02 Kinematics

MOTION IN A STRAIGHT LINE

Kinematics

Kinematics deals with study of motion of a particle without any regard to cause of motion.

Motion and Rest

Motion is a combined property of the object and the observer. There is no meaning of rest or motion without the observer. Nothing is in absolute rest or in absolute motion.

An object is said to be in motion with respect to an observer, if its position changes with time with respect to that observer. It may happen by both ways either observer moves or object moves.

Mechanics is classified under two streams namely kinematics and dynamics.

MECHANICS

- **Kinematics** → The word kinematics means 'science of motion'. Branch of mechanics which deals with study of motion without going into the cause of motion, i.e. force, torque etc.
- **Dynamics** → Study of forces and their effect on objects in motion

Types of Motion

1-D Motion: The motion which can be captured along a straight line.

Example – A train moving on a straight track.

2-D Motion: The motion which can be captured on a plane but not on a straight line.

Example – Motion of a cricket ball.

3-D Motion: The motion which cannot be captured even on a plane.

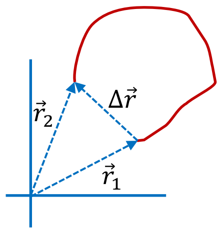
Example – Motion of a kite or bird.

Parameters of Motion

Position

The position of a particle refers to its location in the space at a certain moment of time. It is concerned with the question– “where is the particle at a particular moment of time?”

Displacement



$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$

The change in the position of a moving object is known as displacement. It is the vector joining the initial position $(\vec{r}_1)$ of the particle to its final position $(\vec{r}_2)$ during an interval of time.

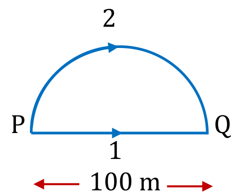
Displacement can be negative, positive or zero.

Distance

The length of the actual path travelled by a particle during a given time interval is called as distance. The distance travelled is a scalar quantity which is quite different from displacement. In general, the distance travelled between two points may not be equal to the magnitude of the displacement between the same points. Distance can never be negative.

Illustration 1:

Ram takes path 1 (straight line) to go from P to Q and Shyam takes path 2 (semicircle).



(a) Find the distance travelled by Ram and Shyam.

(b) Find the displacement of Ram and Shyam.

Solution:

(a) Distance travelled by Ram = 100 m

Distance travelled by Shyam = $\pi(50\ m) = 50\pi\ m$

(b) Displacement of Ram = 100 m

Displacement of Shyam = 100 m

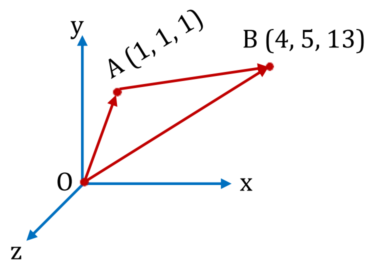
Illustration 2:

A particle starts from point (1, 1, 1) m and reaches at point (4, 5, 13) m. Find:

(i) Initial position vector

(ii) Final position vector

Solution:



Initial position vector $(\overrightarrow{OA})$

$$\overrightarrow{OA} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

Final position vector $(\overrightarrow{OB})$

$$\overrightarrow{OB} = 4\hat{\imath} + 5\hat{\jmath} + 13\hat{k}$$

Displacement, $\Delta\vec{r} = (4-1)\hat{\imath} + (5-1)\hat{\jmath} + (13-1)\hat{k} = 3\hat{\imath} + 4\hat{\jmath} + 12\hat{k}$

Average and instantaneous rate of change

Average:

Average rate of change is defined for an interval. Average rate of change of y with respect to x is written as $\dfrac{\Delta y}{\Delta x}$.

Instantaneous:

Instantaneous rate of change is defined for a given instant. Instantaneous rate of change of y with respect to (w.r.t.) x is written as $\frac{dy}{dx}$ which is also known as differentiation of y with respect to x.

(eg.)	**Average**	**Instantaneous**
Current	$<I>=\frac{\Delta q}{\Delta t}$	$I=\frac{dq}{dt}$
Force	$<\vec{F}>=\frac{\Delta\vec{P}}{\Delta t}$	$\vec{F}=\frac{d\vec{P}}{dt}$
Velocity	$<\vec{v}>=\frac{\Delta\vec{r}}{\Delta t}$	$\vec{v}=\frac{d\vec{r}}{dt}$

Derivatives of Commonly Used Functions

- $y = \text{constant} \Rightarrow \frac{dy}{dx}=0$
- $y = x^n \Rightarrow \frac{dy}{dx}=nx^{n-1}$
- $y = e^x \Rightarrow \frac{dy}{dx}=e^x$
- $y = \ln x \Rightarrow \frac{dy}{dx}=\frac{1}{x}$
- $y = \sin x \Rightarrow \frac{dy}{dx}=\cos x$
- $y = \cos x \Rightarrow \frac{dy}{dx}=-\sin x$
- $y = \tan x \Rightarrow \frac{dy}{dx}=\sec^2 x$
- $y = \cot x \Rightarrow \frac{dy}{dx}=-\text{cosec}^2 x$
- $y = \sec x \Rightarrow \frac{dy}{dx}=\sec x\tan x$
- $y = \text{cosec}\, x \Rightarrow \frac{dy}{dx}=-\text{cosec}\, x\cot x$

Method of Differentiation :

If $y = f(x)$, let us denote $\frac{dy}{dx}=f'(x)$

- Sum or Subtraction of two functions

$$y=f(x)\pm g(x)\Rightarrow\frac{dy}{dx}=f'(x)\pm g'(x)$$

- Product of two functions (Product rule)

$$y=f(x)\cdot g(x)\Rightarrow\frac{dy}{dx}=g(x)f'(x)+f(x)g'(x)$$

- Division of two functions (Quotient rule)

$$y=\frac{f(x)}{g(x)}\Rightarrow\frac{dy}{dx}=\frac{g(x)f'(x)-f(x)g'(x)}{\{g(x)\}^2}$$

- Chain Rule

$$y=f\{g(x)\}\Rightarrow\frac{dy}{dx}=g'(x)f'\{g(x)\}$$

Illustration 3:

Find the value of $\frac{dy}{dx}$ if -

(i) $y = 3x^{-2}$ (ii) $y = (x)^{\frac{3}{4}}$ (iii) $y = \frac{1}{x}$

Solution:

(i) $\frac{dy}{dx} = 3\frac{d}{dx}x^{-2} = 3(-2)x^{-2-1} = \frac{-6}{x^3}$

(ii) $\frac{dy}{dx} = \frac{d}{dx}x^{\frac{3}{4}} = \frac{3}{4}x^{\frac{3}{4}-1} = \frac{3}{4}x^{\frac{-1}{4}}$

(iii) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-1-1} = \frac{-1}{x^2}$

Illustration 4:

Find $\frac{dy}{dx}$, when - (i) $y = \sqrt{x}$, (ii) $y = x^5 + x^4 + 7$, (iii) $y = x^2 + 4x^{-1/2} - 3x^{-2}$

Solution:

(i) $\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

(ii) $\frac{dy}{dx} = \frac{d}{dx}(x^5 + x^4 + 7) = \frac{d}{dx}(x^5) + \frac{d}{dx}(x^4) + \frac{d}{dx}(7) = 5x^4 + 4x^3 + 0 = 5x^4 + 4x^3$

(iii) $\frac{dy}{dx} = \frac{d}{dx}(x^2 + 4x^{-1/2} - 3x^{-2}) = \frac{d}{dx}(x^2) + \frac{d}{dx}(4x^{-1/2}) - \frac{d}{dx}(3x^{-2})$

$= \frac{d}{dx}(x^2) + 4\frac{d}{dx}(x^{-1/2}) - 3\frac{d}{dx}(x^{-2}) = 2x + 4\left(-\frac{1}{2}\right)x^{-3/2} - 3(-2)x^{-3} = 2x - 2x^{-3/2} + 6x^{-3}$

Illustration 5:

Find value of $\frac{dx}{dt}$ if : $x = \sqrt{t} + \frac{1}{\sqrt{t}} + 1$

Solution:

$$\frac{dx}{dt} = \frac{d}{dt}(t^{1/2} + t^{-1/2} + 1)$$

$$= \frac{1}{2}t^{\frac{1}{2}-1} + \left(\frac{-1}{2}\right)t^{\frac{-1}{2}-1} + 0$$

$$= \frac{1}{2\sqrt{t}} - \frac{1}{2t^{3/2}} = \frac{1}{2\sqrt{t}}\left(1 - \frac{1}{t}\right)$$

Illustration 6:

Find $\frac{dy}{dx}$, when $y = \frac{x^2}{\sin x}$.

Solution:

$$\frac{dy}{dx} = \frac{\sin x\left(\frac{dx^2}{dx}\right) - x^2\left(\frac{d\sin x}{dx}\right)}{(\sin x)^2} = \frac{(\sin x)2x - x^2\cos x}{(sinx)^2}$$

- If $y = c\sin(ax + b)$

$$\frac{dy}{dx} = [c\cos(ax+b)][a]$$

Illustration 7:

Find $\frac{dy}{dt}$ if -

(a) $y = 3\sin(2t + 4)$ (b) $y = 4\cos(3t)$

Solution:

(a) $\frac{dy}{dt} = [3\cos(2t+4)][2]$

$= 6\cos(2t + 4)$

(b) $\frac{dy}{dt} = 4[-\sin(3t)][3]$

$= -12\sin 3t$

Illustration 8:

Find the value of $\frac{dy}{dx}$ if -

(i) $y = \sin(x^2)$ (ii) $y = e^{\sqrt{x}}$

Solution:

(i) $\frac{dy}{dx} = \frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot\frac{d}{dx}(x^2)$

$= 2x\cos(x^2)$

(ii) $\frac{dy}{dx} = \frac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}}\frac{d}{dx}\sqrt{x}$

$= \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

Average Velocity (in an interval)

The average velocity of a moving particle over a certain time interval is defined as the displacement divided by the lapsed time.

$$\text{Average velocity } \langle\vec{v}\rangle = \frac{\text{Displacement}}{\text{Time interval}}$$

For straight line motion, along x-axis, we have

$$v_{av} = \bar{v} = \langle v \rangle = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

The dimension of velocity is $[LT^{-1}]$ and its SI unit is m/s.

The average velocity is a vector in the direction of displacement. For motion in a straight line, directional aspect of a vector can be taken care of by +ve and -ve sign of the quantity.

Instantaneous Velocity (at an instant)

The velocity at a particular instant of time is known as instantaneous velocity. The term "velocity" usually means instantaneous velocity.

$$V_{inst.} = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}$$

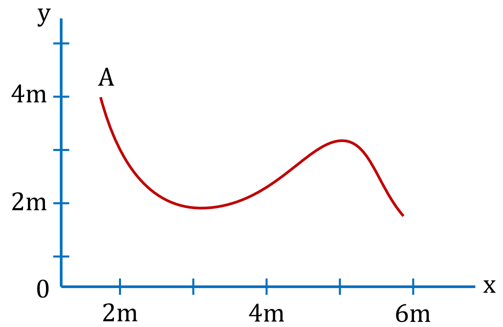
In other words, the instantaneous velocity at a given moment (say, t) is the limiting value of the average velocity as we let Δt approach zero. The limit as $\Delta t \to 0$ is written in calculus notation as dx/dt and is called the derivative of x with respect to t.

Note:

- The magnitude of instantaneous velocity and instantaneous speed are equal.
- The determination of instantaneous velocity by using the definition usually involves calculation of derivative. We can find $v = \frac{dx}{dt}$ by using the standard results from differential calculus.
- Instantaneous velocity is always tangential to the path.

Illustration 9:

A particle starts from a point A and travels along the solid curve shown in figure. Find approximately the position B of the particle such that the average velocity between the positions A and B has the same direction as the instantaneous velocity at B.



Solution:

The given curve shows the path of the particle starting at $y = 4\ m.$

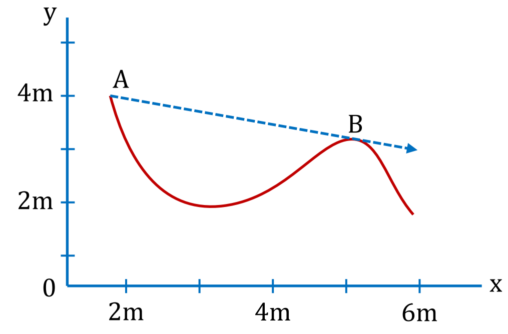
$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time taken}}$$

Where displacement is straight line distance between points.

Instantaneous velocity at any point is the tangent drawn to the curve at that point.

Now, as shown in the graph, line AB shows displacement as well as the tangent to the given curve.

Hence, point B ($x = 5m$, $y = 3m$) is the point at which direction of AB shows average as well as instantaneous velocity.



Average Speed (in an interval)

Average speed is defined as the total path length travelled divided by the total time interval during which the motion has taken place. It helps in describing the motion along the actual path.

$$\text{Average speed } \langle v \rangle = \frac{\text{distance travelled}}{\text{time interval}}$$

The dimension of speed is $[LT^{-1}]$ and its SI unit is m/s.

Note:

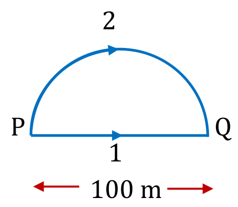
- Average speed is always positive in contrast to average velocity which being a vector, can be positive or negative.
- If the motion of a particle is along a straight line and in same direction then, magnitude of average velocity = average speed.

  $$\langle v \rangle = |\langle \vec{v} \rangle|$$
- Average speed is, in general, greater than or equal to the magnitude of average velocity.

  $$\langle v \rangle \geq |\langle \vec{v} \rangle|$$

Illustration 10:

In the Illustration 1, if Ram takes 4 seconds to go from P to Q by path 1 and Shyam takes 5 seconds to go from P to Q by path 2 find:



(a) Average speed of Ram and Shyam.

(b) Average velocity of Ram and Shyam.

Solution:

(a) Average speed of Ram = $\frac{100}{4}\ m/s = 25\ m/s$

Average speed of Shyam = $\frac{50\pi}{5} m/s = 10\pi\ m/s$

(b) Average velocity of Ram = $\frac{100}{4}\ m/s = 25\ m/s$ (From P to Q)

Average velocity of Shyam = $\frac{100}{5}\ m/s = 20\ m/s$ (From P to Q)

Illustration 11:

A particle travels half of total distance with speed v_1 and next half with speed v_2 along a straight line. Find out the average speed of the particle.

Solution:

Let total distance travelled by the particle be $2s$.

Time taken to travel first half = $\dfrac{s}{v_1}$

Time taken to travel next half = $\dfrac{s}{v_2}$

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}} = \frac{2s}{\dfrac{s}{v_1}+\dfrac{s}{v_2}} = \frac{2v_1v_2}{v_1+v_2}$$

(v_1, average speed and v_2 are in harmonic progression)

Illustration 12:

A person travelling on a straight line moves with a uniform velocity v_1 for some time and with uniform velocity v_2 for the next equal time. The average velocity v is given by -

Solution:

As shown, the person travels from A to B through a distance S, where first part S_1 is travelled in time $t/2$ and next S_2 also in time $t/2$.

So, according to the condition:

$$v_1 = \frac{S_1}{t/2} \text{ and } v_2 = \frac{S_2}{t/2}$$

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time taken}} = \frac{S_1+S_2}{t} = \frac{\dfrac{v_1t}{2}+\dfrac{v_2t}{2}}{t} = \frac{v_1+v_2}{2}$$

Average acceleration (in an interval)

The average acceleration for a finite time interval is defined as :

$$\text{Average acceleration} = \frac{\text{Change in velocity}}{\text{Time interval}}$$

Average acceleration is a vector quantity whose direction is same as that of the change in velocity.

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f-\vec{v}_i}{\Delta t}$$

Since for a straight-line motion the velocities are along a line, therefore

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

(where one has to substitute v_f and v_i with proper signs in one dimensional motion)

Instantaneous Acceleration (at an instant)

The instantaneous acceleration of a particle is its acceleration at a particular instant of time. It is defined as the derivative (rate of change) of velocity with respect to time. We usually mean instantaneous acceleration when we say "acceleration". For straight motion we define instantaneous acceleration as:

$$a = \frac{dv}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta v}{\Delta t}\right) \text{ and in general } \vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{v}}{\Delta t}\right)$$

The dimension of acceleration is $[LT^{-2}]$ and its SI unit is m/s^2.

Illustration 13:

Position of particle moving along x-axis is given as $x = \frac{t^3}{3} - \frac{3}{2}t^2 + 2t + 5$.

Find:(a) position at t = 2 sec. (b) velocity at t = 2 sec.

(c) acceleration at t = 2 sec. (d) acceleration when velocity = 0

(e) velocity when acceleration = 0 (f) avg. acceleration from t = 2 sec. to t = 3 sec.

Solution:

(a) $x_{t=2} = \frac{(2)^3}{3} - \frac{3}{2}(2)^2 + 2\times2+5$

$= \frac{8}{3} - 3\times2+4+5 = \frac{8-18+12+15}{3}$

$= \frac{35-18}{3} = \frac{17}{3}m$

(b) Velocity = $\frac{dx}{dt} = \frac{3t^2}{3} - \frac{3}{2}2t + 2$

Velocity = $t^2 - 3t + 2$

$v_{t=2} = 4 - 6 + 2$

$= 0\ m/s$

(c) Acceleration = $\frac{dv}{dt} = 2t - 3$

$a = 2t - 3$

$a_{t=2} = 2 \times 2 - 3$

$a_{t=2} = 1\ m/s^2$

(d) $v = t^2 - 3t + 2$

$0 = t^2 - 3t + 2$

$t^2 - 2t - t + 2 = 0$

$(t - 2)(t - 1) = 0$

$t = 2, t = 1$

$\therefore\ a = 2t - 3$

$= 2 \times 2 - 3$

$= 1 m/s^2$

$a = 2t - 3$

$= 2 \times 1 - 3$

$= -1\ m/s^2$

(e) $a = 2t - 3$

$0 = 2t - 3$

$2t = 3$

$t = \frac{3}{2}$

$v = t^2 - 3t + 2$

$= \left(\frac{3}{2}\right)^2 - 3 \times \frac{3}{2} + 2 = \frac{9}{4} - \frac{9}{2} + 2$

$\Rightarrow v = \frac{9 - 18 + 8}{4} = \frac{-1}{4}\ m/s$

(f) $< a > = \frac{v_3 - v_2}{3 - 2}$ $\quad \begin{bmatrix} v_3 = 3^2 - 3 \times 3 + 2 = 9 - 9 + 2 = 2 \\ v_2 = (2)^2 - 3 \times 2 + 2 = 4 - 6 + 2 = 0 \end{bmatrix}$

$= \frac{2 - 0}{1}$

$= 2 m/s^2$

Illustration 14:

Position of a particle as function of time is given by

$$\vec{r} = t^2\,\hat{i} + 2t\,\hat{j} + (\sin t)\hat{k}$$

Find velocity and acceleration as function of time.

Solution:

(a) Velocity, $\vec{v} = \frac{d\vec{r}}{dt} = 2t\,\hat{i} + 2\,\hat{j} + \cos t\,\hat{k}$

(b) Acceleration, $\vec{a} = \frac{d\vec{v}}{dt} = 2\hat{i} + (-\sin t)\hat{k}$

Illustration 15:

Position of a particle moving along x-axis is given as:

$x = 8t - t^2$

Find displacement and distance travelled from 0 to 5 sec.

Solution:

$s = x_f - x_i = x_5 - x_0 = (8\,(5) - 5^2) - (0 - 0)$

Displacement, $s = 15\ m$

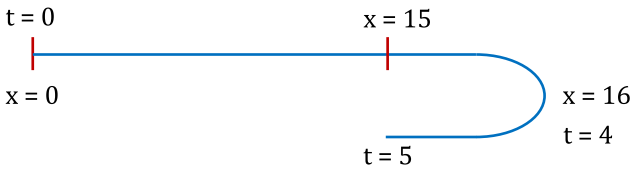
To determine the distance first let us find the turning point.

$$v = \frac{dx}{dt} = 8 - 2t$$

$v = 2(4 - t)$

$0 = 2(4 - t)$

$\therefore \quad t = 4$ sec. turning point



$x_4 = 8 \times 4 - 4^2 = 16m$

Distance $= |x_4 - x_0| + |x_5 - x_4|$

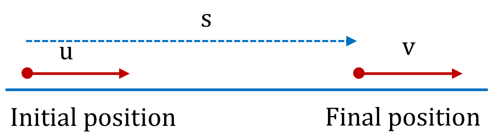
$= |16 - 0| + |15 - 16|$

$= 16 + 1 = 17\ m$

Uniformly accelerated motion

If a particle is accelerated with constant acceleration in an interval of time, then the motion is termed as uniformly accelerated motion in that interval of time.

For uniformly accelerated motion along a straight line (x-axis) during a time interval of t seconds, the following important results can be used.



First equation of motion:

$$a = \frac{v - u}{t} \qquad \text{or} \qquad \boxed{v = u + at}$$

Second equation of motion:

$$S = (v_{av})t \qquad v_{av} = \frac{v+u}{2}$$

$$S = \left(\frac{v+u}{2}\right)t \qquad \text{or} \qquad \boxed{s = ut + \frac{1}{2}at^2}$$

or $\quad s = vt - \frac{1}{2}\,at^2 \qquad$ or $\qquad x_f = x_i + ut + \frac{1}{2}\,at^2$

Third equation of motion:

$$\boxed{v^2 = u^2 + 2as}$$

Displacement in nth second:

$s_{n^{th}}$ = displacement in n seconds – displacement in $(n–1)$ seconds

$$s_{n^{th}} = s_n - s_{n-1}$$

$$s_{n^{th}} = \left(un + \frac{1}{2}an^2\right) - \left(u(n-1) + \frac{1}{2}a(n-1)^2\right)$$

$$\boxed{s_{n^{th}} = u + \frac{a}{2}(2n-1)}$$

u = Initial velocity (at the beginning of interval)

a = Acceleration

v = Final velocity (at the end of interval)

s = Displacement $(x_f - x_i)$

x_f = Final coordinate (position)

x_i = Initial coordinate (position)

$s_{n^{th}}$ = Displacement during the n^{th} sec

Motion with uniform velocity (a = 0):

Consider a particle moving along x-axis with uniform velocity u starting from the point $x = x_i$ at $t = 0$.

Equations of x, v, a are: $x(t) = x_i + ut$; $v(t) = u$; $a(t) = 0$

Directions of vectors in straight line motion:

In straight line motion, all the vectors (position, displacement, velocity and acceleration) will have only one component (along the line of motion) and there will be only two possible directions for each vector. Select any one direction as positive and another as negative.

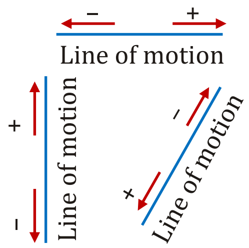
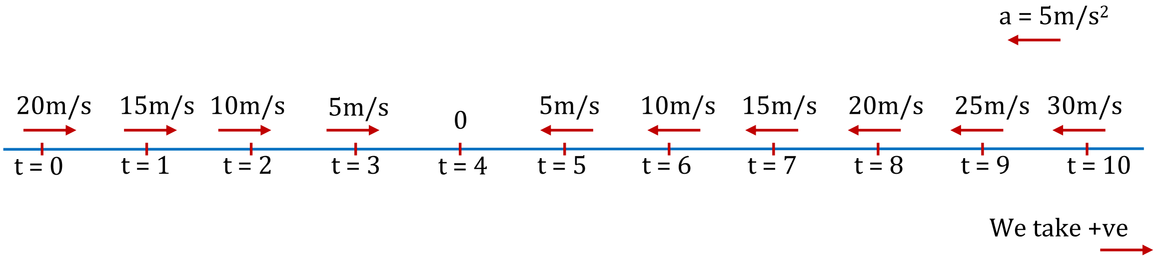


Illustration 16:

Initial velocity of a particle is 20m/s. Its acceleration is 5m/s^2 in the opposite direction as shown in the figure. Fill the table for the asked time intervals.



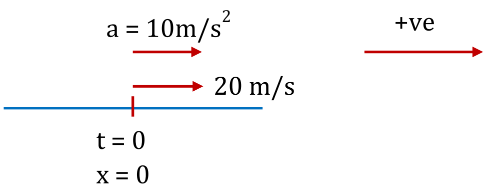
	u	***v***	***a***	***t***	***S***
t = 0 sec. to t = 3 sec.					
t = 2 sec. to t = 4 sec.					
t = 1 sec. to t = 5 sec.					
t = 1 sec. to t = 7 sec.					
t = 4 sec. to t = 7 sec.					
t = 1 sec. to t = 9 sec.					

Solution:

	u	***v***	***a***	***t***	***s***
t = 0 sec. to t = 3 sec.	+20	+5	–5	3	+ve
t = 2 sec. to t = 4 sec.	+10	0	–5	2	+ve
t = 1 sec. to t = 5 sec.	+15	–5	–5	4	+ve
t = 1 sec. to t = 7 sec.	+15	–15	–5	6	0
t = 4 sec. to t = 7 sec.	0	–15	–5	3	–ve
t = 1 sec. to t = 9 sec.	+15	–25	–5	8	–ve

Illustration 17:

Initial velocity of a particle is 20m/s. Its acceleration is 10m/s^2 in the same direction as shown in the figure. Find its velocity and position at t = 1 and 3 sec.



Solution:

0 to 1 sec.	**0 to 3 sec.**
$u = +20$	$u = +20$
$a = +10$	$a = +10$
$t = 1$	$t = 3$
$v = u + at$	$v = u + at$

$v = 20 + 10 \times 1$	$v = 20 + 10 \times 3$
$v = 30 m/s$	$v = 50\ m/s$
$s = \left(\frac{v+u}{2}\right)t$	$s = \left(\frac{v+u}{2}\right)t$
$= \left(\frac{30+20}{2}\right)\times 1$	$= \left(\frac{50+20}{2}\right)\times 3 = 35 \times 3$
$= 25m$	$= 105\ m$
$\therefore\ s = x_f - x_i$	$s = x_f - x_i$
$25 = x_f - 0$	$105 = x_f - 0$
$x_f = 25\ m$	$x_f = 105\ m$

Illustration 18:

A car starting from rest travels $10m$ in first second. Find distance travelled from $5s$ to $7s$ if car has constant acceleration.

Solution:

0 to 1 sec.

$u = 0$

$s = 10\ m$

$t = 1$ sec

$$s = ut + \frac{1}{2}at^2$$

$$10 = 0\times 1 + \frac{1}{2}\times a \times 1$$

$$10 = 0 + \frac{a}{2}$$

$a = 20 m/s^2$

0 to 5 sec.

$u = 0$

$t = 5$ sec.

$a = 20$

$v = 0 + 20 \times 5$

$= 100\ m/s$

5 to 7 sec.

$t = 2$

$a = 20\ m/s$

$u = 100\ m/s$

$$s = 100\times 2 + \frac{1}{2}\times 20\times 2\times 2$$

$= 200 + 40$

$= 240\ m$

Reaction Time:

When a situation demands our immediate action. It takes some time before we really respond. Reaction time is the time a person takes to observe, think and act.

Illustration 19:

A car is moving with 54 km/hr. Driver sees the traffic signal changing from green to red and applies the brakes after a reaction time of 0.3 sec. If retardation of the car is 3 m/s^2 then find the distance travelled by the car before it stops.

Solution:

$v = 54\ km/hr = 15\ m/s$

$s_1 = 15\times 0.3 = 4.5m$ ...(i)

$v^2 = u^2 + 2as_2$

$$0 = 225 - 2 \times 3 \times s_2$$

$$s_2 = \frac{225}{6} = 37.5m \qquad \text{...(ii)}$$

$$s = s_1 + s_2 = 42m$$

Illustration 20:

A driver of a car which is going with 20 m/s and was at $x = 0$ at $t = 0$ applies brake on seeing a red signal. Signal is at $x = 100m$. Find the distance travelled by car after 6 sec. if brakes produced retardation of 4 m/s^2.

Solution:

First let's check when the car stops

$v = u + at$

$0 = 20 + (-4t)$

$4t = 20$

$t = 5$ sec.

So, distance travelled by the car in 6 sec. is same as travelled in 5 sec.

0 to 5 sec.

$u = 20\ m/s$

$a = -4m/s^2$

$t = 5$ sec.

$$s = ut + \frac{1}{2}at^2$$

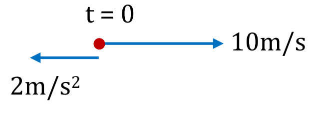
$$s = 20 \times 5 + \frac{1}{2} \times (-4) \times 5 \times 5$$

$s = 100 - 50$

$s = 50\ m$

Illustration 21:

A particle is moving with a speed of $10m/s$. A constant acceleration of $2m/s^2$ acts in the opposite direction. Find distance travelled from



(i) 0 to 3 sec., (ii) 3 to 6 sec.

Solution:

Let's check the turning point-

$v = u + at$

$0 = 10 - 2 \times t$

$$\frac{-10}{-2} = t \qquad \Rightarrow \quad t = 5 \text{ sec.}$$

(i) 0 to 3 sec.

$t = 3$

$a = -2$

$u = 10$

Displacement = distance

$$s = 10\times3+\frac{1}{2}\times-2\times(3)^2$$

$s_3 = 30 - 9$

$s_3 = 21\ m$

(ii) $s_5 = 10\times5+\frac{1}{2}\times-2\times5\times5$

$\Rightarrow\ s_5 = 50-25$

$s_5 = 25m$

$s_6 = 10\times6+\frac{1}{2}\times(-2)\times6\times6 = 60 - 36 = 24\ m$

$d_{3-6} = |25 - 21| + |24 - 25| = 5m$

Note:

- If acceleration is in same direction as velocity, then speed of the particle increases.
- If acceleration is in opposite direction to the velocity then speed decreases i.e., the particle slows down. This situation is known as retardation.
- If particle is retarding or decelerating, then it means speed is decreasing. It does not mean negative acceleration.

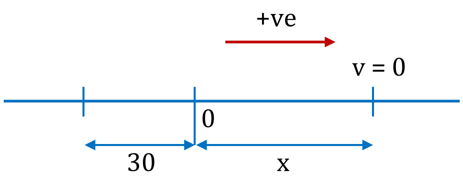
Illustration 22:

A particle starts with velocity 10 ms^{-1} and deceleration of 5 ms^{-2}. Find displacement and distance travelled in 6 sec.

Solution:

Displacement uniform acceleration $\Rightarrow$ equations of motion can be used

$u = 10\ ms^{-1}$; $a = -5\ ms^{-2}$; $t = 6$ sec; $s = ?$



$$s = ut + \frac{1}{2}at^2$$

$$s = 10 \times 6 + \frac{1}{2}\times(-5)\times 6\times 6 = 60 - 90 = -30\ m$$

For finding distance travelled we should find displacement for $v = 0$

$u = 10,\ a = -5,\ v = 0$

$\therefore\quad v^2 = u^2 + 2ax$

$\Rightarrow \quad x = \dfrac{v^2 - u^2}{2a} = \dfrac{(0)^2 - (10)^2}{2(-5)} = 10\ m$

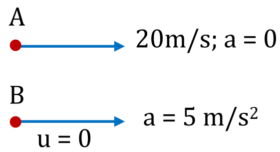
Total distance =|10–0 |+|–30-10|= 50 m

Objective:

To emphasise that in equation of motion *'S'* stands for displacement not for distance. To calculate distance, first find the turning point and break the motion.

Illustration 23:

Two particles A and B are situated at same place initially. Particle A moves with constant speed 20 m/s while B starts moving with constant acceleration 5 m/s^2 in the same direction. Find where and when they will cross each other.



Solution:

$s_1 = s_2$

$$u_1 t + \frac{1}{2} a_1 t^2 = u_2 t + \frac{1}{2} a_2 t^2$$

$$20 \times t + \frac{1}{2} \times 0 \times t^2 = 0 \times t + \frac{1}{2} \times 5 \times t^2 \quad \Rightarrow \quad 20t = \frac{5t^2}{2}$$

$t^2 = 8t \qquad \Rightarrow \quad t^2 - 8t = 0 \qquad \Rightarrow \quad t(t-8) = 0$

$\therefore \quad t = 0, \quad t = 8 \qquad t = 0$ means both particles were together initially.

$$s_A = 20 \times 8 + \frac{1}{2} \times 0 \times 8 \times 8 = 20 \times 8 = 160\ m$$

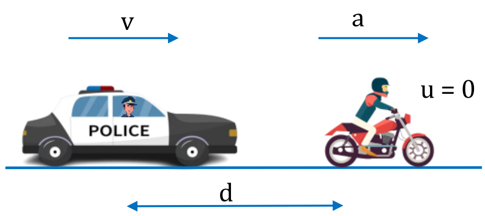
Illustration 24:

A police inspector in a jeep is chasing a pickpocket on a straight road. The jeep is going at its maximum speed v (assumed uniform). The pickpocket rides on the motorcycle of a waiting friend when the jeep is at a distance d away, and the motorcycle starts with a constant acceleration a. Show that the pick pocket will be caught if $v \geq \sqrt{2ad}$.

Solution:

Suppose the pickpocket is caught at a time t after motorcycle starts. The distance travelled by the motorcycle during this interval is

$$s = \frac{1}{2} a t^2 \qquad \text{...(1)}$$



During this interval the jeep travels a distance

$$s + d = vt \qquad \text{...(2)}$$

By (1) and (2),

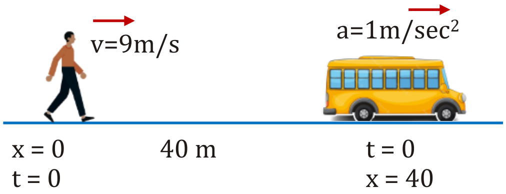
$$\frac{1}{2} a t^2 + d = vt \qquad \text{or,} \qquad t = \frac{v \pm \sqrt{v^2 - 2ad}}{a}$$

The pickpocket will be caught if t is real and positive. This will be possible if

$$v^2 \geq 2ad \qquad \text{or,} \qquad v \geq \sqrt{2ad}$$

Illustration 25:

A man is standing 40 m behind the bus. Bus starts with 1 m/sec^2 constant acceleration and also at the same instant the man starts moving with constant speed 9 m/s. Find the time taken by man to catch the bus.



Solution:

Let after time 't' man will catch the bus

For bus

$$x = x_0 + ut + \frac{1}{2}at^2, \qquad x = 40 + 0(t) + \frac{1}{2}(1)t^2$$

$$x = 40 + \frac{t^2}{2} \qquad \text{...(i)}$$

For man, $x = 9t$ ...(ii)

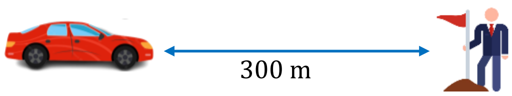
From (i) and (ii)

$$40 + \frac{t^2}{2} = 9t \quad \Rightarrow t = 8\,s \quad \text{or} \quad t = 10s.$$

Man catches the bus at $t = 8s.$

Illustration 26:

A car can travel at maximum speed of 180 km/hr and can have maximum acceleration of 5 m/s^2 and retardation of 3 m/s^2. How fast can it start from rest and come to rest in travelling 300 m?



Solution:

$v_{max} = 180\ km/hr = 50\ m/s$

$a = 5\ m/s^2; \quad r = 3\ m/s^2; \ v = 0; \quad u = 0; \quad s = 300\ m$

Part-I : Let's assume car accelerates for x $m.$

$u = 0;\ a = 5m/s^2$

$v = v_1$

$v_1 = \sqrt{10x}\ m/s$

$$t_1 = \frac{\sqrt{10x}}{5}\ \text{sec}$$

Part-II: Retarding

$s = (300 - x)$ meter

$u = \sqrt{10x}$; $v = 0$; $a = -3$

$v^2 = u^2 + 2as$

$0 = 10x + 2(300 - x)(-3)$

$\Rightarrow 10x = 6(300 - x)$ $\therefore\ x = 112.5$

Solving for t_1

$t_1 = 3\sqrt{5}$

$$t_2 = \frac{\sqrt{10x}}{3} = \frac{\sqrt{10\times 112.5}}{3} = \sqrt{\frac{1125}{9}} = \sqrt{125} = 5\sqrt{5}$$

Total time taken = $3\sqrt{5} + 5\sqrt{5} = 8\sqrt{5}$ sec

Illustration 27:

A particle moving rectilinearly with constant acceleration is having initial velocity of 10 m/s. After some time, its velocity becomes 30 m/s. Find out velocity of the particle at the mid-point of its path.

Solution:

Let the total distance be $2x$.

$\therefore$ Distance up to midpoint = x

Let the velocity at the mid-point be v and acceleration be a.

From equations of motion

$v^2 = 10^2 + 2ax$ ...(1)

$30^2 = v^2 + 2ax$ ...(2)

Equation (2) – (1) gives

$v^2 - 30^2 = 10^2 - v^2$

$\Rightarrow$ $v^2 = 500$ $\Rightarrow$ $v = 10\sqrt{5}\ m/s$

Illustration 28:

Mr. Sharma brakes his car with constant acceleration from a velocity of 25 m/s to 15 m/s over a distance of 200 m.

(a) How much time elapses during this interval?

(b) What is the acceleration?

(c) If he has to continue braking with the same constant acceleration, how much longer would it take for him to stop and how much additional distance would he cover?

Solution:

(a) We select positive direction for our coordinate system to be the direction of the velocity and choose the origin so that $x_i = 0$ when the braking begins. Then the initial velocity is $u_x = +25\ m/s$ at $t = 0$, and the final velocity and position are $v_x = +15\ m/s$ and $x = 200\ m$ at time t.

Since the acceleration is constant, the average velocity in the interval can be found from the average of the initial and final velocities.

$\therefore \quad v_{av,x} = \frac{1}{2}(u_x + v_x) = \frac{1}{2}(15 + 25) = 20\ m/s.$

The average velocity can also be expressed as $v_{av,x} = \frac{\Delta x}{\Delta t}$. With $\Delta x = 200\ m$

and $\quad \Delta t = t - 0$, we can solve for t : $t = \frac{\Delta x}{v_{av,x}} = \frac{200}{20} = 10\ s.$

(b) We can now find the acceleration using $v_x = u_x + a_x t$

$$a_x = \frac{v_x - u_x}{t} = \frac{15-25}{10} = -1\ m/s^2.$$

The acceleration is negative, which means that the positive velocity is becoming smaller as brakes are applied (as expected).

(c) Now with known acceleration, we can find the total time for the car to go from velocity $u_x = 25\ m/s$ to $v_x = 0$. Solving for t, we find

$$t = \frac{v_x - u_x}{a_x} = \frac{0-25}{-1} = 25\ s.$$

The total distance covered is $x = x_i + u_x t + \frac{1}{2} a_x t^2$

$$= 0 + (25)(25) + \frac{1}{2}(-1)(25)^2 = 625 - 312.5 = 312.5\ m.$$

Additional distance covered = 312.5 – 200 = 112.5 m.

Motion Under Gravity (Free Fall)

In the absence of air, it is found that all bodies (irrespective of the size, weight or composition) fall with the same acceleration near the surface of the earth. This motion of a body falling towards the earth from a small altitude is called motion under gravity. Free fall means acceleration of body is equal to acceleration due to gravity.

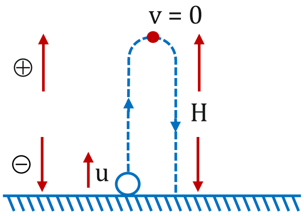
Acceleration produced in the body by the force of gravity, is called acceleration due to gravity. It is represented by the symbol g.

Value of $g = 9.8\ \frac{m}{s^2}$

or $\quad g = 980\ \frac{cm}{s^2}$

or $\quad g = 32\ \frac{ft}{s^2}$

If a Body is Projected Vertically Upward:



Positive / Negative directions are matter of choice. Here upward direction is taken positive.
You may take another choice.

(i) Equation of motion:

Taking initial position as origin and direction of motion (*i.e.* vertically up) as positive

$a = -g$ [As acceleration is downwards while motion upwards]

So, if the body is projected with velocity u and after time t it reaches up to height h then

$$v = u - gt$$

$$h = ut - \frac{1}{2}gt^2$$

$$v^2 = u^2 - 2gh$$

$$h_{n^{th}} = u - \frac{g}{2}(2n - 1)$$

(ii) For maximum height $v = 0$

So, from above equation

$$u = g\,t$$

It is called time of ascent $(t_1) = u/g$ ...(i)

For total time –

$$s = ut + \frac{1}{2}at^2$$

$$0 = uT - \frac{g}{2}T^2$$

$$T = \frac{2u}{g} \qquad \text{...(ii)}$$

If time of descent is t_2 then-

Total time of flight $T = t_1 + t_2 = \frac{2u}{g}$

So, $t_2 = u/g$

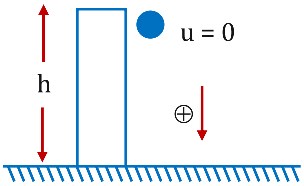
So, in case of motion under gravity, time taken to go up is equal to the time taken to fall down through the same distance.

Time of descent (t_2) = time of ascent $(t_1) = u/g$

$$H = \frac{1}{2}gT^2 \text{ and } u^2 = 2gH$$

$$\Rightarrow H = \frac{u^2}{2g}$$

If a body is dropped from some height (initial velocity zero):



Equations of motion:

Taking initial position as origin and direction of motion (i.e., downward direction) as a positive, here we have

$u = 0$ [As body starts from rest]

$a = +g$ [As acceleration is in the direction of motion]

$v = g\,t$ ...(i)

$h = \frac{1}{2}gt^2$ ...(ii)

Illustration 29:

Analyse the motion of a particle thrown vertically upwards with initial velocity of 30 m/s.

Solution:

O to C

$u = +30$

$v = 0$

$a = -10$

$s = H$

$\therefore \quad v^2 = u^2 + 2as$

$0^2 = 30^2 + 2(-10)H$

$H = \frac{30^2}{2\times 10} = 45m$

$t = ?$

$v = u + at$

$0 = 30 + (-10)t_1$

$t = \frac{30}{10} = 3\text{sec.}$

$v = u + at$

$v = 30 + (-10)t$

$v_1 = 30 - 10 = 20\ m/s$

$v_2 = 30 - 20 = 10\ m/s$

O to F

$u = +30$

$v = ?$

$a = -10$

$s = 0$

$\therefore \quad v^2 = u^2 + 2as$

$v^2 = (30)^2 + 2 \times (-10) \times 0$

$v = -30$

$t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 30t + \frac{1}{2}(-10)t^2$

$t = \frac{2\times 30}{10} = 6\text{sec.}$

$s = \left(\frac{u+v}{2}\right)t$

$s_{0\,to\,1} = \left(\frac{30+20}{2}\right)1 = 25m$

$s_{1\,to\,2} = \left(\frac{20+10}{2}\right)1 = 15m$

$v_3 = 30 - 30 = 0\ m/s$ $\qquad$ $s_{2to3} = \left(\frac{10+0}{2}\right)1 = 5m$

$v_4 = 30 - 40 = -10\ m/s$ $\qquad$ $s_{3to4} = \left(\frac{0-10}{2}\right)1 = -5m$

$v_5 = 30 - 50 = -20\ m/s$ $\qquad$ $s_{4to5} = \left(\frac{-10-20}{2}\right)1 = -15m$

$v_6 = 30 - 60 = -30\ m/s$ $\qquad$ $s_{5to6} = \left(\frac{-20-30}{2}\right)1 = -25m$

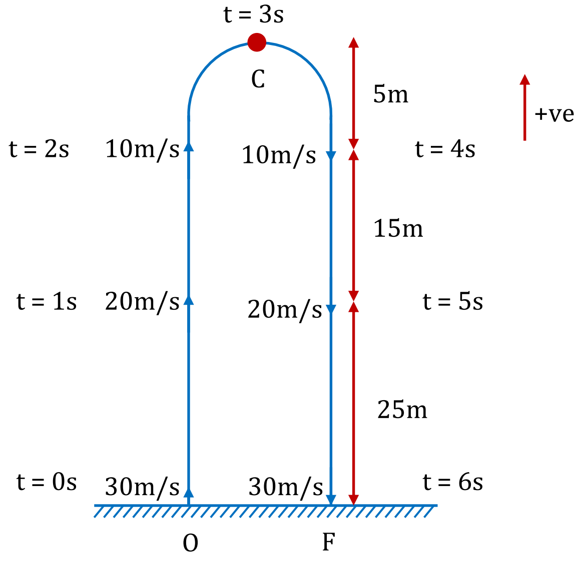
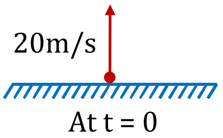


Illustration 30:

If a particle is thrown vertically upwards with initial velocity of 20 m/s, then find :



(a) time of flight $\qquad$ (b) maximum height reached $\qquad$ (c) displacement in 2*nd* sec.

Solution:

(a) A to B

$u = +20$

$a = -10$

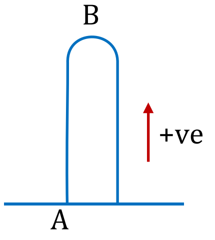
$v = 0$

$v = u + at$

$0 = 20 + (-10)t$

$t = 2$ sec.

$\therefore\ t = 2,\ T = 4$ sec.



(b) s in 0 to 2 sec

$t = 2$

$u = 20$

$$s = \left(\frac{u+v}{2}\right)t = \left(\frac{20+0}{2}\right)2 = 20m$$

(c) s in 1 to 2 sec

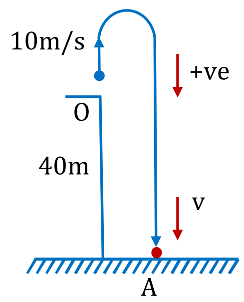
$v = 0$

$a = -10$

$$\therefore\ s = vt - \frac{1}{2}at^2 = 0\times1 - \frac{1}{2}\times(-10)\times1 = \frac{10}{2} = 5\ m$$

Illustration 31:

A particle is thrown vertically upward with a velocity of 10 m/s from a tower of height 40 m. Find time of flight and velocity just before hitting the ground.



Solution:

$u = -10$

$a = +10$

$s = +40$

$$s = ut + \frac{1}{2}at^2$$

$$\Rightarrow\quad 40 = -10\times t + \frac{1}{2}\times 10\times t^2$$

$$40 + 10t = \frac{10}{2}t^2$$

$10t^2 = 80 + 20t$

$10t^2 - 20t - 80 = 0$

$t^2 - 2t - 8 = 0$

$t^2 - 4t + 2t - 8 = 0$

$(t-4)(t+2) = 0$

$t = 4, t = -2$ (reject)

$\therefore\quad v = u + at$

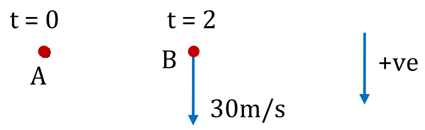
$v = -10 + 10\times 4$

$v = -10 + 40$

$v = 30\ m/s$

Illustration 32:

Particle A is dropped at t = 0. Another particle B is dropped from the same point at t = 2s. When and where will they cross?



Solution:

$s_A = s_B$

$$ut+\frac{1}{2}at^2=u(t-2)+\frac{1}{2}a(t-2)^2$$

$$\frac{1}{2}\times10\times t^2=30\times(t-2)+\frac{1}{2}\times10\times(t-2)^2$$

$5t^2 = 30t - 60 + 5(t^2 + 4 - 4t)$

$5t^2 = 30t - 60 + 5t^2 + 20 - 20t$

$40 = 10t$

t = 4 sec.

$$s=ut+\frac{1}{2}at^2=0\times4+\frac{1}{2}\times10\times16 = 80\ m$$

Illustration 33:

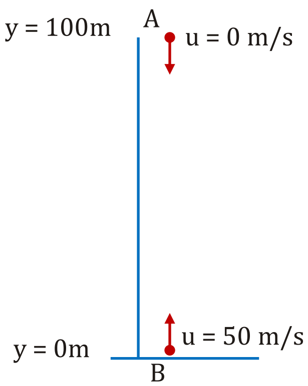
A particle is dropped from height 100 m and another particle is projected vertically up with velocity 50 m/s from the ground along the same line. Find out the position where two particles will meet.

(Take $g = 10\ m/s^2$)

Solution:

Let the upward direction is positive.

Let the particles meet at a distance y from the ground.



For particle A,

$y_0 = +100\ m$

$u = 0\ m/s$

$a = -10\ m/s^2$

$y = 100 + 0(t) - \frac{1}{2}10 \times t^2$ $\quad [y = y_0 + ut + \frac{1}{2}at^2]$

$= 100 - 5t^2$ ...(1)

For particle B,

$y_0 = 0\ m$

$u = +50\ m/s$

$a = -10\ m/s^2$

$y = 50(t) - \frac{1}{2}10t^2$

$= 50t - 5t^2$ ...(2)

According to the Illustration;

$50t - 5t^2 = 100 - 5t^2$

$t = 2$ sec

Putting $t = 2$ sec in equation (1),

$y = 100 - 20 = 80\ m$

Hence, the particles will meet at a height 80 m above the ground.

Illustration 34:

A particle is dropped from a tower. It is found that it travels 45 m in the last second of its journey. Find out the height of the tower? (Take $g = 10\ m/s^2$)

Solution:

Let the total time of journey be n seconds.

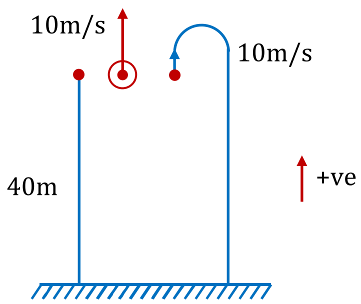
Using; $s_{n^{th}} = u + \frac{a}{2}(2n-1) \quad \Rightarrow \quad 45 = 0 + \frac{10}{2}(2n-1)$

$n = 5$ sec

Height of tower; $h = \frac{1}{2}gt^2 = \frac{1}{2} \times 10 \times 5^2 = 125\ m$

Illustration 35:

A balloon is rising with constant velocity of 10 m/s. When balloon was at 40m from ground, a particle is dropped from balloon find time of flight of particle after dropping?



Solution:

$s = -40m, \quad u = 10\ m/s, \quad a = -10$

$$s = ut + \frac{1}{2}at^2$$

$$-40 = 10 \times t + \frac{1}{2} \times -10 \times t^2$$

$-40 = 10t - 5t^2$

$5t^2 - 10t - 40 = 0$

$t^2 - 2t - 8 = 0$

$(t - 4)(t + 2) = 0$

$t = 4 \qquad t = -2$

$\therefore$ Time = 4 sec.

Note:

As the particle is detached from the balloon it is having the same velocity as that of balloon, but its acceleration is only due to gravity and is equal to g.

Illustration 36:

A stone is dropped from a balloon going up with a uniform velocity of 5 m/s. If the balloon was 60 m high when the stone was dropped, find its height when the stone hits the ground. Take g = 10 m/s^2.

Solution:

$S = ut + \frac{1}{2}at^2$

$-60 = 5(t) + \frac{1}{2}(-10)t^2$

$-60 = 5t - 5t^2$

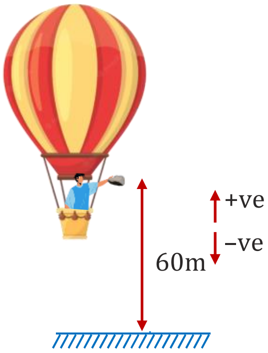
$5t^2 - 5t - 60 = 0$

$t^2 - t - 12 = 0$

$t^2 - 4t + 3t - 12 = 0$

$(t - 4)(t + 3) = 0$

$\therefore \quad t = 4$



Height of balloon from ground at this instant

$= 60 + 4 \times 5 = 80\ m$

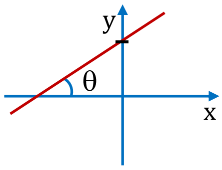
Graphical Interpretation of Differentiation

Slope:

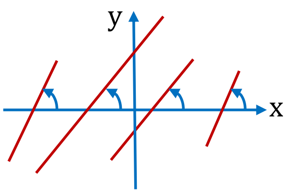
y = (m)x + c → slope

y = (tanθ) x + c

tan θ = slope of straight line



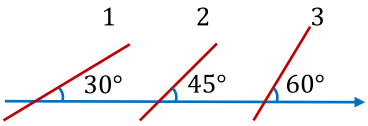
Case-1 : $0° < \theta < 90°$



$\tan\theta = +ve$

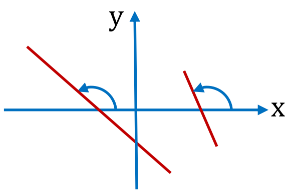
$\Rightarrow$ slope = $+ve$

(eg.)



$|slope|_3 > |slope|_2 > |slope|_1$

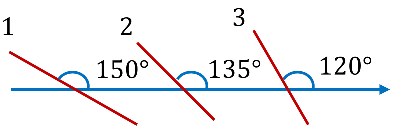
Case-2 : $90° < \theta < 180°$



$\tan\theta = -ve$

$\Rightarrow$ slope = $-ve$

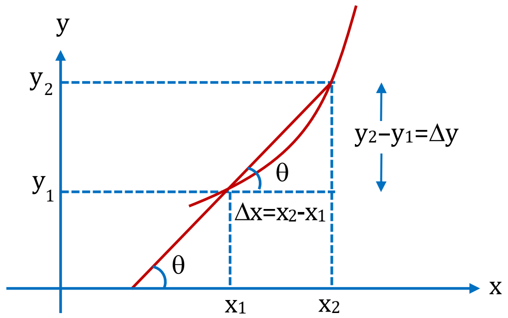
(eg.)



$|slope|_3 > |slope|_2 > |slope|_1$

Slope of secant:

$$\text{Slope} = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

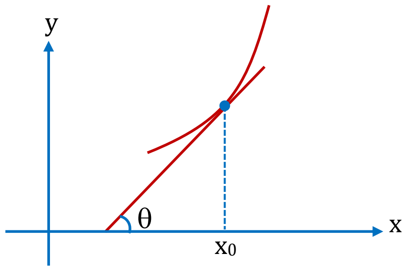


Slope of secant gives avg. rate of change of y w.r.t. x.

Slope of tangent:

$$\text{Slope} = \tan\theta = \frac{dy}{dx}$$

Slope of tangent at point in y-x graph gives inst. rate of change of y *w.r.t.* to x.



Key point

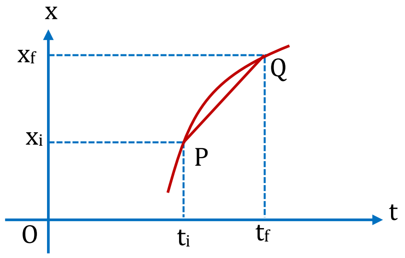
- If $\frac{dy}{dx}$ is positive then y will increase on increasing x.
- If $\frac{dy}{dx}$ is negative they y will decrease on increasing x.

Graphical interpretation of some quantities

Average Velocity:

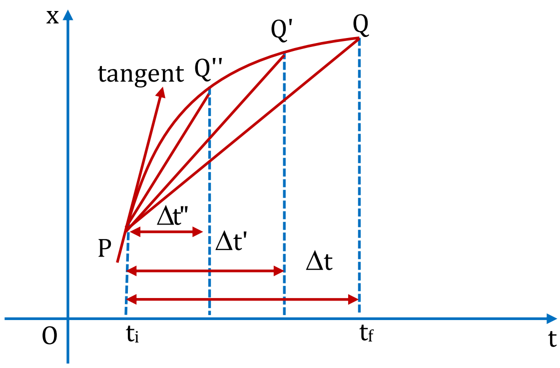
If a particle passes a point P (x_i) at time $t = t_i$ and reaches Q (x_f) at a later time $t = t_f$, its average velocity in the interval PQ is $V_{av} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$

This expression suggests that the average velocity is equal to the slope of the line (chord) joining the points corresponding to P and Q on the x-t graph.



Instantaneous Velocity:

Consider the motion of the particle between the two points P and Q on the x-t graph shown. As the point Q is brought closer and closer to the point P, the time interval between PQ (Δt, $\Delta t'$, $\Delta t''$, ...) get progressively smaller. The average velocity for each time interval is the slope of the appropriate line (PQ, PQ', PQ''......).

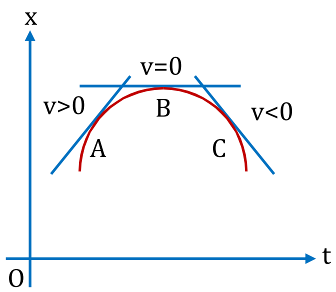


As the point Q approaches P, the time interval approaches zero, but at the same time the slope of the line approaches that of the tangent to the curve at the point P.

As $\Delta t \to 0$, V_{av} $(=\Delta x/\Delta t) \to V_{inst.}$

Geometrically, as $\Delta t \to 0$, chord $PQ \to$ tangent at P.

Hence the instantaneous velocity at P is the slope of the tangent at P in the $x - t$ graph. When the slope of the $x - t$ graph is positive, v is positive (as at the point A in figure). At C, v is negative because the tangent has negative slope. The instantaneous velocity at point B (turning point) is zero as the slope is zero.



Instantaneous Acceleration:

The derivative of velocity with respect to time is the slope of the tangent in velocity-time (v–t) graph.

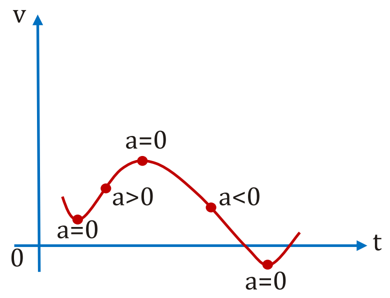
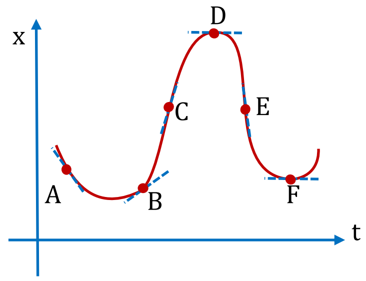


Illustration 37:

Position-time graph of a particle moving along the x-axis is given. Fill the sign of average velocity and instantaneous velocity in the following table:



Average velocity		Instantaneous velocity	
A to B		A	
A to C		B	
B to C		C	
C to E		D	
E to F		E	
		F	

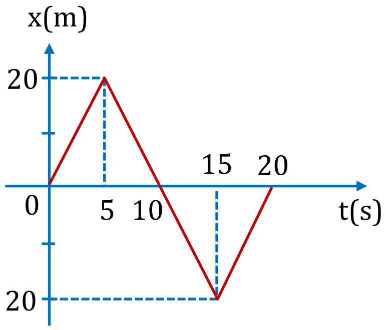
Solution:

Slope of secant gives average velocity and slope of tangent gives instantaneous velocity.

Average velocity		Instantaneous velocity	
A to B	–ve	A	–
A to C	+ve	B	+
B to C	+ve	C	+
C to E	0	D	0
E to F	–ve	E	–
		F	0

Illustration 38:

Position time graph of a particle moving along x-axis is given. Find –



(i) Velocity at t = 1s, 2s, 6s, 11s and 16s

(ii) Average velocity from t = 0s to t = 15s

(iii) Average velocity from t = 5s to t = 15s

(iv) Average speed from 0s to 15s

(v) Average acceleration from t = 2s to t = 6s

(vi) Average velocity from t = 6s to t = 16s

Solution:

(i) v_{ins} = slope = tanθ

$v_1 = 20/5 = 4m/s$ $v_2 = 20/5 = 4m/s$

$v_6 = -20/5 = -4m/s$ $v_{11} = -20/5 = -4m/s$ $v_{16} = 20/5 = 4\ m/s$

(ii) $< v >$ from $t = 0$ *to* $t = 15$

$$<v>=\frac{\Delta x}{\Delta t}=\frac{x_f-x_i}{t_f-t_i}=\frac{-20-0}{15-0}=\frac{-4}{3}m/s$$

(iii) $< v >$ from $t = 5$ *to* $t = 15$

$$<v>=\frac{\Delta x}{\Delta t}=\frac{x_f-x_i}{t_f-t_i}=\frac{-20-20}{15-5}=-4m/s$$

(iv) avg. speed from 0 to 15 sec.

$$\text{avg. speed}=\frac{\text{distance travelled}}{\text{time taken}}=\frac{60}{15}=4m/s$$

(v) avg. acceleration from 2 to 6 sec.

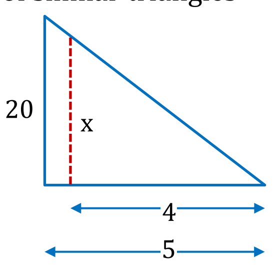
$$\text{avg. acceleration}=\frac{\text{change in velocity}}{\text{time taken}}=\frac{-4-4}{4}=-2m/s^2$$

(vi) From the properties of similar triangles -

$$\frac{20}{5}=\frac{x_6}{4}$$

$x_6 = 16m$

Similarly $x_{16} = -16m$



$< v >$ from t = 6 sec. *to* t = 16 sec.

$$\langle v\rangle=\frac{x_{16}-x_6}{t_{16}-t_6}=\frac{-16-(16)}{10}=\frac{-32}{10}=-3.2m/s$$

Illustration 39:

Position of a particle as a function of time is given as $x = 5t^2 + 4t + 3$. Find the velocity and acceleration of the particle at $t = 2\ s$.

Solution:

Velocity; $v = \dfrac{dx}{dt} = 10t + 4$

At $\quad t = 2\ s$

$\quad v = 10(2) + 4$

$\quad v = 24\ m/s$

Acceleration; $a = \dfrac{dv}{dt} = 10$

Acceleration is constant, so at $t = 2\ s \quad a = 10\ m/s^2$

Double differentiation and higher order derivates

$\dfrac{dy}{dx}$ = differentiation of y w.r.t. x

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) \Rightarrow$ differentiation of $\left(\dfrac{dy}{dx}\right)$ w.r.t. x

$\Rightarrow$ double differentiation of y w.r.t. x

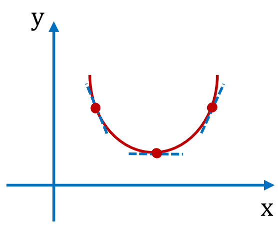
$\rightarrow$ It is represented as $\left(\dfrac{d^2y}{dx^2}\right) \rightarrow$ It is also known as 2^{nd} order derivative of y w.r.t. x

(eg.) Velocity is 1^{st} order derivative of position w.r.t. time i.e $\dfrac{dx}{dt} = v$

Acceleration is 2^{nd} order derivative of position w.r.t. time i.e $\dfrac{d^2x}{dt^2} = a$

Acceleration is 1^{st} order derivative of velocity w.r.t. time i.e $\dfrac{dv}{dt} = a$

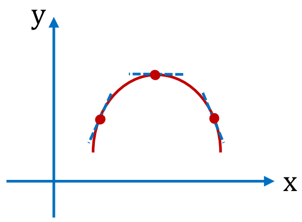
Graphical interpretation of double differentiation:



$\dfrac{dy}{dx}$ is $\uparrow$ with $\uparrow$ x

$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = +ve$

$\Rightarrow \quad \frac{d^2y}{dx^2}$ is +ve for concave upwards



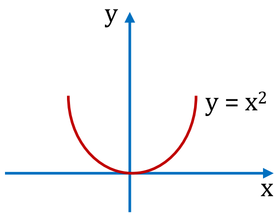
$\frac{dy}{dx}$ is $\downarrow$ with $\uparrow$ x

$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = -ve$

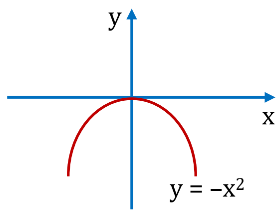
$\Rightarrow \quad \frac{d^2y}{dx^2}$ is –ve for concave downwards

(eg.)

(i)



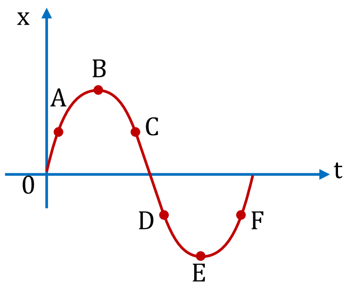
(ii)



$\frac{dy}{dx} = 2x$ $\qquad$ $\frac{dy}{dx} = -2x$

$\frac{d^2y}{dx^2} = 2$ $\qquad$ $\frac{d^2y}{dx^2} = -2$

(eg.)

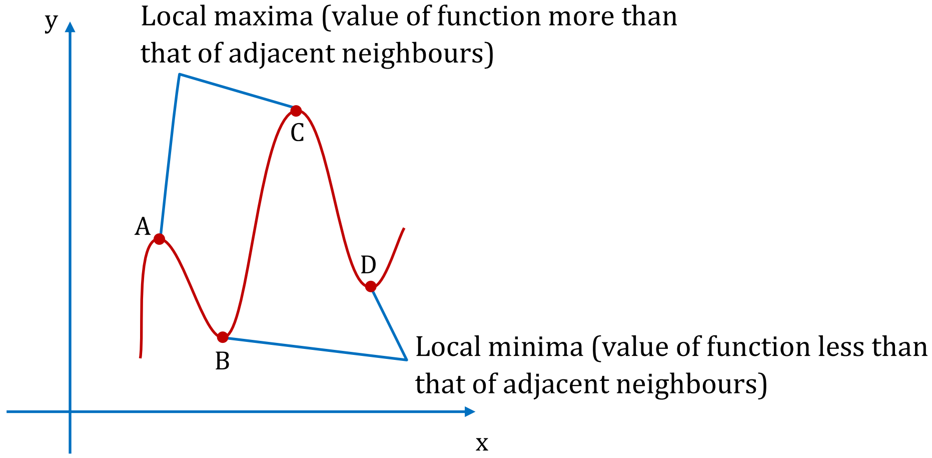


	Velocity	Acceleration
A	+	–
B	0	–
C	–	–
D	–	+
E	0	+
F	+	+

- If concave is upward then double differentiation is positive and if concave is downward then it is negative.
- If tangent at a point makes an angle θ greater than 90°, then slope (differentiation) will be negative and if θ less than 90°, then slope (differentiation) will be positive.

  For angle 180° slope will be zero.

Local maxima and Local minima:



Slope at local maxima and local minima is always zero.

$$\frac{dy}{dx}=0$$

	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
Local maxima	0	–
Local minima	0	+

Illustration 40:

Locate local maxima and local minima of the *y*-*x* curve-

$$y=\frac{x^3}{3}-\frac{3x^2}{2}+2x+5$$

Solution:

$$\frac{dy}{dx}=\frac{3x^2}{3}-\frac{3\cdot 2x}{2}+2$$

$$\frac{dy}{dx}=x^2-3x+2=0$$

Then $x = 1, 2$

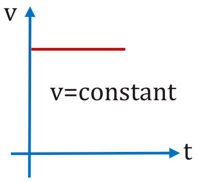
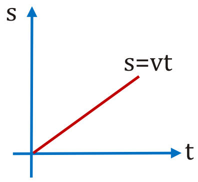
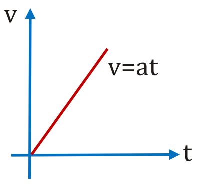
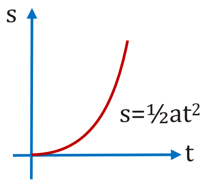
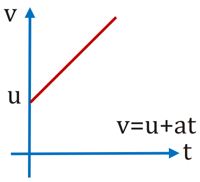
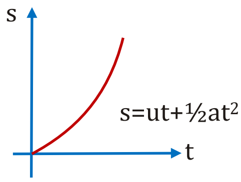
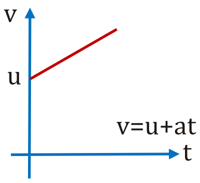
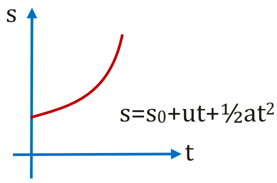
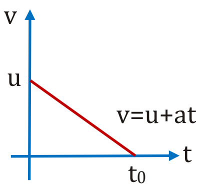
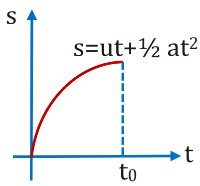
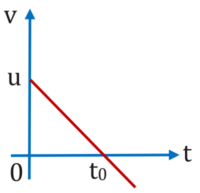
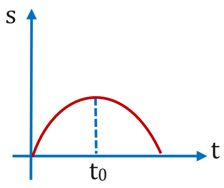
$$\frac{d^2y}{dx^2}=2x-3$$

$\therefore$ $$\left(\frac{d^2y}{dx^2}\right)_{x=1}=2-3=-1$$

$\Rightarrow$ Local maxima at $x = 1$

$$\left(\frac{d^2y}{dx^2}\right)_{x=2}=2\times 2-3=1$$

$\Rightarrow$ Local minima at $x = 2$

S.N.	Different Cases	v–t graph	s–t graph	Important Points
1.	Uniform motion			(i) Slope of s–t graph = v = constant (ii) In s–t graph $s = 0$ at $t = 0$
2.	Uniformly accelerated motion with $u = 0$ at $t = 0$			(i) $u = 0$, i.e. $v = 0$ at $t = 0$ (ii) $u = 0$, i.e., slope of s–t graph at $t = 0$, should be zero (iii) a or slope of $v - t$ graph is constant
3.	Uniformly accelerated with $u \neq 0$ at $t = 0$			(i) $u \neq 0$, i.e., v or slope of s –t graph at $t = 0$ is not zero (ii) v or slope of $s - t$ graph gradually goes on increasing.
4.	Uniformly accelerated motion with $u \neq 0$ and $s = s_0$ at $t = 0$			(i) $s = s_0$ at $t = 0$
5.	Uniformly retarded motion till velocity becomes zero			(i) Slope of s–t graph at $t = 0$ gives u (ii) Slope of s–t graph at $t = t_0$ becomes zero (iii) In this case u can't be zero.
6.	Uniformly retarded then accelerated in opposite direction			(i) At time $t=t_0$, $v=0$ or slope of s–t graph is zero (ii) Slope of s–t graph or velocity first decreases then increases with opposite sign.

Integration

In integral calculus, the differential coefficient of a function is given. We are required to find the function.

Integration is basically used for summation. $\sum$ is used for summation of discrete values, while $\int$ sign is used for continuous function.

If I is integration of $f(x)$ with respect to x then $I = \int f(x)\,dx$ [we can check $\frac{dI}{dx} = f(x)$]

Therefore, $\int f'(x)\,dx = f(x) + c$

where c = an arbitrary constant

Let us proceed to obtain integral of x^n *w.r.t. x.*

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n$$

Since the process of integration is the reverse process of differentiation,

$$\int (n+1)x^n dx = x^{n+1}$$

or $\quad (n+1)\int x^n dx = x^{n+1}$

or $\quad \int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

The above formula holds for all values of n, except $n = -1$.

It is because, for $n = -1$,

$$\int x^n dx = \int x^{-1} dx = \int \frac{1}{x} dx$$

$\because \quad \dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

$\therefore \quad \int \dfrac{1}{x} dx = \log_e x + c$

Similarly, the formulae for integration of some other functions can be obtained if we know the differential coefficients of various functions.

Few basic formulae of integration:

Following are a few basic formulae of integration:

1. $\int k\,dx = kx + c$, k is constant
2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$, Provided $n \neq -1$
3. $\int \sin x\,dx = -\cos x + c \qquad \because \dfrac{d}{dx}(\cos x) = -\sin x$
4. $\int \cos x\,dx = \sin x + c \qquad \because \dfrac{d}{dx}(\sin x) = \cos x$
5. $\int \dfrac{1}{x} dx = \log_e x + c \qquad \because \dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$
6. $\int e^x\,dx = e^x + c \qquad \because \dfrac{d}{dx}(e^x) = e^x$
7. $\int f'(ax+b)\,dx = \dfrac{f(ax+b)}{a} + c \qquad \because \dfrac{d}{dx}(f(ax+b)) = af'(ax+b)$

Illustration 41:

Integrate w.r.t. x:

(i) $x^{11/2}$ (ii) x^{-7} (iii) $x^{p/q}$

Solution:

(i) $\int x^{11/2}dx = \frac{x^{11/2+1}}{\frac{11}{2}+1} + c = \frac{2}{13} x^{13/2} + c$ (ii) $\int x^{-7}dx = \frac{x^{-7+1}}{-7+1}+c = -\frac{1}{6}x^{-6}+c$

(iii) $\int x^{\frac{p}{q}}dx = \frac{x^{\frac{p}{q}+1}}{\frac{p}{q}+1} + c = \frac{q}{p+q} x^{(p+q)/q} + c,\ \frac{p}{q} \neq -1$

Rules of Integration:

1. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

eg.$\to \int (x^2+x^3)dx = \int x^2dx + \int x^3dx = \frac{x^3}{3}+\frac{x^4}{4}+C$

2. $\int kf(x)dx = k\int f(x)dx$

Illustration 42:

Evaluate $\int (x^2 - \cos x + \frac{1}{x})\, dx$

Solution:

$I = \int x^2dx - \int \cos x dx + \int \frac{1}{x}dx = \frac{x^{2+1}}{2+1} - \sin x + log_e\, x + c = \frac{x^3}{3} - \sin x + log_e\, x + c$

Illustration 43:

Find- (i) $\int (3x^2+4x^5)dx$ (ii) $\int 3dx = 3\int dx$ (iii) $\int 3x^5dx$

Solution:

(i) $\int (3x^2+4x^5)dx = 3\int x^2dx + 4\int x^5dx = 3\frac{x^3}{3}+4\frac{x^6}{6}+C = x^3+\frac{2x^6}{3}+C$

(ii) $\int 3dx = 3\int dx = 3x + C$

(iii) $\int 3x^5dx = 3\int x^5dx = 3\frac{x^6}{6}+C = \frac{x^6}{2}+C$

Definite Integrals:

When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

If $\frac{d}{dx}(f(x)) = f'(x)$, then $\int f'(x)dx$ is called indefinite integral

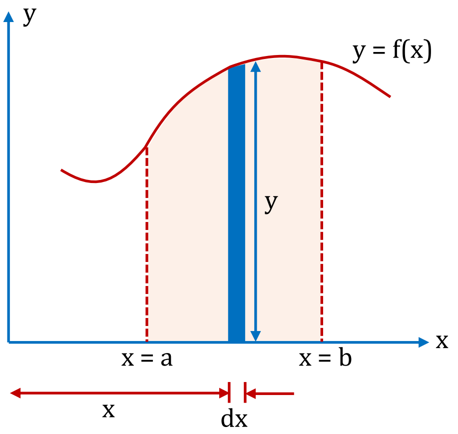
and $\int_a^b f'(x)dx$ is called definite integral

Here, a and b are called lower and upper limits of the variable x.

After carrying out integration, the result is evaluated between upper and lower limits as explained below:

$$\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$$

Area under a curve and definite integration:



Area of small shown element = $ydx = f(x)\, dx$

If we sum up all areas between $x = a$ and $x = b$

then $\int_a^b f(x)dx$ = shaded area between curve and x-axis.

Illustration 44:

Evaluate $\int_{x=0}^{x=2} (2x+3x^2)dx$

Solution:

$$\int_{x=0}^{x=2} 2x\,dx + \int_{x=0}^{x=2} 3x^2\,dx = [x^2]_{x=0}^{x=2} + [x^3]_{x=0}^{x=2} = 2^2 + 2^3 = 12$$

Illustration 45:

Evaluate $\int_{x=2}^{x=5} dx$

Solution:

$$[x]_{x=2}^{x=5} = 5 - 2 = 3$$

Illustration 46:

Evaluate $\int_{t=0}^{t=2} (2t+1)dt$

Solution:

$$\left[2\frac{t^2}{2}+t\right]_{t=0}^{t=2} = [t^2+t]_{t=0}^{t=2} = [2^2 + 2] - [0^2 + 0] = 6$$

Illustration 47:

Evaluate $\int\limits_{x=1}^{x=2} \frac{1}{x^2}\,dx$

Solution:

$$\int\limits_{x=1}^{x=2} \frac{1}{x^2}\,dx = \int\limits_{x=1}^{x=2} x^{-2}\,dx = \left[\frac{-1}{x}\right]_{x=1}^{x=2}$$

$$= \left[\frac{-1}{2}\right] - \left[\frac{-1}{1}\right] = \frac{-1}{2} + 1 = \frac{1}{2}$$

Displacement from *v-t* Graph and Change in Velocity From *a-t* Graph:

Displacement = Δx = area under *v-t* graph. Since a negative velocity causes a negative displacement, areas below the time axis are taken negative. In similar way, can see that $\Delta v = a\,\Delta t$ leads to the conclusion that **area under $a - t$ graph gives the change in velocity Δv during that interval.**

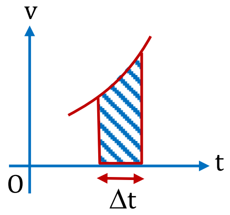
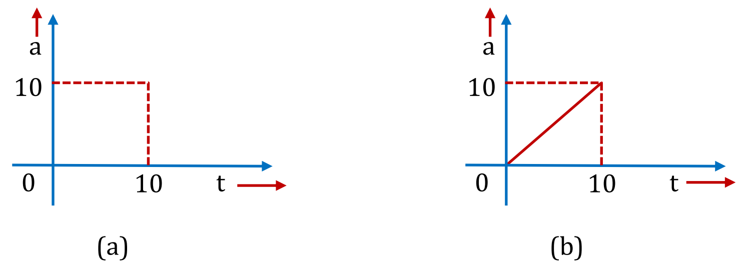


Illustration 48:

For a particle moving along x-axis, following graphs are given. Find the final velocity of the particle after 10 s in each case if initial velocity is zero.



Solution:

Δv = area under a-t curve with t-axis

(a) $v_f - v_i = 10 \times 10$

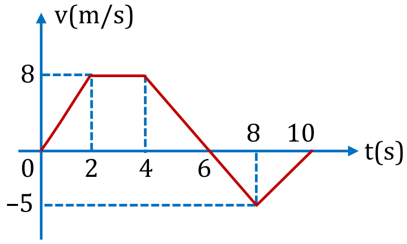
$v_f = 100\ m/s$

(b) $v_f - v_i = \frac{1}{2} \times 10 \times 10$

$v_f = 50\,m/s$

Illustration 49:

For a particle moving along x-axis, velocity-time graph is as shown in figure. Find the distance travelled and displacement of the particle.



Solution:

Distance travelled = Area under v-t graph

(taking all areas as +ve.)

Distance travelled = Area of trapezium + Area of triangle

$$= \frac{1}{2}(2+6)\times 8 + \frac{1}{2}\times 4\times 5 = 32 + 10 = 42\ m$$

Displacement = Area under $v - t$ graph (taking areas below time axis as –ve.)

Displacement = Area of trapezium – Area of triangle

$$= \frac{1}{2}(2+6)\times 8 - \frac{1}{2}\times 4\times 5 = 32 - 10 = 22\ m$$

Hence, distance travelled = 42 m and displacement = 22 m.

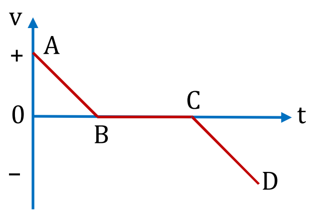
Important Points to Remember:

- For uniformly accelerated motion ($a \neq 0$), $x - t$ graph is a parabola (opening upwards if $a > 0$ and opening downwards if $a < 0$). The slope of tangent at any point of the parabola gives the velocity at that instant.
- For uniformly accelerated motion ($a \neq 0$), $v - t$ graph is a straight line whose slope gives the acceleration of the particle.
- In general, the slope of tangent in $x - t$ graph is velocity and the slope of tangent in $v - t$ graph is the acceleration.
- The area under $a - t$ graph gives the change in velocity.
- The area under the $v - t$ graph gives the distance travelled by the particle, if we take all areas as positive.
- Area under $v - t$ graph gives displacement, if areas below the t–axis are taken negative.

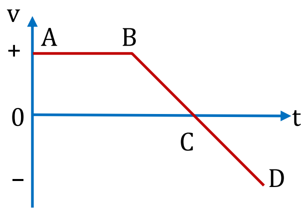
Illustration 50:

Describe the motion shown by the following velocity-time graphs.

(a)



(b)



Solution:

(a) **During interval AB:** velocity is +ve so the particle is moving in +ve direction, but it is slowing down as acceleration (slope of v-t curve) is negative. **During interval BC:** particle remains at rest as velocity is zero. Acceleration is also zero. **During interval CD:** velocity is -ve so the particle is moving in -ve direction and is speeding up as acceleration is also negative.

(b) **During interval AB:** particle is moving in +ve direction with constant velocity and acceleration is zero. **During interval BC:** particle is moving in +ve direction as velocity is +ve, but it slows down until it comes to rest as acceleration is negative. **During interval CD:** velocity is -ve so the particle is moving in -ve direction and is speeding up as acceleration is also negative.

Motion with Non-Uniform Acceleration (Use of Definite Integrals)

$\Delta x = \int_{t_i}^{t_f} v(t)dt$ (Displacement in time interval $t = t_i$ to t_f)

The expression on the right hand side is called the definite integral of $v(t)$ between $t = t_i$ and $t = t_f$. Similarly change in velocity

$$\Delta v = v_f - v_i = \int_{t_i}^{t_f} a(t)dt$$

Solving Illustrations which Involves Non-uniform Acceleration:

(i) Acceleration depending on velocity *v* or time *t*

By definition of acceleration, we have $a = \frac{dv}{dt}$. If a is in terms of t,

$$\int_{v_0}^{v} dv = \int_0^t a(t)dt$$

If a is in terms of v,

$$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_0^t dt$$

On integrating, we get a relation between v and t, and then using $\int_{x_0}^{x} dx = \int_0^t v(t)dt$, x and t can also be related.

(ii) Acceleration depending on position x

$$a = \frac{dv}{dt} \Rightarrow a = \frac{dv}{dx}\frac{dx}{dt} \Rightarrow a = \frac{dx}{dt}\frac{dv}{dx} \Rightarrow a = v\frac{dv}{dx}$$

This is another important expression for acceleration. If a is in terms of x,

$$\int_{v_0}^{v} vdv = \int_{x_0}^{x} a(x)dx$$

On integrating, we get a relation between x and v.

Using $\int_{x_0}^{x} \frac{dx}{v(x)} = \int_0^t dt$, we can relate x and t.

Illustration 51:

$v = 2t + 1$; find displacement from $t = 0$ to $t = 2$ sec.

Solution:

$$v = \frac{dx}{dt}$$

$vdt = dx$

$(2t + 1)\, dt = dx$

$$\int_{t=0}^{t=2} (2t+1)dt = \int_{x=x_i}^{x=x_f} dx$$

$$\Rightarrow \quad \left[\frac{2t^2}{2} + t\right]_{t=0}^{t=2} = [x]_{x_i}^{x_f}$$

$\Rightarrow \quad [2^2 + 2] - [0^2 + 0] = x_f - x_i = s$

$\Rightarrow \quad s = 6$

Illustration 52:

$v = 3t^2$; find x at $t = 3$, if $x = 2$ at $t = 0$.

Solution:

$$v = \frac{dx}{dt}$$

$vdt = dx$

$3t^2dt = dx$

$$\int_{t=0}^{t=3} 3t^2 dt = \int_{x=2}^{x=x_f} dx \Rightarrow \left(\frac{3t^3}{3}\right)_{t=0}^{t=3} = (x)_2^{x_f}$$

$3^3 - 0^3 = x_f - 2$

$27 = x_f - 2$

$x_f = 27 + 2$

$x_f = 29$

Illustration 53:

$a = 3t^2 + 2t$; at $t = 1$ sec., velocity = 3 m/s. Find velocity at $t = 2$ sec.

Solution:

$$a = \frac{dv}{dt}$$

$adt = dv$

$(3t^2 + 2t)\, dt = dv$

$$\int_{t=1}^{t=2}(3t^2+2t)dt = \int_{v=3}^{v=v_f} dv$$

$$\left[\frac{3t^3}{3}+\frac{2t^2}{2}\right]_{t=1}^{t=2} = [v]_{v=3}^{v_f}$$

$[2^3 + 2^2] - [1^3 + 1^2] = v_f - 3$

$[8 + 4] - [2] = v_f - 3$

$12 - 2 = v_f - 3$

$v_f = 10 + 3$

$v_f = 13$ m/s

Illustration 54:

Prove $v^2 = u^2 + 2as$ for constant 'a'. Where symbols have their usual meaning.

Solution:

$$a = v\frac{dv}{dx}$$

$adx = vdv$

$$\int_{x_i}^{x_f} a\,dx = \int_u^v v\,dv$$

If 'a' constant

$$a\int_{x_i}^{x_f} dx = \int_u^v v\,dv$$

$$a[x]_{x_i}^{x_f} = \left[\frac{v^2}{2}\right]_u^v$$

$$a\left[x_f - x_i\right] = \frac{v^2}{2} - \frac{u^2}{2}$$

$2as = v^2 - u^2$

$v^2 = u^2 + 2as$

Illustration 55:

An object starts from rest at $t = 0$ and accelerates at a rate given by $a = 6t$. What is

(a) its velocity and (b) its displacement at any time t?

Solution:

As acceleration is given as a function of time,

$$\therefore \quad \int_{v(t_0)}^{v(t)} dv = \int_{t_0}^{t} a(t)dt$$

Here $t_0 = 0$ and $v(t_0) = 0$ $\quad \therefore \quad v(t) = \int_0^t 6tdt = 6\left(\frac{t^2}{2}\right)_0^t = 6\left(\frac{t^2}{2} - 0\right) = 3t^2$

So, $v(t) = 3t^2$

As $\Delta x = \int_{t_0}^{t} v(t)dt$ $\quad\therefore\quad$ $\Delta x = \int_0^t 3t^2 dt = 3\left(\frac{t^3}{3}\right)_0^t = 3\left(\frac{t^3}{3} - 0\right) = t^3$

Hence, velocity $v(t) = 3t^2$ and displacement $\Delta x = t^3$.

Illustration 56:

For a particle moving along +ve x-axis, acceleration is given as $a = x$. Find the position as a function of time. Given that at $t = 0$, $x = 1$ $v = 1$.

Solution:

$a = x \Rightarrow \frac{vdv}{dx} = x \Rightarrow \frac{v^2}{2} = \frac{x^2}{2} + C$

$t = 0,\ x = 1$ and $v = 1 \Rightarrow \frac{1^2}{2} = \frac{1^2}{2} + C$

$\therefore\ C = 0 \Rightarrow v^2 = x^2$

$v = \pm x$ But given that $x = 1$ when $v = 1$

$\therefore\ v = x \Rightarrow \frac{dx}{dt} = x \Rightarrow \frac{dx}{x} = dt$

$\ell nx = t + C \Rightarrow 0 = 0 + C \Rightarrow \ell nx = t$

$x = e^t$

Illustration 57:

For a particle moving along x-axis, acceleration is given as $a = v$. Find the position as a function of time. Given that at $t = 0$, $x = 0$ $v = 1$.

Solution:

$a = v \Rightarrow \frac{dv}{dt} = v \Rightarrow \int \frac{dv}{v} = \int dt$

$\ell nv = t + c \Rightarrow 0 = 0 + c$ (at $t = 0$, $v=1$)

$v = e^t \Rightarrow \frac{dx}{dt} = e^t \Rightarrow \int dx = \int e^t dt$

$\Rightarrow x = e^t + c \Rightarrow 0 = 1 + c$ (at $t = 0$, $x=0$)

$x = e^t - 1$

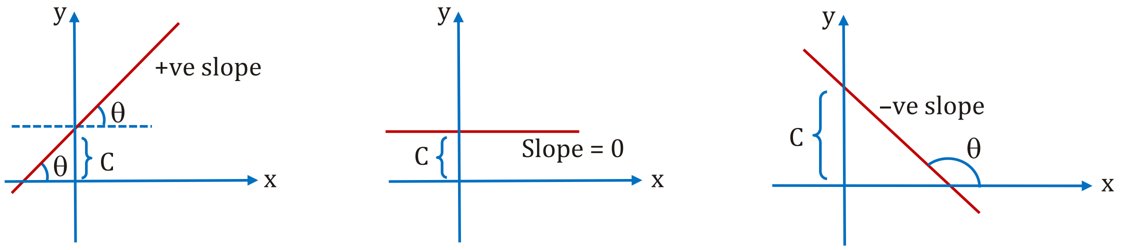
Graph:

Straight Line-Equation, Graph, Slope (+ve, –ve, zero slope):

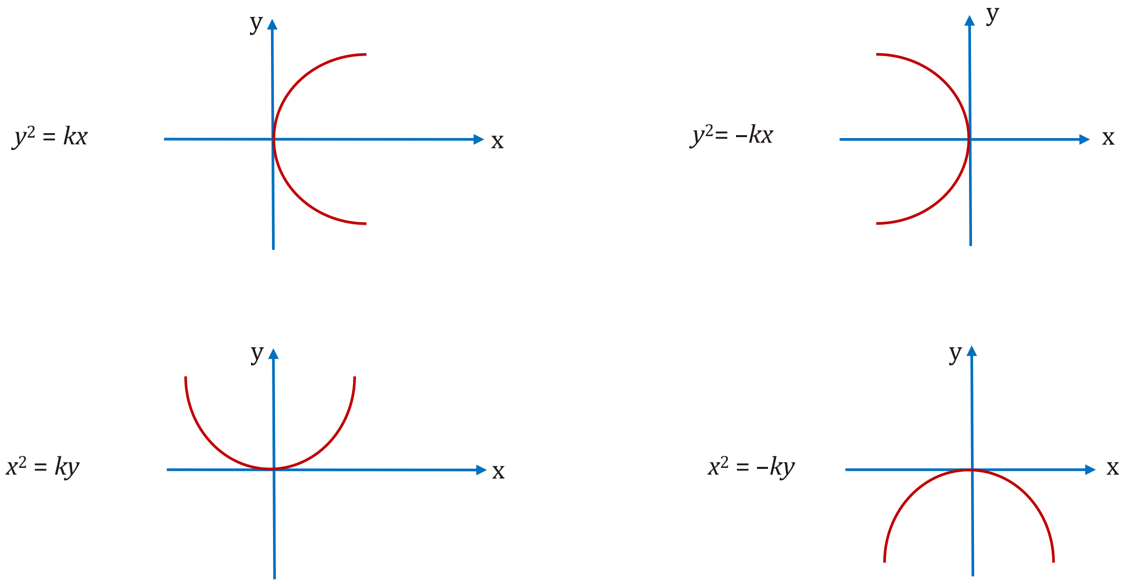
If θ is the angle at which a straight line is inclined to the positive direction of x-axis and $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m does not exist, but the line is parallel to the y-axis. If $\theta = 0$, then $m = 0$ and the line is parallel to the x-axis.

Slope – intercept form: $y = mx + c$ is the equation of a straight line whose slope is m and which makes an intercept c on the y-axis.

m = slope = $\tan\theta = \dfrac{dy}{dx}$



Parabolic Curve-Equation, Graph (Various Situations Up, Down, Left, Right with Conditions):



Where k is a positive constant.

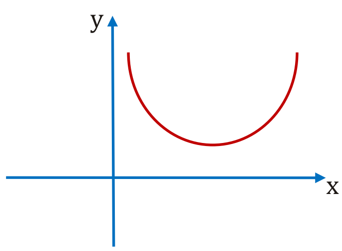
Equation of parabola:

Case (i) : $y = ax^2 + bx + c$

For $a > 0$

The nature of the parabola will be like that of the of nature $x^2 = ky$

Minimum value of y exists at the vertex of the parabola.



$$y_{min} = \frac{-D}{4a} \text{ where, } D = b^2 - 4ac$$

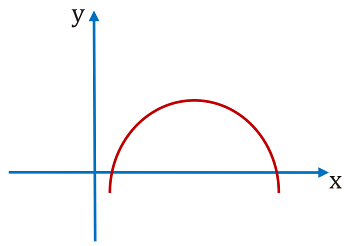
$$\text{Coordinates of vertex} = \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$$

Case (ii) : $a < 0$

The nature of the parabola will be like that of the nature of $x^2 = -ky$

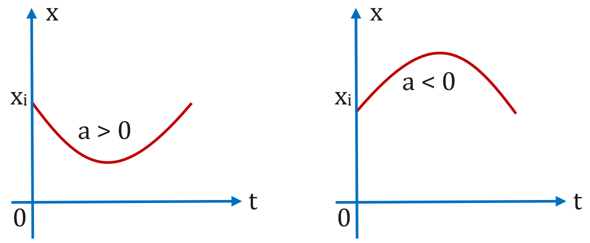
Maximum value of y exists at the vertex of parabola.

$y_{max} = \frac{-D}{4a}$, where, $D = b^2 - 4ac$



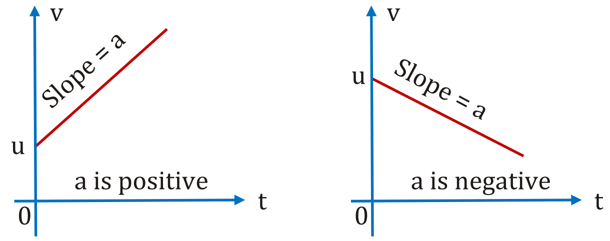
Graphs in Uniformly Accelerated Motion (a ≠ 0):

- x is a quadratic polynomial in terms of t. Hence $x - t$ graph is a parabola.



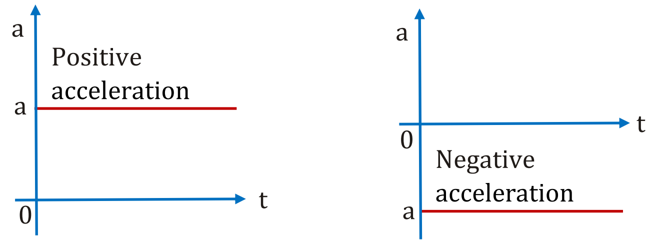
***x-t* graph**

- v is a linear polynomial in terms of t. Hence v–t graph is a straight line of slope a.



***v-t* graph**

- a–t graph is a horizontal line because a is constant.



***a-t* graph**

Interpretation of some more graphs:

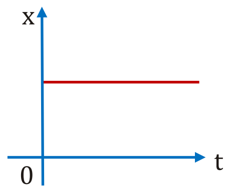
Position vs Time graph

(i) Zero Velocity

As position of particle is fixed at all the time, so the body is at rest.

Slope; $\frac{dx}{dt} = \tan\theta = \tan 0° = 0$

Velocity of particle is zero

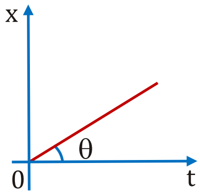


(ii) Uniform Velocity

Here tan θ is constant, $\tan\theta = \frac{dx}{dt}$

$\therefore$ $\frac{dx}{dt}$ is constant.

$\therefore$ velocity of particle is constant.



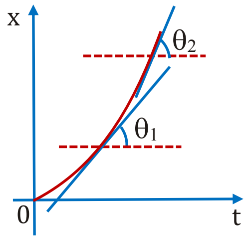
(iii) Non-uniform velocity (increasing with time)

In this case;

As time is increasing, θ is also increasing.

$\therefore$ $\frac{dx}{dt} = \tan\theta$ is also increasing

Hence, velocity of particle is increasing.



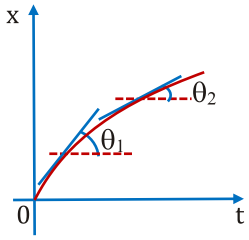
(iv) Non-uniform velocity (decreasing with time)

In this case;

As time increases, θ decreases.

$\therefore$ $\frac{dx}{dt} = \tan\theta$ also decreases.

Hence, velocity of particle is decreasing.



Velocity vs time graph:

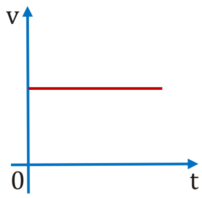
(i) Zero acceleration

Velocity is constant.

$\tan\theta = 0$

$\therefore$ $\frac{dv}{dt} = 0$

Hence, acceleration is zero.

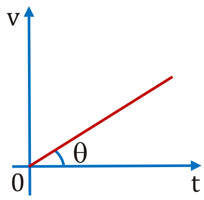


(ii) Uniform acceleration

tanθ is constant.

$\therefore$ $\frac{dv}{dt}$ = constant

Hence, it shows constant acceleration.



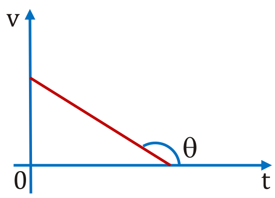
(iii) Uniform retardation

Since θ > 90°

$\therefore$ tanθ is constant and negative.

$\therefore$ $\frac{dv}{dt}$ = negative constant

Hence, it shows constant retardation.



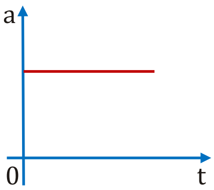
Acceleration vs time graph:

(i) Constant acceleration

$\tan\theta = 0$

$\therefore \dfrac{da}{dt} = 0$

Hence, acceleration is constant.



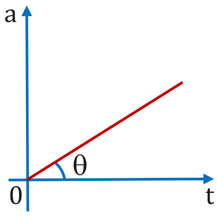
(ii) Uniformly increasing acceleration

θ is constant.

$0° < \theta < 90° \Rightarrow \tan\theta > 0$

$\therefore \dfrac{da}{dt} = \tan\theta = \text{constant} > 0$

Hence, acceleration is uniformly increasing with time.



(iii) Uniformly decreasing acceleration

Since $\theta > 90°$

$\therefore$ $\tan\theta$ is constant and negative.

$\therefore \dfrac{da}{dt} = \text{negative constant}$

Hence, acceleration is uniformly decreasing with time.

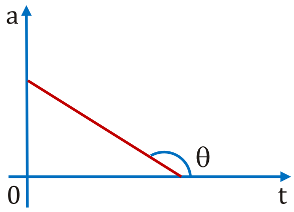
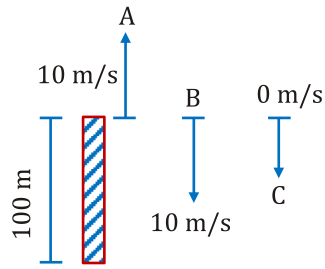


Illustration 58:

At the height of 100 m, a particle A is thrown up with $v = 10\ m/s$, B particle is thrown down with $v = 10 m/s$ and C particle is released with $v = 0\ m/s$. Draw graphs of each particle.

(i) Displacement–time (ii) Speed–time

(iii) Velocity–time (iv) Acceleration–time



Solution:

For particle A:

(i) Displacement vs time graph is

$y = ut + \dfrac{1}{2}at^2$

$u = +10\ m/\text{sec}$

$y = 10t - \dfrac{1}{2} \times 10t^2 = 10\,t - 5t^2$

$v = \dfrac{dy}{dt} = 10 - 10\,t = 0$

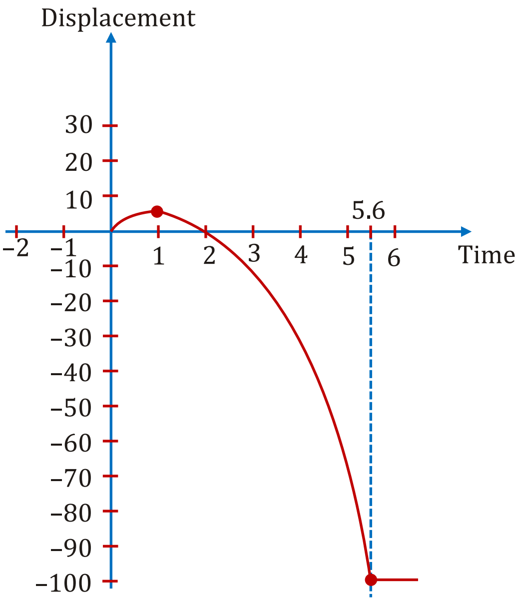
$t = 1$; hence, velocity is zero at $t = 1$

$10\,t - 5\,t^2 = -100$

$t^2 - 2t - 20 = 0$

$t = 5.6$ sec. $\left(t = 1 + \sqrt{21}\ s\right)$

i.e. particle travels till 5.6 seconds.



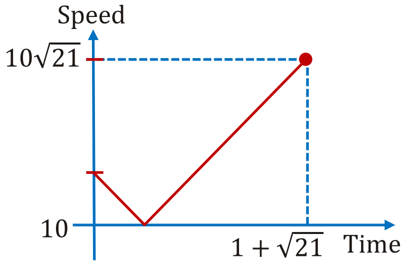
(ii) Speed vs time graph: Particle has constant acceleration = $g \downarrow$ throughout the motion, so v-t curve will be straight line.

when moving up, $v = u + at$

$0 = 10 - 10\,t$ or $t = 1$ is the time at which speed is zero.

there after speed increases at constant rate of $10\ m/s^2$.

Resulting Graph is: (speed is always positive).

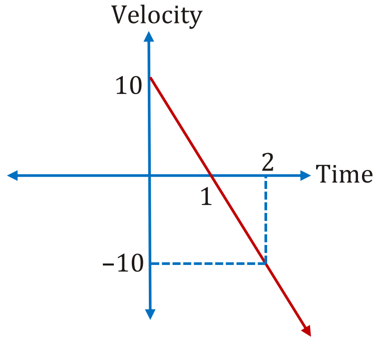


This shows that particle travels till a time of $1+\sqrt{21}$ seconds.

(iii) Velocity vs time graph:

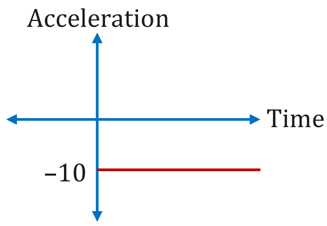
$V = u + at$

$V = 10 - 10t$; this shows that velocity becomes zero at $t = 1$ sec and thereafter the velocity is negative with slope g.



(iv) Acceleration vs time graph:

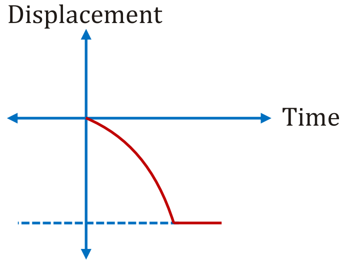
Throughout the motion, particle has constant acceleration = $-10\ m/s^2$.



For particle B:

$$u = -10\ m/s.\ y = -10t - \frac{1}{2}(10)\,t^2 = -10t - 5t^2$$

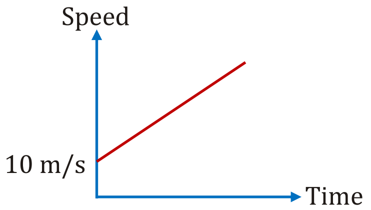
(i) Displacement time graph:



$y = -10t - 5t^2$; $\dfrac{dy}{dt} = -10 - 10t$

This shows that slope becomes more negative with time.

(ii) Speed time graph:

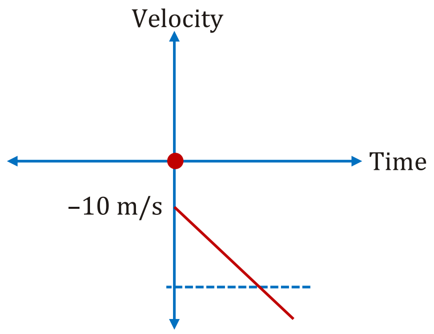


$\dfrac{dy}{dt} = -10 - 10t$

Hence, speed is linearly dependent on time with slope of 10 initial speed = 10 m/s

(iii) Velocity time graph:

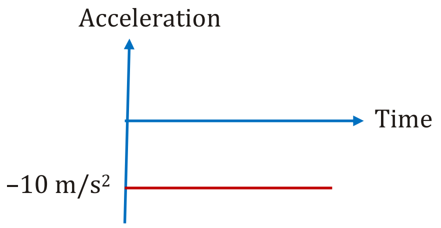
$\dfrac{dy}{dt} = -10 - 10t$



Hence, velocity is linearly dependent on time with slope of –10. Initial velocity = –10 m/s

(iv) Acceleration vs time graph:

Throughout the motion, particle has constant acceleration = –10 m/s^2.

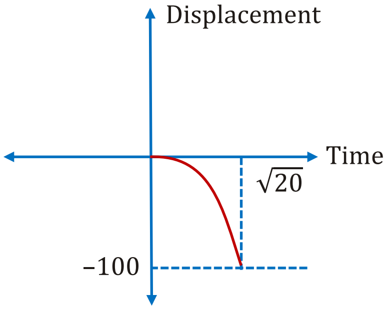


$a = \dfrac{dv}{dt} = -10$

For Particle C:

(i) Displacement time graph:

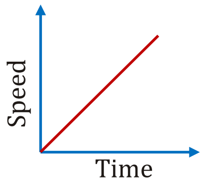
$u = 0, y = -\frac{1}{2} \times 10t^2 = -5t^2$



This shows that slope becomes more negative with time.

(ii) Speed vs time graph:

$v = \frac{dy}{dt} = -10\,t$

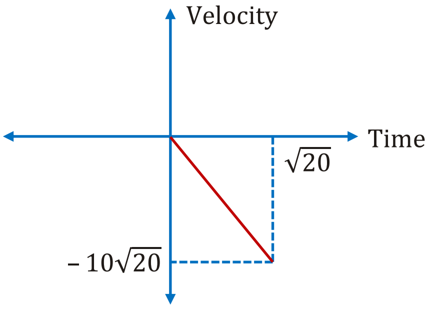


Hence, speed is directly proportional to time with slope of 10.

(iii) Velocity time graph:

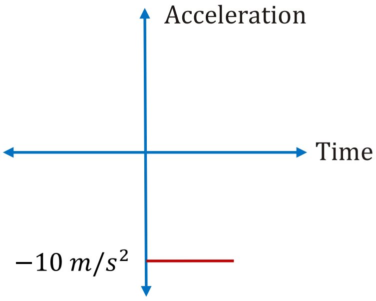
$V = u + at$

$V = -10t;$



Hence, velocity is directly proportional to time with slope of −10.

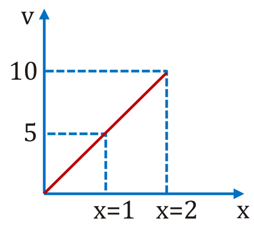
(iv) Acceleration vs time graph:



Throughout the motion, particle has constant acceleration = $-10\ m/s^2$.

Illustration 59:

Velocity v/s position graph is given. Find acceleration at $x = 1\ m$.



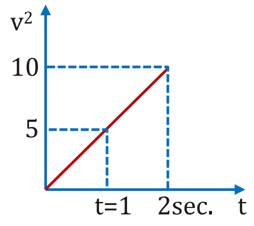
Solution:

$$\frac{dv}{dx} = \tan\theta = 5$$

$$a = v\frac{dv}{dx} = 5 \times 5\ = 25 m/s^2$$

Illustration 60:

Find acceleration at $t = 1$ sec for the given graph.



Solution:

$$\frac{dv^2}{dt} = \tan\theta = 5$$

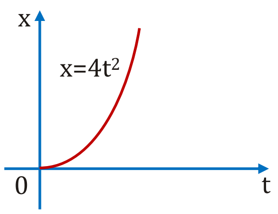
$$\frac{dv^2}{dv} \times \frac{dv}{dt} = 5$$

$$\Rightarrow\quad 2v \times a = 5 \Rightarrow 2\sqrt{5} \times a = 5 \Rightarrow a = \frac{\sqrt{5}}{2}$$

Interconversion of graphs

Illustration 61:

The displacement vs time graph of a particle moving along a straight line is shown in the figure. Draw velocity vs time and acceleration vs time graph.

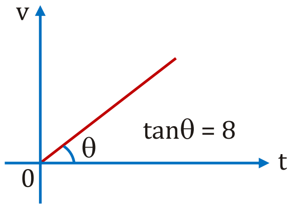


Solution:

$x = 4t^2$

$$v = \frac{dx}{dt}\ = 8t$$

Hence, velocity-time graph is a straight-line having slope i.e. tanθ = 8.



$$a = \frac{dv}{dt} = 8$$

Hence, acceleration is constant throughout and is equal to 8.

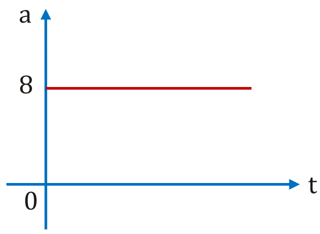
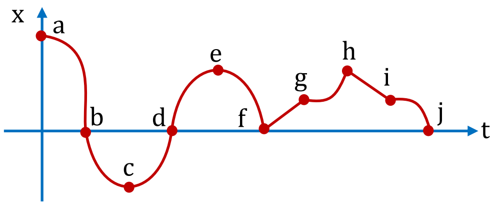


Illustration 62:

If all parts of the *x-t* graph are either parabolic or straight line then draw corresponding *v-t* and *a-t* graph.



Solution:

Slope of *x-t* graph gives velocity and slope of *v-t* graph gives acceleration.

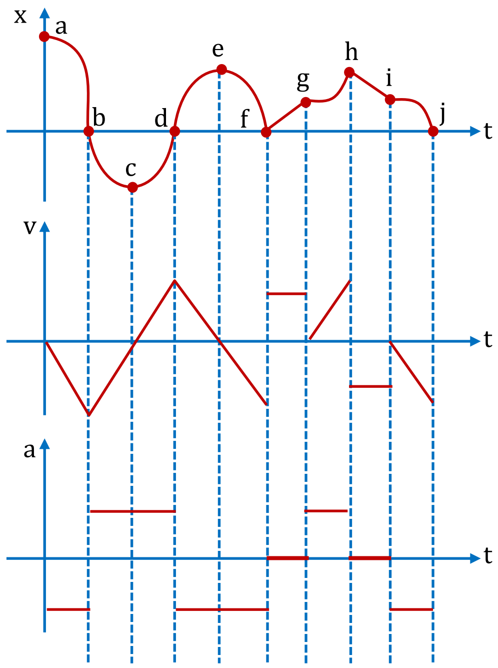
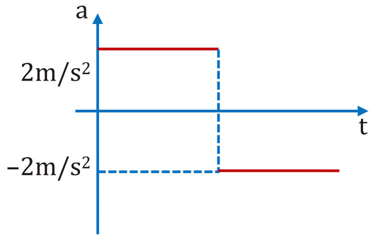


Illustration 63:

Plot velocity-time, position-time and distance-time graph for the given *a*-*t* graph if at $t = 0$; $v = 0$ and $x = 0$.



Solution:

Area of *a*-*t* curve gives change in velocity and area of *v*-*t* curve gives displacement.

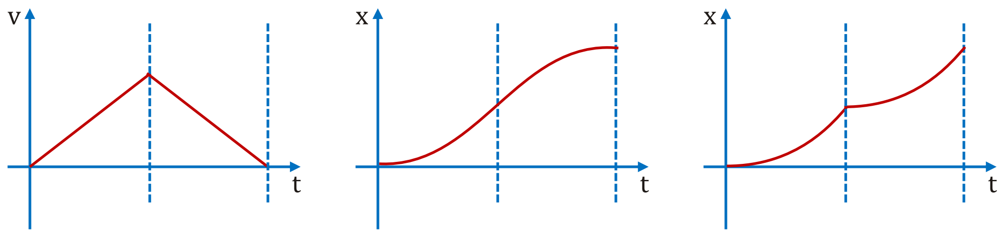
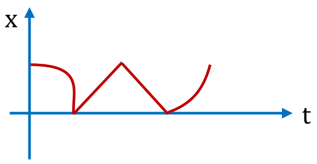
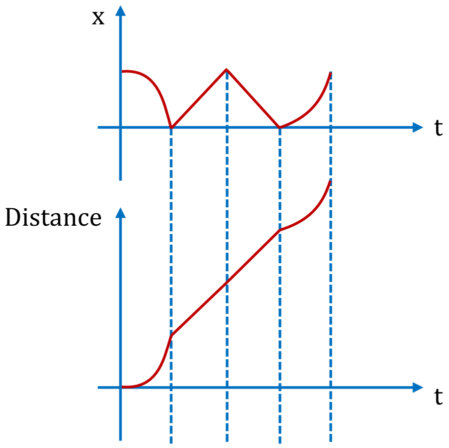


Illustration 64:

Plot distance time graph for the given position time graph.



Solution:



Note:

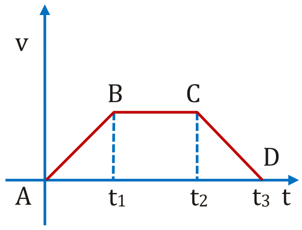
(i) Distance-time and displacement-time graph always starts from origin.

(ii) Distance never decreases. So, to convert position time graph into distance time graph, take the mirror image of the graph when position start decreasing.

(iii) Speed can decrease but it cannot be negative. So, to convert velocity time graph into speed time graph, take the mirror image of the graph when velocity becomes negative.

Illustration 65:

Draw displacement–time and acceleration–time graph for the given velocity–time graph.



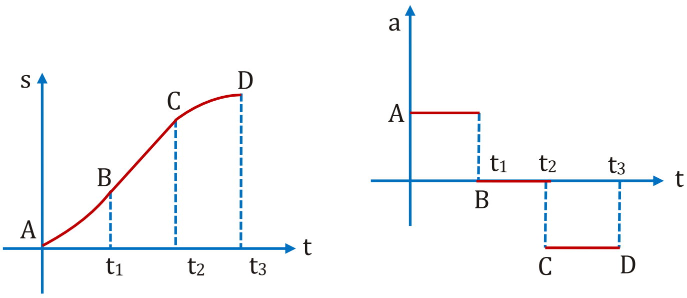
Solution:

Part AB : v-t curve shows constant slope
i.e., constant acceleration or Velocity increases at constant rate with time.
Hence, s-t curve will show constant increase in slope
And a-t curve will be a straight line (with zero slope).

Part BC : v-t curve shows zero slope i.e. constant velocity. So, s-t curve will show constant slope and acceleration will be zero.

Part CD : v-t curve shows negative slope i.e. velocity is decreasing with time or acceleration is negative.
Hence, s-t curve will show decrease in slope becoming zero in the end.
And a-t curve will be a straight line with negative intercept on y-axis (with zero slope).

Resulting graphs are:



Miscellaneous Problems

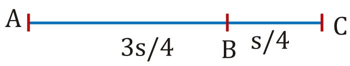
Illustration 66:

A particle covers $\frac{3}{4}$ of total distance with speed v_1 and next $\frac{1}{4}$ with v_2. Find the average speed of the particle.

Solution:

Let the total distance be s

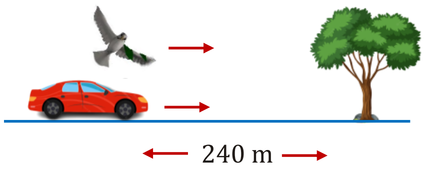
$$\text{Average speed } (< v >) = \frac{\text{Total distance}}{\text{Total time taken}}$$



$$< v > = \frac{s}{\frac{3s}{4v_1} + \frac{s}{4v_2}} = \frac{1}{\frac{3}{4v_1} + \frac{1}{4v_2}} = \frac{4v_1v_2}{v_1 + 3v_2}$$

Illustration 67:

A car is moving with speed 60 km/h and a bird is moving with speed 90 km/h along the same direction as shown in figure. Find the distance travelled by the bird till the time car reaches the tree.



Solution:

Time taken by a car to reach the tree (t) = $\dfrac{240\ \ m}{60\ \ km/hr}$ = $\dfrac{0.24}{60}hr$

Now, the distance travelled by the bird during this time interval

$= 90\times\dfrac{0.24}{60}$ = 0.12 × 3 km = 360 m.

Illustration 68:

The position of a particle moving on X-axis is given by $x = At^3 + Bt^2 + Ct + D$. The numerical values of A, B, C, D are 1, 4, –2 and 5 respectively and SI units are used. Find (a) the dimensions of A, B, C and D, (b) the velocity of the particle at t = 4s, (c) the acceleration of the particle at t = 4s, (d) the average velocity during the interval t = 0 to t = 4s, (e) the average acceleration during the interval t = 0 to t = 4 s.

Solution:

As $x = At^3 + Bt^2 + Ct + D$

(a) Dimensions of A, B, C and D,

$[At^3] = [x]$ (by principle of homogeneity)

$[A] = [LT^{-3}]$

Similarly, $[B] = [LT^{-2}]$, $[C] = [LT^{-1}]$ and $[D] = [L]$;

(b) As $v = \dfrac{dx}{dt} = 3At^2 + 2Bt + C$

Velocity at t = 4 sec.

$v = 3(1)\,(4)^2 + 2(4)\,(4) - 2 = 78\ m/s$.

(c) Acceleration (a) = $\dfrac{dv}{dt} = 6At + 2B$; $a = 32\ m/s^2$

(d) $x = At^3 + Bt^2 + Ct + D$

Position at t = 0 *is* $x = D = 5m$.

Position at t = 4 sec *is* (1)(64) + (4)(16) – (2) (4) + 5 = 125 m

Thus, the displacement during 0 to 4 sec. *is* 125 – 5 = 120 m

$\therefore \langle v \rangle = 120 / 4 = 30\ m/s$

(e) $v = 3At^2 + 2Bt + C$, velocity at t = 0 is $C = -2\ m/s$

Velocity at t = 4 sec is 78 m/s $\therefore \langle a \rangle = \dfrac{v_2 - v_1}{t_2 - t_1} = 20\ m/s^2$

Illustration 69:
For a particle moving along x-axis, velocity is given as a function of time as $v = 2t^2 + \sin t$. At $t = 0$, particle is at origin. Find the position as a function of time.

Solution:

$v = 2t^2 + \sin t \quad \Rightarrow \quad \frac{dx}{dt} = 2t^2 + \sin t$

$\int_0^x dx = \int_0^t (2t^2 + \sin t)dt \quad \Rightarrow x = \frac{2}{3}t^3 - \cos t + 1$

Illustration 70:
A car decelerates from a speed of 20 m/s to rest in a distance of 100 m. What was its acceleration, assumed constant?

Solution:

$v = 0, \quad u = 20\ m/s, \quad s = 100\ m$

$\Rightarrow$ as $v^2 = u^2 + 2\,as$

$0 = 400 + 2a \times 100 \quad \Rightarrow a = -2\ m/s^2$

$\therefore$ Acceleration $= -2\ m/s^2$

Illustration 71:
A 150 m long train accelerates uniformly from rest. If the front of the train passes a railway worker 50 m away from the station at a speed of 25 m/s, what will be the speed of the back part of the train as it passes the worker?

Solution:

$v^2 = u^2 + 2as$

$25 \times 25 = 0 + 100\,a$

$a = \frac{25}{4}\ m/s^2$

Now, for the back end of the train to pass the worker

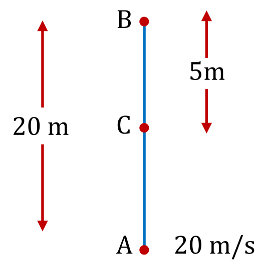
We have $v'^2 = v^2 + 2al = (25)^2 + 2 \times \frac{25}{4} \times 150$

$v'^2 = 25 \times 25 \times 4$

$v' = 50\ m/s.$

Illustration 72:
A particle is thrown vertically upwards with velocity 20 m/s. Find (a) the distance travelled by the particle in first 3 seconds, (b) displacement of the particle in 3 seconds.

Solution:



Highest point say B

$V_B = 0$

$v = u - gt$

$0 = 20 - 10\,t$

$t = 2$ sec.

$\therefore$ Distance travel in first 3 seconds.

$s = s(t = 0 \text{ to } 2\text{sec}) + s\,(2 \text{ sec. to } 3 \text{ sec.})$

$s = [ut + 1/2\, at^2]_{t=0 \text{ to } t=2s} + [ut + 1/2 at^2]_{t=2 \text{ to } t=3s}$

$s = (20 \times 2 - 1/2 \times 10 \times 4) + (1/2 \times 10 \times 1^2)$

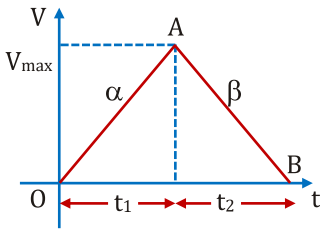
$= (40 - 20) + 5 = 25\ m.$

And displacement = $20 - 5 = 15\ m.$

Illustration 73:

A car accelerates from rest at a constant rate α for some time after which it decelerates at a constant rate β to come to rest. If the total time elapsed is t. Find the maximum velocity acquired by the car.

Solution:



$t = t_1 + t_2$

Slope of OA curve = $\tan\theta = \alpha = \dfrac{v_{max}}{t_1}$

Slope of AB curve = $\beta = \dfrac{v_{max}}{t_2}$

$t = t_1 + t_2$

$\Rightarrow \quad t = \dfrac{v_{max}}{\alpha} + \dfrac{v_{max}}{\beta} \qquad \Rightarrow \quad v_{max} = \left(\dfrac{\alpha\ \beta}{\alpha + \beta}\right)t$

Illustration 74:

In the above question find total distance travelled by the car in time 't'.

Solution:

Method I:

$$v_{max} = \frac{\alpha\beta}{(\alpha + \beta)}t$$

$\Rightarrow \quad t_1 = \dfrac{v_{max}}{\alpha} = \dfrac{\beta t}{(\alpha + \beta)} \qquad \Rightarrow \quad t_2 = \dfrac{v_{max}}{\beta} = \dfrac{\alpha t}{(\alpha + \beta)}$

∴ Total distance travelled by the car in time 't'

$$= \left(\frac{1}{2}\alpha t_1^2\right) + v_{\max} t_2 - \frac{1}{2}\beta t_2^2 = \frac{1}{2}\frac{\alpha\beta^2 t^2}{(\alpha+\beta)^2} + \frac{\alpha^2\beta t^2}{(\alpha+\beta)^2} - \frac{1}{2}\frac{\beta\alpha^2 t^2}{(\alpha+\beta)^2} = \frac{\alpha\beta t^2}{2(\alpha+\beta)}$$

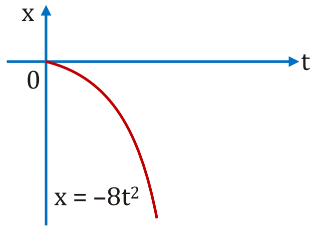
Method II:

Area under graph (directly) $= \frac{1}{2}\frac{\alpha\beta t^2}{(\alpha+\beta)} = \frac{\alpha\beta t^2}{2(\alpha+\beta)}$

Illustration 75:

The displacement vs time graph of a particle moving along a straight line is shown in the figure. Draw velocity vs time and acceleration vs time graph.

Upwards direction is taken as positive, downwards direction is taken as negative.



Solution:

(a) The equation of motion is: $x = -8t^2$

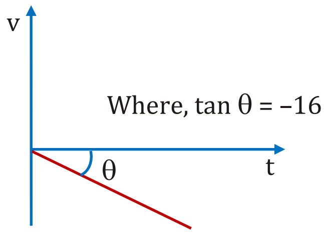
$\therefore\ v = \frac{dx}{dt} = -16\,t$; this shows that velocity is directly proportional to time and slope of velocity-time curve is negative i.e. – 16.

Hence, resulting graph is (i)

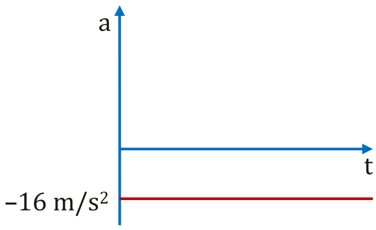
(b) Acceleration of particle is: $a = \frac{dv}{dt} = -16.$

This shows that acceleration is constant but negative.

Resulting graph is (ii)



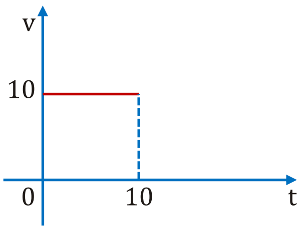
(i)



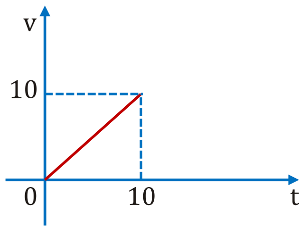
(ii)

Illustration 76:

For a particle moving along x-axis, following graphs are given. Find the distance travelled by the particle in 10 s in each case.



(a)



(b)

Solution:

(a) Distance = area under the v-t curve

$\therefore$ Distance = 10 × 10 = 100 m

(b) Distance = area under the v-t curve

$\therefore$ Distance = $\frac{1}{2}$ × 10 × 10 = 50 m

Illustration 77:

For a particle moving along x-axis, acceleration is given as $a = 2v^2$. If the speed of the particle is v_0 at $x = 0$, find speed as a function of x.

Solution:

$$a = 2v^2 \quad \Rightarrow \quad \frac{dv}{dt} = 2v^2 \quad \Rightarrow \quad \frac{dv}{dx} \times \frac{dx}{dt} = 2v^2$$

$$v\frac{dv}{dx} = 2v^2 \quad \Rightarrow \quad \frac{dv}{dx} = 2v$$

$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^x 2\ dx \Rightarrow [\ell nv]_{v_0}^{v} = [2x]_0^x$$

$$\ell n\frac{v}{v_0} = 2x \quad \Rightarrow \quad v = v_0 e^{2x}$$

MOTION IN A PLANE

Introduction to 2D motion

2D Motion:

- When the motion of an object is restricted within a plane, it is said to undergo a motion in 2D.
- 2D motion can be studied as two independent 1D motions. (One along x-axis and the other along y-axis).

  Example: Motion of a carrom coin.

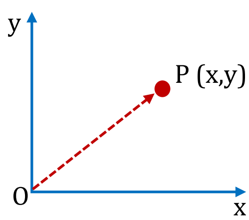
3D Motion:

- When the motion of an object is permitted all over the space, it is said to undergo a motion in 3D.
- 3D motion can be studied as three independent 1D motions. (Along x-axis, y-axis, and z-axis).

  Example: Motion of a fish in an aquarium.

Note:

Projection of x-axis on the y-axis is zero and vice versa. Hence, two vectors which are perpendicular to each other have no effect on each other. In other words, they are independent of each other. The motion along x-axis and y-axis are connected by time which is a scalar quantity. In other words, time is the only parameter common for both x and y components of motion.



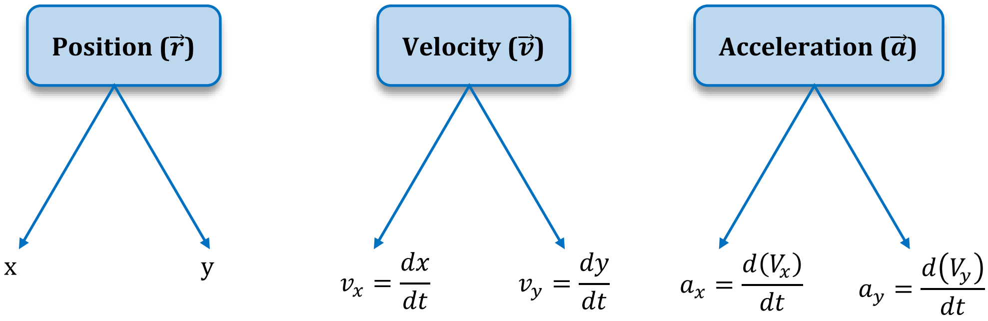
$\vec{r} = x\hat{\imath} + y\hat{\jmath}$

- 2D motion is a vector superimposition of two 1D motions along x and y directions.
- Position of a body under 2D motion can be specified by independent x and y coordinates. Position of point P is (x, y) and the position vector of P is $x\hat{\imath} + y\hat{\jmath}$.

Galileo's Statement

Two perpendicular directions of motion are independent from each other. In other words any vector quantity directed along a direction remains unaffected by a vector perpendicular to it.

- So, in 2D motion, position, velocity, and acceleration are divided into x and y components which are studied separately.



Steps to Analyse 2D motion:

(i) Choose an origin and establish the positive sense of x and y axes.

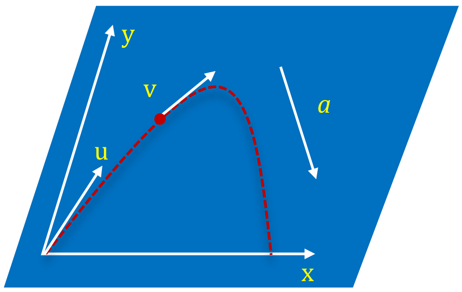
(ii) Resolve every given vector along the x and y axes.

(iii) Tabulate every single data separately for x and y.

(iv) Apply suitable equations of motion along x and y separately.

(v) Relate the two, as the time variable is common for both.

- Consider a 2D motion as shown in the figure. A particle is following a parabolic path with an initial velocity u and constant acceleration a. Let v be the instantaneous velocity at any time t.



- Acceleration can be resolved along x and y direction as a_x and a_y respectively.
- Equations of motion should be applied separately for x and y directions.

x-axis	y-axis
$v_x = u_x + a_x t$	$v_y = u_y + a_y t$
$\Delta x = u_x t + \frac{1}{2} a_x t^2$	$\Delta y = u_y t + \frac{1}{2} a_y t^2$
$v_x^2 = u_x^2 + 2a_x \Delta x$	$v_y^2 = u_y^2 + 2a_y \Delta y$

- Velocity and position can be dealt separately in x and y directions as shown in the table given below:

x-axis	y-axis	Resultant
$v_x = u_x + a_x t$	$v_y = u_y + a_y t$	$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}; \lvert\vec{v}\rvert = \sqrt{v_x^2 + v_y^2}$
$\Delta x = u_x t + \frac{1}{2}a_x t^2$	$\Delta y = u_y t + \frac{1}{2}a_y t^2$	$\vec{r} = x\hat{\imath} + y\hat{\jmath};\ \lvert\vec{r}\rvert = \sqrt{x^2 + y^2}$

- **Similar to 1D motion, in 2D motion, we have,**

$v_x = \frac{dx}{dt}$	$v_y = \frac{dy}{dt}$
$\frac{d(v_x)}{dt} = a_x = v_x\frac{dv_x}{dx}$	$\frac{d(v_y)}{dt} = a_y = v_y\frac{dv_y}{dy}$

Illustration 78:

A particle has initial velocity, $\vec{u} = 2\hat{\imath} + 3\hat{\jmath}$ and acceleration, $\vec{a} = 4\hat{\imath} + 2\hat{\jmath}$ then, Find,

(a) $\vec{v}$ at t = 2 sec. (b) $\vec{s}$ from 0 to 2 sec.

Solution:

x | y

$u_x = 2$ | $u_y = 3$

$a_x = 4$ | $a_y = 2$

$t = 2$ | $t = 2$

(a) $v_x = u_x + a_x t$ | $v_y = u_y + a_y t$

$v_x = 2 + 4 \times 2 = 10$ | $v_y = 3 + 2 \times 2 = 7$

$\vec{v} = 10\hat{\imath} + 7\hat{\jmath}$

(b) $s_x = \left(\frac{u_x+v_x}{2}\right)t$ | $s_y = \left(\frac{u_y+v_y}{2}\right)t$

$s_x = \left(\frac{2+10}{2}\right)2 = 12$ | $s_y = \left(\frac{3+7}{2}\right)2 = 10$

$\vec{s} = 12\hat{\imath} + 10\hat{\jmath}$

Illustration 79:

Position vector of a particle varying with time is given as, $\vec{r} = 6t\hat{\imath} + (8t - 4t^2)\hat{\jmath}$ then find,

(a) $\vec{v}$ (b) $\vec{a}$

(c) Speed of projection (d) $\vec{v}$ when $y = 0$ after $t = 0$

Solution:

$\vec{r} = x\hat{\imath} + y\hat{\jmath}$

$x = 6t$ | $y = 8t - 4t^2$

$v_x = 6$ | $v_y = 8 - 8t$

$a_x = 0$ | $a_y = -8$

(a) $\vec{v} = 6\hat{\imath} + (8 - 8t)\hat{\jmath}$ (b) $\vec{a} = -8\hat{\jmath}$

(c) $|\vec{v}_0| = |6\hat{\imath} + (8 - 8 \times 0)\hat{\jmath}|$

$= |6\hat{\imath} + 8\hat{\jmath}|$

$= 10\ m/s$

(d) $y = 8t - 4t^2 = 0$

$t = 0,\ 2$ (reject) (✓)

$\vec{v}_0 = 6\hat{\imath} + (8 - 8 \times 2)\hat{\jmath}$

$= 6\hat{\imath} - 8\hat{\jmath}$

Illustration 80:

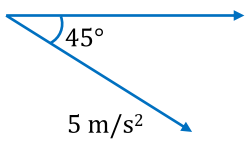
A particle has varying acceleration with time along x-axis as, $a_x = 6t$ and along y-axis as, $a_y = 3t^2$ then find velocity vector $\vec{v}$ and position vector $\vec{r}$ of particle, if at $t = 0$ particle was at rest at origin.

Solution:

	x	y
	$a_x = 6t$	$a_y = 3t^2$
	$v_x = 3t^2 + C_1$	$v_y = t^3 + C_2$
at	$t = 0$,	
	$v_x = 0 \Rightarrow C_1 = 0$	$v_y = 0 \Rightarrow C_2 = 0$
So,	$v_x = 3t^2$	$v_y = t^3$
		$\vec{v} = 3t^2\hat{\imath} + t^3\hat{\jmath}$
	$x = t^3$	$y = \frac{t^4}{4}$
		$\vec{r} = t^3\hat{\imath} + \frac{t^4}{4}\hat{\jmath}$

Illustration 81:

A roller coaster goes down along 45° incline with an acceleration of 5.0 ms^{-2}. (Starts from rest)



(a) How far will the roller coaster travel in 10 seconds horizontally?

(b) How far will the roller coaster travel in 10 seconds vertically?

Solution:

The motion of the roller coaster can be considered as a 1D motion along the direction of acceleration. Taking the direction of acceleration as r-direction.

$$x = x_0 + ut + \frac{1}{2}at^2$$

In r-direction,

$$r = r_0 + ut + \frac{1}{2}at^2$$

$$= 0 + 0 \times t + \frac{1}{2}(5)(10)^2$$

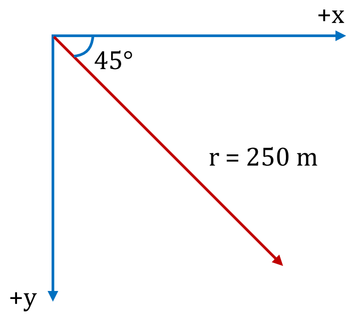
$= 250\ m$

Taking right direction as positive x axis and downwards as positive y axis, Displacement along y-direction

$\Delta y = r \sin\theta$

$= 250 \sin 45°$

$= \frac{250}{\sqrt{2}} m$



Displacement along x-direction

$\Delta x = r \cos\theta$

$= 250 \cos 45°$

$= \frac{250}{\sqrt{2}} m$

Illustration 82:

The position of a particle at time, $t = 0$ is P (−1, 2, −1). It starts moving with an initial velocity $\vec{u} = 3\hat{\imath} + 4\hat{\jmath}\, ms^{-1}$ and with a uniform acceleration $\vec{a} = (-4\hat{\imath} + 3\hat{\jmath})\, ms^{-2}$. Find the position and the magnitude of displacement after 4 seconds.

Solution:

$u_x = 3ms^{-1}$	$u_y = 4ms^{-1}$
$a_x = -4ms^{-2}$	$a_y = 3ms^{-2}$

Initial position vector of the particle = $-\hat{\imath} + 2\hat{\jmath} - \hat{k}$

Final position of particle after 4 seconds,

$\Delta\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$

$\vec{s}_f = \vec{s}_i + \vec{u}t + \frac{1}{2}\vec{a}t^2$

$= \left(-\hat{\imath} + 2\hat{\jmath} - \hat{k}\right) + [(3\hat{\imath} + 4\hat{\jmath}) \times 4] + \left[\frac{1}{2}(-4\hat{\imath} + 3\hat{\jmath}) \times 4^2\right]$

$= -21\hat{\imath} + 42\hat{\jmath} - \hat{k}$

Displacement = $\vec{s}_f - \vec{s}_i = -20\hat{\imath} + 40\hat{\jmath}$

Displacement magnitude= $\sqrt{(-20)^2 + 40^2} = 20\sqrt{5}m$

PROJECTILE MOTION

Basic concept

Projectile:

Any object that is given an initial velocity obliquely, and that subsequently follows a path determined by the net constant force, (In this chapter constant force is gravitational force) acting on it is called a projectile.

Examples of projectile motion :

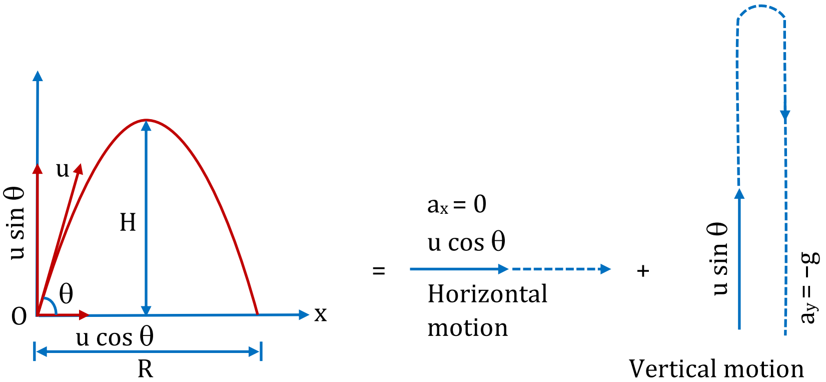
- A cricket ball hit by the batsman for a six.
- A bullet fired from a gun.
- A packet dropped from a plane; but the motion of the aeroplane itself is not projectile motion because there are forces other than gravity acting on it due to the thrust of its engine.

Assumptions of Projectile Motion:

- We shall consider only trajectories that are of sufficiently short range so that the gravitational force can be considered constant in both magnitude and direction.
- All effects of air resistance will be ignored.
- As range is very short compared to the radius of the earth, the part of the earth can be considered to be flat.

Projectile Motion:

- The motion of projectile is known as projectile motion.
- It is an example of two-dimensional motion with constant acceleration.
- Projectile motion is considered as combination of two simultaneous motions in mutually perpendicular directions which are completely independent from each other i.e. horizontal motion and vertical motion.

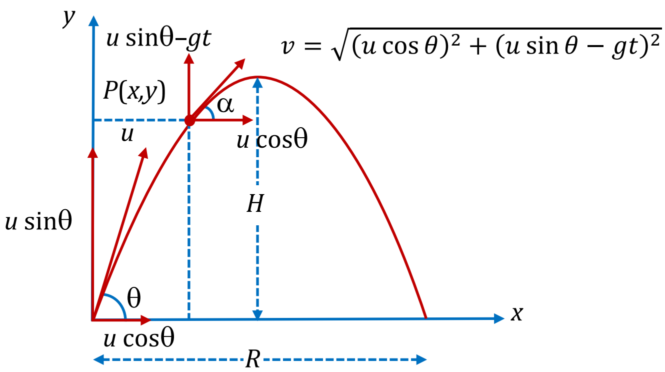


Parabolic path = vertical motion + horizontal motion.

Projectile Thrown at an Angle with Horizontal

- Consider a projectile thrown with a velocity u making an angle θ with the horizontal.
- Initial velocity u is resolved in components in a coordinate system in which horizontal direction is taken as x-axis, vertical direction as y-axis and point of projection as origin.

  $u_x = u \cos\theta$; $u_y = u \sin\theta$



- Again this projectile motion can be considered as the combination of horizontal and vertical motion. Therefore,

Horizontal direction	**Vertical direction**
(a) Initial velocity $u_x = u\cos\theta$	Initial velocity $u_y = u\sin\theta$
(b) Acceleration $a_x = 0$	Acceleration $a_y = -g$
(c) Velocity after time t, $v_x = u\cos\theta$	Velocity after time t, $v_y = u\sin\theta - gt$

Resultant velocity:

$$\vec{V}_R = (u\cos\theta)\hat{i} + (u\sin\theta - gt)\hat{j}$$

$$|\vec{V}_R| = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$$

and $\tan\alpha = \dfrac{u\sin\theta - gt}{u\cos\theta}$

Where α is angle that velocity vector makes with horizontal. Also known as direction or angle of motion.

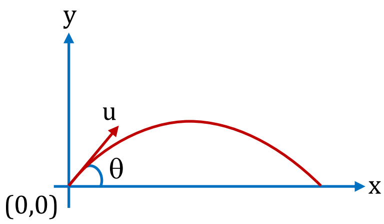
Vectorial treatment (Optional):

Lets say a particle is projected at an angle θ from horizontal with a velocity. Now if we take point of projection as origin and take vertically upward as positive y-axis and horizontal direction as x-axis.

$$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$$

$$\vec{a} = -g\hat{j}$$

Now since acceleration is uniform



Velocity after time t

$$\vec{V} = \vec{u} + \vec{a}t$$

$\Rightarrow$ $\vec{V} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} + (-g\hat{j})t$

$\Rightarrow$ $\vec{V} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\hat{j}$

Displacement after time t

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$\Rightarrow$ $\vec{S} = (u\cos\theta\,\hat{i})t + (u\sin\theta\,\hat{j})t + \frac{1}{2}(-g\hat{j})t^2$

$\Rightarrow$ $\vec{S} = (u\cos\theta\,t)\hat{i} + (u\sin\theta\,t - \frac{1}{2}gt^2)\hat{j}$

Time of Flight

The displacement along vertical direction is zero for the complete flight.

Hence, along vertical direction net displacement = 0.

$\Rightarrow$ $(u\sin\theta)\,T - \frac{1}{2}gT^2 = 0$ $\quad\Rightarrow\quad$ $T = \dfrac{2u\sin\theta}{g}$

Horizontal Range

$R = u_x . T$ $\quad\Rightarrow\quad$ $R = u\cos\theta . \dfrac{2u\sin\theta}{g}$

$$R = \frac{u^2\sin 2\theta}{g}$$

Maximum Height

At the highest point of its trajectory, particle moves horizontally, and hence vertical component of velocity is zero.

Using 3rd equation of motion i.e.

$v^2 = u^2 + 2as$

We have for vertical direction

$0 = u^2 \sin^2 \theta - 2gH$

$\Rightarrow \quad H = \dfrac{u^2 \sin^2 \theta}{2g}$

Illustration 83:

A body is thrown with initial velocity $10m/sec.$ at an angle $37°$ from horizontal.

Find (i) Time of flight (ii) Maximum height.

(iii) Range (iv) Position vector after $t = 1$ sec.

Solution:

$u_x = u\cos\theta = 10\cos 37° = 8m/s$

$u_y = u\sin\theta = 10\sin 37° = 6m/s$

(i) $T = \dfrac{2u\sin\theta}{g} = \dfrac{2\times10\times3}{10\times5} = \dfrac{6}{5} = 1.2$ sec

(ii) $H_{max} = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{10^2\times3\times3}{2\times10\times5\times5} = \dfrac{9}{5} = 1.8\ m$

(iii) $R = u_xT = 8\times1.2 = 9.6\ m$

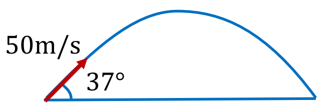
(iv) $x_{t=1} = u_xt = 8\times1 = 8\ m$

$y_{t=1} = u_yt - \frac{1}{2}gt^2 = 6\times1 - 5(1)^2 = 1\ m$

Position of particle at $t = 1$ sec is $(8\hat{i} + \hat{j})m$

Illustration 84:

A body is projected with speed 50 m/s at an angle $37°$ from horizontal then, find T and H and R ($g = 10\ m/s^2$).



Solution:

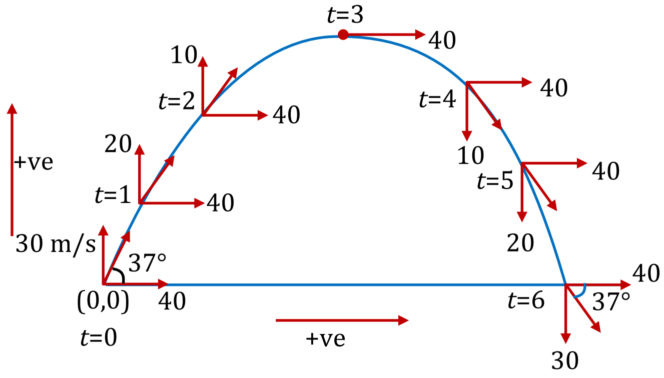
$T = \dfrac{2u\sin\theta}{g} = \dfrac{2\times50\times\sin 37°}{10} = 10\times\dfrac{3}{5} = 6$ sec.

$H = \dfrac{(u\sin\theta)^2}{2g} = \dfrac{(50\times\sin 37°)^2}{2\times10} = \dfrac{\left(50\times\frac{3}{5}\right)^2}{2\times10} = \dfrac{900}{2\times10} = 45m$

$R = \dfrac{u^2\sin 2\theta}{g} = (50)^2\dfrac{2\sin 37°\cos 37°}{10} = \dfrac{2500\times2\times\frac{3}{5}\times\frac{4}{5}}{10} = 240m$

Analysis :

x	y
$a_x = 0$	$a_y = -g$
$u_x = 50 \cos 37° = 40$	$u_y = 50 \sin 37° = 30$
$v_x = 40\ m/s$	$v_y = u_y + a_y t$
$x = u_x t = 40\ t$	$v_y = 30 - 10t$
	$y = 30t + \frac{1}{2}(-10)t^2$



t	$v_x(m/s)$	v_y	x	$y = 30t - \frac{1}{2}10t^2$	$\tan\alpha = \frac{v_y}{v_x} = \frac{u_y - gt}{u_x}$
0	40	30	0	$30(0) - \frac{1}{2}10(0)^2 = 0$	$\tan\alpha = \frac{30}{40} = \frac{3}{4}$
1	40	20	40	25m	$\tan\alpha = \frac{20}{40} = \frac{1}{2}$
2	40	10	80	40m	$\tan\alpha = \frac{1}{4}$
3	40	0	120	45m	$\tan\alpha = 0$
4	40	–10	160	40m	$\tan\alpha = -\frac{1}{4}$
5	40	–20	200	25m	$\tan\alpha = -\frac{1}{2}$
6	40	–30	240	0	$\tan\alpha = -\frac{3}{4}$

Illustration 85:

A particle is given an initial velocity of 25 m/s at an angle 53° with horizontal. Find height of projectile when horizontal distance is 45 m?

Solution:

x	y
$u_x = 25 \cos 53°$	$u_y = 25 \sin 53°$
$= 15\ m/s$	$= 20\ m/s$
$a_x = 0$	$a_y = -10$
$s_x = 45$	$s_y = 20 \times 3 - \frac{1}{2} \times 10 \times 3^2$
$\frac{S_x}{u_x} = t = 3$ sec.	$s_y = 60 - 45$
	$s_y = 15\ m$

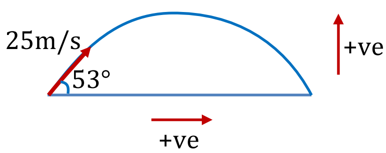
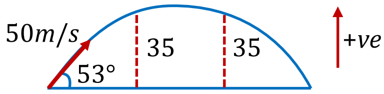


Illustration 86:

A projectile is given an initial velocity of 50 m/s at an angle 37° with verticle. Find horizontal distance (x), when height = 35 m.

Solution:



x		y
$u_x = 50 \cos 53°$		$u_y = 50 \sin 53°$
$= 30\ m/s$		$= 40\ m/s$
$a_x = 0$		$a_y = -10$

$S_y = u_y t - \frac{1}{2} g t^2 = 35$

$\Rightarrow \quad 40t - 5t^2 = 35$

$\Rightarrow \quad 5t^2 - 40t + 35 = 0$

$\Rightarrow \quad t^2 - 8t + 7 = 0$

$\Rightarrow \quad t = 7, 1$ for $S_y = 35$

$t = 1$		$t = 7$
$s_x = 30;$		$s_x = 30 \times 7$
$s_x = 30\ m$	and	$s_x = 210 m$

Illustration 87:

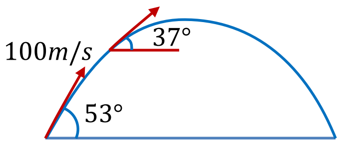
A projectile is given an initial velocity of 100 m/s at an angle 53° with horizontal.

(a) Find time and speed when velocity makes 37° above the horizontal.

(b) Find time and speed when velocity makes 37° below the horizontal.

(c) Find time when velocity is perpendicular to initial direction of motion.

Solution:



(a)

x	y
$u_x = 100 \cos 53°$	$u_y = 100 \sin 53°$
$= 60\ m/s$	$= 80\ m/s$
$a_x = 0$	$a_y = -10$

$v_y = 80 - 10t$

$\therefore \tan \alpha = \frac{v_y}{v_x}$

$\tan 37° = \frac{80 - 10t}{60}$

$\frac{3}{4} = \frac{80 - 10t}{60}$

$18 = 32 - 4t$

$4t = 14$

$t = \frac{14}{4} = \frac{7}{2} = 3.5$ sec.

$v_y = 80 - 10 \times 3.5 = 80 - 35 = 45\ m/s$

$v_x = 60\ m/s$

v or speed = $\sqrt{(45)^2 + (60)^2} = 75\ m/s$

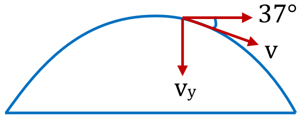
(b) $\frac{v_y}{v_x} = \tan 37°$

$\frac{v_y}{60} = \frac{3}{4}$

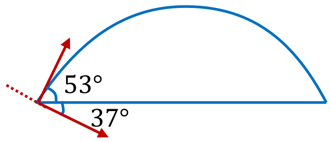
$v_y = 45\ m/s$

$\therefore\ -45 = 80 - 10 \times t$

$t = 12.5$ sec.



(c) Solution is same as (b) part.



General Result

- For maximum range $\theta = 45°$

  $R_{max} = \frac{u^2}{g}$, $\quad H_{max} = \frac{u^2}{2g} \quad \Rightarrow \quad H_{max} = \frac{R_{max}}{2}$

- We get the same range for two angle of projections α and $(90° - \alpha)$ but in both cases, maximum heights attained by the particles are different.

  This is because, $R = \frac{u^2 \sin 2\theta}{g}$ and $\sin 2\,(90° - \alpha) = \sin\,(180° - 2\alpha) = \sin 2\alpha$

- If $R = H$

  i.e., $\frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin^2 \theta}{2g} \quad \Rightarrow \quad \tan\theta = 4$

- Range can also be expressed as $R = \frac{u^2 \sin 2\theta}{g} = \frac{2u \sin\theta . u\cos\theta}{g} = \frac{2u_x u_y}{g}$

Illustration 88:

A body is projected with a speed of 30 ms^{-1} at an angle of 30° with the vertical. Find the maximum height, time of flight and the horizontal range of the motion. [Take $g = 10\ m/s^2$]

Solution:

Here $u = 30\ ms^{-1}$, Angle of projection, $\theta = 90° - 30° = 60°$

Maximum height, $H = \frac{u^2 \sin^2\theta}{2g} = \frac{30^2 \sin^2 60°}{2\times10} = \frac{900}{20} \times \frac{3}{4} = \frac{135}{4} m$

Time of flight, $T = \frac{2u\sin\theta}{g} = \frac{2\times30\times\sin 60^0}{10} = 3\sqrt{3}$ sec.

Horizontal range $= R = \frac{u^2 \sin 2\theta}{g} = \frac{30\times30\times2\sin 60°\cos60°}{10} = 45\sqrt{3}m$

Illustration 89:

A projectile is thrown with a speed of 100 m/s making an angle of 60° with the horizontal. Find the minimum time after which its inclination with the horizontal is 45°?

Solution:

$u_x = 100 \times \cos 60° = 50$

$u_y = 100 \times \sin 60° = 50\sqrt{3}$

$v_y = u_y + a_y t = 50\sqrt{3} - gt$ and $v_x = u_x = 50$

When angle is 45°,

$\tan 45° = \dfrac{v_y}{v_x} \quad \Rightarrow \quad v_y = v_x$

$\Rightarrow \quad 50\sqrt{3} - gt = 50 \quad \Rightarrow \quad 50(\sqrt{3} - 1) = gt \quad \Rightarrow \quad t = 5(\sqrt{3} - 1)\ s$

Illustration 90:

A large number of bullets are fired in all directions with the same speed v. What is the maximum area on the ground on which these bullets will spread ?

Solution:

Maximum distance up to which a bullet can be fired is its maximum range, therefore

$R_{max} = \dfrac{v^2}{g}$

Maximum area $= \pi(R_{max})^2 = \dfrac{\pi v^4}{g^2}$

Illustration 91:

The velocity of projection of a projectile is given by: $\vec{u} = 5\hat{\imath} + 10\hat{\jmath}$. Find

(a) Time of flight, (b) Maximum height, (c) Range

Solution:

We have $u_x = 5$ $u_y = 10$

(a) Time of flight $= \dfrac{2u\sin\theta}{g} = \dfrac{2u_y}{g} = \dfrac{2\times10}{10} = 2s$

(b) Maximum height $= \dfrac{u^2\sin^2\theta}{2g} = \dfrac{{u_y}^2}{2g} = \dfrac{10\times10}{2\times10} = 5m$

(c) Range $= \dfrac{2u\sin\theta . u\cos\theta}{g} = \dfrac{2u_x u_y}{g} = \dfrac{2\times10\times5}{10} = 10m$

Illustration 92:

A particle is projected at an angle of 30° w.r.t. horizontal with speed 20 m/s :

(i) Find the position vector of the particle after 1s.

(ii) Find the angle between velocity vector and position vector at $t = 1s$.

Solution:

(i) $x = u \cos \theta\, t = 20 \times \frac{\sqrt{3}}{2} \times t = 10\sqrt{3}m$

$y = u \sin \theta t - \frac{1}{2} \times 10 \times t^2 = 20 \times \frac{1}{2} \times (1) - 5\,(1)^2 = 5m$

Position vector, $\vec{r} = 10\sqrt{3}\,\hat{\imath} + 5\hat{\jmath}, |\vec{r}| = \sqrt{(10\sqrt{3})^2 + 5^2}$

(ii) $v_x = 10\sqrt{3}\hat{\imath}$

$v_y = u_y + a_y t = 10 - g\,t = 0$

$\therefore\ \vec{v} = 10\sqrt{3}\hat{\imath}, |\vec{v}| = 10\sqrt{3}$

$\vec{v} \cdot \vec{r} = (10\sqrt{3}\hat{\imath}) \cdot (10\sqrt{3}\hat{\imath} + 5\hat{\jmath}) = 300$

$\vec{v}.\vec{r} = |\vec{v}|\,|\vec{r}| \cos \theta$

$\Rightarrow \cos \theta = \frac{\vec{v}\cdot\vec{r}}{|\vec{v}||\vec{r}|} = \frac{300}{10\sqrt{3}\sqrt{325}}$

$\Rightarrow \theta = \cos^{-1}\left(2\sqrt{\frac{3}{13}}\right)$

Equation of Trajectory

The path followed by a particle (here projectile) during its motion is called its **Trajectory**. Equation of trajectory is the relation between instantaneous coordinates (Here x and y coordinate) of the particle.

If we consider the horizontal direction,

$x = u_x.t$

$x = u \cos \theta.\, t$ ...(1)

For vertical direction :

$y = u_y.\, t - 1/2\, gt^2$

$= u \sin \theta.\, t - 1/2\, gt^2$ ...(2)

Eliminating 't' from equation (1) and (2)

$y = u \sin \theta. \frac{x}{u \cos \theta} - \frac{1}{2} g\left(\frac{x}{u \cos \theta}\right)^2 \quad \Rightarrow \quad y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}$

This is an equation of parabola called as trajectory equation of projectile motion.

Other forms of trajectory equation:

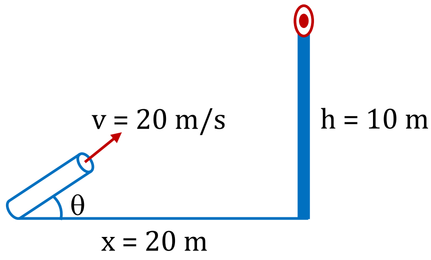
$\Rightarrow \quad y = x \tan \theta - \frac{gx^2(1+\tan^2 \theta)}{2u^2} \qquad \left(\because\ y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta}\right)$

and $y = x \tan \theta \left[1 - \frac{gx}{2u^2 \cos^2 \theta \tan \theta}\right] \quad \Rightarrow \quad y = x \tan \theta \left[1 - \frac{gx}{2u^2 \sin \theta \cos \theta}\right]$

$\Rightarrow \quad y = x \tan \theta \left[1 - \frac{x}{R}\right]$

Illustration 93:

Find the value of θ in the diagram given below so that the projectile can hit the target.



Solution:

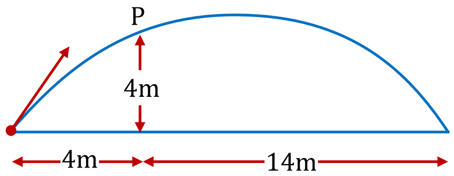
$$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2} \Rightarrow 10 = 20\tan\theta - \frac{5\times(20)^2}{(20)^2}(1+\tan^2\theta)$$

$$\Rightarrow 2 = 4\tan\theta - (1+\tan^2\theta) \Rightarrow \tan^2\theta - 4\tan\theta + 3 = 0$$

$$\Rightarrow (\tan\theta - 3)(\tan\theta - 1) = 0 \Rightarrow \tan\theta = 3, 1 \Rightarrow \theta = 45°, \tan^{-1}(3)$$

Illustration 94:

A ball is thrown from ground level so as to just clear a wall 4 *m* high at a distance of 4 *m* and falls at a distance of 14 *m* from the wall. Find the magnitude and direction of initial velocity of the ball. Figure is given below ($g = 9.8\ m/s^2$).



Solution:

The ball passes through the point $P(4, 4)$. Also, range = 4 + 14 = 18 *m*.

The trajectory of the ball is, $y = x\tan\theta\left(1 - \frac{x}{R}\right)$

Now $x = 4m$, $y = 4m$ and $R = 18\ m$

$$\therefore\quad 4 = 4\tan\theta\left[1 - \frac{4}{18}\right] = 4\tan\theta\left(\frac{7}{9}\right) \Rightarrow \tan\theta = \frac{9}{7} \Rightarrow \theta = \tan^{-1}\frac{9}{7}$$

$$\text{And } R = \frac{2u^2\sin\theta\cos\theta}{g} \Rightarrow 18 = \frac{2}{9.8}\times u^2\times\frac{9}{\sqrt{130}}\times\frac{7}{\sqrt{130}} \Rightarrow u = \sqrt{182}\,m/s$$

Illustration 95:

Derive the expression for range of projectile using equation of trajectory.

Solution:

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$0 = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \qquad (y = 0)$$

$$\frac{gx}{2u^2\cos^2\theta} = \frac{\sin\theta}{\cos\theta}$$

$$\frac{gx}{2u^2\cos\theta} = \sin\theta$$

$$gx = u^2\,2\sin\theta\cos\theta$$

$$x = \frac{u^2\sin 2\theta}{g}$$

Illustration 96:

A ground to ground projectile has equation of trajectory, $y = x - \frac{x^2}{40}$. Find angle of projection and speed of projection.

Solution:

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

By comparing

$\tan\theta = 1;\ \theta = 45°$

$$\frac{g}{2u^2\cos^2\theta} = \frac{1}{40}$$

$\therefore \quad u = 20\ m/s$

$\left(y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}\right)$ is valid when we take starting point as origin and $\xrightarrow{+ve} \uparrow +ve$

Illustration 97:

A projectile is projected horizontally with initial speed of 20 m/s from a height 45 m from ground. Find range of this horizontal projectile.

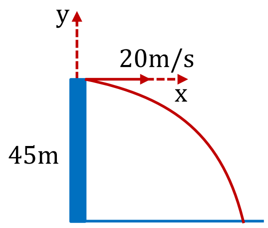
Solution:

$$-45 = x\tan 0° - \frac{10x^2}{2\times20\times20\times\cos^2 0°}$$

$$-45 = 0 - \frac{x^2}{80}$$

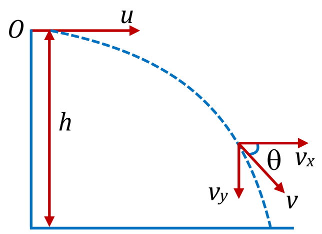
$$x^2 = 45 \times 80$$

$$x = \sqrt{45\times 80} = \sqrt{3600} = 60m$$



Projectile Thrown Parallel to the Horizontal from Some Height

Consider a projectile thrown from point O at some height h from the ground with a velocity u. Now we shall study the characteristics of projectile motion by resolving the motion along horizontal and vertical directions.



Horizontal direction	**Vertical direction**
(i) Initial velocity $u_x = u$	Initial velocity $u_y = 0$
(ii) Acceleration $a_x = 0$	Acceleration $a_y = g$ (downward)

Time of Flight:

This is equal to the time taken by the projectile to return to ground.

From equation of motion

$S = ut + \frac{1}{2}at^2$, along vertical direction, we get

$$-h = u_y t + \frac{1}{2}(-g)t^2 \quad \Rightarrow \quad h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}$$

Horizontal Range

Distance covered by the projectile along the horizontal direction between the point of projection to the point on the ground.

$R = u_x.t$

$\Rightarrow \quad R = u\sqrt{\frac{2h}{g}}$

Velocity at a general point *P* (*x*, *y*)

$v = \sqrt{v_x^2 + v_y^2}$

Here horizontal velocity of the projectile after time t

$v_x = u$

Velocity of projectile in vertical direction after time t

$v_y = 0 + (-g)t = -gt = gt$ (downward)

$\therefore \quad v = \sqrt{u^2 + g^2t^2}$ and $\tan\theta = v_y/v_x$

Velocity with Which the Projectile Hits the Ground

$v_x = u$

$v_y^2 = 0^2 - 2g(-h)$

$v_y = \sqrt{2gh}$

$v = \sqrt{v_x^2 + v_y^2}$

$\Rightarrow \quad v = \sqrt{u^2 + 2gh}$

Trajectory Equation

The path traced by projectile is called the trajectory.

After time t,

$x = ut$ ...(1)

$y = -\frac{1}{2}gt^2$ ...(2)

From equation (1)

$t = x/u$

Put the value of t in equation (2)

$y = -\frac{1}{2}g.\frac{x^2}{u^2}$

This is trajectory equation of the particle projected horizontally from some height.

Illustrations based on horizontal projection from some height

Illustration 98:

A projectile is fired horizontally with a speed of 98 ms^{-1} from the top of a hill 490 m high. Find

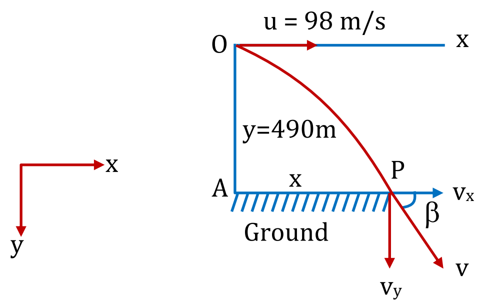
(i) the time taken to reach the ground

(ii) the distance of the target from the hill and

(iii) the velocity with which the projectile hits the ground. (Take $g = 9.8\ m/s^2$)

Solution:

(i) The projectile is fired from the top O of a hill with speed $u = 98\ ms^{-1}$ along the horizontal as shown as OX. It reaches the target P at vertical depth OA, in the coordinate system as shown,



$OA = y = 490\ m$

As, $y = \frac{1}{2}gt^2$

$\therefore\quad 490 = \frac{1}{2} \times 9.8\ t^2$

or $t = \sqrt{100} = 10\ s.$

(ii) Distance of the target from the hill is given by, $AP = x$ = Horizontal velocity × time = 98 × 10 = 980 m.

(iii) The horizontal and vertical components of velocity v of the projectile at point P are

$v_x = u = 98\ ms^{-1}$

$v_y = u_y + gt = 0 + 9.8 \times 10 = 98\ ms^{-1}$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{98^2 + 98^2} = 98\sqrt{2}ms^{-1}$$

Now if the resultant velocity v makes an angle β with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1 \qquad \therefore\ \beta = 45°$$

Illustration 99:

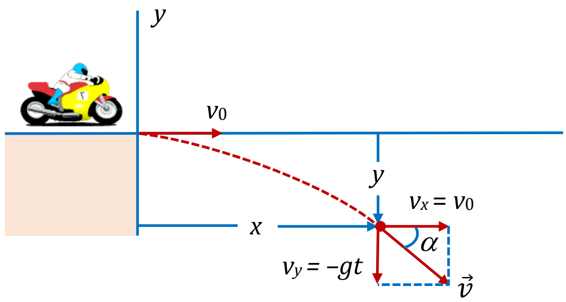
A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m/s. Find the motorcycle's position, distance from the edge of the cliff and velocity after 0.5 s.

Solution:

At $t = 0.50\ s$, the x and y-coordinates are, $x = v_0t = (9.0\ m/s)\ (0.50\ s) = 4.5\ m$

$$y = -\frac{1}{2}gt^2 = -\frac{1}{2}(10\ m/s^2)\ (0.50\ s)^2 = -\frac{5}{4}\ m$$

The negative value of y shows that this time the motorcycle is below its starting point.



The motorcycle's distance from the origin at this time,

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{5}{4}\right)^2} = \frac{\sqrt{349}}{4}\ m.$$

The components of velocity at this time are $v_x = v_0 = 9.0\ m/s$

$v_y = -gt = (-10\ m/s^2)\ (0.50\ s) = -5\ m/s.$

The speed (magnitude of the velocity) at this time is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(9.0 m/s)^2 + (-5 m/s)^2} = \sqrt{106} m/s$$

Illustration 100:

An object is thrown between two tall buildings 180 m from each other. The object is thrown horizontally from a window 55 m above ground from one building which passes through a window 10.9 m above ground in the other building. Find out the speed of projection. (Use $g = 9.8\ m/s^2$)

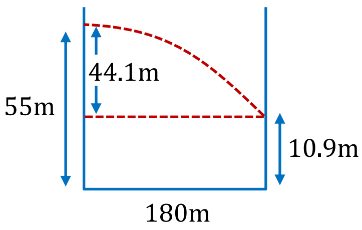
Solution:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 44.1}{9.8}}$$

$t = 3$ sec.

$R = uT$

$\frac{180}{3} = u$; $u = 60\ m/s$

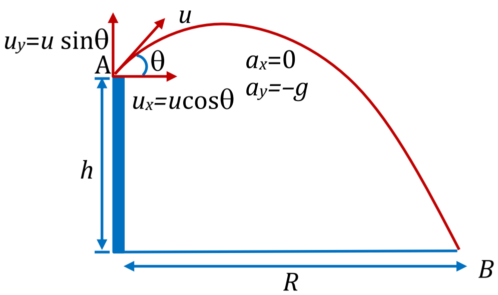


Projection from a Tower

Case (i): Horizontal projection

$u_x = u$; $u_y = 0$; $a_y = -g$

Case (ii): Projection at an angle θ above horizontal



$u_x = u\cos\theta$; $u_y = u\sin\theta$; $a_y = -g$

Equation of motion between A and B (in Y direction)

$S_y = -h$, $u_y = u\sin\theta$, $a_y = -g$, $t = T$

$$S_y = u_y t + \frac{1}{2} a_y t^2 \quad \Rightarrow \quad -h = u\sin\theta\, t - \frac{1}{2} g t^2$$

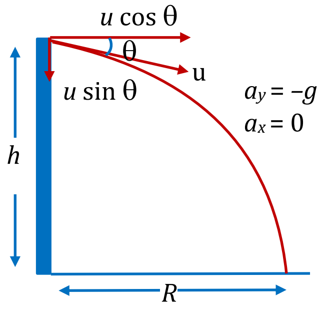
Solving this equation, we will get time of flight, T.

And range, $R = u_x T = u\cos\theta\, T$

Also, $v_y^2 = u_y^2 + 2a_yS_y = u^2\sin^2\theta + 2gh$; $v_x = u\cos\theta$

$$v_B = \sqrt{v_y^2 + v_x^2} \quad \Rightarrow \quad v_B = \sqrt{u^2 + 2gh}$$

Case (iii): Projection at an angle θ below horizontal



$u_x = u \cos\theta \quad ; \quad u_y = -u \sin\theta \quad ; \quad a_y = -g$

$S_y = u_y t + \frac{1}{2} a_y t^2$

$S_y = -h,\ u_y = -u\sin\theta,\ t = T,\ a_y = -g$

$\Rightarrow -h = -u\sin\theta\, T - \frac{1}{2} gT^2 \Rightarrow h = u\sin\theta\, T + \frac{1}{2} gT^2$

Solving this equation, we will get time of flight, T.

And range, $R = u_x T = u\cos\theta\, T$

$v_x = u\cos\theta$

$v_y^2 = u_y^2 + 2a_y S_y = u^2\sin^2\theta + 2(-g)(-h)$

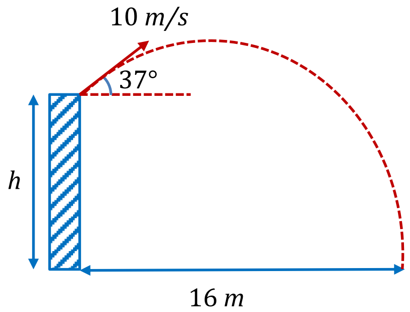
$v_y^2 = u^2\sin^2\theta + 2gh$

$v_B = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$

Illustration 101:

A ball is thrown from the top of a tower with an initial velocity of $10 m/s$ at an angle 37° above the horizontal, hits the ground at a distance $16 m$ from the base of tower. Calculate height of tower.

$[g = 10\ m/s^2]$



Solution:

$S_x = 10\cos 37° \times t = 16$

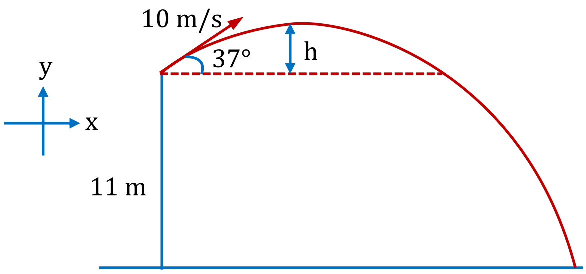
$\Rightarrow \quad 8t = 16 \Rightarrow t = 2 \sec$

$-h = 10\sin 37° \times t - \frac{1}{2} gt^2$

$\Rightarrow \quad h = -6 \times 2 + 5 \times 2^2 = 8m$

Illustration 102:

From the top of a 11 m high tower a stone is projected with speed 10 m/s, at an angle of 37° as shown in figure. Find:



(a) Speed after 2s.

(b) Time of flight.

(c) Horizontal range.

(d) The maximum height attained by the particle.

(e) Speed just before striking the ground.

Solution:

(a) Initial velocity in horizontal direction = 10 cos 37 = 8 m/s

Initial velocity in vertical direction = 10 sin 37° = 6 m/s

Speed after 2 seconds

$\vec{v} = v_x\hat{i} + v_y\hat{j} = 8\hat{i} + (u_y + a_y t)\,\hat{j} = 8\hat{i} + (6 - 10 \times 2)\hat{j} = 8\hat{i} - 14\hat{j}$

$|\vec{v}| = v = \sqrt{8^2 + 14^2} = 2\sqrt{65} = m/s$

(b) $S_y = u_y t + \frac{1}{2} a_y t^2 \Rightarrow -11 = 6 \times t + \frac{1}{2} \times (-10) t^2$

$5t^2 - 6t - 11 = 0 \Rightarrow (t+1)(5t - 11) = 0 \Rightarrow t = \frac{11}{5}$ sec.

(c) Range $= 8 \times \frac{11}{5} = \frac{88}{5} m$

(d) Maximum height above the level of projection, $h = \frac{u_y^2}{2g} = \frac{6^2}{2\times10} = 1.8\ m$

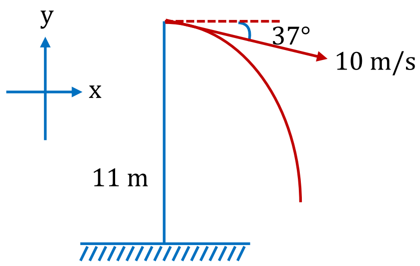
$\therefore$ Maximum height above ground = 11 + 1.8 = 12.8 m

(e) $v = \sqrt{u^2 + 2gh} = \sqrt{100 + 2 \times 10 \times 11}$

$\Rightarrow v = 8\sqrt{5} m/s$

Illustration 103:

From the top of a 11 m high tower a stone is projected with speed 10 m/s, at an angle of 37° as shown in figure. Find



(a) Time of flight.

(b) Horizontal range.

(c) Speed just before striking the ground.

Solution:

$u_x = 10 \cos 37° = 8\ m/s$, $u_y = -10 \sin 37° = -6\ m/s$

(a) $S_y = u_y t + \frac{1}{2} a_y t^2 \Rightarrow -11 = -6 \times t + \frac{1}{2} \times (-10) t^2 \Rightarrow 5t^2 + 6t - 11 = 0$

$\Rightarrow (t-1)(5t + 11) = 0 \Rightarrow t = 1$ sec

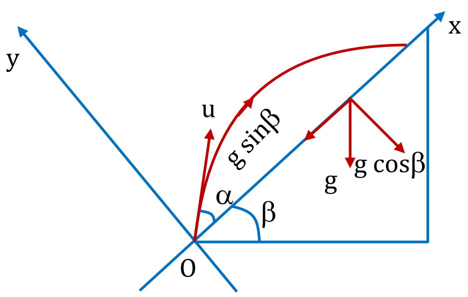
(b) Range = 8 × 1 = 8 m

(c) $v = \sqrt{u^2 + 2gh} = \sqrt{100 + 2 \times 10 \times 11} \Rightarrow v = \sqrt{320}\ m/s = 8\sqrt{5}\ m/s$

Projection on an Inclined Plane

Case (i) : Particle is projected up the incline

Here α is angle of projection w.r.t. the inclined plane. x and y axis are taken along and perpendicular to the incline as shown in the diagram.



In this case:

$a_x = -g \sin \beta$

$u_x = u \cos \alpha$

$a_y = -g \cos \beta$

$u_y = u \sin \alpha$

Time of flight (T):

When the particle strikes the inclined plane, y becomes zero.

$$y = u_y t + \frac{1}{2} a_y T^2$$

$$\Rightarrow \quad 0 = u \sin \alpha T - \frac{1}{2} g \cos \beta T^2$$

$$\Rightarrow \quad T = \frac{2u \sin \alpha}{g \cos \beta} = \frac{2u_\perp}{g_\perp}$$

Where $u_\perp$ and $g_\perp$ are component of u and g perpendicular to the incline.

Maximum height (H):

When half of the time is elapsed, y coordinate is equal to maximum distance from the inclined plane of the projectile.

$$H = u \sin\alpha \left(\frac{u \sin \alpha}{g \cos \beta}\right) - \frac{1}{2} g \cos \beta \left(\frac{u \sin \alpha}{g \cos \beta}\right)^2$$

$$\Rightarrow \quad H = \frac{u^2 \sin^2 \alpha}{2g \cos \beta} = \frac{u^2{}_\perp}{2g_\perp}$$

Range along the inclined plane (R):

When the particle strikes the inclined plane x coordinate is equal to range of the particle.

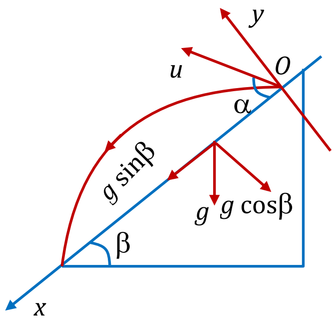
$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow \quad R = u \cos \alpha \left(\frac{2u \sin \alpha}{g \cos \beta}\right) - \frac{1}{2} g \sin\beta \left(\frac{2u \sin \alpha}{g \cos \beta}\right)^2$$

$$\Rightarrow \quad R = \frac{2u^2 \sin \alpha \cos(\alpha+\beta)}{g \cos^2 \beta}$$

Case (ii) : Particle is projected down the incline

In this case :



$a_x = g \sin \beta$; $u_x = u \cos \alpha$

$a_y = -g \cos \beta$

$u_y = u \sin \alpha$

Time of flight (T) :

When the particle strikes the inclined plane y coordinate becomes zero.

$$y = u_y t + \frac{1}{2} a_y T^2$$

$$\Rightarrow \quad 0 = u \sin \alpha T - \frac{1}{2} g \cos \beta T^2$$

$$\Rightarrow \quad T = \frac{2u \sin \alpha}{g \cos \beta} = \frac{2u_\perp}{g_\perp}$$

Maximum height (H) :

When half of the time is elapsed, y coordinate is equal to maximum height of the projectile.

$$H = u \sin\alpha \left(\frac{u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2} g \cos\beta \left(\frac{u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad H = \frac{u^2\sin^2\alpha}{2g\cos\beta} = \frac{u^2{}_\perp}{2g_\perp}$$

Range along the inclined plane (R):

When the particle strikes the inclined plane, x coordinate is equal to range of the particle.

$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow \quad R = u \cos\alpha \left(\frac{2u\sin\alpha}{g\cos\beta}\right) + \frac{1}{2} g \sin\beta \left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad R = \frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$$

Standard results for projectile motion on an inclined plane:

	Up the Incline	Down the Incline
Range	$\frac{2u^2\sin\alpha\cos(\alpha+\beta)}{g\cos^2\beta}$	$\frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$
Time of flight	$\frac{2u\sin\alpha}{g\cos\beta}$	$\frac{2u\sin\alpha}{g\cos\beta}$
Angle of projection for maximum range	$\frac{\pi}{4}-\frac{\beta}{2}$	$\frac{\pi}{4}+\frac{\beta}{2}$
Maximum Range	$\frac{u^2}{g(1+\sin\beta)}$	$\frac{u^2}{g(1-\sin\beta)}$

Here, α is the angle of projection with the incline and β is the angle of incline. $\left[\frac{1}{2}\left(\frac{\pi}{2}\pm\beta\right)\right]$

Note:

For a given speed, the direction which gives the maximum range of the projectile on an incline, bisects the angle between the incline and the vertical, for upward or downward projection.

Illustration 104:

A bullet is fired from the bottom of the inclined plane at angle $\theta = 37°$ with the inclined plane. The angle of incline is $30°$ with the horizontal. Find

(i) The position of the maximum height of the bullet from the inclined plane.

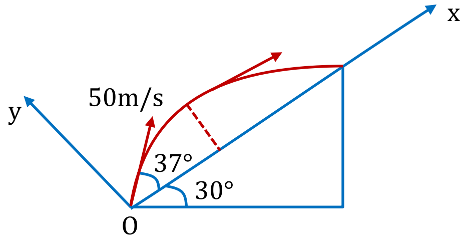
(ii) Time of flight

(iii) Horizontal range along the incline.

(iv) For what value of θ will range be maximum.

(v) Maximum range.

Solution:



(i) Taking axis system as shown in figure

At highest point $v_y = 0$

$v_y^2 = u_y^2 + 2a_y y$

$0 = (30)^2 - 2g\cos 30° y$

$y = 30\sqrt{3}\ m$ (maximum height)

(ii) Again, for x coordinate, $v_y = u_y + a_y t$

$0 = 30 - g\cos 30° \times t \Rightarrow t = 2\sqrt{3}$,

$x = u_x t + \frac{1}{2}a_x t^2 = 40 \times 2\sqrt{3} - \frac{1}{2} \times 5\left(2\sqrt{3}\right)^2 = \left(80\sqrt{3} - 30\right)m$

Time of flight, $T = 2 \times 2\sqrt{3}$ sec $= 4\sqrt{3}$ sec

(iii) $x = u_x t + \frac{1}{2}a_x t^2$

$x = 40 \times 4\sqrt{3} - \frac{1}{2} g \sin 30° \times \left(4\sqrt{3}\right)^2$

Range, $R = 40\ (4\sqrt{3} - 3)\ m$

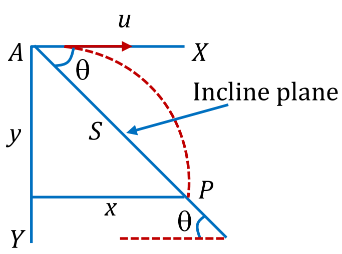
(iv) $\frac{\pi}{4} - \frac{30°}{2} = 45° - 15° = 30°$

(v) $\frac{u^2}{g(1+\sin\beta)} = \frac{50\times 50}{10\left(1+\frac{1}{2}\right)} = \frac{2500}{15} = \frac{500}{3} m$

Illustration 105:

A particle is projected horizontally with a speed u from the top of a plane inclined at an angle θ with the horizontal. How far from the point of projection will the particle strike the plane?

Solution:



Take X, Y-axis as shown in figure. Suppose that the particle strikes the plane at a point P with coordinates (x, y). Consider the motion between A and P.

Suppose distance between A and P is S

Then position of P is,

$x = S\cos\theta$

$y = -S\sin\theta$

Using equation of trajectory (For ordinary projectile motion)

$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$

Here, $y = -S\sin\theta$

$x = S\cos\theta$

θ = angle of projection with horizontal = 0°

$-S\sin\theta = S\cos\theta\,(0) - \frac{g(S\cos\theta)^2}{2u^2}$; $S = \frac{2u^2\sin\theta}{g\cos^2\theta}$

Alter:

$R = \frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$; Here α = β = θ $\Rightarrow R = \frac{2u^2\sin\theta}{g\cos^2\theta}$

Illustration 106:

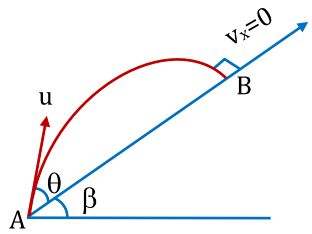
A projectile is thrown at an angle θ with an inclined plane of inclination β as shown in figure. Find the relation between β and θ if :

(a) Projectile strikes the inclined plane perpendicularly, to the inclined plane

(b) Projectile strikes the inclined plane horizontally to the ground

Solution:

(a) If projectile strikes perpendicularly.



$v_x = 0$ when projectile strikes

$v_x = u_x + a_x t$

$0 = u\cos\theta - g\sin\beta T \quad \Rightarrow \quad T = \dfrac{u\cos\theta}{g\sin\beta}$

We also know that $T = \dfrac{2u\sin\theta}{g\cos\beta}$

$\Rightarrow \quad \dfrac{u\cos\theta}{g\sin\beta} = \dfrac{2u\sin\theta}{g\cos\beta} \quad \Rightarrow \quad 2\tan\theta = \cot\beta$

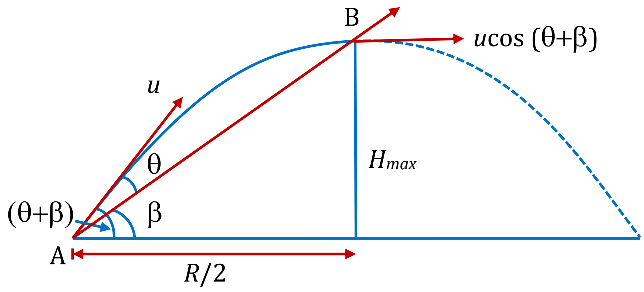
(b) If projectile strikes horizontally, then at the time of striking the projectile will be at the maximum height from the ground.

Therefore, time taken to move from A to B, t_{AB} = 1/2 time of flight over horizontal plane

$= \dfrac{2u\sin(\theta+\beta)}{2\times g}$

Also, t_{AB} = time of flight over incline = $\dfrac{2u\sin\theta}{g\cos\beta}$

Equating for t_{AB},



$$\frac{2u\sin\theta}{g\cos\beta} = \frac{2u\sin(\theta+\beta)}{2g}$$

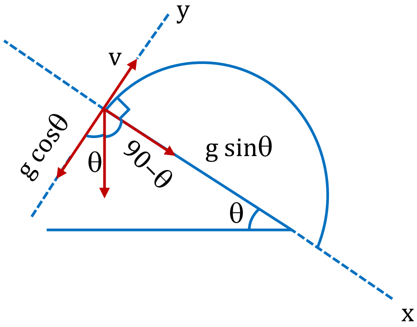
$\Rightarrow \quad 2\sin\theta = \sin(\theta+\beta)\cos\beta.$

Illustration 107:

A projectile is projected with an initial velocity v perpendicular to a plane which is inclined at an angle θ with horizontal. Find time of flight and range on inclined plane?

Solution:

x	y
$u_x = 0$	$u_y = v$
$a_x = g \sin \theta$	$a_y = -g \cos \theta$



$s_y = 0$

$$0 = u_y t + \frac{1}{2} a_y t^2$$

$$0 = vt + \left(-\frac{g \cos \theta t^2}{2}\right)$$

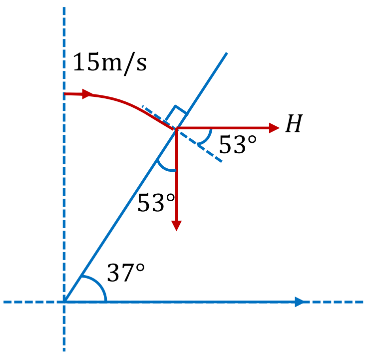
$$g\frac{\cos\theta\, t}{2} = v$$

$\therefore \quad T = \dfrac{2v}{g \cos \theta}$

$$s_x = \frac{1}{2} g \sin \theta \left(\frac{2v}{g \cos \theta}\right)^2$$

Illustration 108:

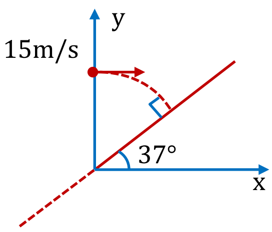
A particle is thrown horizontally with a speed of 15 m/s. It strikes an inclined plane perpendicularly as shown in figure. Find the time of flight of the particle.



Solution:

$$\tan 53° = \frac{v_y}{v_x}$$

$$\frac{4}{3} = \frac{v_y}{15}$$



$v_y = 20\ m/s$

$\because \quad v_y = a_y t \qquad (u_y = 0)$

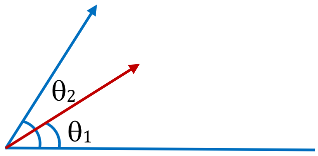
$20 = 10 \times T$

$\therefore \quad T = 2$ sec.

Illustration 109:

Two projectiles are thrown with different speeds and at different angles so as to cover the same maximum height. Find out the sum of the times taken by each to reach the highest point, if time of flight is T.

Solution:



$H_1 = H_2$ (given)

$$\frac{u_1^2 \sin^2\theta_1}{2g} = \frac{u_2^2 \sin^2\theta_2}{2g}$$

$u_1^2 \sin^2\theta_1 = u_2^2 \sin^2\theta_2 \Rightarrow u_1 \sin\theta_1 = u_2 \sin\theta_2$

$\because$ Time of flight $(T) = \dfrac{2u\sin\theta}{g}$

$\therefore$ $T_1 = \dfrac{2u_1\sin\theta_1}{g}$, and $T_2 = \dfrac{2u_2\sin\theta_2}{g}$ $\Rightarrow$ $T_1 = T_2 = T$

$\therefore$ $\dfrac{T_1}{2} + \dfrac{T_2}{2} = \dfrac{T}{2} + \dfrac{T}{2} = T$

Illustration 110:

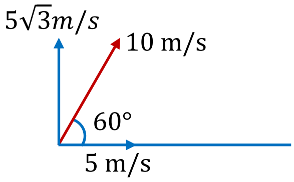
A particle is projected with speed 10 m/s at an angle 60° with horizontal. Find:

(a) Time of flight (b) Range (c) Maximum height

(d) Velocity of particle after one second. (e) Velocity when height of the particle is 1 m

Ans. (a) $\sqrt{3}$ sec. (b) $5\sqrt{3}m$ (c) $\dfrac{15}{4}m$ (d) $10\sqrt{2-\sqrt{3}}\ m/s$ (e) $\vec{v} = 5\hat{\imath} \pm \sqrt{55}\hat{\jmath}$

Solution:



(a) $T = \dfrac{2u\sin\theta}{g} = \dfrac{2\times5\sqrt{3}}{10} = \sqrt{3}$ sec.

(b) Range $= \dfrac{u^2\sin 2\theta}{g} = \dfrac{10\times10\times2\times\sin 60°\cos 60°}{10}$

$= 20\times\dfrac{\sqrt{3}}{2}\times\dfrac{1}{2} = 5\sqrt{3}m.$

(c) Maximum height $H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{10\times10\times3}{2\times10\times4} = \dfrac{15}{4}m$

(d) Velocity at any time 't'

$\because$ $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ ($\because$ $v_x = u_x = 5$, and $v_y = u_y + a_y t = 5\sqrt{3} - 10\times1$)

$\therefore$ $\vec{v} = 5\hat{\imath} + (5\sqrt{3} - 10)\hat{\jmath}$ $\Rightarrow$ $v = 10\left(\sqrt{2-\sqrt{3}}\right)m/s$

(e) Velocity at any height 'h' is $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$; $v_x = u_x = 5$

$v_y^2 = u_y^2 - 2gh = \left(5\sqrt{3}\right)^2 - 2\times10\times1$

$v_y = \sqrt{55}$ $\Rightarrow$ $\vec{v} = 5\hat{\imath} \pm \sqrt{55}\hat{\jmath}$

Illustration 111:

A stone is thrown with a velocity v at an angle θ with horizontal. Find its speed when it makes an angle β with the horizontal.

Solution:

$v \cos\theta = v' \cos\beta$

$v' = \dfrac{v\cos\theta}{\cos\beta}$

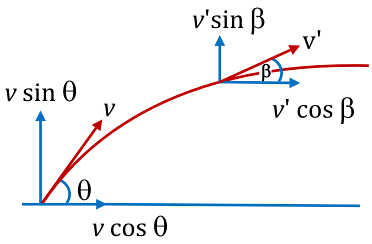


Illustration 112:

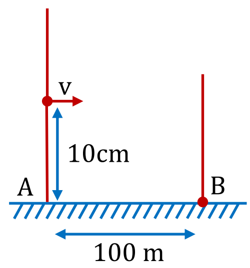
Two paper screens A and B are separated by a distance of 100 m. A bullet pierces A and then B. The hole in B is 10 cm below the hole in A. If the bullet is travelling horizontally at the time of hitting the screen A, calculate the velocity of the bullet when it hits the screen A. Neglect the resistance of paper and air.

(Use, $g = 9.8\ m/s^2$)

Solution:

Equation of motion in x direction $100 = v \times t$

$t = \dfrac{100}{v}$ ...(1)



In y direction

$0.1 = 1/2 \times 9.8 \times t^2$ ...(2)

From equation (1) and (2)

$0.1 = 1/2 \times 9.8 \times (100/v)^2$

On solving we get, $v = 700\ m/s$

Illustration 113:

Two stones A and B are projected simultaneously from the top of a 100 m high tower. Stone B is projected horizontally with speed 10 m/s, and stone A is dropped from the tower. Find out the following ($g = 10\ m/s^2$)

(a) Time of flight of the two stone

(b) Distance between two stones after 3 sec.

(c) Angle of strike with ground for stone B

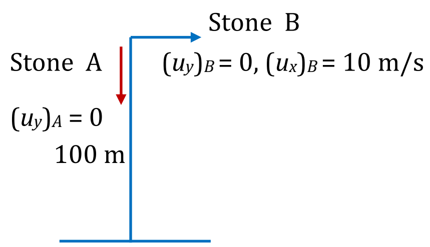
(d) Horizontal range of stone B

Solution:

(a) To calculate time of flight (for both stone) apply equation of motion in y direction

$100 = 1/2\ gt^2$

$t = 2\sqrt{5}\ sec.$



(b) $X_{B/A} = 10 \times 3 = 30\ m$

$Y_{B/A} = 1/2 \times g \times t^2 - 1/2 \times g \times t^2 = 0$

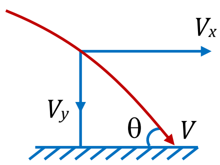
Distance between two stones after 3 $sec.$ = 30m

(c) Angle of striking with ground

$v_y^2 = u_y^2 + 2gh = 0 + 2 \times 10 \times 100$

$v_y = 20\sqrt{5} m/s$ and $v_x = 10\ m/s$

$\because\ \vec{v} = v_x\hat{i} + v_y\hat{j} \Rightarrow \tan\theta = \frac{v_y}{v_x}$



$\Rightarrow \theta = \tan^{-1}\left(\frac{20\sqrt{5}}{10}\right) = \tan^{-1}(2\sqrt{5})$

(d) Horizontal range of stone 'B'

$X_B = 10 \times (2\sqrt{5}) = 20\sqrt{5}\ m$

Illustration 114:

Two particles are projected simultaneously with the same speed v in the same vertical plane with angles of elevation θ and 2θ, where θ < 45°. At what time will their velocities be parallel.

Solution:

Velocity of particle projected at angle 'θ' after time t

$\vec{v}_1 = (v\cos\theta\hat{i} + v\sin\theta\hat{j}) - (gt\hat{j})$

Velocity of particle projected at angle '2θ' after time t

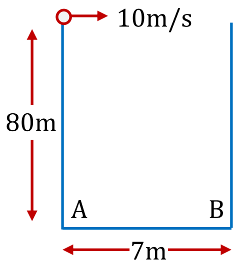
$\vec{v}_2 = (v\cos 2\theta\hat{i} + v\sin 2\theta\hat{j}) - (gt\hat{j})$

Since velocities are parallel so $\frac{v_x}{v_x'} = \frac{v_y}{v_y'} \Rightarrow \frac{v\cos\theta}{v\cos 2\theta} = \frac{v\sin\theta - gt}{v\sin 2\theta - gt}$

Solving above equation we can get result. $\frac{v}{g}\left(\frac{\cos\frac{\theta}{2}}{\cos\frac{3\theta}{2}}\right)$

Illustration 115:

A ball is projected horizontally from top of an 80 m deep well with velocity 10 m/s. Then particle will fall on the bottom at a distance of (all the collisions with the wall are elastic and wall is smooth).



(A) 5 m from A (B) 5 m from B (C) 2 m from A (D) 2 m from B

Solution:

Total time taken by the ball to reach at bottom = $\sqrt{\frac{2H}{g}} = \sqrt{\frac{2\times 80}{10}} = 4$ sec

Let time taken in one collision is t

Then $t \times 10 = 7$

$t = 0.7$ sec.

No. of collisions = $\dfrac{4}{0.7} = 5\dfrac{5}{7}$ (5th collisions from wall B)

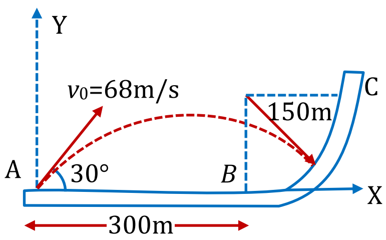
Horizontal distance travelled in between 2 successive collisions = 7 m

$\therefore$ Horizontal distance travelled in 5/7 part of collisions = $\dfrac{5}{7} \times 7 = 5\ m$

Distance from A is 2 m.

Illustration 116:

A projectile is launched from point 'A' with the initial conditions shown in the figure. BC part is circular with radius 150 m. Determine the 'x' and 'y' co-ordinates of the point of impact.



Solution:

Let the projectile strikes the circular path at (x, y) and 'A' to be taken as origin. From the figure co-ordinates of the centre of the circular path is (300, 150). Then the equation of the circular path is

$$(x - 300)^2 + (y - 150)^2 = (150)^2 \quad \text{...(1)}$$

And the equation of the trajectory is

$$y = x\tan 30° - \frac{1}{2}\frac{gx^2}{(68)^2\cos^2 30°}$$

$$y = \frac{x}{\sqrt{3}} - \frac{4x^2 g}{9248\times 3} \quad \text{...(2)}$$

From Equation (1) and (2) we get

$x = 373\ m$; $y = 18.75\ m$

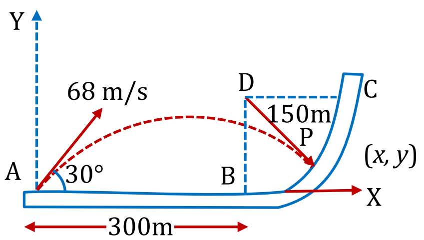
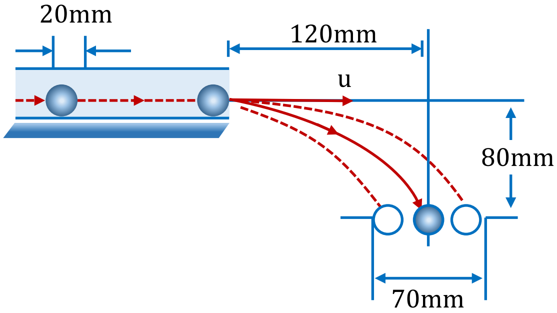


Illustration 117:

Ball bearings leave the horizontal through with a velocity of magnitude 'u' and fall through the 70 *mm* diameter hole as shown. Calculate the permissible range of 'u' which will enable the balls to enter the hole. Take the dotted positions to represent the limiting conditions.



Solution:

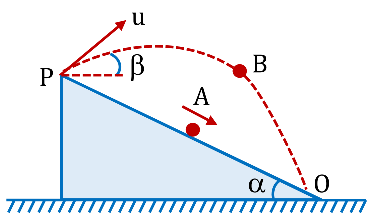
$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times0.08}{9.8}} = 0.13s$$

$$u_{min} = \frac{(120-25+10)\times10^{-3}}{0.13} = 0.807\ m/s$$

and $$u_{max} = \frac{(120+25-10)\times10^{-3}}{0.13} = 1.038\ m/s.$$

Illustration 118:

Particle A is released from a point P on a smooth inclined plane inclined at an angle α with the horizontal. At the same instant another particle B is projected with initial velocity u making an angle β with the horizontal. Both the particles meet again on the inclined plane. Find the relation between α and β.



Solution:

Consider motion of B along the plane

Initial velocity = $u \cos(\alpha + \beta)$

Acceleration = $g \sin \alpha$

$\therefore \quad OP = u\cos(\alpha+\beta)\, t + \frac{1}{2} g \sin(\alpha) t^2$ ...(i)

For motion of particle A along the plane,

Initial velocity = 0

Acceleration = $g \sin \alpha$

$\therefore \quad OP = \frac{1}{2} g \sin \alpha\, t^2$ ...(ii)

From equation (i) and (ii)

$u \cos(\alpha + \beta)\, t = 0$

So, either $t = 0$ or $\alpha + \beta = \frac{\pi}{2}$

Thus, the condition for the particles to collide again is $\alpha + \beta = \frac{\pi}{2}$.

RELATIVE MOTION

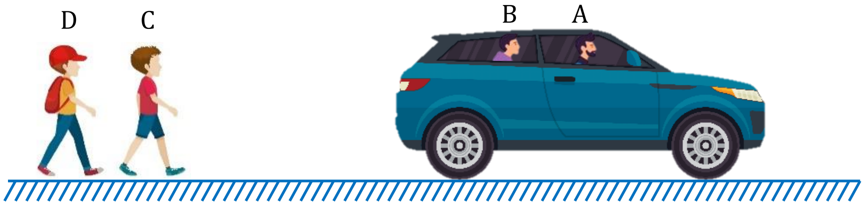
Introduction

Motion is a combined property of the object under study as well as the observer. It is always relative; there is no such thing like absolute motion or absolute rest. Motion is always defined with respect to an observer or reference frame.

Reference Frame

Reference frame is an axis system from which motion is observed along with a clock attached to the axis, to measure time. Reference frame can be stationary or moving.

- Suppose there are two persons A and B sitting in a car moving at constant speed. Two stationary persons C and D observe them from the ground.



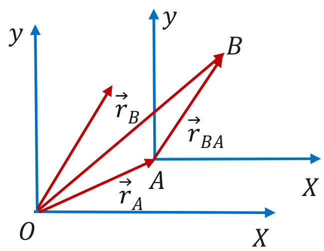
Here B appears to be moving for C and D, but at rest for A. Similarly, C appears to be at rest for D but moving backward for A and B.

Relative Motion Analysis

In two dimensional situation, Consider three observers at Rest O, A and B. Position of B from reference frame of A is $\vec{r}_{BA}$. We must understand that if A wants to go and meet B he needs to look only at $\vec{r}_{BA}$ and decide direction of his velocity which will take him to B. For A, origin of reference frame is not O it is A itself. From our knowledge of vectors we can deduce that

$$\vec{r}_B = \vec{r}_A + \vec{r}_{BA}$$

thus, $\vec{r}_{BA} = \vec{r}_B - \vec{r}_A$ ...(i)



Position vector of B w.r.t A is defined as $\vec{r}_{BA}$

Differentiating this equation W.R.T. time we get

$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$ ...(ii)

On further differentiating we get

$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A$ ...(iii)

Let's see how we use this concept in our daily life.

Suppose that a car A travelling on a straight road at 80 $km\ h^{-1}$ passes a car B going in the same direction at 60 kmh^{-1}, (Fig. a). Then, velocity of A relative to B is given by

$v_A - v_B = 80\ km\ h^{-1} - 60\ km\ h^{-1}$

$= 20\ km\ h^{-1}$ (to the right in fig. b)

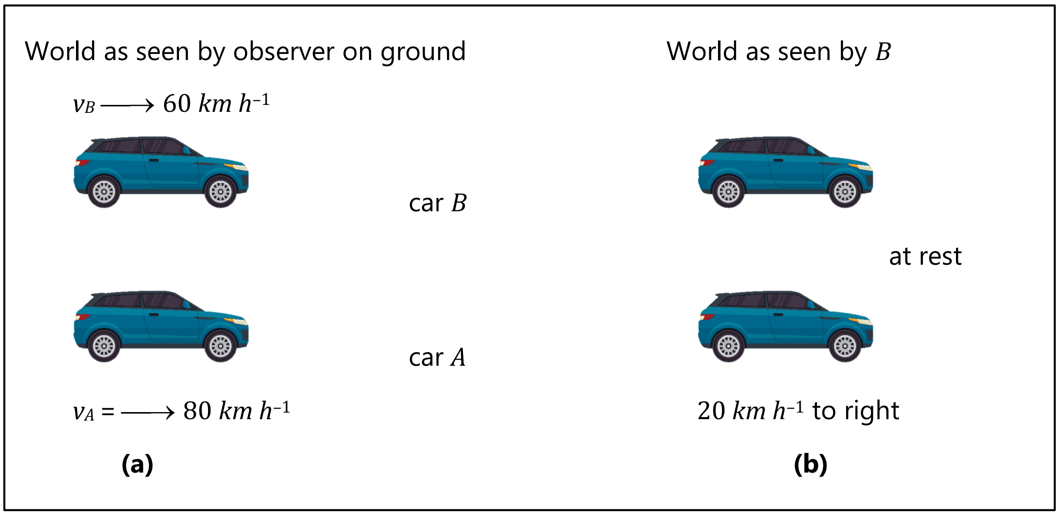
We know that if somebody passes us by in same direction we don't find them moving very fast.

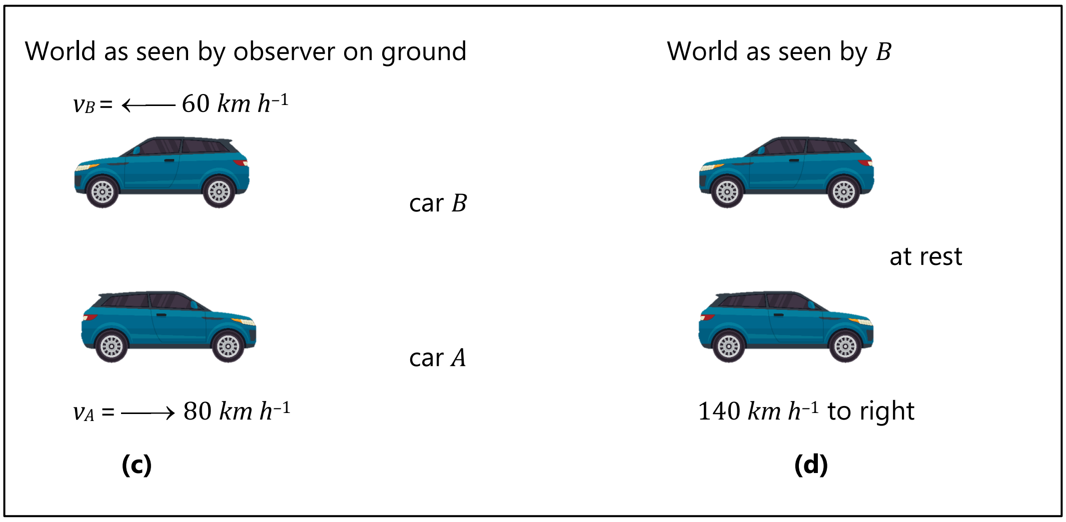
If A and B are travelling in opposite directions, Fig. c, we show this by giving one velocity + sign, say v_A, and the other – sign. Hence, we can write

$$v_A - v_B = +80\ km\ h^{-1} - (-60\ km\ h^{-1})$$
$$= (80 + 60)\ km\ h^{-1}$$
$$= 140\ km\ h^{-1} \qquad \text{(to the right in fig. d)}$$

We know that if somebody passes us by in opposite direction we find them moving really fast.

In effect, in both cases, to find the velocity of A relative to B, we have applied B's velocity reversed to both cars. It is then just as if B is at rest and A has two velocities v_A and v_B, which are subtracted when v_A and v_B are in the same direction and added when they are in opposite directions.





Equations of Motion When Relative Acceleration is Constant

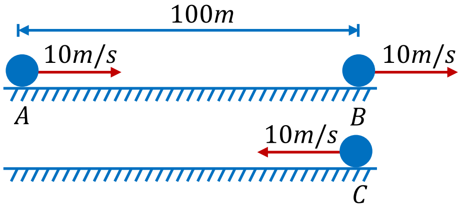
$$v_{rel} = u_{rel} + a_{rel}\, t$$

$$s_{rel} = u_{rel}\, t + \frac{1}{2} a_{rel}\, t^2$$

$$v^2_{rel} = u^2_{rel} + 2a_{rel}\, s_{rel}$$

Illustration 119:

A particle A is moving with a speed of 10 m/s towards right, particle B is moving at a speed of 10 m/s towards right and another particle C is moving at speed of 10 m/s towards left. The separation between A and B is 100 m. Find the time interval between C meeting B and C meeting A.

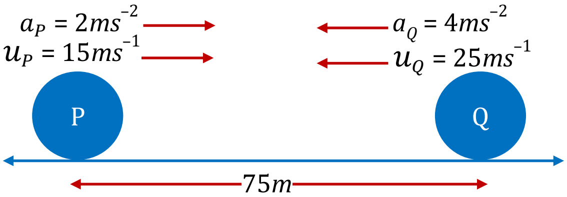


Solution:

$$t = \frac{\text{Seperation between } A \text{ and } C}{V_{app} \text{ of } A \text{ and } C} = \frac{100}{10-(-10)} = 5 \text{ sec.}$$

Illustration 120:

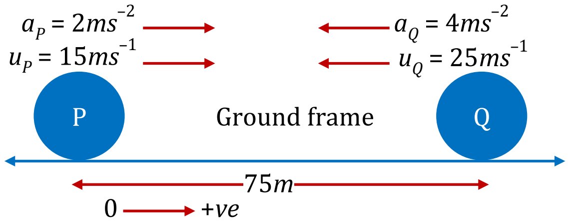
Two particles P and Q, initially separated by 75 m, are moving towards each other along a straight line as shown in figure with a_P = 2 ms^{-2}, u_P = 15 ms^{-1}, and a_Q = 4 ms^{-2}, u_Q = 25 ms^{-1}. Calculate the time when they meet.



Solution:

In ground frame, choosing the origin at P and right as positive

Ground frame	
u_P = +15 ms^{-1}	u_Q = −25 m^{-1}
a_P = +2 ms^{-2}	a_Q = −4 ms^{-2}



At convergence,

Position of P = Position of Q

$\Rightarrow \quad x_P = x_Q$

$$\Rightarrow \quad \left(x_O + ut + \frac{1}{2}at^2\right)_P = \left(x_O + ut + \frac{1}{2}at^2\right)_Q$$

$\Rightarrow \quad \left(0 + 15t + \frac{1}{2} \times 2 \times t^2\right)_P = \left(75 - 25t + \frac{1}{2} \times -4 \times t^2\right)_Q$

$\Rightarrow \quad 15t + t^2 = 75 - 25t - 2t^2$

$\Rightarrow \quad 3t^2 + 40t - 75 = 0$

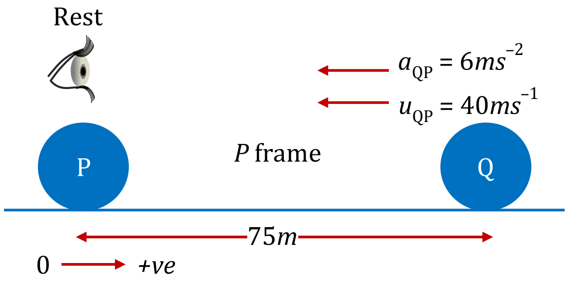
On solving, we get

$\Rightarrow \quad t = -15\ s \qquad$ or $\qquad t = \frac{5}{3}\ s$

Neglecting negative value of time, $t = \frac{5}{3}\ s$

P Frame

Let the frame be P as shown in figure. Here, P acts as the origin and it is at rest. Position, velocity, and acceleration of Q is a relative quantity with respect to P.



Now,

P frame	
$u_{PP} = u_P - u_P = 0\ ms^{-1}$	$u_{QP} = u_Q - u_P = -25 - 15 = -40\ ms^{-1}$
$a_{PP} = a_P - a_P = 0\ ms^{-2}$	$a_{QP} = a_Q - a_P = -4 - 2 = -6\ ms^{-2}$

Now, P and Q will collide when Q covers a distance of 75 m and hits P, which is at rest in this reference frame.

$s = ut + \frac{1}{2}at^2$

$\Rightarrow \quad -75 = -40t - \frac{1}{2}6t^2$

$\Rightarrow \quad 3t^2 + 40t - 75 = 0$

Solve to get,

$t = \frac{5}{3}\ s$ or $t = -15\ s,$

Neglecting negative value of time,

time (t) $= \frac{5}{3}\ s$

Note:

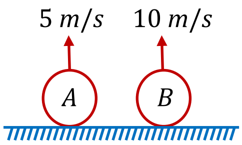
(1) All parameters like position, velocity, and acceleration are 'frame dependent'.

(2) The time duration for an event however is 'frame independent' under non-relativistic circumstances.

Illustration 121:

A and B are thrown vertically upward with velocity, 5 m/s and 10 m/s respectively (g = 10 m/s^2). Find separation between them after one second.

Solution:



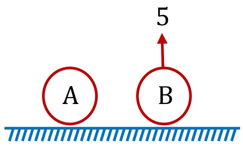
$$S_A = ut - \frac{1}{2}gt^2$$

$$= 5t - \frac{1}{2} \times 10 \times t^2 = 5 \times 1 - 5 \times 1^2 = 5 - 5 = 0$$

$$S_B = ut - \frac{1}{2}gt^2 = 10 \times 1 - \frac{1}{2} \times 10 \times 1^2 = 10 - 5 = 5$$

$\therefore$ $S_B - S_A$ = separation = 5m.

Aliter:



$$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = (-10) - (-10) = 0$$

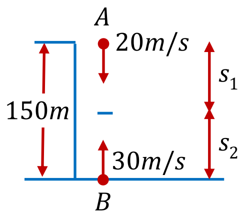
Also $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = 10 - 5 = 5m/s$

$\therefore$ $\vec{s}_{BA}(in\ 1\ \text{sec}\ ) = \vec{v}_{BA} \times t = 5 \times 1 = 5m$

$\therefore$ Distance between A and B after 1 sec = 5 m.

Illustration 122:

A ball is thrown downwards with a speed of 20 m/s from the top of a building 150 m high and simultaneously another ball is thrown vertically upwards with a speed of 30 m/s from the foot of the building. Find the time after which both the balls will meet. (g = 10 m/s^2)



Solution:

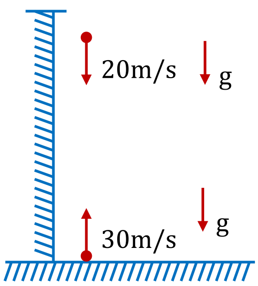
$$S_1 = 20\,t + 5\,t^2$$

$$S_2 = 30\,t - 5\,t^2$$

$$S_1 + S_2 = 150m$$

$\Rightarrow$ $150 = 50\,t \Rightarrow t = 3\,s$

Aliter:



Relative acceleration of both is zero since both have same acceleration in downward direction

$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = (-g) - (-g) = 0$$

$$\vec{v}_{BA} = 30 - (-20) = 50m/s$$

$$s_{BA} = v_{BA} \times t$$

$$t = \frac{s_{BA}}{v_{BA}} = \frac{150}{50} = 3\,s$$

Illustration 123:

A lift is moving down with constant velocity 20 m/s. A person in lift drops a coin from height 5m from floor of the lift. Find:

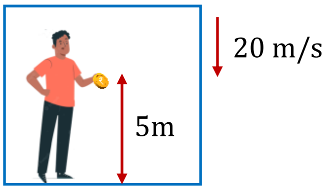
(a) time taken by coin to strike the floor. (b) distance travelled by coin in ground frame.

Solution:

(a) $u_L = 20\ m/s$ $u_C = 20$

$a_L = 0$ $a_C = 10\ m/s^2$

w.r.t. lift :



$u_{C/L} = u_C - u_L = 20 - 20 = 0$

$a_{C/L} = a_C - a_L = 10 - 0 = 10\ m/s^2$

$s_{C/L} = 5\ m$ $t = ?$

$s_{C/L} = 0t + \frac{1}{2} \times 10 \times t^2$

$t^2 = 1$

$t = 1$ sec

(b) $s_C = u_C t + \frac{1}{2} a_c t^2$

$s_C = 20 \times 1 + \frac{1}{2} \times 10 \times 1$

$s_C = 25\ m$

Illustration 124:

A lift is moving down with velocity 10m/s and have acceleration 2m/s^2 downward. A person in lift drops a coin from height 16m from floor of the lift.

(a) Find time taken by coin to strike the ground.

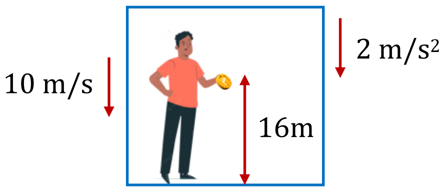
(b) Distance travelled by coin ?

Solution :

(a) $u_L = 10\ m/s$ $u_C = 10\ m/s$

$a_L = 2\ m/s^2$ $a_C = 10\ m/s^2$

w.r.t. lift :



$a_{C/L} = a_C - a_L = 10 - 2 = 8$

$u_{C/L} = u_C - u_L = 10 - 10 = 0$

$s_{C/L} = 16\ m$

$\therefore$ $16 = \frac{1}{2} \times 8 \times t^2$

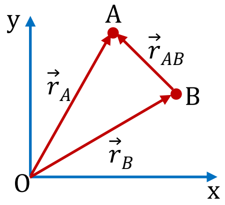
$t^2 = 4$

$t = 2$ sec.

(b) $s_C = u_C t + \frac{1}{2} a_c t^2$

$s_C = 10 \times 2 + \frac{1}{2} \times 10 \times 4 = 20 + 20 = 40\ m$

Relative Motion in Two Dimension



$\vec{r}_A$ = position of A with respect to O

$\vec{r}_B$ = position of B with respect to O

$\vec{r}_{AB}$ = position of A with respect to B

$\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$

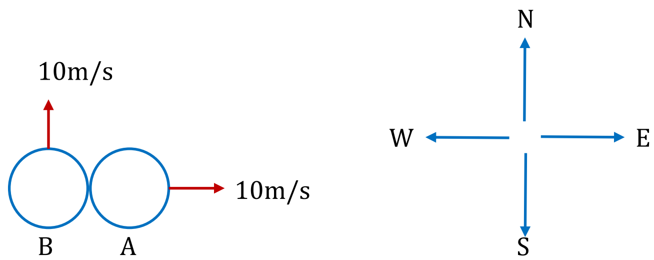
(The vector sum $\vec{r}_A - \vec{r}_B$ can be done by Δ law of addition or resolution method)

$\therefore \quad \dfrac{d(\vec{r}_{AB})}{dt} = \dfrac{d(\vec{r}_A)}{dt} - \dfrac{d(\vec{r}_B)}{dt}$

$\Rightarrow \quad \vec{v}_{AB} = \vec{v}_A - \vec{v}_B\ ;\ \dfrac{d(\vec{v}_{AB})}{dt} = \dfrac{d(\vec{v}_A)}{dt} - \dfrac{d(\vec{v}_B)}{dt} \Rightarrow \vec{a}_{AB} = \vec{a}_A - \vec{a}_B$

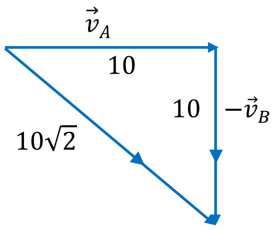
Illustration 125:

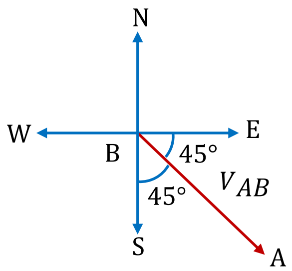
Object A and B both have speed of 10 m/s. A is moving towards East while B is moving towards North starting from the same point as shown. Find velocity of A relative to B.



Solution:

Method 1: $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$ $\qquad\therefore\ |\vec{v}_{AB}| = 10\sqrt{2}$





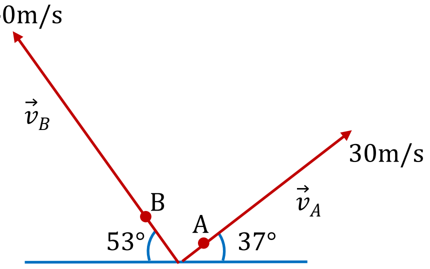
Method 2: $\vec{v}_A = 10\hat{i}, \vec{v}_B = 10\hat{j}$

$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = 10\hat{i} - 10\hat{j}$ $\qquad\therefore\ |\vec{v}_{AB}| = 10\sqrt{2}$

Note:

$|\vec{v}_A - \vec{v}_B| = \sqrt{v_A^2 + v_B^2 - 2v_Av_B\cos\theta}$, where θ is angle between $\vec{v}_A$ and $\vec{v}_B$.

Illustration 126:



Two particles A and B are projected in air. A is thrown with a speed of 30 m/sec and B with a speed of 40 m/sec as shown in the figure. What is the separation between them after 1 sec.

Solution:

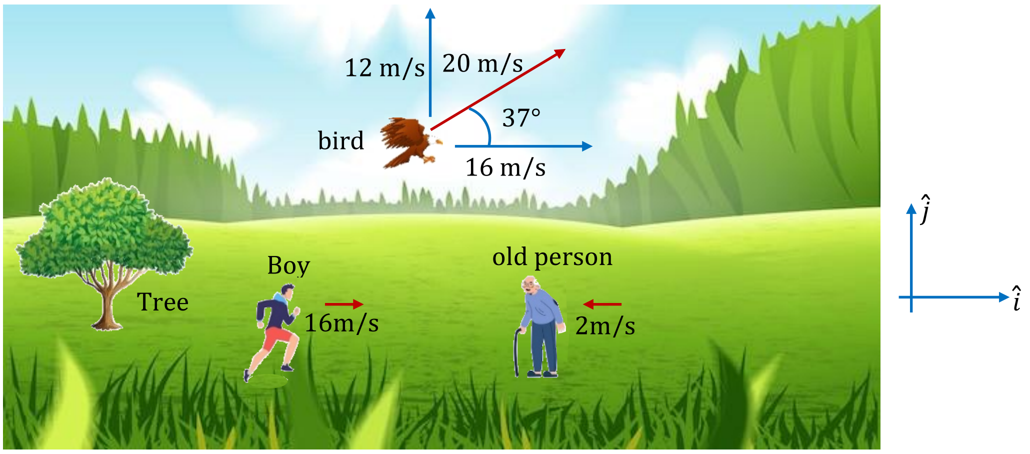
$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = \vec{g} - \vec{g} = 0$$

$\therefore$ $$|\overrightarrow{v_{AB}}| = \sqrt{30^2 + 40^2} = 50$$

$\therefore$ $$s_{AB} = v_{AB}\, t = 50\, t = 50\ m$$

Illustration 127:

An old man and a boy are walking towards each other and a bird is flying over them as shown in the figure.



(a) Find the velocity of tree, bird and old man as seen by boy.

(b) Find the velocity of tree, bird and boy as seen by old man.

(c) Find the velocity of tree, boy and old man as seen by bird.

Solution:

(a) With respect to boy:

$v_{\text{tree}} = 16\ m/s\ (\leftarrow)$ or $-16\hat{i}$

$v_{\text{bird}} = 12\ m/s\ (\uparrow)$ or $12\hat{j}$

$v_{\text{old man}} = 18\ m/s\ (\leftarrow)$ or $-18\hat{i}$

(b) With respect to old man:

$v_{\text{boy}} = 18\ m/s\ (\rightarrow)$ or $18\hat{i}$

$v_{\text{tree}} = 2\ m/s\ (\rightarrow)$ or $2\hat{i}$

$v_{\text{bird}} = 18\ m/s\ (\rightarrow)$ and $12\ m/s\ (\uparrow)$ or $18\hat{i} + 12\hat{j}$

(c) With respect to Bird:

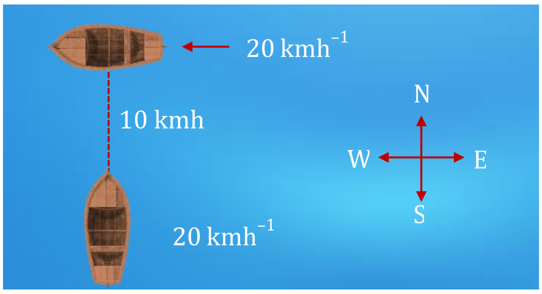
$v_{\text{tree}} = 12\ m/s\ (\downarrow)$ and $16\ m/s\ (\leftarrow)$ or $-12\hat{j} - 16\hat{i}$

$v_{\text{old man}} = 18\ m/s\ (\leftarrow)$ and $12\ m/s\ (\downarrow)$ or $-18\hat{i} - 12\hat{j}$

$v_{\text{boy}} = 12\ m/s\ (\downarrow)$ $or\ -12\hat{j}$

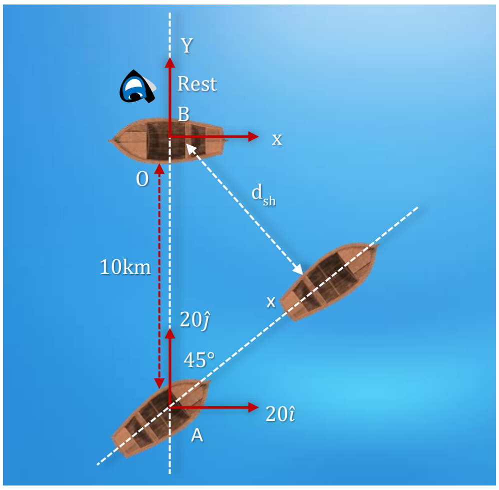
Illustration 128:

Two ships are 10 km apart on north-south vertical at an instant. The one farther north is streaming west at 20 $km\ h^{-1}$. The other is streaming north at 20 $km\ h^{-1}$, as shown. What is their distance of closest approach? How long do they take to reach it?



Solution:

Considering ship B as the frame of reference, ship B is now at rest and is the origin as shown in the figure. Ship A will appear to move as shown in the figure. Given,



$$\vec{v}_B = -20\hat{\imath}\ km\ h^{-1}$$

$$\vec{v}_A = 20\hat{\jmath}\ km\ h^{-1}$$

$$\Rightarrow \quad \vec{v}_{AB} = \vec{v}_A - \vec{v}_B = 20\hat{\jmath} - (-20\hat{\imath})\ km\ h^{-1}$$

$$\Rightarrow \quad \vec{v}_{AB} = 20\hat{\imath} + 20\hat{\jmath}\ km\ h^{-1}$$

$$\Rightarrow \quad |\vec{v}_{AB}| = \sqrt{20^2 + 20^2} = 20\sqrt{2}\ km\ h^{-1}$$

$$\tan(\theta) = \frac{20}{20} = 1 \Rightarrow \theta = 45^\circ$$

$$\vec{v}_{BB} = \vec{0}\ km\ h^{-1}$$

Now, for distance to be shortest, we drop a perpendicular as shown in figure. From the resulting right triangle, we have,

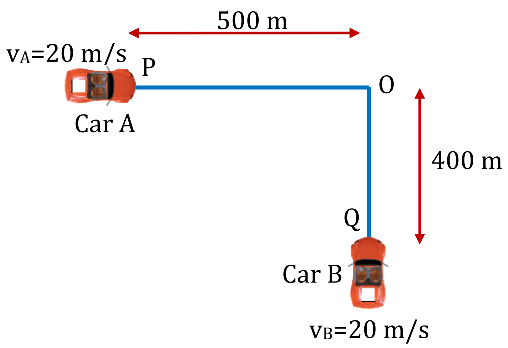
$$\sin(45^\circ) = \frac{d_{sh}}{10}$$

$$\Rightarrow \quad d_{sh} = 10 \times \frac{1}{\sqrt{2}} = 5\sqrt{2}\ km$$

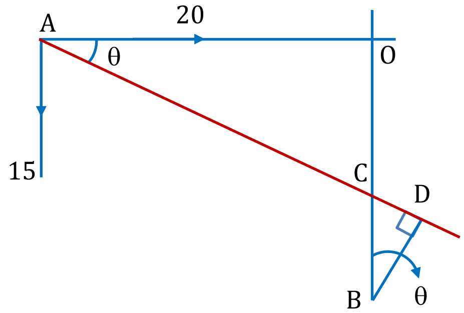
Now, $\quad \text{Time } (t) = \dfrac{d_{sh}}{|\vec{v}_{AB}|} = \dfrac{5\sqrt{2}}{20\sqrt{2}} = \dfrac{1}{4}\ h$

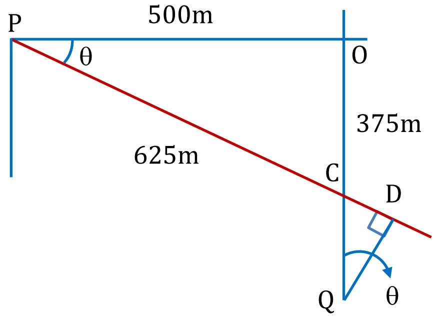
Illustration 129:

Two roads intersect at right angles. Car A is situated at P which is 500m from the intersection O on one of the roads. Car B is situated at Q which is 400m from the intersection on the other road. They start out at the same time and travel towards the intersection at 20m/s and 15m/s respectively. What is the minimum distance between them? How long do they take to reach it?



Solution:





First, we find out the velocity of car A relative to B

As can be seen from (fig.), the magnitude of velocity of B with respect to A

$v_A = 20\ m/s,\ V_B = 15\ m/s,\ OP = 500\ m;\ OQ = 400m$

$\tan\theta = \frac{15}{20} = \frac{3}{4};\ \cos\theta = \frac{4}{5};\ \sin\theta = \frac{3}{5}$

$OC = AC\tan\theta = 500 \times \frac{3}{4} = 375\ m$

$BC = OB - OC = 400 - 375 = 25\ m$

$BD = BC(\cos\theta) = 25 \times \frac{4}{5} = 20\ m$

Shortest distance = $20\ m$

$PD = PC + CD = 625 + 15 = 640$

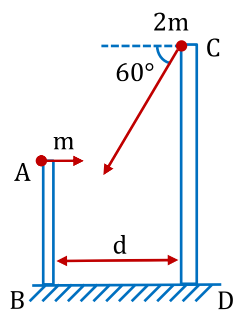
$|\vec{v}_{AB}| = 25\ m/s$

$t = \frac{640}{25} = 25.6$ sec.

Collision of Two Bodies with Help of Relative Motion

Illustration 130:

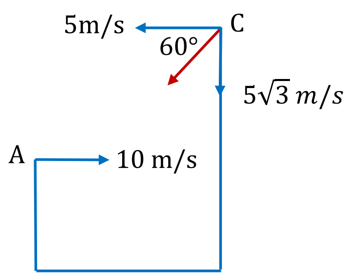
Two towers AB and CD are situated a distance d apart as shown in figure. AB is $20m$ high and CD is $30m$ high from the ground. An object of mass m is thrown from the top of AB horizontally with a velocity of $10\ m/s$ towards CD. Simultaneously another object of mass $2m$ is thrown from the top of CD at an angle of $60°$ to the horizontal towards AB with the same magnitude of initial velocity as that of the first object. The two objects move in the same vertical plane, collide in mid-air and stick to each other.



Calculate the distance d between the towers.

Solution:

Acceleration of A and C both is $9.8\ m/s^2$ downwards.



Therefore, relative acceleration between them is zero i.e., the relative motion between them will be straight line.

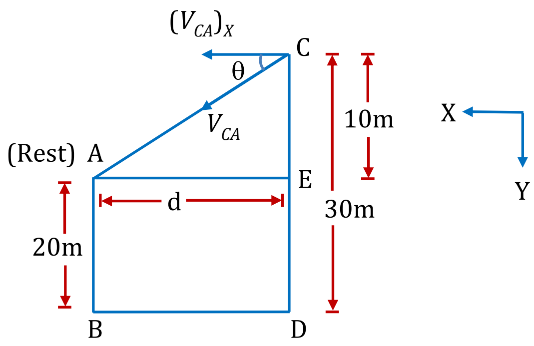
Now assuming A to be at rest, the condition of collision will be that $\vec{V}_{CA} = \vec{V}_C - \vec{V}_A$ relative velocity of C w.r.t. A should be along CA.

$$\vec{V}_A = 10\hat{\imath}$$

$$\vec{V}_C = -5\hat{\imath} - 5\sqrt{3}\hat{\jmath}$$

$$\therefore \quad \vec{V}_{CA} = -5\hat{\imath} - 5\sqrt{3}\hat{\jmath} - 10\hat{\imath}$$

$$\vec{V}_{CA} = -15\hat{\imath} - 5\sqrt{3}\hat{\jmath}$$



Angle made by $\vec{V}_{CA}$ with x-axis, $\tan\theta = \dfrac{-5\sqrt{3}}{-15} = \dfrac{1}{\sqrt{3}}$

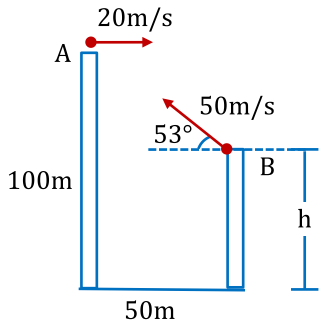
$\Rightarrow \quad \theta = 30°$

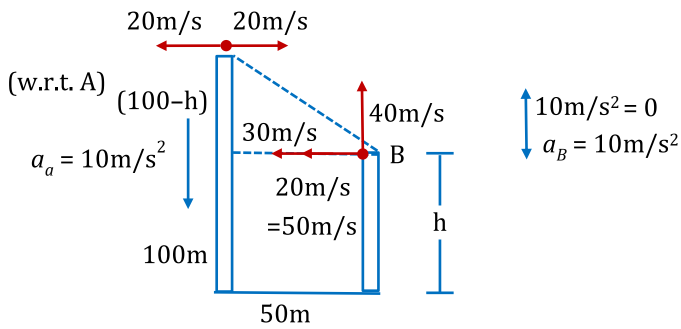
Now, by the geometry of figure $\tan 30° = \dfrac{10}{d} \Rightarrow d = 10\sqrt{3}\,m$.

Illustration 131:

Two particle A and B are projected simultaneously as shown in figure. If they collide in mid air, find h ?

Solution:





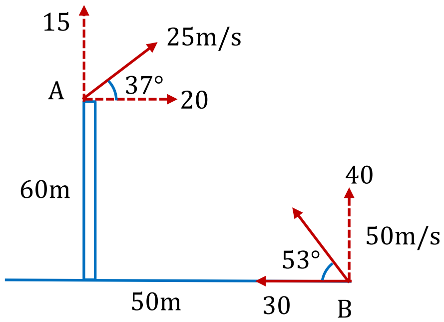
x	y
$v_x = 50$	$v_y = 40$
$s_x = 50$	$s_y = 100 - h$
$a_x = 0$	$a_y = 0$
$t = 1$ sec.	$t = 1$ sec.

$\therefore \quad 100 - h = 40 \times 1 \quad \Rightarrow \quad h = 60\ m$

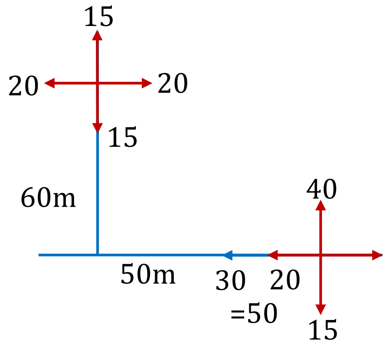
If two particles are projected in gravitation field then during the time of flight of both particles, trajectory of one particle w.r.t. to other particle will be a straight line.

Illustration 132:

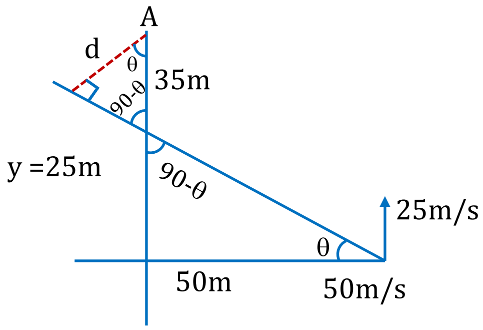
If velocity of A is $(20\hat{\imath} + 15\hat{\jmath})m/s$ and B is $(-30\hat{\imath} + 40\hat{\jmath})m/s$. Will they collide? If no, what is minimum distance between them?



Solution:



$\Rightarrow$



$$\tan\theta = \frac{25}{50} = \frac{1}{2} = \frac{y}{50}$$

$$d_{min.} = 35\cos\theta = 35 \times \frac{2}{\sqrt{5}}$$

Relative Motion in River Flow

If a man can swim relative to water with velocity $\vec{v}_{mR}$ and water is flowing relative to ground with velocity $\vec{v}_R$, velocity of man relative to ground $\vec{v}_m$ will be given by :

$$\vec{v}_{mR} = \vec{v}_m - \vec{v}_R \quad \text{or} \quad \vec{v}_m = \vec{v}_{mR} + \vec{v}_R$$

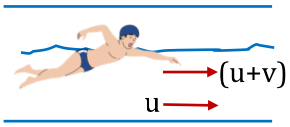
If $\vec{v}_R = 0$, then $\vec{v}_m = \vec{v}_{mR}$ in words, velocity of man in still water = velocity of man w.r.t. river.

River Problem in One Dimension:

Velocity of river is u and velocity of man in still water is v.

Case-1: Man swimming downstream (along the direction of river flow). In this case velocity of river $v_R = +u$ velocity of man w.r.t. river $v_{mR} = +v$

now, $|\vec{v}_m| = |\vec{v}_{mR} + \vec{v}_R| = u + v$



Case-2: Man swimming upstream (opposite to the direction of river flow). In this case velocity of river $\vec{v}_R = -u$ velocity of man w.r.t. river $\vec{v}_{mR} = +v$

now, $|\vec{v}_m| = |\vec{v}_{mR} + \vec{v}_R| = (v - u)$

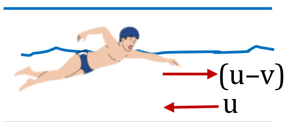


Illustration 133:

A swimmer capable of swimming with velocity 'v' relative to water jumps in a flowing river having velocity 'u'. The man swims a distance d down stream and returns back to the original position. Find out the time taken in complete motion.

Solution:

Total time = time of swimming downstream + time of swimming upstream

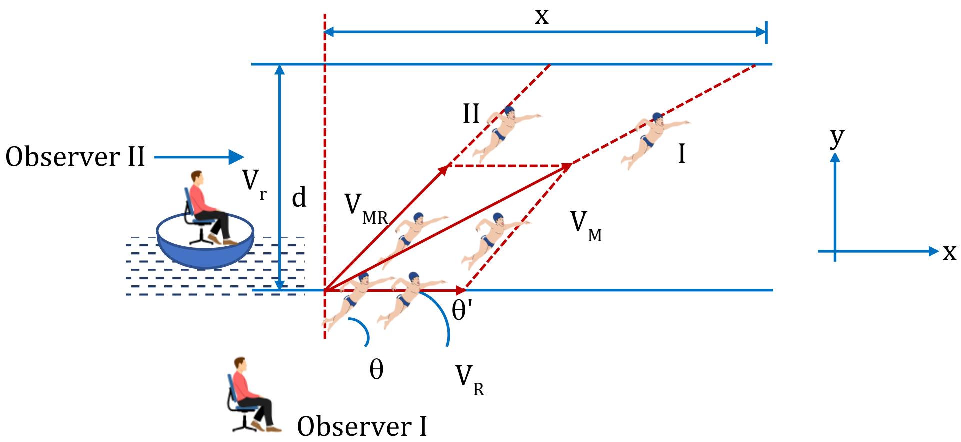
$$t = t_{down} + t_{up} = \frac{d}{v+u} + \frac{d}{v-u} = \frac{2dv}{v^2-u^2}$$

Motion of Man Swimming in a River

Consider a man swimming in a river with a velocity of $\vec{v}_{MR}$ relative to river at an angle of θ with the river flow. The velocity of river is $\vec{v}_R$. Let there be two observers I and II, observer I is on ground and observer II is on a raft floating along with the river and hence moving with the same velocity as that of river. Hence motion w.r.t. observer II is same as motion w.r.t. river. i.e., the man will appear to swim at an angle θ with the river flow for observer II.

For observer I the velocity of swimmer will be $\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$.

Hence the swimmer will appear to move at an angle θ' with the river flow.



Ⓘ : Motion of swimmer for observer I   Ⓘⓘ : Motion of swimmer for observer II

River Problem in Two Dimension (Crossing River)

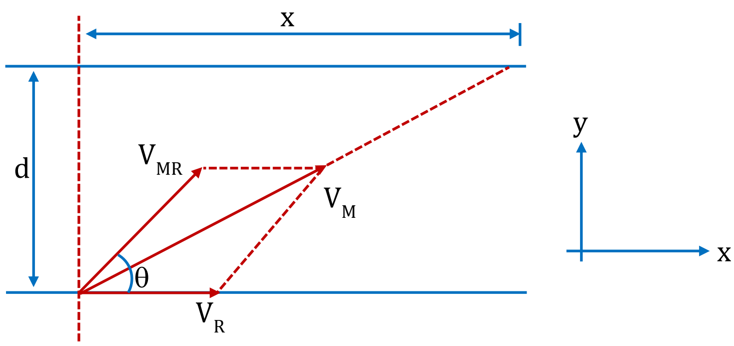
Consider a man swimming in a river with a velocity of $\vec{v}_{MR}$ relative to river at an angle of θ with the river flow. The velocity of river is V_R and the width of the river is d

$$\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$$

$\Rightarrow$ $\vec{v}_M = (v_{MR}\cos\theta\hat{i} + v_{MR}\sin\theta\hat{j}) + v_R\hat{i}$

$\Rightarrow$ $\vec{v}_M = (v_{MR}\cos\theta + v_R)\hat{i} + v_{MR}\sin\theta\hat{j}$

Here $v_{MR}\sin\theta$ is the component of velocity of man in the direction perpendicular to the river flow. This component of velocity is responsible for the man crossing the river. Hence if the time to cross the river is t, then $t = \dfrac{d}{v_y} = \dfrac{d}{v_{MR}\sin\theta}$.



Drift

It is defined as the displacement of man in the direction of river flow. (See the figure). It is simply the displacement along x-axis, during the period the man crosses the river. $(v_{MR}\cos\theta + v_R)$ is the component of velocity of man in the direction of river flow and this component of velocity is responsible for drift along the river flow. If drift is x then,

Drift = $v_x \times t$

$$x = (v_{MR}\cos\theta + v_R) \times \frac{d}{v_{MR}\sin\theta}$$

Crossing the River in Shortest Time

As we know that $t = \frac{d}{v_{MR}\sin\theta}$. Clearly t will be minimum when θ = 90° i.e. time to cross the river will be minimum if man swims perpendicular to the river flow. Which is equal to $\frac{d}{v_{MR}}$.

Crossing the River in Shortest Path, Minimum Drift

The minimum possible drift is zero. In this case the man swims in the direction perpendicular to the river flow as seen from the ground. This path is known as **shortest path**

Here, $x_{min} = 0 \Rightarrow (v_{MR}\cos\theta + v_R) = 0$ or $\cos\theta = -\frac{v_R}{v_{MR}}$

Since cos θ is –ve

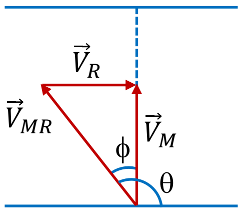
∴ θ > 90°, i.e. for minimum drift the man must swim at some angle ϕ with the perpendicular in backward direction. Where $\sin\phi = \frac{v_R}{v_{MR}}$

$$\theta = \cos^{-1}\left(\frac{-v_R}{v_{MR}}\right)$$

∴ $\left|\frac{v_R}{v_{MR}}\right| \le 1$ i.e. $v_R \le v_{MR}$

i.e. minimum drift is zero if and only if velocity of man in still water is greater than or equal to the velocity of river.

Time to cross the river along the shortest path



$$t = \frac{d}{v_{MR}\sin\theta} = \frac{d}{\sqrt{v_{MR}^2 - v_R^2}}$$

Note:

If $v_R > v_{MR}$ then it is not possible to have zero drift. In this case the minimum drift (corresponding to shortest possible path is non-zero and the condition for minimum drift can be proved to be $\cos\theta = -\left(\frac{v_{MR}}{v_R}\right)$ or $\sin\phi = \left(\frac{v_{MR}}{v_R}\right)$ for minimum but non-zero drift.

Illustration 134:

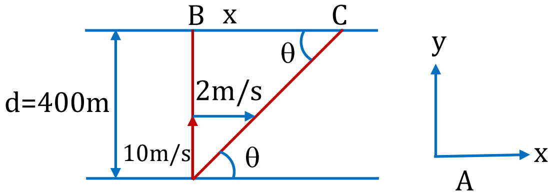
A 400 m wide river is flowing at a rate of 2.0 m/s. A boat is sailing with a velocity of 10 m/s with respect to the water, in a direction perpendicular to the river.

(a) Find the time taken by the boat to reach the opposite bank.

(b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?

(c) In what direction does the boat actually move, with river flow (downstream)?

Solution:



(a) Time taken to cross the river, $t = \frac{d}{v_y} = \frac{400m}{10m/s} = 40s$

(b) Drift $(x) = (v_x)(t) = (2m/s)\ (40s) = 80\ m$

(c) Actual direction of boat, $\theta = \tan^{-1}\left(\frac{10}{2}\right) = \tan^{-1} 5$, (downstream) with the river flow.

Illustration 135:

A man can swim at the rate of 5 km/h in still water. A 1 km wide river flows at the rate of 3 km/h. The man wishes to swim across the river directly opposite to the starting point.

(a) Along what direction must the man swim?

(b) What should be his resultant velocity?

(c) How much time will he take to cross the river?

Solution:

(a) The velocity of man with respect to river $v_{mR} = 5\ km/hr$, this is greater than the river flow velocity, therefore, he can cross the river directly (along the shortest path). The angle of swim must be

$$\theta = \frac{\pi}{2} + \sin^{-1}\left(\frac{v_r}{v_{mR}}\right) = 90° + \sin^{-1}\left(\frac{v_r}{v_{mR}}\right) = 90° + \sin^{-1}\left(\frac{3}{5}\right) = 90° + 37°$$

$= 127°$ w.r.t. the river flow or $37°$ w.r.t. perpendicular in backward direction

(b) Resultant velocity will be

$$v_m = \sqrt{v_{mR}^2 - v_R^2} = \sqrt{5^2 - 3^2} = 4km/hr$$

Along the direction perpendicular to the river flow.

(c) Time taken to cross the river,

$$t = \frac{d}{\sqrt{v_{mR}^2 - v_R^2}} = \frac{1km}{4km/hr} = \frac{1}{4}h = 15\,min$$

Illustration 136:

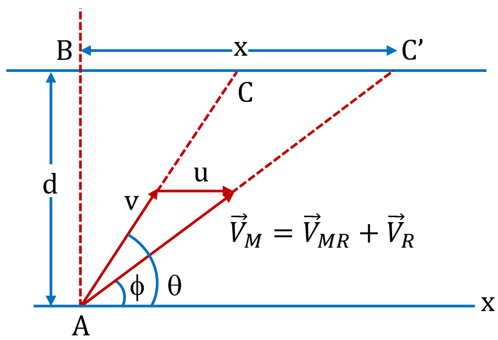
A man wishing to cross a river flowing with velocity u jumps at an angle θ with the river flow.

(i) Find the net velocity of the man with respect to ground if he can swim with speed v in still water.

(ii) In what direction does the man actually move?

(iii) Find how far from the point directly opposite to the starting point does the man reach the opposite bank, if the width of the river is d. (i.e. drift).

Solution:



(i) $v_{MR} = v,\ v_R = u\ ;\ \ \vec{v}_M = \vec{v}_{MR} + \vec{v}_R$

∴ Velocity of man,

$v_M = \sqrt{u^2 + v^2 + 2vu\cos\theta}$

(ii) $\tan\phi = \frac{v\sin\theta}{u + v\cos\theta}$

(iii) $(v \sin\theta)\ t = d \quad \Rightarrow\ t = \frac{d}{v\sin\theta}$

$x = (u + v\cos\theta)\ t = (u + v\cos\theta)\frac{d}{v\sin\theta}$

Illustration 137:

A boat moves relative to water with a velocity v which is $\frac{1}{n}$ times of the river flow velocity u. At what angle to the stream direction must the boat move to minimize drifting?

Solution:

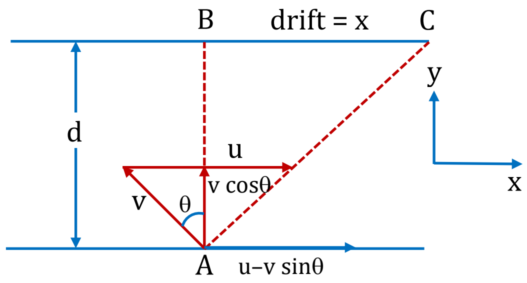
(In this problem, one thing should be carefully noted that the velocity of boat is less than the river flow velocity. Hence boat cannot reach the point directly opposite to its starting point. i.e. drift can never be zero)

Suppose boat starts at an angle θ from the normal direction up stream as shown.

Component of velocity of boat along the river, $v_x = u - v\sin\theta$ and velocity perpendicular to the river,

$v_y = v\cos\theta$.

Time taken to cross the river is



$$t = \frac{d}{v_y} = \frac{d}{v\cos\theta}$$

Drift $\quad x = (v_x)t = (u - v\sin\theta)\frac{d}{v\cos\theta}$

$$= \frac{ud}{v}\sec\theta - d\tan\theta$$

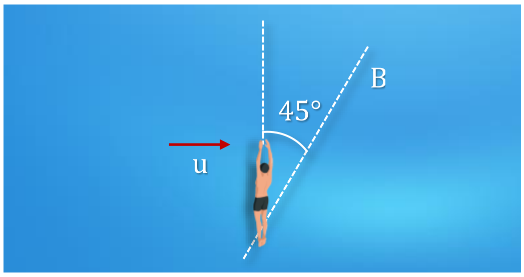
The drift x is minimum, when $\frac{dx}{d\theta} = 0$,

or $\quad \left(\frac{ud}{v}\right)(\sec\theta \,.\, \tan\theta) - d\sec^2\theta = 0$

or $\quad \frac{u}{v}\sin\theta = 1 \quad$ or $\quad \sin\theta = \frac{v}{u}$

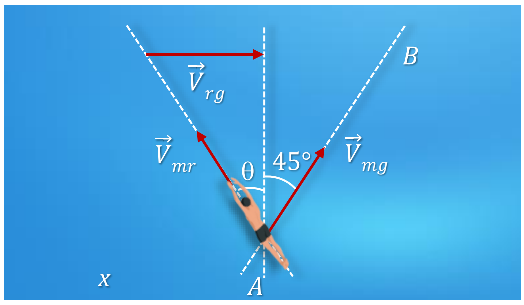
So, for minimum drift, the boat must move at an angle $\theta = \sin^{-1}\left(\frac{u}{v}\right) = \sin^{-1}\frac{1}{n}$ from normal direction.

Illustration 138:



A man wants to reach point B on the opposite bank of a river flowing at a speed u as shown in the figure. What minimum speed relative to water should the man have, so that he can reach point B? In which direction should he swim?

Solution:



To find $(v_{mr})_{min}$ and direction.

Let, $\quad$ Speed of the river = u

$\quad$ Speed of man in still water = v

Let velocity of man w.r.t. ground = $\vec{v}_{mg}$

and $\quad \vec{v}_{mr} = (-v\sin\theta)\hat{\imath} + (v\cos\theta)\hat{\jmath}$

$\quad \vec{v}_r = u\hat{\imath}$

Now, $\quad \vec{v}_{mg} = \vec{v}_{mr} + \vec{v}_r = \{(-v\sin\theta)\hat{\imath} + (v\cos\theta)\hat{\jmath}\} + u\hat{\imath}$

$\Rightarrow \quad \vec{v}_{mg} = (u - v\sin\theta)\hat{\imath} + (v\cos\theta)\hat{\jmath}$

Now, for AB to make 45° with the vertical, $\vec{v}_{mg}$ must be along AB

So, $\tan 45° = \dfrac{v\cos\theta}{u - v\sin\theta} = 1$

$\Rightarrow \quad v\cos\theta = u - v\sin\theta$

$\Rightarrow \quad \dfrac{u}{(\sin\theta + \cos\theta)} = v$

We need to find minimum v.

For $\quad v = v_{min}$

$\Rightarrow \quad \dfrac{u}{\sin\theta + \cos\theta}$ should be minimum

$\Rightarrow \quad \sin\theta + \cos\theta$ must be maximum

$$\frac{d}{d\theta}\{\sin\theta + \cos\theta\} = 0$$

$\Rightarrow \quad \cos\theta - \sin\theta = 0 \qquad \Rightarrow \quad \cos\theta = \sin\theta$

$\Rightarrow \quad \tan\theta = 1 \qquad \Rightarrow \quad \theta = 45°$ = Direction

Put in v_{min} to get,

$$v_{min} = \frac{u}{\sin 45° + \cos 45°} = \frac{u}{\sqrt{2}}$$

So, the man should swim in a direction 45° counter-clockwise with the vertical.

Wind Aeroplane Problems

This is very similar to boat river flow problems. The only difference is that boat is replaced by aeroplane and river is replaced by wind.

Thus, velocity of aeroplane with respect to wind

$$\vec{v}_{aw} = \vec{v}_a - \vec{v}_w$$

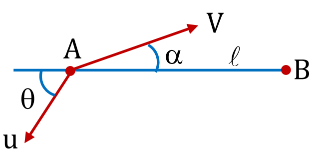
where, $\vec{v}_a$ = velocity of aeroplane w.r.t. ground and, $\vec{v}_w$ = velocity of wind.

Illustration 139:

An aeroplane flies along a straight path A to B and returns back again. The distance between A and B is ℓ and the aeroplane maintains the constant speed v w.r.t. wind. There is a steady wind with a speed u at an angle θ with line AB. Determine the expression for the total time of the trip.

Solution:

Suppose plane is oriented at an angle α w.r.t. line AB while the plane is moving from A to B:



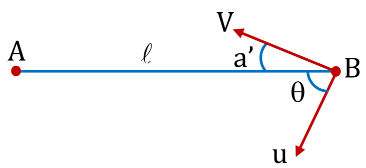
Velocity of plane along $AB = v\cos\alpha - u\cos\theta$,

and for no-drift from line AB ; $v\sin\alpha = u\sin\theta$

$\Rightarrow \quad \sin\alpha = \dfrac{u\sin\theta}{v} \qquad$ ...(i)

time taken from A to B:

$$t_{AB} = \frac{\ell}{v\cos\alpha - u\cos\theta}$$



Suppose plane is oriented at an angle α' w.r.t. line AB while the plane is moving from B to A:

Velocity of plane along $BA = v\cos\alpha' + u\cos\theta$ and for no drift from line AB; $v\sin\alpha' = u\sin\theta$

$$\Rightarrow \quad \sin\alpha' = \frac{u\sin\theta}{v} \quad \Rightarrow \quad \alpha = \alpha'$$

Time taken from B to A:

$$t_{BA} = \frac{\ell}{v\cos\alpha + u\cos\theta}$$

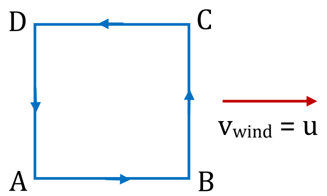
Total time taken $= t_{AB} + t_{BA} = \dfrac{\ell}{v\cos\alpha - u\cos\theta} + \dfrac{\ell}{v\cos\alpha + u\cos\theta}$

$$= \frac{2v\ell\cos\alpha}{v^2\cos^2\alpha - u^2\cos^2\theta} = \frac{2v\ell\sqrt{1-\frac{u^2\sin^2\theta}{v^2}}}{v^2-u^2} \quad \text{[Using equation (i)]}$$

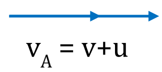
Illustration 140:

Find the time an aeroplane having velocity v, takes to fly around a square with side a if the wind is blowing at a velocity u along one side of the square.

Solution:

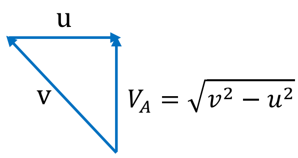


Velocity of aeroplane while flying through AB



$$v_A = v + u \quad ; \qquad t_{AB} = \frac{a}{v+u}$$

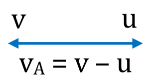
Velocity of aeroplane while flying through BC



$$v_A = \sqrt{v^2-u^2};$$

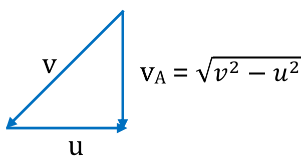
$$t_{BC} = \frac{a}{\sqrt{v^2-u^2}}$$

Velocity of aeroplane while flying through CD



$$v_A = v - u; \quad t_{CD} = \frac{a}{v-u}$$

Velocity of aeroplane while flying through DA



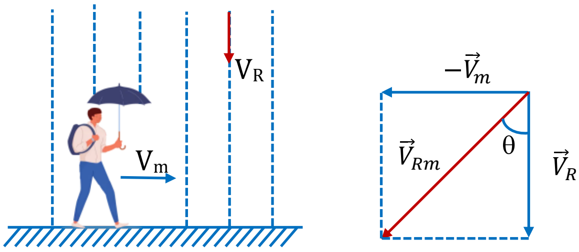
$$v_A = \sqrt{v^2-u^2}\,; \; t_{DA} = \frac{a}{\sqrt{v^2-u^2}}$$

Total time $= t_{AB} + t_{BC} + t_{CD} + t_{DA}$

$$= \frac{a}{v+u} + \frac{a}{\sqrt{v^2-u^2}} + \frac{a}{v-u} + \frac{a}{\sqrt{v^2-u^2}} = \frac{2a}{v^2-u^2}\left(v + \sqrt{v^2-u^2}\right)$$

Rain Problem

If rain is falling vertically with a velocity $\vec{v}_R$ and an observer is moving horizontally with velocity $\vec{v}_m$, the velocity of rain relative to observer will be:



$$\vec{v}_{Rm} = \vec{v}_R - \vec{v}_m$$

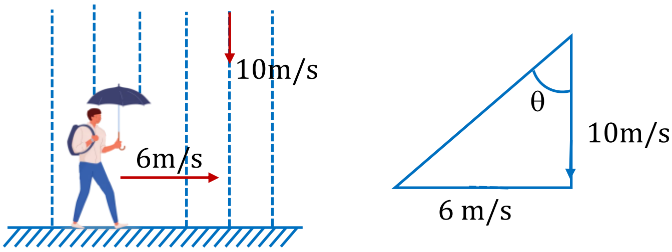
or $\quad v_{Rm} = \sqrt{v_R^2 + v_m^2}$

and direction $\theta = \tan^{-1}\left(\frac{v_m}{v_R}\right)$ with the vertical as shown in figure.

Illustration 141:

Rain is falling vertically at speed of 10 m/s and a man is moving with velocity 6 m/s. Find the angle at which the man should hold his umbrella to avoid getting wet.

Solution:



$\vec{v}_{\text{rain}} = -10\hat{j} \quad \Rightarrow \quad \vec{v}_{man} = 6\hat{i}$

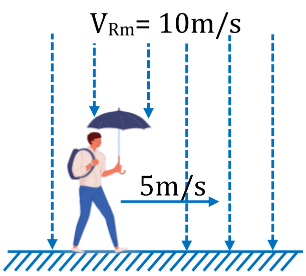
$\vec{v}_{r.w.r.t\ \text{man}} = -10\hat{j} - 6\hat{i}$

$\tan\theta = \frac{6}{10} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{5}\right)$

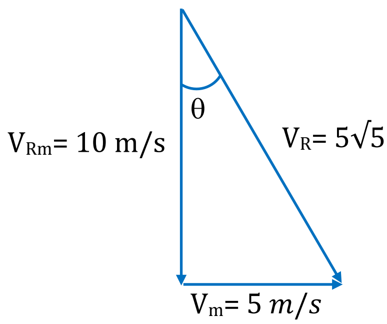
Where θ is angle with vertical.

Illustration 142:

A man moving with 5m/s observes rain falling vertically at the rate of 10 m/s. Find the speed and direction of the rain with respect to ground.



Solution:



$v_{RM} = 10\ m/s,\ v_M = 5\ m/s$

$\vec{v}_{RM} = \vec{v}_R - \vec{v}_M$

$\Rightarrow \quad \vec{v}_R = \vec{v}_{RM} + \vec{v}_M$

$\Rightarrow \quad v_R = 5\sqrt{5}$

$\tan\theta = \frac{1}{2},\ \theta = \tan^{-1}\frac{1}{2}.$

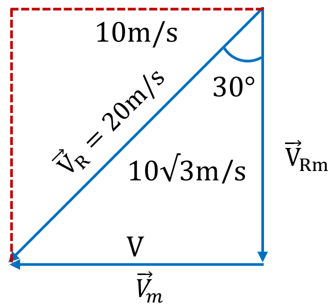
Illustration 143:

A standing man, observes rain falling with velocity of 20 m/s at an angle of 30° with the vertical.

(i) Find the velocity with which the man should move so that rain appears to fall vertically to him.

(ii) Now if he further increases his speed, rain again appears to fall at 30° with the vertical. Find his new velocity.

Solution:



(i) $\vec{v}_m = -v\hat{\imath}\ m/s\ (let)$

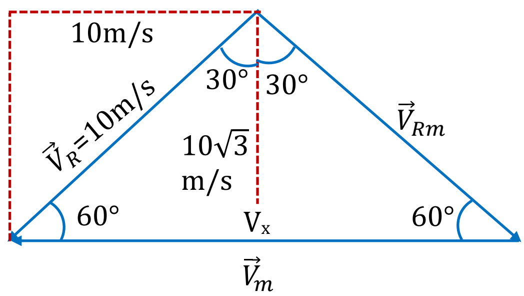
$\vec{v}_R = -10\hat{\imath} - 10\sqrt{3}\hat{\jmath}\ m/s$

$\vec{v}_{RM} = -(10\hat{\imath} - v) - 10\sqrt{3}\hat{\jmath}\ m/s$

$\Rightarrow\quad -(10 - v) = 0$

(For vertical fall, horizontal component must be zero)

or $\quad v = 10\ m/s$



(ii) $\vec{v}_R = -10\hat{\imath} - 10\sqrt{3}\hat{\jmath}$

$\vec{v}_m = -v_x\hat{\imath}$

$\vec{v}_{RM} = -(10 - v_x)\hat{\imath} - 10\sqrt{3}\hat{\jmath}$

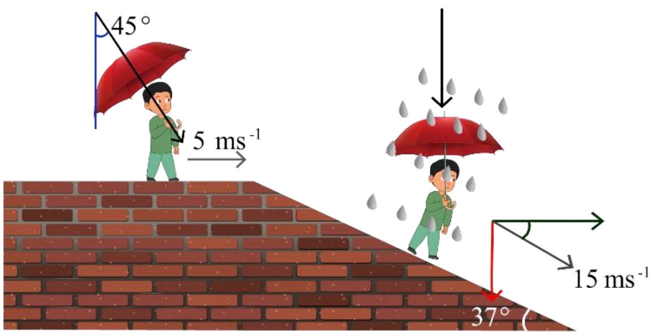
Angle with the vertical = 30°

$\Rightarrow\quad \tan 30° = \dfrac{10 - v_x}{-10\sqrt{3}}$

$\Rightarrow\quad v_x = 20\ m/s$

Illustration 144:

A man moving with a velocity of 5 ms^{-1} on a horizontal road observes that raindrops fall at an angle of 45° with the vertical. When he moves with a velocity of 15 ms^{-1} along a staircase of inclination 37°, he observes raindrops falling vertically as shown in the figure. Find the actual velocity of the rain.

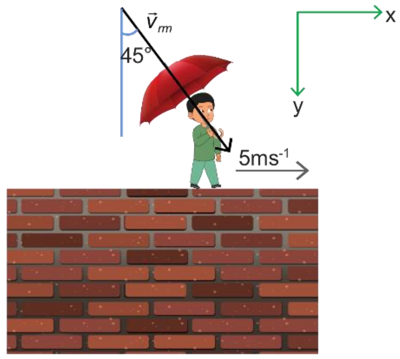


Solution:

Let the actual velocity of the rain w.r.t. the ground be $\vec{v}_{rg}$

Case 1: A man running on a horizontal plane Consider the right direction as positive x-axis and downwards as positive y-axis as shown in the figure.

Let $\vec{v}_{rg} = a\hat{i} + b\hat{j}\ ms^{-1}$



Given velocity of the man w.r.t. the ground, $\vec{v}_{mg} = 5\hat{i}\ ms^{-1}$

Now, the velocity of the rain w.r.t. the man,

$$\vec{v}_{rm} = \vec{v}_{rg} - \vec{v}_{mg} = (a\hat{i} + b\hat{j} - 5\hat{i}\ )ms^{-1}$$

$\Rightarrow \quad \vec{v}_{rm} = \left((a-5)\hat{i} + b\hat{j}\ \right)ms^{-1}$

Now, since $\vec{v}_{rm}$ makes 45° with y axis

$\Rightarrow \quad \tan 45° = \frac{(a-5)}{b}$

$\Rightarrow \quad a - 5 = b$ ...(i)

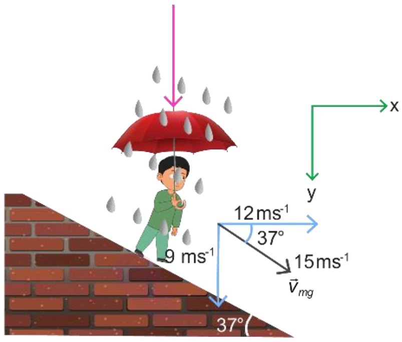
Case 2: A man running on an inclined plane Consider the right direction as positive x-axis and downwards as positive y-axis as shown in figure.

Velocity of the man w.r.t. the ground,

$$\vec{v}_{mg} = (15\cos 37°\hat{i} + 15\sin 37°\hat{j})ms^{-1}$$

$$\vec{v}_{mg} = \left(15 \times \frac{4}{5}\hat{i} + 15 \times \frac{3}{5}\hat{j}\right)ms^{-1}$$

$$\vec{v}_{mg} = (12\hat{i} + 9\hat{j})ms^{-1}$$



Velocity of the rain w.r.t. the man,

$$\vec{v}_{rm} = \vec{v}_{rg} - \vec{v}_{mg} = \left((a\hat{i} + b\hat{j}) - (12\hat{i} + 9\hat{j})\right)ms^{-1}$$

$\Rightarrow \quad \vec{v}_{rm} = \left((a-12)\hat{i} + (b-9)\hat{j}\right)ms^{-1}$

In this case, rain appears to fall vertically, so x component of $\vec{v}_{rm}$ must be 0.

$\Rightarrow \quad a - 12 = 0 \quad \Rightarrow \quad a = 12$

Substitute $a = 12$ in equation (i)

$\Rightarrow \quad b = 12 - 5 = 7$

So, actual velocity of the rain is

$$\vec{v}_{rg} = a\hat{i} + b\hat{j} = (12\hat{i} + 7\hat{j})ms^{-1}$$

Velocity of Approach/Separation

It is the component of relative velocity of one particle w.r.t. another, along the line joining them. If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation. In one dimension, since relative velocity is along the line joining A and B, hence velocity of approach / separation is simply equal to magnitude of relative velocity of A w.r.t. B.

Illustration 145:

A particle A is moving with a speed of 10 m/s towards right and another particle B is moving at speed of 12 m/s towards left. Find their velocity of approach.

Solution:

$V_A = +10$, $V_B = -12 \Rightarrow V_{AB} = V_A - V_B \Rightarrow 10 - (-12) = 22\ m/s$

Since separation is decreasing, hence $V_{app} = |V_{AB}| = 22\ m/s$.

Illustration 146:

A particle A is moving with a speed of 20 m/s towards right and another particle B is moving at a speed of 5 m/s towards right. Find their velocity of approach.

Solution:

$V_A = +20$, $V_B = +5$

$V_{AB} = V_A - V_B$

$20 - (+5) = 15\ m/s$

Since separation is decreasing, hence $V_{app} = |V_{AB}| = 15\ m/s$.

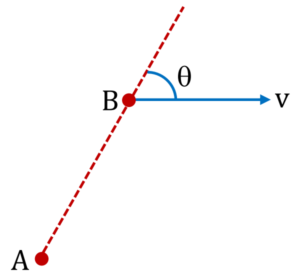
Velocity of Approach / Separation in Two Dimension

It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.

Illustration 147:

Particle A is at rest and particle B is moving with constant velocity v as shown in the diagram at $t = 0$. Find their velocity of separation



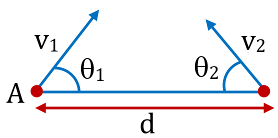
Solution:

$v_{BA} = v_B - v_A = v$

v_{sep} = component of v_{BA} along line $AB = v \cos\theta$

Illustration 148:

Two particles A and B are moving with constant velocities v_1 and v_2. At $t = 0$, v_1 makes an angle θ_1 with the line joining A and B and v_2 makes an angle θ_2 with the line joining A and B. Find their velocity of approach.



Solution:

Velocity of approach is relative velocity along line AB

$$v_{app} = v_1 \cos\theta_1 + v_2 \cos\theta_2$$

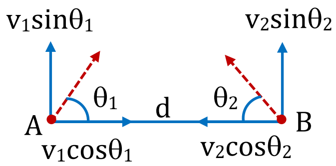
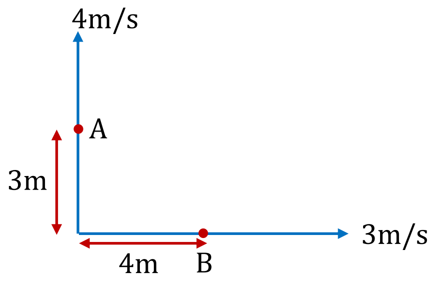


Illustration 149:

Particles A and B are moving as shown in the diagram at $t = 0$. Find their velocity of separation,

(i) at $t = 0$ (ii) at $t = 1$ sec.

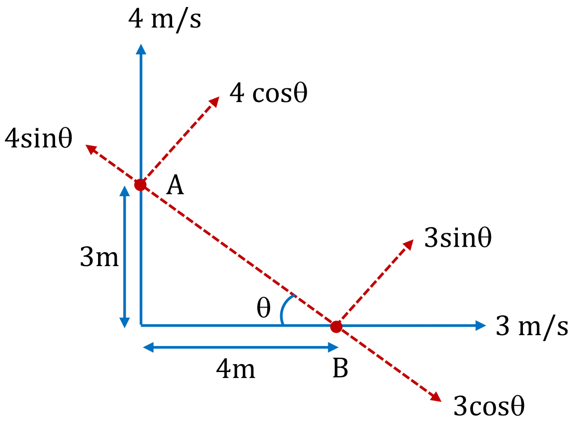


Solution:

(i) $\tan\theta = 3/4$

v_{sep} = Relative velocity along line AB

$$= 3\cos\theta + 4\sin\theta = 3.\frac{4}{5} + 4.\frac{3}{5} = \frac{24}{5} = 4.8\ m/s$$



(ii) $\theta = 45°$

v_{sep} = relative velocity along line AB

$$= 3\cos\theta + 4\sin\theta = 3.\frac{1}{\sqrt{2}} + 4.\frac{1}{\sqrt{2}} = \frac{7}{\sqrt{2}}\ m/s$$

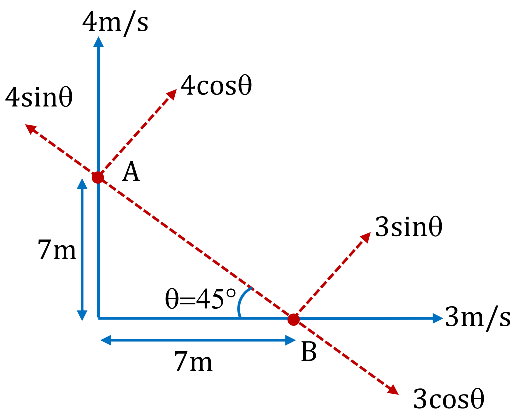
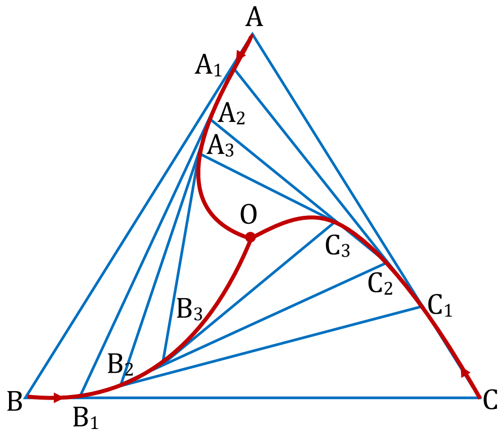


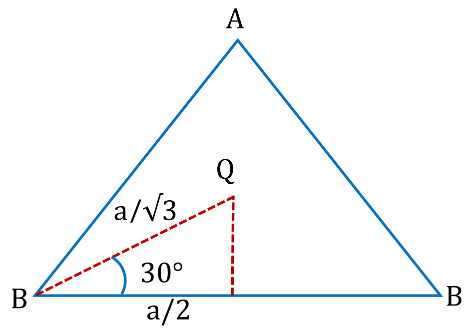
Illustration 150:

There are particles *A, B* and *C* are situated at the vertices of an equilateral triangle *ABC* of side a at $t = 0$. Each of the particles moves with constant speed v. *A* always has its velocity along *AB*, *B* along *BC* and *C* along *CA*. At what time will the particles meet each other?

Solution:



The motion of the particles is roughly sketched in figure. By symmetry they will meet at the centroid *O* of the triangle. At any instant the particles will from an equilateral triangle *ABC* with the same centroid *O*. All the particles will meet at the centre. Concentrate on the motion of any one particle, say *B*. At any instant its velocity makes angle 30° with *BO*. The component of this velocity along *BO* is $v \cos 30°$. This component is the rate of decrease of the distance *BO*.

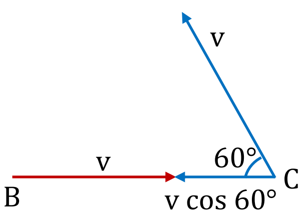


Initially, $BO = \dfrac{a/2}{\cos 30°} = \dfrac{a}{\sqrt{3}}$ = displacement of each particle. Therefore, the time taken for *BO* to become zero is

$$t = \frac{a/\sqrt{3}}{v\cos 30°} = \frac{2a}{\sqrt{3}v\times\sqrt{3}} = \frac{2a}{3v}$$

Aliter :

Velocity of *B* is v along *BC*. The velocity of *C* is along *CA*. Its component along *BC* is $v \cos 60° = v/2$. Thus, the separation *BC* decreases at the rate of approach velocity.



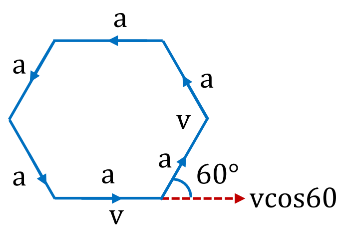
$\therefore$ Approach velocity $= v + \dfrac{v}{2} = \dfrac{3v}{2}$

Since, the rate of approach is constant, the time taken in reducing the separation *BC* from a to zero is $t = \dfrac{a}{\frac{3v}{2}} = \dfrac{2a}{3v}$.

Illustration 151:

Six particles situated at the corners of a regular hexagon of side a move at a constant speed v. Each particle maintains a direction towards the particle at the next corner. Calculate the time the particles will take to meet each other.

Solution:



$V_{app} = V - V\cos 60° = V - V/2 = V/2$

$$t = \frac{a}{V_{app}} = \frac{a}{V/2} = \frac{2a}{V}$$