01 Wave on String and Sound Waves

WAVE ON STRING

Wave

A wave is a disturbance which propagates in space, transport energy from one place to another without the transport of matter.

Example - Ripples on a pond, sound we hear etc.

Classification of wave

Wave

- Based on medium
  - Mechanical wave
  - Electro magnetic wave
- Based on Motion of the particles of medium
  - Longitudinal wave
  - Transverse wave
- Based on Energy propagation
  - Progressive wave
  - Stationary wave
- Based on direction of propagation
  - One dimensional wave
  - Two dimensional wave
  - Three dimensional wave

As shown in the above tree, waves can be primarily classified into four categories–

(i)	Based on medium	(ii)	Based on the motion of particles of the medium
(iii)	Based on energy propagation	(iv)	Based on direction of propagation

Now let us discuss these categories one by one.

Based on medium:

On the basis of medium, waves are of two types (a) Mechanical waves, (b) Electromagnetic waves

(a) **Mechanical waves:** Require material medium for their propagation.

Examples: Sound waves, waves on a string.

Properties of mechanical waves:

(i) Energy and momentum propagates by motion of particles of the medium but medium does not propagate *i.e.* transfer of mass does not takes place.

(ii) Due to elasticity of the medium, particles occupy its original position.

(iii) Due to inertia, medium can transfer kinetic and potential energy.

(b) **Electromagnetic waves:** Material medium is not required for propagation.
Examples: Radio waves, Light waves.

Properties of electromagnetic waves:

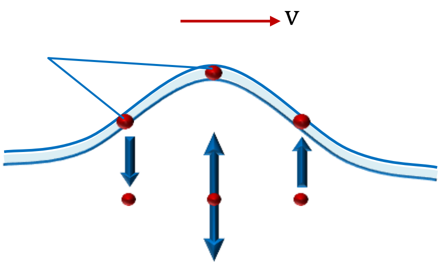
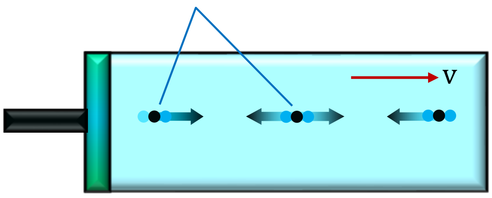
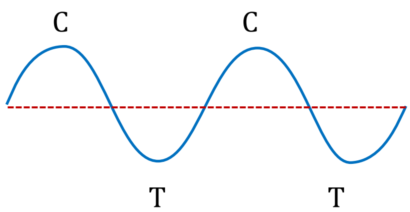
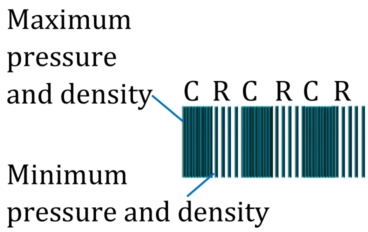
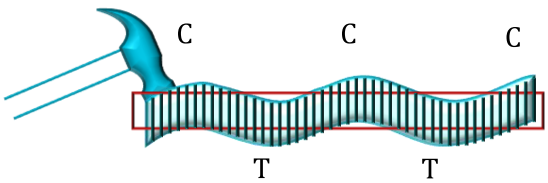
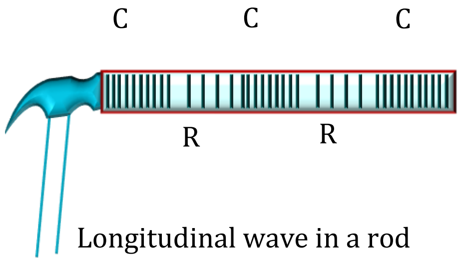
(i) Periodic change takes place in electric and magnetic fields.

(ii) These waves are transverse in nature.

(iii) These waves possess momentum and travels with the speed of light.

Based on the motion of particles of medium:

Waves are of two types on the basis of motion of particles of the medium (a) Longitudinal waves, (b) Transverse waves

Transverse waves	Longitudinal waves
Particles of the medium vibrates in a direction perpendicular to the direction of propagation of wave.  Transverse wave on a string	Particles of a medium vibrate in the direction of wave motion.  Longitudinal wave in a fluid
It travels in the form of crests (C) and troughs (T).  	It travels in the form of compression (C) and rarefaction (R). 
Transverse waves can be transmitted through solids, they can be setup on the surface of liquids. But they can not be transmitted into liquids and gases.  Transverse-wave in a rod	These waves can be transmitted through solids, liquids and gases because for these waves propagation, volume elasticity is necessary.  Longitudinal wave in a rod

Transverse waves	Longitudinal waves
Medium should posses the property of rigidity	Medium should posses the property of elasticity
Transverse waves can be polarised.	Longitudinal waves can not be polarised.
Movement of string of a sitar or violin, movement of the membrane of a Tabla or Dholak, movement of kink on a rope, waves set-up on the surface of water.	Sound waves travel through air, Vibration of air column in organ pipes. Vibration of air column above the surface of water in the tube of resonance apparatus

Based on energy propagation:

Again the waves can be divided into two parts on the basis of energy propagation (a) Progressive waves (b) Stationary waves

(a) **Progressive waves:** The wave propagates with fixed velocity in a medium.

Example: Waves on liquid surface.

Properties:

(i) Every particle attains the maximum displacement at one time or other.

(ii) Energy and momentum propagate in the medium from one particle to another.

(iii) Phase of motion of particles continuously change.

(b) **Stationary Waves:** The particles of the medium vibrate with different amplitude but energy does not prapogate.

Example: Vibration of sonometer wire.

Properties:

(i) There are some particles who always remain at rest, at some points known as nodes where as other particles attain maximum value at other points known as antinodes.

(ii) All particles between two successive nodes are in the same phase.

(iii) At nodes particle velocity is zero while strain is maximum and reverse is the case for antinodes.

(iv) Energy does not propagates.

Based on direction of Propagation:

(a) One dimensional wave: Stretched string.

(b) Two dimensional wave: Waves on liquid surface.

(c) Three dimensional wave: Sound waves.

Examples based on Classification of waves

Illustration 1:

Examples of mechanical waves are -

(A) Waves in spring and water waves
(B) Compression, rarefaction waves in liquid
(C) Sound waves and waves in a string
(D) All of the above

Ans. (D)

Solution:

All of the above needs a material medium to travel.

Illustration 2:

A transverse wave travels along the Z axis. The particles of the medium must move -

(A) along the Z axis (B) along the X axis (C) along the Y axis (D) in the X-Y plane

Ans. (D)

Solution:

In transverse wave, particles of the medium vibrate in a direction perpendicular to the direction of wave propagation and X-Y plane is always perpendicular to Z axis.

Illustration 3:

A progressive wave propagating in a medium carries (through the medium)

(A) Only energy (B) Only momentum

(C) Both energy and momentum (D) Neither energy nor momentum

Ans. (C)

Solution:

Progressive wave is a disturbance in the medium which carries energy and momentum.

Illustration 4:

Is an oscillation a wave? Explain.

Solution:

No an oscillation is not a wave. The term 'wave' implies the transfer of energy through successive vibration of the particles of the medium. So, the oscillation of a body do not constitute a wave.

Illustration 5:

Transverse waves are possible in solids but not in liquids and gases. Why?

Solution:

Transverse mechanical wave require rigidity in the medium. The medium must be elastic so as to provide a restoring force when acted on by shearing stress. But liquids and gases flow when acted on by shearing stress and so cannot transmit waves.

Illustration 6:

How can one distinguish experimentally between longitudinal and transverse waves?

Solution:

One can distinguish between transverse wave and longitudinal waves by conducting experiments on polarization, since longitudinal waves cannot be polarised.

Wave Function

It is a mathematical description of the disturbance created by a wave. For a string, the wave function is displacement, for sound waves, it is a pressure or density fluctuation where as for light waves it is electric or magnetic field.

Now let us consider a one dimensional wave travelling along x-axis. During wave motion, a particle with equilibrium position x is displaced some distance y in the direction perpendicular to the x-axis. In this case y is a function of position (x) and time (t).

i.e. $y = f(x, t)$. This is called wave function

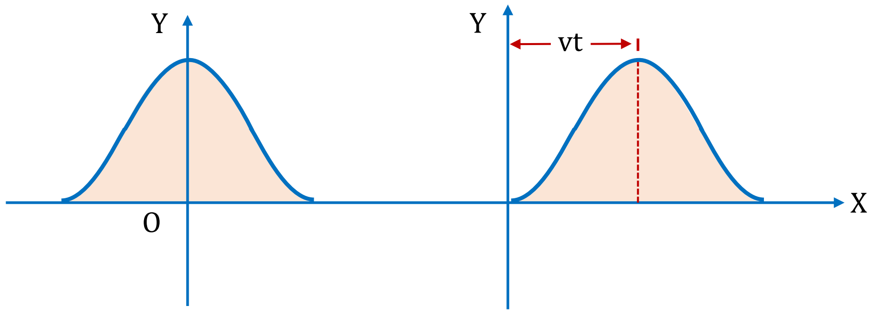
Different physical interpretation of wave function:

(a) Wave function of a wave tells about the displacement of particle of the medium at given time.

(b) If 't' is kept fixed ($t = t_0$) in the wave equation then the resultant equation $y(x, t = t_0) = f(x)$ describes the shape of the wave at given time 't_0'.

(c) If 'x' is kept fixed ($x = x_0$) in the wave equation then the resultant equation $y(x = x_0, t) = f(t)$ represent equation of motion of particle present at $x = x_0$.

Wave function when equation of shape is given: Let the wave pulse is travelling with a speed v, after a time t, the pulse reaches a distance vt along the $+x$-axis as shown. If equation of the shape of pulse at t=0 is given by y=$f(x)$, then after time t this shape (graph) will be shifted towards positive x axis by amount vt. So, the equation of shape at any time t i.e. the wave function now can be represented as $y = f(x - vt)$.



(A) Pulse at time t = 0 **(B) Pulse after time t**

If the wave pulse is travelling along – x-axis then $y = f(x + vt)$

In order of a wave function to represent a wave, the quantities x, t must appear in combinations ($Ax + Bt$), where A and B are the constants such that velocity of the wave is given by

$$v = -\frac{B}{A}$$

$$v = -\frac{\text{Coefficient of } t}{\text{Coefficient of } x}$$

Thus $y = (x - vt)^2$, $\sqrt{(x - vt)}$, $Ae^{-B(x-vt)^2}$ etc. represent travelling waves while,

$y = (x^2 - v^2t^2)$, $(\sqrt{x} - \sqrt{vt})$, $A \sin(4x^2 - 9t^2)$ etc. do not represent a wave.

Illustration 7:

Shape of a pulse at $t = 0$ is given by $y = e^{-(x-1)^2}$ and shape of the same pulse at $t = 2$ sec. is $y = e^{-(x-5)^2}$. Find wave velocity and wave function.

Solution:

At $t = 0$, $y = e^{-(x-1)^2}$

At $t = 2$ sec, $y = e^{-(x-5)^2}$

$y = e^{-[(x-4)-1]^2}$

In 2 sec pulse travels 4 unit

$$v = \frac{4}{2} = 2\text{unit}$$

In t time pulse travels $2t$ unit.

At any time t, $y = e^{-[(x-2t)-1]^2}$

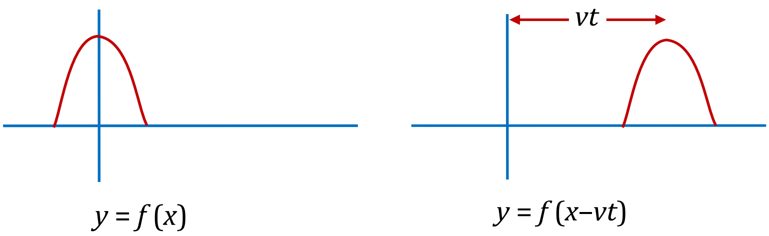


Illustration 8:

$y = f(x)$ at $t = t_0$ and wave is travelling in $+x$ direction with speed v.

Solution:

Wave equation is $y = f(x - v(t - t_0))$.

Illustration 9:

$y = x^2$ at $t = 1$ and wave is travelling in $+x$ direction with speed 3 m/s.

Solution:

Replace x by $x - v\,(t - t_0)$ where $v = 3$ and $t_0 = 1$

Wave equation is $y = \left(x - 3(t-1)\right)^2$

Illustration 10:

$y = x^2 + 2x$ at $t = 0$ and wave is travelling in $-x$ direction with speed 3 m/s.

Solution:

Replace x by $x + vt$ where $v = 3$

Wave equation is $y = (x + 3t)^2 + 2(x + 3t)$.

Illustration 11:

$y = 4x$ at $t = 1$ and wave is travelling in $-x$ direction with speed 3 m/s.

Solution:

Replace x by $x + v\,(t - t_0)$ where $v = 3$ and $t_0 = 1$

Wave equation is $y = 4[x + 3(t-1)]$

Wave function when equation of motion of particle is given:

Imagine a horizontal string stretched in the x direction. Its equilibrium shape is flat and straight. Let y measure the displacement of any particle of the string from its equilibrium position, perpendicular to the string. If the string is plucked on the left end, a pulse will travel to the right. The vertical displacement y of the left end of the string ($x = 0$) is a function of time, i.e., $y\,(x = 0, t) = f\,(t)$.

If there are no frictional losses, the pulse will travel undiminished, retaining its original shape. If the pulse travels with a speed v, the 'position' of the wave pulse is $x = vt$. Therefore, the displacement of the particle at point x at time t was originated at the left end at time $t - \frac{x}{v}$. [$y\,(x, t)$ is function of both x and t]. But the displacement of the left end at time t is $f\,(t)$ thus at time $t - \frac{x}{v}$, it is $f\,(t - \frac{x}{v})$.

Therefore :

$$y(x, t) = y\,(x = 0,\ t - \frac{x}{v}) = f\,(t - \frac{x}{v})$$

This can also be expressed as

$$\Rightarrow \quad \frac{f}{v}(vt - x)$$

$$\Rightarrow \quad -\frac{f}{v}(x - vt)$$

$$y(x, t) = g(x - vt)$$

Using any fixed value of t (i.e. at any instant), this shows shape of the string. If the wave is travelling in $-x$ direction, then wave equation is written as

$$y(x, t) = f\left(t + \frac{x}{v}\right)$$

The quantity $x - vt$ is called phase of the wave function. As phase of the pulse has fixed value $x - vt$ = const.

Taking the derivative w.r.t. time

$$\frac{dx}{dt} = v$$

where v is the phase velocity although often called wave velocity. It is the velocity at which a particular phase of the disturbance travels through space

Illustration 12:

A transverse wave is travelling towards $+x$ axis. At $t = 0$, particle of $x = 0$ is found at $y = +\frac{A}{2}$, while moving towards its mean position. If velocity of wave is v and particles are performing SHM with amplitude A and angular frequency ω. Find equation of wave.

Solution:

Let $y = A\sin(\omega t + \phi)$

At $t = 0$ and $x = 0$, $y = +\frac{A}{2}$

$$+\frac{A}{2} = A\sin(\phi)$$

$$\sin\phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

As at $t = 0$ particle is moving towards mean position, so $\phi = \frac{5\pi}{6}$

Hence, at $x = 0$, $y = A\sin\left(\omega t + \frac{5\pi}{6}\right)$

For wave equation replace t by $t - \frac{x}{v}$

$$y = A\sin\left[\omega\left(t - \frac{x}{v}\right) + \frac{5\pi}{6}\right]$$

Travelling Sine wave in one Dimension (wave on string)

The wave equation $y = f\left(t - \frac{x}{v}\right)$ or $y = f(x - vt)$ is quite general. It holds for arbitrary wave shapes, and for transverse as well as for longitudinal waves.

A complete description of the wave requires specification of $f(x)$. The most important case, by far, in physics and engineering is when $f(x)$ is sinusoidal, that is, when the wave has the shape of a sine or cosine function. This is possible when the source, that is moving the left end of the string, vibrates the left end $x = 0$ in a simple harmonic motion. For this, the source has to continuously do work on the string and energy is continuously supplied to the string.

Let us consider, the equation of a travelling sinusoidal wave is

$$y = A\sin(\omega t \pm kx)$$

The wave equation $y = A \sin(\omega t \pm kx)$ says that at $x = 0$ and $t = 0$, $y = 0$. This is not necessarily the case, of source. For the same condition, y may not equal to zero. Therefore, the most general expression would involve a phase constant ϕ, which allows for other possibilities,

$$y = A \sin(\omega t \pm kx + \phi)$$

A suitable choice of ϕ allows any initial condition to be met.

Analysis of sinusoidal wave $y = A \sin(\omega t - kx)$:

1. According to given wave function equation of motion of particle at $x = x_0$

$$y_{(x_0,t)} = A\sin(\omega t - kx_0)$$
$$= A\sin(\omega t + \phi)$$

Which is the equation of SHM of particle present at $x = x_0$. It can be concluded from the above result that all the particle present on the wave perform SHM with same amplitude A and same angular frequency ω and frequency $f = \frac{\omega}{2\pi}$. Therefore, A is called amplitude of wave and f is called frequency of wave. ϕ, which is the initial phase of particle, depends on location (x) of particle. Suppose ϕ_1 and ϕ_2 are the phases of particles at distance x_1 and x_2 at any time t, then

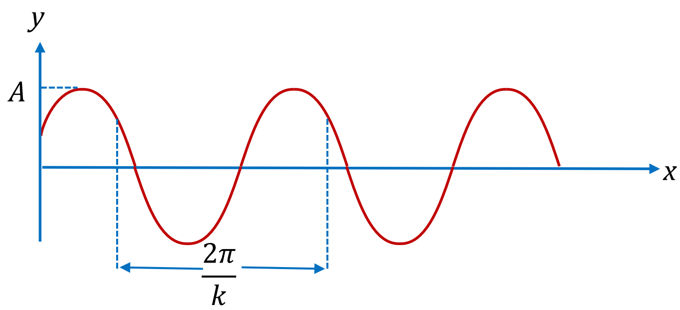
$$\phi_1 - \phi_2 = k(x_2 - x_1)$$
$$|\Delta\phi| = k|\Delta x| \qquad \text{...(1)}$$

2. Shape of the wave on string at $t = t_0$ will be given by

$$y = A\sin(\omega t_0 - kx)$$
$$y = A\sin(kx - \omega t_0 + \pi)$$
$$y = A\sin(kx + \phi) \qquad \text{...(2)}$$

Where $\phi = \pi - \omega t_0$

Below is the graph of equation 2



$\frac{2\pi}{k}$ is the period of the function represented by equation 2, which is the minimum length after which wave repeats itself and is called **wavelength (λ) of** the wave and k is called **wave number**.

Now the equation (1) can be written as

$$\frac{|\Delta\phi|}{2\pi} = \frac{\Delta x}{\lambda} \qquad \text{...(3)}$$

Velocity of wave:

Since velocity of wave, $v = -\frac{\text{Coefficient of } t}{\text{Coeffient of } x}$,

Therefore, $v = \frac{\omega}{k}$ or $v = f\lambda$

Velocity and acceleration of particle present on wave:

Velocity of particle will be given by $v = \frac{\partial y}{\partial t} = A\omega\cos(\omega t - kx)$

And acceleration will be given by $a = \frac{\partial^2 y}{\partial^2 t} = -A\omega^2\sin(\omega t - kx)$

Illustration 13:

A source of frequency 500 *Hz* emit waves of wavelength 0.2 *m*. How long does it take to travel 300 *m*?

(A) 70 *s* (B) 60 *s* (C) 12 *s* (D) 3 *s*

Ans. (D)

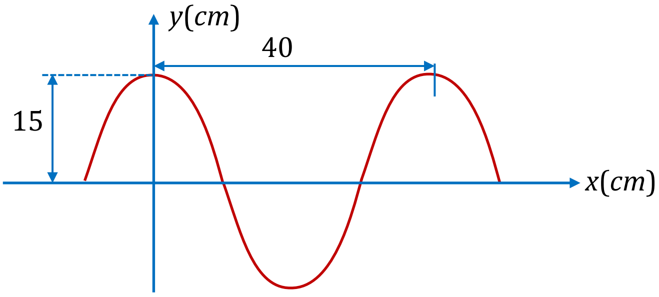
Solution:

Velocity of the wave is $v = f\lambda = 500 \times 0.2 = 100\ m/s$.

Time to travel 300 *m* is $t = \frac{300}{v} = \frac{300}{100} = 3$ sec

Illustration 14:

A sinusoidal wave travelling in the positive *x* direction has an amplitude of 15 *cm*, wavelength 40 *cm* and frequency 8 *Hz*. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15 *cm*, as shown.



(a) Find the angular wave number, period, angular frequency and speed of the wave.

(b) Determine the phase constant ϕ, and write a general expression for the wave function.

Solution:

(a) $k = \frac{2\pi}{\lambda} = \frac{2\pi\ rad}{40\ cm} = \frac{\pi}{20}\ rad/cm$

$T = \frac{1}{f} = \frac{1}{8}s$; $\omega = 2\pi f = 16\pi\ s^{-1}$

$v = f\lambda = 320\ cm/s$

(b) It is given that $A = 15\ cm$ and also $y = 15\ cm$ at $x = 0$ and $t = 0$

Using $y = A\sin(\omega t - kx + \phi)$

$15 = 15\sin\phi$

$\sin\phi = 1$

$\phi = \frac{\pi}{2}\ rad.$

Therefore, the wave function is

$$y = A\sin(\omega t - kx + \frac{\pi}{2}) = (15\ cm)\sin\left[(16\pi s^{-1})t - \left(\frac{\pi}{20}\frac{rad}{cm}\right).x + \frac{\pi}{2}\right]$$

Illustration 15:

A sinusoidal wave is travelling along a rope. The oscillator that generates the wave completes 60 vibrations in 30 *s*. Also, a given maximum travels 425 *cm* along the rope in 10.0 *s*. What is the wavelength?

Solution:

$$v = \frac{425}{10} = 42.5\ cm/s$$

$$f = \frac{60}{30} = 2\ Hz$$

$$\lambda = \frac{v}{f} = 21.25\ cm.$$

Illustration 16:

A simple harmonic progressive wave is represented by the equation $y = 8 \sin 2\pi\,(0.1x - 2t)$ where x and y are in *cm* and t is in seconds. At any instant the phase difference between two particles separated by 2.0 *cm* in the x-direction is

(A) 18° (B) 36° (C) 54° (D) 72°

Ans. (D)

Solution:

$y = 8 \sin 2\pi\left(\frac{x}{10} - 2t\right)$ given by comparing with standard equation $y = A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$

$\lambda = 10cm$

So, Phase Difference $= \frac{2\pi}{\lambda} \times$ path difference $= \frac{2\pi}{10} \times 2 = \frac{2}{5} \times 180° = 72°$

Illustration 17:

A wave represented by the given equation $y = A \sin\left(10\pi x + 15\pi t + \frac{\pi}{3}\right)$ where x is in meter and t is in second. The expression represents

(A) A wave travelling in the positive x-direction with a velocity of 1.5 m/sec

(B) A wave travelling in the negative x-direction with a velocity of 1.5 m/sec

(C) A wave travelling in the negative x-direction with a wavelength of 0.2 m

(D) A wave travelling in the positive x-direction with a wavelength of 0.2 m

Ans. (BC)

Solution:

By comparing with standard equation $y = A \sin(kx + \omega t + \pi/3)$

$$k = 10\,\pi,\ \ \omega = 15\pi$$

We know that: $v = \frac{\omega}{k} = 1.5\ m/sec;\ \lambda = \frac{2\pi}{k} = 0.2$ meter.

Illustration 18:

A transverse wave is described by the equation $y = y_0 \sin 2\pi\left(ft - \frac{x}{\lambda}\right)$. The maximum particle velocity is four times the wave velocity if

(A) $\lambda = \frac{\pi y_0}{4}$ (B) $\lambda = \frac{\pi y_0}{2}$ (C) $\lambda = \pi y_0$ (D) $\lambda = 2\pi y_0$

Ans. (B)

Solution:

Maximum particle velocity = 4 wave velocity

$A\omega = 4f\lambda$

$y_0 2\pi f = 4f\lambda$

$\lambda = \frac{\pi y_0}{2}$

Illustration 19:

The equation of a wave travelling in a string can be written as $y = 3 \cos \pi(100t - x)$ Its wavelength

(A) 100 cm (B) 2 cm (C) 5 cm (D) None of these

Ans. (B)

Solution:

Standard equation : $y = A \cos(\omega t - kx)$

Given equation : $y = 3 \cos(100\pi t - \pi x)$

So, $k = \pi$ and $\lambda = \frac{2\pi}{k} = 2\ cm$

Illustration 20:

A plane wave is represented by $x = 1.2 \sin(314t + 12.56y)$ where x and y are distances measured along x and y direction in meter and t is time in seconds. This wave has

(A) A wave length of 0.25 m and travels in + ve x-direction

(B) A wavelength of 0.25 m and travels in + ve y-direction

(C) A wavelength of 0.5 m and travels in – ve y-direction

(D) A wavelength of 0.5 m and travels in – ve x-direction

Ans. (C)

Solution:

From given equation $k = 12.56$

$\lambda = \frac{2\pi}{k} = 0.5\ m$ direction is along – y axis

Illustration 21:

A transverse wave travelling along a string is described by $y(x, t) = 0.00327 \sin(72.1x - 2.72t)$, in which the numerical constants are in SI units.

(A) What is the amplitude of this wave?

(B) What are the wavelength, period, and frequency of this wave?

(C) What is the velocity of this wave?

(D) What is the displacement y at $x = 22.5\ cm$ and $t = 18.9\ s$?

(E) What is the transverse velocity u of this element of the string, at that place and at that time?

(F) What is the transverse acceleration ay at that position and at that time?

Solution:

(A) $y_m = 0.00327 = 3.27\ mm$

(B) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{72.1} = 0.0871 = 8.71\ cm$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.72} = 2.31\ s$

$f = \frac{1}{T} = \frac{1}{2.31} = 0.433\ Hz$

(C) $v = \frac{\omega}{k} = \frac{2.72}{72.1} = 0.0377 = 3.77\ cms^{-1}$

(D) $y(x,t) = 0.00327\sin(72.1 \times 0.225 - 2.72 \times 18.9)$

$= 0.00192 = 1.92\ mm$

(E) $u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t)$

$= -(2.72)(0.00327)\sin(72.1 \times 0.225 - 2.72 \times 18.9)$

$= 7.20\ mm/s$

(F) $a_y = \frac{\partial u}{\partial t} = -\omega^2 y_m \sin(kx - \omega t) = -\omega^2 y$

$= -(2.72)^2(0.00192) = -0.0142$

$= -14.2\ mm/s^2$

The Linear Wave Equation

By using wave function $y = A \sin(\omega t - kx + \phi)$, we can describe the motion of any point on the string. Any point on the string moves only vertically, and so its x coordinate remains constant. The transverse velocity v_y of the point and its transverse acceleration a_y are

$$v_y = \left[\frac{dy}{dt}\right]_{x=\text{constant}} \Rightarrow \frac{\partial y}{\partial t} = \omega A \cos(\omega t - kx + \phi) \quad ...(1)$$

$$a_y = \left[\frac{dv_y}{dt}\right]_{x=\text{constant}} \Rightarrow \frac{\partial v_y}{\partial t} = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(\omega t - kx + \phi) \quad ...(2)$$

and hence $v_{y,\,max} = \omega A$

$a_{y,\,max} = \omega^2 A$

The transverse velocity and transverse acceleration of any point on the string do not reach their maximum value simultaneously. Infact, the transverse velocity reaches its maximum value (ωA) when the displacement is $y = 0$, whereas the transverse acceleration reaches its maximum magnitude ($\omega^2 A$) when $y = \pm A$.

Now $\left[\frac{dy}{dx}\right]_{t=constant} \Rightarrow \frac{\partial y}{\partial x}$

$$= -kA \cos(\omega t - kx + \phi) \quad ...(3)$$

And $$\frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(\omega t - kx + \phi) \quad ...(4)$$

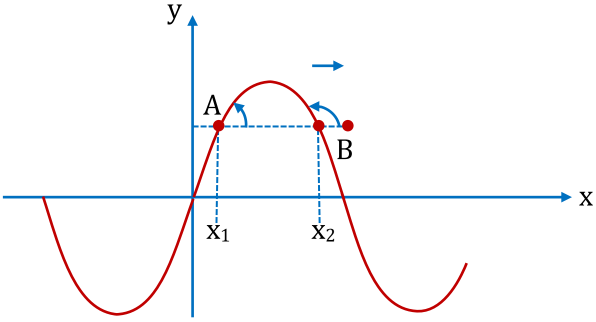
From (1) and (3)

$$\frac{\partial y}{\partial t} = -\frac{\omega}{k}\frac{\partial y}{\partial x}$$

$$v_P = -v_w \times \text{slope}$$

i.e. if the slope at any point is negative, particle velocity is positive and vice-versa, for a wave moving along positive x axis i.e. v_w is positive.

For example, consider two points A and B on the y-x curve for a wave, as shown. The wave is moving along positive x-axis.



Slope at A is positive therefore at the given moment, its velocity is negative. That means it is coming downward. Reverse is the situation for particle at point B.

Now using equation (2) and (4)

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

This is known as the linear wave equation or differential equation representation of the travelling wave model. We have developed the linear wave equation from a sinusoidal mechanical wave travelling through a medium, but it is much more general. The linear wave equation successfully describes waves on strings, sound waves and also electromagnetic waves.

Illustration 22:

Verify that wave function $y = \frac{2}{(x-3t)^2+1}$ is a solution to the linear wave equation. x and y are in cm.

Solution:

By taking partial derivatives of this function w.r.t. x and t

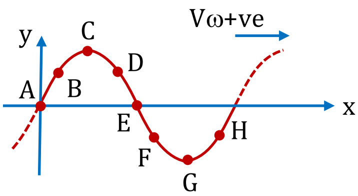
$$\frac{\partial^2 y}{\partial x^2} = \frac{12(x-3t)^2-4}{[(x-3t)^2+1]^3}$$

and $$\frac{\partial^2 y}{\partial t^2} = \frac{108(x-3t)^2-36}{[(x-3t)^2+1]^3}$$ or $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{9}\frac{\partial^2 x}{\partial t^2}$$

Comparing with linear wave equation, we see that the wave function is a solution to the linear wave equation if the speed at which the pulse moves is 3 cm/s. It is apparent from wave function therefore it is a solution to the linear wave equation.

Illustration 23:

For the points marked in the graph, find the signs of v_P and a_P



Solution:

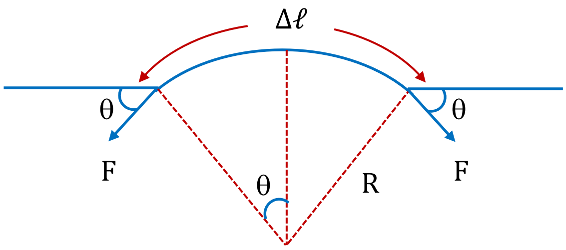
	y_P	Slope of y–x	v_P	$a_P = -\omega^2 y$
A	0	+	-	0
B	+	+	-	-
C	+	0	0	-
D	+	-	+	-
E	0	-	+	0
F	-	-	+	+
G	-	0	0	+
H	-	+	-	+

The Speed of Transverse waves on strings

This velocity depends on two properties of the string.

(a) Elasticity (measured by tension F in the string)

(b) Inertia (measured by mass per unit length)



Take a small element $\Delta\ell$ of the string in which a wave is moving with speed v. Suppose this element forms an arc of radius R. A force of tension pulls this element tangentially by which horizontal components cancel and vertical components add to form a restoring force i.e.

$$F_r = F\sin\theta + F\sin\theta$$

$$F_r = 2F\sin\theta \approx 2F.\frac{\Delta\ell/2}{R} = \frac{F.\Delta\ell}{R}$$

If μ is mass per unit length of string and Δm is the mass of element then downward acceleration is given by

$$a = \frac{F_r}{\Delta m} = \frac{F.\Delta\ell/R}{\Delta m} = \frac{F}{\mu R} \text{ where } \mu = \frac{\Delta m}{\Delta\ell}$$

But element is moving in a circle of radius R. So Centripetal acceleration $a = \frac{v^2}{R}$

$$\therefore \quad \frac{v^2}{R} = \frac{F}{\mu R}$$

$$v = \sqrt{\frac{F}{\mu}}$$

Note:

If A is area of cross-section and ρ is density

$$\mu = \rho A$$

So, velocity $v = \sqrt{\frac{F}{\rho A}}$

Illustration 24:

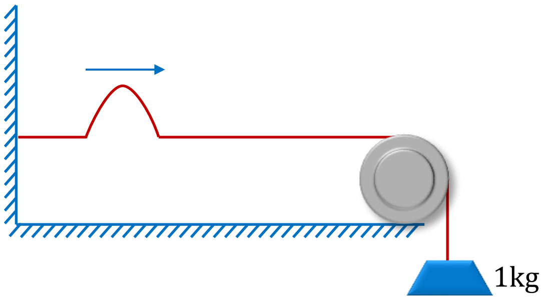


Fig. shows a string of linear mass density 1 *gm/cm* on which a wave pulse is travelling. Find the time taken by the pulse in travelling through a distance of 50 *cm* on the string. Take $g = 10\ m/s^2$.

Solution:

Tension in the string

$$F = mg = 10\ N$$

Mass per unit length = 1 g/cm = 0.1 kg/m

Velocity of wave $v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{10}{0.1}} = 10 m/s$

Time taken by pulse in travelling a distance of 50 *cm* = $\frac{0.5}{10}$ = .05 seconds

Illustration 25:

The diameter of an iron wire is 1.2 *mm*. If the speed of transverse wave in the wire be 50 m/s then what is the tension in the wire. The density of iron is $7.7 \times 10^3\ kg/m^3$.

Solution:

Diameter of wire = 1.2 *mm* = $1.2 \times 10^{-3} m$; Radius of wire = $0.6 \times 10^{-3} m$

Speed of wave = 50 m/s ; Density of wire $\rho = 7.7 \times 10^3\ Kg/m^3$

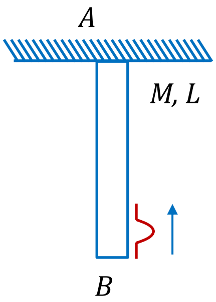
Speed of transverse wave is given by

$$v = \sqrt{\frac{F}{\mu}}$$

But $\mu = \rho A$

$\therefore$ $F = v^2 . \rho A = (50)^2 \times 7.7 \times 10^3 \times 3.14 \times (0.6 \times 10^{-3})^2 = 21.76\ N$

Illustration 26:



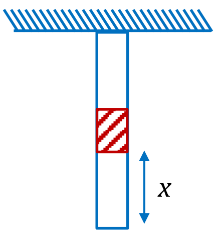
Find: (a) v_ω when wave passes through point at distance x from B?

(b) Time taken by wave to go from B to A.

(c) Ball is dropped at $t = 0$ from the point A, where will they cross?

Solution:

(a)



As $T = \frac{M}{L} xg$

$$v_\omega = \sqrt{\frac{\frac{M}{L}xg}{M/L}} = \sqrt{xg}$$

(b) $v_\omega = \frac{dx}{dt}$; $\sqrt{xg} = \frac{dx}{dt} = v$

$\int_0^t \frac{dx}{\sqrt{x}} = \int_0^t \sqrt{g}dt$; $t = 2\sqrt{\frac{L}{g}}$

(c) They both take same time

$t_{ball} = t_{wave}$

$$\sqrt{\frac{2(L-d)}{g}} = 2\sqrt{\frac{d}{g}}$$

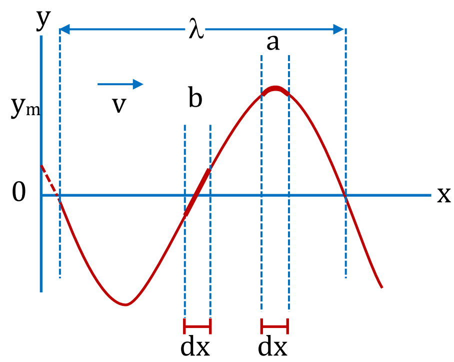
$$d = \frac{L}{3}$$

They cross at $\frac{L}{3}$ from lower end.

Energy density in travelling wave on a string

Snapshot of a travelling wave on a string at time $t = 0$.

Consider a string element of mass dm.



Kinetic energy: $dK = \frac{1}{2}dmu^2$.

Since $y(x, t) = y_m \sin(kx - \omega t)$,

$$u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t).$$

Since $dm = \mu dx$,

$$dK = \frac{1}{2}(\mu dx)(-\omega y_m)^2 \cos^2(kx - \omega t)$$

Potential energy:

Potential energy is carried in the string when it is stretched. Stretching is largest when the displacement has the largest gradient.

Hence, the potential energy is also maximum at the $y = 0$ position. This is different from the harmonic oscillator, in which case energy is conserved.

Consider the extension Δs of a string element.

$$\Delta s = \sqrt{(dx)^2 + [y(x+dx,t) - y(x,t)]^2} - dx$$

$$\approx \sqrt{(dx)^2 + \left(\frac{\partial y}{\partial x}dx\right)^2} - dx = \left[\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right]dx$$

Using approximation

$$\Delta s \approx \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right]dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

The potential energy of the string element is given by the work done in extending the string element,

$$dU = dW = T\Delta s \approx \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{T}{2}k^2 y_m^2 \cos^2(kx - \omega t)dx$$

Note:

$dU = dKE$ only valid for travelling wave.

Proof :

$$dU = \frac{1}{2}T\left(\frac{v_P}{v_\omega}\right)^2 dx = \frac{1}{2}(\mu v_\omega^2)\left(\frac{v_P}{v_\omega}\right)^2 dx = \frac{1}{2}\mu v_P^2 dx$$

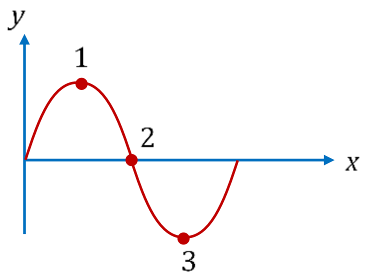
$$dKE = \frac{1}{2}\mu v_P^2 dx$$

$$\frac{dE}{dx} = \frac{dU}{dx} + \frac{dKE}{dx} = \mu v_P^2$$

or $$\frac{dE}{dx} = \mu\left(\frac{dy}{dt}\right)^2 = T\left(\frac{dy}{dx}\right)^2 \qquad \left(\because \frac{dE}{dx} = 2\frac{dKE}{dx} = \frac{\mu dx v_P^2}{dx}\right)$$

Illustration 27:

Compare the mechanical energy ($KE + PE$) of elements 1, 2 and 3 as shown in figure.



Solution:

$|\text{slope}|_2 > |\text{slope}|_3 = |\text{slope}|_1$

$dU_2 \; > \; dU_3 = dU_1$

As $dU = dKE$

$dKE_2 > dKE_3 = dKE_1$

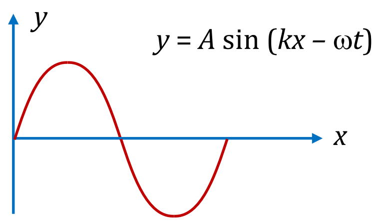
$\therefore$ $dE_2 > dE_3 = dE_1$

Key Points for travelling wave in string only

- KE and PE of an element are always equal to each other. Mechanical energy ($KE + PE$) of an element is not constant.
- At an instant when element is passing through mean position its KE and PE are maximum.
- At an instant when element is at extreme position its KE and PE are minimum.

Illustration 28:

Find energy of string of linear mass density μ from $x = 0$ to $x = \lambda$ at a given instant ($t = 0$)?



Solution:

$$\int dE = \int \mu\left(\frac{dy}{dt}\right)^2 dx$$

$$E = \int_0^{\lambda} \mu A^2\omega^2 \cos^2(kx - \omega t)\, dx$$

$$= \mu A^2\omega^2 \int_0^{\lambda} \cos^2(kx)\, dx$$

$$= \mu A^2\omega^2 \int_0^{\lambda} \left(\frac{1+\cos 2kx}{2}\right) dx$$

$$= \frac{1}{2}\mu A^2\omega^2 \left[x + \frac{\sin 2kx}{2k}\right]_0^{\lambda}$$

$$= \frac{1}{2}\mu A^2\omega^2[\lambda + 0]$$

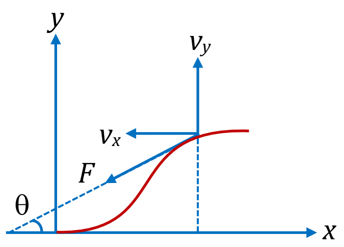
$$\boxed{E = \frac{1}{2}\mu\lambda A^2\omega^2}$$ energy in one wavelength

Power Transmitted along the string by a Sine Wave

Suppose a simple harmonic progressive wave is propagating in x direction

$$y = A\sin(\omega t - kx)$$

The string on the left of the point x exerts a force on to the right of position x. This force acts along the tangent to the string at x.



Rate of change of energy of the particle = $F\, v_y \cos(90 + \theta)$

{Here we are taking velocity component in the direction of force}

$$= -Fv_y \sin\theta$$

This is the equation of instantaneous power transmitted to the neighbouring particles. i.e.

$$P = -Fv_y \sin\theta$$

But $\quad v_y = \dfrac{\partial y}{\partial t}$

And $\quad \sin\theta \approx \theta \approx \tan\theta = \dfrac{\partial y}{\partial x}$

Hence, $P = -T\dfrac{\partial y}{\partial x}\dfrac{\partial y}{\partial t}$

$$P = F\, Ak\cos(\omega t - kx) . \frac{\partial}{\partial t}[A\sin(\omega t - kx)]$$

$$= F\,[Ak\cos(\omega t - kx)]\,[A\,\omega\cos(\omega t - kx)]$$

$$= \frac{\omega^2 A^2 F}{v}\cos^2(\omega t - k\,x)$$

$$< P > = P_{avg} = \frac{\omega^2 A^2 F}{v}\langle \cos^2(\omega t - k\,x)\rangle$$

$$= \frac{1}{2}\frac{\omega^2 A^2 F}{v} = \frac{1}{2}\omega^2 A^2 \mu v$$

$$< P > = 2\pi^2\, \mu\, v\, A^2 f^2$$

Hence $\quad P_{avg} = 2\,\pi^2\,\mu\, v\, A^2 f^2$

i.e $\quad P_{avg} \propto A^2 f^2$

Intensity of wave on string

Intensity of transverse wave on string

$$I = \frac{\text{Power}}{\text{Area}}$$

Since power is variable for a point and we have taken its average value

So, Intensity at any point is also taken average.

$$\langle I \rangle = \frac{\langle P \rangle}{\text{Area}} = \frac{1}{2}\frac{\mu v A^2 \omega^2}{\text{Area}}$$

$$\boxed{\langle I \rangle = \frac{1}{2}\rho v A^2 \omega^2}$$

Where ρ = Density of the material of string.

Illustration 29:

A string with linear mass density $m = 5.00 \times 10^{-2}\ kg/m$ is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?

Solution:

The wave speed on the string is

$$v = \sqrt{\frac{T}{\mu}} = \left(\frac{80.0\ N}{5.00 \times 10^{-2}\ kg/m}\right)^{1/2} = 40.0\ m/s$$

Because f = 60 Hz, the angular frequency ω of the sinusoidal waves on the string has the value

$$\omega = 2\pi f = 2\pi(60.0\ Hz) = 377\ s^{-1}$$

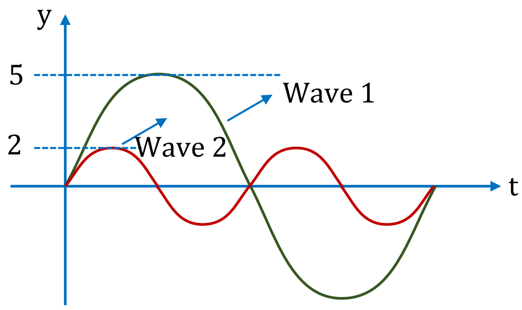
Using these values in following equation for the power, with $A = 6.00 \times 10^{-2}\ m$, gives

$$P = \frac{1}{2}\ \mu\omega^2 A^2 v$$

$$= \frac{1}{2}(5.00 \times 10^{-2}\ kg/m)\ (377 s^{-1})^2 \times (6.00 \times 10^{-2}\ m)^2\ (40.0\ m/s) = 512\ W.$$

Illustration 30:

Two waves in the same medium are represented by y-t curves in the figure. Find ratio of their average intensities?

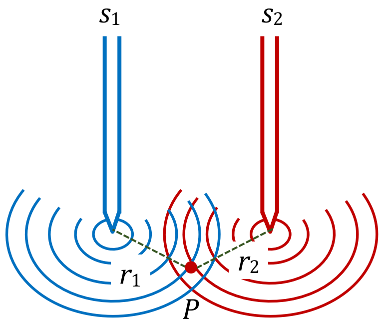


Solution:

$$\frac{I_1}{I_2} = \frac{\omega_1^2 A_1^2}{\omega_2^2 A_2^2} = \frac{f_1^2 \cdot A_1^2}{f_2^2 \cdot A_2^2} = \frac{1 \times 25}{4 \times 4} = \frac{25}{16}$$

Interference and the principle of superposition

When two of more than two waves pass through the same region simultaneously, we say that waves interfere in that region.



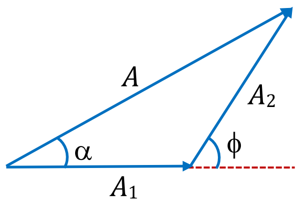
In the event of interference, waves follow the principle of superposition according to which when two or more wave pass through a point, the disturbance at this point is given by the sum of disturbances each wave could produce in absence of other wave.

Let us consider two needles performing SHM on the surface of water due to which ripples are generated.

$$y_1 = A_1 \sin(\omega t - kr_1) \text{ in absence of } S_2$$

$$y_2 = A_2 \sin(\omega t - kr_2) \text{ in absence of } S_1$$

In presence of both S_1 and S_2, displacement of particle at point P is given by



$$y = y_1 + y_2$$

$$y = A_1 \sin(\omega t - kr_1) + A_2 \sin(\omega t - kr_2)$$

$$\boxed{y = A \sin(\omega t + \alpha)}$$

If two sinusoidal waves of same frequency interfere with each other, then resultant wave or resultant motion of particle will be SHM of same frequency.

Nature of interference –

$$y_1 = A_1 \sin(\omega t + \phi_1)$$

$$y_2 = A_2 \sin(\omega t + \phi_2)$$

$$\Delta\phi = \phi_2 - \phi_1$$

$$A = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2 \cos \Delta\phi}$$

For maximum resultant amplitude, $\Delta\phi = 2n\pi$

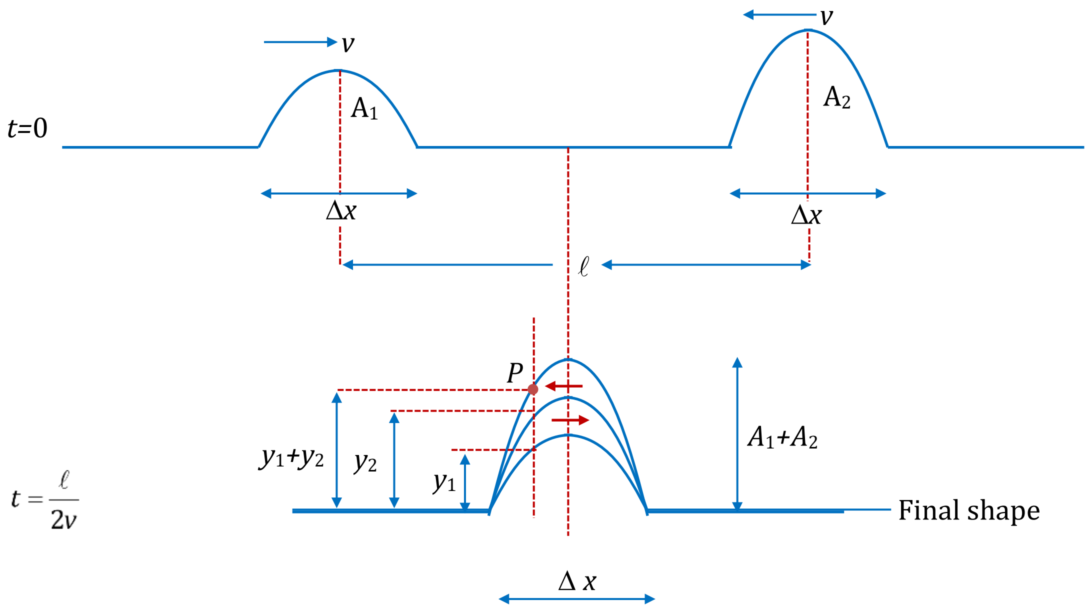
$$A = A_1 + A_2 \equiv \text{Constructive interference}$$

And if amplitude is minimum i.e.

$$A = |A_1 - A_2| \text{ when } \Delta\phi = (2n+1)\pi \equiv \text{Destructive interference}$$

Interference on a string

Two pulses of different amplitude but same wave speed a moving towards each other as shown in figure. As they interfere the net displacement (y_{net}) of any point P is given by $y_{net} = y_1 + y_2$.



Velocity analysis in interference

We know that, $y = y_1 + y_2$

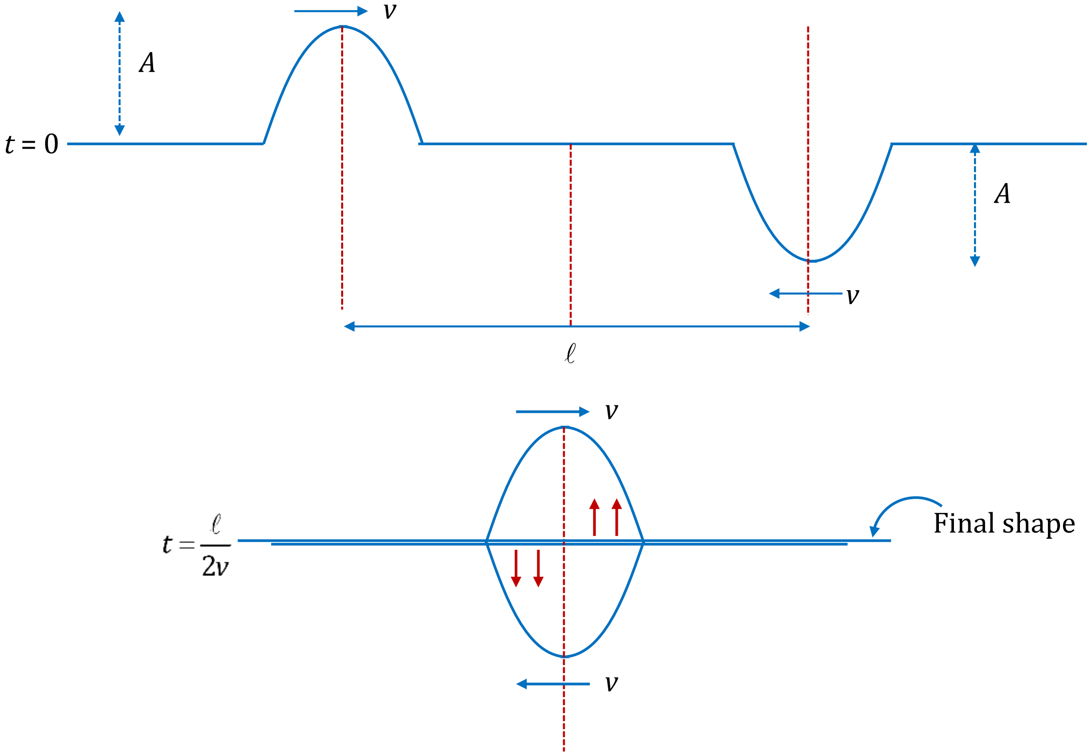
Now differentiating w.r.t. time

$$\frac{\partial y}{\partial t} = \frac{\partial y_1}{\partial t} + \frac{\partial y_2}{\partial t}$$

$$\boxed{v_{\text{resultant}} = v_1 + v_2}$$

Here we can not apply $\frac{\partial y}{\partial t} = -v_w\left(\frac{\partial y}{\partial x}\right)$ in resultant wave.

Let us consider two pulse of same magnitude moving towards each other with same wave speed.



Potential energy of any string element is given by

$$dU = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

Due to interference of pulses, string will be straight $t = \frac{\ell}{2v}$

At $\quad t = \frac{\ell}{2v}, \frac{\partial y}{\partial x} = 0$

So, $\quad dU = 0$ for any string element

All the energy is in the form of *K.E.*

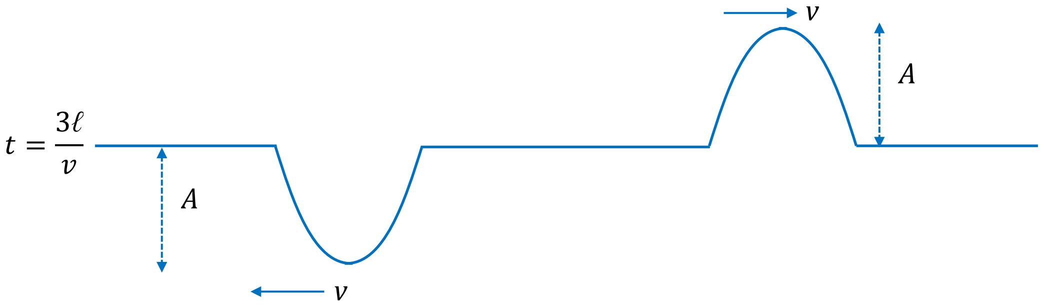
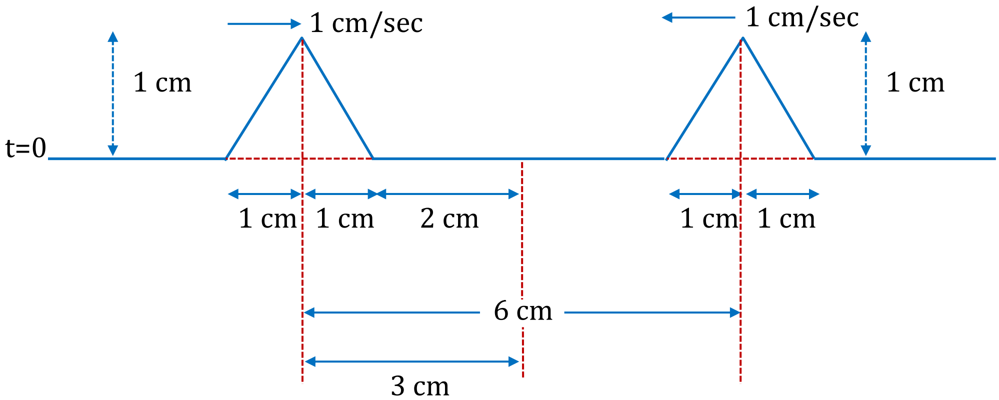


Illustration 31:

Shape of string at $t = 0$ is shown in fig. Draw the shape of the string at $t = 3$ sec and $t = 2.5$ sec.



Solution:

Net displacement of any point on string is given by

$$y = y_1 + y_2$$

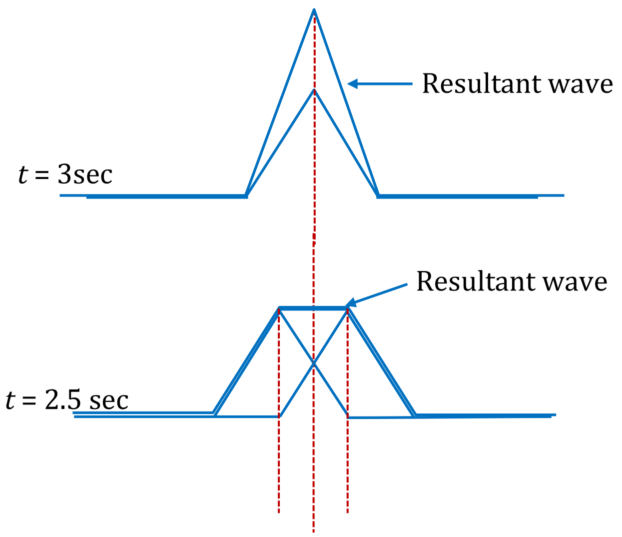
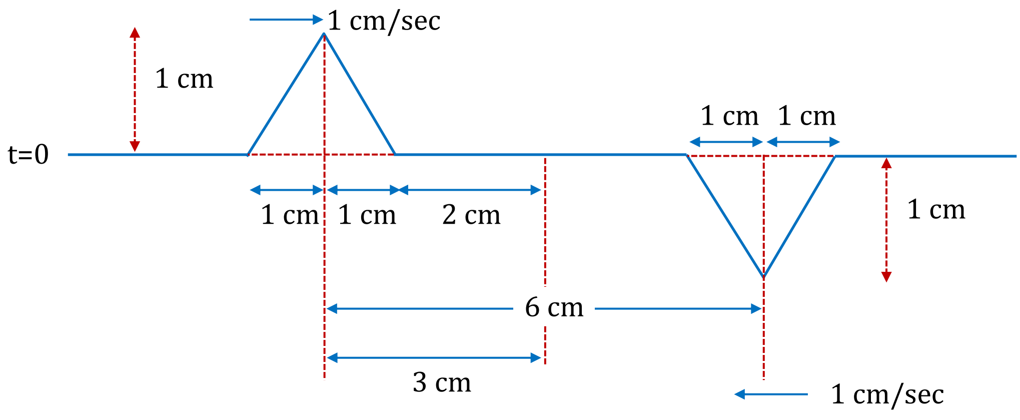
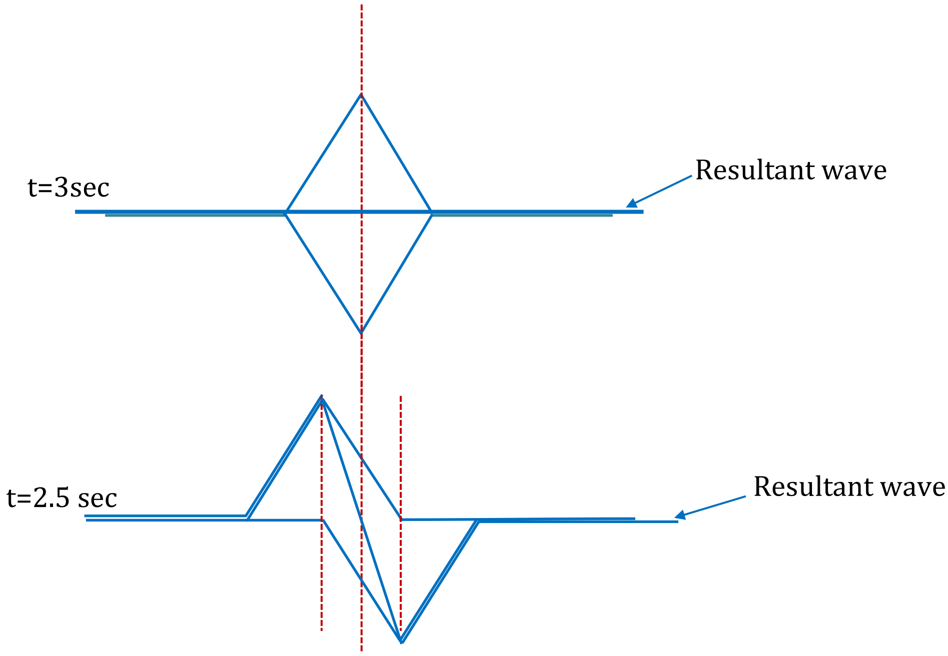


Illustration 32:

Shape of string at t = 0 is shown in fig. Draw the shape of the string at t = 3 sec and t = 2.5 sec.

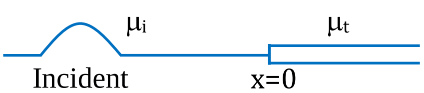


Solution:



Reflection & transmission of wave on composite string:

Consider a wave travelling in a string of linear mass density μ_i as shown in the figure. At x = 0, string is joint with another string of different material of linear mass density μ_t.



Let us assume equation of incident wave, reflected wave and transmitted wave as given by

$y_i = A_i \sin(\omega t - k_i x)$ $\rightarrow$ Incident wave

$y_r = A_r \sin(\omega t - k_r x)$ $\rightarrow$ Reflected wave

$y_t = A_t \sin(\omega t - k_t x)$ $\rightarrow$ Transmitted wave

At interface ($x = 0$), the wave should be continuous so

$y_i + y_r = y_t$

$A_i \sin(\omega t) + A_r \sin(\omega t) = A_t \sin(\omega t)$

$\boxed{A_i + A_r = A_t}$ ...(1)

Now conserving rate of flow energy

$P_i = P_r + P_t$

$$\frac{1}{2}\sqrt{T\mu_i}A_i^2\omega^2 = \frac{1}{2}\sqrt{T\mu_i}A_r^2\omega^2 + \frac{1}{2}\sqrt{T\mu_t}A_t^2\omega^2$$

$$(A_i + A_r)\sqrt{\mu_i}(A_i - A_r) = \sqrt{\mu_t}A_tA_t$$

$$\boxed{A_i - A_r = \sqrt{\frac{\mu_t}{\mu_i}}A_t} \quad ...(2)$$

Adding (1) and (2)

$$2A_i = \left(\frac{\sqrt{\mu_t}+\sqrt{\mu_i}}{\sqrt{\mu_i}}\right)A_t$$

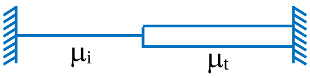
$$\boxed{A_t = \left(\frac{2\sqrt{\mu_i}}{\sqrt{\mu_i}+\sqrt{\mu_t}}\right)A_i} \text{ and } \boxed{A_r = \left(\frac{\sqrt{\mu_i}-\sqrt{\mu_t}}{\sqrt{\mu_i}+\sqrt{\mu_t}}\right)A_i}$$

In terms of wave velocity.

$$\boxed{A_t = \left(\frac{2v_t}{v_t+v_i}\right)A_i} \text{ and } \boxed{A_r = \left(\frac{v_t-v_i}{v_t+v_i}\right)A_i}$$

Illustration 33:

If 64% energy is reflected back on reflection from heavier string. Find $\frac{\mu_i}{\mu_t}$?



Solution:

$$\frac{1}{2}\sqrt{T\mu_i}(A_r)^2\omega^2 = \frac{64}{100} \times \frac{1}{2}\sqrt{T\mu_i}(A_i)^2\omega^2$$

$A_r = -(0.8A_i)$

(because wave gets inverted after reflection from heavy string)

$$-0.8A_i = \left(\frac{\sqrt{\mu_i}-\sqrt{\mu_t}}{\sqrt{\mu_i}+\sqrt{\mu_t}}\right)A_i$$

$$\frac{\mu_i}{\mu_t} = \frac{1}{81}$$

Reflection of wave on string

(1) **Reflection from fixed or rigid end:** At this end, particle of the string always has zero displacement due to rigid wall attached to it, therefor following points can be concluded.

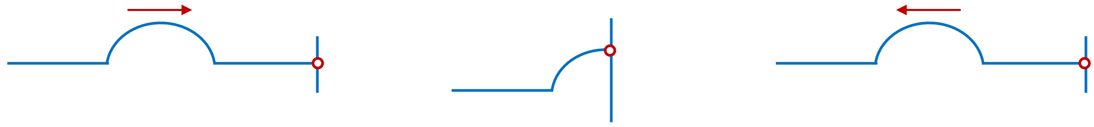
(a) After reflection from fixed end, shape of the wave/pulse gets inverted as shown in the figure

Incident pulse Intermediate shape of string Reflected pulse

(b) There is a phase difference of π between incident wave and reflected wave at point of reflection.

(2) **Reflection from free end:** To understand the free end, we can imagine a massless rind attached to this end and can move freely on a vertical pole as shown in the figure. At this end, particle oscillate with double amplitude, therefore following point can be concluded.

(a) After reflection from free end, shape of the pulse remain same as shown in the figure.

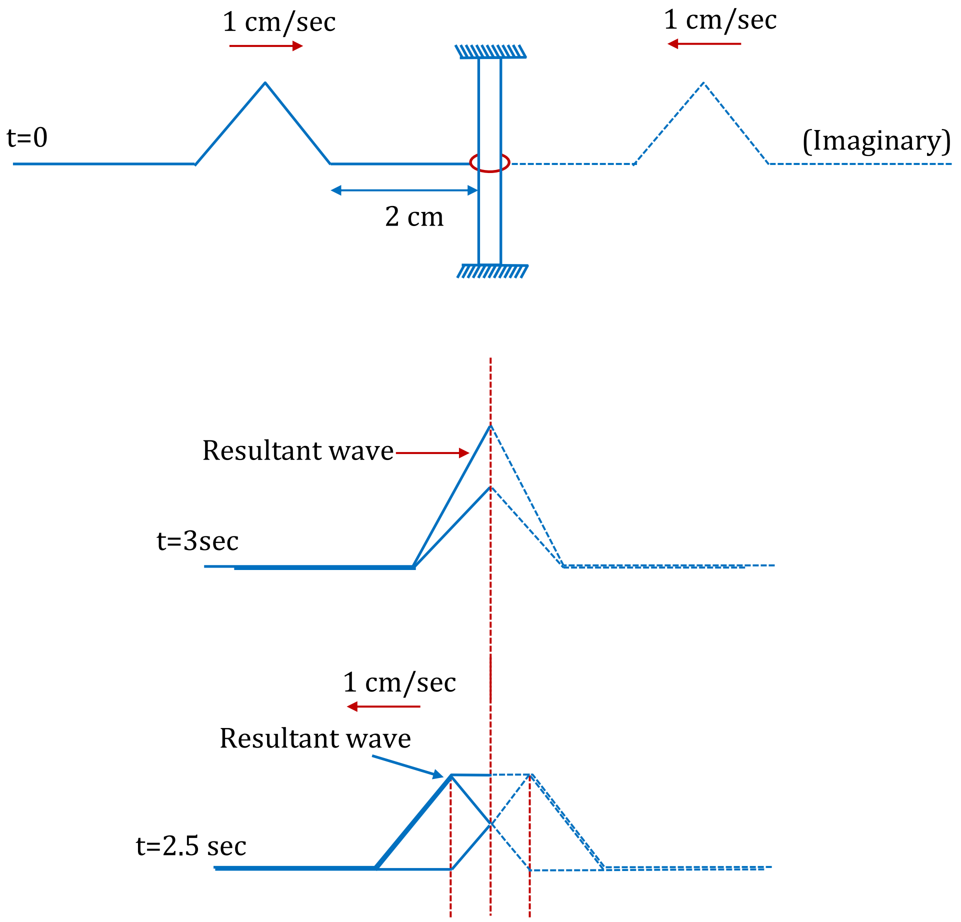


Incident pulse   Intermediate state   Reflected pulse

(b) There is no phase difference between incident and reflected wave at the point of reflection.

Example for free end :

A wave pulse is moving towards right with wave speed 1 cm/sec. Let us consider an imaginary wave moving towards left with same speed.



Example for fixed end :

A wave pulse is moving towards right with wave speed 1 cm/sec. Let us consider an imaginary wave moving towards left with same speed. The imaginary wave is inverted because there is a phase difference of π between incident wave and reflected wave at point of reflection.

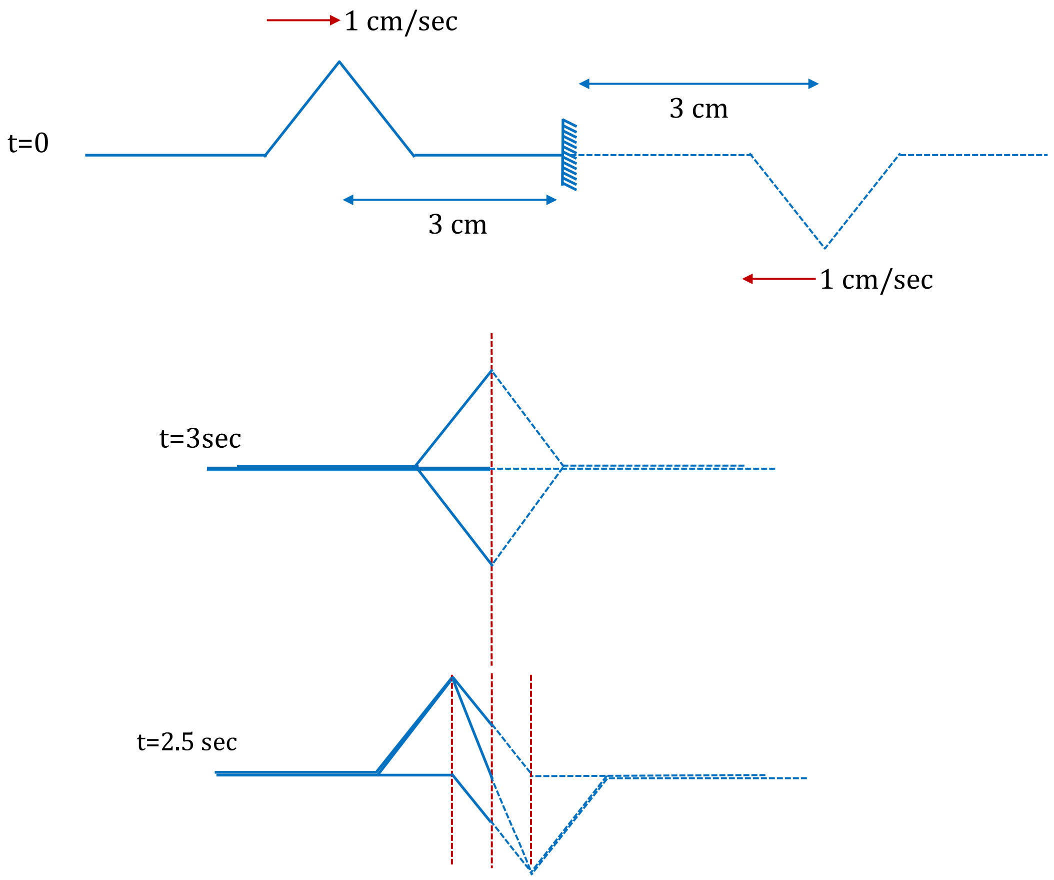
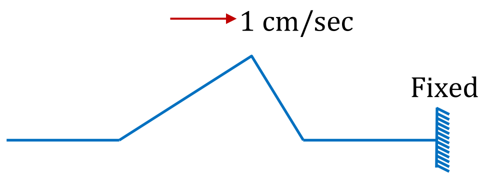


Illustration 34:

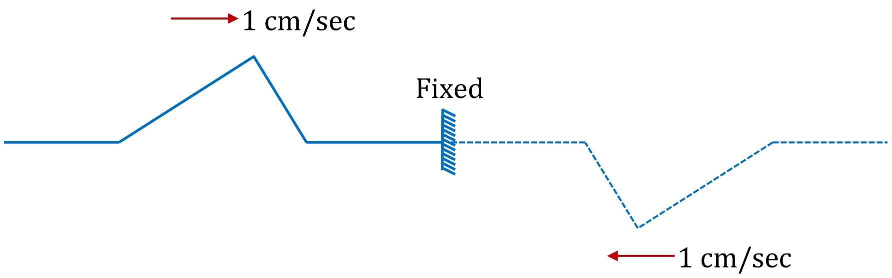
A pulse travelling with speed $v = 1\ cm/\text{sec}$ in a string is shown at $t = 0$, moving towards fixed end. Draw the pulse after it completely gets reflected from fixed end.



Solution:

Let us consider an imaginary inverted pulse moving towards left with same speed.

Shape of pulse before reflection from fixed end



Shape of pulse after reflection from fixed end

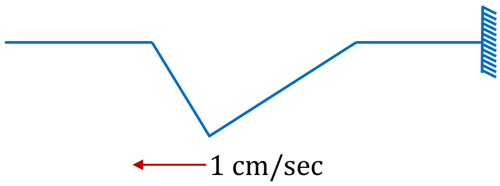
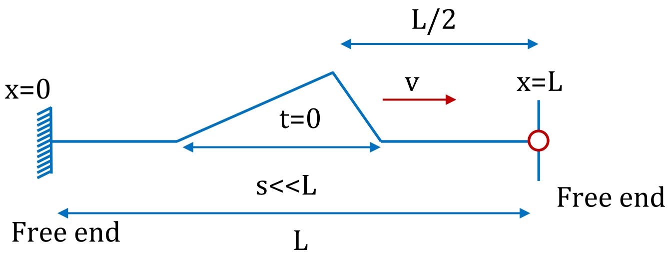
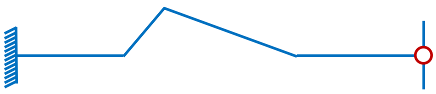
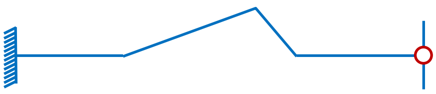


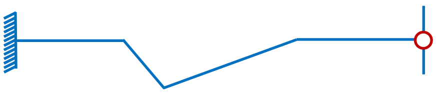
Illustration 35:

A small pulse travelling with speed v in a string is shown at $t = 0$, moving towards free end. Which of these is not CORRECTLY matched?



(i) $t = \frac{L}{v}$ (P) 

(ii) $t = \frac{2L}{v}$ (Q) 

(iii) $t = \frac{3L}{v}$ (R) 

(A) (i) (B) (ii) (C) (iii) (D) None of these

Ans. (B)

Illustration 36:

Consider a incident wave $y = A\sin(kx - \omega t)$. It gets reflected at $x = 0$ which is fixed end

(a) Find equation of reflected wave?

(b) If loss of energy is 19%, find equation of reflected wave.

Solution:

(a) If we put x=0 in the wave equation we will get the equation of motion of particle i.e., $y = A\sin(-\omega t)$

Or $y = A\sin(\omega t + \pi)$

Therefor equation of the motion of particle present on reflected wave at $x = 0$ is

Also $y = A\sin(\omega t + \pi + \pi)$ we will get the equation of reflected wave if we put $t + \frac{x}{v}$ in place of t, therefore equation of reflected wave will be given by

$$y = A\sin(\omega t + kx + 2\pi)$$

(b) In this case, amplitude become 0.9A.

So, equation of reflected wave will be given by

$$y = 0.9A\sin(\omega t + kx + 2\pi)$$

Standing Waves

Suppose two sine waves of equal amplitude and frequency propagate on a long string in opposite directions. The equations of the two waves are given by

$$y_1 = A\sin(\omega t - kx)$$

and $$y_2 = A\sin(\omega t + kx + \phi)$$

These waves interfere to produce standing waves. To understand these waves, let us discuss the special case when $\phi = 0$.

The resultant displacements of the particles of the string are given by the principle of superposition as

$$y = y_1 + y_2$$

$$= A[\sin(\omega t - kx) + \sin(\omega t + kx)]$$

$$= 2A\sin\omega t\cos kx$$

or, $$y = (2A\cos kx)\sin\omega t$$

This is the required result and from this we can say that:

1. As this equation satisfies the wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

it represents a wave. However, as it is not of the form $f(ax \pm bt)$, the wave is not travelling and so is called standing or stationary wave.

2. The amplitude of the wave $A_s = 2A\cos kx$ is not constant but varies periodically with position (and not with time as in beats).

3. The points for which amplitude is minimum are called nodes and for these

$$\cos kx = 0$$

i.e., $$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

i.e., $$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots \qquad \left[\text{as } k = \frac{2\pi}{\lambda}\right]$$

i.e., in a stationary wave, nodes are equally spaced.

4. The points for which amplitude is maximum are called antinodes and for these,

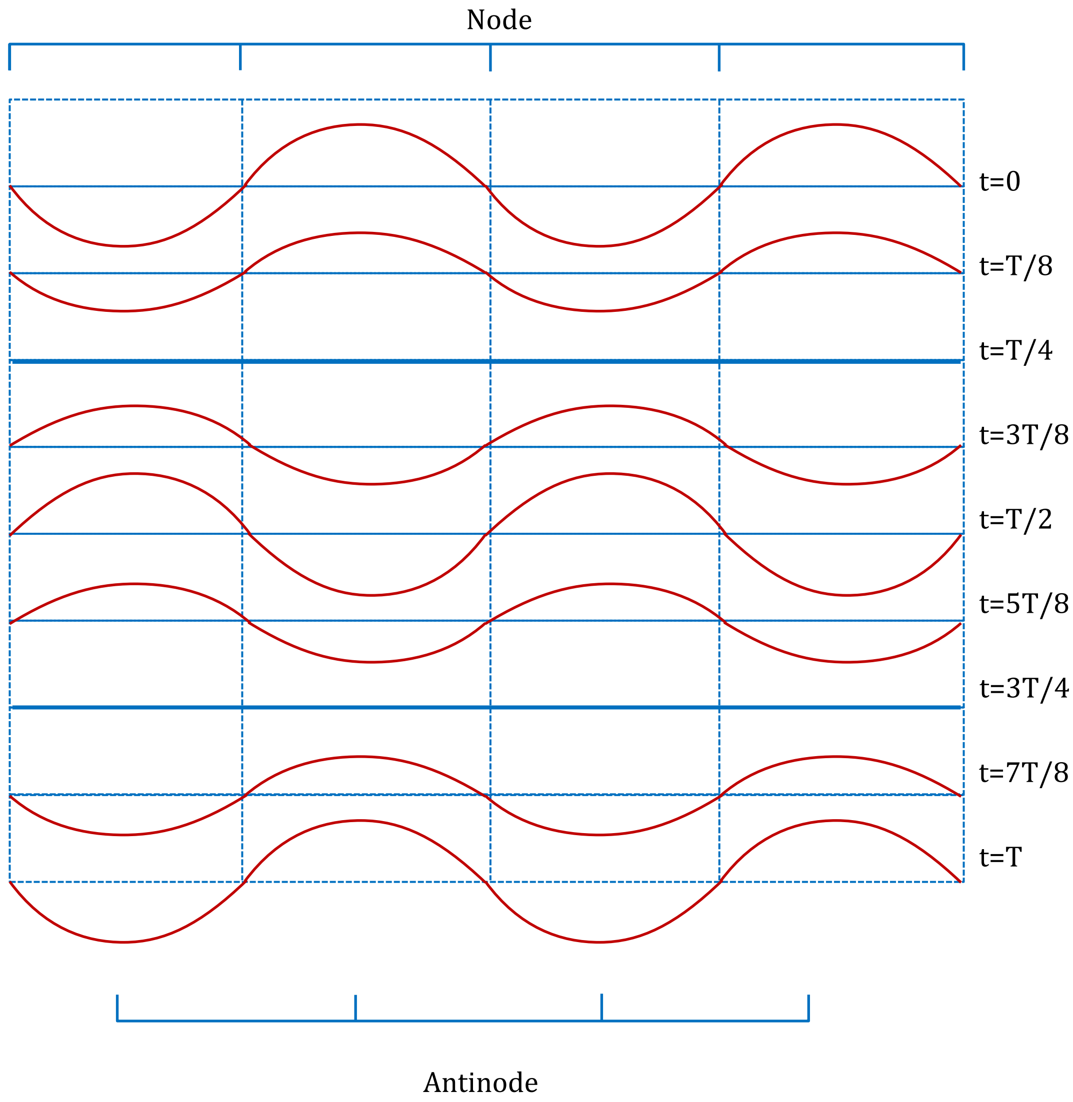
$\cos kx = \pm 1$

i.e., $kx = 0, \pi, 2\pi, 3\pi, ...$

i.e., $x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, ...$ $\left[\text{as } k = \frac{2\pi}{\lambda}\right]$

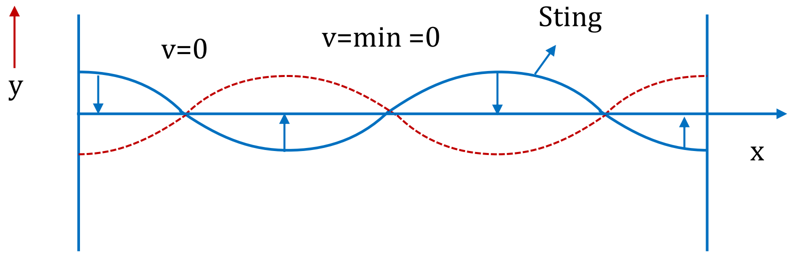
i.e., like nodes, antinodes are also equally spaced with spacing $(\lambda/2)$ and $A_{max} = \pm 2A$. Furthermore, nodes and antinodes are alternate with spacing $(\lambda/4)$.

5. The nodes divide the medium into segments (or loops). All the particles in a segment vibrate in same phase, but in opposite phase with the particles in the adjacent segment. Twice in one period all the particles pass through their mean position simultaneously with maximum velocity $(A_s\omega)$, the direction of motion being reversed after each half cycle.

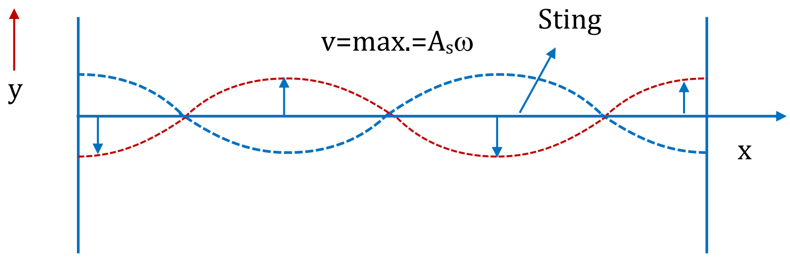


6. Standing waves can be transverse or longitudinal, e.g., in strings (under tension) if reflected wave exists, the waves are transverse-stationary, while in organ pipes waves are longitudinal-stationary.

7. As in stationary waves nodes are permanently at rest, so no energy can be transmitted across them, i.e., energy of one region (segment) is confined in that region. However, this energy oscillates between elastic potential energy and kinetic energy of the particles of the medium. When all the particles are at their extreme positions *KE* is minimum while elastic *PE* is maximum (as shown in figure A), and when all the particles (simultaneously) pass through their mean position *KE* will be maximum while elastic *PE* minimum (Figure B). The total energy confined in a segment (elastic *PE* + *KE*), always remains the same.



Elastic PE = max = E
Kinetic energy = min = 0
(A)



Elastic PE = min = 0
Kinetic energy = max = E
(B)

Illustration 37:

Two waves travelling in opposite directions produce a standing wave. The individual wave functions are

$y_1 = (4\ cm)\sin(3x - 2t)$

$y_2 = (4\ cm)\sin(3x + 2t)$

where *x* and *y* are in centimeter.

(a) Find the maximum displacement of a particle of the medium at $x = 2.3\ cm$.

(b) Find the position of the nodes and antinodes.

Solution:

(a) When the two waves interfere, the result is a standing wave whose mathematical representation is given by equation, with $A = 4\ cm$ and $k = 3$ rad/cm;

$$y = (2A\sin kx)\cos\omega t = [(8\ cm)\sin 3\ x]\cos 2\ t$$

Thus, the maximum displacement of a particle at the position $x = 2.3\ cm$ is

$$y_{max} = [(8\ cm)\sin 3x]_{x = 2.3\ cm} = (8\ m)\sin(6.9\text{ rad}) = 4.6\ cm$$

(b) Because $k = 2\pi/\lambda = 3$ rad/cm, we see that $\lambda = 2\pi/3cm$. Therefore, the antinodes are located at

$$x = n\left(\frac{\pi}{6}\right) cm \quad (n = 1, 3, 5, ...)$$

and the nodes are located at

$$x = n\left(\frac{\pi}{3}\right) cm \quad (n = 1, 2, 3, ...)$$

Illustration 38:

Two travelling waves of equal amplitudes and equal frequencies move in opposite direction along a string. They interfere to produce a standing wave having the equation.

$$y = A \cos kx \sin \omega t$$

in which $A = 1.0\ mm$, $k = 1.57\ cm^{-1}$ and $\omega = 78.5\ s^{-1}$. (a) Find the velocity and amplitude of the component travelling waves. (b) Find the node closest to the origin in the region $x > 0$. (c) Find the antinode closest to the origin in the region $x > 0$. (d) Find the amplitude of the particle at $x = 2.33\ cm$.

Solution:

(a) The standing wave is formed by the superposition of the waves

$$y_1 = \frac{A}{2}\sin(\omega t - kx) \quad \text{and} \quad y_2 = \frac{A}{2}\sin(\omega t + kx).$$

The wave speed of either of the waves is

$$v = \frac{\omega}{k} = \frac{78.5\ s^{-1}}{1.57\ cm^{-1}} = 50\ cm/s\ ;\ \text{Amplitude} = 0.5\ mm.$$

(b) For a node, $\cos kx = 0$.

The smallest positive x satisfying this relation is given by

$$kx = \pi/2 \quad \text{or,} \quad x = \frac{\pi}{2k} = \frac{3.14}{2\times 1.57 cm^{-1}} = 1\ cm$$

(c) For an antinode, $|\cos kx| = 1$.

The smallest positive x satisfying this relation is given by

$$kx = \pi \quad \text{or,} \quad x = \frac{\pi}{k} = 2\ cm$$

(d) The amplitude of vibration of the particle at x is given by $|A \cos kx|$. For the given point,

$$kx = (1.57\ cm^{-1})(2.33\ cm) = \frac{7}{6}\pi = \pi + \frac{\pi}{6}$$

Thus, the amplitude will be $(1\ mm)\ |\cos(\pi + \pi/6)| = \frac{\sqrt{3}}{2} mm = 0.86\ mm.$

Vibration of string:

(a) Fixed at both ends : Suppose a string of length L is kept fixed at the ends $x = 0$ and $x = L$. In such a system suppose we send a continuous sinusoidal wave of a certain frequency, say, toward the right. When the wave reaches the right end. It gets reflected and begins to travel back. The left-going wave then overlaps the wave, which is still travelling to the right. When the left-going wave reaches the left end, it gets reflected again and the newly reflected wave begins to travel to the right. overlapping the left-going wave. This process will continue and, therefore, very soon we have many overlapping waves, which interfere with one another. In such a system, at any point x and at any time t, there are always two waves, one moving to the left and another to the right. We, therefore, have

$y_1(x, t) = A \sin (kx - \omega t)$ (wave travelling in the positive direction of x-axis)

and $y_2(x, t) = A \sin (kx + \omega t)$ (wave travelling in the negative direction of x-axis).

The principle of superposition gives, for the combined wave

$$y'(x, t) = y_1(x, t) + y_2(x, t)$$
$$= A \sin (kx - \omega t) + A \sin (kx + \omega t)$$
$$= (2A \sin kx) \cos \omega t$$

It is seen that the points of maximum or minimum amplitude stay at one position.

Nodes: The amplitude is zero for values of kx that give $\sin kx = 0$ i.e. for,

$kx = n\pi$, for $n = 0, 1, 2, 3, ...$

Substituting $k = 2\pi/\lambda$ in this equation, we get

$x = n\frac{\lambda}{2}$, for $n = 0, 1, 2, 3, ...$

The positions of zero amplitude are called the **nodes.** Note that a distance of $\frac{\lambda}{2}$ or half a wavelength separates two consecutive nodes.

Antinodes : The amplitude has a maximum value of $2A$, which occurs for the values of kx that give $|\sin kx| = 1$. Those values are

$kx = (n + 1/2)\pi$ for $n = 0, 1, 2, 3, ...$

Substituting $k = 2\pi/\lambda$ in this equation, we get.

$x = (n + 1/2)\frac{\lambda}{2}$ for $n = 0, 1, 2, 3, ...$

as the positions of maximum amplitude. These are called the **antinodes**. The antinodes are separated by $\lambda/2$ and are located half way between pairs of nodes.

For a stretched string of length L, fixed at both ends, one end is chosen as position $x = 0$, and the other end is $x = L$. In order that this end is a node;

$L = n\frac{\lambda}{2}$, for $n = 1, 2, 3, ...$

This condition shows that standing waves on a string of length L have restricted wavelength given by

$\lambda = \frac{2L}{n}$, for $n = 1, 2, 3, ...$

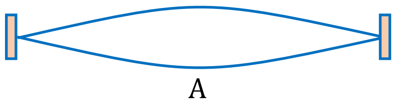
The frequencies corresponding to these wavelengths follow from Eq. as

$f = n\frac{v}{2L}$, for $n = 1, 2, 3, ...$

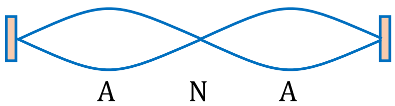
where v is the speed of travelling waves on the string. The set of frequencies given by equation are called the natural frequencies or **modes** of oscillation of the system. This equation tells us that the natural frequencies of a string are integral multiples of the lowest frequency $f_0 = \frac{v}{2L}$, which corresponds to $n = 1$.

The oscillation mode with that lowest frequency is called the fundamental mode or the first harmonic. The second harmonic or first overtone is the oscillation mode with $n = 2$. The third harmonic and second overtone corresponds to $n = 3$ and so on. The frequencies associated with these modes are often labeled as f_1, f_2, f_3 and so on. The collection of all possible modes is called the harmonic series and n is called the harmonic number.

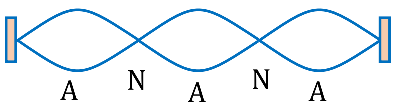
Some of the harmonic of a stretched string fixed at both the ends are shown in figure.



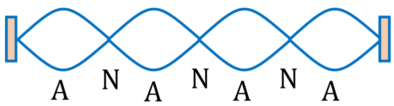
(a)
Fundamental or
first harmonic
$f_0 = v/2L$



(b)
Second harmonic
or first overtone
$f_1 = 2f_0 = 2v/2L$



(c)
Third harmonic
or 2nd overtone
$f_2 = 3f_0 = 3v/2L$



(d)
Fourth harmonic
or 3rd overtone
$f_3 = 4f_0 = 4v/2L$

Illustration 39:

A middle C string on a piano has a fundamental frequency of 262 Hz, and the A note has fundamental frequency of 440 Hz. (a) Calculate the frequencies of the next two harmonics of the C string. (b) If the strings for the A and C notes are assumed to have the same mass per unit length and the same length, determine the ratio of tensions in the two strings.

Solution:

(a) Because $f_1 = 262\ Hz$ for the C string, we can use Equation to find the frequencies f_2 and f_3

$$f_2 = 2f_1 = 524\ Hz$$

$$f_3 = 3f_1 = 786\ Hz$$

(b) Using Equation for the two strings vibrating at their fundamental frequencies gives

$$f_{1A} = \frac{1}{2L}\sqrt{\frac{T_A}{\mu}}$$

$$f_{1C} = \frac{1}{2L}\sqrt{\frac{T_C}{\mu}}$$

$$\therefore \quad \frac{f_{1A}}{f_{1C}} = \sqrt{\frac{T_A}{T_C}}$$

$$\Rightarrow \quad \frac{T_A}{T_C} = \left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{440\ Hz}{262\ Hz}\right)^2 = 2.82$$

Illustration 40:

A wire having a linear mass density 10^{-3} kg/m is stretched between two rigid supports with a tension of 90 N. The wire resonates at a frequency of 350 Hz. The next higher frequency at which the same wire resonates is 420 Hz. Find the length of the wire.

Solution:

Suppose the wire vibrates at 350 Hz in its nth harmonic and at 420 Hz in its $(n + 1)$th harmonic.

$$350 = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{...(i)}$$

and $$420 = \frac{(n+1)}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{...(ii)}$$

This gives $\frac{420}{350} = \frac{n+1}{n}$ or $n = 5$

Putting the value in (i),

$$350 = \frac{5}{2\ell}\sqrt{\frac{90}{10^{-3}}}$$

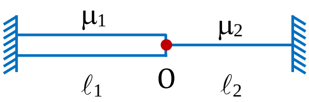
$$\Rightarrow \quad 350 = \frac{5}{2\ell} \times 300$$

$$\lambda = \frac{1500}{700} = \frac{15}{7}\,m = 2.1\,m$$

Illustration 41:

Given $\frac{\mu_1}{\mu_2} = 4$ and 'O' is node.

Find $\frac{\ell_2}{\ell_1}$ such that when the strings resonate there is 2nd harmonic in string-1 and 3rd harmonic in string-2.



Solution:

Tension and f same in both string

$$\frac{2}{2\ell_1}\sqrt{\frac{T}{\mu_1}} = \frac{3}{2\ell_2}\sqrt{\frac{T}{\mu_2}}$$

$$\frac{\ell_2}{\ell_1} = \frac{3}{2}\sqrt{\frac{\mu_1}{\mu_2}}$$

$$\frac{\ell_2}{\ell_1} = 3$$

(b) **Fixed at one end :** Standing waves can be produced on a string which is fixed at one end and whose other end is free to move in a transverse direction. Such a free end can be nearly achieved by connecting the string to a very light thread.

If the vibrations are produced by a source of "correct" frequency, standing waves are produced. If the end $x = 0$ is fixed and $x = L$ is free, the equation is again given by

$$y = 2A \sin kx \cos \omega t$$

with the boundary condition that $x = L$ is an antinode. The boundary condition that $x = 0$ is a node is automatically satisfied by the above equation. For $x = L$ to be an antinode,

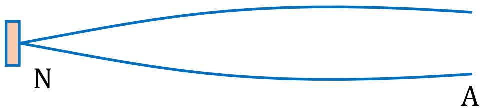
$$\sin kL = \pm 1$$

or, $kL = \left(n + \frac{1}{2}\right)\pi$ or, $\frac{2\pi L}{\lambda} = \left(n + \frac{1}{2}\right)\pi$

or, $\frac{2Lf}{v} = n + \frac{1}{2}$ or, $f = \left(n + \frac{1}{2}\right)\frac{v}{2L} = \frac{n+\frac{1}{2}}{2L}\sqrt{T/\mu}$ .....

These are the normal frequencies of vibration. The fundamental frequency is obtained when $n = 0$, i.e.,

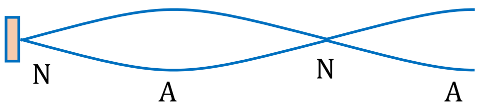
$f_0 = v/4L$ (a)



Fundamental

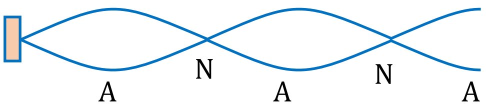
The overtone frequencies are

$f_1 = \frac{3v}{4L} = 3f_0$ (b)



First overtone

$f_2 = \frac{5v}{4L} = 5f_0$ (c)



Second overtone

We see that all the harmonic of the fundamental are not the allowed frequencies for the standing waves. Only the odd harmonics are the overtones. Figure shows shapes of the string for some of the normal modes.

Laws of transverse vibrations of a string - sonometer wire

(a) Law of length $f \propto \frac{1}{L}$ so $\frac{f_1}{f_2} = \frac{L_2}{L_1}$; if T and μ are constant

(b) Law of tension $f \propto \sqrt{T}$ so $\frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$; L and μ are constant

(c) Law of mass $f \propto \frac{1}{\sqrt{\mu}}$ so $\frac{f_1}{f_2} = \sqrt{\frac{\mu_2}{\mu_1}}$; T and L are constant

Energy density of string carrying standing wave: we have already evaluated the expressions of potential energy and kinetic energy of a differential element present on string in case of travelling wave, which can be used here also but function representing displacement of particle $y = f(x, t)$ will change. Let us consider a standing wave $y = A \sin(kx) \sin(\omega t)$ to calculate energy density.

Then, $\frac{\partial y}{\partial t} = A\omega \sin(kx) \cos(\omega t)$ ...(1)

$\frac{\partial y}{\partial x} = Ak \cos(kx) \sin(\omega t)$ ...(2)

Since in case of standing wave $\frac{\partial y}{\partial x} \neq -\frac{1}{v}\frac{\partial y}{\partial t}$, therefore in this case kinetic and potential energy of an element are not equal.

Kinetic energy of an element for the above assumed wave is

$$dK = \frac{1}{2}\mu(dx)\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu A^2\omega^2 \sin^2(kx)\cos^2(\omega t)\,(dx)$$

Where μ is linear mass density of string.

Potential energy of the element will be given by

$$dU = \frac{1}{2}T(dx)\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}TA^2k^2 \cos^2(kx)\sin^2(\omega t)\,(dx)$$

Therefor total energy density of element will be given by

$$\frac{dE}{dx} = \frac{dK}{dx} + \frac{dU}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

$$= \frac{1}{2}\mu A^2\omega^2\sin^2(kx)\cos^2(\omega t) + \frac{1}{2}TA^2k^2\cos^2(kx)\sin^2(\omega t) \qquad ...(3)$$

Few important points regarding energy of standing wave in strings:

(1) Since $\frac{\partial y}{\partial x}$ (slope) at antinode is always zero, therefor potential energy of an element at this point is always zero.

(2) Since $\frac{\partial y}{\partial t}$ (velocity) of element at node is always zero therefor kinetic energy of an element at node is always zero.

(3) If we integrate the total mechanical energy from equation (3) for one loop, we will find that this energy is coming independent of time and is equal to

$$E = \frac{1}{8}\mu\lambda A^2\omega^2$$

Therefor energy of a loop remains conserved in standing wave.

Power analysis in standing wave on string:

We have already derived the expression of power in case of travelling wave which can be used here also.

$$P = -T\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right) \qquad ...(4)$$

If the equation of standing wave is $y = A\sin(kx)\sin(\omega t)$

Then $P = -TA^2k\omega\cos(kx)\sin(\omega t)\sin(kx)\cos(\omega t)$

$$= -\frac{1}{4}TA^2k\omega\sin(2kx)\sin(2\omega t)$$

From the above equation, we can find value of power transmission at any point.

Few important points regarding power transmission in standing wave on string:

(1) Since value of $\frac{\partial y}{\partial x}$ (slope) at antinode is always zero, therefore according to equation (4), power transmission through antinode is always zero.

(2) Since value of $\frac{\partial y}{\partial t}$ (velocity) of element at node is always zero, therefor according to equation (4), power transmission through node is always zero.

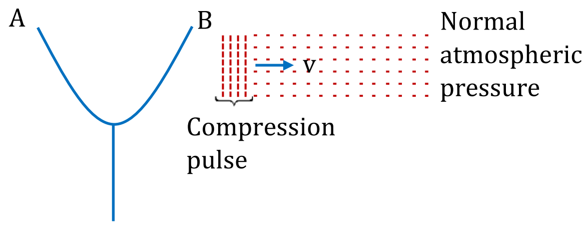
(3) On combining above two points it can be concluded that energy stored in one loop or even in half loop remain conserved.

SOUND WAVES

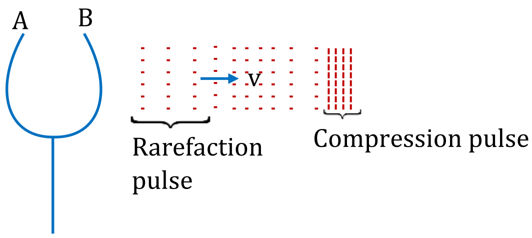
Sound is a mechanical longitudinal wave that is created by a vibrating source such as a guitar string, the human vocal cords, the prongs of a tuning fork or the diaphragm of a loudspeaker. Being a mechanical wave, sound needs a medium having properties of inertia and elasticity for its propagation. Sound waves propagate in any medium through a series of periodic compressions and rarefactions of pressure, which is produced by the vibrating source.

Consider a tuning fork producing sound waves

When Prong *B* moves outward towards right it compresses the air in front of it, causing the pressure to rise slightly. The region of increased pressure is called a compression pulse and it travels away from the prong with the speed of sound.



After producing the compression pulse, the prong *B* reverses its motion and moves inward. This drags away some air from the region in front of it, causing the pressure to dip slightly below the normal pressure. This region of decreased pressure is called a rarefaction pulse. Following immediately behind the compression pulse, the rarefaction pulse also travels away from the prong with the speed of sound.



If the prongs vibrate in *SHM*, the pressure variations in the layer close to the prong also varies simple harmonically.

Q.1 Which of the following is the longitudinal wave?

(A) Sound waves (B) Waves on plucked string

(C) Water waves (D) Light waves

Ans. (A)

Q.2 Which of following can not travel through vacuum?

(A) Electromagnetic wave (B) Sound wave

(C) Light wave (D) X-ray

Ans. (B)

Q.3 A longitudinal wave consists of:

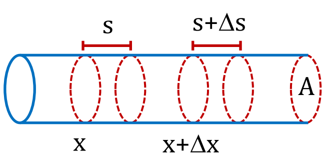
(A) rarefactions and compressions (B) only compressions

(C) only rarefactions (D) crest and trough

Ans. (A)

Displacement Wave and Pressure Wave in One Dimension (Plane Wave)

A longitudinal wave in a fluid (liquid or gas) can be described either in terms of the longitudinal displacement suffered by the particles of the medium or in terms of the excess pressure generated due to the compression or rarefaction. Let us see how the two representations are related to each other.



Consider a wave going in the x-direction in a fluid. Suppose that at a time t, the particle at the undisturbed position x suffers a displacements in the x-direction. The wave can then be described by the equation.

$$s = s_0 \sin \omega(t - x / v) \qquad \text{...(i)}$$

Consider the element of the material which is contained within x and $x + \Delta x$ (as shown in figure) in the undisturbed state. Considering a cross-sectional area A, the volume of the element in the undisturbed state is $A\Delta x$ and its mass is $\rho A\Delta x$. As the wave passes, the ends at x and $x + \Delta x$ are displaced by amounts s and $s + \Delta s$ according to equation (i) above. The increase in volume of this element at time t is

$$\Delta V = A\Delta s = A\frac{\partial s}{\partial x}.\Delta x = As_0(-\omega/v)\cos(t - x/v)\Delta x$$

Where Δs has been obtained by differentiating equation (i) with respect to x. The element is, therefore, under a volume strain.

$$\frac{\Delta V}{V} = \frac{-As_0\omega\cos\omega(t - x/v)\Delta x}{vA\Delta x} = \frac{-s_0\omega}{v}\cos\omega(t - x/v)$$

The corresponding stress i.e., the excess pressure developed in the element at x at time t is,

$$P = B\left(\frac{-\Delta V}{V}\right)$$

Where B is the bulk modulus of the material. Thus,

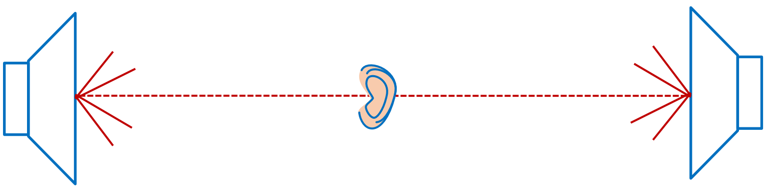
$$P = B\frac{s_0\omega}{v}\cos\omega(t - x/v) \qquad \text{...(ii)}$$

The pressure amplitude P_0 and the displacement amplitude s_0 are related as

$$P_0 = \frac{B\omega}{v}s_0 = Bks_0$$

Where k is the wave number. Also, we see from (i) and (ii) that the pressure wave differs in phase by $\pi/2$ from the displacement wave. The pressure maxima occur where the displacement is zero and displacement maxima occur where the pressure is at its normal level.

The fact that, displacement is zero where the pressure-change is maximum and vice versa, puts the two descriptions on different footings. The human ear or an electronic detector responds to the change in pressure and not to the displacement in a straight forward way. Suppose two audio speakers are driven by the same amplifier and are placed facing each other. A detector is placed midway between them.



The displacement of the air particles near the detector will be zero as the two sources drive these particles in opposite directions. However, both send compression waves and rarefaction waves together. As a result, pressure increases at the detector simultaneously due to both the sources. Accordingly, the pressure amplitude will be doubled, although the displacement remains zero here. A detector detects maximum intensity in such a condition. Thus, the description in terms of pressure wave is more appropriate than the description in terms of the displacement wave as far as sound properties are concerned.

Illustration 42:

Find the following for given wave equation.

$P = 0.02 \sin(3000\,t - 9\,x)$ [all quantities are in S.I. units.]

(A) Frequency (B) Wavelength (C) Speed of sound wave

(D) If the equilibrium pressure of air is in $10^5\ Pa$ then find maximum and minimum pressure.

Solution:

Comparing with the standard form of a travelling wave

$$P = P_0 \sin[\omega(t - x/v)]$$

We see that $\omega = 3000\ s^{-1}$.

$$f = \frac{\omega}{2\pi} = \frac{3000}{2\pi}\ Hz$$

Also, from the same comparison, $\omega/v = 9.0\ m^{-1}$.

or, $$v = \frac{\omega}{9.0\ m^{-1}} = \frac{3000\ s^{-1}}{9.0\ m^{-1}} = \frac{1000}{3}\ m/s$$

The wavelength is $\lambda = \dfrac{v}{f} = \dfrac{1000\ /3m/s}{3000/2\pi\ \ Hz} = \dfrac{2\pi}{9}m$

The pressure amplitude is $P_0 = 0.02\ N/m^2$. Hence, the maximum and minimum pressures at a point in the wave motion will be $(1.0 \times 10^5 \pm 0.02)\ N/m^2$

Illustration 43:

A sound wave of wavelength 40 cm travels in air. If the difference between the maximum and minimum pressures at a given point is $2.0 \times 10^{-3}\ N/m^2$, find the amplitude of vibration of the particles of the medium. The bulk modulus of air is $1.4 \times 10^5\ N/m^2$.

Solution:

The pressure amplitude is $P_0 = \dfrac{2.0\times10^{-3}N/m^2}{2} = 10^{-3}\ N/m^2$.

The displacement amplitude s_0 is given by $P_0 = B\,k\,s_0$

or, $$s_0 = \frac{P_0}{B\ \ k} = \frac{P_0\lambda}{2\ \ \pi\ \ B} = \frac{10^{-3}N/m^2\times(40\times10^{-2}m)}{2\times\pi\times14\times10^4 N/m^2} = \frac{100}{7\pi}\text{Å} = 4.54\ \text{Å}$$

Illustration 44:

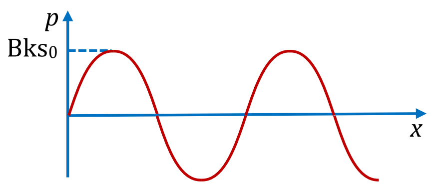
A sound wave is travelling and its equation is given by $s = s_0 \cos(kx - \omega t)$

(i) Draw $P - x$ graph at $t = 0$ (ii) Draw $P - t$ graph at $x = 0$

Solution:

$$\Delta P = -B\left(\frac{\partial s}{\partial x}\right) = Bs_0k\sin(kx - \omega t)$$

(i) At $t = 0$, $P = Bks_0 \sin(kx)$



(ii) At $x = 0$, $P = Bk\, s_0 \sin(-\omega t)$

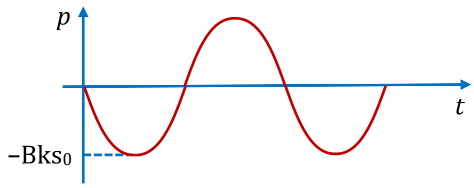
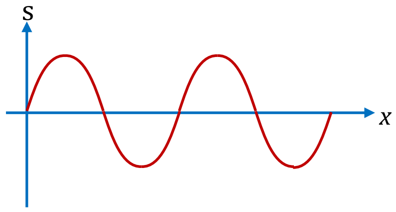


Illustration 45:

A sound wave is travelling towards right and snapshot at $t = 0$ is as shown in figure. Draw its $P - x$ graph at $t = 0$



Solution:

$S = S_0 \sin(kx)$

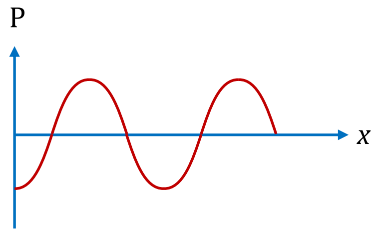
Replace x by $x-vt$

$S = S_0 \sin k(x - vt)$

$S = S_0 \sin(kx - \omega t)$

$$\Delta P = -B\left(\frac{\partial s}{\partial x}\right)$$

$$\Delta P = -BkS_0 \cos(kx - \omega t)$$



Speed of longitudinal (Sound) waves

In a solid medium $v = \sqrt{\dfrac{k + \frac{4}{3}\eta}{\rho}}$ where k = Bulk modulus; η = Modulus of rigidity; ρ = Density.

When the solid is in the form of long bar $v = \sqrt{\dfrac{Y}{\rho}}$ where Y = Young's modulus of material of rod.

Velocity of sound waves in a fluid medium (liquid or gas) is given by

$$v = \sqrt{\frac{B}{\rho}}$$

where, ρ = density of the medium and B = Bulk modulus of the medium given by,

$$B = -V\,\frac{dP}{dV}$$

Newton's formula : Newton assumed propagation of sound through a gaseous medium to be an isothermal process.

PV = constant

$$\frac{dP}{dV} = \frac{-P}{V}$$

and hence $B = P$ using equation $B = -V\dfrac{dP}{dV}$

and thus velocity of sound in a gas,

$$v = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{RT}{M}} \text{ where } M = \text{molar mass}$$

the density of air at 0° and pressure 76 cm of Hg column is $\rho = 1.293\ kg/m^3$. This temperature and pressure is called standard temperature and pressure at STP. Speed of sound in air is 280 m/s. This value is less than measured speed of sound in air 332 m/s then Laplace suggested the correction.

Laplace's correction : Later Laplace established that propagation of sound in a gas is not an isothermal but an adiabatic process and hence PV^{γ} = constant.

$$\frac{dP}{dV} = -\gamma\frac{P}{V}$$

where, $B = -V\dfrac{dP}{dV} = \gamma P$

and hence speed of sound in a gas,

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

Factors affecting speed of sound in atmosphere.

(a) **Effect of temperature:** as temperature (T) increases velocity (v) increases.

$$v \propto \sqrt{T}$$

For small change in temperature above room temperature v increases linearly by 0.6 m/s for every 1°C rise in temp.

$$v = \sqrt{\frac{\gamma R}{M}} \times T^{1/2}$$

$$\frac{\Delta v}{v} = \frac{1}{2}\frac{\Delta T}{T}$$

$$\Delta v = \left(\frac{1}{2}\frac{v}{T}\right)\ \Delta T$$

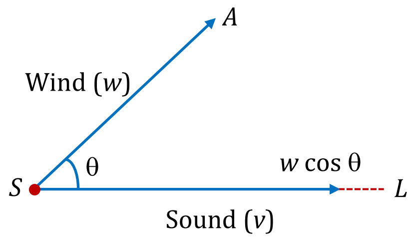
$$\Delta v = (0.6)\ \ \Delta T$$

(b) **Effect of pressure:** The speed of sound in a gas is given by $v = \sqrt{\dfrac{\gamma P}{\rho}} = \sqrt{\dfrac{\gamma RT}{M}}$

So at constant temperature, if P changes then ρ also changes in such a way that P/ρ remains constant. Hence pressure does not have any effect on velocity of sound as long as temperature is constant.

(c) **Effect of humidity:** With increase in humidity density decreases. This is because the molar mass of water vapour is less than the molar mass of air.

(d) **Effect of wind velocity:** Because wind drifts the medium (air) along its direction of motion therefore the velocity of sound in a particular direction is the algebraic sum of the velocity of sound and the component of wind velocity in that direction. Resultant velocity of sound along $SL = v + w\cos\theta$.



Some points regarding velocity of sounds:

(1) As solids are most elastic while gases least *i.e.* $E_S > E_L > E_G$. So, the velocity of sound is maximum in solids and minimum in gases

$$v_{\text{steel}} > v_{\text{water}} > v_{\text{air}}$$

$$5000\ m/s > 1500\ m/s > 330\ m/s$$

As for sound $v_{\text{water}} > v_{\text{Air}}$ while for light $v_w < v_A$.

Water is rarer than air for sound and denser for light.

The concept of rarer and denser media for a wave is through the velocity of propagation (and not density). Lesser the velocity, denser is said to be the medium and vice-versa.

(2) Sound of any frequency or wavelength travels through a given medium with the same velocity.

For a given medium velocity remains constant. All other factors like phase, loudness pitch, quality *etc.* have practically no effect on sound velocity.

(3) Relation between velocity of sound and root mean square velocity.

$$v_{\text{sound}} = \sqrt{\frac{\gamma RT}{M}} \quad \text{and} \quad v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

So, $$\frac{v_{\text{rms}}}{v_{\text{sound}}} = \sqrt{\frac{3}{\gamma}} \quad \text{or} \quad v_{\text{sound}} = [\gamma/3]^{1/2}\, v_{\text{rms}}.$$

(4) There is no atmosphere on moon, therefore propagation of sound is not possible there. To do conversation on moon, the astronaut uses an instrument which can transmit and detect electromagnetic waves.

Illustration 46:

Find the speed of sound in H_2 at temperature T, if the speed of sound in O_2 is 450 m/s at this temperature.

Solution:

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Since temperature, T is constant,

$$\frac{v_{H_2}}{v_{O_2}} = \sqrt{\frac{M_{O_2}}{M_{H_2}}} = \sqrt{\frac{32}{2}} = 4$$

$$v_{H_2} = 4 \times 450 = 1800\ m/s$$

Illustration 47:

If the density of oxygen is 16 times that of hydrogen, what will be the ratio of their corresponding velocities of sound waves

(A) 1 : 4 (B) 4 : 1 (C) 16 : 1 (D) 1 : 16

Ans. (A)

Solution:

(A) $v=\sqrt{\frac{\gamma P}{\rho}} \Rightarrow \frac{v_{O_2}}{v_{H_2}}=\sqrt{\frac{\rho_{H_2}}{\rho_{o_2}}}=\sqrt{\frac{1}{16}}=\frac{1}{4}$

Illustration 48:

At which temperature the speed of sound in hydrogen will be same as that of speed of sound in oxygen at 100°C

(A) – 148°C (B) – 212.5°C (C) – 317.5°C (D) – 249.7°C

Ans. (D)

Solution:

Speed of sound in gases is $v=\sqrt{\frac{\gamma RT}{M}}$

As v and γ are same for both gases in this example

$\therefore \quad T \propto M$

$$\frac{T_{H_2}}{T_{O_2}}=\frac{M_{H_2}}{M_{O_2}}$$

$$\frac{T_{H_2}}{(273+100)}=\frac{2}{32} \Rightarrow T_{H_2}=23.2K=-249.7^\circ C$$

Illustration 49:

A gas mixture has 24% of Argon, 32% of oxygen, and 44% of CO_2 by mass. Find the velocity of sound in the gas mixture at 27 °C. Given R = 8.4 in S.I. units.

Molecular weight of Ar = 40, O_2 = 32, CO_2 = 44 and $\gamma_{Ar}=5/3,\ \gamma_{O_2}=7/5,\ \gamma_{CO_2}=4/3$

Solution:

$$M_{mix}=\frac{M_{Total}}{\frac{m_{Ar}}{M_{Ar}}+\frac{m_{o_2}}{M_{o_2}}+\frac{m_{co_2}}{M_{co_2}}}=\frac{100}{\frac{24}{40}+\frac{32}{32}+\frac{44}{44}}=\frac{100}{2.6}$$

$$f_{mix}=\frac{n_1f_1+n_2f_2+n_3f_3}{n_1+n_2+n_3}=\frac{0.6\times 3+1\times 5+1\times 6}{2.6}=\frac{12.8}{2.6}$$

$$\gamma_{mix}=\frac{f_{mix}+2}{f_{mix}}=\frac{\frac{12.8}{2.6}+2}{\frac{12.8}{2.6}}=\frac{18}{12.8}$$

$$v=\sqrt{\frac{\gamma_{mix}RT}{M_{mix}}}=\sqrt{\frac{18\times 8.4\times 300}{12.8\times\frac{100}{2.6}\times 10^{-3}}}=\sqrt{\frac{18\times 8.4\times 300\times 2.6}{12.8\times 100\times 10^{-3}}}$$

$$\approx 303.5\ m/s$$

Intensity of Sound Waves:

Like any other progressive wave, sound waves also carry energy from one point of space to the other. This energy can be used to do work, for example, forcing the eardrums to vibrate or in the extreme case of a sonic boom created by a supersonic jet, can even cause glass panes of windows to crack.

The amount of energy carried per unit time by a wave is called its power and power per unit area held perpendicular to the direction of energy flow is called intensity.

For a sound wave travelling along positive x-axis described by the equation.

$$s = s_0 \sin(\omega t - kx + \phi)$$

$$p = p_0 \cos(\omega t - kx + \phi)$$

$$\frac{\partial s}{\partial t} = \omega s_0 \cos(\omega t - kx + \phi)$$

Instantaneous power $P = F.v = pA\,\frac{\partial s}{\partial t}$

$$P = p_0 \cos(\omega t - kx + \phi).\, A\omega s_0 \cos(\omega t - kx + \phi)$$

$$P_{\text{average}} = \langle P \rangle$$

$$= p_0 A\omega s_0 \langle \cos^2(\omega t - kx + \phi) \rangle$$

$$= \frac{p_0 \omega s_0 A}{2}$$

We know that

$$v = \sqrt{\frac{B}{\rho}}$$

$\Rightarrow$ $B = \rho v^2$

Hence $p_0 = Bks_0 = \rho v^2 ks_0$

$$P_{\text{average}} = \frac{1}{2}\omega p_0 A\left(\frac{p_0}{\rho v^2 k}\right) = \frac{p_0^2 A}{2\rho v} = \frac{\rho A v \omega^2 s_0^2}{2}$$

Maximum power = $P_{max} = \dfrac{p_0^2 A}{\rho v} = \rho v A\; v_{p,\,\max}^2 = \rho A v \omega^2 s_0^2$

Total energy transfer = $P_{avg} \times t = \dfrac{\rho A v \omega^2 s_0^2}{2} \times t$

Average intensity = $\dfrac{\text{Average power}}{\text{Area}}$

So the average intensity at position x is given by

$$\langle I \rangle = \frac{1}{2}\,\frac{\omega^2 s_0^2 B}{v} = \frac{p_0^2 v}{2B}$$

Substituting $B = \rho v^2$, intensity can also be expressed as

$$\langle I \rangle = \frac{P_0^2}{2\rho v}$$

Note:

- If the source is a point source then

  $I \propto \frac{1}{r^2}$ and $s_0 \propto \frac{1}{r}$

  $\Rightarrow \quad s = \frac{a}{r}\sin(\omega t - kr + \phi)$

- If a sound source is a line source then

  $I \propto \frac{1}{r}$ and $s_0 \propto \frac{1}{\sqrt{r}} \qquad \Rightarrow \quad s = \frac{a}{\sqrt{r}}\sin(\omega t - kr + \phi)$

- In general Intensity $(I) = -B\left(\frac{\partial s}{\partial x}\right)\left(\frac{\partial s}{\partial t}\right)$

Illustration 50:

The pressure amplitude in a sound wave from a radio receiver is $2 \times 10^{-3}\ N/m^2$ and the intensity at a point is $10^{-6}\ W/m^2$. If by turning the "Volume" knob the pressure amplitude is increased to $3 \times 10^{-3}\ N/m^2$, evaluate the intensity.

Solution:

The intensity is proportional to the square of the pressure amplitude.

Thus, $\frac{I'}{I} = \left(\frac{p'_0}{p_0}\right)^2$

$I' = \left(\frac{p'_0}{p_0}\right)^2 I = \left(\frac{3}{2}\right)^2 \times 10^{-6}\ W/m^2 = 2.25 \times 10^{-6}\ W/m^2$

Illustration 51:

A circular plate of area $0.4\ cm^2$ is kept at distance of $2m$ from source of power $\pi\ W$. Find the amount of energy received by plate in 5 secs.

Solution:

The energy emitted by the speaker in one second is $\pi\ J$. Let us consider a sphere of radius $2.0\ m$ centered at the speaker. The energy $\pi\ J$ falls normally on the total surface of this sphere in one second. The energy falling on the area $0.4\ cm^2$ of the plate in one second

$= \frac{0.4 \times 10^{-4}}{4\pi \times 2^2} \times \pi = 2.5 \times 10^{-6} J$

The energy falling on the microphone in 5.0 sec. is $2.5 \times 10^{-6} J \times 5 = 12.5\ \mu J$.

Illustration 52:

Find the displacement amplitude of particles of air of density $1.2\ kg/m^3$, if intensity, frequency and speed of sound are $8 \times 10^{-6}\ W/m^2$, $5000\ Hz$ and $330\ m/s$ respectively.

Solution:

The relation between the intensity of sound and the displacement amplitude is

$I = \frac{P_0^2}{2\rho v}$

$P_0 = \sqrt{2\rho v I}$

$$s_0 = \frac{P_0}{Bk} = \frac{\sqrt{2\rho v I}}{\rho v^2} \times \frac{v}{2\pi f} = \sqrt{\frac{I}{2\rho v}}\ \frac{1}{\pi f}$$

$s_0 = 6.4\ nm$

Pitch and Frequency

Frequency is an objective property measured in units of *Hz* and which can be assigned a unique value. However, a person's perception of frequency is subjective. The brain interprets frequency primarily in terms of a subjective quality called Pitch. A pure note of high frequency is interpreted as high-pitched sound and a pure note of low frequency as low-pitched sound.

Pitch of a sound is that sensation by which we differentiate a buffalo voice, a male voice and a female voice. Buffalo voice is of low pitch, a male voice has higher pitch and a female voice has much higher pitch. This sensation primarily depends on the dominant frequency present in the sound. Higher the frequency, higher will be the pitch and vice versa. The dominant frequency of a buffalo voice is smaller than that of a male voice which in turn is smaller than that of a female voice.

Loudness and Intensity

Audible intensity range for humans

The ability of human to perceive intensity at different frequency is different. The perception of intensity is maximum at 1000 *Hz* and perception of intensity decreases as the frequency decreases or increases from 1000 *Hz*.

- For a 1000 *Hz* tone, the smallest sound intensity that a human ear can detect is $10^{-12}\ W/m^2$. On the other hand, continuous exposure to intensities above $1 W/m^2$ can result in permanent hearing loss.
- The overall perception of intensity of sound to human ear is called **loudness**.
- Human ear do not perceives loudness on a linear intensity scale rather it perceives loudness on logarithmic intensity scale.

For example:

If intensity is increased 10 times human ear does not perceive 10 times increase in loudness. It roughly perceived that loudness is doubled where intensity increased by 10 times. Hence it is prudent to define a logarithmic scale for intensity.

Decibel scale:

The logarithmic scale which is used for comparing two sound intensity is called **decibel scale**.

The intensity level β described in terms of decibels is defined as $\beta = 10\ log\left(\frac{I}{I_0}\right)\ dB$

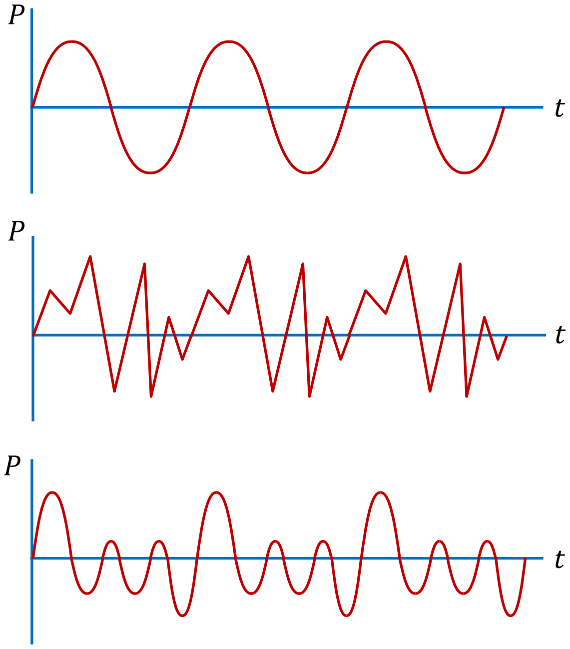
Here I_0 is the threshold intensity of hearing for human ear

i.e. $I_0 = 10^{-12}\ W/m^2$.

- In terms of decibel threshold of human hearing is 1 *dB*.
- Note that intensity level β is a dimensionless quantity and is not same as intensity expressed in W/m^2.

Quality and Waveform:

A sound generated by a source may contain a number of frequency components in it. Different frequency components have different amplitudes and superposition of them results in the actual waveform. The appearance of sound depends on this waveform apart from the dominant frequency and intensity. Figure shows waveforms for a tuning fork, a clarinet and a cornet playing the same note (fundamental frequency = 440 Hz) with equal loudness.



We differentiate between the sound from a tabla and that from a mridang by saying that they have different quality. A musical sound has certain well-defined frequencies which have considerable amplitude. These frequencies are generally harmonics of a fundamental frequency. Such a sound is particularly pleasant to the ear. On the other hand, a noise has frequencies that do not bear well-defined relationship among themselves.

Illustration 53:

If the intensity is increased by a factor of 20, by how many decibels is the intensity level increased.

Solution:

Let the initial intensity be I and the intensity level be β_1 and when the intensity is increased by 20 times, the intensity level increases to β_2.

Then $\quad \beta_1 = 10 \log (I / I_0)$

and $\quad \beta_2 = 10 \log (20I / I_0)$

Thus, $\quad \beta_2 - \beta_1 = 10 \log (20I / I) = 10 \log 20 = 13\ dB$

Illustration 54:

How many times the pressure amplitude is increased, if sound level is increased by 40 dB.

Solution:

The sound level in dB is

$$\beta = 10 \log_{10} \left(\frac{I}{I_0}\right)$$

If β_1 and β_2 are the sound levels and I_1 and I_2 are the intensities in the two cases,

$$\beta_2 - \beta_1 = 10 \left[\log_{10}\left(\frac{I_2}{I_0}\right) - \log_{10}\left(\frac{I_1}{I_0}\right)\right]$$

$$40 = 10 \log_{10} \left(\frac{I_2}{I_1}\right)$$

$$\frac{I_2}{I_1} = 10^4$$

As the intensity is proportional to the square of the pressure amplitude,

we have $\dfrac{P_{02}}{P_{01}} = \sqrt{\dfrac{I_2}{I_1}} = \sqrt{10000} = 100.$

Illustration 55:

Density and bulk modulus of medium is given $\rho = 1 kg/m^3$ and $B = 4 \times 10^4\ N/m^2$. If loudness at a point is $20dB$. Find p_0 at that point?

Solution:

$$20 = 10\ log_{10}\left(\frac{I}{I_0}\right)$$

$$10^2 = \frac{I}{10^{-12}}$$

$$I = 10^{-10}$$

As $$\langle I \rangle = \frac{1}{2} \times \frac{p_0^2}{\sqrt{B\rho}}$$

$$10^{-10} = \frac{1}{2} \times \frac{p_0^2}{\sqrt{1 \times 4 \times 10^4}}$$

$$10^{-10} = \frac{p_0^2}{4 \times 10^2}$$

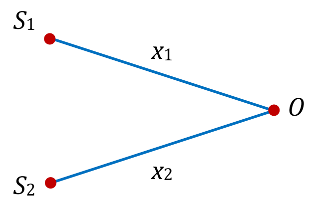
$$p_0 = 2 \times 10^{-4}$$

Interference of Sound Waves:

The combination of separate waves in the same region of space to produce a resultant wave is called Interference. To analyse such wave combination, we make use of **superposition principle**.

Superposition principle: If two or more travelling waves are moving through a medium, the resultant value of the wave function at any point is the algebraic sum of the values of the wave functions of the individual waves.

Suppose the two sources are vibrating. Also, let P_{01} and P_{02} be the amplitudes of the waves from S_1 and S_2 respectively. Let us examine the resultant change in pressure at a point O. The pressure changes at P due to the two waves are described by



If $\quad P_1 = P_{01} \sin(\omega t - kx_1 + \theta_1)$

and $\quad P_2 = P_{02} \sin(\omega t - kx_2 + \theta_2)$

Resultant excess pressure at point O is

$$P = P_1 + P_2$$

$\Rightarrow \quad P = P_0 \sin(\omega t - kx + \theta)$

where, $P_0 = \sqrt{P_{01}^2 + P_{02}^2 + 2P_{01}P_{02}\cos\Delta\phi}$ and $\Delta\phi = |\ k\ (x_1 - x_2) + (\theta_2 - \theta_1)\ |$ ...(1)

(i) For constructive interference

$$\Delta\phi = 2n\pi \quad \Rightarrow \quad P_0 = P_{01} + P_{02}$$

(ii) For destructive interference

$$\Delta\phi = (2n+1)\pi \Rightarrow P_0 = |\ P_{01} - P_{02}\ |$$

If $\Delta\phi$ is only due to path difference, then $\Delta\phi = \frac{2\pi}{\lambda}\Delta x$, and

Condition for constructive interference : $\Delta x = n\lambda$, $n = 0, \pm 1, \pm 2$

Condition for destructive interference : $\Delta x = (2n + 1)\frac{\lambda}{2}$, $n = 0, \pm 1, \pm 2$

from equation (1)

$$P_0^2 = P_{01}^2 + P_{02}^2 + 2P_{01}P_{02}\cos\Delta\phi$$

Since intensity $I \propto$ (Pressure amplitude)^2,

For resultant intensity, $\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\Delta\phi}$

If $I_1 = I_2 = I_0$

$I = 2I_0 \ (1 + \cos\Delta\phi)$

$$\boxed{I = 4I_0\cos^2\left(\frac{\Delta\phi}{2}\right)}$$

Hence in this case,

For constructive interference : $\Delta\phi = 0, 2\pi, 4\pi ....$ and $I_{max} = 4I_0$

And for destructive interference : $\Delta\phi = \pi, 3\pi ...$ and $I_{min} = 0$

Coherence : Two sources are said to be coherent if the phase difference between them does not change with time. In this case their resultant intensity at any point in space remains constant with time. Two independent sources of sound are generally incoherent in nature, i.e. phase difference between them changes with time and hence the resultant intensity due to them at any point in space changes with time.

Reflection of Sound Waves:

The two conditions under which reflection of sound takes place are :

(a) Discontinuity in the medium i.e, wave is continuous at the boundary. When a sound wave is reflected from a rigid boundary, the particles are not able to vibrate at the boundary. So, a reflected wave is generated. This wave interferes with the incident wave so that net displacement is zero giving rise to maximum variation in pressure. Hence phase of reflected wave remains same i.e, a compression pulse is reflected as compression pulse and vice versa.

(b) **Low pressure region:** When the sound wave travels in open tube, it gets reflected from the open end by which force is exerted on the particles due to outside air. This force is of small magnitude and so the particles vibrate with increased amplitude. So, pressure remains at average value by which the phase change of *p* occurs. Hence compression pulse is reflected as rarefaction pulse and vice versa.

(c) Echo is an example of reflection of sound from a distant object. If there is a sound reflector at a distance d from the source then time taken to hear the echo is $2d/v$. Where d is distance of object to source.

Note:

Multiple or successive echoes will be produced where there are two or more reflectors.

Illustration 56:

When two waves with same frequency and constant phase difference interfere-

(A) there is a gain of energy

(B) there is a loss of energy

(C) the energy is redistributed and the distribution changes with time.

(D) the energy is redistributed and the distribution remains constant with time.

Ans. (D)

Solution:

The phenomenon of interference is in accordance with law of conservation of energy i.e. energy can neither be created nor be destroyed, only transformation of one form to other takes place. So, energy is redistributed and the distribution remain constant with time.

Illustration 57:

If the ratio of the amplitude of two interfering beams be 2 : 3, the ratio of the minimum and maximum intensities of sound would be-

(A) 4 : 9 (B) $\sqrt{2}:\sqrt{3}$ (C) 1 : 5 (D) 1 : 25

Ans. (D)

Solution:

$$\frac{s_1}{s_2}=\frac{2}{3}$$

Let $s_1 = 2x$ and $s_2 = 3x$

$$\frac{I_{\min}}{I_{\max}}=\left(\frac{s_1-s_2}{s_1+s_2}\right)^2=\left(\frac{2x-3x}{2x+3x}\right)^2=\frac{1}{25}$$

Illustration 58:

Two waves of same frequency but of amplitudes a and $2a$ respectively superimpose over each other. The amplitude at a point where the phase difference is $\frac{3\pi}{2}$ will be-

(A) $9a$ (B) $\sqrt{3}a$ (C) a (D) $\sqrt{5}a$

Ans. (D)

Solution:

Here $s_1 = a$, $s_2 = 2a$ and $\Delta\phi = \frac{3\pi}{2}$

$$s_0 = \sqrt{s_{01}^2+s_{02}^2+2s_{01}s_{02}\cos\Delta\phi} = \sqrt{a^2+(2a)^2+2\times a\times 2a\times\cos\frac{3\pi}{2}}$$

$$= \sqrt{5}a$$

Illustration 59:

A man standing between two cliffs hears the first echo of a sound after 2 *sec* and second echo 3 sec. after the initial sound. If the speed of sound be 330 m/s the distance between two cliffs should be -

(A) 1650 m (B) 990 m (C) 825 m (D) 660 m

Ans. (C)

Solution:

Let a man M be at a distance x from c_1 cliff and y from c_2 cliff. So, time interval between original sound and echoes from c_1 and c_2 will be

$$t_1 = \frac{2x}{v} \text{ and } t_2 = \frac{2y}{v}$$

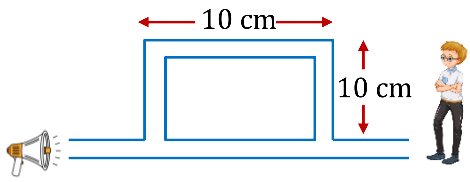
Distance between two cliffs

$$t_1 + t_2 = \frac{2(x+y)}{v}$$

$$(x+y) = \frac{v(t_1+t_2)}{2} = \frac{330\times(2+3)}{2} = 825m$$

Illustration 60:

Figure shows a tube having sound source at one end and observer at other end. Source produces frequencies upto 10000 Hz. Speed of sound is 400 m/s. Find the frequencies at which person hears maximum intensity.



Solution:

The sound wave bifurcates at the junction of the straight and the rectangular parts. The wave through the straight part travels a distance x_1 = 10 cm and the wave through the rectangular part travels a distance x_2 = 3 ×10 cm = 30 cm before they meet again and travel to the receiver. The path difference between the two waves received is, therefore.

$$\Delta x = x_2 - x_1 = 30\ cm - 10\ cm = 20\ cm$$

The wavelength of either wave is $\lambda = \frac{v}{f} = \frac{400}{f}$. For constructive interference, $\Delta x = n\lambda$, where n is an integer.

$$\Delta x = n.\frac{400}{f} \quad \Rightarrow \quad f = \frac{400n}{\Delta x}$$

$$\Rightarrow \quad f = \frac{400n}{0.2} = 2000\ n$$

Thus, the frequencies within the specified range which cause maximum of intensity are 2000 Hz and 8000Hz.

Illustration 61:

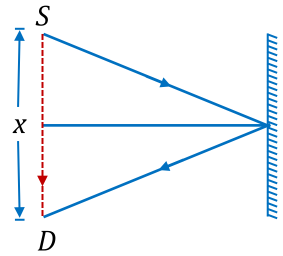
A source emitting sound of frequency 165 Hz is placed in front of a wall at a distance of 2 m from it. A detector is also placed in front of the wall at the same distance from it. Find the distance between the source and the detector for which the detector detects phase difference of 2π between the direct and reflected wave. Speed of sound in air = 330 m/s.

Solution:

The situation is shown in figure. Suppose the detector is placed at a distance of x meter from the source. The direct wave received from the source travels a distance of x meter.

The wave reaching the detector after reflection from the wall has travelled a distance of

$$\ell = 2\sqrt{2^2 + \left(\frac{x}{2}\right)^2} \text{ meter}$$



The path difference between the two waves is

$\Delta x = \ell - x$

$$\Delta x = 2\sqrt{2^2 + \left(\frac{x}{2}\right)^2} - x \quad \text{...(i)}$$

$\Delta x = \lambda$ for $\Delta\phi = 2\pi$

The wavelength is $\lambda = \frac{v}{f} = \frac{330 \;\; m/s}{165 \;\; s^{-1}} = 2 \; m.$

Thus, by (i) $2\sqrt{2^2 + \left(\frac{x}{2}\right)^2} - x = 2$

$$\sqrt{4 + \frac{x^2}{4}} = 1 + \frac{x}{2}$$

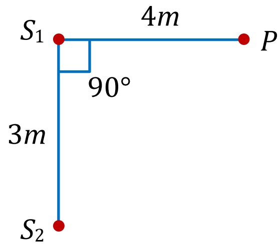
$$4 + \frac{x^2}{4} = 1 + \frac{x^2}{4} + x$$

$x = 3$

Thus, the detector should be placed at a distance of 3 m from the source. Note that there is no abrupt phase change.

Illustration 62:

S_1 and S_2 are two coherent sources of sound of frequency 110Hz each. They have no initial phase difference. The intensity at a point P due to S_1 is I_0 and due to S_2 is $4I_0$. If the velocity of sound is 330 m/s then the resultant intensity at P is



(A) I_0 (B) $9I_0$

(C) $3I_0$ (D) $8I_0$

Ans. (C)

Solution:

The wavelength of sound source = $\frac{330}{110}$ = 3 metre.

The phase difference between interfering waves at P is

$$= \Delta\phi = \frac{2\pi}{\lambda}(S_2P - S_1P)$$

$$= \frac{2\pi}{3}(5 - 4) = \frac{2\pi}{3}$$

$\therefore$ Resultant intensity at $P = I_0 + 4I_0 + 2\sqrt{I_0}\sqrt{4I_0}\cos\frac{2\pi}{3} = 3\,I_0$

Interference in time : Beats

When two sound waves of same amplitude travelling in same direction with different frequency superimpose, then intensity varies periodically with time. This effect is called Beats.

Consider two sound waves of frequency f_1 and f_2 propagating in the same direction so we have

$$y_1 = A \sin(2\pi f_1 t - kx)$$

$$y_2 = A \sin(2\pi f_2 t - kx) \text{ where } f_1 - f_2 = \Delta f$$

By principle of superposition

$$y = y_1 + y_2$$

$$= A[\sin(2\pi f_1 t - kx) + \sin(2\pi f_2 t - kx)]$$

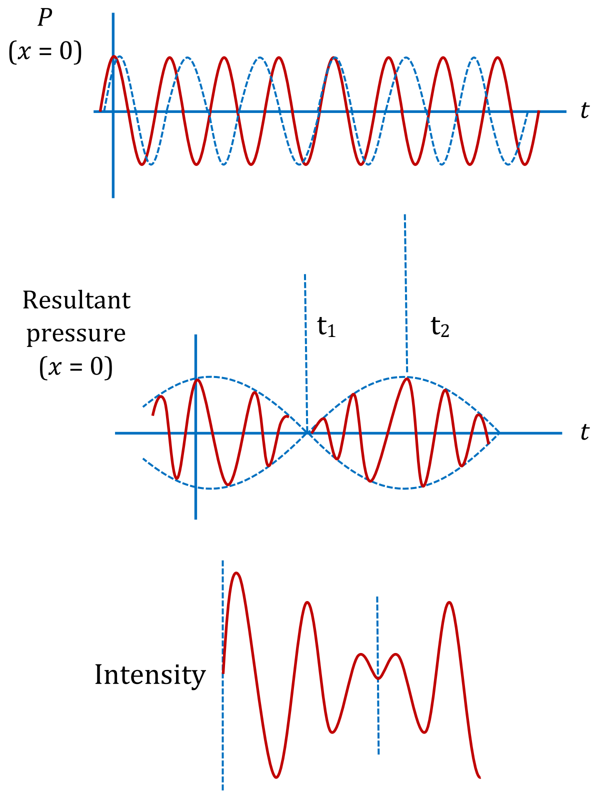
$$= A[\sin\{2\pi(f_2 + \Delta f)t - kx\} + \sin(2\pi f_2 t - kx)]$$

But $\quad \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$$y = [2A\cos(\Delta f \pi t)]\sin[2\pi f_2 t + \pi \Delta f t - kx]$$

Thus, frequency of variation of amplitude $= \frac{\Delta f}{2}$

$$= \frac{|f_1 - f_2|}{2}$$



We can explain the formation of beats graphically also. 1st figure shows the variation of pressure as a function of time at $x = 0$. At $t = 0$, two waves produce a change in pressure by which phase difference grows. At time t_1 the two pressure changes becomes out of phase and the resultant becomes zero. At time t_2 the phase difference increases and hence the formation of beats occur.

frequency of variation of intensity

$$\Delta f = |f_1 - f_2| = \text{Beat frequency}$$

Beat Time Period:

Time interval between two successive maxima or minima is called Beat time period (T).

Beat Frequency:

Number of beats per second is called Beat frequency. If frequency of parent waves are f_1 and f_2, then Beat frequency $= |f_1 - f_2|$

Beats time period $= \frac{1}{|f_1 - f_2|}$

Note:

- The frequency $|f_1 - f_2|$ should not be very large otherwise beats are not audible. It should be less than 16 Hertz.
- Determination of unknown frequency with the help of beats.

(a) If tuning fork of unknown frequency (f_2) oscillates with tuning fork of known frequency (f_1) producing f beats then $f_2 = f_1 \pm f$

(b) If wax is applied to arm of tuning fork of unknown frequency. Then its frequency decreases. Hence it oscillate with tuning fork of unknown frequency if

(i) Beat frequency increases then $f_2 = f_1 - f$ (ii) Beat frequency decreases then $f_2 = f_1 + f$

(c) If arm of tuning fork of unknown frequency is filed then its frequency increases. Hence it oscillates with tuning fork of known frequency if

(i) Beat frequency increases then $f_2 = f_1 + f$ (ii) Beat frequency decreases then $f_2 = f_1 - f$

(d) If n tuning fork are arranged in series that each tuning fork produces N beats with the previous one and frequency of last tuning fork is P times of first tuning fork then frequency of first tuning fork.

$$= \frac{(n-1)N}{P-1}$$

Illustration 63:

The phenomenon of beats can take place-

(A) for longitudinal waves only (B) for transverse waves only

(C) for both longitudinal and transverse waves (D) for sound waves only

Ans. (C)

Solution:

Beats is the phenomenon of periodic variation of intensity of sound. When two waves of different frequencies interfere it occurs for both longitudinal and transverse waves.

Illustration 64:

The speed of sound in a gas in which two waves of wavelength $1m$ and $1.01m$ produces 10 beats in 3 seconds is-

(A) 330 m/s (B) 33.6 m/s (C) 336.7 m/s (D) 3367 m/s

Ans. (C)

Solution:

If v is the speed of sound in the gas and f_1 and f_2 be the frequencies of the two waves. Then

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{1} \quad \text{and} \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{1.01}$$

Number of beats per second $f_1 - f_2 = \dfrac{10}{3}$

$$f_1 - f_2 = \frac{v}{1} - \frac{v}{1.01} = \frac{10}{3} \Rightarrow v = \frac{10}{3} \times \frac{1.01}{.01} = 336.7 m/s$$

Illustration 65:

16 tuning forks are arranged in the order of increasing frequency. Any two successive forks give 8 beats per second when sounded together. If the last fork give the octave of the first, then the frequency of the first fork will be -

(A) 120 (B) 60 (C) 40 (D) 240

Ans. (A)

Solution:

If frequency of first tuning fork is f then second will be $(f + 8)$ and third will be $(f + 16)$........ and 16th will be

$= f + (16 - 1)\,8$

$= f + 120$

But according to the question

$f + 120 = 2f$

$f = 120$

Illustration 66:

A tuning fork when vibrating along with a sonometer produces 6 beats per sec. When the length of wire is either 20 *cm* or 21 *cm* find the frequency of the tuning fork.

(A) 240 *Hz* (B) 246 *Hz* (C) 242 *Hz* (D) 248 *Hz*

Ans. (B)

Solution:

Let frequency of fork be f. It produces 6 beats per sec which implies that frequency of sonometer will be either $(f + 6)$ or $(f - 6)$. With increase in length frequency decreases so for 20 *cm* frequency will be $(f + 6)$ and for 21 *cm* frequency will be $(f - 6)$

But $\quad \nu$ = constant

$\therefore \quad f_1 \ell_1 = f_2 \ell_2$

$(f + 6) \times 20 = (f - 6) \times 21$

$f = 246\ Hz$

Illustration 67:

Two strings X and Y of a sitar produces a beat of frequency 4*Hz*. When the tension of string Y is slightly increased, the beat frequency is found to be 2*Hz*. If the frequency of X is 300*Hz*, then the original frequency of Y was.

(A) 296 *Hz* (B) 298 *Hz* (C) 302 *Hz* (D) 304 *Hz*.

Ans. (A)

Solution:

$|f - 300| = 4 \quad \Rightarrow \quad f = 296 \ \text{ or } \ 304$

when tension in string Y increases slightly then its frequency also increases

As beat frequency decreases

$\Rightarrow \quad f < 300 \quad \Rightarrow \quad f = 296\ Hz.$

Illustration 68:

A string 25 *cm* long fixed at both ends and having a mass of 2.5 *g* is under tension. A pipe closed from one end is 40 *cm* long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decreases the beat frequency. If the speed of sound in air is 320 *m/s*. Find tension in the string.

Solution:

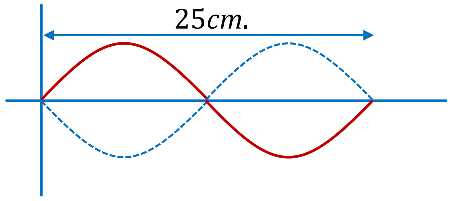
$$\mu = \frac{2.5}{25} = 0.1\ g/cm = 10^{-2}\ Kg/m$$

$$\lambda_s = 25\ cm = 0.25\ m$$

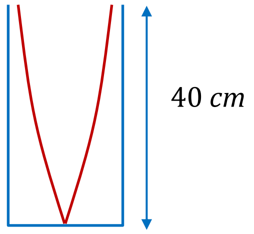
$$\Rightarrow \quad f_s = \frac{1}{\lambda_s}\sqrt{\frac{T}{\mu}}$$

$$\lambda_p = 160\ cm = 1.6\ m$$

$$\Rightarrow \quad f_p = \frac{v}{\lambda_p}$$



string in 1st overtone



pipe in fundamental mode

By decreasing the tension, beat frequency is decreased

$\therefore \quad f_s > f_p$

$$\Rightarrow \quad f_s - f_p = 8 \Rightarrow \frac{1}{0.25}\sqrt{\frac{T}{10^{-2}}} - \frac{320}{1.6} = 8$$

$$\Rightarrow \quad T = 27.04\ N$$

Illustration 69:

The wavelength of two sound waves are $49cm$ and $50cm$ respectively. If the room temperature is $30\ °C$ then the number of beats produced by them is approximately (velocity of sound in air at $0°C = 332\ m/s$).

(A) 6 (B) 10 (C) 14 (D) 18

Ans. (C)

Solution:

$$v = 332\sqrt{\frac{303}{273}}$$

$$\Rightarrow \quad \text{Beat frequency} = f_1 - f_2 = v\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$$

$$= 332\sqrt{\frac{303}{273}}\left(\frac{1}{49} - \frac{1}{50}\right) \times 100 \cong 14$$

Longitudinal Standing Waves:

Two longitudinal waves of same frequency and amplitude travelling in opposite directions interfere to produce a standing wave.

If the two interfering waves are given by

$$P_1 = P_0 \sin(\omega t - kx)$$

and $\quad P_2 = P_0 \sin(\omega t + kx + \phi)$

then the equation of the resultant standing wave would be given by

$$P = P_1 + P_2 = 2P_0 \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right)$$

$$\Rightarrow \quad P = P'_0 \sin\left(\omega t + \frac{\phi}{2}\right)$$

This is equation of SHM in which the amplitude P'_0 depends on position as

$$P'_0 = 2P_0 \cos\left(kx + \frac{\phi}{2}\right)$$

Points where pressure remains permanently at its average value; i.e. pressure amplitude is zero is called a pressure node, and the condition for a pressure node would be given by

$$P'_0 = 0$$

i.e. $\cos\left(kx + \frac{\phi}{2}\right) = 0$

i.e. $kx + \frac{\phi}{2} = 2n\pi \pm \frac{\pi}{2}$, $\quad n = 0, 1, 2.....$

Similarly points where pressure amplitude is maximum is called a pressure antinode and condition for a pressure antinode would be given by

$$P'_0 = \pm\, 2P_0$$

i.e. $\cos\left(kx + \frac{\phi}{2}\right) = \pm 1$

or $\left(kx + \frac{\phi}{2}\right) = n\pi$, $\quad n = 0, 1, 2.....$

If the sound waves were represented in terms of displacement waves, then the equation of standing wave corresponding to the above pressure equation would be

$$s = s'_0 \cos\left(\omega t + \frac{\phi}{2}\right) \qquad \text{where } s'_0 = 2s_0 \sin\left(kx + \frac{\phi}{2}\right)$$

This can be easily observed to be an equation of SHM. It represents the medium particles moving simple harmonically about their mean position at x.

Note:

A pressure node in a standing wave would correspond to a displacement antinode. Similarly, a pressure anti-node would correspond to a displacement node.

Vibration of Air Columns:

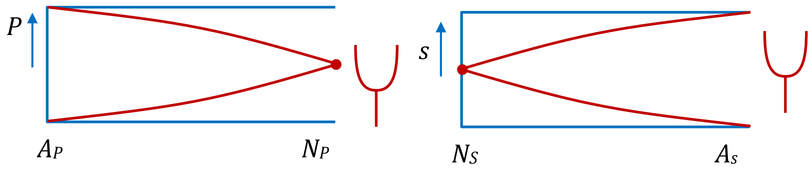
Standing waves can be set up in air-columns trapped inside cylindrical tubes if frequency of the tuning fork sounding the air column matches one of the natural frequency of air columns. In such a case the sound of the tuning fork becomes markedly louder, and we say there is resonance between the tuning fork and air-column. To determine the natural frequency of the air-column, notice that there is a displacement node (pressure antinode) at each closed end of the tube as air molecules there are not free to move, and a displacement antinode (pressure-node) at each open end of the air-column.

In reality antinodes do not occurs exactly at the open end but a little distance outside. However, if diameter of tube is small compared to its length, this end correction can be neglected.

Closed Organ Pipe

In the diagram, A_p = Pressure antinode, A_s = displacement antinode, N_p = pressure node, N_s = displacement node.

Fundamental mode:



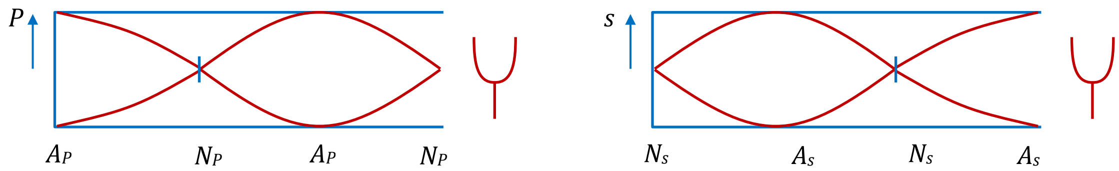
The smallest frequency (largest wavelength) that satisfies the boundary condition for resonance (i.e. displacement node at left end and antinode at right end is $\lambda_0 = 4\ell$, where ℓ = length of closed pipe the corresponding frequency.

$$\frac{\lambda_0}{4} = \ell \Rightarrow \lambda_0 = 4\ell$$

$f_0 = \dfrac{v}{\lambda_0} = \dfrac{v}{4\ell}$ is called the fundamental frequency.

First Overtone : Here there is one node and one antinode apart from the nodes and antinodes at the ends.

$$\frac{3\lambda}{4} = \ell \Rightarrow \lambda = \frac{4\ell}{3}$$



$$\lambda_1 = \frac{4\ell}{3} = \frac{\lambda_0}{3}$$

and corresponding frequency,

$$f_1 = \frac{v}{\lambda_1} = 3f_0$$

This frequency is 3 times the fundamental frequency and hence is called the 3rd harmonic.

nth overtone:

In general, the nth overtone will have n nodes and n antinodes between the two ends. The corresponding wavelength is

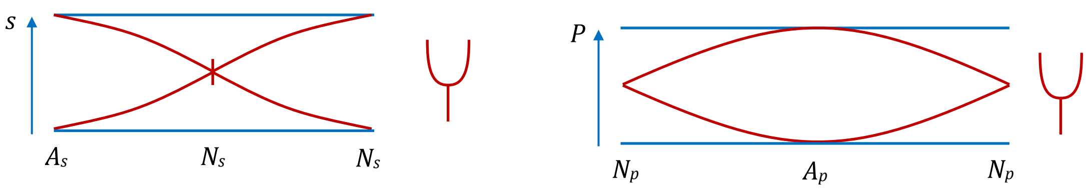
$$(2n + 1)\frac{\lambda}{4} = \ell$$

$$\lambda_n = \frac{4\ell}{2n+1} = \frac{\lambda_0}{2n+1} \qquad \text{and} \qquad f_n = (2n + 1)f_0$$

This corresponds to the $(2n + 1)^{th}$ harmonic. Clearly only odd harmonic are allowed in a closed pipe.

Open Organ Pipe:

Fundamental mode :



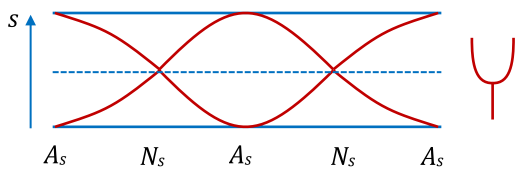
The smallest frequency (largest wave length) that satisfies the boundary condition for resonance (i.e. displacement antinodes at both ends) is,

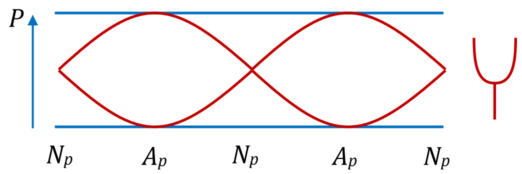
$$\lambda_0 = 2\ell$$

corresponding frequency, is called the fundamental frequency

$$f_0 = \frac{v}{2\ell}$$

1st Overtone : Here there is one displacement antinode between the two antinodes at the ends.





$$\lambda_1 = \frac{2\ell}{2} \qquad \Rightarrow \lambda_1 = \frac{\lambda_0}{2}$$

and, corresponding frequency

$$f_1 = \frac{v}{\lambda_1} = 2f_0$$

This frequency is 2 times the fundamental frequency and is called the 2*nd* harmonic.

nth overtone : The nth overtone has n displacement antinodes between the two antinode at the ends.

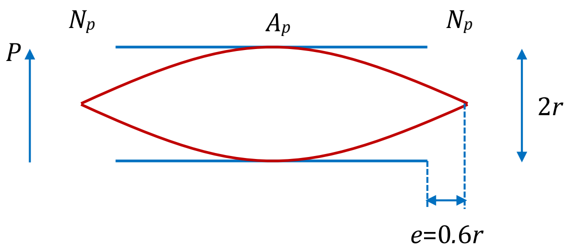
$$(n+1)\,\frac{\lambda}{2} = \ell$$

$$\lambda_n = \frac{2\ell}{n+1} = \frac{\lambda_0}{n+1} \qquad \text{and} \qquad f_n = (n+1)\,f_0$$

This correspond to $(n + 1)^{th}$ harmonic. Clearly both even and odd harmonics are allowed in an open pipe.

End correction:

As mentioned earlier the displacement antinode at an open end of an organ pipe lies slightly outside the open end. The distance of the antinode from the open end is called end correction and its value is given by $e = 0.6\,r$.



where r = radius of the organ pipe.

Effective length of closed organ pipe is $\ell' = \ell + e$

and effective length of open organ pipe is $\ell' = \ell + 2e$

with end correction, the fundamental frequency of a closed pipe (f_c) and an open organ pipe (f_o) will be given by

$$f_c = \frac{v}{4(\ell + 0.6\,r)} \qquad \text{and} \qquad f_o = \frac{v}{2(\ell + 1.2\,r)}$$

Illustration 70:

Fundamental frequency of an organ pipe filled with N_2 is 1000 Hz. Find the fundamental frequency if N_2 is replaced by H_2.

Solution:

Suppose the speed of sound in hydrogen is v_H and that in nitrogen is v_N. The fundamental frequency of an organ pipe is proportional to the speed of sound in the gas contained in it. If the fundamental frequency with hydrogen in the tube is f, we have

$$\frac{f}{1000\ Hz} = \frac{v_H}{v_N} = \sqrt{\frac{M_N}{M_H}}$$ (Since both N_2 and H_2 are diatomic, γ is same for both)

or, $$\frac{f}{1\ \ kHz} = \sqrt{\frac{28}{2}} \Rightarrow f = 1000\sqrt{14}\ Hz$$

Illustration 71:

A tube open at only one end is cut into two tubes of non-equal lengths. The piece open at both ends has fundamental frequency of 450 Hz and other has fundamental frequency of 675 Hz. What is the 1^{st} overtone frequency of the original tube.

Solution:

$$450 = \frac{v}{2\ell_1} \quad ; \quad 675 = \frac{v}{4\ell_2}$$

Length of original tube = $(\ell_1 + \ell_2)$

Its fundamental frequency,

$$f_1 = \frac{v}{4(\ell_1 + \ell_2)} = \frac{v}{4\left(\frac{v}{900} + \frac{v}{675 \times 4}\right)}$$

$$= \frac{(2700 \times 900)}{4(2700 + 900)} = 168.75$$

I^{st} overtone frequency = $3f_1$ = 506.25

Illustration 72:

The range of audible frequency for humans is 20 Hz to 20,000 Hz. If speed of sound in air is 336 m/s. What can be the maximum and minimum length of a musical instrument, based on resonance pipe.

Solution:

For an open pipe, $f = \frac{v}{2\ell} n$

$$\Rightarrow \qquad \ell = \frac{v}{2f}.n$$

$$\ell_{\min} = \frac{336}{2 \times 20{,}000} = 8.4\,mm$$

$$\ell_{max} = \frac{v}{2f_{\min}}\, n_{max} = \frac{v}{2 \times 20}\, n_{max} = 8.4 \times n_{max}$$

Similarly for a closed pipe,

$$\ell = \frac{v}{4f}(2n+1)$$

$$\ell_{min} = \frac{v}{4f_{max}}(2n+1)_{min} = \frac{336}{4\times 20000} = 4.2\ mm$$

$$\ell_{max} = \frac{v}{4f_{min}}(2n+1)_{max} = \frac{v}{2\times 20}(2n+1)_{max}$$

$$= 8.4 \times (2n+1)_{max}$$

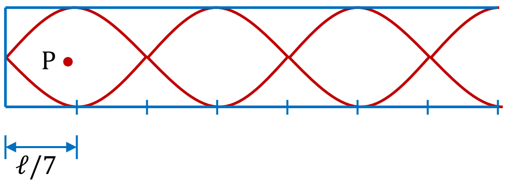
Clearly there is no upper limit on the length of such a musical instrument.

Illustration 73:

A closed organ pipe has length 'ℓ'. The air in it is vibrating in 3rd overtone with maximum amplitude 'A'. Find the amplitude at a distance of $\ell/7$ from closed end of the pipe.

Solution:

The figure shows variation of displacement of particles in a closed organ pipe for 3rd overtone.



For third overtone $\ell = \frac{7\lambda}{4}$ or $\lambda = \frac{4\ell}{7}$

$\Rightarrow \quad \frac{\lambda}{4} = \frac{\ell}{7}$

Hence the amplitude at P at a distance $\frac{\ell}{7}$ from closed end is 'A' because there is an antinode at that point.

Illustration 74:

When a closed pipe is suddenly opened then the second overtone of closed pipe and first overtone of open pipe differ by 100 Hz. The fundamental frequency of closed pipe will be -

(A) 200 Hz (B) 100 Hz (C) 300 Hz (D) 400 Hz

Ans. (B)

Solution:

Second overtone of closed pipe = $\frac{5v}{4L}$

First overtone of open pipe = $\frac{v}{L}$

But $\quad \frac{5v}{4L} - \frac{v}{L} = 100$

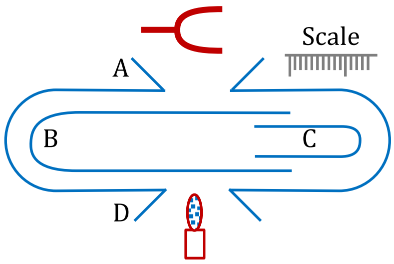
$$\frac{v}{4L} = 100$$

$$f = 100\ Hz$$

Apparatus for determining speed of sound

Quinck's tube:

It consists of two *U* shaped metal tubes. Sound waves with the help of tuning fork are produced at *A* which travel through *B* & *C* and comes out at *D* where a sensitive flame is present. Now the two waves coming through different paths interfere and flame flares up. But if they are not in phase, destructive interference occurs and flame remains undisturbed.

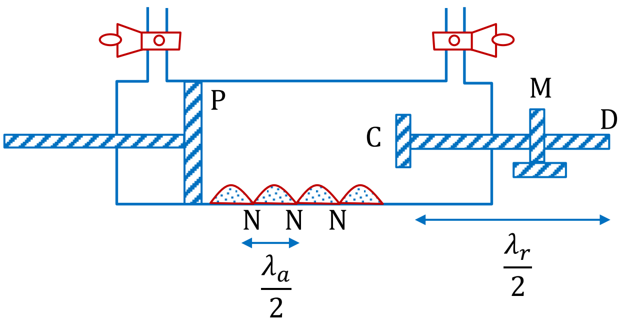


Suppose destructive interference occurs at *D* for some position of *C*. If now tube *C* is moved so that interference condition is disturbed and again by moving a distance *x*, destructive interference occurs so that $2x = \lambda$ similarly if the distance moved between successive constructive and destructive interference is *y* then $2y = \frac{\lambda}{2}$ Now by having value of *x* or *y*, speed of sound is given by

$$v = f\lambda$$

Kundt's Tube:

It is a tube used to determine speed of sound in different gases. It consists of a glass tube in which a small quantity of lycopodium powder is spread. The tube is rotated so that powder starts slipping. Now rod *CD* is rubbed at end *D* so that stationary waves form. The disc *C* vibrates so that air column also vibrates with the frequency of the rod. The piston *P* is adjusted so that frequency of air column become same as that of rod. So, resonance occurs and column is thrown into stationary waves. The powder sets into oscillations at antinodes while heaps of powder are formed at nodes.



For rod $\ell_r = \frac{\lambda_r}{2}$

$$\lambda_r = 2\ell_r$$

and $v_r = f\lambda_r = f(2\ell_r)$

For air $\ell_a = \frac{\lambda_a}{2}$

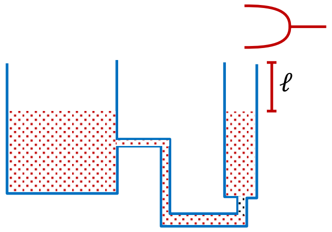
$$\lambda_a = 2\ell_a$$

and $v_a = f\lambda_a = f.(2\ell_a)$

so $\frac{v_r}{v_a} = \frac{\ell_r}{\ell_a}$

$$v_r = v_a \cdot \frac{\ell_r}{\ell_a}$$

Resonance Tube:



It is a closed organ pipe having air column of variable length. If the frequency of a tuning fork is same as the frequency of vibration of air column in the tube then intensity of sound becomes maximum. This is said to be the condition of resonance. The frequency is given by

$$f = \frac{(2n+1)v}{4\ell} \quad \text{or} \quad \ell = \frac{(2n+1)v}{4f}$$

Here frequency is constant and length is variable. When water level in resonance tube is lowered, it gives resonance with same frequency at different length.

On neglecting the end correction, if ℓ is the minimum length of pipe for resonance, then other length for resonance are $3\ell, 5\ell, 7\ell, \ldots\ldots\ldots$

If we include the end correction, if ℓ is minimum length of pipe for resonance and x is end correction. Then other lengths of pipe for resonance are

$$(\ell_1 + x) : (\ell_2 + x) : (\ell_3 + x)\ldots\ldots\ldots = 1 : 3 : 5\ldots\ldots$$

Doppler's Effect

When there is relative motion between the source of a sound/light wave & an observer along the line joining them, the actual frequency observed is different from the frequency of the source. This phenomenon is called Doppler's Effect. If the observer and source are moving towards each other, the observed frequency is greater than the frequency of the source. If the observer and source move away from each other, the observed frequency is less than the frequency of source.

(v = velocity of sound w.r.t. ground, c = velocity of sound with respect to medium, v_m = velocity of medium, v_0 = velocity of observer, v_s = velocity of source.)

(a) Sound source is moving and observer is stationary :

If the source emitting a sound of frequency f is travelling with velocity v_s along the line joining the source and observer,

observed frequency, $f' = f\left(\frac{v}{v - v_s}\right)$

and Apparent wavelength $\lambda' = \lambda\left(\frac{v - v_s}{v}\right)$

- In the above expression, the positive direction is taken along the velocity of sound, i.e. from source to observer. Hence v_s is positive if source is moving towards the observer, and negative if source is moving away from the observer.

(b) Sound source is stationary and observer is moving with velocity v_o along the line joining them :

The source (at rest) is emitting a sound of frequency 'f' travelling with velocity v so that wavelength is $\lambda = v/f$, i.e. there is no change in wavelength. However since the observer is moving with a velocity v_0 along the line joining the source and observer, the observed frequency is

$$f' = f\left(\frac{v+v_0}{v}\right)$$

- In the above expression, the positive direction is taken along the velocity of sound, i.e. from source to observer. Hence v_o is positive if observer is moving towards the source, and negative if observer is moving away from the source.

(c) The source and observer both are moving with velocities v_s and v_o along the line joining them :

The observed frequency, $\quad f' = f\left(\frac{v+v_0}{v-v_s}\right)$

and Apparent wavelength $\quad \lambda' = \lambda\left(\frac{v-v_s}{v+v_0}\right)$

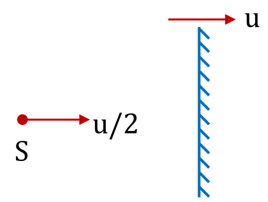
- In the above expression also, the positive direction is taken along the velocity of sound, i.e. from source to observer. Hence v_o is positive if observer is moving towards the source, and negative if observer is moving away from the source.

  Hence v_s is positive if source is moving towards the observer, and negative if source is moving away from the observer.
- In all of the above expression, v stands for velocity of sound with respect to ground.

  If velocity of sound with respect to medium is c and the medium is moving in the direction of sound from source to observer with speed v_m, $v = c + v_m$ and if the medium is moving opposite to the direction of sound from observer to source with speed v_m, $v = c - v_m$

Illustration 75:

A wall is moving with velocity u and a source of sound moves with velocity $u/2$ in the same direction as shown in figure. Assuming that the sound travels with velocity $10u$. The ratio of incident sound wavelength on the wall to the reflected sound wavelength by the wall, is equal to (assume observer for reflected sound is at rest) :



(A) 9 : 11 (B) 11 : 19 (C) 4 : 5 (D) 5 : 4

Ans. (A)

Solution:

Frequency received by the wall

$$f_1 = \frac{10u - u}{10u - u/2} f = \frac{9u}{9.5u} f$$

$$\Rightarrow \qquad \lambda_1 = \frac{9.5u}{f}$$

Frequency reflected by the wall

$$f_2 = \frac{10u}{10u + u} f_1 = \frac{10u}{11u} \times \frac{9u}{9.5u} f$$

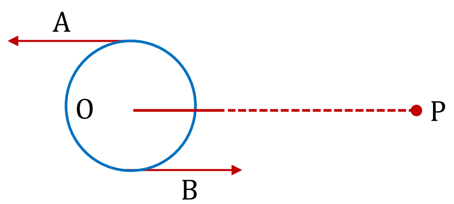
$$\Rightarrow \qquad \lambda_2 = \frac{11u \times 9.5}{9f} \quad = \frac{\lambda_1}{\lambda_2} = \frac{9.5}{11 \times 9.5} \times 9 = \frac{9}{11}$$

Illustration 76:

A whistle of frequency 540 *Hz* is moving in a circle of radius 2 *ft* at a constant angular speed of 15 *rad/s*. What are the lowest and highest frequencies heard by a listener standing at rest, a long distance away from the centre of the circle? (velocity of sound in air is 1100 *ft/sec.*)

Solution:

The whistle is moving along a circular path with constant angular velocity ω. The linear velocity of the whistle is given by



$v_s = \omega R$ where, R is radius of the circle.

At points A and B, the velocity v_s of whistle is parallel to line OP; i.e., with respect to observer at P, whistle has maximum velocity v_s away from P at point A, and towards P at point B. (Since distance OP is large compared to radius R, whistle may be assumed to be moving along line OP).

Observer, therefore, listens maximum frequency when source is at B moving towards observer:

$$f_{max} = f\left(\frac{v}{v - v_s}\right)$$

where, v is speed of sound in air. Similarly, observer listens minimum frequency when source is at A, moving away from observer:

$$f_{\min} = f\left(\frac{v}{v + v_s}\right)$$

For $f = 540$ *Hz*, $v_s = 2\,ft \times 15\,rad/s = 30\,ft/s$, and $v = 1100\,ft/s$,

We get (approx.)

$f_{max} = 555$ *Hz*

and $\quad f_{min} = 525$ *Hz.*

Illustration 77:

A train approaching a hill at a speed of 40 *km/hr* sounds a whistle of frequency 600 *Hz* when it is at a distance of 1 *km* from a hill. A wind with a speed of 40 *km/hr* is blowing in the direction of motion of the train. Find,

(A) the frequency of the whistle as heard by an observer on the hill.

(B) the distance from the hill at which the echo from the hill is heard by the driver and its frequency. (Velocity of sound in air = 1200 *km/hr*.)

Solution:

A train is moving towards a hill with speed v_s with respect to the ground. The speed of sound in air, i.e. the speed of sound with respect to medium (air) is c, while air itself is blowing towards hill with velocity v_m (as observed from ground). For an observer standing on the ground, which is the inertial frame, the speed of sound towards hill is given by

$$v = c + v_m$$

(A) The observer on the hill is stationary while source is approaching him. Hence, frequency of whistle heard by him is

$$f' = f\left(\frac{v}{v - v_s}\right)$$

for f = 600 *Hz*, v_s = 40 *km/hr*, and V = (1200 + 40) *km/hr*, we get

$$f' = 600\left(\frac{1240}{1240 - 40}\right) = 620\ Hz.$$

(B) The train sounds the whistle when it is at distance x from the hill. Sound moving with velocity v with respect to ground, takes time t to reach the hill, such that,

$$t = \frac{x}{v} = \frac{x}{c + v_m} \qquad \text{...(i)}$$

After reflection from hill, sound waves move backwards, towards the train. The sound is now moving opposite to the wind direction. Hence, its velocity with respect to the ground is

$$v' = c - v_m$$

Suppose when this reflected sound (or echo) reaches the train, it is at distance x' from hill. The time taken by echo to travel distance x' is given by

$$t' = \frac{x'}{v} = \frac{x'}{c - v_m} \qquad \text{...(ii)}$$

Thus, total time $(t + t')$ elapses between sounding the whistle and echo reaching back. In the same time, the train moves a distance $(x - x')$ with constant speed v_s, as observed from ground. That is,

$$x - x' = (t + t')\, v_s$$

Substituting from (i) and (ii), for t and t', we find

$$x - x' = \frac{v_s}{c + v_m}x + \frac{v_s}{c - v_m}x'$$

or $$\frac{c + v_m - v_s}{c + v_m}\,x = \frac{v_s + c - v_m}{c - v_m}\,x'$$

For $x = 1\ km$, $c = 1200\ km/hr$, $v_s = 40\ km/hr$, and $v_m = 40\ km/hr$, we get

$$\frac{1200+40-40}{1200+40}\times 1 = \frac{40+1200-40}{1200-40}\ x'$$

or, $$x' = \frac{1160}{1240} = 0.935\ km$$

Thus, the echo is heard when train is 935 m from the hill.

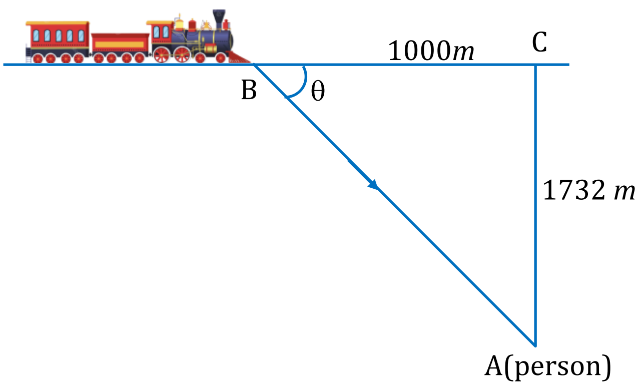
Now, for the observer moving along with train, echo is a sound produced by a stationary source, i.e., the hill. Hence as observed from ground, source is stationary and observer is moving towards source with speed 40 km/hr. Hence $v_o = -40\ km/hr$. On the other hand, reflected sound travels opposite to wind velocity. That is, velocity of echo with respect to ground is v'. Further, the source (hill) is emitting sound of frequency f'' which is the frequency observed by the hill.

Thus, frequency of echo as heard by observer on train, is given by

$$f''' = f''\left(\frac{v'+v_0}{v'}\right) \Rightarrow f''' = \frac{(1160-(-40))}{1160} \times 620 = 641\ Hz$$

Illustration 78:

A train producing frequency of 640 Hz is moving towards point C with speed 72 km/hr. A person is sitting 1732 m from point C as shown. Find the frequency heard by person when sound generated at B reaches to the person, if speed of sound is 330 m/s.



Solution:

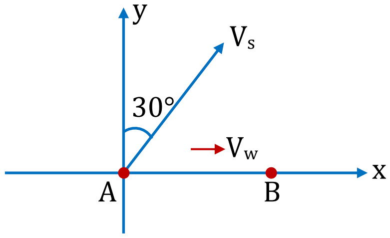
The observer A is at rest with respect to the air and the source is travelling at a velocity of 72 km/h i.e., 20 m/s.

As it is clear from the figure, the person receives the sound of the whistle in a direction BA making an angle θ with the track where $\tan\theta = \frac{1732}{1000} = \sqrt{3}$, i.e. $\theta = 60°$. The component of the velocity of the source (i.e., of the train) along this direction is $20\cos\theta = 10\ m/s$. As the source is approaching the person with this component, the frequency heard by the observer is

$$u' = \frac{v}{v-u\cos\theta}\upsilon = \frac{330}{330-10} \times 640\ Hz = 660\ Hz.$$

Illustration 79:

In the figure shown a source of sound of frequency 510 *Hz* moves with constant velocity $v_s = 20\ m/s$ in the direction shown. The wind is blowing at a constant velocity $v_w = 20\ m/s$ towards an observer who is at rest at point *B*. The frequency detected by the observer corresponding to the sound emitted by the source at initial position *A*, will be (speed of sound relative to air = 330 m/s)



(A) 485 *Hz* (B) 500 *Hz* (C) 512 *Hz* (D) 525 *Hz*

Ans. (D)

Solution:

Apparent frequency

$$f' = f\frac{(u+v_w)}{(u+v_w-v_s\cos 60°)} = \frac{510\ (330+20)}{330+20-20\ \cos 60°} = 510 \times \frac{350}{340} = 525\ Hz$$