02 Rotational Motion

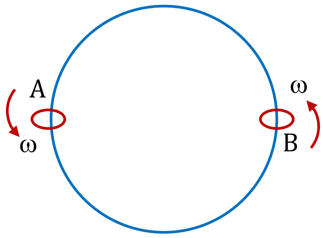
Rigid Body

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time). Remember, rigid body is a mathematical concept and any system which satisfies the above condition is said to be rigid as long as it satisfies it.

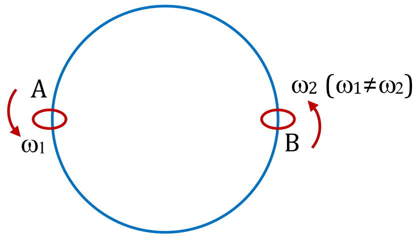
System behaves as a rigid body

System behaves as a non-rigid body

A and *B* are beads which move on a circular fixed ring



A+B is a rigid body system
But A+B+ ring is non-rigid system

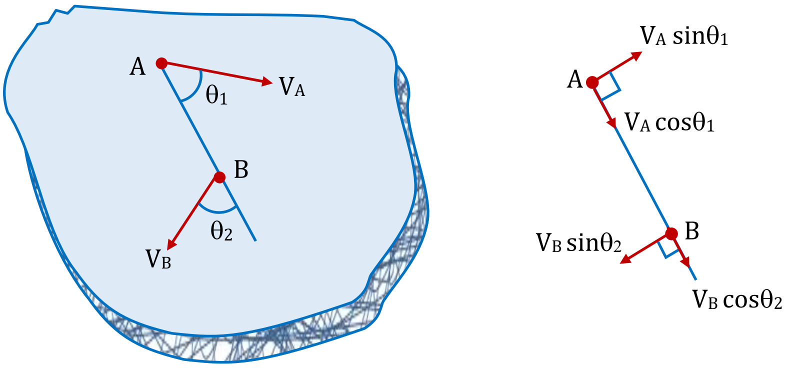


A + B is non-rigid system

- If a system is rigid, since there is no change in the distance between any pair of particles of the system, shape and size of system remains constant. Hence, we intuitively feel that while a stone or cricket ball are rigid bodies, a balloon or elastic string is non-rigid.

  But any of the above system is rigid as long as relative distance does not change, whether it is a cricket ball or a balloon. But at the moment when the bat hits the cricket ball or if the balloon is squeezed, relative distance changes and now the system behaves like a non-rigid system.
- For every pair of particles in a rigid body, there is no velocity of separation or approach between the particles. i.e. any relative motion of a point *B* on a rigid body with respect to another point *A* on the rigid body will be perpendicular to line joining *A* to *B*, hence with respect to any particle *A* of a rigid body the motion of any other particle *B* of that rigid body is circular motion.

Let velocities of A and B with respect ground be $\vec{V}_A$ and $\vec{V}_B$ respectively in the figure below.



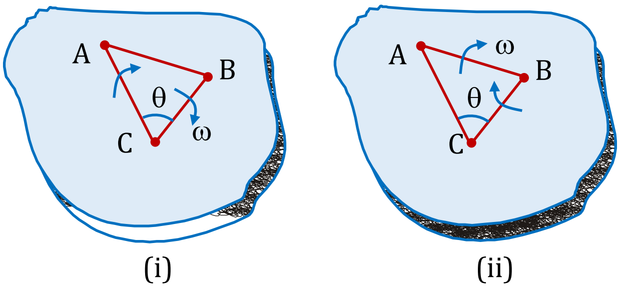
If the above body is rigid $V_A \cos \theta_1 = V_B \cos \theta_2$ (velocity of approach / separation is zero)

V_{BA} = relative velocity of B with respect to A.

$V_{BA} = V_A \sin \theta_1 + V_B \sin \theta_2$ (which is perpendicular to line AB)

B will appear to move in a circle to an observer fixed at A.

- W.r.t. any point of the rigid body the angular velocity of all other points of that rigid body is same.
- Suppose A, B, C is a rigid system hence during any motion sides AB, BC and CA must rotate through the same angle. Hence all the sides rotate by the same rate.

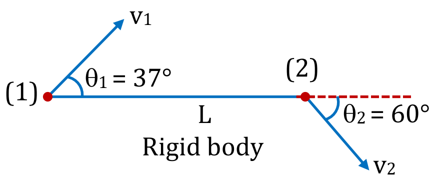


From figure (i) angular velocity of A and B w.r.t. C is ω.

From figure (ii) angular velocity of A and C w.r.t. B is ω.

Illustration 1:

For a rigid body system, the velocity vectors of particles 1 and 2 are represented by $\vec{V}_1$ and $\vec{V}_2$ respectively in the given figure. Where $V_1 = 10 m/s$ and $L = 2\ m$.



Then find:

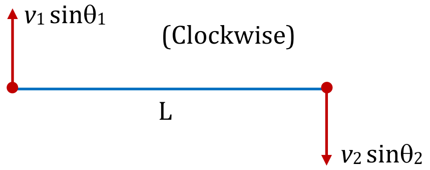
(a) Relation between v_1, v_2, θ_1, θ_2 ?

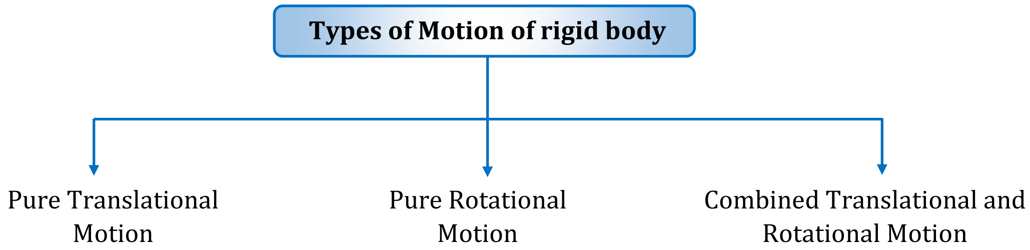
(b) Find angular velocity ω of rod ?

Solution:

(a) $v_1 \cos \theta_1 = v_2 \cos \theta_2$

(b) $\omega = \frac{(v_\perp)_{rel.}}{R} \Rightarrow \frac{v_1 \sin\theta_1 + v_2 \sin\theta_2}{L}$





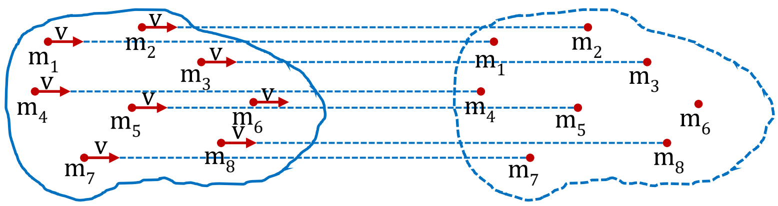
(I) Pure Translational Motion:

A body is said to be in pure translational motion, if the displacement of each particle of the system is same during any time interval. During such a motion, all the particles have same displacement $(\vec{s})$, velocity $(\vec{v})$ and acceleration $(\vec{a})$ at an instant.

Consider a system of n particle of mass m_1, m_2, m_3, ... m_n under going pure translation then from above definition of translational motion

$$\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \cdots \vec{a}_n = \vec{a} \text{ (say)}$$

and $\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = ...\vec{v}_n = \vec{v}$ (say)



From Newton's laws for a system.

$$\vec{F}_{ext} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots$$

$$\vec{F}_{ext} = M\vec{a}$$

Where M = Total mass of the body

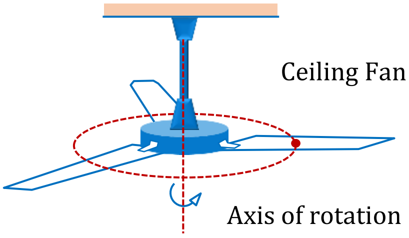
$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots$$

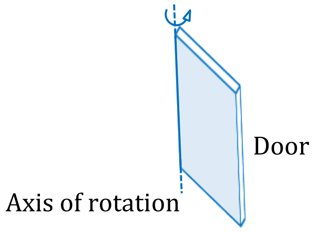
$$\vec{P} = M\vec{v}$$

Total Kinetic Energy of body = $\frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2 + \text{.................} = \frac{1}{2} Mv^2$

Rotation about a fixed axis (Pure rotation):

Rotation of ceiling fan, opening and closing of doors and needles of a wall clock etc. come into this category.

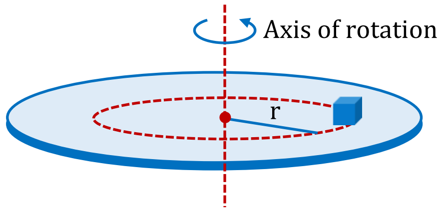




A door rotates about a vertical line that passes through its hinges. This vertical line is the axis of rotation. In the figure, the axis of rotation is shown by dashed line.

Axis of rotation:

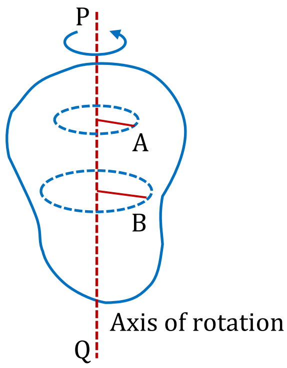
An imaginary line perpendicular to plane of circular paths of particles of a rigid body in rotation and containing the centers of all these circular paths is known as axis of rotation.



It is not necessary that the axis of rotation pass through the body. Consider system shown in the figure, where a block is fixed on a rotating disc. The axis of rotation passes through the center of the disc but not through the block.

Important observations :

Let us consider a rigid body of arbitrary shape rotating about a fixed axis *PQ* passing through the body. Two of its particles *A* and *B* shown are moving on their circular paths.



All of its particles, not on the axis of rotation, move on circular paths with centers on the axis or rotation. All these circular paths are in parallel planes that are perpendicular to the axis of rotation.

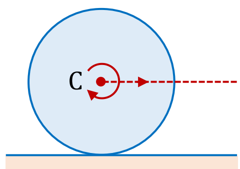
All the particles of the body cover same angular displacement in the same time interval, therefore all of them move with the same angular velocity and angular acceleration.

Particles moving on circular paths of different radii move with different speeds and different magnitudes of linear acceleration. Furthermore, no two particles in the same plane perpendicular to the axis of rotation have same velocity and acceleration vectors.

Rotation about an axis in translation :

Rotation about an axis in translation includes a broad category of motions. Rolling is an example of this kind of motion. A rod lying on table when pushed from its one end in its perpendicular direction also executes this kind of motion.

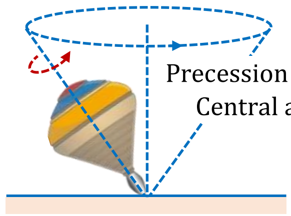
Consider rolling of wheels of a vehicle, moving on straight level road.



Relative to a reference frame, moving with the vehicle, wheel appears rotating about its stationary axel. The rotation of the wheel from this frame is rotation about fixed axis. Relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axel, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions – rotation about the axel fixed with the vehicle and translation of the axel together with the vehicle.

Rotation about an axis in rotation :

In this kind of motion, the body rotates about an axis that also rotates about some other axis. Analysis of rotation about rotating axes is not in our scope, therefore we will discuss it to have an elementary idea only.



As an example consider a rotating top. The top rotates about its central axis of symmetry and this axis sweeps a cone about a vertical axis. The central axis continuously changes its orientation, therefore it is in rotational motion. This type of rotation in which the axis of rotation also rotates and sweeps out a cone is known as precession.

Another example of rotation about axis in rotation is a table-fan swinging while rotating. Table-fan rotates about its horizontal shaft along which axis of rotation passes. When the rotating table-fan swings, its shaft rotates about a vertical axis.

Kinematics of Rotational Motion:

Angular Displacement (θ) :

- When a particle moves in a curved path, the angle subtended by its position vector about a fixed point is known as angular displacement.
- Unit : radian
- Dimension : $M^0L^0T^0$ i.e. dimensionless.
- Small angular displacement is a vector whereas large angular displacement is a scalar.

Angular Velocity (ω) :

- The angular displacement per unit time is defined as angular velocity.

  $\omega = \frac{\Delta\theta}{\Delta t}$, where $\Delta\theta$ is the angular displacement during the time interval Δt.

- Instantaneous angular velocity, $\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

  Average angular velocity, $\vec{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$

- Unit : rad/s
- Dimension : $[M^0L^0T^{-1}]$, which is same as that of frequency.
- Instantaneous angular velocity is a vector quantity, whose direction is normal to the rotational plane (or axial vector) and its direction is given by right hand screw rule.
- If ω be the angular velocity, v be the linear velocity and r be the radius of path, we have the following relation.

  $\vec{v} = \vec{\omega} \times \vec{r}$

- If n be the frequency then $\omega = 2\pi n$, if T be the time period then $\omega = 2\pi/T$

- The angular velocity of a rotating rigid body can be either positive or negative, depending on whether body is rotating in the direction of increasing θ (anticlockwise) or of decreasing θ (clockwise).
- The magnitude of an angular velocity is called the angular speed which is also represented by ω.

Angular Acceleration (α) :

- The rate of change of angular velocity is defined as angular acceleration $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$.
- Suppose at time t_1, a particle has angular velocity $\vec{\omega}_1$ and $\vec{\omega}_2$ at time t_2 then average angular acceleration, $\vec{\alpha} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$
- It is a vector quantity, whose direction is along the change in direction of angular velocity.
- Unit : rad/s^2
- Dimension : $M^0L^0T^{-2}$
- Relation between angular acceleration $\vec{\alpha}$ and tangential acceleration $\vec{a}_t$, is $\vec{a}_t = \vec{\alpha} \times \vec{r}$
- Radial acceleration, $\vec{a}_r = \vec{\omega} \times \vec{v}$. Its direction is along the radius.
- Total acceleration $\vec{a} = \vec{a}_t + \vec{a}_r = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$.

Comparison of Linear Motion and Rotational Motion :

	Linear Motion		Rotational Motion
(i)	If acceleration is 0, v = constant and $s = vt$	(i)	If angular acceleration is 0, ω = constant and $\theta = \omega t$
(ii)	If acceleration a = constant,	(ii)	If angular acceleration α = constant then
	(a) $s = \frac{(u+v)}{2}t$		(a) $\theta = \frac{(\omega_0 + \omega)}{2}t$
	(b) $a = \frac{v-u}{t}$		(b) $\alpha = \frac{\omega - \omega_0}{t}$
	(c) $v = u + at$		(c) $\omega = \omega_0 + \alpha t$
	(d) $s = ut + \frac{1}{2}at^2$		(d) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
	(e) $v^2 = u^2 + 2as$		(e) $\omega^2 = \omega_0^2 + 2\alpha\theta$
	(f) $S_{nth} = u + \frac{a}{2}(2n-1)$		(f) $\theta_{nth} = \omega_0 + \frac{\alpha}{2}(2n-1)$
(iii)	If acceleration is not constant, the above equation will not be applicable. In this case.	(iii)	If angular acceleration is not constant, the above equation will not be applicable. In this case.
	(a) $\vec{v} = \frac{d\vec{s}}{dt}$		(a) $\vec{\omega} = \frac{d\vec{\theta}}{dt}$
	(b) $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{s}}{dt^2} = \vec{v}\frac{d\vec{v}}{ds}$		(b) $\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2} = \vec{\omega}\frac{d\vec{\omega}}{d\theta}$

Illustration 2:

If angular displacement of a particle moving on a curved path be given as, $\theta = 1.5t + 2t^2$, where t is in sec., the angular velocity at $t = 2$ sec., will be (in rad/sec.)

(A) 1.5 (B) 2.5 (C) 9.5 (D) 8.5

Answer: (C)

Solution:

$$\theta = 1.5\,t + 2t^2$$

$$\therefore \quad \frac{d\theta}{dt} = 1.5 + 4t$$

Now, $\left(\frac{d\theta}{dt}\right)$ at $t = 2$ sec $= 1.5 + 4 \times 2$

$= 1.5 + 8 = 9.5$ rad/sec.

Illustration 3:

A disc rotates with a uniform angular acceleration of 2.0 rad/s^2 about its axis. If the disc starts from rest, how many revolutions will it make in the first 10 seconds?

Solution:

The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(2.0\ rad/s^2)\,(10s)^2 = 100\ rad.$$

As the wheel turns by 2π radian in each revolution, the number of revolutions in $10s$ is

$$n = \frac{100}{2\pi} = 16$$

Illustration 4:

Find ω_2, If there is no slipping between these wheels.

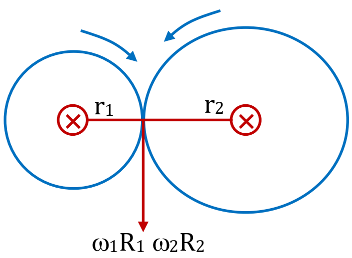
Given $\omega_1 = 400$ rpm

$r_1 = 2cm$

$r_2 = 6cm$

Solution:

(e.g.) No slipping



$\omega_1 R_1 = \omega_2 R_2$ for no slipping

Illustration 5:

The motor of an engine is rotating about its axis with an angular velocity of 120 rev/minute. It comes to rest in 10*s*, after being switched off the engine. Assuming uniform angular deceleration, find the number of revolutions made by it before coming to rest.

Solution:

The initial angular velocity = 120 rev/minute = (4π) rad/s.

Final angular velocity = 0.

Time interval = 10 *s*.

Let the angular acceleration be α. Using the equation $\omega = \omega_0 + \alpha t$, we obtain

$$\alpha = (-4\pi/10)\ \text{rad/s}^2$$

The angle rotated by the motor during this motion is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \left(4\pi\frac{rad}{s}\right)(10s) - \frac{1}{2}\left(\frac{4\pi}{10}\frac{rad}{s^2}\right)(10s)^2$$

$$= 20\pi \text{ rad} = 10 \text{ revolutions. (no. of revolution} = \frac{20\pi}{2\pi})$$

Hence the motor rotates through 10 revolutions before coming to rest.

Energy of A Rigid Body

Gravitational P.E. :

$$U = m_1gh_1 + m_2gh_2 + m_3gh_3 + ...$$

$$U = (m_1h_1 + m_2h_2 + m_3h_3 + ...)\, g$$

$$U = m_t\left(\frac{m_1h_1 + m_2h_2 + m_3h_3 + ...}{m_1 + m_2 + m_3 + ...}\right)g$$

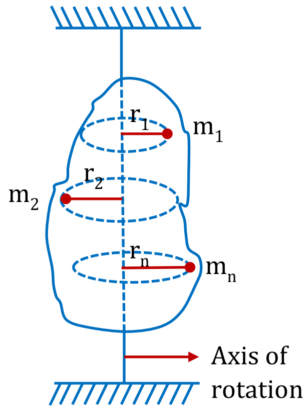
$U = m_t h_{cm}\, g$ $\quad$ (m_t = total mass)

(II) Pure Rotational Motion:

Figure shows a rigid body of arbitrary shape in rotation about a fixed axis, called the axis of rotation. Every point of the body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval. Such a motion is called pure rotation.

We know that each particle has same angular velocity

(since the body is rigid.)



So, $\quad v_1 = \omega r_1,\ v_2 = \omega r_2,\ v_3 = \omega r_3 \ ......\ v_n = \omega r_n$

Total Kinetic Energy = $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ...$

$$= \frac{1}{2}[m_1r_1^2 + m_2r_2^2 + ...]\,\omega^2$$

$= \frac{1}{2}I\omega^2$, where $I = m_1r_1^2 + m_2r_2^2 + ...$ (is called moment of inertia)

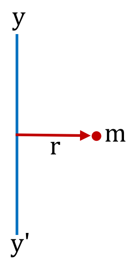
ω = angular speed of body.

Moment of Inertia:

Definition : Property of a rigid body by virtue of which it opposes change in its rotational velocity (angular velocity) is known as *MI*.

- This is always written with respect to an axis of rotation.
- This plays same role in rotational motion as mass plays in translational motion
- Difference between mass and *MI* is that mass is property of body and is independent of any reference chosen but *MI* depends on the
  (i) axis of rotation
  (ii) shape of the body
  (iii) size of the body
  (iv) density of the material of the body.

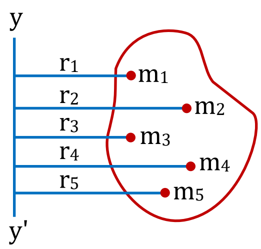
1. *MI* of a point mass :



$I_{yy}{'} = mr^2$

r is perpendicular distance from mass to axis of rotation yy'.

2. *MI* for point mass distribution :



Where r_i is $\perp$ distance of M_i from axis of rotation yy' then $I_{yy'} = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...$

Illustration 6:

Two heavy particles having masses m_1 and m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2 respectively.

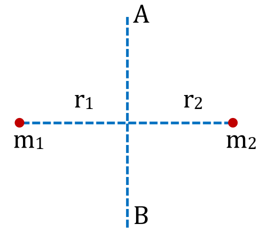
(i) What is the moment of inertia of the system about axis AB ?

(ii) What is the moment of inertia of the system about an axis passing through m_1 and perpendicular to the line joining m_1 and m_2 ?

(iii) What is the moment of inertia of the system about an axis passing through m_1 and m_2 ?

Solution:

(i) Moment of inertia of particle on left is $I_1 = m_1r_1^2$.

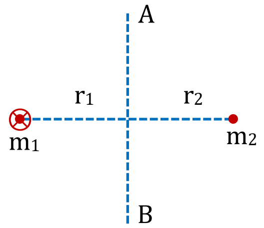


Moment of Inertia of particle on right is $I_2 = m_2r_2^2$.

Moment of Inertia of the system about AB is

$$I = I_1 + I_2 = m_1r_1^2 + m_2r_2^2$$

(ii) Moment of inertia of particle on left is $I_1 = 0$

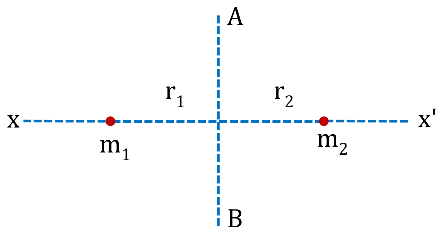


Moment of Inertia of particle on right is $I_2 = m_2(r_1 + r_2)^2$.

Moment of Inertia of the system about axis passing through m_1

$$I = I_1 + I_2 = 0 + m_2(r_1 + r_2)^2$$

(iii) Moment of inertia of particle on left is $I_1 = 0$



Moment of Inertia of particle on right is $I_2 = 0$

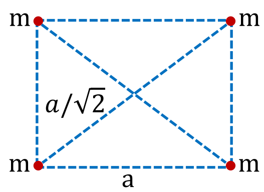
Moment of Inertia of the system about XX',

$$I_{xx'} = I_1 + I_2 = 0 + 0$$

Illustration 7:

Four particles each of mass m are kept at the four corners of a square of edge a.

Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.



Solution:

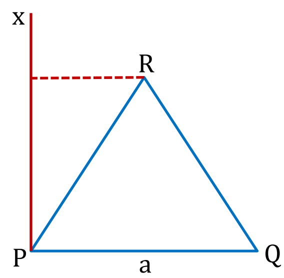
The perpendicular distance of every particle from the given line is $a/\sqrt{2}$.

The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$.

The moment of inertia of the system is, therefore, $4 \times \frac{1}{2}ma^2 = 2ma^2$.

Illustration 8:

Three particles, each of mass m, are situated at the vertices of an equilateral triangle PQR of side a as shown in figure. Calculate the moment of inertia of the system about



(i) The line PX perpendicular to PQ in the plane of PQR.

(ii) One of the sides of the triangle PQR

(iii) About an axis passing through the centroid and perpendicular to plane of the triangle PQR.

Solution:

(i) Perpendicular distance of P from $PX = 0$

Perpendicular distance of Q from $PX = a$

Perpendicular distance of R from $PX = a/2$

Thus, the moment of inertia of the particle at $P = 0$, of the particle at $Q - ma^2$, and of the particle at $R = m(a/2)^2$. The moment of inertia of the three-particle system about PX is

$$0 + ma^2 + m(a/2)^2 = \frac{5ma^2}{4}$$

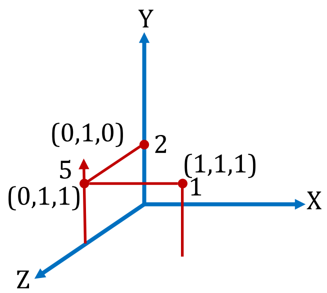
Note that the particles on the axis do not contribute to the moment of inertia.

(ii) Moment of inertia about the side PR = mass of particle Q × square of perpendicular distance of Q from side PR, $I_{PR} = m\left(\frac{\sqrt{3}}{2}a\right)^2 = \frac{3ma^2}{4}$

(iii) Distance of centroid from all the particle is $\frac{a}{\sqrt{3}}$, so moment of inertia about an axis and passing through the centroid perpendicular plane of triangle $PQR = I_R = 3m\left(\frac{a}{\sqrt{3}}\right)^2 = ma^2$

Illustration 9:

If masses $2kg$, $5kg$ and $1kg$ are kept at (0, 1, 0), (0, 1, 1) and (1, 1, 1) respectively. Then find MI of system



(a) w.r.t. x-axis

(b) w.r.t. y-axis

(c) w.r.t. z-axis

Solution:

$$I_{xx}' = 2(1)^2 + 5(\sqrt{2})^2 + 1(\sqrt{2})^2 = 14\ kgm^2$$

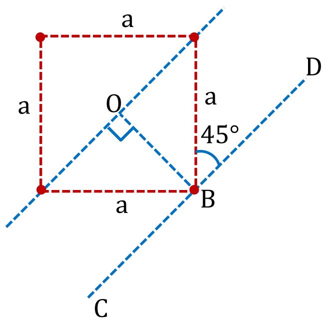
$$I_{yy}' = 2(0)^2 + 5(1)^2 + 1(\sqrt{2})^2 = 7\ kgm^2$$

$$I_{zz}' = 2(1)^2 + 5(1)^2 + 1(\sqrt{2})^2 = 9\ kgm^2$$

So, MI of the distribution depends on axis for which MI is being written.

Illustration 10:

Four point masses each of mass m kept at the four corners of a square of side length 'a' find the moment of inertia of system about axis CD.



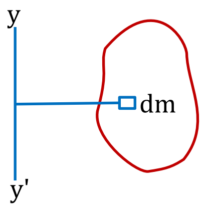
Solution:

$$I = 0 + m\left(\frac{a}{\sqrt{2}}\right)^2 + m\left(\frac{a}{\sqrt{2}}\right)^2 + m\left(\sqrt{2}a\right)^2 = 3ma^2$$

3. For continuous mass distribution :

$I_{yy}' = \int dm\, r^2$

where r is $\perp$ distance of any element from axis of rotation.



Important point :

MI of two or more than two bodies can be added or subtracted only when all *MI* are written w.r.t. same axis.

Note:

Moment of inertia of a large object can be calculated by integrating *M.I.* of an element of the object**:**

$I = \int dI_{element}$ where dI = moment of inertia of a small element.

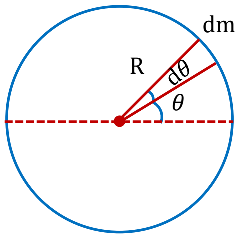
Element chosen should be such that : either perpendicular distance of axis from each point of the element is same or the moment of inertia of the element about the axis of rotation is known.

Illustration 11:

Calculate the moment of inertia of a ring having mass *M*, radius *R* and having uniform mass distribution about an axis passing through the centre of ring and perpendicular to the plane of ring ?

Solution:

$I = \int (dm) r^2$



Because each element is equally distanced from the axis so $r = R$

$= R^2 \int dm = MR^2$

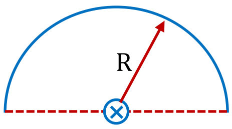
$I = MR^2$

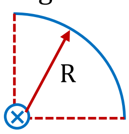
Note:

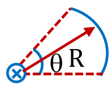
Answer will remain same even if the mass is nonuniformly distributed because always $\int dm = M$.

Illustration 12:

Find out the moment of Inertia of figures shown each having mass *M*, radius *R* and having uniform mass distribution about an axis passing through the centre and perpendicular to the plane ?

(a) 

(b) 

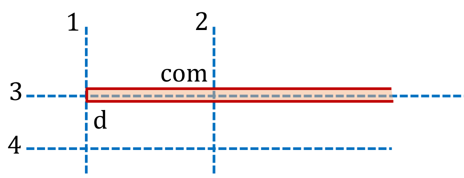
(c) 

Solution:

MR^2 (infact *M.I.* of any part of mass *M* of a ring of radius *R* about axis passing through geometrical centre and perpendicular to the plane of the ring is = MR^2).

Illustration 13:

Calculate the moment of inertia of a uniform rod of mass *M* and length ℓ about an axis 1, 2, 3 and 4.



Solution:

$$(I_1) \ = \int (dm) r^2 = \int_0^{\ell} \left(\frac{M}{\ell} dx\right) \ x^2 = \frac{M\ell^2}{3}$$

$$(I_2) \ = \int (dm) r^2 = \int_{-\ell/2}^{\ell/2} \left(\frac{M}{\ell} dx\right) \ x^2 = \frac{M\ell^2}{12}$$

(I_3) = 0 (axis 3 passing through the axis of rod)

$$(I_4) \ = d^2 \int (dm) \ = Md^2$$

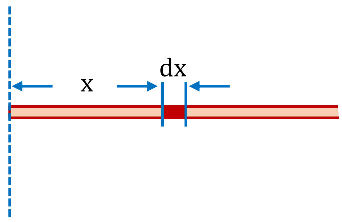
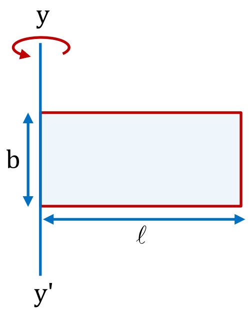


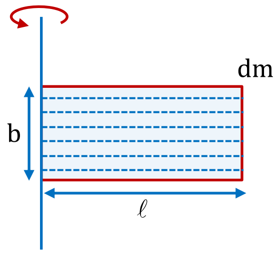
Illustration 14:

Determine the moment of inertia of a uniform rectangular plate of mass 'M', side 'b' and Length 'l' about an axis passing through the edge 'b' and in the plane of plate.



Solution:

Each section of dm mass rod in the rectangular plate has moment of inertia dI about an axis passing through edge 'b'; $dI = \frac{l^2}{3} dm$

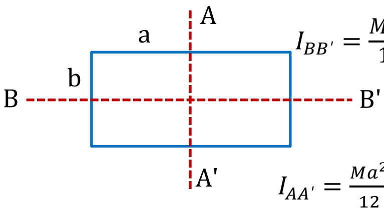


So, $\quad I_{yy'} = \int dI = \frac{\ell^2}{3} \int dm = \frac{M\ell^2}{3}$

Key Point:

Moving mass parallel to AOR does not change moment of inertia.

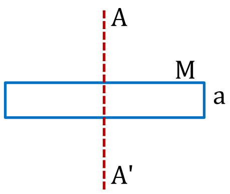
(eg.)



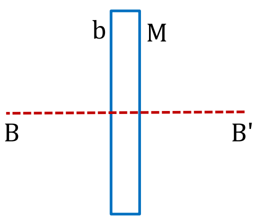
$I_{BB'} = \frac{Mb^2}{12}$

$I_{AA'} = \frac{Ma^2}{12}$

$\Rightarrow$



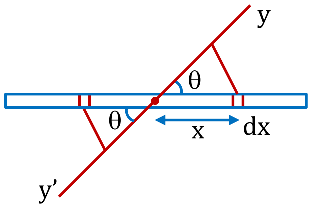
$I_{AA'} = \frac{Ma^2}{12}$



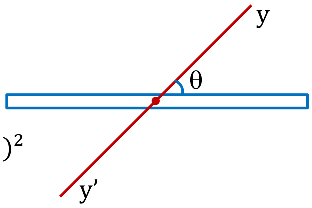
$\Rightarrow dI = dmr^2$

(e.g.)

Uniform rod (about an axis yy' passing through centre of rod and making an angle θ to the rod as shown).



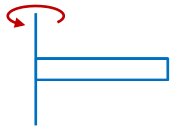
$dI = \frac{m}{\ell} dx (x \sin\theta)^2$



$$I = \frac{m}{\ell}\sin^2\theta \int_{-\ell/2}^{\ell/2} x^2\,dx = \frac{m\ell^2}{12}\sin^2\theta$$

Illustration 15:

Find *MOI* about an axis passing through the end perpendicular to rod ?



$$\lambda = \lambda_0\left(1+\frac{x}{L}\right)$$

Solution:

$$\int dI = \int x^2 dm$$

$$I = \int_0^L x^2 \lambda dx$$

$$\Rightarrow \quad \int_0^L x^2\lambda_0\left(1+\frac{x}{L}\right)dx$$

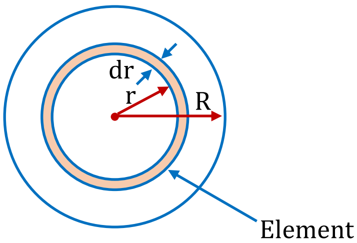
$$\Rightarrow \quad \lambda_0\int_0^L\left(x^2+\frac{x^3}{L}\right)dx$$

$$\Rightarrow \quad \lambda_0\left[\int_0^L x^2dx+\int_0^L\frac{x^3}{L}dx\right]$$

$$\Rightarrow \quad \lambda_0\left[\frac{L^3}{3}+\frac{L^3}{4}\right]=\frac{7L^3}{12}$$

Illustration 16:

Determine the moment of Inertia of a uniform disc having mass *M*, radius *R* about an axis passing through centre and perpendicular to the plane of disc ?



Solution:

$$I = \int dI_{ring}$$

element ring $dI = dmr^2$

$$dm = \frac{M}{\pi R^2}2\pi r dr \quad \text{(here we have used the uniform mass distribution)}$$

$$\therefore \quad I = \int_0^R \frac{M}{\pi R^2}.(2\pi r dr).r^2 \Rightarrow I = \frac{MR^2}{2}$$

Two Important Theorems on Moment of Inertia :

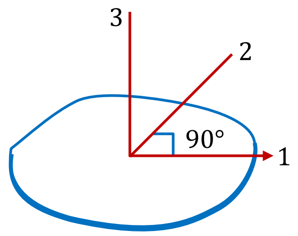
(i) Perpendicular Axis Theorem :

[Only applicable to plane laminar bodies (i.e. for 2 dimensional objects only)].

If axis 1 and 2 are in the plane of the body and perpendicular to each other.

Axis 3 is perpendicular to plane of body.

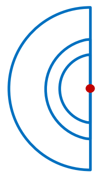
Then, $I_3 = I_1 + I_2$



Body is in 1-2 plane

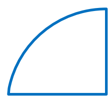
- The point of intersection of the three axis need not be center of mass, it can be any point in the plane of body which lies on the body or even outside it.

Half disc of mass *m* and radius *R* about same axis as previous.



$$\Rightarrow \quad dI = dmr^2 = \left(\frac{M}{\pi R^2/2} \times \pi r dr\right) r^2 \quad ; \quad I = \int dI = \frac{2M}{R^2}\left(\frac{r^4}{4}\right)_0^R = \frac{MR^2}{2}$$

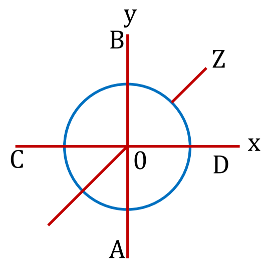
Quarter disc of mass *m* and radius *R* about same axis as above.



$$\Rightarrow \quad dI = dmr^2 = \left(\frac{M}{\pi R^2/4} \times \frac{\pi r dr}{2}\right) r^2 \quad ; \quad I = dI = \frac{2M}{R^2}\left(\frac{r^4}{4}\right)_0^R = \frac{MR^2}{2}$$

Illustration 17:

Calculate the moment of inertia of a uniform disc of mass *M* and radius *R* about a diameter.



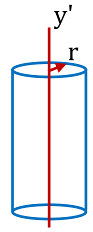
Solution:

Let *CD* and *AB* be two mutually perpendicular diameters of the disc. Take them as *X* and *Y*-axis and the line perpendicular to the plane of the disc through the centre as the *Z*-axis. The moment of inertia of the ring about the *Z*-axis is $I = \frac{1}{2}MR^2$. As the disc is uniform, all of its diameters are equivalent and so $I_x = I_y$, From perpendicular axes theorem,

$$I_z = I_x + I_y$$

Hence $I_x = \frac{I_Z}{2} = \frac{MR^2}{4}$

Ex. **Hollow cylinder** (about yy' axis)



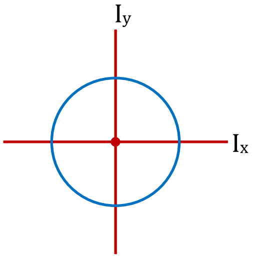
$dI = dmr^2$

$I = \int dI = r^2 \int dm = Mr^2$

Note :

Moment of Inertia is Independent from length of cylinder.

Ex. **Ring (about a diameter)**



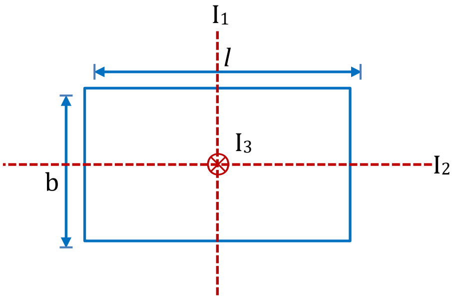
$I_x + I_y = MR^2$

and $I_x = I_y$ (by symmetry)

$I_x = \dfrac{MR^2}{2}$

Ex. **Rectangular plate**

(about an axis $\perp$ to plane and passing through centre)



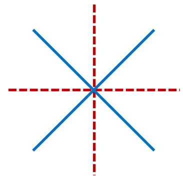
By using perpendicular axis theorem

$I_3 = I_1 + I_2$

$I_3 = \dfrac{Mb^2}{12} + \dfrac{Ml^2}{12}$

Illustration 18:

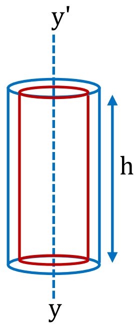
Two uniform identical rods each of mass M and length L are joined to form a cross as shown in figure. Find the moment of inertia of the cross about a bisector as shown dotted in the figure.



Solution:

Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\dfrac{ML^2}{12}$ and hence the moment of inertia of the cross is $\dfrac{ML^2}{6}$. The moment of inertia of the cross about the two bisector are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\dfrac{ML^2}{6} = 2I \Rightarrow I = \dfrac{ML^2}{12}$.

Ex. **Solid cylinder** (about yy' axis)



$dI = dm\, r^2$

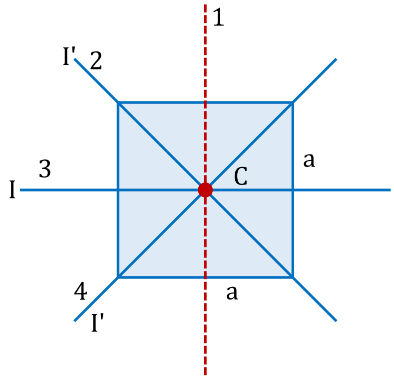
$= \dfrac{M}{\pi R^2 h} \times 2\pi r\, dr\, h r^2$

$I = \int dI = \dfrac{2M}{R^2}\int_0^R r^3\, dr$

$= \dfrac{MR^2}{2}$

Illustration 19:

In the figure shown find the moment of inertia of square plate having mass m and sides a. About an axis 2 passing through point C (centre of mass) and in the plane of plate.



Solution:

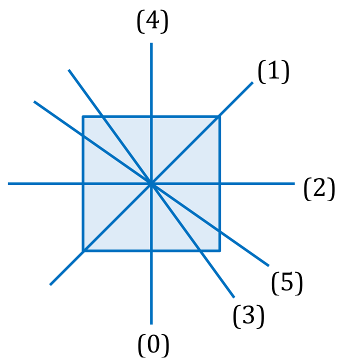
Using perpendicular axis theorems, $I_C = I_4 + I_2 = 2I'$

Using perpendicular theorems, $I_C = I_3 + I_1 = I + I = 2I$

$2I' = 2I$

$I' = I$

$I_C = 2I = \dfrac{ma^2}{6} \quad \Rightarrow \quad I' = \dfrac{ma^2}{12}$



Ex. **Hollow sphere** (about a diameter)

$r = R\,Cos\,\theta$

$$dm = \frac{M}{4\pi R^2} \times 2\pi(R\cos\theta)Rd\theta$$

$$dl = dm(R\cos\theta)^2$$

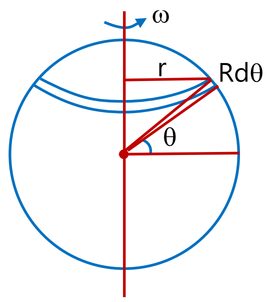
$$dl = \frac{M}{2}\cos\theta\, d\theta\, R^2 \cos^2\theta$$

$$I = \frac{MR^2}{2}\int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta$$

$$\cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}$$

$$I = \frac{MR^2}{8}\left[\int_{-\pi/2}^{\pi/2} \cos 3\theta\, d\theta + 3\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta\right]$$

$$= \frac{MR^2}{8}\left[\frac{-2}{3} + 6\right] = \frac{2}{3}MR^2$$



Ex. **Square plate** (all axis lies in the plane of plate and passing through *C.M.*) moment of inertia about each axis will be same.

$$I_1 = \frac{ma^2}{12} = I_2 = I_3 = I_5 = I_4$$

$$I_2 + I_4 = \frac{ma^2}{6} \quad \Rightarrow \quad I_2 = \frac{ma^2}{12}$$

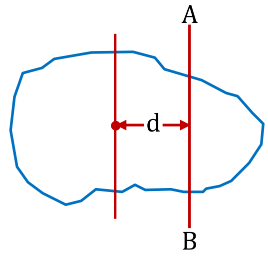
(ii) Parallel Axis Theorem (Applicable to planer as well as 3 dimensional objects):

If I_{AB} = Moment of Inertia of the object about axis AB.

I_{cm} = Moment of Inertia of the object about an axis passing through centre of mass and parallel to axis AB.

M = Total mass of object.

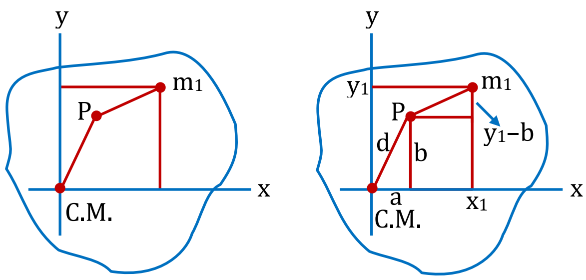
d = Perpendicular distance between axis AB about which moment of Inertia is to be calculated and the one passing through the centre of mass and parallel to it.



$$I_{AB} = I_{CM} + Md^2$$

Proof of Parallel-Axis Theorem :

Used to find moment of inertia about an axis which parallel to the axis passing through *C.M.*



C.M. is at origin

$I_P = I_{CM} + Md^2$

$I_{CM} \to MI$ of the rigid body about an axis passing through CM

$I_P \to MI$ of the rigid body about an axis which is parallel to the above axis through CM and is at distance d from the axis through CM

$$I_{CM} = \sum_i m_i\left(x_i^2 + y_i^2\right)$$

$$I_p = \sum_i m_i\left[(x_i - a)^2 + (y_i - b)^2\right]$$

z coordinates are not involved so m_i can be replaced by sum of mass of all particles placed on z-axis with co-ordinate (x_i, y_i)

$$= \Sigma m_i\begin{bmatrix} x_i^2 + y_i^2 + a^2 + b^2 \\ -2ax_i - 2by_i \end{bmatrix}$$

$$= \Sigma m_i\left(x_i^2 + y_i^2\right) + \Sigma m_i\left(a^2 + b^2\right) - \Sigma 2am_ix_i - \Sigma 2bm_iy_i$$

$$= I_{CM} + \left(a^2 + b^2\right)\Sigma m_i - 2a\,\Sigma m_ix_i - 2b\Sigma m_iy_i = I_{CM} + Md^2 - O - O$$

$$I_p = I_{CM} + Md^2$$

Ex. Solid sphere (about a diameter)

$$dI = \frac{2}{3}dmr^2$$

$$dm = \frac{M}{\frac{4}{3}\pi R^3} \times 4\pi r^2\,dr = \frac{3M}{R^3}r^2\,dr$$

$$dI = \frac{2}{3} \times \frac{3M}{R^3}r^2(dr)r^2 = \frac{2M}{R^3}\int_0^R r^4\,dr = \frac{2M}{5}R^2$$

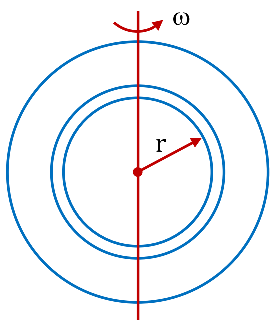
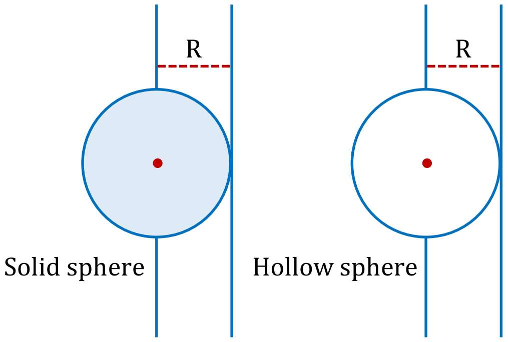


Illustration 20:

Find the moment of inertia of a uniform sphere of mass m and radius R about a tangent if the spheres (i) solid (ii) hollow.

Solution:



(i) Using parallel axis theorem,

$$I = I_{CM} + md^2$$

For solid sphere,

$$I_{CM} = \frac{2}{5}mR^2, d = R$$

$$I = \frac{7}{5}mR^2$$

(ii) Using parallel axis theorem,

$$I = I_{CM} + md^2$$

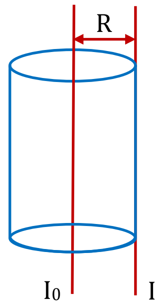
For hollow sphere

$$I_{CM} = \frac{2}{3}mR^2, d = R$$

$$I = \frac{5}{3}mR^2$$

Illustration 21:

Calculate the moment of inertia of a hollow cylinder of mass M and radius R about a line parallel to the axis of the cylinder and on the surface of the cylinder.



Solution:

The moment of inertia of the cylinder about its axis = MR^2.

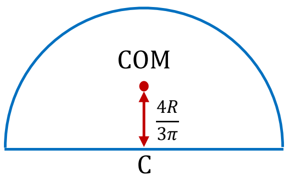
Using parallel axes theorem,

$$I = I_0 + MR^2 = MR^2 + MR^2 = 2MR^2$$

Illustration 22:

Find out the moment of inertia of a semi circular disc about an axis passing through its centre of mass and perpendicular to the plane?

Solution:



Moment of inertia of a semi circular disc about an axis passing through centre and perpendicular to plane of disc, $I = \frac{MR^2}{2}$, $d = \frac{4R}{3\pi} \Rightarrow \frac{MR^2}{2} = I_{CM} + M\left(\frac{4R}{3\pi}\right)^2$

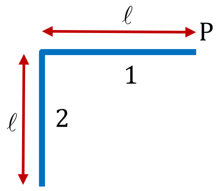
Using parallel axis theorem $I = I_{CM} + Md^2$, d is the perpendicular distance between two parallel axis passing through centre C and COM.

$$I = \frac{MR^2}{2}, d = \frac{4R}{3\pi} \Rightarrow \frac{MR^2}{2} = I_{CM} + M\left(\frac{4R}{3\pi}\right)^2$$

$$I_{CM} = \left[\frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2\right]$$

Illustration 23:

Find the moment of inertia of the two uniform joint rods having mass m each about point P as shown in figure. Using parallel axis theorem.



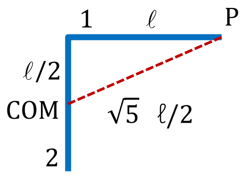
Solution:

Moment of inertia of rod 1 about axis P,

$$I_1 = \frac{m\ell^2}{3}$$

Moment of inertia of rod 2 about axis P,

$$I_2 = \frac{m\ell^2}{12} + m\left(\sqrt{5}\frac{\ell}{2}\right)^2$$



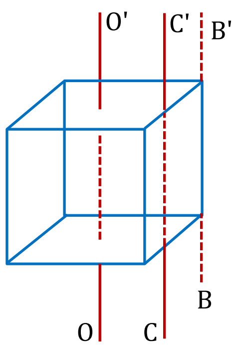
So, moment of inertia of a system about axis P,

$$I = I_1 + I_2 = \frac{ml^2}{3} + \frac{ml^2}{12} + m\left(\sqrt{5}\frac{\ell}{2}\right)^2$$

$$I = \frac{m\ell^2}{3}$$

Illustration 24:

A cube of mass M and sides a is shown in figure. Find MOI about axis OO', CC' and BB'.



Solution:

$$I_{OO'} = \frac{Ma^2}{6}$$

(By parallel axis theorem)

$$I_{BB'} = \frac{Ma^2}{6} + M\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{2Ma^2}{3}$$

$$I_{CC'} = \frac{Ma^2}{6} + M\left(\frac{a}{2}\right)^2 = \frac{5Ma^2}{12}$$

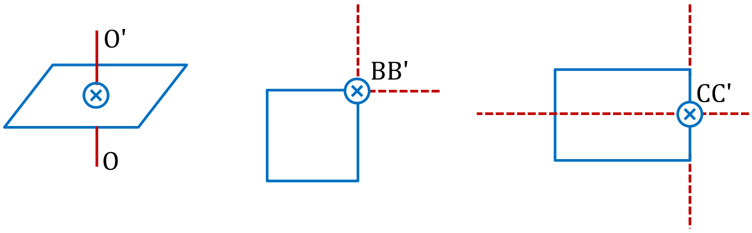
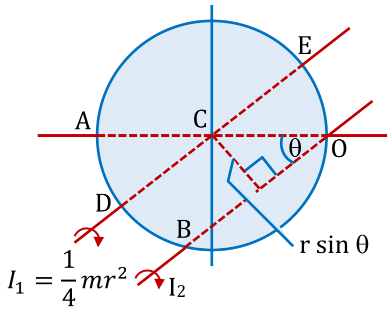


Illustration 25:

Find expression for moment of inertia of a uniform disc of mass m, radius r about one of its secant making an angle θ with one of its diameter.

Solution:

A disc, the secant OB and diameter OA are shown in the adjoining figure. The secant OB is parallel to another diameter DE. Moment of inertia of the disc about one of its diameter is $\frac{1}{4}mr^2$ and hence moment of inertia I_1 about the diameter DE. Perpendicular distance between the secant OB and the parallel diameter DE is r sin θ. Substituting above information in the theorem of parallel axes, we have

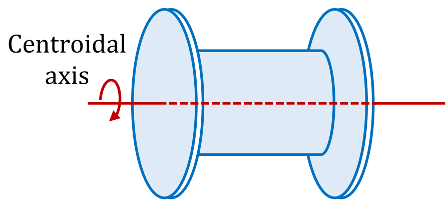


$$I_2 = I_1 + m(r\sin\theta)^2$$

$$\Rightarrow \qquad I_2 = mr^2\left(\frac{1}{4} + \sin^2\theta\right)$$

Illustration 26:

Find moment of inertia about centroidal axis of a bobbin, which is constructed by joining coaxially two identical discs each of mass m and radius $2r$ to a cylinder of mass m and radius r as shown in the figure.



Solution:

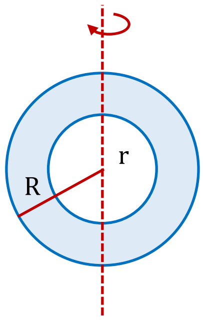
The bobbin is a composite body made by joining two identical discs coaxially to cylinder. The moment of inertia I of the bobbin equals to the sum of moment of inertias of the two discs and moment of inertia of the cylinder about their centroidal axes. Using expressions for the moment of inertia for disc and cylinder, we have,

$$I = 2I_{disk} + I_{cylinder}$$

$$\Rightarrow \qquad I = 2\left\{\frac{1}{2}m(2r)^2\right\} + \frac{1}{2}mr^2 = \frac{9}{2}mr^2$$

Illustration 27:

Find moment of inertia about one of diameter of a hollow sphere of mass m, inner radius r and outer radius R.



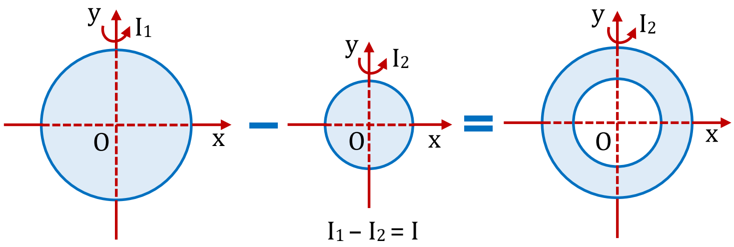
Solution:

The hollow sphere is assumed as if a concentric smaller sphere of radius r is removed from a larger sphere of radius R. Thus the moment of inertia of the hollow sphere about any axes can be obtained by subtracting moment of inertia of the smaller sphere from that axis of the larger sphere. As shown in the following figure.

Let the mass of the hollow sphere is m.

Density of the material used is

$$\rho = \frac{3m}{4\pi\left(R^3 - r^3\right)}$$



Masses m_1 and m_2 of the smaller spheres are

$$m_1 = \rho\left(\frac{4}{3}\pi R^3\right) = \frac{mR^3}{R^3 - r^3}$$

and $$m_2 = \rho\left(\frac{4}{3}\pi r^3\right) = \frac{mr^3}{R^3 - r^3}$$

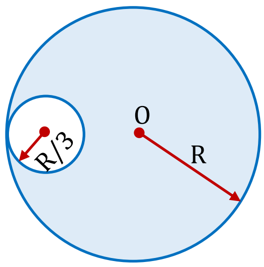
Subtracting I_2 from I_1 we have I.

$$I = I_1 - I_2$$

$\Rightarrow$ $$I = \frac{2m_1R^2}{5} - \frac{2m_2r^2}{5} = \frac{2m\left(R^5 - r^5\right)}{5\left(R^3 - r^3\right)}$$

Illustration 28:

A uniform disc of radius R has a round disc of radius $R/3$ cut as shown in Fig. The mass of the remaining (shaded) portion of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical centre of original disc and perpendicular to the plane of the disc.



Solution:

Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density $-\sigma$.

Now $I_0 = I_\sigma + I_{-\sigma}$

$I_\sigma = (\sigma\pi R^2)R^2/2 = M.I.$ of complete disc about o

$$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2]\,(2R/3)^2$$

$\therefore \quad I_0 = \frac{4}{9}MR^2$ **Ans.**

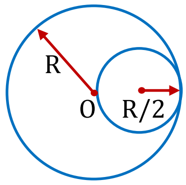
Key Point :

(i) Among the given parallel axis *MOI* will be minimum about the axis passing through *COM*.

(ii) *MOI* about the parallel axis having same distance from *COM* will be equal.

Illustration 29:

A hole of radius $R/2$ is cut from a solid sphere of radius R. If the mass of the remaining part of sphere is M, then find moment of inertia of the body about an axis passing through O.



Solution:

$$\rho = \frac{M}{\frac{4}{3}\pi\left(R^3 - \left(\frac{R}{2}\right)^3\right)}$$

$$\rho = \frac{6M}{7\pi R^3}$$

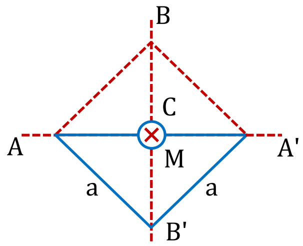
$$I_{0(\text{due to complete sphere})} = \frac{2}{5}\times\frac{6M}{7\pi R^3}\times\frac{4}{3}\pi R^3.R^2 = \frac{16MR^2}{35}$$

$$I_{(\text{due to cutted part})} = -\frac{2}{5}\times\frac{6M}{7\pi R^3}\times\frac{4\pi}{3}\frac{R^3}{8}\times\frac{R^2}{4} - \frac{6M}{7\pi R^3}\times\frac{4\pi}{3}\frac{R^3}{8}\times\frac{R^2}{4} = -\frac{MR^2}{20}$$

$$I_{\text{net}} = \frac{16MR^2}{35} - \frac{MR^2}{20} = \frac{57MR^2}{140}$$

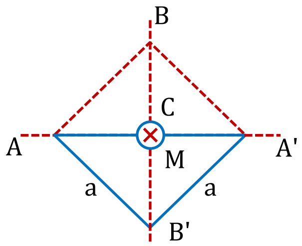
Illustration 30:

A triangular plate having two sides a, as shown in figure. Find *MOI* about *AA'*, *BB'* and *CC'*.



Solution:

(a)



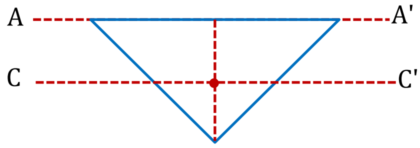
$$I_{AA'} = I_{BB'} = I$$

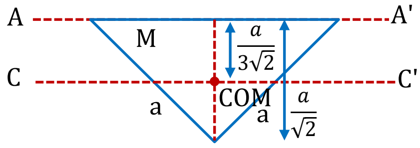
By the perpendicular axis theorem $I_{AA'} + I_{BB'} = I_C$

$$I_C = 2I = \frac{2Ma^2}{12}$$

$$\Rightarrow \quad I_{AA'} = \frac{Ma^2}{12}$$

(b)



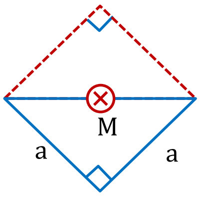


Find $I_{CC'}$ = ?

$$I_{AA'} = I_{CC'} + M\left(\frac{a}{3\sqrt{2}}\right)^2$$

$$\Rightarrow \quad \frac{Ma^2}{12} - \frac{Ma^2}{18} = I_{CC'} \quad ; \quad \boxed{I_{CC'} = \frac{Ma^2}{36}}$$

(c)



$$2I = \frac{2Ma^2}{6}$$

$$\Rightarrow \quad I = \frac{Ma^2}{6}$$

Radius of Gyration (K):

It is the radial distance from a rotational axis at which the mass of an object could be concentrated without altering the moment of inertia of the body about that axis.

If the mass *m* of the body were actually concentrated at *a* distance *k* from the axis, the moment of inertia about that axis would be mk^2.

$$k = \sqrt{\frac{I}{m}}$$

K has no meaning without axis of rotation
K is scalar quantity

The radius of gyration has dimensions of length and is measured in appropriate units of length such as meters.

Ex. : If $m_1 = m_2 = m_3 = m$ then, $M = mn$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + ... r_n^2}{n}} \quad (n = \text{total number of particles})$$

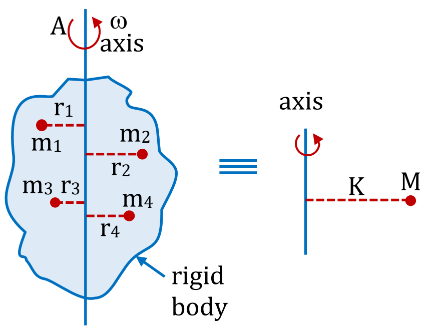


Illustration 31:

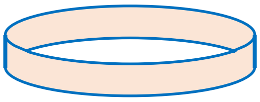
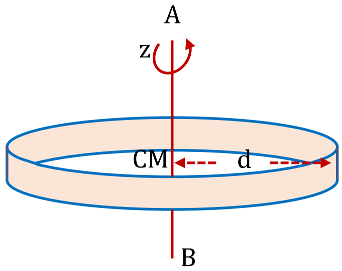
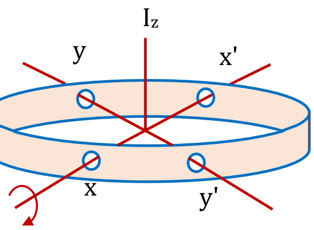
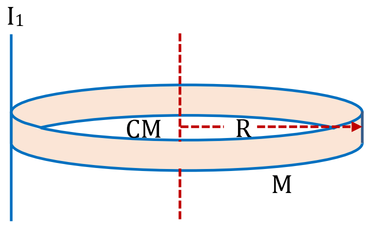
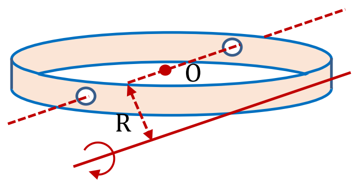
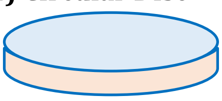
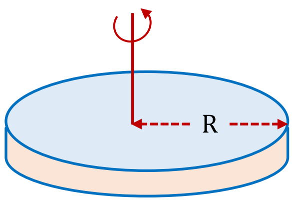
Find the radius of gyration of a solid uniform sphere of radius R about its tangent.

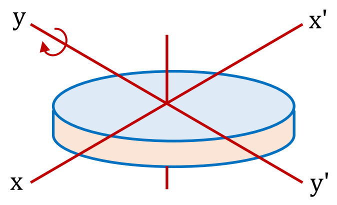
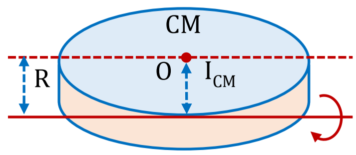
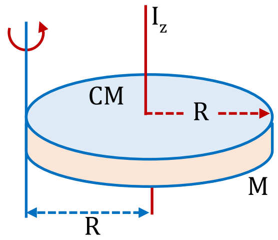
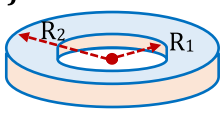
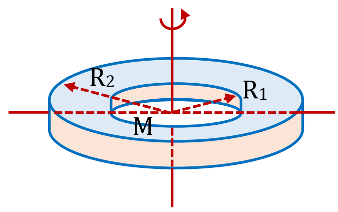
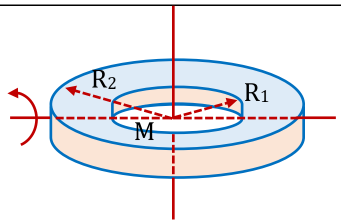
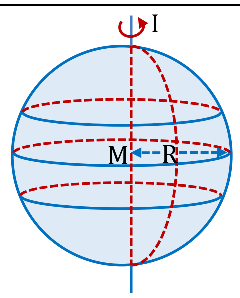
Solution:

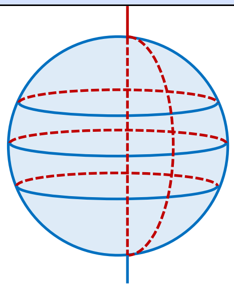
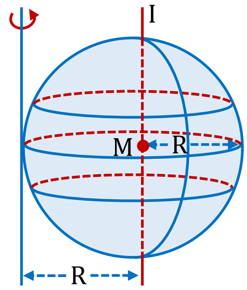
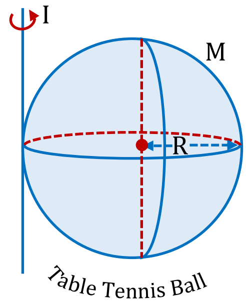
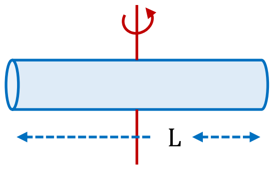
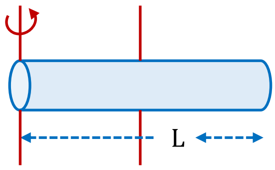
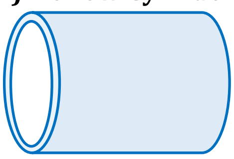
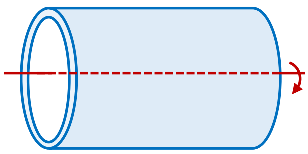
$$I = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2 = mK^2$$

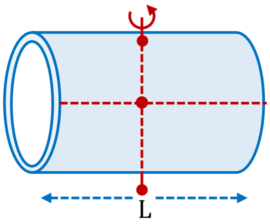
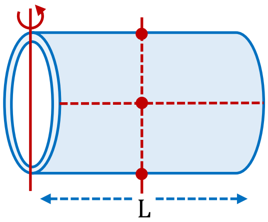
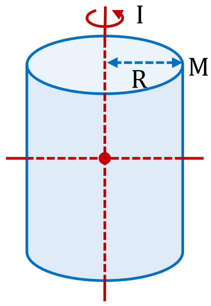
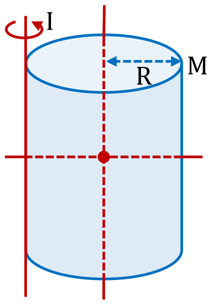
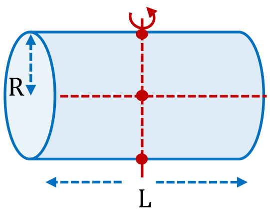
$$\Rightarrow \qquad K = \sqrt{\frac{7}{5}}R$$

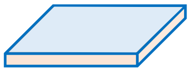
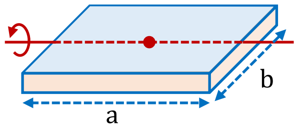
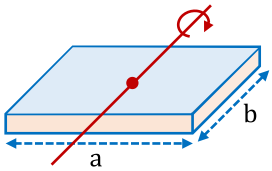
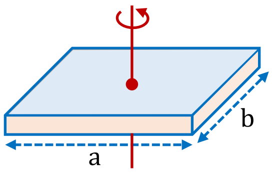
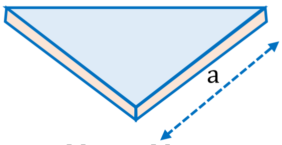
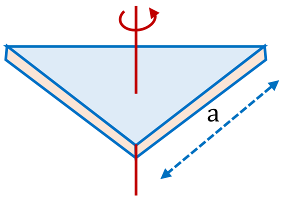
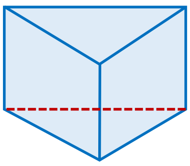
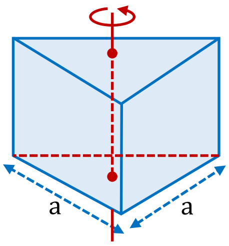
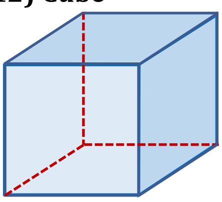
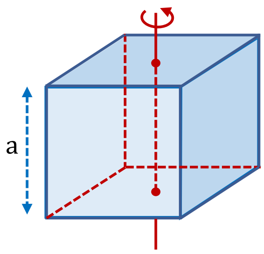
Moment of Inertia of Some Regular Bodies :

Shape of the body	Position of the axis of rotation		Figure	Moment of Inertia (I)	Radius of gyration (K)
(1) Circular Ring 	(a)	About an axis perpendicular to the plane and passes through the centre		MR^2	R
	(b)	About the diametric axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c)	About an axis tangential to the rim and perpendicular to the plane of the ring		$2MR^2$	$\sqrt{2}R$
	(d)	About an axis tangential to the rim and lying in the plane of ring		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
(2) Circular Disc  M = Mass R = Radius	(a)	About an axis passing through the centre and perpendicular to the plane		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$

Shape of the body	Position of the axis of rotation		Figure	Moment of Inertia (I)	Radius of gyration (K)
	(b)	About a diametric axis		$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c)	About an axis tangential to the rim and lying in the plane of the disc		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	(d)	About an axis tangential to the rim and perpendicular to the plane of disc		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
(3) Annular disc  M = Mass R_1 = Internal Radius R_2 = Outer Radius	(a)	About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}\left[R_1^2+R_2^2\right]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	(b)	About a diametric axis		$\frac{M}{4}\left[R_1^2+R_2^2\right]$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
(4) Solid Sphere M = Mass R = Radius	(a)	About its diametric axis which passes through its centre of mass		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$

Shape of the body	Position of the axis of rotation		Figure	Moment of Inertia (I)	Radius of gyration (K)
	(b)	About a tangent to the Sphere		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
	(b)	About a tangent to the surface		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
(6) Thin Rod  Thickness is negligible w.r.t. length	(a)	About an axis passing through centre of mass and perpendicular to its length		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$
	(b)	About an axis passing through one end and perpendicular to length of the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(7) Hollow Cylinder  M = Mass R = Radius L = Length	(a)	About its geometrical axis which is parallel to its Length		MR^2	R

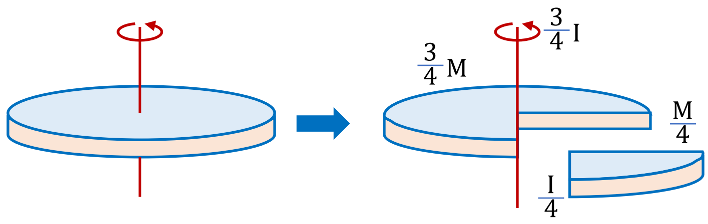
Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
	(b) About an axis which is perpendicular to its length and passes through its centre of mass.		$\frac{MR^2}{2}+\frac{ML^2}{12}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$
	(c) About an axis perpendicular to its length and passing through one end of the Cylinder.		$\frac{MR^2}{2}+\frac{ML^2}{3}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$
(8) Solid Cylinder  M = Mass R = Radius L = Length	(a) About its geometrical axis, which is parallel to its length.		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b) About an axis tangential to the cylindrical surface and parallel to its geometrical axis.		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
	(c) About an axis passing through the centre of mass and perpendicular to its length.		$\frac{ML^2}{12}+\frac{MR^2}{4}$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$

Shape of the body	Position of the axis of rotation		Figure	Moment of Inertia (I)	Radius of gyration (K)
(9) Rectangular Plate  M = Mass a = Length b = Breadth	(a)	About an axis passing through centre of mass and perpendicular to side b in its plane.		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
	(b)	About an axis passing through centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
	(c)	About an axis passing through centre of mass and perpendicular to its plane.		$\frac{M\left(a^2+b^2\right)}{12}$	$\sqrt{\frac{a^2+b^2}{12}}$
(10) Triangular Lamina  Mass = M		About an axis passing through centre of mass and perpendicular to its plane.		$\frac{Ma^2}{12}$	$\frac{a}{\sqrt{12}}$
(11) Triangular Prism  each side of base = a		About an axis passing through centre of mass and perpendicular to its plane.		$\frac{Ma^2}{12}$	$\frac{a}{\sqrt{12}}$
(12) Cube  Side a		About an axis passes through centre of mass and perpendicular to face.		$\frac{Ma^2}{6}$	$\frac{a}{\sqrt{6}}$

- Moment of inertia is not a vector quantity as direction *CW* or *ACW* is not to be specified and also not a scalar as it has different values in different directions (i.e. about different axes). It is a tensor quantity.
- If a wheel is made by two different materials then for larger moment of inertia, higher density material should be at outer side.
- Radius of gyration depends on:

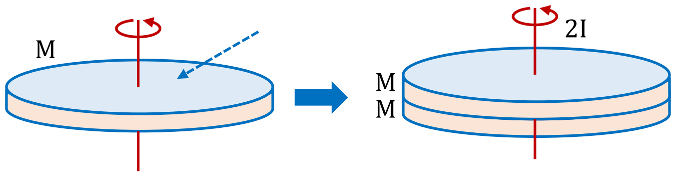
  (i) Axis of rotation (ii) Distribution of mass of the body
- Radius of gyration does not depend on:

  (i) Mass of the body (ii) Angular quantities (angular displacement, angular velocity etc.)
- For Symmetrical separation radius of gyration remains unchanged.



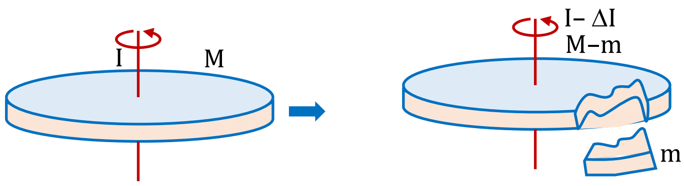
$\Rightarrow$ *K* remains unchanged

- For Symmetrical attachment radius of gyration remains unchanged.



$\Rightarrow$ *K* remains unchanged

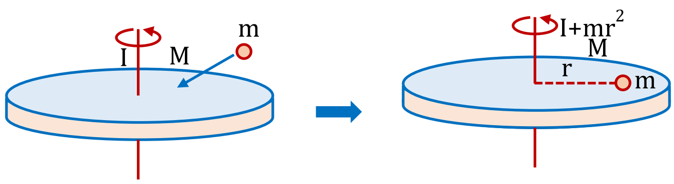
- For Unsymmetrical separation radius of gyration will be changed.



$\Rightarrow K = \sqrt{\dfrac{I - \Delta I}{M - m}}$ $\Rightarrow$ *K* changed

Here ΔI = *M. I.* of detached mass w.r.t. same axis.

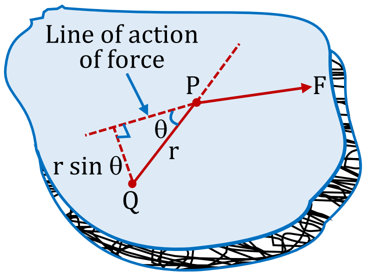
- For Unsymmetrical attachment radius of gyration will be changed.



$\Rightarrow K = \sqrt{\dfrac{I + mr^2}{M + m}}$ $\Rightarrow$ *K* changed

Torque

Torque represents the capability of a force to produce change in the rotational motion of the body.



Torque about a point :

Torque of force $\vec{F}$ about a point $\vec{\tau} = \vec{r} \times \vec{F}$

Where $\vec{F}$ = force applied

P = point of application of force

Q = Point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force w.r.t. the point about which we want to determine the torque.

$|\vec{\tau}| = r\,F \sin\theta = r_{\perp}F = rF_{\perp}$

Where θ = angle between the direction of force and the position vector of P with respect to Q.

$r_{\perp} = r \sin\theta$ = perpendicular distance of line of action of force from point Q, it is also called force arm.

$F_{\perp} = F \sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

SI unit of torque is Nm.

Torque is a vector quantity and it's direction is determined using right hand thumb rule and it is always perpendicular to the plane of rotation of the body.

Illustration 32:

A particle of mass m is released in vertical plane from a point P at $x = x_0$ on the x-axis it falls vertically along the y-axis. Find the torque τ acting on the particle at a time t about origin ?

Solution:

Torque is produced by the force of gravity.

$$\vec{\tau} = -\,r\,F \sin\theta\hat{k}$$

or $$\tau = r_{\perp}F = x_0 mg$$

$$\vec{\tau} = -mgx_0\hat{k}$$

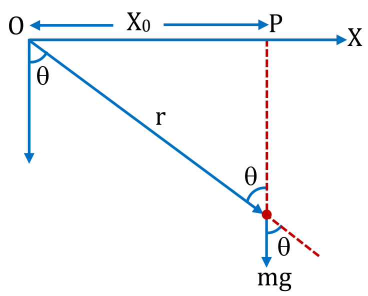
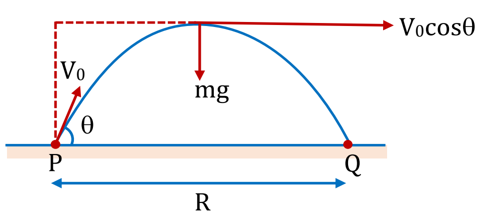


Illustration 33:

A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Find out the magnitude of torque about the point of projection acting on the particle when it is at its maximum height ?



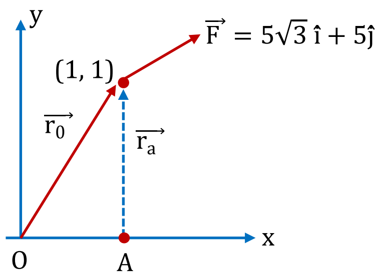
Solution:

$$\tau = rF\sin\theta = \frac{R}{2}mg = \frac{v_0^2 \sin 2\theta}{2g}\, mg$$

$$\Rightarrow \qquad \tau = \frac{mv_0{}^2 \sin 2\theta}{2}$$

Illustration 34:

Find the torque about point O and A due to given force $\vec{F}$.



Solution:

Torque about point O,

$$\vec{\tau} = \vec{r}_0 \times \vec{F}, \vec{r}_0 = \hat{\imath} + \hat{\jmath}, \vec{F} = 5\sqrt{3}\hat{\imath} + 5\hat{\jmath}$$

$$\vec{\tau} = (\hat{\imath} + \hat{\jmath}) \times (5\sqrt{3}\hat{\imath} + 5\hat{\jmath}) = 5(1 - \sqrt{3})\hat{k}$$

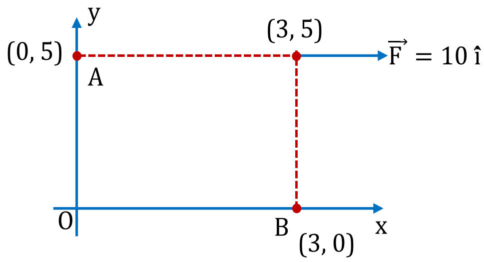
Torque about point A,

$$\vec{\tau} = \vec{r}_a \times \vec{F}, \vec{r}_a = \hat{\jmath}, \vec{F} = 5\sqrt{3}\hat{\imath} + 5\hat{\jmath}$$

$$\vec{\tau} = \hat{\jmath} \times (5\sqrt{3}\hat{\imath} + 5\hat{\jmath}) = 5(-\sqrt{3})\hat{k}$$

Illustration 35:

Find out torque about point A, O and B due to given force $\vec{F} = 10\hat{\imath}$.



Solution:

Torque about point A,

$$\vec{\tau}_A = \vec{r}_A \times \vec{F}, \vec{r}_A = 3\hat{\imath}, \vec{F} = 10\hat{\imath}$$

$$\vec{\tau}_A = 3\hat{\imath} \times 10\hat{\imath} = 0$$

Torque about point B,

$$\vec{\tau}_B = \vec{r}_B \times \vec{F}, \vec{r}_B = 5\hat{\jmath}, \vec{F} = 10\hat{\imath}$$

$$\vec{\tau}_B = 5\hat{\jmath} \times 10\hat{\imath} = -50\hat{k}$$

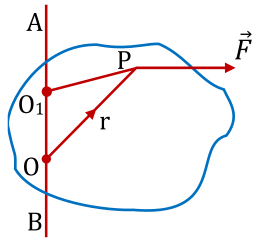
Torque about point O,

$$\vec{\tau}_O = \vec{r}_O \times \vec{F}, \vec{r}_O = 3\hat{\imath} + 5\hat{\jmath}, \vec{F} = 10\hat{\imath}$$

$$\vec{\tau}_O = (3\hat{\imath} + 5\hat{\jmath}) \times 10\hat{\imath} = -50\hat{k}$$

Torque about an axis :

The torque of a force $\vec{F}$ about an axis AB is defined as the component of torque of $\vec{F}$ about any point O on the axis AB, along the axis AB.



In the given figure torque of $\vec{F}$ about O is $\vec{\tau_0} = \vec{r} \times \vec{F}$

The torque of $\vec{F}$ about AB, τ_{AB} is component of $\vec{\tau_0}$ along line AB.

There are four cases of torque of a force about an axis.:

Case I : Force is parallel to the axis of rotation $\vec{F} \parallel AB$

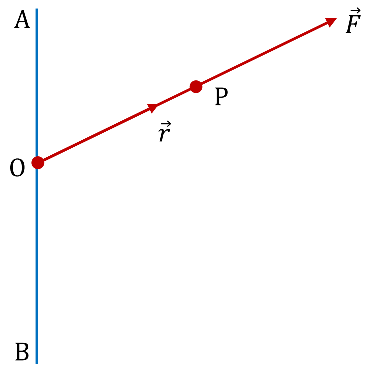
AB is the axis of rotation about which torque is required

$\vec{r} \times \vec{F}$ is perpendicular to $\vec{F}$, but $\vec{F} \parallel AB$, hence $\vec{r} \times \vec{F}$ is perpendicular toAB.

The component of $\vec{r} \times \vec{F}$ along AB is, therefore, zero.

Case II : The line of force intersects the axis of rotation (F intersect AB) $\vec{F}$ intersects AB along $\vec{r}$, then $\vec{F}$and $\vec{r}$are along the same line. The torque about O is $\vec{r} \times \vec{F} = 0$.

Hence component this torque along line AB is also zero.

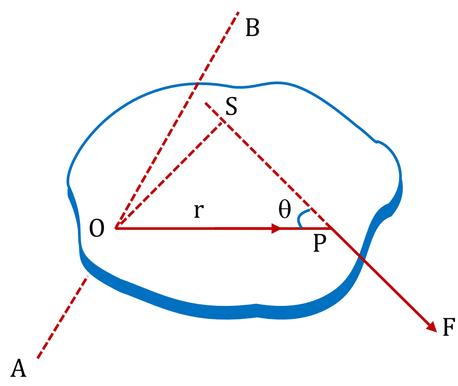


Case III : $\vec{F}$ perpendicular to ABbut $\vec{F}$ and AB do not intersect.

In three dimensions, two lines can be perpendicular without intersecting each other.

Two nonparallel and nonintersecting lines are called skew lines.

Figure shows the plane through the point of application of force P that is perpendicular to the axis of rotation AB. Suppose the plane intersects the axis at the point O. The force F is in this plane (since F is perpendicular to AB). Taking the origin at O,



Torque = $\vec{r} \times \vec{F} = \overrightarrow{OP} \times \vec{F}$.

Thus, torque = $rF \sin\theta = F(OS)$

where OS is the perpendicular from O to the line of action of the force $\vec{F}$. The line OS is also perpendicular to the axis of rotation. It is thus the length of the common perpendicular to the force and the axis of rotation.

The direction of $\vec{\tau} = \overrightarrow{OP} \times \vec{F}$ is along the axis AB because $\overrightarrow{AB} \perp \overrightarrow{OP}$and $\overrightarrow{AB} \perp \vec{F}$. The torque about AB is therefore equal to the magnitude of $\vec{\tau}$ that is $F.(OS)$.

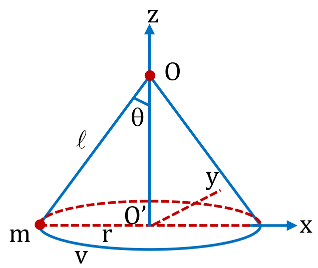
Thus, the torque of $\vec{F}$ about AB = magnitude of the force F × length of the common perpendicular to the force and the axis. The common perpendicular OS is called the lever arm or moment arm of this torque.

Case IV : $\vec{F}$ and AB are skew but not perpendicular.

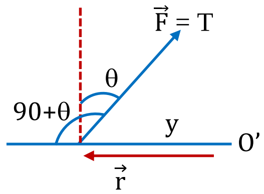
Here we resolve $\vec{F}$ into two components, one is parallel to axis and other is perpendicular to axis. Torque of the parallel part is zero and that of the perpendicular part can be found, by using the result of **case (III)**.

Illustration 36:

A bob of mass m is suspended at point O by string of length ℓ. Bob is moving in a horizontal circle find out: (i) Torque of gravity and tension about point O and O', (ii) Net torque about axis OO'.



Solution:



(i) Torque about point O

Torque of tension (T), $\tau_T = 0$ (tension is passing through point O)

Torque of gravity, $\tau_{mg} = mg\ell \sin\theta$

Torque about point O'

Torque of gravity, $\tau_{mg} = mgr$, $r = \ell \sin\theta$

$$\tau_{mg} = mg\ell \sin\theta \text{ (along positive } \hat{j})$$

Torque of tension, $\tau_T = Tr\sin(90+\theta)$ $(T\cos\theta = mg)$

$\tau_T = Tr\cos\theta$

$\tau_T = \dfrac{mg}{\cos\theta}(\ell \sin\theta)\cos\theta = mg\ell \sin\theta$ (along negative $\hat{j}$)

(ii) Torque about axis OO'

Torque of gravity about axis, OO' $\tau_{mg} = 0$ (Force mg parallel to axis OO')

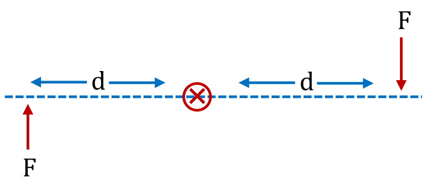
Torque of tension about axis, OO' $\tau_T = 0$ (Force T is passing through the axis OO')

Net torque about axis OO', $\tau_{net} = 0$

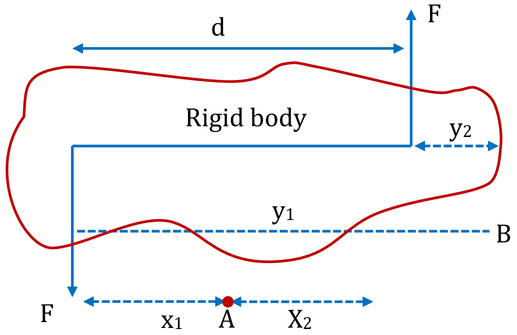
Force Couple

A pair of forces each of same magnitude and acting in opposite direction is called a force couple. Torque due to couple = Magnitude of one force × perpendicular distance between their lines of action.

Magnitude of torque = $\tau = F(2d)$



- A couple does not exert a net force on an object even though it exerts a torque.
- Net torque due to a force couple is same about any point.



Torque about A $= x_1F + x_2F = F(x_1 + x_2) = Fd$

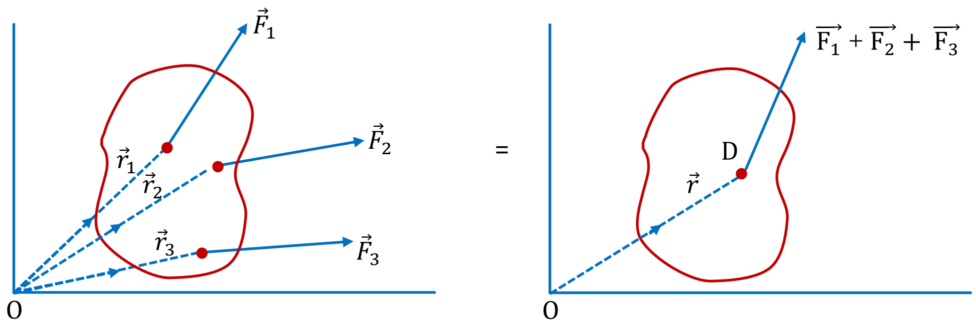
Torque about B $= y_1F - y_2F = F(y_1 - y_2) = Fd$

- If net force acting on a system is zero, torque is same about any point.
- A consequence is that, if $F_{net} = 0$ and $\tau_{net} = 0$ about one point, then $\tau_{net} = 0$ about any point.

Point of Application of Force :

Point of Application of force is the point at which, if net force is assumed to be acting, then it will produce same translational as well as rotational effect, as was produced earlier.

We can also define point of application of force as a point about which torque of all the forces is zero.

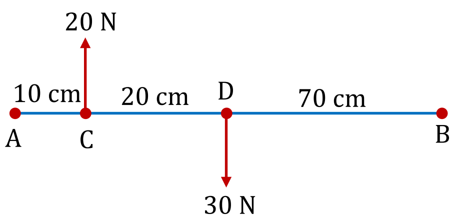


Consider three forces $\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$ acting on a body if D is point of application of force then torque of $\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$ acting at a point D about O is same as the original torque about O.

$$\left[\overrightarrow{r_1} \times \overrightarrow{F_1} + \overrightarrow{r_2} \times \overrightarrow{F_2} + \overrightarrow{r_3} \times \overrightarrow{F_3}\right] = \vec{r} \times \left(\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}\right)$$

Illustration 37:

Determine the point of application of force, when forces of 20 N and 30 N are acting on the rod as shown in figure.



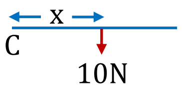
Solution:

Net force acting on the rod, $F_{net} = 10N$

Net torque acting on the rod about point C

$$\tau_c = (20 \times 0) + (30 \times 20) = 600 \text{ clockwise}$$

Let the point of application be at a distance x from point C



$$600 = 10\,x$$

$\Rightarrow$ $x = 60\ cm$

$\therefore$ 70 cm from A is point of Application of force.

Note :

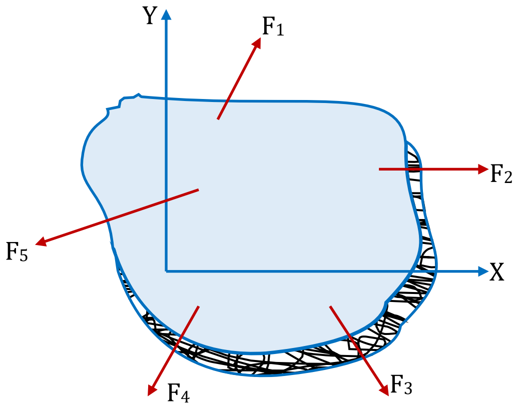
(i) Point of application of gravitational force is known as the centre of gravity.

(ii) Centre of gravity coincides with the centre of mass if value of force is assumed to be constant.

(iii) Concept of point of application of force is imaginary, as in some cases it can lie outside the body.

Equilibrium

A system is in mechanical equilibrium if it is in translational as well as rotational equilibrium.



For this : $\overrightarrow{F_{net}} = 0$

$\vec{\tau}_{net} = 0$ (about every point)

if $\vec{F}_{\text{net}} = 0$ then $\vec{\tau}_{\text{net}}$ is same about every point

Hence necessary and sufficient condition for equilibrium is

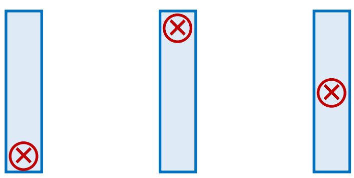
$\vec{F}_{\text{net}} = 0,$

$\Rightarrow$ $\vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} + \vec{F_5} = 0$

$\vec{\tau}_{\text{net}} = 0$ about any one point, which we can choose as per our convenience.

($\vec{\tau}_{\text{net}}$ will automatically be zero about every point)

- The equilibrium of a body is called **stable** if the body tries to regain its equilibrium position after being slightly displaced and released. It is called **unstable** if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in **neutral** equilibrium.



Unstable equilibrium   Stable equilibrium   Neutral equilibrium

Illustration 38:

Two small kids weighing 10 kg and 15 kg are trying to balance a seesaw of total length 5.0 m, with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit ?

Solution:

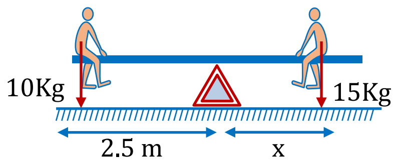
It is clear that the 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is x. As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are :

(a) (15 kg) g downward by the 15 kg kid,

(b) (10 kg) g downward by the 10 kg kid,

(c) Weight of the seesaw and

(d) The normal force by the fulcrum.

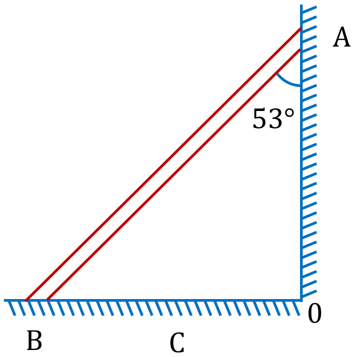


Taking torques about the fulcrum,

$(15\text{ kg})g\,x = (10\text{ kg})g\,(2.5\ m)$ or $x = 1.7\ m$

Illustration 39:

A uniform ladder of mass m = 10 kg leans against a smooth vertical wall making an angle $\theta = 53°$ with it. The other ends rests on a rough horizontal floor. Find the normal force and the friction force that the floor exerts on the ladder.



Solution:

The forces acting on the ladder are shown in figure. They are:

(a) Its weight W,

(b) Normal force N_1 by the vertical wall,

(c) Normal force N_2 by the floor and

(d) Frictional force f by the floor.

Taking horizontal and vertical components,

$N_1 = f$ ...(i)

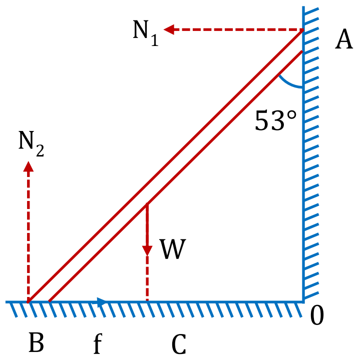
and $N_2 = mg$ ...(ii)

Taking torque about B,

$N_1(AO) = mg(CB)$

or $N_1(AB)\cos\theta = mg\dfrac{AB}{2}\sin\theta$

or $N_1\dfrac{3}{5} = \dfrac{W}{2}\,\dfrac{4}{5}$

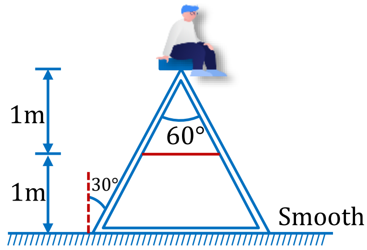


or $\quad N_1 = \frac{2}{3} W \qquad$ ...(iii)

The normal force by the floor is $N_2 = W = (10\ kg)\ (9.8\ m/s^2) = 98\ N.$

The frictional force is $f = N_1 = \frac{2}{3} W = 65\ N.$

Illustration 40:



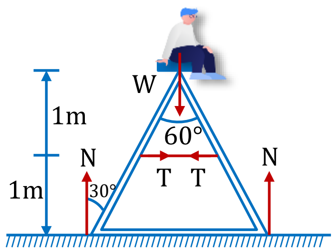
The ladder shown in figure has negligible mass and rests on a frictionless floor. A crossbar connects the two legs of the ladder at the centre as shown. The angle between the two legs is 60°. The fat person sitting on the ladder has a mass of 80 kg. Find the constant forces exerted by the floor on each leg and the tension in the crossbar.

Solution:

The forces acting on different parts are shown in figure. Consider the vertical equilibrium of "the ladder plus the person" system. The forces acting on this system are its weight (80 kg)g and the contact force $N + N = 2N$ due to the floor. Thus

$$2\ N = (80\ kg)\ g$$

or $\quad N = (40\ kg)\ (9.8\ m/s^2) = 392\ N.$



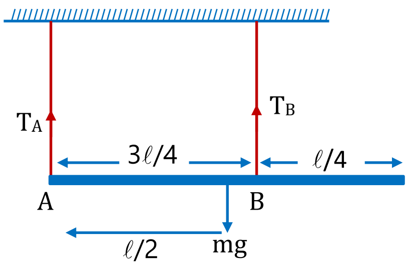
Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

$$N\ (2m)\ \tan 30° = T(1\ m)$$

or $\quad T = N \frac{2}{\sqrt{3}} = (392\ N) \times \frac{2}{\sqrt{3}} = 450\ N$

Illustration 41:

A uniform rod of length ℓ, mass m is hung from two strings of equal length from a ceiling as shown in figure. Determine the tensions in the strings ?



Solution:

$T_A + T_B = mg \qquad$ ...(i)

Net torque about point A is zero

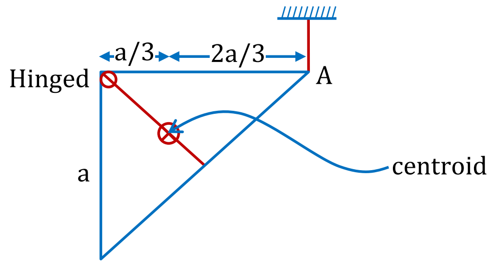
So, $\quad T_B \times \frac{3\ell}{4} = mg\ \frac{\ell}{2} \qquad$ ...(ii)

From equation (i) and (ii),

$$T_A = mg/3,\ \ T_B = 2mg/3$$

Illustration 42:

A right angle triangular plate of mass *m* is hinged from a corner and hangs from other corner with help of string from a ceiling and is in equilibrium. Find tension in string.



Solution:

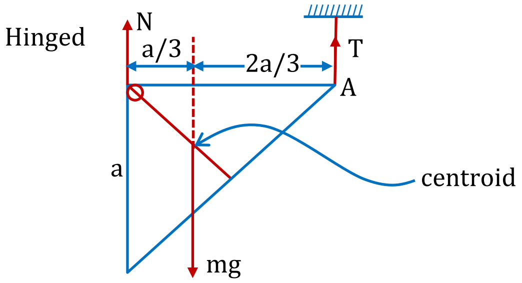
$$\sum F = 0$$

$\Rightarrow$ $T + N = mg$

$$\sum \tau_A = 0$$

$\Rightarrow$ $N \times a = \dfrac{2mga}{3}$

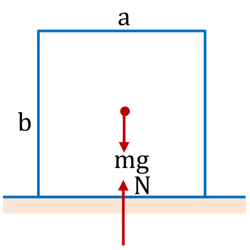
$$T = \frac{mg}{3}$$



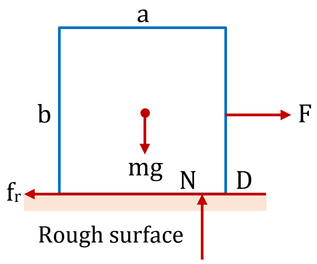
Toppling

In many situations an external force is applied to a body to cause it to slide along a surface. In certain cases, the body may tip over before sliding starts. This is known as toppling.

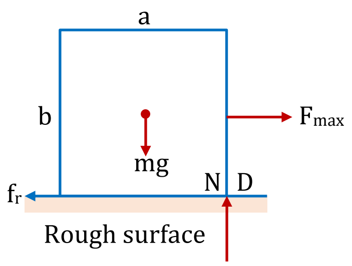
(1) There is no horizontal force so pressure at bottom is uniform and normal is collinear with *mg*.



(2) If a force is applied at COM, pressure is not uniform, normal reaction shifts right so that torque of *N* can counter balance torque of friction.



(3) If F is continuously increased N keeps shifting towards right until it reaches the right most point D. Here we have assumed that the surface is sufficiently rough so that there is no sliding as F is increase to F_{max}. If force is increased any further, then torque of N can not counter balance torque of friction f_r and body will topple. The value of force now is the maximum value for which toppling will not occur F_{max}.



$F_{max} = f_r$

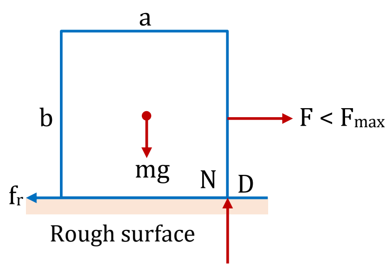
$N = mg$

Torque about COM

$f_r \cdot b/2 = N \cdot a/2$

$\Rightarrow \quad f_r = Na/b = mg\, a/b,\ F_{max} = mg\, a/b$

(4) If surface is not sufficiently rough and the body starts sliding before F is increased to $F_{max} = (mg\, a/b)$ then body will slide before toppling. Once body starts sliding friction becomes constant and hence no toppling. This is the case if



$F_{max} > f_{limiting}$

$\Rightarrow \quad mg\, a/b > \mu mg$

$\mu < a/b$

Condition for toppling when $\mu \geq a/b$ in this case body will topple if $F > mg\, a/b$ but, if $\mu < a/b$, body will not topple at any value of F applied on COM.

Golden Key Points

- For shown situation (A) and (B), more chances of toppling is in (A) then (B). In case of toppling, normal reaction must passes through end points.
- If block kept on smooth horizontal surface and a force F is applied at centre of mass in horizontal direction, for any value of force F, block can't topple.

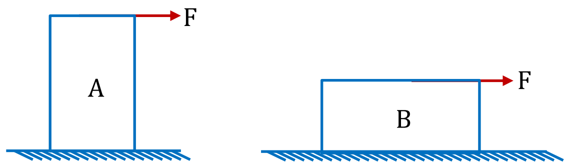
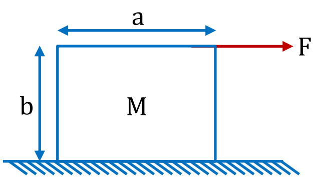


Illustration 43:

There is block of mass M with sides a and b. A force F is applied to the right upper corner of the block. Find the minimum value of F to topple the block about an edge.



Solution:

In case of toppling

Taking torque about O

$$F(b) = Mg\left(\frac{a}{2}\right)$$

$$\Rightarrow \qquad F_{min} = \frac{Mga}{2b}$$

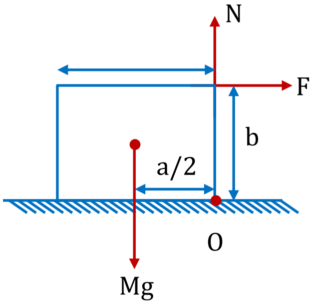
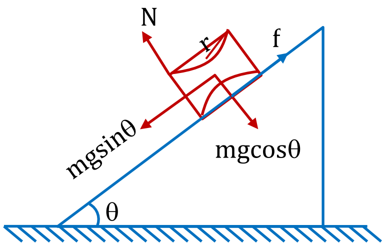


Illustration 44:

A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will:

(a) slide before toppling (b) topple before sliding

Solution:



(a) The cylinder will slide if $mg \sin\theta > \mu mg \cos\theta \Rightarrow \tan\theta > \mu$ ...(i)

The cylinder will topple if $(mg\sin\theta)\frac{h}{2} > (mg\cos\theta)r \Rightarrow \tan\theta > \frac{2r}{h}$ ...(ii)

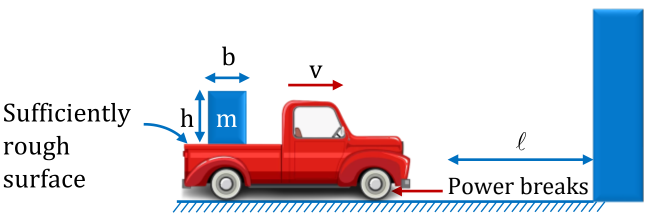
Thus, the condition of sliding is $\tan\theta > \mu$ and condition of toppling is $\tan\theta > \frac{2r}{h}$

Hence, the cylinder will slide before toppling if $\mu < \frac{2r}{h}$

(b) The cylinder will topple before sliding if $\mu > \frac{2r}{h}$

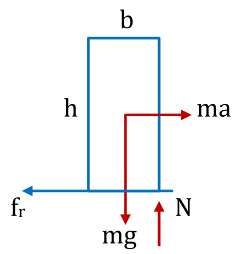
Illustration 45:

Find minimum value of ℓ so that truck can avoid the dead end, without toppling the block kept on it.



Solution:

$$ma\frac{h}{2} \le mg\frac{b}{2} \quad \Rightarrow \; a \le \frac{b}{h}g$$



Final velocity of truck is zero. So that

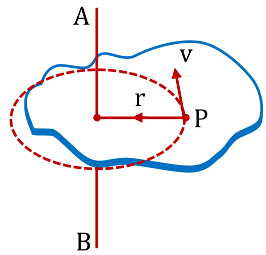
$$0 = v^2 - 2(\frac{b}{h}g)\,\ell$$

$$\ell = \frac{h}{2b}\frac{v^2}{g}$$

Rotation About a Fixed Axis

Consider a rigid body rotating about a fixed axis AB (figure). Consider a particle P of mass m rotating in a circle of radius r.

The radial acceleration of the particle = $\frac{v^2}{r} = \omega^2 r$.



(Where v is speed of particle, ω is angular speed of body and r is distance of particle from axis of rotation)

Thus, the radial force on it = $m\omega^2 r$

The tangential acceleration of the particle = $\frac{d|\vec{v}|}{dt}$

Thus, the tangential force on it $= m\frac{d|\vec{v}|}{dt} = mr\frac{d\omega}{dt} = mr\alpha$

The torque of $m\omega^2 r$ about AB is zero as it intersects the axis and that of $mr\alpha$ is $mr^2\alpha$ as the force and the axis are skew and perpendicular. Thus, the torque of the resultant force acting on P is $mr^2\alpha$. Summing over all the particles, the total torque of all the forces acting on all the particles of the body is

$$\tau_{total} = \sum_i m_i r_i^2 \alpha = I\alpha \qquad \text{...(i)}$$

where $I = \sum_i m_i r_i^2$

The quantity I is called the moment of inertia of the body about the axis of rotation. Note that m_i is the mass of the i^{th} particle and r_i is its perpendicular distance from the axis.

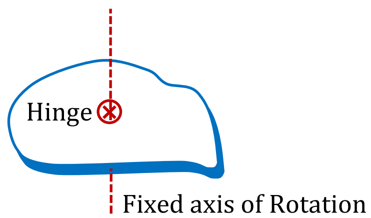
If I_{Hinge} = moment of inertia about the axis of rotation (since this axis passes through the hinge, hence the name given as I_{Hinge}

$\vec{\tau}_{ext}$ = Resultant external torque acting on the body about axis of rotation

α = Angular acceleration of the body.

$\vec{\tau}_{\text{ext hinge}} = I_{\text{Hinge}}\,\vec{\alpha}$

Rotational Kinetic Energy = $\frac{1}{2}.I.\omega^2$



$\vec{P} = M\vec{v}_{CM}$

$\vec{F}_{\text{external}} = M\vec{a}_{CM}$

Net external force acting on the body has two components tangential and centripetal.

⇒ $F_C = ma_C$

$$= m\frac{v^2}{r_{cm}} = m\omega^2 r_{cm}$$

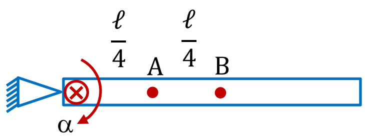
⇒ $F_t = ma_t = m\alpha\, r_{CM}$

- Total momentum of body is given by

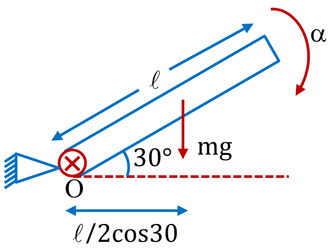
  $\vec{P} = M\vec{V}_{cm}$ (where M = Total mass of body)
- $\vec{P} = M\vec{V}_{cm}$
- If body is rotating about an axis passing through centre of mass, then net force on the body is zero. In this situation it is possible that angular velocity of body is variable.
- If body is rotating about an axis which is not passing through centre of mass, then net force on the body is not equal to zero in this situation it is possible that angular velocity of body is constant.

Illustration 46:

(a) A rod of Mass *'m'* and length *'l'* is hinged at one end, placed horizontal as shown in figure. Find angular acceleration just after releasing the rod and also find tangential acceleration of *A, B*.

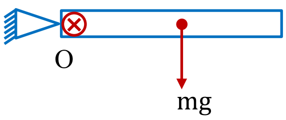


(b) A rod of Mass *'m'* and length *'l'* hinged at one end, as shown in figure. Find angular acceleration just after releasing the rod.



Solution:

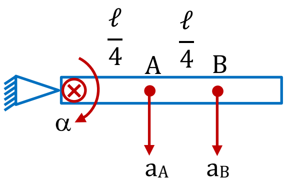
(a)



$$\tau_{H/0} + \tau_{mg/0} = I\alpha$$

$$0 + \frac{mg\ell}{2} = \frac{ml^2}{3}\times\alpha$$

$$\alpha = \frac{3g}{2\ell}$$



$$a_A = \frac{\alpha l}{4} = \frac{3g}{2l} \times \frac{l}{4} = \frac{3g}{8}$$

$$a_B = \frac{\alpha l}{2} = \frac{3g}{2l} \times \frac{l}{2} = \frac{3g}{4}$$

(b) $\tau_{H/0} + \tau_{mg/0} = I\alpha$

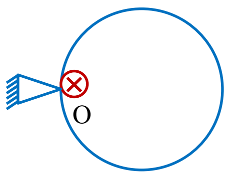
$$0 + \frac{mg\ell}{2}\cos 30° = \frac{m\ell^2}{3} \times \alpha$$

$$\alpha = \frac{3\sqrt{3}g}{4\ell}$$

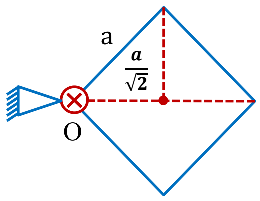
Illustration 47:

Find angular acceleration of the given bodies just after releasing them.

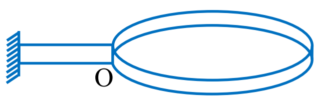
(1) Disc (M, R)



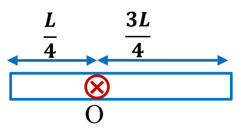
(2) Square plate (M, a)



(3) Disc (M, R)



(4) Rod (M, L)



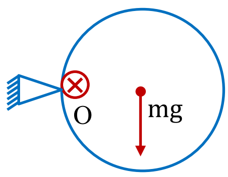
Solution:

(In all the cases taking clockwise direction as positive)

(1) $\tau_{H/0} + \tau_{mg/0} = I\alpha$

$$0 + mgR = \frac{3}{2}mR^2 \times \alpha$$

$$\frac{2g}{3R} = \alpha$$

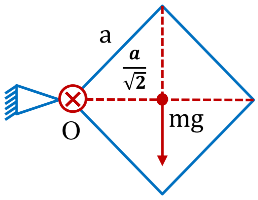


(2) $\tau_{H/0} + \tau_{mg/0} = I\alpha$

$$0+\frac{mga}{\sqrt{2}}=\left(\frac{ma^2}{6}+m\left(\frac{a}{\sqrt{2}}\right)^2\right)\alpha$$

$$\frac{mga}{\sqrt{2}}=\frac{4ma^2}{6}\times\alpha$$

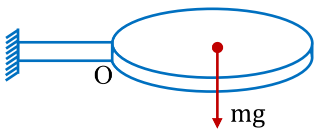
$$\frac{3g}{2\sqrt{2}a}=\alpha$$



(3) $\tau_{H/0} + \tau_{mg/0} = I\alpha$

$$0+mgR=\frac{5}{4}mR^2\alpha$$

$$\alpha=\frac{4g}{5R}$$

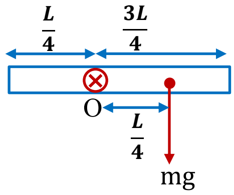


(4) $\tau_{H/0} + \tau_{mg/0} = I\alpha$

$$0+\frac{mgL}{4}=\left(\frac{ML^2}{12}+M\left(\frac{L}{4}\right)^2\right)\alpha$$

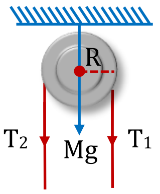
$$\frac{mgL}{4}=\frac{7ML^2}{48}\times\alpha$$

$$\frac{12g}{7L}=\alpha$$



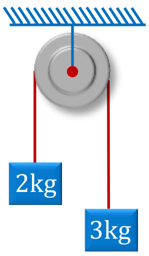
Angular acceleration of inertial pulley :

Rough pulley (M = Mass, R = Radius, I = Moment of Inertia.)

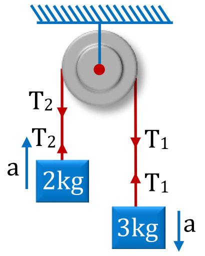


$T_2R - T_1R = I\alpha$

(eg.) Two block of masses 2 Kg and 3 Kg are hanging through of pulley of mass M, radius R, moment of inertia I as shown in the given figure. Find acceleration of block and angular acceleration of pulley. (Assume there is no slipping between pulley and string).



Solution:



$T_2 - 2g = 2a$ ...(1)

$3g - T_1 = 3a$ ...(2)

$T_1R - T_2R = I\alpha$ ...(3)

$\alpha R = a$ ...(4)

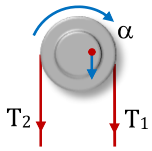
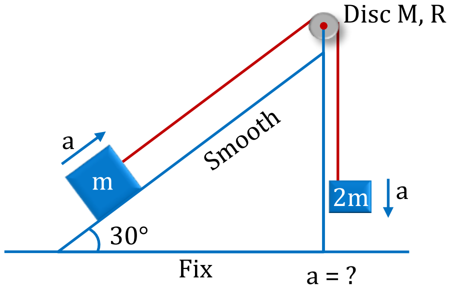


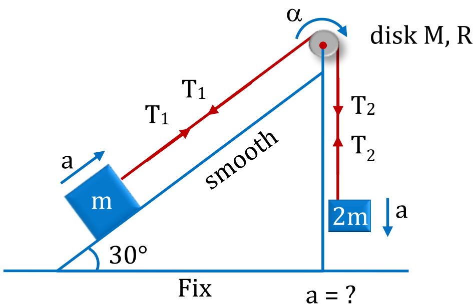
Illustration 48:

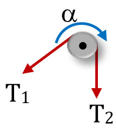
A block of mass $2m$ is connected with a block of mass m via a pulley of mass M, radius R, which is connected across an inclined as shown in the figure. Find acceleration of blocks ?

(Assume there is no slipping between pulley and string).



Solution:





$T_1 - mg\sin 30° = ma$ ...(1)

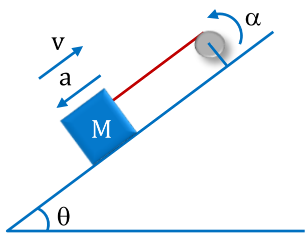
$2mg - T_2 = 2ma$ ...(2)

$T_1R\text{-}T_2R = I\alpha$ ...(3)

$a = R\alpha$ ...(4)

Illustration 49:

A wheel of radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in figure. A string is wrapped round the wheel and its free end supports a block of mass M which can slide on the plane. Initially, the wheel is rotating at a speed ω in a direction such that the block slides up the plane. How far will the block move before stopping ?



Solution:

Suppose the deceleration of the block is a.

The linear deceleration of the rim of the wheel is also a.

The angular deceleration of the wheel is $\alpha = a/r$.

If the tension in the string is T, the equations of motion are as follows:

$$Mg \sin\theta - T = Ma \quad \text{and} \quad Tr = I\alpha = Ia/r$$

Eliminating T from these equations,

$$Mg \sin\theta - I\frac{a}{r^2} = Ma$$

Giving, $a = \dfrac{Mgr^2 \sin\theta}{I + Mr^2}$

The initial velocity of the block up the incline is $v = \omega r$. Thus, the distance moved by the block before stopping is

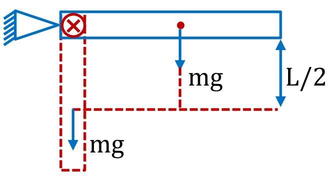
$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + Mr^2)}{2Mgr^2 \sin\theta} = \frac{(I + Mr^2)\omega^2}{2Mg\sin\theta}$$

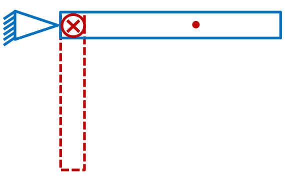
Application of work energy theorem in pure rotation :

KE of Rigid Body performing pure rotation about $AOR = \dfrac{1}{2} I_{AOR}\omega^2$

Illustration 50:

A rod of mass m, length L is hinged at one and released from horizontal position in a vertical plane. Find its angular velocity when it becomes vertical ?





Solution:

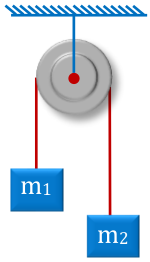
$$W_g + W_H = KE_f - KE_i$$

$$mg\frac{L}{2} + 0 = \frac{1}{2}I\omega^2 - 0$$

We know $I = \dfrac{mL^2}{3}$

$$\frac{mgL}{2} = \frac{mL^2}{6} \times \omega^2 \Rightarrow \omega = \sqrt{\frac{3g}{L}}$$

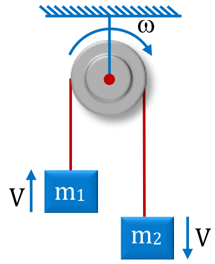
Illustration 51:



In the given figure two blocks of masses m_1 and m_2 ($m_2 > m_1$) are connected via pulley. The system is released from rest, after m_2 have fallen a height h find velocity of blocks and angular velocity of the disc (The disc has mass M and radius R).

Solution:

(Three as a system)



$$W_g + W_T = KE_f - KE_i$$

$$(-m_1gh + m_2gh) + 0 = \frac{1}{2}I\omega^2 + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 - 0$$

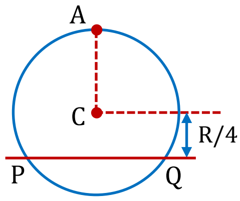
$$(m_2 - m_1)gh = \frac{1}{2}I\frac{v^2}{R^2} + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2$$

$$(m_2 - m_1)gh = \left(m_1 + m_2 + \frac{I}{R^2}\right)\frac{v^2}{2}$$

$$\sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + \frac{I}{R^2}}} = v$$

Illustration 52:

A uniform circular disc has radius R and mass m. A particle also of mass m is fixed at a point A on the edge of the disc as shown in figure. The disc can rotate freely about a fixed horizontal chord PQ that is at a distance $R/4$ from the centre C of the disc. The line AC is perpendicular to PQ. Initially the disc is held vertical with the point A at its highest position. It is then allowed to fall so that it starts rotating about PQ. Find the linear speed of the particle when it reaches its lowest position.



Solution:

$$I_{PQ} = \frac{mR^2}{4} + \frac{mR^2}{16} = \frac{5mR^2}{16}$$

Applying energy conservation

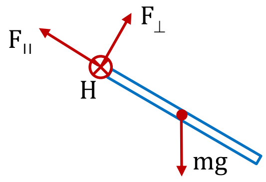
$$mg\,\frac{5R}{2} + mg\,\frac{R}{2} = \left[\frac{1}{2}\times\frac{5mR^2}{16} + \frac{1}{2}m\frac{25R^2}{16}\right]\omega^2$$

$$\omega = \sqrt{\frac{48g}{15R}}$$

$$V = \frac{5R}{4}\times\omega$$

Calculation of Hinge Force

Parallel i.e. along the rod



$$\sum F_{along} = m(a_{CM})_n = m\omega^2 \frac{L}{2}$$

Flow : $\omega \rightarrow E \rightarrow \omega \rightarrow (a_{CM})_n \rightarrow F_{||}$

Perpendicular to rod

$$\sum F_t = m(a_{CM})_t = m\alpha \frac{L}{2}$$

Flow : $\tau = I\alpha \rightarrow \alpha \rightarrow a_t \rightarrow F_{\perp}$

$$F_H = \sqrt{F_{||}^2 + F_{\perp}^2}$$

Illustration 53:

A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.

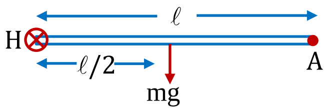
(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest ?

(ii) Calculate the acceleration (tangential and radial) of point A at this moment.

(iii) Calculate net hinge force acting at this moment.

Solution:

(i) $\tau_H = I_H \alpha$



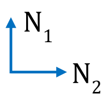
$$mg\frac{\ell}{2} = \frac{m\ell^2}{3} \times \alpha \Rightarrow \alpha = \frac{3g}{2\ell}$$

(ii) $a_{tA} = \alpha\ell = \frac{3g}{2\ell}\ell = \frac{3g}{2}$

$a_{CA} = \omega^2 r = 0$ ($\omega = 0$ just after release)

(iii) Suppose hinge exerts normal reaction in component form as shown

In vertical direction $F_{ext} = ma_{CM}$



$\Rightarrow$ $mg - N_1 = m.\frac{3g}{4}$

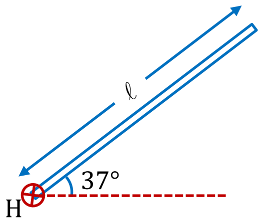
$\Rightarrow$ $a_{cm} = \frac{\ell}{2}\alpha \Rightarrow N_1 = \frac{mg}{4}$

In horizontal direction

$F_{ext} = ma_{CM} \Rightarrow N_2 = 0$ (a_{CM} in horizontal = 0 as $\omega = 0$ just after release).

Illustration 54:

A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find angular acceleration α of the rod just after it is released from initial position making an angle of 37° with horizontal from rest? Find force exerted by the hinge just after the rod is released from rest.



Solution:

Torque about hinge = $\tau_H = I\alpha$

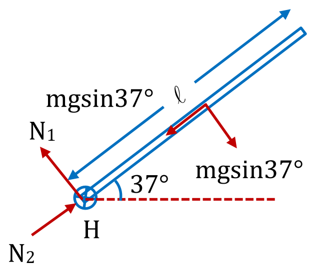
$$mg\cos 37°\,\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha$$

$$\alpha = 6g/5\ell$$

$$a_t = \alpha\,\frac{\ell}{2} = \frac{3g}{5}$$

$$mg\cos 37° - N_1 = ma_t$$

$$N_1 = \frac{mg}{5}$$

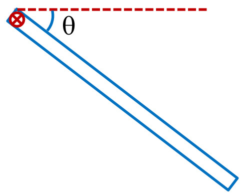


Angular velocity of rod is zero. so $N_2 = mg\sin 37° = 3mg/5$

$$N = \sqrt{N_1^2 + N_2^2} = \sqrt{\left(\frac{mg}{5}\right)^2 + \left(\frac{3mg}{5}\right)^2} = \frac{mg\sqrt{10}}{5}$$

Illustration 55:

Solve previous eg. when rod is making an angle θ with horizontal.



Solution:

$$mg\cos\theta \times \frac{\ell}{2} = \frac{m\ell^2}{3} \times \alpha$$

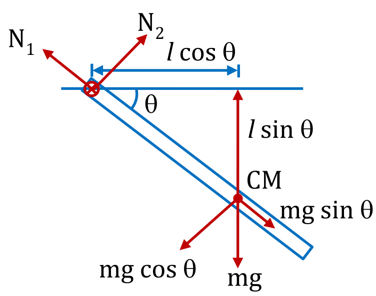
$$\alpha = \frac{3g\cos\theta}{2\ell}$$

Energy conservation

$$mg\frac{\ell}{2}\sin\theta = \frac{1}{2}\frac{m\ell^2}{3}\omega^2$$

$$\omega = \sqrt{\frac{3g\sin\theta}{\ell}}$$

$$a_r = \omega^2\,\frac{\ell}{2} = \frac{3g\sin\theta}{\ell} \times \frac{\ell}{2} = \frac{3g\sin\theta}{2}$$



$$a_t = \alpha\,\frac{\ell}{2} = \frac{3g\cos\theta}{4}$$

$$N_1 - mg\sin\theta = m\,\frac{3g\sin\theta}{2}$$

$$N_1 = \frac{5mg\sin\theta}{2}$$

$$mg\cos\theta - N_2 = m\,\frac{3g\cos\theta}{4}$$

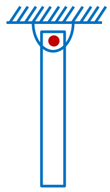
$$N_2 = \frac{mg\cos\theta}{4}$$

$$N = \sqrt{N_1^2 + N_2^2}$$

$$|\vec{P}| = m\omega\,\frac{\ell}{2} = \frac{m\sqrt{3g\ell\sin\theta}}{2}$$

Illustration 56:

A rod of mass m and length ℓ is hinged at it's upper end. It can rotate in vertical plane. It is given angular velocity ω so that it can complete vertical circle. Find (a) angular velocity of rod (b) Tension at centre of rod at initial moment.

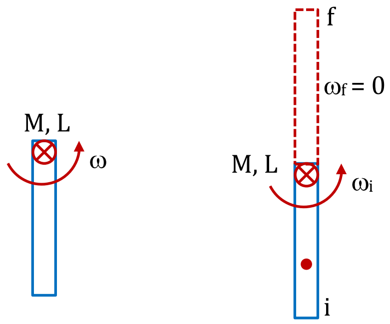


Solution:

(a) $W_g + W_H = K_f - K_i$

$$-mg\ell + 0 = 0 - \frac{1}{2}\frac{mL^2}{3}\omega^2$$

$$\omega = \sqrt{\frac{6g}{\ell}}$$



(b) T at mid point of rod ?

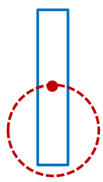
$$T - m'g = m'\omega^2 R$$

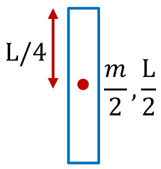
$$m' = \frac{m}{2}$$

$$T - \frac{m}{2}g = \frac{m}{2}\frac{6g}{L}\times\frac{3L}{4}$$

$$T = \frac{9mg}{4} + \frac{mg}{2}$$

$$T = \frac{11mg}{4}$$





Angular Momentum ($\vec{L}$)

Angular momentum of a particle about a point.

$$\vec{L} = \vec{r} \times \vec{P} \qquad \Rightarrow \qquad L = rp\sin\theta$$

$$|\vec{L}| = r_\perp \times P \qquad \text{or} \qquad |\vec{L}| = P_\perp \times r$$

Where $\vec{P}$ = Momentum of particle

$\vec{r}$ = Position vector of particle with respect to point O about which angular momentum is to be calculated.

θ = Angle between vectors $\vec{r}$ and $\vec{P}$

$r_\perp$ = Perpendicular distance of line of motion of particle from point O.

$P_\perp$ = Component of momentum perpendicular to $\vec{r}$.

SI unit of angular momentum is kgm^2/sec.

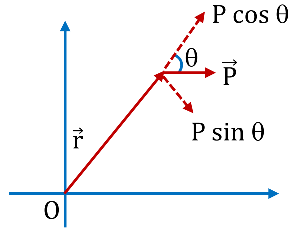


Illustration 57:

A particle of mass $'m'$ is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal. Find out the angular momentum of particle about the point of projection when

(i) it just starts its motion (ii) it is at highest point of path (iii) it just strikes the ground

Answer: (i) 0 ; (ii) $mu\cos\theta\,\dfrac{u^2\sin^2\theta}{2g}$; (iii) $mu\sin\theta\,\dfrac{u^2\sin 2\theta}{g}$

Solution :

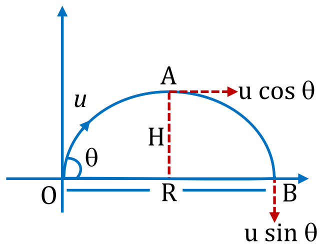
(i) Angular momentum about point O is zero.

(ii) Angular momentum about point A.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L = H \times mu\cos\theta$$

$$L = mu\cos\theta\,\frac{u^2\sin^2\theta}{2g}$$



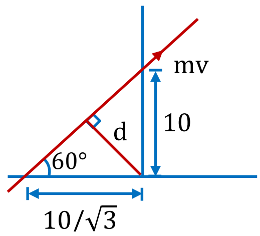
(iii) Angular momentum about point B.

$$L = R \times mu\sin\theta$$

$$= mu\sin\theta\,\frac{u^2\sin 2\theta}{g}$$

Illustration 58:

A particle of mass m is moving along line $y = \sqrt{3}x + 10$ with velocity v. Find its angular momentum about origin.



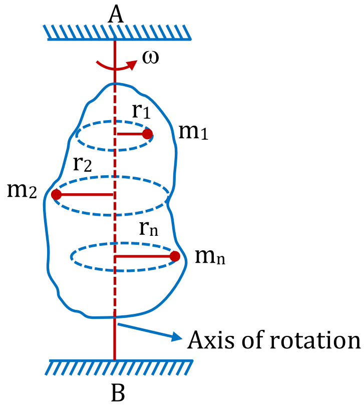
Solution:

$$d = \frac{10}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 5$$

$$L = 5\ mv$$

Angular momentum of a rigid body rotating about fixed axis :

Angular momentum of a rigid body about the fixed axis AB is



$$L_{AB} = L_1 + L_2 + L_3 + \ldots\ldots + L_n$$

$$L_1 = m_1 r_1\omega r_1,\ L_2 = m_2 r_2\omega r_2,\ L_3 = m_3 r_3\omega r_3,\ L_n = m_n r_n\omega r_n$$

$$L_{AB} = m_1 r_1\omega r_1 + m_2 r_2\omega r_2 + m_3 r_3\omega r_3 \ldots\ldots + m_n r_n\omega r_n$$

$$L_{AB} = \sum_{n=1}^{n=n} m_n \quad (r_n)^2 \times \omega \qquad \left[\sum_{n=1}^{n=n} m_n \quad (r_n)^2 = I_H\right]$$

$$L_{AB} = I_H\ \omega$$

$$L_H = I_H\ \omega$$

L_H = Angular momentum of object about axis of rotation.

I_H = Moment of Inertia of rigid body about axis of rotation.

ω = Angular velocity of the object.

Illustration 59:

A uniform circular ring of mass 400 g and radius 10 cm is rotated about one of its diameter at an angular speed of 20 rad/s. Find the kinetic energy of the ring and its angular momentum about the axis of rotation.

Solution:

The moment of inertia of the circular ring about its diameter is

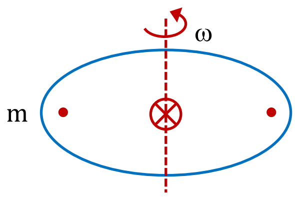
$$I = \frac{1}{2}\ Mr^2 = \frac{1}{2}\ (0.400\ kg)\ (0.10\ m)^2 = 2 \times 10^{-3}\ kg\text{-}m^2$$

The kinetic energy is $K = \frac{1}{2} I\omega^2 = \frac{1}{2}(2 \times 10^{-3}\ kg - m^2)\ (400\ rad^2/s^2) = 0.4\ J$ and the angular momentum about the axis of rotation is

$$L = I\omega = (2 \times 10^{-3}\ kg\text{-}m^2)\ (20\ rad/s) = 0.04\ kg\text{-}m^2/s = 0.04\ J\text{-}s$$

Illustration 60:

A disc of mass m and radius R is rotating about axis passing through centre with angular velocity ω. Two particles of same mass m are attached diametrically opposite at it's circumference, then calculate angular momentum of the system ?



Solution:

$$L_{sys} = L_{disk} + \ell_{m_1} + \ell_{m_2}$$

$$L_{sys} = \left(\frac{mR^2}{2}\omega\right) + m(\omega R)R + m(\omega R)R = \frac{5}{2} mR^2\omega$$

Note:

$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$ is valid only in inertial frame.

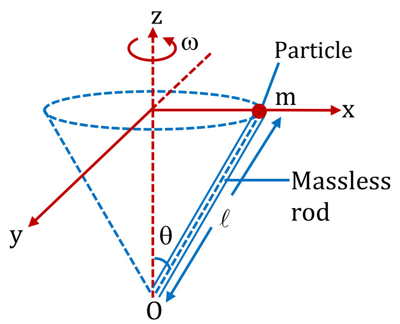
Angular momentum of an inverted conical pendulum :

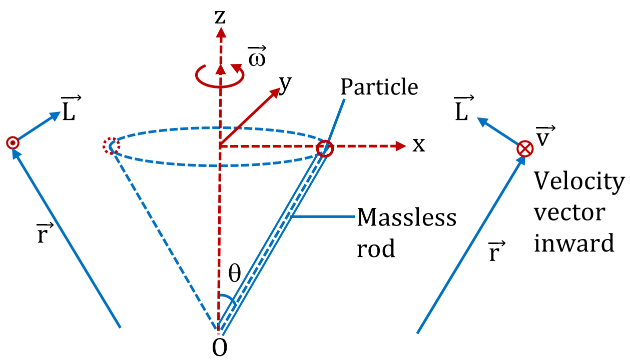
Angular momentum about O,

$\vec{r} = \ell \sin\theta\,\hat{\imath} + \ell\cos\theta\,\hat{k}$

$\vec{u} = (\ell\sin\theta)\omega\hat{\jmath}$

$\vec{L} = \vec{r}\times m\vec{v} = m\ell^2\sin^2\theta\,\omega\hat{k} - m\ell^2\omega\sin\theta\cos\theta\,\hat{\imath}$





Concept : Angular momentum vector $\vec{L}$ is perpendicular to position vector $\vec{r}$ as well as momentum vector $\vec{p}$. The magnitude of $\vec{L}$ is constant but its direction is continuously varying. As the particle swings, $\vec{L}$ vector sweeps out a cone. The z-component of $\vec{L}$ is constant but the horizontal component travels around the circle with the particle.

Relation between torque and angular momentum :

$\vec{L} = \vec{r}\times\vec{P}$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{P}) = \left(\frac{d\vec{r}}{dt}\right)\times\vec{P} + \vec{r}\times\frac{d\vec{P}}{dt}$$

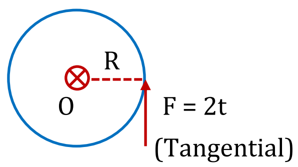
$$\frac{d\vec{L}}{dt} = \vec{V}\times\vec{P} + \vec{r}\times\vec{F} \quad ; \quad \text{since, } (\vec{V}\times\vec{P} = \vec{0}), (\vec{r}\times\vec{F} = \vec{\tau}_{ext})$$

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau}_{\text{axis}} = \frac{dL_{\text{axis}}}{dt}$$

Illustration 61:

A Disc of mass m and radius R is free to rotate about it's axis passing through the centre and perpendicular to the plane of disc. If a force F is applied in tangential direction as shown in figure then calculate angular velocity at time t.



Solution:

$$\tau_{net} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau}_{net} = \vec{R} \times \vec{F} = RF$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt}$$

$$\Rightarrow \quad RF = I\frac{d\omega}{dt}$$

$$\Rightarrow \quad R\int_0^t Fdt = \int_0^\omega \left(\frac{MR^2}{2}\right) d\omega$$

$$\Rightarrow \quad \int_0^t 2tdt = \frac{MR}{2}\int_0^\omega d\omega \Rightarrow t^2 = \frac{MR\omega}{2}$$

$$\Rightarrow \quad \omega = \frac{2t^2}{MR}$$

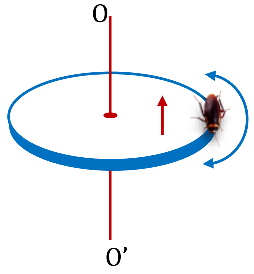
Conservation of angular momentum :

If external torque about an axis is zero then angular momentum of the system about that axis will remain conserved. So, if $\tau_{axis} = 0 \Rightarrow L_{axis}$ = constant.

Two Bodies Rotatory System :

Figure shows an insect at the rim of a disc that can turn about a frictionless axis. Initially the system is at rest. The insect crawls along the edge of the disc ; due to the friction force between disc and insect, the torque of friction rotates the system (disc + insect).

Since the net torque of the paired internal forces about the axis of rotation OO' is zero, therefore applying conservation of angular momentum about OO'.



$$\vec{L}_{\text{insect}} + \vec{L}_{\text{disc}} = 0$$

or $$|\vec{L}_{\text{insect}}| = |\vec{L}_{\text{disc}}|$$

$$I_1\omega_1 = I_2\omega_2 \qquad ...(A)$$

Differentiating the above with respect to time, we get

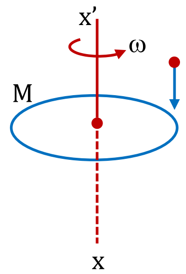
$$I_1\alpha_1 = I_2\alpha_2$$

Integrating equation (A), we get

$$\int I_1\frac{d\theta_1}{dt} = \int I_2\frac{d\theta_2}{dt} \quad ; \quad I_1\theta_1 = I_2\theta_2$$

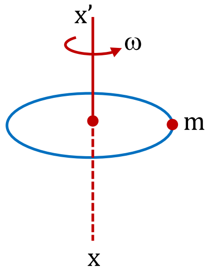
Illustration 62:

A disc of mass M and radius r is rotating about its axis with angular velocity ω. Now if a mass m falls vertically on its rim and sticks to it. Then find final angular velocity of the combined system.



Solution:

Applying angular momentum conversion,



$$I_1\omega_1 = I_2\omega_2$$

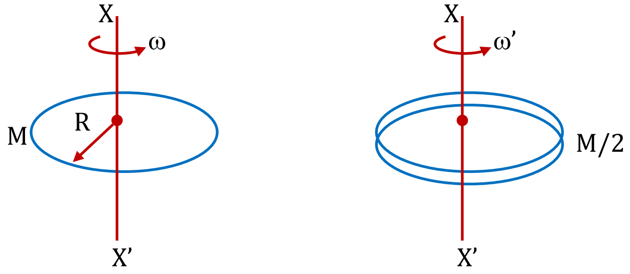
$$M\frac{R^2}{2}\omega = \left(\frac{MR^2}{2} + mR^2\right)\omega'$$

$$\omega' = \left(\frac{M}{M+2m}\right)\omega$$

Illustration 63:

In previous question if instead of particle a disc of mass $M/2$, and radius R, is kept on the first one, with axis coinciding, find final angular velocity of the system.

Solution:



Angular momentum conservation

$$\frac{MR^2}{2}\omega = \left(\frac{MR^2}{2} + \frac{M}{2}\frac{R^2}{2}\right)\omega' \quad \Rightarrow \omega' = \frac{2}{3}\omega$$

Illustration 64:

A star in form of sphere of (M, R) is rotating about its axis with angular velocity ω_0. If due to internal forces its radius becomes half uniformly. Find its new average angular velocity ω ?

Solution:

$$(\tau_{Net}) = 0$$

$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

$$\frac{2}{5}MR^2\omega_0 = \frac{2}{5}M\left(\frac{R}{2}\right)^2 \times \omega_f$$

$$\omega_f = 4\omega_0$$

Illustration 65:

A man of mass 100 kg stands at the rim of a turn-table of radius 2m, moment of inertia 4000 kg-m^2 mounted on a vertical frictionless shaft at its centre. The whole system is initially at rest. The man walks along the outer edge of the turn-table with a velocity of 1 m/s relative to the earth. Calculate following -

(a) With what angular velocity and in what direction does the turn-table rotate?

(b) Through what angle will it have rotated when the man reaches his initial position on the turn-table?

(c) Through what angle will it have rotated when the man reaches his initial position relative to earth?

Solution:

Let the man be moving anti-clockwise.

(a) By conservation of angular momentum on the man-table system,

$$\vec{L}_i = \vec{L}_f \text{ or } 0 + 0 = I_m\omega_m + I_t\omega_t$$

$$\omega_t = -\frac{I_m\omega_m}{I_t} \qquad \text{where } \omega_m = \frac{v}{r} = \frac{1}{2}\, rad/s$$

$$= -100\,(2)^2 \times \frac{1/2}{4000} = -\frac{1}{20}\, rad/s$$

Thus the table rotates clockwise (opposite to man) with angular velocity 0.05 *rad/s.*

(b) If the man complete one revolution relative to the table, then

$$\theta_{mt} = 2\pi \;;\quad 2\pi = \theta_m - \theta_t$$

$$2\pi = \omega_m t - \omega_t t \qquad \text{(where } t \text{ is the time taken)}$$

$$t = \frac{2\pi}{(\omega_m - \omega_t)} = \frac{2\pi}{0.5 + 0.05}$$

Angular displacement of table is

$$\theta_t = \omega_t t = -0.05 \times \frac{2\pi}{0.55} = -\frac{2\pi}{11} \text{ radian}$$

The table rotates through $2\pi/11$ radians clockwise.

(c) If the man completes one revolution relative to the earth,

then $\theta_m = 2\pi$

$$\text{Time} = \frac{2\pi}{\omega_m} = \frac{2\pi}{0.5}$$

During this time, angular displacement of the table,

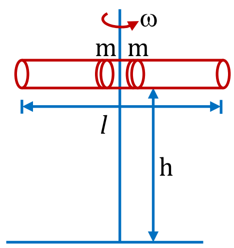
$$\theta_t = \omega_t\,(\text{time}) = -0.05 \times \frac{2\pi}{0.5}$$

$$\theta_t = -\frac{\pi}{5} \text{ radian}$$

$$\theta_t = 36° \text{ in clockwise direction}$$

Illustration 66:

Mass of rod is *M*. The rod is rotating with angular velocity ω when both rings of mass *m* are situated on axis. Find



(i) Angular velocity of rod when rings has reached the ends

(ii) Velocity of rings when they leave the rod

Solution:

Angular momentum of the system should remain constant

$$\frac{ML^2}{12} \times \omega + 0 \times \omega + 0 \times \omega = \left(\frac{ML^2}{12} + \frac{mL^2}{4} + \frac{mL^2}{4}\right) \times \omega_1$$

$$\omega_1 = \frac{\frac{M}{12}\omega}{\left(\frac{M}{12}+\frac{m}{2}\right)} = \frac{M\omega}{(M+6m)}$$

Conserving energy

$$\frac{1}{2}\frac{ML^2}{12} \times \omega^2 + 0 + 0 = \frac{1}{2}\frac{ML^2}{12}\omega_1^2 + \left(\frac{1}{2}mV^2\right)\times 2$$

$$\frac{ML^2\omega^2}{12}\left[1-\frac{M^2}{(M+6m)^2}\right] = 2mV^2$$

$$V^2 = \frac{ML^2\omega^2}{24m}\left[\frac{(M+6m-M)(M+6m+M)}{(M+6m)^2}\right]$$

$$= \frac{ML^2\omega^2}{24m}\left[\frac{12m(M+3m)}{(M+6m)^2}\right] = \frac{ML^2\omega^2(M+3m)}{2(M+6m)^2}$$

$$V^2 = \frac{ML^2\omega^2(M+3m)}{2(M+6m)^2}$$

$$V_{\perp}^2 + V_{||}^2 = V^2$$

$$V_{\perp} = \frac{\omega L}{2}$$

$$V_{||} = \sqrt{V^2 - \frac{(\omega L)^2}{4}}$$

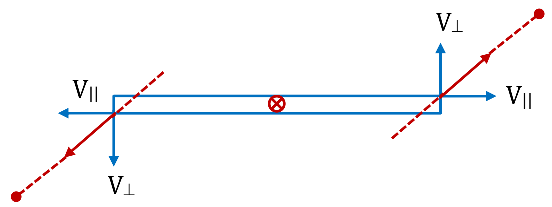
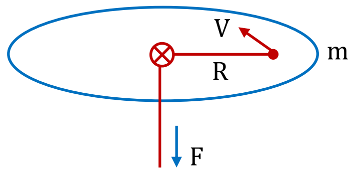


Illustration 67:

In the following figure string is pulled such that radius becomes half then calculate the value of v ?



Solution:

$$\tau_{net} = 0$$

$$\tau_{T/0} + (\tau_{mg} + t_N)_{/O} = 0$$

$$0 + 0 = 0$$

$\Rightarrow$ $\vec{L}$ conservation about O

$\therefore$ $L_i = L_f$

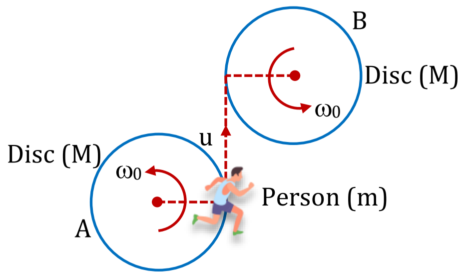
W_T on particle $= K_f - K_i$

$$= \frac{1}{2}m(2v_0)^2 - \frac{1}{2}mv_0^2$$

W_T on particle $= \frac{3}{2}mv_0^2 = W_f$

Illustration 68:

Two identical discs each of mass $M(= 4m)$ and radius R are rotating in opposite sense with equal angular speed ω_0 about vertical axis passing through their centres, (as shown in figure). A person of mass m sitting on circumference of disc A jumps with a tangential relative velocity u (after jumping) w.r.t one rotating disc (A) and lands on other disc (B) also tangential. Now the second disc (B) comes to a stop. Find the relative velocity u.



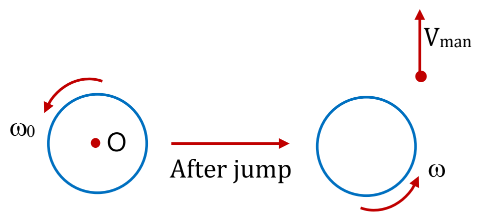
(A) $u=\frac{\omega_0 R}{2}$ (B) $u = \omega_0 R$

(C) $u=\frac{3\omega_0 R}{2}$ (D) $u = 2\omega_0 R$

Ans. (C)

Solution:

(a) For Disc A



$V_{man/disc} = u = v_{man} - v_{disc}$

if $= v_{disc} = \omega R$ (After jump)

$u = v_{man} - \omega R$

$v_{man} = u + \omega R$ ...(A)

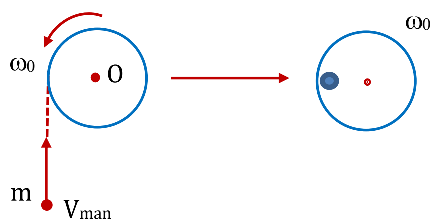
Applying angular momentum conservation:

$(L_{\text{initial}}) = (L_{\text{final}})$

$$\frac{MR^2}{2}\omega_0 + mR^2\omega_0 = \frac{MR^2}{2}\omega + mv_{man} \times R \quad \text{...(i)}$$

(b) For disc B

Applying conservation of angular momentum:



$L_{\text{initial}} = L_{\text{final}}$

$$(I_c\omega_0) - mv_{man}R = 0$$

$$\frac{MR^2}{2}\omega_0 = mv_{man}R$$

$$\frac{4mR^2\omega_0}{2} = mv_{man}R$$

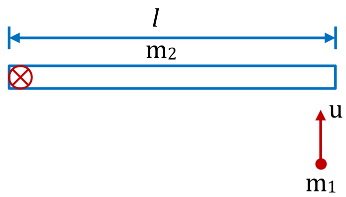
$$\Rightarrow \quad V_{man} = 2\omega_0 R \quad \text{...(ii)}$$

On Solving (i) and (ii)

$$\omega = \frac{\omega_0}{2}$$

$$u = \frac{3}{2}\omega_0 R$$

Illustration 69:



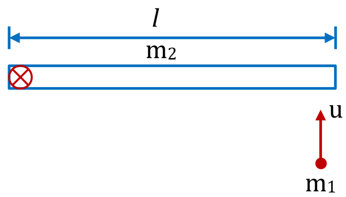
A rod AB of length l and mass m_2 is hinged at end A. A particle of mass m_1 sticks after collision to the end B then find :-

(a) Final angular velocity

(b) The impulse of force due to hinge during collision

(c) The impulse on particle

Solution:

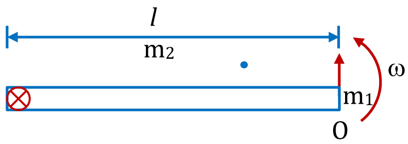


(a) $\tau_H = 0$

See L about hinge is conserved

Initially, $L_H = m_1 u\, l$

Finally, $L_H = \left(\dfrac{m_2 l^2}{3} + m_1 l^2\right)\omega$



Now, $m_1 u\ell = \left(\dfrac{3m_1\ell^2 + m_2\ell^2}{3}\right)\omega$

$$\omega = \frac{3m_1 u}{(m_2 + 3m_1)\ell}$$

(b) Change in linear momentum

$= F_H\,\Delta t$

$F_H\Delta t = -\,[(m_2\, v_{cm} + m_1\omega l) - m_1 u\,]$

$$= -\left[\frac{m_2\,\omega l}{2} + m_1\,\omega l - m_1 u\right]$$

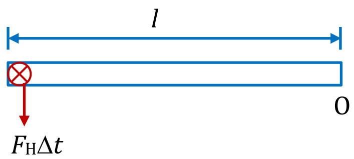
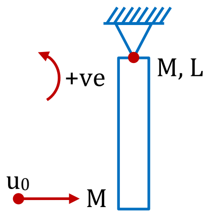


Illustration 70:



A rod of mass M and length l is hinged at point O and a particle of same mass travelling along the end of rod with velocity u_0 sticks to it. Find:

(a) Angular velocity of the combined system after collision.

(b) Impulse on the particle.

(c) Impulse of Hinge force.

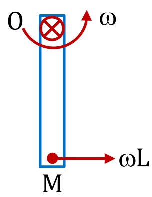
Solution:

(a) $(\tau_{Net})_O$ on rod + particle = 0

$L_i = L_f$

$Mu_0L + 0 = M(\omega L)L + \dfrac{ML^2}{3}\omega$

$\Rightarrow \quad \omega = \dfrac{3u_0}{4L}$



(b) $\vec{I} = P_f - P_i = M\omega L - Mu_0$

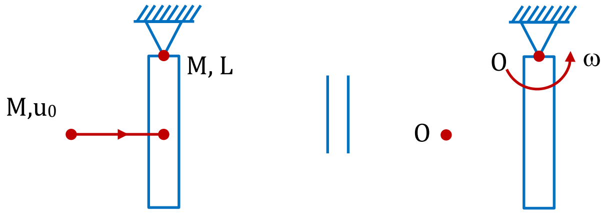
(c) (Rod + particle) as system

$$\vec{I}_H = (\vec{P}_f)_{sys.} - (\vec{P}_i)_{sys.} = \left[M\omega L + \frac{M\omega L}{2}\right] - [Mu_0 + 0]$$

$$= M\frac{3}{4}u_0 + M\frac{3}{8}u_0 - Mu_0 = \frac{Mu_0}{8}\hat{i}$$

Illustration 71:

A rod of mass M and length l is hinged at point O and a particle of same mass travelling along the midpoint of rod with velocity u_0 sticks to it. Find angular velocity of the combined system after collision.



Solution:

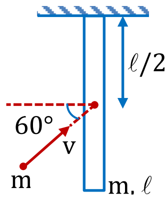
$(\tau_{Net})_O = 0$

$L_i = L_f$

$$\frac{Mu_0L}{2} + 0 = 0 + \frac{ML^2}{3}\omega \quad \Rightarrow \omega = \frac{3u_0}{2L}$$

Illustration 72:

A thin rod of mass m and length ℓ is hinged to a ceiling and it is free to rotate in a vertical plane. A particle of mass m, moving with speed v strikes it as shown in the figure and gets stick with the rod. The value of v, for which the rod becomes horizontal after collision is



(A) The value of v, for which rod becomes horizontal after collision is $\sqrt{\frac{168}{9}g\ell}$

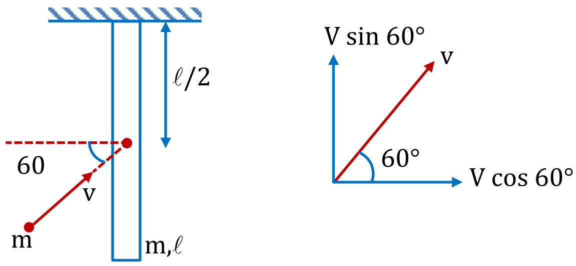
(B) The value of v, for which rod becomes horizontal after collision is $\sqrt{\frac{53}{3}g\ell}$

(C) Angular momentum of (rod + particle) system will remain constant about hinge just before and after collision

(D) Angular momentum of (rod + particle) system will remain same about centre of mass just before and after collision

Answer : (AC)

Solution:



$$m.\frac{v}{2}.\frac{\ell}{2} = \left(m\frac{\ell^2}{3} + m\frac{\ell^2}{4}\right)\omega$$

$$\frac{v}{4} = \frac{7}{12}\ell\omega \Rightarrow \omega = \frac{3}{7}\frac{v}{\ell} \qquad \text{...(i)}$$

$$\frac{1}{2}\left(\frac{7}{12}m\ell^2\right)\omega^2 = 2mg\frac{\ell}{2}$$

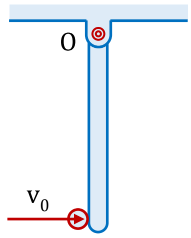
$$\Rightarrow \qquad \omega = \sqrt{\frac{24}{7}\frac{g}{\ell}} \qquad \text{...(ii)}$$

From equation (i) and (ii)

$$v = \sqrt{\frac{168}{9}g\ell}$$

Illustration 73:

A uniform rod of mass M and length ℓ is suspended from a fixed support and can rotate freely in the vertical plane. A small ball of mass m moving horizontally with velocity v_o strikes elastically the lower end of the rod as shown in the figure. Find the angular velocity of the rod and velocity of the ball immediately after the impact.



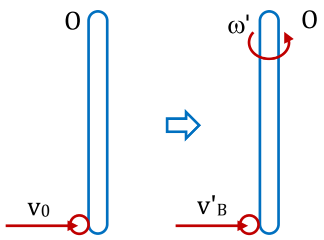
Solution:

The rod is hinged and the ball is free to move. External forces acting on the rod ball system are their weights and reaction from the hinge. Weight of the ball as well as the rod are finite and contribute negligible impulse during the impact, but impulse of reaction of the hinge during impact is considerable and cannot be neglected. Obviously linear momentum of the system is not conserved. The angular impulse of the reaction of hinge about the hinge is zero. Therefore angular momentum of the system about the hinge is conserved.

Let velocity of the ball after the impact becomes v'_B and angular velocity of the rod becomes ω'.

We denote angular momentum of the ball and the rod about the hinge before the impact by L_{B1} and L_{R1} and after the impact by L_{B2} and L_{R2}.

Applying conservation of angular momentum about the hinge, we have



$$\vec{L}_{B1} + \vec{L}_{R1} = \vec{L}_{B2} + \vec{L}_{R2} \rightarrow mv_o\ell + 0 = mv'_B\ell + I_o\omega'$$

Substituting $\frac{1}{3}M\ell^2$ for I_o, we have

$$3mv'_B + M\ell\omega' = 3mv_o \qquad ...(1)$$

The velocity of the lower end of the rod before the impact was zero and immediately after the impact it becomes $\ell\omega'$ towards right. Employing these facts we can express the coefficient of restitution according to equation

$$e = \frac{v'_{Qn} - v'_{Pn}}{v_{pn} - v_{Qn}} \rightarrow \ \ell\omega' - v'_B = ev_o \qquad ...(2)$$

From equation (1) and (2), we have

Velocity of the ball immediately after the impact, $v'_B = \frac{(3m - eM)v_o}{3m + M}$

Angular velocity of the rod immediately after the impact, $\omega' = \frac{3(1+e)mv_o}{(3m+M)\ell}$

Angular impulse :

$$\vec{J} = \int \vec{\tau}dt \ \ (\text{Unit - } Nsm)$$

$$\vec{J}_{net} = \vec{L}_f - \vec{L}_i$$

$$\boxed{\vec{J} = \vec{r} \times \vec{I}}$$

Illustration 74:

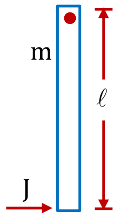
A thin uniform rod of mass m and length ℓ is free to rotate about its upper end. When it is at rest, it receives an impulse J at its lowest point, normal to its length. Immediately after impact

(A) the angular momentum of the rod is $J\ell$.

(B) the angular velocity of the rod is $\frac{3J}{m\ell}$

(C) the kinetic energy of the rod is $\frac{3J^2}{2m}$

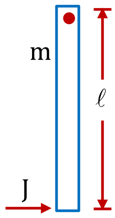
(D) the linear velocity of the midpoint of the rod is $\frac{3J}{2m}$



Answer : (ABCD)

Solution:

By impulse momentum theorem,



$$J\ell = \frac{m\ell^2}{3}\omega \Rightarrow \omega = \frac{3J}{m\ell}$$

$$KE \text{ of rod} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\left(\frac{3J}{m\ell}\right)^2 = \frac{3J^2}{2m}$$

$$\text{Linear velocity of midpoints} = \omega\left(\frac{\ell}{2}\right) = \frac{3J}{2m}$$

Combined Translational and Rotational Motion of a Rigid Body :

The general motion of a rigid body can be thought of as a sum of two independent motions. A translation of some point of the body plus a rotation about this point. A most convenient choice of the point is the centre of mass of the body as it greatly simplifies the calculations.

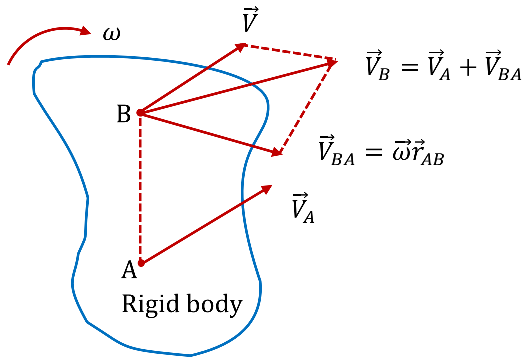
Consider a fan inside a train, and an observer A on the platform.

- If the fan is switched off while the train moves, the motion of fan is pure translation at each point on the fan undergoes same translation in any time interval.
- If fan is switched on while the train is at rest the motion of fan is pure rotation about its axis ; as each point on the axis is at rest, while other points revolve about it with equal angular velocity.
- If the fan is switched on while the train is moving, the motion of fan to the observer on the platform is neither pure translation nor pure rotation. This motion is an example of general motion of a rigid body.
- Now if there is an observer B inside the train, the motion of fan will appear to him as pure rotation.
- Hence we can see that the general motion of fan w.r.t. observer A can be resolved into pure rotation of fan as observed by observer B plus pure translation of observer B (w.r.t. observer A)
- Such a resolution of general motion of a rigid body into pure rotation and pure translation is not restricted to just the fan inside the train, but is possible for motion of any rigid system.

Kinematics of general motion of a rigid body :

For a rigid body as earlier stated value of angular displacement (θ), angular velocity (ω), angular acceleration (α) is same for all points on the rigid body about any other point on the rigid body.

Hence if we know velocity of any one point (say A) on the rigid body and angular velocity of any point on the rigid body about any other point on the rigid body (say ω), velocity of each point on the rigid body can be calculated.



Since distance AB is fixed $\vec{V}_{BA} \perp \overrightarrow{AB}$

We know that $\omega = \dfrac{V_{BA\perp}}{r_{BA}}$

$$V_{BA\perp} = V_{BA} = \omega r_{BA}$$

In vector form $\vec{V}_A$

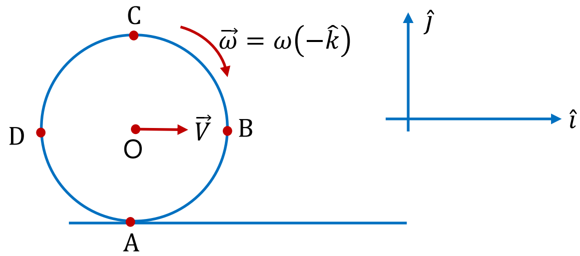
Now from relative velocity : $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$

$$\vec{V}_B = \vec{V}_A + \vec{V}_{BA} \Rightarrow \vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{BA}$$

Similarly, $\vec{v}_{CO} = \omega r\left(-\hat{i}\right)$; $\vec{v}_{DO} = \left(\hat{j}\right)$ [for any rigid system]

Illustration 75:

Consider the general motion of a wheel (radius r) which can be view on pure translation of its center O (with the velocity v) and pure rotation about O (with angular velocity ω). Find out $\vec{v}_{AO}, \vec{v}_{BO}, \vec{v}_{CO}, \vec{v}_{DO}$ and $\vec{v}_A, \vec{v}_B, \vec{v}_C, \vec{v}_D$.



Solution:

$$\vec{v}_{AO} = (\vec{\omega} \times \vec{r}_{AO})$$

$$\vec{v}_{AO} = \left(\omega(-\hat{k}) \times O\vec{A}\right)$$

$$\vec{v}_{AO} = \left(\omega(-\hat{k}) \times r(-\hat{\jmath})\right)$$

$$\vec{v}_{AO} = -\omega r\hat{\imath}$$

Similarly,

$$\vec{v}_{BO} = \omega r(-\hat{\jmath})$$

$$V_{CO} = \vec{\omega} \times \vec{r}_{CO} = \omega(-\hat{k}) \times r\hat{\jmath}$$

$$\vec{V}_{CO} = \omega r\hat{\imath}$$

$$\vec{V}_{DO} = \omega(-\hat{k}) \times r(-\hat{\imath})$$

$$\vec{V}_{DO} = \omega r\hat{\jmath}$$

$$\vec{V}_C = \vec{V}_O + \vec{V}_{CO} = V\hat{\imath} + \omega r\hat{\imath}$$

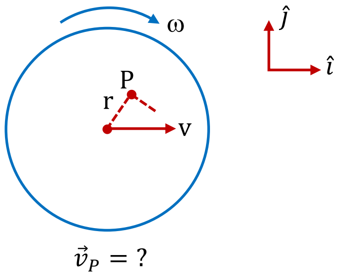
Similarly, $\vec{V}_A = \vec{V}_O + \vec{V}_{AO} = V\hat{\imath} - \omega r\hat{\imath}$

$$\vec{V}_B = \vec{V}_O + \vec{V}_{BO} = V\hat{\imath} - \omega r\hat{\jmath}$$

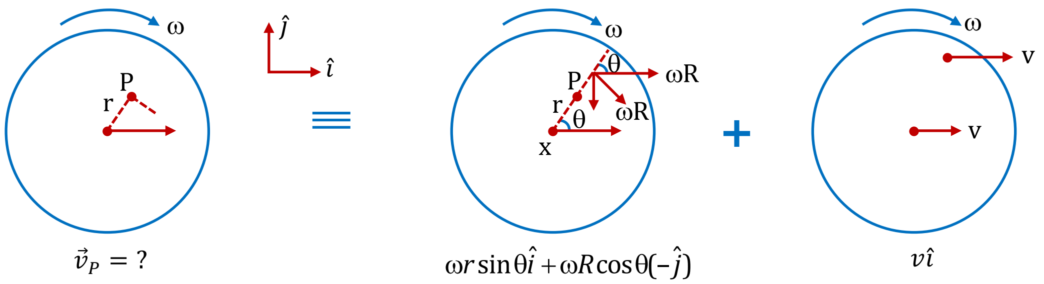
$$\vec{V}_D = \vec{V}_O + \vec{V}_{DO} = V\hat{\imath} + \omega r\hat{\imath}$$

Illustration 76:

A wheel of radius R is performing CRTM if $V_{CM} = V\hat{\imath}$ and angular velocity ω as shown in figure then calculate the velocity of point *'P'*.



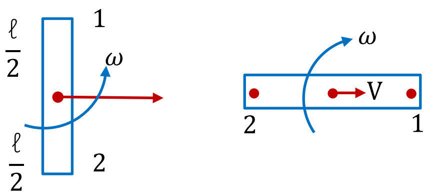
Solution:



$\therefore \qquad \vec{v}_P = (\omega r \sin\theta + v)\hat{\imath} + \omega R \cos\theta\,(-\hat{\imath})$

Illustration 77:

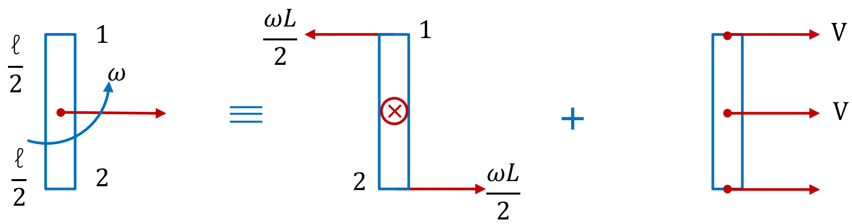
A rectangular block of length ℓ in CRTM whose velocity of COM is $V\hat{\imath}$ and angular velocity is ω as shown in figure, then calculate velocity of 1 and 2 in (a and b)



(a) **(b)**

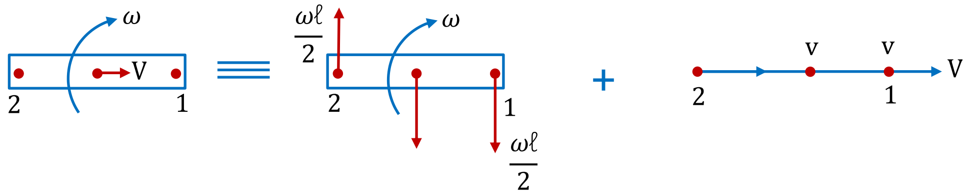
Solution:

(a)



$$\vec{v}_1 = \left(v - \frac{\omega L}{2}\right)\hat{i} \quad ; \quad \vec{v}_2 = \left(v + \frac{\omega L}{2}\right)\hat{i}$$

(b)



$$\vec{v}_1 = \frac{\omega L}{2}\left(-\hat{j}\right) + v\hat{i}$$

$$\vec{v}_2 = \frac{\omega \ell}{2}\left(\hat{j}\right) + v\hat{i}$$

Acceleration analysis :

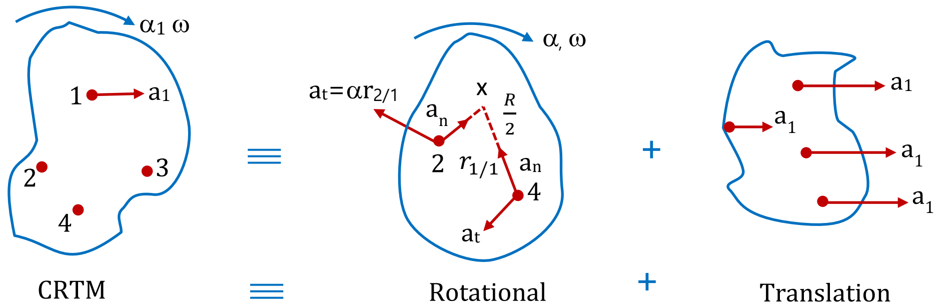
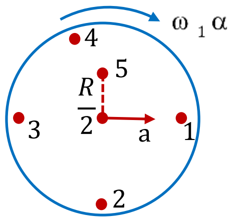


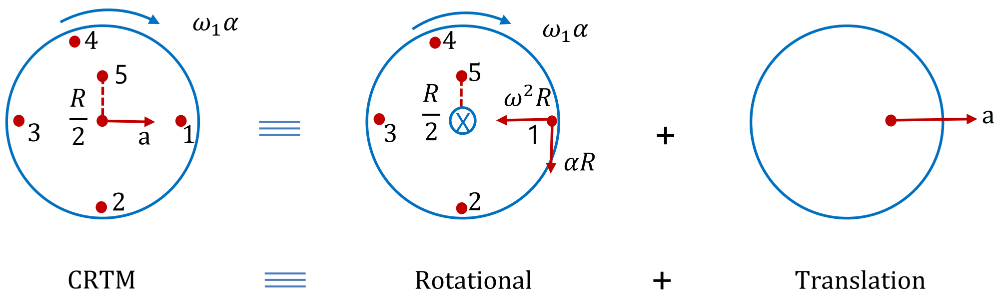
Illustration 78:

A disc is performing CRTM whose acceleration of COM is $a\hat{i}$ and angular acceleration is α as shown in figure. Calculate $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ and $\vec{a}_5$.



CRTM

Solution:



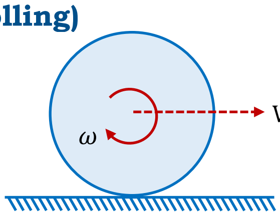
	$\omega^2 R$	αR	a
$\vec{a}_1$	$-\hat{i}$	$-\hat{j}$	i
$\vec{a}_2$	$\hat{j}$	$-\hat{i}$	i
$\vec{a}_3$	$\hat{i}$	$\hat{j}$	i
$\vec{a}_4$	$-\hat{j}$	$\hat{i}$	i
$\vec{a}_5$	$\frac{\omega^2 R}{2}(-\hat{j})$	$\frac{\hat{i}}{2}$	i

$\Rightarrow \quad \vec{a}_1 = \omega^2 R(-\hat{\imath}) + \alpha R(-\hat{\jmath}) + a\hat{\imath}$

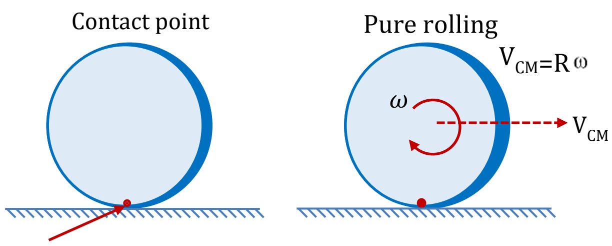
Rolling Motion

When a body perform translatory motion as well as rotatory motion then it is known as rolling. The velocity of centre of mass represents linear motion while angular velocity represents rotatory motion.
Total energy in rolling = Translatory kinetic energy + Rotatory kinetic energy.

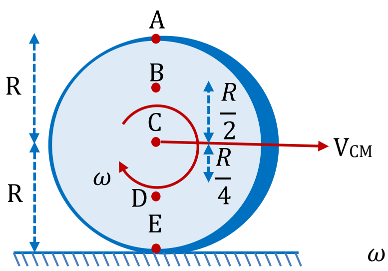
Rolling without slipping (Pure rolling)

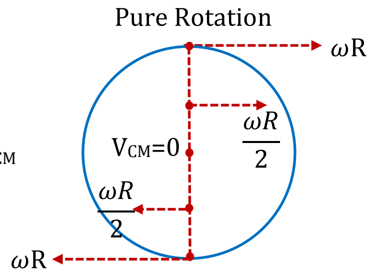


- If the velocity of point of contact with respect to the surface is zero then it is known as pure rolling.



- If a body is performing rolling then the velocity of any point of the body with respect to the surface is given by $\vec{v} = \vec{v}_{CM} + \vec{\omega} \times \vec{R}$





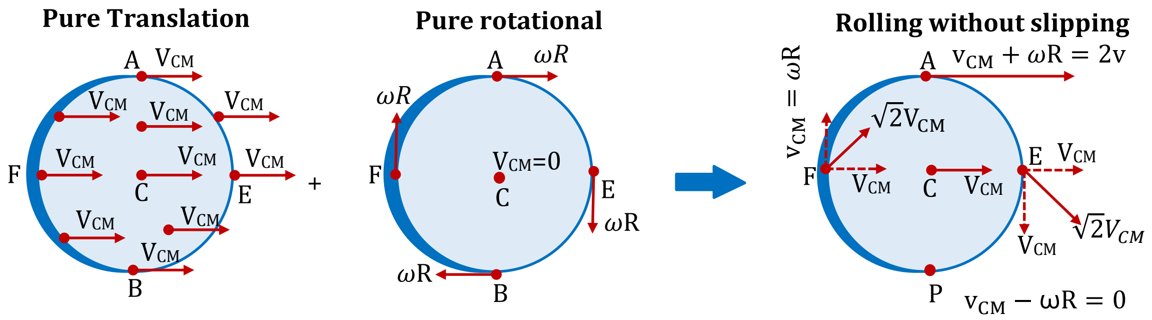
$$v_A = v_{CM} + \omega R = v_{CM} + v_{CM} = 2v_{CM}$$

$$v_B = v_{CM} + \frac{\omega R}{2} = v_{CM} + \frac{v_{CM}}{2} = \frac{3}{2}v_{CM}$$

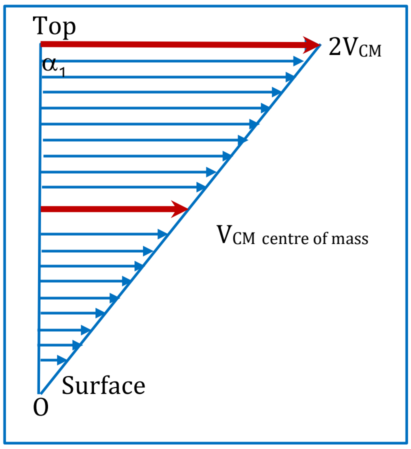
$$v_C = v_{CM} + 0 = v_{CM}$$

$$v_D = v_{CM} - \frac{\omega R}{4} = v_{CM} - \frac{v_{CM}}{4} = \frac{3}{4}v_{CM}$$

$$v_E = v_{CM} - \omega R = v_{CM} - v_{CM} = 0$$



Pure Translatory motion + Pure Rotatory Motion = Rolling motion



For pure rolling above body

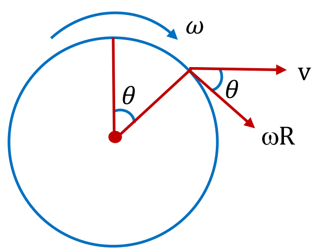
$V_A = 2V_{CM}$

$V_E = \sqrt{2}\, V_{CM}$

$V_F = \sqrt{2}\, V_{CM}$

$V_P = 0$

Velocity at a point on rim of sphere



At point 'p', $v_{net} = \sqrt{v^2 + R^2\omega^2 + 2vR\omega\cos\theta}$

For pure rolling, $v = R\omega$

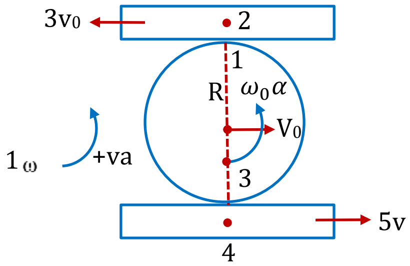
$$= \sqrt{v^2 + v^2 + 2v^2\cos\theta} = \sqrt{2v^2 + 2v^2\cos\theta}$$

$$= \sqrt{2v^2(1+\cos\theta)} = 2v\cos\left(\frac{\theta}{2}\right) \left(\text{since } (1+\cos\theta) = 2\cos^2\left(\frac{\theta}{2}\right)\right)$$

$$\boxed{v_{net} = 2v\cos\frac{\theta}{2}}$$

Illustration 79:

A disc is placed between two rods as shown in figure. Find out v_0 and ω_0 in terms of v. If no slipping any where.



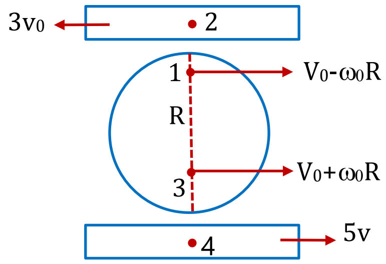
Solution:

Taking upper rod and disc into picture. If rolling $v_1 = v_2$

$v_0 - \omega_0 R = -3v$ ...(1)

Taking lower rod and disc into picture. If rolling $v_3 = v_4$

$v_0 + \omega_0 R = 5v$ ...(2)



Adding equation (1) and (2)

$2v_0 = 2v$

$v_0 = v$

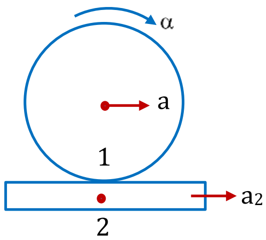
$\therefore \quad v_0 + \omega_0 R = 5v$

$v + \omega_0 R = 5v$

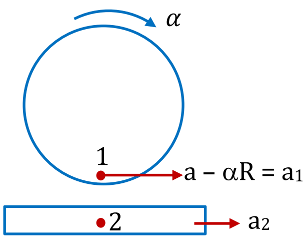
$$\boxed{\omega_0 = \frac{4v}{R}}$$

Illustration 80:

A disc of radius R is in pure rolling with acceleration of COM is a and angular acceleration is α as shown in figure. Find relation between all three variables. a, a_2 and α.



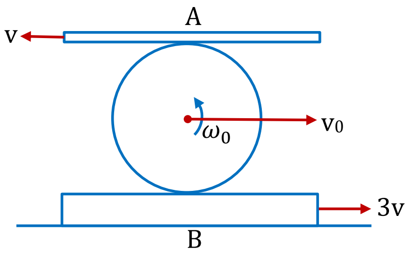
Solution:



$a - \alpha R = a_2$

Illustration 81:

The disc of radius r is confined to roll without slipping at A and B. If the plates have the velocities shown in below figure, then which option is / are correct.



(A) linear velocity $v_0 = v$

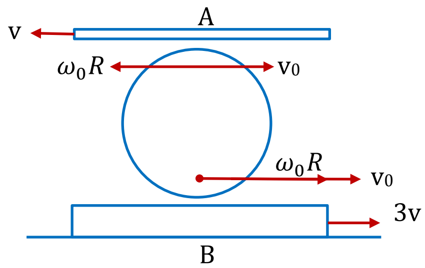
(B) angular velocity of disc is $\dfrac{3v}{2r}$

(C) angular velocity of disc is $\dfrac{2v}{r}$

(D) None of these

Answer: (AC)

Solution:



$v_A = \omega_0 r - v_0 = v$

$\omega_0 r - v_0 = v$ ...(i)

$v_B = \omega_0\, r + v_0 = 3v$ ...(ii)

From equation (i) and (ii)

$2\omega_0 r = 4V$

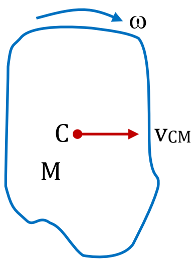
$\Rightarrow \quad \omega_0 r = 2v$

$\omega_0 = \dfrac{2v}{r}$

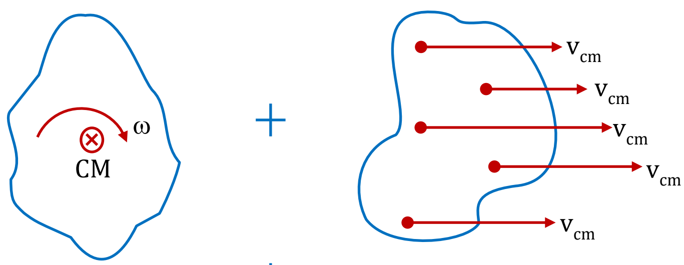
From equation (i)

$v_0 = v$

Kinetic Energy of Rigid Body Performing CRTM



$$\boxed{KE = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2}$$



CRTM = Rotational KE + Translational KE

$$KE = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$$

Proof :

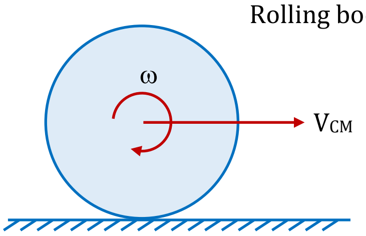
$$KE = \frac{1}{2}\sum m_i(\vec{v}_i \cdot \vec{v}_i) = \frac{1}{2}\sum m_i(\vec{v}_{i/CM} \cdot + \vec{v}_{CM}) \cdot (\vec{v}_{i/CM} \cdot + \vec{v}_{CM})$$

$$= \frac{1}{2}\sum m_i(v^2_{i/CM} \cdot + 2\vec{v}_{i/CM} \cdot \vec{v}_{CM} + \vec{v}^2_{CM}) = \frac{1}{2}I_{CM}\omega^2 + \underbrace{\sum m_i \vec{v}_{i/CM} \cdot \vec{v}_{CM}}_{\text{0 because momentum w.r.t. COM is zero.}} + \frac{1}{2}Mv^2_{CM}$$

Rolling Kinetic Energy :

Rolling Kinetic Energy,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$



Rolling Kinetic Energy,

$$E = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)$$

$$E_{\text{translation}} : E_{\text{rotation}} : E_{\text{Total}} = 1 : \frac{K^2}{R^2} : 1 + \frac{K^2}{R^2}$$

Body	$\frac{K^2}{R^2}$	$\frac{E_{trans}}{E_{rotation}} = \frac{1}{\frac{K^2}{R^2}}$	$\frac{E_{trans}}{E_{total}} = \frac{1}{1+\frac{K^2}{R^2}}$	$\frac{E_{rotation}}{E_{total}} = \frac{\frac{K^2}{R^2}}{1+\frac{K^2}{R^2}}$
Ring	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Disc	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5}{2}$	$\frac{5}{7}$	$\frac{2}{7}$
Spherical shell	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Solid cylinder	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Hollow cylinder	1	1	$\frac{1}{2}$	$\frac{1}{2}$

Illustration 82:

A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its centre moves at a speed of 2.00 cm/s. Find its kinetic energy.

Solution:

As the sphere rolls without slipping on the plane surface, its angular speed about the centre is $\omega = \frac{V_{CM}}{r}$. The kinetic energy is

$$= K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{CM^2} = \frac{1}{2}.\frac{2}{5}Mr^2\omega^2 + \frac{1}{2}Mv^2_{cm}$$

$$= \frac{1}{5}Mv^2_{cm} + \frac{1}{2}Mv^2_{cm} = \frac{7}{10}Mv^2_{cm} = \frac{7}{10}(0.200kg)(0.02m/s)^2 = 5.6\times10^{-5}J$$

Dynamics of general motion of a rigid body :

This motion can be viewed as translation of centre of mass and rotation about an axis passing through centre of mass

If I_{CM} = Moment of inertia about this axis passing through COM

τ_{cm} = Net torque about this axis passing through COM

$\vec{a}_{cm}$ = Acceleration of COM

$\vec{V}_{CM}$ = Velocity of COM

$\vec{F}_{ext}$ = Net external force acting on the system.

$\vec{P}_{\text{system}}$ = Linear momentum of system.

$\vec{L}_{CM}$ = Angular momentum about centre of mass.

$\vec{r}_{CM}$ = Position vector of COM w.r.t. point A.

Then (i) $\vec{\tau}_{cm} = I_{cm}\vec{\alpha}$ (ii) $\vec{F}_{ext} = m\vec{a}_{cm}$

(iii) $\vec{P}_{\text{system}} = M\vec{v}_{cm}$ (vi) Total $K.E.=\frac{1}{2}Mv_{cm}^2+\frac{1}{2}I_{cm}\omega^2$

(v) $\vec{L}_{CM} = I_{CM}\omega^2$

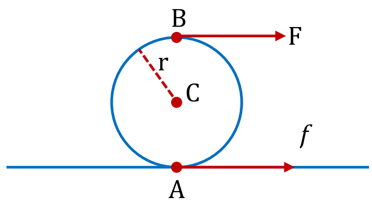
(vi) Angular momentum about point $A = \vec{L}$ of $C.M.$ about A

$\vec{L}_A = I_{CM}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$

- $\frac{dL_A}{dt} = \frac{d}{dt}\left(I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}\right) \neq I_A\frac{d\vec{\omega}}{dt}$. Notice that torque equation can be applied to a rigid body in a general motion only and only about an axis through centre of mass.

Illustration 83:

A constant force F acts tangentially at the highest point of a uniform disc of mass m kept on a rough horizontal surface as shown in figure. If the disc rolls without slipping, calculate the acceleration of the centre (C) and point A and B of the disc.



Solution:

The situation is shown in figure. As the force F rotates the disc, the point of contact has a tendency to slip towards left so that the static friction on the disc will act towards right. Let r be the radius of the disc and a be the linear acceleration of the centre of the disc. The angular acceleration about the centre of the disc is $\alpha = a/r$, as there is no slipping.

For the linear motion of the centre,

$$F + f = ma \quad ...(i)$$

and for the rotational motion about the centre,

$$Fr - fr = I\alpha = \left(\frac{1}{2}mr^2\right)\left(\frac{a}{r}\right) \quad \text{or} \quad F - f = \frac{1}{2}ma \quad ...(ii)$$

From (i) and (ii),

$$2F = \frac{3}{2}ma \quad \text{or} \quad a = \frac{4F}{3m}$$

Acceleration of point A is zero.

Acceleration of point B is – $a_B = a_C + \alpha r$

$$a_B = a + \frac{a}{r} \times r = 2a$$

$$2a = 2\left(\frac{4F}{3m}\right) = \left(\frac{8F}{3m}\right) \text{ **Ans.**}$$

Rolling motion on an inclined plane:

Illustration 84:

A circular rigid body of mass m, radius R and radius of gyration (k) rolls without slipping on an inclined plane of a inclination θ. Find the linear acceleration of the rigid body and force of friction on it. What must be the minimum value of coefficient of friction so that rigid body may roll without sliding?

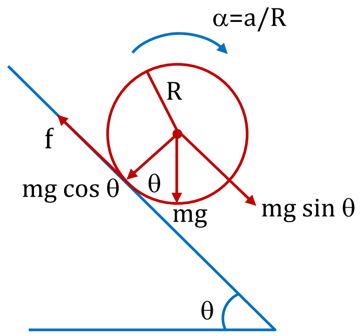
Solution:

If a is the acceleration of the centre of mass of the rigid body and f the force of friction between sphere and the plane, the equation of translatory and rotatory motion of the rigid body will be.

$mg \sin\theta - f = ma$ (Translatory motion)

$fR = I\alpha$ (Rotatory motion)

$$f = \frac{I\alpha}{R}$$



$I = mk^2$, due to pure rolling $a = \alpha R$

$$mg\sin\theta - \frac{I\alpha}{R} = m\alpha R$$

$$mg\sin\theta = m\alpha R + \frac{I\alpha}{R}$$

$$mg\sin\theta = m\alpha R + \frac{mk^2\alpha}{R} = ma + \frac{mk^2 a}{R^2}\left(\alpha = \frac{a}{R}\right)$$

$$mg\sin\theta = ma + \frac{mk^2 a}{R^2}$$

$$g\sin\theta = a\left[\frac{R^2+k^2}{R^2}\right]$$

$$a = \frac{g\sin\theta}{\left[\frac{R^2+k^2}{R^2}\right]}\ ;\ a = \frac{g\sin\theta}{\left[1+\frac{k^2}{R^2}\right]}$$

$$f = \frac{I\alpha}{R}$$

$$f = \frac{mk^2 a}{R^2} \Rightarrow = \frac{mg\,k^2\sin\theta}{R^2+k^2}$$

$$f \le \mu N$$

$$\frac{mk^2}{R^2}a \le \mu\, mg\cos\theta$$

$$R^2\frac{k^2}{R^2}\times\frac{g\sin\theta}{\left(k^2+R^2\right)}\le\mu g\cos\theta$$

$$\mu\ge\frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]}\Rightarrow\mu_{min}=\frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]}$$

Note:

From above example if rigid bodies are solid cylinder, hollow cylinder, solid sphere and hollow sphere.

(1) Increasing order of acceleration.

$$a_{\text{solid sphere}} > a_{\text{solid cylinder}} > a_{\text{hollow sphere}} > a_{\text{hollow cylinder}}$$

(2) Increasing order of required friction force for pure rolling.

$$f_{\text{hollow cylinder}} > f_{\text{hollow sphere}} > f_{\text{solid cylinder}} > f_{\text{solid sphere}}$$

(3) Increasing order of required minimum friction coefficient for pure rolling.

$$\mu_{\text{hollow cylinder}} > \mu_{\text{hollow sphere}} > \mu_{\text{solid cylinder}} > \mu_{\text{solid sphere}}$$

Rolling Motion on an inclined plane :

(i) Velocity at bottom of inclined plane

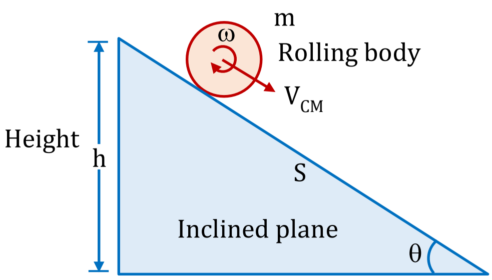
Applying Conservation of energy

$$mgh = \frac{1}{2}mv^2+\frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2+\frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

$$mgh = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right) \quad ...(1)$$

$$h = s\sin\theta \quad ...(2)$$



From (1) and (2)

$$\boxed{v_{Rolling}=\sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}}=\sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}}$$

When body slide $\frac{K^2}{R^2}=0$

Velocity when body slides $\boxed{v_{sliding}=\sqrt{2gh}=\sqrt{2gs\sin\theta}}$ $\Rightarrow$ $v_S > v_R$

(ii) Acceleration of body

$$v^2=u^2+2as \qquad (\because\ u=0)$$

$$\therefore \qquad v^2=2as$$

$$\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}=2as \qquad \Rightarrow \qquad \boxed{a_R=\frac{g\sin\theta}{1+\frac{K^2}{R^2}}}$$

When body slides $\frac{K^2}{R^2}=0$

Acceleration when body slides, $\boxed{a_S=g\sin\theta}$, $a_S > a_R$

(iii) Time taken by the rolling body to reach the bottom

$$S=\frac{1}{2}at^2 \;\Rightarrow\; t=\sqrt{\frac{2s}{a}},\;\; s=\frac{h}{\sin\theta}$$

$$\boxed{t_R=\sqrt{\frac{2S}{g\sin\theta}\left(1+\frac{K^2}{R^2}\right)}=\sqrt{\frac{2h}{g\sin^2\theta}\left(1+\frac{K^2}{R^2}\right)}=\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}}$$

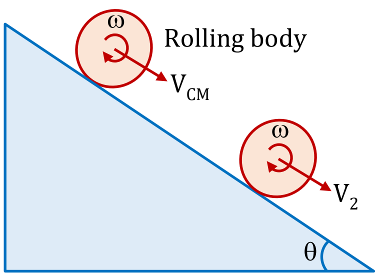
When body slides, $\frac{K^2}{R^2}=0$

Time of descend when body slides, $\boxed{t_S=\sqrt{\frac{2s}{g\sin\theta}}=\sqrt{\frac{2h}{g\sin^2\theta}}=\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}}$

So, $t_R > t_S$

Note:

If different body are rolled down on an inclined plane then body which has



(i) $\frac{K^2}{R^2}$ Least, will reach first

(ii) $\frac{K^2}{R^2}$ Maximum, will reach last

(iii) $\frac{K^2}{R^2}$ Equal, will reach together

(iv) Change in kinetic energy due to rolling, $v_2 > v_1$

$$=\frac{1}{2}mv_2^2\left(1+\frac{K^2}{R^2}\right)-\frac{1}{2}mv_1^2\left(1+\frac{K^2}{R^2}\right)=\frac{1}{2}m\left(1+\frac{K^2}{R^2}\right)\left(v_2^2-v_1^2\right)$$

(v) Rolling vs Sliding

$$\boxed{V_R=\frac{V_S}{\sqrt{1+\frac{K^2}{r^2}}}},\quad \boxed{a_R=\frac{a_S}{1+\frac{K^2}{r^2}}},\quad \boxed{t_R=t_S\sqrt{1+\frac{K^2}{r^2}}}$$

$V_R < V_s$ $\qquad a_R < a_S \qquad t_R > t_S$

Work-Energy Theorem Practice [Applications]

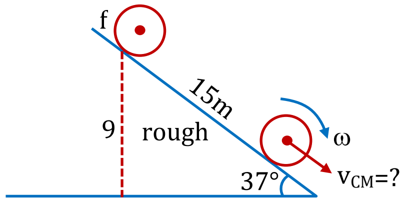
Key point :

For a body performing rolling motion over a fixed surface, work done by friction force on the body will be zero as velocity of point of application of friction is always zero.

Illustration 85:

A disc of mass M and radius R starts to rotate on an inclined plane of inclination angle 37° with horizontal when it has fallen down 15 m. then calculate velocity of COM in the following cases ;

(a) If incline plane is rough. (b) If incline plane is smooth.



Solution:

(a) $W_g + W_N + W_f = KE_f - KE_i$

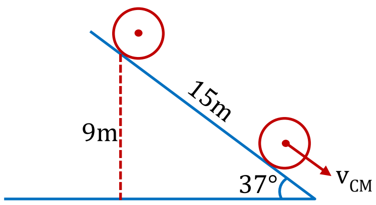
$$Mg9+0+0=\left(\frac{1}{2}\frac{MR^2}{2}\omega^2+\frac{1}{2}Mv_{CM}^2\right)-(0+0)$$

$$9Mg=\frac{1}{2}M\left(\frac{v_{CM}^2}{2}+v_{CM}^2\right) = \frac{1}{2}M\left(\frac{3}{2}V_{cm}^2\right)$$

$$6g=\frac{v_{CM}^2}{2}$$

$$v_{CM}=\sqrt{12g}$$

(b)



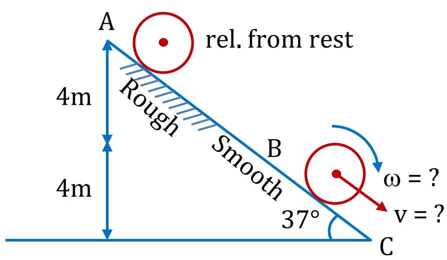
$W_g + W_N = KE_f - KE_i$

$$Mg9+0+0=\left(0+\frac{1}{2}Mv_{CM}^2\right)-(0+0)$$

$$v_{CM} = \sqrt{18g}$$

Illustration 86:

In previous illustration at any instant disc is at 8 m. height from the ground, if half of the inclined plane is rough and rest are smooth. Then calculate final velocity and angular velocity of disc when it reaches at point C.



Solution:

W-E from A to B

$W_g + W_N + W_f = KE_f - KE_i$

$$Mg4+0+0=\left(\frac{1}{2}\frac{MR^2}{2}\omega_B^2+\frac{1}{2}Mv_B^2\right)-(0+0)$$

$$4Mg = \frac{1}{2}\left(\frac{(\omega_B R)^2}{2} + v_B^2\right)M \qquad [\omega_B R = V_B]$$

$$8g = \frac{3v_B^2}{2}$$

$\Rightarrow$ $$\frac{16g}{3} = v_B^2$$

$\Rightarrow$ $$v_B = \sqrt{\frac{16g}{3}}$$

$\Rightarrow$ $$v_B = 4\sqrt{\frac{g}{3}}$$

From B to C 'ω' doesn't change :

$$\omega_B = \omega_C = v_B r = 4r\sqrt{\frac{g}{3}}$$

And then W-E from B to C and get,

$$v_C = \sqrt{\frac{40g}{3}} \text{ and } \omega_B = \omega_C = 4r\sqrt{\frac{g}{3}}$$

Note:

Consider a rigid body performing *CRTM*. For analysing its motion we apply two equation :

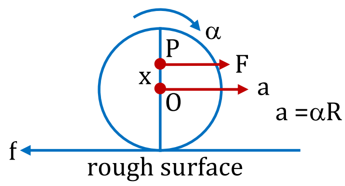
(1) $\sum \vec{F}_{ext} = M\vec{a}_{CM}$

(2) $\sum \vec{\tau}_{ext} = I\vec{\alpha}$

Second equation is for a rigid body and valid in inertial frame. To apply second equation about non-inertial point, pseudo force is applied at COM of body and τ of pseudo force is also taken into account. Most suitable point to apply second equation is about COM of the body because τ of pseudo force about COM is "zero".

Illustration 87:

A rigid body of mass m and radius r rolls without slipping on a rough surface. A force is acting on a rigid body at x distance from the centre as shown in figure. Find the value of x so that static friction is zero.



Solution:

Torque about centre of mass,

$Fx = I_{cm}\alpha$ ...(1)

$F = ma$ ...(2)

From equation (1) and (2)

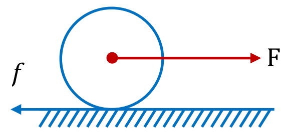
$$max = I_{cm}\,\alpha \qquad (a = \alpha R)\ ; \quad x = \frac{I_{CM}}{mR}$$

Note:

For pure rolling if any friction is required then friction force will be static friction. It may be zero, backward direction or forward direction depending on value of x. If F below the point P then friction force will act in backward direction or above the point P friction force will act in forward direction. This point P is called point of percussion.

Illustration 88:

A horizontal force F acts on the sphere at its centre as shown. Coefficient of friction between ground and sphere is μ. What is maximum value of F, for which there is no slipping ?



Solution:

For linear motion, $F - f = Ma$

and for rotational motion, $\tau = I\,\alpha$

$\Rightarrow \quad f.R = \frac{2}{5}MR^2 \cdot \frac{a}{R}$

$\Rightarrow \quad f = \frac{2}{5}Ma$

$\therefore \quad F - f = \frac{5}{2}f$ or $F = \frac{7}{2}f$

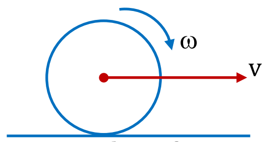
$\because \quad f \le \mu Mg$

So $\quad F \le \frac{7}{2}\mu Mg$

Note:

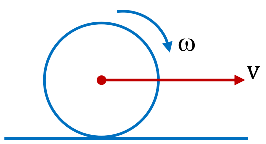
Friction in rolling.

(i)



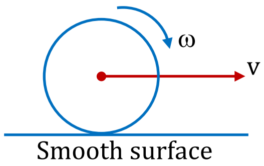
No friction and pure rolling.

(ii) $v = \omega R$



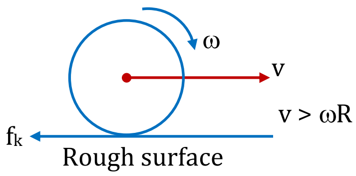
No friction and pure rolling (If the body is not perfectly rigid, then there is a small friction acting in this case which is called rolling friction)

(iii) $v > \omega R$ or $v < \omega R$



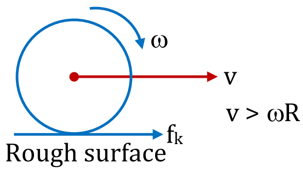
No friction force but not pure rolling.

(iv) $v > \omega R$



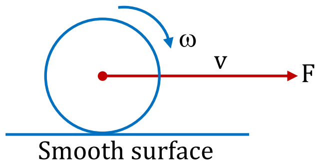
There is relative motion at point of contact, so kinetic friction, $f_k = \mu N$ will act in backward direction. This kinetic friction decrease v and increase ω, so after some time $v = \omega R$ and pure rolling will resume like in case (ii).

(v) $v < \omega R$



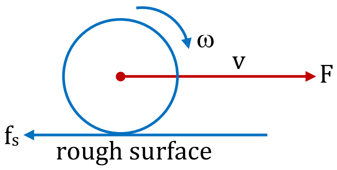
There is Relative Motion at point of contact so Kinetic Friction, $f_k = \mu N$ will act in forward direction. This kinetic friction increase v and decrease ω, so after some time $v = \omega R$ and pure rolling will resume like in case (ii).

(vi) $v = \omega R$ (initial)



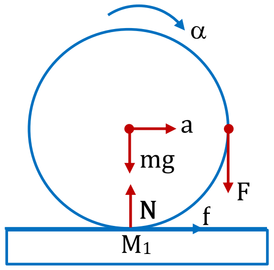
No friction and no pure rolling.

(vii) $v = \omega R$ (initial)



Static friction whose value can lie between zero and $\mu_s N$ will act in backward direction. If coefficient of friction is sufficiently high, then f_s compensates for increasing v due to F by increasing ω and body may continue in pure rolling with increases v as well as ω.

Illustration 89:



A disc of mass M and radius R is in CRTM on plank of mass M_1 whose angular acceleration α and velocity of COM is a_{cm} then calculate a_{cm}, α of disc and acceleration of mass M_1.

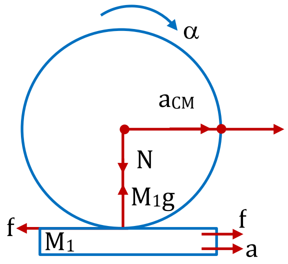
Solution:

$F - f = Ma_{CM}$ ...(1)

$fR = \frac{MR^2}{2} \times \alpha$ ...(2)

$a_{CM} - \alpha R = a$ ...(3)

$f = M_1 a$ ...(4)



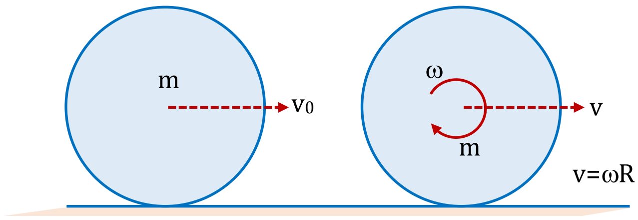
By solving equation 1, 2, 3 and 4

$$\alpha = \frac{2}{3R}\left(\frac{F}{M} - \frac{f}{M_1}\right)$$

$$a_{cm} = \frac{2}{3}\frac{F}{M} + \frac{f}{3M_1} \quad ; \quad a = \frac{f}{M_1}$$

Illustration 90:

A body of radius R and mass m is placed on horizontal rough surface with linear velocity v_0, after some time it comes in the condition of pure rolling then determine :



(i) Time t at which body starts pure rolling.

(ii) Linear velocity of body at time t.

(iii) Work done by frictional force in this time t.

Solution:

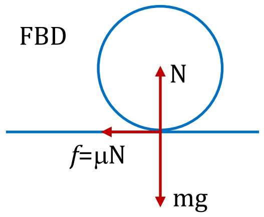
For translatory motion, $v = u + at$

initial velocity, $u = v_0$

Let after time t pure rolling starts and at this time t final velocity = v and

Acceleration = a

From FBD :



Normal Reaction, $N = mg$

Friction force, $f = \mu N = \mu mg \Rightarrow ma = \mu mg \quad [\because f = ma]$

Retardation, $a = \mu g$

$v = v_0 - at$ (–*ve* sign for retardation)

$v = v_0 - \mu gt$ ...(i)

For rotatory motion $\omega = \omega_0 + \alpha t$ (Initial angular velocity $\omega_0 = 0$)

$$\omega = \alpha t \qquad \text{...(ii)}$$

$\because$ $\tau = I\alpha$

$\Rightarrow$ $\alpha = \frac{\tau}{I} = \frac{fR}{mK^2} = \frac{\mu mgR}{mK^2}$

$\Rightarrow$ $\alpha = \frac{\mu gR}{K^2} \qquad \text{...(iii)}$

(i) From equation (ii) and equation (iii)

$$\omega = \frac{\mu gR}{K^2}t \qquad \text{...(iv)}$$

$\because$ For pure rolling, $v = \omega R$

$\Rightarrow$ $\omega = \frac{v}{R} \qquad \text{...(v)}$

From equation (iv) and (v)

$$\frac{v}{R} = \frac{\mu gR}{K^2}t$$

or $v = \frac{\mu gR^2t}{K^2} \qquad \text{...(vi)}$

Substitute v from equation (vi) into equation (i)

$$\frac{\mu gR^2t}{K^2} = v_0 - \mu gt$$

$\Rightarrow$ $t = \frac{v_0}{\mu g\left[1+\frac{R^2}{K^2}\right]}$

(ii) Putting the value of t in equation (i)

$$v = v_0 - \mu g\frac{v_0}{\mu g\left[1+\frac{R^2}{K^2}\right]}$$

$$v = v_0 - \frac{v_0}{1+\frac{R^2}{K^2}} = \frac{v_0\left[1+\frac{R^2}{K^2}\right]-v_0}{1+\frac{R^2}{K^2}} = v_0\frac{\left[1+\frac{R^2}{K^2}-1\right]}{\left[1+\frac{R^2}{K^2}\right]}$$

$$v = \frac{v_0 + v_0\frac{R^2}{K^2} - v_0}{1+\frac{R^2}{K^2}} \quad \frac{v_0}{\frac{k^2}{R^2}\left(1+\frac{R^2}{K^2}\right)} = \frac{v_0}{\left(1+\frac{K^2}{R^2}\right)}$$

Velocity of body when pure rolling starts,

$$v = \frac{v_0}{1+\frac{K^2}{R^2}}$$

(iii) Work done in sliding by frictional force :

Work done = Initial kinetic energy – Final kinetic energy

Work done by friction

$$W_f = \frac{1}{2}Mv_0^2 - \frac{1}{2}\frac{Mv_0^2}{\left(1+\frac{K^2}{R^2}\right)} = \frac{Mv_0^2}{2\left(1+\frac{R^2}{K^2}\right)}$$

Illustration 91:

A cylinder is given angular velocity ω_0 and kept on a horizontal rough surface the initial velocity is zero. Find out distance travelled by the cylinder before it performs pure rolling and work done by friction force.

Solution:

$$\tau = I\alpha \Rightarrow f \times R = I\alpha$$

$$\mu Mg\,R = \frac{MR^2\alpha}{2}$$

$$\alpha = \frac{2\mu g}{R} \qquad ...(1)$$

Initial velocity $u = 0$

$$v^2 = u^2 + 2as$$

$$v^2 = 2as \qquad ...(2)$$

$$f_K = Ma$$

$$\mu Mg = Ma$$

$$a = \mu g \qquad ...(3)$$

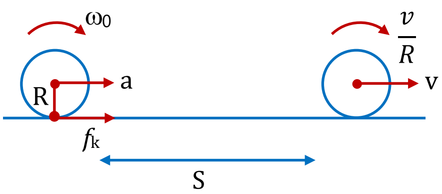
$$\omega = \omega_0 - \alpha t$$

From equation (1)

$$\omega = \omega_0 - \frac{2\mu g}{R}t$$

$$v = u + at$$

From equation (3)



$$v = \mu g t$$

$$\omega = \omega_0 - \frac{2v}{R}$$

$$\omega = \omega_0 - 2\omega$$

$$\omega = \frac{\omega_0}{3}$$

From equation (2)

$$\left(\frac{\omega_0 R}{3}\right)^2 = (2as) = 2\mu gs$$

$$s = \frac{\omega_0{}^2R^2}{18\,\mu g}$$

Work done by the friction force

$$W = (-f_k\,R\,\Delta\theta + f_k\Delta s) = -\,\mu mg\,R\,\Delta\theta + \frac{\mu mg \times \omega_0^2 R^2}{18\,\mu g}$$

$$\Delta\theta = \omega_0 \times t - \frac{1}{2}\alpha t^2 = \omega_0 \times \left(\frac{\omega_0 R}{3\mu g}\right) - \frac{1}{2} \times \frac{2\mu g}{R}\left(\frac{\omega_0 R}{3\mu g}\right)^2 = \left(\frac{\omega_0^2 R}{3\mu g}\right) - \frac{\omega_0^2 R}{9\mu g} = \frac{2\omega_0^2 R}{9\mu g}$$

$$W_f = -\mu mg \times R\frac{2\omega_0^2 R}{9\mu g} + \mu mg \times \frac{\omega_0^2 R^2}{18\mu g} = -\frac{2m\omega_0^2 R^2}{9} + \frac{m\omega_0^2 R^2}{18} = \frac{-3m\omega_0^2 R^2}{18} = -\frac{m\omega_0^2 R^2}{6}$$

Alternative Solution

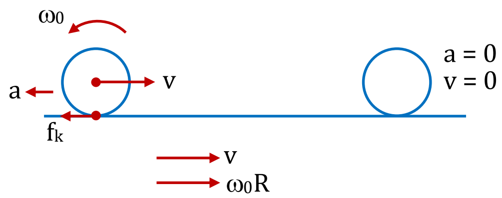
Using work energy theorem,

$$W_g + W_a + W_{f_k} = \Delta K$$

$$w_{f_k} = \left[\frac{1}{2}m\left(\frac{\omega_0 R}{3}\right)^2 + \frac{1}{2}\frac{mR^2}{2} \times \left(\frac{\omega_0}{3}\right)^2\right] - \left[\frac{1}{2}\frac{mR^2}{2} \times \omega_0^2\right] = \left(\frac{-m\omega_0^2 R^2}{6}\right)$$

Illustration 92:

A hollow sphere is projected horizontally along a rough surface with speed v and angular velocity ω_0 find out the ratio. So that the sphere stops moving after some time.



Solution:

Torque about lowest point of sphere.

$$f_k \times R = I\alpha$$

$$\mu mg \times R = \frac{2}{3}mR^2\alpha$$

$\alpha = \dfrac{3\mu g}{2R}$ (Angular acceleration in opposite direction of angular velocity.)

$\omega = \omega_0 - \alpha t$ (Final angular velocity $\omega = 0$)

$$\omega_0 = \frac{3\mu g}{2R} \times t$$

$$t = \frac{\omega_0 \times 2R}{3\mu g}$$

Acceleration '$a = \mu g$'

$v_f = v - at$ (Final velocity $v_f = 0$)

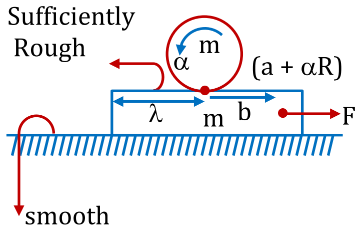
$v = \mu g \times t$; $t = \dfrac{v}{\mu g}$

To stop the sphere, time at which v and ω are zero, should be same.

$$\frac{v}{\mu g} = \frac{2\omega_0 R}{3\mu g} \Rightarrow \frac{v}{\omega_0} = \frac{2R}{3}$$

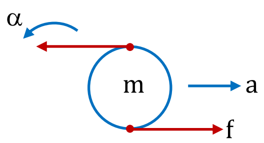
Illustration 93:

A cylinder is lying on a rough plank which is kept on a smooth surface. Force F is applied on the plank. Find time taken by the cylinder to fall off the plank.
(Friction is sufficient for pure rolling.)



Solution:

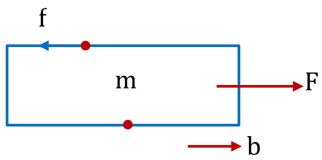
Friction on the plate is backward or on cylinder friction is forward, so cylinder move forward.



Because of pure rolling static friction *f*.

$$fR = \frac{mR^2}{2}\alpha$$

$$\alpha = \frac{2f}{mR}$$



$$f = ma$$

$$F - f = mb$$

$$F = m(a+b)$$

$$a = \frac{\alpha R}{2}$$

At contact point

$$b = a + \alpha R$$

$$b = \frac{3\alpha R}{2}$$

$$b = 3a$$

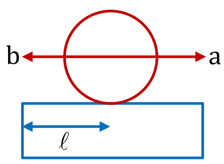
$$F = 4ma$$

$$a = \frac{F}{4m}$$

$$b = \frac{3F}{4m}$$

w.r.t. plate distance is covered = ℓ

and acceleration w.r.t. plate is $(b - a)$

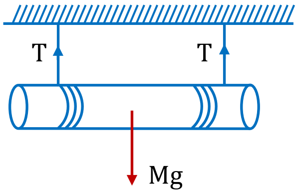


So, $$\ell = \frac{1}{2}(a-b)t^2$$

$$\ell = \frac{1}{2}\times 2at^2 \Rightarrow t = \sqrt{\frac{l}{a}} = 2\sqrt{\frac{ml}{F}}$$

Illustration 94:

A hollow cylinder of mass m is suspended through two light strings wrapped around it as shown in figure. Calculate (a) the tension T in the string and (b) the speed of the cylinder as it falls through a distance ℓ.



Solution:

The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is $\alpha = a/R$, as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$$Mg - 2T = Ma \qquad \text{...(i)}$$

and for the motion about the centre of mass, it is

$$2\,Tr = (Mr^2\alpha) = Mra \qquad \text{or} \qquad 2T = Ma$$

From (i) and (ii),

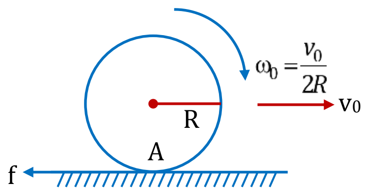
$$a = \frac{g}{2} \qquad \text{and} \qquad T = \frac{Mg}{4}$$

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance ℓ is given by

$$v^2 = 2\left(\frac{g}{2}\right)\ell \qquad \text{or} \qquad v = \sqrt{g\ell}$$

Illustration 95:

A hollow sphere of mass M and radius R as shown in figure slips on a rough horizontal plane. At some instant it has linear velocity v_0 and angular velocity about the centre $\frac{v_0}{2R}$ as shown in figure. Calculate the linear velocity after the sphere starts pure rolling.



Solution:

Velocity of the centre = v_0 and the angular velocity about the centre = $\frac{v_0}{2R}$. Thus $v_0 > \omega_0 R$. The sphere slips forward and thus the friction by the plane on the sphere will act backward. As the friction is kinetic, its value is $\mu N = \mu Mg$ and the sphere will be decelerated by $a_{cm} = f/M$. Hence,

$$v(t) = v_0 - \frac{f}{M}t \qquad \text{...(i)}$$

This friction will also have a torque $I' = fR$ about the centre. This torque is clockwise and in the direction of ω_0. Hence the angular acceleration about the centre will be

$$\alpha = f\frac{R}{(2/3)MR^2} = \frac{3f}{2MR}$$

and the clockwise angular velocity at time t will be

$$\omega(t) = \omega_0 + \frac{3f}{2MR}t = \frac{v_0}{2R} + \frac{3f}{2MR}t$$

Pure rolling starts when

$$v(t) = R\omega(t) \text{ i.e., } v(t) = \frac{v_0}{2} + \frac{3f}{2M}t \quad \text{...(ii)}$$

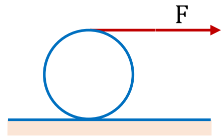
On solving (i) and (ii)

$$v(t) = \left(\frac{v_0}{2} + \frac{3v_0}{10}\right)$$

Thus, the sphere rolls with linear velocity $4v_0/5$ in the forward direction.

Illustration 96:

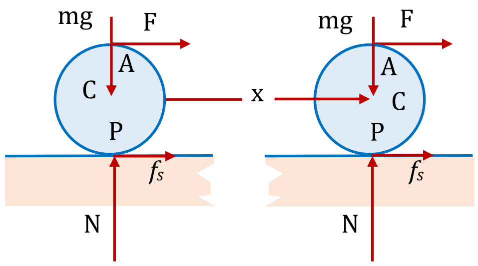
A thin light cord is wound around a uniform cylinder placed on a rough horizontal ground. When free end of the cord is pulled by a constant force F the cylinder rolls. Denote sradius of the cylinder by r and obtain expression for work done by each of the forces acting on the cylinder when center of the cylinder shifts by distance x.



Solution:

Forces acting on the cylinder are its weight W, the normal reaction from the ground N, the tension T in the cord and the force of static friction f_s. The tension in the cord equals to the applied force F. These forces are shown in the adjacent figure.

In rolling point of contact P is at instantaneous rest, the center C moves with velocity $v_C = \omega r$ and the top point moves with velocity $v_A = 2v_C = 2\omega r$ both parallel to the surface on which body rolls. Since the cord is inextensible displacement of the top point equals to the displacement of the free end of the cord. These facts suggest that during displacement x of the center, the free end of the cord shift through a distance $2x$.



Work done by the weight of the cylinder :

$W_g = 0$, The weight is assumed to act on the center of gravity which coincides with the mass center in uniform gravitational field near the ground. The displacement x of the mass center and weight both are perpendicular to each other so the work done by gravity is zero.

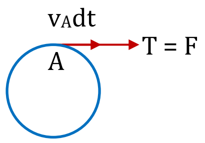
Work done by the normal reaction on the cylinder :

$W_N = 0$, The normal reaction acts on the particle of the body which is in contact with the ground. The particles making contact continuously change and remain at instantaneous rest during contact. Therefore, normal reaction does no work.

Work done by the force of static friction :

$W_{fs} = 0$, The force of static friction f_s acts on the particle of the body which is in contact with the ground. The particles making contact continuously change and remain at instantaneous rest during contact. Therefore, force of static friction f_s does no work.

Work done by the tension in the cord :



$W_T = W_F$, the particle of the wheel on which the tension in the cord acts is at the top point.

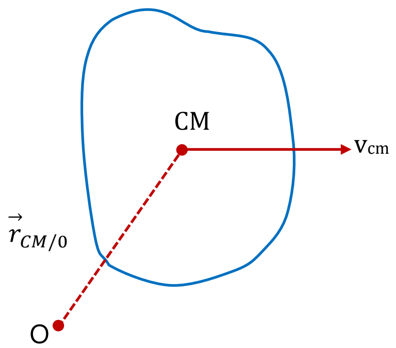
Though this particle is also continuously changing but it is not at instantaneous rest and has velocity v_A. So, in every infinitesimally small time interval displacement of this particle is $v_A dt = 2v_C dt$, thus work done dW_T by the tension during a time interval dt

$$dW_T = T(2v_C dt) = 2F(v_C dt)$$

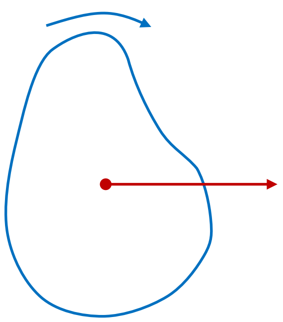
When the center shifts by a distance x the work done by the tension becomes,

$$W_T = W_F = 2Fx$$

Angular momentum of Rigid Body Performing CRTM

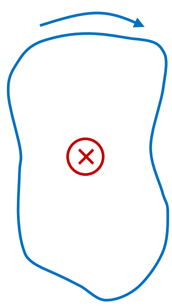


$$\boxed{L_{sys/O} = I_{CM}\omega + m(\vec{r}_{CM/O} \times \vec{v}_{CM})}$$

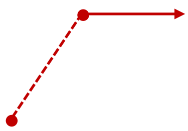


Rotational

$\equiv$



Rotational as Rigid body pure rotation about CM

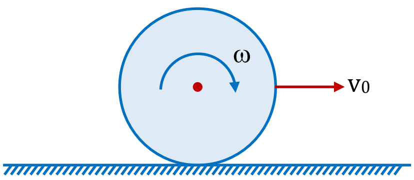


Translation as a particle

Illustration 97:

A solid sphere rolls without slipping on a rough surface and the centre of mass has constant speed v_0. If mass of the sphere is M and its radius is R, then find the angular momentum of the sphere about the point of contact.

Solution:



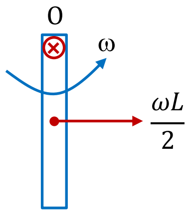
$\because \quad \vec{L}_P = \vec{L}_{cm} + \vec{r} \times \vec{p}_{cm} = I_{cm}\vec{\omega} + \vec{R} \times M\vec{v}_{cm}$

Since sphere is in pure rolling motion hence

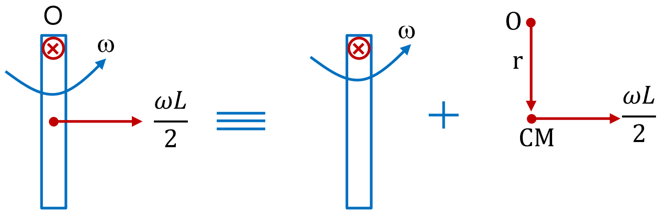
$$\omega = \frac{v_0}{R} \Rightarrow \vec{L}_p = \left(\frac{2}{5}MR^2\frac{v_0}{R}\right)\left(-\hat{k}\right) + Mv_0R(-\hat{k}) = \frac{7}{5}Mv_0R\left(-\hat{k}\right)$$

Illustration 98:

A rod of length L and mass M hinged at point O is rotating about O with angular velocity ω. Find angular momentum of system with respect to O.



Solution:

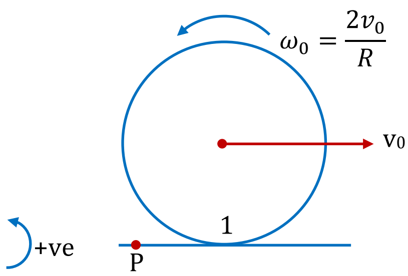


$$L_{sys/0} = \frac{ML^2}{12}\omega + \frac{ML}{2}\frac{\omega L}{2} = \left(\frac{ML^2}{12} + \frac{ML}{4}\right)\omega = \frac{ML^2}{3}\omega = I_{AOR}\omega$$

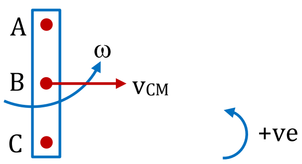
Example:

(A) Sphere (m, R)

A solid sphere (M, R) is doing CRTM with velocity of COM v_0 and angular velocity $\frac{2v_0}{R}$. Find angular momentum with respect to point P.



(B) If a rod (M, L) is doing CRTM and angular velocity and velocity of centre of rod is ω and v_{cm} respectively then find L about A, B and C ?



Solution:

(A) $L_{sys/P} = \frac{2}{5}MR^2 \times \frac{2v_0}{R} - Mv_0R$

$$= \frac{4v_0MR}{5} - Mv_0R = -\frac{Mv_0R}{5} = \frac{Mv_0R}{5} \text{ (clockwise)}$$

(B) $L_{sys/B} = \frac{ML^2}{12}\omega + 0 = \frac{ML^2\omega}{12}$

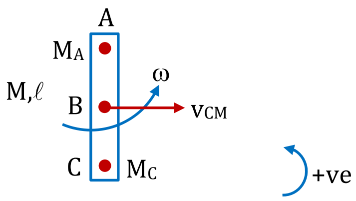
$$L_{sys/A} = \frac{ML^2}{12}\omega + \frac{MLv_{CM}}{2}$$

$$L_{sys/C} = \frac{ML^2}{12}\omega - \frac{ML}{2}v_{CM}$$

Illustration 99:

If a rod (M, L) is doing CRTM and angular velocity and velocity of centre of rod is ω and v_{cm} respectively then find L of the system about A, B and C.

(Assuming anticlockwise direction to be positive)



Solution:

$$L_{sys/A} = \left(\frac{ML^2}{12}\omega + \frac{ML}{2}v_{CM}\right) + M\left(v_{CM} + \omega\frac{L}{2}\right)L$$

$$L_{sys/B} = \left(\frac{ML^2}{12}\omega\right) - M\left(v_{CM} - \frac{\omega L}{2}\right)\frac{L}{2} + M\left(v_{CM} + \frac{\omega L}{2}\right)\frac{L}{2}$$

$$L_{sys/C} = \left(\frac{ML^2}{12}\omega - Mv_{CM}\frac{L}{2}\right) - M\left(v_{CM} - \frac{\omega L}{2}\right)L$$

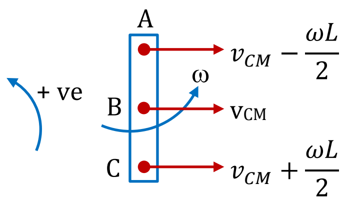


Illustration 100:

A hollow sphere of radius R and mass m is fully filled with non viscous liquid of mass m. It is rolled down a horizontal plane such that its centre of mass moves with a velocity v. If it purely rolls

(A) Kinetic energy of the sphere is $\frac{5}{6}mv^2$

(B) Kinetic energy of the sphere is $\frac{4}{5}mv^2$

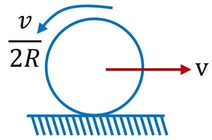
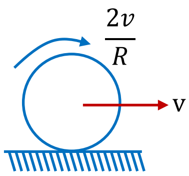
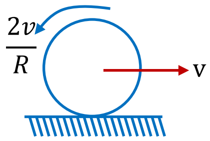
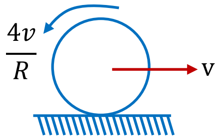
(C) Angular momentum of the sphere about a fixed point on ground is $\frac{8}{3}mvR$

(D) Angular momentum of the sphere about a fixed point on ground is $\frac{14}{5}mvR$

Answer: (C)

Illustration 101:

Four different situations of a moving disc are shown in column I and predictions about its final motion and forces acting on it are given in column - II.

Column- I	Column- II
(A) 	(P) Finally disc will roll along the initial direction of velocity (v)
(B) 	(Q) Finally, disc will roll in direction opposite to the initial direction of velocity (v)
(C) 	(R) Finally, disc stops
(D) 	(S) Initially friction force acts in the direction opposite to that of initial velocity
	(T) None of these

Answer: (A-S,P), (B-P), (C-S,R), (D-S,Q)

Solution:

Final direction of pure rolling will be in direction of initial angular momentum about point of contact $L = mv_{cm}R + I_{cm}\,\omega$

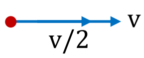
For (A) : $mvR - \frac{MR^2}{2}\left(\frac{v}{2R}\right) = \frac{3mvR}{4}$, hence in clockwise direction.

For (B) : $mvR + \frac{mR^2}{2}\left(\frac{2v}{R}\right) = 2mvR$, hence in clockwise direction.

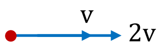
For (C) : $mvR - \frac{mR^2}{2}\left(\frac{2v}{R}\right) = 0$

For (D) : $mvR - \frac{mR^2}{2}\left(\frac{4v}{R}\right) = -mRv,$ hence in anticlockwise direction.

Direction of friction force :

For (A) : Velocity of point of contact  $\Rightarrow$ Friction will be opposite to velocity

For (B) : Velocity of point of contact  $\Rightarrow$ Friction will be in the direction of velocity

For (C) : Velocity of point of contact  $\Rightarrow$ Friction will be opposite to the velocity

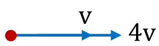
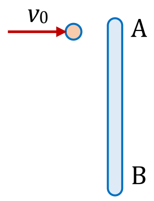
For (D) : Velocity of point of contact  $\Rightarrow$ Friction will be opposite to the velocity

Illustration 102:

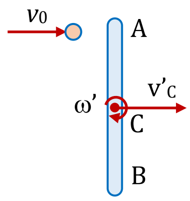
A uniform rod AB of mass M and length ℓ is kept at rest on a smooth horizontal plane.



A particle P of mass m_o moving perpendicular to the rod with velocity v_o strikes the rod at one of its ends as shown in the figure. Derive suitable expressions for the coefficient of restitution, velocity of mass center of the rod and angular velocity of the rod immediately after the impact. Assume e is the coefficient of restitution.

Solution:

Both the bodies can move freely in the horizontal plane, therefore no horizontal external force acts on the particle-rod system. The linear momentum as well as angular momentum about any axis normal to the plane is conserved.



Let the velocity of the particle, angular velocity of the rod and velocity of the mass center of the rod immediately after the impact are v'_P towards right, ω' in clockwise sense and v'_C towards right as shown in the adjacent figure. Using relative motion equation, we can express the velocity of the end A of the rod.

$$v'_A = v'_C + \frac{1}{2}\omega'\ell \qquad \text{...(1)}$$

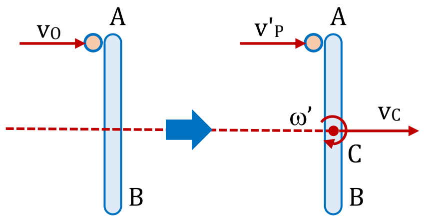
We denote linear momentum of the particle and rod before the impact by $\vec{p}_{P1}$ and $\vec{p}_{C1}$ and immediately after the impact by $\vec{p}_{P2}$ and $\vec{p}_{C2}$ respectively.

Applying conservation of linear momentum, we have

$$\vec{p}_{P2} + \vec{p}_{C2} = \vec{p}_{P1} + \vec{p}_{C1} \qquad \Rightarrow \quad mv'_P + Mv'_C = mv_o \qquad \text{...(2)}$$

The above equation shows that the mass center of the rod will move towards the right. If we write angular momentum of the rod about a stationary point O, which is in line with the velocity v'_C, the first term involving moment of momentum of rod vanishes and only angular momentum due to its centroidal rotation remains in the expression.

We denote angular momentum of the particle and the rod about the point O before the impact by L_{P1} and L_{R1} and after the impact by L_{P2} and L_{R2}.



Before the impact Immediately after the impact

Applying conservation of angular momentum about the hinge, we have

$$\vec{L}_{P2} + \vec{L}_{R2} = \vec{L}_{P1} + \vec{L}_{R1}$$

$$\Rightarrow \quad \frac{1}{2}mv'_P\ell + I_C\omega' = \frac{1}{2}mv_o\ell$$

Substituting $\frac{1}{12}M\ell^2$ for I_C, we have

$$6mv'_P + M\ell\omega' = 6mv_o \qquad \text{...(3)}$$

The velocity of the end A of the rod before the impact was zero and immediately after the impact it becomes v'_A towards right. Employing these facts we can express the coefficient of restitution as

$$e = \frac{v'_A - v'_P}{v_P - v_A} \rightarrow v'_A - v'_P = ev_o$$

Substituting v'_A form equation (1), we have

$$2v'_C + \omega'\ell - 2v'_P = 2ev_o \qquad ...(4)$$

Equation (2), (3) and (4) involves three unknowns v'_C, ω' and v'_P, which can be obtained by solving these equations.

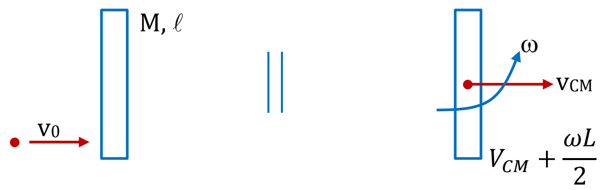
Velocity of the ball immediately after the impact, $v'_P = \left(\frac{4m - eM}{4m + M}\right)v_o$

Velocity of mass center of rod immediately after the impact, $v'_C = \left\{\frac{m(1+e)}{4m+M}\right\}v_o$

Angular velocity of the rod immediately after the impact, $\omega' = \left\{\frac{6m(1+e)}{4m+M}\right\}\frac{v_o}{\ell}$

Illustration 103:

A particle of mass M_1 strikes at one end of a stationary rod of mass M, length ℓ. After collision particle sticks to the rod. Then find ω and v_{cm} of rod just after collision.



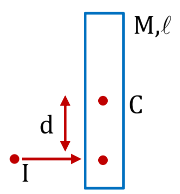
Solution:

As $F_{ext} = 0$

$P_i = P_f$

$$M(0) + M_1v_0 = Mv_{CM} + M_1\left(v_{CM} + \frac{\omega L}{2}\right)$$

Illustration 104:



An impulse I acts on the lower half of the rod of mass M and length ℓ at a distance d from the centre as shown in the figure.

Find v_{cm} and ω of the rod just after the impulse.

Solution:

$I = P_f - P_i$

$I = Mv_{CM} - 0$

$(\tau_{net})_C = 0$

$(L_i)_C = (L_f)_C$

$$0 + 0 = \frac{ML^2\omega}{12} - Mv_{CM}d$$

$$MV_{CM}d = \frac{M\omega L^2}{12}$$

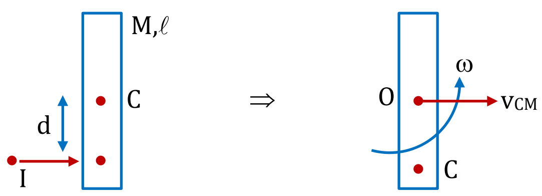
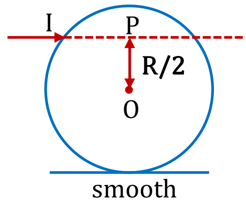


Illustration 105:



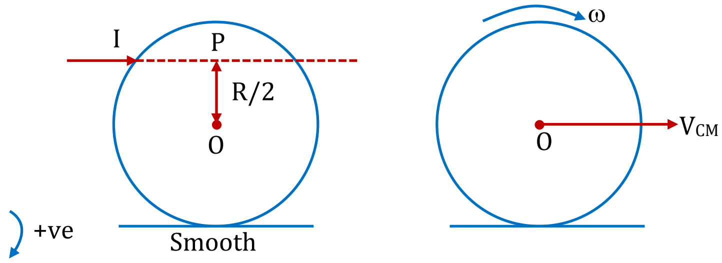
An impulse I acts on solid sphere of mass M and radius R at $\frac{R}{2}$ distance from centre O. Find v_{CM} and ω of sphere after impulse about O and P.

Solution:

$I = P_f - P_i$

$I = Mv_{CM}$

$V_{CM} = \frac{I}{M}$



$\therefore$ $(\tau)_0 = (L_f)_0 - (L_i)_0$

$$I\frac{R}{2} = \left(\frac{2}{5}MR^2\omega + 0\right) - (0+0)$$

$$\boxed{\omega = \frac{5I}{4MR}}$$

For check take about 'P'.

$(L_i)_P = (L_f)_P$

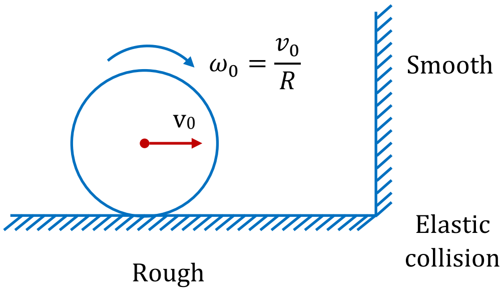
$$0 + 0 = \frac{2}{5}MR^2\omega - M\frac{R}{2}v_{CM}$$

$$0 + 0 = \frac{2}{5}MR^2\frac{5I}{4MR} - \frac{MR}{2}\times\frac{I}{M} = \frac{IR}{2} - \frac{IR}{2}$$

$0 = 0$

Illustration 106:

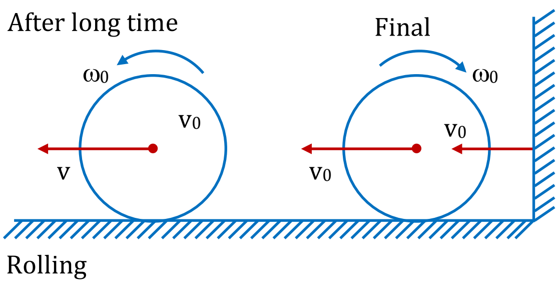
Solid sphere (M, R) is doing rolling motion with velocity v_0 and angular velocity ω_0 collides elastically with the smooth wall. Find $\vec{v}_{CM}$ of ball just after collision and after long time.



Solution:

Just after collision, $v_{cm} = v_0$

After long time,



$$Mv_0R - \frac{2}{5}Mv_0R = MvR + \frac{2}{5}M\omega R^2$$

$$\frac{3v_0}{5} = \frac{7v}{5}$$

$$v = \frac{3v_0}{7}$$

Illustration 107:

A bicycle is in motion. The force of friction exerted by the ground on its wheel is such that it acts:

(A) in backward direction on front wheel and in forward direction on rear wheel when it is accelerating

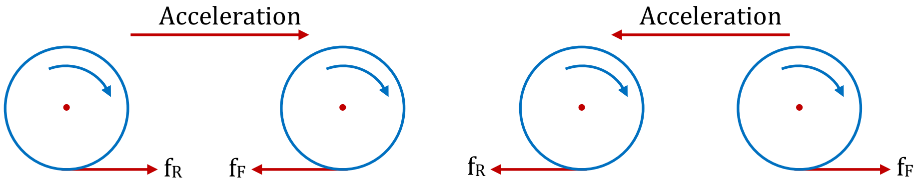
(B) in forward direction on front wheel and in backward direction on rear wheel when brakes are applied on rear wheel only.

(C) in backward direction on front wheel and in forward direction on rear wheel when brakes are applied on rear wheel only.

(D) in backward direction on both the wheels when brakes are applied on front wheel.

Answer: (AB)

Solution:



Instantaneous Axis of Rotation (IAR) :

It is a mathematical line about that a body in combined translation and rotation can be conceived in pure rotation at an instant. It continuously changes its location. This axis is always perpendicular to the plane used to represent the motion and the intersection of the axis with this plane defines the location of instantaneous centre of zero velocity (*IC*).

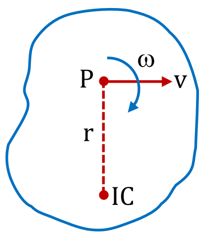
Location of the Instantaneous Centre of Rotation :

If the location of the *IC* is unknown, it may be determined by using the fact that the relative position vector extending from the *IC* to a point is always perpendicular to the velocity of the point. Following three possibilities exist.

(i) Given the velocity of a point (normally the centre of mass) on the body and the angular velocity of the body.

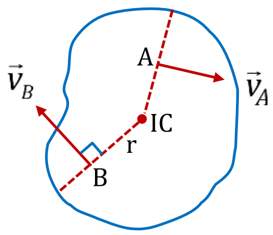
If v and ω are known, the *IC* is located along the line drawn perpendicular to $\vec{v}$ at *P*, such that the distance from *P* to *IC* is, $r = \frac{v}{\omega}$.

Note that *IC* lie on that side of *P* which causes rotation about the *IC*, which is consistent with the direction of motion caused by $\vec{\omega}$ and $\vec{v}$.



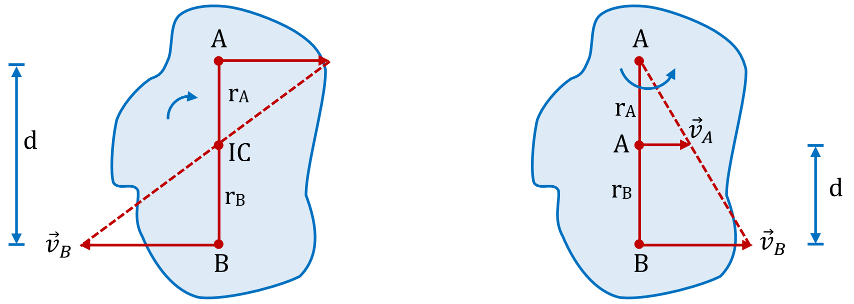
(ii) Given the lines of action of two non-parallel velocities

Consider the body shown in figure where the line of action of the velocities $\vec{v}_A$ and $\vec{v}_B$ are known. Draw perpendiculars at *A* and *B* to these lines of action. The point of intersection of these perpendiculars as shown locates the *IC* at the instant considered.



(iii) Given the magnitude and direction of two parallel velocities

When the velocities of points A and B are parallel and have known magnitudes v_A and v_B then the location of the IC is determined by proportional triangles as shown in figure.



In both the cases, $r_A = \dfrac{v_A}{\omega}$ and $r_B = \dfrac{v_B}{\omega}$

In fig. (a) $r_A + r_B = d$

and in fig. (b) $r_B - r_A = d$

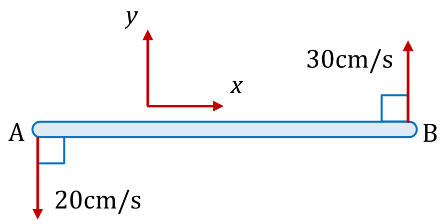
As a special case, if the body is translating, $v_A = v_B$ and the IC would be located at infinity, in which case $\omega = 0$.

Illustration 108:

A 100 cm rod is moving on a horizontal surface. At an instant, when it is parallel to the x-axis its ends A and B have velocities 30 cm/s and 20 cm/s as shown in the figure.

(a) Find its angular velocity and velocity of its center.

(b) Locate its instantaneous axis of rotation.



Solution:

(a) Velocity vector of point B

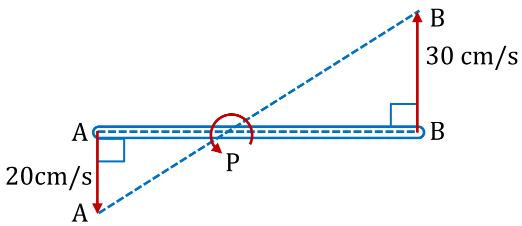
$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \overrightarrow{AB}$$

$\Rightarrow$ $\omega = 0.5\ rad/s$

Velocity vector of the center C of the rod.

$$\vec{v}_C = \vec{v}_A + \vec{\omega} \times \overrightarrow{AC}$$

$\Rightarrow$ $\vec{v}_C = -20\hat{\jmath} + 0.5\hat{k} \times 50\hat{\imath} = 5.0\hat{\jmath}\ cm/s$



(b) Here velocity vectors of the particles A and B are antiparallel, therefore the instantaneous axis of rotation passes through intersection of the common perpendicular to their velocity vectors and a line joining tips of the velocity vectors. The required geometrical construction is shown in the following figure.

Since triangles $AA'P$ and $BB'P$ are similar and $AB = 100\ cm$, we have $AP = 40\ cm$.

The instantaneous axis of rotation passes through the point P, which is 40 cm from A.

Analytical Approach:

The instantaneous center of rotation is at instantaneous rest.

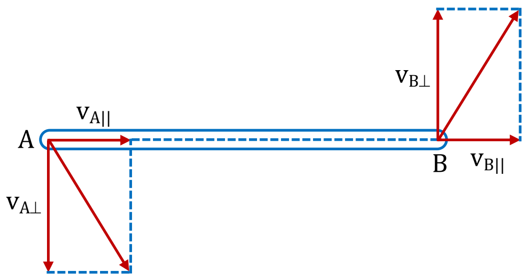
$$\vec{v}_P = \vec{v}_A + \vec{\omega} \times \overrightarrow{AP} \quad \Rightarrow \quad \vec{0} = -20\hat{\jmath} + 0.5\hat{k} \times (AP)\hat{\jmath} \quad \Rightarrow \quad AP = 40cm$$

Illustration 109:

Can you suggest a quick way to find angular velocity of a rod, if velocities of two of its points are known?

Solution:

When distance between two points and their velocity components perpendicular to the line joining them are known.



Angular velocity of the rod,

$$\omega = \frac{|\vec{v}_{BA}|}{AB} = \frac{|\vec{v}_{B\perp} - \vec{v}_{A\perp}|}{AB}$$

Illustration 110:

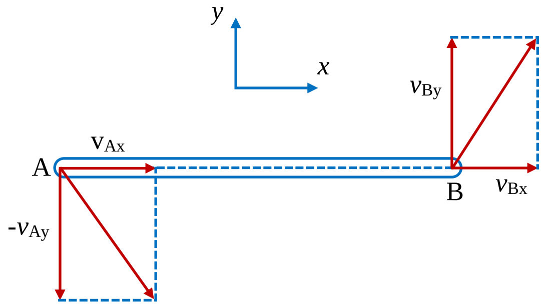
A 50 cm long rod AB is in combined translation and rotation motion on a table. At an instant velocity component of point A perpendicular to the rod is 10 cm/s, velocity component of point B parallel to the rod is 6.0 cm/s and angular velocity of the rod is 0.4 rad/s in anticlockwise sense as shown in the given figure.

(a) Find velocity vectors of point A and B.

(b) Locate the instantaneous axis of rotation.

Solution:

Let x-y plane of a coordinate system coincides with the table top and the rod is parallel to the x-axis at the instant considered. The rod is shown in this coordinate frame in the following figure.



(a) Since distance between any two points remains unchanged, the velocity components of any two points parallel to the line joining them must be equal. Therefore, we have

$$v_{Ax} = v_{Bx} = 6.0\ cm/s \qquad ...(1)$$

Velocities of points A and B must satisfy following equation

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \overrightarrow{AB}$$

$$\Rightarrow \quad 6.0\hat{\imath} + v_{By}\hat{\jmath} = 6.0\hat{\imath} - 10\hat{\jmath} + 0.4\hat{k} \times 50\hat{\imath}$$

Equating y-components of both the sides, we have

$$v_{By} = 10\ cm/s \qquad ...(2)$$

From equation (1), (2) and the given information, we can express the velocity vectors of the points A and B.

$$\vec{v}_A = 6.0\hat{\imath} - 10\hat{\jmath}\ cm/s, \text{ and } \vec{v}_B = 6.0\hat{\imath} + 10\hat{\jmath}\ cm/s$$

(b) The instantaneous axis of rotation is at instantaneous rest. Let the end A of the rod is at the origin and coordinates of the point P in the x-y plane through which the IAR passes is (x, y).

$$\vec{v}_P = \vec{v}_A + \vec{\omega} \times \overrightarrow{AP}$$

$$\Rightarrow \quad \vec{0} = 6.0\hat{\imath} - 10\hat{\jmath} + 0.4\hat{k} \times (x\hat{\imath} + y\hat{\jmath})$$

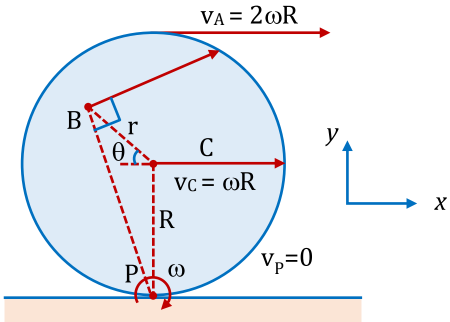
Equating coefficients of x and y-components of both the sides, we have

$$x = 25\, cm \text{ and } y = 15\, cm$$

Therefore, coordinates of the point P through which IAR passes the x-y plane are (25, 15).

Velocity relations by Use of IAR :

In rolling without slipping on stationary surface the point of contact is at instantaneous rest, therefore the IAR and the IAR passes through it. We will see how velocity of the center C, the top point A and an arbitrarily chosen point B can be calculated by assuming the body in state of pure rotation about the IAR.



Velocity of center C

$$\vec{v}_C = \vec{\omega} \times \overrightarrow{PC} = \omega R\hat{\imath}$$

Velocity of the top point A

$$\vec{v}_A = \vec{\omega} \times \overrightarrow{PA} = 2\omega R\hat{\imath}$$

Velocity of the point B

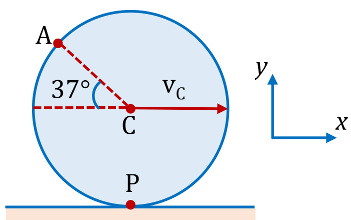
$$\vec{v}_B = \vec{\omega} \times \overrightarrow{PB} = (v_C + \omega r \sin\theta)\hat{\imath} + \omega r \cos\theta\, \hat{\jmath}$$

Illustration 111:

A cylinder of radius 5 m rolls on a horizontal surface. Velocity of its center is 25 m/s. Find its angular velocity and velocity of the point A.

Solution:

In rolling the angular velocity $\vec{\omega}$ and velocity of the center of a round section body satisfy condition,



$$\vec{v}_C = \vec{\omega} \times \vec{r}_{C/P} \Rightarrow 25\hat{\imath} = \omega\hat{k} \times 5\hat{\jmath} \Rightarrow \vec{\omega} = -5\hat{k}\ \ rad/s$$

Angular velocity vector points in the negative z-axis so the cylinder rotates in clockwise sense.

Velocity of the point A can be calculated by either analytical method, superposition method or by using method of IAR.

Analytical Method :

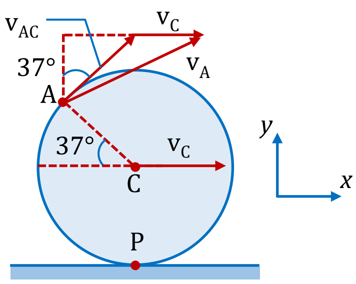
$$\vec{v}_A = \vec{v}_C + \vec{\omega} \times \overrightarrow{CA}$$

$$\Rightarrow \quad \vec{v}_A = 25\hat{\imath} + \left(-5\hat{k}\right) \times (-5\cos 37°\hat{\imath} + 5\sin 37°\hat{\jmath})$$

$$\vec{v}_A = (40\hat{\imath} + 20\hat{\jmath})m/s$$

Superposition Method :

In rolling $v_C = v_{AC} = \omega R = 25 m/s$. The superposition i.e. vector addition of the terms of equation $\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$ are shown in the following figure. Resolving $v_{AC} = \omega R = 25 m/s$ into its Cartesian components and adding to $\vec{v}_C$, we obtain

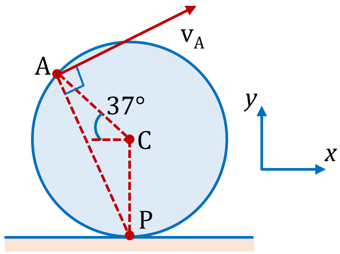


$$\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$$

$$\Rightarrow \quad \vec{v}_A = 25\hat{\imath} + 15\hat{\imath} + 20\hat{\jmath} = (40\hat{\imath} + 20\hat{\jmath}) m/s$$

Use of IAR

The contact point P is the IAR in rolling. The cylinder is in pure rotation about the IAR at the instant under consideration, so from the relative motion equation, we have



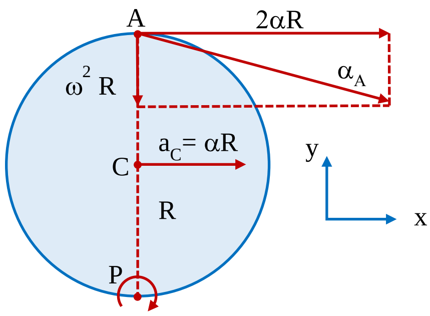
$$\vec{v}_A = \vec{\omega} \times \vec{r}_{AP}$$

$$\Rightarrow \quad \vec{v}_A = \vec{\omega} \times \left(\overrightarrow{PC} + \overrightarrow{CA}\right) = \left(-5\hat{k}\right) \times \{(5\hat{\jmath}) + (-4\hat{\imath} + 3\hat{\jmath})\}$$

$$\vec{v}_A = (40\hat{\imath} + 20\hat{\jmath}) m/s$$

Acceleration relations by use of IAR :

Acceleration relations can also be obtained by assuming the body in pure rotation about the IAR. Here we will use relative motion equation. Always keep in mind that the acceleration of the IAR is not zero, it has value $\omega^2 R$ and points towards the center of the body. Now we will see how acceleration of the center C, the top point A and an arbitrarily chosen point B can be calculated by assuming the body in state of pure rotation about the IAR.



Acceleration of center C,

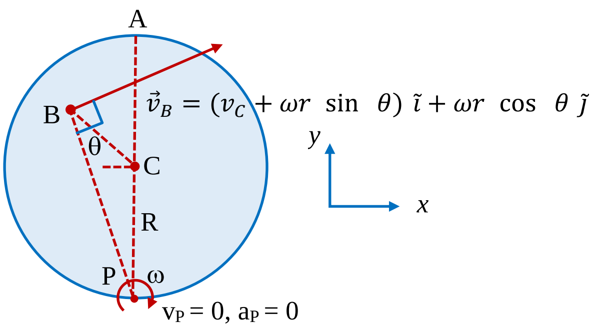
$$\vec{a}_C = \vec{a}_P + \vec{\alpha} \times \overrightarrow{PC} - \omega^2 \overrightarrow{PC}$$

$$\vec{a}_C = \alpha R\hat{\imath} - \omega^2 R\hat{\jmath}$$

Acceleration of the top point A

$$\vec{a}_A = \vec{a}_p + \vec{\alpha} \times \overrightarrow{PA} - \omega^2 \overrightarrow{PA}\hat{\jmath}$$

$$\vec{a}_A = 2\alpha R\hat{\imath} - 2\omega^2 R\hat{\jmath}$$



Acceleration of the point B

$$\vec{a}_B = \vec{a}_P + \vec{\alpha} \times \overrightarrow{PB} - \omega^2 \overrightarrow{PB}$$

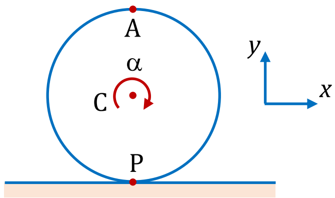
$$\vec{a}_B = \{\alpha(R + r\sin\theta) + \omega^2\}\hat{\imath}$$
$$+\{\alpha r\cos\theta - \omega^2 r\sin\theta\}\hat{\imath}$$

Illustration 112:

A body of round section of radius 10 *cm* starts rolling on a horizontal stationary surface with uniform angular acceleration 2 rad/s^2.

(a) Find initial acceleration of the center C and top point A.

(b) Find expression for acceleration of the top point A as function of time.



Solution:

Initially when the body starts, it has no angular velocity; therefore, $\vec{a}_A = \vec{a}_P + \vec{\alpha} \times \overrightarrow{PA}$

The angular acceleration vector is $\vec{\alpha} = -2\hat{k}\ rad/s^2$.

(a) Acceleration of the center C is obtained by using condition for rolling without slipping.

$$\vec{a}_C = \vec{\alpha} \times \overrightarrow{PC} \quad \Rightarrow \quad \vec{a}_C = -2\hat{k} \times 10\hat{j} = 20\hat{\imath}\ cm/s^2$$

Acceleration of the point A can be obtained either by analytical method, superposition method or by use of IAR. These methods for calculation of acceleration of the top point are already described; therefore, we use the result.

$$\vec{a}_A = 2\alpha R\hat{\imath} \quad \Rightarrow \quad \vec{a}_A = 40\hat{\imath}\ cm/s^2$$

(b) Initially at the instant t = 0, when the body starts, its angular velocity is zero. Later, it acquires angular velocity $\vec{\omega}$, therefore acceleration of any point on the body, other than its center, has an additional component of acceleration.

Angular velocity acquired by the body at time t is

$$\vec{\omega} = \vec{\omega}_o + \vec{\alpha}t$$

$\Rightarrow$ Substituting $\omega_o = 0$

We have $\vec{\omega} = -2t\hat{k}$

Analytical Method :

Using the relative motion equation for the pair of points C and A, we have

$$\vec{a}_A = \vec{a}_C + \vec{\alpha} \times \overrightarrow{CA} - \omega^2\overrightarrow{CA}$$

$$\Rightarrow \quad \vec{a}_A = \alpha R\hat{\imath} + \left(-\alpha\hat{k}\right) \times R\hat{j} - \omega^2 R\hat{j} = 2\alpha R\hat{\imath} - \omega^2 R\hat{j}$$

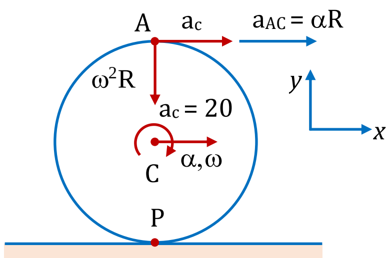
Substituting the known values

$$\vec{\alpha} = -2\hat{k}\ \text{rad/s}^2, \vec{\omega} = -2t\hat{k}\ \text{rad/s and } R = 10\ cm$$

We have $\vec{a}_A = 40\hat{\imath} - 40t^2\hat{j}\ cm/s$

Superposition Method :

We superimpose translation motion of the center and rotation motion about the center. In fact it is vector addition of terms of above equation used in analytical method.



From the above figure, we have

$$\vec{a}_A = (a_C + \alpha R)\hat{\imath} - \omega^2 R\hat{j}$$

Substituting known values

$$\vec{\alpha} = -2\hat{k}\ \text{rad/s}^2,\ \vec{\omega} = -2t\hat{k}\ \text{rad/s and } R = 10\ cm,$$

We have $\vec{a}_A = 40\hat{\imath} - 40t^2\hat{j}\ cm/s^2$

Use of IAR :

The point of contact P is the IAR, because the body is rolling without slipping. We use relative motion equation for pairs of points P and A.

$$\vec{a}_A = \vec{a}_P + \vec{\alpha} \times \overrightarrow{PA} - \omega^2 \overrightarrow{PA}$$

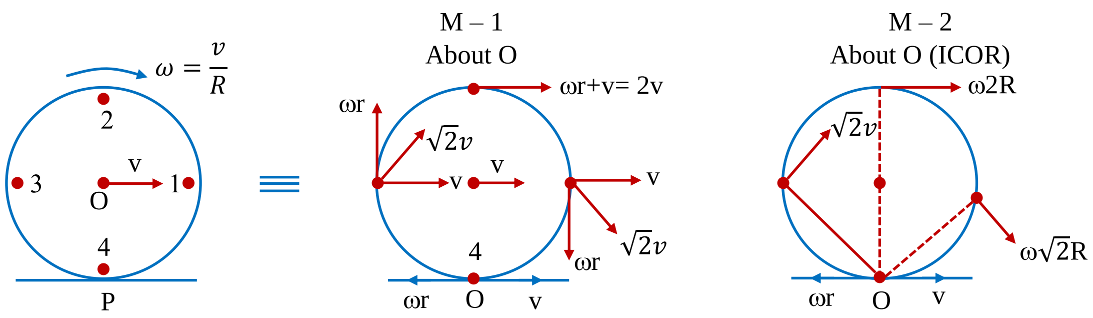
$$\Rightarrow \quad \vec{a}_A = \omega^2 R\hat{j} + 2\alpha R\hat{i} - 2\omega^2 R\hat{j} = 2\alpha R\hat{i} - \omega^2 R\hat{j}$$

Substituting known values

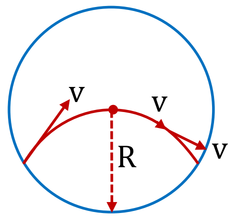
$$\alpha = 2 \text{ rad}/s^2, \omega = 2t \text{ rad}/s \text{ and } R = 10cm,$$

We have, $\vec{a}_A = 40\hat{i} - 40t^2\hat{j} cm/s^2$

(eg.) A sphere is rolling with v velocity and angular velocity ω.



(a) Which all points have speed v *and* locus ?



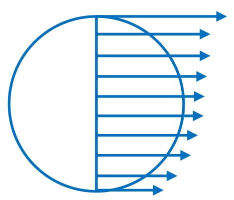
Points → ∞ points

Locus → arc

(b) Which all points have speed $2v$?

Ans. Top most

(c) Which points have horizontal velocity ?



(d) Which points have vertical velocity?

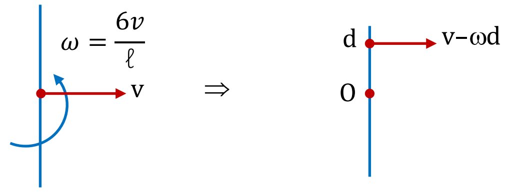
Ans. No point

(e) Range of speed ?

Ans. [0-2v]

Locate (ICOR) :

(eg.) Rod (M, ℓ) having velocity v and angular velocity ω. Locate ICOR.

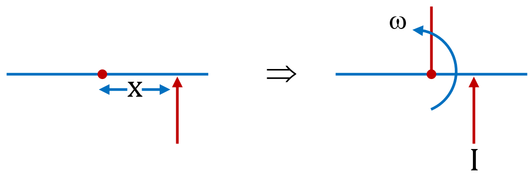


$v - \omega d = 0$

$$d = \frac{v}{\omega} = \frac{v\ell}{6v} = \frac{\ell}{6}$$

$\therefore$ ICOR is $\frac{\ell}{6}$ from center.

(eg.) Rod (M, ℓ) gets an impulse I at x distance from centre of the rod.



Locate ICOR

Just after I

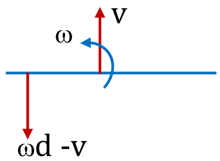
$I = Mv - 0$

$$v = \frac{I}{M}$$

$$\Rightarrow \quad Ix = \frac{ML^2}{12}\omega + 0 - (0 - 0)$$

$$\omega = \frac{Ix12}{ML^2}$$

$\omega d - v = 0$



$$\Rightarrow \quad d = \frac{v}{\omega} = \frac{I \times ML^2}{M \times Ix12} = \frac{L^2}{12x}$$

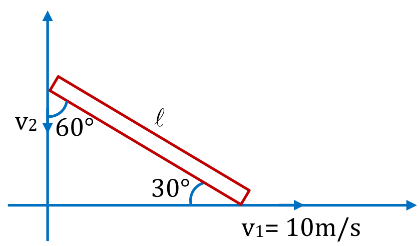
Illustration 113:

A rod of length l is lying on a smooth wall with one end on wall and other on horizontal surface as shown in the figure.

At the instant shown find:

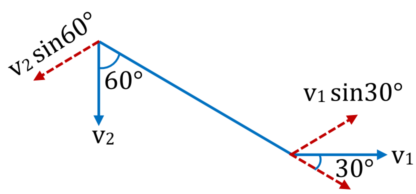
(a) v_2 (b) ω

(c) ICOR location (d) v_{CM} of rod



Solution:

(a)



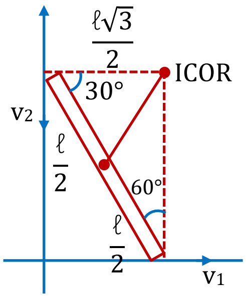
$v_2 \cos 60° = v_1 \cos 30°$

$$\frac{v_2}{2}=10\times\frac{\sqrt{3}}{2}$$

$$v_2=10\sqrt{3}$$

(b) $\omega=\frac{(v_{rel})_\perp}{R}=\frac{v_1\sin 30°-(-v_2\sin 60°)}{\ell}=\frac{5+15}{\ell}=\frac{20}{\ell}$

(c)



ICOR is $\frac{\ell}{2}$ distance from centre.

(d) $\frac{\omega\ell}{2}=v_1 \Rightarrow \omega=\frac{10\times 2}{\ell}=\frac{20}{\ell}$

$$v_{CM}=\frac{\omega\ell}{2}=\frac{20}{\ell}\times\frac{\ell}{2}=10m/s$$

Alternate method:

$$r_{CM}=\frac{x}{2}\hat{i}+\frac{y}{2}\hat{j}$$

$$\vec{v}_{CM}=\frac{\frac{dx}{dt}}{2}\hat{i}+\frac{\frac{dy}{dt}}{2}\hat{j}=\frac{v_1}{2}\hat{i}+\frac{v_2}{2}\hat{j}$$

$$=5\hat{i}-5\sqrt{3}\hat{j}$$

v_{CM} = 10 m/s

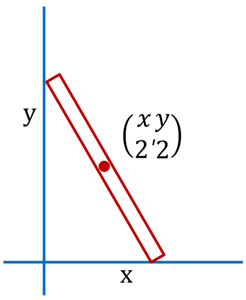
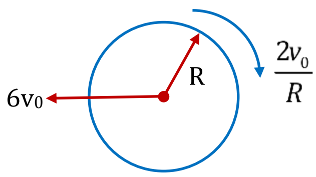


Illustration 114:

A disc of radius R = 2m moves as shown in the figure, with translation velocity of 6 v_0 of its centre of mass and an angular velocity of $\frac{2v_0}{R}$. The distance (in m) of instantaneous axis of rotation from its centre of mass is



(A) 3 (B) 4 (C) 5 (D) 6

Answer: (D)

Solution:

Instantaneous axis of rotation lies above the centre of mass where $v-\omega r=0$

$\Rightarrow$ $r=\frac{v}{\omega}=\frac{6v_0}{\frac{2v_0}{R}}=3R$

$\therefore$ $r=2\times 3=6m$