendicular to the surface and the field inside is zero.

Consider two points *a* and *b* on the surface of a charged conductor. Along a surface path connecting these points, $\vec{E}$ is always perpendicular to the displacement $d\vec{\ell}$. Therefore $\vec{E}.d\vec{\ell} = 0$. Thus, we conclude that the potential difference between *a* and *b* is necessarily zero.

$$V_b - V_a = -\int_a^b \vec{E}.d\vec{\ell} = 0$$

The surface of any charged conductor in electrostatic equilibrium is an equipotential surface. Because the electric field is zero inside the conductor, the electric potential is constant everywhere inside the conductor and equal to its value at the surface.



Because of the constant value of the potential, no work is required to move a test charge from the interior of a charged conductor to its surface.

The potential everywhere inside the conductor, including the surface, has the same value, which may or may not be zero, depending on where the zero of potential is defined.

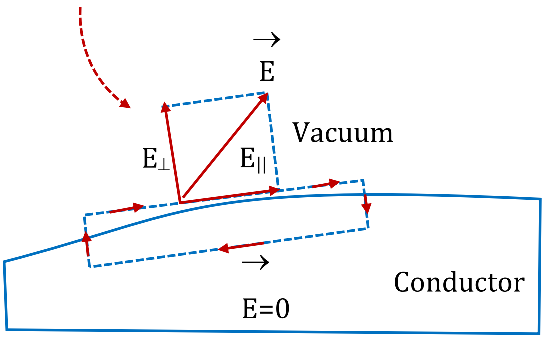
Consider the conductor shown if figure with surface *S*, and consider two points on the surface, *a* and *b*. We can use equation along any path leading from *a* to *b* to obtain $\Delta V = V_b - V_a$, including the path shown through the conductor.

For the path chosen, which is fully in the conductor. *E* is zero everywhere along the path. Therefore, $\Delta V = 0 \rightarrow V_b = V_a$. Since this is true for all points *a* and *b* on the surface, the surface must be an equipotential. (Indeed, the whole conductor is an equipotential by the same argument.)

4. Since the electric field $\vec{E}$ is always perpendicular to an equipotential surface. When all charges are at rest, the electric field just outside a conductor must be perpendicular to the surface at every point. We know that $\vec{E}$ = 0 everywhere inside the conductor, otherwise, charges would move. At any point just inside the surface the component of $\vec{E}$ tangent to the surface is zero. The tangential component of $\vec{E}$ is also zero just outside the surface a charge could move around a rectangular path partly inside and partly outside and return to its starting point with a net amount of work having been done on it. This would violate the conservative nature of electrostatic fields, so the tangential component of $\vec{E}$ just outside the surface must be zero at every point on the surface. Thus $\vec{E}$ perpendicular to the surface at each point, proving our statement.

An Impossible electric field

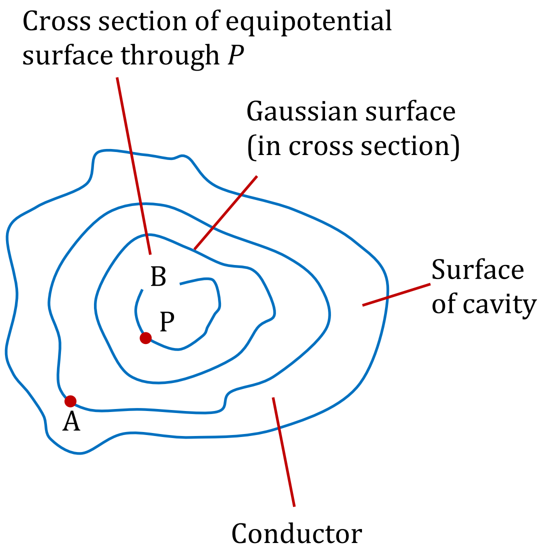
If the electric field just outside a conductor had a tangential component $E_{||}$, a charge could move in a loop with net work done.



5. In an electrostatic situation if a conductor contains a cavity and if no charge is present inside the cavity, then there can be no net charge anywhere on the surface of the cavity. Every point in the cavity is at the same potential. In figure the conducting surface *A* of the cavity is an equipotential surface. Suppose point *P* in the cavity is at a different potential, then we can construct a different equipotential surface *B* including point *P*.

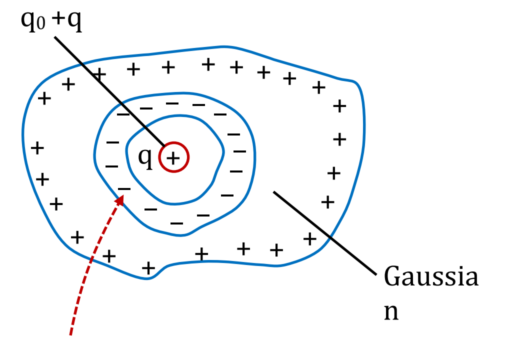
Now consider a Gaussian surface, shown in figure between the two equipotential surfaces. The field at every point between the equipotential is from *A* toward *B*, or from *B* toward *A*, depending on which equipotential surface is at higher potential. In either case the flux through this Gaussian surface is certainly not zero. Then in accordance with Gauss's law the charge enclosed by the Gaussian surface cannot be zero. This contradicts our initial assumption that there is no charge in the cavity. So, the potential at *P* must be same as that the cavity wall.

The entire region of the cavity must therefore be at the same potential. But for this to be true, the electric field inside the cavity must be zero everywhere.

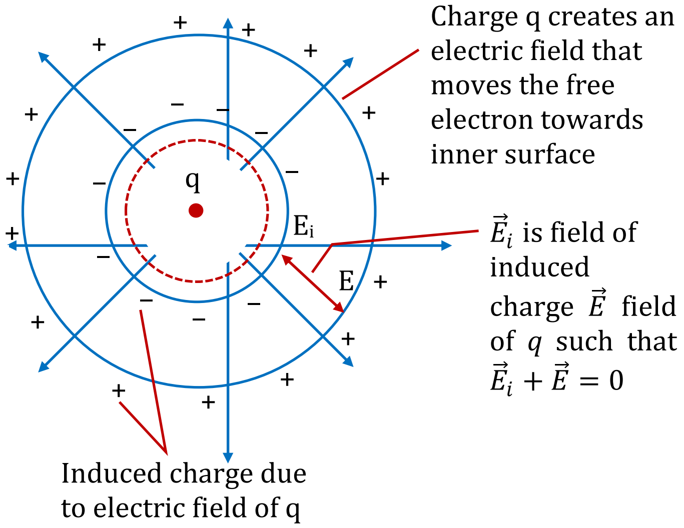


6. Suppose we place a small body with a charge *q* inside a cavity within a conductor. Consider a Gaussian surface in the material of conductor the conductor is uncharged and is insulated from the charge *q*. Again $\vec{E}$ =0 everywhere on surface *A*, so according to Gauss's law the total charge inside this surface must be zero. Therefore, there must be a charge $-q$ distributed on the surface of the cavity. This charge appears due to induction of charge inside cavity. The total charge on the conductor must remain zero, so a charge $+q$ must appear either on its outer surface or inside the material. But in an electrostatic situation there can't be any excess charge within the material of a conductor. So, we conclude that the charge $+q$ must appear on the outer surface. By the same reasoning, if the conductor originally had a charge q_c, then the total charge on the outer surface must be $q_c + q$ after the charge *q* is inserted into the cavity.

An isolated charge q placed inside



For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity



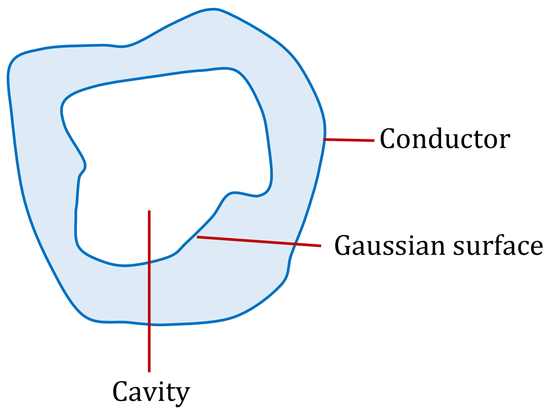
Concept: In an electrostatic situation, the electric field inside a conductor is zero. This means that the interior of a conductor is an example of an equipotential volume.

The same is true if the conductor is hollow and there are no charge inside the hollow. To show this, imagine a Gaussian surface within the conductor surrounding the cavity, as shown in figure.

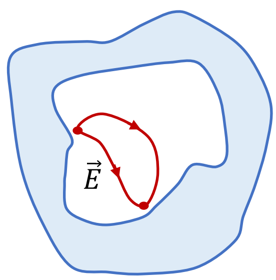
Since the static electric field is zero within the conductor, the left-hand side of Gauss's law is zero, implying there is zero.

Total charge within the Gaussian surface. We imagine some positive charge to be on one side of the inside surface of the hollow conductor and some negative charge on the other, as shown in figure. So, the total charge is zero, consistent with the prediction of Gauss's law. If such charge separation exists on the inside surface of the cavity, there will be electric field lines within the hollow from the positive to the negative charges.

If we use equation $dV = -\vec{E}.d\vec{r}$ and integrate the electric field along one of these field lines from a positive charge to a negative charge, the result will not be zero since $d\vec{r}$ is parallel to $\vec{E}$ over the whole path. This implies that there is a potential difference.



Gaussian surface surrounding a hollow part in a conductor

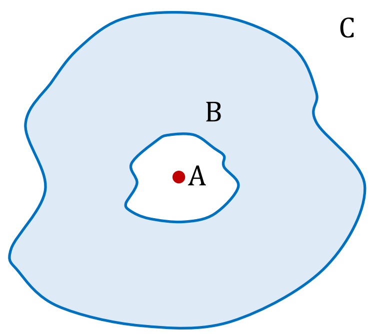


Electric field lines from a positive to a negative charge

Concept: The charge on a conductor flows to the surface, because of their mutual repulsion, the charges naturally spread out as much as possible, but this is true regardless of the size or shape of the conductor. The charge on a conductor will seek the configuration that minimizes its potential energy. The electrostatic energy of a solid object (with specified shape and total charge) is a minimum when that charge is spread over the surface.

For instance, the energy of a sphere is $\left(\frac{1}{8\pi\varepsilon_0}\right)\left(\frac{q^2}{R}\right)$ if the charge is uniformly distributed over the surface, but it is greater, $\left(\frac{3}{20\pi\varepsilon_0}\right)\left(\frac{q^2}{R}\right)$, if the charge is uniformly distributed throughout the volume.

Concept: The electric field due to charges on the outer surface of conductor is zero for all the points inside the conductor separately.



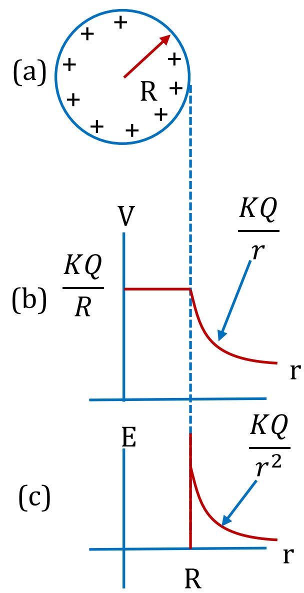
Consider a charged conductor having charge $+q_1$ and Q is kept inside the cavity. Lets call charge Q inside cavity as A, the induced charge $-Q$ on the surface of the cavity as B and the charge on the surface of the conductor $Q+q_1$ as C.

Now field inside the conductor is, $\vec{E}_{net}$ and $\vec{E}_A, \vec{E}_B$ and $\vec{E}_C$ are fields due to charge A, B and C inside the conductor. and $\vec{E}_{net} = \vec{E}_A + \vec{E}_B + \vec{E}_C = 0$

Now the electric field due to charges on the outer surface of conductor is zero for all the points inside the conductor separately and the $\vec{E}_B + \vec{E}_A$ is zero separately.

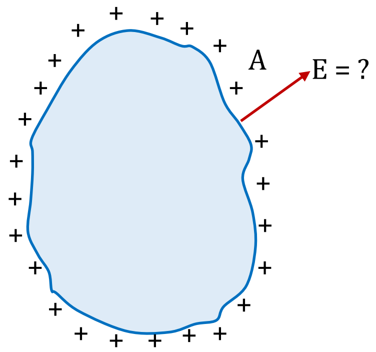
Concept: Consider a solid metal conducting sphere of radius R and total positive charge Q as shown in figure. The electric field outside the sphere is KQ/r^2 and points radially outward. Because the field outside of a spherically symmetric charge distribution is identical to that of a point charge, we expect the potential to also be that of a point charge, KQ/r. At the surface of the conducting sphere in figure the potential must be KQ/r.

Because the entire sphere must be at the same potential, the potential at any point within the sphere must also be KQ/R. figure (b) is a plot of the electric potential as a function of r, and figure (c) shows how the electric field varies with r.

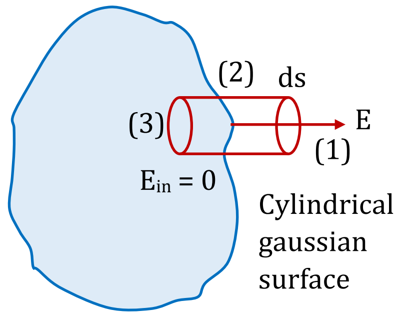


Finding field due to a conductor

Suppose we have a conductor and at any *'A'*, local surface charge density = σ. We have to find electric field just outside the conductor surface.



For this, let's consider a small cylindrical Gaussian surface, which is partly inside and partly outside the conductor surface, as shown in figure. It has a small cross section area ds and negligible height.



Applying Gauss's theorem for this surface :

$$\Phi_{net} = \frac{q_{in}}{\varepsilon_0} = \frac{\sigma ds}{\varepsilon_0}$$

Flux through surface (1): $\phi_1 = Eds$ (because $\vec{E}$ is normal to the surface of conductor)

Flux through surface (2): $\phi_2 = 0$ ($\vec{E}$ is normal to curved Gaussian surface)

Flux through surface (3): $\phi_3 = 0$ (as E inside the conductor=0)

So, $$Eds = \frac{\sigma ds}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{\sigma}{\varepsilon_0}$$

Electric field just outside the surface of conductor:

$$E = \frac{\sigma}{\varepsilon_0}$$

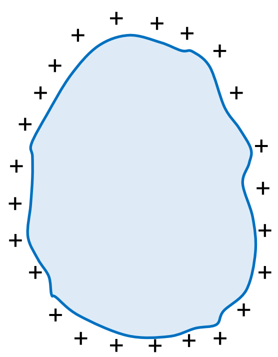
(direction will be normal to the surface)

In vector form: $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$

(Here, $\hat{n}$ = unit vector normal to the conductor surface)

Electrostatic pressure at the surface of the conductor:

Suppose a conductor is given some charge. Due to repulsion, all the charges will reach the surface of the conductor. But the charges will still repel each other. So, an outward force will be felt by each charge due to others. Due to this force, there will be some pressure at the surface, which is called electrostatic pressure.



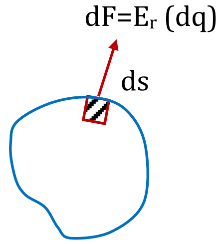
To find the electrostatic pressure, let's take a small surface element having Area *'ds'*.

Force on this element due to the remaining charges :

$$dF = \begin{pmatrix}\text{electric field at} \\ \text{that place due to} \\ \text{remainig charges}\end{pmatrix}\begin{pmatrix}\text{charge of} \\ \text{the small} \\ \text{element}\end{pmatrix}$$

Let electric field at that point due to the remaining charges = E_r

and charge of the small element = $dq = \sigma ds$



$\Rightarrow \quad dF = (E_r)\,(dq) = (E_r)\,(\sigma ds)$

So, pressure on this small element

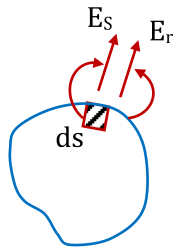
$$P = \frac{dF}{ds} = \frac{(E_r)(\sigma ds)}{ds}$$

$\Rightarrow \quad P = (E_r)\,(\sigma) \qquad ...(1)$

Now to find pressure, we have to find E_r (electric field at that position due to the remaining charges)

Suppose, Electric field due to the small element near the surface = E_s

Electric field due to the remaining part near the surface = E_r



At a point just outside the surface, electric field due to the small element (E_s) will be normally out wards, and electric field due to the remaining part (E_r) will also be normally out wards.

So Net electric field just outside the surface = $E_s + E_r$ and we have proved that

electric field just outside the conductor surface = $\dfrac{\sigma}{\varepsilon_o}$

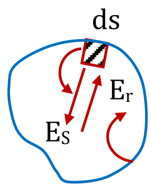
$$\Rightarrow \quad E_s + E_r = \frac{\sigma}{\varepsilon_o} \qquad ...(2)$$

Now, lets see the electric field just inside the metal surface. Here, electric field due to the remaining charges (E_r) will be in the same direction (normally outward), but the electric field due to the small element will be in opposite direction (normally inward)

So net electric field just inside the metal surface = $E_r - E_s$ and we know that electric field inside a conductor = 0

So, $\quad E_r - E_s = 0 \quad \Rightarrow \quad E_r = E_s \qquad ...(3)$

From eqn. (2) and eqn. (3), we can say that :



$$2E_r = \frac{\sigma}{\varepsilon_o} \quad \Rightarrow \quad E_r = \frac{\sigma}{2\varepsilon_o}$$

Now, we can easily find the pressure from equation (1)

$$P = (E_r)\,(\sigma) = \frac{\sigma}{2\varepsilon_o}\,(\sigma) = \frac{\sigma^2}{2\varepsilon_o}$$

So, electrostatic pressure at the surface of the conductor $P = \dfrac{\sigma^2}{2\varepsilon_o}$

where, σ = local surface charge density.

Illustration 94:

Prove that if an isolated (isolated means no charges are near the sheet) large conducting sheet is given a charge then the charge distributes equally on its two surfaces.

Solution:

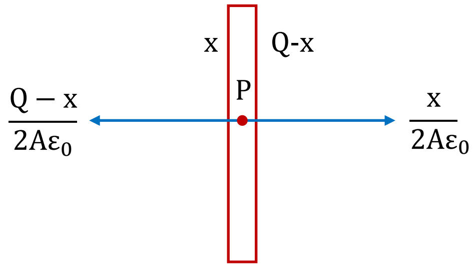
Let there is x charge on left side of sheet and $Q - x$ charge on right side of sheet.

Since, point P lies inside the conductor, so $E_P = 0$

or $\quad \dfrac{x}{2A\varepsilon_0} - \dfrac{Q-x}{2A\varepsilon_0} = 0$

$\Rightarrow \quad \dfrac{2x}{2A\varepsilon_0} = \dfrac{Q}{2A\varepsilon_0}$

$\Rightarrow \quad x = \dfrac{Q}{2}$ and $Q - x = \dfrac{Q}{2}$



So, charge is equally distributed on both sides.

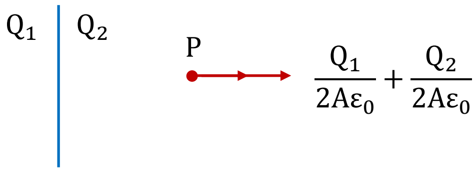
Illustration 95:

If an isolated infinite sheet contains charge Q_1 on its one surface and charge Q_2 on its other surface, then prove that electric field intensity at a point in front of sheet will be $\dfrac{Q}{2A\varepsilon_0}$, where $Q = Q_1 + Q_2$

Solution:

Electric field at point P:

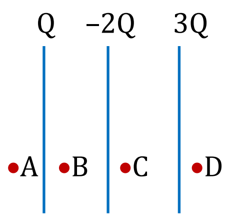
$$\vec{E} = \vec{E}_{Q_1} + \vec{E}_{Q_2} = \frac{Q_1}{2A\varepsilon_0}\hat{n} + \frac{Q_2}{2A\varepsilon_0}\hat{n} = \frac{Q_1+Q_2}{2A\varepsilon_0}\hat{n} = \frac{Q}{2A\varepsilon_0}\hat{n}$$



[This shows that the resultant field due to a sheet depends only on the total charge of the sheet and not on the distribution of charge on individual surfaces].

Illustration 96:

Three large conducting sheets placed parallel to each other at finite distance contain charges Q, $-2Q$ and $3Q$ respectively. Find electric field at points A, B, C and D.



Solution:

(i) $\quad E_A = E_Q + E_{-2Q} + E_{3Q}$. Here E_Q means electric field due to 'Q'.

$E_A = \dfrac{(Q-2Q+3Q)}{2A\varepsilon_0} = \dfrac{2Q}{2A\varepsilon_0} = \dfrac{Q}{A\varepsilon_0}$, towards left

(ii) $\quad E_B = \dfrac{Q-(-2Q+3Q)}{2A\varepsilon_0} = 0$

(iii) $\quad E_C = \dfrac{(Q-2Q)-(3Q)}{2A\varepsilon_0} = \dfrac{-4Q}{2A\varepsilon_0} = \dfrac{-2Q}{A\varepsilon_0}$, towards right $\quad \Rightarrow \dfrac{2Q}{A\varepsilon_0}$ towards left

(iv) $\quad E_D = \dfrac{(Q-2Q+3Q)}{2A\varepsilon_0} = \dfrac{2Q}{2A\varepsilon_0} = \dfrac{Q}{A\varepsilon_0}$, towards right

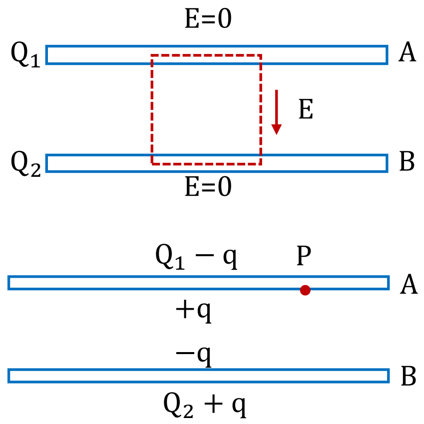
Illustration 97:

Two conducting plates A and B are placed parallel to each other. A is given a charge Q_1 and B a charge Q_2. Prove that the charges on the inner facing surfaces are of equal magnitude and opposite sign.

Solution:

Consider a Gaussian surface as shown in figure. Two faces of this closed surface lie completely inside the conductor where the electric field is zero. The flux through these faces is, therefore, zero. The other parts of the closed surface which are outside the conductor are parallel to the electric field and hence the flux on these parts is also zero. The total flux of the electric field through the closed surface is, therefore zero. From Gauss's law, the total charge inside this closed surface should be zero. The charge on the inner surface of *A* should be equal and opposite to that on the inner surface of *B*.

The distribution should be like the one shown in figure. To find the value of *q*, consider the field at a point *P* inside the plate *A*. Suppose, the surface area of the plate (one side) is *A*. Using the equation, $E = \sigma / (2\varepsilon_0)$, the electric field at *P*.



Due to the charge $Q_1 - q = \dfrac{Q_1 - q}{2A\varepsilon_0}$ (downward)

Due to the charge $+q = \dfrac{q}{2A\varepsilon_0}$ (upward),

Due to the charge $-q = \dfrac{q}{2A\varepsilon_0}$ (downward),

and due to the charge $Q_2 + q = \dfrac{Q_2+q}{2A\varepsilon_0}$ (upward).

The net electric field at *P* due to all the four charged surfaces is (in the downward direction)

$$E_p = \frac{Q_1-q}{2A\varepsilon_0} - \frac{q}{2A\varepsilon_0} + \frac{q}{2A\varepsilon_0} - \frac{Q_2+q}{2A\varepsilon_0}$$

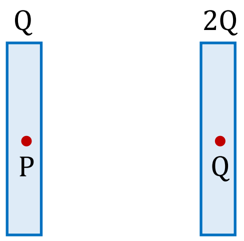
As the point *P* is inside the conductor, this field should be zero. Hence,

$$Q_1 - q - q + q - Q_2 - q = 0 \qquad \text{or} \qquad q = \frac{Q_1 - Q_2}{2}$$

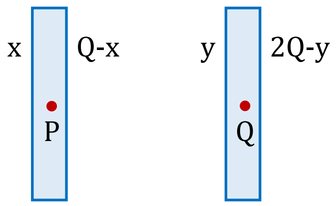
This result is a special case of the following result. When charged conducting plates are placed parallel to each other, the two outermost surfaces get equal charges and the facing surfaces get equal and opposite charges.

Illustration 98:

Two large parallel conducting sheets (placed at finite distance) are given charges *Q* and 2*Q* respectively. Find out charges appearing on all the surfaces.



Solution:



Let there is x amount of charge on left side of first plate. So, on its right side charge will be $Q–x$. Similarly, for second plate there is y charge on left side and $2Q - y$ charge is on right side,

$E_p = 0$ (By property of conductor)

$$\Rightarrow \quad \frac{x}{2A\varepsilon_o} - \left\{\frac{Q-x}{2A\varepsilon_o} + \frac{y}{2A\varepsilon_o} + \frac{2Q-y}{2A\varepsilon_o}\right\} = 0$$

We can also say that charge on left side of P = charge on right side of P

$$x = Q - x + y + 2Q - y$$

$$\Rightarrow \quad x = \frac{3Q}{2},\ Q - x = \frac{-Q}{2}$$

Similarly, for point $Q : x + Q - x + y = 2Q - y$

$$\Rightarrow \quad y = Q/2,\ 2Q - y = 3Q/2$$

So, final charge distribution of plates is

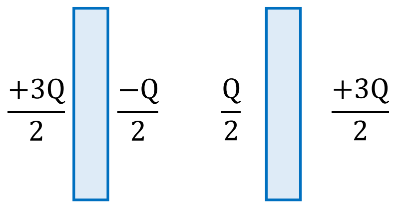
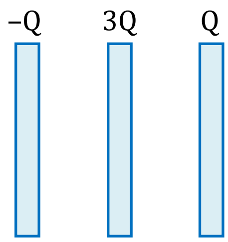


Illustration 99:

Figure shows three large metallic plates with charges $- Q$, $3Q$ and Q respectively. Determine the final charges on all the surfaces.



Solution:

We assume that charge on surface 2 is x. Following conservation of charge, we see that surfaces 1 has charge $(- Q - x)$. The electric field inside the metal plate is zero. So, field at P is zero.

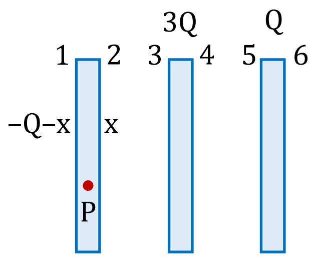
Resultant field at P

$$E_P = 0$$

$$\Rightarrow \quad \frac{-Q-x}{2A\varepsilon_0} = \frac{x+3Q+Q}{2A\varepsilon_0}$$

or $\quad - Q - x = x + 4Q$

$$\therefore \quad x = \frac{-5Q}{2}$$



Note:

We see that charges on the facing surfaces of the plates are of equal magnitude and opposite sign. This can be in general proved by Gauss theorem also. Remember this, it is important result. Thus the final charge distribution on all the surfaces is as shown in figure :

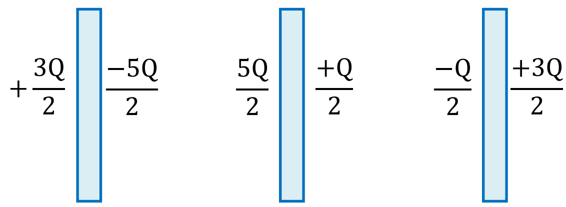
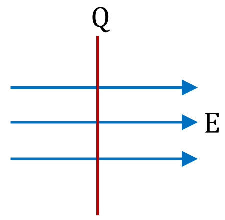


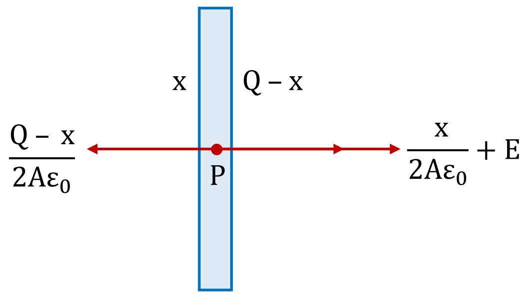
Illustration 100:

An isolated conducting sheet of area A and carrying a charge Q is placed in a uniform electric field E, such that electric field is perpendicular to sheet and covers all the sheet. Find out charges appearing on its two surfaces.



Solution:

Let there is x charge on left side of plate and $Q - x$ charge on right side of plate



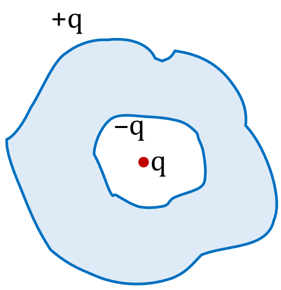
$\because \quad E_P = 0$

$$\therefore \quad \frac{x}{2A\varepsilon_0} + E = \frac{Q-x}{2A\varepsilon_0} \quad \text{or} \quad \frac{x}{A\varepsilon_0} = \frac{Q}{2A\varepsilon_0} - E$$

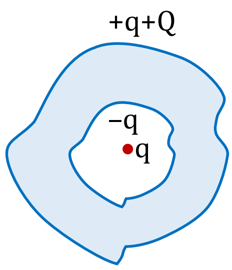
$$\therefore \quad x = \frac{Q}{2} - EA\varepsilon_0 \quad \text{and} \quad Q - x = \frac{Q}{2} + EA\varepsilon_0$$

Some other important results for a closed conductor:

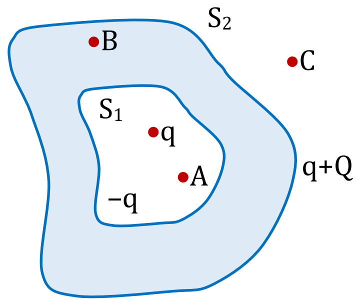
(i) If a charge q is kept in the cavity then $-q$ will be induced on the inner surface and $+q$ will be induced on the outer surface of the conductor (it can be proved using Gauss theorem)



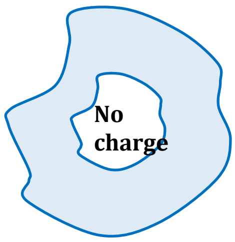
(ii) If a charge q is kept inside the cavity of a conductor and conductor is given a charge Q then $-q$ charge will be induced on inner surface and total charge on the outer surface will be $q + Q$. (it can be proved using Gauss theorem)



(iii) Resultant field, due to q (which is inside the cavity) and induced charge on S_1, at any point outside S_1 (like B, C) is zero. Resultant field due to $q + Q$ on S_2 and any other charge outside S_2, at any point inside of surface S_2 (like A, B) is zero

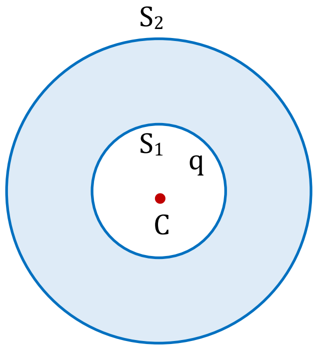


(iv) Resultant field in a charge free cavity in a closed conductor is zero. There can be charges outside the conductor and on the surface also. Then also, this result is true. No charge will be induced on the inner most surface of the conductor.



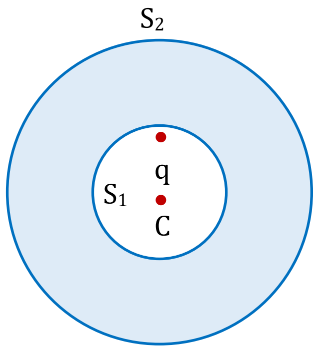
(v) Charge distribution for different types of cavities in conductors

(A)



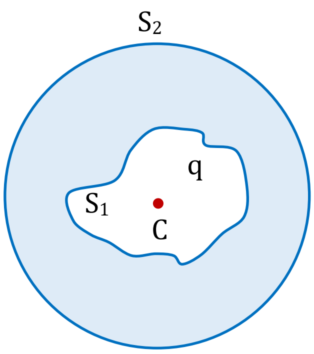
Charge is at the common centre
(S_1, S_2 $\rightarrow$spherical)

(B)



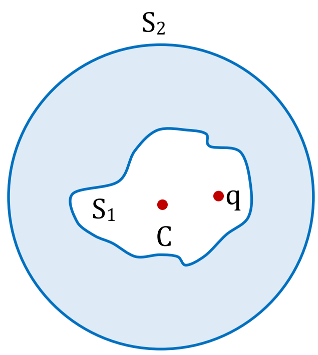
Charge is not at the common centre
(S_1, S_2 $\rightarrow$spherical)

(C)



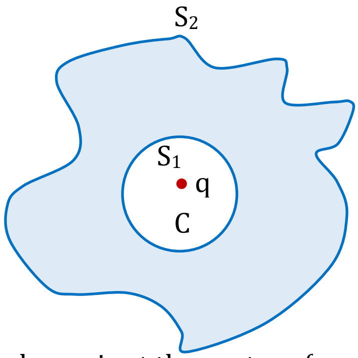
charge is at the centre of S_2
($S_2 \rightarrow$ spherical)

(D)



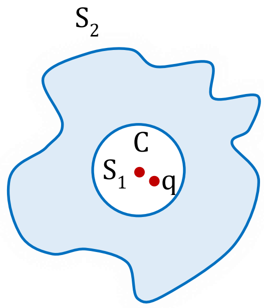
charge is not at the centre of S_2
($S_2 \rightarrow$ spherical)

(E)



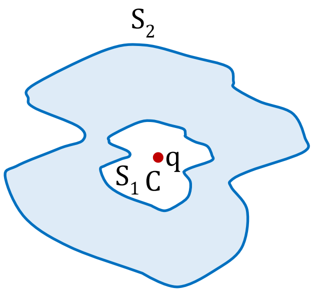
charge is at the centre of S_1 (Spherical)

(F)



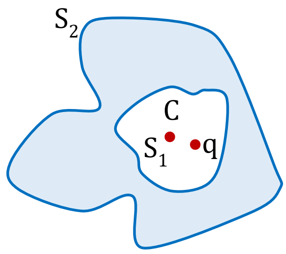
charge not at the centre of S_1 (Spherical)

(G)



charge is at the geometrical centre

(H)



charge is not at the geometrical centre

Using the result that $\vec{E}_{res}$ in the conducting material should be zero and using result (iii) we can show that

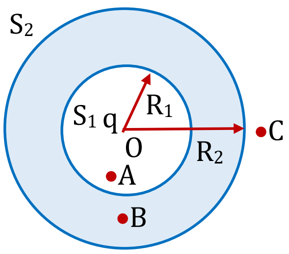
Case	A	B	C	D	E	F	G	H
S_1	Uniform	Nonuniform	Nonuniform	Nonuniform	Uniform	Nonuniform	Uniform	Nonuniform
S_2	Uniform	Uniform	Uniform	Uniform	Nonuniform	Nonuniform	Nonuniform	Nonuniform

Note:

In all cases, charge on inner surface $S_1 = -q$ and on outer surface $S_2 = q$. The distribution of charge on 'S_1' will not change even if some charges are kept outside the conductor (i.e. outside the surface S_2). But the charge distribution on 'S_2' may change if some charges(s) is/are kept outside the conductor

Illustration 101:

An uncharged conductor of inner radius R_1 and outer radius R_2 contains a point charge q at the centre as shown in figure.



(i) Find $\vec{E}$ and V at points A, B and C

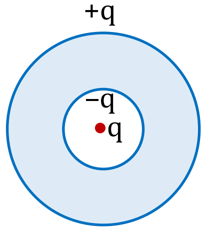
(ii) If a point charge Q is kept outside the sphere at a distance 'r' ($>>R_2$) from centre, then find out resultant force on charge Q and charge q.

Solution:

(i) At point A :

$V_A = \frac{Kq}{OA} + \frac{Kq}{R_2} + \frac{K(-q)}{R_1}$

$\vec{E}_A = \frac{Kq}{OA^3}\overrightarrow{OA}$



Note:

Electric field at 'A' due to $-q$ of S_1 and $+q$ of S_2 is zero individually because they are uniformly distributed

At point B : $V_B = \frac{Kq}{OB} + \frac{K(-q)}{OB} + \frac{Kq}{R_2} = \frac{Kq}{R_2}$, $E_B = 0$

At point C : $V_C = \frac{Kq}{OC}$, $\vec{E}_C = \frac{Kq}{OC^3}\overrightarrow{OC}$

(ii) Force on point charge Q :

Note: Here, force on 'Q' will be only due to 'q' of S_2 [see result (iii)]

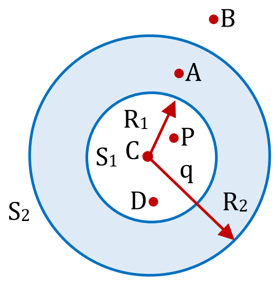
$\vec{F}_Q = \frac{KqQ}{r^2}\hat{r}$ (r = distance of 'Q' from centre 'O')

Force on point charge q:

$\vec{F}_q = 0$ (using result (iii) & charge on S_1 uniform)

Illustration 102:

An uncharged conductor of inner radius R_1 and outer radius R_2 contains a point charge q placed at point P (not at the centre) as shown in figure. Find out the following :



(i) V_C (ii) V_A

(iii) V_B (iv) E_A

(v) E_B (vi) Force on charge Q, if it is placed at B.

Solution:

(i) $V_C = \frac{Kq}{CP} + \frac{K(-q)}{R_1} + \frac{Kq}{R_2}$

Note:

$-q$ on S_1 is non-uniformly distributed. Still it produces potential $\frac{K(-q)}{R_1}$ at 'C' because 'C' is at distance 'R_1' from each point of 'S_1'.

(ii) $V_A = \frac{Kq}{R_2}$

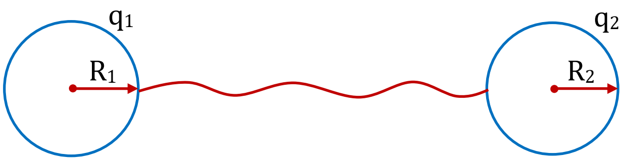
(iii) $V_B = \dfrac{Kq}{CB}$

(iv) $E_A = 0$ (point is inside metallic conductor)

(v) $E_B = \dfrac{Kq}{CB^2}\hat{CB}$

(vi) $F_Q = \dfrac{KQq}{CB^2}\hat{CB}$

(vi) Sharing of charges : Two conducting hollow spherical shells of radii R_1 and R_2 having charges Q_1 and Q_2 respectively and separated by large distance & are joined by a conducting wire



Let final charges on spheres are q_1 and q_2 respectively. Potential on both spherical shell becomes equal after joining. Therefore,

$$\frac{Kq_1}{R_1} = \frac{Kq_2}{R_2} \Rightarrow \frac{q_1}{q_2} = \frac{R_1}{R_2} \quad ...(i)$$

and, $q_1 + q_2 = Q_1 + Q_2$ ...(ii)

From (i) and (ii) : $q_1 = \dfrac{(Q_1+Q_2)R_1}{R_1+R_2}$; $q_2 = \dfrac{(Q_1+Q_2)R_2}{R_1+R_2}$

Ratio of charges : $\dfrac{q_1}{q_2} = \dfrac{R_1}{R_2} \Rightarrow \dfrac{\sigma_1 4\pi R_1^2}{\sigma_2 4\pi R_2^2} = \dfrac{R_1}{R_2}$

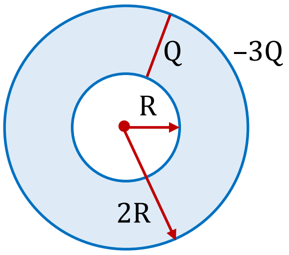
$\therefore$ Ratio of surface charge densities : $\dfrac{\sigma_1}{\sigma_2} = \dfrac{R_2}{R_1}$

Ratio of final charges : $\dfrac{q_1}{q_2} = \dfrac{R_1}{R_2}$

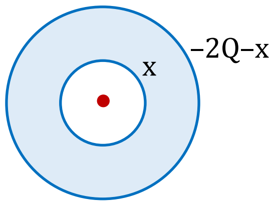
Ratio of final surface charge densities : $\dfrac{\sigma_1}{\sigma_2} = \dfrac{R_2}{R_1}$

Illustration 103:

The two conducting spherical shells are joined by a conducting wire which is cut after some time when charge stops flowing. Find out the charge on each sphere after that.



Solution:



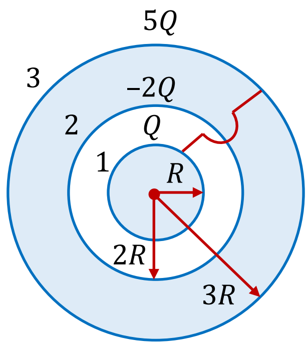
After cutting the wire, the potential of both the shells is equal

Thus, potential of inner shell, $V_{in} = \dfrac{Kx}{R} + \dfrac{K(-2Q-x)}{2R} = \dfrac{k(x-2Q)}{2R}$

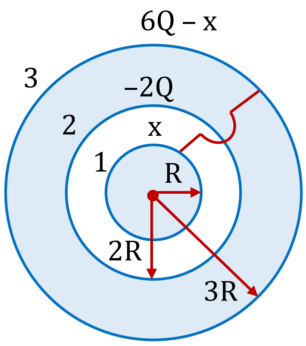
and potential of outer shell, $V_{out} = \dfrac{Kx}{2R} + \dfrac{K(-2Q-x)}{2R} = \dfrac{-KQ}{R}$

Illustration 104:

Find charge on each spherical shell after joining the inner most shell and outer most shell by a conducting wire. Also find charges on each surface.



Solution:



Let the charge on the innermost sphere be x.

Finally, potential of shell 1 = Potential of shell 3

$\therefore \quad \dfrac{Kx}{R} + \dfrac{K(-2Q)}{2R} + \dfrac{K(6Q-x)}{3R} = \dfrac{Kx}{3R} + \dfrac{K(-2Q)}{3R} + \dfrac{K(6Q-x)}{3R}$

$3x - 3Q + 6Q - x = 4Q$

$2x = Q$

$x = \dfrac{Q}{2}$

$\therefore$ Charge on innermost shell = $\dfrac{Q}{2}$

and Charge on outermost shell = $\dfrac{11Q}{2}$

Charge on middle shell = $-2Q$

$\therefore$ Final charge distribution is as shown in figure.

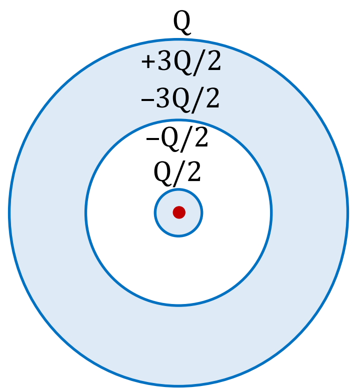
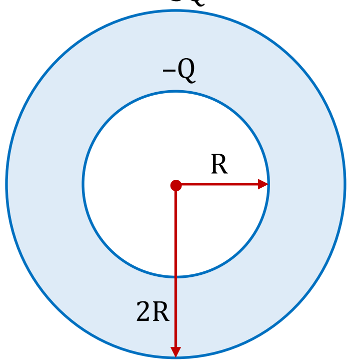
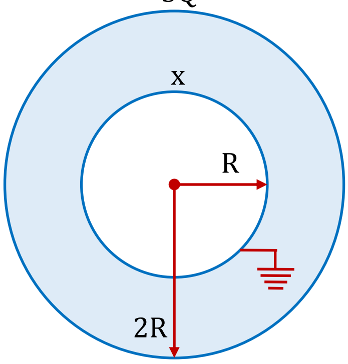


Illustration 105:

Two conducting hollow spherical shells of radii R and $2R$ carry charges $-Q$ and $3Q$ respectively. How much charge will flow into the earth if inner shell is grounded?



Solution:



When inner shell is grounded to the Earth then the potential of inner shell will become zero because potential of the Earth is taken to be zero.

$$\frac{Kx}{R} + \frac{K3Q}{2R} = 0$$

or $\quad x = \dfrac{-3Q}{2}$ (the charge that has appeared on inner shells after grounding)

$\Rightarrow \quad \dfrac{-3Q}{2} - (-Q) = \dfrac{-Q}{2}$ [hence, charge flown into the Earth = $\dfrac{Q}{2}$]

Illustration 106:

An isolated conducting sphere of charge Q and radius R is connected to a similar uncharged sphere (kept at a large distance) by using a high resistance wire. After a long time, what is the amount of heat loss?

Solution:

When two conducting spheres of equal radii are connected, charge is equally distributed on them (Result VI). So, we can say that heat loss of system :

$$\Delta H = U_i - U_f$$

$$= \left(\frac{Q^2}{8\pi\varepsilon_0 R} - 0\right) - \left(\frac{Q^2/4}{8\pi\varepsilon_0 R} + \frac{Q^2/4}{8\pi\varepsilon_0 R}\right) = \frac{Q^2}{16\pi\varepsilon_0 R}$$

GRAVITATION

The Discovery of the Law of Gravitation

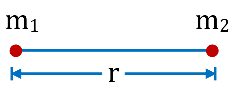
The way the law of universal gravitation was discovered is often considered the paradigm of modern scientific technique. The major steps involved were

- The hypothesis about planetary motion given by **Nicolaus Copernicus (1473–1543)**.
- The careful experimental measurements of the positions of the planets and the Sun by **Tycho Brahe (1546–1601)**.
- Analysis of the data and the formulation of empirical laws by **Johannes Kepler (1571–1630)**.
- The development of a general theory by **Isaac Newton (1642–1727)**.

Universal Law of Gravitation: Newton's Law

According to this law "Each particle attracts every other particle. The force of attraction between them is directly proportional to the product of their masses and inversely proportional to square of the distance between them".

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{or} \quad F = G\frac{m_1 m_2}{r^2}$$

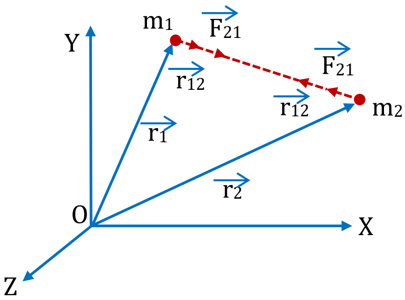


where $G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$ is the universal gravitational constant.

Dimensional Formula of G :

$$G = \frac{Fr^2}{m_1 m_2} = \frac{[MLT^{-2}][L^2]}{[M^2]} = [M^{-1}L^3T^{-2}]$$

Vector Form of Newton's Law of Gravitation



Let $\vec{r}_{12}$ = Displacement vector from m_1 to m_2

$\vec{r}_{21}$ = Displacement vector from m_2 to m_1

$\vec{F}_{21}$ = Gravitational force exerted on m_2 by m_1

$\vec{F}_{12}$ = Gravitational force exerted on m_1 by m_2

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{21}^2}\hat{r}_{21} = -\frac{Gm_1m_2}{r_{21}^3}\vec{r}_{21}$$

Negative sign shows that :

(i) The direction of $\vec{F}_{12}$ is opposite to that $\vec{r}_{21}$

(ii) The gravitational force is attractive in nature

Similarly, $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$ or $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12} \Rightarrow \vec{F}_{12} = -\vec{F}_{21}$

The gravitational force between two bodies are equal in magnitude and opposite in direction.

Important characteristics of gravitational force:

(i) Gravitational force between two bodies form an action and reaction pair i.e. the forces are equal in magnitude but opposite in direction.

(ii) Gravitational force is a central force i.e. it acts along the line joining the centers of the two interacting bodies.

(iii) Gravitational force between two bodies is independent of the nature of the medium, in which they lie.

(iv) Gravitational force between two bodies does not depend upon the presence of other bodies.

(v) Gravitational force is negligible in case of light bodies but becomes appreciable in case of massive bodies like stars and planets.

(vi) Gravitational force is long range-force i.e., gravitational force between two bodies is effective even if their separation is very large.

For Illustration gravitational force between the Sun and the Earth is of the order of 10^{27} N although distance between them is 1.5×10^7 *km.*

Illustration 107:

Two particles of masses 1 *kg* and 2 *kg* are placed at a separation of 50 *cm*. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial acceleration of heavier particle.

Solution:

Force exerted by one particle on another, $F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{(0.5)^2} = 5.3\times10^{-10}\,N$

Acceleration of heavier particle = $\frac{F}{m_2} = \frac{5.3\times10^{-10}}{2} = 2.65\times10^{-10}\,ms^{-2}$

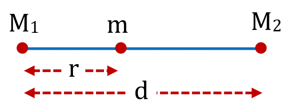
This Illustration shows that gravitation is very weak but only this force keep bind our solar system and also this universe, all galaxies and other interstellar system.

Illustration 108:

Two stationary particles of masses M_1 and M_2 are at a distance '*d*' apart. A third particle lying on the line joining the particles, experiences no resultant gravitational forces. What is the distance of this particle from M_1.

Solution:

The force on m towards M_1 is $F_1 = \frac{GM_1m}{r^2}$



The force on m towards M_2 is $F_2 = \frac{GM_2m}{(d-r)^2}$

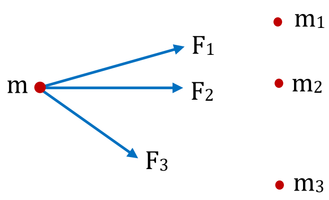
According to question, net force on m is zero i.e. $F_1 = F_2$

$$\Rightarrow \quad \frac{GM_1m}{r^2}=\frac{GM_2m}{(d-r)^2} \quad \Rightarrow \quad \left(\frac{d-r}{r}\right)^2=\frac{M_2}{M_1}$$

$$\Rightarrow \quad \frac{d}{r}-1=\frac{\sqrt{M_2}}{\sqrt{M_1}} \quad \Rightarrow \quad r = d\left[\frac{\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}}\right]$$

Principle of superposition

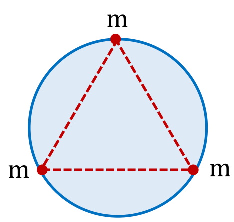
The force exerted by a particle on other particle remains unaffected by the presence of other nearby particles in space. Total force acting on a particle is the vector sum of all the forces acted upon by the individual masses when they are taken alone.



$$\vec{F}=\vec{F}_1+\vec{F}_2+\vec{F}_3+.......$$

Illustration 109:

Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side $'a'$. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining their original separation $'a'$. Determine the initial velocity that should be given to each particle and time period of circular motion.



Solution:

The resultant force on particle at A due to other two particles is

$$F_A=\sqrt{F_{AB}^2+F_{AC}^2+2F_{AB}F_{AC}\cos 60^\circ}=\sqrt{3}\frac{Gm^2}{a^2} \qquad ...(i) \qquad \left[\because F_{AB}=F_{AC}=\frac{Gm^2}{a^2}\right]$$

Radius of the circle $r=\frac{2}{3}\times a\sin 60^\circ=\frac{a}{\sqrt{3}}$

If each particle is given a tangential velocity v,

so that F acts as the centripetal force,

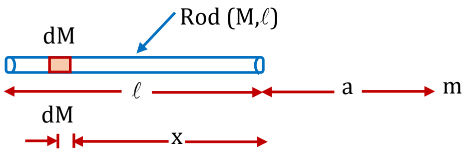
Now $\quad \frac{mv^2}{r}=\sqrt{3}\frac{mv^2}{a} \qquad ...(ii)$

From (i) and (ii) $\sqrt{3}\frac{mv^2}{a}=\frac{Gm^2\sqrt{3}}{a^2} \quad \Rightarrow \quad v=\sqrt{\frac{Gm}{a}}$

Time period, $T=\frac{2\pi r}{v}=\frac{2\pi a}{\sqrt{3}}\sqrt{\frac{a}{Gm}}=2\pi\sqrt{\frac{a^3}{3Gm}}$

Illustration 110:

Find gravitational force exerted by point mass *'m'* on uniform rod (mass *'M'* and length *'ℓ'*).



Solution:

dF = force on element in horizontal direction = $\dfrac{G.dM.m}{(x+a)^2}$

where $dM = \dfrac{M}{\ell}dx.$

$$\therefore \qquad F=\int dF=\int_0^{\ell}\frac{G.Mmdx}{\ell(x+a)^2}=\frac{G.Mm}{\ell}\int_0^{\ell}\frac{dx}{(x+a)^2}=\frac{G.Mm}{\ell}\left[-\frac{1}{(\ell+a)}+\frac{1}{a}\right]=\frac{GMm}{(\ell+a)a}$$

Gravitational Field

The space surrounding the body within which its gravitational force of attraction is experienced by other bodies is called gravitational field. Gravitational field is very similar to electric field in electrostatics where charge *'q'* is replaced by mass *'m'* and electric constant *'K'* is replaced by gravitational constant *'G'*. The intensity of gravitational field at a point is defined as the force experienced by a unit mass placed at that point.

$$\vec{E}=\frac{\vec{F}}{m}$$

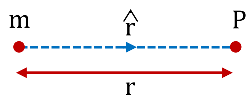
The unit of the intensity of gravitational field is $N\,kg^{-1}$.

Intensity of Gravitational Field due to Point Mass

The force due to mass m on test mass m_0 placed at point P is given by :

$$F=\frac{GMm_0}{r^2}$$

Hence $E=\dfrac{F}{m_0} \quad \Rightarrow \quad E=\dfrac{GM}{r^2}$



In vector form, $\vec{E}=-\dfrac{GM}{r^2}\hat{r}$

Dimensional formula of intensity of gravitational field = $\dfrac{F}{m}=\dfrac{[MLT^{-2}]}{[M]}=[M^0LT^{-2}]$

Illustration 111:

Find the distance of a point from the Earth's centre where the resultant gravitational field due to the Earth and the moon is zero. The mass of the Earth is 6.0×10^{24} *kg* and that of the moon is 7.4×10^{22} *kg*. The distance between the Earth and the moon is 4.0×10^5 *km.*

Solution:

The point must be on the line joining the centres of the Earth and the moon and in between them. If the distance of the point from the Earth is x, the distance from the moon is $(4.0 \times 10^5\, km - x)$. The magnitude of the gravitational field due to the Earth is

$$E_1 = \frac{GM_e}{x^2} = \frac{G \times 6 \times 10^{24}\, kg}{x^2}$$

and magnitude of the gravitational field due to the moon is

$$E_2 = \frac{GM_m}{(4.0 \times 10^5\, km - x)^2} = \frac{G \times 7.4 \times 10^{22}\, kg}{(4.0 \times 10^5\, km - x)^2}$$

These fields are in opposite directions. For the resultant field to be zero $E_1 = E_2$.

or, $$\frac{6 \times 10^{24}\, kg}{x^2} = \frac{7.4 \times 10^{22}\, kg}{(4.0 \times 10^5\, km - x)^2}$$

or, $$\frac{x}{4.0 \times 10^5\, km - x} = \sqrt{\frac{6 \times 10^{24}}{7.4 \times 10^{22}}} = 9$$

or, $x = 3.6 \times 10^5\, km.$

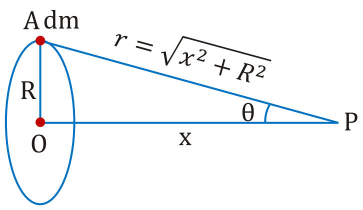
Illustration 112:

Calculate gravitational field intensity due to a uniform ring of mass M and radius R at a distance x on the axis from center of ring.

Solution:

Consider any particle of mass dm. Gravitational field at point P due to dm

$$dE = \frac{Gdm}{r^2} \text{ along } PA$$



Component along PO is $dE \cos\theta = \dfrac{Gdm}{r^2} \cos\theta$

Net gravitational field at point P is

$$E = \int \frac{Gdm}{r^2}\cos\theta = \frac{G\cos\theta}{r^2}\int dm$$

$$= \frac{GMx}{(R^2 + x^2)^{3/2}} \text{ towards the center of ring}$$

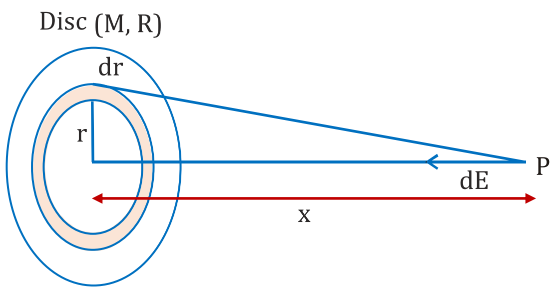
Illustration 113:

Calculate gravitational field intensity at a distance x on the axis from centre of a uniform disc of mass M and radius R.

Solution:

Consider a elemental ring of radius r and thickness dr on surface of disc as shown in figure.

Gravitational field due to elemental ring



$$dE = \frac{G(dM)x}{(x^2+r^2)^{3/2}}$$

Here $\quad dM = \frac{M}{\pi R^2}.2\pi r dr = \frac{2M}{R^2} r dr$

$\therefore \quad dE = \frac{G.2Mxrdr}{R^2(x^2+r^2)^{3/2}}$

$\therefore \quad E = \int_0^R \left(\frac{2GMx}{R^2}\right)\frac{rdr}{(x^2+r^2)^{3/2}}$

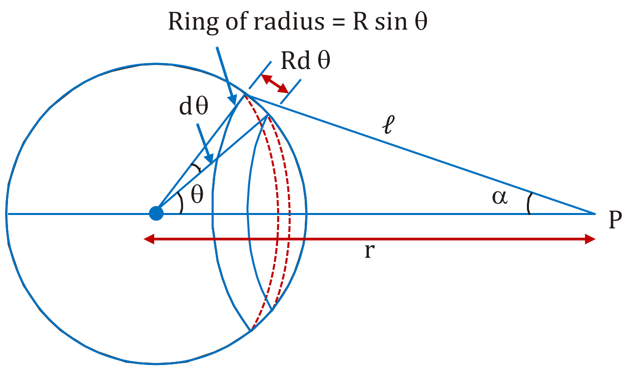
$\therefore \quad E = \frac{2GMx}{R^2}\left[\frac{1}{x} - \frac{1}{\sqrt{x^2+R^2}}\right]$

Illustration 114:

For a given uniform spherical shell of mass M and radius R, find gravitational field at a distance r from centre in following two cases:

(a) $r \geq R$ (b) $r < R$

Solution:



$$dE = \frac{GdM}{\ell^2}.\cos\alpha \qquad r \geq R$$

$$dM = \frac{M}{4\pi R^2} \times 2\pi R \sin\theta R d\theta$$

$$dM = \frac{M}{2}\sin\theta\, d\theta$$

$\therefore \quad dE = \frac{GM\sin\theta\cos\alpha\, d\theta}{2\ell^2}$

Now $\quad \ell^2 = R^2 + r^2 - 2rR\cos\theta \qquad$ ...(i)

$\quad R^2 = \ell^2 + r^2 - 2\ell r\cos\alpha \qquad$ ...(ii)

$\therefore \quad \cos\alpha = \frac{\ell^2 + r^2 - R^2}{2\ell r}$

$\quad \cos\theta = \frac{R^2+r^2-\ell^2}{2rR}$

Differentiating (1)

$\therefore \quad 2\ell\, d\ell = 2rR\sin\theta\, d\theta$

$\therefore \quad dE = \frac{GM}{2\ell^2} \cdot \frac{\ell\, d\ell}{Rr} \cdot \frac{\ell^2 + r^2 - R^2}{(2\ell\, r)} \qquad \Rightarrow dE = \frac{GM}{4Rr^2}\left[1 + \frac{r^2 - R^2}{\ell^2}\right] d\ell$

$\therefore \quad E = \int dE = \frac{GM}{4Rr^2}\left[\int_{r+R}^{r-R} d\ell + (r^2 - R^2)\int_{r+R}^{r-R} \frac{d\ell}{\ell^2}\right] \qquad \Rightarrow E = \frac{GM}{r^2}, r \geq R$

If point is inside the shell limit changes to [($R - r$) *to* $R + r$]

$E = 0$ when $r < R$.

Illustration 115:

Find the relation between the gravitational field on the surface of two planets A and B of masses m_A, m_B and radius R_A and R_B respectively if

(i) they have equal mass (ii) they have equal (uniform) density

Solution:

Let E_A and E_B be the gravitational field intensities on the surface of planets A and B.

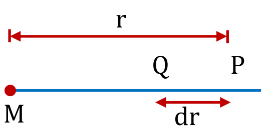
then, $E_A = \frac{Gm_A}{R_A^2} = \frac{G\frac{4}{3}\pi R_A^3 \rho_A}{R_A^2} = \frac{4G\pi}{3}\rho_A R_A$

Similarly, $E_B = \frac{GM_B}{R_{B^2}} = \frac{4G}{3}\pi\ \rho_B R_B$

(i) For $m_A = m_B, \frac{E_A}{E_B} = \frac{R_B^2}{R_A^2}$ (ii) For $\rho_A = \rho_B, \frac{E_A}{E_B} = \frac{R_A}{R_B}$

Gravitational Potential

The gravitational potential at a point in the gravitational field of a body is defined as the amount of work done by an external agent in bringing a body of unit mass from infinity to that point, slowly (no change in kinetic energy). Gravitational potential is very similar to electric potential in electrostatics.



Gravitational Potential Due to a Point Mass

Let the unit mass be displaced through a distance dr towards mass M, then work done is given by

$$dW = F\,dr = \frac{GM}{r^2}dr$$

Total work done in displacing the particle from infinity to point P is

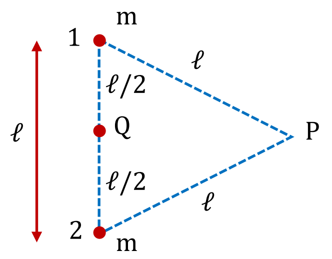
$$W = \int dw = \int_{\infty}^{r} \frac{GM}{r^2}dr = \frac{-GM}{r}$$

Thus, gravitational potential, $\boxed{V = -\frac{GM}{r}}$.

The unit of gravitational potential is $J\ kg^{-1}$. Dimensional Formula of gravitational potential

$$=\frac{\text{Work}}{\text{Mass}}=\frac{[ML^2T^{-2}]}{[M]}=[M^0L^2T^{-2}]$$

Illustration 116:



Find out potential at P and Q due to the two point mass system. Find out work done by external agent in bringing unit mass from P to Q. Also find work done by gravitational force.

Solution:

(i) V_{P1} = potential at P due to mass 'm' at '1' = $-\dfrac{Gm}{\ell}$

$$V_{P2}=-\frac{Gm}{\ell}$$

$\therefore$ $V_P = V_{P1} + V_{P2} = -\dfrac{2Gm}{\ell}$

(ii) $V_{Q1}=-\dfrac{Gm}{\ell/2}$ $\Rightarrow$ $V_{Q2}=-\dfrac{Gm}{\ell/2}$

$\therefore$ $V_Q = V_{Q1} + V_{Q2} = -\dfrac{Gm}{\ell/2}-\dfrac{Gm}{\ell/2}=-\dfrac{4Gm}{\ell}$

Force at point $Q = 0$

(iii) work done by external agent = $(V_Q - V_P) \times 1 = -\dfrac{2GM}{\ell}$

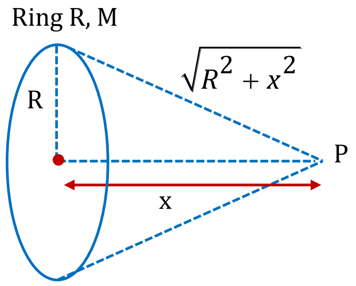
(iv) work done by gravitational force = $V_P - V_Q = \dfrac{2GM}{\ell}$

Illustration 117:

Find potential at a point 'P' at a distance 'x' on the axis away from centre of a uniform ring of mass M and radius R.

Solution:

Ring can be considered to be made of large number of point masses (m_1, m_2 .... etc)



$$V_P=-\frac{Gm_1}{\sqrt{R^2+x^2}}-\frac{Gm_2}{\sqrt{R^2+x^2}}-.......$$

$$=-\frac{G}{\sqrt{R^2+x^2}}(m_1+m_2..........)=\frac{GM}{\sqrt{R^2+x^2}},$$

where $M = m_1 + m_2 + m_3 + ...$

Potential at centre of ring = $-\dfrac{G.M}{R}$

Relation between Gravitational Field and Potential

The work done by an external agent to move unit mass from a point to another point in the direction of the field E, slowly through an infinitesimal distance dr = Force by external agent × Distance moved = – Edr.

Thus $dV = -Edr$

$$\Rightarrow \quad E = -\frac{dV}{dr}$$

Therefore, gravitational field at any point is equal to the negative gradient at that point.

Illustration 118:

The gravitational field in a region is given by $\vec{E} = 20(\hat{i} + \hat{j}) N/kg$. Find the gravitational potential at the origin (0, 0) (*in J/kg*)

(A) Zero (B) $20\sqrt{2}$ (C) $-20\sqrt{2}$ (D) can not be defined

Ans. (A)

Solution:

$$V = -\int E.dr = \left[\int E_x.dx + \int E_y.dy\right] = 20x + 20y$$

At origin $V = 0$.

Illustration 119:

In above problem, find the gravitational potential at a point whose co-ordinates are (5, 4) : (*in J/kg*)

(A) – 180 (B) 180 (C) – 90 (D) zero

Ans. (B)

Solution:

$V = 20 \times 5 + 20 \times 4 = 180\ J/kg$

Illustration 120:

In the above problem, find the work done in shifting a particle of mass 1 *kg* from origin (0, 0) to a point (5, 4): (In *J*)

(A) – 180 (B) 180 (C) – 90 (D) zero

Ans. (B)

Solution:

$$W = m(V_f - V_i) = 1(180 - 0) = 180\ J$$

Illustration 121:

Find gravitational field at a point (x, y, z).

$$v = 2x^2 + 3y^2 + zx,$$

Solution:

$$\vec{E} = -\nabla V$$

$$E_x = \frac{-\partial v}{\partial x} = -4x - z \quad ; \quad E_y = -6y \quad ; \quad E_z = -x$$

$$\therefore \quad \text{Field} = \vec{E} = -[(4x + z)\hat{\imath} + 6y\hat{\jmath} + x\hat{k}]$$

Analogy between Electrostatics and Gravitation

(1) Point Charge | **Point Mass**

(a) $E = \dfrac{kQ}{r^2}$ | $g = \dfrac{GM}{r^2}$

(b) $V = \dfrac{kQ}{r}$ | $V_G = \dfrac{-GM}{r}$

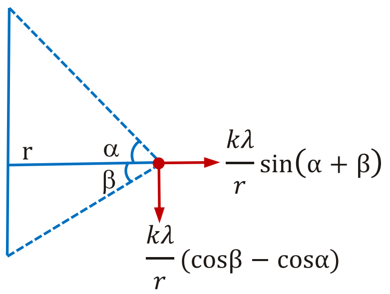
(2) Uniform charged ring | **Ring of uniform mass distribution**

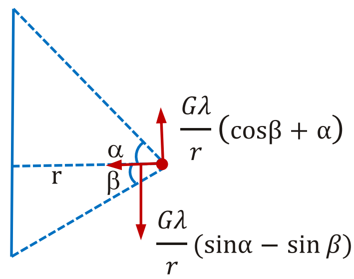
(a) $E = \dfrac{kQx}{(r^2+x^2)^{3/2}}$ on axis | $g = \dfrac{GMx}{(r^2+x^2)^{3/2}}$ on axis

E is max. when $x = \dfrac{r}{\sqrt{2}}$ | g is max. when $x = \dfrac{r}{\sqrt{2}}$

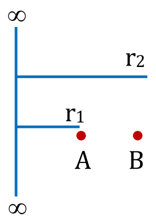
(b) $V = \dfrac{kQ}{\sqrt{r^2+x^2}}$ on axis, $\dfrac{kQ}{r}$ at center | $V_G = \dfrac{-GM}{\sqrt{r^2+x^2}}$ on axis, $\dfrac{-GM}{r}$ at center

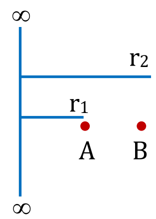
(3) Uniform linear charge | **Uniform linear mass**





(4) Infinite Linear charge | **Infinite linear mass**





(a) $E = \dfrac{2K\lambda}{r}$ | $g = \dfrac{2G\lambda}{r}$

(b) $V_B - V_A = -\,2K\lambda \ln \dfrac{r_2}{r_1}$ | $V_B - V_A = 2G\lambda \ln \left(\dfrac{r_2}{r_1}\right)$

(5) Infinite Sheet of charge

$E = \dfrac{\sigma}{2\varepsilon_0}$

$= 2\pi K\sigma$

Infinite Sheet of mass

$g = \dfrac{\sigma}{2} \times 4\pi G = 2\pi G\sigma$

(σ = Mass per unit area)

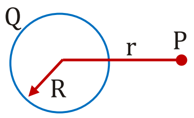
- Notice gravitational force is always attractive and hence gravitational potential is always $-ve$ (for a repulsive force potential is positive). This can be explained from the sign of W_{ext} in moving the test charge from ∞ to the point under consideration.
- Since $\vec{g}$ points from B towards A potential increases as we move from A to B. Just like electric potential, gravitational potential also increases opposite to field direction.

(6) Uniformly charged hollow sphere

Charge Q, radius R

distance of field point from center r

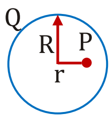
Case I. $r > R$



$E = \dfrac{kQ}{r^2}$

$V = \dfrac{kQ}{r}$

Case II. $r < R$



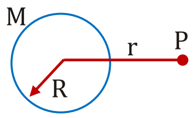
$E = 0$

$V = \dfrac{kQ}{R}$

Hollow sphere of uniform mass

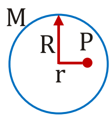
Mass M, radius R

distance of field point from center r



$g = \dfrac{GM}{r^2}$

$V_G = -\dfrac{GM}{r}$



$g = 0$

$V_G = -\dfrac{GM}{R}$

(7) Electrostatics self energy of uniformly charged thin spherical shell.

$U = \dfrac{KQ^2}{2R}$

Gravitational self energy of uniform thin spherical shell.

$U = -\dfrac{GM^2}{2R}$

(8) Uniformly charged solid sphere

mass M, radius R

$E = \dfrac{kQ}{r^2}, r > R$

$\dfrac{kQr}{R^3}, r < R$

$V = \dfrac{kQ}{r}, r > R$

$\dfrac{kQ}{2R^3}\left(3R^2 - r^2\right), r < R$

Uniformly solid sphere

mass M, radius R

$g = \dfrac{GM}{r^2}, r > R$

$\dfrac{GM}{R^3}r, r < R$

$V_a = -\dfrac{GM}{r}, r > R$

$\dfrac{-GM}{2R^3}\left(3R^2 - r^2\right), r < R$

(9) **Electrostatics self energy of uniformly charged solid sphere.**

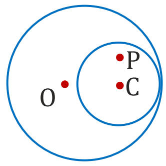
$$U=\frac{3}{5}\frac{KQ^2}{R}$$

Gravitational self energy of uniform solid sphere.

$$U=-\frac{3}{5}\frac{GM^2}{R}$$

Illustration 122:

A uniform solid sphere of density ρ and radius R has a spherical cavity of radius r inside it as shown. Find gravitation field at



(a) O (b) C (c) P (prove that field inside cavity is uniform)

Ans. (a) $\dfrac{k\left(\frac{4}{3}\pi r^3\rho\right)}{(OC)^2}\hat{CO}$ (b) $\dfrac{4\pi G\rho\overrightarrow{OC}}{3}$ (c) $\dfrac{4}{3}\pi G\rho\overrightarrow{OC}$

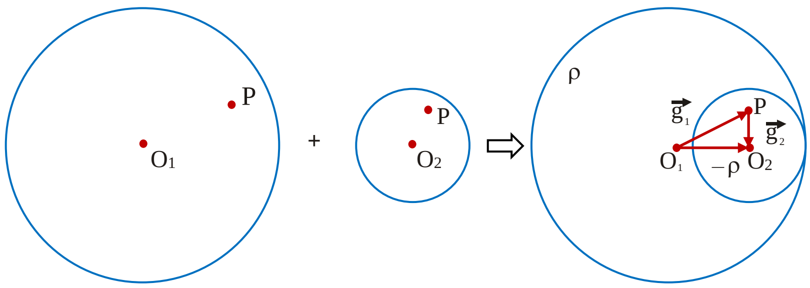
Solution:

$\vec{g}$ = Gravitational field at any point inside sphere

$$\vec{g}=\frac{GM}{R^3}\vec{r}\ =\frac{G}{R^3}\frac{4}{3}\pi R^3\rho\vec{r}$$

$$\vec{g}=\frac{4}{3}\pi G\rho\vec{r}$$

Let the sphere with cavity is formed by superimposing it with a small sphere of density (−ρ) as shown



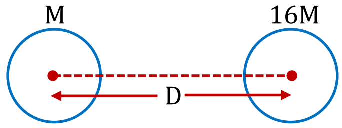
Resultant field $\vec{g}=\vec{g}_1+\vec{g}_2$

$$=\left(\frac{4}{3}\pi G\rho\right)\overrightarrow{O_1P}+\left(\frac{4}{3}\pi G\rho\right)\overrightarrow{PO_2}$$

$$=\frac{4}{3}\pi G\rho\overrightarrow{O_1O_2}\quad\left[\overrightarrow{O_1O_2}=\overrightarrow{OC}\right]$$

It is independent of position of point inside cavity.

Illustration 123:



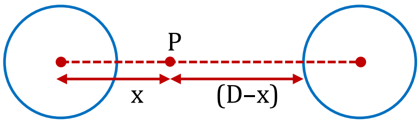
(a) Find where is gravitational field intensity is zero.

(b) Find gravitational potential at P.

Solution:

(a) $$\frac{GM}{x^2} = \frac{G(16M)}{(D-x)^2}$$

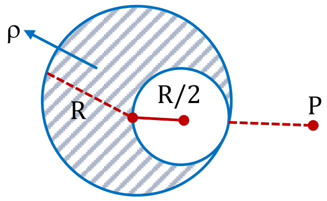
$$4 = \frac{D-x}{x} \quad \Rightarrow \quad x = \frac{D}{5}$$



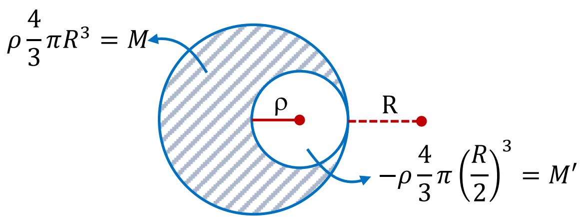
(b) $$v_P = \frac{-GM}{\frac{D}{5}} + \left(\frac{-G(16M)}{\frac{4D}{5}}\right) = -\frac{25GM}{D}$$

Illustration 124:

GFI at *P* = ?



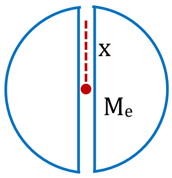
Solution:



$$E_P = \frac{GM}{(2R)^2} + \frac{GM'}{(3R/2)^2}$$

Illustration 125:

A particle is released at a distance *x* from centre of Earth as shown in figure. Show that it performs SHM and find *T*.



Solution:

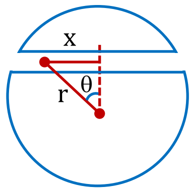
$$F = m \times GFI$$

$$ma = m\left(G\frac{M_e x}{R^3}\right)$$

$$a = \underbrace{\left[\frac{GM_e}{R^3}\right]}_{\omega^2} x$$

$$\boxed{T = 2\pi\sqrt{\frac{R^3}{GM_e}}}$$

Illustration 126:



A particle is released in a tunnel across Earth as shown in figure. Show that it performs SHM and find T.

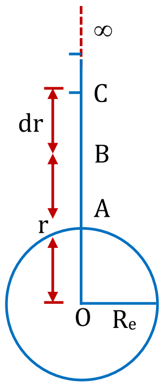
Solution:

$$F_x = \left(\frac{GM_e r}{R^3}\sin\theta\right)m = ma$$

$$a = \underbrace{\left(\frac{GM_e}{R^3}\right)}_{\omega^2}x$$

$$\boxed{T = 2\pi\sqrt{\frac{R^3}{GM_e}}} \quad \text{(as } x = r\sin\theta\text{)}$$

Gravitational Potential Energy



Gravitational potential energy of two mass system is equal to the work done by an external agent in assembling them, while their initial separation was infinity. Consider a body of mass m placed at a distance r from another body of mass M. The gravitational force of attraction between them is given by, $F = \frac{GMm}{r^2}$

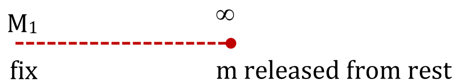
Now, Let the body of mass m is displaced from point. C to B through a distance 'dr' towards the mass M, then work done by internal conservative force (gravitational) is given by,

$$dW = F\,dr = \frac{GMm}{r^2}\,dr \quad \Rightarrow \quad \int dW = \int_{\infty}^{r}\frac{GMm}{r^2}dr$$

$\therefore$ Gravitational potential energy, $\boxed{U = -\frac{GMm}{r}}$

Illustration 127:

Mass M_1 is fixed and mass, m is released from rest at infinity, find its velocity, when separation between them is 'r'.



Solution:

$$W_G + \underbrace{W_{ext}}_{0} = KE_f - \underbrace{KE_i}_{0}$$

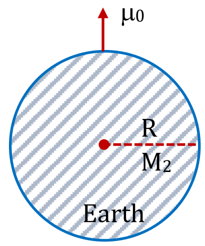
$$U_i - U_f = KE_f - 0$$

$$-\frac{GM_1 m}{\infty} - \left(-\frac{GM_1 m}{r}\right) = \frac{1}{2}mv^2 - 0$$

$$\frac{GM_1 m}{r} = \frac{1}{2}mv^2$$

$$\boxed{v = \sqrt{\frac{2GM_1}{r}}}$$

Illustration 128:



Particle is projected vertically with speed μ_0, particle reaches maximum height of $3R$, find μ_0 (where R is radius of Earth)

Solution:

Height is always measured from surface.

$$\underset{\substack{\Downarrow \\ 0}}{W_G} = KE_f - KE_i$$

$$U_i - U_f = 0 - KE_i$$

$$-\frac{GM_e m}{R} - \left(-\frac{GM_e m}{4R}\right) = -\frac{1}{2}m\mu_0^2$$

$$-\frac{GM_e m}{R} + \frac{GM_e m}{4R} = -\frac{1}{2}m\mu_0^2$$

$$\frac{1}{2}m\mu_0^2 = \frac{3GM_e m}{4R}$$

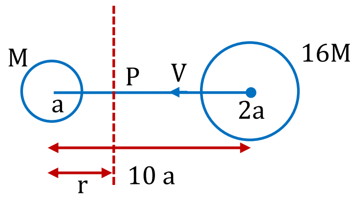
$$\mu_0 = \sqrt{\frac{3GM_e}{2R}}$$

Illustration 129:

Distance between centres of two stars is 10 a. The masses of these stars are M and 16 M and their radii are a and $2a$ respectively. A body is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star?

Solution:

Let P be the point on the line joining the centres of the two planets and the net field at it is zero



Then, $\dfrac{GM}{r^2} - \dfrac{G.16M}{(10a-r)^2} = 0 \quad \Rightarrow \quad (10\,a-r)^2 = 16\,r^2$

$\Rightarrow \quad 10a - r = 4r \quad \Rightarrow \quad r = 2a$

Potential at point P,

$$V_P = \frac{-GM}{r} - \frac{G.16M}{(10a-r)^2} = \frac{-GM}{2a} - \frac{2GM}{a} = \frac{-5GM}{2a}$$

Now if the particle projected from the larger planet has enough energy to cross this point, it will reach the smaller planet.

For this, the $K.E.$ imparted to the body must be just enough to raise its total mechanical energy to a value which is equal to $P.E.$ at point P.

i.e., $\dfrac{1}{2}mv^2 - \dfrac{G(16M)m}{2a}\dfrac{-GMm}{8a} = mV_P$

or $\dfrac{v^2}{2} - \dfrac{8GM}{a} - \dfrac{GM}{8a} = -\dfrac{5GMm}{2a}$

or $v^2 = \dfrac{45GM}{4a}$

or $v_{\min} = \dfrac{3}{2}\sqrt{\dfrac{5GM}{a}}$

Gravitational Self-Energy

The gravitational self-energy of a body (or a system of particles) is defined as the work done by an external agent in assembling the body (or system of particles) from infinitesimal elements (or particles) that are initially an infinite distance apart.

Gravitational Self Energy of a System of n Particles

Potential energy of n particles at an average distance 'r' due to their mutual gravitational attraction is equal to the sum of the potential energy of all pairs of particle, i.e.,

$$U_s = -G\sum_{\substack{\text{all pairs}\\ j\neq i}} \frac{m_i m_j}{r_{ij}}$$

This expression can be written as

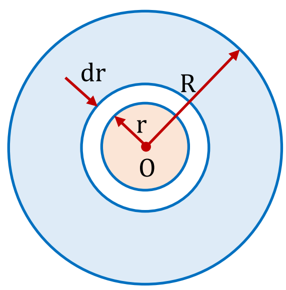
$$U_s = -\frac{1}{2}G\sum_{i=1}^{i=n}\sum_{\substack{j=1\\ j\neq i}}^{j=n}\frac{m_i m_j}{r_{ij}}$$

If consider a system of 'n' particles, each of same mass 'm' and separated from each other by the same average distance 'r', then self energy

or $$U_s = -\frac{1}{2}G\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{m^2}{r}\right)_{ij}$$

Thus, on the right hand side 'i' comes 'n' times while 'j' comes $(n-1)$ times. Thus $U_s = -\frac{1}{2}Gn(n-1)\frac{m^2}{r}$.

Gravitational Self Energy of a Uniform Sphere (Star)



$$U_{sphere} = -G\frac{\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2 dr\rho\right)}{r} \text{ where } \rho = \frac{M}{\left(\frac{4}{3}\right)\pi R^3} = -\frac{1}{3}G(4\pi\rho)^2 r^4 dr$$

$$U_{star} = -\frac{1}{3}G(4\pi\rho)^2\int_0^R r^4 dr = -\frac{1}{3}G(4\pi\rho)^2\left[\frac{r^5}{5}\right]_0^R = -\frac{3}{5}G\left(\frac{4\pi}{3}R^3\rho\right)^2\frac{1}{R}$$

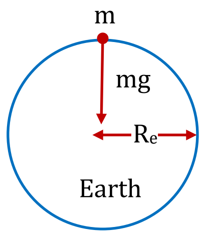
$$\therefore \qquad U_{star} = -\frac{3}{5}\frac{GM^2}{R}$$

Acceleration Due to Gravity

It is the acceleration, a freely falling body near the Earth's surface acquires due to the Earth's gravitational pull. The property by virtue of which a body experiences or exerts a gravitational pull on another body is called **gravitational mass m_G**, and the property by virtue of which a body opposes any change in its state of rest or uniform motion is called its **inertial mass** m_I, thus if $\vec{E}$ is the gravitational field intensity due to the Earth at a point P, and $\vec{g}$ is acceleration due to gravity at the same point, then $m_I\vec{g} = m_G\vec{E}$.

Now the value of inertial and gravitational mass happen to be exactly same to a great degree of accuracy for all bodies. Hence, $\vec{g} = \vec{E}$

The gravitational field intensity on the surface of Earth is therefore numerically equal to the acceleration due to gravity (g), there. Thus, we get,



where, M_e = Mass of Earth

R_e = Radius of Earth

Note:

Here the distribution of mass in the Earth is taken to be spherical and symmetrical so that its entire mass can be assumed to be concentrated at its center for the purpose of calculation of g.

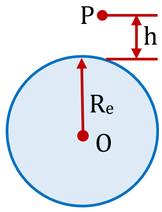
Variation of Acceleration Due to Gravity

(a) Effect of Altitude:

Acceleration due to gravity on the surface of the Earth is given by, $g = \frac{GM_e}{R_e^2}$.

Now, consider the body at a height 'h' above the surface of the Earth, then the acceleration due to gravity at height 'h' is given by

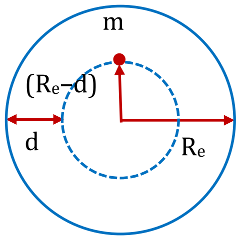
$$g_h = \frac{GM_e}{(R_e + h)^2} = g\left(1 + \frac{h}{R_e}\right)^{-2} \simeq g\left(1 - \frac{2h}{R_e}\right) \text{ when } h << R.$$



The decrease in the value of 'g' with height $h = g - g_h = \frac{2gh}{R_e}$.

Then percentage decrease in the value of 'g' $= \frac{g - g_h}{g} \times 100 = \frac{2h}{R_e} \times 100\%$.

(b) Effect of Depth:



The gravitational pull on the surface is equal to its weight i.e. $mg = \frac{GM_e m}{R_e^2}$

$$\therefore \quad mg = \frac{G \times \frac{4}{3}\pi R_e^3 \rho m}{R_e^2} \text{ or } g = \frac{4}{3}\pi G R_e \rho \qquad \text{...(i)}$$

When the body is taken to a depth d, the mass of the sphere of radius ($R_e - d$) will only be effective for the gravitational pull and the outward shall will have no resultant effect on the mass. If the acceleration due to gravity on the surface of the solid sphere is g_d, then

$$g_d = \frac{4}{3}\pi G(R_e - d)\rho \qquad \text{...(ii)}$$

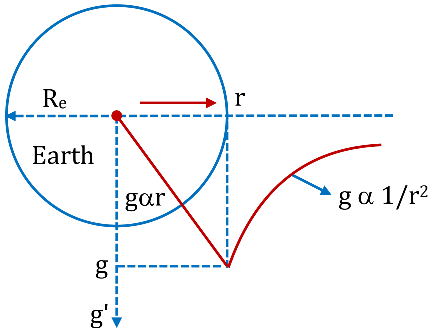
By dividing equation (ii) by equation (i)

$$\Rightarrow \quad g_d = g\left(1 - \frac{d}{R_e}\right)$$

Important Points:

(i) At the center of the Earth, $d = R_e$, so $g_{centre} = g\left(1-\frac{R_e}{R_e}\right)=0$.

Thus weight (mg) of the body at the centre of the Earth is zero.

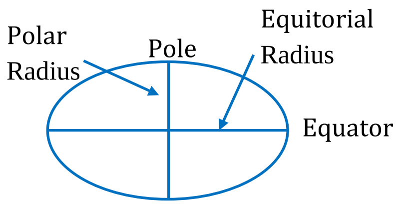


(ii) Percentage decrease in the value of 'g' with the depth = $\left(\frac{g-g_d}{g}\right)\times 100=\frac{d}{R_e}\times 100$.

(c) Effect of the Surface of Earth

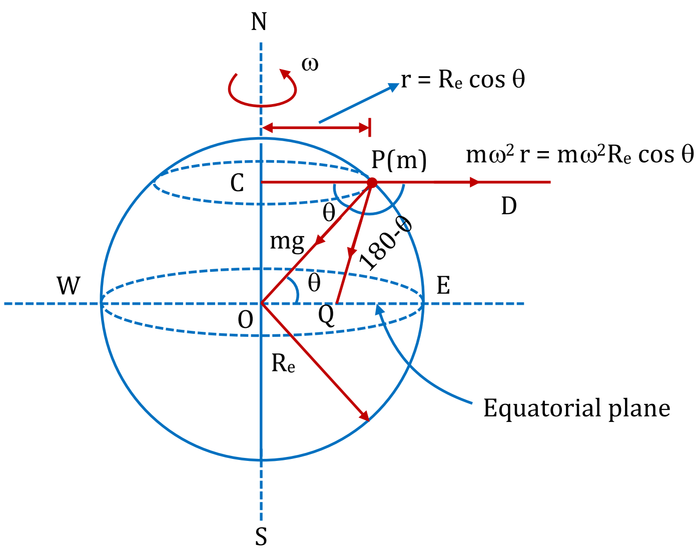
The equatorial radius is about 21 km longer than its polar radius.

We know, $g=\frac{GM_e}{R_e^2}$. Hence $g_{pole} > g_{equator}$. The weight of the body increases as the body taken from the equator to the pole.



(d) Effect of Rotation of the Earth

The Earth rotates around its axis with angular velocity ω. Consider a particle of mass m at latitude θ. The angular velocity of the particle is also ω.



According to parallelogram law of vector addition, the resultant force acting on mass m along PQ is

$$F = [(mg)^2 + (m\omega^2 R_e \cos\theta)^2 + \{2mg \times m\omega^2 R_e \cos\theta\} \cos(180 - \theta)]^{1/2}$$

$$= [(mg)^2 + (m\omega^2 R_e \cos\theta)^2 - (2m^2 g\omega^2 R_e \cos\theta)\cos\theta]^{1/2}$$

$$= mg\left[1 + \left(\frac{R_e\omega^2}{g}\right)^2 \cos^2\theta - 2\frac{R_e\omega^2}{g}\cos^2\theta\right]$$

At pole $\theta = 90° \Rightarrow g_{pole} = g$, At equator $\theta = 0 \Rightarrow g_{equator} = g\left[1 - \frac{R_e\omega^2}{g}\right]$.

Hence $g_{pole} > g_{equator}$

If the body is taken from pole to the equator, then $g' = g\left[1 - \frac{R_e\omega^2}{g}\right]$.

Hence % change in weight = $\dfrac{mg - mg\left(1 - \frac{R_e\omega^2}{g}\right)}{mg} \times 100 = \dfrac{mR_e\omega^2}{mg} \times 100 = \dfrac{R_e\omega^2}{g} \times 100$

Key Points:

1. Apparent weight is equal to normal reaction is equal to reading of weighing machine is equal to $m \times g_{apparent}$.
   [Apparent weight = Normal reaction = Reading of weighing machine = $m \times g_{apparant}$]
2. Motion of an object is referred as free fall if its acceleration is same as acceleration due to gravity.

Illustration 130:

At some planet, g_{app} at equator is equal to half of g_{app} at poles.

Find ω in terms of g and R.

Solution:

$$g_e = \frac{1}{2}g_p \quad \Rightarrow \quad g - R\omega^2 = \frac{1}{2}g \quad \Rightarrow \quad \omega^2 R = \frac{g}{2} \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{2R}}$$

Weightlessness

State of the free fall $\left(\vec{a} = -\frac{GM}{r^2}\vec{r}\right)$ is called state of weightlessness. If a body is in a satellite (which does not produce its own gravity) orbiting the Earth at a height h above its surface then

True weight = $mg_h = \dfrac{mGM}{(R+h)^2} = \dfrac{mg}{\left(1 + \frac{h}{R}\right)^2}$; Apparent weight = $m(g_h - a)$

But, $a = \dfrac{v_0^2}{r} = \dfrac{GM}{r^2} = \dfrac{GM}{(R+h)^2} = g_h$; Apparent weight = $m(g_h - g_h) = 0$

Note:

The condition of weightlessness can be overcome by creating artificial gravity by rotating the satellite in addition to its revolution.

Illustration 131:

Find ratio of gravitational field on the surface of two planets which are of uniform mass density and have radius R_1 and R_2 if

(a) they are of same mass (b) they are of same density.

Ans. (a) $\frac{g_1}{g_2} = \frac{R_2^2}{R_1^2}$ (b) $\frac{g_1}{g_2} = \frac{R_1}{R_2}$

Solution:

$$g = \frac{GM}{R^2} = \frac{G\frac{4}{3}\pi R^3 \rho}{R^2} = \left(\frac{4\pi GR}{3}\right)\rho$$

Escape Speed

The minimum speed required to send a body out of the gravity field of a planet (send it to $r \to \infty$).

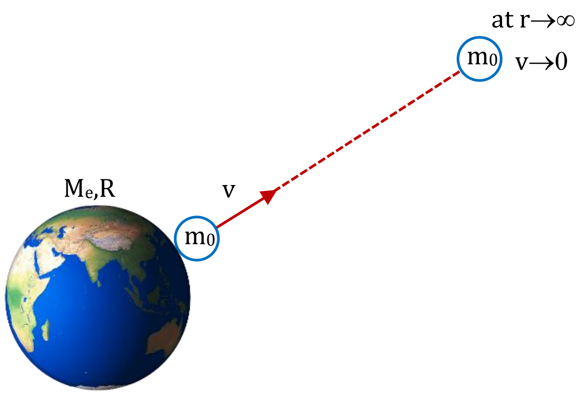
Escape Speed at Earth's Surface

Suppose a particle of mass m is on Earth's surface. We project it with a velocity V from the Earth's surface, so that it just reaches $r \to \infty$ (at $r \to \infty$, its velocity become zero). Applying energy conservation between initial position (when the particle was at Earth's surface) and find positions (when the particle just reaches to $r \to \infty$)

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}m_0 v^2 + m_0\left(-\frac{GM_e}{R}\right) = 0 + m_0\left(-\frac{GM_e}{(r \to \infty)}\right)$$

$$\Rightarrow \quad v = \sqrt{\frac{2GM_e}{R}}$$



Escape speed from Earth is surface $v_e = \sqrt{\frac{2GM_e}{R}}$

If we put the values of G, Me, R then we get $V_e = 11.2\ km/s$.

Escape Speed Depends on:

(i) Mass (M_e) and size (R) of the planet.

(ii) Position from where the particle is projected.

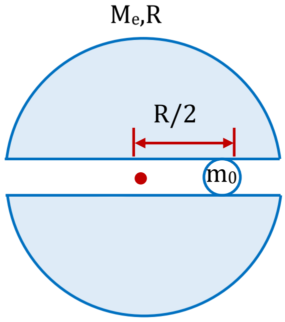
Escape speed does not depend on:

(i) Mass of the body which is projected (m_0)

(ii) Angle of projection.

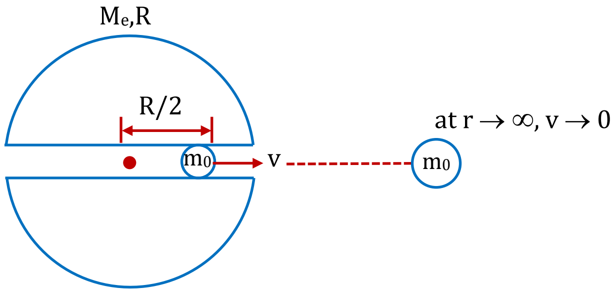
If a body is thrown from Earth's surface with escape speed, it goes out of Earth's gravitational field and never returns to the Earth's surface. But it starts revolving around the sun.

Illustration 132:



A very small groove is made in the Earth, and a particle of mass m_0 is placed at $R/2$ distance from the centre. Find the escape speed of the particle from that place.

Solution:



Suppose we project the particle with speed v, so that it just reaches at $(r \to \infty)$.

Applying energy conservation $K_i + U_i = K_f + U_f$

$$\frac{1}{2}m_0v^2 + m_0\left(-\frac{GM_e}{2R^3}\left(3R^2 - \left(\frac{R}{2}\right)^2\right)\right) = 0 + 0$$

$$v = \sqrt{\frac{11GM_e}{4R}} = v_e \text{ at that position.}$$

Illustration 133:

Find radius of such planet on which the man escapes through jumping. The capacity of jumping of person on Earth is 1.5 m. Density of planet is same as that of Earth.

Solution:

For a planet: $\frac{1}{2}mv^2 - \frac{GM_pm}{R_p} = 0 \qquad \Rightarrow \frac{1}{2}mv^2 = \frac{GM_pm}{R_p}$

On Earth $\to \frac{1}{2}mv^2 = m\left(\frac{GM_E}{R_E^2}\right)h$

$$\therefore \qquad \frac{GM_pm}{R_p} = \frac{GM_E.m}{R_E^2}\cdot h \Rightarrow \frac{M_p}{R_p} = \frac{M_Eh}{R_E^2}$$

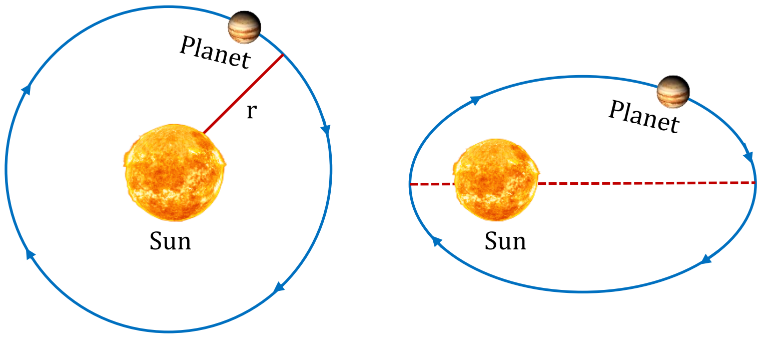
$$\therefore \qquad \text{Density } (\rho) \text{ is same} \Rightarrow \frac{4/3\pi R_p^3\rho}{R_p} = \frac{4/3\pi R_E^3\rho}{R_E^2}h \Rightarrow R_P = \sqrt{R_Eh}$$

Kepler's Law for Planetary Motion

Suppose a planet is revolving around the sun, or a satellite is revolving around the Earth, then the planetary motion can be studied with help of Kepler's three laws.

Kepler's Law of Orbit

Each planet moves around the Sun in a circular path or elliptical path with the Sun at its focus. (In fact circular path is a subset of elliptical path)



Law of Areal Velocity

To understand this law, lets understand the angular momentum conservation for the planet.

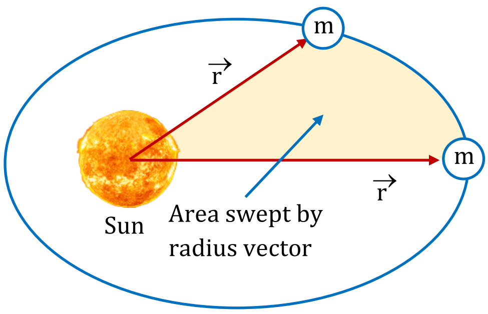
If a planet moves in an elliptical orbit, the gravitation force acting on it always passes through the centre of the sun. So, torque of this gravitation force about the centre of the Sun will be zero. Hence, we can say that angular momentum of the planet about the centre of the Sun will remain conserved (constant)

τ about the Sun = 0

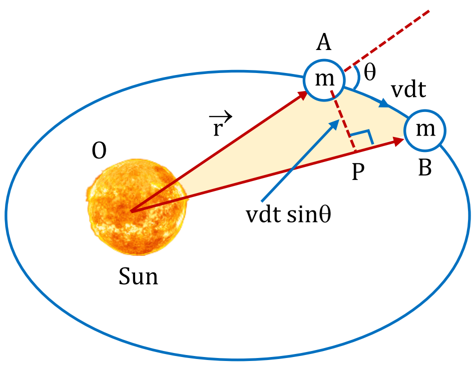
$\Rightarrow \quad \frac{dL}{dt}=0 \qquad \Rightarrow \quad L_{\text{planet}}$ / Sun = constant $\qquad \Rightarrow \quad mvr\sin\theta$ = constant

Now we can easily study the Kepler's law of aerial velocity.

If a planet moves around the sun, the radius vector ($\vec{r}$) also rotates and sweeps area as shown in figure. Now let's find rate of area swept by the radius vector ($\vec{r}$).



Suppose a planet is revolving around the Sun and at any instant its velocity is v, and angle between radius vector ($\vec{r}$) and velocity ($\vec{v}$). In dt time, it moves by a distance vdt, during this dt time, area swept by the radius vector will be OAB which can be assumed to be a triangle



dA = 1/2 (Base) (Perpendicular height)

dA = 1/2 (r) ($vdt\sin\theta$)

So, rate of area swept $\frac{dA}{dt}=\frac{1}{2}vr\sin\theta$

We can write $\frac{dA}{dt}=\frac{1}{2}\frac{mvr\sin\theta}{m}$

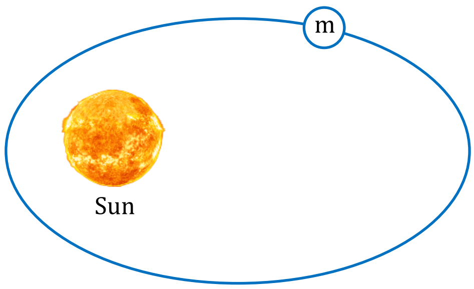
Where mvr sin θ = Angular momentum of the planet about the sun, which remains conserved (constant)

$$\Rightarrow \quad \frac{dA}{dt}=\frac{L_{planet/sun}}{2m}=\text{constant}$$

So, **rate of area swept by the radius vector is constant.**

Illustration 134:

Suppose a planet is revolving around the Sun in an elliptical path given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Find time period of revolution. Angular momentum of the planet about the Sun is L.



Solution:

Rate of area swept $\frac{dA}{dt}=\frac{L}{2m}$ = constant

$$\Rightarrow \quad dA=\frac{L}{2m}dt\ ;\ \int_{A=0}^{A=\pi ab}dA=\int_{t=0}^{t=T}\frac{L}{2m}dt \quad \Rightarrow \quad \pi ab=\frac{L}{2m}T \quad \Rightarrow T=\frac{2\pi mab}{L}$$

Kepler's Law of Time Period

Suppose a planet is revolving around the Sun in circular orbit then $\frac{m_0v^2}{r}=\frac{GM_Sm_0}{r^2}$

$$v=\sqrt{\frac{GM_s}{r}}$$

Time period of revolution is $T=\frac{2\pi r}{v}=2\pi r\sqrt{\frac{r}{GM_s}}$

$$T^2=\left(\frac{4\pi^2}{GM_s}\right)r^3 \quad \Rightarrow \quad T^2=\alpha r^3$$

For all the planet of a Sun, $T^2\propto r^3$

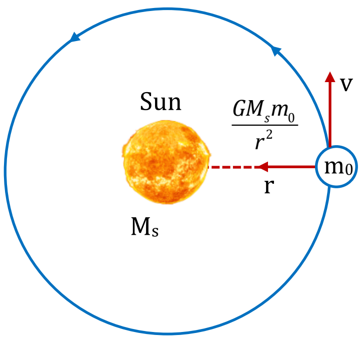
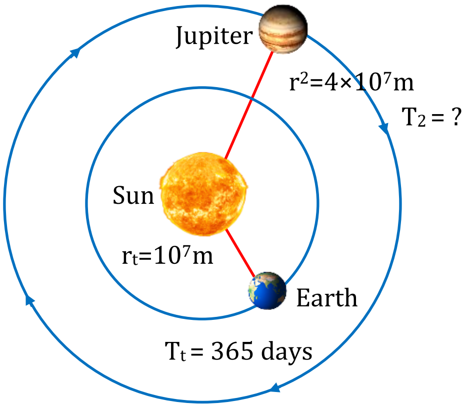


Illustration 135:

The Earth and Jupiter are two planets of the sun. The orbital radius of the Earth is 10^7 *m* and that of Jupiter is 4×10^7 *m*. If the time period of revolution of Earth is $T = 365$ days, find time period of revolution of the Jupiter.



Solution:

For both the planets

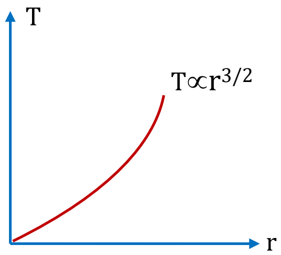
$$T^2 \propto r^3$$

$$\left(\frac{T_{\text{Jupiter}}}{T_{\text{Earth}}}\right)^2 = \left(\frac{r_{\text{Jupiter}}}{r_{\text{Earth}}}\right)^3$$

$$\Rightarrow \quad \left(\frac{T_{\text{Jupiter}}}{365\text{ days}}\right)^2 = \left(\frac{4\times 10^7}{10^7}\right)^3$$

$$T_{\text{jupiter}} = 8 \times 365 \text{ days}$$

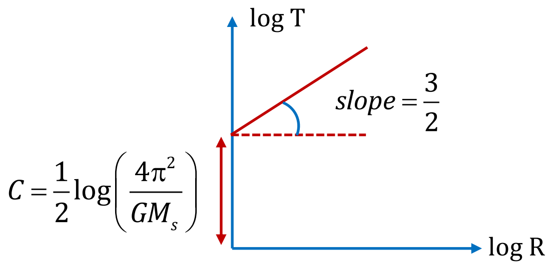
Graph of T vs r :



Graph of log T v/s log R :

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right)R^3 \Rightarrow 2\log T = \log\left(\frac{4\pi^2}{GM_s}\right) + 3\log R$$

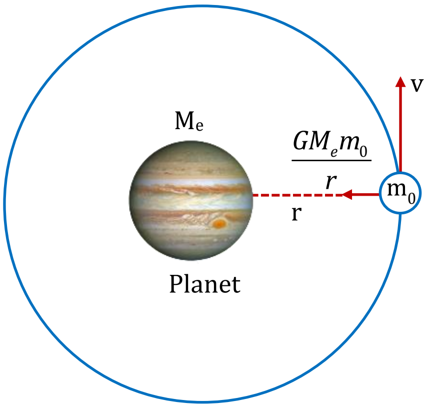
$$\log T = \frac{1}{2}\log\left(\frac{4\pi^2}{GM_s}\right) + \frac{3}{2}\log R$$



- If planets are moving in elliptical orbit, then $T^2 \propto a^3$ where a = semi major axis of the elliptical path.

Circular Motion of a Satellite Around a Planet

Suppose at satellite of mass m_0 is at a distance r from a planet.
If the satellite does not revolve, then due to the gravitational attraction, it may collide to the planet.
To avoid the collision, the satellite revolve around the planet, for circular motion of satellite.



$\Rightarrow \qquad \frac{GM_e m_0}{r^2} = \frac{m_0 v^2}{r}$ ....(i)

$\Rightarrow \qquad v = \sqrt{\frac{GM_e}{r}}$ this velocity is called orbital velocity (v_0)

$$v_0 = \sqrt{\frac{GM_e}{r}}$$

Total Energy of the Satellite Moving in Circular Orbit

(i) $KE = \frac{1}{2} m_0 v^2$ and from equation (i)

$$\frac{m_0 v^2}{r} = \frac{GM_e m_0}{r^2}$$

$$\Rightarrow \quad m_0 v^2 = \frac{GM_e m_0}{r}$$

$$\Rightarrow \quad KE = \frac{1}{2} m_0 v^2 = \frac{GM_e m_0}{2r}$$

(ii) Potential energy $U = -\frac{GM_e m_0}{r}$

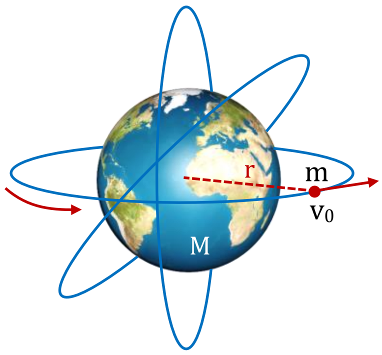
Total energy = $KE + PE = \left(\frac{GM_e m_0}{2r}\right) + \left(\frac{-GM_e m_0}{r}\right)$

$$TE = -\frac{GM_e m_0}{2r}$$

Total energy is negative. It shows that the satellite is still bounded with the planet.

Essential Condition's for Satellite Motion

- Centre of satellite's orbit coincide with centre of Earth.
- Plane of orbit of satellite is passing through centre of Earth.



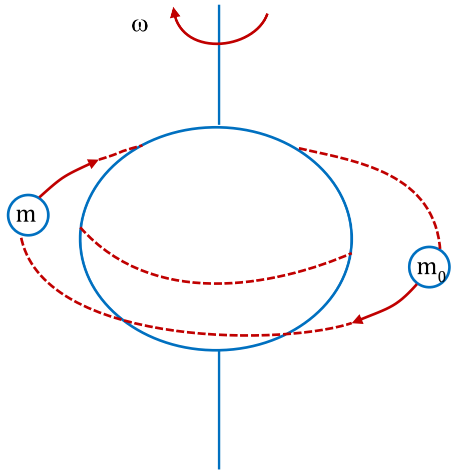
Geo-Stationary Satellite

We know that the Earth rotates about its axis with angular velocity ω_{Earth} and time period T_{Earth} = 24 hours.

Suppose a satellite is set in an orbit which is in the plane of the equator, whose ω is equal to ω_{Earth}, (or its T is equal to T_{Earth} = 24 hours) and direction is also same as that of Earth. Then as seen from Earth, it will appear to be stationary. This type of satellite is called geo- stationary satellite. For a geo-stationary satellite,

$w_{satellite} = w_{Earth}$

$\Rightarrow \quad T_{satellite} = T_{Earth}$ = 24 hr.



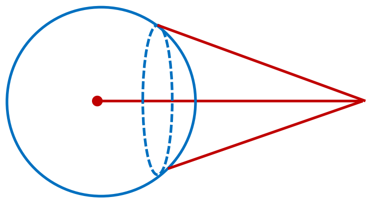
So, time period of a geo-stationery satellite must be 24 hours. To achieve T = 24 hour, the orbital radius geo-stationary satellite:

$$T^2 = \left(\frac{4\pi^2}{GM_e}\right) r^3$$

Putting the values, we get orbital radius of geo stationary satellite $r = 6.6\ R_e$ (here R_e = radius of the Earth)

height from the surface $h = 5.6\ R_e$.

Special Points about Geo–Stationary Satellite

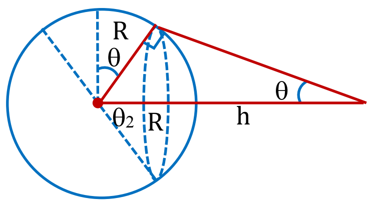


- All three essential conditions for satellite motion should be followed.
- It rotates in equatorial plane.
- Its height from Earth surface is 36000 *km*. ($\sim 6R_e$)
- Its angular velocity and time period should be same as that of Earth.
- Its rotating direction should be same as that of Earth (West to East).
- Its orbit is called parking orbit and its orbital velocity is 3.1 *km*./sec.
- Maximum latitude at which message can be received by geostationary satellite is $\theta = \cos^{-1}\left(\frac{R_e}{R_e + h}\right)$
- The area of Earth's surface covered by geostationary satellite is $S = \Omega R_e^2 = \frac{2\pi h R_e^2}{R_e + h}$
- The area of Earth's surface escaped by geostationary satellite is
- The area of Earth's surface escaped by geostationary satellite is

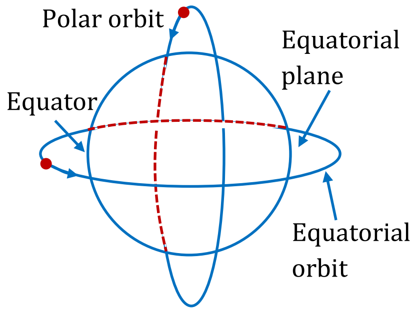
$$\cos\theta_2 = \frac{R}{R+h}$$

$$4\pi R^2 - 2\pi R^2(1-\cos\theta_2) = 2\pi r^2 + 2\pi r^2\cos(90° - \theta)$$

$$= 2\pi R^2(1+\sin\theta)$$



Polar Satellite (Sun–Synchronous Satellite)



It is that satellite which revolves in polar orbit around Earth. A polar orbit is that orbit whose angle of inclination with equatorial plane of Earth is 90° and a satellite in polar orbit will pass over both the north and south geographic poles once per orbit. Polar satellites are Sun–synchronous satellites. Every location on Earth lies within the observation of polar satellite twice each day. The polar satellites are used for getting the cloud images, atmospheric data, ozone layer in the atmosphere and to detect the ozone hole over Antarctica.

Only the equatorial orbits are stable for a satellite. For any satellite to orbit around in a stable orbit, it must move in such an orbit so that the centre of Earth lies at the centre of the orbit.

Binding Energy

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is closed or different parts of the system are bound to each other.

Binding Energy of Satellite (System):

$$B.E. = -T.E.$$

$$B.E. = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} = \frac{L^2}{2mr^2}$$

Hence $B.E. = K.E. = -T.E. = \frac{-\text{P.E.}}{2}$

Work Done in Changing the Orbit of Satellite:

W = Change in mechanical energy of system but $E = \frac{-GMm}{2r}$, so $W = E_2 - E_1 = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

Illustration 136:

A satellite moves eastwards very near to the surface of the Earth in equatorial plane with speed (v_0). Another satellite moves at the same height with the same speed in the equatorial plane but westwards. If R = radius of the Earth and ω be its angular speed of the Earth about its own axis. Then find the approximate difference in the two time period as observed on the Earth.

Solution:

$$T_{west} = \frac{2\pi R}{v_0 + R\omega}$$

and $$T_{east} = \frac{2\pi R}{v_0 - R\omega}$$

$$\Rightarrow \quad \Delta T = T_{east} - T_{west} = 2\pi R\left[\frac{2R\omega}{v_0^2 - R^2\omega^2}\right] = \frac{4\pi\omega R^2}{v_0^2 - R^2\omega^2}$$

Illustration 137:

Find W_{ext} to change orbital radius of satellite from r_1 to r_2.

Solution:

$$W_{ext} + W_G = KE_f - KE_i$$

$$W_{ext} + U_i - U_f = KE_f - KE_i$$

$$W_{ext} = (K_f + U_f) - (K_i + U_i) = TE_f - TE_i$$

$$W_{ext} = \frac{-GM_pM_S}{2r_2} - \left(\frac{-GM_pM_S}{2r_1}\right)$$

Illustration 138:

A satellite of mass m, initially at rest on the Earth, is launched into a circular orbit at a height equal to the radius of the Earth. The minimum energy required is

(A) $\frac{\sqrt{3}}{4}mgR$ (B) $\frac{1}{2}mgR$ (C) $\frac{1}{4}mgR$ (D) $\frac{3}{4}mgR$

Ans. (D)

Solution:

From law of energy conservation

$$U_i + K_i = U_f + K_f$$

$$\frac{-GMm}{R} + K_i = \frac{-GMm}{2R} + \frac{1}{2}mV_0^2$$

$$K_i = \frac{GMm}{2R} + \frac{1}{2}m\left(\sqrt{\frac{Gm}{2R}}\right)^2$$

$$K_i = \frac{3GMm}{4R} = \frac{3}{4}mgR$$

Illustration 139:

An artificial satellite (mass m) of a planet (mass M) revolves in a circular orbit whose radius is n times the radius R of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming the force of resistance on satellite to depend on velocity as $F = av^2$ where 'a' is a constant, calculate how long the satellite will stay in the space before it falls onto the planet's surface.

Solution:

Air resistance $F = -av^2$, where orbital velocity $v = \sqrt{\frac{GM}{r}}$

r = The distance of the satellite from planet's centre $\Rightarrow F = -\frac{GMa}{r}$

The work done by the resistance force $dW = Fdx = Fvdt = \frac{GMa}{r}\sqrt{\frac{GM}{r}}dt = \frac{(GM)^{3/2}a}{r^{3/2}}dt$ ...(i)

The loss of energy of the satellite = dE $\therefore \frac{dE}{dr} = \frac{d}{dr}\left[-\frac{GM\text{ m}}{2r}\right] = \frac{GMm}{2r^2}$ $\Rightarrow dE = \frac{GMm}{2r^2}dr$ ...(ii)

Since, $dE = -\,dW$ (work energy theorem) $-\dfrac{GMm}{2r^2}\,dr = \dfrac{(GM)^{3/2}}{r^{3/2}}\,adt$

$$\Rightarrow \qquad t = -\frac{m}{2a\sqrt{GM}}\int_{nR}^{R}\frac{dr}{\sqrt{r}} = \frac{m\sqrt{R}\left(\sqrt{n}-1\right)}{a\sqrt{GM}} = \left(\sqrt{n}-1\right)\frac{m}{a\sqrt{gR}}$$

Illustration 140:

Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 h and 8h respectively. The radius of the orbit of S_1 is 10^4 km. When S_2 is closest to S_1, find:

(a) the speed of S_2 relative to S_1 and

(b) the angular speed of S_2 as observed by an astronaut in S_1.

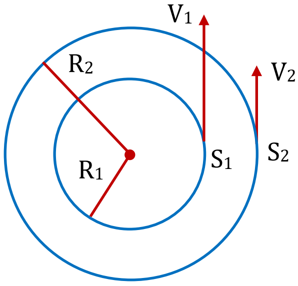
Solution:

Let the mass of the planet be M, that of S_1 be m_1 and S_2 be m_2. Let the radius of the orbit of S_1 be $R_1(= 10^4\ km)$ and of S_2 be R_2.

Let v_1 and v_2 be the linear speeds of S_1 and S_2 with respect to the planet. Figure shows the situation.

As the square of the time period is proportional to the cube of the radius.

$$\left(\frac{R_2}{R_1}\right)^3 = \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{8h}{1h}\right)^2 = 64$$



or, $\dfrac{R_2}{R_1} = 4$

or, $R_2 = 4R_1 = 4 \times 10^4\ m.$

Now the time period of S_1 is 1 h. So,

$$\frac{2\pi R_1}{v_1} = 1h$$

or, $v_1 = \dfrac{2\pi R_i}{1h} = 2\pi \times 10^4\ km/h$

Similarly, $v_2 = \dfrac{2\pi R_2}{8h} = \pi \times 10^4\ km/h$

(a) At the closest separation, they are moving in the same direction. Hence the speed of S_2 with respect to S_1 is $|v_2 - v_1| = \pi \times 10^4\ km/h.$

(b) As seen from S_1, the satellite S_2 is at a distance $R_2 - R_1 = 3 \times 10^4\ km$ at the closest separation. Also, is moving at $\pi \times 10^4\ km/h$ in a direction perpendicular to the line joining them. Thus, the angular speed of S_2 as observed by S_1 is

$$\omega = \frac{\pi \times 10^4\ km/h}{3 \times 10^4\ km/h} = \frac{\pi}{3}\,rad/h.$$

Illustration 141:

A space–ship is launched into a circular orbit close to the Earth's surface. What additional speed should now be imparted to the spaceship so that orbit to overcome the gravitational pull of the Earth.

Solution:

Let ΔK be the additional kinetic energy imparted to the spaceship to overcome the gravitation pull then by energy conservation

$$-\frac{GMm}{2R} + \Delta K = 0 + 0$$

$$\Rightarrow \quad \Delta K = \frac{GMm}{2R}$$

Total kinetic energy = $\frac{GMm}{2R} + \Delta K = \frac{GMm}{2R} + \frac{GMm}{2R} = \frac{GMm}{R}$

Then, $\frac{1}{2}mv_2^2 = \frac{GMm}{R}$

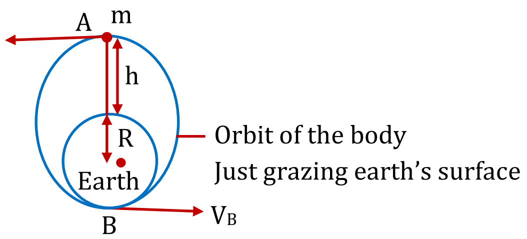
$$\Rightarrow \quad v_2 = \sqrt{\frac{2GM}{R}}$$

But, $v_1 = \sqrt{\frac{GM}{R}}$

So, Additional velocity = $v_2 - v_1 = \sqrt{\frac{2GM}{R}} - \sqrt{\frac{GM}{R}} = \left(\sqrt{2}-1\right)\sqrt{\frac{GM}{R}}$

Illustration 142:

For a particle projected in a transverse direction from a height h above Earth's surface, find the minimum initial velocity so that it just grazes the surface of Earth path of this particle would be an ellipse with center of Earth as the farther focus, point of projection as the apogee and a diametrically opposite point on Earth's surface as perigee.



Solution:

Suppose velocity of projection at point A is v_A and at point B, the velocity of the particle is v_B.

then applying Newton's 2nd law at point A and B, we get, $\frac{mv_A^2}{r_A} = \frac{GM_e m}{(R+h)^2}$ and $\frac{mv_B^2}{r_B} = \frac{GM_e m}{R^2}$

Where r_A and r_B are radius of curvature of the orbit at points A and B of the ellipse,

But $r_A = r_B = r$(say).

Now applying conservation of energy at points A and B

$$\frac{-GM_e m}{R+h} + \frac{1}{2}mv_A^2 = \frac{-GM_e m}{R} + \frac{1}{2}mv_B^2$$

$$\Rightarrow \quad GM_e m\left(\frac{1}{R} - \frac{1}{(R+h)}\right) = \frac{1}{2}\left(mv_B^2 - mv_A^2\right) = \left(\frac{1}{2}rGM_e m\left(\frac{1}{R^2} - \frac{1}{(R+h)^2}\right)\right)$$

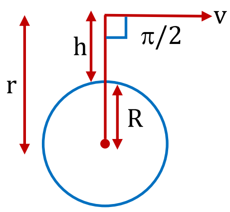
or, $r = \dfrac{2R(R+h)}{2R+h} = \dfrac{2Rr}{R+r}$

$\therefore$ $V_A^2 = \dfrac{rGM_e}{(R+h)^2} = 2GM_e \dfrac{R}{r(r+R)}$

where r = Distance of point of projection from Earth's centre = $R + h$.

Conditions for Different Trajectory

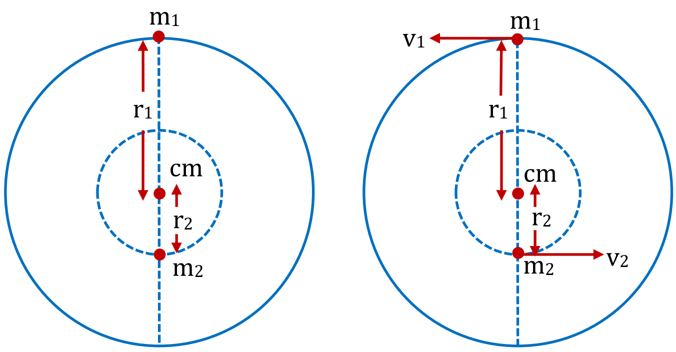
For a body being projected tangentially from above Earth's surface, say at a distance r from Earth's center, the trajectory would depend on the velocity of projection v.



	Velocity	Orbit
1.	Velocity, $v < \sqrt{\dfrac{GM}{r}\left(\dfrac{2R}{r+R}\right)}$	Body returns to Earth
2.	$\sqrt{\dfrac{GM}{r}} > v > \sqrt{\dfrac{GM}{r}\left(\dfrac{2R}{r+R}\right)}$	Body acquires an elliptical orbit with Earth as the far-focus w.r.t. the point of projection.
3.	Velocity is equal to the critical velocity of the orbit, $v = \sqrt{\dfrac{GM_e}{r}}$	Circular orbit with radius r
4.	Velocity is between the critical and escape velocity of the orbit $\sqrt{\dfrac{2GM_e}{r}} > v > \sqrt{\dfrac{GM_e}{r}}$ i.e $TE < 0$	Body acquires an elliptical orbit with Earth as the near focus w.r.t. the point of projection.
5.	$v = v_{esc} = \sqrt{\dfrac{2GM_e}{r}}$ i.e $TE = 0$	Body just escapes Earth's gravity, along a parabolic path.
6.	$v > v_{esc} = \sqrt{\dfrac{2GM_e}{r}}$ i.e $TE > 0$	Body escape Earth's gravity along a hyperbolic path.

Double (Binary) Star System

If two stars of comparable mass revolve about their common centre of mass under the influence of mutual gravitational force then this system is called binary star system.



For centre of mass to remain at rest both the stars should rotate with same 'ω'.

Let distance of centre of mass for m_1 is r_1 and m_2 is r_2. Then

$m_1r_1 = m_2r_2$ ...(i)

$r_1 + r_2 = r$ ...(ii)

From (i) and (ii)

$$r_1 = \frac{m_2 r}{m_1 + m_2}$$

$$r_2 = \frac{m_1 r}{m_1 + m_2}$$

$T_1 = T_2 \quad \Rightarrow \quad \omega_1 = \omega_2$ ...(iii)

$$\frac{Gm_1m_2}{r^2} = F = m_1\omega_1^2 r_1 \quad \text{...(iv)}$$

$$\frac{Gm_1m_2}{r^2} = m_2\omega_2^2 r_2 \quad \text{...(v)}$$

$$\therefore \quad \frac{Gm_1(m_1+m_2)}{r^2 m_1 r} = \omega_2^2$$

$$\frac{G(m_1+m_2)}{r^3} = \omega_1^2$$

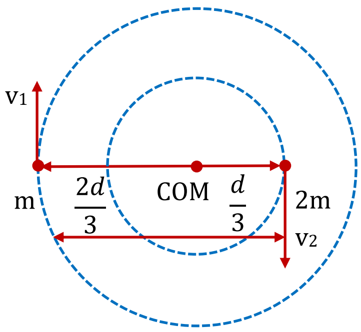
Angular momentum of system

$L = m_1v_1r_1 + m_2v_2r_2$

$$L = \left(\frac{m_1m_2r}{m_1+m_2}\right)\omega(r)$$

$l = \mu\omega r^2$

where $\mu = \dfrac{m_1m_2}{m_1+m_2}$ is the reduced mass of system.



A pair of stars rotating about their COM (centre of mass)

$$F = \frac{GM(2m)}{d^2}$$

	Star-1	Star-2	Ratio
1.	$F=\dfrac{mv_1^2}{2d/3}$	$F=\dfrac{mv_1^2}{d/3}$	$\dfrac{v_1}{v_2}=\dfrac{2}{1}$
	$G\dfrac{M_2m}{d^2}=\dfrac{mv_1^2}{2d/3}$	$G\dfrac{M_2m}{d^2}=\dfrac{mv_2^2}{d/3}$	
	$v_1=2\sqrt{\dfrac{Gm}{3d}}$	$v_2=\sqrt{\dfrac{Gm}{3d}}$	
2.	$T_1=\dfrac{2\pi}{v_1}\dfrac{2d}{3}$	$T_2=\dfrac{2\pi}{v_2}\dfrac{d}{3}$	$\dfrac{T_1}{T_2}=\dfrac{1}{1}$
3.	$KE_1=\dfrac{1}{2}mv_1^2$	$KE_2=\dfrac{1}{2}mv_2^2$	$\dfrac{KE_1}{KE_2}=\dfrac{2}{1}$
4.	$L_1=mv_1\dfrac{2d}{3}$	$L_2=2mv_2\dfrac{d}{3}$	$\dfrac{L_1}{L_2}=\dfrac{2}{1}$

Illustration 143:

In a double star, two stars (one of mass m and the other of $2m$) distant d apart rotate about their common centre of mass. Deduce an expression of the period of revolution. Show that the ratio of their angular momentum about the centre of mass is the same as the ratio of their kinetic energies.

Solution:

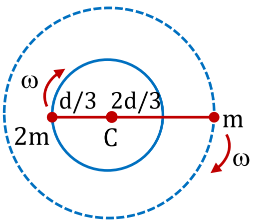
The centre of mass C will be at distances $d/3$ and $2d/3$ from the masses $2m$ and m respectively. Both the stars rotate round C in their respective orbits with the same angular velocity ω. The gravitational force acting on each star due to the other supplies the necessary centripetal force.

The gravitational force on either star is $\dfrac{G(2m)m}{d^2}$. If we consider the rotation of the smaller star, the centripetal force ($mr\omega^2$) is $\left[m\left(\dfrac{2d}{3}\right)\omega^2\right]$ and for bigger star $\left[\dfrac{2md\omega^2}{3}\right]$ i.e. same

$$\therefore \quad \frac{G(2m)m}{d^2}=m\left(\frac{2d}{3}\right)\omega^2 \quad \text{or} \quad \omega=\sqrt{\left(\frac{3Gm}{d^3}\right)}$$

Therefore, the period of revolution is given by

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\left(\frac{d^3}{3Gm}\right)}$$



The ratio of the angular momentum is

$$\frac{(I\omega)_{big}}{(I\omega)_{small}}=\frac{I_{big}}{I_{small}}=\frac{(2m)\left(\frac{d}{3}\right)^2}{m\left(\frac{2d}{3}\right)^2}=\frac{1}{2}$$

Since ω is same for both. The ratio of their kinetic energies is $\dfrac{\left(\frac{1}{2}I\omega^2\right)_{\text{big}}}{\left(\frac{1}{2}I\omega^2\right)_{\text{small}}}=\dfrac{I_{\text{big}}}{I_{\text{small}}}=\dfrac{1}{2}$, which is the same as the ratio of their angular momentum.

Gravitational Pressure

Illustration 144:

A uniform sphere has a mass M and radius R. Find the pressure p inside the sphere, caused by gravitational compression, as a function of the distance r from its centre. Evaluate p at the centre of the Earth, assuming it to be a uniform sphere.

Ans. $p = \frac{3}{8}(1 - r^2/R^2)\, GM^2/\pi R^4$. About 1.8×10^8 atmospheres.

Solution:

Consider an element of a cylinder extending from centre to surface.

$$p = \int \frac{dmg}{A} = -\int_R^r \frac{GMr}{R^3} \times 4\pi r^2 \rho dr$$

$$= \frac{4\pi G\rho^2}{3}\frac{\left[R^2 - r^2\right]}{2} = \frac{2\pi G\rho^2}{3}\left[R^2 - r^2\right]$$

Illustration 145:

A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If $P(r)$ is the pressure at $r(r < R)$, then the correct option(s) is(are) :-

(A) $P(r = 0) = 0$

(B) $\frac{P(r = 3R/4)}{P(r = 2R/3)} = \frac{63}{80}$

(C) $\frac{P(r = 3R/5)}{P(r = 2R/5)} = \frac{16}{21}$

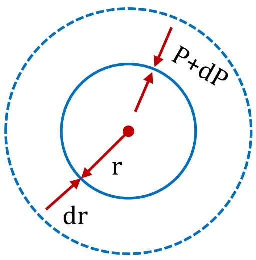
(D) $\frac{P(r = R/2)}{P(r = R/3)} = \frac{20}{27}$

Ans. (BC)

Solution:

$$dF = [P - (P + dP)]A$$

$$\Rightarrow \quad \frac{Gm}{r^2} dm = -(dP)A$$



$$\Rightarrow \quad \int_0^P dP = -\int_R^r \frac{G\frac{Mr^3}{R^3} 4ar^2 dr\rho}{r^2(4\pi r^2)}$$

$$\therefore \quad P = \frac{G\rho M}{2R}\left(1 - \frac{r^2}{R^2}\right)$$