02 Circular Motion and Work, Energy and Power

CIRCULAR MOTION

When a particle moves in a plane such that its distance from a fixed (or moving) point remains constant, then its motion is known as circular motion with respect to that fixed (or moving) point. The fixed point is called centre, and the distance of particle from fixed point is called radius.

Kinematics of circular motion:

Variables of Motion:

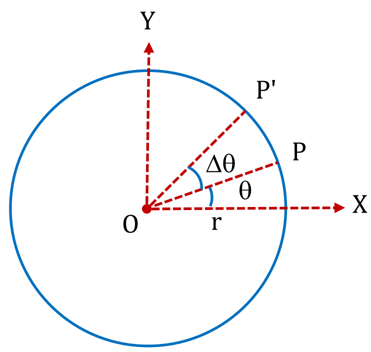
(a) Angular Position: To decide the angular position of a point in space we need to specify (i) origin and (ii) reference line.

The angle made by the position vector w.r.t. origin, with the reference line is called angular position.

Clearly angular position depends on the choice of the origin as well as the reference line.

Circular motion is a two dimensional motion or motion in a plane. Suppose a particle P is moving in a circle of radius r and centre O.

The angular position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle.



(b) Angular Displacement:

Definition : Angle through which the position vector of the moving particle rotates in a given time interval is called its angular displacement. Angular displacement depends on origin, but it does not depends on the reference line. As the particle moves on above circle its angular position θ changes. Suppose the point rotates through an angle Δθ in time Δt, then Δθ is angular displacement.

Important points:

- Angular displacement is a dimensionless quantity. Its SI unit is radian, some other units are degree and revolution.

  2π rad = 360° = 1 *rev*

- Infinitesimally small angular displacement is a vector quantity, but finite angular displacement is a scalar, because while the addition of the Infinitesimally small angular displacements is commutative, addition of finite angular displacement is not.

  $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ but $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$

- Direction of small angular displacement is decided by right hand thumb rule. When the fingers are directed along the motion of the point then thumb will represents the direction of angular displacement.

(c) Angular Velocity ω

(i) Average Angular Velocity

$$\omega_{av} = \frac{\text{Angular displacement}}{\text{Total time taken}}$$

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2. Since angular displacement is a scalar, average angular velocity is also a scalar.

(ii) Instantaneous Angular Velocity: It is the limit of average angular velocity as Δt approaches zero. i.e. $\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$

Since infinitesimally small angular displacement $\overrightarrow{d\theta}$ is a vector quantity, instantaneous angular velocity $\vec{\omega}$ is also a vector, whose direction is given by right hand thumb rule.

Important points:

- Angular velocity has dimension of $[T^{-1}]$ and SI unit rad/s.
- For a rigid body, as all points will rotate through same angle in same time, angular velocity is a characteristic of the body as a whole, e.g., angular velocity of all points of earth about earth's axis is $(2\pi/24)$ rad/hr.
- If a body makes *'n'* rotations in *'t'* seconds then average angular velocity in radian per second will be

$$\omega_{av} = \frac{2\pi n}{t}$$

If T is the period and *'f'* the frequency of uniform circular motion $\omega_{av} = \frac{2\pi}{T} = 2\pi f$.

(d) Angular Acceleration α:

(i) Average Angular Acceleration: Let ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2 respectively, then the average angular acceleration α_{av} is defined as

$$\vec{\alpha}_{av} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1} = \frac{\Delta\vec{\omega}}{\Delta t}$$

(ii) Instantaneous Angular Acceleration : It is the limit of average angular acceleration as Δt approaches zero, i.e., $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$

Since $\vec{\omega} = \frac{d\vec{\theta}}{dt}$

$$\therefore \quad \vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$$

Also $\vec{\alpha} = \omega \frac{d\vec{\omega}}{d\theta}$

Important points :

- Both average and instantaneous angular acceleration are axial vectors with dimension $[T^{-2}]$ and unit rad/s^2.
- If $\alpha = 0$, circular motion is said to be uniform.

Motion with constant angular velocity

$\theta = \omega t$, $\alpha = 0$

Motion with constant angular acceleration

$\omega_0 \Rightarrow$ Initial angular velocity

$\omega \Rightarrow$ Final angular velocity

$\alpha \Rightarrow$ Constant angular acceleration

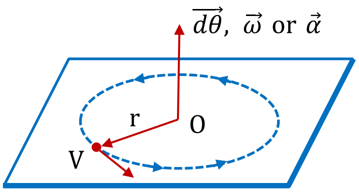
$\theta \Rightarrow$ Angular displacement

Circular motion with constant angular acceleration is analogous to one dimensional translational motion with constant acceleration. Even here equation of motion has same form.

$\omega = \omega_0 + \alpha t \quad ; \quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega^2 = \omega_0^2 + 2\alpha\,\theta \quad ; \quad \theta = \left(\frac{\omega+\omega_0}{2}\right)t$

$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$



Equations of circular motion

Translatory / Linear Motion	**Rotational Motion**
• Initial velocity (u)	Initial angular velocity (ω_0)
• Final velocity (v)	Final angular velocity (ω)
• Displacement (s)	Angular displacement (θ)
• Acceleration (a)	Angular Acceleration (α)
If a = constant, then	If α = constant, then
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s_{n^{th}} = u + \frac{a}{2}(2n-1)$	$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$
$s = \left(\frac{u+v}{2}\right)t$	$\theta = \left(\frac{\omega_0+\omega}{2}\right)t$

Illustration 1:

A particle is moving with constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \frac{\pi}{2}$.

Solution:

Time taken to describe angle θ, $t = \frac{\theta}{\omega} = \frac{\theta R}{v} = \frac{\pi R}{2v}$

Average velocity = $\frac{\text{Total displacement}}{\text{Total time}} = \frac{\sqrt{2}R}{\pi R/2v} = \frac{2\sqrt{2}}{\pi}v$

Instantaneous velocity = v

The ratio of average velocity to its instantaneous velocity = $\frac{2\sqrt{2}}{\pi}$.

Illustration 2:

A fan is rotating with angular velocity 100 rev/sec. Then it is switched off. It takes 5 minutes to stop.

(a) Find the total number of revolutions made before it stops. (Assume uniform angular retardation)

(b) Find the value of angular retardation.

(c) Find the average angular velocity during this interval.

Solution:

(a) $\theta = \left(\frac{\omega+\omega_0}{2}\right)t = \left(\frac{100+0}{2}\right) \times 5 \times 60 = 15000$ revolution.

(b) $\omega = \omega_0 + \alpha t \quad \Rightarrow \quad 0 = 100 - \alpha(5 \times 60) \quad \Rightarrow \quad \alpha = \frac{1}{3}$ rev./sec^2

(c) $\omega_{av} = \frac{\text{Total Angle of Rotation}}{\text{Total time taken}} = \frac{15000}{5\times60} = 50$ rev./sec.

Illustration 3:

A fan rotating with ω = 100 rad/s, is switched off. After $2n$ rotation its angular velocity becomes 50 rad/s. Find the angular velocity of the fan after n rotations.

Solution:

$\omega^2 = \omega_0^2 + 2\alpha\theta$

$50^2 = (100)^2 + 2\alpha(2\pi \,.\, 2n)$ ...(1)

If angular velocity after n rotation is ω_n

$\omega_n^2 = (100)^2 + 2\alpha(2\pi \,.\, n)$ ...(2)

From equation (1) and (2)

$$\frac{50^2 - 100^2}{\omega_n^2 - 100^2} = \frac{2\alpha(2\pi.2n)}{2\alpha 2\pi n} = 2$$

$\Rightarrow \quad \omega_n^2 = \frac{50^2+100^2}{2}$

$\Rightarrow \quad \omega = 25\sqrt{10}$ rad/s

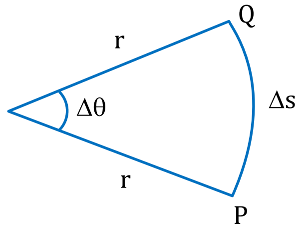
Relation between speed and angular velocity:

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Here, $\vec{v}$ is velocity of the particle, $\vec{\omega}$ is angular velocity about centre of circular motion and '$\vec{r}$' is position of particle w.r.t. center of circular motion.

Since $\vec{\omega} \perp \vec{r}$

$v = \omega r$ for circular motion.

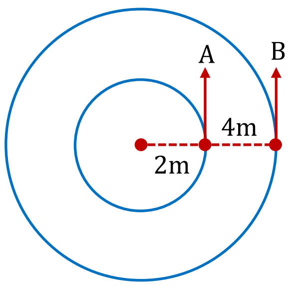


$$\text{Angle }(\Delta\theta) = \frac{\text{arc}}{\text{radius}} = \frac{\Delta s}{r} \Rightarrow \frac{\Delta s}{r} = \Delta\theta \Rightarrow \Delta s = r\Delta\theta$$

$$\therefore \quad \frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} \text{ if } \Delta t \to 0 \text{ then } \frac{ds}{dt} = r\frac{d\theta}{dt} \quad \Rightarrow \boxed{v = \omega r}$$

Illustration 4:

At $t = 0$, angular velocity of particle A and B are 10 rad/sec and 5 rad/sec. respectively show in a figure. Find relative velocity of B wrt A after $\frac{\pi}{5}$ sec. ?



Solution:

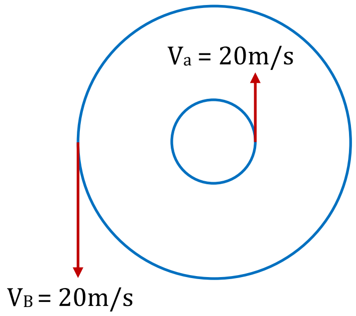
$V_A = R_A\omega_A = 2 \times 10 \Rightarrow 20 m/s$

$V_B = R_B\omega_B = 4 \times 5 \Rightarrow 20 m/s$

After $\frac{\pi}{5}$ sec.

$\therefore \quad \theta_A = 10 \times \frac{\pi}{5} = 2\pi \;;\; \theta_B = 5 \times \frac{\pi}{5} = \pi$

$|V_{B/A}| = 40 m/s$



Acceleration of a particle in uniform circular motion

Uniform circular motion: If a particle is moving in a circle with constant speed then motion is called Uniform circular motion (UCM).

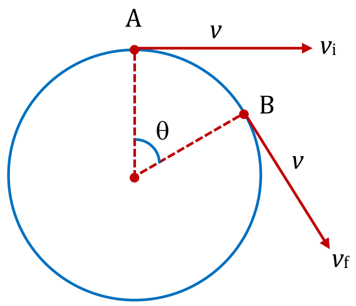
$$\Delta\vec{v} = \vec{v}_B - \vec{v}_A$$

Magnitude of change in velocity

$$|\Delta\vec{v}| = |\vec{v}_B - \vec{v}_A| = \sqrt{v_B^2 + v_A^2 + 2v_A v_B \cos\ (\pi - \theta)}$$

($v_A = v_B = v$, since speed is same)

$$\therefore \quad |\Delta\vec{v}| = 2v\sin\frac{\theta}{2}$$



Distance travelled by particle between A and $B = r\,\theta$

Hence time taken, $\Delta t = \frac{r\theta}{v}$

Average acceleration,

$$|\vec{a}| = \left|\frac{\overrightarrow{\Delta v}}{\Delta t}\right| = \frac{2v\sin\frac{\theta}{2}}{\frac{r\theta}{v}} = \frac{v^2}{r}\frac{2\sin\frac{\theta}{2}}{\theta}$$

If $\Delta t \to 0$, then θ is small, $\sin(\theta/2) = \theta/2$

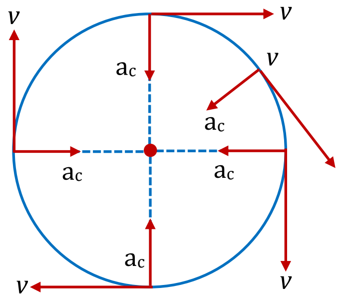
$$\lim_{\Delta t\to 0}\left|\frac{\Delta\vec{v}}{\Delta t}\right| = \left|\frac{d\vec{v}}{dt}\right| = \frac{v^2}{r}$$

i.e., instantaneous acceleration is $\frac{v^2}{r}$.

$\therefore \quad a = a_c = \frac{v^2}{r}$

Though we have derived the formula of centripetal acceleration under condition of constant speed, the same formula is applicable even when the speed is variable.

Key points:-



(1) The above acceleration is due to change in direction of velocity. It is called centripetal acc. or normal acceleration (a_N).

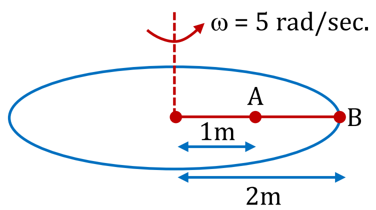
(2) a_c is always directed towards the centre i.e. $\perp$ to the velocity.

It acts always towards the centre. So, it is centripetal.

(3) In U.C.M, magnitude of a_c is constant but direction is changing.

Illustration 5:

Find centripetal acceleration of given points (A and B) as shown in figure.



Solution:

$(a_c)_A = \omega^2 R = 5 \times 5 \times 1 = 25\ m/\text{sec}^2$

$(a_c)_B = \omega^2 R = 5 \times 5 \times 2 = 50\ m/\text{sec}^2$

Non – Uniform Circular Motion

i.e. speed not constant.

If a particle is moving in a circle with variable speed then motion is called Non-Uniform circular motion.

There are two types of acceleration in circular motion; Tangential acceleration and centripetal acceleration.

$$a_{net} = \sqrt{(a_t)^2 + (a_n)^2}$$

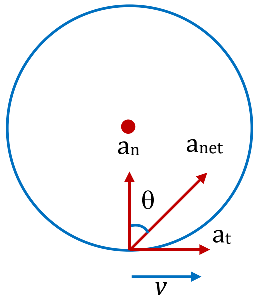
$$\tan\theta = \frac{a_t}{a_n}$$

$$\boxed{S = \theta R}$$

Differentiation $\boxed{|V| = \omega R}$

Differentiation $\frac{d|V|}{dt} = R\frac{d\omega}{dt}$

$$\boxed{a_t = \alpha R}$$



(a) **Tangential acceleration :** Component of acceleration directed along tangent of circle is called tangential acceleration. It is responsible for changing the speed of the particle. It is defined as,

$$a_t = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = \text{Rate of change of speed.}$$

$$a_t = \alpha r$$

Important Point:

(i) In vector form $\vec{a}_t = \vec{\alpha} \times \vec{r}$

(ii) If tangential acceleration is directed in direction of velocity then the speed of the particle increases.

(iii) If tangential acceleration is directed opposite to velocity then the speed of the particle decreases.

(b) **Centripetal acceleration :** It is responsible for change in direction of velocity. In circular motion, there is always a centripetal acceleration.

Centripetal acceleration is always variable because it changes in direction.

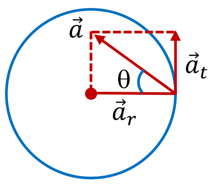
Centripetal acceleration is also called radial acceleration or normal acceleration.

(c) **Total acceleration :** Total acceleration is vector sum of centripetal acceleration and tangential acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_r + \vec{a}_t$$

$$a = \sqrt{{a_t}^2 + {a_r}^2}$$

$$\tan\theta = \frac{a_t}{a_r}$$



Important Point:

(i) Differentiation of speed gives tangential acceleration.

(ii) Differentiation of velocity ($\vec{v}$) gives total acceleration.

(iii) $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d|\vec{v}|}{dt}$ are not same physical quantity. $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of rate of change of velocity, i.e. magnitude of total acceleration and $\frac{d|\vec{v}|}{dt}$ is a rate of change of speed, i.e. tangential acceleration.

Illustration 6:

A particle is moving in a circle of radius 10 cm with uniform speed completing the circle in 4s, find the magnitude of its acceleration.

Solution:

The distance covered in completing the circle is $2\pi r = 2\pi \times 10\ cm$. The linear speed is

$$v = 2\pi r/t = \frac{2\pi \times 10 cm}{4s} = 5\pi\ cm/s.$$

The acceleration is $a = \frac{v^2}{r} = \frac{(5\pi cm/s)^2}{10cm} = 2.5\pi^2\ cm/s^2$

Illustration 7:

A particle moves in a circle of radius 2.0 cm at a speed given by $v = 4t$, where v is in cm/s and t is in seconds.

(a) Find the tangential acceleration at $t = 1s$.

(b) Find total acceleration at $t = 1s$.

Solution:

(a) Tangential acceleration

$$a_t = \frac{dv}{dt} \quad \text{or} \quad a_t = \frac{d}{dt}(4t) = 4\ cm/s^2$$

$$a_C = \frac{v^2}{R} = \frac{(4)^2}{2} = 8\ cm/s^2$$

(b) $a = \sqrt{a_t^2 + a_C^2} = \sqrt{(4)^2 + (8)^2} = 4\sqrt{5}\ cm/s^2$

Illustration 8:

A particle begins to move with a tangential acceleration of constant magnitude 0.6 m/s^2 in a circular path. It slips when its total acceleration becomes 1 m/s^2. Find the angle through which it would have turned before it starts to slip.

Solution:

$a_{Net} = \sqrt{a_t^2 + a_c^2}$ $\Rightarrow$ $\omega^2 = \omega_0^2 + 2\alpha\theta$

$\because$ $\omega_0 = 0$ so $\omega^2 = 2\alpha\theta$

$a_c = \omega^2 R = 2\ (\alpha R\theta)$

$a_c = \omega^2 R = 2a_t\theta$

$1 = \sqrt{0.36 + (1.2\theta)^2}$ $\Rightarrow$ $1 - 0.36 = (1.2\ \theta)^2$

$\Rightarrow$ $\frac{0.8}{1.2} = \theta$ $\Rightarrow$ $\theta = \frac{2}{3}$ radian

Illustration 9:

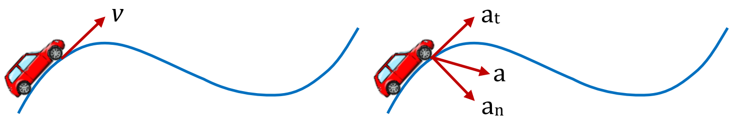
A particle is moving with a constant angular acceleration of 4 rad/sec² in a circular path. At time t = 0 particle was at rest. Find the time at which the magnitudes of centripetal acceleration and tangential acceleration are equal.

Solution:

$$a_t = \alpha R \quad \Rightarrow \quad v = 0 + \alpha R t \quad \Rightarrow \quad a_c = \frac{v^2}{R} = \frac{\alpha^2 R^2 t^2}{R} = \frac{\alpha^2 R^2 t^2}{R}$$

$$\because \quad |a_t| = |a_c| \quad \Rightarrow \alpha R = \frac{\alpha^2 R^2 t^2}{R} \quad \Rightarrow \quad t^2 = \frac{1}{\alpha} = \frac{1}{4} \quad \Rightarrow \quad t = \frac{1}{2}\text{ sec.}$$

Acceleration of particle in curvilinear motion:



Consider a particle moving over a curve as shown in figure. Let acceleration of particle (a_{net}) be making angle θ with velocity ($\vec{v}$) at point P.

$$\boxed{\vec{a}_{net} = \frac{d\vec{v}}{dt}}$$

Component of a_{net} in direction of velocity is called tangential acceleration [a_t] of particle. It is responsible for changing the magnitude of velocity i.e. speed of particle.

$$\boxed{a_t = \frac{d|\vec{v}|}{dt}}$$

Component of a_{net} perpendicular to the velocity i.e. along the normal is called normal acc. [a_n].

It is responsible for changing the direction of velocity.

Its [a_n] direction is towards the CONCAVE side.

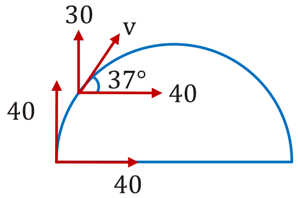
Illustration 10:

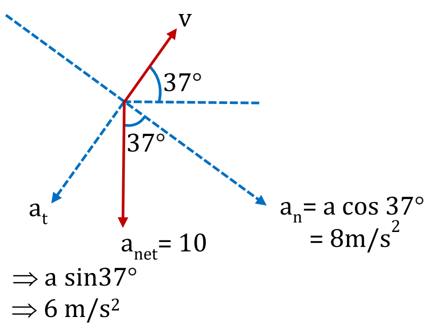
A particle is projected with velocity $40\sqrt{2}\ m/s$ at angle of 45° with horizontal at t = 0. Find 'a_t' and 'a_n' at.

(a) t = 1 sec. (b) t = 4 sec. (c) t = 7 sec.

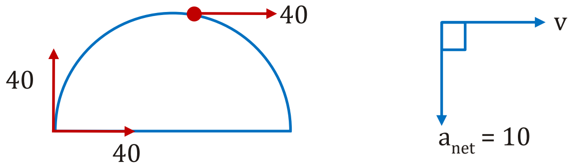
Solution:

(a)





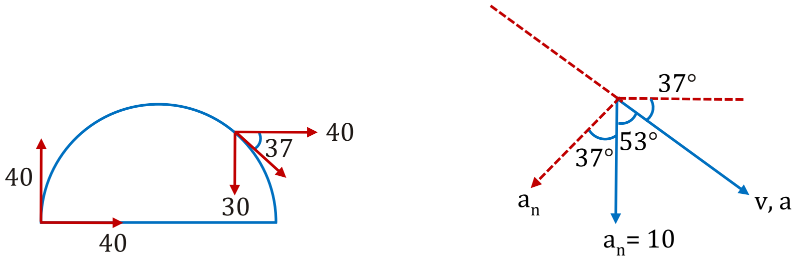
(b)



$\therefore \quad a_t = 0$

$a_n = 10$

(c)



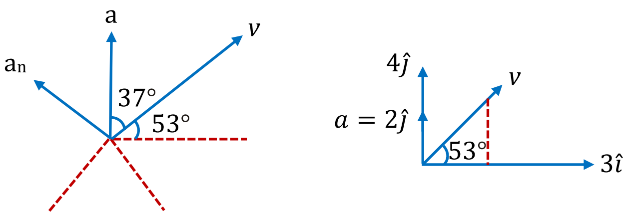
$a_t = 10 \cos 53° = 10 \times \dfrac{3}{5} = 6m/s^2$

$a_n = 10 \cos 37° = 10 \times \dfrac{4}{5} = 8m/s^2$

Illustration 11:

$\vec{v} = 3\hat{i} + 2t\hat{j}$, find a_{net}, a_t, a_n at $t = 2$ sec. ?

Solution:



$$a_{net} = \frac{d\vec{v}}{dt} = 2\hat{j}$$

Component of $\vec{a}_{net}$ along $(\vec{V}_2(3\hat{i} + 4\hat{j})) = a_t$

Component of $\vec{a}_{net} \perp \vec{v}_2(3\hat{i} + 4\hat{j}) = \overrightarrow{a_n}$

$a_t = a \cos 37° = \dfrac{2\times4}{5} = \dfrac{8}{5} m/s^2$

$a_n = a \sin 37° = 2 \times 3/5 = 6/5\ m/s^2$

We know that a_t change the magnitude of velocity i.e. speed.

$\vec{v} = 3\hat{i} + 2t\hat{j}$

(v)speed $= \sqrt{9 + 4t^2}$

$$a_t = \frac{dv}{dt} = \frac{d\left(\sqrt{9 + 4t^2}\right)}{dt} = \frac{1}{2\sqrt{9 + 4t^2}}(8t)$$

At $\quad t = 2 \quad ; \quad a_t = \dfrac{8}{\sqrt{25}} = \dfrac{8}{5}$

Radius of curvature:

Any curved path can be assumed to be made of infinite circular arcs. Radius of curvature at a point is the radius of the circular arc at a particular point which fits the curve at that point.

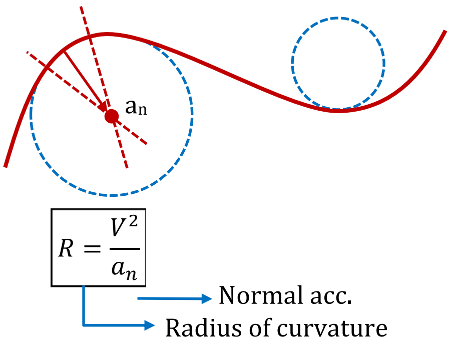
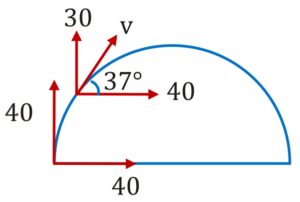


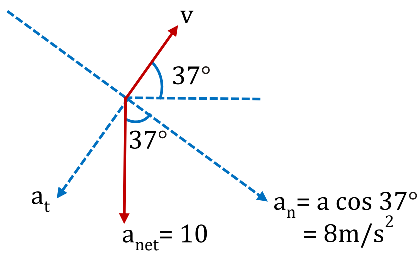
Illustration 12:

A particle is projected with velocity $40\sqrt{2}m/s$ at angle of 45° with horizontal at $t = 0$ of projectile find radius of curvature at $t = 1, 4, 7$.

Solution:

(a)

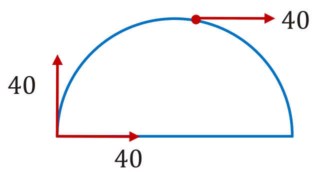


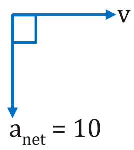


$\Rightarrow$ a sin37°
$\Rightarrow$ 6 m/s²

$$R = \frac{v^2}{a_n} = \frac{50 \times 50}{8} = \frac{625}{2}m$$

(b)



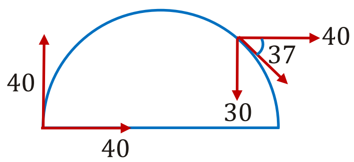


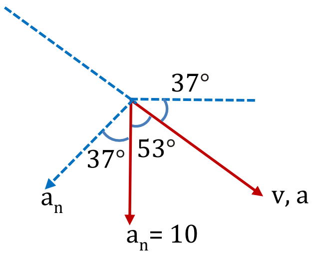
$\therefore$ $a_t = 0$

$a_n = 10$

$$R = \frac{v^2}{a_n} = \frac{40 \times 40}{10} = 160m$$

(c)

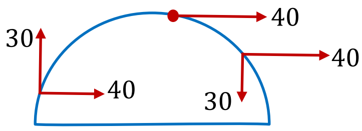




$a_t = 10 \cos 53° = 10 \times \frac{3}{5} = 6 m/s^2$

$a_n = 10 \cos 37° = 10 \times \frac{4}{5} = 8 m/s^2$

t	v	a_n	R
1	50	8	$\frac{(50)^2}{8} = \frac{2500}{8} m$
4	40	10	$\frac{(40)^2}{10} = 160m$
7	50	8	$\frac{(50)^2}{8} = \frac{2500}{8} m$



$$R = \frac{v^2}{a_n} = \frac{50 \times 50}{8} = \frac{625}{2} m$$

Illustration 13:

A bee travels along a circular path (projectile motion) with constant speed of $25 m/s$ and having radius $160\ m$. Find a_n of bee at highest point ?

Solution:

Radius of curvature = $160m$

$$R = \frac{V^2}{a_n}$$

$$160 = \frac{(25)^2}{a_n}$$

$$a_n = \frac{625}{160} m/s^2$$

Note:

Radius of curvature is property of curve and not of motion of particle.

Dynamics of circular motion:

If there is no force acting on a body it will move in a straight line (with constant speed). Hence if a body is moving in a circular path or any curved path, there must be some force acting on the body.

If speed of body is constant, the net force acting on the body is along the inside normal to the path of the body and it is called centripetal force.

$$\text{Centripetal force } (F_c) = ma_c = \frac{mv^2}{r} = m\omega^2 r$$

However, if speed of the body varies then, in addition to above centripetal force which acts along inside normal, there is also a force acting along the tangent of the path of the body which is called tangential force.

$$\text{Tangential force } (F_t) = ma_t = m\frac{dv}{dt} = m\alpha r \quad ; \quad \text{where } \alpha \text{ is the angular acceleration}$$

Important Point:

Remember $\frac{mv^2}{r}$ is not a force itself. It is just the value of the net force acting along the inside normal which is responsible for circular motion. This force may be friction, normal, tension, spring force, gravitational force or a combination of them.

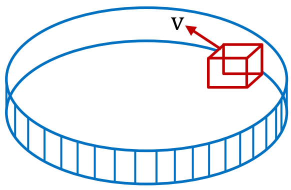
So, to solve any problem in uniform circular motion we identify all the forces acting along the normal (towards center), calculate their resultant and equate it to $\frac{mv^2}{r}$.

If circular motion is non-uniform then in addition to above step we also identify all the forces acting along the tangent to the circular path, calculate their resultant and equate it to $\frac{mdv}{dt}$ or $\frac{md|\vec{v}|}{dt}$.

Circular motion in horizontal plane:

Illustration 14:

A block of mass m moves with speed v against a smooth, fixed vertical circular groove of radius r kept on smooth horizontal surface.



Find :

(i) Normal reaction of the floor on the block.

(ii) Normal reaction of the vertical wall on the block.

Solution:

Here centripetal force is provided by normal reaction of vertical wall.

(i) Normal reaction of floor, $N_F = mg$

(ii) Normal reaction of vertical wall, $N_W = \frac{mv^2}{r}$.

Illustration 15:

A block of mass m is kept on the edge of a horizontal turn table of radius R, which is rotating with constant angular velocity ω (along with the block) about its axis. If coefficient of friction is μ, find the friction force between block and table.

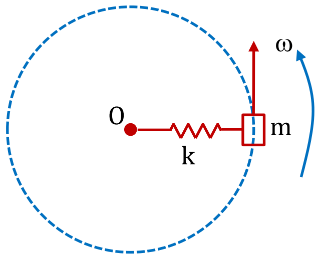
Solution:

Here centripetal force is provided by friction force.

Friction force = centripetal force = $m\omega^2R$

Illustration 16:

A block of mass m is tied to a spring of spring constant k, natural length ℓ, and the other end of spring is fixed at O. If the block moves in a circular path on a smooth horizontal surface with constant angular velocity ω, find tension in the spring.



Solution:

Assume extension in the spring is x.

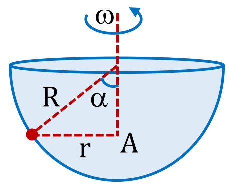
Here centripetal force is provided by spring force.

Centripetal force, $kx = m\omega^2(\ell + x) \Rightarrow x = \frac{m\omega^2\ell}{k-m\omega^2}$

Therefore, tension = $kx = \frac{km\omega^2\ell}{k-m\omega^2}$

Illustration 17:

A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical as shown. A small ball kept in the bowl rotates with the bowl without slipping on its smooth surface and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.



Solution:

Let ω be the angular speed of rotation of the bowl. Two force are acting on the ball.

1. Normal reaction N
2. Weight mg

The ball is rotating in a circle of radius r (= $R \sin \alpha$) with centre at A at an angular speed ω. Thus,

$N \sin \alpha = mr\omega^2 = mR \sin \alpha\ \omega^2$

$N = mR\omega^2$ ...(i)

$N \cos \alpha = mg$ ...(ii)

Dividing equations (i) by (ii), we get

$$\frac{1}{\cos \alpha} = \frac{\omega^2 R}{g}$$

$$\therefore \quad \omega = \sqrt{\frac{g}{R \cos \alpha}}$$

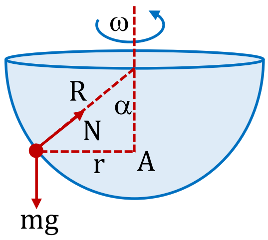
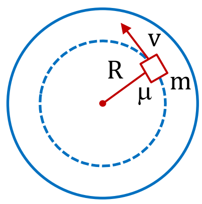


Illustration 18:

The coefficient of friction between block and table is μ. Find the tension in the string if the block moves on the horizontal table with speed v in circle of radius R.



Solution:

The magnitude of centripetal force is $\frac{mv^2}{R}$.

(i) If limiting friction is greater than or equal to $\frac{mv^2}{R}$, then static friction alone provides centripetal force, so tension is equal to zero.

$T = 0$

(ii) If limiting friction is less than $\frac{mv^2}{R}$, then friction as well as tension both combine to provide the necessary centripetal force.

$$T + f_r = \frac{mv^2}{R}$$

In this case friction is equal to limiting friction, $f_r = \mu mg$

$$\therefore \quad \text{Tension} = T = \frac{mv^2}{R} - \mu mg$$

Conical Pendulum:

If a small particle of mass m tied to a string is whirled along a horizontal circle, as shown in figure then the arrangement is called a 'conical pendulum'. In case of conical pendulum, the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force.

Thus, $T\sin\theta = \frac{mv^2}{r}$ and $T\cos\theta = mg$

$\Rightarrow \quad v = \sqrt{rg\tan\theta}$

Angular speed,

$$\omega = \frac{v}{r} = \sqrt{\frac{g\tan\theta}{r}}$$

So, the time period of pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

$$= 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

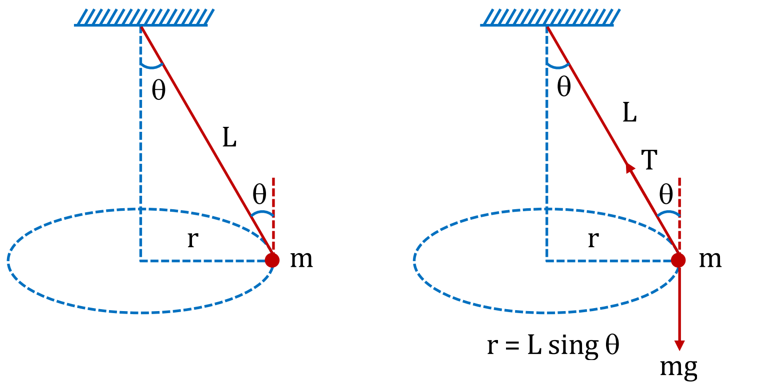


Illustration 19:

Consider a conical pendulum having bob of mass m is suspended from a ceiling through a string of length L. The bob moves in a horizontal circle of radius r. Find (a) Angular speed of the bob and (b) Tension in the string.

Solution:

The situation is shown in figure below. The angle θ made by the string with the vertical is given by

$$\sin\theta = r / L,\ \cos\theta = h/L = \frac{\sqrt{L^2-r^2}}{L} \qquad \text{...(i)}$$

The forces on the particle are

(a) the tension T along the string and

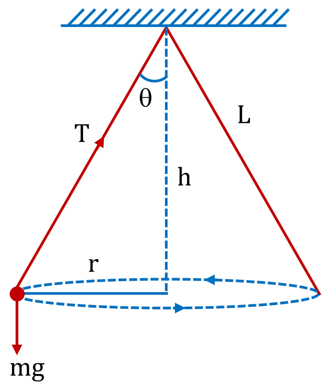
(b) the weight mg vertically downward.

The particle is moving in a circle with a constant speed v. Thus, the radial acceleration towards the centre has magnitude v^2/r. Resolving the forces along the radial direction and applying Newton's second law,

$$T\sin\theta = m(v^2 / r) \qquad \text{...(ii)}$$

As there is no acceleration in vertical direction, we have from Newton's law,

$$T\cos\theta = mg \qquad \text{...(iii)}$$



Dividing (ii) by (iii),

$$\tan\theta = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{rg\tan\theta}$$

$$\Rightarrow \quad \omega = \frac{v}{r} = \sqrt{\frac{g\tan\theta}{r}} = \sqrt{\frac{g}{h}} = \sqrt{\frac{g}{L\cos\theta}} = \sqrt{\frac{g}{(L^2-r^2)^{\frac{1}{2}}}}$$

And from (iii),

$$T = \frac{mg}{\cos\theta} = \frac{mgL}{(L^2-r^2)^{\frac{1}{2}}} \text{ d}$$

Illustration 20:

A string breaks under a load of 50 kg. A mass of 1 kg is attached to one end of the string 10 m long and is rotated in horizontal circle. Calculate the greatest number of revolutions that the mass can make in one second without breaking the string.

Solution:

$\omega = 2\pi n,\quad T_{max} = 500\ N,\quad r = L\sin\theta,\quad T\sin\theta = m\omega^2 r$

$\Rightarrow\quad T = m\omega^2 L$

$\Rightarrow\quad T_{max} = m\omega_{max}^2 L$

$\Rightarrow\quad T_{max} = m(2\pi n_{max})^2 L$

$$n_{max} = \frac{1}{2\pi}\sqrt{\frac{T_{max}}{mL}} = \frac{1}{2\pi}\sqrt{\frac{500}{1\times10}} = \frac{\sqrt{50}}{2\pi}\ \text{revolution per second.}$$

- **Dynamics of Non-uniform Circular Motion (NUCM)**

(1) Along radial

$$\sum F_C = ma_c = \frac{mv^2}{R} = mw^2R$$

(2) Along tangential

$$\sum F_t = ma_t$$

$$F_{net} = m\,a_{net} = m\sqrt{(a_t)^2 + (a_c)^2} = \sqrt{(\sum F_C)^2 + (\sum F_t)^2}$$

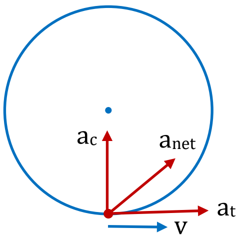


Illustration 21:

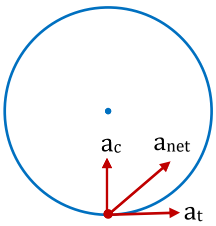
A particle of mass 'm' starts from rest at $t = 0$ in a circle of R with constant magnitude. Find F_{net} as a function of time 't'?

Solution:

$F_{net} = m\,a_{net}$

$\quad = m\sqrt{(a_t)^2 + (a_c)^2}$

$a_c = \dfrac{v^2}{R} = \dfrac{a^2t^2}{R}\qquad [\therefore v = at]$



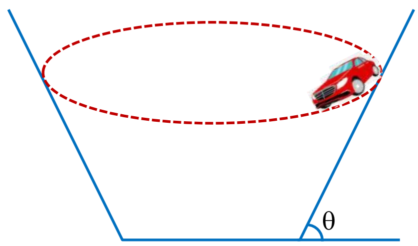
Dynamics of circular motion

Circular Turning of Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which provides the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by the following three ways :

- By friction only.
- By banking of roads only.
- By friction and banking of roads both.

In real life the necessary centripetal force is provided by friction and banking of roads both.



By Friction only

Suppose a car of mass m is moving with a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force will be provided to the car by the force of friction f acting towards centre of the circular path.

Thus, $f = \frac{mv^2}{r}$; $f_{max} = \mu N = \mu mg$

Therefore, for a safe turn without skidding $\frac{mv^2}{r} \le f_{max} \Rightarrow \frac{mv^2}{r} \le \mu mg \Rightarrow v \le \sqrt{\mu rg}$.

By Banking of Roads only

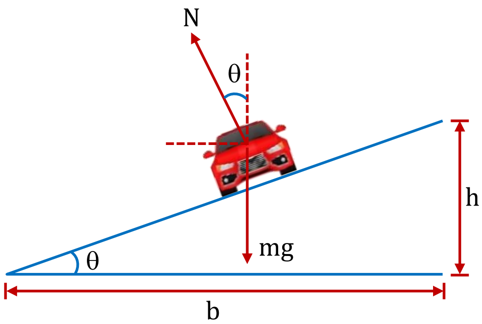
Friction is not always reliable at turns particularly when high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn in the sense that the outer part of the road is some what lifted compared to the inner part.

$N \sin\theta = \frac{mv^2}{r}$ and $N\cos\theta = mg$

$\Rightarrow \quad \tan\theta = \frac{v^2}{rg}$

$\Rightarrow \quad v = \sqrt{rg\tan\theta}$

$\because \quad \tan\theta = \frac{h}{b}$



Note:

$\tan\theta = \frac{v^2}{rg} = \frac{h}{b}$

Friction and Banking of Road both

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, i.e., weight (mg) is fixed both in magnitude and direction. The direction of second force, i.e., normal reaction N is also fixed (perpendicular to road) while the direction of the third force, i.e., friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_{max} = \mu N$).

So, direction and the magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is towards the centre.

(a) If speed of the vehicle is small then friction acts outwards.

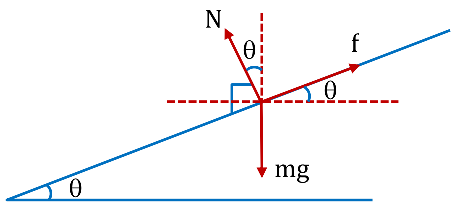
In this case, $N\cos\theta + f\sin\theta = mg$ ...(i)

and $N\sin\theta - f\cos\theta = \frac{mv^2}{R}$ ...(ii)

For minimum speed $f = \mu N$ so by dividing equation (i) by equation (ii)

$$\frac{N\cos\theta + \mu N\sin\theta}{N\sin\theta - \mu N\cos\theta} = \frac{mg}{mv_{min}^2/R}$$

Therefore, $v_{min} = \sqrt{Rg\left(\frac{\tan\theta - \mu}{1 + \mu\tan\theta}\right)}$



If we assume μ = tan ϕ, then

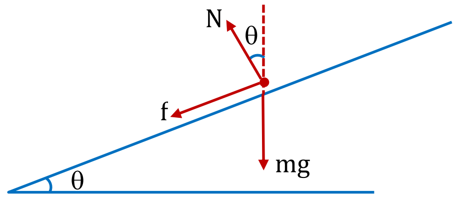
$$v_{min} = \sqrt{Rg\left(\frac{\tan\theta - \tan\phi}{1+\tan\phi\tan\theta}\right)} = \sqrt{Rg\tan(\theta - \phi)}$$

(b) If speed of vehicle is high then friction force act inwards.

in this case for maximum speed

$N\cos\theta - \mu N\sin\theta = mg$

and $N\sin\theta + \mu N\cos\theta = \frac{mv_{max}^2}{R}$



which gives $v_{max} = \sqrt{Rg\left(\frac{\tan\theta+\mu}{1-\mu\tan\theta}\right)} = \sqrt{Rg\tan(\theta+\phi)}$

Hence for successful turning on a rough banked road, velocity of vehicle must satisfy following relation

$$\sqrt{Rg\tan(\theta-\phi)} \le v \le \sqrt{Rg\tan(\theta+\phi)}$$

where θ = banking angle and $\phi = \tan^{-1}(\mu)$.

Illustration 22:

What should be the angle of banking of a circular track of radius 600 m which is designed for cars at an average speed of 180 km/hr ?

Solution:

Let the angle of banking be θ. The forces on the car are (figure)

(a) weight of the car Mg downward and

(b) normal force N

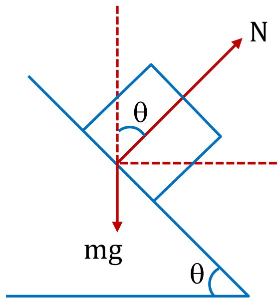
For proper banking, static frictional force is not needed.

For vertical direction the acceleration is zero.

So, $N\cos\theta = Mg$ ...(i)

For horizontal direction, the acceleration is v^2/r towards the centre, so that

$N\sin\theta = Mv^2/r$ ...(ii)



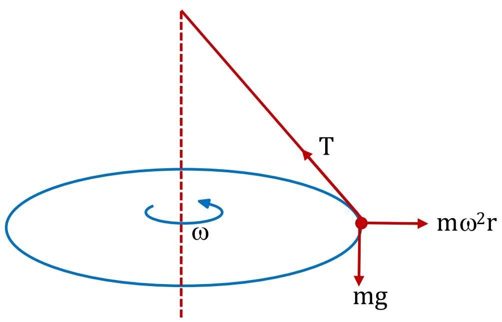
From (i) and (ii),

$\tan\theta = v^2/rg$

Putting the values, $\tan\theta = \frac{(180(km/h))^2}{(600m)(10m/s^2)} = 0.4167 \Rightarrow \theta = 22.6°$.

Centrifugal force

When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer A who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release. To another observer B, who is sharing the motion along the circular path (i.e., the observer B is also rotating with the body which is released, it appears to B), as if it has been thrown off along the radius away from the centre by some force. This inertial force is called centrifugal force.



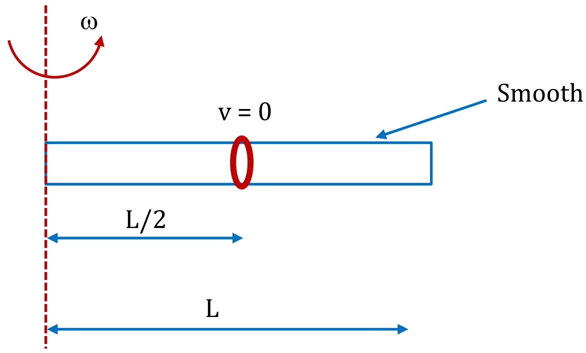
Its magnitude is equal to that of the centripetal force $\frac{mv^2}{r}$ = $m\omega^2 r$. Direction of centrifugal force, it is always directed radially outward.

Centrifugal force is a fictitious force which has to be applied as a concept only in a rotating frame of reference to apply Newton's law of motion in that frame. *FBD* of ball w.r.t. non inertial frame rotating with the ball.

Suppose we are working from a frame of reference that is rotating at a constant, angular velocity ω with respect to an inertial frame. If we analyse the dynamics of a particle of mass *m* kept at a distance *r* from the axis of rotation, we have to assume that a force $mr\omega^2$ react radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

Illustration 23:

A ring which can slide along the rod are kept at mid point of a smooth rod of length *L*. The rod is rotated with constant angular velocity ω about vertical axis passing through its one end. Ring is released from mid point. Find the velocity of the ring when it just leave the rod.



Solution:

Centrifugal force $m\omega^2 x = ma$

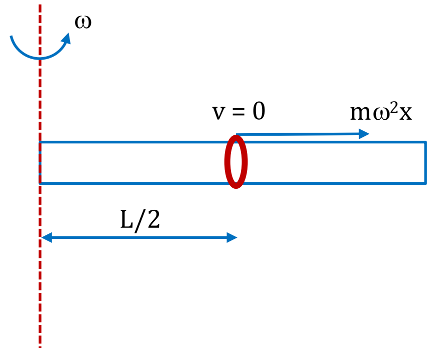
$$\omega^2 x = \frac{vdv}{dx}$$

$$\int_{L/2}^{L} \omega^2 x dx = \int_0^v vdv \quad \text{(integrate both side)}$$

$$\omega^2 \left(\frac{x^2}{2}\right)_{L/2}^{L} = \left(\frac{v^2}{2}\right)_0^v$$

$$\omega^2 \left(\frac{L^2}{2} - \frac{L^2}{8}\right) = \frac{v^2}{2}$$

$$v = \frac{\sqrt{3}}{2}\omega L$$



Velocity at time of leaving the rod

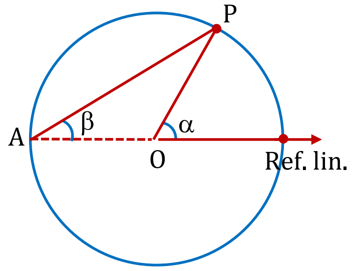
$$v' = \sqrt{(\omega L)^2 + \left(\frac{\sqrt{3}}{2}\omega L\right)^2} = \frac{\sqrt{7}}{2}\omega L$$

Relative Angular Velocity

Just as velocities are always relative, similarly angular velocity is also always relative. There is no such thing as absolute angular velocity. Angular velocity is defined with respect to origin, the point from which the position vector of the moving particle is drawn.

Consider a particle *P* moving along a circular path shown in the figure.

Here angular velocity of the particle *P* w.r.t. *'O'* and *'A'* will be different



Angular velocity of a particle *P* w.r.t. *O*, $\omega_{PO} = \frac{d\alpha}{dt}$

Angular velocity of a particle *P* w.r.t. *A*, $\omega_{PA} = \frac{d\beta}{dt}$

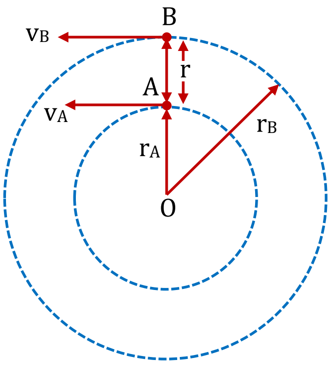
Definition:

Angular velocity of a particle '*A*' with respect to the other moving particle '*B*' is the rate at which position vector of '*A*' with respect to '*B*' rotates at that instant. (or it is simply, angular velocity of *A* with origin fixed at *B*). Angular velocity of *A* w.r.t. *B*, ω_{AB} is mathematically define as

$$\omega_{AB} = \frac{\text{Component of relative velocity of } A \text{ w.r.t. } B \text{, perpendicular to line}}{\text{Seperation between } A \text{ and } B} = \frac{(V_{AB})_\perp}{r_{AB}}$$

Important points:

- If two particles are moving on two different concentric circles with different velocities then angular velocity of *B* as observed by *A* will depend on their positions and velocities. Consider the case when *A* and *B* are closest to each other moving in same direction as shown in figure. In this situation



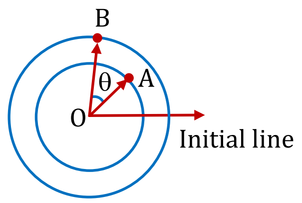
$$(V_{BA})_\perp = v_{rel} = |\vec{v}_B - \vec{v}_A| = v_B - v_A$$

Separation between *A* and *B* is

$$r_{BA} = r_B - r_A \text{ so, } \quad \omega_{BA} = \frac{(V_{BA})_\perp}{r_{BA}} = \frac{v_B - v_A}{r_B - r_A}$$

- If two particles are moving on the same circle or different coplanar concentric circles in same direction with different uniform angular speed ω_A and ω_B respectively, the rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is

$$\frac{d\theta}{dt} = \omega_B - \omega_A$$



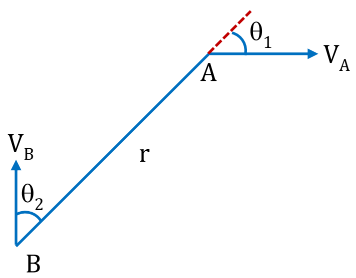
So, the time taken by one to complete one revolution around *O* w.r.t. the other

$$T = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1 T_2}{T_1 - T_2}$$

- $\omega_B - \omega_A$ is rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. This is not angular velocity of *B* w.r.t. *A*. (Which is rate at which line *AB* rotates)

Illustration 24:

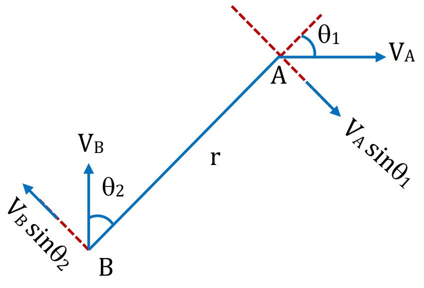
Find the angular velocity of *A* with respect to *B* in the figure given below:



Solution:

Angular velocity of A with respect to B;

$$\omega_{AB} = \frac{(V_{AB})_\perp}{r_{AB}}$$



$\Rightarrow \quad (V_{AB})_\perp = V_A \sin\theta_1 + V_B \sin\theta_2$

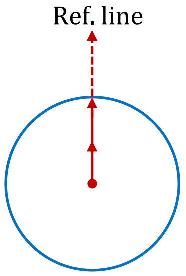
$\Rightarrow \quad r_{AB} = r$

$$\omega_{AB} = \frac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$$

Illustration 25:

Find the time period of meeting of minute hand and second hand of a clock.

Solution:



$\omega_{min} = \frac{2\pi}{60} rad/min., \quad \omega_{sec} = \frac{2\pi}{1} rad/sec.$

$\theta_{sec} - \theta_{min} = 2\pi$ (for second and minute hand to meet again)

$(\omega_{sec} - \omega_{min})\, t = 2\pi$

$2\pi(1 - 1/60)\, t = 2\pi$

$\Rightarrow \quad t = \frac{60}{59}$ min.

Illustration 26:

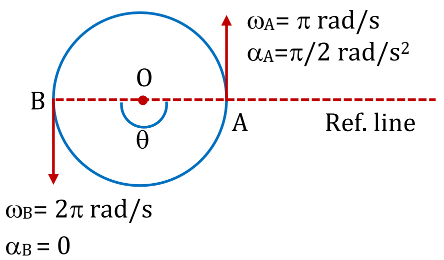
Two particle A and B move on a circle. Initially Particle A and B are diagonally opposite to each other. Particle A move with angular velocity π rad/sec., angular acceleration $\pi/2$ rad/sec^2 and particle B moves with constant angular velocity 2π rad/sec in same sense. Find the time after which both the particle A and B will collide.

Solution:

Suppose angle between OA and $OB = \theta$ then, rate of change of θ,

$\dot{\theta} = \omega_B - \omega_A = 2\pi - \pi = \pi$ rad/sec

$\ddot{\theta} = \alpha_B - \alpha_A = -\frac{\pi}{2}$ rad/sec^2



If angular displacement is $\Delta\theta$,

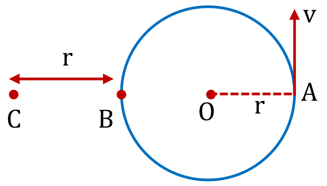
$$\Delta\theta = \dot{\theta}t + \frac{1}{2}\ddot{\theta}t^2$$

For A and B to collide angular displacement $\Delta\theta = \pi$

$\Rightarrow \quad \pi = \pi t + \frac{1}{2}\left(\frac{-\pi}{2}\right)t^2 \quad \Rightarrow \quad t^2 - 4t + 4 = 0 \quad \Rightarrow \quad t = 2$ sec.

Illustration 27:

A particle is moving with constant speed in a circle as shown, find the angular velocity of the particle A with respect to fixed point B and C if angular velocity with respect to O is ω.



Solution:

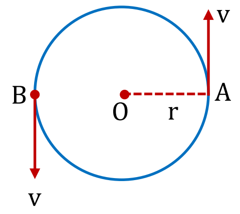
Angular velocity of A with respect to O is ; $\omega_{AO} = \dfrac{(v_{AO})_\perp}{r_{AO}} = \dfrac{v}{r} = \omega$

$\therefore \quad \omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}} = \dfrac{v}{2r} = \dfrac{\omega}{2}$

and $\quad \omega_{AC} = \dfrac{(v_{AC})_\perp}{r_{AC}} = \dfrac{v}{3r} = \dfrac{\omega}{3}$

Illustration 28:

Particles A and B move with constant and equal speeds in a circle as shown, find the angular velocity of the particle A with respect to B, if angular velocity of particle A w.r.t. O is ω.



Solution:

Angular velocity of A with respect to O is ; $\omega_{AO} = \dfrac{(v_{AO})_\perp}{r_{AO}} = \dfrac{v}{r} = \omega$

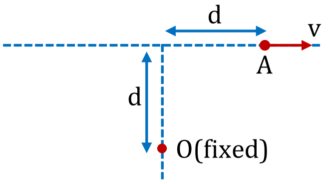
Now, $\omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}} \Rightarrow v_{AB} = 2v$, since v_{AB} is perpendicular to r_{AB},

$\therefore \quad (v_{AB})_\perp = v_{AB} = 2v$; $r_{AB} = 2r$

$\Rightarrow \quad \omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}} = \dfrac{2v}{2r} = \omega$

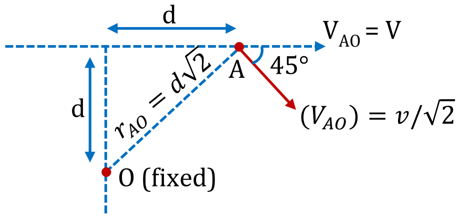
Illustration 29:

Find angular velocity of A with respect to O at the instant shown in the figure.



Solution:

Angular velocity of A with respect to O is ;



$\omega_{AO} = \dfrac{(v_{AO})_\perp}{r_{AO}}$

$v_{AO} = v$, $(v_{AO})_\perp = \dfrac{v}{\sqrt{2}}$

$r_{AO} = d\sqrt{2}$

$\omega_{AO} = \dfrac{(v_{AO})_\perp}{r_{AO}} = \dfrac{v/\sqrt{2}}{d\sqrt{2}} = \dfrac{v}{2d}$

WORK, ENERGY AND POWER

Work:

Whenever a force is applied to an object, it is ready to do work but work will be done only if the object has displacement in the direction of force. In work always at least two bodies are involved. One who is doing work (whose energy is decreasing). The other on which work is being done (whose energy is increasing).

Work done by force $\vec{F}$ on an object is defined as

$$\boxed{W_F = \int \vec{F} \cdot d\vec{s}}$$

where $\vec{F}$ is force on object and $d\vec{s}$ is displacement of point of application of force ($\vec{F}$).

Units:

SI Unit : Joule (J)

Joule : One joule of work is said to be done when a force of one newton displaces a body by one meter in the direction of force.

1 joule = 1 newton × 1 meter = 1 kgm^2s^{-2} [$1N = 1\ kg\ m/s^2$]

erg : One erg of work is said to be done when a force of one dyne displaces a body by one cm. in the direction of force. C.G.S. Unit

1 erg = 1 dyne × 1 *cm.* = 1 *gm. cm*^{2} *s*$^{-2}$

Other Units:

(a) 1 joule = 10^7 *erg*

(b) 1 *erg* = 10^{-7} joule

(c) 1 $eV = 1.6 \times 10^{-19}$ joule

(d) 1 joule = 6.25×10^{18} eV

(e) 1 $MeV = 1.6 \times 10^{-13}$ J

(f) 1 $J = 6.25 \times 10^{12}$ MeV

(g) 1 kilo watt hour (kWh) = 3.6×10^6 joule

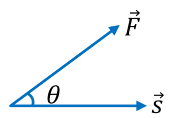
- For a particular displacement, work is independent of time, work will be same for same displacement whether the time taken is small or large.
- When several forces act, work done by a force, for a particular displacement, is independent of other forces.

Dimensions:

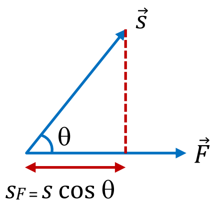
[Work] = [Force] [Displacement]= $[MLT^{-2}][L] = [ML^2T^{-2}]$

Work done by a constant force :

If the direction and magnitude of a force applied on a body is constant then the force is said to be constant.



[θ angle between $\vec{F}$ and $\vec{s}$]



$W_F = \int \vec{F} \cdot d\vec{s} = \vec{F} \cdot \int d\vec{s} = \vec{F} \cdot \vec{s} = F(s\cos\theta)$

$\boxed{W_F = F s_F}$

where s_F is component of $\vec{s}$ along $\vec{F}$ i.e. displacement along $\vec{F}$.

Work done by multiple forces:

If several forces act on a particle, then we can replace $\vec{F}$ by the net force $\Sigma\vec{F}$ in equation where

$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$

$\therefore \quad W = \left[\sum \vec{F}\right] \cdot \vec{s}$ ...(i)

This gives the work done by the net force during a displacement $\vec{S}$ of the particle. We can rewrite equation (i) as:

$W = \vec{F}_1.\vec{S} + \vec{F}_2.\vec{S} + \vec{F}_3.\vec{S} + \cdots$

or $\quad W = W_1 + W_2 + W_3 + \cdots$

So, the work done on the particle is the sum of the individual work done by all the forces acting on the particle.

Nature of work done :

Work done is a scalar quantity, its value may be positive, negative or even zero.

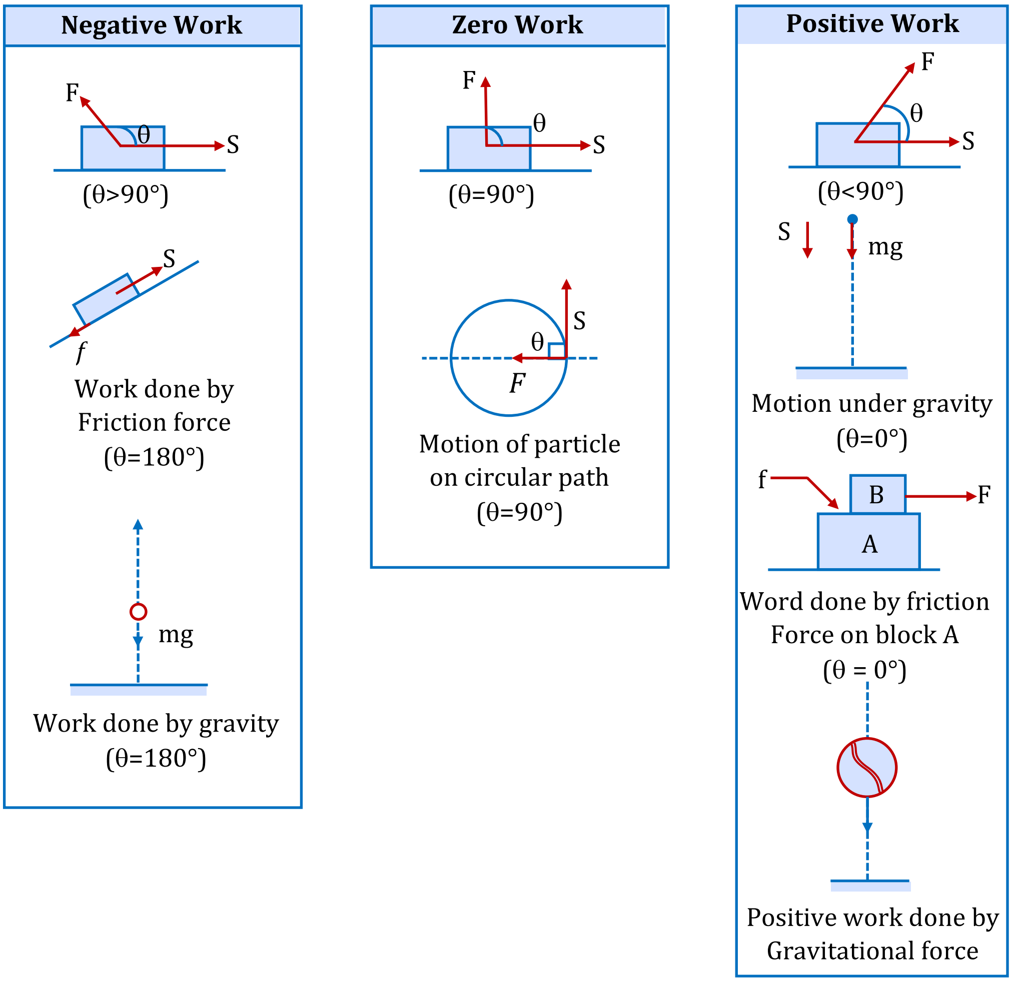
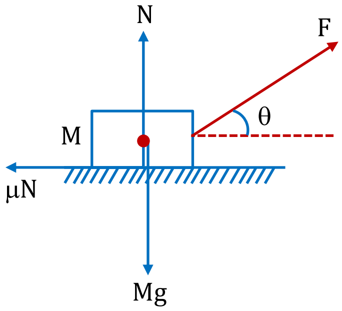


Illustration 30:

A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with horizontal. Coefficient of friction between block and surface is μ. If the block travels with uniform velocity, find the work done by this applied force during a displacement d of the block.

Solution:

The forces acting on the block are shown in figure. As the block moves with uniform velocity the resultant force on it is zero.



∴ $F\cos\theta = \mu N$ ...(i)

$F\sin\theta + N = Mg$ ...(ii)

Eliminating N from equations (i) and (ii),

$F\cos\theta = \mu\,(Mg - F\sin\theta)$

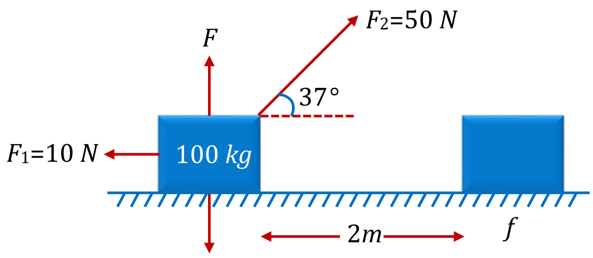
$$F = \frac{\mu Mg}{\cos\theta + \mu\sin\theta}$$

Work done by this force during a displacement d

$$W = F.d\cos\theta = \frac{\mu Mgd\cos\theta}{\cos\theta + \mu\sin\theta}$$

Illustration 31:

Find the work done by each force acting on the block.



Solution:

$$W_{F_2} = 50 \times 2 \times \cos 37° = 80J$$

$$W_{F_1} = 10 \times 2 \times \cos 180° = -20J$$

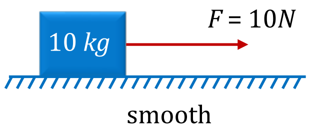
$W_g = 0$, because θ = 90°

$W_N = 0$, because θ = 90°

Illustration 32:

At $t = 0$ block was at rest? Find W_F in 2 sec.?

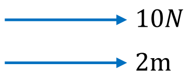
Solution:



$a = 1\ m/s^2$, $t = 2$, $u = 0$

$s = \frac{1}{2} \times 1 \times 4$

$s = 2\ m$



$\therefore$ $W_F = 10 \times 2 \times \cos 0°$

$W_F = 20$ Joule

Illustration 33:

A force of 21 N is acting along $\left(2\hat{\imath} + 2\hat{\jmath} + \hat{k}\right)$. Object gets displaced from (1, 2, 5) to (2, 4, 2). Find work done by 21 N ?

Solution:

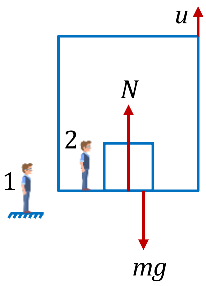
Displacement $(\vec{s}) = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

$\therefore$ $W_F = \vec{F} \cdot \vec{s} = 21 \times \frac{(2\hat{i}+2\hat{j}+\hat{k})}{\sqrt{9}} \cdot \left(\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) = \left(14\hat{\imath} + 14\hat{\jmath} + 7\hat{k}\right) \cdot \left(\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$

$= 14 + 28 - 21 = 21$ Joule

Illustration 34:

A lift is going up with constant velocity. We will calculate work done by normal force (N) from two reference frames, one in the ground frame and second one is on the lift frame as shown in the figure.



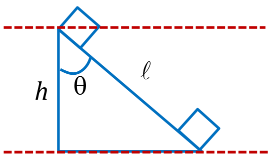
$N - mg = 0 \Rightarrow N = mg$

Reference frame 1 (ground frame)	Reference frame 2 (lift frame)
Work Done in 1st ref. frame = $N \cdot ut$	Work Done in 2nd ref. frame = 0

(*We can observe that work is dependent on reference frame.*)

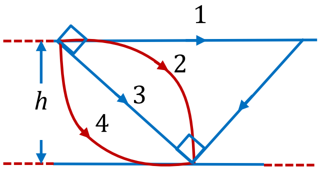
Illustration 35:

Find work done by the gravity (W_g)?



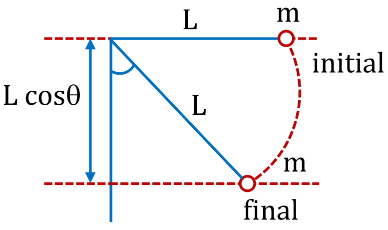
Solution:

$W_g = mg(\ell \cos\theta) = mgh$



$W_{g_1} = W_{g_2} = W_{g_3} = W_{g_4} = mgh$

Illustration 36:



$W_g = ?$

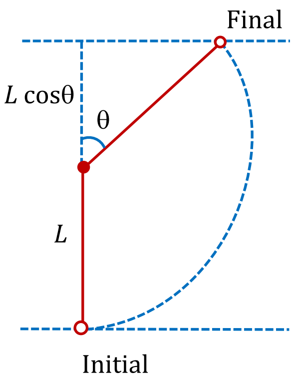
Solution:

$mg \downarrow \;\; L\cos\theta \downarrow$

$\therefore \quad W_g = mgL\cos\theta$

Illustration 37:

$W_g = ?$



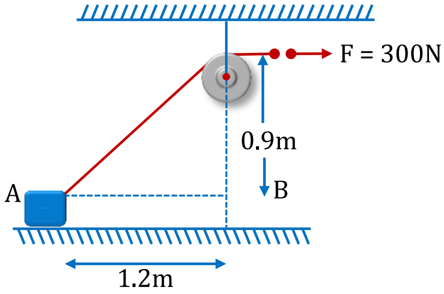
Solution:

$mg \downarrow \;\; L(1 + \cos\theta) \uparrow$

$\therefore \quad W_g = -mgL(1 + \cos\theta)$

Illustration 38:

Find the work done by force when body moves from A to B.



Solution:

I[st] method:

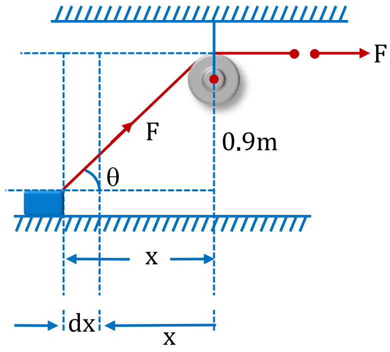
$$W = -\int_{x_i}^{x_f} F\cos\theta\, dx$$

$$W = -\int_{1.2}^{0} f\frac{x}{\sqrt{x^2+(0.9)^2}}dx$$

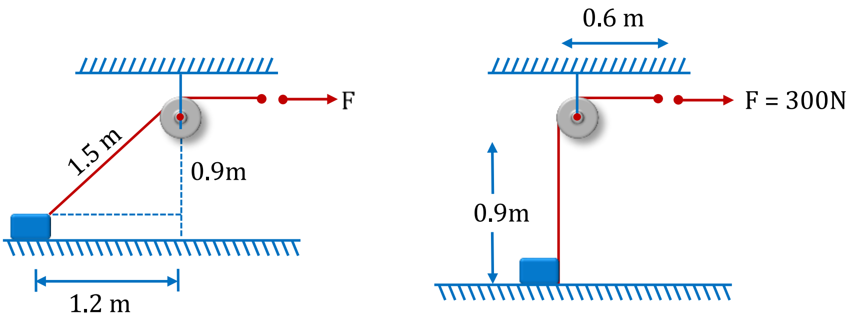
$$W = -\int_{0}^{1.2} x\frac{dx}{\sqrt{x^2+(0.9)^2}}$$

$W = F \times 0.6$

$W = 300 \times 0.6 = 180\ J$



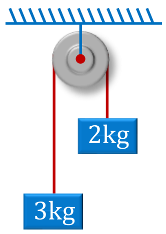
II[nd] method :



$W = FS = 300 \times 0.6 = 180\ J$

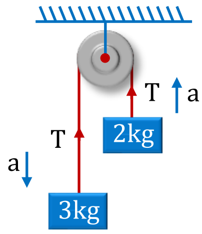
Illustration 39:

System released from rest at $t = 0$. Find in 2 sec.



(A) W_g on 3 kg (B) W_T on 3kg (C) W_g on 2kg (D) W_T on 2kg

Solution:



For (2 kg) : $T - 20 = 2a$ ...(i)

For (3 kg) : $30 - T = 3a$ ...(ii)

(i) + (ii) $10 = 5a$

$a = 2m/s^2$, $t = 2$ sec, $u = 0$

$\therefore$ $s = \frac{1}{2} \times 2 \times 4 = 4m$ and $T = 24\ N$

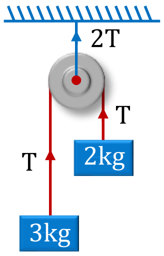
(A) $W_g = 30 \times 4 \times \cos 0° = 120$ joule

(B) W_T on 3 kg = $24 \times 4 \times \cos 180° = -96$ joule

(C) W_g on 2kg = $20 \times 4 \times \cos 180° = -80$ joule

(D) W_T on 2 kg = $24 \times 4 \times \cos 0° = 96$ joule

[W_T on system = 0 $\because$ $F = 0$]



Work done by a variable force :

If the force applying on a body is changing its direction or magnitude or both, the force is said to be variable. Suppose a variable force causes displacement in a body from position P_1 to position P_2. To calculate the work done by the force the path from P_1 to P_2 can be divided into infinitesimal element, each element is so small that during displacement of body through it, the force is supposed to be constant. If $d\vec{r}$ be small displacement of point of application is $d\vec{r}$ and $\vec{F}$ be the force applied on the body, the work done by force is $dW = \vec{F} \cdot d\vec{r}$.

The total work done in displacing body from P_1 to P_2 is given by, $\int dW = \int_{P_1}^{P_2} \vec{F}.d\vec{r} \Rightarrow W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$

If $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of the points P_1 and P_2 respectively, the total work done $W = \int_{r_1}^{r_2} \vec{F}.d\vec{r}$.

Illustration 40:

A particle of mass 2kg moves on x–axis under 3 forces such that its x co–ordinate is given by $x = (3t^2 + 2)m$. Find W_{net} from $t = 0$ to $t = 1$ sec?

Solution:

$x = 3t^2 + 2$ $\quad$ $F_{net} = ma$

$v = 6t$ $\quad$ $F_{net} = 2 \times 6$

$a = 6$ $\quad$ $= 12\ N$

$\therefore$ $s = x_f - x_i \Rightarrow 5 - 2 = 3m$

$W_{net} = 12 \times 3 \times \cos 0° = 36$ joule

Work done by variable force:

$$\boxed{W = \int \vec{F}.d\vec{r}}$$

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\boxed{W = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{Z_i}^{Z_f} F_z dz}$$

If particle goes from (x_i, y_i, z_i) to (x_f, y_f, z_f).

Illustration 41:

$\vec{F} = 2\hat{i} + 2y\hat{j} + 3z^2\hat{k}$ and particle goes from (1, 2, 1) to (2, 3, 0), find W_{net}?

Solution:

$$W \int_1^2 2dx + \int_2^3 2ydy + \int_1^0 3z^2 dz = [2x]_1^2 + \left[\frac{2y^2}{2}\right]_2^3 + \left[\frac{3z^3}{3}\right]_1^0$$

$= [2 \times 2 - 2 \times 1] + [3^2 - 2^2] + [0^3 - 1^3]$

$= 2 + 5 - 1 = 6J$

Illustration 42:

Particle moves on x-axis from $x = 0$ to 2 under $F_x = (4x - 2)N.W_F$?

Solution:

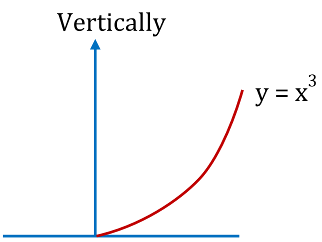
$$\int F_x\, dx = \int_0^2 (4x - 2)dx = \int_0^2 4x\, dx - \int_0^2 2dx$$

$= [2x^2]_0^2 - [2x]_0^2 = [2 \times 4 - 2 \times 0] - [(2 \times 2) - (2 \times 0)]$

$= [8 - 0] - [4 - 0] = 4J$

Illustration 43:

Particle of $2kg$ moves along curve such that x changes from 0 to 2 m. Force $(2x\hat{i} + 30\hat{j})$ acts along with mg. Find W_{net}?



Solution:

$x = 0, y = 0$

$x = 2, y = 8$

$$F_{net} = 2x\hat{i} + 30\hat{j} - \underbrace{20\hat{j}}_{mg} = 2x\hat{i} + 10\hat{j} = \int_0^2 2x\, dx + \int_0^8 10\, dy$$

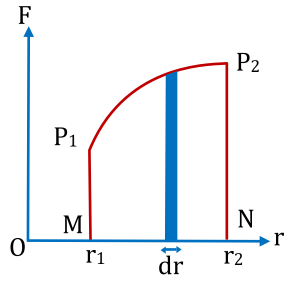
$= [x^2]_0^2 + [10y]_0^8$

$= [4 - 0] + [80 - 0]$

$= 4 + 80 = 84$ joule

Work as area under curve:

Suppose a body, whose initial position is r_1, is acted upon by a variable force $\vec{F}$ and consequently the body acquires its final position r_2. From position r to $r + dr$ or for small displacement dr, the work done will be $\vec{F}.d\vec{r}$ whose value will be the area of the shaded strip of width dr. The work done on the body in displacing it from position r_1 to r_2 will be equal to the sum of areas of all such strips.



Thus, total work done, $W = \sum_{r_1}^{r_2} dW = \sum_{r_1}^{r_2} F.dr$ = Area of P_1P_2NM.

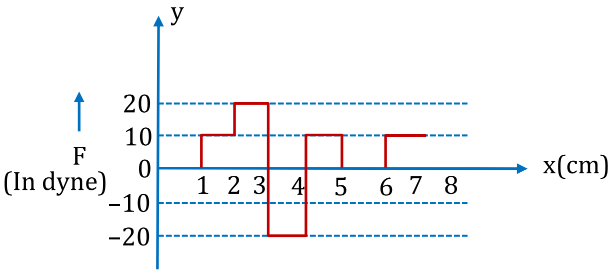
The area of the graph between curve and displacement axis is equal to the work done.

Note:

To calculate the work done by graphical method, for the sake of simplicity, here we have assumed the direction of force and displacement as same, but if they are not in same direction, the graph must be plotted between $F\cos\theta$ and r.

Illustration 44:

For the force displacement diagram shown in adjoining diagram. Calculate the work done by the force in displacing the body from $x = 1\ cm$ to $x = 5\ cm$.

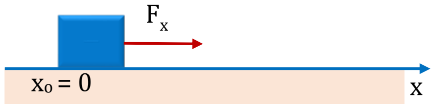


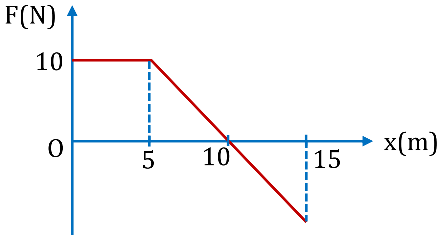
Solution:

Work = Area under the curve and displacement axis = 10 + 20 – 20 + 10 = 20 *erg*.

Illustration 45:

A horizontal force F is used to pull a box placed on floor. Variation in the force with position coordinate x measured along the floor is shown in the graph.



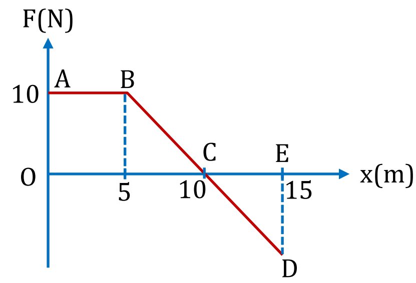


(a) Calculate work done by the force in moving the box from $x = 0\ m$ to $x = 10\ m$.

(b) Calculate work done by the force in moving the box from $x = 10\ m$ to $x = 15\ m$.

(c) Calculate work done by the force in moving the box from $x = 0\ m$ to $x = 15\ m$.

Solution:



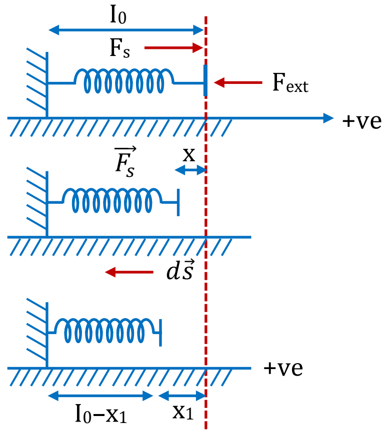
In rectilinear motion work done by a force equals to area under the force–position graph and the position axis

(a) $W_{0\to10}$ = Area of trapazium $OABC = 75J$

(b) $W_{10\to15}$ = −Area of triangle $CDE = -25\,J$

(c) $W_{0\to15}$ = Area of trapazium $OABC$ − Area of triangle $CDE = 50\,J$

Spring Force:



Natural length of spring is l_0.

Similarly, when we **compress** spring by x_1 from natural length, then work done by spring force.

$\vec{F} = kx\hat{\imath}$

$d\vec{S} = (dx)(-\hat{\imath})$ {dx is $+ve$ as x is increasing}

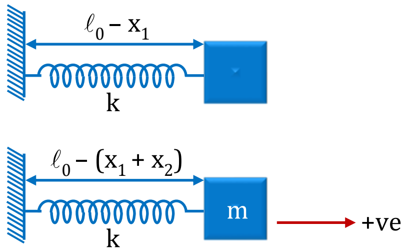
$dW = \vec{F}\cdot d\vec{S}$

$\int dW = -\int_0^{x_1} kx\,dx = -\frac{1}{2}kx_1^2$

Illustration 46:

Find work done by spring if we compress it further by x_2.

Solution:

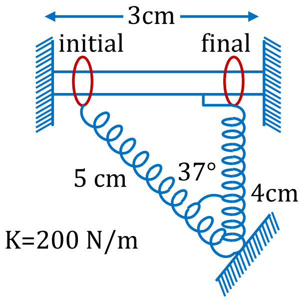


$\vec{F} = kxi,\ \ d\vec{S} = dx(-\hat{\imath})$

$dW = \vec{F}\cdot d\vec{S} = -kx\,dx$

$W = -k\int_{x_1}^{(x_2+x_1)} x\,dx = -\frac{1}{2}k[(x_2+x_1)^2 - x_1^2]$

Illustration 47:



W_{sp} from initial to final = ? If natural length = 3cm.

Solution:

In right Δ,

$$W_{sp} = \frac{1}{2} \times 200 \left[\left(\frac{5-3}{100}\right)^2 - \left(\frac{4-3}{100}\right)^2\right]$$

$$= 100 \left[\frac{4}{10000} - \frac{1}{10000}\right] = 100 \times \frac{3}{10000} = \frac{3}{100} J$$

Work done by tangential force:

$$W_F = \int \vec{F}.d\vec{s} = \int F ds \cos 0°$$

$$W_F = F\int ds = F.s$$

→ Length of curve

Now this 'ds' became magnitude of displacement i.e. distance.

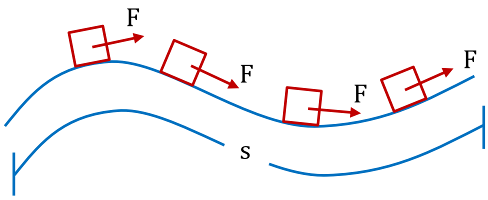
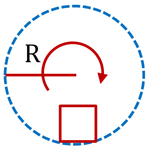


Illustration 48:

A block move in a horizontal circle of radius r. Find work done by friction.

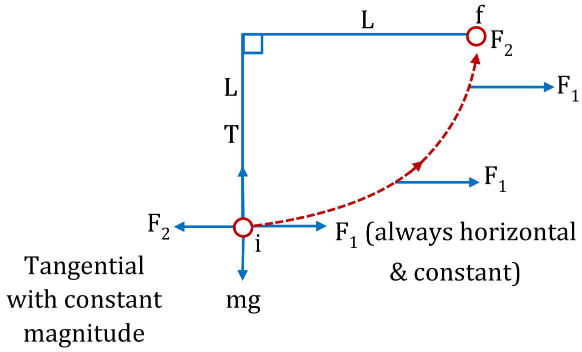


Solution:

$$W_{\text{friction}} = -\mu mg\ 2\pi r$$

Illustration 49:

Find work done by force F_2.



Solution:

$W_T = 0 \qquad (\because T \perp d\vec{s})$

$W_{mg} = -mgL$

$W_{F_1} = F_1 L$

$W_{F_2} = -F_2 \times (\text{Length of curve})$

$$= -F_2 \times \frac{\pi}{2} L$$

Energy:

- The energy of a body is defined as the capacity of doing work.
- There are various forms of energy:
  (i) mechanical energy (ii) chemical energy (iii) electrical energy (iv) sound energy
  (v) light energy (vi) magnetic energy (vii) nuclear energy etc.
- Energy of an isolated system always remains constant it can neither be created nor it can be destroyed however it may be converted from one form to another

  Examples:

  Electrical energy $\xrightarrow{\text{Motor}}$ Mechanical energy

  Mechanical energy $\xrightarrow{\text{Generator}}$ Electrical energy

  Light energy $\xrightarrow{\text{Photocell}}$ Electrical energy

  Electrical energy $\xrightarrow{\text{Heater}}$ Heat energy

  Electrical energy $\xrightarrow{\text{Radio/Speaker}}$ Sound energy

  Nuclear energy $\xrightarrow{\text{Nuclear}\rightleftarrows\rightleftarrows\text{Reactor}}$ Electrical energy

  Chemical energy $\xrightarrow{\text{Cell}}$ Electrical energy

  Electrical energy $\xrightarrow{\text{Secondary CellChargeable}}$ Chemical energy
- Energy is a scalar quantity.
- Unit : Its unit is same as that of work or torque. In MKS: joule, watt second; In CGS : erg.

Note:

$1\ eV = 1.6 \times 10^{-19}$ joule; $1\ kWh = 3.6 \times 10^{6}$ joule; $10^{7}\ erg = 1$ joule.

- Dimension $[M^1L^2T^{-2}]$.
- According to Einstein's mass energy equivalence principle mass and energy are inter convertible i.e. they can be changed into each other. Energy equivalent of mass m is, $E = mc^2$ where, m : mass of the particle, c : velocity of light , E : equivalent energy corresponding to mass m.
- In mechanics we are concerned with mechanical energy only which is of two type :
  (i) kinetic energy (ii) potential energy

Kinetic energy

- The energy possessed by a body by virtue of its motion is called kinetic energy.
- If a body of mass m is moving with velocity v, its kinetic energy $KE = 1/2\ mv^2$.
- Kinetic energy is always positive.
- If linear momentum of body is p, the kinetic energy for translatory motion is $KE = \dfrac{p^2}{2m} = \dfrac{1}{2}mv^2$.

  $K.E.$ of a body can be calculated by the amount of work done in stopping the moving body or by the amount of work done in imparting the present velocity to the body from the state of rest.

Work–Kinetic Energy Theorem :

Work done by all the forces (conservative or non–conservative) acting on a particle or an object is equal to the change in its kinetic energy. So, work done by all the forces = change in kinetic energy.

$$W = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Derivation:

For a particle $\Sigma\vec{F} = m\vec{a}$

$$\Sigma\vec{F}\cdot d\,\vec{s} = m\frac{d\vec{v}}{dt}\cdot d\,\vec{s} = m\vec{a}\cdot d\vec{s} = ma_t ds = m\frac{dv}{dt}ds = mvdv$$

Work Done by resultant Force = $\int\left(\Sigma\vec{F}\right)\cdot d\vec{s} = m\int_{v_i}^{v_f} v\,dv$

Summation of work by all the forces = $\Sigma(\int \vec{F}\cdot d\,\vec{s}) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$

$\Sigma W = k_f - k_i$

i.e. Sum of work done by all the forces on a particle is equal to change in kinetic energy of the particle.

So, the kinetic energy of a particle is equal to the total work that was done to accelerate it from rest to its present speed.

The kinetic energy of a particle is equal to the total work that particle can do in the process of being brought to rest. This is why you pull your hand and arm backward when you catch a ball. As the ball comes to rest, it does an amount of work (force times distance) on your hand equal to the ball's initial kinetic energy. By pulling your hand back, you maximize the distance over which the force acts and so minimize the force on your hand.

How to apply work kinetic energy theorem:

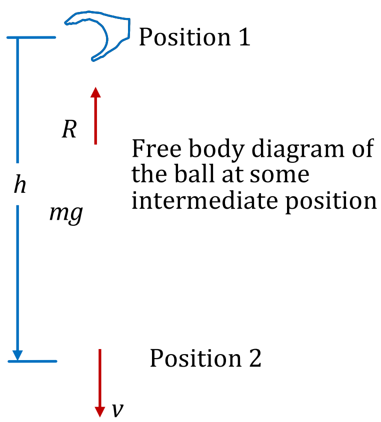
The work kinetic energy theorem is deduced here for a single body moving relative to an inertial frame, therefore it is recommended at present to use it for a single body in inertial frame. To use work kinetic energy theorem the following steps should be followed.

- Identify the initial and final positions as position 1 and 2 and write expressions for kinetic energies, whether known or unknown.
- Draw the free body diagram of the body at any intermediate stage between positions 1 and 2. The forces shown will help in deciding their work. Calculate work by each force and add them to obtain work done $W_{1\to 2}$ by all the forces.
- Use the work obtained in step 2 and kinetic energies in step 1 into $W_{1\to 2} = K_2 - K_1$.

Illustration 50:

A 5kg ball when falls through a height of 20 m acquires a speed of 10 m/s. Find the work done by air resistance.

Solution:



The ball starts falling from position 1, where its speed is zero; hence, kinetic energy is also zero.

$K_1 = 0\,J$ ...(i)

During downwards motion of the ball constant gravitational force mg acts downwards and air resistance R of unknown magnitude acts upwards as shown in the free body diagram. The ball reaches position 20 m below the position–1 with a speed v = 10 m/s, so the kinetic energy of the ball at position 2 is

$K_2 = \frac{1}{2}mv^2 = 250J$ ...(ii)

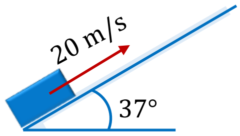
The work done by gravity

$$W_{g,1\to2} = mgh = 1000\,J \qquad ...(iii)$$

Denoting the work done by the air resistance $W_{R,1\to2}$ and making use of eq. 1, 2, and 3 in work kinetic energy theorem, we have $W_{1\to2} = K_2 - K_1 \Rightarrow W_{g,1\to2} + W_{R,1\to2} = K_2 - K_1 \Rightarrow W_{R,1\to2} = -750J$

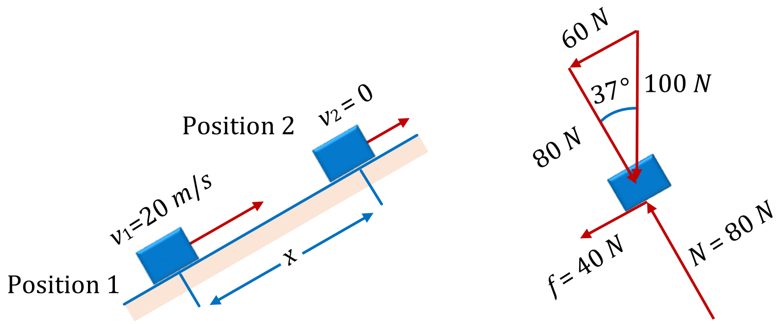
Illustration 51:

A box of mass m = 10 kg is projected up an inclined plane from its foot with a speed of 20 m/s as shown in the figure. The coefficient of friction μ between the box and the plane is 0.5. Find the distance travelled by the box on the plane before it stops first time.



Solution:

The box starts from position 1 with speed v_1 = 20 m/s and stops at position 2.



Kinetic energy at position 1: $K_1 = \frac{1}{2}mv_1^2 = 2000\,J$

Kinetic energy at position 2: $K_2 = 0\,J$

Work done by external forces as the box moves from position 1 to position 2:

$$W_{1\to2} = W_{g,1\to2} + W_{f,1\to2} = -60x - 40x = -100xJ$$

Applying work energy theorem for the motion of the box from position 1 to position 2, we have

$$W_{1\to2} = k_2 - k_1 \Rightarrow -100\,x = 0 - 2000 \Rightarrow x = 20\,m$$

Illustration 52:

A 60 gm tennis ball thrown vertically up at 24 m/s rises to a maximum height of 26 m. What was the work done by resistive forces?

Solution:

$$W_g + W_{res} = (0 - \frac{1}{2}mu^2)$$

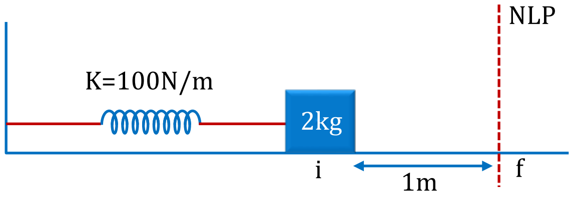
$$W_{res} = mgh - \frac{1}{2}mu^2$$

$$W_{res} = \left[\frac{60}{1000} \times 10 \times 26 - \frac{1}{2} \times \frac{60}{1000} \times (24)^2\right] = -1.68\,J$$

Illustration 53:

(a) Block from rest. Find its speed at *NLP* ?

(b) How far from *NLP* will it stop ?



Solution:

(a) $W_{sp} + W_{mg} + W_N = \frac{1}{2} \times m \times v_f^2 - \frac{1}{2} \times m \times v_1^2$

$\frac{1}{2} \times 100[1^2 - 0^2] + 0 + 0 = \frac{1}{2} \times 2\left[v_f^2 - 0\right]$

$50 \times 1 = v_f^2$

$v_f = \sqrt{50} = 5\sqrt{2}\ m/s$

(b) $W_N = 0,\ W_{mg} = 0$

$\frac{1}{2} \times 100[1^2 - x^2] = \frac{1}{2} \times 2[0]$

$50[1 - x^2] = 0$

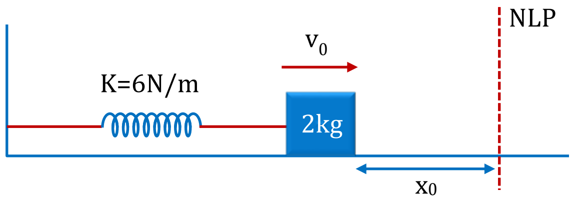
$x^2 = 1$

$x = 1m$

Illustration 54:

Given : $v_0 = 6m/s,\ x_0 = 4m$

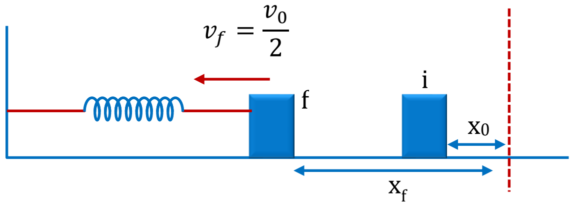
(a) Find compression when $v = \frac{v_0}{2}$ (b) Speed at compression of $\frac{x_0}{2}$



Solution:

(a) $W_{sp} + W_g + W_N = K_f - K_i$

$\frac{1}{2} \times K\left[x_0^2 - x_f^2\right] + 0 + 0 = \frac{1}{2} m \left(\frac{v_0}{2}\right)^2 - \frac{1}{2}(v_0)^2$



$\frac{1}{2} \times 6\left[(4)^2 - x_f^2\right] = \frac{1}{2} \times 2\left[\left(\frac{6}{2}\right)^2 - 6^2\right]$

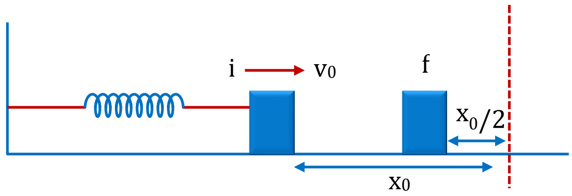
$3\left[16 - x_f^2\right] = 9 - 36$

$16 - x_f^2 = \frac{-27}{3}$

$x_f^2 = 25$

$x_f = 5\ m$

(b) $W_g + W_N + W_{sp} = K_f - K_i$



$$0 + 0 + \frac{1}{2}K\left[x_0^2 - \left(\frac{x_0}{2}\right)^2\right] = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$$

$$\frac{1}{2} \times 6\left[(4)^2 - (2)^2\right] = \frac{1}{2} \times 2\left[v_f^2 - 6^2\right]$$

$$36 = v_f^2 - 36$$

$$v_f = \sqrt{72}$$

$$v_f = 6\sqrt{2}$$

Illustration 55:

A uniform chain of mass M and length L is lying on a frictionless table in such a way that its 1/3 part is hanging vertically down. Calculate the work done in pulling the chain up the table.

Solution:

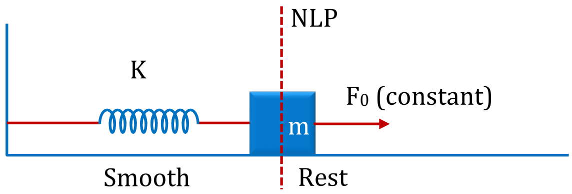
If length x of the chain is pulled up on the table, then the length of hanging part of the chain would be $\left(\frac{L}{3} - x\right)$and its weight would be $\frac{M}{L}\left(\frac{L}{3} - x\right)g$.

If it is pulled up further by a distance dx, the work done in pulling up.

$\Rightarrow \quad dW = \frac{M}{L}\left(\frac{L}{3} - x\right)gdx \qquad \Rightarrow \quad W = \int_0^{L/3}\frac{M}{L}\left(\frac{L}{3} - x\right)gdx = \frac{MgL}{18}$

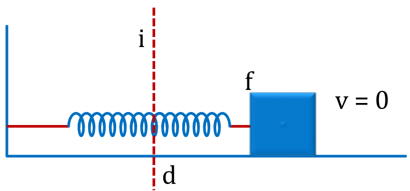
Illustration 56:

Find maximum elongation in spring ?



Solution:

$$W_g + W_N + W_{sp} + W_{F_b} = K_f - K_i$$

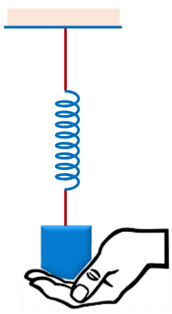


$$0 + 0 + \frac{1}{2}K[0^2 - d^2] + F_0 d = 0 - 0$$

$$\frac{1}{2}Kd^2 = F_0 d$$

$$d = \frac{2F_0}{K}$$

Illustration 57:



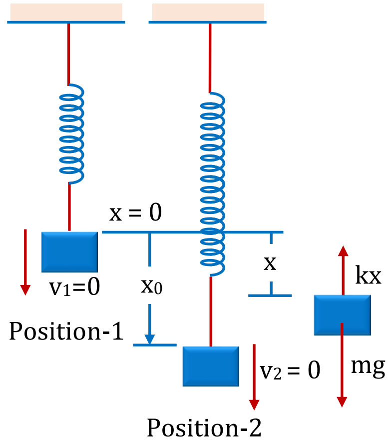
A block of mass m is suspended from a spring of force constant k. It is held to keep the spring in its relaxed length as shown in the figure.

(a) The applied force is decreased gradually so that the block moves downward at negligible speed. How far below the initial position will the block stop?

(b) The applied force is removed suddenly. How far below the initial position, will the block come to an instantaneous rest?

Solution:

(a) As the applied force (F) is decreased gradually, everywhere in its downward motion the block remains in the state of translational equilibrium and moves with negligible speed. Its weight (mg) is balanced by the upward spring force (kx) and the applied force. When the applied force becomes zero the spring force becomes equal to the weight and the block stops below a distance x_o from the initial position. The initial and final positions and free body diagram of the block at any intermediate position are shown in the adjoining figure. Applying the conditions of equilibrium, we have $x_o = \frac{mg}{k}$.



(b) In the previous situation the applied force was decreased gradually keeping the block everywhere in equilibrium. If the applied force is removed suddenly, the block will accelerate downwards. As the block moves, the increase in spring extension increases the upward force, due to which acceleration decreases until extension becomes x_0. At this extension, the block will acquire its maximum speed and it will move further downward. When extension becomes more than x_0 spring force becomes more than the weight (mg) and the block decelerates and ultimately stops at a distance x_m below the initial position. The initial position–1, the final position–2, and the free body diagram of the block at some intermediate position when spring extension is x are shown in the adjoining figure.

Kinetic energy in position–1 is

$K_1 = 0$

Kinetic energy in position–2 is

$K_2 = 0$

Work done $W_{1\to2}$ by gravity and the spring force is

$$W_{1\to2} = W_{g,1\to2} + W_{\text{spring},1\to2} = mgx_m - \frac{1}{2}kx_m^2$$

Using above values in the work energy theorem, we have

$$W_{1\to2} = K_2 - K_1 \to mgx_m - \frac{1}{2}kx_m^2 = 0$$

$x_m = 2mg/k$

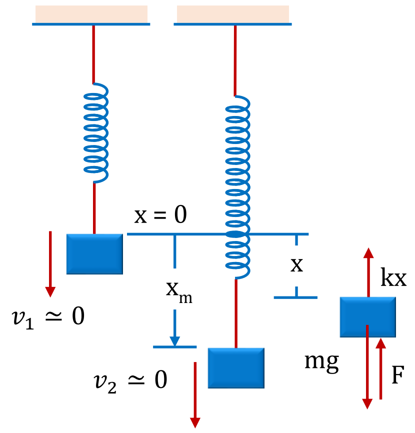
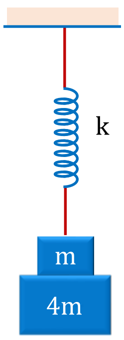


Illustration 58:

Find how much m will rise if $4m$ falls away.

Blocks are at rest and in equilibrium.



Solution:

Applying *WET* on block of mass m

$$w_g + w_{sp} = k_f - k_i$$

Let finally displacement of block from equilibrium is x

$$-mg\left(\frac{5mg}{k}+x\right)+\frac{1}{2}k\left(\frac{25m^2g^2}{k^2}\right)-\frac{1}{2}kx^2 = 0$$

$$\frac{1}{2}kx^2 + mgx - \frac{15m^2g^2}{2k} = 0$$

$$x = \frac{3mg}{k}$$

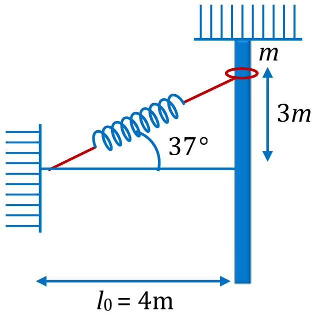
Displacement from initial is $\frac{5mg}{k}+\frac{3mg}{k}=\frac{8mg}{k}$

Illustration 59:

Find velocity of ring when spring becomes horizontal.

$m = 10\ kg$

$k = 400\ Nm^{-1}$



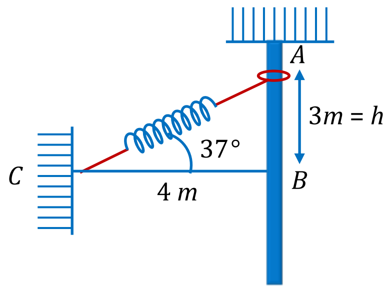
Solution:

$m = 10\ kg$

$k = 400\ N/m$

Natural length of spring = $4m$

Decrease in *PE* = Increase in *KE*



$$\frac{1}{2}k \times x^2 + mgh = \frac{1}{2}mv^2$$

$$\frac{1}{2}\times 400 \times 1^2 + 10 \times 10 \times 3 = \frac{1}{2}\times 10V^2$$

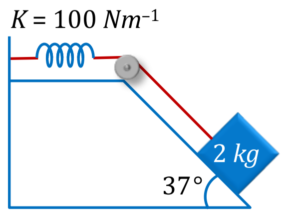
$$200 + 300 = 5V^2$$

$$5V^2 = 500$$

$$V = \sqrt{100}m/s = 10\ m/s$$

Illustration 60:

(a) A 2 kg block situated on a smooth fixed incline is connected to a spring of negligible mass, with spring constant $k = 100\ Nm^{-1}$, via a frictionless pulley. The block is released from rest when the spring is unstretched. How far does the block move down the incline before coming (momentarily) to rest? What is its acceleration at its lowest point?



(b) The experiment is repeated on a rough incline. If the block is observed to move 0.20 m down along the incline before it comes to instantaneous rest, calculate the coefficient of kinetic friction.

Ans. (a) $s = 0.24\ m$, $a = 6\ m/s^2$, (b) $x = 1/8$]

Solution:

(a) Applying work–energy theorem

$$mgs \sin 37° = \frac{1}{2} ks^2$$

$$2 \times 10 \times s \times \frac{3}{5} = \frac{1}{2} \times 100 \times s^2 \text{ on solving } s = 0.24\ m$$

Accelerating at its lowest point

$$a = \frac{ks - mg \sin 37°}{m} = \frac{100 \times 0.24 - 2 \times 10 \times \frac{3}{5}}{2} = 6\ m/s^2\ ;\ a = 6\ m/s^2$$

(b) Work done by gravity + work done by friction = Energy stored in spring

$$mg\ s \sin 37° - \mu mg \cos 37° \times s = \frac{1}{2} ks^2$$

$$mg \sin 37° - \frac{1}{2} ks = \mu mg \cos 37°$$

$$2 \times 10 \times \frac{3}{5} - \frac{1}{2} \times 100 \times s = \mu \times 2 \times 10 \times \frac{4}{5}$$

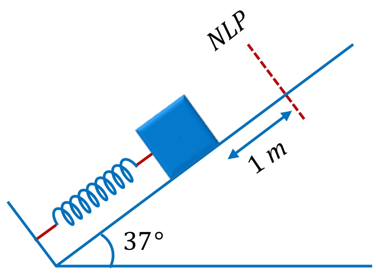
Given $s = 0.20\ m$

$$\frac{12 - 50s}{16} = \mu$$

$$\mu = \frac{1}{8}$$

Illustration 61:

Find maximum distance travelled by block along incline after releasing from rest at $1m$ compression at spring. Spring constant $k = 324\ N/m$ and coefficient of friction = 0.5. Spring is not attached to the block.



Solution:

$$W_n + W_g + W_f + W_{sp} = k_f - k_i$$

$$0 + (-mg \sin \theta \times x) + (-\mu mg \cos \theta \times x) + \left[\frac{1}{2} \times k(1^2 - 0^2)\right] = 0$$

$$\left(-9 \times 10 \times \frac{3}{5} \times x\right) - \left(0.5 \times 9 \times 10 \times \frac{4}{5} \times x\right) + \left(\frac{1}{2} \times 324\right) = 0$$

$$-54x - 36x + 162 = 0$$

$$90x = 162$$

$$x = \frac{162}{90}$$

$$x = 1.8\ m$$

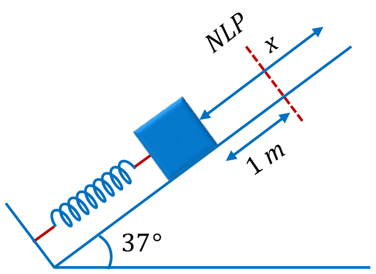
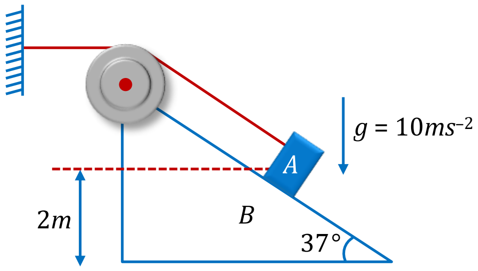


Illustration 62:

Find velocity of A and B when A is about to touch the ground. Also verify that work done by tension on the whole system and normal between A and B is zero.

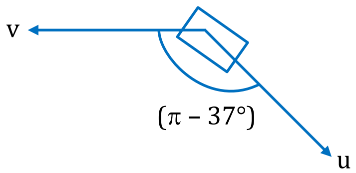


$m_A = 5\ kg \quad m_B = 10\ kg$

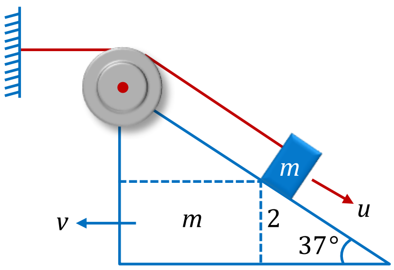
Solution:

$|\vec{V}| = |\vec{u}|$

Net speed of block



$$V_b = \sqrt{u^2 + u^2 - 2u^2 \cos 37°} = \sqrt{2u^2 - 2u^2\frac{4}{5}} = u\sqrt{\frac{2}{5}}$$

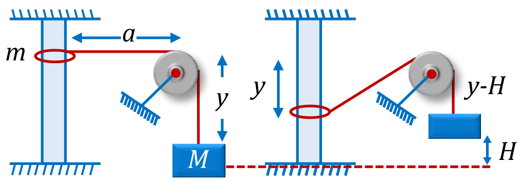


By energy conservation

Decrease in *P.E.* of block = Increase in *KE* of wedge + block.

Illustration 63:

In the figure shown, the ring starts moving down from rest. What will be the relation between the velocity of the ring and the velocity of the block at any position? What will be the distance that the ring moves before coming to rest?



Solution:

To find the relationship between the velocities of the block and the ring, we will use the concept of virtual work. We have already studied that the total work done by tension on a system is always zero. Assuming small displacements of the bodies when the angle made by the string is θ (displacement is assumed to be small so that the angle made by the string does not change appreciably)

$$-TS_R \cos\theta - TS_B = 0$$

$$S_R \cos\theta + S_B = 0$$

$$\Rightarrow \quad v_R \cos\theta + v_B = 0$$

Here we should be careful, the relationship between the small displacements is same as that of the velocities because we have divided the entire relationship by small time interval dt. But to obtain the relationship between accelerations, we have to differentiate this expression which will involve derivative of cosθ also because θ is also a variable.

Since tension does not do any work, only work here is done by the force of gravity.

$$W_g = \Delta K_1 + \Delta K_2$$

$$mgh - MgH = \frac{1}{2}mv_R^{\ 2} + \frac{1}{2}mv_B^{\ 2}$$

The interesting thing to note here is that even if mass of the ring is less than that of the block, the ring will go down. Explain.

At the position of rest, $v_B = 0$. So, from the equation of constrained motion, v_R is also 0.

$$mgh - MgH = 0.$$

Also, from the geometry, the original length of the string is

$$a + y = \ell.$$

$$\sqrt{a^2 + h^2} + y - H = \ell$$

$$\Rightarrow \quad \sqrt{a^2 + h^2} = a + H$$

$$\Rightarrow \quad h^2 = H^2 + 2aH$$

$$\Rightarrow \quad \left(\frac{MH}{m}\right)^2 = H^2 + 2aH$$

$$\Rightarrow \quad H = \frac{2aM^2}{M^2 - m^2}$$

$$h = \frac{2aMm}{M^2 - m^2}$$

It is clear from the equation that the physically admissible solution is available only when $M > m$. If $M < mh$ comes out to be negative which is not possible. This means that if $M<m$, the block and the ring will never come to rest.

Illustration 64:

The displacement x of a body of mass 1 kg on horizontal smooth surface as a function of time t is given by $x = \frac{t^3}{3}$. Find the work done by the external agent for the first one second.

Solution:

$\because \quad x = \dfrac{t^3}{3}$

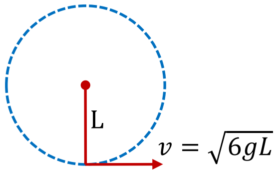
$\therefore \quad v = \dfrac{dx}{dt} = t^2$, Velocity at $t = 0$, $u = 0$ and at $t = 1s$ $v = 1\ m/s$

Using work energy theorem :

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}1(1)^2 = 0.5\ J$$

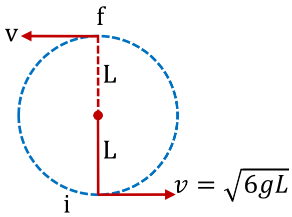
Illustration 65:

Find v and T at top most point?



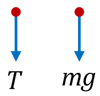
Solution:

$$-mg \times 2L + 0 = \frac{1}{2}mv^2 - \frac{1}{2}m6gL$$



$$v = \sqrt{2gL}$$

at top



$$T + mg = \frac{mv^2}{L}$$

$$T = mg$$

Tension maximum at bottom point and Tension minimum at top point.

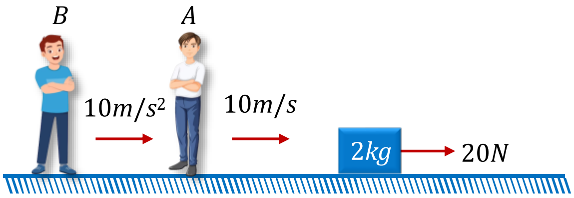
Frame dependency:

Work and kinetic energy depends on reference frame. Work energy theorem is valid in inertial frame but we can use its equation in non–inertial frame by considering work of Psuedo force.

Work done by Psuedo Force:

Illustration 66:

A block of mass 2 kg is dragged by a force of 20 N on a smooth horizontal surface. It is observed from three reference frames ground, observer A and observer B. Observer A is moving with constant velocity of 10 m/s and B is moving with constant acceleration of 10 m/s^2. The observer B and block starts simultaneously at $t = 0$.



	Column I		Column II
(A)	Work energy theorem is applicable in	(P)	100 *J*
(B)	Work done on block in 1*s* as observed by ground is	(Q)	– 100 *J*
(C)	Work done on block is 1*s* as observed by observer *A* is	(R)	Zero
(D)	Work done on block in 1*s* as observed by observer *B* is	(S)	Only ground and *A*
		(T)	All frames ground, *A* and *B*

Ans. (A→T); (B→P); (C→Q); (D→R)

Solution:

For (A) : Work energy theorem is applicable in all reference frames.

For (B) : w.r.t. ground : At $t=0$, $u=0$ and $t = 1\ s$, $v = at = \left(\frac{20}{2}\right)(1) = 10\ m/s$

Work done = change in kinetic energy = $\frac{1}{2}(2)(10)^2 - \frac{1}{2}(2)(0)^2 = 100\ J$

For (C) : w.r.t. observer *A* : Initial velocity = 0 – 10 = – 10 *m/s*. Final velocity = 10 – 10 = 0 *m/s*

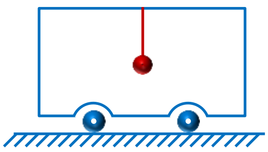
Work done = $\frac{1}{2}(2)(0)^2 - \frac{1}{2}(2)(-10)^2 = -100\ J$

For (D) : w.r.t. observer *B* : Initial velocity = 0 – 0 = 0 *m/s*

Final velocity = 10 – 10 = 0 *m/s* ; Work done = 0 *J*

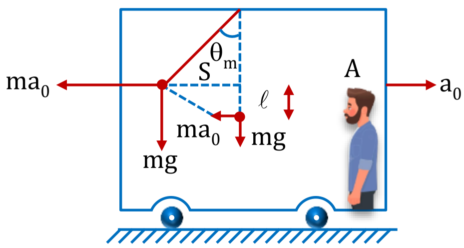
Illustration 67:

The trolley is accelerated horizontally with acceleration a_0. Find the maximum angular displacement of pendulum.



Solution:

With respect to observer *A* in trolley,



$$W_G + W_{Pseudo} = \Delta K$$

$$-mgh + ma_0 S = \Delta K$$

$$-mg\ell(1 - \cos\theta_m) + ma_0\ell\sin\theta_m = 0$$

$$g\left(2\sin^2\frac{\theta_m}{2}\right) = a_0\left(2\sin\frac{\theta_m}{2}\cos\frac{\theta_m}{2}\right)$$

$$\tan\frac{\theta_m}{2} = \frac{a_0}{g}$$

$$\theta_m = 2\tan^{-1}\left(\frac{a_0}{g}\right)$$

Central Force:

Force depending only on the distance between interacting particles and directed along the straight line connecting them is referred as central force.

$$\vec{F} = F(r)\hat{r}$$

All forces following inverse square law are called central forces.

$$\vec{F} = \frac{k}{r^2}\hat{r} \text{ is a central force such as Gravitational force and Coulomb force.}$$

- Central forces are functions of position only.

Conservative and Non–conservative Forces:

Gravitational, electrostatic, and restoring force of a spring are some of the natural forces with a property in common that work done by them in moving a particle from one point to another depends only on the locations of the initial and final points and not on the path followed irrespective of pair of points selected. On the other hand, there are forces such as friction, whose work depends on path followed. Accordingly, forces are divided into two categories – one whose work is path independent and other whose work is path dependant. The forces of the former category are known as **conservative forces** and of the later one as **non–conservative** forces.

A force, whose finite non–zero work $W_{1\to2}$ expended in moving a particle from a position 1 to another position 2 is independent of the path followed, is defined as a conservative force.

Consider a particle moving from position 1 to position 2 along different paths A, B, and C under the action of a conservative force F as shown in figure. If work done by the force along path A, B, and C are $W_{1\to2,A}$, $W_{1\to2,B}$ and $W_{1\to2,C}$ respectively, we have

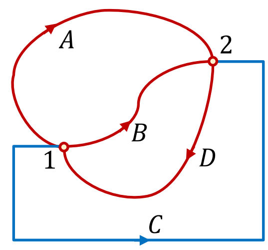
$$W_{1\to2,A} = W_{1\to2,B} = W_{1\to2,C}$$

If these works are positive, the work done $W_{2\to1,D}$ by the same force in moving the particle from position 2 to 1 by any other path say D will have the same magnitude but negative sign. Hence, we have

$$W_{1\to2,A} = W_{1\to2,B} = W_{1\to2,C} = -W_{2\to1,D}$$

$$W_{1\to2,A} + W_{2\to1,D} = W_{1\to2,B} + W_{2\to1,D} = W_{1\to2,C} + W_{2\to1,D} = 0$$

The above equations are true for any path between any pair of points 1 and 2.



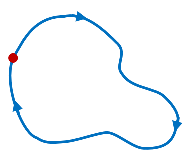
Representing the existing conservative force by $\vec{F}$, an infinitely path increment by $d\vec{r}$ and integration over a closed path by $\oint(\,).d\vec{r}$, the above equation can be represented in an alternative form as $\oint \vec{F}\,.\,d\vec{r} = 0$.

This equation shows that the total work done by a conservative force in moving a particle from position 1 to another position 2 and moving it back to position 1 i.e. around a closed path is zero. It is used as a fundamental property of a conservative force.

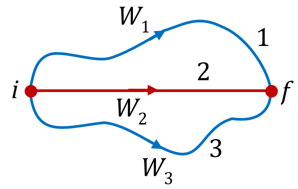
Key Points:

- Force is conservative when work done by it in any **closed path** (Loop) is **zero**.

  Work done by conservative force is independent of path. It depends only on initial & final position.



$W_F = 0 \quad \Rightarrow \quad F$ is conservative



$W_1 = W_2 = W_3$ by F

Examples of Conservative Force:

- Constant forces and central forces are conservative forces. (gravitational, electrostatic, elastic force, restoring force due to spring etc.)
- In presence of conservative forces mechanical energy remains constant.
- Work done along a closed path or in a cyclic process is zero. i.e. $\oint \vec{F}.d\vec{r} = 0$
- If $\vec{F}$ is a conservative force then $\vec{\nabla} \times \vec{F} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k} = 0$$

- A conservative force must be function only of position not of velocity or time.
- All forces, whose magnitude or direction depends on the velocity, are non–conservative. Sliding friction, which acts in opposite direction to that of velocity, and viscous drag of fluid depends in magnitude of velocity and acts in opposite direction to that of velocity are non–conservative.

Illustration 68:

$\vec{F} = y\hat{i} + x\hat{j}$

Solution:

$F_x = y\,;\ \frac{\partial F_x}{\partial y} = 1$

$F_y = x\,;\ \frac{\partial F_y}{\partial x} = 1$

$1 = 1$

$\therefore \quad \vec{F}$ is conservative

Non–conservative force

A force is said to be non–conservative if work done by or against the force in moving a body depends upon the path between the initial and final positions.

Work done in a closed path is not zero in a non–conservative force field.

The frictional forces are non–conservative forces. This is because the work done against friction depends on the length of the path along which a body is moved. It does not depend on the initial and final positions. The work done by frictional force in a round trip is not zero.

Examples of non–conservative force:

- Kinetic friction
- The velocity–dependent forces such as air resistance, viscous force etc. are non–conservative forces.

Conservative Forces	Non–conservative Forces
• Work done does not depend upon path.	• Work done depends upon path.
• Work done in a round trip is zero.	• Work done in a round trip is not zero.
• Central forces, spring forces etc. conservative forces	• Force are velocity–dependent and retarding in nature e.g. friction, viscous force etc.
• When only a conservative force acts within a system, the kinetic energy and potential energy can change. However, their sum, the mechanical energy of the system, does not change.	• Work done against a non–conservative force may be dissipated as heat energy.
• Work done is completely recoverable.	• Work done is not completely recoverable.

Potential Energy:

The difference in potential energy between two points A and B is equal to the work done by external force $\left(\vec{F}_{ext}\right)$ against the conservative forces in moving a particle **slowly** (that is without developing kinetic energy) between those two points.

$$U_B - U_A = \Delta U = W_{ext.\left(\substack{A\to B \\ Slowly}\right)} = \int_A^B \vec{F}_{ext.} . d\vec{r}$$

$$\vec{F}_{ext.} + \vec{F}_C = m\vec{a} \quad \Rightarrow \quad \vec{F}_{ext.} + \vec{F}_C \simeq 0 \quad \Rightarrow \quad \vec{F}_{ext.} = -\vec{F}_C$$

$$\Delta U = -\int_A^B \vec{F}_C \cdot d\vec{r} \quad \Rightarrow \quad \Delta U = -W_C$$

- Change in potential energy between two points is equal to the negative of work done by conservative forces.
- $\boxed{\Delta U = -W_{\text{internal conservative}}}$

1. PE is defined only for conservative forces.

2. The energy which a body has by virtue of its position or configuration in a conservative force field.

3. Potential energy is a relative quantity.

4. Change in PE is negative of work done by internal conservative force.

5. PE cannot be defined for single particle system. It is always defined for system of more than one particle.

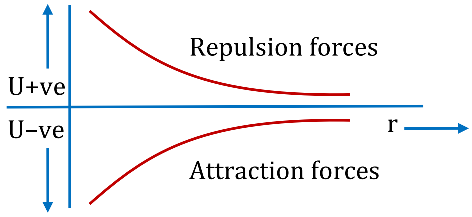
6. As only change is defined for potential energy so to tell PE at same position we have to select PE at one position independently as reference value.

7. At reference position, the potential energy of the body is zero or the body has lost the capacity of doing work.

8. Potential energy may be positive or negative

(i) Potential energy is positive, if force field is repulsive in nature.

(ii) Potential energy is negative, if force field is attractive in nature.



9. If r (separation between body and force centre), U, force field is attractive or vice–versa.

10. If r, $U\downarrow$, force field is repulsive in nature.

Purpose: By defining *PE* we can avoid repeated calculation of work for conservative forces and since *PE* depends only on position (initial and final), we can directly write effect of conservative forces in terms of their respective *PE's.*

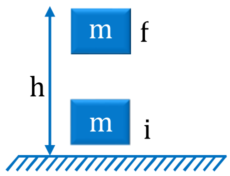
We will define *PE* for gravity and spring.

Gravitational Potential energy :

$\Delta U = -W_g$

$U_f - U_i = mgh$

$U_f = U_i + mgh$



If we take U at ground = 0 as reference value then

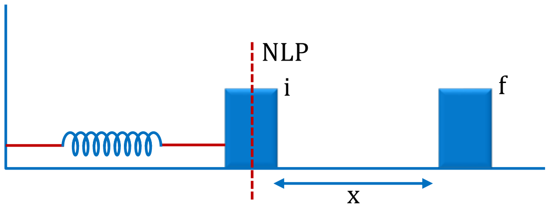
$U = mgh$

Spring potential energy:

$U_f - U_i = -W_{sp}$

$U_f - U_i = -\left(\frac{1}{2}K(0^x - x^2)\right)$

$U_f - U_i = \frac{1}{2}Kx^2$



If we take $U_{NLP} = 0$

Then $\boxed{U_{sp} = \frac{1}{2}Kx^2}$ compr. or elong.

Hence we can say that

The potential energy associated with a spring force of an ideal spring when compressed or elongated by a distance *x* from its natural length is defined by the following equation

$U = \frac{1}{2}kx^2$

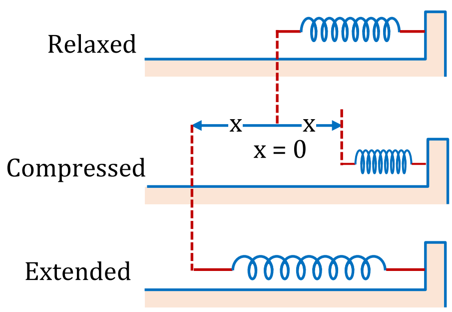
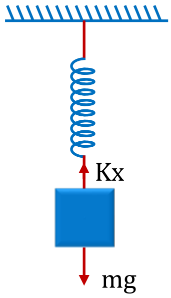


Illustration 69:

Final *PE* of spring at equilibrium.

Solution:



At equilibrium $Kx = mg$

$$x = \frac{mg}{K}$$

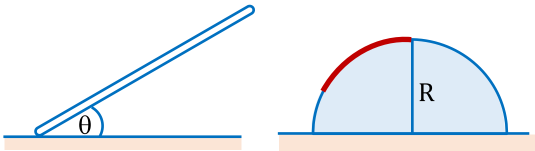
$$U = \frac{1}{2} \times K(x^2)$$

$$= \frac{1}{2} \times K \times \frac{m^2g^2}{K^2} = \frac{m^2g^2}{2K}$$

Illustration 70:

Find the gravitational potential energies in the following physical situations. Assume the ground as the reference potential energy level.

(a) A thin rod of mass *m* and length *L* kept at angle θ with one of its end touching the ground.



(b) A flexible rope of mass m and length *L* placed on a smooth hemisphere of radius *R* and one of the ends of the rope is fixed at the top of the hemisphere.

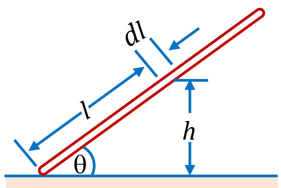
Solution:

In both the above situations, mass is distributed over a range of position coordinates. In such situations calculate potential energy of an infinitely small portion of the body and integrate the expression obtained over the entire range of position coordinates covered by the body.

(a) Assume a small portion of length $d\ell$ of the rod at distance ℓ from the bottom end and height of the midpoint of this portion from the ground is *h*. Mass of this portion is *dm*. When $d\ell$ approaches to zero, the gravitational potential energy *dU* of the assumed portion becomes

$$dU = \frac{m}{L} ghd\ell = \frac{m}{L} g\ell \sin\theta \, d\ell$$

The gravitational potential energy *U* of the rod is obtained by carrying integration of the above equation over the entire length of the rod.



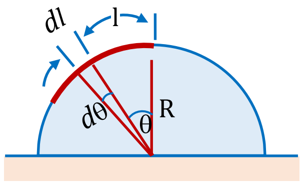
$$U = \int_{\ell=0}^{L} \frac{m}{L} g\ell \sin\theta \, d\ell$$

$$= \frac{1}{2} mgL \sin\theta$$

(b) The gravitational potential energy *dU* of a small portion of length $d\ell$ shown in the adjoining figure, when $d\ell$ approaches to zero is

$$dU = \frac{m}{L} gR^2 \cos\theta \, d\theta$$

The gravitational potential energy U of the rope is obtained by carrying integration of the above equation over the entire length of the rope.



$$U = \int_{\theta=0}^{\theta=\frac{L}{R}} \frac{m}{L} gR^2 \cos\theta \, d\theta = \frac{m}{L} gR^2 \left\{ \sin\left(\frac{L}{R}\right) \right\}$$

Law of Conservation of Energy:

Energy can neither be created nor be destroyed, it only can be converted from one form to another.

Conservation of Mechanical Energy:

The total potential energy of a system and the total kinetic energy of all the constituent bodies together is known as the mechanical energy of the system. If E, K and U respectively denote the total mechanical energy, total kinetic energy, and the total potential energy of a system in any configuration then we have

$E = K + U$

Consider a system on which no external force acts and all the internal forces are conservative. If we apply work–kinetic energy $(W_{1\to2} = K_2 - K_1)$ theorem, the work $W_{1\to2}$ will be the work done by internal conservative forces, negative of which equals the change in potential energy. Rearranging the kinetic energy and potential energy terms, we have

$E = K_1 + U_1 = K_2 + U_2$

The above equation takes the following form

$E = K + U = \text{constant}$

$\Delta E = 0 \Rightarrow \Delta K + \Delta U = 0$

Above equations, express the principle of conservation of mechanical energy.

If there is no net work done by any external force or any internal non–conservative force, then the total mechanical energy of a system is conserved.

The principle of conservation of mechanical energy is developed from the work–kinetic energy principle for systems where change in configuration takes place under internal conservative forces only. Therefore, in physical situations, where external forces or non–conservative internal forces are involved, the use of work–kinetic energy principle should be preferred.

In systems, where external forces or internal nonconservative forces do work, the net work done by these forces becomes equal to the change in the mechanical energy of the system.

Total Mechanical Energy & Conservation:

Hence, we can say that

Mechanical Energy (ME) = Potential Energy (U) + Kinetic energy (KE)

$W_{\text{net}} = KE_f - KE_i$

$W_{\text{ext}} + W_{\text{int.}} = KE_f - KE_i$

$W_{\text{ext}} + W_{int,C} + W_{\text{int,NC}} = KE_f - KE_i$

$W_{\text{ext}} + (U_i - U_f) + W_{\text{int, NC}} = KE_f - KE_i$

$W_{\text{ext}} + W_{\text{int, NC}} = (KE_f + U_f) - (KE_i + U_i)$

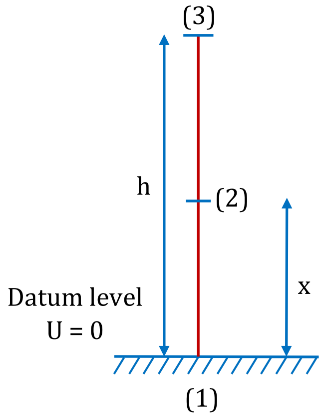
$W_{\text{ext}} + W_{\text{int,NC}} = ME_f - ME_i$

If $W_{\text{ext}} + W_{\text{int, NC}} = 0$

$ME = U + KE = \text{conserved i.e. constant}$

$$\left\{ \begin{array}{l} W_{\text{net}} = W_{\text{ext}} + W_{\text{int.}} \\ W_{\text{int}} = W_{\text{int},C} + W_{\text{int,NC}} \\ W_{\text{int}} \to W_{\text{internal}} \\ W_{\text{int},\,C} \to W_{\text{internal conservative}} \\ W_{\text{int,NC}} \to W_{\textit{internal nonconservative}} \end{array} \right.$$

Application to Gravitational Field:



Let a ball of mass m be dropped from position (3) (as shown in figure)

At point 1:

$$PE = 0\,; \qquad KE = \frac{1}{2}mv^2 \quad \because\ v = \sqrt{2gh} \qquad \text{so, } KE = mgh$$

At point 2:

$$PE = mgx\,; \qquad KE = \frac{1}{2}mv^2 \quad \because\ v = \sqrt{2g(h-x)} \qquad \text{so, } KE = mg(h-x)$$

At point 3:

$$PE = mgh\,;\ KE = 0$$

So, during the motion of the ball at any position,

$$\boxed{ME_{(1)} = ME_{(2)} = ME_{(3)}} \text{ and } \boxed{PE = mgx}\,;\ \boxed{KE = mg(h-x)}$$

Illustration 71:

In presence of conservative forces only PE is defined as $U = x^2 - 2x$ for particle travelling along x–axis. If $v = 0$ at origin find-

(a) KE *at* $x = 1$

(b) Find where will KE maximum and also find $KE_{max.}$

(c) Find KE at $x = 4$

(d) Range of x where particle can be moved ?

Solution:

(a) $U_0 + KE_0 = U_1 + KE_1$

$$(0^2 - 2 \times 0) + \left(\frac{1}{2} \times m \times 0^2\right) = (1^2 - 2 \times 1) + KE_1$$

$0 + 0 = -1 + KE_1$

$KE_1 = 1$

(b) $ME = U + KE$

$0 = x^2 - 2x + KE$

$KE = 2x - x^2$

$KE_{max.} = 2x - x^2$

$KE_{max.}$ when $\dfrac{dKE}{dx} = 0 \quad \Rightarrow \quad 2 - 2x = 0$

$x = 1m$

$KE_{max.} = 1$ joule

(c) At x = 4 m

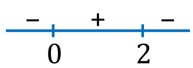
$KE = 2x - x^2$

$KE_4 = 2 \times 4 - (4)^2$

$= -8$ joule Not possible

$\therefore$ Particle can never be at $x = 4$

(d)

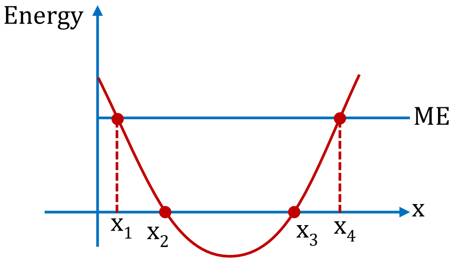


$KE \geq 0$

$2x - x^2 \geq 0$

$x \in [0, 2]$

Illustration 72:



Solution:

$KE \geq 0$

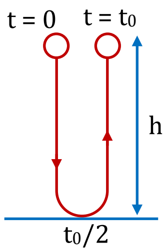
$ME - U \geq 0$

$ME \geq U$

Range of particle = $[x_1, x_4]$

Illustration 73:

A particle is dropped at $t = 0$ sec., draw energies v/s time graph ?



Solution:

At $t = 0$

U_{max} and $KE_{min.}$

At $t = \frac{t_0}{2}$

$U_{min.}$ and $KE_{max.}$

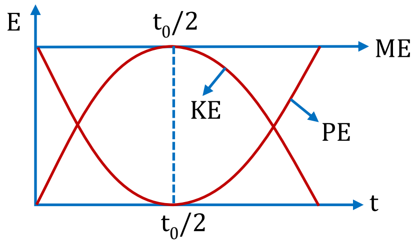
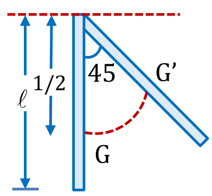


Illustration 74:

A meter scale of mass m initially vertical is displaced at 45° keeping the upper end fixed. Find out the change in potential energy.



Solution:

$$\Delta U = mg\,\Delta h_{cm} = mg\frac{\ell}{2}(1 - \cos\theta) = mg \times \frac{1}{2}(1 - \cos 45^0) = \frac{mg}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Illustration 75:

A uniform rod of length $4m$ and mass $20kg$ is lying horizontal on the ground. Calculate the work done in keeping it vertical with one of its ends touching the ground.

Solution:

As the rod is kept in vertical position the shift in the centre of gravity is equal to the half the length

$= \ell/2$ Work done $W = mgh = mg\dfrac{\ell}{2} = (20)(9.8)\left(\dfrac{4}{2}\right) = 392\,J$

Potential energy in terms of forces (Conservative):

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

$\Delta U = -W$; $\quad i \to (x_i, y_i, z_i)$

$\qquad\qquad f \to (x_f, y_f, z_f)$

$$U_f - U_i = -\left[\int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz\right]$$

Illustration 76:

$\vec{F} = 2x\hat{\imath} + 2\hat{\jmath} + z\hat{k}$ if U at (2, 3, 0) = 10 joule, then find U at (3, 2, 2) = ?

Solution:

$$U_{(3,2,2)} - U_{(2,3,0)} = -\left[\int_2^3 2xdx + \int_3^2 2dy + \int_0^2 zdz\right]$$

$$= -\left[3^2 - 2^2 + 2(2-3) + \frac{2^2}{2} - \frac{0^2}{2}\right]$$

$U_{(3,2,2)} - 10 = -[5 - 2 + 2]$

$U_{(3,2,2)} = 5J$

Differential Relation between conservative force and potential energy :

$$\Delta U = -\int_A^B \vec{F}.d\vec{r} \quad \Rightarrow \quad \Delta U = -\int_{x_A}^{x_B} F_x dx - \int_{y_A}^{y_B} F_y dy - \int_{z_A}^{z_B} F_z dz$$

In one dimension, $\quad \Delta U = -\int_{x_A}^{x_B} F_x dx$

In differential form, $\quad F_x = -\dfrac{dU}{dx}$

In three dimensions, $\quad F_x = -\dfrac{\partial U}{\partial x},\ F_y = -\dfrac{\partial U}{\partial y},\ F_z = -\dfrac{\partial U}{\partial z} \Rightarrow \vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$

Where $F_x \to$ diff. of U w.r.t x (keeping y and z constant)

Similarly, F_y = differentiation of U w.r.t y be keeping x and z constant.

$$\vec{F} = -\left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]U \quad \Rightarrow \quad \vec{F} = -\vec{\nabla}U$$

$$\Rightarrow \quad \vec{F} = -\left[\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right]$$

where $\vec{\nabla} = \left[\hat{\imath}\dfrac{\partial}{\partial x} + \hat{\jmath}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right]$

Illustration 77:

$U = 3x^2yz^3$, find $\vec{F}$.

Solution:

$$\vec{F} = -\left[3yz^3(2x)\hat{\imath} + 3x^2z^3\hat{\jmath} + 3x^2y(3z^2)\hat{k}\right]$$

Illustration 78:

3 sin $(x + y)$

Find $\vec{F}$ at $\left(0, \frac{\pi}{4}\right)$?

Solution:

$$\vec{F_x} = -\left[\frac{-\partial U}{dx}\hat{\imath} + \frac{-\partial U}{dy}\hat{\jmath}\right]$$

$$= -[3\cos(x + y)\hat{\imath} + 3\cos(x + y)\hat{\jmath}]$$

Illustration 79:

$U = 15 + (x - 4)^2$

ME = constant = 20 joule

Find: (a) $U_{min.}$ and where, (b) $U = 15 + (0)^2$, (c) Equation position (mean position)

Solution:

(a) $U = 15 + (0)^2$

$U = 15$ and at $x = 4$

(b) $KE_{max.}$ when $U_{min.}$

$\therefore$ $ME = U_{min.} + KE_{max.}$

$20 - 15 = KE_{max}$

$KE_{max} = 5$

(c) $F_x = \frac{-dU}{dx} = -[2(x - 4)(1)] = 0$

Illustration 80:

A particle of mass m is moving in a horizontal circle of radius r, under a centripetal force equal to $(-k/r^2)$, where k is constant. Calculate the total energy of the particle.

Solution:

As the particle is moving in a circle, so $\frac{mv^2}{r} = \frac{k}{r^2}$.

Now $K.E. = \frac{1}{2}mv^2 = \frac{k}{2r}$

As $U = -\int_{\infty}^{r} F.dr = \int_{\infty}^{r}\left(\frac{k}{r^2}\right)dr = -\frac{k}{r}$.

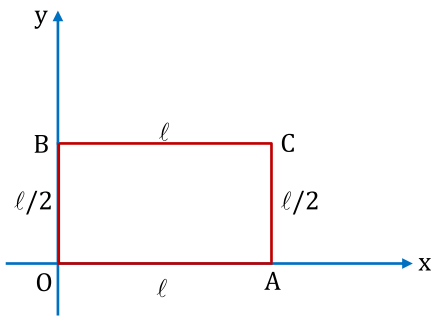
So, total energy = $U + K.E. = -\frac{k}{r} + \frac{k}{2r} = -\frac{k}{2r}$

Negative energy means that particle is in bound state.

Illustration 81:

Check whether force $\vec{F} = x^2y\hat{i} + xy^2\hat{j}$ is conservative.

Solution:



$$W = \int_i^f F_x\,dx + \int_i^f F_y\,dy$$

$$W = \int x^2y\,dx + \int xy^2\,dy$$

OAC path :–

For *OA* path, $y = 0 \Rightarrow dy = 0$

$W_{OA} = 0$

For *AC* path, $x = \ell \Rightarrow dx = 0$

$$W_{AC} = \int_0^{\ell/2} \ell\,y^2\,dy = \ell\left(\frac{y^3}{3}\right)_0^{\ell/2} \Rightarrow W_{AC} = \frac{\ell^4}{24} \Rightarrow W_{OAC} = \frac{\ell^4}{24}$$

OBC path

For *OB* path, $x = 0 \Rightarrow dx = 0$

$W_{OB} = 0$

For *BC* path, $y = \frac{\ell}{2} \Rightarrow dy = 0$

$$W_{BC} = \int_0^{\ell} \frac{\ell}{2}x^2\,dx = \frac{\ell}{2}\left(\frac{x^3}{3}\right)_0^{\ell} = \frac{\ell^4}{6} \Rightarrow W_{OBC} = \frac{\ell^4}{6} \Rightarrow W_{OAC} \neq W_{OBC}$$

Hence force $\vec{F}$ is non–conservative.

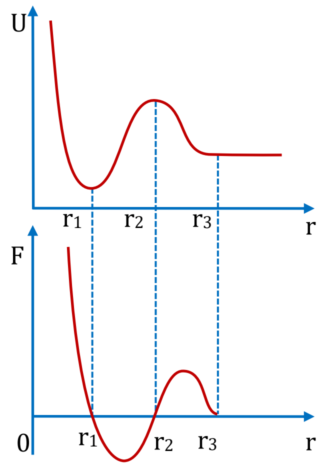
A force $\vec{F}$ is conservative when

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \Rightarrow \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \Rightarrow \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

Potential energy and nature of equilibrium :

The above equation suggests that on every location where the potential energy function assumes a minimum or a maximum value or in every region where the potential energy function assumes a constant value, the associated conservative force becomes zero and a body under the action of only this conservative force must be in the state of equilibrium. Different status of potential energy function in the state of equilibrium suggests us to define three different types of equilibriums – the stable, unstable and neutral equilibrium.

The state of stable and unstable equilibrium is associated with a point location, where the potential energy function assumes a minimum and maximum value respectively, and the neutral equilibrium is associated with region of space, where the potential energy function assumes a constant value.



Force is negative of the slope of the potential energy function

For the sake of simplicity, consider a one-dimensional potential energy function U of a central force F. Here r is the radial coordinate of a particle. The central force F experienced by the particle equals to the negative of the slope of the potential energy function. Variation in the force with r is also shown in the figure.

At locations $r = r_1$, $r = r_2$, and in the region $r \geq r_3$, where potential energy function assumes a minimum, a maximum, and a constant value respectively, the force becomes zero and the particle is in the state of equilibrium.

Stable Equilibrium:

At $r = r_1$ the potential energy function is a minima and the force on either side acts towards the point $r = r_1$. If the particle is displaced on either side and released, the force tries to restore it at $r = r_1$. At this location the particle is in the state of stable equilibrium. The dip in the potential energy curve at the location of stable equilibrium is known as potential well. A particle when disturbed from the state of stable within the potential well starts oscillations about the location of stable equilibrium. At the locations of stable equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \text{ ; and } \frac{\partial F}{\partial r} < 0 \text{ ; and } \frac{\partial^2 U}{\partial r^2} > 0$$

Unstable Equilibrium:

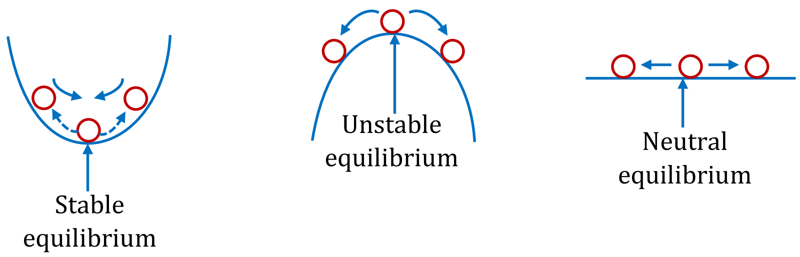
At $r = r_2$ the potential energy function is a maxima, the force acts away from the point $r = r_2$. If the particle is displaces slightly on either side, it will not return to the location $r = r_2$. At this location, the particle is in the state of unstable equilibrium. At the locations of unstable equilibrium we have

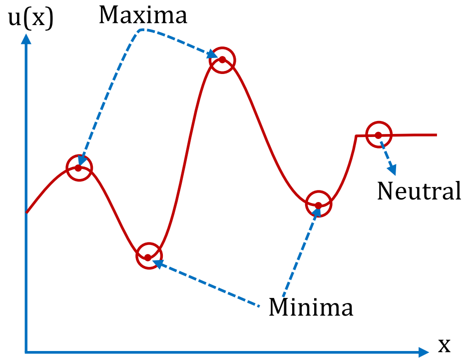
$$F(r) = -\frac{\partial U}{\partial r} = 0 \text{ therefore } \frac{\partial F}{\partial r} > 0 \text{ ; and } \frac{\partial^2 U}{\partial r^2} < 0$$

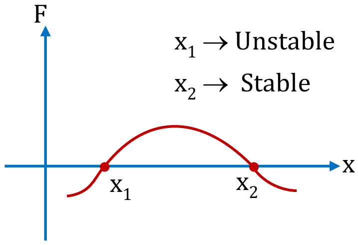
Neutral Equilibrium:

In the region $r \geq r_3$, the potential energy function is constant and the force is zero everywhere. In this region, the particle is in the state of neutral equilibrium. At the locations of neutral equilibrium we have

$$F(r) = -\frac{\partial U}{\partial r} = 0 \text{; therefore } \frac{\partial F}{\partial r} = 0 \text{; and } \frac{\partial^2 U}{\partial r^2} = 0$$

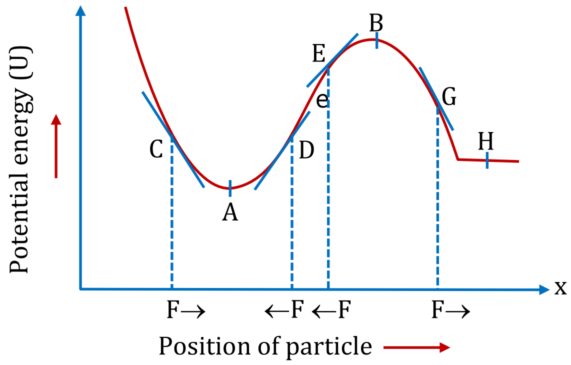






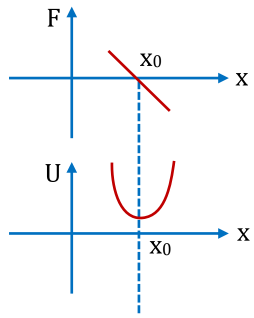
Potential energy curve and equilibrium:

It is a curve which shows change in potential energy with position of a particle.



Stable Equilibrium:

When a particle is slightly displaced from equilibrium position and it tends to come back towards equilibrium then it is said to be in stable equilibrium.



At point **C :** slope $\frac{dU}{dx}$ is negative, so F is positive.

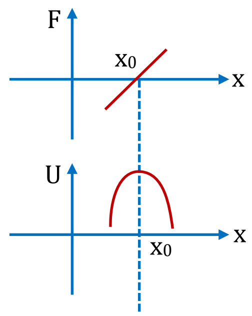
At point **D :** slope $\frac{dU}{dx}$ is positive, so F is negative.

At point **A :** it is the point of stable equilibrium.

At **A :** $U = U_{min}$, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ = positive.

Unstable equilibrium:

When a particle is slightly displaced from equilibrium and it tends to move away from equilibrium position then it is said to be in unstable equilibrium.



At point **E** : slope $\frac{dU}{dx}$ is positive, so F is negative.

At point **G** : slope $\frac{dU}{dx}$ is negative, so F is positive.

At point **B** : it is the point of unstable equilibrium.

At **B :** $U = U_{max}$, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ = negative.

Neutral equilibrium :

When a particle is slightly displaced from equilibrium position and no force acts on it then equilibrium is said to be neutral equilibrium.

Point **H** is at neutral equilibrium $\Rightarrow U$ = constant ; $\frac{dU}{dx} = 0,\ \frac{d^2U}{dx^2} = 0$

	Stable equilibrium	Unstable equilibrium	Neutral equilibrium
F	0	0	0
$\frac{dU}{dx} = -F$	0	0	0
$\frac{dF}{dx}$	–	+	0
$\frac{d^2U}{dx^2} = -\frac{dF}{dx}$	+	–	0

Illustration 82:

Find nature of equilibrium along x, y and z axis.

Solution:

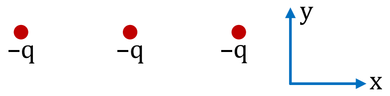
Stable equilibrium along x-axis.

Unstable equilibrium along y-axis.

Unstable equilibrium along z-axis.

Illustration 83:

Find nature of equilibrium along x, y and z axis.



Solution:

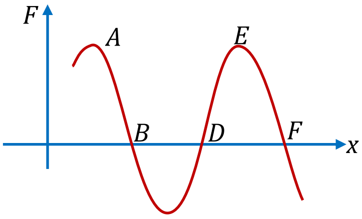
Unstable equilibrium along x-axis.

Stable equilibrium along y-axis.

Stable equilibrium along z-axis.

Illustration 84:

Find equilibrium position and their nature?



Solution:

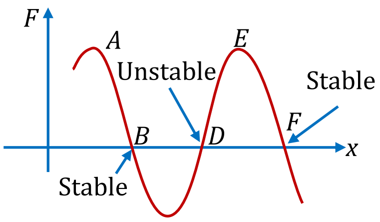


Illustration 85:

Find equilibrium position and their nature?

$$U = \frac{x^3}{3} - \frac{5x^2}{2} + 6x + 8$$

Solution:

$$\frac{dU}{dx} = x^2 - 5x + 6 = 0$$

$(x - 2)(x - 3) = 0$

$x = 2, 3$

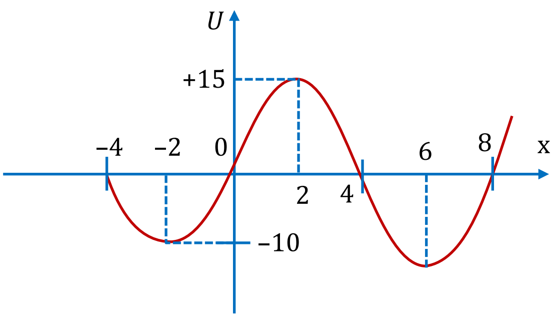
$\frac{d^2U}{dx^2} \Rightarrow$ At, $x = 2$ $\Rightarrow$ -1 $\Rightarrow$ $-ve$ unstable

$\Rightarrow$ At, $x = 3$ $\Rightarrow$ $+1$ $\Rightarrow$ $+ve$ stable

Illustration 86:

(a) If particle released from rest at $x = 0$ in which direction will it move ?

(b) If particle released from $x = 2^+$. Find maximum KE of particle till $x = 8$.



Solution:

(a) $\frac{dU}{dx} = -F$

$+ve = -F$

$F = -ve$

$\therefore$ Move along $-ve$ x-$axis$

(b) $U = KE =$ constant

$15 + 0 = -10 + KE_{x=6}$

$KE_{x=6} = 25$ joule

Illustration 87:

Force between the atoms of a diatomic molecule has its origin in the interactions between the electrons and the nuclei present in each atom. This force is conservative and associated potential energy $U(r)$ is, to a good approximation, represented by the Lennard – Jones potential.

$$U(r) = U_0\left\{\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right\}$$

Here r is the distance between the two atoms and U_o and a are positive constants. Develop expression for the associated force and find the equilibrium separation between the atoms.

Solution:

Using equation, $F = -\frac{dU}{dr}$, we obtain the expression for the force,

$$F = \frac{6U_0}{a}\left\{2\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right\}$$

At equilibrium the force must be zero. Therefore, the equilibrium separation r_0 is

$$r_0 = 2^{\frac{1}{6}}a$$

Power:

Average power & instantaneous power:

- When we purchase a car or jeep we are interested in the horsepower of its engine. We know that, usually an engine with large horsepower is most effective in accelerating the automobile.

  In many cases, it is useful to know not just the total amount of work being done, but also how fast the work is being done. We define power as the rate at which work is being done.

  $$\text{Average Power} = \frac{\text{Work done}}{\text{Time taken to do work}}$$

  If ΔW is the amount of work done in a time interval Δt, then $P = \frac{\Delta W}{\Delta t} = \frac{W_2 - W_1}{t_2 - t_1}$

- Instantaneous power is the scalar product of force and velocity at any instant.

  When work is measured in joules and t is in seconds, the unit for power is joules per second, which is called watt. For motors and engines, power is usually measured in horsepower, where 1 hp = 746 W. The definition of power is applicable to all types of energies also like mechanical, electrical, thermal.

  Instantaneous power $P = \frac{dW}{dt} = \frac{\vec{F}.d\vec{r}}{dt} = \vec{F}.\vec{v}$

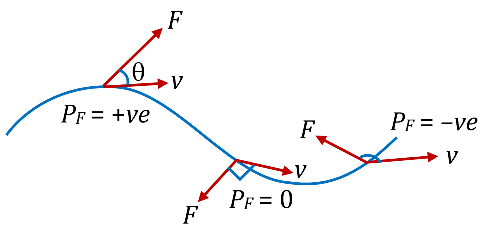
  where v is the instantaneous velocity of the particle. Here dot product is used because only that component of force will contribute to power which is acting in the direction of instantaneous velocity.

  $$P = \frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{s}}{dt} = \vec{F}.\vec{v} = Fv\cos\theta$$

  $P = F(v\cos\theta)$

  $= Fv_F$ (v_F is component of $\vec{v}$ along $\vec{F}$)

  $= F_v v$ (F_v is component of $\vec{F}$ along $\vec{v}$)



- For a system of varying mass, $F = \dfrac{d}{dt}(mv) = m\dfrac{dv}{dt} + v\dfrac{dm}{dt}$

  If v = constant, then $F = v\dfrac{dm}{dt}$, then $P = \vec{F}.\vec{v} = v^2\dfrac{dm}{dt}$
- Power is a scalar quantity with dimensions $M^1L^2T^{-3}$.
- SI unit of power is J/s or watt.
- 1 horsepower = 746 watts = 550 *ft–lb/s.*

  $\text{Power}_{\text{avg}} = \dfrac{\Delta W}{\Delta t} = \dfrac{\text{Total work}}{\text{Total time}}$

  $\text{Power}_{\text{inst}} = \dfrac{dW}{dt} = \dfrac{d}{dt}(\vec{F}\cdot d\vec{S}) = \vec{F}\cdot\dfrac{d\vec{S}}{dt} = \vec{F}\cdot\vec{V}$

  $P = (|\vec{F}|\cos\theta)|\vec{V}|$ = (component of $\vec{F}$ along $\vec{V}$)·Speed

Illustration 88:

$x = t^2 + \dfrac{t^3}{3}$, $m = 2kg$. Find power at $t = 2$ sec. of net force.

Solution:

$v = 2t + t^2 \quad\Rightarrow\quad v_2 = 8\ m/s$

$a = 2t + 2 \quad\Rightarrow\quad a_2 = 6\ m/s^2$

$\therefore \quad F_{net} = ma = 12N$

$\therefore \quad P = Fv = 96W$

Graphical Analysis :

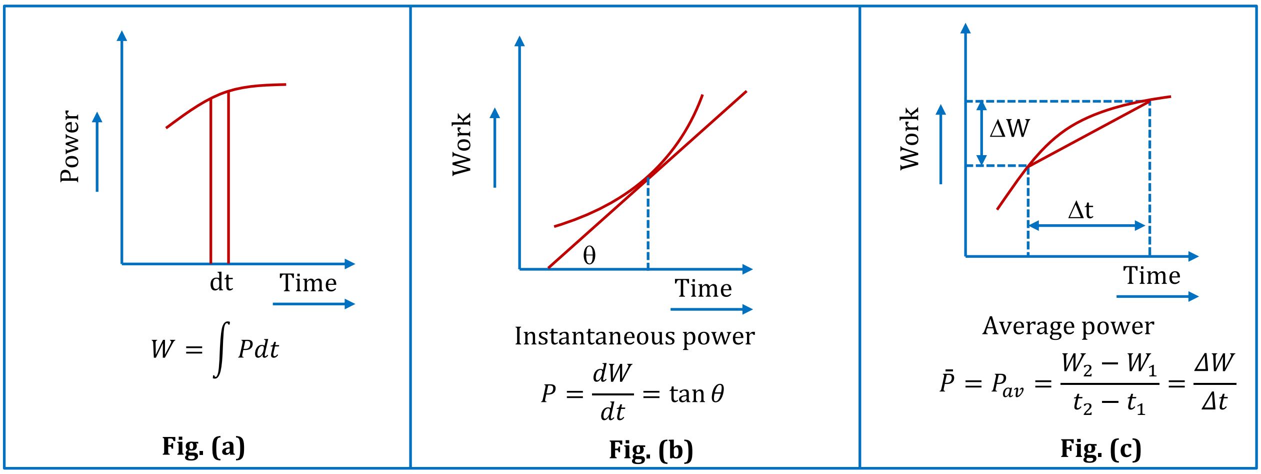


Fig. (a) **Fig. (b)** **Fig. (c)**

- Area under power–time graph gives the work done. $W = \int Pdt$ (See Fig. a)
- The slope of tangent at a point on work–time graph, gives instantaneous power. (See Fig. b)
- The slope of a straight line joining two points on work–time graph gives average power between two points. (See Fig. c)

Efficiency:

Machines are designed to convert energy into useful work, however because of frictional effects and other dissipative forces work performed by the machine is always less than the energy supplied to the machine.

The efficiency of a machine is given by $\eta = \dfrac{\text{Work done}}{\text{Energy input}}$

Illustration 89:

A truck pulls a mass of 1200 kg at constant speed of 10m/s on a level road. The tension in coupling is 1000 N. What is the power spent on the mass. Find tension when truck moves up a road with inclination 1 in 6.

Solution:

Force applied by truck f = 1000 N

Power spent in pulling the mass $P = fv = 1000 \times 10 = 10^4\ W$

Here sinθ = 1/6, the required force for truck to move up is

$$F = f + Mg \sin\theta$$

$$F = 1000\ N + 1200 \times 9.8 \times \frac{1}{6} = 2960\ N$$

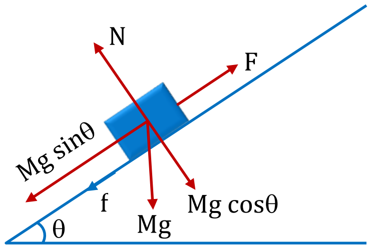


Illustration 90:

In unloading grains from the hold of a ship, an elevator lifts the grains through a distance of 12 m. Grains are discharged at the top of the elevator at a rate of 2.0 kg per second and the discharge speed of each grain particle is 3.0 m/s. Find the minimum horsepower of the motor that can elevate grains in this way.

Solution:

Work done by the motor each second, i.e.

$$\text{Power} = mgh + \frac{1}{2}mv^2$$

Here the mass of grain discharged (and lifted) in one second m = 2.0 kg, v = 3.0 m/s and h = 12 m

$\therefore$ Power = 249 W = 0.33 $H.P.$

The motor must have an output of at least 0.33 $H.P.$

Illustration 91:

A pump can take out 7200 kg of water per hour from a 100 m deep well. Calculate the power of the pump, assuming that its efficiency is 50%. (g = 10 m/s^2)

Solution:

$$\text{Output power} = \frac{mgh}{t} = \frac{7200\times10\times100}{3600} = 2000\ W$$

$$\text{Efficiency, } \eta = \frac{\text{Output power}}{\text{Input power}}$$

$$\text{Input power} = \frac{\text{Output power}}{\eta} = \frac{2000\times100}{50} = 4\ kW$$

Illustration 92:

A pump is used to deliver water at a certain rate from a given pipe. To obtain n times water from same pipe in the same time by what amount should power of motor be increased ?

Solution:

Amount of water per second

$$\frac{dm}{dt} = Av\rho$$

v = Velocity of flow, A is area of cross–section, ρ = Density of liquid

To get n times water in same time:

$$\left(\frac{dm}{dt}\right)' = n\frac{dm}{dt}$$

$\Rightarrow \quad Av'\rho = nAv\rho \Rightarrow v' = nv$

$$P = v^2\frac{dm}{dt}$$

So, $\dfrac{P'}{P} = \dfrac{v'^2(dm/dt)'}{v^2(dm/dt)} = \dfrac{n^2v^2 n dm/dt}{v^2 dm/dt} = n^3 \qquad \Rightarrow P' = n^3P$

Thus, to get n times water the power must be increased n^3 times.

Illustration 93:

A constant power P is applied on a body of mass m by a variable force in same direction of displacement. Find relation of

(i) Velocity with respect to time.

(ii) Displacement with respect to time.

Solution:

$P = Fv\cos\theta \qquad \because \ \theta = 0 \qquad \therefore \ P = Fv$

(i) $P = mav \qquad (F = ma)$

$$\Rightarrow \quad P = \frac{mdv}{dt}v \quad \Rightarrow \quad \frac{Pdt}{m} = vdv \quad \Rightarrow \quad \int vdv = \frac{P}{m}\int 1dt$$

$$\Rightarrow \quad \frac{v^2}{2} = \frac{Pt}{m} \quad \Rightarrow \quad v = \sqrt{\frac{2Pt}{m}}$$

(ii) $\dfrac{dx}{dt} = \sqrt{\dfrac{2Pt}{m}} \quad \Rightarrow \quad \int dx = \int\sqrt{\dfrac{2P}{m}}t^{1/2}dt \quad \Rightarrow \quad x = \left(\dfrac{2}{3}\sqrt{\dfrac{2P}{m}}\right)t^{3/2}$

Illustration 94:

A body of mass m accelerates uniformly from rest to v_1 in time t_1. The instantaneous power delivered to the body as a function of time t is–

(A) $\dfrac{mv_1t}{t_1}$ (B) $\dfrac{mv_1^2t}{t_1^2}$ (C) $\dfrac{mv_1t^2}{t_1}$ (D) $\dfrac{mv_1^2t}{t_1}$

Solution:

$$P = Fv = mav = m\left(\frac{v_1}{t_1}\right)\left(\frac{v_1}{t_1}\right)t = \frac{mv_1^2}{t_1^2}t$$

Illustration 95:

An engine pumps water of density ρ, through a hose pipe. Water leaves the hose pipe with a velocity v. Find the

(i) rate at which kinetic energy is imparted to water.

(ii) power of the engine.

Solution:

(i) Rate of change of kinetic energy

$$= \frac{dE_k}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}v^2\frac{dm}{dt} = \frac{1}{2}v^2\frac{d}{dt}(\rho Ax)$$

$$= \frac{1}{2}\rho Av^2\frac{dx}{dt} = \frac{1}{2}\rho Av^3$$

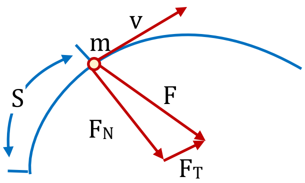
(ii) Power $= Fv = \left(v\frac{dm}{dt}\right)v = v^2\frac{dm}{dt} = v^2(\rho Av) = \rho Av^3$

Illustration 96:

A vehicle of mass m starts moving such that its speed v varies with distance travelled s according to the law $v = k\sqrt{s}$, where k is a positive constant. Deduce a relation to express the instantaneous power delivered by its engine.

Solution:

Let the particle is moving on a curvilinear path. When it has travelled a distance s, the force F acting on it and its speed v are shown in the adjoining figure.



Instantaneous power delivered by the engine:

$$P = \vec{F}.\vec{v} = (\vec{F}_T + \vec{F}_N).\vec{v} = F_Tv = ma_Tv$$

Tangential acceleration of the vehicle,

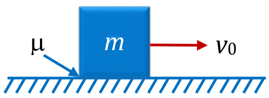
$$a_T = v\frac{dv}{ds} = \frac{k^2}{2}$$

From above equations, we have,

$$P = \frac{mk^3}{2}\sqrt{s}$$

Illustration 97:

A body of mass 'm' starts moving on rough surface with speed v_0. Coefficient of kinetic friction between surface and block is μ. Find



(i) the instantaneous power developed by friction force.

(ii) the average power developed by friction force over the whole time of motion.

Solution:

(a) $f_k = \mu_k mg$

$v = v_0 - \mu_k gt$

$P = \vec{F}.\vec{v}$

$P = -f_k v$

Instantaneous power $P = -\mu_k mg(v_0 - \mu_k gt)$

(b) Average power $(P_{av}) = \dfrac{\text{Total work}}{\text{Total time}}$

$$P_{av} = \frac{\int_0^{t_0} Pdt}{t_0}$$

$v = 0 \quad t = \dfrac{v_0}{\mu_k g}$

$$P_{av} = \frac{-\frac{1}{2}mv_0^2}{\frac{v_0}{\mu_k g}} = -\frac{1}{2}mv_0\mu_k g$$

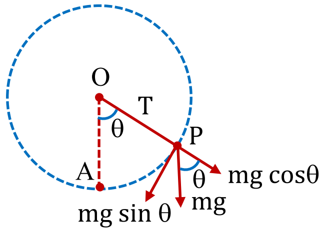
Circular Motion in Vertical Plane

Suppose a particle of mass m is attached to an inextensible light string of length R. The particle is moving in a vertical circle of radius R about a fixed–point O. It is imparted a velocity u in horizontal direction at lowest point A. Let v be its velocity at point P of the circle as shown in figure.

Here, $h = R(1 - \cos\theta)$ ...(i)

From conservation of mechanical energy,

$\frac{1}{2}m(u^2 - v^2) = mgh \Rightarrow v^2 = u^2 - 2gh$ ....(ii)



The necessary centripetal force is provided by the resultant of tension,

T and $mg\cos\theta$

$T - mg\cos\theta = \dfrac{mv^2}{R}$ ...(iii)

Since speed of the particle decreases with height, hence tension is maximum at the bottom, where $\cos\theta = 1$ (as $\theta = 0°$).

$\Rightarrow \quad T_{max} = \dfrac{mv^2}{R} + mg;\ T_{min} = \dfrac{mv'^2}{R} - mg$ at the top. Here, v' = speed of the particle at the top.

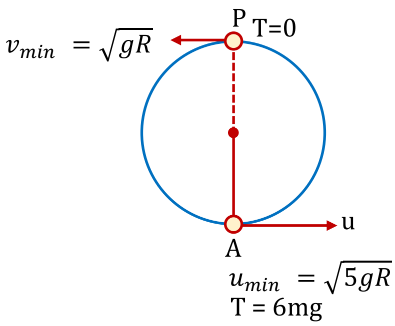
Condition of Looping the Loop $(u \geq \sqrt{5gR})$:

The particle will complete the circle if the string does not slack even at the highest point $(\theta = \pi)$. Thus, tension in the string should be greater than or equal to zero $(T \geq 0)$ at $\theta = \pi$. In critical case substituting $T = 0$ and in equation (iii), we get $mg = \dfrac{mv_{min}^2}{R} \Rightarrow v_{min} = \sqrt{gR}$ (at highest point)

Substituting in equation (i), Therefore, from equation (ii)

$u_{min}^2 = v_{min}^2 + 2gh = gR + 2g(2R) = 5gR$

$\Rightarrow \quad u_{min} = \sqrt{5gR}$



Thus, if $u \geq \sqrt{5gR}$, the particle will complete the circle. At $u = \sqrt{5gR}$, velocity at highest point is $v = \sqrt{gR}$ and tension in the string is zero.

Substituting $\theta = 0^0$ and $v = \sqrt{5gR}$ in equation (iii), we get $T = 6\,mg$ or in the critical condition tension in the string at lowest position is $6\,mg$. This is shown in figure. If $u < \sqrt{5gR}$, following two cases are possible.

Condition of Leaving the Circle $\left(\sqrt{2gR} < u < \sqrt{5gR}\right)$

If $u < \sqrt{5gR}$, the tension in the string will become zero before reaching the highest point. From equation (iii), tension in the string becomes zero ($T = 0$) where,

$$\cos\theta = \frac{-v^2}{Rg} \qquad \Rightarrow \cos\theta = \frac{2gh-u^2}{Rg}$$

Substituting, this value of cos θ in equation (i), we get

$$\frac{2gh-u^2}{Rg} = 1 - \frac{h}{R} \qquad \Rightarrow h = \frac{u^2+Rg}{3g} = h_1 \text{ (say)} \qquad \text{...(iv)}$$

or we can say that at height h_1 tension in the string becomes zero. Further, if $u < \sqrt{5gR}$, velocity of the particle becomes zero when

$$0 = u^2 - 2gh \qquad \Rightarrow h = \frac{u^2}{2g} = h_2 \text{(say)} \qquad \text{...(v)}$$

i.e., at height h_2 velocity of particle becomes zero.

Now, the particle will leave the circle if tension in the string becomes zero but velocity is not zero, or $T = 0$ but $v \neq 0$. *This* is possible only when $h_1 < h_2$

$$\Rightarrow \quad \frac{u^2+Rg}{3g} < \frac{u^2}{2g} \qquad \Rightarrow \quad 2u^2 + 2Rg < 3u^2 \qquad \Rightarrow \quad u^2 > 2Rg \qquad \Rightarrow \quad u > \sqrt{2Rg}$$

Therefore, if $\sqrt{2gR} < u < \sqrt{5gR}$, the particle leaves the circle.

From equation (iv), we can see that $h > R$ if $u^2 > 2gR$. Thus, the particle, will leave the circle when $h > R$ or $90^0 < \theta < 180^\circ$. This situation is shown in the figure

$$\sqrt{2gR} < u < \sqrt{5gR} \quad \text{or} \quad 90^\circ < \theta < 180^\circ$$

Note:

After leaving the circle, the particle will follow a parabolic path.

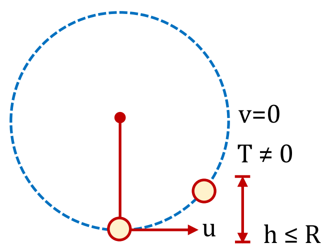
Condition of Oscillation $\left(0 < u \leq \sqrt{2gR}\right)$:

The particle will oscillate if velocity of the particle becomes zero but tension in the string is not zero or $v = 0$, but $T \neq 0$. This is possible when $h_2 < h_1$

$$\Rightarrow \quad \frac{u^2}{2g} < \frac{u^2+Rg}{3g} \Rightarrow 3u^2 < 2u^2 + 2Rg \Rightarrow u^2 < 2Rg \Rightarrow u < \sqrt{2Rg}$$

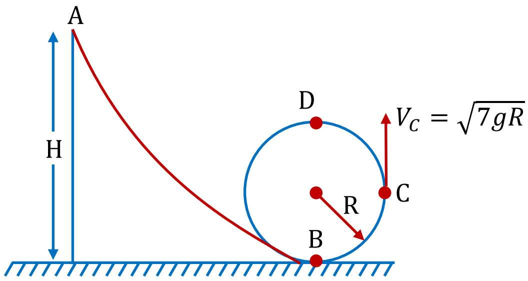
Moreover, if $h_1 = h_2$, $u = \sqrt{2Rg}$ and tension and velocity both becomes zero simultaneously. Further, from equation (iv), we can see that $h \leq R$ if $u \leq \sqrt{2Rg}$.

Thus, for $0 < u \leq \sqrt{2Rg}$, particle oscillates in lower half of the circle ($0^\circ < \theta \leq 90^\circ$).



This situation is shown in the figure. $0 < u \leq \sqrt{2gR}$ or $0^\circ < \theta \leq 90^\circ$.

Illustration 98:



Calculate following for shown situation: –

(a) Speed at D

(b) Normal reaction at D

(c) Height H

Solution:

(a) $v_D^2 = v_C^2 - 2gR = 5gR \quad \Rightarrow v_D = \sqrt{5gR}$

(b) $mg + N_D = \dfrac{mv_D^2}{R} \quad \Rightarrow N_D = \dfrac{m(5gR)}{R} - mg = 4mg$

(c) By energy conservation between point A and C

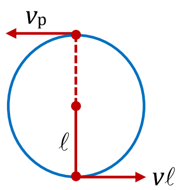
$$mgH = \frac{1}{2}mv_C^2 + mgR = \frac{1}{2}mv_D^2 + mg2R = \frac{1}{2}m(5gR) + mg2R = \frac{9}{2}mgR \Rightarrow H = \frac{9}{2}R$$

Illustration 99:

A stone of mass 1 kg tied to a light string of length $\ell = \dfrac{10}{3}\,m$ is whirling in a circular path in vertical plane. If the ratio of the maximum to minimum tension in the string is 4, find the speed of the stone at the lowest and highest points.

Solution:

$$\because \quad \frac{T_{max}}{T_{min}} = 4 \qquad \therefore \quad \frac{\frac{mv_\ell^2}{\ell}+mg}{\frac{mv_P^2}{\ell}-mg} = 4 \qquad \Rightarrow \frac{v_\ell^2+g\ell}{v_P^2-g\ell} = 4$$



We know $v_\ell^2 = v_p^2 + 4g\ell$

$\Rightarrow \quad \dfrac{v_P^2+5g\ell}{v_P^2-g\ell} = 4 \qquad \Rightarrow 3v_P^2 = 9g\ell$

$\Rightarrow \quad v_P = \sqrt{3g\ell} = \sqrt{3 \times 10 \times \dfrac{10}{3}} = 10\ ms^{-1}$

$\Rightarrow \quad v_\ell = \sqrt{7g\ell} = \sqrt{7 \times 10 \times \dfrac{10}{3}} = 15.2\ ms^{-1}$

$\Rightarrow \quad T = (mg + ma) + 2m(g + a)\,(1 - \cos\theta) = m(g + a)\,(3 - 2\cos\theta)$

Illustration 100:

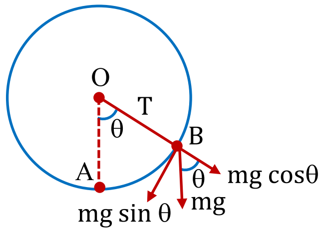
A heavy particle hanging from a fixed point by a light inextensible string of length ℓ, is projected horizontally with speed $\sqrt{(g\ell)}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of the motion when the tension in the string is equal to the weight of the particle.

Solution:

Let tension in the string becomes equal to the weight of the particle when particle reaches the point B and deflection of the string from vertical is θ. Resolving mg along the string and perpendicular to the string, we get net radial force on the particle at B i.e.

$$F_R = T - mg\cos\theta \quad ...(i)$$

If v_B be the speed of the particle at B, then



$$F_R = \frac{mv_B^2}{\ell} \quad ...(ii)$$

From (i) and (ii), we get,

$$T - mg\cos\theta = \frac{mv_B^2}{\ell} \quad ...(iii)$$

Since at B, $T = mg$; $mg\ (1-\cos\theta) = \dfrac{mv_B^2}{\ell}$

$$\Rightarrow \quad v_B^2 = g\ell(1-\cos\theta) \quad ...(iv)$$

Applying conservation of mechanical energy of the particle at point A and B, we have

$$\frac{1}{2}mv_A^2 = mg\ell\ (1-\cos\theta) + \frac{1}{2}mv_B^2\ ;\ \text{where } v_A = \sqrt{g\ell} \text{ and } v_B = \sqrt{g\ell(1-\cos\theta)}$$

$$\Rightarrow \quad g\ell = 2g\ell\ (1-\cos\theta) + g\ell\ (1-\cos\theta)$$

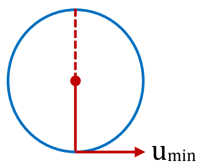
$$\Rightarrow \quad \cos\theta = \frac{2}{3} \quad \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Putting the value of cosθ in equation (iv), we get,

$$v_B = \sqrt{\frac{g\ell}{3}}$$

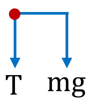
Illustration 101:

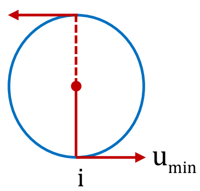
Find $u_{min.}$ for complete circle ?



Solution:

At top





$$\underset{0}{\underset{\downarrow}{T}} + mg = \frac{mv^2}{L}$$

Work–energy theorem

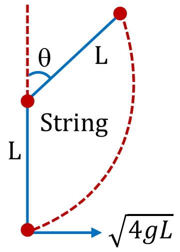
$$-2mgL + 0 = \frac{1}{2}mgL - \frac{1}{2}m \times u^2$$

$$u_{min} = \sqrt{5gL}$$

u at bottom $\geq \sqrt{5gL}$ in case of string then complete circle.

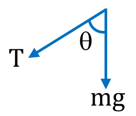
Illustration 102:

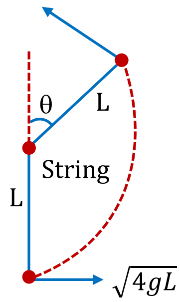
Find θ at which string will slack and maximum height from bottom most point??



Solution:

$$\underset{0}{\underset{\downarrow}{T}} + mg\cos\theta = \frac{mv^2}{L}$$





$$\frac{mg\cos\theta L}{m} = v^2$$

$$v = \sqrt{gL\cos\theta}$$

W–E theorem

$$W_g + W_T = K_f - K_i$$

$$-mgL(1 + \cos\theta) + 0 = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$-2gL(1 + \cos\theta) = v^2 - u^2$$

$$-gL - 2gL\cos\theta = gL\cos\theta - 4gL$$

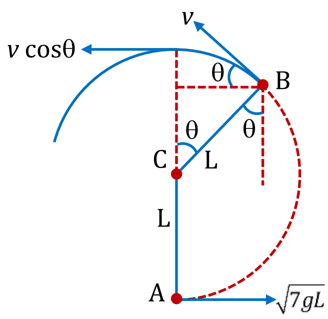
$$-2 - 2\cos\theta = \cos\theta - 4$$

$$2 = 3\cos\theta$$

∴ $\cos\theta = \frac{2}{3}$

Maximum height from bottom most point.

$$v\cos\theta = \sqrt{Rg\frac{2}{3}} \times \frac{2}{3}$$



$$W_T + W_g = K_f - K_i$$

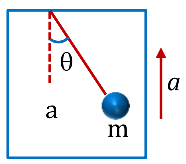
$$0 + (-mgH) = \frac{1}{2} \times m \times L \times g \times \frac{2}{3} \times \frac{4}{9} - \frac{1}{2} \times m \times 4gL$$

$$H = \frac{-4L}{27} + 2L$$

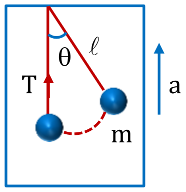
$$H = \frac{50L}{27}$$

Illustration 103:

A particle of mass m is attached to the ceiling of a cabin with an inextensible light string of length ℓ. The cabin is moving upward with an acceleration $'a'$. The particle is taken to a position such that the string makes an angle θ with vertical. When string becomes vertical, find the tension in the string.



Solution:

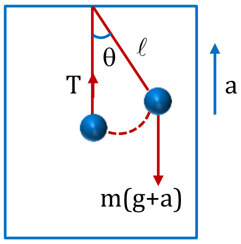


In a frame associated with cabin work done on the particle when it comes in the vertical position $mg\ell(1-\cos\theta)$ = $+ma(1-\cos\theta)$

By work energy theorem,

$$\frac{mv^2}{2} = \left(mg\ell + ma\ell\right)(1-\cos\theta)$$

$$\Rightarrow \quad \frac{v^2}{2} = (g+a)\,\ell\,(1-\cos\theta)$$



At vertical position, $T - (mg + ma) = \dfrac{mv^2}{\ell}$

$$\Rightarrow \quad T = (mg+ma) + 2m(g+a)\,(1-\cos\theta) = mg\,(g+a)\,(3-2\cos\theta)$$

Illustration 104:

The kinetic energy of a particle continuously increase with time. Then

(A) the magnitude of its linear momentum is increasing continuously.

(B) its height above the ground must continuously decrease.

(C) the work done by all forces acting on the particle must be positive.

(D) the resultant force on the particle must be parallel to the velocity at all times.

Ans. (AC)

Solution:

For (A) : $p = \sqrt{2mK}$ if K then p

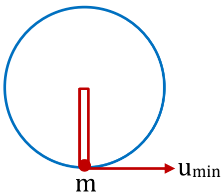
For (B) : Its height may or ↓

For (C) : $W = \Delta K$ if ΔK = positive then W = positive

For (D) : The resultant force on the particle must be at an angle less than 90° all times.

Illustration 105:

A massless rod having a particle of mass 'm' attached to its end as shown in figure. Find $u_{min.}$ for complete circular motion ?



Solution:

$$-2mgL + 0 = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$-2mgL = -\frac{1}{2}m \times u^2$$

$$u = \sqrt{4gL}$$

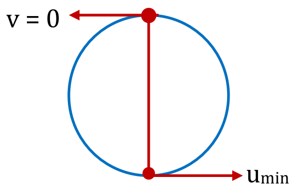
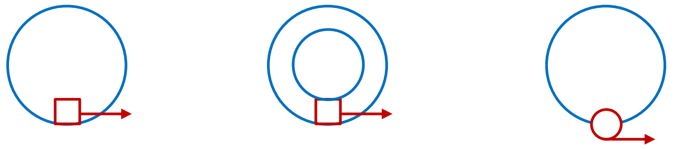


Illustration 106:

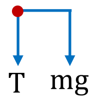
Find $u_{min.}$ for complete circular motion?



$u_{min} = \sqrt{5gL}$ (A) $u_{min} = \sqrt{4gL}$ (B) $u_{min} = \sqrt{4gL}$ (C)

Solution:

(A) At top



$$\underset{\substack{\downarrow \\ 0}}{T} + mg = \frac{mv^2}{L}$$

$$v_{min.} - \sqrt{gL}$$

Work–energy theorem

$$-2mgL + 0 = \frac{1}{2}mgL - \frac{1}{2}m \times u^2$$

$$u_{min} = \sqrt{5gL}$$

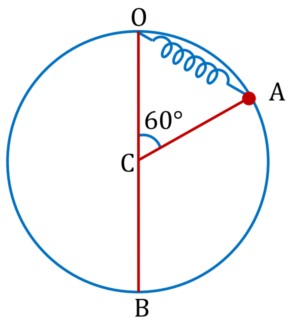
u at bottom $\geq \sqrt{5gL}$ in case of string then complete circle.

(B/C) From energy conservation :

$\frac{1}{2}mu^2 = mg(2L)$ (Block is only needed to reach to the top most point for complete circular motion.)

$$u = \sqrt{4gL}$$

Illustration 107:



A particle of mass 5 *kg* is free to slide on a smooth ring of radius r = 20 *cm* fixed in a vertical plane. The particle is attached to one end of a spring whose other end is fixed to the top point O of the ring. Initially the particle is at rest at a point A of the ring such that $OCA = 60^0$, C being the centre of the ring. The natural length of the spring is also equal to r = 20 *cm*. After the particle is released and slides down the ring the contact force between the particle and the ring becomes zero when it reaches the lowest position B. Determine the force constant of the spring.

Solution:

COME : $K_1 + U_1 = K_2 + U_2$

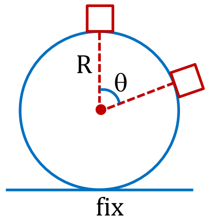
$$\frac{3mgr}{2} = \frac{1}{2}mv^2 + \frac{1}{2}kr^2 \qquad \text{...(i)}$$

Force equation, $kr = mg + \frac{mv^2}{r}$

Solving we get, $k = \frac{2mg}{r} = 500\ N/m$

Illustration 108:

A block from the top of a smooth sphere is released. Find θ at which block will leave contact?



Solution:

$$mg\cos\theta - \underset{\underset{0}{\downarrow}}{N} = \frac{mv^2}{R}$$

W–E theorem,

$$W_g + W_N = K_f - K_i$$

$$mgR - mgR\cos\theta = \frac{1}{2}\times m\times Rg\cos\theta$$

$$mRg(1-\cos\theta) = \frac{1}{2}\times mgR\cos\theta$$

$$1-\cos\theta = \frac{\cos\theta}{2}$$

$$1 = \frac{3\cos\theta}{2}$$

$$\cos\theta = \frac{2}{3}$$

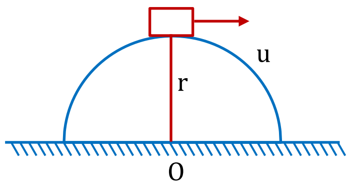
Illustration 109:

A small block of mass m is projected horizontally from the top of the smooth hemisphere of radius r with speed u as shown. For values of $u \geq u_0$, it does not slide on the hemisphere (i.e. leaves the surface at the top itself).

(i) For $u = 2u_0$, it lands at point P on the ground Find OP.

(ii) For $u = u_0/3$, find the height from the ground at which it leaves the hemisphere.

(iii) Find its net acceleration at the instant it leaves the hemisphere.



Solution:

For speed u_0, contact at top is lost.

$\Rightarrow \quad N + \dfrac{mu_0^2}{r} = mg \Rightarrow (N = 0)\ u_0 = \sqrt{gr}$

(a) For vertical motion; $t = \sqrt{\dfrac{2r}{g}}$

∴ Horizontal distance,

$$s = 2u_0.t = 2\sqrt{gr}\times\sqrt{\frac{2r}{g}} = 2\sqrt{2}r$$

(b) COME : $\dfrac{1}{2}\dfrac{m(u_0)^2}{3} + mgr = \dfrac{1}{2}mv^2 + mgr\cos\theta$ ...(i)

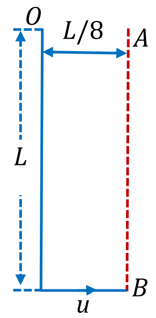
Force equation : $N + \dfrac{mv^2}{r} = mg\cos\theta$ ...(ii)

∴ $h = r\cos\theta = \dfrac{19}{27}r$

(c) $|\vec{a}_{net}| = |\vec{a}_r + \vec{a}_t| = \sqrt{(g\sin\theta)^2 + (g\cos\theta)^2} = g$

Illustration 110:

A particle is suspended vertically from a point O by an inextensible massless string of length L. A vertical line AB is at a distance $\frac{L}{8}$ from O as shown in figure. The object is given a horizontal velocity u. At some point, its motion ceases to be circular and eventually the object passes through the line AB. At the instant of crossing AB, its velocity is horizontal. Find u.



Solution:

COME : $\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgL(1 + \sin\theta)$ ...(i)

For equation $\Rightarrow T + mg\sin\theta = \frac{mv^2}{L}$ ...(ii)

Since the particle crosses the $\frac{L}{8}$

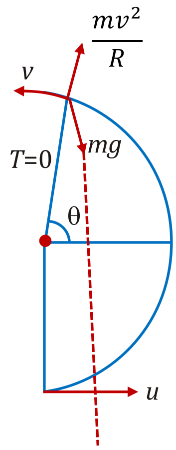
Line at its half of its range

$\therefore \quad \frac{v^2 \sin\theta . \cos\theta}{g} = L\cos\theta - \frac{L}{8}$ ...(iii)

$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$

From equation (i)

$$u = \sqrt{gL\left(2 + \frac{3\sqrt{3}}{2}\right)}$$



Work Energy Theorem for System of Particle:

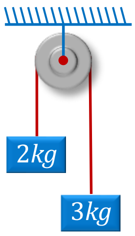
$(W_{net})_{sys} = (KE_f)_{sys} - (KE_i)_{sys}$

Work by F on system = $\sum W_F$

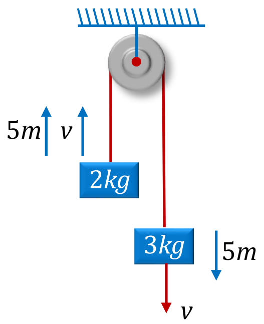
KE of system = $\sum KE$ of each particle of system.

Illustration 111:

Released from rest. Find speed of each, after $3kg$ goes $\downarrow 5m$?



Solution:



(2 + 3)

$(W_g)_{sys} \quad + \quad (W_T)_{sys} \quad = \quad (KE_f)_{sys} \quad - \quad (KE_i)_{sys}$

$W_{g3} \quad W_{g2} \quad W_{T2} \quad W_{T3} \quad = \quad KE_{f3} \quad + \quad KE_{f3} \quad - \quad KE_{3i} \quad + \quad KE_{2i}$

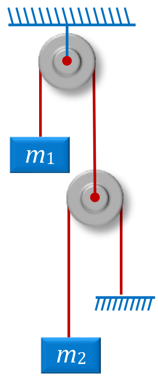
$$+(-100) \quad + \quad 0 \quad = \frac{1}{2} \times 3v^2 + \frac{1}{2} \times 2v^2 - 0$$

$$50 = \frac{5v^2}{2}$$

$$v = \sqrt{20} m/s$$

Illustration 112:

System is released from rest. Find velocity of block m_2 when it move by h?



Solution:

$$Th(-1) + 2Tx(1) = 0$$

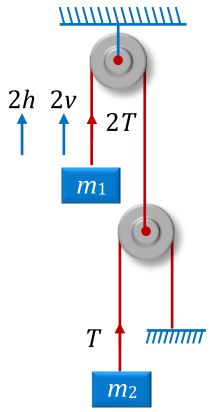
$$x = \frac{h}{2}$$

$$(W_g)_{Sys.} = m_2gh - m_1g\frac{h}{2}$$

$$(KE)_{Sys.} = \frac{1}{2} \times m_2 \times v^2 + \frac{1}{2} \times m \times \left(\frac{v}{2}\right)^2$$

$$(2m_2 - m_1)g\frac{h}{2} = \frac{v^2}{2}\left(\frac{4m_2 + m_1}{4}\right)$$

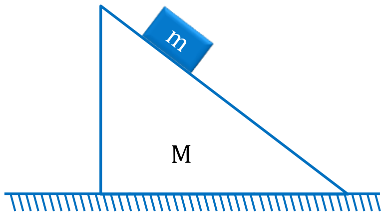
$$v = \sqrt{\left(\frac{8m_2 - 4m_1}{4m_2 + m_1}\right)gh}$$



- Total work done by internal force on the system is independent of reference frame.
- If distance between particles in a system along internal force remains unchanged always then work done by internal force on the system is 'zero'.

Illustration 113:

A block of mass m lies on wedge of mass M. The wedge in turn lies on smooth horizontal surface. Friction is absent everywhere. The wedge block system is released from rest. All situation given in column–I are to be estimated in duration the block undergoes a vertical displacement *'h'* starting from rest (assume the block to be still on the wedge, g is acceleration due to gravity).



	Column I		Column II
(A)	Work done by normal reaction acting on the block is	(p)	Positive
(B)	Work done by normal reaction (exerted by block) acting on wedge is	(q)	Negative
(C)	The sum of work done by normal reaction on block and work done by normal reaction (exerted by block) on wedge is	(r)	Zero
(D)	Net work done by all forces on block is	(s)	Less than mgh in magnitude

Ans. (A→Q,S), (B→P,S), (C→R), (D→P,S)

Solution:

By applying conservation of momentum wedge will acquire some velocity $= -\frac{mv_x}{M+m}$ where v_x is velocity of block w.r.t wedge in negative x-direction.

(A) Work done by normal on block is $= -\frac{1}{2}M\left(\frac{mv_x}{M+m}\right)^2$

(B) Work done by normal on wedge is $= \frac{1}{2}M\left(\frac{mv_x}{M+m}\right)^2$ is positive.

(C) Net work done by normal is = 0

(D) Less than mgh as $K.E.$ is $< \frac{1}{2}m2gh$,

$KE_f > KE$ is positive.