01 Center of Mass and Collision

Introduction:

Every physical system has a certain point associated with it whose motion characterises the motion of the whole system. When the system moves under some external forces, then this point moves as if the entire mass of the system is concentrated at this point and also the external force is applied at this point for translational motion. This point is called the center of mass of the system.

COM is an imaginary point, which may or may not be located on the system, and in **many (not always)** cases of mechanics, whole mass can be assumed to be concentrated on it. If external force is applied at Center of Mass then only translational motion is possible not rotational.

Center of Mass of a System of 'N' Discrete Particles

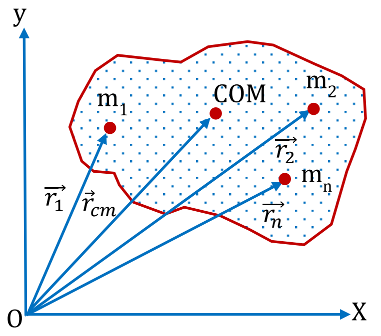
Consider a system of N point masses $m_1, m_2, m_3, ..., m_n$ whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...$, respectively. Then the position vector of the center of mass C of the system is given by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n} \; ; \; \vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i}$$

$$\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

where $m_i\vec{r}_i$ is called the moment of mass of the particle w.r.t O.

$M = (\sum_{i=1}^{n} m_i)$ is the total mass of the system.

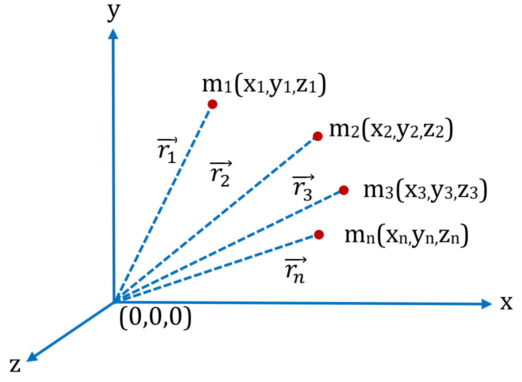


Note:

If the origin is taken at the center of mass then $\sum_{i=1}^{n} m_i\vec{r}_i = 0$. Hence, the COM is the point about which the sum of "mass moments" of the system is zero.

In cartesian coordinate system:

If co-ordinates of particles of mass $m_1, m_2, ...,$ are $(x_1, y_1, z_1), (x_2, y_2, z_2)...$



then position vector of their centre of mass is

$$\vec{R}_{CM} = x_{cm}\,\hat{i} + y_{cm}\,\hat{j} + z_{cm}\,\hat{k}$$

$$= \frac{m_1(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+m_2(x_2\hat{i}+y_2\hat{j}+z_2\hat{k})+m_3(x_3\hat{i}+y_3\hat{j}+z_3\hat{k})+...}{m_1+m_2+m_3+...}$$

$$= \frac{(m_1x_1+m_2x_2+....)\hat{i}+(m_1y_1+m_2y_2...)\hat{j}+(m_1z_1+m_2z_2+..)\hat{k}}{m_1+m_2+m_3+\cdots}$$

So, $x_{cm} = \left(\dfrac{m_1x_1+m_2x_2+.......}{m_1+m_2+......}\right)$, $y_{cm} = \left(\dfrac{m_1y_1+m_2y_2+.......}{m_1+m_2+..........}\right)$, $z_{cm} = \left(\dfrac{m_1z_1+m_2z_2+.......}{m_1+m_2+.........}\right)$

$x_{cm} = \dfrac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$, $y_{cm} = \dfrac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$, $z_{cm} = \dfrac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$

Illustration 1:

Four particles of mass 1 *kg*, 2 *kg*, 3 *kg* and 4 *kg* are placed at the four vertices *A*, *B*, *C* and *D* of a square of side 1 *m*. Find the position of center of mass of the particles.

Solution:

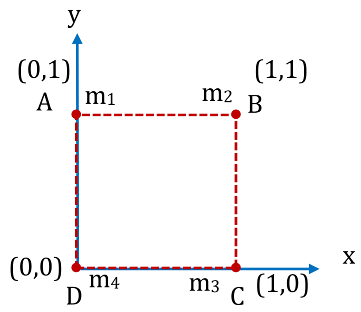
Assuming *D* as the origin, *DC* as *x*-axis and *DA* as *y*-axis, we have

$m_1 = 1\ kg$, $(x_1, y_1) = (0, 1m)$

$m_2 = 2\ kg$, $(x_2, y_2) = (1m, 1m)$

$m_3 = 3\ kg$, $(x_3, y_3) = (1m, 0)$

and $m_4 = 4\ kg$, $(x_4, y_4) = (0, 0)$



Co-ordinates of their COM are

$$x_{CM} = \frac{m_1x_1+m_2x_2+m_3m_3+m_4x_4}{m_1+m_2+m_3+m_4}$$

$$= \frac{(1)(0)+2(1)+3(1)+4(0)}{1+2+3+4} = \frac{5}{10} = 0.5\ m$$

Similarly,

$$y_{CM} = \frac{m_1y_1+m_2y_2+m_3y_3+m_4y_4}{m_1+m_2+m_3+m_4}$$

$$= \frac{(1)(1)+2(1)+3(0)+4(0)}{1+2+3+4} = \frac{3}{10} = 0.3\ m$$

$\therefore$ $(x_{CM}, y_{CM}) = (0.5\ m, 0.3\ m)$

Thus, position of CM of the four particles is as shown in figure.

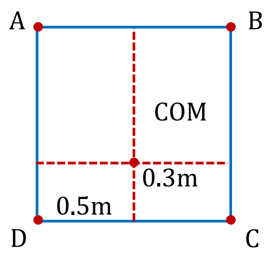
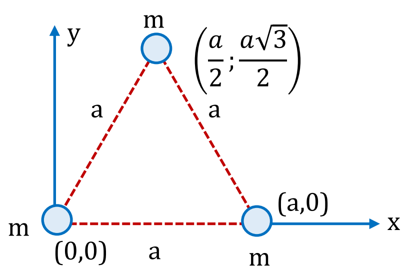


Illustration 2:

Three bodies of equal masses are placed at vertices of an equilateral triangle of side length *a*, (as shown in figure). Find out the co-ordinates of centre of mass.



Solution:

$$x_{CM} = \frac{0\times m + a\times m + \frac{a}{2}\times m}{m+m+m} = \frac{a}{2}$$

$$y_{CM} = \frac{0\times m + 0\times m + \frac{a\sqrt{3}}{2}\times m}{m+m+m} = \frac{a\sqrt{3}}{6}$$

Position of COM of Two Particles

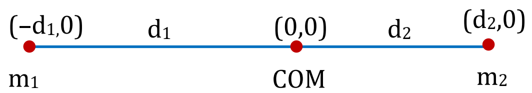
Center of mass of two particles of masses m_1 and m_2 separated by a distance r lies in between the two particles. The distance of center of mass from any of the particle (r) is inversely proportional to the mass of the particle (m)

or

COM of two particles divides internally the line joining two particles in inverse ratio of their masses.

Proof: $x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$

$$0 = \frac{m_1(-d_1) + m_2(d_2)}{m_1+m_2}$$



$m_1d_2 = m_2d_2$

$$\frac{d_1}{d_2} = \frac{m_2}{m_1}$$

$$d \propto \frac{1}{m}$$

Here, d_1 = distance of COM from m_1

and d_2 = distance of COM from m_2

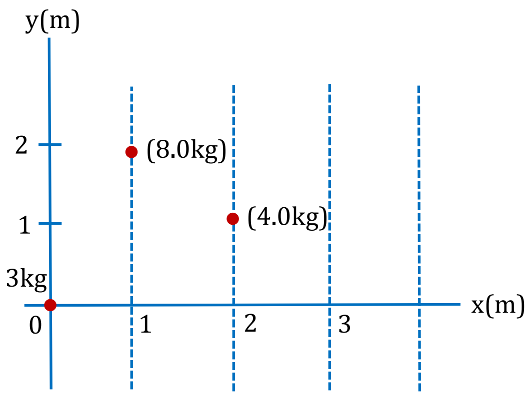
From the above discussion, we see that

$d_1 = d_2 = 1/2$ if $m_1 = m_2$, i.e., COM lies midway between the two particles of equal masses.

Similarly, $d_1 > d_2$ if $m_1 < m_2$ and $d_1 < d_2$ if $m_2 < m_1$ i.e. COM is always nearer to the particle having larger mass.

Illustration 3:

What are the co–ordinates of the centre of mass of the three particles system shown in figure.



Ans. 1.1*m*, 1.3 *m*

Solution:

$$X_{CM} = \frac{3\times0+8\times1+4\times2}{3+8+4} = \frac{16}{15} = 1.1m$$

$$Y_{CM} = \frac{3\times0+8\times2+4\times1}{3+8+4} = \frac{20}{15} = 1.3m$$

Illustration 4:

The position vector of three particles of masses $m_1 = 1\ kg$, $m_2 = 2\ kg$ and $m_3 = 3kg$ are $\vec{r}_1 = (\hat{\imath} + 4\hat{\jmath} + \hat{k})m$, $\vec{r}_2 = (\hat{\imath} + \hat{\jmath} + \hat{k})m$ and $\vec{r}_3 = (2\hat{\imath} - \hat{\jmath} - 2\hat{k})m$ respectively. Find the position vector of their center of mass.

Solution:

The position vector of COM of the three particles will be given by $\vec{r}_{COM} = \frac{m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3}{m_1+m_2+m_3}$

Substituting the values, we get

$$\vec{r}_{CM} = \frac{(1)(\hat{\imath}+4\hat{\jmath}+\hat{k})+(2)(\hat{\imath}+\hat{\jmath}+\hat{k})+(3)(2\hat{\imath}-\hat{\jmath}-2\hat{k})}{1+2+3} = \frac{1}{2}(3\hat{\imath} + \hat{\jmath} - \hat{k})m$$

Center of Mass of a Continuous Mass Distribution

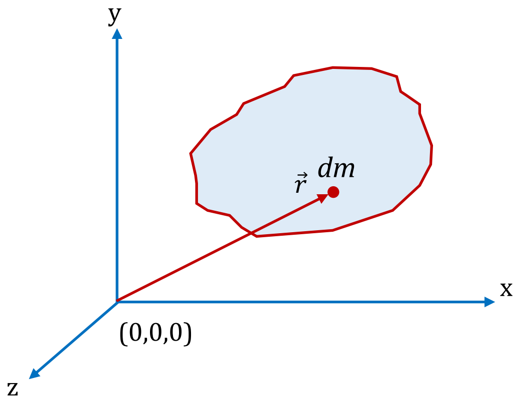
For continuous mass distribution the center of mass can be located by replacing summation sign with an integral sign. Proper limits for the integral are chosen according to the situation

$$x_{cm} = \frac{\int xdm}{\int dm}, y_{cm} = \frac{\int ydm}{\int dm}, z_{cm} = \frac{\int zdm}{\int dm}$$

$\int dm = M$ (mass of the body)

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}dm$$

x, y and z are the coordinates of center of mass of dm mass.



Note:

(i) If an object has symmetric mass distribution about x axis then y coordinate of COM is zero and vice-versa.

(ii) Center of mass of uniform bodies lies on their geometric centre's.

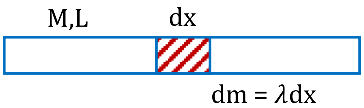
Types of mass distributions:

Linear mass density (λ):

Mass per unit length = λ

If uniform then $\lambda = \frac{M}{L}$

Generally defined for rod, ring, arc etc.

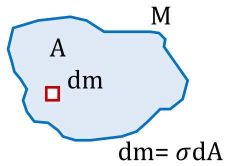


Surface mass density (σ):

Mass per unit area = σ

If uniform then $\sigma = \frac{M}{A}$

Generally defined for Lamina's like triangular plate, rectangular plate, disc, square plate etc.

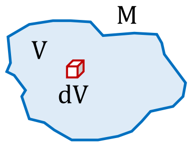


Volume mass density (ρ):

Mass per unit length = ρ

If uniform then $\rho = \frac{M}{V}$

Generally defined for solid sphere, solid cylinder etc.



Center of Mass of a Uniform Rod

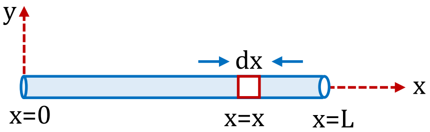
Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$. Mass per unit length of the rod = $\frac{M}{L}$.

Hence dm (the mass of the element dx situated at $x = x$ is) = $\frac{M}{L}dx$.

The coordinates of the element dx are (x, 0, 0). Therefore, x-coordinate of COM of the rod will be

$$x_{CM} = \frac{\int_0^L x dm}{\int dm} = \frac{\int_0^L (x)\left(\frac{M}{L}dx\right)}{M} = \frac{1}{L}\int_0^L x dx = \frac{L}{2}$$

The y-coordinate of COM is $y_{CM} = \frac{\int y dm}{\int dm} = 0$. Similarly, $z_{CM} = 0$



i.e. the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$ i.e. it lies at the center of the rod.

Illustration 5:

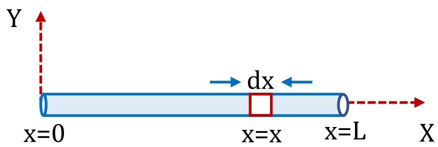
A rod of length L is placed along the x-axis between $x = 0$ and $x = L$. The linear density (mass/length) λ of the rod varies with the distance x from the origin as $\lambda = Rx$. Here, R is a positive constant. Find the position of center of mass of this rod.

Solution:

Mass of element dx situated at $x = x$ is $dm = \lambda\, dx = Rx\, dx$

The COM of the element has coordinates (x, 0, 0)

Therefore, x-coordinate of COM of the rod will be



$$x_{COM} = \frac{\int_0^L x dm}{\int dm} = \frac{\int_0^L (x)(Rx)dx}{\int_0^L (Rx)dx} = \frac{R\int_0^L x^2 dx}{R\int_0^L x dx} = \frac{\left[\frac{x^3}{3}\right]_0^L}{\left[\frac{x^2}{2}\right]_0^L} = \frac{2L}{3}$$

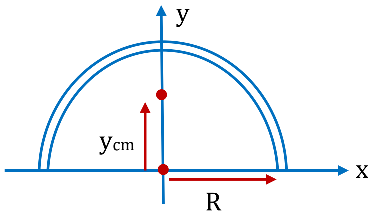
The y-coordinate of COM of the rod is $y_{COM} = \frac{\int y dm}{\int dm} = 0$ (as $y = 0$)

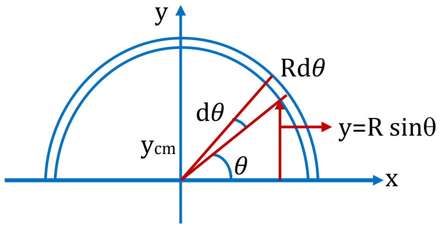
Similarly, $z_{COM} = 0$

Hence, the center of mass of the rod lies at $\left[\frac{2L}{3}, 0, 0\right]$.

Center of Mass of a Semicircular Ring

Figure shows the object (semi circular ring). By observation we can say that the x-coordinate of the center of mass of the ring is zero as the half ring is symmetrical about y-axis on both sides of the origin. We are only required to find the y-coordinate of the center of mass.





To find y_{cm} we use $y_{cm} = \frac{1}{M}\int dmy$ ...(i)

$$\lambda = \frac{M}{\pi R}$$

Here for dm we consider an elemental arc of the ring at an angle θ from the x-direction of angular width $d\theta$. If radius of the ring is R then its y coordinate will be R sinθ, here dm is given as

$$dm = \frac{M}{\pi R} \times R\, d\theta$$

So, from equation (i), we have

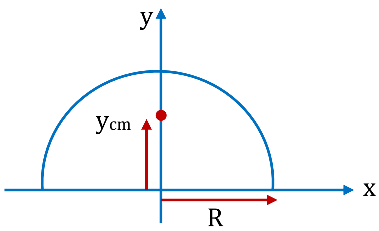
$$y_{cm} = \frac{1}{M}\int_0^{\pi} \frac{M}{\pi R} R d\theta\, (R \sin\theta) = \frac{R}{\pi}\int_0^{\pi} \sin\theta\, d\theta$$

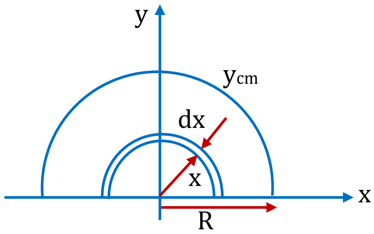
$$y_{cm} = \frac{2R}{\pi} \quad \text{...(ii)}$$

Center of Mass of Semicircular Disc

Figure shows the half disc of mass M and radius R. Here, we are only required to find the y-coordinate of the center of mass of this disc as center of mass will be located on its half vertical diameter. Here to find y_{cm}, we consider a small elemental ring of mass dm of radius x on the disc (disc can be considered to be made up of such thin rings of increasing radii) which will be integrated from 0 to R. Here dm is given as

$dm = \frac{2M}{\pi R^2}(\pi x)\, dx$ and $\sigma = \frac{2M}{\pi R^2}$





Now the y-coordinate of the COM of element is taken as $\frac{2x}{\pi}$, as in previous section, we have derived that the center of mass of a semi-circular ring is concentrated at $\frac{2R}{\pi}$.

Here y_{cm} of semi-circular disc is given as

$$y_{cm} = \frac{1}{M}\int_0^R dm \frac{2x}{\pi} = \frac{1}{M}\int_0^R \frac{4M}{\pi R^2} x^2 dx$$

$$y_{cm} = \frac{4R}{3\pi}$$

Center of Mass of a Solid Hemisphere

The hemisphere is of mass M and radius R. To find its center of mass (only y-coordinate), we consider an element disc of width dy, mass dm at a distance y from the center of the hemisphere. The radius of this elemental disc will be given as

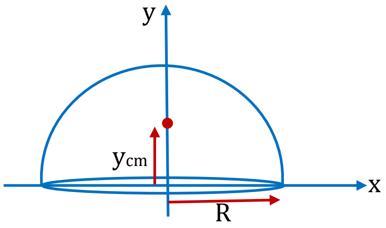
(r = radius of elemental disc)

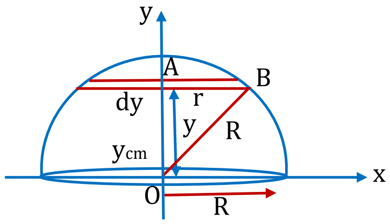
By ΔOAB

$$R^2 = y^2 + r^2$$

$$r^2 = R^2 - y^2$$

$$\rho = \frac{3M}{2\pi R^3}$$





The mass dm of this disc can be given as

$$dm = \frac{3M}{2\pi R^3} \times \pi r^2 dy = \frac{3M}{2R^3}(R^2 - y^2)dy$$

y_{cm} of the hemisphere is given as

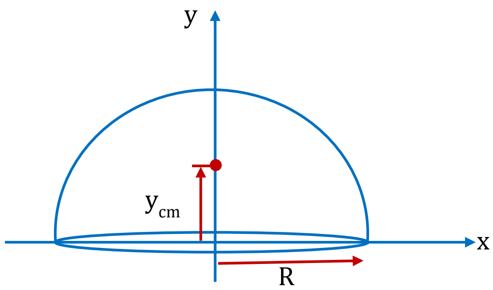
$$y_{cm} = \frac{1}{M}\int_0^R dmy = \frac{1}{M}\int_0^R \frac{3M}{2R^3}(R^2 - y^2)\, dy\, y = \frac{3}{2R^3}\int_0^R (R^2 - y^2)y\, dy$$

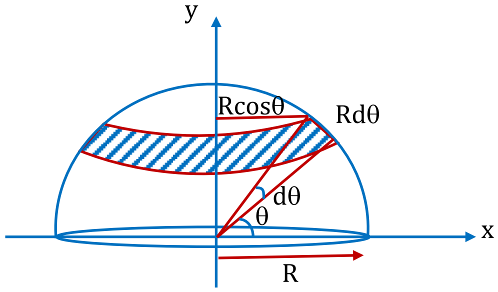
$$y_{cm} = \frac{3R}{8}$$

Center of Mass of a Hollow Hemisphere

A hollow hemisphere of mass M and radius R. Now we consider an elemental circular strip of angular width $d\theta$ at an angular distance θ from the base of the hemisphere. This strip will have an area. $\left(\sigma = \frac{M}{2\pi R^2}\right)$

$$dS = 2\pi R \cos\theta\, Rd\theta$$





Its mass dm is given as

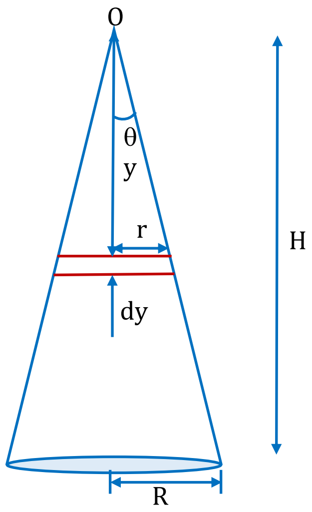
$$dm = \frac{M}{2\pi R^2} 2\pi R \cos\theta\, Rd\theta$$

Here y-coordinate of this strip of mass dm can be taken as $R \sin\theta$. Now we can obtain the centre of mass of the system as.

$$y_{cm} = \frac{1}{M}\int_0^{\frac{\pi}{2}} dmR\sin\theta = \frac{1}{M}\int_0^{\frac{\pi}{2}}\left(\frac{M}{2\pi R^2} 2\pi R^2 \cos\theta\, d\theta\right) R\sin\theta$$

$$= R\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \qquad \Rightarrow \quad y_{cm} = \frac{R}{2}$$

Center of Mass of a Solid Cone



A solid cone has mass M, height H and base radius R. Obviously the center of mass of this cone will lie somewhere on its axis, at a height less than $H/2$. To locate the center of mass we consider an elemental disc of width dy and radius r, at a distance y from the apex of the cone. Let the mass of this disc be dm, which can be given as $\left(\rho = \dfrac{M}{\frac{1}{3}\pi R^2 H}\right)$

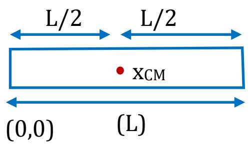
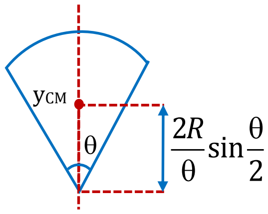
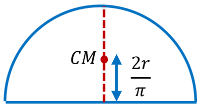
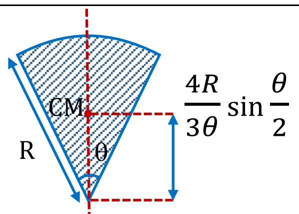
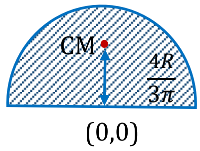
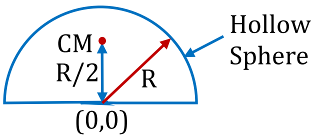
$$dm = \frac{3M}{\pi R^2 H} \times \pi r^2\, dy$$

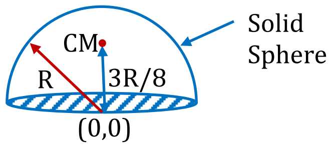
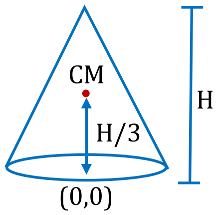
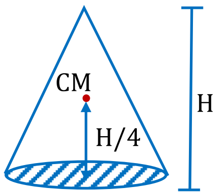
$$tan\theta = \frac{R}{H} = \frac{r}{y}$$

$$\left(r = \frac{Ry}{H}\right)$$

Here y_{cm} can be given as

$$y_{cm} = \frac{1}{M}\int_0^H y\,dm = \frac{1}{M}\int_0^H \left(\frac{3M}{\pi R^2 H}\pi\left(\frac{Ry}{H}\right)^2 dy\right) y = \frac{3}{H^3}\int_0^H y^3 dy = \frac{3H}{4}$$

S.No.	Bodies		Location of COM
1.	Rod		$\left[\frac{L}{2}, 0\right]$
2.	Circular Arc		$\left[0, \frac{2R}{\theta} sin\frac{\theta}{2}\right]$
3.	Semi-circular arc [Half-ring]		$\left[0, \frac{2R}{\pi}\right]$
4.	Disc		$\left[0, \frac{4R}{3\theta} sin\frac{\theta}{2}\right]$
5.	Semi-disc		$\left[0, \frac{4R}{3\pi}\right]$
6.	Hollow sphere		$\left[0, \frac{R}{2}\right]$

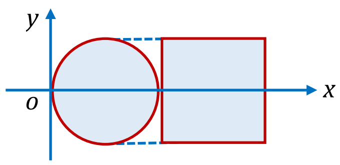
S.No.	Bodies		Location of COM
7.	Solid sphere		$\left[0, \frac{3R}{8}\right]$
8.	Hollow cone		$\left[0, \frac{H}{3}\right]$
9.	Solid cone		$\left[0, \frac{H}{4}\right]$

Mass Center of composite bodies

Illustration 6:

A composite body is made of joining two or more bodies. Find mass center of the following composite body made by joining a uniform disk of radius r and a uniform square plate of the same mass per unit area (σ) and side of square plate is '$2r$'.

Solution:



To find mass center, the component bodies are assumed as particles of masses equal to corresponding bodies located on their respective mass centers. Then we use equation to find coordinates of the mass center of the composite body.

To find mass center of the composite body, we first have to calculate masses of the bodies because their mass distribution is given.

If we denote surface mass density (mass per unit area) by σ, masses of the bodies are

Mass of the disk, m_d = Mass per unit area $\times$ Area $= \sigma\pi r^2$

Mass of the square plate, m_p = Mass per unit area $\times$ Area $= \sigma(4r^2) = 4\sigma r^2$

Location of mass center of the disk, x_d = Center of the disk $= r$ and $y_d = 0$

Location of mass center of the square plate, x_p = Center of the surface of the plate $= 3r$ and $y_p = 0$

Using equation, we obtain coordinates (x_c, y_c) of the composite body.

$$x_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} = \frac{r(\pi+12)}{(\pi+4)} \quad \text{and} \quad y_c = \frac{m_d y_d + m_s y_s}{m_d + m_s} = 0$$

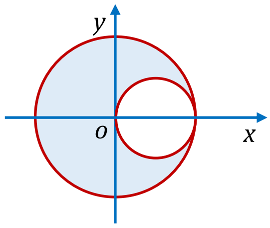
Coordinates of the mass center are $\left(\frac{r(\pi+12)}{(\pi+4)}, 0\right)$.

Mass Center of truncated bodies (Cavity Problems)

Illustration 7:

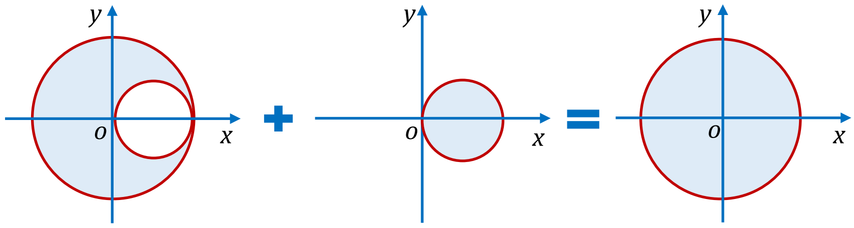
A truncated body is made by removing a portion of a body.

Find mass center of the following truncated disk made by removing disk of radius equal to half of the original disk as shown in the figure. Radius of the original uniform disk is r. (Mass Density = σ)



Solution:

To find mass center of truncated bodies we can make use of superposition principle that is, if we add the removed portion in the same place we obtain the original body. The idea is illustrated in the following figure.



The removed portion is added to the truncated body keeping their location unchanged relative to the coordinate frame.

Denoting masses of the truncated body, removed portion and original body by m_{tb}, m_{rp} and m_{ob} and location of their mass centers by x_{tb}, x_{rp} and x_{ob}, we can write $m_{tb}x_{tb} + m_{rb}x_{rp} = m_{ob}x_{ob}$

From the above equation we obtain position co-ordinate x_{tb} of the mass center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rb}x_{rp}}{m_{tb}} \quad \text{...(1)}$$

Denoting mass per unit area by σ, we can express the masses m_{tb}, m_{rp} and m_{ob}.

Mass of truncated body, $m_{tb} = \sigma\left\{\pi\left(r^2 - \frac{r^2}{4}\right)\right\} = \frac{3\sigma\pi r^2}{4}$

Mass of the removed portion, $m_{rp} = \frac{\sigma\pi r^2}{4}$

Mass of the original body, $m_{ob} = \sigma\pi r^2$

Mass center of the truncated body, x_{tb}

Mass center of the removed portion, $x_{rp} = \frac{r}{2}$

Mass center of the original body, $x_{ob} = 0$

Substituting the above values in equation (1), we obtain the mass the center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rb}x_{rp}}{m_{tb}} = \frac{(\sigma\pi r^2)\times 0 - \left(\frac{\sigma\pi r^2}{4}\right)\left(\frac{r}{2}\right)}{\frac{3\sigma\pi r^2}{4}} = -\frac{r}{6}$$

Mass center of the truncated body is at point $x_{tb} = -\frac{r}{6}$.

Motion of Center of Mass and Conservation of Linear Momentum

Displacement of COM due to Displacement of Particles of System:

$$\vec{r}_{cm}=\frac{m_1\vec{r}_1+m_2\vec{r}_2+.....}{m_1+m_2+.....}$$

$$d\vec{r}_{cm}=\frac{m_1d\vec{r}_1+m_2d\vec{r}_2+.....}{m_1+m_2+.....}$$

Displacement of COM $\boxed{\Delta\vec{r}_{cm}=\frac{m_1\Delta\vec{r}_1+m_2\Delta\vec{r}_2+.....}{m_1+m_2+.....}}$

Velocity of Center of Mass of system

$$M\vec{v}_{cm}=m_1\frac{d\vec{r}_1}{dt}+m_2\frac{d\vec{r}_2}{dt}+m_3\frac{d\vec{r}_3}{dt}+\ldots+m_n\frac{d\vec{r}_n}{dt}=m_1\vec{v}_1+m_2\vec{v}_2+m_3\vec{v}_3+\ldots+m_n\vec{v}_n$$

Here numerator of the right hand side term is the total momentum of the system i.e., summation of momentum of the individual component (particle) of the system

Hence velocity of center of mass of the system is the ratio of momentum of the system to the mass of the system.

$\therefore\quad \vec{P}_{system}=M\vec{v}_{cm}=M\vec{v}_{cm}$

Acceleration of Center of Mass of system

$$M\vec{a}_{cm}=m_1\frac{d\vec{v}_1}{dt}+m_2\frac{d\vec{v}_2}{dt}+m_3\frac{d\vec{v}_3}{dt}+\ldots+m_n\frac{d\vec{v}_n}{dt}=m_1\vec{a}_1+m_2\vec{a}_2+m_3\vec{a}_3+\ldots+m_n\vec{a}_n$$

$$\Rightarrow\quad \vec{a}_{cm}=\frac{\text{Net force on system}}{M}=\frac{\text{Net Exeternal Force + Net internal Force}}{M}=\frac{\text{Net External Force}}{M}$$

($\because$ Action and reaction both of an internal force must be within the system. Vector summation will cancel all internal forces and hence net internal force on system is zero)

$\therefore\quad \vec{F}_{ext}=M\vec{a}_{cm}$

where $\vec{F}_{ext}$ is the sum of the 'external' forces acting on the system. The internal forces which the particles exert on one another play absolutely no role in the motion of the center of mass.

If no external force is acting on a system of particles, the acceleration of center of mass of the system will be zero. If a_{cm} = 0, it implies that v_{cm} must be a constant and if v_{cm} is a constant, it implies that the total momentum of the system must remain constant. It leads to the principal of conservation of momentum in absence of external forces.

If $\quad \vec{F}_{ext}=0$ then $\vec{v}_{cm}=$ constant

"If resultant external force is zero on the system, then the net momentum of the system must remain constant".

Motion of COM in a moving system of particles :

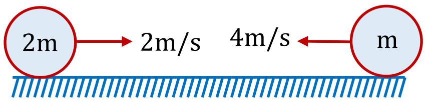
(1) COM at rest :

If F_{ext} = 0 and V_{cm} = 0, then COM remains at rest. Individual components of the system may move and have non-zero momentum due to mutual forces (internal), but the net momentum of the system remains zero.

(i) All the particles of the system are at rest.

(ii) Particles are moving such that their net momentum is zero.

Example:



(iii) A bomb at rest suddenly explodes into various smaller fragments, all moving in different directions, then, since the explosive forces are internal and there is no external force on the system for explosion, therefore, the COM of the bomb will remain at the original position and the fragment fly such that their net momentum remains zero.

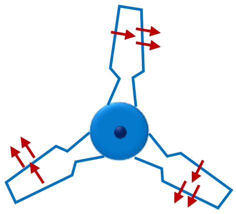
(iv) Two men standing on a frictionless platform, push each other, then their net momentum remains zero because the push forces are internal for the two men system.

(v) A boat floating in a lake also has net momentum zero if the people on it changes their position, because the friction force required to move the people is internal in the boat system.

(vi) Objects initially at rest, if move under mutual forces (electrostatic or gravitation) also have net momentum zero.

(vii) A light spring of spring constant k kept compressed between two blocks of masses m_1 and m_2 on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions such that the net momentum is zero.

(viii) In a fan, all particles are moving but COM is at rest



(2) COM moving with uniform velocity :

If F_{ext} = 0, then V_{cm} remains constant, therefore, net momentum of the system also remains conserved. Individual components of the system may have variable velocities and momentum due to mutual forces (internal), but the net momentum of the system remains constant and COM continues to move with the initial velocity.

(i) All the particles of the system are moving with same velocity.

E.g.: A car moving with uniform speed on a straight road, has its COM moving with a constant velocity.

(ii) Internal explosions / breaking does not change the motion of COM and net momentum remains conserved. A bomb moving in a straight line suddenly explodes into various smaller fragments, all moving in different directions then, since the explosive forces are internal and there is no external force on the system for explosion therefore, the COM of the bomb will continue the original motion and the fragments fly such that their net momentum remains conserved.

(iii) Man jumping from cart or buggy also exert internal forces therefore net momentum of the system does not change and hence, motion of COM remains conserved.

(iv) Two moving blocks connected by a light spring on a smooth horizontal surface. If the acting forces are only due to spring then COM will remain in its motion and momentum will remain conserved.

(v) Particles colliding in absence of external impulsive forces also have their momentum conserved.

(3) COM moving with acceleration :

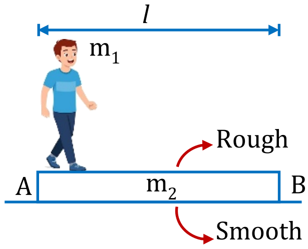
If an external force is present then COM continues its original motion as if the external force is acting on it, irrespective of internal forces.

Example:

Projectile motion : An axe thrown in air at an angle θ with the horizontal will perform a complicated motion of rotation as well as parabolic motion under the effect of gravitation.

Illustration 8:

If man walks from A to B find displacement of man and plank.



Ans. $S_2 = \dfrac{-m_1}{m_1+m_2}\ell$

Solution:

Initial momentum of system man and plank is zero. Net ext. force on this system is zero.

Thus, $\vec{P}_{sys} = 0 \quad \Rightarrow \quad \vec{v}_{cm} = 0 \quad \Rightarrow \quad \Delta\vec{s}_{cm} = 0$

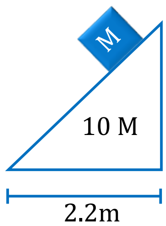
If the man moves towards right, the plank will move towards left by a distance (X) (taking the two blocks together as the system).

The horizontal position of CM remains same $\Rightarrow m_1(\ell - x) = m_2 x$

$$x = \frac{m_1\ell}{m_1+m_2} \text{ (left)}$$

Illustration 9:

A block of mass M is placed on the top of a bigger block of mass 10 M as shown in figure. All the surfaces are frictionless. The system is released from rest. Find the distance moved by the bigger block at the instant the smaller block reaches the ground.



Solution:

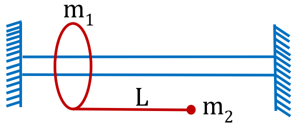
If the bigger block moves towards right by a distance (X), the smaller block will move towards left by a distance (2.2 –X) (taking the two blocks together as the system).

The horizontal position of CM remains same $\Rightarrow M(2.2 - X) = 10\,MX$

$\Rightarrow X = 0.2\ m.$

Illustration 10:

System released from rest. Find how much m_1 moved when m_2 is below m_1 ?



Solution:

$(m_1 + m_2)_{system}$

Same logic,

$\Delta r_{cm} = 0$

Let m_1 moved x

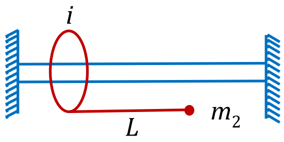
$\therefore$ m_2 moved $(L - x)$

$\vec{r}_1 = x\hat{i}$

$\therefore$ $$\Delta r_{cm} = \frac{m_1 x\hat{i} - m_2(L-x)\hat{i}}{m_1+m_2}$$

$0 = m_1x - m_2L + m_2x$

$$\left.\begin{aligned} x &= \frac{m_2L}{m_1+m_2} \\ L - x &= \frac{m_1L}{m_1+m_2} \end{aligned}\right\}$$



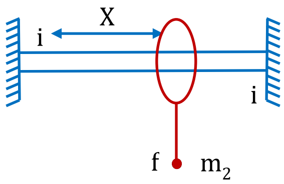
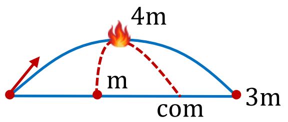


Illustration 11:

A projectile is fired at a speed of 100 m/s at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1 : 3, the lighter piece coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

Solution:



Internal force do not effect the motion of the centre of mass, the centre of mass hits the ground at the position where the original projectile would have landed. The range of the original projectile is,

$$x_{COM} = \frac{2u^2 \sin\theta \cos\theta}{g} = \frac{2\times10^4\times\frac{3}{5}\times\frac{4}{5}}{10} m = 960\ m$$

The center of mass will hit the ground at this position. As the smaller block comes to rest after breaking, it falls down vertically and hits the ground at half of the range, i.e., at $x = 480\ m$. If the heavier block hits the ground at x_2, then

$$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$

$$960 = \frac{(m)(480)+(3m)(x_2)}{(m+3m)}$$

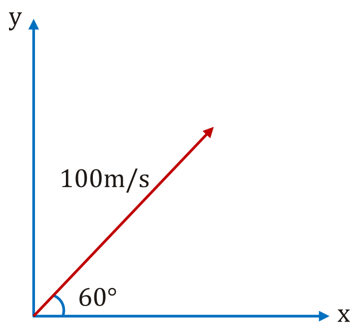
$x_2 = 1120\ m$

Illustration 12:

A shell is fired from a cannon with a speed of 100 m/s at an angle 60° with the horizontal (positive x-direction). At the highest point of its trajectory, the shell explodes into two equal fragments. One of the fragments moves along the negative x-direction with a speed of 50 m/s. What is the speed of the other fragment at the time of explosion.

Solution:

As we know in absence of external force the motion of centre of mass of a body remains unaffected. Thus, here the centre of mass of the two fragments will continue to follow the original projectile path. The velocity of the shell at the highest point of trajectory is



$$v_M = u\cos\theta = 100 \times \cos 60° = 50\ m/s$$

Let v_1 be the speed of the fragment which moves along the negative x-direction and the other fragment has speed v_2. Which must be along positive x-direction. Now from momentum conservation, we have

$$mv = \frac{-m}{2}v_1 + \frac{m}{2}v_2$$

or $\quad 2v = v_2 - v_1$

or $\quad v_2 = 2v + v_1 = (2 \times 50) + 50 = 150\ m/s$

Illustration 13:

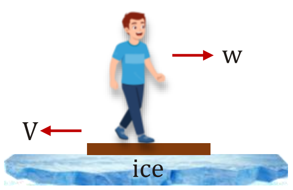
A man of mass m is standing on a platform of mass M kept on smooth ice. If the man starts moving on the platform with a speed v relative to the platform, with what velocity relative to the ice does the platform recoil ?

Solution:

Consider the situation shown in figure. Suppose the man moves at a speed w towards right and the platform recoils at a speed V towards left, both relative to the ice. Hence, the speed of the man relative to the platform is $V + w$. By the question,

$$V + w = v, \quad \text{or} \quad w = v - V \qquad ...(i)$$

Taking the platform and the man to be the system, there is no external horizontal force on the system. The linear momentum of the system remains constant. Initially both the man and the platform were at rest. Thus,



$$0 = MV - mw$$

or $\quad MV = m(v - V)$ [Using (i)]

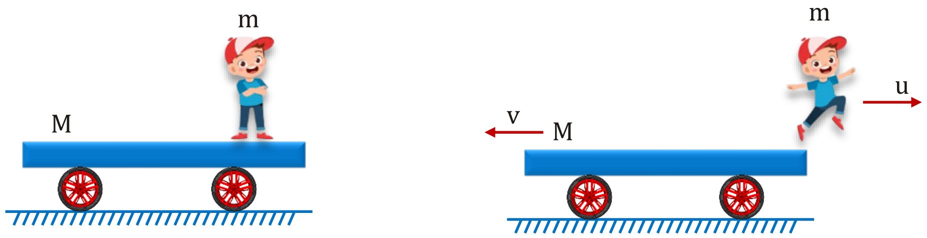
or, $\quad V = \dfrac{mv}{M+m}$

Illustration 14:

A flat car of mass M is at rest on a frictionless floor with a child of mass m standing at its edge. If child jumps off from the car towards right with an initial velocity u, with respect to the car, find the velocity of the car after its jump.

Solution:

Let car attains a velocity v, and the net velocity of the child with respect to earth will be $u - v$, as u is its velocity with respect to car.



Initially, the system was at rest, thus according to momentum conservation, momentum after jump must be zero, as

$$m(u - v) = Mv$$

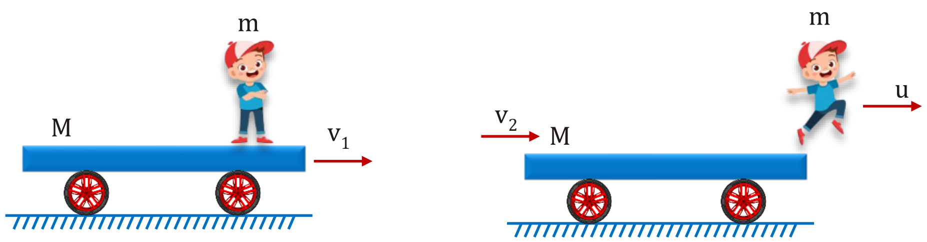
$$v = \frac{mu}{m+M}$$

Illustration 15:

A flat car of mass M with a child of mass m is moving with a velocity v_1 on a friction less surface. The child jumps in the direction of motion of car with a velocity u with respect to car. Find the final velocities of the child and that of the car after jump.

Solution:

This case is similar to the previous example, except now the car is moving before jump. Here also no external force is acting on the system in horizontal direction, hence momentum remains conserved in this direction. After jump car attains a velocity v_2 in the same direction, which is less than v_1, due to backward push of the child for jumping. After jump child attains a velocity $u + v_2$ in the direction of motion of car, with respect to ground.



According to momentum conservation $(M + m)v_1 = Mv_2 + m\,(u + v_2)$

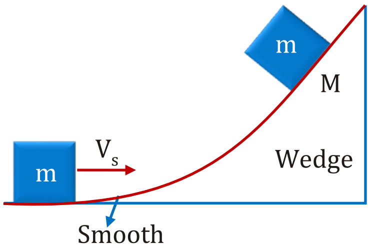
Velocity of car after jump is, $v_2 = \dfrac{(M+m)v_1 - mu}{M+m}$

Velocity of child after jump is, $u + v_2 = \dfrac{(M+m)v_1 + (M)u}{M+m}$

Questions based on conservation of linear momentum and work-energy theorem:

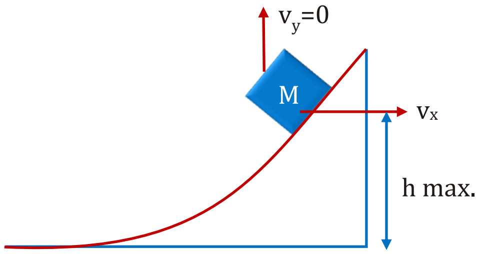
Illustration 16:

A block of mass m is kept at the edge of movable wedge (mass M). If velocity V_s given to the block then find maximum height reached by block. Find maximum height reached by block ?



Solution:

Here external net force is 0. So, momentum is conserved.



$F_{ext} = 0$

$P_i = P_f$

$mV_s + 0 = mv_x + Mv_x$

$$v_x = \frac{mV_s}{(M+m)} \quad \text{...(i)}$$

Let h be the maximum height achieved at this height both ($m + M$) move horizontally with same velocity.

At maximum height vertical component of velocity = $v_y = 0$

Applying work energy theorem,

$W_{all} = \Delta KE$

$$-mgh_{max} = \frac{1}{2}mv_x{}^2 + \frac{1}{2}Mv_x{}^2 - \frac{1}{2}mV_s^2$$

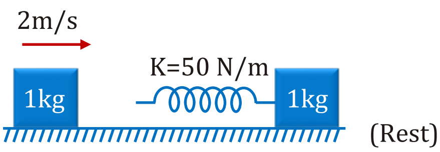
$$-mgh_{max} = \frac{1}{2}v_x{}^2[m + M] - \frac{1}{2}mV_s^2 \quad \text{...(ii)}$$

$h_{max.} = ?$

$$h_{max} = \frac{-m^2V_s^2(M+m)}{2(M+m)2\times mg} + \frac{V_s^2}{2g}$$

Illustration 17:

Each of the blocks shown in figure has mass 1 kg. The rear block moves with a speed of 2 m/s towards the front block kept at rest. The spring attached to the front block is light and has a spring constant 50 N/m. Find the maximum compression of the spring. (Assume surface is frictionless)



Solution:

Maximum compression will take place when the blocks move with equal velocity. As no net external horizontal force acts on the system of the two blocks, the total linear momentum will remain constant. If V is the common speed at maximum compression, we have,

$$(1\ kg)\ (2\ m/s) = (1\ kg)V + (1\ kg)V \qquad \text{or} \qquad V = 1\ m/s$$

Initial kinetic energy $= \frac{1}{2}(1\ kg)\ (2\ m/s)^2 = 2\ J.$

Final kinetic energy $= \frac{1}{2}(1\ kg)\ (1 m/s)^2 + \frac{1}{2}(1\ kg)\ (1\ m/s)^2 = 1\ J$

The kinetic energy lost is stored as the elastic energy in the spring.

Hence, $\frac{1}{2}(50\ N/m)\ x^2 = 2J - 1J = 1\ J \qquad \text{or} \qquad x = 0.2\ m$

Impulse

Impulse of a force $\vec{F}$ acting on a body for the time interval $t = t_1$ to $t = t_2$ is defined as :

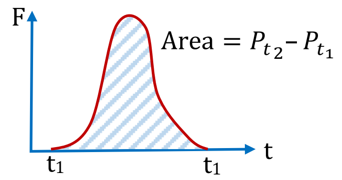
$$\vec{I} = \int_{t_1}^{t_2} \vec{F} dt \Rightarrow \quad \vec{I} = \int \vec{F}\, dt = \int m\frac{d\vec{v}}{dt} dt = \int m d\vec{v}$$

$$\vec{I} = m(\vec{v}_2 - \vec{v}_1) = \Delta\vec{P} = \text{change in momentum due to force}$$

Also $\quad \vec{I}_{Res} = \int_{t_1}^{t_2} \vec{F}_{Res} dt = \Delta\vec{P}$ **(impulse - momentum theorem)**

Note:

Impulse applied to an object in a given time interval can also be calculated from the area under force time (F-t) graph in the same time interval.



Instantaneous Impulse :

There are many cases when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts, may each be unknown, but the value of their product (i.e., impulse) can be known by measuring the initial and final momenta. Thus, we can write.

$$\vec{I} = \int \vec{F}\, dt = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$$

Important Points :

(1) It is a vector quantity.

(2) Dimensions = $[MLT^{-1}]$.

(3) SI unit = $kg\ m/s$.

(4) Direction is along change in momentum.

(5) Magnitude is equal to area under the F-t. graph.

(6) $\vec{I} = \int \vec{F}\, dt = \vec{F}_{av} \int dt = \vec{F}_{av} \Delta t$

(7) It is not a property of a particle, but it is a measure of the degree to which an external force changes the momentum of the particle.

Illustration 18:

The hero of a stunt film fires 50 g bullets from a machine gun, each at a speed of 1.0 km/s. If he fires 20 bullets in 4 seconds, what average force does he exert against the machine gun during this period.

Solution:

The momentum of each bullet

$$= (0.050\ kg)\ (1000\ m/s) = 50\ kg\text{-}m/s$$

The gun has been imparted this much amount of momentum by each bullet fired. Thus, the rate of change of momentum of the gun

$$= \int N\,dt = \frac{nmu}{t} = 250N$$

In order to hold the gun, the hero must exert a force of 250 N against the gun.

Impulsive force :

A force, of relatively higher magnitude and acting for relatively shorter time, is called impulsive force.

An impulsive force can change the momentum of a body in a finite magnitude in a very short time interval.

Impulsive force is a relative term. There is no clear boundary between an impulsive and Non-Impulsive force.

Note:

Usually colliding forces are impulsive in nature.

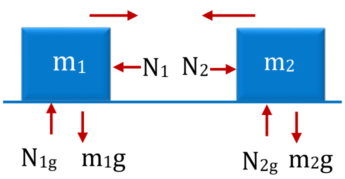
Since, the application time is very small, hence, very little motion of the particle takes place.

Important points :

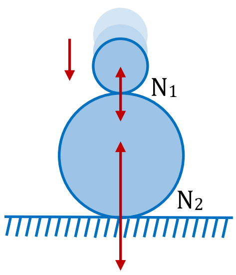
1. Gravitational force and spring force are always non-Impulsive.
2. Normal, tension and friction are case dependent.
3. An impulsive force can only be balanced by another impulsive force.

1. **Impulsive Normal :** In case of collision, normal forces at the surface of collision are always impulsive

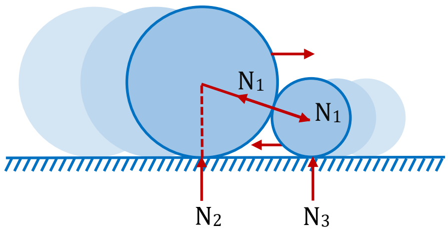
eg.



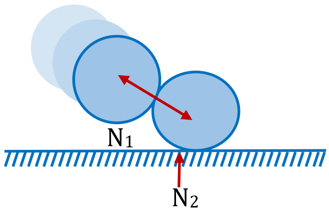
N_i = Impulsive; N_g = Non-impulsive



Both normals are Impulsive

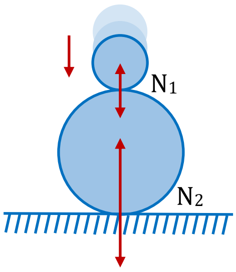


N_1, N_3 = Impulsive; N_2 = non-impulsive

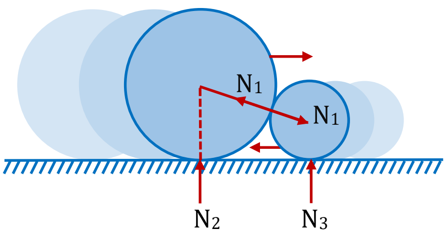


Both normals are Impulsive

2. **Impulsive Friction :** If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.



Friction at both surfaces is impulsive

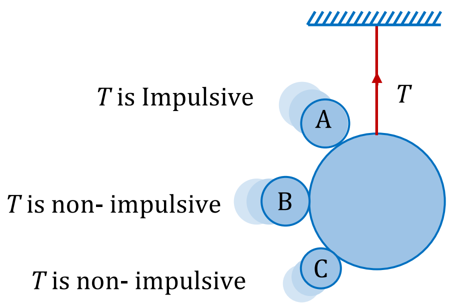


Friction due to N_2 is non-impulsive and due to N_3 and N_1 are impulsive.

3. **Impulsive Tensions :** When a string jerks, equal and opposite tension act suddenly at each end. Consequently equal and opposite impulses act on the bodies attached with the string in the direction of the string. There are two cases to be considered.

(a) **One end of the string is fixed :** The impulse which acts at the fixed end of the string cannot change the momentum of the fixed object there. The object attached to the free end however will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string where no impulsive forces act.

(b) **Both ends of the string attached to movable objects :** In this case equal and opposite impulses act on the two objects, producing equal and opposite changes in momentum. The total momentum of the system therefore remains constant, although the momentum of each individual object is changed in the direction of the string. Perpendicular to the string, however, no impulse acts and the momentum of each particle in this direction is unchanged.

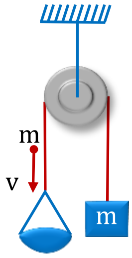


All normal are impulsive but tension T is impulsive only for the ball A

For this example :

In case of rod, Tension is always impulsive and in case of spring, Tension is always non-impulsive.

Illustration 19:



A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v, find the speed with which the system moves just after the collision.

Solution:

Let the required speed is V.

Further, let J_1 = impulse between particle and pan

and J_2 = impulse imparted to the block and the pan by the string

Using, Impulse = change in momentum

For particle $J_1 = mv - mV$ ...(i)

For pan $J_1 - J_2 = mV$ ...(ii)

For block $J_2 = mV$ ...(iii)

Solving, these three equation, we get $V = \frac{v}{3}$

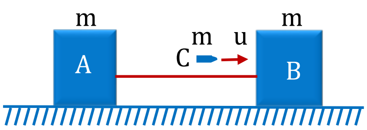
Alternative Solution:

Applying conservation of linear momentum along the string;

$mv = 3mV$; we get, $V = \frac{v}{3}$

Illustration 20:

Two identical blocks A and B, connected by a massless string, are placed on a frictionless horizontal plane. A bullet having same mass, moving with speed u strikes block B from behind as shown. If the bullet gets embedded into the block B then find:



(a) The velocity of A, B, C after collision.

(b) Impulse on A due to tension in the string.

(c) Impulse on C due to normal force of collision.

(d) Impulse on B due to normal force of collision.

Solution:

(a) By Conservation of linear momentum $v = \frac{u}{3}$

(b) $\int T dt = \frac{mu}{3}$

(c) $\int N\,dt = m\left(\frac{u}{3} - u\right) = \frac{-2mu}{3}$

(d) $\int (N - t)dt = \int N dt - \int T\,dt = \frac{mu}{3}$

$\int N\,dt = \frac{2mu}{3}$

Collision or Impact

Collision is an event in which an impulsive force acts between two or more bodies for a short time, which results in change of their velocities.

Note:

(a) In a collision, particles may or may not come in physical contact.

(b) The duration of collision, Δt is negligible as compared to the usual time intervals of observation of motion.

(c) In a collision the effect of external non-impulsive forces such as gravity are not taken into a account as due to small duration of collision (Δt) average impulsive force responsible for collision is much larger than external forces acting on the system.

The collision is infact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles.

Line of Impact

The line passing through the common normal to the surfaces in contact during impact is called line of impact. The force during collision acts along this line on both the bodies.

Direction of Line of impact can be determined by:

(a) Geometry of colliding objects like spheres, discs, wedge etc.

(b) Direction of change of momentum.

If one particle is stationary before the collision then the line of impact will be along its motion after collision.

Classification of collisions

(a) On the basis of line of impact:

(i) Head-on collision : If the velocities of the colliding particles are along the same line before and after the collision.

(ii) Oblique collision : If the velocities of the colliding particles are along the different lines before and after the collision.

(b) On the basis of energy:

(i) Elastic collision : In an elastic collision, the colliding particles regain their shape and size completely after collision. i.e., no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus, kinetic energy of system after collision is equal to kinetic energy of system before collision. Thus, in addition to the linear momentum, kinetic energy also remains conserved before and after collision.

(ii) Inelastic collision : In an inelastic collision, the colliding particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles after collision is not equal to that of before collision. However, in the absence of external forces, law of conservation of linear momentum still holds good.

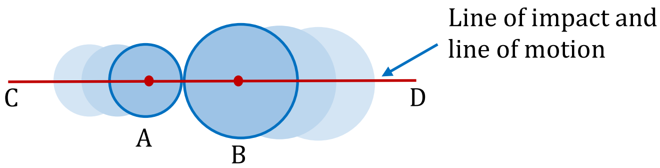
(iii) Perfectly inelastic : In a perfectly inelastic collision maximum amount of kinetic energy is lost and both the particles stick together after collision and move with same velocity.

Note:

Actually, collision between all real objects are neither perfectly elastic nor perfectly inelastic, its inelastic in nature.

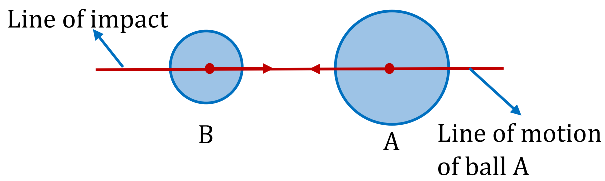
Examples of line of impact and collisions based on line of impact

(i) Two balls A and B are approaching each other such that their centers are moving along line CD.



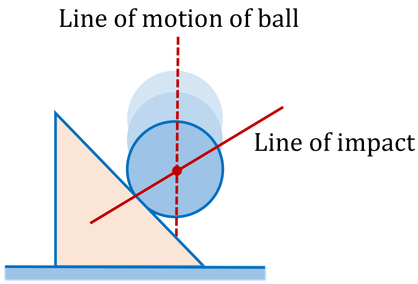
Head on Collision (If line of motion and line of impact is same)

(ii) Two balls A and B are approaching each other such that their center are moving along dotted lines as shown in figure.



Oblique Collision

(iii) Ball is falling on a stationary wedge.



Coefficient of Restitution (e)

The coefficient of restitution is defined as the ratio of the impulses of reformation and deformation of either body.

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r ft}{\int F_d dt}$$

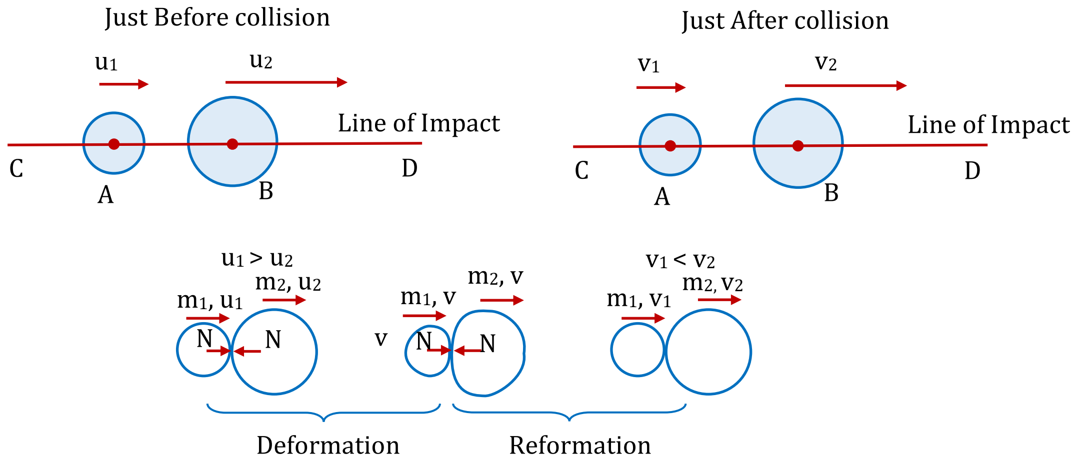
$$= \frac{\text{Velocity of seperation along line of impact}}{\text{Velocity of approach of point of contact along line of impact}}$$

The most general expression for coefficient of restitution is

$$e = \frac{\text{Velocity of seperation of points of contact along line of impact}}{\text{Velocity of approach of point of contact along line of impact}}$$

Example for calculation of *e*

Two smooth balls A and B approaching each other such that their centers are moving along line CD in absence of external impulsive force. The velocities of A and B just before collision be u_1 and u_2 respectively. The velocities of A and B just after collision be v_1 and v_2 respectively.



$\because$ $F_{ext} = 0$ momentum is conserved for the system.

$\Rightarrow$ $m_1u_1 + m_2u_2 = (m_1 + m_2)v$ in between (during collision) $= m_1v_1 + m_2v_2$

$\Rightarrow$ $$v = \frac{m_1u_1+m_2u_2}{m_1+m_2} = \frac{m_1v_1+m_2v_2}{m_1+m_2} \quad ...(1)$$

Impulse of Deformation :

J_D = change in momentum of any one body during deformation.

$= m_2(v - u_2)$ for m_2

$= m_1(-v + u_1)$ for m_1

Impulse of Reformation :

J_R = change in momentum of any one body during Reformation.

$= m_2(v_2 - v)$ for m_2

$= m_1(v - v_1)$ for m_1

$$e = \frac{\text{Impulse of Reformation }(\vec{J}_R)}{\text{Impulse of Deformation }(\vec{J}_D)} = \frac{v_2-v_1}{u_1-u_2} = e = \frac{\text{Velocity of separation along } LOI}{\text{Velocity of approach along } LOI}$$

Note:

e is independent of shape and mass of object but depends on the material. The coefficient of restitution is constant for a pair of materials.

(a) $e = 1$ Impulse of Reformation = Impulse of Deformation

Velocity of separation = Velocity of approach

Kinetic energy of particles after collision is equal to that of before collision.

Collision is elastic.

(b) $e = 0$ Impulse of Reformation = 0

Velocity of separation = 0

Kinetic energy of particles after collision is not equal to that of before collision.

Collision is perfectly inelastic.

(c) $0 < e < 1$ Impulse of Reformation < Impulse of Deformation

Velocity of separation < Velocity of approach

Kinetic energy of particles after collision is not equal to that of before collision.

Collision is Inelastic.

Note:

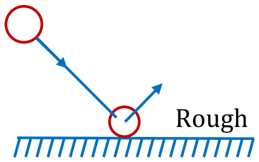
In case of contact collisions e is always less than unity.

$\therefore \quad 0 \le e \le 1$

Important Point:

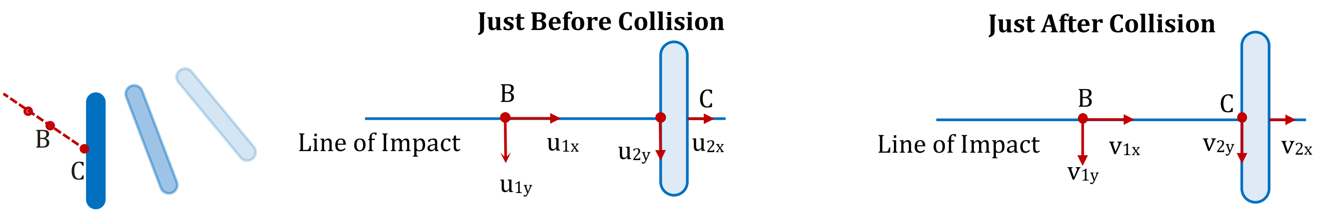
In case of elastic collision, if rough surface is present then $k_f < k_i$ (because friction is impulsive).

Where, k is Kinetic Energy.



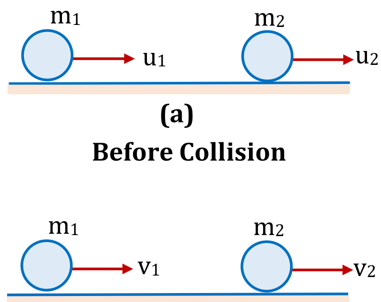
A particle 'B' moving along the dotted line collides with a rod also in state of motion as shown in the figure. The particle B comes in contact with point C on the rod.

To write down the expression for coefficient of restitution e, we first draw the line of impact. Then we resolve the components of velocities of points of contact of both the bodies along line of impact just before and just after collision.



Then $\quad e = \dfrac{V_{2x} - V_{1x}}{u_{1x} - u_{2x}}$

Collision in one dimension (Head on)



(a)
Before Collision

(b)
After Collision

$e = \dfrac{v_2 - v_1}{u_1 - u_2} \Rightarrow (u_1 - u_2)\, e = (v_2 - v_1)$

By momentum conservation,

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$v_2 = v_1 + e(u_1 - u_2)$ and $v_1 = \dfrac{m_1u_1 + m_2u_2 - m_2e(u_1 - u_2)}{m_1 + m_2}$

$v_2 = \dfrac{m_1u_1 + m_2u_2 + m_1e(u_1 - u_2)}{m_1 + m_2}$

Special Case:

(1) $e = 0$

$\Rightarrow \quad v_1 = v_2$

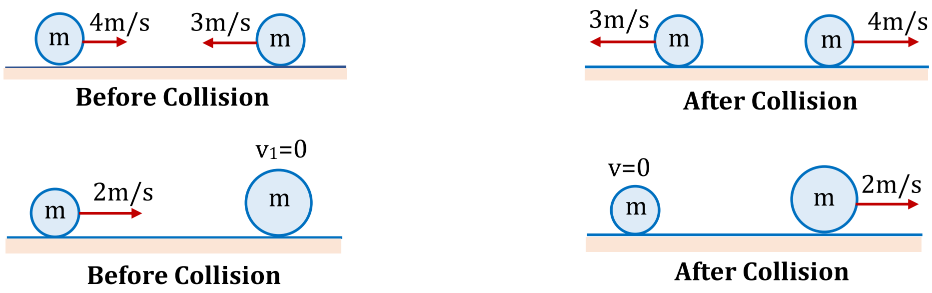
$\Rightarrow$ for perfectly inelastic collision, both the bodies, move with same vel. after collision.

(2) $e = 1$

and $m_1 = m_2 = m$,

we get $v_1 = u_2$ and $v_2 = u_1$

i.e., when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities., e.g.



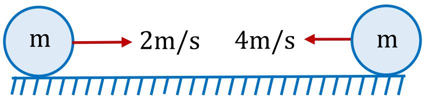
(3) $m_1 >> m_2$

$m_1 + m_2 = m_1$ and $\dfrac{m_2}{m_1} = 0$

$\Rightarrow$ $v_1 = u_1$ No change and $v_2 = u_1 + e(u_1 - u_2)$

Illustration 21:

Two identical balls are approaching towards each other on a straight line with velocity 2 m/s and 4 m/s respectively. Find the final velocities, after elastic collision between them.



Solution:

The two velocities will be exchanged and the final motion is reverse of initial motion for both.

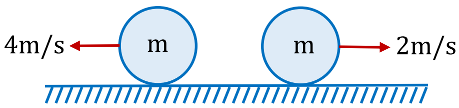
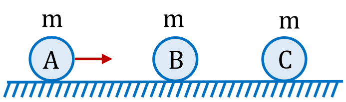


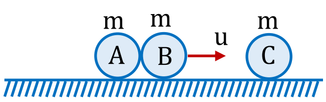
Illustration 22:

Three balls A, B and C of same mass *'m'* are placed on a frictionless horizontal plane in a straight line as shown. Ball A is moved with velocity u towards the middle ball B. If all the collisions are elastic then, find the final velocities of all the balls.

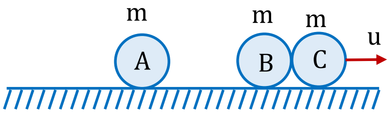


Solution:

A collides elastically with B and comes to rest but B starts moving with velocity u



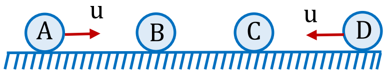
After a while B collides elastically with C and comes to rest but C starts moving with velocity u



$\therefore$ Final velocities, $V_A = 0$; $V_B = 0$ and $V_C = u$

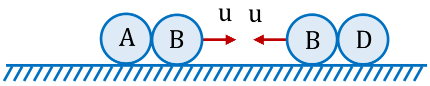
Illustration 23:

Four identical balls A, B, C and D are placed in a line on a frictionless horizontal surface. A and D are moved with same speed 'u' towards the middle as shown. Assuming elastic collisions, find the final velocities.

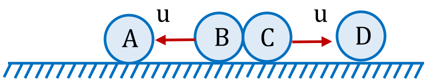


Solution:

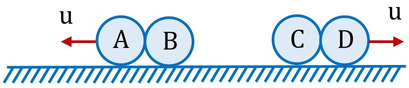
A and D collides elastically with B and C respectively and come to rest but B and C starts moving with velocity u towards each other as shown



B and C collides elastically and exchange their velocities to move in opposite directions



Now, B and C collides elastically with A and D respectively and come to rest but A and D starts moving with velocity u away from each other as shown



$\therefore$ Final velocities $V_A = u\ (\leftarrow)$; $V_B = 0$; $V_C = 0$ and $V_D = u\ (\rightarrow)$

Illustration 24:

Two particles of mass m and $2m$ moving in opposite directions on a frictionless surface collide elastically with velocity v and $2v$ respectively. Find their velocities after collision, also find the fraction of kinetic energy lost by the colliding particles.

Solution:

Let the final velocities of m and $2m$ be v_1 and v_2 respectively as shown in the figure:

By conservation of momentum :

$$m(2v) + 2m(-v) = m(v_1) + 2m\,(v_2)$$

or $\quad 0 = mv_1 + 2mv_2$

or $\quad v_1 + 2v_2 = 0 \qquad$ ...(1)

and since the collision is elastic:

$$v_2 - v_1 = 2v - (-v)$$

or $\quad v_2 - v_1 = 3v \qquad$ ...(2)

Solving the above two equations, we get,

$v_2 = v$ and $v_1 = -2v$

i.e., the mass $2m$ returns with velocity v while the mass m returns with velocity $2v$ in the direction shown in figure:

The collision was elastic therefore, no kinetic energy is lost, KE loss = $KE_i - KE_f$

or $\left(\frac{1}{2}m(2v)^2 + \frac{1}{2}(2m)(-v)^2\right) - \left(\frac{1}{2}m(2v)^2 + \frac{1}{2}(2m)v^2\right) = 0$

Illustration 25:

On a frictionless surface, a ball of mass m moving at a speed v makes a head on collision with an identical ball at rest. The kinetic energy of the balls after the collision is 3/4th of the original. Find the coefficient of restitution.

Solution:

As we have seen in the above discussion, that under the given conditions :

By using conservation of linear momentum and equation of e, we get,

$$v_1' = \left(\frac{1+e}{2}\right)v \qquad \text{and} \qquad v_2' = \left(\frac{1-e}{2}\right)v$$

Given that

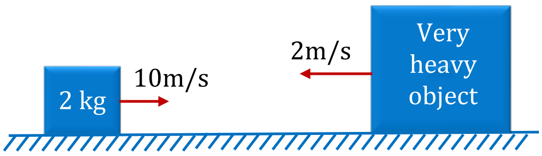
$$K_f = \frac{3}{4}K_i \qquad \text{or} \qquad \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 = \frac{3}{4}\left(\frac{1}{2}mv^2\right)$$

Substituting the value, we get

$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4} \text{ or } e = \frac{1}{\sqrt{2}}$$

Illustration 26:

A block of mass 2 kg is pushed towards a very heavy object moving with 2 m/s closer to the block (as shown). Assuming elastic collision and frictionless surfaces, find the final velocities of the blocks.



Solution:

Let v_1 and v_2 be the final velocities of $2kg$ block and heavy object respectively then,

$v_1 = u_1 + 1\,(u_1 - u_2) = 2u_1 - u_2 = -14\ m/s$

$v_2 = -2m/s$

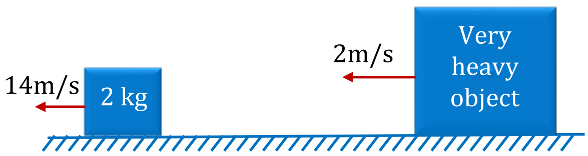
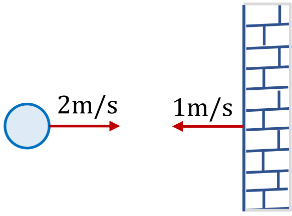


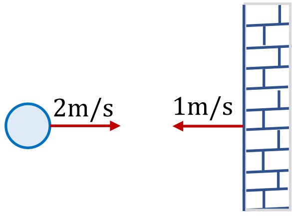
Illustration 27:

A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in fig. Assuming collision to be elastic, find the velocity of the ball immediately after the collision.

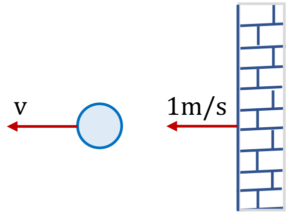


Solution:

The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in figure. Since collision is elastic (e = 1),



Before Collision



After Collision

Separation speed = Approach speed

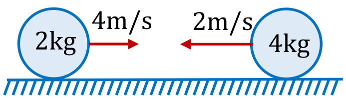
or $v - 1 = 2 + 1$

or $v = 4\ m/s$

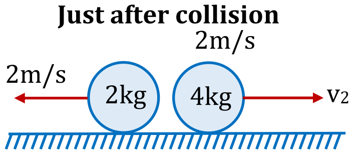
Illustration 28:

Two balls of masses 2 kg and 4 kg are moved towards each other with velocities 4 m/s and 2 m/s respectively on a frictionless surface. After colliding the 2 kg ball returns back with velocity 2m/s.

Just before collision



Just after collision



Then find:

(a) Velocity of 4 kg ball after collision

(b) Coefficient of restitution e

(c) Impulse of deformation J_D

(d) Maximum potential energy of deformation

(e) Impulse of reformation J_R

Solution:

(a) By momentum conservation, $2(4) - 4(2) = 2(-2) + 4(v_2)$ $\Rightarrow v_2 = 1$ m/s

(b) $e = \dfrac{\text{Velocity of separation}}{\text{Velocity of approach}} = \dfrac{1-(-2)}{4-(-2)} = \dfrac{3}{6} = 0.5$

(c) At maximum deformed state, by conservation of momentum, common velocity is $v = 0$.

$$J_D = m_1(v - u_1) = m_2(v - u_2) = 2(0 - 4) = -8\ N - s = 4(0 - 2) = -8\ N - s$$

or $= 4(0 - 2) = -8\ N - s$

(d) Potential energy at maximum deformed state U = loss in kinetic energy during deformation

or $U = \theta$ or $U = 24$ Joule

(e) $J_R = m_1(v_1 - v) = m_2(v - v_2) = 2(-2 - 0) = -4$ N-s

or $4(0 - 1) = -4$ N-s

or $e = \dfrac{J_R}{J_D}$

$\Rightarrow$ $J_R = eJ_D = (0.5)(-8) = -4$ N-s

Collision in two dimension (oblique)

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.

2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.

3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remain conserved before and after collision in any direction.

4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction we can apply

Relative speed of separation = e (Relative speed of approach)

Illustration 29:

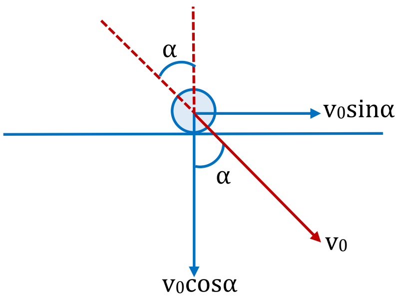
A ball of mass m hits a floor with a speed v_0 making an angle of incidence a with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.

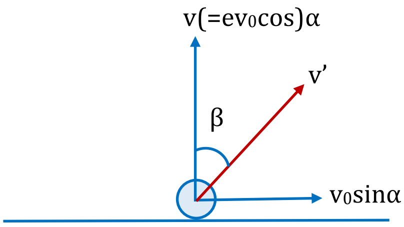
Solution:

The component of velocity v_0 along common tangential direction $v_0 \sin \alpha$ will remain unchanged. Let v be the component along common normal direction after collision. Applying,

Relative speed of separation = e (Relative speed of approach) along common normal direction,

We get $v = ev_0 \cos \alpha$



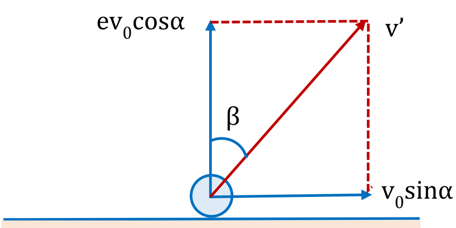


Thus, after collision components of velocity v' are $v_0 \sin \alpha$ and $ev_0 \cos \alpha$

$\therefore$ $v' = \sqrt{(v_0 \sin \alpha)^2 + (ev_0 \cos \alpha)^2}$

and $\tan \beta = \dfrac{v_0 \sin \alpha}{ev_0 \cos \alpha}$

or $\tan \beta = \dfrac{\tan \alpha}{e}$



Note:

For elastic collision,

$e = 1$

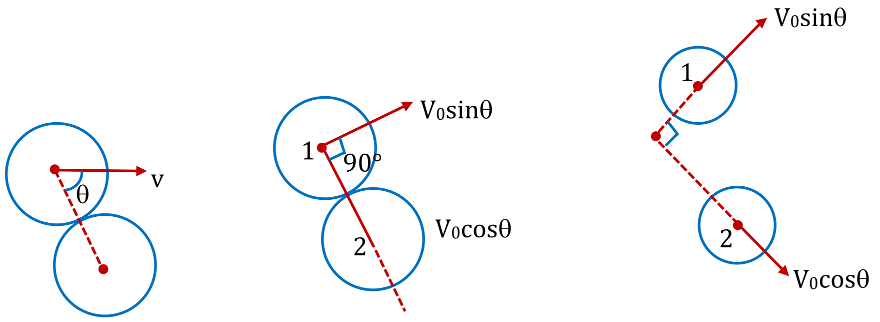
$\therefore$ $v' = v_0$ and $\beta = \alpha$

Illustration 30:

A ball of mass m makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.

Solution:

In head on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along common normal direction, $v \cos \theta$



Before Collision **After Collision**

becomes zero after collision, while that of 2 becomes $v \cos \theta$. While the components along common tangent direction of both the particles remain unchanged. Thus, the components along common tangent and common normal direction of both the balls in tabular form are given below.

Ball	Component along common tangent direction		Component along common normal direction	
	Before collision	After collision	Before collision	After collision
1	$v \sin \theta$	$v \sin \theta$	$v \cos \theta$	0
2	0	0	0	$v \cos \theta$

From the above table and figure, we see that both the balls move at right angle after collision with velocities $v \sin \theta$ and $v \cos \theta$.

Note:

When two identical bodies have an oblique elastic collision, with one body at rest before collision, then the two bodies will go in $\perp$ directions.

Illustration 31:

Two spheres are moving towards each other. Both have same radius but their masses are 2 kg and 4 kg. If the velocities are 4 m/s and 2 m/s respectively and coefficient of restitution is $e = 1/3$, find.

(a) The common velocity along the line of impact.

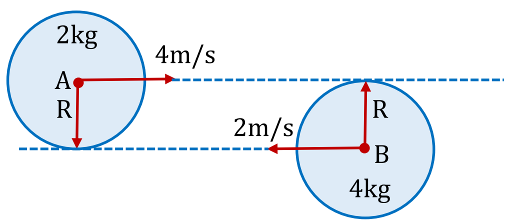
(b) Final velocities along line of impact.

(c) Impulse of deformation.

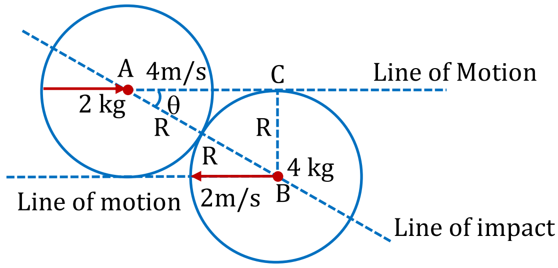
(d) impulse of reformation.

(e) Maximum potential energy of deformation.

(f) Loss in kinetic energy due to collision.

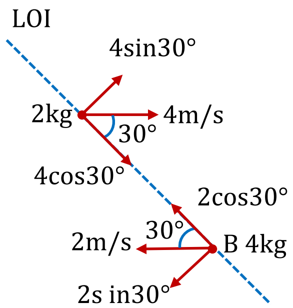


Solution:

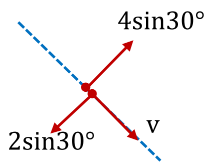


In ΔABC $\sin\theta = \dfrac{BC}{AB} = \dfrac{R}{2R} = \dfrac{1}{2}$ or $\theta = 30°$

(a) By conservation of momentum along line of impact.



Just before Collision Along LOI

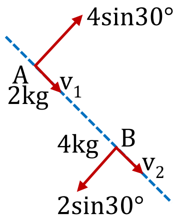


Maximum deformed State

$2(4\cos 30°) - 4(2\cos 30°) = (2+4)v$

or $v = 0$ (common velocity along LOI)

(b)



Just After Collision Along LOI

Let v_1 and v_2 be the final velocity of A and B respectively then, by conservation of momentum along line of impact,

$$2(4\cos 30°) - 4(2\cos 30°) = 2(v_1) + 4(v_2)$$

or $0 = v_1 + 2v_2$ ...(1)

By coefficient of restitution,

$$e = \frac{\text{Velocity of separation along } LOI}{\text{Velocity of approach along } LOI}$$

or $= \dfrac{1}{3} = \dfrac{v_2 - v_1}{4\cos 30° + 2\cos 30°}$ or $v_2 - v_1 = \sqrt{3}$ ...(2)

From the above two equations,

$$v_1 = \frac{-2}{\sqrt{3}}\,m/s \quad \text{and} \quad v_2 = \frac{1}{\sqrt{3}}\,m/s$$

(c) $J_D = m_1(v - u_1) = 2(0 - 4\cos 30°)$

(d) $J_R = eJ_D = \frac{1}{3}(-4\sqrt{3}) = -\frac{4}{\sqrt{3}} N - s$

(e) Maximum potential energy of deformation is equal to loss in kinetic energy during deformation upto maximum deformed state,

$$U = \frac{1}{2} m_1(u_1 \cos \theta)^2 + \frac{1}{2} m_2(u_2 \cos \theta)^2 - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \frac{1}{2} 2(4 \cos 30°)^2 + \frac{1}{2} 4(-2\cos 30°)^2 - \frac{1}{2}(2 + 4)(0)^2 \quad \text{or} \quad U = 18 \text{ Joule}$$

(f) Loss in kinetic energy,

$$\Delta KE = \frac{1}{2} m_1(u_1 \cos \theta)^2 + \frac{1}{2} m_2(u_2 \cos \theta)^2 - \left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right)$$

$$= \frac{1}{2} 2(4 \cos 30°)^2 + \frac{1}{2} 4(-2 \cos 30°)^2 - \left(\frac{1}{2}2\left(\frac{2}{\sqrt{3}}\right)^2 + \frac{1}{2}4\left(\frac{1}{\sqrt{3}}\right)^2\right)$$

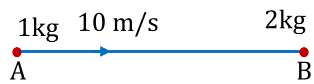
$\Delta KE = 16$ Joule

Illustration 32:

Two point particles A and B are placed in line on a frictionless horizontal plane. If particle A (mass 1 kg) is moved with velocity 10 m/s towards stationary particle B (mass 2 kg) and after collision the two move at an angle of 45° with the initial direction of motion, then find:

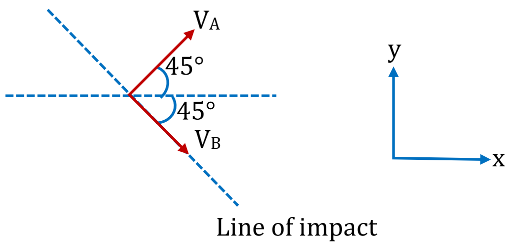
(a) Velocities of A and B just after collision.

(b) Coefficient of restitution.



Solution:

The very first step to solve such problems is to find the line of impact which is along the direction of force applied by A on B, resulting the stationary B to move. Thus, by watching the direction of motion of B, line of impact can be determined. In this case line of impact is along the direction of motion of B. i.e. 45° with the initial direction of motion of A.



(a) By conservation of momentum, along x direction:

$m_A u_A = m_A v_A \cos 45° + m_B v_B \cos 45°$

or $1(10) = 1(v_A \cos 45°) + 2(v_B \cos 45°)$

or $v_A + 2v_B = 10\sqrt{2}$ ...(1)

Along y direction

$0 = m_A v_A \sin 45° + m_B v_B \sin 45°$

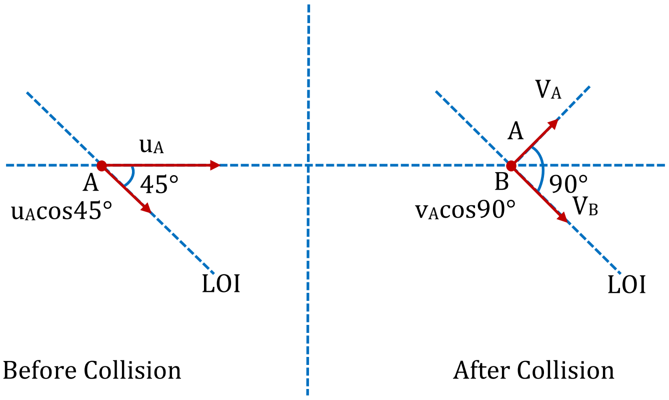
or $0 = 1(v_A \sin 45°) - 2(v_B \sin 45°)$

or $v_A = 2v_B$ ...(2)

Solving the two equations,

$$V_A = \frac{10}{\sqrt{2}} m/s$$

(b) $e = \dfrac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$



or $e = \dfrac{v_B - v_A \cos 90^\circ}{u_A \cos 45^\circ} = \dfrac{\frac{5}{\sqrt{2}} - 0}{\frac{10}{\sqrt{2}}} = \dfrac{1}{2}$

Illustration 33:

A smooth sphere of mass m is moving on a horizontal plane with a velocity $3\hat{i} + \hat{j}$ when it collides with a vertical wall which is parallel to the vector $\hat{j}$. If the coefficient of restitution between the sphere and the wall is $\frac{1}{2}$, find:

(a) the velocity of the sphere after impact.

(b) the loss in kinetic energy caused by the impact.

(c) the impulse $\vec{J}$ that acts on the sphere.

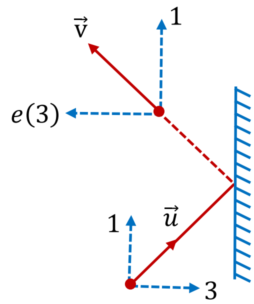
Solution:

Let $\vec{v}$ be the velocity of the sphere after impact.

To find $\vec{v}$ we must separate the velocity components parallel and perpendicular to the wall.

Using the law of restitution, the component of velocity parallel to the wall remains unchanged while component perpendicular to the wall becomes e times in opposite direction.

Thus, $\vec{v} = \frac{3}{2}\hat{i} + \hat{j}$



(a) Therefore, the velocity of the sphere after impact is

$= -3\hat{i} + \hat{j}$

(b) The loss in K.E.

$$= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}m(3^2 + 1^2) - \frac{1}{2}m\left(\left\{\frac{3}{2}\right\}^2 + 1^2\right) = \frac{27}{8}m$$

(c) $J = mv_f - mv_i = m\left(\dfrac{-3\hat{i}}{2} - 3\hat{i}\right) = \dfrac{-9m}{2}\hat{i}$

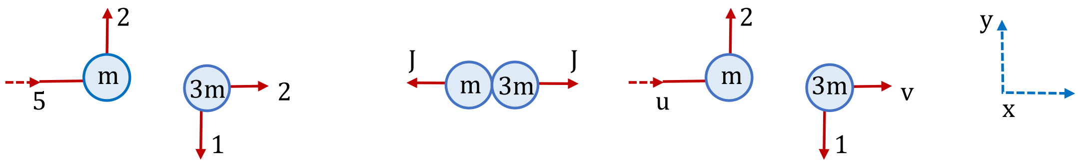
Illustration 34:

Two smooth spheres, A and B, having equal radii, lie on a horizontal table. A is of mass m and B is of mass $3m$. The spheres are projected towards each other with velocity vector $5\hat{\imath}+2\hat{\jmath}$ and $2\hat{\imath}-\hat{\jmath}$ respectively and when they collide the line joining their centers is parallel to the vector $\hat{\imath}$.

If the coefficient of restitution between A and B is $\frac{1}{3}$, find the velocities after impact and the loss in kinetic energy caused by the collision. Find also the magnitude of the impulses that act at the instant of impact.

Solution:

The line of centers at impact, is parallel to the vector, the velocity components of A and B perpendicular to are unchanged by the impact.



Applying conservation of linear momentum and the law of restitution, we have in x direction

$5m + (3m)(2) = mu + 3\,mv$ ...(i)

and $\frac{1}{3} = \frac{v-u}{5-2}$ ...(ii)

Solving these equations, we have $u = 2$ and $v = 3$

The velocities of A and B after impact are therefore,

$2\hat{\imath}+2\hat{\jmath}$ and $3\hat{\imath}-\hat{\jmath}$ respectively

Before impact the kinetic energy of A is $\frac{1}{2}m\left(5^2+2^2\right) = \frac{29}{2}m$

and of B is $\frac{1}{2}(3m)\left(2^2+1^2\right) = \frac{15}{2}m$

After impact the kinetic energy of A is $\frac{1}{2}m\left(2^2+2^2\right) = 4m$

and of B is $\frac{1}{2}(3m)\left(3^2+1^2\right) = 15m$

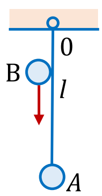
Therefore, the loss in $K.E.$ at impact is $\frac{29}{2}m + \frac{15}{2}m - 4m - 15m = 3m$

To find value of J, we consider the change in momentum along $\hat{\imath}$ for one sphere only.

For sphere B, $J = 3m(3-2)$ or $J = 3m$

Illustration 35:

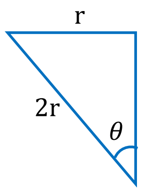
A small steel ball A is suspended by an inextensible thread of length $\ell = 1.5$ from O. Another identical ball is thrown vertically downwards such that its surface remains just in contact with thread during downward motion and collides elastically with the suspended ball. If the suspended ball just completes vertical circle after collision, calculate the velocity of the falling ball just before collision. ($g = 10\ ms^{-2}$)

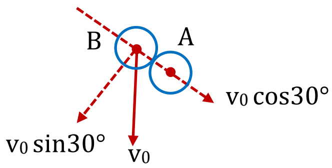


Solution:

Velocity of ball A just after collision is $\sqrt{5gl}$

Let radius of each ball be r and the joining centers of the two balls makes an angle θ with the vertical at the instant of collision, then





$$\sin\theta = \frac{r}{2r} = \frac{1}{2} \text{ or } \theta = 30°$$

Let velocity of ball B (just before collision) be v_0. This velocity can be resolved into two components, (i) v_0 cos 30°, along the line joining the center of the two balls and (ii) v_0 sin 30° normal to this line. Head-on collision takes place due to v_0 cos 30° and the component v_0 sin 30°of velocity of ball B remains unchanged.

Since, ball A is suspended by an inextensible string, therefore, just after collision, it can move along horizontal direction only. Hence, a vertically upward impulse is exerted by thread on the ball A. This means that during collision two impulses act on ball A simultaneously. One is impulsive interaction J between the balls and the other is impulsive reaction J' of the thread.

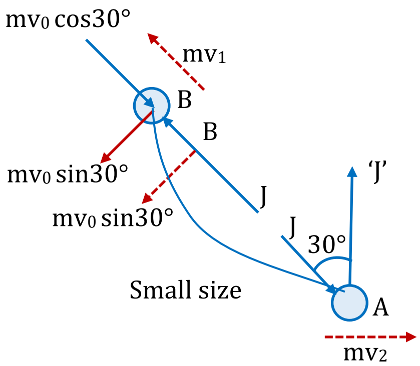
Velocity v_1 of ball B along line of collision is given by

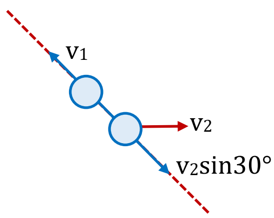
$$J - mv_0 \cos 30° = mv_1$$

or $\quad v_1 = \frac{J}{m} - v_0 \cos 30° \quad$ ...(i)

Horizontal velocity v_0 of ball A is given by $J \sin 30° = mv_2$

or $\quad v_2 = \frac{J}{2m} \quad$ ...(ii)





Since, the balls collide elastically, therefore, coefficient of restitution is $e = 1$.

Hence, $e = \frac{v_2 \sin 30° - (-v_1)}{v_0 \cos 30° - 0} = 1 \quad$ ...(iii)

Solving equations (i), (ii), and (iii),

$$J = 1.6\, mv_0 \cos 30°$$

$\therefore \quad v_1 = 0.6\, v_0 \cos 30° \quad$ and $\quad v_2 = 0.8\, v_0 \cos 30°$

Since, ball A just completes vertical circle, therefore $v_2 = \sqrt{5g\ell}$

$\therefore \quad 0.8\, v_0 \cos 30° = \sqrt{5g\ell} \quad$ or $\quad v_0 = 12.5\ ms^{-1}$

Variable Mass System :

If a mass is added or ejected from a system, at rate μ kg/s and relative velocity $\vec{v}_{rel}$ (w.r.t. the system), then the force exerted by this mass on the system has magnitude $\mu|\vec{V}_{rel}|$.

Thrust Force ($\vec{F_t}$)

$$\vec{F_t} = \vec{v}_{rel}\left(\frac{dm}{dt}\right)$$

Suppose at some moment $t = t$ mass of a body is m and its velocity is $\vec{v}$. After some time at $t = t + dt$ its mass becomes $(m - dm)$ and velocity becomes $\vec{v} + d\vec{v}$. The mass dm is ejected with relative velocity $\vec{v}_r$. Absolute velocity of mass *'dm'* is therefore $(\vec{v} + \vec{v}_r)$. If no external forces are acting on the system, the linear momentum of the system will remain conserved, or $\vec{P}_i = \vec{P}_f$

or $\quad m\vec{v} = (m - dm)(\vec{v} + d\vec{v}) + dm(\vec{v} + \vec{v}_r)$

or $\quad m\vec{v} = m\vec{v} + md\vec{v} - (dm)\vec{v} - (dm)(d\vec{v}) + (dm)\vec{v} + \vec{v}_r dm$

The term (dm) $(d\vec{v})$ is too small and can be neglected.

$\therefore \quad md\vec{v} = -\vec{v}_r dm \quad$ or $\quad m\left(\frac{d\vec{v}}{dt}\right) = \vec{v}_r\left(-\frac{dm}{dt}\right)$

Here, $\quad m\left(-\frac{d\vec{v}}{dt}\right)$ = thrust force $(\vec{F}_t)$

and $\quad -\frac{dm}{dt}$ = rate at which mass is ejecting

or $\quad \vec{F}_t = \vec{v}_r\left(\frac{dm}{dt}\right)$

Problems related to variable mass can be solved in following four steps:

1. Make a list of all the forces acting on the main mass and apply them on it.

2. Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{V}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing.

3. Find net force on the mass and apply $\vec{F}_{net} = m\frac{d\vec{v}}{dt}$

(m = mass at the particular instant)

4. Integrate it with proper limits to find velocity at any time t.

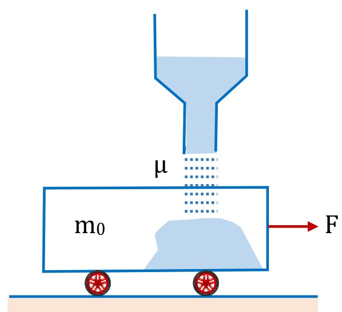
Note:

Problems of one-dimensional motion (which are mostly asked in JEE) can be solved in easier manner just by assigning positive and negative signs to all vector quantities. Here are few example in support of the above theory.

Illustration 36:

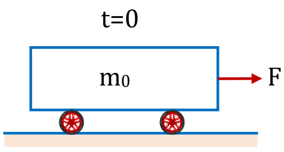
A flat car of mass m_0 starts moving to the right due to a constant horizontal force F. Sand spills on the flat car from a stationary hopper. The rate of loading is constant and equal to μ kg/s.

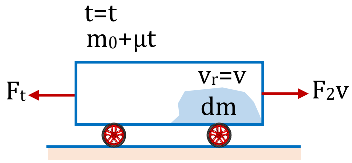
Find the time dependence of the velocity and the acceleration of the flat car in the process of loading. The friction is negligibly small.



Solution:

Initial velocity of the flat car is zero. Let v be its velocity at time t and m its mass at that instant. Then





At $t = 0$, $v = 0$ and $m = m_0$ at $t = t$, $v = v$ and $m = m_0 + \mu t$

Here, $v_r = v$ (backwards)

$$\frac{dm}{dt} = \mu$$

$\therefore$ $F_t = V_r \frac{dm}{dt} = \mu v$ (backwards)

Net force on the flat car at time t is $F_{net} = F - F_t$

or $m\frac{dv}{dt} = F - \mu v$ ...(i)

or $(m_0 + \mu t)\frac{dv}{dt} = F - \mu v$

or $\int_0^v \frac{dv}{F-\mu v} = \int_0^t \frac{dt}{m_0+\mu t}$

$$-\frac{1}{\mu}[ln(F - \mu v)]_0^v = \frac{1}{\mu}[ln(m_0 + \mu t)]_0^t$$

$\Rightarrow$ $ln\left(\frac{F}{F-\mu v}\right) = ln\left(\frac{m_0+\mu t}{m_0}\right)$

$\therefore$ $\frac{F}{F-\mu v} = \frac{m_0+\mu t}{m_0}$

or $v = \frac{Ft}{m_0+\mu v}$

From equation (i),

$\frac{dv}{dt}$ = acceleration of flat car at time t

or $= \frac{F-\mu v}{m}$

$$a = \left(\frac{F - \frac{F\mu t}{m_0+\mu t}}{m_0+\mu t}\right)$$

or $a = \frac{Fm_0}{(m_0+\mu t)^2}$

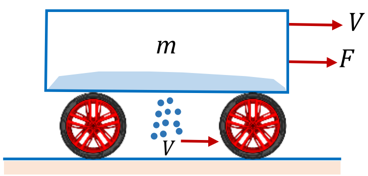
Illustration 37:

A cart loaded with sand moves along a horizontal floor due to a constant force F coinciding in direction with the cart's velocity vector. In the process sand spills through a hole in the bottom with a constant rate μkg/s. Find the acceleration and velocity of the cart at the moment t, if at the initial moment $t = 0$ the cart with loaded sand had the mass m_0 and its velocity was equal to zero. Friction is to be neglected.

Solution:

In this problem the sand spills through a hole in the bottom of the cart. Hence, the relative velocity of the sand v_r will be zero because it will acquire the same velocity as that of the cart at the moment.

$v_r = 0$



Thus, $F_t = 0 \quad \left(as F_t = v_r \frac{dm}{dt}\right)$

and the net force will be F only.

$\therefore \quad F_{net} = F$

or $\quad m\left(\frac{dv}{dt}\right) = F \quad$ ...(i)

But here $m = m_0 - \mu t$

$\therefore \quad (m_0 - \mu t)\frac{dv}{dt} = F \quad$ or $\quad \int_0^v dv = \int_0^t \frac{F dt}{m_0 - \mu t}$

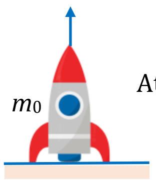
$\therefore \quad v = \frac{F}{-\mu}[ln(m_0 - \mu t)]_0^t \quad$ or $\quad v = \frac{F}{-\mu} ln\left(\frac{m_0}{m_0 - \mu t}\right)$

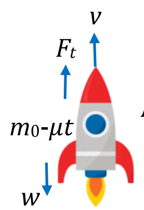
From equation (i), acceleration of the cart

$a = \frac{dv}{dt} = \frac{F}{m} \quad$ or $\quad a = \frac{F}{m_0 - \mu t}$

Rocket propulsion:

Let m_0 be the mass of the rocket at time $t = 0$. m its mass at any time t and v its velocity at that moment. Initially, let us suppose that the velocity of the rocket is u.





Exhaust velocity = v_t

Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases with respect to rocket. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time $t = t$,

1. Thrust force on the rocket, $\quad F_t = v_r\left(\frac{-dm}{dt}\right)$ (upwards)

2. Weight of the rocket, $\quad W = mg \quad$ (downwards)

3. Net force on the rocket, $\quad F_{net} = F_t - W \quad$ (upwards) $\quad$ or $\quad F_{net} = v_r\left(\frac{-dm}{dt}\right) - mg$

4. Net acceleration of the rocket, $a = \frac{F}{m}$

or $\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$

or $dv = \frac{v_r}{m}(-dm) - gdt$

or $\int_u^v dv = v_r \int_{m_0}^{m} \frac{-dm}{m} - g\int_0^t dt$

Thus, $v = u - gt + v_r\, \ell n\left(\frac{m_0}{m}\right)$ ...(i)

Note:

1. $F_t = v_r\left(-\frac{dm}{dt}\right)$ is upwards, as v_r is downwards and $\frac{dm}{dt}$ is negative.

2. If gravity is ignored and initial velocity of the rocket $u = 0$, equation (i) reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$

C- Frame

We may attach a frame of reference, designated X_C, Y_C, Z_C, to the center of mass of a system. Relative to this frame, the centre of mass is at rest ($v_{CM} = 0$). This is called the **centre of mass** or C-frame of reference. In view of equation $P = Mv_{cm}$, the total momentum of a system of particles referred to the C-frame of reference is always zero.

$\vec{P} = \sum_i \vec{P}_i = 0$ in the C-frame of reference

For that reason, the C-frame is sometimes called the **zero momentum frame**. The C-frame is important because many problems can be more simply analyzed in the C-frame compared to ground frame. It is clear that the C-frame moves with a velocity v_{CM} relative to the ground frame. When no external forces act on a system, the C-frame can be considered as inertial.

Illustration 38:

The velocities of two particles of masses m_1 and m_2 relative to an inertial observer are v_1 and v_2. Determine the velocity of the centre of mass relative to the observer and the velocity of each particle relative to the centre of mass.

Solution:

From equation $v_{cm} = \frac{dr_{cm}}{dt} = \frac{1}{M}\sum_i m_i \frac{dr_i}{dt} = \frac{\sum_i m_i v_i}{M}$ the velocity of the centre of mass relative to the observer is

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

The velocities of each particle relative to the centre of mass,

$$v'_{1COM} = v_1 - v_{cm} = v_1 - \left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)$$

$$= \frac{m_2(v_1 - v_2)}{m_1 + m_2} = \frac{m_2 v_{12}}{m_1 + m_2}$$

and $\quad v'_{2COM} = v_2 - v_{cm}$

$$= \frac{m_1(v_2 - v_1)}{m_1 + m_2} = -\frac{m_1 v_{21}}{m_1 + m_2}$$

where $v_{12} = v_1 - v_2$ is the relative velocity of the two particles. Thus, in the C-frame, the two particles appear to be moving in opposite directions, with velocities inversely proportional to their masses.

Let us find relation between kinetic energy of a system from ground frame and C-frame we have a system consisting of many particles, lets say speed of the i^{th} particle is v_i. Then kinetic energy of system, K in ground frame will be summation of individual kinetic energy.

$$K_s = \sum\left(\frac{1}{2}m_i v_i^2\right)$$

Now $\quad \vec{v}_i = \vec{v}_{i/c} + \vec{v}_c$

where $\vec{v}_i$ is velocity of the i^{th} particle in ground, $\vec{v}_{i/c}$ is velocity of the i^{th} particle in reference frame attached to the CM and $\vec{v}_c$ is velocity of CM in ground frame.

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$k_a = \frac{1}{2}\sum m_i \vec{v}_{i/c}^{\,2} + \frac{1}{2}\sum m_i \vec{v}_c^{\,2} + \frac{1}{2}\cdot 2\left(\sum m_i \cdot \vec{v}_{i/c} \cdot \vec{v}_c\right)$$

$$k_a = \sum \frac{1}{2} m_i \vec{v}_{i/c}^{\,2} + \frac{1}{2}\left(\sum m_i\right)\vec{v}_c^{\,2} + \left(\sum m_i \cdot \vec{v}_{i/c}\right)\vec{v}_c$$

we can take $\vec{v}_c$ out of summation in second and third term as it is constant. Now third term becomes zero, as $\sum m_i \vec{v}_{i/c} = M\vec{v}_{c/c}$.

$\vec{v}_{c/c}$ is velocity of COM in frame of COM, which is zero. Also, it represents momentum of system in C-frame which is zero

$$\left(\frac{1}{2}\sum m_i \vec{v}_{i/c}^{\,2}\right) = k_{s/c}$$

Thus, we get,

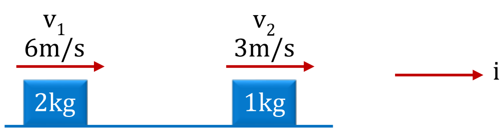
$$k_a = k_{s/c} + \frac{1}{2}mv_c^2$$

where $k_{s/c}$ means kinetic energy of system in C-frame. This important conclusion will be again useful in rotational dynamics we can do little manipulation to write the equation as

$$k_a = k_{s/c} + \frac{P_C^2}{2M}$$

Two Particles in COM Frame

Illustration 39:



(a) $\quad \vec{v}_{cm}\,;\ \vec{v}_{2/COM}\,;\ \vec{v}_{1/COM}$

(b) $\quad KE_2$ and KE_1 in ground frame and $KE_{sys.}$ in ground frame.

(c) KE_2 and KE_1 in *COM* frame and KE_{sys} in *COM* frame.

(d) $\vec{P}_{1/cm}$ and $\vec{P}_{2/COM}$ and $\vec{P}_{sys/COM}$

(e) $\mu = \frac{m_1 m_2}{m_1+m_2}$

(f) $\frac{1}{2}\mu(3)^2$

(g) $\frac{1}{2}mv_{cm}^2$

Solution:

(a) $\vec{v}_{cm} = \frac{6\times2+1\times3}{2+1} = \frac{15}{3} = 5m/s = 5\hat{\imath}$

$\vec{v}_{2/COM} = 6\hat{\imath} - 5\hat{\imath} = 1\hat{\imath}$

$\vec{v}_{1/COM} = 3\hat{\imath} - 5\hat{\imath} = -2\hat{\imath}$

(b) $KE_2 = \frac{1}{2}\times 2 \times (6)^2 = 36$ Joule

$KE_1 = \frac{1}{2}\times 1 \times (3)^2 = 4.5$ Joule

$KE_{sys} = 36 + 4.5 = 40.5$ Joule

(c) In *COM* frame $v_2 = 1$, $v_1 = -2$

$KE_2 = \frac{1}{2}\times 2 \times (1)^2 = 1$ Joule

$KE_1 = \frac{1}{2}\times 1 \times (2)^2 = 2$ Joule

$KE_{sys/COM} = 1 + 2 = 3$ Joule

(d) $P_{1/cm} = 1 \times -2 = -2$ Joule

$P_{2/cm} = 2 \times 1 = 2$ Joule

$P_{sys/COM} = 2 \times 1 - 1 \times 2 = 0$

(e) $\mu = \frac{m_1 m_2}{m_1+m_2} = \frac{2\times1}{2+1} = \frac{2}{3}kg$

(f) $\frac{1}{2}\mu(3)^2 = \frac{1}{2}\times\frac{2}{3}\times 3 \times 3 = 3$

(g) $\frac{1}{2}mv_{cm}^2 \Rightarrow \frac{1}{2}\times 3 \times (5)^2 = 37.5$

Illustration 40:

Solution:

Reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$

$$\left|P_{1/cm}\right| = m_1 v_{1/cm} = \frac{m_1 m_2}{m_1+m_2}(V_{rel.}) = |P_{2/cm}|$$

$$\boxed{KE_{sys/COM} = \frac{1}{2}\mu(V_{rel.})^2}$$

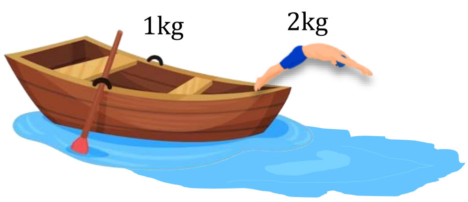
$$\boxed{KE_{sys/ground} = \frac{1}{2}\mu(V_{rel.})^2 + \frac{1}{2}mv_{cm}^2}$$

total mass

$$KE_{sys/g} = KE_{sys/cm} + KE_{cm}$$

Illustration 41:

Man jumps at $6m/s$ w.r.t. final velocity of boat. Find W_{man} ?



Solution:

There is no external force

So, momentum is conserved

$$m_1v_1 + m_2v_2 = 0 \qquad ...(i)$$

Let final velocity of Boat be $v(-\hat{i})$

$$6 = v_M - v_B$$

$$(v_M = 6 - v) \qquad ...(ii)$$

From equation (i) and (ii)

$$v_M = 2m/s \quad ; \quad v_B = 4m/s$$

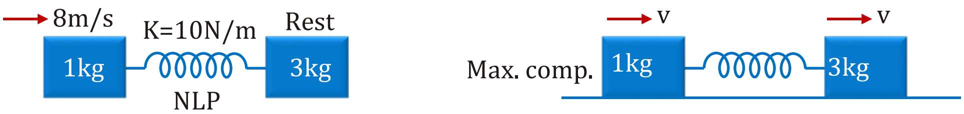
$$KE_f = \frac{1}{2} \times 1 \times (4)^2 + \frac{1}{2} \times 2 \times 2^2 = 12 \text{ Joule}$$

$$\mu = \frac{2}{3}$$

$$W_{man} = KE_f - KE_i$$

$$\left(\frac{1}{2} \times \frac{2}{3} \times 6^2 + \frac{1}{2} \times mv_{cm}^2\right) - \left(\frac{1}{2} \times \frac{2}{3} \times 0^2 + \frac{1}{2} \times mv_{cm}^2\right) = 12 \text{ Joule}$$

Illustration 42:



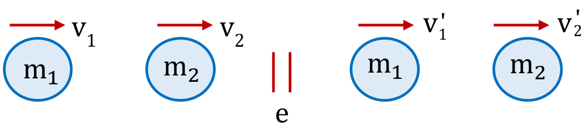
Solution:

$$W_{sp} = KE_f - KE_i$$

$$\frac{1}{2}k\left(0 - x^2\right) = \left(\frac{1}{2} \times \frac{3}{4} \times 0^2 + \frac{1}{2} \times mv_{cm}^2\right) - \left(\frac{1}{2} \times \frac{3}{4} \times 8^2 + \frac{1}{2} \times mv_{cm}^2\right)$$

$$x = \sqrt{\frac{24}{5}}$$

***KE* loss in collision :**



$KE_{loss} = KE_i - KE_f$

$$= \left(\frac{1}{2}\mu v_{app.}^2 + \frac{1}{2}Mv_{cm}^2\right) - \left(\frac{1}{2}\mu v_{sep}^2 + \frac{1}{2}\times Mv_{cm}^2\right)$$

$$= \frac{1}{2}\mu v_{\text{app.}}^2 - \frac{1}{2}\mu v_{\text{sep}}^2$$

$$\boxed{KE_{\text{loss}} = \frac{1}{2}\mu v_{\text{app.}}^2(1-e^2)}$$

$$\frac{v_{\text{sep}}}{v_{\text{app}}} = e$$

A system of two particles :

Suppose the masses of the particles are equal to m_1 and m_2 and their velocities in the K reference frame to $\vec{v}_1$ and $\vec{v}_2$, respectively. Let us find the expressions defining their momenta and the total kinetic energy in the C-frame.

The momentum of the first particle in the C-system is

$$P_{1/c} = m_1\ \vec{v}_{1/c} = m_1(\ \vec{v}_1 - \vec{V}_c\ )$$

where $\vec{V}_c$ is the velocity of the centre of mass (of the C system) in the K reference frame. Substituting in this formula expression $\vec{V}_c = \frac{1}{m}\sum m_i\vec{v}_i = \frac{1}{m}\sum\vec{p}_i$, we obtain

$$\vec{p}_i = \mu(\vec{v}_1 - \vec{v}_2)$$

where μ is the so called **reduced mass** of the system.

$$\mu = \frac{m_1m_2}{m_1+m_2}$$

Similarly, the momentum of the second particle in the C frame is

$$\vec{p}_{2/c} = \mu(\vec{v}_2 - \vec{v}_1)$$

The momenta of the two particles in the C-frame are equal in magnitude and opposite in direction; the modulus of the momentum of each particle is

$$\vec{p}_{1/c} = \mu v_{rel}$$

where $v_{rel} = |\vec{v}_1 - \vec{v}_2|$ is the velocity of one particle relative to another.

Finally, let us consider kinetic energy. The total kinetic energy of the two particles in the C-frame is

$$K_{s/c} = K_1 + K_2 = \frac{\vec{P}^2}{2m_1} + \frac{\vec{P}^2}{2m_2}$$

Since in accordance with equation,

$$\mu = \frac{m_1m_2}{m_1+m_2}, \frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}, \text{ then}$$

$$K_{s/c} = \frac{p^2}{2\mu} = \frac{\mu v_{rel}^2}{2}$$

If the particles interact, then total mechanical energy in the C frame is

$$E = T + U$$

where U is the potential energy of interaction of the given particles.

e.g.

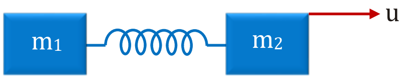
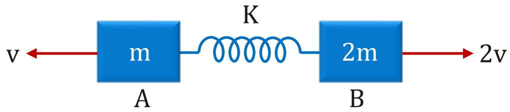


Illustration 43:

Two blocks A and B of masses m and $2m$ placed on smooth horizontal surface are connected with a light spring. The two blocks are given velocities as shown when spring is at natural length.



(i) Find velocity of centre of mass.

(ii) Maximum extension in the spring.

Solution:

Velocity of $C.M.$ $v_{cm} = \dfrac{3mv - mv}{3m} = v$

In $C.O.M.$ frame. Initial momentum = 0

At the time of maximum elongation both the masses will be moving in same direction with same speed.

Initial relative velocity, $v_{rel} = 3v$

Dec. in KE = Increase in PE of spring

$$\frac{1}{2}\mu\, v_{rel}^2 = \frac{1}{2}kx^2$$

$$\frac{1}{2}\frac{m\times 2m}{3m}(3v)^2 = \frac{1}{2}kx^2$$

$$3mv^2 = \frac{1}{2}kx^2$$

$$x = v\sqrt{\frac{6m}{k}}$$

Objective : To learn how to apply *C* - frame.

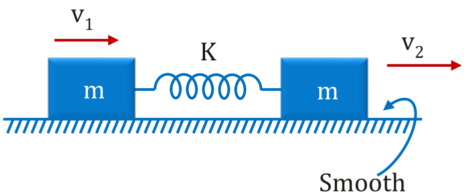


Illustration 44:

Two identical blocks of mass m, each are connected by a spring as shown in the figure. At any instant of time t = 0, one block is given a velocity v_1 and other is given a velocity v_2 ($v_1 > v_2$) in the same direction simultaneously as shown in the figure. Find the maximum energy stored in the spring.

Ans. $\mathbf{\dfrac{1}{4}m(v_1 - v_2)^2}$

Solution:

$$v_{COM} = \frac{mv_1+mv_2}{2m} = \frac{v_1+v_2}{2}$$

$$v_{1_{COM}} = v_1 - \frac{(v_1+v_2)}{2} = \frac{2v_1-v_1-v_2}{2} = \frac{v_1-v_2}{2}$$

$$v_{2_{COM}} = v_2 - \left(\frac{v_1+v_2}{2}\right) = \frac{2v_2-v_1-v_2}{2} = \frac{v_2-v_1}{2}$$

$$KE_{S/COM} = \frac{1}{2}\mu(v_1 - v_2)^2 + \frac{1}{2}\mu(v_1 - v_2)^2$$

$$= \frac{m}{4}(v_1 - v_2)^2$$

Maximum kinetic energy of system (COM) = Maximum potential energy is stored in spring.