02 Simple Harmonic Motion

Periodic Motion

- Any motion which repeats itself after regular intervals of time is called periodic motion.
- The constant interval of time after which the motion is repeated is called time period.

  Example:

  (1) Motion of planets around the sun. (2) Motion of the pendulum of wall clock.

Oscillatory Motion

- The motion of a body is said to be oscillatory or vibratory motion if it moves back and forth (to and from) about a fixed point after certain intervals of time.
- The fixed point about which the body oscillates is called mean position or equilibrium position.

  Example:

  (1) Vibration of the wire of 'Sitar'.

  (2) Oscillation of the mass suspended from spring.
- Every oscillatory motion need not to be periodic also every periodic motion need not to be oscillatory.

Harmonic Functions

The trigonometric function of constant amplitude and single frequency is defined as harmonic function. (Among all the trigonometrical functions only "sin" and "cos" functions are taken as harmonic function in basic form")

$$\left.\begin{array}{l} y = A\sin\theta = A\sin\omega t \\ y = A\cos\theta = A\cos\omega t \end{array}\right\}\text{Harmonic function}$$

Note:

The function will be non-harmonic if:

(i) Its amplitude is not constant. (ii) It is basically formed by "tan", "cot", "sec", "cosec" functions.

Period

The smallest interval of time after which the motion is repeated is called period.

Example:

(1) Period of oscillation of Earth is 1 year or 365 days.

(2) The Halley's comet appears after every 76 years, so its period is 76 years.

- $f(t) = A\cos(\omega t)$

If the argument of this function, ωt, is increased by an integral multiple of 2π radians the value of the function remains the same. The function $f(t)$ is then periodic and its period, T is given by

$$T = \frac{2\pi}{\omega}$$

The same result is obviously correct if we consider a sine function, $f(t) = a\sin(\omega t)$. Further, a linear combination of sine and cosine functions like $f(t) = A\cos(\omega t) + B\sin(\omega t)$ is also a periodic function with the same period T.

Illustration 1:

Which of the following functions of time represent (a) periodic and (b) non-periodic motion? Give the period for each case of periodic motion [ω is any positive constant].

(i) $\sin(\omega t) + \cos(\omega t)$ (ii) $\sin(\omega t) + \cos(2\omega t) + \sin(4\omega t)$

Solution:

(i) $\sin(\omega t) + \cos(\omega t)$ is a periodic function, it can also be written as $\sqrt{2}\sin(\omega t + \frac{\pi}{4})$

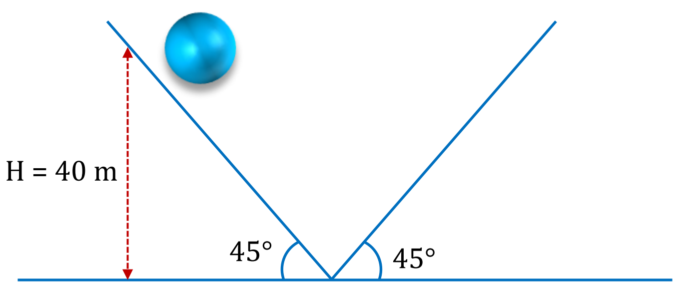
Now $\sqrt{2}\sin(\omega t + \frac{\pi}{4}) = \sqrt{2}\sin(\omega t + \frac{\pi}{4} + 2\pi) = \sqrt{2}\sin\{\omega(t + \frac{\pi}{4\omega} + \frac{2\pi}{\omega})\}$

The periodic time of the function $\frac{2\pi}{\omega}$

(ii) Each term represents a periodic function with a different angular frequency, $\sin(\omega t)$ has a period $T = \frac{2\pi}{\omega}$; $\cos(2\omega t)$ has a period $t = \frac{\pi}{\omega} = \frac{T}{2}$; and $\sin(4\omega t)$ has a period $\frac{2\pi}{4\omega} = \frac{T}{4}$. The period of the first term is a multiple of the periods of the last two terms. Therefore, the smallest interval of time after which the sum of the three terms repeats is T, and thus, the sum is a periodic function with a period $\frac{2\pi}{\omega}$.

Illustration 2:

Ball is released from rest. What is Time Period of Oscillation? All surfaces are frictionless and neglect energy losses at the connecting link of two inclined planes.



Solution:

By Energy conservation

$$mgH = mgh$$

$$H = h$$

Let T_1 be the time for ball to slide down Incline -1 and T_2 for ball to climb up Incline – 2.

$$\frac{H}{\sin(45)} = \frac{1}{2} g \sin(45^0) T_1^2$$

$$T_1 = \sqrt{\frac{2H}{g \sin^2(45^0)}}$$

Similarly,

$$T_2 = \sqrt{\frac{2H}{g \sin^2(45^0)}}$$

Time of oscillation = $2(T_1 + T_2)$

Time of oscillation = 16 second

Frequency

The reciprocal of T gives the number of repetitions that occur per unit time. This quantity is called the frequency of the periodic motion. It is represented by the symbol f. The relation between f and T is

$$f = \frac{1}{T}$$

Simple Harmonic Motion

Simple harmonic motion is the simplest form of vibratory or oscillatory motion.

The conditions of S.H.M are-

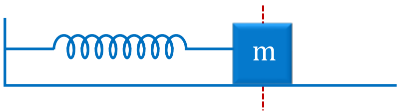
- The magnitude of force (F) is directly proportional to the distance of the body from a fixed point.
- The direction of force is always towards the fixed point.

S.H.M. are of Two Types

(a) Linear S.H.M.

When a particle moves to and fro about a fixed-point (called equilibrium position) along a straight line then its motion is called linear simple harmonic motion.

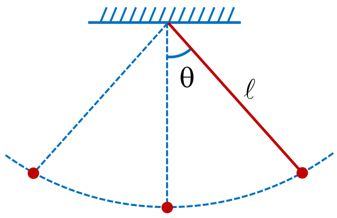
Example : Motion of a mass connected to spring.



(b) Angular S.H.M.

When a system oscillates angularly with respect to a fixed axis then its motion is called angular simple harmonic motion.

Example : Motion of a bob of simple pendulum.



Linear S.H.M

The restoring force (or acceleration) acting on the particle should always be proportional to the displacement of the particle from equilibrium position and directed towards the equilibrium position

$\therefore \quad F \propto -x \quad$ or $\quad a \propto -x$

Negative sign shows that direction of force and acceleration is towards equilibrium position and x is displacement of particle from equilibrium position.

$F = -kx$

$k \rightarrow$ Force constant of the S.H.M.

Units: S.I. Unit: N/m

CGS Unit: $dyne/cm$

Equation of SHM

The necessary and sufficient condition for SHM is

$F = -kx$

where k = positive constant for a SHM = Force constant

x = displacement from mean position

$\Rightarrow \quad -kx = ma$

$\Rightarrow \quad m \times \frac{d^2x}{dt^2} = -kx \quad \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad$ [differential equation of SHM]

$\Rightarrow \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad$ where $\omega = \sqrt{\frac{k}{m}}$

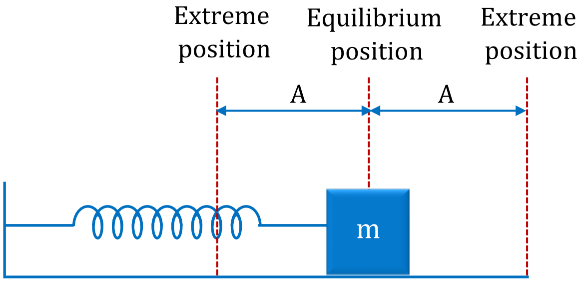
Its solution is $x = A\sin(\omega t + \phi)$

A = Amplitude

ω = Angular Frequency

ϕ = Initial Phase

Important Terms of S.H.M.



(1) Amplitude (A)

The maximum displacement of a particle performing S.H.M. from its mean position on either side is called amplitude.

S.I. unit : *m*

C.G.S. unit : *cm*

Dimensions : $[M^0L^1T^0]$

(2) Range or path length of S.H.M. ($2A$)

The distance between two extreme positions of a particle performing S.H.M. is called as path length or range of S.H.M.

S.I. unit : *m*

C.G.S. unit : *cm*

Dimensions : $[M^0L^1T^0]$

- It is the distance equal to twice the amplitude i.e. $2A$.

(3) Period (T)

The time taken by a particle performing S.H.M to complete one oscillation is called as period of S.H.M.

S.I. Unit : second

Dimensions : $[M^0L^0T^1]$

- $T = \dfrac{2\pi}{\omega}$

(4) Frequency (n)

The number of oscillations of a particle performing S.H.M. in one second is called as frequency.

S.I. Unit : hertz (*Hz*)

Dimensions : $[M^0L^0T^{-1}]$

- $f = \dfrac{1}{T}$

Kinematics of SHM

Velocity Analysis

It is the rate of change of particle displacement with respect to time at that instant.

Let the displacement from mean position is given

By $x = A \sin(\omega t + \phi)$ ...(A)

velocity $v = \dfrac{dx}{dt} = A\omega \cos(\omega t + \phi)$

$v = A\omega \cos(\omega t + \phi)$ ...(B)

At mean position ($x = 0$), velocity is maximum.

$V_{\max} = \pm\omega A$

At extreme position ($x = A$), velocity is minimum.

$v_{\min} = \text{Zero}$

Relation between v and x :

From equation (A) and (B)

$$\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)=1$$

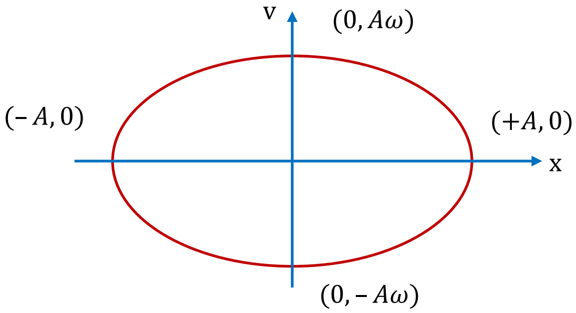
$$\frac{x^2}{A^2}+\frac{V^2}{(A\omega)^2}=1$$

$$\frac{V^2}{A^2\omega^2}=1-\frac{x^2}{A^2}\Rightarrow V^2=\omega^2\left(A^2-x^2\right)$$

$$V=\omega\sqrt{A^2-x^2}$$

V_{max} at x = 0 (mean position)

$$V_{max}=\pm A\omega$$



Acceleration Analysis:

It is the rate of change of particle's velocity w.r.t. time at that instant.

$x = A\sin(\omega t + \phi)$ ...(A)

$V = A\omega\cos(\omega t + \phi)$ ...(B)

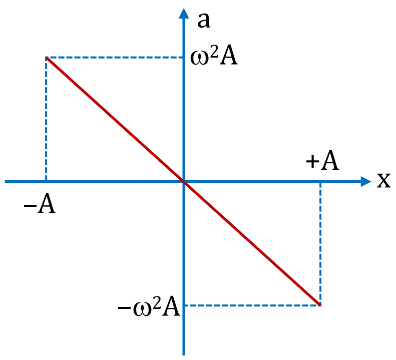
Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d}{dt}[A\omega\cos(\omega t + \phi)]$

$a=-\omega^2 A\sin(\omega t+\phi)$ ...(C)

$$a=-\omega^2 x$$

$$a_{max}=\mp\omega^2 A$$

Graph between acceleration and position:



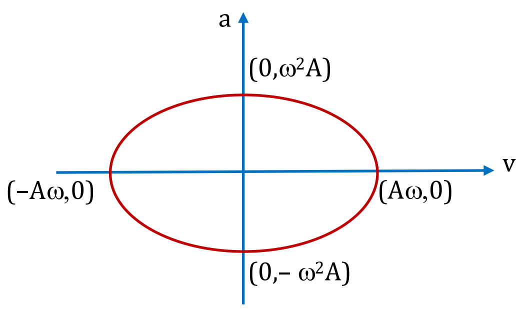
Relation between velocity and acceleration:

From equation (B) and (C)

$$\frac{V^2}{A^2\omega^2}+\frac{a^2}{\left(A\omega^2\right)^2}=1$$

$$a_{max}=\mp\omega^2 A$$

Graph between velocity and acceleration:



Note:

Negative sign shows that acceleration is always directed towards the mean position. At mean position (x = 0), acceleration is minimum.

$$a_{min}=\text{zero}$$

At extreme position ($x = A$), acceleration is maximum.

$$|a_{max}|=\omega^2 A$$

Graphical Representation of Displacement, Velocity & Acceleration in SHM

Displacement, $x = A \sin \omega t$

Velocity, $v = A\,\omega \cos \omega T = A\,\omega \sin\left(\omega t + \frac{\pi}{2}\right)$

or $\quad v = \omega\sqrt{A^2 - x^2}$

Acceleration, $a = -\omega^2 A \sin \omega t = \omega^2 A \sin(\omega t + \pi)$

or $\quad a = -\omega^2 x$

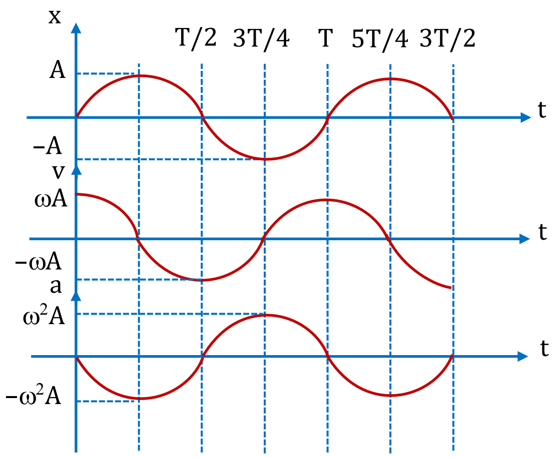
Note:

- $v = \omega\sqrt{A^2 - x^2}$

  $a = -\omega^2 x$

These relations are true for any equation of *x*.

Time, t	0	T/4	T/2	3T/4	T
Displacement, *x*	0	A	0	–A	0
Velocity, *v*	Aω	0	–Aω	0	Aω
Acceleration, *a*	0	$-\omega^2$A	0	ω^2A	0



1. All the three quantities displacement, velocity and acceleration vary harmonically with time, having same period.
2. The velocity amplitude is ω times the displacement amplitude $(v_{max} = \omega A)$.
3. The acceleration amplitude is ω^2 times the displacement amplitude $(a_{max} = \omega^2 A)$.
4. In SHM, the velocity is ahead of displacement by a phase angle of $\frac{\pi}{2}$.
5. In SHM, the acceleration is ahead of velocity by a phase angle of $\frac{\pi}{2}$.

Illustration 3:

The equation of particle executing simple harmonic motion is $x = (5\ m)\ sin\left[(\pi s^{-1})t + \frac{\pi}{3}\right]$. Write down the amplitude, time period and maximum speed. Also find the velocity at $T = 1s$.

Solution:

Comparing with equation $x = A \sin(\omega t + \phi)$, we see that the amplitude = 5 m and

$$\omega = \pi s^{-1}$$

The velocity at time, $t \Rightarrow \frac{dx}{dt} = A\omega \cos(\omega t + \delta)$

At $\quad t = 1\ s,$

$$V = (5\ m)\ (\pi\ s^{-1})\ cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2} m/s$$

Illustration 4:

A particle is performing SHM of amplitude, *"A"* and time period *"T"*. Find the time taken by the particle to go from 0 to $A/2$.

Solution:

Let equation of SHM be $x = A \sin \omega t$

when $\quad x = 0,\ t = 0$

when $\quad x = A/2\ ;\ A/2 = A \sin \omega t$

or $\quad \sin \omega t = 1/2\ ;\ \omega t = \pi/6$

$$\frac{2\pi}{T} t = \pi/6\ ;\ t = T/12$$

Hence, time taken is $T/12$ where T is time period of SHM.

Note:

If mean position is not at the origin, then we can replace x by $x - x_0$ and the equation becomes $x - x_0 = -A \sin \omega t$, where x_0 is the position co-ordinate of the mean position.

Illustration 5:

A particle of mass 2 kg is moving on a straight line under the action of a force $F = (8 - 2x)\ N$. It is released at rest from $x = 6m$.

(A) Is the particle moving simple harmonically?

(B) Find the equilibrium position of the particle.

(C) Write the equation of motion of the particle.

(D) Find the time period of SHM.

Solution:

$$F = 8 - 2x \quad \text{or} \quad F = -2(x - 4)$$

For equilibrium position $F = 0$

$\Rightarrow \quad x = 4m$ is equilibrium position.

Hence the motion of particle is SHM with force constant 2 and equilibrium position $x = 4$.

(A) Yes, motion is SHM.

(B) Equilibrium position is $x = 4\ m$.

(C) At $x = 6\ m$, particle at rest i.e. it is one of the extreme position. Hence amplitude is $A = 2\ m$ and initially particle at the extreme position.

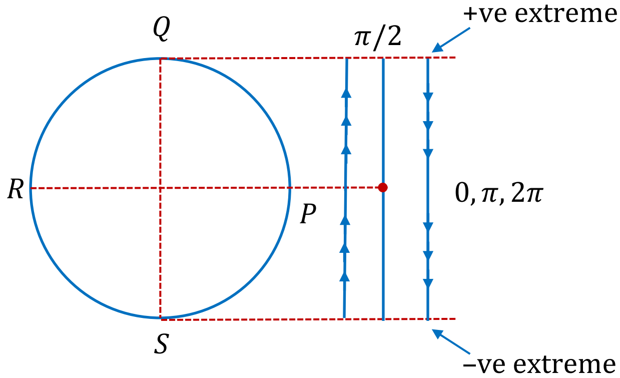
$\therefore$ Equation of SHM can be written as

$$x - 4 = 2\cos\omega t, \text{ where } \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{2}} = 1\ (\text{sec})^{-1} \quad \text{i.e. } x = 4 + 2\cos t$$

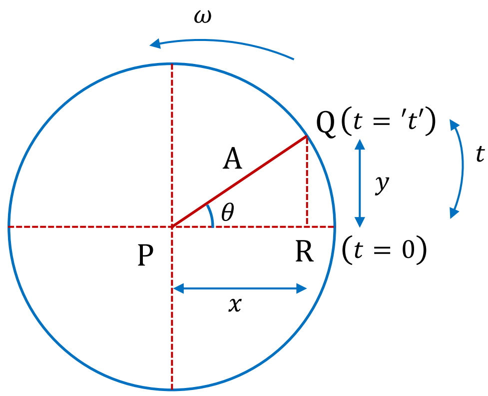
(D) Time period, $T = \dfrac{2\pi}{\omega} = 2\pi\,\text{sec}.$

SHM as a Projection of Uniform Circular Motion

A particle is moving in anticlockwise direction in a circle whose projection is performing SHM in vertical diameter. Let particle starts from position P (that is a mean position) and moves in anticlockwise direction. As it moves from position P to position Q on circle, then its projection moves from mean position to positive extreme position. It moves from position Q to position R on circle, its projection moves from positive extreme to mean position. If particle moves from position R to position S, then its projection moves from mean position to negative extreme. Now particle moves from position S to position P, its projection moves from negative extreme to mean position.



UCM	SHM
From P to Q	MP to the extreme
From Q to R	Positive extreme to MP
From R to S	MP to negative extreme
From S to P	Negative extreme to MP



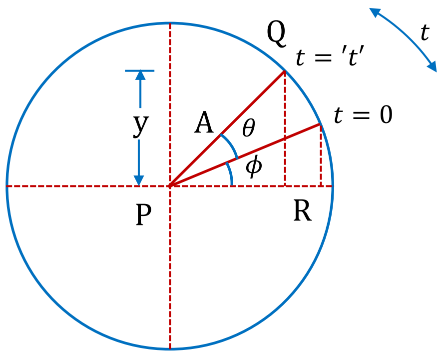
$(\theta = \omega t)$

In triangle PQR

$\sin\theta = \dfrac{QR}{PQ}$ & $\cos\theta = \dfrac{PR}{PQ}$

$\sin\theta = \dfrac{y}{A}$ & $PR = PQ\cos\theta$

$y = A\sin\theta$ & $x = A\cos\theta$



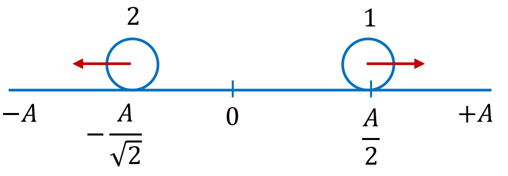
In ΔPQR

$\sin(\theta + \phi) = \dfrac{y}{A}$

$\boxed{y = A\sin(\theta + \phi)}$

Illustration 6:

Write the equation of SHM for particle 1 and 2 as shown in figure. It is given that angular frequency is ω and given diagram is at $t = 0$.



Solution:

Method-1:

Particle 1 is going from mean to positive extreme and is at displacement $A/2$ initially (at $t = 0$) so its equation is

$$x = A\sin(\omega t + \phi)$$

$\Rightarrow \quad \frac{A}{2} = A\sin\phi \quad \Rightarrow \phi = \frac{\pi}{6}$

So, equation of SHM for particle 1 is

$$\left(x = A\sin\left(\omega t + \frac{\pi}{6}\right)\right)$$

For particle 2, at $t = 0$

$$x = A \sin(\omega t + \phi + \pi)$$

$$x = -A \sin(\omega t + \phi)$$

$\Rightarrow \quad -\frac{A}{\sqrt{2}} = -A\sin\phi$

$\Rightarrow \quad \phi = \frac{\pi}{4}$

So, its phase is $\left(\pi + \frac{\pi}{4}\right) = \frac{5\pi}{4}$

$\therefore \quad$ Equation for particle 2 is $x = A\sin\left(\omega t + \frac{5\pi}{4}\right)$

Method-2:

For particle -1

$$\phi_1 = \frac{\pi}{6}$$

For particle -2

$$\phi_2 = \frac{5\pi}{4}$$

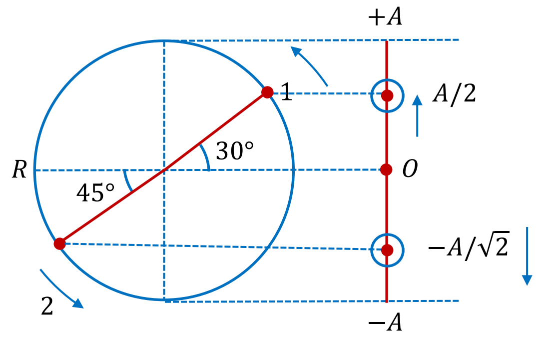
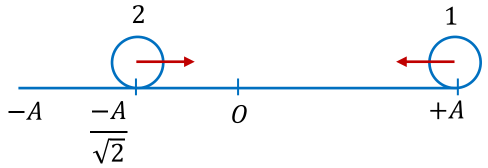


Illustration 7:

Find time taken by particle in moving from position (1) to position (2)? It is given that angular frequency is ω and the situation shown here is at t = 0. Time period of SHM is T.



Solution:

Phase difference between 1 and 2 is

$$\Delta\phi = \frac{5\pi}{4}$$

$$\frac{2\pi}{T}\cdot t = \frac{5\pi}{4}$$

$$t = \frac{5T}{8}.$$

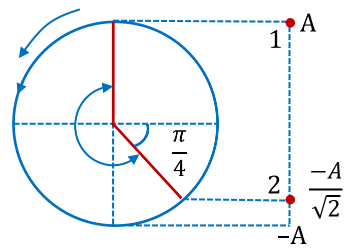
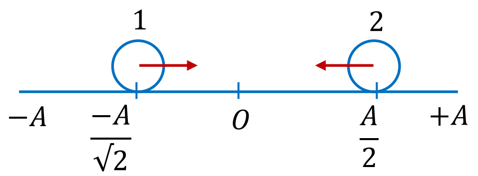


Illustration 8:

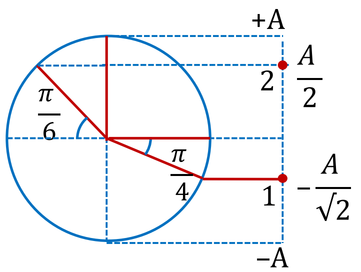
Find phase difference between two particles of phase as shown in figure. It is given that angular frequency is ω and the situation shown here is at t = 0.



Solution:

$$\Delta\phi = \frac{\pi}{6} + \frac{3\pi}{4}$$

$$\Delta\phi = \frac{11\pi}{12}$$



Two Particles doing SHM on Same Path

(Assume 'ω' same and A same)

ϕ different ⇒ Δϕ= constant

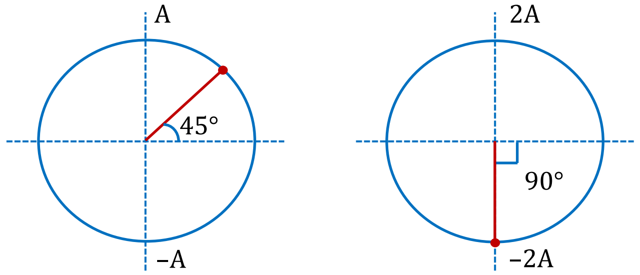
Illustration 9:

Two particles are in SHM on same straight line with amplitude A and $2A$ and with same angular frequency ω. It is observed that when first particle is at a distance $\frac{A}{\sqrt{2}}$ from origin and going toward mean position, other particle is at extreme position on other side of mean position. Find phase difference between the two particles.

(A) 45° (B) 90° (C) 135° (D) 180°

Ans. (C)

Solution:



$\Delta\phi = 135°$

Two Particles doing SHM on Same Path

(Assume 'ω' same and A same)

ϕ different $\Rightarrow \Delta\phi =$ constant

(A) Condition of Crossing:

$X_P = X_Q$ (when they both are at the same position w.r.t to mean position)

Particles will meet when their phasors are on same horizontal line.

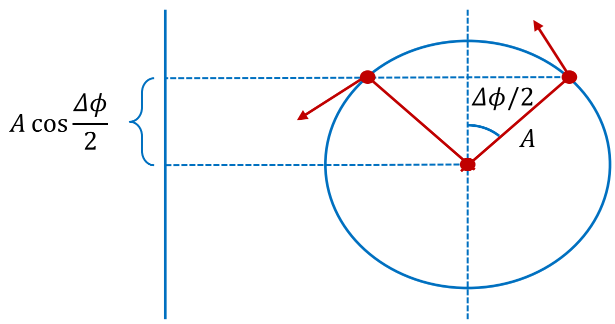


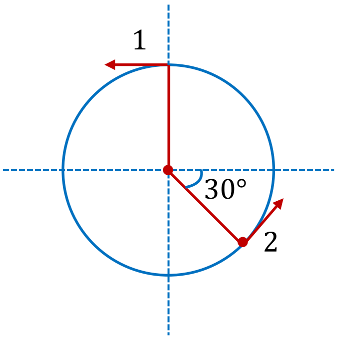
Illustration 10:

Two particles are in SHM in a straight line about same equilibrium position. Amplitude A and time period T of both the particles are equal. At time $t = 0$, one particle is at displacement $y_1 = +A$ and the other at $y_2 = -A/2$, and they are approaching towards each other. After what time they cross each other?

(A) $T/3$ (B) $T/4$ (C) $5T/6$ (D) $T/6$

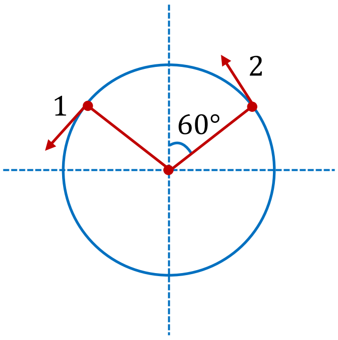
Solution:

They will cross each other when the phasor will lie on same Horizontal line.



at $t = 0$

$\Delta\phi = \frac{\pi}{3}$



at $t = t$

$\therefore \quad \Delta t = \frac{T}{6}$

(B) Maximum separation between them:

$\frac{dr_{rel.}}{dt} = v_{rel.}$ if r_{rel} is maximum

$\Rightarrow \quad \frac{dr_{rel.}}{dt} = v_{rel.} = 0$

$V_P = V_Q$

$\Rightarrow \quad X_P = -X_Q$

$\therefore$ Same distance from mean position.

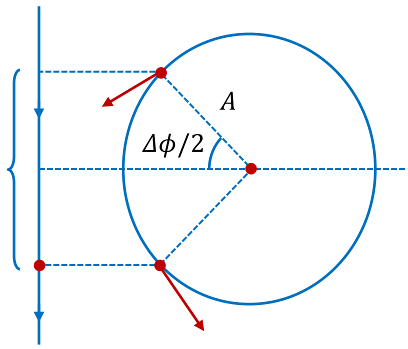


Illustration 11:

Two particles A and B perform SHM along the same straight line with the same amplitude 'a', same frequency 'f' and same equilibrium position 'O'. The greatest distance between them is found to be $\frac{3a}{2}$. At some instant of time they have the same displacement from mean position. What is the displacement?

(A) $a/2$ (B) $a\sqrt{7}/4$ (C) $\sqrt{3}a/2$ (D) $3a/4$

Solution:

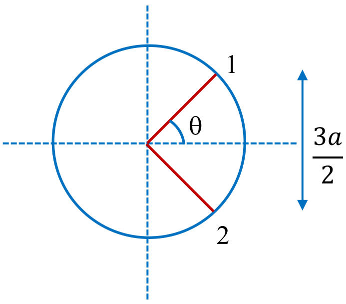
$a\sin\theta = \frac{3a}{4}$

$\sin\theta = \frac{3}{4}$

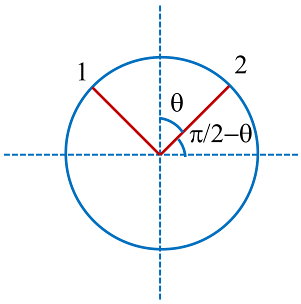
$\Rightarrow \quad \cos\theta = \frac{\sqrt{7}}{4}$

$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{\sqrt{7}}{4}$

$\Rightarrow \quad x = \frac{a\sqrt{7}}{4}$



Initially



At the time of collision

Relation between Displacement and Velocity

$\because \quad v = \omega\sqrt{A^2 - x^2} \Rightarrow v^2 = \omega^2\left(A^2 - x^2\right) \quad \Rightarrow \dfrac{v^2}{\omega^2} = A^2 - x^2$

$\Rightarrow \quad \dfrac{x^2}{A^2} + \dfrac{v^2}{(A\omega)^2} = 1$ (equation of an ellipse)

So, the graph between displacement and velocity is an ellipse.

If $\omega > 1 (A\omega > A)$ If $\omega < 1 (A\omega < A)$ If $\omega = 1 (A\omega = A)$

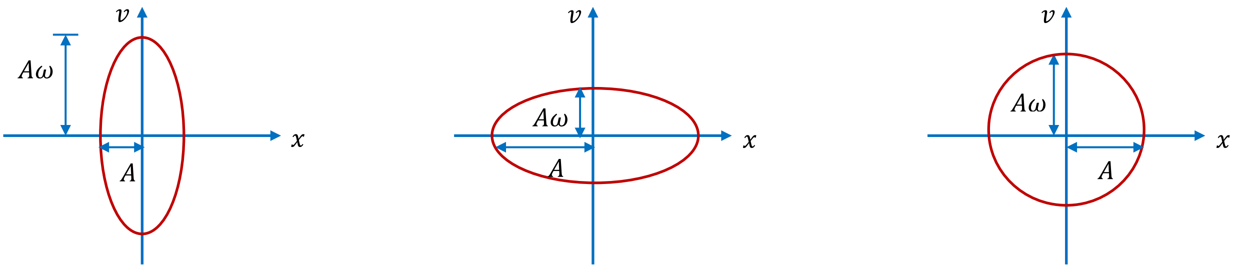
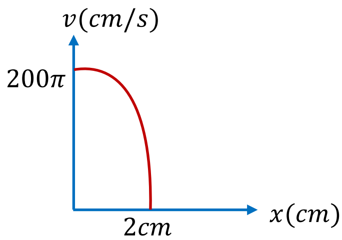


Illustration 12:

Find frequency of SHM for which graph between velocity and displacement is given:



Solution:

$A = 2cm$

$V_{max} = A\omega = 200\,\pi \quad \Rightarrow 2 \times \omega = 200\pi \quad \Rightarrow (2\pi f) = 100\pi \quad \Rightarrow f = 50\ Hz$

Illustration 13:

$v - x$ relation for a SHM is given as $\dfrac{x^2}{4} + \dfrac{v^2}{400\pi^2} = 1$, where x is in cm and v is in cm/s. Find the time period and amplitude of this SHM.

Solution:

$$\frac{x^2}{4} + \frac{v^2}{400\pi^2} = 1$$

Comparing this equation with $\dfrac{x^2}{A^2} + \dfrac{v^2}{(A\omega)^2} = 1$ gives

$$A^2 = 4 \text{ and } (A\omega)^2 = 400\pi^2$$

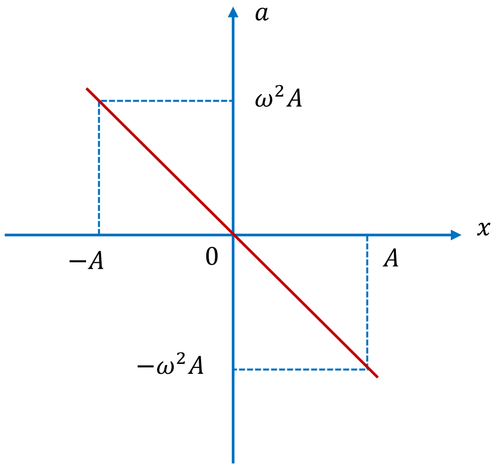
$\Rightarrow \quad A = 2$

and $\quad A\omega = 20\,\pi$

$\therefore \quad 2\omega = 20\pi \qquad \therefore \quad 2\omega = 20\pi$

$\Rightarrow \quad \dfrac{2\pi}{T} = 10\pi \qquad \Rightarrow \quad T = \left(\dfrac{1}{5}\right)$ *sec.* and A = 2 *cm.*

Relation between Displacement and Acceleration:

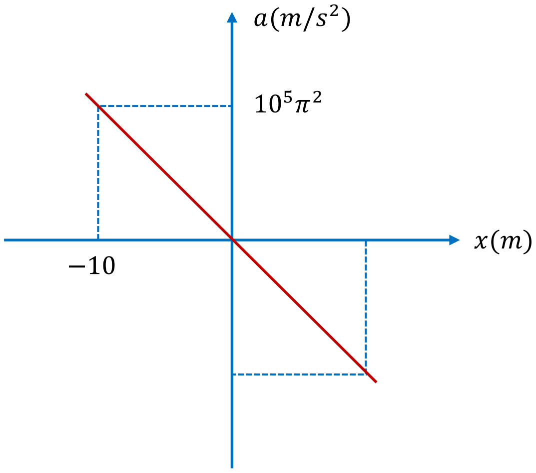


$\because \quad a = -\omega^2 x$

Which is an equation of straight line with negative slope.
So, the graph between acceleration and displacement is a straight line.

Illustration 14:

The acceleration displacement graph for a particle executing SHM is given by:



Find the frequency of SHM.

Solution:

$\because \quad a = -\omega^2 x$ and $a_{max} = \omega^2 A \quad \Rightarrow \quad 10^5\pi^2 = \omega^2(10)$

$\Rightarrow \quad \omega = 100\pi \qquad \Rightarrow \quad f = 50\ Hz$

Potential Energy (U or P.E.)

The potential energy is related to conservative force by the relation

$$F = -\frac{dU}{dx} \Rightarrow \int dU = -\int F dx$$

For S.H.M. $F = -kx$

So $\quad \int dU = -\int (-kx)dx = \int kxdx \quad \Rightarrow \quad U = \dfrac{1}{2}kx^2 + C$

At $\quad x = 0,\ U = U_0 \qquad \Rightarrow \ C = U_0 \qquad$ So $U = \dfrac{1}{2}Kx^2 + U_0$

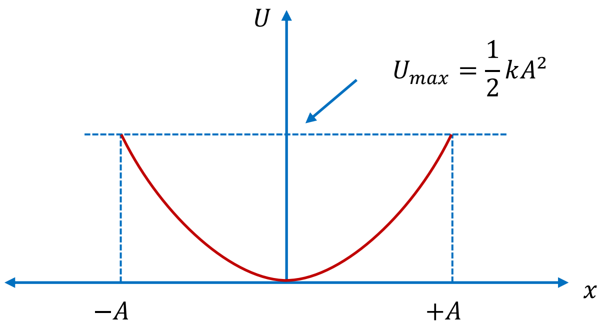
If at $x = 0$ potential energy is zero then $U_0 = 0$

(i) at $x = 0$ (Mean position)

$P.E._{min} = 0$

(ii) at $x = \pm A$ (extreme position)

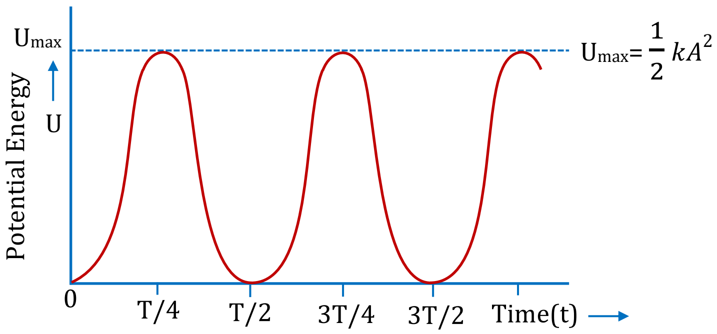
$$P.E._{max} = \frac{1}{2}KA^2$$



In Terms of Time

Since $x = A\sin(\omega t + \phi)$

$$U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$



If initial phase ϕ is zero, then

$$U = \frac{1}{2}kA^2\sin^2\omega t = \frac{1}{2}m\omega^2A^2\sin^2\omega t$$

Kinetic Energy (K or K.E.)

(i) In Terms of Displacement:

If mass of the particle executing S.H.M. is m and its velocity is v, then kinetic energy at any instant will be

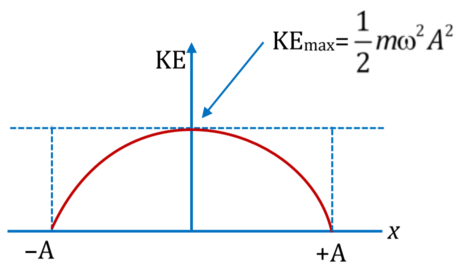
$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

(a) At mean position ($x = 0$)

$$K.E_{max} = \frac{1}{2}kA^2$$

(b) At extreme position ($x = \pm A$)

$$K.E_{min} = 0$$



(ii) In Terms of Time:

$$v = A\omega\cos(\omega t + \phi)$$

$$K = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi)$$

If initial phase ϕ is zero

$$K = \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t$$

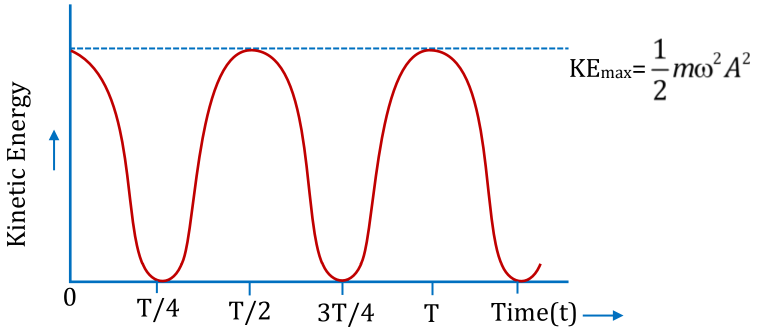
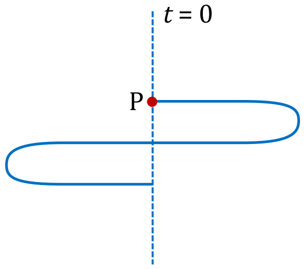


Illustration 15:



After what minimum time $KE = PE$? (If time period is T.)

Solution:

$$KE = PE$$

$$\frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kx^2$$

$$2x^2 = A^2$$

$$x^2 = \frac{A^2}{2}$$

$\Rightarrow \quad x = +\frac{A}{\sqrt{2}} \quad$ or $\quad -\frac{A}{\sqrt{2}}$

$\therefore \quad$ Phase covered $= \frac{\pi}{4}$

$$\omega t = \frac{\pi}{4}$$

$\Rightarrow \quad \frac{2\pi}{T} \times t = \frac{\pi}{4}$

$\Rightarrow \quad t = \frac{T}{8}$ sec

Illustration 16:

The P.E. of a particle in SHM is 2.5 J when its displacement is half of the amplitude. Determine the total energy of particle.

Solution:

$$T.E. = \frac{1}{2}kA^2 \text{ and } P.E. = \frac{1}{2}kx^2 \Rightarrow 2.5 = \frac{1}{2}k\left(\frac{A^2}{4}\right) \Rightarrow \frac{1}{2}kA^2 = T.E. = 10J$$

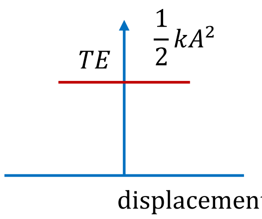
Total energy in S.H.M. is given by ; E = potential energy + kinetic energy = $U + K$

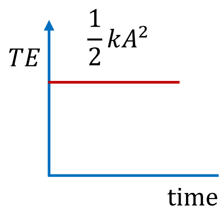
(i) w.r.t. position $E=\frac{1}{2}kx^2+\frac{1}{2}k\left(A^2-x^2\right)\Rightarrow E=\frac{1}{2}kA^2$ = constant

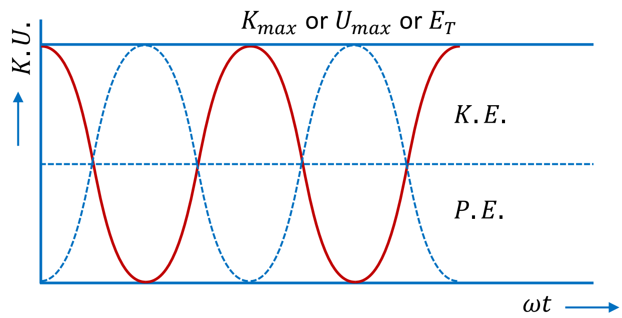
(ii) w.r.t. time

$$E=\frac{1}{2}m\omega^2A^2\omega^2A^2\sin^2\omega t+\frac{1}{2}m\omega^2A^2\cos^2\omega t$$

$$=\frac{1}{2}m\omega^2A^2\left(\sin^2\omega t+\cos^2\omega t\right)=\frac{1}{2}m\omega^2A^2=\frac{1}{2}kA^2 \text{ = constant}$$







Average Energy in S.H.M.

(i) The time average of P.E. and K.E. over one cycle is

(a) $<KE>_t=<\frac{1}{2}m\omega^2A^2\cos^2\omega t>=\frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right)=\frac{1}{4}m\omega^2A^2=\frac{1}{4}kA^2$

(b) $<PE>_t=<\frac{1}{2}m\omega^2A^2\sin^2\omega t>=\frac{1}{2}m\omega^2A^2<\sin^2\omega t>=\frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right)=\frac{1}{4}m\omega^2A^2=\frac{1}{4}kA^2$

(c) $<TE>_t=<\frac{1}{2}m\omega^2A^2>=\frac{1}{2}m\omega^2A^2=\frac{1}{2}kA^2$

(ii) The position average of P.E. and K.E. between $x = -A$ to $x = A$

(a) $$<KE>_x=\frac{\int_{-A}^{A}\frac{1}{2}m\omega^2\left(A^2-x^2\right)dx}{\int_{-A}^{A}dx}=\frac{\frac{1}{2}m\omega^2\left(A^2x-\frac{x^3}{3}\right)_{-A}^{A}}{2A}=\frac{1}{3}kA^2$$

(b) $$<PE>_x=\frac{\int_{-A}^{A}(PE)dx}{\int_{-A}^{A}dx}=\frac{\int_{-A}^{A}\left(\frac{1}{2}kx^2\right)dx}{\int_{-A}^{A}dx}=\frac{\frac{1}{2}k\left(\frac{x^3}{3}\right)_{-A}^{A}}{2A}=\frac{1}{6}kA^2$$

(c) $$<KE>_x=\frac{\int_{-A}^{A}\frac{1}{2}(TE)dx}{\int_{-A}^{A}dx}=\frac{\int_{-A}^{A}\left(\frac{1}{2}kA^2\right)dx}{\int_{-A}^{A}dx}=\frac{1}{2}kA^2$$

Illustration 17:

P.E. at mean position of a particle is 3*J.* If the mean K.E. over time is 4 *J*, find its total energy.

Solution:

$$U_0 = 3J \text{ , Mean } K.E. = \frac{1}{4}kA^2 = 4J \quad \Rightarrow \quad \frac{1}{2}kA^2 = 8J$$

So total energy is = $\frac{1}{2}kA^2 + U_0 = 8J + 3J = 11J$

Illustration 18:

The potential energy of a particle of mass 0.1 *kg* moving along the *x*-axis is given by $U = 5x\,(x - 4)\,J$, where *x* is in meters. It can be concluded that:

(A) the particle is acted upon by a constant force

(B) the speed of the particle is maximum at $x = 2m$

(C) the particle executes simple harmonic motion

(D) the angular frequency of oscillation of the particle is 10 s^{-1}

Solution:

$$F = -\frac{dU}{dx}$$

$$F = -10x + 20 = -10(x-2)$$

$F = 0$ at $x = 2$ *m* so *v* is maximum at $x = 2$ *m*

$$a = \frac{-10(x-2)}{0.1} \text{ and also } a = -\omega^2(x-2)$$

$$\omega = \sqrt{\frac{10}{0.1}}$$

Therefore, the correct options are (BCD).

Illustration 19:

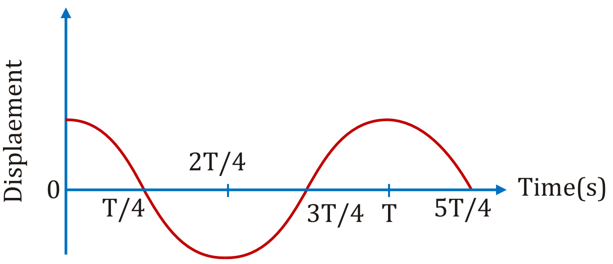
The displacement-time graph of a particle executing S.H.M. is shown in figure. Which of the following statement is/are TRUE?

(A) The force is zero at $t = \frac{3T}{4}$

(B) The acceleration is maximum at $T = T$

(C) The velocity is maximum at $t = \frac{T}{4}$

(D) The *P.E.* is equal to *K.E.* of oscillation at $t = \frac{T}{2}$



Solution:

Force $|F| = kx$, zero at $x = 0$

$|a| = \omega^2 x$, maximum at $x = \pm A$

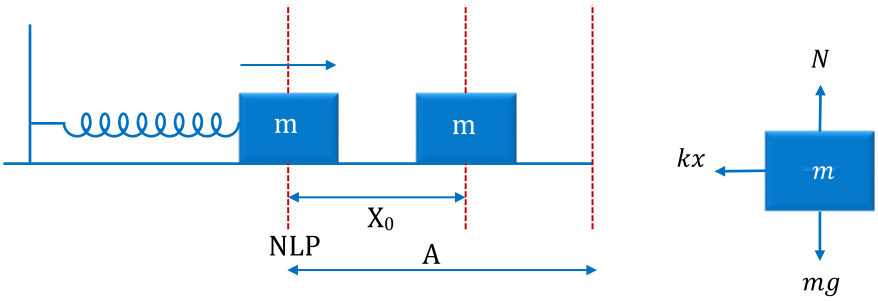
v is maximum at mean position, $x = 0$

$PE = KE$ at $x = \pm\frac{A}{\sqrt{2}}$

Therefore, the correct option are (ABC).

Dynamics of SHM with Spring Mass System

Horizontal Spring-Mass System



If it is pulled on one side and is released, it executes to and fro motion about the mean position.

- Let 'x' be the displacement of the mass. At any instant of time 't' the restoring force 'F' is given by $F = -kx$ where 'k' is force constant.
- According to Newton's second law,

$$ma = -kx$$

$$a = \left(\frac{-k}{m}\right)x \qquad ...(1)$$

- When k and m are constants,

$$a \propto -x$$

Hence, the oscillations of a loaded spring are simple harmonic motion.

- Acceleration of the particle in S.H.M at any time is given by

$$a = -\omega^2 x \qquad ...(2)$$

By comparing (1) and (2)

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \qquad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$$

Vertical Spring-Mass System

- Initially particle was at rest at equilibrium position then it is displaced slightly and release, calculate time period of oscillation.

In equilibrium position

$$mg = kx_0 \Rightarrow x_0 = \frac{mg}{k}$$

When spring is further stretched by x then

$$F_{net} = mg - k(x + x_0) = mg - kx - kx_0$$

$$\Rightarrow \quad F_{net} = -kx \Rightarrow ma = -kx$$

$$\Rightarrow \quad a = -\frac{k}{m}x \qquad \text{...(i)}$$

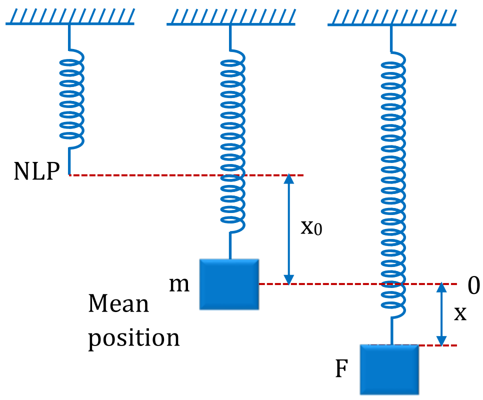
From standard equation of SHM

$$a = -\omega^2 x \qquad \text{...(ii)}$$

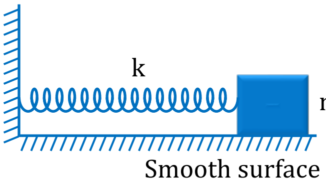
From (i) and (ii)

$$\omega = \sqrt{\frac{k}{m}}$$

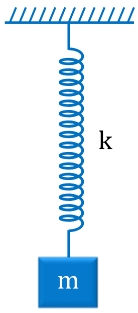
$$T = 2\pi\sqrt{\frac{m}{k}}$$



(a) (b) (c)

(1) 

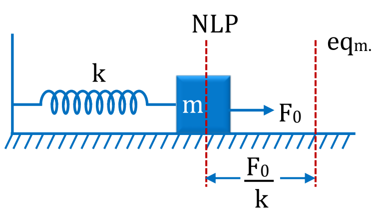
$$T = 2\pi\sqrt{\frac{m}{k}}$$

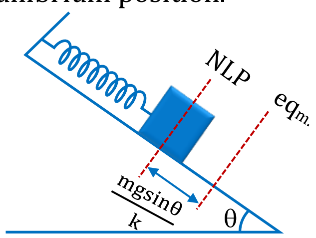
(2) 

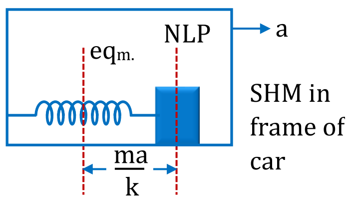
$$T = 2\pi\sqrt{\frac{m}{k}}$$

Spring Block System in Presence of Constant Force

1. In a spring block system, if an additional constant force applied on the block then time period remains same. It only shifts the equilibrium position.







$$\text{Time period} = 2\pi\sqrt{\frac{m}{k}}$$

2. Time period of spring pendulum doesn't depend on acceleration due to gravity. So, a watch based on spring pendulum shows correct time everywhere.

Variation of Different Parameters During SHM

Case I.

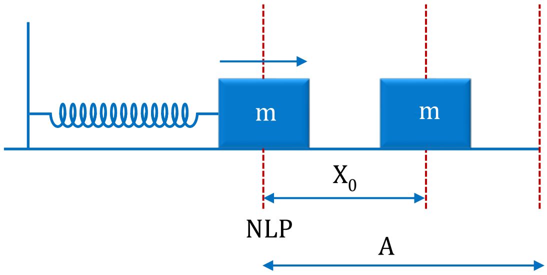
Velocity doubles at $X = X_0$. What will be new amplitude and time period ?

$V_0 = \omega\sqrt{A^2 - X_0^2}$ ...(1)

$2V_0 = \omega\sqrt{A_1^2 - X_0^2}$ ...(2)

From 1 & 2

$4 = \dfrac{A_1^2 - x_0^2}{A^2 - x_0^2}$ so $A_1 = \sqrt{4A^2 - 3X_0^2}$



Time period remains same.

Case II.

A block performing SHM collide with another block on its way at a distance of X_0.

Amplitude (original) = A

Calculate new amplitude and time period?

New time period = $2\pi\sqrt{\dfrac{2m}{k}}$, $\omega' = \sqrt{\dfrac{k}{2m}}$

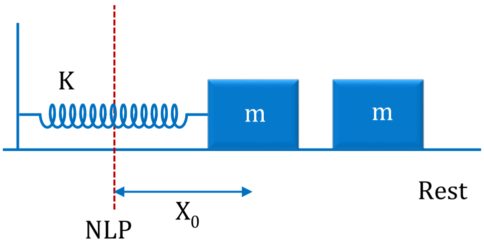
$V_0 = \sqrt{\dfrac{k}{m}}\sqrt{A^2 - X_0^2}$ ...(1)

$\Rightarrow \dfrac{V_0}{2} = \sqrt{\dfrac{K}{2m}}\sqrt{A_1^2 - X_0^2}$ ...(2)

From 1 and 2

$\sqrt{2}\sqrt{A_1^2 - X_0^2} = \sqrt{A^2 - X_0^2}$

$\therefore \quad A_1 = \sqrt{\dfrac{A^2 + X_0^2}{2}}$



Case III :

Initially block is at rest. Initially lift is at rest. Suddenly lift starts going up with acceleration *'a'*. Calculate Amplitude?

New equilibrium position

$= \dfrac{m(g+a)}{K}$

$A = \dfrac{m(g+a)}{K} - \dfrac{mg}{K} = \dfrac{ma}{K}$

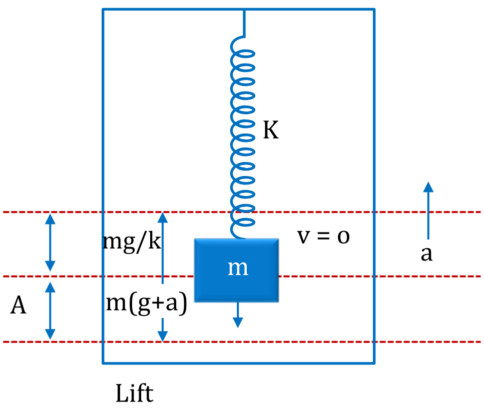


Illustration 20:

A particle of mass 200 g executes a simple harmonic motion. The restoring force is provided by a spring of spring constant 80 N/m. Find the time period.

Solution:

The time period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{200\times10^{-3}\,kg}{80N/m}} = 2\pi \times 0.05\ s = 0.31\ s.$$

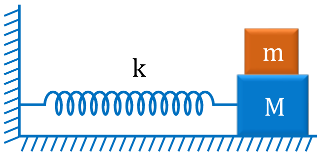
Illustration 21:

The friction coefficient between the two blocks shown in figure is μ and the horizontal plane is smooth.

(a) If the system is slightly displaced and released, find the time period.

(b) Find the magnitude of the frictional force between the blocks when the displacement from the mean position is x.

(c) What can be the maximum amplitude if the upper block does not slip relative to the lower block?



Solution:

(a) For small amplitude, the two blocks oscillate together. The angular frequency is

$$\omega = \sqrt{\frac{k}{M+m}} \text{ and so the time period } T = 2\pi\sqrt{\frac{M+m}{k}}$$

(b) The acceleration of the blocks at displacement x from the mean position is

$$a = -\omega^2 x = \left(\frac{-kx}{M+m}\right)$$, the resultant force on the upper block is, therefore, $ma = \left(\frac{-mkx}{M+m}\right)$

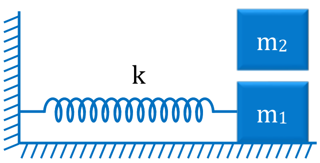
This force is provided by the friction of the lower block. Hence, the magnitude of the frictional force is $\left(\frac{mkx}{M+m}\right)$.

(c) Maximum force of friction required for simple harmonic motion of the upper block is $\frac{mkA}{M+m}$ at the extreme positions. But the maximum frictional force can only be μmg.

Hence, $\frac{mkA}{M+m} = \mu mg$ or, $A = \frac{\mu(M+m)g}{K}$

Illustration 22:

A block of mass 4kg attached with spring of spring constant 100 N/m executing SHM of amplitude 0.1m on smooth horizontal surface as shown in figure. If another block of mass 5 kg is gently placed on it, at the instant it passes through the mean position then find the frequency and amplitude of the motion assuming that the two blocks always move together.



Solution:

Frequency, $n_2 = \frac{1}{2\pi}\sqrt{\frac{K}{m_1 + m_2}}$

$$= \frac{1}{2\pi}\sqrt{\frac{100}{4+5}} = \frac{1}{2\pi} \times \frac{10}{3} = \frac{5}{3\pi}$$

By conservation of linear momentum at mean position,

$$P_i = P_f$$

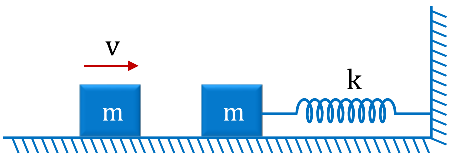
$$\Rightarrow \quad m_1\omega_1 A_1 = (m_1 + m_2)\omega_2 A_2 \qquad \Rightarrow \quad m_1\sqrt{\frac{k}{m_1}}A_1 = (m_1 + m_2)\sqrt{\frac{k}{m_1 + m_2}}A_2$$

$$\Rightarrow \quad \sqrt{km_1}A_1 = \sqrt{k(m_1 + m_2)}A_2 \qquad \Rightarrow \quad A_2 = \frac{2}{30}m$$

Ans. $\frac{5}{3\pi}Hz, \frac{2}{30}m$

Illustration 23:

The left block in figure collides inelastically with the right block and sticks to it. Find the amplitude of the resulting simple harmonic motion.



Solution:

Since the collision is perfectly inelastic. So, by applying principle of conservation of momentum the velocity of both blocks will become $\frac{v}{2}$.

The kinetic energy just after the collision = $\frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2$. This is also the total energy of vibration as the spring is unstretched at this moment. If the amplitude is *A*, the total energy can also be written as $\frac{1}{2}KA^2$.

$$\frac{1}{2}KA^2 = \frac{1}{4}mv^2$$

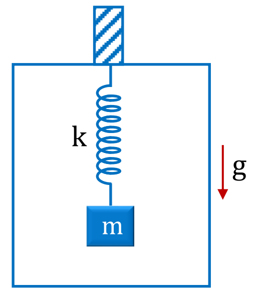
$$A = \sqrt{\frac{m}{2k}}v$$

Illustration 24:

A block of mass *m* is suspended from the ceiling of a stationary elevator through a spring of spring constant *k* and suddenly, the cable breaks and the elevator starts falling freely. Show that block now executes a simple harmonic motion of amplitude *mg/k* in the elevator.

Solution:

When the elevator is stationary, the spring is stretched to support the block. If the extension is *x*, the tension is *kx* which should balance the weight of the block.

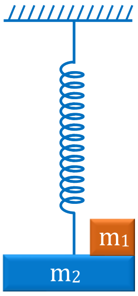


Thus, $x = mg/k$. As the cable breaks, the elevator starts falling with acceleration *'g'*. We shall work in the frame of reference of the elevator. Now we have to use a psuedo force mg upward on the block. This force will 'balance' the weight.

Thus, the block is subjected to a net force *kx* by the spring when it is at a distance *x* from the position of unstretched spring. Hence, its motion in the elevator is simple harmonic with its mean position corresponding to the unstretched spring.

Illustration 25:

The system is in equilibrium and at rest. Now mass m_1 is removed from m_2. Find the time period and amplitude of resultant motion. Spring constant is *K*.

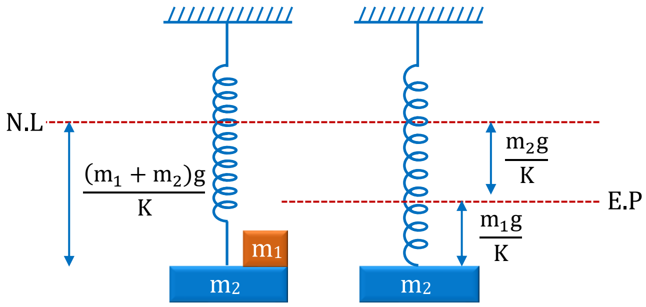


Solution:

Initial extension in the spring

$$x = \frac{(m_1 + m_2)g}{K}$$

Now, if we remove m_1, Equilibrium Position (E.P.) of m_2 will be $\frac{m_2 g}{K}$ below natural length of spring.



At the initial position, since velocity is zero i.e. it is the extreme position.

Hence Amplitude = $\frac{m_1 g}{K}$

Time period = $2\pi\sqrt{\frac{m_2}{K}}$

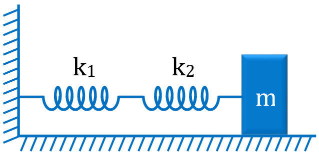
Combination of Springs

Series Combination:

Here force is same in both springs

$$k_2 x_2 = k_1 x_1 = F \qquad \text{...(i)}$$

Also $x = x_1 + x_2$ ...(ii)



From (i) and (ii)

$$x_2 = \left(\frac{k_1}{k_1 + k_2}\right)x \quad ...(iii)$$

As $k_2 x_2 = -ma$

$$\Rightarrow \quad a = -\frac{k_2 x_2}{m}$$

$$\Rightarrow \quad a = -\frac{1}{m}\left(\frac{k_1 k_2}{k_1 + k_2}\right)x \quad ...(iv)$$

From standard equation of SHM

$$a = -\omega^2 x \quad ...(v)$$

From (iv) and (v)

$$\omega = \sqrt{\frac{k_1 k_2}{(k_1 + k_2)m}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}} = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Note:

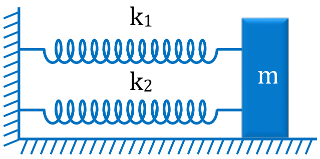
- In series combination, tension is same in all the springs and extension will be different. (If k is same then deformation is also same)
- In series combination, extension of springs will be reciprocal of its spring constant.
- Spring constant of spring is inversely proportional to its natural length

  $\because \quad k \propto 1/\ell$

  $\therefore \quad k_1 \ell_1 = k_2 \ell_2 = k_3 \ell_3$
- If a spring is cut in *'n'* pieces then spring constant of one piece will be *nk*.

Parallel combination:

Extension is same for both springs but force acting will be different.



Force acting on the system = F

$$k_1 x + k_2 x = -ma$$

$$a = -\frac{(k_1 + k_2)x}{m} \quad ...(i)$$

From standard equation of SHM

$$a = -\omega^2 x \quad ...(ii)$$

From (i) and (ii)

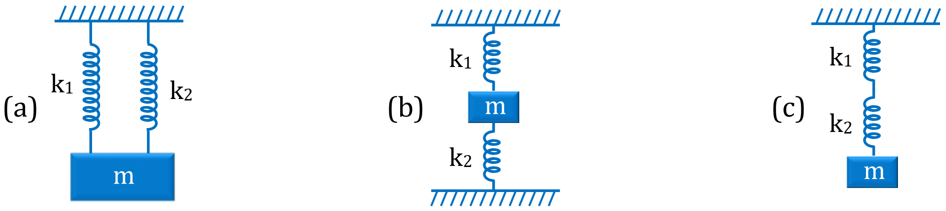
$$\omega = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \Rightarrow 2\pi\sqrt{\frac{m}{k_{eq}}}$$

$$k_{eq} = k_1 + k_2$$

Illustration 26:

Three spring mass systems are shown in figure. Assuming gravity free space, find the time period of oscillations in each case. What should be the answer if space is not gravity free ?



Solution:

(a) When spring is stretched by x then restoring force.

$$F = k_1 x + k_2 x$$

$$F = k_{eq} x$$

$$k_{eq} x = k_1 x + k_2 x$$

$$k_{eq} = k_1 + k_2$$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

(b) When block is displaced by x from mean position then restoring force.

$$F = k_1 x + k_2 x$$

$$k_{eq} x = k_1 x + k_2 x$$

$$k_{eq} = k_1 + k_2$$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

(c) When block is displaced by x and extension in upper spring is x_1, extension in lower spring is x_2

then $F = k_1 x_1 \Rightarrow x_1 = \frac{F}{k_1}$

$$F = k_2 x_2 \Rightarrow x_2 = \frac{F}{k_2}$$

$$F = k_{eq} x \Rightarrow x = \frac{F}{k_{eq}}$$

$$x = x_1 + x_2 \Rightarrow \frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$$

$$\Rightarrow \qquad k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1 k_2}}$$

When space is not gravity free then answers do not change as time period of spring mass system is independent of gravity.

Illustration 27:

Infinite springs with force constants k, $2k$, $4k$, $8k$, respectively are connected in series. Calculate the effective force constant of the spring.

Solution:

$$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + ...\infty$$

(For infinite G.P. $S_\infty = \dfrac{a}{1-r}$ where a = First term, r = common ratio)

$$\frac{1}{k_{eff}} = \frac{1}{k}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...\right] = \frac{1}{k}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{2}{k}$$

so $\qquad k_{eff} = k/2$

Calculation of Time Period for Linear SHM

Steps to Calculate Time Period for SHM

1. Find out equilibrium position and write down condition for that (if any).
2. Displace the particle slightly by x from equilibrium position and draw FBD at that point.
3. Apply Newton's second law and try to show proportionality between net force (F_{net}) and x like $F_{net} = -k_{eff.}(x)$
4. As $F_{net} = ma \Rightarrow ma = -k_{eff.}(x) \Rightarrow a = \dfrac{-k_{eff.}x}{m}$ ...(i)
5. From standard equation of SHM $a = -\omega^2 x$ ...(ii)
6. Comparing the equation (i) and (ii) we will get

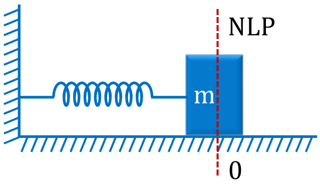
$$\omega^2 = \frac{k_{eff}}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{k_{eff}}{m}} = \frac{2\pi}{T} = 2\pi f$$

Note:

After proving $F_{net} = -k_{eff.}(x)$, we can directly write $T = 2\pi\sqrt{\dfrac{m}{k_{eff.}}}$.

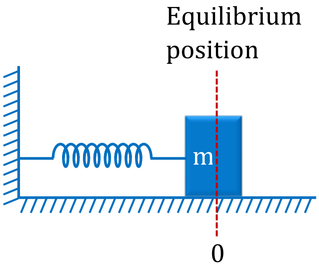
Illustration 28:

Initially particle was at rest, then it is displaced slightly and released. Calculate time period of oscillation.

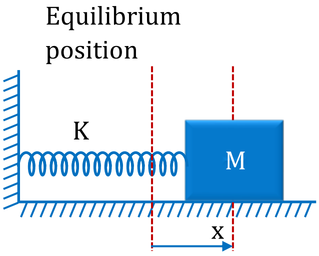


Solution:

Step 1.

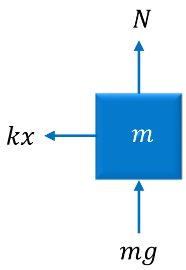


Step 2.



$kx = -ma$

$a = -\dfrac{kx}{m}$ ...(i)



From standard equation of SHM

$a = -\omega^2 x$ ...(ii)

From (i) and (ii)

$\omega = \sqrt{\dfrac{k}{m}}$

$\boxed{T = 2\pi\sqrt{\dfrac{m}{k}}}$

Illustration 29:

A particle of mass m is located in a potential field given by $U_{(x)} = U_o(1 - \cos ax)$ where U_o and a are constants. The period of small oscillations is

(A) $2\pi\sqrt{\dfrac{U_0}{ma^2}}$ (B) $2\pi\sqrt{\dfrac{mU_0}{a^2}}$ (C) $2\pi\sqrt{\dfrac{a_0}{mU_0}}$ (D) $2\pi\sqrt{\dfrac{m}{U_0a^2}}$

Ans. (D)

Solution:

$U_{(x)} = U_o(1-\cos ax)$

as $\quad F = \dfrac{-dU}{dx}$

$\Rightarrow \quad F = -U_0\left(0-(-\sin ax)a\right) \quad \Rightarrow F = -U_0 a \sin ax$

For small amplitude ax is very small, hence

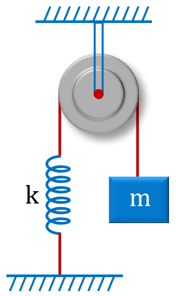
$$F \approx -U_0\, a(ax) = -U_0\, a^2 x$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{U_0 a^2}}$$

Illustration 30:

Figure shows a system consisting of a massless pulley, a spring of force constant k = 4000 N/m and a block of mass $m = 1kg$.

If the block is slightly displaced vertically down from its equilibrium position and released, find the frequency of its vertical oscillation.



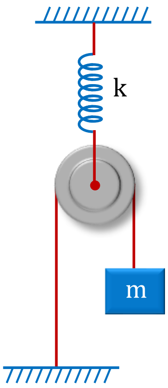
Solution:

As the pully is fixed and string is inextensible, if mass m is displaced by y the spring will stretch by y, and as there is no mass between string and spring (as pully is massless) $F = T = ky$ i.e., restoring force is linear and so motion of mass m will be linear simple harmonic with frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4000}{1}} \approx 10Hz$$

Illustration 31:

Figure shows a system consisting of a massless pulley, a spring of force constant k = 4000 N/m and a block of mass m = 1 kg. If the block is slightly displaced vertically down from its equilibrium position and released, find the frequency of its vertical oscillation.



Solution:

The pulley is movable and string is inextensible, so if mass m moves down a distance y, the pulley will move down by $(y/2)$. So, the force in the spring $F = k(y/2)$.

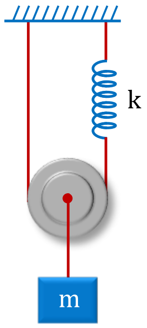
Now as pully is massless $F = 2T, \Rightarrow F/2 = (k/4)y$. So, the restoring force on the mass m

$$T = \frac{1}{4}ky = k'y \Rightarrow k' = \frac{1}{4}k$$

So, $\quad f = \dfrac{1}{2\pi}\sqrt{\dfrac{k'}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{4m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{4000}{4}} = 5Hz$

Illustration 32:

Figure shows a system consisting of a massless pulley, a spring of force constant k = 4000 N/m and a block of mass m = 1 kg. If the block is slightly displaced vertically down from its equilibrium position and released find the frequency of its vertical oscillation.



Solution:

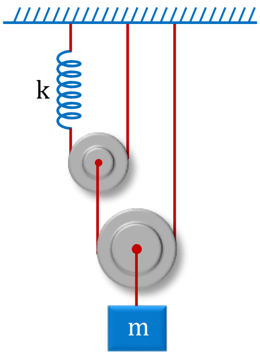
In this situation if the mass m moves by y the pulley will also move by y and so the spring will stretch by $2y$ (as string is inextensible) and so $T' = F = 2ky$. Now as pulley is massless so $T = F + T' = 4ky$, i.e., the restoring force on the mass m.

$$T = 4ky = k'y \Rightarrow k' = 4k$$

So $$f = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{4k}{m}} = 20Hz$$

Illustration 33:

What is the period of small oscillations of the block of mass m if the spring is ideal and pulleys are massless?



Solution:

On displacing the block by x, the spring will stretch by $4x$, so force on spring will be $4kx$ and force on block will be

$$F_{net} = -(4k)(4x) = -16kx$$

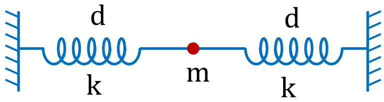
$\Rightarrow$ $$k_{eq} = 16K$$

as $$T = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

$$T = 2\pi\sqrt{\frac{m}{16k}}$$

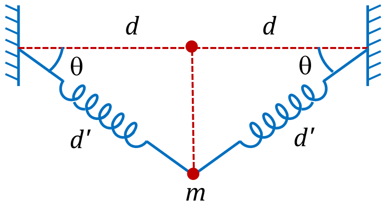
Illustration 34:

A small body of mass m is connected to two horizontal springs of elastic constant k, natural length $\frac{3d}{4}$. In the equilibrium position both springs are stretched to length d, as shown in figure. What will be the ratio of period of the motion $\left(\frac{T_a}{T_b}\right)$ if the body is displaced horizontally by a small distance where



T_a is time period when the particle oscillates along the line of springs and T_b is time period when the particle oscillates perpendicular to the plane of the figure? Neglect effects of gravity.

Solution:



Initial stretch in both springs = $d-\frac{3d}{4}=\frac{d}{4}$

$$F_{\text{restoring}}=k\left(\frac{d}{4}+x\right)-k\left(\frac{d}{4}-x\right)=2kx$$

$\Rightarrow$ $$T_a=2\pi\sqrt{\frac{m}{2k}}$$

$d' = d\sec\theta$

$x' = d\sec\theta - \frac{3d}{4} = d\left(\frac{1}{\cos\theta}-\frac{3}{4}\right)$

Force towards equilibrium position ($kx'\sin\theta$)

$=dk\left(\tan\theta-\frac{3\sin\theta}{4}\right)$ due to one spring and net force $=2dk\left(\tan\theta-\frac{3\sin\theta}{4}\right)$

For small θ, force $=2dk\left[\theta-\frac{3\theta}{4}\right]=kd\left(\frac{\theta}{2}\right)$

$d(\theta)$ = displacement from mean position

$\Rightarrow$ $$F=-\frac{kx}{2}\Rightarrow T_b=2\pi\sqrt{\frac{2m}{k}}$$

$\Rightarrow$ $$\frac{T_A}{T_B}=2\pi\sqrt{\frac{m}{2k}}\div 2\pi\sqrt{\frac{2m}{k}}=\frac{1}{2}$$ **Ans.**

Illustration 35:

A wooden cube (density of wood '$\frac{\rho}{3}$') of side 'ℓ' floats in a liquid of density 'ρ' with its upper and lower surfaces horizontal. If the cube is pushed slightly down and released, it performs simple harmonic motion of period 'T'. Then, 'T' is equal to :-

(A) $2\pi\sqrt{\frac{3\ell}{2g}}$ (B) $\pi\sqrt{\frac{\ell}{3g}}$ (C) $2\pi\sqrt{\frac{3\ell}{g}}$ (D) $2\pi\sqrt{\frac{2\ell}{g}}$

Ans. (B)

Solution:

$F_{net} = \rho Axg$

$$F_{net} = \rho Axg \Rightarrow \frac{\rho}{3} A\ell \quad a = \rho Axg \Rightarrow a = g\frac{3x}{\ell}$$

$$\text{Time period} = 2\pi\sqrt{\frac{\ell}{3g}}.$$

Illustration 36:

If rod of length 4 *m*, area 4 cm^2 and young modulus 2×10^{10} N/m^2 is attached with mass 200 *kg*, then angular frequency of SHM (*rad/sec.*) of mass is equal to

(A) 1000 (B) 10 (C) 100 (D) 10 π

Ans. (C)

Solution:

$$K = \frac{AY}{\ell} = \frac{4\times10^{-4}\times2\times10^{10}}{4} = 2 \times 106$$

$$\omega = \sqrt{\frac{K}{m}} = 100$$

Energy Method to Calculate Time Period

Steps to Solve Problem

(i) Find out mean position and write down condition for that. (if any)

(ii) Write total energy of particle at some displacement *'x'* from mean position. Assume mean position as reference level for writing gravitational potential energy if required.

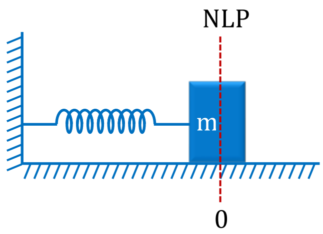
(iii) Differentiate energy equation w.r.t. time and calculate acceleration from that.

(iv) As $a = -\omega^2 x$

Compare and get ω and *T.*

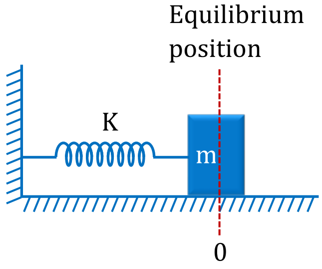
Illustration 37:

Initially particle was at rest then it is displaced slightly and released. Calculate the time period of oscillation.

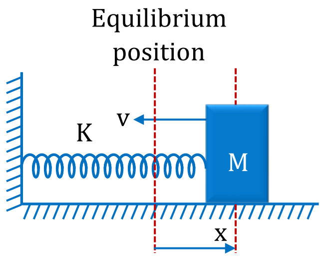


Solution:

Step 1.



Step 2.



$$E = K + U$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \qquad \text{...(i)}$$

Differentiate equation (i) w.r.t. time

$$\frac{dE}{dt} = \frac{1}{2}m2va + \frac{1}{2}k2xv$$

As total mechanical energy in SHM remains constant

$$\frac{dE}{dt} = 0$$

$$mva + kxv = 0$$

$$a = -\frac{k}{m}x \qquad \text{...(ii)}$$

From standard equation of SHM

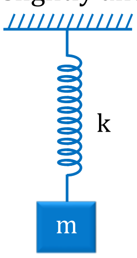
$$a = -\omega^2 x \qquad \text{...(iii)}$$

From (ii) and (iii)

$$\omega^2 = \frac{k}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}}$$

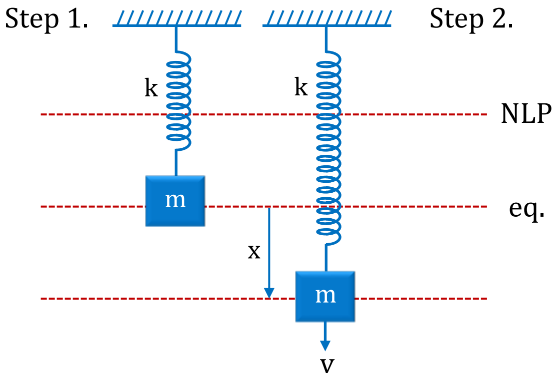
Illustration 38:

Initially particle was at rest then it is displaced slightly and released calculate time period for oscillation.



Solution:

In equilibrium position



$$Mg = kx_0 \qquad \Rightarrow x_0 = \frac{mg}{k}$$

Total energy

$$E = \frac{1}{2}mv^2 + \frac{1}{2}k(x_0 + x)^2 - mgx$$

$$\frac{dE}{dt} = \frac{1}{2}m2va + \frac{1}{2}k2(x_0 + x)(0 + v) - mgv$$

$$0 = mva + k(x_0 + x)v - mgv$$

$$0 = ma + kx_0 + kx - mg$$

as $kx_o = mg$

$$0 = ma + kx$$

$\Rightarrow$ $a = \dfrac{-kx}{m}$ ...(i)

as $a = -\omega^2 x$ ...(ii)

From (i) and (ii)

$$\omega^2 = \frac{k}{m}$$

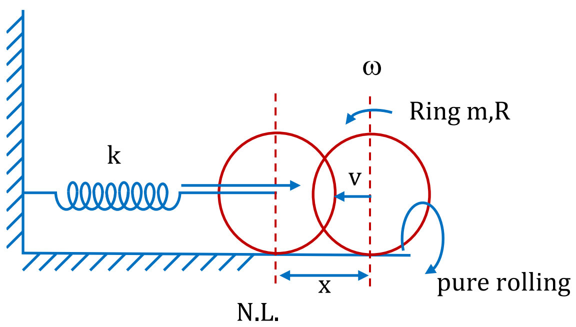
$$\Rightarrow \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

Illustration 39:

Initially a ring of mass *m* and radius *R* was at rest on rough horizontal surface, then it is displaced slightly and released, calculate time period for oscillation. (Assume pure rolling)

Solution:



$$E = U + K$$

$$= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 + \frac{1}{2}mr^2\left(\frac{v}{r}\right)^2$$

$$= \frac{1}{2}kx^2 + mv^2$$

$$\Rightarrow \frac{dE}{dt} = \frac{1}{2}k2xv + m2va = 0$$

$$\Rightarrow kxv + 2mva = 0$$

$a = -\left(\dfrac{k}{2m}\right)x$ ...(i)

From standard equation of SHM

$$a=-\omega^2 x \qquad \text{...(ii)}$$

From (i) and (ii)

$$\omega^2=\frac{k}{2m}\Rightarrow \omega=\sqrt{\frac{k}{2m}}=\frac{2\pi}{T}\Rightarrow 2\pi\sqrt{\frac{2m}{k}}$$

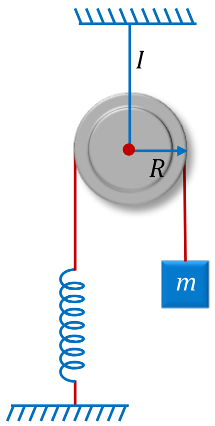
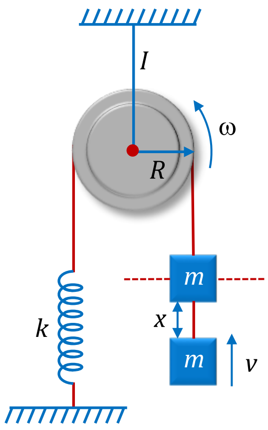


Illustration 40:

Figure shows a pulley block system in equilibrium. If the block is displaced down slightly from its equilibrium position and released. Find the time period of oscillation of the system. Assume there is sufficient friction present between pulley and string so that string will not slip over pulley surface.

Solution:

If m is in equilibrium tension in string must be mg and spring is stretched by h so that $mg=kh$. If we displace the block downward by a distance x and released, it starts executing SHM with amplitude A. During its oscillation we consider the block at a displacement x below the equilibrium position, if it is moving at a speed v at this position, the pulley will be rotating at an angular speed ω given as $\omega=\frac{v}{r}$.



Thus, at this position the total energy of oscillating system is

$$E_T=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2+\frac{1}{2}k(x+h)^2-mgx$$

Differentiating with respect to time, we get

$$\frac{dE_T}{dt}=\frac{1}{2}m\left(2v\frac{dv}{dt}\right)+\frac{1}{2}I\left(\frac{1}{r^2}\right)\left(2v\frac{dv}{dt}\right)+\frac{1}{2}k\left[\frac{2(x+h)dx}{dt}\right]-mg\left(\frac{dx}{dt}\right)=0$$

or $$mva+\frac{I}{r^2}va+k(x+h)v-mgv=0$$

or $$a=-\left(\frac{k}{m+\frac{I}{r^2}}\right)x \quad [\text{as } mg=kh]$$

Comparing above equation with standard differential equation of SHM we get $\omega=\sqrt{\left(\frac{k}{m+\frac{I}{r^2}}\right)}$.

Thus, time period of oscillation is $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m+\frac{I}{r^2}}{k}}$.

Angular SHM

To and fro motion about a fixed axis or position angularly is known as angular SHM.

In angular SHM the restoring torque (or angular acceleration) acting on the particle should always be proportional to the angular displacement of the particle and directed towards the equilibrium position.

$\therefore$ $\tau \propto -\theta$ or $\alpha \propto -\theta$

$\tau = -c\theta$ (where c = restoring torque constant)

$I\alpha = -c\theta$ α = angular acceleration and I = Moment of inertia

$$\alpha = -\frac{c}{I}\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad \left[\because \omega^2 = \frac{c}{I}\right]$$

and $T = 2\pi\sqrt{\frac{I}{C}}$

Solution of above differential equation is $\theta = \theta_0 \sin(\omega t + \phi)$. Which is an equation of angular displacement.

$\therefore$ Angular velocity

$$\frac{d\theta}{dt} = \theta_0\omega\cos(\omega t + \phi)$$

And angular acceleration,

$$\alpha = \frac{d^2\theta}{dt} = -\theta_0\omega^2\sin(\omega t + \phi)$$

Comparison between linear and angular S.H.M.

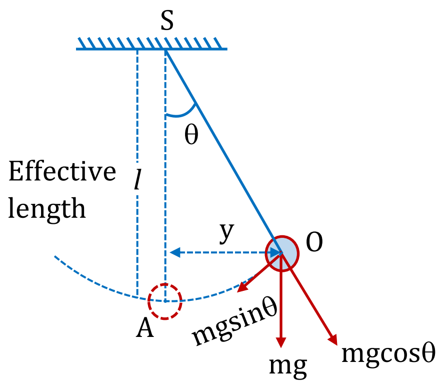
Linear S.H.M.	Angular S.H.M.
$F = -kx$	$\tau = -C\theta$
Where k is the restoring force constant	Where C is the restoring torque constant
$a = -\frac{k}{m}x$	$\alpha = -\frac{c}{I}\theta$
$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$	$\frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0$
It is known as differential equation of linear S.H.M.	It is known as differential equation of angular S.H.M.
$x = A\sin\omega t$	$\theta = \theta_0 \sin\omega t$
$a = -\omega^2 x$	$\alpha = -\omega^2\theta$
where ω is the angular frequency	
$\omega^2 = \frac{k}{m} = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = 2\pi n$	$\omega^2 = \frac{c}{I} = \sqrt{\frac{c}{I}} = \frac{2\pi}{T} = 2\pi n$
Where T is time period and n is frequency	
$T = 2\pi\sqrt{\frac{m}{k}}$	$T = 2\pi\sqrt{\frac{I}{C}}$
$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$	$n = \frac{1}{2\pi}\sqrt{\frac{c}{I}}$
This concept is valid for all types of linear SHM	This concept is valid for all types of angular SHM

Simple Pendulum

If a heavy point mass is suspended by a weightless and inextensible string from a rigid support, then this arrangement is called a simple pendulum.

Expression for time period:

Method-I (Force Method)



Restoring force acting on pendulum, $F = -mg\sin\theta$

For small angle, sin $\theta \approx \frac{OA}{SA} = \frac{y}{l}$

$\therefore \quad Ma = -mg \times \frac{y}{l} \Rightarrow a = -\frac{g}{l}y$

It proves that if displacement is small then simple pendulum performs S.H.M.

$\because \quad |a| = \omega^2 y \Rightarrow \omega^2 = \frac{g}{l} \Rightarrow \sqrt{\frac{g}{l}}$ and $T = 2\pi\sqrt{\frac{l}{g}}$

Method-II (Torque Method)

Bob of pendulum moves along the arc of circle in vertical plane. Here motion involved is angular and oscillatory where restoring torque is provided by gravitational force.

$\tau = -(mg)(\ell \sin\theta)$ (Negative sign shows opposite direction of τ and angular displacement)

$\tau = -mg\ell\theta$ (If angular displacement is small, then $\sin\theta \simeq \theta$)

$I\alpha = -mg\ell\theta$

$\alpha = -\frac{mg\ell}{I}\theta$ (where I = moment of inertia of bob about point of suspension so $I = ml^2$)

Now $\quad \alpha = -\frac{mg\ell}{m\ell^2}\theta \Rightarrow \alpha = -\frac{g}{\ell}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$

It is differential equation of angular SHM of simple pendulum

Comparing with standard differential equation of angular SHM $\left(\frac{d^2\theta}{dt^2} + \omega^2\theta = 0\right)$

Therefore $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$

Note:

Simple pendulum is the example of SHM but only when its angular displacement is very small.

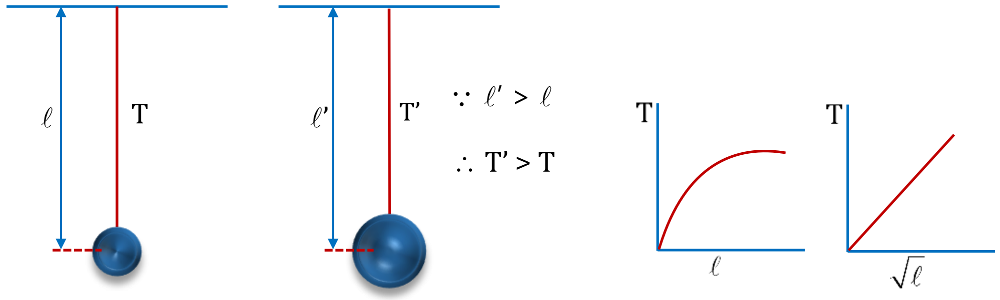
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\text{length of simple pendulum}}{\text{acceleration}}}$$

- $T = 2\pi\sqrt{\frac{l}{g}}$ is valid when length of simple pendulum (ℓ) is negligible as compare to radius of earth ($\ell << R_e$) but if l is comparable to radius of earth then time period, $T = 2\pi\sqrt{\frac{R_e}{\left[1+\frac{R_e}{l}\right]g}}$

- The time period of oscillation of simple pendulum of infinite length ($\ell \to \infty$)

  $T = 2\pi\sqrt{\frac{R}{g}} = 84.6$ minute $\approx 1\frac{1}{2}$ hour (it is maximum time period)

- If a simple pendulum of density ρ is made to oscillate in a liquid of density σ then its time period will increase as compare to that of air and is given by $T = 2\pi\sqrt{\frac{l}{\left[1-\frac{\sigma}{\rho}\right]g}}$
- The time period of simple pendulum is independent of mass of the bob but it depends on size of bob (position of centre of mass). So in simple pendulum when a solid iron bob is replaced by light aluminium bob of same radius then time period remains unchanged.
- Time period of simple pendulum is directly proportional to square root of length.



- When a person sitting on an oscillating swing comes in standing position then centre of mass raises upwards and length decreases, so time period decreases and frequency increases means swing oscillates faster.
- When a hollow spherical bob of simple pendulum is completely filled with water and a small hole is made in bottom of it, then as water drain out, at first its time period increases, after that it decrease and when sphere becomes empty then finally it becomes as before (T).
- If simple pendulum is shifted to poles, equator or hilly areas, then its time period may be different $\left(T \propto \frac{1}{\sqrt{g}}\right)$
- If a clock based on oscillation of simple pendulum is shifted from earth to moon then it becomes slow because its time period increases and becomes $\sqrt{6}$ times compare to earth $\frac{g_M}{g_E} = \frac{1}{6} \Rightarrow T_M = \sqrt{6}T_E$.

Second's Pendulum

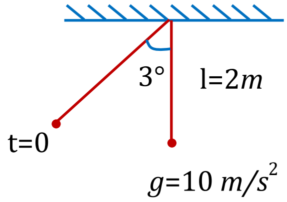
If the time period of a simple pendulum is 2 seconds then it is called second's pendulum. Second's pendulum take one second to go from one extreme position to other extreme position.

For second's pendulum, time period, $T = 2 = 2\pi\sqrt{\frac{l}{g}}$

At the surface of Earth, $g = 9.8 m/s^2 \approx 2\pi\sqrt{\frac{l}{g}}$

So length of second pendulum at the surface of earth $\ell \approx 1$ meter.

Illustration 41:



A bob is released from rest at $t = 0$, making an angle of 3° from the vertical. Find angular displacement at any time t.

Solution:

$$\theta = \theta_0 \sin(\omega t + \phi)$$

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{5}$$

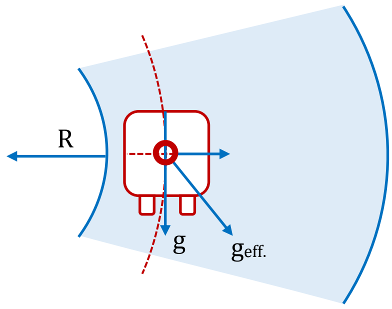
$$\theta = 3^0 \sin\left(\sqrt{5}\,t + \frac{\pi}{2}\right)$$

However, take $\phi = \frac{\pi}{2}$ because we are taking clockwise as $+ve$.

Illustration 42:

A simple pendulum of length L and mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes oscillation in a radial direction about its equilibrium position, then calculate its time period.

Solution:



Centripetal acceleration $a_c = \frac{v^2}{R}$ and Acceleration due to gravity = g

So, $g_{eff} = \sqrt{g^2 + \left(\frac{v^2}{R}\right)^2}$

$\Rightarrow$ Time period $T = 2\pi\sqrt{\frac{L}{g_{eff}}} = 2\pi\sqrt{\frac{L}{\sqrt{g^2 + \frac{v^4}{R^2}}}}$

Illustration 43:

A simple pendulum is suspended from the ceiling of a lift. When the lift is at rest, its time period is T. With what acceleration should lift be accelerated upwards in order to reduce its time period to $\frac{T}{3}$.

Solution:

In stationary lift,

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{...(i)}$$

In accelerated lift,

$$\frac{T}{3} = T' = 2\pi\sqrt{\frac{l}{g+a}} \quad \text{...(ii)}$$

Divide (i) by (ii) $3 = \sqrt{\frac{g+a}{g}} \Rightarrow g + a = 9g \Rightarrow a = 8g$

Periodic time of simple pendulum in reference system.

$$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

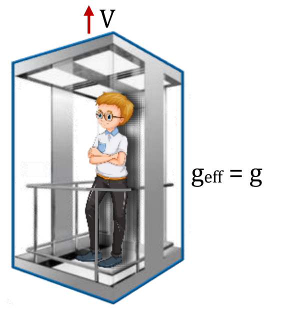
Where, $g_{eff.}$ = effective gravity acceleration in reference system or total downwards acceleration.

(a) If reference system is lift

(i) If velocity of life v = constant

acceleration $a = 0$ and $g_{eff.} = g$

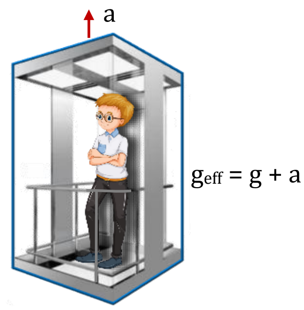
$\therefore \quad T = 2\pi\sqrt{\frac{l}{g}}$



(ii) If lift is moving upwards with acceleration a

$g_{eff.} = g + a$

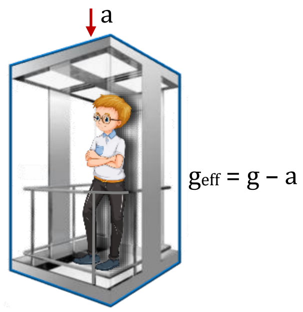
$T = 2\pi\sqrt{\frac{l}{g+a}}$, hence T decreases.



(iii) If lift is moving downwards with acceleration a

$g_{eff.} = g - a$

$\therefore \quad T = 2\pi\sqrt{\frac{l}{g-a}}$,hence T increases.



(iv) If lift falls downwards freely

$g_{eff.} = g - g = 0$

$T = \infty$ simple pendulum will not oscillate

(b) A simple pendulum is mounted on a moving truck

(i) If truck is moving with constant velocity, no pseudo force acts on the pendulum hence T remains same

$T = 2\pi\sqrt{\frac{l}{g}}$

(ii) If truck accelerates forward with acceleration a then a pseudo force acts in opposite direction. So effective acceleration, $g_{eff} = \sqrt{g^2 + a^2}$

and $\quad T' = 2\pi\sqrt{\frac{l}{g_{eff}}}$

Time period $T' = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a^2}}}$

$\Rightarrow \quad T'$ decreases

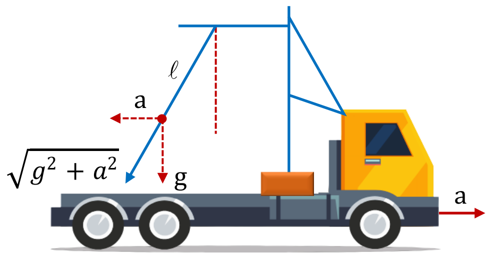


Illustration 44:

The bob of simple pendulum is suspended by a metallic wire. If a is the coefficient of linear expansion and change in temperature is $d\theta$, the effective length of wire becomes $\ell' = \ell\,(1 + a\,d\theta)$. Find percentage change in it's time period.

Solution:

$$T' = 2\pi\sqrt{\frac{\ell'}{g}} \text{ and } T = 2\pi\sqrt{\frac{\ell}{g}}$$

Hence $\dfrac{T'}{T} = \sqrt{\dfrac{\ell'}{\ell}} = (1 + a d\theta)^{1/2} = 1 + \dfrac{1}{2} a d\theta$

$\therefore$ Percentage increase in time period

$$= \left[\frac{T' - T}{T}\right] \times 100 = \left[\frac{T'}{T} - 1\right] \times 100 = \left[1 + \frac{a d\theta}{2} - 1\right] \times 100 = 50 a d\theta$$

Illustration 45:

The length of simple pendulum is increased by 2% then find the percentage change in time period.

Solution:

$$\left(\frac{\Delta T}{T} \times 100\right) = \frac{1}{2}\left(\frac{\Delta \ell}{\ell} \times 100\right)$$

$$\Rightarrow \quad \left(\frac{\Delta T}{T} \times 100\right) = \frac{1}{2} \times 2 = 1\%$$

Illustration 46:

If the length of simple pendulum is increased by 21% then find the percentage change in time period

Solution:

$$\ell_1 = \ell$$

$$\ell_2 = \ell + \frac{21}{100}\ell = \frac{121}{100}\ell = 1.21\ell$$

$$\frac{T_1}{T_2} = \sqrt{\frac{\ell_1}{\ell_2}} = \sqrt{\frac{\ell}{1.21\ell}} = \sqrt{\frac{100}{121}} = \frac{10}{11} \Rightarrow T_2 = \frac{11}{10} T_1$$

So, percentage change = $\dfrac{T_2 - T_1}{T_1} \times 100 = \left(\dfrac{11 - 10}{11}\right) \times 100 = 10\%$

Illustration 47:

Find the length of second's pendulum (i) On earth (ii) On moon

Solution:

(i) $T = 2\pi\sqrt{\dfrac{\ell}{g}}$

$\Rightarrow 2^2 = 4\pi^2\left(\dfrac{\ell}{g}\right)$

$\Rightarrow \ell = 1\ m$

(ii) $T = 2\pi\sqrt{\dfrac{\ell}{g} \times 6}$

$\Rightarrow 2^2 = 4\pi^2\left(\dfrac{6\ell}{g}\right)$

$\Rightarrow \dfrac{1}{6} m$

Physical Pendulums

In real life, point masses do not exist instead we deal with rigid bodies. These bodies also oscillate and if appropriate conditions are met ($\tau \propto -\theta$ or $\alpha \propto -\theta$) they will also perform angular SHM.

Based on restoring torque there are 2 types of physical pendulums

(1) Compound Pendulum

(2) Torsional Pendulum

Compound Pendulum

When a rigid body is suspended from an axis and made to oscillate about that then it is called compound pendulum.

C = Initial position of center of mass

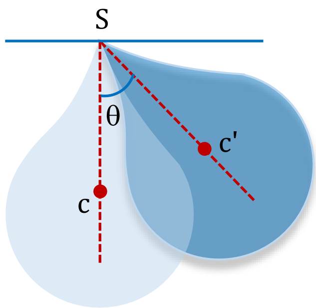
C' = Position of center of mass after time t

S = Point of suspension

ℓ = Distance between point of suspension and **center of mass** (it remains constant during motion)

For small angular displacement θ from mean position

The restoring torque is given by



$$\tau = -mg\ell\theta$$

$$I\alpha = -mg\ell\theta$$

$$\alpha = -\frac{mg\ell}{I}\theta \Rightarrow \omega^2 = \frac{mg\ell}{I} \Rightarrow \omega = \sqrt{\frac{mg\ell}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{mg\ell}} \quad \therefore \; I = I_{com} + m\ell^2$$

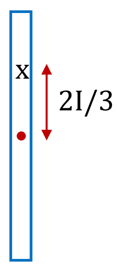
$$\Rightarrow \quad T = 2\pi\sqrt{\frac{I_{com} + m\ell^2}{mg\ell}}$$

Here $I_{com} = mk^2$

$\therefore$ k is radius of gyration about axis passing from centre of mass

Illustration 48:

Find time period for small oscillation of rod about axis passing through a point $\frac{2\ell}{3}$.



Solution:

As we know

$$T = 2\pi\sqrt{\frac{I}{mgd}} \qquad \therefore \quad I = I_{com} + md^2$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{I_{com} + md^2}{mgd}} \qquad \Rightarrow \quad T = 2\pi\sqrt{\frac{(m\ell^2/12) + (4m\ell^2/9)}{mg2\ell/3}} = \sqrt{\frac{19\ell}{24g}}$$

Illustration 49:

A disc of mass M (=2 kg) and radius R (=5 m) is pivoted at the centre. The disc lies in the vertical plane and is free to rotate about the pivot (see figure). A particle of mass m (=1 kg) is attached to the periphery of the disc as shown in the figure. The time period of small oscillation of the system is: [Taking g = 10 m/s^2]

(A) 2π sec (B) π sec (C) 4π sec (D) (π/2) sec

Ans. (A)

Solution:

$x = (5 \times 1)/3 = 5/3$

$$T = 2\pi\sqrt{\frac{I_{com} + Md^2}{Mgd}}$$

$$I = \frac{2\times5^2}{2} + 1\times5^2 = 50$$

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{50}{\frac{3\times10\times5}{3}}} = 2\pi$$

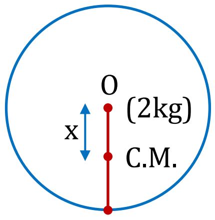
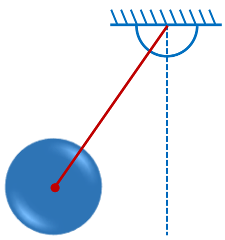


Illustration 50:

A metal rod of length 1.25 m and mass 'm' is pivoted at one end. A solid sphere of same mass and radius 0.25 m is attached at its centre to the free end of the rod and the sphere is free to rotate about its centre. The rod-sphere system performs SHM in vertical plane after being released from the same displaced position. Angular frequency of small oscillations is ω rad/s. Find the value of ω^2.



Solution:

$$\tau = \left(mg\frac{\ell}{2} + mg\ell\right)\sin\theta$$

$$I = \frac{m\ell^2}{3} + m\ell^2 \; ; \; \alpha = \frac{t}{I} = \frac{3mg\ell\theta}{2\left[\frac{4m\ell^2}{3}\right]} = \frac{9}{8}\frac{g\theta}{\ell}$$

$$\omega^2 = \frac{9g}{8\ell} = 9$$

Torsional Pendulum

In torsional pendulum, an extended object is suspended at the center of mass by a light torsion wire.

Torsional wire is a special kind of wire which produces restoring torque proportional to angle by which it is rotated (only for small value of θ).

$\therefore$ $\tau \propto -\theta$ or $\alpha \propto -\theta$

$\Rightarrow$ $\tau = -C\theta$ (where C = Torsional constant)

$\Rightarrow$ $I\alpha = -C\theta$ (I = Moment of inertia about axis passing through **c.o.m**)

$$\Rightarrow \alpha = -\frac{C}{I}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0 \Rightarrow \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

$$\Rightarrow \left[\because \omega^2 = \frac{C}{I}\right] \qquad \Rightarrow T = 2\pi\sqrt{\frac{I}{C}}$$

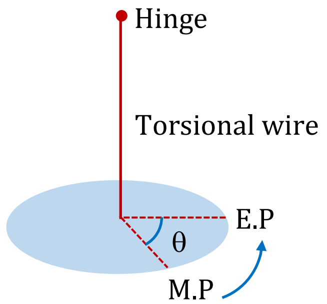


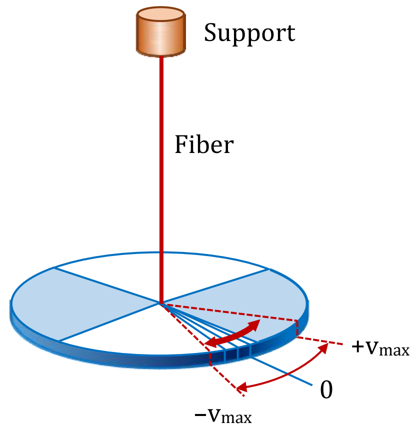
Illustration 51:

A uniform disc of radius 5.0 *cm* and mass 200 *g* is fixed at its center to a metal wire, the other end of which is fixed to a ceiling. The hanging disc is rotated about the wire through an angle 2° and is released. If the disc makes torsional oscillations with time period 0.20 *s*, find

(a) The torsional constant of the wire.

(b) Maximum angular velocity of disc

Solution:



(a) The situation is shown in figure.

The moment of inertia of the disc about the wire is

$$I = \frac{mr^2}{2} = \frac{(0.200kg)(5.0\times10^{-2}m)^2}{2} = 2.5\times10^{-4}\,kg-m^2$$

The time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \text{ or } C = \frac{4\pi^2 I}{T^2}$$

$$\frac{4\pi^2(2.5\times10^{-4}\,kg-m^2)}{(0.20s)^2} = 0.25\frac{kg-m^2}{s^2}$$

(b) $\omega_{disc} = \theta_o\omega_{SHM}\cos(\omega_{SHM}t+\phi)$

θ_o = Angular amplitude = 2°

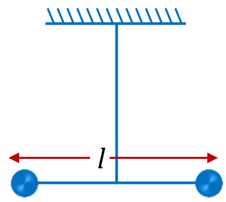
$$\omega_{SHM} = \frac{2\pi}{T} = 10\pi\ \sec^{-1}$$

$$\omega_{Disc(Maximum)} = \theta_o\omega_{SHM}$$

$$\omega_{Disc(Maximum)} = \frac{4\pi}{360}\times10\pi = 1.09\ rad/\sec$$

Illustration 52:

Two masses m and $\frac{m}{2}$ are connected at the two ends of a massless rigid rod of length l. The rod is suspended by a thin wire of torsional constant k at the centre of mass of the rod-mass system (see figure). Because of torsional constant k, the restoring torque is $\tau = k\theta$ for angular displacement θ. If the rod is rotated by θ_0 and released, the tension in it when it passes through its mean position will be:

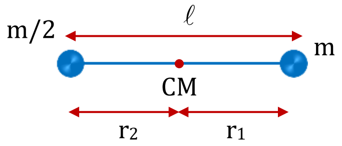


(A) $\frac{3k\theta_0^2}{l}$ (B) $\frac{k\theta_0^2}{2l}$ (C) $\frac{2k\theta_0^2}{l}$ (D) $\frac{k\theta_0^2}{l}$

Ans. (D)

Solution:

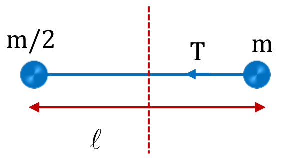
$$\frac{r_1}{r_2}=\frac{1}{2}\Rightarrow r_1=\frac{\ell}{3}$$



$$\omega=\sqrt{\frac{k}{I}}$$

$$I=\frac{ml^2}{9}+\frac{4ml^2}{9}=\frac{ml^2}{3}$$

$$\omega=\sqrt{\frac{3k}{m\ell^2}}$$



$\Omega = \omega\theta_0$ = angular velocity at mean position

$T = m\Omega^2 r_1$

$$T=m\Omega^2\frac{\ell}{3}=m\omega^2\theta_0^2\frac{\ell}{3}=m\frac{3k}{m\ell^2}\theta_0^2\frac{\ell}{3}=\frac{k\theta_0^2}{l}$$

Series & Parallel Connection:

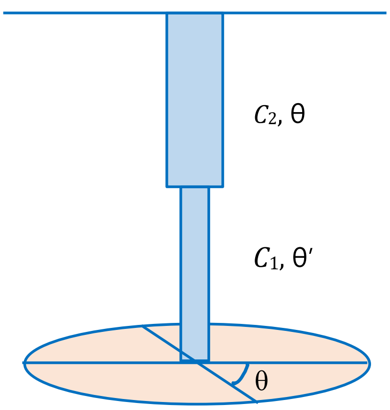
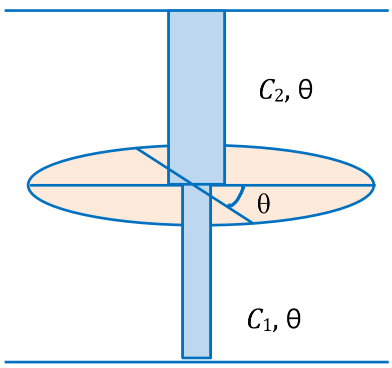
Series	Parallel
	
$\tau_1=\tau_2\Rightarrow C_1\theta'=C_2\theta$	$\tau_{net}=\tau_1+\tau_2=C_{eqv}\theta$
$\frac{1}{C_{eqv}}=\frac{1}{C_1}+\frac{1}{C_2}$	$C_{eqv}=C_1+C_2$

Illustration 53:

A disc of mass m and radius R is fastened with two rods of torsional constant $6C$ and $3C$ as shown. Time period of small oscillation of disc about its axis in case-I and case-II are respectively T_1 and T_2. Value of $\frac{T_1^2}{T_2^2}$ is:

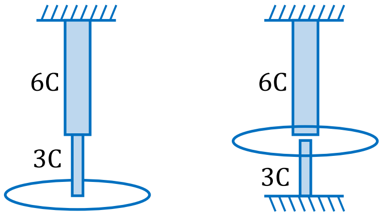


Figure – 1 Figure – 2

(A) $\frac{9}{4}$ (B) $\frac{2}{9}$ (C) $\frac{9}{2}$ (D) $\frac{3}{4}$

Ans. (C)

Solution:

For figure-1

$$C_1 = \frac{6C \times 3C}{6C + 3C} = 2C$$

For figure-2

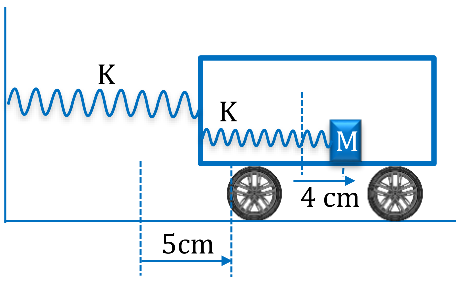
$$C_2 = 6C + 3C = 9C$$

C_1 and C_2 are equivalent torsional constants

$$\frac{T_1}{T_2} \propto \sqrt{\frac{C_2}{C_1}} \Rightarrow \left(\frac{T_1}{T_2}\right)^2 = \frac{9}{2}$$

Introduction of Superposition

Let us consider a situation in which a block performing SHM is placed inside a cart performing SHM If block is displaced towards right 4 *cm* and cart is also displaced towards right by 5 *cm*. Displacement of block with respect to ground = $\vec{S}_{B/C} + \vec{S}_C$ = 4 *cm* + 5 *cm* = 9 *cm*.



Displacement of block with respect to car

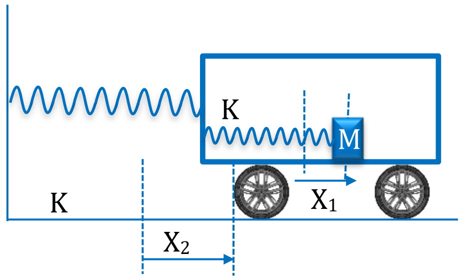
$$x_1 = A_1 \sin(\omega_1 t + \phi_1)$$

Displacement of cart with respect to ground

$$x_2 = A_2 \sin(\omega_2 t + \phi_2)$$

Then, displacement of block with respect to ground is given by $x_1 + x_2$.

Superposition or Combination of SHM



The resultant acceleration of the block is given by:

$$a=\frac{d^2x}{dt^2}=a_1+a_2$$

$$a=\frac{d^2x}{dt^2}=-\omega_1^2A_1\sin(\omega_1t+\phi_1)-\omega_2^2A_2\sin(\omega_2t+\phi_2)$$

$$a=-\omega_1^2x_1-\omega_2^2x_2$$

We can't write it as

$$a=-\omega^2x \quad \text{as } (\omega_1\neq\omega_2)$$

So overall motion is not SHM (Periodic and Oscillatory)

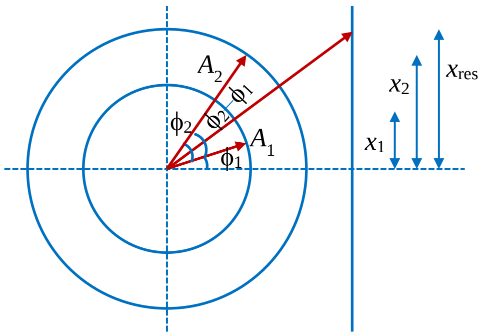
If $\omega_1 = \omega_2$

$$a=-\omega^2(x_1+x_2)=-\omega^2x_{res}$$

$$x_{res}=(x_1+x_2)$$

So overall motion is SHM (Periodic and Oscillatory).

Superposition of SHM of same using Phasor



Note:

Superposition of SHM of same frequency is also a SHM

Superposition of Two Shm's

Principle of Superposition:

$$F_1=-kx_1$$

$$x_1=A_1\sin(\omega t+\phi_1)$$

$$F_2=-kx_2$$

$$x_2=A_2\sin(\omega t+\phi_2)$$

If both acts

$$F_{res.}=F_1+F_2=-k(x_1+x_2)=kx_{res}$$

$$x_{res.}=x_1+x_2$$

In same direction and of same frequency:

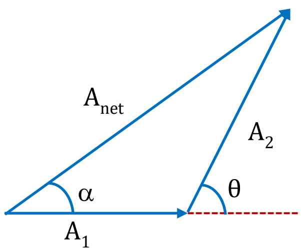
$$x_1=A_1\sin(\omega t+\phi_1)$$

$$x_2=A_2\sin(\omega t+\phi_2)$$

$$x=x_1+x_2$$

$$x=A_1\sin(\omega t+\phi_1)+A_2\sin(\omega t+\phi_2)$$

$$x=A_{net}\sin((\omega t+\phi_1)+\alpha)$$



Where, $A_{net}^2=A_1^2+A_2^2+2A_1A_2\cos\theta$

$$\theta=(\omega t+\phi_2)-(\omega t+\phi_1)$$

$$=\phi_2-\phi_1\text{, and } \quad \tan\alpha=\frac{A_2\sin\phi}{A_1+A_2\sin\phi}$$

Note:

If $\theta = 0$, both SHM's are in phase and $A = A_1 + A_2$

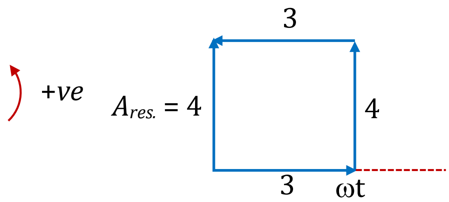
If $\theta = \pi$, both SHM's are out of phase and $A = |A_1 - A_2|$

The resultant amplitude due to superposition of two or more than two SHM's of this case can also be found by phasor diagram also.

Illustration 54:

$$x = 3\sin(\omega t) + 4\sin\left(\omega t + \frac{\pi}{2}\right) + 3\sin(\omega t + \pi)$$

Solution:



$$x = 4\sin\left(\omega t + \frac{\pi}{2}\right)$$

Illustration 55:

Is $x = A\left(\sin\omega t - \sqrt{3}\cos\omega t\right)$ equation of SHM? If yes find amplitude and ϕ ?

Solution:

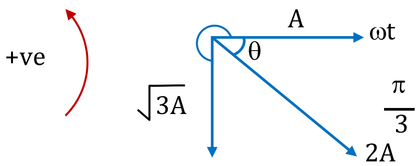
Method-1: $$x = 2A\left(\frac{1}{2}\sin\omega t - \frac{\sqrt{3}}{2}\cos\omega t\right)$$

$$x = 2A\left(-\cos(\omega t + 30^\circ)\right)$$

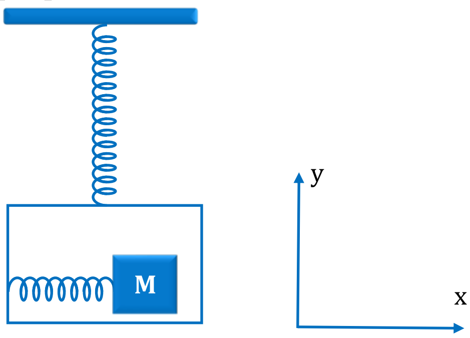
Method-2: $$x_1 = A\sin(\omega t), x_2 = \sqrt{3}A\sin\left(\omega t + \frac{3\pi}{2}\right)$$

$$x_{res.} = 2A\sin\left(\omega t - \frac{\pi}{3}\right)$$

Amplitude = $2A$ and $\phi = \frac{\pi}{3}$



Superposition of two SHM in perpendicular directions



$$x = A\sin(\omega_1 t + \phi_1)$$
$$y = B\sin(\omega_2 t + \phi_2)$$

Particles subjected to in two perpendicular directions (Lissajous Figures)

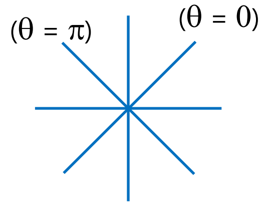
$$x = A\sin\omega_1 t$$
$$y = B\sin(\omega_2 t + \theta)$$

If $\omega_1 = \omega_2$

Case.1 If θ = 0 or π then $y = \pm(B/A)x$ So, path will be straight line & resultant displacement will be

$$r = (x^2 + y^2)^{1/2} = (A^2 + B^2)^{1/2}\sin\omega t$$

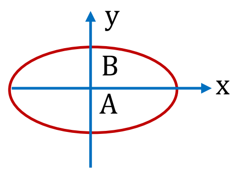
which is equation of SHM having amplitude $\sqrt{A^2 + B^2}$



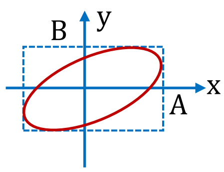
Case.2 If $\theta = \frac{\pi}{2}$ then $x = A\sin\omega t$

$$y = B\sin(\omega t + \pi/2) = B\cos\omega t$$

So, resultant will be $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ i.e. equation of an ellipse and if $A = B$, then superposition will be an equation of circle.



Case.3 If $\theta \neq 0, \frac{\pi}{2}, \pi$



Superposition of SHM's along the same direction (using diagram)
If two or more SHM's are along the same line, their resultant can be obtained by vector addition by making phasor diagram.

1. Amplitude of SHM is taken as length (magnitude) of vector
2. Phase difference between the vectors is taken as the angle between these vectors. The magnitude of resultant of vectors give resultant amplitude of SHM and angle of resultant vector gives phase constant of resultant SHM.

Illustration 56:

$$x_1 = 3\sin\omega t \quad ; \quad x_2 = 4\cos\omega t$$

Find (i) amplitude of resultant SHM. (ii) equation of the resultant SHM.

Solution:

First write all SHM's in terms of sine functions with positive amplitude.

$\therefore$ $x_1 = 3\sin\omega t$

$$x_2 = 4\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$A = \sqrt{3^2 + 4^2 + 2\times3\times4\cos\frac{\pi}{2}} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\tan\phi = \frac{4\sin\frac{\pi}{2}}{3 + 4\cos\frac{\pi}{2}} \quad \phi = 53^\circ$$

Equation $x = 5(\omega t + 53^0)$

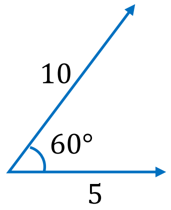
Illustration 57:

Two SHM's whose equations are given by $x_1 = 5\sin(\omega t+30°)$ and $x_2 = 10\sin\left(\omega t+\frac{\pi}{2}\right)$ respectively are superimposed on each other. Find the amplitude of the resulting SHM.

Solution:

$$x_1 = 5\sin\left(\omega t+30^0\right)$$

$$x_2 = 10\sin\left(\omega t+\frac{\pi}{2}\right)$$



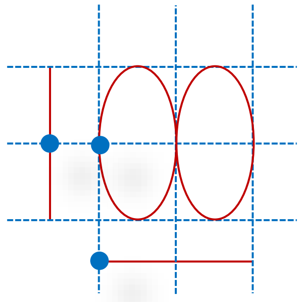
Phasor diagram

$$A=\sqrt{\left(5^2+10^2+2\times5\times10\cos60^0\right)}$$

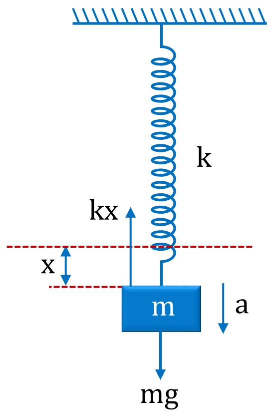
$$=\sqrt{25+100+50}=\sqrt{175}=5\sqrt{7}$$

Note:

- Superposition of two SHM in same direction if $\omega_1=\omega_2$ then resultant is SHM
  and we use phasor diagram tool to solve.
- Superposition of two S.H.M. in perpendicular direction if $\omega_1=\omega_2$ doesn't implies their resultant is SHM.
  If $\omega_1=\omega_2$ and $\phi=0$ or $\phi=\pi$ the resultant will be SHM.
  Here particle completes two oscillations in y directions while it completes one oscillation in x direction.
  $\therefore\quad \omega_y=2(\omega_x)$



Damped Oscillations



In real life the motion of a simple pendulum, swinging in air, dies out eventually. Why does it happen? This is because the air drag and the friction at the support oppose the motion of the pendulum and dissipate its energy gradually. The pendulum is said to execute **damped oscillations**.

In damped oscillations, the energy of the system is dissipated continuously; but, for small damping, the oscillations remain approximately periodic. The dissipating forces are generally the frictional forces.

To understand the effect of such external forces on the motion of an oscillator, let us consider a system as shown in figure. Here a block of mass m connected to an elastic spring of spring constant k oscillates vertically.

If the block is pushed down a little and released, its angular frequency of oscillation is $\sqrt{\frac{k}{m}}$.

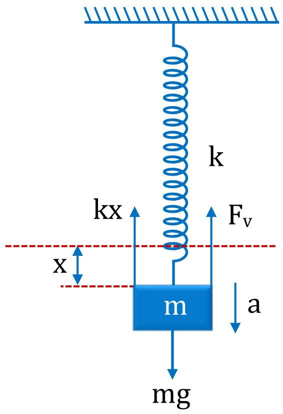
However, in practice, the surrounding medium (air) will exert a damping force on the motion of the block and the mechanical energy of the block-spring system will decrease. The energy loss will appear as heat of the surrounding medium (and the block also).

The damping force depends on the nature of the surrounding medium. If the block is immersed in a liquid, the magnitude of damping will be much greater and the dissipation of energy much faster. The damping force is generally proportional to velocity of the bob and acts opposite to the direction of velocity. If the damping force is denoted by F_d,

$$F_d = -bv$$

Here b is damping coefficient

where the positive constant b depends on characteristics of the medium (viscosity, for example) and the size and shape of the block, etc.



Relation $F_d = -bv$ is usually valid only for small velocity.

When the mass m is attached to the spring and released, the spring will elongate a little and the mass will settle at some height. This position is the equilibrium position of the mass. If the mass is pulled down or pushed up a little, the restoring force on the block due to the spring is $F_S = -kx$, where x is the displacement of the mass from its equilibrium position.

Thus, the total force acting on the mass at any time t, is

$$F = mg - kx - bv.$$

If a is the acceleration of mass at time t, then by Newton's Law of

Motion applied along the direction of motion, we have

$$m\,a = mg - k\,x - b\,v$$

Here we have dropped the vector notation because we are discussing one-dimensional motion.

Using the first and second derivatives of $x(t)$ for $v(t)$ and $a(t)$ respectively, we have

$$m\frac{d^2x}{dt^2} = mg - b\frac{dx}{dt} - k\,x$$

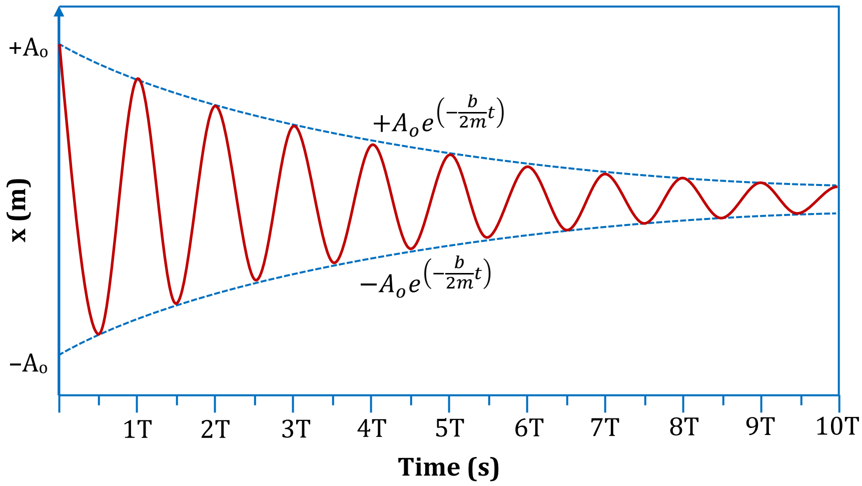
The solution of Eq. (iii) describes the motion of the block under the influence of a damping force which is proportional to velocity. The solution is found to be of the form

$$x(t) = A_0\, e^{-bt/2m} \cos(\omega t + \phi) \qquad \text{...(i)}$$

where A_0 is the initial amplitude and ω is the angular frequency of the damped oscillator given by, In this function, the cosine function has a period $2\pi/\omega$ but the function $x(t)$ is not strictly periodic because of the factor $e^{-bt/2m}$ which decreases continuously with time.

However, if the decrease is small in one time period T, the motion represented by Eq. (i) is approximately periodic.

The solution, Eq. (i), can be graphically represented as shown in figure.



Angular Frequency (ω)

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$\omega = \sqrt{\omega_0^2 - \lambda^2}\text{, where damping constant } \lambda = \frac{b}{2m}$$

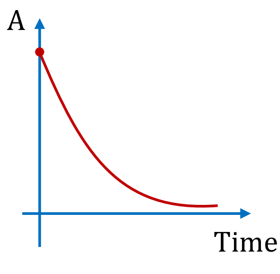
Natural angular frequency $\omega_0 = \sqrt{\frac{k}{m}}$

Time Period of Oscillation (T)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}} \text{ and natural time period } T_0 = \frac{2\pi}{\omega_0}$$

Amplitude(A)

$$A = A_0 e^{-\frac{b}{2m}t} = A_0 e^{-\lambda t}$$



Mechanical Energy

The mechanical energy of the undamped oscillator is $\frac{1}{2}kA^2$.

For a damped oscillator, the amplitude is not constant but depends on time. For small damping, we may use the same expression but regard the amplitude as $A\,e^{-bt/2m}$.

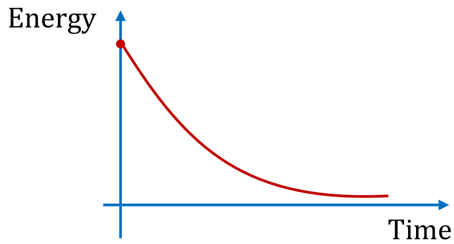
$$E = \frac{1}{2}k\ A^2 = \frac{1}{2}k\ A_0^2 e^{\frac{-b}{m}t}$$

$$E = E_0 e^{-\frac{b}{m}t} = E_0 e^{-2\lambda t}$$

Here E_0 is initial mechanical energy

$$E_0 = \frac{1}{2}k\ A_0^2$$

(Damping constant for Energy is 2λ)



Above Equation shows that the total energy of the system decreases exponentially with time.

Half Time (t_h)

Half time is time in which amplitude decreases to half of initial Amplitude

$$A = A_0 e^{-\lambda t} = \frac{A_0}{2} \quad \Rightarrow \quad e^{-\lambda t} = 0.5 \quad \Rightarrow \lambda t = \ln(2)$$

$$t_h = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$$

Mean or average Time (t_a)

Average time is time in which amplitude decreases to $\frac{1}{e}$ times of initial Amplitude.

$$A = A_e e^{-\lambda t} = \frac{A_0}{e} \quad \Rightarrow \quad e^{-\lambda t} = e^{-1} \Rightarrow \lambda t = 1$$

$$t_a = \frac{1}{\lambda}$$

Illustration 58:

When an oscillator completes 100 oscillations its amplitude reduced to $\frac{1}{3}$ of initial value. What will be its amplitude, when it completes 200 oscillations :

(A) $\frac{1}{8}$ (B) $\frac{2}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{9}$

Solution:

In case of damped vibration, amplitude at any instant is

$a = a_0\ e^{-\lambda t}$

where a_0 = initial amplitude

λ = damping constant

1st Case :

$t = 100\ T$ and $a = \frac{a_0}{3}$

$\therefore \quad \frac{a_0}{3} = a_0\ e^{-\lambda(100T)} \quad \Rightarrow e^{-100\lambda T} = \frac{1}{3}$

2nd Case :

$t = 200\ T$

$$a = a_0\ e^{-\lambda t} = a_0\ e^{-\lambda(200T)} = a_0\left(e^{-100\lambda T}\right)^2 = a_0 \times \left(\frac{1}{3}\right)^2 = \frac{a_0}{9}$$

Illustration 59:

The damping force on an oscillator is directly proportional to the velocity. The units of the constant of proportionality are :

(A) $kgms^{-1}$ (B) $kgms^{-2}$ (C) kgs^{-1} (D) kgs

Ans. (C)

Solution:

$F \propto v \quad \Rightarrow F = kV$

$$k = \frac{F}{v} \quad \Rightarrow [k] = \frac{[kgms^{-2}]}{[ms^{-1}]} = kg\ s^{-1}$$

Illustration 60:

The amplitude of a damped oscillator decreases to 0.9 times its original magnitude in $5s$. In another $10s$ it will decrease to α times its original magnitude, where α equals.

(A) 0.7 (B) 0.81 (C) 0.729 (D) 0.6

Ans. (C)

Solution:

$$A = A_0\ e^{-\frac{b(t)}{2m}}$$

After 5 second

$$0.9A_0 = A_0 e^{-\frac{b(5)}{2m}} \qquad \text{...(i)}$$

After 10 more second

$$A = A_0 e^{-\frac{b(15)}{2m}} \qquad \text{...(ii)}$$

From (i) and (ii)

$A = 0.729\ A_0$

Hence **Ans. (C)**

Illustration 61:

For the damped oscillator, the mass m of the block is $200g$, $k = 90\ N\ m^{-1}$ and the damping constant b is $40\ gs^{-1}$. Calculate: (a) the period of oscillation, (b) time taken for its amplitude of vibrations to drop to half of its initial value and (c) the time taken for its mechanical energy to drop to half its initial value.

Solution:

(a) The time period T from equation is given by

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}} = 0.3\,\text{sec}$$

(b) Now, from equation, the time, $T_{1/2}$ for the amplitude to drop to half of its initial value is given by

$$A(t) = Ae^{-bt/2m}$$

$$\frac{A}{2} = Ae^{-\frac{bT_{1/2}}{2m}} \quad \Rightarrow \quad T_{1/2} = \frac{2m\ell n2}{b} = 6.93\ s$$

(c) For calculating the time $t_{1/2}$, for its mechanical energy to drop to half of its initial value we make use of equation $E = E_0 e^{-\frac{b}{m}t} = E_0 e^{-2\lambda t}$. From this equation we have

$$E(t_{1/2}) = \frac{E(0)}{2}$$

$$\Rightarrow \quad E(0)e^{-bt/m} = \frac{E(0)}{2}$$

$$t_{1/2} = \frac{(\ell n2)m}{b} = \frac{6.93}{2}$$

This is just half of the decay period for amplitude. This is not surprising, because, according to Equation $E = E_0 e^{-\frac{b}{m}t} = E_0 e^{-2\lambda t}$, energy depends on the square of the amplitude. Notice that there is a factor of 2 in the exponents of the two exponentials.

Types of Oscillations:

On the basis of the value of λ $\left(\lambda = \frac{b}{2m}\right)$, the damped oscillation can be categorized as

1. Undamped (λ = 0)

Resistive force is zero, b = 0 and λ = 0

$\Rightarrow$ $A = A_0, E = E_0$ Amplitude and Energy remain constant

$\Rightarrow$ $\omega = \omega_0 = \sqrt{\frac{k}{m}}$ and $T = T_0\sqrt{\frac{m}{k}}$

The system oscillates at its natural angular frequency (ω_0).

2. Under Damped ($\lambda \neq 0$ and $\lambda < \omega_0$)

Resistive force is non-zero, $b \neq 0$ and $\lambda \neq 0$

$\Rightarrow$ Amplitude and Energy decreases

$\Rightarrow$ $A = A_e e^{\lambda t}, E = E_0 e^{-2\lambda t}$

$\Rightarrow$ $\omega = \sqrt{\omega_0^2 - \lambda^2} = +ive$

The system oscillates (at reduced frequency compared to the undamped case) with the amplitude gradually decreasing to zero.

3. Over Damped ($\lambda \neq 0$ and $\lambda > \omega_0$)

Resistive force is non-zero, $b \neq 0$ and $\lambda \neq 0$

$\Rightarrow$ Amplitude and Energy decreases

$\Rightarrow$ $A = A_0 e^{-\lambda t}, E = E_0 e^{-2\lambda t}$

$\Rightarrow$ As $\omega_0 < \lambda \Rightarrow \omega = \sqrt{\omega_0^2 - \lambda^2}$ is an imaginary number

$\Rightarrow$ Particle finally come to rest without performing any Oscillatory motion

The system returns (exponentially decays) to equilibrium without oscillating. For larger values of the damping constant **λ,** block return to equilibrium more slowly.

4. Critically Damped ($\lambda = \omega_0$)

$\Rightarrow$ Particle finally come to rest without performing any Oscillatory motion

$\Rightarrow$ But It is fastest over damped Oscillation (Particle take smallest time to come to rest)

The system returns to equilibrium as quickly as possible without oscillating.

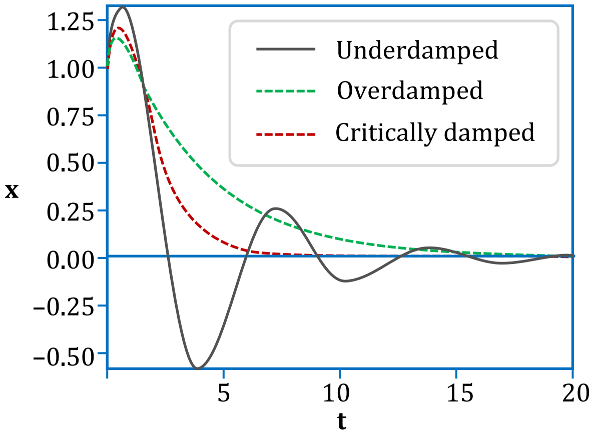


Illustration 62:

Show that the system $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x = 0$ is under damped, find its damped angular frequency

Solution:

$m = 1, b = 1, k = 3$

$b^2 < 4mk$

So, the system is under damped

$$\omega_d = \sqrt{\left(\frac{k}{m} - \frac{b^2}{4m^2}\right)} = \frac{\sqrt{11}}{2} s^{-1}$$

Illustration 63:

Determine whether the system is underdamped, overdamped or critically damped.

(i) $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0$

(ii) $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$

Solution:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

If $b^2 < 4mk$ then the system is underdamped

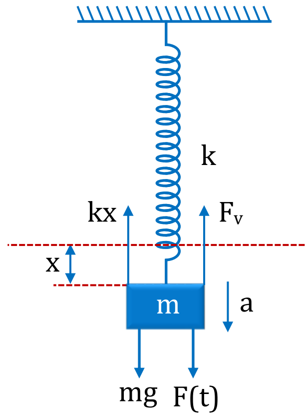
If $b^2 > 4mk$ then the system is overdamped

If $b^2 = 4mk$ then the system is critically damped

(i) overdamped

(ii) Critically damped

Forced Oscillations



- An external driving force starts oscillations in a stationary system.
- The amplitude remains constant (or grows) if the energy input per cycle exactly equals (or exceeds) the energy loss from damping.
- Eventually, $E_{driving} = E_{lost}$ and a steady-state condition is reached.
- Oscillations then continue with constant amplitude.
- Oscillations are at the driving frequency ω_d.

$$F(t) = F_0\, cos(\omega_d t + \phi')$$

Oscillating driving force applied to a damped oscillator

Equation for Forced (Driven) Oscillations:

ω_0 = natural frequency $\quad \omega_0 = \sqrt{\frac{k}{m}}$

ω_d = driving frequency of external force

External driving force function:

$$F = F_0\, cos(\omega_d t + \phi')$$

$$F_{net} = F - bv - kx + mg = ma$$

$$F_{net} = F - b\frac{dx}{dt} - kx + mg = m\frac{d^2x}{dt^2}$$

Solution for forced (Driven) Oscillations:

$$F_{net} = F_0\cos(\omega_d t + \phi') - b\frac{dx}{dt} - kx + mg = m\frac{d^2x}{dt^2}$$

Solution of this differential equation is

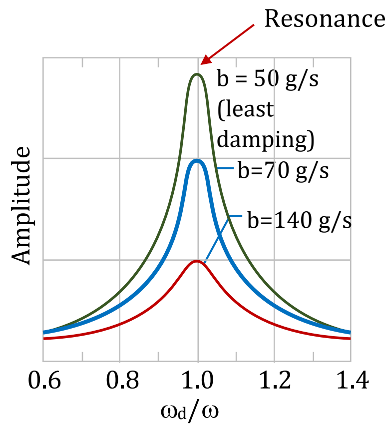
$$X(t) = A\cos(\omega_d t + \phi)$$

Where $A = \dfrac{F_0/m}{\sqrt{(\omega_d^2-\omega_0^2)^2+\left(\dfrac{b\omega_d}{m}\right)^2}}$

The system always oscillation at the driving always oscillates at the driving frequency ω_d in steady-state.

Resonance Condition in Forced Oscillations

For given values of *m, b* and F_0 amplitude of motion of block is maximum when $\omega_d = \omega_0$. This is known as Forced Oscillation in Resonance condition



- The largest amplitude oscillations occur at or near resonance ($\omega_D \sim \omega_D$)
  As damping becomes weaker → resonance sharpens and Amplitude at resonance increases.
- At resonance, the applied force is in phase with the velocity and the power F_0v transferred to the oscillator is a maximum.
- The amplitude of resonant oscillations can become enormous when the damping is weak, storing enormous amounts of energy.

Applications:

- Buildings driven by earthquakes
- Bridges under wind load
- All kinds of radio devices, microwave
- Other numerous applications

Illustration 64:

A particle of mass *m* is attached to a spring (of spring constant *k*) and has a natural angular frequency ω_0. An external force $F(t)$ proportional to cos ωt ($\omega \neq \omega_0$) is applied to the oscillator. The time displacement of the oscillator will be proportional to :

(A) $\dfrac{m}{\omega_0^2 - \omega^2}$

(B) $\dfrac{1}{m\ (\omega_0^2 - \omega^2)}$

(C) $\dfrac{1}{m\ (\omega_0^2 + \omega^2)}$

(D) $\dfrac{m}{\omega_0^2 + \omega^2}$

Ans. (B)

Solution:

$F(t) = F_0 A \cos(\omega t + \phi_0)$

where $A = \dfrac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 b^2}}$

For small damping

$$A \approx \frac{F_0}{[m(\omega_0^2 - \omega^2)]}$$

$$\Rightarrow \quad x(t) = \left[\frac{F_0}{m(\omega_0^2 - \omega^2)}\right]\cos(\omega + \phi_0)$$

Illustration 65:

In forced oscillation of a particle, the amplitude is maximum for a frequency ω_1 of the force, while the energy is maximum for a frequency ω_2 of the force, then :

(A) $\omega_1 = \omega_2$

(B) $\omega_1 > \omega_2$

(C) $\omega_1 < \omega_2$ when damping is small and $\omega_1 > \omega_2$ when damping is large

(D) $\omega_1 < \omega_2$

Ans. (A)

Solution:

Amplitude and energy both are maximum at resonance, when driving frequency is equal to the natural frequency of oscillation. Hence $\omega_1 = \omega_2 = \omega_0$

Illustration 66:

A simple pendulum has a time period T_0 if there is no air resistance. If a small air resistance is acting on the bob as it oscillates,

(A) The time period will be initially more than T_0 and decreases with time.

(B) The time period will be less than T_0 initially and increases with time.

(C) The time period will be less than T_0 and remains constant.

(D) The time period will be more than T_0 and remains constant.

Ans. (D)

Solution:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}} \text{ and } T_0 = \frac{2\pi}{\omega_0}$$

$T \geq T_0$ As λ is non-negative

Due to damping Time period of simple pendulum increases in comparison to undamped motion but remains constant with time.

Illustration 67:

In damped oscillations, the amplitude after 10 sec. is 0.6 A_0, where A_0 is the initial amplitude, then determines the amplitude after another 10 sec.

Solution:

$$A = A_0 e^{-\lambda t}$$

At $t = 10$ $A = A_0 e^{-10\lambda} = 0.6 A_0$

$\Rightarrow$ $e^{-10\lambda} = 0.6$

At $t = 20$; $A = A_0 e^{-20\lambda}$

At $t = 20$; $A = A_0 (0.6)^2 = 0.36 A_0$

Illustration 68:

In damped oscillations, the amplitude after 10 sec. 0.6 A_0, where A_0 is the initial amplitude, then determine the value of damping constant (in s^{-1}). Given ln(5) = 1.61, ln (3) = 1.1.

Solution:

$$A = A_0 e^{-\lambda t}$$

At $t = 10$; $A = A_0 e^{-10\lambda} = 0.6A_0$

$\Rightarrow$ $e^{-10\lambda} = 0.6$

$\Rightarrow$ $-10\lambda = \ln(0.6) = \ln\frac{3}{5} = \ln 3 - \ln 5$

$\Rightarrow$ $-10\lambda = 1.1 - 1.61 = -0.5 -10\lambda$

$\Rightarrow$ $\lambda = 0.05 s^{-1}$

Illustration 69:

Damped harmonic oscillator consists of a block (m= 2 kg), a spring $\left(k = 50\pi^2 N/m\right)$ and a damping force $\left(F = -12\pi v\, N\right)$. Find angular frequency and time period of motion. Also find angular frequency and time period if damping force is removed.

Solution:

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \lambda^2}$$

m = 2 Kg, $k = 50\pi^2$ N/m and $b = 12\,\pi$

$$\omega = \sqrt{\frac{50\pi^2}{2} - \left(\frac{12\pi}{4}\right)^2} = \sqrt{25\pi^2 - 9\pi^2} = 12\pi\, Rad/s$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{12\pi} = \frac{1}{6}\text{sec}.$$

For undamped motion,

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\pi^2}{2}} = 5\pi\, Rad/s$$

$$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{5\pi} = \frac{2}{5}\text{sec}$$

Illustration 70:

If the differential equation given by the oscillatory motion of body in a dissipative medium under the influence of a periodic force, then the state of maximum amplitude of the oscillation is a measure of

(A) Free vibration (B) Damped vibration (C) Forced vibration (D) Resonance

Ans. (D)

Solution:

In Resonance condition amplitude of forced oscillation is maximum.

Illustration 71:

When an automobile move over an uneven path, it experience jerks. To minimize the effect of the jerks the suspensions used in should satisfy the condition of

(A) Undamped Oscillation
(B) Over-damped Oscillation
(C) Under-damped Oscillation
(D) Critical damped Oscillation

Ans. (D)

Solution:

When an automobile move over an uneven path, it experience jerks. To minimize the effect of the jerks the suspensions used in should satisfy the condition of Critical damped Oscillation, because Automobile require to come its natural state as soon as possible.