01 Units, Dimensions and Vectors

UNITS AND DIMENSIONS

Physics is the study of nature and its laws. Feynman has given a wonderful description of what is "understanding the nature". Suppose we do not know the rules of chess but are allowed to watch the moves of the players. If we watch the game for a long time, we may make out some of the rules. With the knowledge of these rules we may try to understand why a player played a particular move. However, this may be a very difficult task. Even if we know all the rules of chess, it is not so simple to understand all the complications of a game in a given situation and predict the correct move. Knowing the basic rules is, however, the minimum requirement if any progress is to be made.

One may guess at a wrong rule by partially watching the game. The experienced player may make use of a rule for the first time and the observer of the game may get surprised. Because of the new move some of the rules guessed at may prove to be wrong and the observer will frame new rules.

Physics goes the same way. The nature around us is like a big chess game played by Nature. The events in the nature are like the moves of the great game. We are allowed to watch the events of nature and guess at the basic rules according to which the events take place. We may come across new events which do not follow the rules guessed earlier and we may have to declare the old rules inapplicable or wrong and discover new rules.

Physical Quantities:

All quantities that can be measured and used to define laws of nature are called physical quantities. eg. time, length, mass, force, work done, etc. In physics we study about physical quantities and their inter relationship.

Magnitude of a physical quantity = (Its Numerical value) (unit) = (n) (u)

Magnitude of a physical quantity always remains constant, it will not change if we express it in some other unit. So,

$$(n)(u) = \text{constant}$$

$$n \propto \frac{1}{u} \qquad\qquad n_1u_1 = n_2u_2$$

$$\text{Numerical value} \propto \frac{1}{\text{unit}}$$

Measurement:

Measurement is the comparison of a quantity with a standard of the same physical quantity.

There are two type of physical quantities.

(i) Fundamental quantities (ii) Derived quantities

(i) Fundamental (Basic) Quantities:

- These are the elementary quantities which cover the entire span of physics.
- Any other quantity can be derived from these.
- All the basic quantities are chosen such that they are independent of each other (i.e., distance (d), time (t) and velocity (v) cannot be chosen as basic quantities, because they are related as $V = \frac{d}{t}$).

An International Organization named CGPM : General Conference on weights and Measures, choose seven physical quantities as basic or fundamental.

Length	Time	Mass	Temperature	Electrical current	Luminous Intensity	Amount of Substance
[L]	[T]	[M]	[K]	[A]	[Cd]	[mol]

These are the elementary quantities (in our planet) that's why chosen as basic quantities.

In fact, any set of independent quantities can be chosen as basic quantities by which all other physical quantities can be derived. i.e.,

(A)	(V)	(ρ)	(S)	(R)	(Cd)	(mol)
Area	Velocity	Density	Sp. Heat Capacity	Resistance	Luminous Intensity	mole

Can be chosen as basic quantities (on some other planet, these might also be used as basic quantities)

But,

(L)	(A)	(V)
Length	Area	Velocity

Cannot be used as basic quantities as

Area = $(\text{Length})^2$ so they are not independent.

(ii) Derived Quantities:

Physical quantities which depend upon fundamental quantities or which can be derived from fundamental quantities are known as derived quantities.

Examples:- Volume, Pressure, Density etc.

Units:

All physical quantities are measured w.r.t. standard magnitude of the same physical quantity and these standards are called UNITS. eg. second, meter, kilogram, etc.

So, the four basic properties of units are:

1. They must be well defined.
2. They should be easily available and reproducible.
3. They should be invariable e.g. step as a unit of length is not invariable.
4. They should be accepted by all.

FPS (Foot Pound Second) or British Engineering System:

In this system length, mass and time are taken as fundamental quantities and their base units are foot (ft), pound (lb) and second (s) respectively.

CGS (Centimeter Gram Second) or Gaussian System:

In this system the fundamental quantities are length, mass and time and their respective units are centimetre (cm), gram (g) and second (s).

MKS (Meter Kilogram Second) System:

In this system the fundamental quantities are length, mass and time but their fundamental units are metre (m), kilogram (kg) and second (s) respectively.

International System (SI) of Units:

This system is modification over the MKS system, so it is also known as Rationalised MKS system. Besides the three base units of MKS system seven fundamental and two supplementary units are also included in this system.

Classification of Units:

The units of physical quantities can be classified as follows :

Fundamental or Base Units:

The units of fundamental quantities are called *base units*. In SI there are seven base units.

S.No.	Physical Quantity	SI unit	Symbol
1.	Length	Metre	*m*
2.	Mass	Kilogram	*kg*
3.	Time	Second	*s*
4.	Temperature	Kelvin	*K*
5.	Electric current	Ampere	*A*
6.	Luminous intensity	Candela	*cd*
7.	Amount of substance	Mole	*mol*

Derived units:

The units of derived quantities or the units that can be expressed in terms of the base units are called derived units

eg. → Area = $\ell \times b = m^2$

Volume = $\ell \times b \times h = m^3$

Density = $\frac{M}{V} = \frac{kg}{m^3} = kg/m^3 = kg - m^{-3}$

Force = $M \times a = kg \times \frac{m}{s^2} = kg - m - s^{-2}$

Pressure = $\frac{\text{Force}}{\text{Area}} = \frac{kg - m - s^2}{m^2} = kg - m^{-1} - s^{-2}$

Illustration 1:

Find the unit of speed.

Solution:

$$\text{Unit of speed} = \frac{\text{Unit of distance}}{\text{Unit of time}} = \frac{\text{Metre}}{\text{Second}} = ms^{-1}$$

Some derived units are named in honour of great scientists.

- Unit of force – Newton (N) ⇒ ($kgms^{-2}$)
- Unit of frequency – Hertz (Hz) ⇒ (s^{-1}) etc.

Supplementary units:

The units defined for the supplementary quantities namely plane angle and solid angle are called the supplementary units. The unit for plane angle is radian and the unit for the solid angle is steradian.

Dimensions:

Dimensions of a physical quantity are the powers to which the fundamental quantities must be raised to represent the given physical quantity.

Illustration 2:

Find the dimensions of force

Where, Force (Quantity) = Mass × Acceleration

Solution:

Force (Quantity) = Mass × Acceleration

$$= \text{Mass} \times \frac{\text{Velocity}}{\text{Time}} = \text{Mass} \times \frac{\text{Length}}{(\text{Time})^2}$$

$$= \text{Mass} \times \text{Length} \times (\text{Time})^{-2}$$

So, dimensions of force : 1 in mass

1 in length

–2 in time

Dimensional formula:

The physical quantity that is expressed in terms of the base quantities is enclosed in square brackets to remind that the equation is among the dimensions and not among the magnitudes. Thus, above equation may be written as [force] = $[MLT^{-2}]$.

Such an expression for a physical quantity in terms of the base quantities is called the dimensional formula. Thus, the dimensional formula of force is $[MLT^{-2}]$. The two versions given below are equivalent and are used interchangeably.

(a) The dimensional formula of force is $[MLT^{-2}]$.

(b) The dimensions of force are 1 in mass, 1 in length and –2 in time.

The *dimensional formula* of any physical quantity is that expression which represents how and which of the base quantities are included in that quantity.

a.	Mass $\rightarrow$ M	b.	Length $\rightarrow$ L
c.	Time $\rightarrow$ T	d.	Temp. $\rightarrow$ θ, K
e.	Current $\rightarrow$ I, A	f.	Amount of substance $\rightarrow$ Mol

Illustration 3:

Find dimensional formula of power.

Solution:

$$\text{Power } (P) = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Work}}{\text{Time}} = \frac{kg{-}m^2{-}s^{-2}}{s}$$

$$= \frac{M^1L^2T^{-2}}{T^1} = kg - m^2 - s^{-3} = [M^1L^2T^{-3}]$$

Illustration 4:

Torque is defined as moment of force and is given by

Torque = force × perpendicular distance

Find dimensional formula of Torque.

Solution:

Torque, τ (Tau) = Force × Distance

$= kg\text{-}m\text{-}s^{-2} \times m$

$= kg\text{-}m^2\text{-}s^{-2}$

$= [M^1L^2T^{-2}]$

Illustration 5:

Find dimensional formula of Angular momentum.

Solution:

Angular Momentum (L) = $M \times V \times r$

$= kg\text{-}m\text{-}s^{-1}\text{-}m$

Unit Dimension = $kg\text{-}m^2\text{-}s^{-1}$

$= [M^1L^2T^{-1}]$

Illustration 6:

Find dimensions of coefficient of viscosity. Viscous force is given by

$F_v = 6\pi\eta rv$

Where, F_v= Viscous force

η = coefficient of viscosity

r = radius of sphere

v = velocity of object

Solution:

Coefficient of Viscosity (η)

$$[\eta] = \left[\frac{F_v}{6\pi rV}\right] = \frac{[MLT^{-2}]}{[L][LT^{-1}]} = [ML^{-1}T^{-1}]$$

Illustration 7:

Find dimensional formula of pressure.

Solution:

$$\text{Pressure } (P) = \frac{\text{Force}}{\text{Area}} = \frac{M^1L^1T^{-2}}{L^2}$$

Dimension of $P = [M^1L^{-1}T^{-2}]$

Unit = $kg\text{-}m^{-1} - s^{-2}$ = 1 pascal

Illustration 8:

Surface tension of a liquid is defined as force on imaginary line per unit length. Find its dimensional formula.

Solution:

$$\text{Surface Tension } (S) = \frac{\text{Force}}{\text{Length}}$$

$$= \frac{M^1L^1T^{-2}}{L}$$

Dimension of $(S) = [M^1L^0T^{-2}]$

Unit = $kg\text{-}s^{-2}$

Illustration 9:

Find dimensional formula for universal gravitational constant (G).

Solution:

Universal Gravitational Constant (G)

$$g = \frac{GM}{r^2}$$

$$G = \frac{gr^2}{M}$$

$$G = \frac{m\,s^{-2}\,m^2}{kg}$$

$G = [M^{-1}L^3T^{-2}]$

Unit of $G = kg^{-1}\text{-}m^3\text{-}s^{-2}$

Units and Dimensions of Physical Quantities

Quantity	Common Symbol	SI unit	Dimension
Displacement	s	METRE (m)	L
Mass	m, M	KILOGRAM (kg)	M
Time	t	SECOND (s)	T
Area	A	m^2	L^2
Volume	V	m^3	L^3
Density	ρ	kg/m^3	M/L^3
Velocity	v, u	m/s	L/T
Acceleration	a	m/s^2	L/T^2
Force	F	Newton (N)	ML/T^2
Work	W	Joule (J) (=$N-m$)	ML^2/T^2
Energy	E, U, K	Joule (J)	ML^2/T^2
Power	P	Watt (W) (=J/s)	ML^2/T^3
Momentum	p	kg-m/s	ML/T
Gravitational constant	G	N-m^2/kg^2	L^3/MT^2
Angle	θ, φ	Radian	
Angular velocity	ω	Radian/s	T^{-1}
Angular acceleration	α	Radian/s^2	T^{-2}
Angular momentum	L	kg-m^2/s	ML^2/T
Moment of inertia	I	kg-m^2	ML^2
Torque	τ	N-m	ML^2/T^2
Angular frequency	ω	Radian/s	T^{-1}
Frequency	υ	Hertz (Hz)	T^{-1}
Period	T	s	T
Young's modulus	Y	N/m^2	M/LT^2
Bulk modulus	B	N/m^2	M/LT^2
Shear modulus	η	N/m^2	M/LT^2
Surface tension	S	N/m	M/T^2
Coefficient of viscosity	η	N-s/m^2	M/LT
Pressure	P, p	N/m^2, Pa	M/LT^2
Wavelength	λ	m	L
Intensity of wave	I	W/m^2	M/T^3
Temperature	T	$KELVIN$ (K)	K
Specific heat capacity	c	$J/kg-K$	L^2/T^2K
Stefan's constant	σ	W/m^2-K^4	M/T^3K^4
Heat	Q	J	ML^2/T^2
Thermal conductivity	K	$W/m-K$	ML/T^3K

Illustration 10:

(a) Can there be a physical quantity which has no unit and dimensions

(b) Can a physical quantity have unit without having dimensions

Solution:

(a) Yes, strain (b) Yes, angle with unit radian

Illustration 11:

If velocity (V), time (T) and force (F) were chosen as basic quantities, find the dimensions of mass.

Solution:

Dimensionally :

Force = Mass × Acceleration

$$\text{Force} = \text{Mass} \times \frac{\text{Velocity}}{\text{Time}}$$

$$\text{Mass} = \frac{\text{Force} \times \text{Time}}{\text{Velocity}}$$

$[\text{Mass}] = [FTV^{-1}]$

Principle of homogeneity

Two physical quantities having same dimensions can be added or subtracted but there is no such restriction in division and multiplication.

- Suppose in any formula, $(L + \alpha)$ term is coming (where L is length). As length can be added only with a length, so α should also be a kind of length.

  So $[\alpha] = [L]$

- Similarly consider a term $(F - \beta)$ where F is force. A force can be added/subtracted with a force only and give rise to a third force. So β should be a kind of force and its result $(F - \beta)$ should also be a kind of force.

$F - \beta$

A third force and its dimension will also be $M^1L^1T^{-2}$

β should be a kind of force $\Rightarrow [\beta] = [M^1L^1T^{-2}]$

Illustration 12:

$$\frac{\alpha}{t^2} = Fv + \frac{\beta}{x^2}$$

Find dimensional formula for $[\alpha]$ and $[\beta]$ (here t = time, F = force, v = velocity, x = distance).

Solution:

Since dimension of $Fv = [Fv] = [M^1L^1T^{-2}]\ [L^1T^{-1}] = [M^1L^2T^{-3}]$,

So $\left[\frac{\beta}{x^2}\right]$ should also be $[M^1L^2T^{-3}]$; $\frac{[\beta]}{[x^2]} = [M^1 L^2T^{-3}]$; $[\beta] = [M^1L^4T^{-3}]$

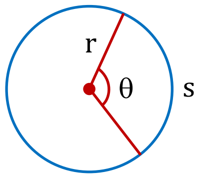
And $\left[Fv + \frac{\beta}{x^2}\right]$ will also have dimension $M^1L^2T^{-3}$, so L.H.S. should also have the same dimension $M^1L^2T^{-3}$

so, $\frac{[\alpha]}{[t^2]} = [M^1L^2T^{-3}] \Rightarrow [\alpha] = [M^1L^2T^{-1}]$

Key Points about Dimension:

(i) Angles and trigonometric functions are 'Dimensionless Quantities'.

Proof: Angle are 'dimensionless' $[\theta] = M^0L^0T^0$



$$[\theta] = \left[\frac{s}{r}\right] = \left[\frac{\text{Arc length}}{\text{Radius length}}\right] = \frac{L}{L} = L^0$$

Proof: 'Trigonometric' functions are dimensionless as they are angle function.

$\therefore$ $[\theta] = M^0L^0T^0$

$[\sin\theta] = M^0L^0T^0$

Illustration 13:

$\cos(\alpha x)$, where x = distance

Tell the dimension of α.

Solution:

$\therefore$ $[\alpha x] = [M^0L^0T^0] = 1$ (Trigonometric function)

$[\alpha]\,[x] = [M^0L^0T^0] = 1$

or $[\alpha] = [L^{-1}]$

Illustration 14:

$$\sin\left(\frac{kx}{t}\right)$$

Find the dimension of k where x = distance.

Solution:

$$\sin\left(\frac{kx}{t}\right)$$

$[k] = [TL^{-1}]$

(ii) All exponents are 'Dimensionless' (Base should be a numerical value).

$e^{-\alpha t^2}$ $\Rightarrow$ $[-\alpha t^2] = [M^0L^0T^0] = 1$ (Both base and exponent must be dimensionless)

$\left[e^{-\alpha t^2}\right] = [M^0L^0T^0]$

(iii) Logarithm functions and its argument are dimensionless.

$\log(a)$ $\Rightarrow$ $[a] = [M^0L^0T^0] = 1$

$[\log a] = [M^0L^0T^0]$

Illustration 15:

$\alpha = \frac{F}{v^2}\sin(\beta t)$ (Here v = velocity, F = force, t = time). Find the dimension of α and β.

Solution:

$$\alpha = \frac{F}{v^2}\sin(\beta t)$$

Dimensionless

Dimensionless

$\Rightarrow [\beta][t] = 1$

$[\beta] = [T^{-1}]$

So $[\alpha] = \frac{[F]}{[v^2]} = \frac{[M^1L^1T^{-2}]}{[L^1T^{-1}]^2} = M^1L^{-1}T^0$

Illustration 16:

$\alpha = \frac{Fv^2}{\beta^2}\log_e\left(\frac{2\pi\beta}{v^2}\right)$; Where F = force, v = velocity. Find the dimensions of α and β.

Solution:

$$\alpha = \frac{Fv^2}{\beta^2}\log_e\left(\frac{2\pi\beta}{v^2}\right)$$

Dimensionless

Dimensionless

$\Rightarrow \frac{[2\pi][\beta]}{[v^2]} = 1$

$\Rightarrow \frac{[1][\beta]}{[L^2T^{-2}]} = 1 \qquad \Rightarrow \qquad [\beta] = [L^2T^{-2}]$

as $[\alpha] = \frac{[F][v^2]}{[\beta^2]} \qquad \Rightarrow \qquad [\alpha] = \frac{[M^1L^1T^{-2}][L^2T^{-2}]}{[L^2T^{-2}]^2} \qquad \Rightarrow \qquad [\alpha] = [M^1L^{-1}T^0]$

Illustration 17:

The time dependence of physical quantity P is found to be of the form

$P = P_0e^{-\alpha t^2}$

Where 't' is the time and α is some constant.

Then the constant α will

(A) be dimensionless

(B) have dimensions of T^{-2}

(C) have dimensions of P

(D) have dimensions of P multiplied by T^{-2}

Ans. (B)

Solution:

Since in e^x, x is dimensionless

∴ αt^2 should be dimensionless

$[\alpha t^2] = [M^0L^0T^0]$

$[\alpha] = [M^0L^0T^{-2}]$

Applications of dimensional analysis:

First application

To find the dimensions of unknown physical quantities.

Illustration 18:

Using the theory of dimensions, determine the dimensions of constants 'a' and 'b' in Vander Wall's equation

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT.$$

Solution:

$\frac{a}{V^2}$ must have the same dimensions as that of P (because it is added to P)

Dimension of b must be same as that of V.

$$\frac{[a]}{L^6} = [ML^{-1}T^{-2}]$$

$$[a] = [ML^5T^{-2}]$$

$$[b] = [L^3]$$

Second Application:

To check correctness of an equation using principle of homogeneity of dimension :

Principle of homogeneity : According to this, only same physical quantities can be added or subtracted

eg. (i) $x = A \pm B \quad \Rightarrow [x] = [A] = [B]$

$$\underset{L}{s} = \underset{\substack{LT^{-1}T\\=L}}{ut} + \underset{\substack{LT^{-2}T^2\\=L}}{\frac{1}{2}at^2}$$

$\therefore$ dimensionally correct

Illustration 19:

Find the correct relation

$$F = \frac{mv^2}{r^2} \text{ or } \frac{mv^2}{r}$$

Solution:

Checking the dimensional correctness of relation

$$F = \frac{mv^2}{r^2}$$

$$[L.H.S.] = [MLT^{-2}]$$

$$[R.H.S]. = \frac{[M(LT^{-1})^2]}{[L^2]} = [ML^0T^{-2}]\ ;\ [LHS] \neq [RHS]$$

$$F = \frac{Mv^2}{r}$$

$$[LHS] = [MLT^{-2}]$$

$$[RHS] = \frac{[M(LT^{-1})^2]}{[L]} = [MLT^{-2}]\ ;\ [LHS] = [RHS]$$

Hence dimensionally second relation is correct.

Limitation:

Every physically correct equation must be correct dimensionally, but dimensionally correct equation may or may not be correct physically.

Illustration 20:

Check whether this equation may be correct or not.

$$P_r = \frac{3\,Fv^2}{\pi^2 t^2 x}$$

Solution:

Pressure, $P_r = \frac{3\,Fv^2}{\pi^2 t^2 x}$ (where P_r = Pressure, F = force , v = velocity , t = time , x = distance)

Dimension of $L.H.S$ = $[P_r] = [M^1L^{-1}T^{-2}]$

Dimension of $R.H.S = \frac{[3]\,[F]\,[v^2]}{[\pi^2]\,[t^2]\,[x]} = \frac{[M^1L^1T^{-2}]\,[L^2T^{-2}]}{[T^2]\,[L]} = [M^1L^2T^{-6}]$

Dimension of $L.H.S$ and $R.H.S$ are not same. So, the relation cannot be correct.

Note:

Sometimes a question is asked which is beyond our syllabus, then certainly it must be the question of dimensional analyses.

Third Application:

To derive relationship between different physical quantities:

Using the same principle of homogeneity of dimensions, new relations among physical quantities can be derived if the dependent quantities are known.

Illustration 21:

It is known that the time of revolution T of a satellite around the earth depends on the universal gravitational constant G, the mass of the earth M, and the radius of the circular orbit R. Obtain an expression for T using dimensional analysis.

Solution:

We have $[T] = [G]^a\,[M]^b\,[R]^c$

$$[M]^0\,[L]^0\,[T]^1 = [M]^{-a}\,[L]^{3a}\,[T]^{-2a} \times [M]^b \times [L]^c = [M]^{b-a}\,[L]^{c+3a}\,[T]^{-2a}$$

Comparing the exponents

For [T] : $1 = -2a \quad \Rightarrow a = -\frac{1}{2}$

For [M] : $0 = b - a \quad \Rightarrow b = a = -\frac{1}{2}$

For [L] : $0 = c + 3a \quad \Rightarrow c = -3a = \frac{3}{2}$

Putting the values, we get $T = G^{-1/2}\,M^{-1/2}\,R^{3/2} = \sqrt{\frac{R^3}{GM}}$

$\Rightarrow \quad T = K\sqrt{\frac{R^3}{GM}}$ (Where K is proportionality constant)

So, the actual expression is $T = 2\pi\sqrt{\frac{R^3}{GM}}$ (Experimentally $K = 2\pi$).

Illustration 22:

When a small sphere moves at low speed through a fluid, the viscous force F opposing the motion, is found experimentally to depend on the radius 'r', the velocity v of the sphere and the viscosity η of the fluid. Find the force F (Stoke's law).

Solution:

$F = f(\eta\,;\, r\,;\, v)$

$F = k\,.\,\eta^x.\, r^y.\, v^z$

$[F] = [\eta]^x.\,[r]^y.\,[v]^z$

$[MLT^{-2}] = [ML^{-1}T^{-1}]^x\,[L]^y\,[LT^{-1}]^z$

$[MLT^{-2}] = [M^x\,L^{-x+y+z}\,T^{-x-z}]$

Comparing coefficients

$x = 1\,,\, -x + y + z = 1\,;\, -x - z = -2$

$x = y = z = 1$

So, $F = k\eta vr$

$F = 6\pi\eta vr$ (As through experiments : $k = 6\pi$)

Illustration 23:

If velocity (V), force (F) and time (T) are chosen as fundamental quantities, express (i) mass and (ii) energy in terms of V, F and T.

Solution:

Let M = (some Number) $(V)^a\,(F)^b\,(T)^c$

Equating dimensions of both the sides

$[M^1L^0T^0] = [L^1T^{-1}]^a\,[M^1L^1T^{-2}]^b\,[T^1]^c$

$[M^1L^0T^0] = [M^bL^{a+b}\,T^{-a-2b+c}]$

Get $a = -1$, $b = 1$, $c = 1$

M = (Some Number) $(V^{-1}\,F^1\,T^1)$ $\Rightarrow$ $[M] = [V^{-1}\,F^1\,T^1]$

Similarly, we can also express energy in terms of V, F, T

Let $[E]$ = [some Number] $[V]^a\,[F]^b\,[T]^c$

$\Rightarrow$ $[ML^2T^{-2}] = [M^0L^0T^0]\,[LT^{-1}]^a\,[MLT^{-2}]^b\,[T]^c$

$\Rightarrow$ $[M^1L^2T^{-2}] = [M^b\,L^{a+b}\,T^{-a-2b+c}]$

$\Rightarrow$ $1 = b;\, 2 = a + b\,;\, -2 = -a - 2b + c$

Get $a = 1\,;\, b = 1\,;\, c = 1$

$\therefore$ E = (some Number) $V^1F^1T^1$ or $[E] = [V^1][F^1][T^1]$.

Illustration 24:

In a new system, force (F), momentum (p) and length (L) are taken as fundamental quantities. Find dimension of energy in this new system ?

Solution:

$$E = k\,F^a\,p^b\,L^c$$

$$[M^1\,L^2\,T^{-2}] = [M^1\,L^1\,T^{-2}]^a\,[M^1\,L^1\,T^{-1}]^b\,[L]^c$$

$$[M^1\,L^2\,T^{-2}] = [M^{a+b}\,L^{a+b+c}\,T^{-2a-b}]$$

By comparing

$1 = a + b$ ...(i)

$2 = a + b + c$ ...(ii)

$2 = 2a + b$ ...(iii)

From equation (i), (ii) and (iii)

$\therefore$ $a = 1$

$b = 0$

$c = 1$

So, $E = k\,F^1\,P^0\,L^1$

$E = k\,F\,L$

Conversion of units:

If units are u_1, u_2 and u_3 and numerical values of physical quantity are n_1, n_2 and n_3 then,

$$n_1u_1 = n_2u_2 = n_3u_3$$

$$nu = \text{constant}$$

For example : 1 m = 100 cm = 1000 mm

When unit become smaller, numerical value increase.

Illustration 25:

Find value of 1N in C.G.S. system.

Solution:

$$[F] = [MLT^{-2}]$$

$$n_1u_1 = n_2u_2$$

$$1\frac{kg-m}{s^2} = n_2\frac{g-cm}{s^2}$$

$$1\underbrace{\frac{kg\text{-}m}{sec^2}}_{\text{Newton}} = 10^5\underbrace{\frac{g\text{-}cm}{sec^2}}_{\text{Dyne}}$$

$$n_2 = 1\frac{kg}{g}\frac{m}{cm}\frac{s^2}{s^2}$$

$$n_2 = 10^5$$

$\therefore$ $\boxed{1\ N = 10^5 \text{ dyne}}$

Illustration 26:

Find the value of 1 *Joule* energy in C.G.S. system.

Solution:

$[E] = [M^1L^2T^{-2}]$

$n_1u_1 = n_2u_2$

$1\dfrac{kg-m^2}{s^2} = n_2\dfrac{g-cm^2}{s^2}$

$n_2 = 1\dfrac{kg}{g}\dfrac{m^2}{cm^2}\dfrac{s^2}{s^2}$

$n_2 = 1\dfrac{10^3 g}{g}\left(\dfrac{100cm}{cm}\right)^2$

$n_2 = 10^3 \times 10^4$

$n_2 = 10^7$

$\therefore \quad \boxed{1 \text{ Joule} = 10^7 \text{ Erg}}$

$$1\underbrace{\frac{kg\text{-}m^2}{s^2}}_{\text{Joule}} = 10^7\underbrace{\frac{g\text{-}cm^2}{s^2}}_{\text{Erg}}$$

Illustration 27:

Convert G from SI system to C.G.S. system.

Dimensional formula = $[M^{-1}L^3T^{-2}]$

Universal gravitational constant G is given as

$G = 6.67 \times 10^{-11} \times \dfrac{m^3}{kg.s^2}$

Solution:

$G = 6.67 \times 10^{-11} \times \dfrac{(100cm)^3}{1000g.(1\,s)^2}$

$G = 6.67 \times 10^{-11} \times \dfrac{10^6}{10^3}\dfrac{cm^3}{g.s^2}$

$G = 6.67 \times 10^{-8}\dfrac{cm^3}{g.s^2}$

Conversion of a Quantity from a Given System to New Hypothetical System:

Illustration 28:

In a particular system, the unit of length, mass and time are chosen to be 10cm, 10gm and 0.1s respectively. The unit of force in this system will be equivalent to

(A) $\dfrac{1}{10}N$ (B) $1N$ (C) $10N$ (D) $100\ N$

Ans. (A)

Solution:

Dimensionally

$[F] = [MLT^{-2}]$

In C.G.S system

$1 \text{ dyne} = 1g\ 1\ cm\ (1s)^{-2}$

In new system

$1x = (10g)\,(10\ cm)\,(0.1s)^{-2}$

$$\frac{1\text{ dyne}}{1x} = \frac{1g}{10g} \times \frac{1cm}{10cm}\left(\frac{10s}{1s}\right)^{-2}$$

$$1\text{ dyne} = \frac{1}{10{,}000} \times 1x$$

10^4 dyne = $1x$

$10\,x = 10^5$ dyne = $1\,N$

$$x = \frac{1}{10}N$$

Illustration 29:

If the units of length and force are increased four times, then the unit of energy will

(A) increase 8 times (B) increase 16 times (C) decreases 16 times (D) increase 4 times

Ans. (B)

Solution:

Dimensionally

$[E] = [ML^2T^{-2}]$

$[E] = [MLT^{-2}]\,[L]$

$E' = (4)\,(MLT^{-2})\,(4L)$

$E' = 16\,(ML^2T^{-2})$

Limitations of Dimensional Analysis:

- It yields only empirical formula

  From Dimensional analysis we get, $T = (\text{Some Number})\sqrt{\dfrac{\ell}{g}}$

  So, the expression of T can be

$$T = 2\sqrt{\frac{\ell}{g}} \qquad T = \sqrt{\frac{\ell}{g}}\sin(.....)$$

or   or

$$T = 50\sqrt{\frac{\ell}{g}} \qquad T = \sqrt{\frac{\ell}{g}}\,log\,(......)$$

or   or

$$T = 2\pi\sqrt{\frac{\ell}{g}} \qquad T = \sqrt{\frac{\ell}{g}} + (t_0)$$

- Dimensional analysis doesn't give information about the "some Number".

- This method is useful only when a physical quantity depends on other quantities by multiplication and power relations.

  ($i.e., f = x^a y^b z^c$)

  It fails if a physical quantity depends on sum or difference of two quantities

  ($i.e., f = x + y - z$)

  i.e., we cannot get the relation

  $$S = ut + \frac{1}{2}at^2$$

  From dimensional analysis.
- This method will not work if a quantity depends on another quantity as sine or cosine, logarithmic or exponential relation. The method works only if the dependence is by power functions.
- In mechanics, we equate the powers of M, L and T hence we get only three equations. So, we can have only three variables (only three dependent quantities)

  So dimensional analysis will work only if the quantity depends only on three parameters, not more than that.

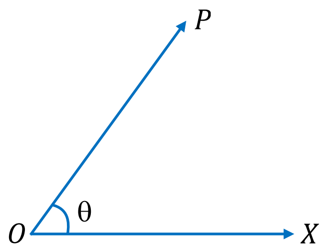
Trigonometry:

Angle:

Consider a revolving line OP.

Suppose that it revolves in anticlockwise direction starting from its initial position OX.

The angle is defined as the amount of revolution that the revolving line makes with its initial position.



From fig. the angle covered by the revolving line OP is $\theta = \angle POX$

The angle is taken positive if it is traced by the revolving line in anticlockwise direction and the angle is taken negative if it is covered in clockwise direction.

$1° = 60'$ (minute)

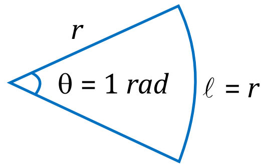
$1' = 60''$ (second)

1 right angle = 90° (degrees)

Also 1 right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle subtended at the centre of a circle by an arc of the circle, whose length is equal to the radius of the circle.

$$1 \text{ rad} = \frac{180°}{\pi} = 57.3°$$

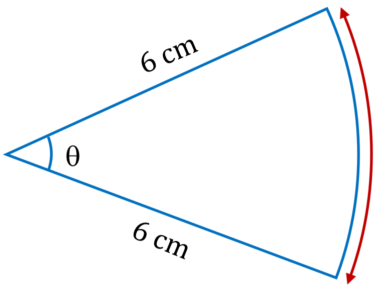


To convert an angle from degree to radian multiply it by $\frac{\pi}{180°}$

To convert an angle from radian to degree multiply it by $\frac{180°}{\pi}$

Illustration 30:

A circular arc is of length π cm. Find angle subtended by it at the centre in radian and degree.



Solution:

$$\theta = \frac{s}{r} = \frac{\pi \text{ cm}}{6 \text{ cm}} = \frac{\pi}{6}\text{rad}$$

As, $1 \text{ rad} = \frac{180°}{\pi}$

So, $\theta = \frac{\pi}{6} \times \frac{180°}{\pi} = 30°$

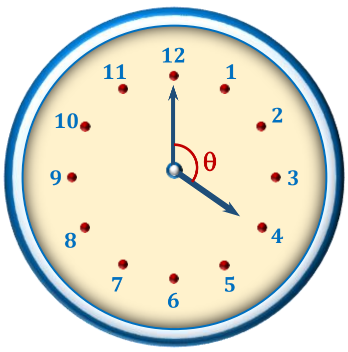
Illustration 31:

When a clock shows 4 o'clock, how much angle do its minute and hour needles make?

(A) 120° (B) $\frac{\pi}{3}$ rad (C) $\frac{2\pi}{3}$ rad (D) 160°

Ans. (AC)

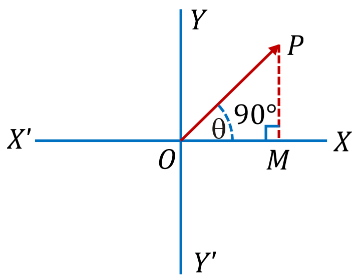
Solution:



From diagram angle $\theta = 4 \times 30° = 120° = \frac{2\pi}{3}$ rad

Trigonometric ratios (or *T* ratios):

Let two fixed lines XOX' and YOY' intersect at right angles to each other at point O.



Then,

(i) Point O is called origin.

(ii) XOX' is known as X-axis and YOY' as Y-axis.

(iii) Portions XOY, YOX', $X'OY'$ and $Y'OX$ are called I, II, III and IV quadrants respectively.

Consider that the revolving line *OP* has traced out angle θ (in I quadrant) in anticlockwise direction. From *P*, draw perpendicular *PM* on *OX*. Then, side *OP* (in front of right angle) is called hypotenuse, side *MP* (in front of angle θ) is called opposite side or perpendicular and side *OM* (making angle θ with hypotenuse) is called adjacent side or base.

The three sides of a right-angled triangle are connected to each other through six different ratios, called trigonometric ratios or simply *T*-ratios :

$\sin\theta = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{MP}{OP}$ $\qquad \cos\theta = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{OM}{OP}$

$\tan\theta = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{MP}{OM}$ $\qquad \cot\theta = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{OM}{MP}$

$\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{OP}{OM}$ $\qquad \text{cosec}\,\theta = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{OP}{MP}$

It can be easily proved that :

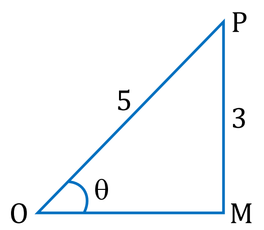
$\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ $\qquad \sec\theta = \dfrac{1}{\cos\theta}$ $\qquad \cot\theta = \dfrac{1}{\tan\theta}$

$\sin^2\theta + \cos^2\theta = 1$ $\qquad 1 + \tan^2\theta = \sec^2\theta$ $\qquad 1 + \cot^2\theta = \text{cosec}^2\theta$

Illustration 32:

Given sin θ = 3/5. Find all the other *T*-ratios, if θ lies in the first quadrant.

Solution:



In Δ *OMP*, $\sin\theta = \dfrac{3}{5}$

So $\quad MP = 3$ and $OP = 5$

∵ $\quad OM = \sqrt{(5)^2 - (3)^2} = \sqrt{25-9} = \sqrt{16} = 4$

Now, $\quad \cos\theta = \dfrac{OM}{OP} = \dfrac{4}{5}$ $\qquad \tan\theta = \dfrac{MP}{OM} = \dfrac{3}{4}$

$\cot\theta = \dfrac{OM}{MP} = \dfrac{4}{3}$ $\qquad \sec\theta = \dfrac{OP}{OM} = \dfrac{5}{4}$ $\qquad \text{cosec}\,\theta = \dfrac{OP}{MP} = \dfrac{5}{3}$

Table: The *T*-ratios of a few standard angles ranging from 0° to 180°

Angle (θ)	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ (not defined)	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0

Four Quadrants and ASTC Rule*:

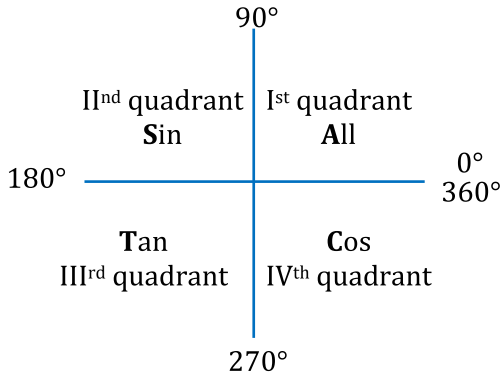
In first quadrant, all trigonometric ratios are positive.

In second quadrant, only sinθ and cosecθ are positive.

In third quadrant, only tanθ and cotθ are positive.

In fourth quadrant, only cosθ and secθ are positive

* Remember as Add Sugar To Coffee or After School To College.



Trigonometrical Ratios of General Angles: (Reduction Formulae)

(i) Trigonometric function of an angle $(2n\pi + \theta)$ where n=0, 1, 2, 3....... will be remain same.

$\sin(2n\pi + \theta) = \sin\theta$ $\quad \cos(2n\pi + \theta) = \cos\theta$ $\quad \tan(2n\pi + \theta) = \tan\theta$

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will remain same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

$\sin(\pi-\theta) = +\sin\theta$ $\quad \cos(\pi - \theta) = -\cos\theta$ $\quad \tan(\pi - \theta) = -\tan\theta$

$\sin(\pi+\theta) = -\sin\theta$ $\quad \cos(\pi + \theta) = -\cos\theta$ $\quad \tan(\pi + \theta) = +\tan\theta$

$\sin(2\pi-\theta) = -\sin\theta$ $\quad \cos(2\pi - \theta) = +\cos\theta$ $\quad \tan(2\pi - \theta) = -\tan\theta$

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos\theta$ $\quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ $\quad \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos\theta$ $\quad \cos\left(\frac{\pi}{2} - \theta\right) = +\sin\theta$ $\quad \tan\left(\frac{\pi}{2} - \theta\right) = +\cot\theta$

(iv) Trigonometric function of an angle $-\theta$ (negative angles)

$\sin(-\theta) = -\sin\theta$ $\quad \cos(-\theta) = +\cos\theta$ $\quad \tan(-\theta) = -\tan\theta$

$\sin(90°+\theta) = \cos\theta$ $\cos(90°+\theta) = -\sin\theta$ $\tan(90°+\theta) = -\cot\theta$	$\sin(180°-\theta) = \sin\theta$ $\cos(180°-\theta) = -\cos\theta$ $\tan(180°-\theta) = -\tan\theta$	$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$	$\sin(90°-\theta) = \cos\theta$ $\cos(90°-\theta) = \sin\theta$ $\tan(90°-\theta) = \cot\theta$
$\sin(180°+\theta) = -\sin\theta$ $\cos(180°+\theta) = -\cos\theta$ $\tan(180°+\theta) = \tan\theta$	$\sin(270°-\theta) = -\cos\theta$ $\cos(270°-\theta) = -\sin\theta$ $\tan(270°-\theta) = \cot\theta$	$\sin(270°+\theta) = -\cos\theta$ $\cos(270°+\theta) = \sin\theta$ $\tan(270°+\theta) = -\cot\theta$	$\sin(360°-\theta) = -\sin\theta$ $\cos(360°-\theta) = \cos\theta$ $\tan(360°-\theta) = -\tan\theta$

Illustration 33:

Find the value of

(i) cos (–60°) (ii) tan 210° (iii) sin 300° (iv) cos 120°

Solution:

(i) $\cos(-60°) = \cos 60° = \frac{1}{2}$

(ii) $\tan 210° = \tan(180° + 30°) = \tan 30° = \frac{1}{\sqrt{3}}$

(iii) $\sin 300° = \sin(270° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

(iv) $\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$

A few important trigonometric formulae:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ $\sin(A-B) = \sin A \cos B - \cos A \sin B$ $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$	$\cos(A+B) = \cos A \cos B - \sin A \sin B$ $\cos(A-B) = \cos A \cos B + \sin A \sin B$ $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
$\sin 2A = 2 \sin A \cos A$ $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$	$\cos 2A = \cos^2 A - \sin^2 A$ $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
$1 + \cos A = 2\cos^2 \dfrac{A}{2}$	$1 - \cos A = 2\sin^2 \dfrac{A}{2}$

Small Angle Approximation

If θ is small (say < 5°) then $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and $\tan\theta \approx \theta$. Here θ must be in radians.

VECTORS

Scalar Quantities:

A physical quantity which can be described completely by its magnitude only and does not require a direction is known as a scalar quantity.

It obeys the ordinary rules of algebra.

Ex: Distance, mass, time, speed, density, volume, temperature etc.

Vector Quantities:

If a physical quantity in addition to magnitude -

(a) It has a specified direction.

(b) It obeys commutative law of additions: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(c) It obeys the rule of vector algebra.

Then only it is said to be a vector.

If any of the above conditions is not satisfied the physical quantity cannot be a vector.

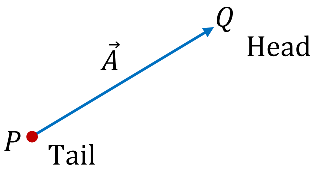
If a physical quantity is a vector it has a direction, but the converse may or may not be true, i.e. if a physical quantity has a direction, it may or may not be a vector. For example, Electric current has direction but is not vector because it does not obey rule of vector algebra.

Ex.: Displacement, velocity, acceleration, force etc.

Representation of Vector

- A vector is represented by a line headed with an arrow.
- Its length is proportional to its magnitude and arrow indicates direction.

$\vec{A}$ is a vector.

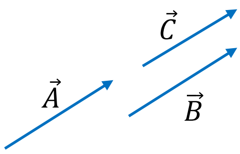


$\vec{A} = \overrightarrow{PQ}$

Magnitude of $\vec{A} = |\vec{A}|$ or A.

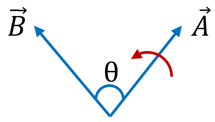
Important Points:

- If a vector is displaced parallel to itself it does not change (see Figure)



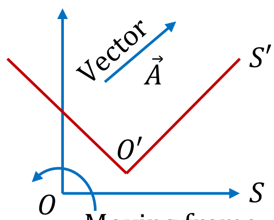
$\vec{A} = \vec{B} = \vec{C}$
Transition of a vector parallel to itself

- If a vector is rotated through an angle other than multiple of 2π (or 360°) it changes (see Figure).



$\vec{A} \neq \vec{B}$ Rotation of a vector

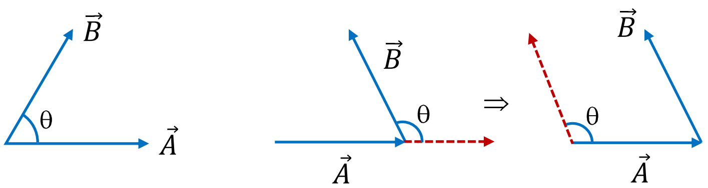
- If the frame of reference is translated or rotated the vector does not change (though its components may change). (see Figure).



Moving frame

Angle between Vectors:

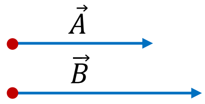
- Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail or head to head by displacing either of the vectors parallel to itself (i.e. $0 \le \theta \le \pi$).



Types of Vector:

- ### Parallel Vectors:

Those vectors which have same direction are called parallel vectors.

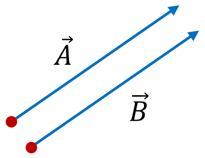


Angle between two parallel vectors is always 0°.

- ### Equal Vectors:

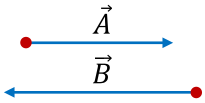
Vectors which have equal magnitude, same direction and they represent same physical quantity are called equal vectors.

$\vec{A} = \vec{B}$



- ### Anti–Parallel Vectors:

Those vectors which have opposite direction are called anti–parallel vector.



Angle between two anti–parallel vectors is always 180°.

- ### Negative (or Opposite) Vectors:

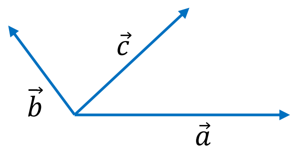
Vectors which have equal magnitude but opposite direction are called negative vectors of each other.

$\overrightarrow{AB}$ and $\overrightarrow{BA}$ are negative vectors $\overrightarrow{AB} = -\overrightarrow{BA}$.

- ### Co-Initial Vectors:

Co-initial vectors are those vectors which have the same initial point.

In figure $\vec{a}$, $\overrightarrow{b}$ and $\vec{c}$ are co-initial vectors.



- **Collinear Vectors:**

  The vectors lying in the same line are known as collinear vectors.

  Angle between collinear vectors is either 0° or 180°.

  Example:

  (i) ← ← (θ = 0°) (ii) → → (θ = 0°) (iii) ← → (θ = 180°) (iv) → ← (θ = 180°)

- **Coplanar Vectors:**

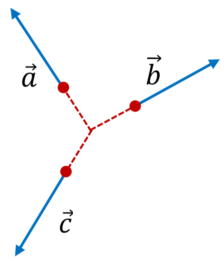
  Vectors located in the same plane are called coplanar vectors.

  Two vectors are always coplanar.

- **Concurrent Vectors:**

  Those vectors which pass through a common point are called concurrent vectors.

  In the given figure $\vec{a}, \vec{b}$ and $\vec{c}$ are concurrent vectors.



- **Null or Zero Vector:**

  A vector having zero magnitude is called null vector. The direction of a null vector is indeterminate.

  Sum of two vectors is always a vector so, $(\vec{A}) + (-\vec{A}) = \vec{0}$.

  $\vec{0}$ is a zero vector or null vector.

- **Unit Vector:**

  A vector having unit magnitude is called unit vector. It is used to specify direction. A unit vector is represented by $\hat{A}$ (Read as A cap or A hat or A caret).

  Unit vector in the direction of $\vec{A}$ is, $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

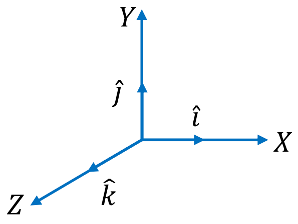
  $$\left(\text{Unit vector} = \frac{\text{Vector}}{\text{Magnitude of the vector}}\right)$$

  $$\vec{A} = A\hat{A} = |\vec{A}|\hat{A}$$

  A unit vector is used to specify the direction of a vector.

Base Vectors:

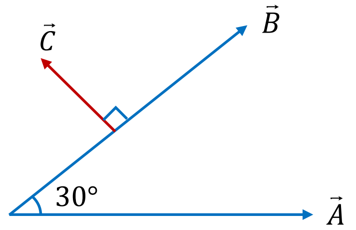
In an XYZ co-ordinate frame there are three unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, these are used to indicate X, Y and Z directions respectively and are called as base vectors.



These three unit vectors are mutually perpendicular to each other.

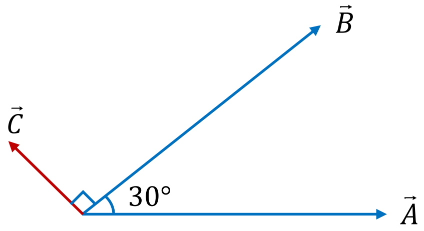
Illustration 34:

Find the angle between $\vec{A}$ and $\vec{C}$.



Solution:

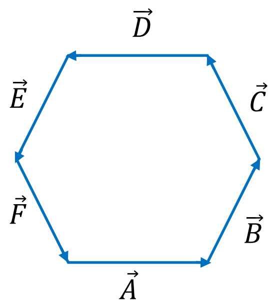
By rearranging vectors, we get



So, angle between $\vec{A}$ and $\vec{C}$ = 30° + 90° = 120°.

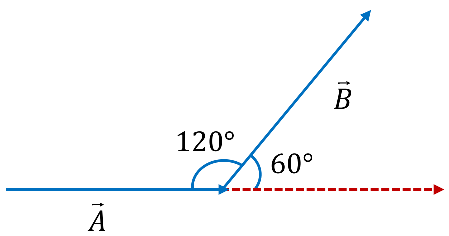
Illustration 35:

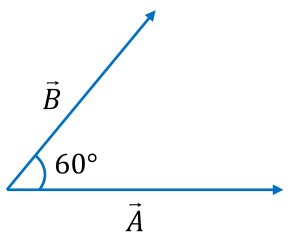
A regular hexagon is shown in figure. Find angle between (a) $\vec{A}$ and $\vec{B}$ (b) $\vec{A}$ and $\vec{C}$.



Solution:

(a)





So, angle between $\vec{A}$ and $\vec{B}$ = 60°.

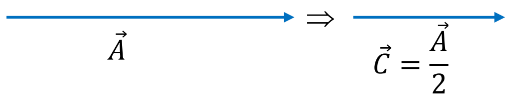
(b) Angle between $\vec{A}$ and $\vec{C}$ = 60° + 60° = 120°.

Multiplication of a Vector by a Scalar:

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $\vec{B}$ (= $\lambda\vec{A}$) whose magnitude is changed by the factor λ but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is λ times.

If, $\vec{B} = 2\vec{A}$

If, $\vec{C} = \frac{\vec{A}}{2}$



If, $\vec{D} = -\vec{A}$

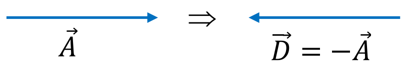


Illustration 36:

A physical quantity (m = 3kg) is multiplied by a vector $\vec{a}$ such that $\vec{F} = m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if

(i) $\vec{a} = 3m/s^2$ East wards (ii) $\vec{a} = -4m/s^2$ North wards

Solution:

(i) $\vec{F} = m\vec{a} = 3 \times 3\ ms^{-2}$ East wards = 9 N East wards

(ii) $\vec{F} = m\vec{a} = 3 \times (-4)\ N$ North wards = – 12N North wards = 12 N South wards.

Addition of Two Vectors:

Vector addition can be performed by using following methods

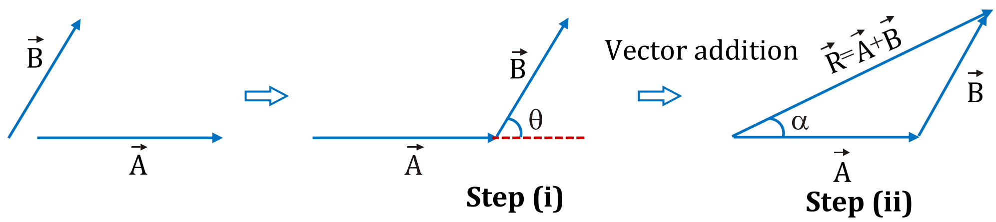
(i) Graphical methods

(ii) Analytical methods

Addition of two vectors is quite different from simple algebraic sum of two numbers.

Triangle Law of Addition of Two Vectors:

If two vectors are represented by two sides of a triangle in same order then their sum or 'resultant vector' is given by the third side of the triangle taken in opposite order of the first two vectors.



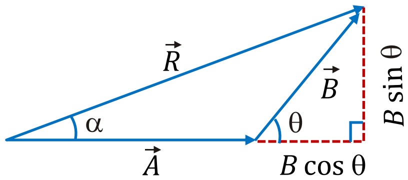
To add two vectors $\vec{A}$ and $\vec{B}$ shift any of the two vectors parallel to itself until the tail of $\vec{B}$ is at the head of $\vec{A}$. The sum $\vec{A} + \vec{B}$ is a vector $\vec{R}$ drawn from the tail of $\vec{A}$ to the head of $\vec{B}$, i.e., $\vec{A} + \vec{B} = \vec{R}$. As the figure formed is a triangle, this method is called 'triangle method' of addition of vectors.

Sum of two vectors is also called resultant vector of these two vectors. Resultant $\vec{R} = \vec{A} + \vec{B}$.

Length of $\vec{R}$ is the magnitude of vector sum i.e. $|\vec{A}+\vec{B}|$.

$\therefore \quad |\vec{R}| = |\vec{A}+\vec{B}| = \sqrt{(A+B\cos\theta)^2+(B\sin\theta)^2}$

$= \sqrt{A^2+B^2+2AB\cos\theta}$

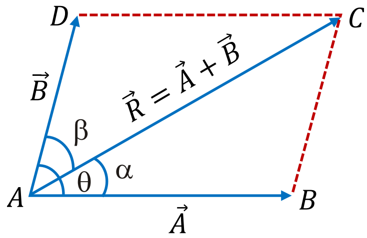


Here θ: angle between $\vec{A}$ and $\vec{B}$.

Let direction of $\vec{R}$ make α angle with $\vec{A}$: $\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta}$

Parallelogram Law of Addition of Two Vectors:

If two vectors are represented by two adjacent sides of a parallelogram which are directed away from their common point then their sum (i.e. resultant vector) is given by the diagonal of the parallelogram passing away through that common point.



$\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC} \Rightarrow \vec{A}+\vec{B}=\vec{R}$

$\Rightarrow \quad R=\sqrt{A^2+B^2+2AB\ \cos\theta}$, $\tan\alpha = \dfrac{B\sin\theta}{A+B\cos\theta}$

and $\quad \tan\beta = \dfrac{A\sin\theta}{B+A\cos\theta}$

Illustration 37:

Two forces of magnitudes $3N$ and $4N$ respectively are acting on a body. Calculate the resultant force if the angle between them is :

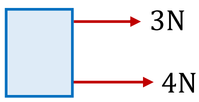
(i) 0° (ii) 180° (iii) 90°

Solution:

(i) θ = 0°, both the forces are parallel, $R = A+B$

Net force or resultant force $R = 3+4 = 7N$

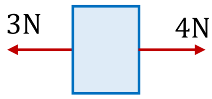
Direction of resultant is along both the forces.



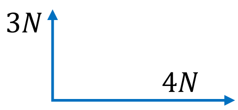
(ii) θ = 180°, both the forces are antiparallel, $R = A-B$

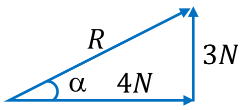
$\therefore$ Net force or resultant force $R = 4-3 = 1N$

Direction of net force is along larger force i.e. along $4N$.



(iii) θ = 90°, both the forces are perpendicular





then $R=\sqrt{A^2+B^2+2AB\cos 90^\circ} = \sqrt{A^2+B^2} = \sqrt{3^2+4^2} = 5N$

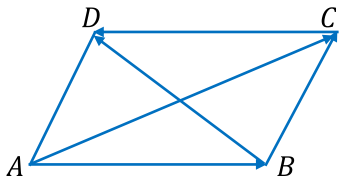
$\tan\alpha = \frac{3}{4} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 37^\circ$

Magnitude of resultant is $5N$ which is acting at an angle of 37° from $4N$ force.

Illustration 38:

Figure shows a parallelogram $ABCD$. Prove that $\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}$.

Solution:



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$ [Applying triangle law of vectors]

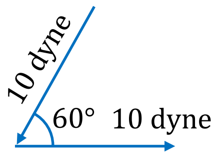
Now $\quad \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AB} + 2\overrightarrow{BC} + \overrightarrow{CD}$

But $\quad \overrightarrow{CD} = -\overrightarrow{AB}$

$\therefore \quad \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AB} + 2\overrightarrow{BC} - \overrightarrow{AB} = 2\overrightarrow{BC}$

Illustration 39:

Two forces each numerically equal to 10 dynes are acting as shown in the figure, then find resultant of these two vectors.



Solution:

The angle θ between the two vectors is 120° and not 60°.

$$R = \sqrt{(10)^2 + (10)^2 + 2(10)(10)(\cos 120°)} = \sqrt{100 + 100 - 100} = 10 \text{ dyne}$$

Important Points:

- Vector addition is commutative $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.
- Vector addition is associative $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$.
- Vectors representing same physical quantity only, can be added.
- Resultant of two vectors will be maximum when they are parallel i.e. angle between them is zero.

$$R_{max} = |\vec{A} + \vec{B}|_{max} = \sqrt{A^2 + B^2 + 2AB\cos 0°} = \sqrt{(A+B)^2} = A + B$$

or $\quad |\vec{A} + \vec{B}|_{max} = |\vec{A}| + |\vec{B}|$

- Resultant of two vectors will be minimum when they are antiparallel i.e. angle between them is 180°.

$$R_{min} = |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos 180°} = \sqrt{(A-B)^2} = |A - B|$$

$\Rightarrow \quad R_{min} = \left||\vec{A}| - |\vec{B}|\right| \Rightarrow |\vec{A} + \vec{B}|_{min} = \left||\vec{A}| - |\vec{B}|\right|$

- The resultant of three non-coplanar vectors can never be zero, or minimum number of non-coplanar vectors whose sum can be zero is four.
- If two vectors have equal magnitude i.e. $|\vec{A}| = |\vec{B}| = a$ and angle between them is θ then resultant will be along the angle bisector of $\vec{A}$ and $\vec{B}$ and its magnitude is equal to $2a\cos\frac{\theta}{2}$.

$$|\vec{R}| = |\vec{A} + \vec{B}| = 2a\cos\frac{\theta}{2}$$

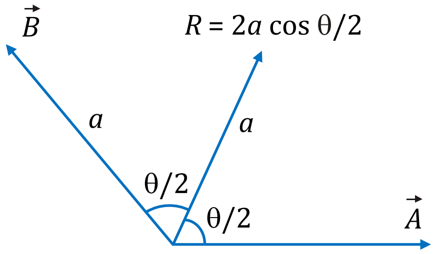
Special Case:

If $\theta = 120°$

then $R = 2a\cos\frac{120°}{2} = a$

i.e., if $\theta = 120°$

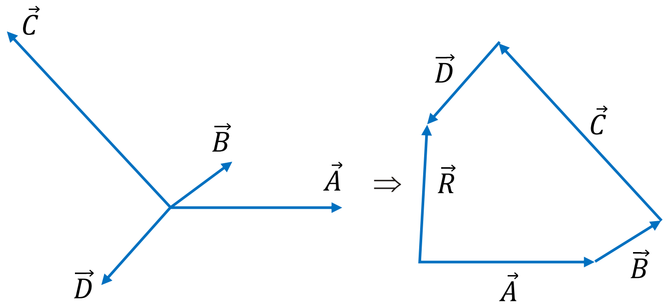
then $|\vec{R}| = |\vec{A}+\vec{B}| = |\vec{A}| = |\vec{B}| = a$



Addition of More Than Two Vectors (Law of Polygon):

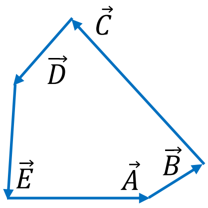
If some vectors are represented by sides of a polygon in same order, then their resultant vector is represented by the closing side of polygon in the opposite order.

$\vec{R} = \vec{A}+\vec{B}+\vec{C}+\vec{D}$



- In a polygon if all the vectors taken in same order are such that the head of the last vector coincides with the tail of the first vector then their resultant is a null vector.

  $\vec{A}+\vec{B}+\vec{C}+\vec{D}+\vec{E} = \vec{0}$



- If n coplanar vectors of equal magnitude are arranged at equal angles of separation then their resultant is always zero.

Subtraction of Two Vectors:

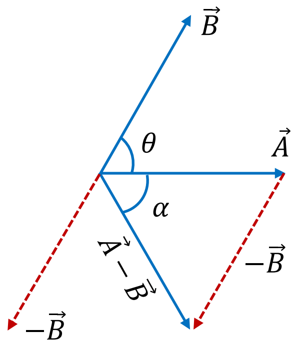
Let $\vec{A}$ and $\vec{B}$ are two vectors. Their difference i.e. $\vec{A}-\vec{B}$ can be treated as sum of the vector $\vec{A}$ and vector $(-\vec{B})$.

$\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$

To subtract $\vec{B}$ from $\vec{A}$, reverse the direction of $\vec{B}$ and add to $\vec{A}$ according to law of triangle.

$$|\vec{A}-\vec{B}| = \sqrt{A^2+B^2+2AB\cos(\pi-\theta)}$$

$$= \sqrt{A^2+B^2-2AB\cos\theta} \ \& \ \tan\alpha = \frac{B\sin\theta}{A-B\cos\theta}$$



Where θ is the angle between $\vec{A}$ and $\vec{B}$.

Important Points:

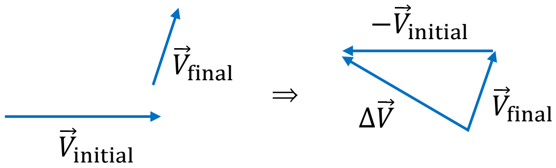
- Vector subtraction does not follow commutative law i.e. $\vec{A}-\vec{B} \neq \vec{B}-\vec{A}$.
- Vector subtraction does not follow associative law i.e. $(\vec{A}-\vec{B})-\vec{C} \neq \vec{A}-(\vec{B}-\vec{C})$.
- If two vectors have equal magnitude, i.e. $|\vec{A}|=|\vec{B}|=a$ and θ is the angle between them, then

$$|\vec{A}-\vec{B}|=\sqrt{a^2+a^2-2a^2\cos\theta}=2a\sin\frac{\theta}{2}$$

Special case:

If $\theta=60°$ then $2a\sin\frac{\theta}{2}=a$ i.e., $|\vec{A}-\vec{B}|=|\vec{A}|=|\vec{B}|=a$ at θ = 60°

Change in a vector physical quantity means subtraction of initial vector from the final vector.



- The parallelogram formed by two vectors $\vec{A}$ and $\vec{B}$ will have two diagonals, one diagonal represents $\vec{A}+\vec{B}$ and another diagonal represents $\vec{A}-\vec{B}$.

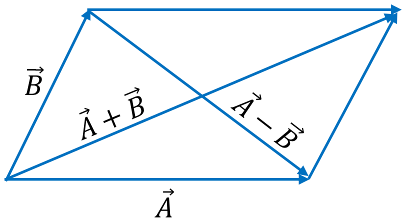


Illustration 40:

The magnitude of pairs of displacement vectors are given. Which pairs of displacement vectors cannot be added to give a resultant vector of magnitude 13 cm?

(A) 4 cm, 16 cm (B) 20 cm, 7 cm (C) 1 cm, 15 cm (D) 6 cm, 8 cm

Ans. (C)

Solution:

Resultant of two vectors $\vec{A}$ and $\vec{B}$ must satisfy $|A-B| \leq R \leq A+B$.

Illustration 41:

If $|\hat{a}+\hat{b}|=|\hat{a}-\hat{b}|$, then find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

Let angle between $\vec{a}$ and $\vec{b}$ be θ

$\because$ $|\hat{a}+\hat{b}|=|\hat{a}-\hat{b}|$

$\therefore$ $\sqrt{1^2+1^2+2(1)(1)\cos\theta}=\sqrt{1^2+1^2-2(1)(1)\cos\theta}$

$\Rightarrow$ $2+2\cos\theta=2-2\cos\theta$

$\Rightarrow$ $\cos\theta=0$

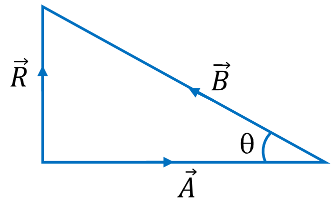
$\therefore$ $\theta=90°$

Illustration 42:

The resultant of two velocity vectors $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$. Magnitude of Resultant is $\vec{R}$ equal to half magnitude of $\vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$.

Solution:

Since $\vec{R}$ is perpendicular to $\vec{A}$. Figure shows the three vectors $\vec{A}, \vec{B}$ and $\vec{R}$ angle between $\vec{A}$ and $\vec{B}$ is $\pi - \theta$



$$\sin\theta = \frac{R}{B} = \frac{B}{2B} = \frac{1}{2}$$

$\Rightarrow$ $\theta = 30°$

$\Rightarrow$ Angle between A and B is 150°.

Illustration 43:

If the sum of two unit vectors is also a unit vector. Find the magnitude of their difference.

Solution:

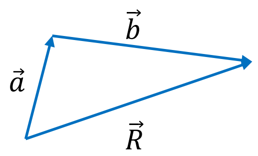
Let $\hat{A}$ and $\hat{B}$ are the given unit vectors and $\hat{R}$ is their resultant then $|\hat{R}| = |\hat{A} + \hat{B}|$.

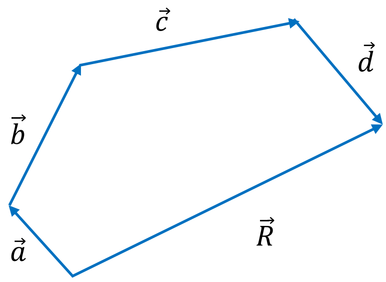
$$1 = \sqrt{|\hat{A}|^2 + |\hat{B}|^2 + 2|\hat{A}||\hat{B}|\cos\theta}$$

$$1 = 1 + 1 + 2\cos\theta \qquad \Rightarrow \quad \cos\theta = -\frac{1}{2}$$

$$|\vec{A} - \vec{B}| = \sqrt{|\hat{A}|^2 + |\hat{B}|^2 - 2|\hat{A}||\hat{B}|\cos\theta} = \sqrt{1 + 1 - 2 \times 1 \times 1\left(-\frac{1}{2}\right)} = \sqrt{3}$$

Components of Vector:



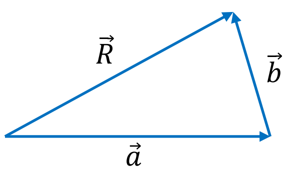


$\vec{R} = \vec{a} + \vec{b}$ $\qquad$ $\vec{R} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$

$\vec{a}$ and $\vec{b}$ are components of $\vec{R}$. $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are components of $\vec{R}$.

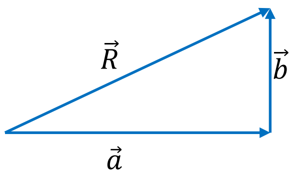
- A vector can be expressed as a vector sum of two or more than two vectors, then these vectors are known as components of vectors.
- Any vector can be expressed in two way:
  - (i) Non-rectangular components
  - (ii) Rectangular components

  $\vec{R} = \vec{a} + \vec{b}$



($\vec{a}$ and $\vec{b}$ are non-rectangular components.)

and $\vec{R} = \vec{a} + \vec{b}$

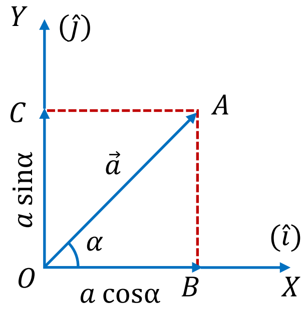


($\vec{a}$ and $\vec{b}$ are rectangular components.)

If the vectors are perpendicular to each other or the angle between the component vector is 90º, then it is known rectangular.

Resolution of Vectors into Rectangular Components in Two Dimensions:

When a vector is splitted into components which are at right angles to each other, then the components are called rectangular or orthogonal components of that vector.



(i) Let vector $\vec{a} = \overrightarrow{OA}$ in $X - Y$ plane, makes angle α from X-axis. Draw perpendiculars AB and AC from A on the X-axis and Y-axis respectively.

(ii) The length OB is called projection of $\overrightarrow{OA}$ on X-axis or component of $\overrightarrow{OA}$ along X-axis and is represented by a_x. Similarly, OC is the projection of $\overrightarrow{OA}$ on Y-axis and is represented by a_y.

According to law of vector addition.

$$\vec{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$$

Thus $\vec{a}$ has been resolved into two parts, one along OX and the other along OY, which are mutually perpendicular.

In $\Delta OAB, \frac{OB}{OA} = \cos\alpha \Rightarrow OB = OA\cos\alpha \Rightarrow a_x = a\cos\alpha$

and $\frac{AB}{OA} = \sin\alpha \Rightarrow AB = OA\sin\alpha = OC \Rightarrow a_y = a\sin\alpha$

$\therefore$ $a_x = a\cos\alpha$ and $a_y = a\sin\alpha$

If $\hat{\imath}$ and $\hat{\jmath}$ denote unit vectors along OX and OY respectively then $\overrightarrow{OB} = a\cos\alpha\hat{\imath}$ and $\overrightarrow{OC} = a\sin\alpha\hat{\jmath}$.

So according to rule of vector addition $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC} \Rightarrow \vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} = a\cos\alpha\,\hat{\imath} + a\sin\alpha\,\hat{\jmath}$

So $|\vec{a}| = \sqrt{a_x^2 + a_y^2}$ and $\tan\alpha = \frac{a_y}{a_x}$.

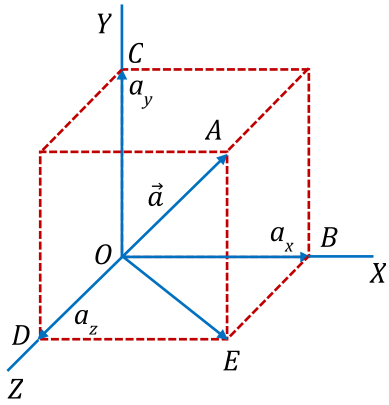
Rectangular Components of a Vector in Three Dimensions:

(i) Consider a vector $\vec{a}$ represented by $\overrightarrow{OA}$, as shown in figure. Consider O as origin and draw a rectangular parallelopiped with its three edges along the X, Y and Z axes.

(ii) Vector $\vec{a}$ is the diagonal of the parallelopiped; its projections on X, Y and Z axes are $\vec{a}_x, \vec{a}_y$ and $\vec{a}_z$ respectively. These are the three rectangular components of $\vec{A}$.

Using triangle law of vector addition $\overrightarrow{OA} = \overrightarrow{OE} + \overrightarrow{EA}$

Using parallelogram law of vector addition $\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{OD}$



$\therefore \quad \overrightarrow{OA} = \left(\overrightarrow{OB} + \overrightarrow{OD}\right) + \overrightarrow{EA}$

$\because \quad \overrightarrow{EA} = \overrightarrow{OC}$

$\therefore \quad \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OD} + \overrightarrow{OC}$

Now $\quad \overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = a_x\hat{\imath}, \overrightarrow{OC} = a_y\hat{\jmath}$ and $\overrightarrow{OD} = a_z\hat{k}$

$\therefore \quad \vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$

Also $\quad (OA)^2 = (OE)^2 + (EA)^2$

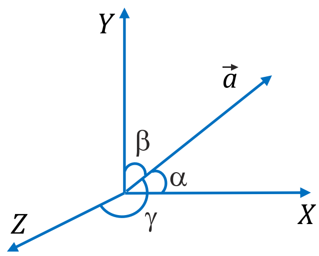
But $\quad (OE)^2 = (OB)^2 + (OD)^2$ and $EA = OC$

$(OA)^2 = (OB)^2 + (OD)^2 + (OC)^2$

or $\quad a^2 = {a_x}^2 + {a_y}^2 + {a_z}^2 \Rightarrow a = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$

Direction Cosines:

Let $\vec{a}$ makes angle : α with X axis, β with Y axis and γ with Z axis



- $\cos\alpha = \dfrac{a_x}{a} \quad \Rightarrow a_x = a\cos\alpha$
- $\cos\beta = \dfrac{a_y}{a} \quad \Rightarrow a_y = a\cos\beta$
- $\cos\gamma = \dfrac{a_z}{a} \quad \Rightarrow a_z = a\cos\gamma$

cos α, cos β and cos γ are **direction cosines** of the vector.

Putting the value of a_x, a_y and a_z in eq. (i) we get $a^2 = a^2\cos^2\alpha + a^2\cos^2\beta + a^2\cos^2\gamma$

$\Rightarrow \quad \boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1} \quad \Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

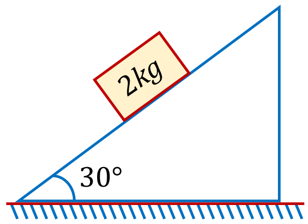
$\Rightarrow \quad 3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1 \quad \Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

- A vector can be resolved into infinite number of components.

  For example: $10\hat{\imath} = \hat{\imath} + \hat{\imath} + \hat{\imath} \ldots 10$ times; or $\frac{\hat{\imath}}{2} + \frac{\hat{\imath}}{2} + \frac{\hat{\imath}}{2} \ldots 20$ times and so on.
- Maximum number of rectangular components of a vector in a plane is two. But maximum number of rectangular components in space (3-dimensions) is three which are along X, Y and Z axes.
- A vector is independent of the orientation of axes but the components of that vector depend upon the orientation of axes.
- The component of a vector along its perpendicular direction is always zero.

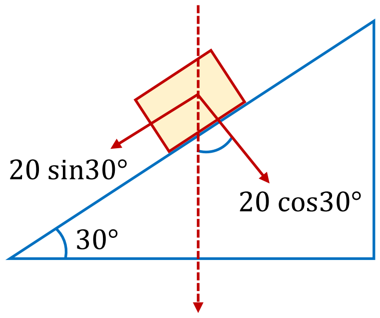
Illustration 44:

A mass of 2 kg lies on an inclined plane as shown in figure. Resolve its weight along and perpendicular to the plane. (Assume $g = 10 m/s^2$)



Solution:

Component along the plane

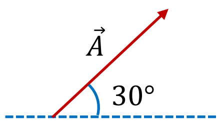


$= 20\ sin\ 30 = 10N$

Component perpendicular to the plane = $20 \cos 30 = 10\sqrt{3}N$.

Illustration 45:

A vector makes an angle of 30° with the horizontal. If horizontal component of the vector is 250. Find magnitude of vector and its vertical component.



Solution:

Let vector is $\vec{A}$

$$A_x = A\cos 30° = 250 = \frac{A\sqrt{3}}{2} \Rightarrow A = \frac{500}{\sqrt{3}}$$

$$A_y = A\sin 30° = \frac{500}{\sqrt{3}} \times \frac{1}{2} = \frac{250}{\sqrt{3}}$$

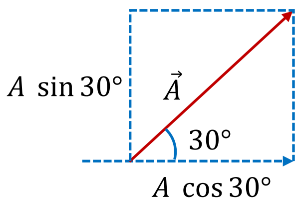


Illustration 46:

Find the angle made by $(\hat{\imath} + \hat{\jmath})$ vector from X and Y axes respectively.

Solution:

$$a = \sqrt{{a_x}^2 + {a_y}^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\cos\alpha = \frac{a_x}{a} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \alpha = 45°$ and $\cos\beta = \frac{a_y}{a} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \beta = 45°$

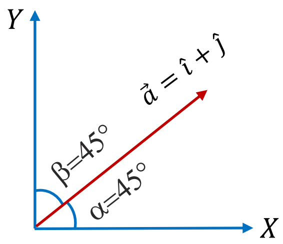


Illustration 47:

Find out the angle made by $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ vector from X, Y and Z axes respectively.

Solution:

Given $A_x = A_y = A_z = 1$

So $A = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2} = \sqrt{1+1+1} = \sqrt{3}$

$\cos\alpha = \frac{A_x}{A} = \frac{1}{\sqrt{3}}$

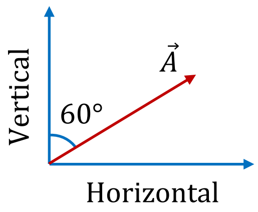
$\Rightarrow \quad \alpha = \cos^{-1}\frac{1}{\sqrt{3}}$; $\cos\beta = \frac{A_y}{A} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \beta = \cos^{-1}\frac{1}{\sqrt{3}}$; $\cos\gamma = \frac{A_z}{A} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \gamma = \cos^{-1}\frac{1}{\sqrt{3}}$

Illustration 48:

A force of $4N$ is applied at an angle of 60° from the vertical. Find out its components along horizontal and vertical directions.



Solution:

Vertical component = 4 cos 60° = $2N$

Horizontal component = 4 sin 60° = $2\sqrt{3}N$.

Addition & Subtraction in Component Form:

Suppose there are two vectors in component form. Then the addition and subtraction between these two are

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\vec{A} \pm \vec{B} = (A_x \pm B_x)\hat{i} + (A_y \pm B_y)\hat{j} + (A_z \pm B_z)\hat{k}$$

Also, if we are having a third vector present in component form and this vector is added or subtracted from the addition or subtraction of above two vectors then

$$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

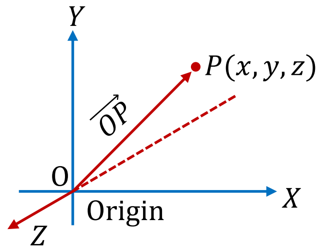
$$\vec{A} \pm \vec{B} \pm \vec{C} = (A_x \pm B_x \pm C_x)\hat{i} + (A_y \pm B_y \pm C_y)\hat{j} + (A_z \pm B_z \pm C_z)\hat{k}$$

Note:

Modulus of vector A is given by -

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

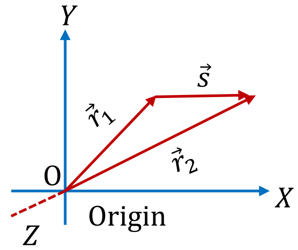
Position vector (P.V.) of a point:



Let a point O in space (or plane) be taken as origin. If P is any point then $\overrightarrow{OP}$ is called the position vector (P.V.) of the point P with respect to O.
If the position coordinates of P are (x, y, z) then the position vector $\overrightarrow{OP}$ is given by

$$\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Displacement vector:



Change in position vector is known as displacement vector.

If $\quad \vec{r}_i = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_f = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

Then displacement vector $(\vec{s}) = \vec{r}_f - \vec{r}_i$

$$= (x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}) - (x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})$$

$$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

Illustration 49:

Determine that vector which when added to the resultant of $\vec{P} = 2\hat{\imath} + 7\hat{\jmath} - 10\hat{k}$ and $\vec{Q} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ gives a unit vector along X-axis.

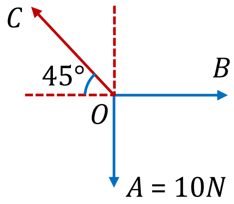
Solution:

Resultant = $(2\hat{\imath} + 7\hat{\jmath} - 10\hat{k}) + (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) = 3\hat{\imath} + 9\hat{\jmath} - 7\hat{k}$

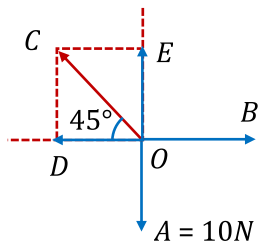
But $\vec{R}$+ required vector = $\hat{\imath}$, so required vector = $\hat{\imath} - \vec{R} = \hat{\imath} - (3\hat{\imath} + 9\hat{\jmath} - 7\hat{k}) = -2\hat{\imath} - 9\hat{\jmath} + 7\hat{k}$

Illustration 50:

The sum of three vectors shown in figure, is zero. What is the magnitude of vector $\overrightarrow{OB}$ & $\overrightarrow{OC}$?



Solution:



Resolve $\overrightarrow{OC}$ into two rectangular components.

$$OD = OC\cos 45° \text{ and } OE = OC\sin 45°$$

For zero resultant $OE = OA$ or $OC\sin 45° = 10N$

$\Rightarrow \quad OC \times \frac{1}{\sqrt{2}} = 10N \qquad \Rightarrow \quad |\overrightarrow{OC}| = 10\sqrt{2}N$

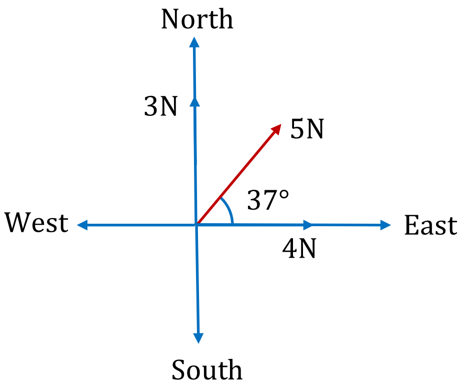
and $\quad OD = OB$

$\Rightarrow \quad OC\cos 45° = OB \qquad \Rightarrow \quad 10\sqrt{2} \times \frac{1}{\sqrt{2}} = OB$

$\Rightarrow \quad OB = 10N$

Illustration 51:

For shown situation, what will be the magnitude of minimum force in newton that can be applied in any direction so that the resultant force is along east direction?



Solution:

Let force be F so resultant is in east direction

$\Rightarrow \quad 4\hat{i} + 3\hat{j} + (5\cos 37°\hat{i} + 5\sin 37°\hat{j}) + \vec{F} = k\hat{i}$

$\Rightarrow \quad 4\hat{i} + 3\hat{j} + 4\hat{i} + 3\hat{j} + \vec{F} = k\hat{i}$

$\Rightarrow \quad 8\hat{i} + 6\hat{j} + \vec{F} = k\hat{i}$

$\Rightarrow \quad \vec{F} = (k-8)\hat{i} - 6\hat{j} \Rightarrow F = \sqrt{(k-8)^2 + (6)^2}$

$\Rightarrow \quad F_{\text{min}} = 6N$

OR

F_{min} = y-components of existing forces = $3N + 5N \sin 37° = 3N + 5N\left(\frac{3}{5}\right) = 6N$

Illustration 52:

Person A is at $(2\hat{i} + 3\hat{j} - 4\hat{k})$ and moving with velocity $(\hat{i} - \hat{j} + \hat{k})$. Person B is at $(-2\hat{i} + 4\hat{j} - \hat{k})$, with what velocity should B move to meet A after 3 *sec*?

Solution:

Final position of A = Final position of B

$$2\hat{i} + 3\hat{j} - 4\hat{k} + 3(\hat{i} - \hat{j} + \hat{k}) = -2\hat{i} + 4\hat{j} - \hat{k} + 3\vec{v}$$

$$\therefore \quad \frac{7\hat{i} - 4\hat{j}}{3} = \vec{v}$$

Multiplication of Vectors:

Scalar Product:

The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \cdot \vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the product of their magnitude with cosine of angle between them. Thus, $\vec{A} \cdot \vec{B} = AB\cos\theta$ (here θ is the angle between the vectors).

Properties:

It is always a scalar which is positive if angle between the vectors is acute (i.e. < 90°) and negative if angle between them is obtuse (i.e. $90° < \theta \le 180°$)

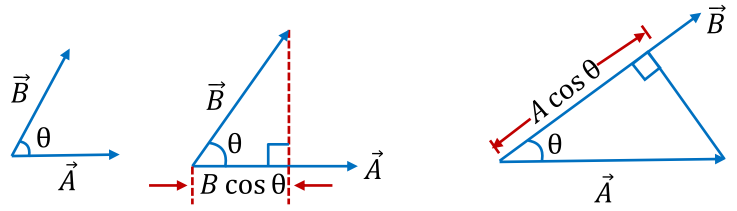
It is commutative, i.e., $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

It is distributive, i.e. $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

As by definition $\vec{A}\cdot\vec{B} = AB\cos\theta$. The angle between the vectors $\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{AB}\right]$

$$\vec{A}\cdot\vec{B} = A(B\cos\theta) = B(A\cos\theta)$$

Geometrically, $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$ and $A\cos\theta$ is the projection of $\vec{A}$ onto $\vec{B}$ as shown. So $\vec{A}.\vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$ and vice versa.



- Scalar product of two vectors will be maximum when cos θ = max = 1, i.e., θ = 0°, i.e., vectors are parallel

  $\Rightarrow \quad (\vec{A}.\vec{B})_{max} = AB$
- If the scalar product of two nonzero vectors is zero then the vectors are perpendicular.
- The scalar product of a vector by itself is termed as self dot product and is given by

  $$(\vec{A})^2 = \vec{A}.\vec{A} = AA\cos\theta = AA\cos 0° = A^2 \quad \Rightarrow \quad A = \sqrt{\vec{A}\cdot\vec{A}}$$
- In case of unit vector $\hat{n}$,

  $$\hat{n}\cdot\hat{n} = 1\times 1\times\cos 0° = 1 \qquad \Rightarrow \quad \hat{n}\cdot\hat{n} = \hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$$

  In case of orthogonal unit vectors $\hat{\imath},\hat{\jmath}$ and $\hat{k}$; $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$

  In general, $\vec{A}\cdot\vec{B} = (A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})\cdot(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}) = [A_xB_x + A_yB_y + A_zB_z]$

Illustration 53:

If the vectors $\vec{P} = a\hat{\imath}+a\hat{\jmath}+3\hat{k}$ and $\vec{Q} = a\hat{\imath}-2\hat{\jmath}-\hat{k}$ are perpendicular to each other. Find the value(s) of a.

Solution:

If vectors $\vec{P}$ and $\vec{Q}$ are perpendicular.

$\Rightarrow \quad \vec{P}\cdot\vec{Q} = 0 \quad \Rightarrow \quad (a\hat{\imath}+a\hat{\jmath}+3\hat{k}).(a\hat{\imath}-2\hat{\jmath}-\hat{k}) = 0 \quad \Rightarrow \quad a^2 - 2a - 3 = 0$

$\Rightarrow \quad a^2 - 3a + a - 3 = 0 \quad \Rightarrow \quad a(a-3) + 1(a-3) = 0 \quad \Rightarrow \quad a = -1, 3$

Illustration 54:

Find angle between $\vec{A} = 3\hat{\imath}+4\hat{\jmath}$ and $\vec{B} = 12\hat{\imath}+5\hat{\jmath}$.

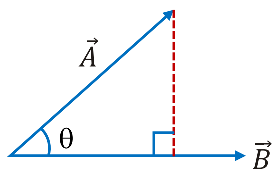
Solution:

We have

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(3\hat{\imath}+4\hat{\jmath})\cdot(12\hat{\imath}+5\hat{\jmath})}{\sqrt{3^2+4^2}\sqrt{12^2+5^2}}$$

$$\cos\theta = \frac{36+20}{5\times 13} = \frac{56}{65} \quad \theta = \cos^{-1}\frac{56}{65}$$

Projection of $\vec{A}$ on $\vec{B}$



(i) In scalar form : Projection of $\vec{A}$ on $\vec{B} = A\cos\theta = A\left(\frac{\vec{A}.\vec{B}}{AB}\right) = \frac{\vec{A}.\vec{B}}{B} = \vec{A}.\hat{B}$

(ii) In vector form : Projection of $\vec{A}$ on $\vec{B} = (A\cos\theta)\hat{B} = \left(\frac{\vec{A}.\vec{B}}{B}\right)\hat{B} = (\vec{A}.\hat{B})\hat{B}$

Illustration 55:

Find the component of $3\hat{\imath} + 4\hat{\jmath}$ along $\hat{\imath} + \hat{\jmath}$.

Solution:

Component of $\vec{A}$ along $\vec{B}$ is given by $\frac{\vec{A}.\vec{B}}{B}$, hence required component

$$= \frac{(3\hat{\imath} + 4\hat{\jmath}) \cdot (\hat{\imath} + \hat{\jmath})}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

In vector form = $\frac{7}{\sqrt{2}}\left(\frac{\hat{\imath} + \hat{\jmath}}{\sqrt{2}}\right) = \frac{7}{2}(\hat{\imath} + \hat{\jmath})$

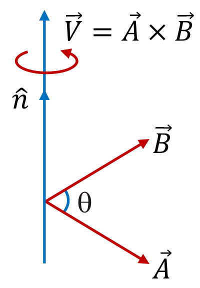
Vector Product:

The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \times \vec{B}$ (read $\vec{A}$ cross $\vec{B}$) is defined as: $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$.

Here θ is the angle between the vectors and the direction $\hat{n}$ is given by the right-hand-thumb rule.

Right-Hand-Thumb Rule:

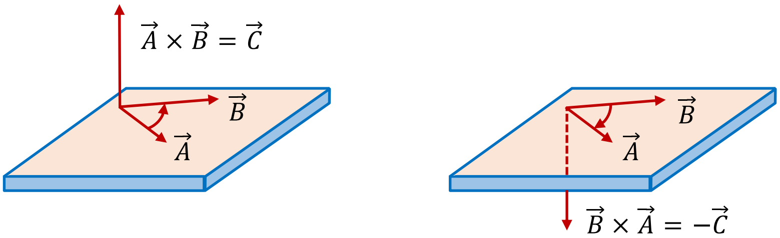
To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding. Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed they go towards $\vec{B}$.



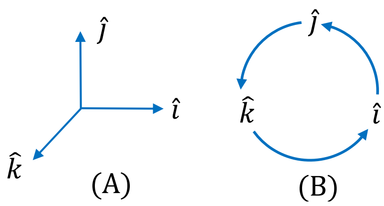
The direction of the thumb gives the direction of $\hat{n}$.

Properties:

- Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors i.e. orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.
- Vector product of two vectors is not commutative i.e. $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$.
  But $|\vec{A} \times \vec{B}| = |\vec{B} \times \vec{A}| = AB \sin\theta$



- The vector product is distributive when the order of the vectors is strictly maintained i.e.
  $A \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.
- The magnitude of vector product of two vectors will be maximum when $\sin\theta_{max} = 1$, i.e. $\theta = 90°$.
  $|\vec{A} \times \vec{B}|_{max} = AB$ i.e., magnitude of vector product is maximum if the vectors are orthogonal.
- The magnitude of vector product of two non–zero vectors will be minimum when $|\sin\theta|$ = minimum = 0, i.e. $\theta = 0°$ or $180°$ and $|\vec{A} \times \vec{B}|_{min} = 0$ i.e. if the vector product of two non–zero vectors vanishes, the vectors are collinear.
- The self cross product i.e. cross product of a vector by itself vanishes i.e. is a null vector.
  $\vec{A} \times \vec{A} = AA \sin 0°\hat{n} = \vec{0}$.
  In case of unit vector $\hat{n}, \hat{n} \times \hat{n} = \vec{0} \Rightarrow \hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$
- In case of orthogonal unit vectors $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ in accordance with right-hand-thumb-rule,
  $\hat{\imath} \times \hat{\jmath} = \hat{k} \quad \hat{\jmath} \times \hat{k} = \hat{\imath} \quad \hat{k} \times \hat{\imath} = \hat{\jmath}$



- In terms of components,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\imath}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{\jmath}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

Examples of Vector Product:

(i) Torque : $\vec{\tau} = \vec{r} \times \vec{F}$

(ii) Angular momentum : $\vec{J} = \vec{r} \times \vec{p}$

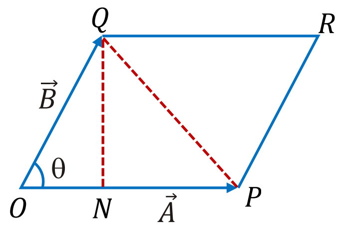
(iii) Velocity : $\vec{v} = \vec{\omega} \times \vec{r}$

(iv) Tangential Acceleration : $\vec{a} = \vec{\alpha} \times \vec{r}$

Geometrical Meaning of Vector Product of Two Vectors:

(i) Consider two vectors $\vec{A}$ and $\vec{B}$ which are represented by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ and $\angle POQ = \theta$

(ii) Complete the parallelogram $OPRQ$. Join P with Q. Here $OP = A$ and $OQ = B$. Draw $QN \perp OP$.



(iii) Magnitude of cross product of $\vec{A}$ and $\vec{B}$

$$|\vec{A}\times\vec{B}| = AB\sin\theta = (OP)\,(OQ\sin\theta) = (OP)\,(NQ) \qquad (\because NQ = OQ\ \sin\theta)$$

$$= \text{Base} \times \text{height} = \text{Area of parallelogram } OPRQ$$

Area of $\Delta POQ = \dfrac{\text{Base}\times\ \text{height}}{2} = \dfrac{(OP)\ (NQ)}{2} = \dfrac{1}{2}|\vec{A}\times\vec{B}|$

$\therefore$ Area of parallelogram $OPRQ$ = 2[area of ΔOPQ] $= |\vec{A}\times\vec{B}|$

Formulae to Find Area:

If $\vec{A}$ and $\vec{B}$ are two adjacent sides of a triangle, then its area $= \frac{1}{2}|\vec{A}\times\vec{B}|$.

If $\vec{A}$ and $\vec{B}$ are two adjacent sides of a parallelogram then its area $= |\vec{A}\times\vec{B}|$.

If $\vec{A}$ and $\vec{B}$ are diagonals of a parallelogram then its area $= \frac{1}{2}|\vec{A}\times\vec{B}|$.

Illustration 56:

$\vec{A}$ is Eastwards and $\vec{B}$ is downwards. Find the direction of $\vec{A}\times\vec{B}$.

Solution:

Applying right hand thumb rule we find that $\vec{A}\times\vec{B}$ is along North.

Illustration 57:

If $\vec{A}\cdot\vec{B} = |\vec{A}\times\vec{B}|$, find angle between $\vec{A}$ and $\vec{B}$.

Solution:

$\vec{A}\cdot\vec{B} = |\vec{A}\times\vec{B}| \Rightarrow AB\cos\theta = AB\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45°$

Illustration 58:

Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ. Find a unit vector which is perpendicular to both $\vec{A}$ and $\vec{B}$.

Solution:

$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$

$\Rightarrow \hat{n} = \dfrac{\vec{A}\times\vec{B}}{AB\sin\theta}$ here $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

$(-\hat{n}) = -\dfrac{\vec{A}\times\vec{B}}{AB\sin\theta} = \dfrac{\vec{B}\times\vec{A}}{AB\sin\theta}$ is also perpendicular to both $\vec{A}$ and $\vec{B}$.

Illustration 59:

Find $\vec{A} \times \vec{B}$, if $\vec{A} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$ and $\vec{B} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

Solution:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 4 \\ 2 & -1 & 2 \end{vmatrix} = \hat{\imath}\,[-4-(-4)] - \hat{\jmath}\,(2-8) + \hat{k}\,[-1-(-4)] = 6\hat{\jmath} + 3\hat{k}.$$

To check co-planarity:

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = 0.$

Illustration 60:

Prove that the following vectors are coplanar.

$$\vec{A} = 2\hat{\imath} - \hat{\jmath} + \hat{k}, \vec{B} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k} \text{ and } \vec{C} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$$

Solution:

Given, $\vec{A} = 2\hat{\imath} - \hat{\jmath} + \hat{k}, \vec{B} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $\vec{C} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$

For three vectors to be coplanar.

$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

So, $(2\hat{\imath} - \hat{\jmath} + \hat{k}) \cdot [(i - 3\hat{\jmath} - 5\hat{k}) \times (3\hat{\imath} - 4\hat{\jmath} - 4\hat{k})]$

$\Rightarrow$ $(2\hat{\imath} - \hat{\jmath} + \hat{k}) \cdot [-8\hat{\imath} - 11\hat{\jmath} + 5\hat{k}]$

$\Rightarrow$ $-16 + 11 + 5 = -16 + 16 = 0$

So, $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar.