CHAPTER 2

Atomic Physics

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Structure of an atom
(b) Thompson's Atom Model
(c) Rutherford's Atom Model
(d) Bohr's Model along with their postulates and drawbacks
(e) The Bohr's Theory applied on Hydrogen like atoms for understanding the concepts of energy and various transitions of an atom
(f) Hypothetical Bohr's Model
(g) X-rays and applications.

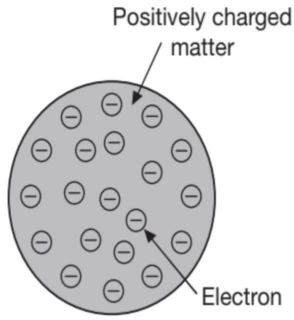
All this is followed by an Exercise Set (fully solved) which contains questions as per the latest JEE pattern. At the end of Exercise Set, a collection of problems asked previously in JEE Main are also given.

STRUCTURE OF AN ATOM: AN INTRODUCTION

All matter is made up of tiny particles known as atoms. There are about 105 different kinds of atoms, and they combine with each other in different ways to form groups called molecules. All matter has been found to be composed of atoms or molecules, and the basic knowledge of atoms and their constitution gives us valuable information about the behaviour of matter.

THOMSON EMPIRICAL MODEL/ THOMSON PLUM PUDDING MODEL

J.J. Thomson gave the, first idea regarding structure of atom. The model is known after him as Thomson's atom model. According to this, entire positive charge is distributed uniformly in the form of a sphere. Negatively charged electrons are arranged within this sphere lying here and there. The model is popularly known as plum-pudding model. Every electron is attracted towards the centre of uniformly charged sphere while they exert a force of repulsion upon each other. The electrons get themselves arranged in such a way that the force of attraction and that of repulsion balance each other. When disturbed, electrons vibrate to and fro within the atom and cause emission of visible, infra-red and ultra-violet light.



Thomson's atom model

Thomson's atom model satisfied the requirements of the atom and the demands of electro-magnetic theory. According to this model, hydrogen can give rise to a single spectral line. Experimentally, hydrogen is found to give several series, each series consisting of several lines. This indicated that Thomson's atom model needed modifications which was modified by Rutherford.

RUTHERFORD'S ATOMIC MODEL

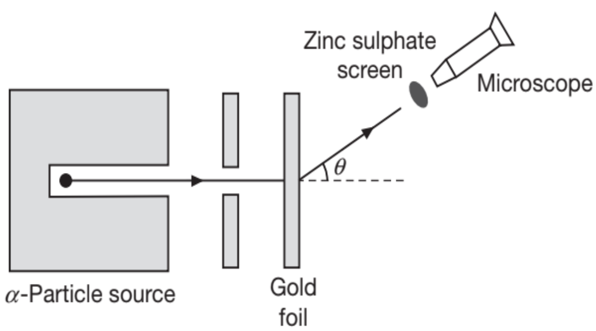
The correct description of the distribution of positive and negative charges within an atom was made in 1911 by a New Zealander while working at Manchester University in England. This was Ernest Rutherford, who was later called as Lord Rutherford for his many scientific achievements. He entered into physics during that crucial period of its development when the phenomenon of natural radioactivity had just been discovered, and he was first to realize that radioactivity represents a spontaneous disintegration of heavy unstable atoms.

Rutherford realized that important information about the inner structure of atoms can be obtained by the study of collisions between the rushing α particles incident on the atoms of various materials that form the target on which the α particle beam is incident.

EXPERIMENTAL ARRANGEMENT

The basic idea of the experimental arrangement used by Rutherford in his studies was explained as follows:

A piece or speck of α-emitting radioactive material is placed in a lead shield with a hole that allows a narrow beam of the α-particles to pass through it. In front of this arrangement is placed a gold thin metal foil to deflect or scatter the α particles. After passing through the gold foil the deflected particles are incident on a pivoted fluorescent screen with a magnifier through which the tiny flashes of light were observed whenever an α-particle struck the screen. The whole apparatus placed in an evacuated chamber, so that the particles would not collide with air molecules.



OBSERVATIONS

Rutherford performed experiments on the scattering of alpha particles by extremely thin metals foils and made the following observations.

(a) Most of the α-particles, either passed straight through the metal foil or suffered only small deflections. This could not be explained by Thomson's atom model.

(b) A few particles were deflected through angles which were less than or equal to 90°.

(c) Very few particles were deflected through angles greater than 90°. It was observed that about only 1 in 8000 particles was found to be deflected greater than 90°. Sometimes a particle was found to be deflected through 180°. In other words, it was sent back in the same direction from where it came. The large angle of scattering came as a greater surprise. It could not be explained by Thomson's atom model. It was one of the main reasons for rejecting Thomson's atom model.

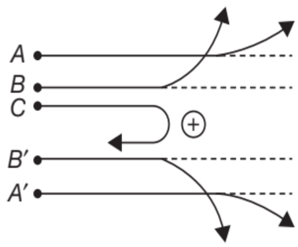
CONCLUSIONS

(a) The fact that most of the α-particles passed undeviated led to the conclusion that an atom has a lot of empty space in it.

(b) α-particles are heavy particles having high initial speeds. These could be deflected through large angles only by a strong electrical force. This led Rutherford to the conclusion that entire positive charge and nearly the entire mass of the atom were concentrated in a tiny central core. Rutherford named this core as Nucleus.

(c) The difference in deflection of various particles can be explained as follows:

α-particles which pass at greater distances away from the nucleus, shown as A and A' in figure, suffer a small deflection due to smaller repulsion exerted by the nucleus upon them. The particles like B and B' which pass close to the nucleus experience a comparatively greater force and hence get deflected through greater angles. A particle C which travels directly towards the nucleus is first slowed down by the repulsive force. Such a particle finally stops and then, is repelled along the direction of its approach. Thus, it gets repelled back after suffering a deviation of 180°.



Different deviations for different α particles

Also, during the experiment following conclusions were made

(i) If ϕ is the angle made by a scattered particle with its original direction of motion and N is the number of particles available in that direction, it was found that,

$$\frac{1}{\sin^4\left(\frac{\phi}{2}\right)} \propto N$$

(ii) If t is the thickness of the foil and N is the number of α-particles scattered in a particular direction $(\phi = \text{constant})$ it was observed that

$$\frac{N}{t} = \text{constant}$$

$$\Rightarrow \quad \frac{N_1}{N_2} = \frac{t_1}{t_2}$$

ILLUSTRATION 1

In Rutherford's scattering experiment, if the number of α particles scattered at an angle of $90°$ is 55, then calculate the number of α particles scattered at an angle of $60°$.

SOLUTION

Since we know that

$$N \propto \frac{1}{\sin^4\left(\frac{\phi}{2}\right)}$$

$$\Rightarrow \quad \frac{N_{60°}}{N_{90°}} = \frac{\sin^4\left(\frac{90°}{2}\right)}{\sin^4\left(\frac{60°}{2}\right)} = \frac{\sin^4(45°)}{\sin^4(30°)} = \frac{\left(\frac{1}{\sqrt{2}}\right)^4}{\left(\frac{1}{2}\right)^4} = 4$$

$$\Rightarrow \quad \frac{N_{60°}}{N_{90°}} = \frac{N_{60°}}{55} = 4$$

$$\Rightarrow \quad N_{60°} = 4(55) = 220$$

RUTHERFORD'S ATOM-MODEL POSTULATES

On the basis of the conclusions drawn from Rutherford's experiment, a new atom model was proposed. This atom model, known as Rutherford's atom model, had the following characteristics.

(a) An atom consists of equal amounts of positive and negative charge so, the atom, as a whole is electrically neutral.

(b) The entire positive charge of the atom and practically its entire mass is concentrated in a small region which forms the core of the atom, called the nucleus.

(c) The negative charge, which is contained in the atom in the form of electrons, is distributed all around the nucleus, but separated from it.

(d) In order to explain the stability of electron at a certain distance from the nucleus, it was proposed by Rutherford that the electrons revolve round the nucleus in circular orbits. The electrostatic force of attraction between the nucleus and the electron provides the centripetal force to the electron to revolve in the orbit.

(e) The nuclear diameter is of the order of 10^{-14} m. This can be calculated by using the concept of distance of closest approach.

DISTANCE OF CLOSEST APPROACH

Let an α-particle (initially far from nucleus) having velocity v_α approach a nucleus (head-on) having a charge $+Ze$. The velocity of the α-particle decreases till it comes to rest at a distance r_0 from the nucleus. It is, then, repelled back along the direction of approach, r_0 gives the radius of nucleus.

Initial K.E. of α-particle $= \frac{1}{2} m_\alpha v_\alpha^2 = K_\alpha$

Initial P.E. of α-particle $= 0$

Final K.E. of α-particle $= 0$

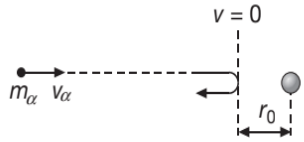
Final P.E. of α-particle $= \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_0}$

By Law of Conservation of Energy

$$(U+K)_{\text{initial}} = (U+K)_{\text{final}}$$

$$\Rightarrow \quad K_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_0}$$

$$\Rightarrow \quad r_0 = \frac{2}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{m_\alpha v_\alpha^2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{K_\alpha}\right)$$



For α-particle, $q_1 = 2e$

$$\Rightarrow \quad r_0 = \frac{2}{4\pi\varepsilon_0}\left(\frac{2Ze^2}{m_\alpha v_\alpha^2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{2Ze^2}{K_\alpha}\right)$$

$$\Rightarrow \quad r_0 = \frac{4}{4\pi\varepsilon_0}\left(\frac{Ze^2}{m_\alpha v_\alpha^2}\right) = 4\times 9\times 10^9\left(\frac{Ze^2}{m_\alpha v_\alpha^2}\right)$$

In one of the experiments, α-particles of velocity 2×10^7 ms^{-1} were bombarded upon gold foil with $Z = 79$

So, for $Z = 79$, $e = 1.59\times 10^{-19}$ C, $m_\alpha = 4\times 1.67\times 10^{-27}$ kg, $v_\alpha = 2\times 10^7$ ms^{-1}, we get,

$$r_0 = 4\times 9\times 10^9 \times \frac{79\times\left(1.59\times 10^{-19}\right)^2}{4\times 1.67\times 10^{-27}\times\left(2\times 10^7\right)^2}$$

$$\Rightarrow \quad r_0 = 2.69\times 10^{-14} \text{ m}$$

This gives the order of the radius of nucleus.

ILLUSTRATION 2

A head-on collision takes place between an α-particle of kinetic energy 5.5 MeV and a gold nucleus $(Z = 79)$. Calculate the distance of closest approach.

SOLUTION

Since, the distance of closest approach is given by

$$r_0 = \frac{1}{4\pi\varepsilon_0}\left(\frac{2Ze^2}{m_\alpha v_\alpha^2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{2Ze^2}{K_\alpha}\right)$$

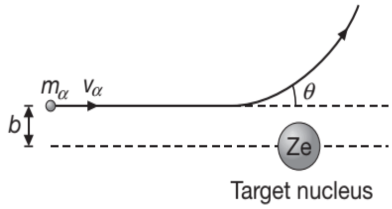
$$\Rightarrow \quad r_0 = 9\times 10^9 \times \frac{2\times 79\times\left(1.6\times 10^{-19}\right)^2}{5.5\times\left(1.6\times 10^{-13}\right)}$$

$$\Rightarrow \quad r_0 = 4.13\times 10^{-14} \text{ m}$$

The radius of gold nucleus must be smaller than r_0, so it may lie between 10^{-14} to 10^{-15} m.

TRAJECTORY OF AN ALPHA PARTICLE AND IMPACT PARAMETER

The perpendicular distance of the initial velocity vector of the α-particle from the centre of the nucleus is called **impact parameter** and is denoted by b.



The angle between the direction of approach of the α-particle and the direction in which it finally goes is defined as the angle of scattering and is denoted by θ. Rutherford concluded that

$$\tan\left(\frac{\theta}{2}\right) = \frac{1}{b}\left(\frac{Ze^2}{4\pi\varepsilon_0 K_\alpha}\right)$$

$$\Rightarrow \quad b = \frac{Ze^2}{4\pi\varepsilon_0 K_\alpha}\cot\left(\frac{\theta}{2}\right)$$

For a given nucleus and α-particle of given energy K

$$\tan\left(\frac{\theta}{2}\right) \propto \frac{1}{b}$$

$$\Rightarrow \quad \cot\left(\frac{\theta}{2}\right) \propto b$$

A graph between b and $\cot\left(\frac{\theta}{2}\right)$ is a straight line.

Therefore θ increases with decrease in value of b which implies that an α-particle, passing closer to the nucleus, is deflected at large angles.

In case of head-on collision, the impact parameter tends to zero and the α-particle rebounds back.

An α-particle close to the nucleus has small impact parameter and suffers large scattering.

For a large impact parameter, the α-particle goes nearly undeviated and has a small deflection.

The fact, that only a small fraction of number of α-particles rebounds back, indicates that the number of α-particles suffering head on collision is very small, which indicates that the mass of the atom is concentrated in a small volume. Therefore, Rutherford scattering is a powerful way to determine the size of the nucleus.

ILLUSTRATION 3

Let a 5 MeV α-particle is scattered by $74°$ when it approaches a gold nucleus $(Z = 79)$. Find the impact parameter.

SOLUTION

Since, $Z = 79$

$$K_\alpha = 5\text{ MeV} = 5\times1.6\times10^{-13}\text{ J} = 8\times10^{-13}\text{ J}$$

$$\theta = 74°$$

$$\Rightarrow \frac{\theta}{2} = 37°$$

$$\Rightarrow \cot\left(\frac{\theta}{2}\right) = \frac{4}{3}$$

Using the relation

$$\tan\left(\frac{\theta}{2}\right) = \frac{1}{b}\left(\frac{Ze^2}{4\pi\varepsilon_0 K_\alpha}\right)$$

$$\Rightarrow b = \left(\frac{Ze^2}{4\pi\varepsilon_0 K_\alpha}\right)\cot\left(\frac{\theta}{2}\right)$$

$$\Rightarrow b = \frac{79\times(1.6\times10^{-19})^2\times(9\times10^9)}{8\times10^{-13}}\times\frac{4}{3}$$

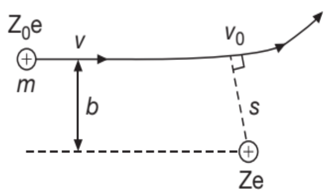
$$\Rightarrow b = 3.03\times10^{-16}\text{ m}$$

ILLUSTRATION 4

A particle of mass m, atomic number Z_0, initial speed v and impact parameter b is scattered by a heavy nucleus of atomic number Z. Use the principle of conservation of angular momentum and energy to obtain a relation between the minimum distance s of the particle from the nucleus in terms of Z, Z_0, v and b. Show that for $b = 0$, s reduces to the distance of closest approach r_0 given by

$$r_0 = \frac{1}{4\pi\varepsilon_0}\left(\frac{2ZZ_0e^2}{mv^2}\right)$$

SOLUTION



By angular momentum conservation, we have

$$mvb = mv_0 s \qquad \ldots(1)$$

By energy conservation, we have

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + \frac{1}{4\pi\varepsilon_0}\left(\frac{ZZ_0e^2}{s}\right) \qquad \ldots(2)$$

Substituting value of v_0 from equation (1) in equation (2), we get

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{vb}{s}\right)^2 + \frac{1}{4\pi\varepsilon_0}\left(\frac{ZZ_0e^2}{s}\right)$$

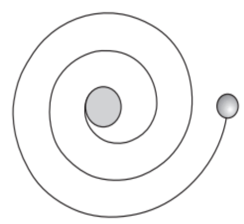
$$\Rightarrow \frac{1}{2}mv^2\left(1-\frac{b^2}{s^2}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{ZZ_0e^2}{s}\right)$$

For $b = 0$, we have

$$s = \frac{1}{4\pi\varepsilon_0}\left(\frac{2ZZ_0e^2}{mv^2}\right)$$

FAILURE OF RUTHERFORD MODEL

(a) According to laws of electro-magnetic theory, a charged particle in accelerated motion must radiate energy in the form of electro-magnetic radiation. As the electron revolves in a circular orbit, it is constantly subjected to centripetal acceleration $\frac{v^2}{r}$.



Electron spiralling inwards

So, it must radiate energy continuously. Bohr calculated that this emission of radiation would cause the electrons in an atom to lose all their energy and fall into the nucleus within a hundred - millionth of a second following a spiral path. Thus, the whole atomic structure should collapse. Since matter composed of atoms exists permanently, as far as we know, there was obviously something wrong here.

(b) According to Rutherford model, electron can revolve in any orbit. So, it must emit continuous radiations of all frequencies. But elements emit spectral lines of only definite frequencies.

BOHR'S ATOMIC MODEL

To rectify the drawbacks of Rutherford Model, Bohr proposed a theory which applies to hydrogen atom and species like $He^{+}(Z=2)$, $Li^{++}(Z=3)$, $Be^{+++}(Z=4)$ etc. Here a single electron revolves around a stationary nucleus of positive charge Ze where $Z=1$ for hydrogen atom, $Z=2$ for He^{+} etc. Bohr in defiance of the well-established laws of classical mechanics and electrodynamics, proposed the following postulates to support his atomic model.

Circular Orbits

The atom consists of central nucleus, containing the entire positive charge and almost all mass of the atom. The electrons revolve around the nucleus in certain discrete circular orbits. The necessary centripetal force for circular orbit is provided by Coulomb's attraction between the electron and nucleus. So,

$$\frac{mv^2}{r}=\frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r^2}$$

where, $m=$ mass of electron,
$r=$ radius of circular orbit,
$v=$ speed of electron in circular orbit,
$Ze=$ charge on nucleus,
$Z=$ atomic number,
$e=$ charge on electron $=-1\cdot6\times10^{-19}$ C

Stationary Orbits

The allowed orbits for electron are those in which the electron does not radiate energy. These orbits are also called stationary orbits.

Stationary Nucleus

The nucleus is so heavy, that its motion may be neglected.

Constancy of Mass

The mass of the electron in motion is assumed to be constant.

Quantum Condition (Bohr's Quantisation Rule)

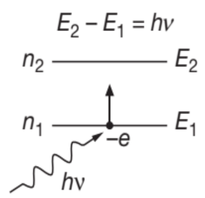
The stationary orbits are those in which angular momentum of electron is an integral multiple of $\frac{h}{2\pi}(=\hbar)$. This condition is also called as Bohr's Quantisation Rule according to which only those orbits are permitted for which the angular momentum of the electron in that orbit is an integral multiple of $\frac{h}{2\pi}$. This rule applies to an electron revolving in a particular orbit.

Mathematically, according to Bohr's Quantisation Rule, we have $L=mvr=n\left(\frac{h}{2\pi}\right)$, where n being an integer or the principle quantum number of the electron in the revolving orbit.

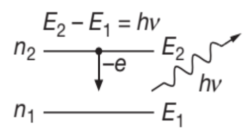
Bohr's Transition Rule

This rule applies to an electron making a transition from one stationary orbit to another. Whenever, an electron makes a transition from one orbit to the other, then a photon is emitted or absorbed having energy equal to the difference of energies between initial and final orbits/states.

So, when a photon of energy equal to the energy difference of two levels (say, $h\nu=E_2-E_1$) is incident on an electron in the lower energy level (n_1), then the electron will get excited to the higher energy level (n_2) as shown.



Similarly, when an electron makes a transition from a higher energy level (n_2) to a lower energy level (n_1), then a photon of energy equal to the energy difference of two levels i.e., $h\nu=E_2-E_1$ is emitted as shown.



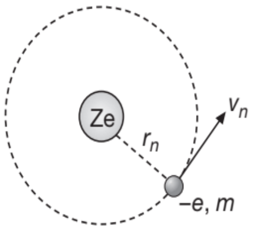
So, according to Bohr's Transition Rule, we have

$$|\Delta E|=|E_i-E_f|=h\nu$$

$$\Rightarrow \quad \nu_{i\to f}=\nu_{f\to i}=\nu=\frac{|E_i-E_f|}{h}$$

BOHR'S THEORY OF THE HYDROGEN LIKE ATOMS

Bohr proposed a theory which applies to hydrogen atom and species like He^+, Li^{++}, Be^{+++} etc. where a single electron revolves around a stationary nucleus of positive charge Ze as shown in figure.



We must note that, $Z=1$ for hydrogen atom, $Z=2$ for He^+, $Z=3$ for Li^{++}, $Z=4$ for Be^{+++}. Bohr applied the well established laws of classical mechanics and electrodynamics, to calculate the following quantities for the hydrogen like atoms.

Radius of Orbit

Since, we have

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r^2} \qquad \ldots(1)$$

and $$mvr = \frac{nh}{2\pi} \qquad \ldots(2)$$

From (2), $v = \frac{nh}{2\pi mr}$. Put in (1), we get

$$r_n = \frac{n^2h^2\varepsilon_0}{\pi me^2 Z}$$

$$\Rightarrow \quad r_n = (0.53)\frac{n^2}{Z}\ \text{Å}$$

So, for H-like atoms, we have

$$r_n \propto \frac{n^2}{Z}$$

Velocity of Electron in *n*th Orbit

Since, $v_n = \frac{nh}{2\pi mr_n}$ and $r_n = \frac{n^2h^2\varepsilon_0}{\pi me^2 Z}$

$$\Rightarrow \quad v_n = \left(\frac{e^2}{2h\varepsilon_0}\right)\frac{Z}{n} = \left(\frac{e^2}{2h\varepsilon_0 c}\right)\left(\frac{cZ}{n}\right)$$

$$\Rightarrow \quad v_n = \alpha\left(\frac{cZ}{n}\right)$$

where $\alpha = \frac{e^2}{2h\varepsilon_0 c}$ is the fine structure constant (a pure number) whose value is $\frac{1}{137}$.

$$\Rightarrow \quad v_n = \left(\frac{1}{137}\right)\frac{cZ}{n}$$

i.e. velocity of electron in Bohr's first orbit is $\frac{c}{137}$, in second orbit is $\frac{c}{274}$ and so on.

Angular Frequency/Velocity

$$\omega_n = \frac{v_n}{r_n} = \frac{\left(\frac{e^2}{2h\varepsilon_0}\right)\frac{Z}{n}}{\left(\frac{h^2\varepsilon_0}{\pi me^2}\right)\frac{n^2}{Z}} = \left(\frac{\pi me^4}{2\varepsilon_0^2h^3}\right)\left(\frac{Z^2}{n^3}\right)$$

$$\Rightarrow \quad \omega_n \propto \frac{Z^2}{n^3}$$

Frequency

$$f_n = \frac{\omega_n}{2\pi} = \left(\frac{me^4}{4\varepsilon_0^2h^3}\right)\left(\frac{Z^2}{n^3}\right)$$

$$\Rightarrow \quad f_n \propto \frac{Z^2}{n^3}$$

Time Period of Revolution

$$T_n = \frac{1}{f_n} = \frac{1}{\left(\frac{me^4}{4\varepsilon_0^2h^3}\right)\left(\frac{Z^2}{n^3}\right)} = \left(\frac{4\varepsilon_0^2h^3}{me^4}\right)\left(\frac{n^3}{Z^2}\right)$$

$$\Rightarrow \quad T_n \propto \frac{n^3}{Z^2}$$

Current

$$i_n = \frac{e}{T_n} = ef_n = \left(\frac{me^5}{4\varepsilon_0^2h^3}\right)\left(\frac{Z^2}{n^3}\right)$$

$$\Rightarrow \quad i_n \propto \frac{Z^2}{n^3}$$

Magnetic Field at the Centre of Atom

$$B_n = \frac{\mu_0 i_n}{2r_n} = \left(\frac{\mu_0}{2}\right)\left[\left(\frac{me^5}{4\varepsilon_0^2h^3}\right)\left(\frac{Z^2}{n^3}\right)\right]\left[\left(\frac{\pi me^2}{h^2\varepsilon_0}\right)\left(\frac{Z}{n^2}\right)\right]$$

$$\Rightarrow \quad B_n = \left(\frac{\pi m^2 Z^3 e^7 \mu_0}{8\varepsilon_0^3 n^5 h^5}\right)\left(\frac{Z^3}{n^5}\right)$$

$$\Rightarrow \quad B_n \propto \frac{Z^3}{n^5}$$

Magnetic Moment of Atom

$$M_n = i_n A_n = i_n\left(\pi r_n^2\right) = \pi\left(\frac{me^5 Z^2}{4\varepsilon_0^2 h^3 n^3}\right)\left(\frac{n^2 h^2 \varepsilon_0}{\pi m e^2 Z}\right)^2$$

$$\Rightarrow \quad M_n = i_n A_n = i_n\left(\pi r_n^2\right) = \pi\left(\frac{me^5 Z^2}{4\varepsilon_0^2 h^3 n^3}\right)\left(\frac{n^2 h^2 \varepsilon_0}{\pi m e^2 Z}\right)^2$$

$$\Rightarrow \quad M_n = n\left(\frac{eh}{4\pi m}\right)$$

The term $\frac{eh}{4\pi m}$ is called the Bohr's Magneton for the atom.

$$\Rightarrow \quad M_n \propto n$$

Angular Momentum

$$L_n = m v_n r_n = m\left(\frac{e^2 Z}{2h\varepsilon_0 n}\right)\left(\frac{n^2 h^2 \varepsilon_0}{\pi m e^2 Z}\right) = n\left(\frac{h}{2\pi}\right)$$

As expected from Bohr's Quantisation Rule.

Kinetic Energy of Electron (E_K)

Since, we have

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r^2}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{Ze^2}{8\pi\varepsilon_0 r}$$

$$\Rightarrow \quad E_K = \frac{1}{2}mv^2 = \frac{Ze^2}{8\pi\varepsilon_0 r}$$

Potential Energy (U) of Electron in nth Orbit

$$U = -\frac{1}{4\pi\varepsilon_0}\frac{(Ze)(e)}{r}$$

$$\Rightarrow \quad U = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

Total Energy (E) of Electron in nth Orbit

Total Energy = K.E. + P.E.

$$\Rightarrow \quad E = \frac{Ze^2}{8\pi\varepsilon_0 r} - \frac{Ze^2}{4\pi\varepsilon_0 r}$$

$$\Rightarrow \quad E = -\frac{Ze^2}{8\pi\varepsilon_0 r}$$

So, we conclude that

$$\text{Total Energy} = -\text{K.E.} = \frac{1}{2}(\text{P.E.})$$

Further, since $r = \frac{n^2 h^2 \varepsilon_0}{\pi m e^2 Z}$

$$\Rightarrow \quad E = -\left(\frac{me^4}{8h^2\varepsilon_0^2}\right)\frac{Z^2}{n^2}$$

$$\Rightarrow \quad E = -(13.6)\frac{Z^2}{n^2}\ \text{eV}$$

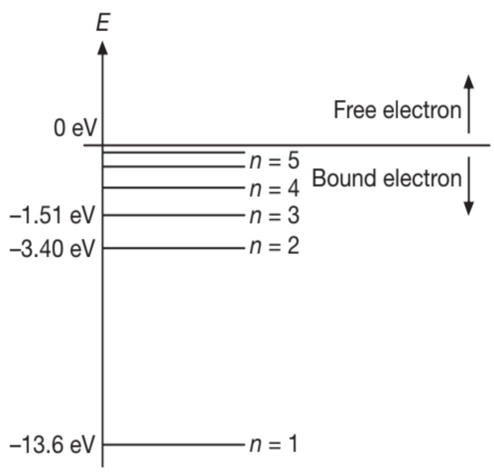
Also $E = -\left(\frac{me^4}{8\varepsilon_0^2 ch^3}\right)ch\frac{Z^2}{n^2}$

$$\Rightarrow \quad E = -(Rch)\frac{Z^2}{n^2}$$

where R = Rydberg's constant $= \frac{me^4}{8\varepsilon_0^2 ch^3} = 1.097 \times 10^7\ \text{m}^{-1}$

and Rydberg's Energy is $Rch \approx 2.17 \times 10^{-18}\ \text{J} \approx 13.6\ \text{eV}$. It is the energy of an electron in the first orbit of H atom.

An energy level diagram for the hydrogen atom $(Z = 1)$ is shown in the figure.



The vertical axis represents energy. The (arbitrary) zero of energy is taken as the energy of a stationary electron, infinitely far from the positive nucleus. The lowest energy level $(n=1)$ is known as the ground state. The energy level corresponding to $n=2$ is called the first excited state and so on. In this diagram zero energy level corresponds to $n=\infty$ which is the ionized state of the atom.

ILLUSTRATION 5

A beam of monochromatic light of wavelength λ ejects photoelectrons from a caesium metal surface having work function 1.7 eV. These photoelectrons are made to collide with hydrogen atoms in ground state. Find the maximum value of λ for which the

(a) hydrogen atoms may be ionized.
(b) hydrogen atom gets excited from the ground state to the first excited state and
(c) excited hydrogen atoms may emit visible light.

SOLUTION

(a) To ionize H-atom, the kinetic energy of photoelectrons must be at least 13.6 eV

$$\Rightarrow \quad \frac{12375}{\lambda}=1.7+13.6$$

$$\Rightarrow \quad \lambda=\frac{12375}{15.3}\approx 809\ \text{Å}$$

(b) To excite H-atom from $n=1$ to $n=2$, the kinetic energy possessed by the photoelectrons must be at least 10.2 eV.

$$\Rightarrow \quad \frac{12375}{\lambda}=1.7+10.2=11.9$$

$$\Rightarrow \quad \lambda=\frac{12375}{11.9}\approx 1040\ \text{Å}$$

(c) To emit visible light photons, H-atom must be excited at least from $n=1$ to $n=3$, so that for Balmer series it can emit visible light. So, the kinetic energy possessed by the photoelectrons must be at least 12.09 eV

$$\Rightarrow \quad \frac{12375}{\lambda}=1.7+12.09=13.79$$

$$\Rightarrow \quad \lambda=\frac{12375}{13.79}\approx 897\ \text{Å}$$

ILLUSTRATION 6

Calculate the angular momentum of an electron in Bohr's hydrogen atom whose energy is -3.4 eV?

SOLUTION

Energy of electron in n^{th} Bohr orbit of hydrogen atom is given by,

$$E=-\frac{13.6}{n^2}\ \text{eV}$$

$$\Rightarrow \quad -3.4=-\frac{13.6}{n^2}$$

$$\Rightarrow \quad n^2=4$$

$$\Rightarrow \quad n=2$$

Since, the angular momentum of an electron in n^{th} orbit is given by

$$L=\frac{nh}{2\pi}$$

$$\Rightarrow \quad L=2\left(\frac{h}{2\pi}\right)=\frac{h}{\pi}$$

ILLUSTRATION 7

Using Bohr's theory show that when n is very large the frequency of radiation emitted by hydrogen atom due to transition of electron from n to $(n-1)$ is equal to frequency of revolution of electron in its orbit.

SOLUTION

Frequency of revolution electron in nth orbit is given by

$$f_{\text{revolution}}=\frac{1}{T}=\frac{v}{2\pi r}=\frac{\left(\frac{e^2}{2h\varepsilon_0 c}\right)\frac{cz}{n}}{2\pi\left(\frac{n^2h^2\varepsilon_0}{\pi me^2 z}\right)}$$

$$\Rightarrow \quad f_{\text{revolution}}=\left(\frac{me^4}{4\varepsilon_0^2h^3}\right)\frac{z^2}{n^3} \qquad \ldots(1)$$

Further, frequency of transition from n to $(n-1)$ is

$$hf=\frac{me^4z^2}{8h^2\varepsilon_0^2}\left[\frac{1}{(n-1)^2}-\frac{1}{n^2}\right]$$

$$\Rightarrow \quad hf=\frac{me^4z^2}{8h^2\varepsilon_0^2}\left[\frac{2n-1}{n^2(n-1)^2}\right]$$

When n is large then

$$2n-1 \cong 2n \text{ and } n-1 \cong n$$

$$\Rightarrow \quad hf \cong \frac{me^4z^2}{8h^2\varepsilon_0^2}\frac{2n}{n^4}$$

$$\Rightarrow \quad f_{\text{transition}} \cong \left(\frac{me^4}{4\varepsilon_0^2h^3}\right)\frac{z^2}{n^3} \qquad \ldots(2)$$

So, from (1) and (2), we observe that for large n,

$$f_{\text{revolution}} = f_{\text{transition}}\,(\text{between adjacent levels})$$

This Principle is also called **"BOHR'S CORRESPONDENCE PRINCIPLE"**.

ILLUSTRATION 8

An electron is orbiting in a circular orbit of radius r under the influence of a constant magnetic field of strength B. Assuming that Bohr's postulate regarding the quantisation of angular momentum holds good for this electron, find

(a) the allowed values of the radius r of the orbit.
(b) the kinetic energy of the electron in orbit.
(c) the potential energy of interaction between the magnetic moment of the orbital current due to the electron moving in its orbit and the magnetic field B.
(d) the total energy of the allowed energy levels.
(e) the total magnetic flux due to the magnetic field B passing through the nth orbit.

(Assume that the charge on the electron is $-e$ and the mass of the electron is m).

SOLUTION

(a) Since, $r = \dfrac{mv}{Be}$ ...(1)

From Bohr's Quantisation Rule, we have

$$mvr = \frac{nh}{2\pi} \qquad \ldots(2)$$

Solving these two equations, we get

$$r = \sqrt{\frac{nh}{2\pi Be}} \text{ and } v = \sqrt{\frac{nhBe}{2\pi m^2}}$$

(b) $K = \dfrac{1}{2}mv^2 = \dfrac{nhBe}{4\pi m}$

(c) $M = iA = \left(\dfrac{e}{T}\right)(\pi r^2) = \dfrac{e}{\left(\dfrac{2\pi r}{v}\right)}(\pi r^2) = \dfrac{evr}{2}$

$$\Rightarrow \quad M = \frac{e}{2}\sqrt{\frac{nh}{2\pi Be}}\sqrt{\frac{nhBe}{2\pi m^2}} = \frac{nhe}{4\pi m}$$

Since, $U = -MB\cos 180^\circ$

$$\Rightarrow \quad U = \frac{nheB}{4\pi m}$$

The angle between $\vec{M}$ and $\vec{B}$ will be 180° because instead of taking electronic current, we have to take conventional current which moves opposite to electronic current.

(d) $E = U + K = \dfrac{nheB}{2\pi m}$

(e) $|\phi| = B\pi r^2 = \dfrac{nh}{2e}$

FREQUENCY OF EMITTED RADIATION

If electron jumps from initial state n_i to a final state n_f, then frequency of emitted or absorbed radiation ν is given by applying Bohr's Transition Rule, according to which

$$E_i - E_f = h\nu$$

$$\Rightarrow \quad \nu = \frac{E_i - E_f}{h} = Z^2Rc\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

If c is the speed of light and λ the wavelength of emitted or absorbed radiation, then

$$\nu = \frac{c}{\lambda} = Z^2Rc\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

So, Wave number $(\bar{\nu})$ is given by

$$\bar{\nu} = \frac{1}{\lambda} = Z^2R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

This relation holds for radiations emitted by hydrogen-like atoms i.e.

$H(Z=1)$, $He^{+}(Z=2)$, $Li^{++}(Z=3)$ and

$Be^{+++}(Z=4)$.

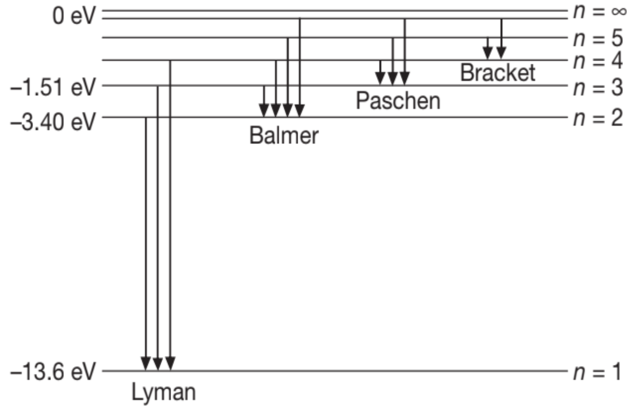
HYDROGEN SPECTRUM

Since, wave number $(\bar{v})$ is given by

$$\bar{v} = \frac{1}{\lambda} = Z^2 R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

where R is the Rydberg constant.

The various transitions for the hydrogen atom are shown in the following figure. All transitions starting from $n = 2$ onwards and ending at $n = 1$ belong to the Lyman Series. Likewise, all transitions starting from $n = 3$ onwards and ending at $n = 2$ belong to the Balmer Series. The other spectral series' names are mentioned in the figure.



	INITIAL STATE	FINAL STATE	WAVELENGTH FORMULA	FIRST MEMBER - SECOND MEMBER	SERIES LIMIT $n_i \to \infty$ TO n_f	MAXIMUM WAVELENGTH $(n_f + 1)$ TO n_f	LINES FOUND IN
LYMAN	$n_i = 2, 3, 4, 5, 6, \ldots$	$n_f = 1$	$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n_i^2}\right)$	$n_i = 2$ to $n_f = 1$ $n_i = 3$ to $n_f = 1$	From ∞ to 1 $\lambda = \frac{1}{R}$ $\lambda = 911$ Å	From 2 to 1 $\lambda = \frac{4}{3R}$ $\lambda = 1216$ Å	UV Region
BALMER	$n_i = 3, 4, 5, 6, 7, \ldots$	$n_f = 2$	$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n_i^2}\right)$	$n_i = 3$ to $n_f = 2$ $n_i = 4$ to $n_f = 2$	From ∞ to 2 $\lambda = \frac{4}{R}$ $\lambda = 3646$ Å	From 3 to 2 $\lambda = \frac{36}{5R}$ $\lambda = 6563$ Å	Visible Region
PASCHEN	$n_i = 4, 5, 6, 7, 8, \ldots$	$n_f = 3$	$\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n_i^2}\right)$	$n_i = 4$ to $n_f = 3$ $n_i = 5$ to $n_f = 3$	From ∞ to 3 $\lambda = \frac{9}{R}$ $\lambda = 8204$ Å	From 4 to 3 $\lambda = \frac{144}{7R}$ $\lambda = 18753$ Å	IR Region
BRACKETT	$n_i = 5, 6, 7, 8, 9, \ldots$	$n_f = 4$	$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n_i^2}\right)$	$n_i = 5$ to $n_f = 4$ $n_i = 6$ to $n_f = 4$	From ∞ to 4 $\lambda = \frac{16}{R}$ $\lambda = 14585$ Å	From 5 to 4 $\lambda = \frac{400}{9R}$ $\lambda = 40515$ Å	IR Region
PFUND	$n_i = 6, 7, 8, 9, 10, \ldots$	$n_f = 5$	$\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n_i^2}\right)$	$n_i = 6$ to $n_f = 5$ $n_i = 7$ to $n_f = 5$	From ∞ to 5 $\lambda = \frac{25}{R}$ $\lambda = 22790$ Å	From 6 to 5 $\lambda = \frac{900}{11R}$ $\lambda = 74583$ Å	Far IR Region

ILLUSTRATION 9

Find the largest and shortest wavelengths in the Lyman series for hydrogen. In what region of the electromagnetic spectrum does each series lie?

SOLUTION

The transition equation for Lyman series is given by,

$$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right),\ n = 2, 3, \ldots..$$

The largest wavelength is corresponding to $n = 2$, so

$$\frac{1}{\lambda_{max}} = 1.097 \times 10^7 \left(\frac{1}{1} - \frac{1}{4}\right)$$

$$\Rightarrow \quad \frac{1}{\lambda_{max}} = 0.823 \times 10^7$$

$$\Rightarrow \quad \lambda_{max} = 1.2154 \times 10^{-7}\ \text{m}$$

$$\Rightarrow \quad \lambda_{max} = 1215\ \text{Å}$$

The shortest wavelength corresponds to $n \to \infty$, so

$$\frac{1}{\lambda_{min}} = 1.097 \times 10^7 \left(\frac{1}{1} - \frac{1}{\infty}\right)$$

$$\Rightarrow \quad \lambda_{min} = 0.911 \times 10^{-7}\ \text{m} = 911\ \text{Å}$$

Both of these wavelengths lie in ultraviolet (UV) region of electromagnetic spectrum.

ILLUSTRATION 10

Electrons of energies 10.20 eV and 12.09 eV can cause radiation to be emitted from hydrogen atoms. Calculate in each case, the principal quantum number of the orbit to which electron in the hydrogen atom is raised and the wavelength of the radiation emitted if it drops back to the ground state. Take $hc = 12375$ eVÅ

SOLUTION

Since the orbital energy of an electron revolving in n^{th} orbit is given by

$$E_n = -\left(\frac{13.6}{n^2}\right)\ \text{eV}$$

When $n = 1$, $E_1 = -13.6$ eV

$n = 2$, $E_2 = -3.4$ eV

$n = 3$, $E_3 = -1.51$ eV

Here we observe that

$$10.20\ \text{eV} = E_2 - E_1$$

and $12.09\ \text{eV} = E_3 - E_1$

So, by absorbing a radiation photon of 10.2 eV the electron will make a transition to $n = 2$ state and by absorbing a 12.09 eV photon the electron will make a transition to $n = 3$ state. Now after the life time of excited states, the electron in $n = 2$ and $n = 3$ will make transitions to lower states and ultimately come back to ground state. In this process, the possibilities of reverse transition are

$n = 3 \rightarrow n = 2$

$n = 3 \rightarrow n = 1$

$n = 2 \rightarrow n = 1$

In above three transitions the amount of energy released will be

$$\Delta E_{32} = (-1.51\ \text{eV}) - (-3.4\ \text{eV}) = 1.89\ \text{eV}$$

$$\Delta E_{31} = (-1.51\ \text{eV}) - (-13.6\ \text{eV}) = 12.09\ \text{eV}$$

$$\Delta E_{21} = (-3.4\ \text{eV}) - (-13.6\ \text{eV}) = 10.20\ \text{eV}$$

Thus, wavelength of radiations of corresponding transition are

$$\lambda_{32} = \frac{12375}{1.89} = 6548\ \text{Å}$$

$$\lambda_{31} = \frac{12375}{12.09} = 1024\ \text{Å}$$

$$\lambda_{21} = \frac{12375}{10.2} = 1213\ \text{Å}$$

ILLUSTRATION 11

A single electron orbits around a stationary nucleus of charge $+Ze$, where Z is a constant and e is the magnitude of electronic charge. It requires **47.2 eV** to excite the electron from second Bohr orbit to the third Bohr orbit.

(a) Find the value of Z

(b) Find the energy required to excite the electron from $n = 3$ to $n = 4$

(c) Find the wavelength of radiation required to remove electron from first Bohr's orbit to infinity.

(d) Find the kinetic energy, potential energy and angular momentum of the electron in the first Bohr orbit.

SOLUTION

(a) Given $\Delta E_{23} = 47.2$ eV

Since, $\Delta E = 13.6Z^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$ eV

$\Rightarrow \quad 47.2 = 13.6Z^2\left(\frac{1}{2^2}-\frac{1}{3^2}\right)$

$\Rightarrow \quad Z = 5$

(b) To find ΔE_{34}, $n_1 = 3$, $n_2 = 4$

$$\Delta E = 13.6Z^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) \text{ eV}$$

$$\Rightarrow \quad \Delta E = 13.6\times 5^2\left(\frac{1}{3^2}-\frac{1}{4^2}\right) = 16.53 \text{ eV}$$

(c) Ionization energy is the energy required to excite the electron from $n = 1$ to $n \to \infty$

Thus, $\Delta E = 13.6\times 5^2\left(\frac{1}{1^2}-\frac{1}{\infty^2}\right) = 340$ eV

The respective wavelength is

$$\lambda = \frac{hc}{\Delta E} = \frac{12400}{\Delta E} = \frac{12400}{340} = 36.47 \text{ Å}$$

(d) $K = -E = +340$ eV

$U = 2E = -680$ eV

$$L = \frac{h}{2\pi} = \frac{6.63\times 10^{-34}}{2\pi} = 1.056\times 10^{-34} \text{ Js}$$

ILLUSTRATION 12

Estimate the average kinetic energy of each hydrogen atom at room temperature. Use the result obtained to explain why nearly all H atoms are in the ground state at room temperature and hence emit no light.

SOLUTION

According to kinetic theory, the average kinetic energy of each H atom is given by,

$$E_{av} = \frac{3}{2}k_B T$$

$$\Rightarrow \quad E_{av} = \frac{3}{2}(1.38\times 10^{-23})(300)$$

$$\Rightarrow \quad E_{av} = 6.2\times 10^{-21} \text{ J}$$

$$\Rightarrow \quad E_{av} = \frac{6.2\times 10^{-21}}{1.6\times 10^{-19}} = 0.04 \text{ eV}$$

The average kinetic energy is thus very small compared to the energy between the ground state and the next higher energy state $(13.6 - 3.4 = 10.2 \text{ eV})$. Any atoms in excited state emit light and eventually fall to the ground state. Once in the ground state, collisions with other atoms can transfer energy of 0.04 eV on the average. A small fraction of atoms can have much more energy (in accordance with the distribution of molecular speeds), but even kinetic energy that is 10 times the average is not nearly enough to excite atoms above the ground state. Thus, at room temperature, nearly all atoms are in the ground state. Atoms can be excited to upper states at very high temperatures or by passing current of high energy electrons through the gas, as in a discharge tube.

ILLUSTRATION 13

The wavelength of the first line of Lyman series for hydrogen is identical to that of the second line of Balmer series for some hydrogen like ion x. Calculate energies of the first four levels, of x.

SOLUTION

Wavelength of the first line of Lyman series for hydrogen atom will be given by the equation

$$\frac{1}{\lambda_1} = R\left(\frac{1}{1^2}-\frac{1}{2^2}\right) = \frac{3R}{4} \quad \ldots(1)$$

The wavelength of second Balmer line for hydrogen like ion x is

$$\frac{1}{\lambda_2} = Rz^2\left(\frac{1}{2^2}-\frac{1}{4^2}\right) = \frac{3Rz^2}{16} \quad \ldots(2)$$

Given that $\lambda_1 = \lambda_2$

$$\Rightarrow \quad \frac{1}{\lambda_1} = \frac{1}{\lambda_2}$$

$$\Rightarrow \quad \frac{3R}{4} = \frac{3Rz^2}{16}$$

$$\Rightarrow \quad z = 2$$

Hence the ion x is actually He^+. The energies of first four levels of ion x are,

$$E_1 = -(13.6)z^2 = -54.4 \text{ eV}, E_2 = \frac{E_1}{(2)^2} = -13.6 \text{ eV}$$

$$E_3 = \frac{E_1}{(3)^2} = -6.04 \text{ eV and } E_4 = \frac{E_1}{(4)^2} = -3.4 \text{ eV}$$

ILLUSTRATION 14

A doubly ionised lithium atom is hydrogen-like with atomic number 3.

(a) Find the wavelength of the radiation required to excite the electron in Li^{2+} from the first to the third Bohr orbit. (Ionisation energy of the hydrogen atom equals 13.6 eV).

(b) How many spectral lines are observed in the emission spectrum of the above excited system?

SOLUTION

Given, $Z = 3$

Since, $E_n \propto \dfrac{Z^2}{n^2}$

(a) To excite the atom from $n = 1$ to $n = 3$, energy of photon required is,

$$E_{1\to 3} = E_3 - E_1 = \frac{(-13.6)(3)^2}{(3)^2} - \left(\frac{(-13.6)(3)^2}{(1)^2}\right)$$

$$\Rightarrow \quad E_{1\to 3} = 108.8 \text{ eV}$$

Corresponding wavelength will be,

$$\lambda(\text{in Å}) = \frac{12375}{E(\text{in eV})} = \frac{12375}{108.8} = 113.74 \text{ Å}$$

(b) From n^{th} orbit, total number of emission lines (N) can be $N = \dfrac{n(n-1)}{2} = \dfrac{3(3-1)}{2} = 3$.

ILLUSTRATION 15

An electron in a hydrogen atom is in a state of binding energy 0.85 eV. The electron makes a transition to a state of excitation energy of **10.2 eV**. Calculate the energy and wavelength of photon emitted.

SOLUTION

Since the binding energy is always negative, therefore,

$$E_i = -0.85 \text{ eV}$$

Let n_i be the initial binding state of the electron, then

$$E_n = -13.6\frac{Z^2}{n_i^2}$$

$$\Rightarrow \quad -0.85 = -13.6\frac{Z^2}{n_i^2}$$

$$\Rightarrow \quad n_i = 4$$

Binding energy is $E_n = -\dfrac{13.6Z^2}{n^2}$

$$\Rightarrow \quad 0.85 \text{ eV} = \frac{-13.6(1)^2}{n_2^2}$$

$$\Rightarrow \quad n_2 = 4$$

Let the electron now goes to an energy level n whose excitation energy is 10.2 eV. Since the excitation energy ΔE is defined with respect to ground state, therefore

$$E = 13.6Z^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\text{eV}$$

$$\Rightarrow \quad 10.2 = 13.6 \times 1^2\left(\frac{1}{1^2} - \frac{1}{n_f^2}\right)$$

$$\Rightarrow \quad n_f = 2$$

So, the electron makes a transition from energy level $n_i = 4$ to $n_f = 2$

Thus, the energy released is $\Delta E = E_4 - E_2$

$$\Rightarrow \quad \Delta E = 13.6\left[\frac{1}{2^2} - \frac{1}{4^2}\right] = 2.55 \text{ eV}$$

Since, $\lambda = \dfrac{hc}{\Delta E} = \dfrac{12400}{2.25 \text{ eV}} = 5511$ Å

ILLUSTRATION 16

A particle of charge equal to that of an electron, $-e$ and mass 208 times the mass of electron (called a μ-meson) moves in a circular orbit around a nucleus of charge $+3e$ (take the mass of the nucleus to be infinite). Assuming that the Bohr model of the atom is applicable to this system:

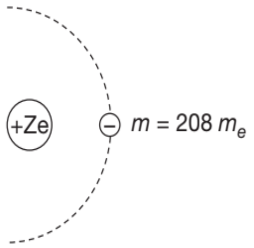
(i) Calculate the radius of nth Bohr orbit

(ii) Find the value of n, for which the radius of orbit is approximately the same as that of first Bohr orbit for the hydrogen atom;

(iii) Find the wavelength of radiation emitted when the μ-meson jumps from the third orbit to first orbit (Rydberg's constant $= 1.097 \times 10^7 \text{ m}^{-1}$).

SOLUTION

If we assume that mass of nucleus is very much mass of mu-meson, then nucleus will be assumed to be at rest, only mu-meson is revolving round it.



(i) In nth orbit the necessary centripetal force to the mu-meson will be provided by the electrostatic force between the nucleus and the mu-meson.

Hence, $\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0} - \frac{(Ze)(e)}{r^2}$ ...(1)

Further, it is given that Bohr model is applicable to this system also. Hence,

Angular momentum in nth orbit is $L = \frac{nh}{2\pi}$

$\Rightarrow \quad mvr = n\frac{h}{2\pi}$ ...(2)

$\Rightarrow \quad v = \frac{nh}{2\pi mr}$

Substituting in (1), we get

$$r = \frac{n^2h^2\varepsilon_0}{\pi me^2 Z}$$

Substituting $Z = 3$ and $m = 208m_e$, we get

$$r_n = \frac{n^2h^2\varepsilon_0}{624\pi m_e e^2}$$

(ii) The radius of the first Bohr orbit for the hydrogen atom is

$$r_1 = \frac{h^2\varepsilon_0}{\pi m_e e^2}$$

Equating this with the radius calculated in part (a), we get

$$n^2 \approx 624$$

$$\Rightarrow \quad n \approx 25$$

(iii) Kinetic energy of atom is

$$K = \frac{1}{2}mv^2 = \frac{Ze^2}{8\pi\varepsilon_0 r}$$

and the potential energy is $U = -\frac{Ze^2}{4\pi\varepsilon_0 r}$

$\Rightarrow$ Total energy $E_n = -\frac{Ze^2}{8\pi\varepsilon_0 r}$

Substituting value of r, calculated in part (a), we get

$$E_n = \frac{1872}{n^2}\left(\frac{m_e e^4}{8\varepsilon_0^2 h^2}\right)$$

But $\left(-\frac{m_e e^4}{8\varepsilon_0^2 h^2}\right)$ is the ground state energy of hydrogen atom and hence is equal to -13.6 eV

$$\Rightarrow \quad E_n = \frac{-1872}{n^2}(13.6)\text{ eV} = -\frac{25459.2}{n^2}\text{ eV}$$

$$\Rightarrow \quad E_3 - E_1 = -25459.2\left(\frac{1}{9} - \frac{1}{1}\right) = 22630.4\text{ eV}$$

So, the corresponding wavelength, is

$$\lambda(\text{in Å}) = \frac{12375}{22630.4}$$

$$\Rightarrow \quad \lambda = 0.546\text{ Å}$$

ILLUSTRATION 17

A hydrogen like atom of atomic number z is in an excited state of quantum number $2n$. It can emit a maximum energy photon of 204 eV. If it makes a transition to quantum state n, a photon of energy 40.8 eV is emitted. Find n, z and the ground state energy (in eV) for this atom. Also, calculate the minimum energy (in eV) that can be emitted by this atom during de-excitation. Ground state energy of hydrogen atom is 13.6 eV.

SOLUTION

Given, $E_{2n} - E_1 = 204$ eV

$\Rightarrow \quad (13.6)z^2\left(1 - \frac{1}{4n^2}\right) = 204$ ...(1)

Also, $E_{2n} - E_n = 40.8$ eV

$\Rightarrow \quad 13.6z^2\left(\frac{1}{n^2} - \frac{1}{4n^2}\right) = 40.8$ ...(2)

Solving equations (1) and (2), we get

$$n = 2 \text{ and } z = 4$$

Since, $E_n = \frac{(-13.6)z^2}{n^2}$ eV, so we have

$$E_1 = (-13.6)(4)^2\text{ eV}$$

$\Rightarrow \quad E_1 = -217.6 \text{ eV}$

During de-excitation, minimum energy emitted is,

$$E_{min} = E_{2n} - E_{2n-1} = E_4 - E_3$$

$$\Rightarrow \quad E_{min} = \frac{-217.6}{4^2} - \left(\frac{-217.6}{3^2}\right) = 10.58 \text{ eV}$$

ILLUSTRATION 18

Calculate the energy of a H-atom in the first excited state, if the potential energy is assumed to be zero in the ground state.

SOLUTION

Since, we know that, in ground state, $n = 1$, we have

$$TE = -KE = \frac{PE}{2}$$

$$\Rightarrow \quad PE = 2(TE) = 2(-13.6 \text{ eV})$$

$$\Rightarrow \quad PE = -27.2 \text{ eV}$$

However, we have assumed this energy to be zero i.e., potential energy is increased by 27.2 eV. Since, kinetic energy in all energy states will remain unchanged whereas potential energy and hence, the total energy in all states will increase by 27.2 eV. Further, first excited state means $n = 2$, so

$$E_2 = -3.4 \text{ eV} \text{ (previously)}$$

$$\Rightarrow \quad E_2' = -3.4 + 27.2 = 23.8 \text{ eV} \text{ (now)}$$

ILLUSTRATION 19

An imaginary particle has a charge equal to that of an electron and mass 100 times the mass of the electron. It moves in a circular orbit around a nucleus of charge $+4e$. Take the mass of the nucleus to be infinite. Assuming that the Bohr's model is applicable to this system.

(a) Derive an expression for the radius of nth Bohr orbit.

(b) Find the wavelength of the radiation emitted when the particle jumps from fourth orbit to the second orbit.

SOLUTION

(a) We have $\dfrac{m_p v^2}{r_n} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{ze^2}{r_n^2}$ ...(1)

The quantization of angular momentum gives,

$$m_p v r_n = \frac{nh}{2\pi} \quad \text{...(2)}$$

Solving equations (1) and (2), we get

$$r = \frac{n^2 h^2 \varepsilon_0}{z\pi m_p e^2}$$

Substituting $m_p = 100 \text{ m}$

where m = mass of electron and $z = 4$

we get, $r_n = \dfrac{n^2 h^2 \varepsilon_0}{400\pi m e^2}$

(b) As we know, $E_1^H = -13.60 \text{ eV}$

and $E_n \propto \left(\dfrac{z^2}{n^2}\right) m$

For the given particle,

$$E_4 = \frac{(-13.60)(4)^2}{(4)^2} \times 100 = -1360 \text{ eV}$$

and $$E_2 = \frac{(-13.60)(4)^2}{(2)^2} \times 100 = -5440 \text{ eV}$$

$$\Rightarrow \quad \Delta E = E_4 - E_2 = 4080 \text{ eV}$$

$$\Rightarrow \quad \lambda(\text{in Å}) = \frac{12375}{\Delta E(\text{in eV})} = \frac{12375}{4080} = 3 \text{ Å}$$

ILLUSTRATION 20

Electrons in hydrogen-like atom $(Z = 3)$ make transitions from the fifth to the fourth orbit and from the fourth to the third orbit. The resulting radiations are incident normally on a metal plate and eject photoelectrons. The stopping potential for the photoelectrons ejected by the shorter wavelength is 3.95 V. Calculate the work function of the metal, and the stopping potential for the photoelectrons ejected by the longer wavelength (Rydberg constant $= 1.094 \times 10^7 \text{ m}^{-1}$)

SOLUTION

The stopping potential for shorter wavelength is 3.95 V i.e., maximum kinetic energy of photoelectrons corresponding to shorter wavelength will be 3.95 eV. Further energy of incident photons corresponding to shorter wavelength will be in transition from $n = 4$ to $n = 3$.

$$E_{4\to3} = E_4 - E_3 = \frac{-(13.6)(3)^2}{(4)^2} - \left(\frac{-(13.6)(3)^2}{(3)^2}\right)$$

$\Rightarrow \quad E_{4\to3} = 5.95 \text{ eV}$

Now, from the equation,

$$K_{max} = E - W$$

we have $W = E - K_{max} = E_{4\to3} - K_{max}$

$\Rightarrow \quad W = (5.95 - 3.95) \text{ eV} = 2 \text{ eV}$

Longer wavelength will correspond to transition from $n = 5$ to $n = 4$. From the relation, we get

$$\frac{1}{\lambda} = Rz^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

The longer wavelength,

$$\frac{1}{\lambda} = (1.094\times10^7)(3)^3\left(\frac{1}{16} - \frac{1}{25}\right)$$

$\Rightarrow \quad \lambda = 4.514\times10^{-7} \text{ m} = 4514 \text{ Å}$

Energy corresponding to this wavelength is

$$E = \frac{12375 \text{ eV Å}}{4514 \text{ Å}} = 2.74 \text{ eV}$$

So, maximum kinetic energy of photo-electrons is

$$K_{max} = E - W = (2.74 - 2) \text{ eV} = 0.74 \text{ eV}$$

Hence, the stopping potential is 0.74 V.

ILLUSTRATION 21

Hydrogen atom in its ground state is excited by means of monochromatic radiation of wavelength 975 Å. How many different lines are possible in the resulting spectrum? Calculate the longest wavelength amongst them. You may assume the ionization energy for hydrogen atom as 13.6 eV.

SOLUTION

Energy corresponding to given wavelength is

$$E(\text{in eV}) = \frac{12375}{\lambda(\text{in Å})} = \frac{12375}{975} = 12.69 \text{ eV}$$

Now, let the electron excites to n^{th} energy state. Then,

$$E_n - E_1 = 12.69$$

$\Rightarrow \quad \frac{(-13.6)}{(n^2)} - (-13.6) = 12.69$

$\Rightarrow \quad n \approx 4$

i.e., electron excites to 4^{th} energy state

Total number of lines in emission spectrum would be

$$N = \frac{n(n-1)}{2} = \frac{4\times3}{2} = 6$$

Longest wavelength will correspond to the minimum energy and minimum energy is released in transition from $n = 4$ to $n = 3$.

$$E_{4\to3} = E_4 - E_3 = \frac{-13.6}{(4^2)} - \left(\frac{-13.6}{(3)^2}\right) = 0.66 \text{ eV}$$

Hence, longest wavelength will be,

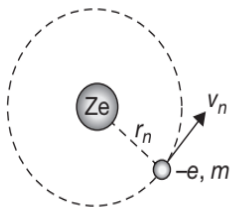
$$\lambda_{max} = \frac{12375}{E(\text{in eV})} = \frac{12375}{0.66} \text{ Å} = 1.875\times10^{-6} \text{ m}$$

$\Rightarrow \quad \lambda_{max} = 1.875\ \mu\text{m}$

HYPOTHETICAL BOHR'S MODEL

Consider a hypothetical hydrogen like atom for which an electron is revolving around the nucleus under the influence of a new potential energy field given as $U = f(r)$. This potential energy field, in this case, can be assumed to be a non-coulomb field. The force acting on the electron is then given by $F = -\frac{dU}{dr}$.

This force is responsible for providing the necessary centripetal force to the electron to revolve in a circle of radius r as shown.



$$\Rightarrow \quad F = \left|-\frac{dU}{dr}\right| = \frac{mv_n^2}{r_n} \qquad \ldots(1)$$

According to Bohr's Quantisation Rule, we have

$$mv_n r_n = \frac{nh}{2\pi} \qquad \ldots(2)$$

Now using equations (1) and (2), we can derive all the properties for electron motion like, radius of n^{th} orbit, velocity of electron in n^{th} orbit, angular velocity, frequency, time period, current, magnetic induction, magnetic moment and the total energy of energy levels for this hypothetical atom in the same way we've derived these for properties for a general hydrogen like atom.

ILLUSTRATION 22

A small particle of mass m moves in such a way that the potential energy $U=-\frac{1}{2}mb^2r^2$, where b is a constant and r is the distance of the particle from the origin taken at the nucleus. Assuming Bohr model of quantization of angular momentum and circular orbits, show that radius of the n^{th} allowed orbit is proportional to $\sqrt{n}$.

SOLUTION

Force on mass m in conservative field is

$$F=-\frac{dU}{dr}=mb^2r$$

For circular orbit of particle, we have

$$mb^2r=\frac{mv^2}{r} \qquad \ldots(1)$$

$$\Rightarrow \quad v=br$$

Also, by Bohr's Quantisation rule, we have

$$mvr=\frac{nh}{2\pi} \qquad \ldots(2)$$

$$\Rightarrow \quad m(br)r=\frac{nh}{2\pi}$$

$$\Rightarrow \quad r=\sqrt{\frac{nh}{2\pi mb}}$$

ILLUSTRATION 23

Assume a hypothetical hydrogen atom in which the potential energy between electron and proton at separation r is given by $U=k\left(\log_e r-\frac{1}{2}\right)$, where k is a constant. For such a hypothetical hydrogen atom, calculate the radius of nth Bohr's orbit and energy levels.

SOLUTION

Force of interaction between electron and proton is given by

$$F=-\frac{dU}{dr}=-\frac{k}{r}$$

Force is negative. It means there is attraction between the particles and they are bound to each other. This force provides the necessary centripetal force for the electron. So, we have

$$\frac{mv^2}{r}=\frac{k}{r} \qquad \ldots(1)$$

According to Bohr's assumption, we have

$$mvr=n\left(\frac{h}{2\pi}\right) \qquad \ldots(2)$$

Solving equations (1) and (2), we get

$$r=\frac{nh}{2\pi\sqrt{mk}} \text{ and } v=\sqrt{\frac{k}{m}}$$

Since, $E=U+\frac{1}{2}mv^2$

$$\Rightarrow \quad E=k\log_e r-\frac{k}{2}+\frac{k}{2}=k\log_e r$$

So, $r_n=\frac{nh}{2\pi\sqrt{mk}}$ and $E_n=k\log_e\left(\frac{nh}{2\pi\sqrt{mk}}\right)$

ILLUSTRATION 24

Suppose that the potential energy between electron and proton at a distance r in a hypothetical hydrogen atom is given by $\frac{-ke^2}{3r^3}$. Use Bohr's theory to obtain energy levels of such a hypothetical atom.

SOLUTION

For the hypothetical atom, the potential energy of the electron revolving in the n^{th} orbit is given by

$$U=-\frac{ke^2}{3r^3}$$

The force on the electron in this potential field is given by

$$F=-\frac{dU}{dr}=\frac{ke^2}{r^4}$$

This force provides the necessary centripetal force to the electron to revolve in a circle of radius r in the n^{th} orbit with a speed v, so we have

$$\frac{mv^2}{r}=\frac{ke^2}{r^4}$$

$$\Rightarrow \quad mv^2=\frac{ke^2}{r^3} \qquad \ldots(1)$$

Applying Bohr's Quantisation Rule i.e. $mvr = \dfrac{nh}{2\pi}$

$$\Rightarrow \quad v = \frac{nh}{2\pi mr} \qquad \ldots(2)$$

Substituting equation (2) in (1), we get

$$m\left(\frac{nh}{2\pi mr}\right)^2 = \frac{ke^2}{r^3}$$

$$\Rightarrow \quad r = r_n = \frac{4\pi^2 ke^2 m}{n^2h^2} \text{ and } v = v_n = \frac{n^3h^3}{8\pi^3 km^2e^2}$$

Now energy in n^{th} orbit is $E = E_n = U + K$

$$\Rightarrow \quad E_n = -\frac{ke^2}{3r^3} + \frac{1}{2}mv^2 \qquad \ldots(3)$$

From equation (1), we get $\dfrac{1}{2}mv^2 = \dfrac{ke^2}{2r^3}$

Substituting this value of kinetic energy in equation (3), we get

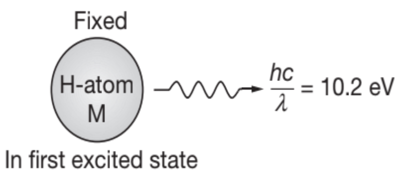
$$E_n = -\frac{ke^2}{3r^3} + \frac{ke^2}{2r^3} = \frac{1}{6}\left(\frac{ke^2}{r^3}\right)$$

$$\Rightarrow \quad E_n = \frac{1}{6}ke^2\left(\frac{n^2h^2}{4\pi^2ke^2m}\right)^3 = \frac{n^6h^6}{384\pi^6k^2e^4m^3}$$

RECOIL OF AN ATOM DUE TO ELECTRON TRANSITION

When an electron makes a transition from a higher energy level to a lower energy level, then a photon of wavelength λ is emitted by the atom. Since, the emitted photon possesses momentum $p = \dfrac{h}{\lambda}$, then the atom may or may not recoil depending upon whether it is fixed (kind of stationary) or is free to move. To understand this completely, let is discuss the following cases.

(a) When the hydrogen atom (of mass M) emitting the photon is fixed, then the entire energy equal to the energy difference of the two levels (between which the electron transition takes place) is given to the photon as shown.



If an electron makes a transition from $n_i = 2$ to $n_f = 1$, then the energy difference between the two levels is $\Delta E = (-3.4) - (-13.6) = 10.2 \text{ eV}$.

So, the emitted photon possesses this energy and hence we have $\dfrac{hc}{\lambda} = 10.2 \text{ eV}$, where $\dfrac{1}{\lambda} = R_H\left(\dfrac{1}{(1)^2} - \dfrac{1}{(2)^2}\right) = \dfrac{3R_H}{4}$.

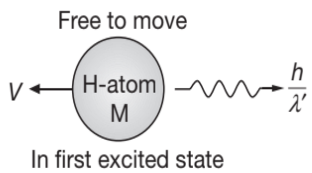
For hydrogen like atoms having atomic number Z, (like He^+ atom, we have $Z = 2$), if the electron de-excites from higher energy level n_i to a lower energy level n_f, then we have

$$\frac{1}{\lambda} = R_H Z^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$\Rightarrow \quad \frac{hc}{\lambda} = (R_H ch)Z^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

where $R_H = 1.097 \times 10^7 \text{ m}^{-1}$

(b) However, when the hydrogen atom (initially at rest) is free to move, then this entire energy equal to the energy difference of the two levels (between which the electron transition takes place) is distributed between the photon and the hydrogen atom, because the hydrogen atom recoils due to the emission of the photon.



Due to this, the energy possessed by the photon is slightly less than the case when the atom was assumed to be fixed. So, if λ' is the new wavelength of the emitted photon, then we have $\lambda' > \lambda$.

If the electron in the hydrogen atom of mass M, de-excites from $n_i = 2$ to $n_f = 1$, then a photon of energy $\Delta E = 10.2 \text{ eV}$ is emitted and hence the hydrogen atom recoils with a speed V. Then according to Conservation of Linear Momentum, we have

$$0 = MV + \frac{h}{\lambda'}$$

$$\Rightarrow \quad |V| = \frac{h}{M\lambda'} \qquad \ldots(1)$$

By Law of Conservation of energy, we have

$$\frac{1}{2}MV^2 + \frac{hc}{\lambda'} = \Delta E = 10.2 \text{ eV}$$

Since mass of hydrogen atom is very large than photon, so the recoil speed of the hydrogen atom is neglected and hence $\frac{1}{2}MV^2$ can also be neglected.

$$\Rightarrow \quad \frac{hc}{\lambda'} = \Delta E = 10.2 \text{ eV}$$

$$\Rightarrow \quad \frac{h}{\lambda'} = \frac{\Delta E}{c} = \frac{10.2}{c} \text{ eV}$$

$$\Rightarrow \quad \frac{h}{\lambda'} = MV = \frac{\Delta E}{c} = \frac{10.2}{c} \text{ eV}$$

$$\Rightarrow \quad V = \frac{\Delta E}{Mc} = \frac{10.2}{Mc} \text{ ms}^{-1}$$

In general, for hydrogen like atoms having atomic number Z, (like He^+ atom, we have $Z = 2$), if the electron de-excites from higher energy level n_i to a lower energy level n_f, then $\Delta E = (R_H ch) Z^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$ and hence the recoil speed is now given by

$$V_{\text{H like}} = \frac{\Delta E}{(M_{\text{H like}})c}$$

ILLUSTRATION 25

An isolated hydrogen atom emits a photon of 10.2 eV.

(a) Determine the momentum of photon emitted

(b) Calculate the recoil momentum of the atom

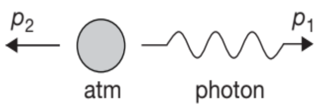
(c) Find the kinetic energy of the recoil atom. [Mass of proton, $m_p = 1.67 \times 10^{-27}$ kg]

SOLUTION

(a) Momentum of the photon is

$$p_1 = \frac{E}{c} = \frac{10.2 \times 1.6 \times 10^{-19}}{3 \times 10^8} = 5.44 \times 10^{-27} \text{ kgms}^{-1}$$

(b) Applying the Law of Conservation of Linear Momentum, we get



$$p_2 = p_1 = 5.44 \times 10^{-27} \text{ kgms}^{-1}$$

(c) $K = \frac{1}{2}mv^2$ (v = recoil speed of atom, m = mass of hydrogen atom)

$$\Rightarrow \quad K = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$$

Substituting the value of the momentum of atom, we get

$$K = \frac{(5.44 \times 10^{-27})^2}{2 \times 1.67 \times 10^{-27}} = 8.86 \times 10^{-27} \text{ J}$$

ILLUSTRATION 26

Light from a discharge tube containing hydrogen atoms falls on the surface of a piece of sodium. The kinetic energy of the fastest photoelectrons emitted from sodium is **0.73 eV**. The work function for sodium is **1.82 eV**. Calculate the

(a) the energy of the photons causing the photoelectrons emission.

(b) the quantum numbers of the two levels involved in the emission of these photons

(c) the change in the angular momentum of the electron in the hydrogen atom, in the above transition, and

(d) the recoil speed of the emitting atom assuming it to be at rest before the transition. (Ionization potential of hydrogen is 13.6 eV).

SOLUTION

(a) From Einstein's equation of photoelectric effect,

$$\begin{pmatrix} \text{Photon} \\ \text{energy} \\ \text{causing} \\ \text{photo-} \\ \text{electric} \\ \text{emission} \end{pmatrix} = \begin{pmatrix} \text{Maximum} \\ \text{kinetic} \\ \text{energy of} \\ \text{emitted} \\ \text{photons} \end{pmatrix} + \begin{pmatrix} \text{Work} \\ \text{function} \\ \text{of sodium} \end{pmatrix}$$

$$\Rightarrow \quad E = K_{max} + W = (0.73 + 1.82) \text{ eV}$$

$$\Rightarrow \quad E = 2.55 \text{ eV}$$

(b) In case of a hydrogen atom,

$$E_1 = -13.6 \text{ eV}, \; E_2 = -3.4 \text{ eV},$$

$$E_3 = -1.5 \text{ eV}, \; E_4 = -0.85 \text{ eV}$$

Since, $\Delta E = E_4 - E_2 = 2.55$ eV, therefore, quantum numbers of the two levels involved in the emission of these photons are 4 and 2 i.e., from $4 \rightarrow 2$.

(c) Change in angular momentum transition from 4 to 2 will be

$$\Delta L = L_2 - L_4 = 2\left(\frac{h}{2\pi}\right) - 4\left(\frac{h}{2\pi}\right)$$

$$\Rightarrow \quad \Delta L = -\frac{h}{\pi}$$

(d) By Law of Conservation of Linear Momentum, we have

$$\left(\begin{array}{c}\text{Momentum of}\\ \text{hydrogen atom}\end{array}\right) = \left(\begin{array}{c}\text{Momentum of}\\ \text{emitted photon}\end{array}\right)$$

$$\Rightarrow \quad Mv = \frac{\Delta E}{c} \quad (M = \text{mass of hydrogen atom})$$

$$\Rightarrow \quad v = \frac{\Delta E}{Mc}$$

$$\Rightarrow \quad v = \frac{(2.55)(1.6\times10^{-19})}{(1.67\times10^{-27})(3\times10^{8})}$$

$$\Rightarrow \quad v = 0.814 \text{ ms}^{-1}$$

ATOMIC COLLISIONS

The electron of an atom can be excited from lower energy level to a higher energy level by making a photon or electromagnetic radiation to fall on the atom (as already discussed). This radiation or photon must have an energy corresponding to the energy difference of the levels between which the electron transition takes place.

However, the electron in an atom can also be excited by colliding the atom with a particle or another atom. This is called as **Atomic Collision**. *In atomic collisions*, the loss in kinetic energy of the system (colliding particle and atom) is possible only when this loss in energy is sufficient enough to

(a) either excite the electron to a higher energy level or

(b) ionise the atom i.e. send an electron to infinity.

So, when an elementary particle collides with an atom, then the atom can be excited to an energy level above its ground state. In this process, a part of their combined kinetic energy (i.e., sum of K.E. of atom and incoming particle) is absorbed by the atom. *Also, this loss in energy is not converted to any other form of energy (like heat etc) during the collision and we have assumed the collisions to be head-on.*

Before we discuss further, we must remember the energies of electron in the different levels of the atom. Since, we know that for hydrogen like atoms, $E_n = -\frac{13.6Z^2}{n^2}$. However, for hydrogen atom, we have $Z = 1$, so energies of electron in different levels is

$$\left.\begin{array}{l}\left.\begin{array}{l}\left.\begin{array}{l}E_1 = -13.6 \text{ eV}\\ E_2 = -3.4 \text{ eV}\end{array}\right] \Delta E_{12} = 10.2\\ E_3 = -1.51 \text{ eV}\end{array}\right] \Delta E_{13} = 12.09\\ E_4 = -0.85 \text{ eV}\end{array}\right] \Delta E_{14} = 12.75$$

$E_5 = -0.55$ eV, $E_6 = -0.38$ eV, $E_7 = -0.28$ eV

$E_8 = -0.20$ eV, $E_9 = -0.17$ eV, $E_{10} = -0.14$ eV

For quickly handling problems, we must keep in mind that

$$\Delta E_{12} = E_2 - E_1 = 10.2 \text{ eV},$$

$$\Delta E_{13} = E_3 - E_1 = 12.09 \text{ eV} \text{ and}$$

$$\Delta E_{14} = E_4 - E_1 = 12.75 \text{ eV}$$

ILLUSTRATION 27

A neutron having kinetic energy (K), collides head-on with a stationary H-atom. Find the nature of collision (elastic/inelastic/perfectly inelastic), when the neutron possesses a

(i) kinetic energy of 12 eV
(ii) kinetic energy of 20.4 eV
(iii) kinetic energy of 22 eV
(iv) kinetic energy of 24.18 eV

SOLUTION

To find nature of collision, we must first know about loss in energy of the system is minimum for an elastic collision and maximum for a perfectly inelastic collision (where the bodies after collision stick to each other and move as one single body). For an elastic collision, $\Delta E = 0$ and for a perfectly inelastic collision, we have $|\Delta E| = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 - u_2)^2$.

$$\Rightarrow \quad 0 \le |\Delta E| \le \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 - u_2)^2$$

$$\Rightarrow \quad 0 \le |\Delta E| \le \frac{1}{2}\left(\frac{m_n m_\mu}{m_n + m_\mu}\right)u^2$$

Since, we know that $m_H \approx m_n = m$ (say), so we get

$$0 \le |\Delta E| \le \frac{1}{2}\left(\frac{m}{2}\right)u^2$$

$$\Rightarrow \quad 0 \le |\Delta E| \le \frac{K}{2} \qquad \left\{\because K = \frac{1}{2}m_n u^2 = \frac{1}{2}mu^2\right\}$$

So, loss in energy, $|\Delta E|$, must lie in the range zero to $\frac{K}{2}$.

(i) When the kinetic energy of neutron is $K = 12$ eV

Since, we have calculated that $0 \le |\Delta E| \le \frac{K}{2}$

$\Rightarrow \quad 0 \le |\Delta E| \le 6$ eV

Maximum Loss in kinetic energy for $K = 12$ eV is 6 eV. This loss in energy is less than 10.2 eV, which is the energy required to excite the electron from ground state $(n_1 = 1)$ to first excited state $(n_2 = 2)$. This loss in energy is not sufficient enough to be absorbed by the electron, so that it can go from ground state to first excited state. So, no energy loss will place and hence for $K = 12$ eV, the collision is elastic.

(ii) When the kinetic energy of neutron is $K = 20.4$ eV

Since, we have calculated that $0 \le |\Delta E| \le \frac{K}{2}$

$\Rightarrow \quad 0 \le |\Delta E| \le 10.2$ eV

When $|\Delta E| = 0$, then the collision is elastic.

For $K = 20.4$ eV, the maximum loss in kinetic energy is $|\Delta E| = 10.2$ eV. This loss in energy is equal to the energy required to excite the electron from ground state $(n_1 = 1)$ to first excited state $(n_2 = 2)$. *So, collision is perfectly inelastic* i.e. the neutron will collide with the hydrogen atom (at rest) and both move together as one single body. In this process the neutron looses 10.2 eV kinetic energy to the electron, so that the electron gets excited from ground state to first excited state and the combined system (neutron and atom) moves with a kinetic energy of $(20.4 - 10.2)$ eV $= 10.2$ eV

(iii) When the kinetic energy of neutron is $K = 22$ eV

Since, we have calculated that $0 \le |\Delta E| \le \frac{K}{2}$

$\Rightarrow \quad 0 \le |\Delta E| \le 11$ eV

$\Rightarrow \quad 0 \le \Delta E \le 11$ eV > 10.2 eV

In this case, two possibilities arise.

Firstly, when the loss in energy is zero i.e. then the collision is elastic.

Secondly, out of 11 eV loss in energy of the system, the electron will take up 10.2 eV energy to get excited from ground state to first excited state and both neutron and hydrogen atom will be moving with kinetic energy equal to the remaining energy i.e. $(11 - 10.2) = 0.8$ eV.

So, the collision is an inelastic collision.

(iv) When the kinetic energy of neutron is K = 24.18 eV

Since, we have calculated that $0 \le |\Delta E| \le \frac{K}{2}$

$\Rightarrow \quad 0 \le |\Delta E| \le 12.09$ eV

In this case, three possibilities arise.

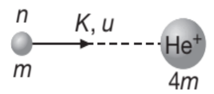
Firstly, when the loss in energy is zero i.e. then the collision is elastic.

Secondly, if $\Delta E = 10.2$ eV, then collision is inelastic because 10.2 eV excites the electron from ground state to first excited state and the remaining energy is shared by both neutron and atom, so that both move independent of each other with different velocity after collision.

Thirdly, if $\Delta E = 12.09$ eV, then collision is perfectly inelastic because 12.09 eV lost by the neutron will just excite the atom from $n = 1$ to $n = 3$ and both will move with a common velocity after collision.

ILLUSTRATION 28

A He^+ ion is at rest and in ground state. A neutron with initial velocity u, kinetic energy K collides head on with the He^+ ion. Calculate minimum value of K so that there can be an inelastic collision between these two particles.



SOLUTION

For He^+, we have $E_n = -\frac{13.6Z^2}{n^2}$ eV

$$\Rightarrow \quad E_n = -\frac{13.6(4)}{n^2} = -\frac{54.4}{n^2} \text{ eV}$$

From this relation, we get

$$E_1 = -54.4 \text{ eV}$$

$$E_2 = 13.6 \text{ eV} \quad \Rightarrow E_2 - E_1 = 40.8 \text{ eV}$$

$$E_3 = -6.04 \text{ eV} \quad \Rightarrow E_3 - E_1 = 48.36 \text{ eV}$$

$$E_4 = -3.4 \text{ eV} \quad \Rightarrow E_4 - E_1 = 51 \text{ eV}$$

$$E_\infty = 0 \quad \Rightarrow E_\infty - E_1 = 54.4 \text{ eV}$$

So, possible losses in energy can only be

$$\Delta E = \{40.8 \text{ eV}, 48.36 \text{ eV}, 51 \text{ eV}, \ldots\ldots, 54 \text{ eV}\}$$

Now minimum loss $(= \text{zero})$ will be for an elastic collision and maximum loss will be for perfectly inelastic collision.

$$\Rightarrow \quad \text{Loss} = \frac{1}{2}\frac{(m)(4m)}{4m+m}(u-0)^2 = \frac{1}{2}\left(\frac{4m}{5}\right)u^2$$

$$\Rightarrow \quad \text{Loss} = \frac{4}{5}\left(\frac{1}{2}mu^2\right) = \frac{4K}{5}$$

$$\Rightarrow \quad 0 \le \Delta E \le \frac{4K}{5}$$

Now, when $\Delta E = 0$, the collision is elastic.

Also, for inelastic collision $\frac{4K}{5} > 40.8$ eV

$$\Rightarrow \quad K > 51 \text{ eV}$$

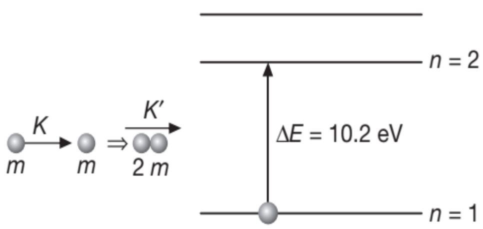
Because at $\Delta E = 40.8$ eV, perfectly inelastic collision will take place.

ILLUSTRATION 29

A moving hydrogen atom makes a head on collision with a stationary hydrogen atom. Before collision both atoms are in ground state and after collision they move together. What is the minimum value of the kinetic energy of the moving hydrogen atom, such that one of the atoms reaches one of the excitation state?

SOLUTION

Let K be the kinetic energy of the moving hydrogen atom and K', the kinetic energy of combined mass after collision.



By Law of Conservation of Linear Momentum, we have

$$p = p'$$

$$\Rightarrow \quad \sqrt{2Km} = \sqrt{2K'(2m)}$$

$$\Rightarrow \quad K = 2K' \qquad \ldots(1)$$

From Conservation of Energy, $K = K' + \Delta E$ ...(2)

Solving equations (1) and (2), we get

$$\Delta E = \frac{K}{2}$$

Now minimum value of ΔE for hydrogen atom is 10.2 eV, so we have

$$\Delta E \ge 10.2 \text{ eV}$$

$$\Rightarrow \quad \frac{K}{2} \ge 10.2$$

$$\Rightarrow \quad K \ge 20.4 \text{ eV}$$

Therefore, the minimum kinetic energy of moving hydrogen is 20.4 eV.

ILLUSTRATION 30

A 100 eV electron collides with a stationary helium ion (He^+) in its ground state and excites to a higher level. After the collision, He^+ ions emit two photons in succession with wavelength 1085 Å and 304 Å. Find the principal quantum number of the excited state. Also calculate the energy of the electron after the collision. Given $h = 6.63 \times 10^{-34}$ Js.

SOLUTION

The energy of the electron in the n^{th} state of He^+ ion of atomic number $Z(=2)$ is given by

$$E_n = -(13.6 \text{ eV})\frac{Z^2}{n^2}$$

$$\Rightarrow \quad E_n = -\frac{(13.6 \text{ eV}) \times (2)^2}{n^2}$$

$$\Rightarrow \quad E_n = -\frac{54.4}{n^2} \text{ eV} \qquad \ldots(1)$$

The energies E_1 and E_2 of the two emitted photons in eV are

$$E_1 = \frac{12431}{1085} \text{ eV} = 11.4 \text{ eV}$$

and $$E_2 = \frac{12431}{304} \text{ eV} = 40.9 \text{ eV}$$

So, total energy is

$$E = E_1 + E_2 = 11.4 + 40.9 = 52.3 \text{ eV}$$

For the transition from $n_i = n$ to $n_f = 1$, we have

$$\Delta E = -(54.4 \text{ eV})\left(\frac{1}{1^2} - \frac{1}{n^2}\right)$$

Since, $\Delta E = 52.3$ eV

$$\Rightarrow \quad 52.3 \text{ eV} = 54.4 \text{ eV} \times \left(1 - \frac{1}{n^2}\right)$$

$$\Rightarrow \quad 1 - \frac{1}{n^2} = \frac{52.3}{54.4} = 0.96$$

$$\Rightarrow \quad n^2 = 25$$

$$\Rightarrow \quad n = 5$$

The energy of the incident electron is given to be 100 eV. The energy supplied to He^+ ion is 52.3 eV. So, the energy of the electron left after the collision is $100 - 52.3 = 47.7$ eV.

ILLUSTRATION 31

A moving H-atom makes a head on perfectly inelastic collision with a stationary Li^{++} ion. Before collision H-atom and Li^{++} ion are both in their first excited states. What is the velocity of the moving H atom if after collision H is found in its ground state and Li^{++} ion in its second excited state. Take mass of hydrogen atom, $m_H = 1.66 \times 10^{-27}$ kg and mass of Li^{++} to be $m_{Li^{++}} = 7m_H$.

SOLUTION

For Li^{++}, we have $Z = 3$

$$\Rightarrow \quad E_2 = \frac{-13.6(3)^2}{(2)^2} = -30.6 \text{ eV}$$

$$\Rightarrow \quad E_3 = \frac{-13.6(3)^2}{(3)^2} = -13.6 \text{ eV}$$

Energy required for Li^{++}, ion to go from first excited state $(n = 2)$ to second excited state $(n = 3)$ is

$$\Delta E = -13.6 - (-30.6) = 17 \text{ eV}$$

Energy released by hydrogen atom to go from first excited state to ground state is

$$\Delta E' = -3.4 - (-13.6) = 10.2 \text{ eV}$$

So, $\Delta E - \Delta E' = 17 - 10.2 = 6.8$ eV is the energy that should come from loss in KE in collision.

From Law of Conservation of Linear Momentum, velocity of combined mass is

$$v = \frac{m_1 u_1}{m_1 + m_2} \qquad (m_2 = 7m_1)$$

$$\Rightarrow \quad \Delta KE = \frac{1}{2}m_1 u_1^2 - \frac{1}{2}(m_1 + m_2)v^2$$

Since, $\Delta K = 6.8$ eV

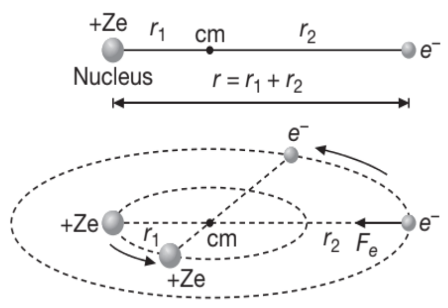
Solving these equations, we get

$$u_1 = 3.9 \times 10^4 \text{ ms}^{-1}$$

EFFECT OF MASS OF NUCLEUS ON BOHR MODEL

Till now, we had studied that in hydrogen atom or hydrogen like atoms, the nucleus is at rest and electron is revolving around this stationary nucleus. *Since no external force is acting on the electron nucleus system, so the centre of mass of the system remains at rest.* Theoretically, mass of electron is negligible or small compared to that of nucleus and hence the centre of mass of the atom is almost situated at nucleus. Due to this, in Bohr's atom model of hydrogen like atoms, the nucleus almost remains at rest and electron revolves around it.

But practically, the situation is a bit different. Actually, centre of mass of electron-nucleus system is close to nucleus (but not at the nucleus) because the nucleus is heavy and so, to keep the centre of mass of electron nucleus system at rest, both electron and nucleus revolve around their centre of mass just like the double star system as shown in figure.



So, in the atom, the nucleus and electron revolve around their centre of mass in concentric circles of radii r_1 and r_2 to keep centre of mass at rest. If r is the distance of electron from nucleus, the distances of nucleus and electron from the centre of mass are given by

$$r_1 = \frac{m_e r}{m_N + m_e} \text{ and } r_2 = \frac{m_N r}{m_N + m_e}$$

For the nucleus, we have

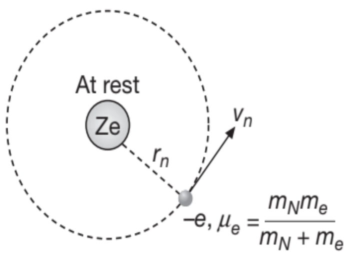
$$m_N r_1 \omega^2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{Ze^2}{r^2}\right)$$

$$\Rightarrow \quad m_N\left(\frac{m_e r}{m_N + m_e}\right)\omega^2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{Ze^2}{r^2}\right)$$

$$\Rightarrow \quad \left(\frac{m_N m_e}{m_N + m_e}\right) r\omega^2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{Ze^2}{r^2}\right)$$

$$\Rightarrow \quad \mu_e r\omega^2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{Ze^2}{r^2}\right)$$

So, in the above system, we can analyse the motion of electron with respect to nucleus by assuming the nucleus to be at rest and then replacing the mass of electron by the reduced mass of electron-nucleus system i.e. $\mu_e = \dfrac{m_N m_e}{m_N + m_e}$. Now the relative picture of atom will be same as that considered earlier as shown in figure. However, we have just replaced the mass of electron by the reduced mass of electron-nucleus system.



Now we can use all those relations which we've derived earlier for Bohr model just by replacing the mass of electron m_e by the reduced mass μ_e.

The radius of electron in n^{th} orbit of Bohr's model is given by

$$r_n = \frac{n^2 h^2 \varepsilon_0}{\pi m_e e^2 Z}$$

But if we consider the motion of nucleus into account, then radius of n^{th} orbit will be given by

$$r_n' = \frac{n^2 h^2 \varepsilon_0}{\pi \mu_e e^2 Z} = r_n\left(\frac{m_e}{\mu_e}\right)$$

Similarly, the speed of electron in n^{th} Bohr orbit is given by

$$v_n = \left(\frac{e^2}{2h\varepsilon_0}\right)\frac{Z}{n}$$

Since we note that, in the above expression of speed no term of m_e (mass of electron) is present, hence speed of electron in an orbit does not depend on electron mass. So, there will be no change in the speed of revolution of an electron in an orbit, if we consider the motion of nucleus into account.

Similarly, the expression of energy of electron in the n^{th} orbit of Bohr's model is given by

$$E_n = -\frac{m_e e^4 Z^2}{8n^2 h^2 \varepsilon_0^2}$$

But if we consider the motion of nucleus into account, then energy of electron in n^{th} orbit of Bohr's model is given by

$$E_n' = -\frac{\mu_e e^4 Z^2}{8n^2 h^2 \varepsilon_0^2} = E_n\left(\frac{\mu_e}{m_e}\right)$$

Thus, we can say that the energy of electron will be slightly less compared to what we've derived earlier. But for numerical calculations this small change can be ignored unless in a given problem, it is asked to consider the effect of motion of nucleus.

ABOUT RYDBERG CONSTANT

Since, we know that the Rydberg constant for hydrogen atom, when the mass of the nucleus (m_N) is very large compared to the mass of the electron (m_e), is given by

$$R_\infty = R = \frac{m_e e^4}{8\varepsilon_0^2 ch^3}$$

Please do not think that the Rydberg constant is same for all elements. The reason is that in Bohr's Theory, the nucleus is assumed infinitely heavy (and hence at rest) as compared to the electron. But if the mass of nucleus is taken into account, then the electron mass m_e has to replaced by reduced mass (μ_e), where $\mu_e = \dfrac{m_e m_N}{m_e + m_N}$.

$$\Rightarrow \quad \mu_e = \frac{m_e m_N}{m_e + m_N} = \frac{m_e}{1 + \dfrac{m_e}{m_N}}$$

Therefore, $R' = \dfrac{\mu_e e^4}{8\varepsilon_0^2 ch^3} = R\left(\dfrac{\mu_e}{m_e}\right)$

For a heavy nucleus, $m_N \to \infty$, then $\mu_e \to m_e$, so the Rydberg's Constant is represented by R_∞. Then we have

$$R_\infty = R = \frac{m_e e^4}{8\varepsilon_0^2 ch^3} = 1.097 \times 10^7 \text{ m}^{-1}$$

Rydberg's constant for an element is given by

$$R' = \frac{R_\infty}{\left(1+\frac{m_e}{m_N}\right)} = \frac{R}{\left(1+\frac{m_e}{m_N}\right)} = \frac{m_e e^4}{8\left(1+\frac{m_e}{m_N}\right)\varepsilon_0^2 ch^3}$$

Clearly this depends on mass of nucleus m_N and so Rydberg constant is different for different elements. Greater is m_N, larger is the value of Rydberg constant R'. Thus, Rydberg constant increases with increase in mass of nucleus.

ILLUSTRATION 32

The nucleus of a deuterium has a mass of 3.34×10^{-27} kg as compared to 1.67×10^{-27} kg for the hydrogen. Calculate the wavelength difference between the first Balmer line emitted by hydrogen and the first Balmer line emitted by deuterium. Given that the mass of electron is $m_e = 9.109 \times 10^{-31}$ kg.

SOLUTION

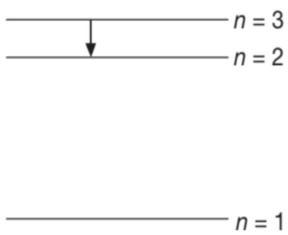
The first Balmer line corresponds to the transition from $n = 3$ to $n = 2$. In case of hydrogen atom, we have

$$\Delta E = E_3 - E_2$$

$$\Rightarrow \quad \Delta E = \left\{-\frac{13.6}{3^2} - \left(\frac{-13.6}{2^2}\right)\right\} \text{ eV} = 1.89 \text{ eV}$$

So, wavelength $\lambda = \dfrac{12375}{\Delta E(\text{in eV})}$

$$\Rightarrow \quad \lambda = \frac{12375}{1.89} = 6547.6 \text{ Å}$$



For ordinary hydrogen reduced mass of proton and electron is,

$$\mu_1 = \frac{(1.67 \times 10^{-27})(9.109 \times 10^{-31})}{(1.67 \times 10^{-27} + 9.109 \times 10^{-31})}$$

$$\Rightarrow \quad \mu_1 = 9.10408 \times 10^{-31} \text{ kg}$$

For deuterium atom reduced mass of nucleus and electron is,

$$\mu_2 = \frac{(3.34 \times 10^{-27})(9.109 \times 10^{-31})}{(3.34 \times 10^{-27} + 9.109 \times 10^{-31})}$$

$$\Rightarrow \quad \mu_2 = 9.10654 \times 10^{-31} \text{ kg}$$

All energies are proportional to μ, whereas the wavelengths are inversely proportional to μ. The wavelength of photon emitted in case of hydrogen is given by

$$\lambda_1 = \frac{(6547.6)(9.109 \times 10^{-31})}{(9.10408 \times 10^{-31})} = 6551 \text{ Å}$$

Similarly, in case of deuterium, wavelength of photon emitted is,

$$\lambda_2 = \frac{(6547.6)(9.109 \times 10^{-31})}{(9.10654 \times 10^{-31})} = 6549 \text{ Å}$$

$$\Rightarrow \quad \Delta\lambda = \lambda_2 - \lambda_1 = 2 \text{ Å}$$

ILLUSTRATION 33

Taking into account the motion of the nucleus of a hydrogen atom, find the expressions for the electron's binding energy in the ground state and for the Rydberg constant. How much (in percent) do the binding energy and the Rydberg constant, obtained without taking into account the motion of the nucleus, differ from the more accurate corresponding value of these quantities?

SOLUTION

If mass of nucleus is considered (not infinity), then the reduced mass of nucleus electron system can be taken as

$$\mu = \frac{mM}{m+M}$$

where, m is mass of electron and M is that of nucleus. The binding energy in ground state of hydrogen atom can now be given as

$$E = \frac{\mu e^4}{8h^2\varepsilon_0^2}$$

$$\Rightarrow \quad E = 13.6 \times \frac{\mu}{m} \text{ eV}$$

$$\Rightarrow \quad E = \frac{13.6M}{m+M} \text{ eV}$$

Since the hydrogen atom Rydberg constant is given by

$$R = \frac{me^4}{8\varepsilon_0^2 ch^3}$$

If effect of mass of nucleus is considered, the new value of Rydberg constant will be

$$R' = \frac{\mu e^4}{8\varepsilon_0^2 ch^3}$$

$$\Rightarrow \quad R' = \frac{RM}{m+M}$$

Percentage difference in the values of R and R' is

$$\frac{\Delta R}{R} = \left(\frac{R'-R}{R}\right)100\%$$

$$\Rightarrow \quad \frac{\Delta R}{R} = \frac{m}{M} \times 100 \approx 0.055\%$$

ILLUSTRATION 34

Calculate the separation between the particles of a system in the ground state, the corresponding binding energy and wavelength of first line in Lyman series if such a system is positronium consisting of an electron and positron revolving round their common centre of mass.

SOLUTION

Reduced mass $\mu = \frac{(m)(m)}{(m+m)} = \frac{m}{2}$

Since, $r \propto \frac{1}{m}$

For, H atom, we have

$$r_1 = 0.53 \text{ Å}$$

$$\Rightarrow \quad r = (2)(0.53) = 1.06 \text{ Å}$$

Since, $E \propto m$ {binding energy}

For, H atom, we have

$$E_1 = 13.6 \text{ eV}$$

$$\Rightarrow \quad E = \frac{13.6}{2} = 6.8 \text{ eV}$$

For, H atom, we have

$$E_{2\to 1} = 10.2 \text{ eV} \qquad \text{\{First line of Lyman series\}}$$

$$\Rightarrow \quad E_{21} = \frac{10.2}{2} = 5.1 \text{ eV}$$

$$\Rightarrow \quad \lambda = \frac{12375}{5.1} \text{ Å} = 2426 \text{ Å}$$

ILLUSTRATION 35

Determine the separation of the first line of the Balmer series in a spectrum of ordinary hydrogen and tritium (mass number 3). Take Rydberg's constant $R = 10967800 \text{ m}^{-1}$

SOLUTION

Since $R' = R_H\left(\frac{m_{\text{nucleus}}}{m_{\text{nucleus}} + m_{\text{electron}}}\right)$

First line of Balmer series of H-atom is

$$\frac{1}{\lambda_H} = R\left(\frac{1}{4} - \frac{1}{9}\right) \times \frac{m_H}{m_e + m_H}$$

First line of Balmer series for T-atom is

$$\frac{1}{\lambda_T} = R\left(\frac{1}{4} - \frac{1}{9}\right) \times \frac{m_T}{m_e + m_T}$$

$$\Rightarrow \quad \lambda_H - \lambda_T = \frac{36(m_e + m_H)}{5Rm_H} - \frac{36(m_e + m_T)}{5Rm_T}$$

$$\Rightarrow \quad \Delta\lambda = \frac{36}{5R}\left[\frac{(m_e + m_H)m_T - (m_e + m_T)m_H}{m_H m_T}\right]$$

$$\Rightarrow \quad \Delta\lambda = \frac{36}{5R}\left[\frac{m_e(m_T - m_H)}{m_H m_T}\right]$$

$$\Rightarrow \quad \Delta\lambda = \frac{36}{5 \times 10967800} \times \frac{9.109 \times 10^{-31} \times 2 \times 1.67 \times 10^{-27}}{3 \times (1.67 \times 10^{-27})}$$

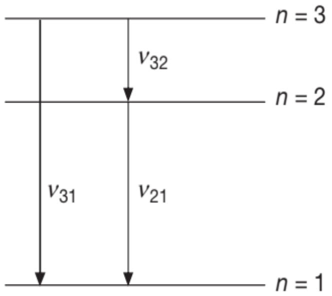
$$\Rightarrow \quad \Delta\lambda = 2.387 \text{ Å}$$

RITZ COMBINATION PRINCIPLE

If an electron is initially in an excited state with say $n = 3$, then it may transit downward from $n = 3$ level to $n = 1$ level directly. Alternatively, it may first transit from $n = 3 \to n = 2$ and subsequently from

$n=2 \rightarrow n=1$. In the first case if ν_{31} be the frequency of the photon emitted

$$h\nu_{31} = E_3 - E_1 \qquad \dots(1)$$



In the second case, two different spectral lines (photons) of frequency ν_{32} and ν_{21} respectively would be emitted given by

$$h\nu_{32} = E_3 - E_2 \text{ and } h\nu_{21} = E_2 - E_1 \qquad \dots(2)$$

(1) can be rewritten as

$$h\nu_{31} = (E_3 - E_2) + (E_2 - E_1)$$

$$\Rightarrow \quad h\nu_{31} = h\nu_{32} + h\nu_{21}$$

$$\Rightarrow \quad \nu_{31} = \nu_{32} + \nu_{21}$$

Ritz made this discovery empirically (1908) long before Bohr proposed his theory and is known as Ritz combination principle.

Generalising, we may write, labelling the photon frequency by appropriate integers, as follows:

$$h\nu_{sm} = E_s - E_m$$

$$\Rightarrow \quad h\nu_{sm} = (E_s - E_n) + (E_n - E_m)$$

$$\Rightarrow \quad h\nu_{sm} = h\nu_{sn} + h\nu_{nm} \quad (m < n < s) \qquad \dots(3)$$

Since all combinations predicted by (3) are not actually observed, there has been an imposition of some rules, called selection rules, to eliminate certain combinations. Bohr's theory provides, as discussed above, a proper explanation of the combination principle.

Conceptual Note(s)

(a) If an electron in a single atom jumps to a level having principal quantum number n, then the maximum number of photons (emitted) will be $(n-1)$.

(b) If in a hydrogen like sample, the electrons of many atoms jump to a level having principal quantum number n, then the maximum number of photons emitted $= {}^nC_2 = \frac{n(n-1)}{2}$.

(c) For example if many of the atoms are in third excited state ($n = 4$), then the number of photons emitted for spectral lines seen is $N = \frac{4 \times 3}{2} = 6$

(d) If an atom goes to excited state by absorbing certain energy, then it may emit a number of photons in succession; the sum of the energies of all emitted photons will be equal to the amount of energy absorbed. For example if an atom absorbing energy E reaches to an excited level and it returns to ground state by emitted wavelength λ_1, λ_2 and λ_3, then

$$E = \frac{hc}{\lambda_1} + \frac{hc}{\lambda_2} + \frac{hc}{\lambda_3}$$

(e) If an energetic electron strikes an electron (target electron) of an atom in ground state then electron jumps to an excited state by absorbing energy equal to the difference of ground state energy and excited state energy and the remaining energy is still carried by the incident (or striking) electron.

(f) The de-Broglie quantum condition: According to de-Broglie, only those orbits are allowed as stationary orbits in which circumference of orbit is equal to the integral multiple of de-Broglie wavelength.

i.e., $2\pi r = n\lambda$

$$\Rightarrow \quad 2\pi r = n\left(\frac{h}{mv}\right)$$

$$\Rightarrow \quad mvr = n\left(\frac{h}{2\pi}\right)$$

This is same as Bohr's quantum condition.

(g) "An atom possesses discrete levels" was verified by Franck Hertz Experiment.

MERITS AND DEMERITS OF BOHR'S THEORY

The merits of Bohr's theory can hardly be overestimated. It saved physics at a time when it was in the grip of severe crisis. But, like every physical theory, this theory also has drawbacks. We enumerate below some of its merits and demerits.

Merits

(a) The determination of the ratio of the mass of an electron to that of a proton in terms of the reduced mass concept of Bohr's theory agrees excellently with the value obtained by other methods.

(b) The general principle used by Bohr has also been successfully applied to a great number of phenomena such as the excitation and ionisation of atoms, X-ray spectra etc.

(c) The validity of the theory is further confirmed by the fact that the theory predicts new undiscovered series lines (spectral) which have later been actually observed.

(d) The theory has been instrumental to the discovery of heavy hydrogen (deuterium) by H.C. Urey.

(e) It gives a convincing explanation and a very simple and elegant picture of the origin of spectral lines.

(f) The agreement between the empirically determined value of the Rydberg constant and that evaluated by Bohr in terms of fundamental constants offers an excellent proof of the truth of Bohr's theory.

Demerits

(a) There is an ad hoc nature in the assumptions of Bohr in that the quantum idea of the stationary orbits is mixed up with the classical idea of coulomb force.

(b) The assumption of only circular orbits is utterly unjustified. In fact, Bohr in his original paper suggested that the orbit might be an ellipse instead of a circle.

(c) The spectral series, though agree excellently in case of hydrogen, are at variance with the theory for multi-electron atomic systems, e.g. the helium, singly ionised lithium etc. In these cases, it becomes necessary to introduce a magnetic quantum number.

(d) It cannot suggest any explanation whatsoever for the origin of the fine structure of the spectral lines.

(e) Bohr's theory is also unable to account for the multiple structure of spectral lines. For example, the doublet of sodium, triplets of magnesium etc. cannot be explained from Bohr's theory.

(f) It cannot make any calculation about the transitions or the selection rules which apply to them.

(g) It could not explain the splitting up of spectral lines when an atom is subjected to electric field (phenomenon called Stark Effect) or magnetic field (phenomenon called Zeeman Effect).

CRITICAL POTENTIAL

The resonance potential, the excitation potentials and the ionisation potentials are all included in the wider term critical potentials.

RESONANCE POTENTIAL

A minimum potential V is required to accelerate the bombarding electron to an energy V (in electron-volt) in order that an atom may be excited from its ground state to the next higher state. This potential is called the resonance potential.

EXCITATION POTENTIAL

The various values of the potential required to impart the necessary energy to excite an atom to different higher states are known as excitation potentials.

IONISATION POTENTIAL

The minimum potential necessary to supply the required energy to ionise an atom is called the ionisation potential or the first ionisation potential.

Problem Solving Technique(s)

Let us illustrate the above definitions by taking the case of hydrogen atom. For H atom, we have

$$E_n = -\frac{me^4}{8\varepsilon_0^2 n^2 h^2} = -\frac{2.17\times 10^{-18}}{n^2} J = -\frac{13.6}{n^2} \text{eV}$$

So the energy of the 1st, 2nd, 3rd,......., ∞-th orbits are respectively –13.6 eV, –3.4 eV, –1.15 eV,...., 0 eV. Hence,

Resonance potential $= -3.4 - (-13.6) = 10.2$ eV

First excitation potential = resonance potential = 10.2 eV

Second excitation potential $= -1.51 - (-13.6) = 12.09$ eV

Ionisation potential $= 0 - (-13.6) = 13.6$ eV

ILLUSTRATION 36

Find the ratio of ionization energy of Bohr's hydrogen atom and hydrogen-like lithium atom.

SOLUTION

Energy of an electron in n^{th} state of Bohr's hydrogen-like atom (of atomic number z) is given by,

$$E = -\left(\frac{13.6Z^2}{n^2}\right)\text{ eV}$$

The ionization energy of this atom is equal to the magnitude of the energy of electron in the ground state i.e. $E_\infty = 13.6Z^2$

$$\Rightarrow \quad \frac{(E_\infty)_H}{(E_\infty)_{Li}} = \frac{(Z_H)^2}{(Z_{Li})^2}$$

$$\Rightarrow \quad \frac{(E_\infty)_H}{(E_\infty)_{Li}} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

ILLUSTRATION 37

Find the quantum number n corresponding to excited state of He^+ ion if on transition to the ground state, the ion emits two photons in succession with wavelengths 108.5 nm and 30.4 nm. The ionization energy of H atom is 13.6 eV.

SOLUTION

The energy transitions for the given wavelengths are

$$\Delta E_1 = \frac{12400}{\lambda_1} = \frac{12400}{1085} = 11.43\text{ eV}$$

$$\Delta E_2 = \frac{12400}{\lambda_2} = \frac{12400}{304} = 40.79\text{ eV}$$

Total energy emitted $\Delta E = \Delta E_1 + \Delta E_2 = 52.22$ eV

$$\Rightarrow \quad \Delta E = 13.6Z^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\text{ eV}$$

where, ΔE is the energy emitted

$$\Rightarrow \quad 52.34 = 13.6 \times 2^2\left(\frac{1}{1^2} - \frac{1}{n^2}\right)$$

Thus, $n = 5$

Test Your Concepts-I

Based on Atomic Structure and Properties

(Solutions on page H.38)

1. A doubly ionised lithium atom is hydrogen like with atomic number 3.
   (a) Find the wavelength of the radiation require to excite the electron in Li^{++} from the first to the third Bohr orbit (Ionisation energy of the hydrogen atom equals 13.6 eV).
   (b) How many spectral lines are observed in the emission spectrum of the above excited system.
2. Find the ionization energy of a doubly ionized lithium atom.
3. An electron and a proton are separated by a large distance and the electron approaches the proton with a kinetic energy of 2 eV. If the electron is captured by the proton to form a hydrogen atom in the ground state, what wavelength photon would be given off?
4. A 12.5 meV α-particle approaching a gold nucleus (Z = 79) is deflected by 180°. How close does it approach the nucleus?
5. If the average life time of an excited state of hydrogen is of the order of 10^{-8} s, estimate how many orbits an electron makes when it is in the state n = 2 before it suffers a transition to state $n = 1$.
6. How many times does the electron go round the first Bohr orbit of hydrogen atom in 1 s?
7. Suppose potential energy between electron and proton at separation r is given by $U = -k\log_e r$, where k is a constant. For such a hypothetical hydrogen atom, calculate the radius of n^{th} Bohr's orbit and its energy levels.
8. Certain gas of identical hydrogen like atoms has all its atoms in a particular upper energy level. The

atoms make transition to a higher energy level when a monochromatic radiation, having wavelength 1654 Å, is incident upon it. Subsequently, the atoms emit radiation of only three different photon energies.

(a) Identify the atom
(b) Obtain the ionization energy for the gas atoms.
(c) If the atoms of the gas are to be excited to such a level which gives radiation of only six different photon energies, what should be energy of incident radiation.

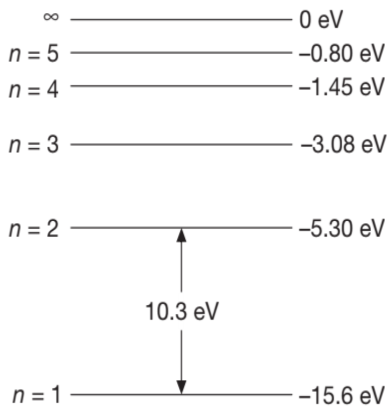
9. Electrons are emitted from an electron gun at almost zero velocity and are accelerated by an electric field E through a distance of 1 m. The electrons are now scattered by an atomic hydrogen sample in grounds state. What should be the minimum value of E so that red light of wavelength 6563 Å may be emitted in the hydrogen?

10. A hot gas emits radiation of wavelengths 460 Å, 831 Å and 1035 Å only. Assume that the atoms have only two excited states and the difference between consecutive energy levels decreases as energy is increased. Taking the energy of the highest energy state to be zero. Find the energies of the ground state and the first excited state.

11. A mixture of hydrogen atoms (in their ground state) and hydrogen like ions (in their first excited state) are being excited by electrons which have been accelerated by same potential difference V volts. After excitation when they come directly into ground state, the wavelengths of emitted light are found in the ratio 5 : 1. Calculate the minimum value of V for which both the atoms get excited after collision with electrons. Also find the atomic number of other ion and the energy of emitted light by hydrogen atoms and ions.

12. A stationary He^+ emitted a photon corresponding to the first line of Lyman series. This photon liberated a photo electron from a stationary hydrogen atom in the ground state. Find the velocity of the photoelectron.

13. The energy levels of a hypothetical one electron atom are shown in the figure.



(a) Find the ionization potential of this atom.
(b) Find the short wavelength limit of the series terminating at $n = 2$.
(c) Find the excitation potential for the state $n = 3$.
(d) Find wave number of the photon emitted for the transition $n = 3$ to $n = 1$.
(e) What is the minimum energy that an electron will have after interacting with this atom in the ground state if the initial kinetic energy of the electron is
(i) 6 eV
(ii) 11 eV

14. Using the known values for hydrogen atom, calculate

(a) radius of third orbit for Li^{+2}
(b) speed of electron in fourth orbit for He^+.

15. Find an expression for the magnetic dipole moment and magnetic field induction at the center of a Bohr's hypothetical hydrogen atom in the nth orbit of the electron in terms of universal constants.

16. A small particle of mass m moves in such a way that the potential energy $U = ar^2$ where a is a constant and r is the distance of the particle from the origin. Assuming Bohr's model of quantization of angular momentum and circular orbits, find the radius of nth allowed orbit.

17. The electron in a hydrogen atom makes a transition $n_1 \to n_2$ where n_1 and n_2 are the principle quantum numbers of the two states. Assume the Bohr model to be valid, the time period of the electron in the initial state is eight times that in the final state. What are the possible values of n_1 and n_2?

18. If potential energy in first orbit is taken to be zero, then find the kinetic energy, potential energy and total energy in first and second orbit of hydrogen atom.

19. Wavelengths belonging to Balmer series lying in the range of 450 nm to 750 nm were used to eject photoelectrons from a metal surface whose work function is 2 eV. Find (in eV) the maximum kinetic energy of the emitted photoelectrons. Take $hc = 1242$ eV nm.

20. The potential energy of a particle varies as:

$$U(x) = \begin{bmatrix} E_0 & 0 \le x \le 1 \\ 0 & x > 1 \end{bmatrix}$$

For $0 \le x \le 1$, the de-Broglie wavelength is λ_1 and for $x > 1$, the de-Broglie wavelength is λ_2. Total energy of the particle is $2E_0$. Find $\frac{\lambda_1}{\lambda_2}$.

21. An electron in a hydrogen like atom is in an excited state. It has a total energy of –3.4 eV. Calculate.

(a) the kinetic energy,

(b) the de-Broglie wavelength of the electron.

22. An electron of energy 20 eV collides with a hydrogen atom in the ground state. As a result of the collision, the atom is excited to a higher energy state and the electron is scattered with reduced velocity. The atom subsequently returns to its ground state with emission of radiation of wavelength 1.216×10^{-7} m. Find the velocity of the scattered electron.

23. Determine the maximum wavelength that hydrogen in its ground state can absorb. What would be the next smaller wavelength that would work? Take $hc = 12400$ eVÅ

24. Find the ratio of minimum to maximum wavelength of radiation emitted by electron in ground state of Bohr's hydrogen atom.

25. Calculate the difference between the ionization potentials of atomic hydrogen and atomic deuterium.

X-RAYS

When fast moving electrons strike a target of high melting point and high atomic weight (like tungsten, platinum molybdenum), electromagnetic radiations called X-rays are produced. A large part of these radiations has wavelength of the order of 0.1 nm and is known as X-rays.

X-rays were discovered by WC Roentgen in 1895, therefore they are also known as Roentgen rays. He found that photographic film wrapped in black paper becomes exposed when placed near a cathode ray tube. He concluded that some invisible radiations were coming from cathode ray tube which penetrated the black paper and exposed the photographic plate. He named these radiations as X-rays because he was unaware about the nature and properties of radiations. A device used to produce X-rays is generally called an X-rays tube or Coolidge tube.

Production

Coolidge modified the Roentgen tube. A modern X-ray tube consists of

(a) An electron source, preferably a filament heated by the passage of an electric current which may be varied.

(b) A heavy target of high melting point inclined at $45°$ to the path of electron beam, kept cooled by circulating cold water internally.

(c) A source of high potential difference applied across the filament and the target, keeping target positive with respect to filament. When the filament is heated, a fine beam of electrons strikes the target to produce X-rays.

Control of Intensity

The intensity of incident electrons determines the intensity of X-rays, i.e., greater is the number of electrons striking the target, more intense are the X-rays produced.

Control of Penetrating Power

The potential difference across the filament and target determines the energy and hence the penetrating power of X-rays.

Need of Cooling Device

Only about 1% of incident electron's energy is converted into X-rays and the remaining 99% is converted into heat, therefore cooling device is essential with an X-ray tube.

Hard and Soft X-rays

X-rays up to 4 Å have high penetrating power and are called hard X-rays while those of $\lambda > 4$ Å are called soft X-rays.

ILLUSTRATION 38

Find the energy, the frequency and the momentum of an X-ray photon of wavelength 0.10 nm. Take $hc = 12400$ eVÅ

SOLUTION

Given wavelength is $\lambda = 0.1 \times 10^{-9}$ nm $= 1$ Å

Photon energy, $E = \frac{12400}{1} = 12.4$ keV

Frequency, $v = \frac{E}{h} = \frac{12.4 \times 10^3 \times 1.6 \times 10^{-19}}{6.63 \times 10^{-34}}$

$\Rightarrow \quad v \approx 3 \times 10^{18}$ Hz

Photon momentum, $p = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34}}{10^{-10}}$

$\Rightarrow \quad p = 6.63 \times 10^{-24}$ Js

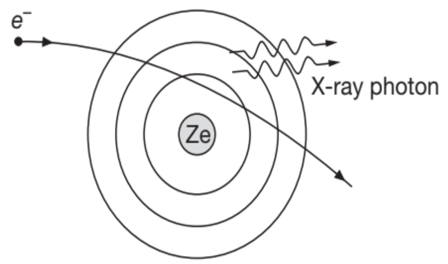
X-RAY SPECTRA CLASSIFICATION

In X-ray tube, when high speed electrons strike the target, they penetrate the target. They loose their kinetic energy and come to rest inside the metal. The electron before finally being stopped makes several collisions with the atoms in the target. At each collision one of the following two types of X-rays may get formed.

A. Continuous X-rays
B. Characteristic X-rays

Continuous X-rays

It consists of radiations of all possible wavelengths within a definite wavelength range having a definite short wavelength limit. These are produced due to deceleration of electrons passing near heavy nucleus.



The loss in energy of electrons during retardation is emitted in form of continuous X-rays. Since the electrons suffer collisions at all angles, right from the glancing collision to the direct hit, so they suffer varying decelerations and hence radiations of all possible wavelengths within a certain range are emitted, forming the continuous spectrum.

The maximum limiting frequency v_{max} or minimum limiting wavelength λ_{min} is obtained when entire kinetic energy of bombarding electron is converted to X-ray energy. If V_0 is the accelerating potential difference, then

$$\frac{1}{2}mv^2 = eV_0 = hv_{max}$$

$$\Rightarrow \quad v_{max} = \frac{eV_0}{h}$$

In terms of λ_{min}, we have

$$eV_0 = \frac{hc}{\lambda_{min}}$$

$$\Rightarrow \quad \lambda_{min} = \frac{hc}{eV_0} = \frac{12375}{V_0} \text{ Å} \approx \frac{12400}{V_0} \text{ Å}$$

This relation is called as **Duane-Hunt Law** and gives the shortest wavelength limit of continuous X-ray spectrum. This wavelength is also called as cut-off wavelength.

If f is the fraction of kinetic energy of electrons converted into X-ray, then wavelength of emitted X-ray photon is given by

$$f(eV_0) = \frac{hc}{\lambda}$$

$$\Rightarrow \quad \lambda = \frac{hc}{feV_0}$$

The intensity vs wavelength graph for X-rays is shown in figure.

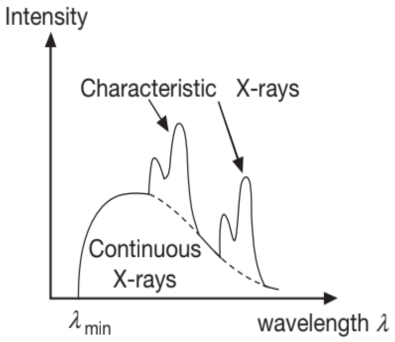


ILLUSTRATION 39

Find the cut-off wavelength of the *X*-rays emitted by an *X*-ray tube operating at 30 kV.

SOLUTION

For minimum wavelength, the total kinetic energy should be converted into an *X*-ray photon.

Thus, $\lambda = \frac{hc}{E} = \frac{12400}{E} = \frac{12400}{30 \times 10^3} = 0.41\ \text{Å}$

ILLUSTRATION 40

If an *X*-ray tube operates at the voltage of 10 kV, find the ratio of the de-Broglie wavelength of the incident electrons to the shortest wavelength of *X*-rays produced. The specific charge of electron is $1.8 \times 10^{11}\ \text{Ckg}^{-1}$.

SOLUTION

de Broglie wavelength (λ_d) when a charge q is accelerated by a potential difference of V_0 volt is given by

$$\lambda_d = \frac{h}{\sqrt{2mqV_0}} \quad \ldots(1)$$

For cut-off wavelength of *X*-rays, we have

$$qV_0 = \frac{hc}{\lambda_m}$$

$$\Rightarrow \quad \lambda_m = \frac{hc}{qV_0} \quad \ldots(2)$$

From equations (1) and (2), we get

$$\frac{\lambda_d}{\lambda_m} = \frac{\sqrt{\frac{qV_0}{2m}}}{c}$$

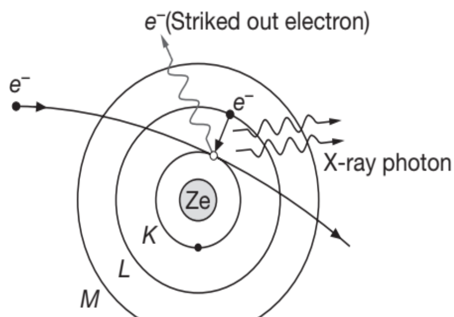
For electron, we have v

$$\frac{q}{m} = 1.8 \times 10^{11}\ \text{Ckg}^{-1}$$

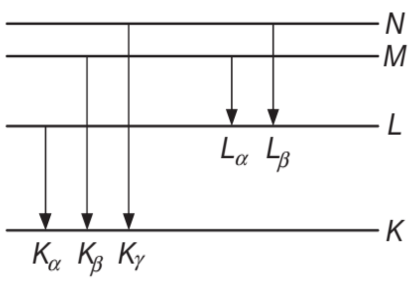
$$\Rightarrow \quad \frac{\lambda_d}{\lambda_m} = \frac{\sqrt{\frac{1.8 \times 10^{11} \times 10 \times 10^3}{2}}}{3 \times 10^8} = 0.1$$

Characteristic X-rays

The minimum wavelength depends on the electron energy, but not on the target material. The line spectrum depends on the element used as target. These characteristic X-rays are produced when an electron knocks out an atomic electron from one of the inner levels. The ejected electron leaves a vacancy, which is then filled by an electron falling from a higher level. When an electron jumps from higher energy orbit E_1 to lower energy orbit E_2, it radiates energy $(E_1 - E_2)$. Thus, this energy difference is radiated in the form of X-rays of very small but definite wavelength which depends upon the target material. The X-ray spectrum consists of sharp lines and is called characteristic X-ray spectrum.



If the transitions are to the $n = 1$ level, the X-rays are labelled K_α, K_β.... If they are to the $n = 2$ level, they are labelled L_α, L_β,.... etc. In the figure shown, the energy level diagram for an atom is drawn. The arrows indicate the transitions that give rise to the different series of X-rays.



$$\lambda_{K_\alpha} = \frac{hc}{E_L - E_K} \text{ for } K_\alpha \text{ wavelength}$$

$$\Rightarrow \quad \lambda_{L_\alpha} = \frac{hc}{E_M - E_L} \text{ for } L_\alpha \text{ wavelength and so on.}$$

ILLUSTRATION 41

Find the maximum potential difference which may be applied across an X-ray tube with tungsten target without emitting any characteristic K or L X-ray. The energy levels of the tungsten atom with an electron knocked out are as follows.

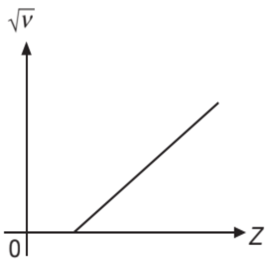
Shell containing vacancy	K	L	M
Energy in keV	69.5	11.3	2.3

SOLUTION

Energy required to knockout L shell electron is 11.3 keV hence the potential difference across the tube must be less than 11.3 kV so that L shell electron does not experience L series X-ray emission.

MOSELEY'S LAW

In 1913, Moseley noted that the characteristic lines shifted systematically as the target material was changed. He plotted the square root of the frequency of the K_α line versus the atomic number Z for many elements. The straight line he obtained is shown in the figure.



Moseley's plot did not pass through the origin, because when one of the two electrons in the $n = 1$ level is ejected, then an electron in the next highest level will drop to the lower state to fill the vacancy and in the process it emits the K_α frequency. For this electron, the electric field due to the nucleus is screened by the remaining electrons in the $n = 1$ level. Moseley estimated that the effective nuclear charge for the K_α transition is $(Z-1)e$.

In general, wave length of characteristic spectrum can be calculated using Bohr model including the concept of screening effect, where the outer electrons are screened by the inner electrons due to which the effective atomic number is $(Z - b)$ and hence we have

$$\frac{1}{\lambda} = R(Z-b)^2\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$$

The energy of X-ray radiations will be given by

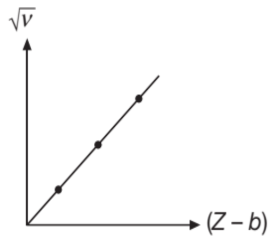
$$\Delta E = \frac{hc}{\lambda} = h\nu = Rhc(Z-b)^2\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$$

$$\Rightarrow \quad \nu = Rc(Z-b)^2\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) \quad \ldots(1)$$

The above relation was first proposed as an empirical relation given below and is called **Moseley's Law.**

$$\sqrt{\nu} = a(Z-b) \quad \ldots(2)$$

where ν is frequency of emitted line, Z is atomic number of target, b is screening constant or shielding constant, $(Z-b)$ is called as effective atomic number. The plot of $\sqrt{\nu}$ vs $(Z-b)$ is a straight line passing through the origin as shown.



The proportionality constant a is obtained by comparing equations (1) and (2).

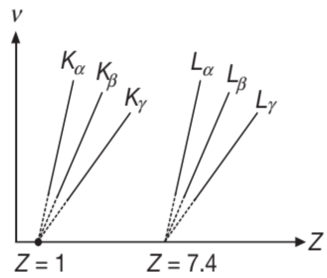
$$\Rightarrow \quad a = \sqrt{Rc\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)}$$

For K_α line, we have $n_2 = 1$ and $n_1 = 4$

$$\Rightarrow \quad a = \sqrt{\frac{3Rc}{4}}$$

The constant b doesn't depend on the nature of target. Different values of b are given below for reference.

1. for K_α-series, $b = 1$
2. for L_α-series, $b = 7.4$
3. for M_α-series, $b = 19.2$



Thus, Moseley's Law for the frequency of the K_α line is

$$\sqrt{\nu_{K_\alpha}} = a(Z-1)$$

where $a=\sqrt{\frac{3}{4}Rc}$, in which R is the Rydberg's constant and c is the speed of light. The wavelength of K-lines is given by

$$\frac{1}{\lambda}=(Z-1)^2\left[1-\frac{1}{n^2}\right] \text{ where } n=2,3,4,.....$$

ILLUSTRATION 42

The energy of a silver atom with a vacancy in K shell is 25.31 keV in L shell is 3.56 keV and in M shell is 0.530 keV higher than the energy of the atom with no vacancy. Calculate the frequency of K_α, K_β and L_α X-rays of silver.

SOLUTION

Energies required for below transition are

$n=1$ to ∞ is $E_1 = 25.31$ KeV

$n=2$ to ∞ is $E_2 = 3.56$ KeV

$n=3$ to ∞ is $E_3 = 0.53$ KeV

Energy of K_α line is $\Delta E_{21} = 25.31-3.56 = 21.75$ KeV

Energy of K_β line is $\Delta E_{31} = 25.31-0.530 = 24.78$ KeV

Energy of L_α line is $\Delta E_{31} = 3.56-0.530 = 3.03$ KeV

$$\Rightarrow \quad \nu_{K_\alpha} = \frac{\Delta E_{21}}{h} = 5.249\times10^{18} \text{ Hz}$$

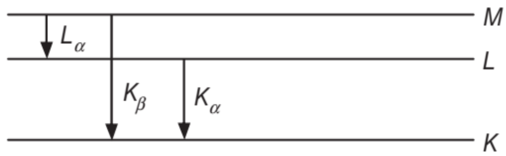
$$\Rightarrow \quad \nu_{K_\beta} = \frac{\Delta E_{31}}{h} = 5.98\times10^{18} \text{ Hz}$$

$$\Rightarrow \quad \nu_{L_\alpha} = \frac{\Delta E_{32}}{h} = 7.312\times10^{17} \text{ Hz}$$

ILLUSTRATION 43

Show that the frequency of K_β X-ray of a material equals to the sum of frequencies of K_α and L_α X-rays of the same material.

SOLUTION



The energy level diagram of an atom with one electron knocked out is shown above.

Energy of K_α X-ray is $E_{K\alpha} = E_L - E_K$ of K_β X-ray is $E_{K_\alpha} = E_M - E_K$

and, of L_α X-ray is $E_{L\alpha} = E_M - E_L$

thus, $E_{K_\beta} = E_{K_\alpha} + E_{L\alpha}$ or $\nu_{K_\beta} = \nu_{K_\alpha} + \nu_{L_\alpha}$

ILLUSTRATION 44

A free atom of iron emits K_α X-rays of energy 6.2 keV. Calculate the recoil kinetic energy of the atom. Mass of an iron atom $= 9.3\times10^{-20}$ kg.

SOLUTION

Wavelength of K_α photon is

$$\lambda_{K_\alpha} = \frac{12400}{6200} = 2 \text{ Å}$$

Using momentum conservation, we get

$$p=\frac{h}{\lambda}=mv_R$$

$\Rightarrow$ Recoil energy E_R of atom is

$$E_R = \frac{p^2}{2m} = \frac{\left(\frac{h}{\lambda}\right)^2}{2m}$$

$$\Rightarrow \quad E_R = \left(\frac{6.63\times10^{-34}}{2\times10^{-10}}\right)\times\frac{1}{2\times9.3\times10^{-20}\times1.6\times10^{-19}}$$

$$\Rightarrow \quad E_R = 3.7\times10^{-10} \text{ eV}$$

ILLUSTRATION 45

The wavelength of K_α X-ray of tungsten is 20 pm. It takes 11.3 keV to knock out an electron from the L shell of a tungsten atom. What should be the minimum accelerating voltage across an X-ray tube having tungsten target which allows production of K_α X-ray? Take $hc = 12400$ eVÅ

SOLUTION

Binding energy of L-shell electron is 11.3 keV

Energy difference of $n=2$ and $n=1$ shell is

$$\Delta E = \frac{12400}{0.2} = 62 \text{ keV}$$

$\Rightarrow$ Binding energy of K-shell electron is

$$E_K = 62 + 11.3 = 73.3 \text{ keV}$$

Thus, accelerating voltage required to knock out K-shell electron is 73.3 V

Moseley's Law: Conclusions

(a) Mosley's Law supported Bohr's theory.

(b) It experimentally determined the atomic number (Z) of elements.

(c) This law established the importance of ordering of elements in periodic table by atomic number and not by atomic weight.

(d) Gaps in Moseley's data for $A = 43$, 61, 72, 75 suggested existence of new elements which were later discovered.

(e) The atomic numbers of Cu, Ag and Pt were established to be 29, 47 and 78 respectively.

(f) When a vacancy occurs in the K-shell, there is still one electron remaining in the K-shell. An electron in the L-shell will feel an effective charge of $(Z-1)e$ due to $+Ze$ from the nucleus and $-e$ from the remaining K-shell electron, because L-shell orbit is well outside the K-shell orbit.

(g) Wave length of characteristic spectrum is

$$\frac{1}{\lambda} = R(Z-b)^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

(h) Energy of X-ray radiations is given by

$$\Delta E = h\nu = \frac{hc}{\lambda} = Rhc(Z-b)^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

(i) If transition takes place from $n_2 = 2$ to $n_1 = 1$ we get the K_α line, for which we have

(i) $a = \sqrt{\frac{3Rc}{4}} = 2.47 \times 10^{15}$ Hz

(ii) Frequency is given by

$$\nu_{K_\alpha} = RC(Z-1)^2\left(1 - \frac{1}{2^2}\right) = \frac{3RC}{4}(Z-1)^2$$

$$\Rightarrow \quad \nu_{K_\alpha} = 2.47 \times 10^{15}(Z-1)^2 \text{ Hz}$$

(iii) $E_{K_\alpha} = 10.2(Z-1)^2$ eV

(iv) In general, the wavelength of all the K-lines are given by $\frac{1}{\lambda_K} = R(Z-1)^2\left(1 - \frac{1}{n^2}\right)$, where $n = 2, 3, 4, \ldots\ldots.$ whereas for K_α line, $\lambda_{K_\alpha} = \frac{1216}{(Z-1)^2}$ Å.

PROPERTIES OF X-RAYS

(a) X-rays are electromagnetic waves of very short wavelength of order of 1 Å. Therefore, they can exhibit properties of reflection, refraction, interference, diffraction, polarisation like ordinary light. Due to this property they help in the study of crystal structure.

(b) They travel in vacuum with speed of light i.e. $c = 3 \times 10^8$ ms^{-1}

(c) They are electrically neutral, hence cannot be deflected by electric and magnetic fields.

(d) They do not possess magnetic moment.

(e) They have ionising power. Therefore, when they pass through a gas, the gas is ionised.

(f) They have penetrating power. They can penetrate light substances like wood, flesh, thick paper, thin sheets of metals, but cannot penetrate heavy substances like lead, calcium, barium sulphate etc.

(g) When incident on certain metals, they liberate electrons. This effect is called photoelectric effect.

(h) They cause fluorescence in many substances like barium, cadmium, zinc sulphide etc.

(i) They have destructive effect on living tissues. Therefore, the persons working with X-rays often wear lead clothes.

ILLUSTRATION 46

In Moseley's equation, we have $\sqrt{f}=a(Z-b)$, where a and b are constants. Find their values with the help of the following data.

Element	Z	Wavelength of K_α X-rays
Mo	42	0.71 Å
Co	27	1.785 Å

SOLUTION

Since, according to Moseley's Law we have

$$\sqrt{f}=a(Z-b)$$

where $f=\dfrac{c}{\lambda}$

So, for the first element, we have

$$\sqrt{\frac{c}{\lambda_1}}=a(Z_1-b) \quad \ldots(1)$$

and for the second element, we have

$$\sqrt{\frac{c}{\lambda_2}}=a(Z_2-b) \quad \ldots(2)$$

From equations (1) and (2), we get

$$\sqrt{c}\left(\frac{1}{\sqrt{\lambda_1}}-\frac{1}{\sqrt{\lambda_2}}\right)=a(Z_1-Z_2) \quad \ldots(3)$$

Now, we know that $c=3\times10^8\ \text{ms}^{-1}$ and further it is given to us that $\lambda_1=0.71\times10^{-10}\ \text{m}$, $\lambda_2=1.785\times10^{-10}\ \text{m}$, $Z_1=42$ and $Z_2=27$. Solving above three equations, we get

$$a=5\times10^7\ (\text{Hz})^{1/2} \text{ and } b=1.37$$

ILLUSTRATION 47

Determine the energy of the characteristic X-ray (K_β) emitted from a tungsten $(Z=74)$ target when an electron drops from the M shell $(n=3)$ to a vacancy in the K shell $(n=1)$.

SOLUTION

Energy associated with the electron in the K shell is approximately

$$E_K=-(74-1)^2(13.6\ \text{eV})=-72474\ \text{eV}$$

An electron in the M shell is subjected to an effective nuclear charge that depends on the number of electrons in the $n=1$ and $n=2$ states because these electrons shield the M electrons from the nucleus.

Since, there are eight electrons in the $n=2$ state and one remaining in the $n=1$ state, so, roughly nine electrons shield M shell electrons from the nucleus, and hence we have $Z_{\text{eff}}=Z-9$. The energy associated with an electron in the M shell is given by

$$E_M=\frac{-13.6Z_{\text{eff}}^2}{3^2}\ \text{eV}=\frac{-13.6(Z-9)^2}{3^2}\ \text{eV}$$

$$\Rightarrow \quad E_M=-\frac{(13.6)(74-9)^2}{9}\ \text{eV}=-6384\ \text{eV}$$

Therefore, emitted X-ray has an energy equal to

$$E_M-E_K=[-6384-(-72474)]\ \text{eV}=66090\ \text{eV}$$

ILLUSTRATION 48

X-rays are incident on a target metal atom having 30 neutrons. The ratio of atomic radius of the target atom and ${}_2^4\text{He}$ is $(14)^{1/3}$.

(a) Find the atomic number of target atom.

(b) Find the frequency of K_α line emitted by this metal.

Assume that the radius $r\propto A^{\frac{1}{3}}$ and $c=3\times10^8\ \text{ms}^{-1}$.

SOLUTION

(a) From the relation $r\propto A^{\frac{1}{3}}$

$$\Rightarrow \quad \frac{r_2}{r_1}=\left(\frac{A_2}{A_1}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \left(\frac{A_2}{4}\right)^{\frac{1}{3}}=(14)^{\frac{1}{3}}$$

$$\Rightarrow \quad A_2=56$$

(b) $Z_2=A_2-\text{number of neutrons}$

$$\Rightarrow \quad Z_2=56-30=26$$

Since, $f_{K_\alpha} = Rc(Z-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right) = \frac{3Rc}{4}(Z-1)^2$

Substituting the given values of R, c and Z, we get

$$f_{K_\alpha} = 1.55\times10^{18}\ \text{Hz}$$

ILLUSTRATION 49

Characteristic X-rays of frequency 4.2×10^{18} Hz are produced when transitions from L-shell, K-shell take place in a certain target material. Use Mosley's Law to determine the atomic number of the target material. Given Rydberg constant $R = 1.1\times10^7\ \text{m}^{-1}$.

SOLUTION

$$\Delta E = h\nu = Rhc(Z-b)^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$$

For K-series, $b = 1$

$$\Rightarrow \quad \nu = Rc(Z-1)^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$$

Substituting the values, we get

$$4.2\times10^{18} = (1.1\times10^7)(3\times10^8)(Z-1)^2\left(\frac{1}{1}-\frac{1}{4}\right)$$

$$\Rightarrow \quad (Z-1)^2 = 1697$$

$$\Rightarrow \quad Z-1 \approx 41$$

$$\Rightarrow \quad Z = 42$$

ILLUSTRATION 50

The wavelength of the characteristics X-ray K_α line emitted from zinc $(Z = 30)$ is 1.415 Å. Find the wavelength of the K_α line emitted from molybdenum $(Z = 42)$.

SOLUTION

According to Moseley's law, the frequency for K series is given by

$$\nu \propto (Z-1)^2$$

$$\Rightarrow \quad \frac{c}{\lambda} \propto (Z-1)^2$$

$$\Rightarrow \quad \frac{1}{\lambda} = k(Z-1)^2 \qquad \ldots(1)$$

where k is a constant. Let λ' be the wavelength of K_α line emitted from molybdenum, then

$$\frac{1}{\lambda'} = k(Z-1)^2 \qquad \ldots(2)$$

Dividing Equation (1) by (2), we get

$$\lambda' = \left(\frac{Z-1}{Z'-1}\right)^2 \lambda$$

$$\Rightarrow \quad \lambda' = \left(\frac{30-1}{42-1}\right)^2 \times 1.415\ \text{Å} = 0.708\ \text{Å}$$

BRAGG'S LAW

When an X-ray beam of wavelength λ is incident on a crystal of inter planar spacing d at grazing angle θ, then the directions of diffraction maxima are given by

$$2d\sin\theta = n\lambda \qquad \ldots(1)$$

where n is an integer, called order of maxima. Equation (1) is called Bragg's equation

$$\lambda = \frac{2d\sin\theta}{n}$$

For maximum wavelength

$$n_{\min} = 1 \text{ and } (\sin\theta)_{\max} = 1$$

$$\Rightarrow \quad \lambda_{\max} = 2d$$

Hence equation (1) has solution only for $\lambda \le 2d$.

INTENSITY OF TRANSMITTED X-RAY

The intensity of monochromatic X-ray beam after penetrating a thickness x of a target material is given by $I = I_0 e^{-\mu x}$, where μ is a constant called Absorption Coefficient. Its value depends upon nature of material. μ increases with increase of λ and atomic number Z of absorbing material. μ is maximum for lead.

Test Your Concepts-II

Based on X-rays and Properties

(Solutions on page H.42)

1. What potential difference should be applied across an X-ray tube to get X-ray of wavelength not less than 0.10 nm? What is the maximum energy of a photon of this X-ray in joule? Take hc = 12400 eVÅ
2. **(a)** An X-ray tube produces a continuous spectrum of radiation with its short-wavelength end at 0.45 Å. What is the maximum energy of a photon in the radiation?
   (b) From your answer to (a), guess what order of accelerating voltage (for electrons) is required in such a tube?
3. If the short series limit of the Balmer series for hydrogen is 3644 Å, find the atomic number of the element which give X-ray wavelengths down to 1 Å. Identify the element.
4. Iron emits K_α X-ray of energy 3.69 keV. Calculate the times taken by an iron K_α photon and a calcium K_α photon to cross through a distance of 3 km.
5. Use Moseley's Law with $b = 1$ to find the frequency of the K_α X-rays of La ($Z = 57$) if the frequency of the K_α X-rays of Cu ($Z = 29$) is known to be 1.88×10^{18} Hz.
6. When the voltage applied to an X-ray tube is increased $\eta = 1.5$ times, the short wave limit of an X-ray continuous spectrum shifts by $\Delta\lambda = 26$ pm. Find the initial voltage applied to the tube.
7. Find the cut-off wavelength for the continuous X-rays coming from an X-ray tube operating at 40 kV.
8. An X-rays tube operates at 20 kV. Find the maximum speed of the electrons striking the anticathode if the charge and mass of electron are 1.6×10^{-19} C and 9×10^{-31} kg.
9. Calculate the wavelength of the emitted characteristic X-ray from a tungsten ($Z = 74$) target when an electron drops from a M shell to a vacancy in the K shell.
10. A material whose K absorption edge is 0.2 Å is irradiated by X-rays of wavelength 0.15 Å. Find the maximum energy of the photoelectrons that are emitted from the K shell. (Take hc = 12400 eVÅ)
11. If two times the λ_{min} of continuous X-ray spectra of target atom A at 34.3 kV is same as the wavelength of K_α line of target atom B at 40 kV, then determine the atomic number of the atom B.
12. Stopping potentials of 24 kV, 100 kV, 110 kV and 115 kV are measured for photoelectrons emitted from a certain element when it is radiated with monochromatic X-ray. If this element is used as a target in an X-ray tube, what will be the wavelength of K_α line?

SOLVED PROBLEMS

PROBLEM 1

A peak emission from a black body at a certain temperature occurs at a wavelength of 9000 Å. On increasing the temperature, the total radiation emitted is increased 81 times. At the initial temperature, when the peak radiation from the black body is incident on a metal surface, it does not cause any photoemission from the surface. After the increase of temperature, the peak radiation from the black body caused photoemission. To bring these photoelectrons to rest, a potential equivalent to the excitation energy between the $n = 2$ and $n = 3$ Bohr levels of hydrogen atom is required. Calculate the work function of the metal.

SOLUTION

Let T be the initial absolute temperature of the black body. The total energy emitted by the body per unit area per second is given according to Stefan's Law by the relation

$$E = \sigma T^4$$

where σ is the Stefan's constant.
When the temperature of the black body is raised to T'. We have

$$E' = \sigma T'^4,$$

$$\Rightarrow \quad \frac{E'}{E} = \frac{T'^4}{T^4} = 81$$

$$\Rightarrow \quad T' = 3T$$

Since it is given that peak emission at temperature T occurs at a wavelength $\lambda_m = 9000$ Å. If λ'_m is the wavelength for peak emission at temperature T', then from Wien's displacement law $\lambda_m T = \text{constant}$, we have

$$\lambda'_m T' = \lambda_m T$$

$$\Rightarrow \quad \lambda'_m = \lambda_m \left(\frac{T}{T'}\right) = 9000\text{ Å}\left(\frac{T}{3T}\right) = 3000\text{ Å}$$

According to the problem, the kinetic energy of the emitted photo-electrons corresponds to the excitation energy for transition $n = 2$ to $n = 3$, which is given by

$$K_{\max} = \Delta E_{32} = (13.6\text{ eV})\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$$

$$\Rightarrow \quad K_{\max} = 13.6\left(\frac{5}{36}\right) = 1.89\text{ eV}$$

The energy of the photon of wavelength $\lambda' = 3000$ Å in eV is

$$E = \frac{12375}{3000}\text{ eV} = 4.125\text{ eV}$$

Now, from Einstein's photoelectric equation, the work function for the metal surface is given by

$$\phi_0 = E - K_{\max}$$

$$\Rightarrow \quad \phi_0 = (4.125 - 1.89)\text{ eV} = 2.235\text{ eV}$$

PROBLEM 2

A gas of hydrogen like ions is prepared in such a way that the ions are only in the ground state and the first excited state. A monochromatic light of wavelength 1216 Å is absorbed by the ions. The ions are lifted to higher excited states and emit radiations of six wavelengths, some higher, some lower or some greater than the incident wavelength.

(a) Find the principle quantum number of the final excited state.
(b) Identify the nuclear charge on the ions.
(c) Calculate the value of the maximum and minimum wavelengths.

SOLUTION

(a) Since, total six wavelengths are obtained in the emission spectrum, hence from

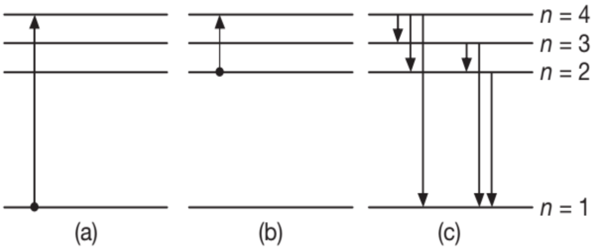
$$\frac{n(n-1)}{2} = 6$$

We have $n = 4$, i.e., after excitation the single electron jumps to 3rd excited state or $n = 4$.

(b) Energy difference corresponding to $\lambda = 1216$ Å is,

$$\Delta E = \frac{12375}{1216}\text{ eV} = 10.177\text{ eV} \cong 10.2\text{ eV}$$

Now it may jump either from $n = 1$ or $n = 2$.



If it jumps from $n = 1$, then in emission spectrum all the six photons have energy equal to or less than the energy of absorbed photon or the wavelength of emitted photon is either equal to or greater than the wavelength of absorbed photon. While in the question it is given that the emitted wavelengths are either less than or greater than or smaller than the wavelength of absorbed photon. Which is possible only in the second case, i.e., when electron jumps from $n = 2$ to $n = 4$.

Hence, $E_4 - E_2 = 10.177$ eV

$$\Rightarrow \quad \frac{-13.6z^2}{4^2} - \left(\frac{-13.6z^2}{2^2}\right) = 10.177$$

Solving this, we get $z \approx 2$

(c) Maximum wavelength corresponds to minimum energy, i.e., a transition from $n = 4$ to $n = 3$. Thus,

$$\Delta E_{\text{min}} = E_4 - E_3$$

$$\Rightarrow \quad \Delta E_{\text{min}} = \frac{(-13.6)(2)^2}{(4)^2} - \left[\frac{-(13.6)(2)^2}{(3)^2}\right]$$

$$\Rightarrow \quad \Delta E_{\text{min}} = 2.64 \text{ eV}$$

$$\Rightarrow \quad \lambda_{\text{max}} = \frac{hc}{\Delta E_{\text{min}}} = \frac{12375}{2.64} = 4687 \text{ Å}$$

Minimum wavelength corresponds to maximum energy, i.e., a transition from $n = 4$ to $n = 1$.

Hence, $\Delta E_{\text{max}} = E_4 - E_1$

$$\Rightarrow \quad \Delta E_{\text{max}} = \frac{(-13.6)(2)^2}{(4)^2} - \left[\frac{(-13.6)(2)^2}{(1)^2}\right]$$

$$\Rightarrow \quad \Delta E_{\text{max}} = 51 \text{ eV}$$

$$\Rightarrow \quad \lambda_{\text{min}} = \frac{12375}{51} = 242 \text{ Å}$$

PROBLEM 3

A gas of hydrogen like atoms can absorb radiations of 68 eV. Consequently, the atoms emit radiations of only three different wavelengths. All the wavelengths are equal or smaller than that of the absorbed photon.

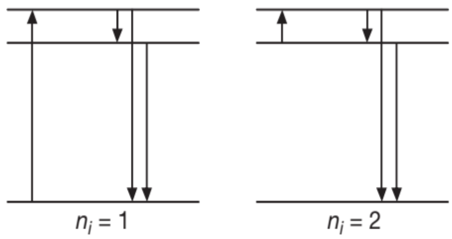
(a) Find the initial state of the gas atoms.
(b) Identify the gas atoms.
(c) Calculate the minimum wavelength of the emitted radiations.
(d) Find the ionization energy and the respective wavelength for the gas atoms.

SOLUTION

(a) Since, $N = \dfrac{n(n-1)}{2} = 3$

$\Rightarrow \quad n = 3$

i.e., after excitation atom jumps to second excited state. Hence $n_f = 3$. So n_i can be 1 or 2



If $n_i = 1$ then energy emitted is either equal to, greater than or less than the energy absorbed. Hence the emitted wavelength is either equal to, less than or greater than the absorbed wavelength. Hence $n_i \neq 1$.

If $n_i = 2$, then $E_e \geq E_a$ and hence $\lambda_e \leq \lambda_b$

So, $n_i = 2$

(b) Since, $E_3 - E_2 = 68$ eV

$$\Rightarrow \quad (13.6)(Z^2)\left(\frac{1}{4} - \frac{1}{9}\right) = 68$$

$$\Rightarrow \quad Z = 6$$

(c) $$\lambda_{\text{min}} = \frac{12375}{E_3 - E_1} = \frac{12375}{(13.6)(6)^2\left(1 - \frac{1}{9}\right)} = 28.43 \text{ Å}$$

(d) Ionization energy is

$$IE = (13.6)(6)^2 = 489.6 \text{ eV}$$

$$\Rightarrow \quad \lambda = \frac{12375}{489.6} = 25.3 \text{ Å}$$

PROBLEM 4

From a metal surface, photoelectrons are emitted when 4000 Å radiation is incident on the surface having work function 1.9 eV. These photoelectrons then pass through a region containing α-particles. It is observed that a maximum energy photoelectron combines with an α-particle to form a He^+ ion, emitting a single photon in this process. He^+ ions thus formed are in their fourth excited state. Find the energies (in eV) of the photons, lying in the 2 eV to 4 eV range, that are likely to be emitted during and after the combination. Take $h = 4.14 \times 10^{-15}$ eVs.

SOLUTION

The energy of the incident photon is

$$E = h\nu = \frac{hc}{\lambda} = \frac{12375}{4000}\text{ eV} \approx 3.1\text{ eV}$$

From Einstein's photoelectric equation, the maximum kinetic energy of the emitted electrons is

$$K_{max} = h\nu - \phi_0 = 3.1\text{ eV} - 1.9\text{ eV} = 1.2\text{ eV}$$

Since, it is given that

$$e^- + \text{He} \rightarrow \text{He}^+ + \text{photon}$$

where the electron has maximum kinetic energy K_{max} and the He^+ ion $(Z = 2)$ is in the fourth excited state which corresponds to $n = 5$. Since we know that the energy of the electron in the n^{th} state for a hydrogen like atom is

$$E_n = -(13.6)\frac{Z^2}{n^2}\text{ eV} \qquad \ldots(1)$$

$$\Rightarrow \quad E_5 = -(13.6\text{ eV}) \times \frac{(2)^2}{(5)^2} = -2.18\text{ eV}$$

The energy of the emitted photon in the above combination process is

$$E = K_{max} + (-E_5)$$

$$\Rightarrow \quad E = 1.2\text{ eV} + 2.18\text{ eV} = 3.38\text{ eV}$$

and this energy lies well within the range 2 eV to 4 eV.

After the recombination process, the electron may undergo transitions from a higher level to a lower level, thus emitting photons. Using Equation (1), the energies in the lower energy levels of He^+ ions are

$$E_4 = \frac{(-13.6\text{ eV}) \times (2)^2}{(4)^2} = -3.4\text{ eV}$$

$$E_3 = \frac{(-13.6\text{ eV}) \times (2)^2}{(3)^2} = -6.04\text{ eV}$$

$$E_2 = \frac{(-13.6\text{ eV}) \times (2)^2}{(2)^2} = -13.6\text{ eV}$$

and $$E_1 = \frac{(-13.6\text{ eV}) \times (2)^2}{(1)^2} = -54.4\text{ eV}$$

Since, $E_5 = -2.18$ V, so the energies of the emitted photons are given by the differences of these energies which must lie in the range 2 eV to 4 eV.

We observe that the following difference of energies (in addition to $E = 3.38$ eV) lie well within the asked range of 2 eV to 4 eV.

(1) $\Delta E_{43} = E_4 - E_3 = -3.4 - (-6.04)$

$\Rightarrow \quad \Delta E_{43} = 2.64\text{ eV}$

(2) $\Delta E_{53} = E_5 - E_3 = -2.18 - (-6.04)$

$\Rightarrow \quad \Delta E_{53} = 3.86\text{ eV}$

Hence, the energies of the photons that are likely to be emitted with energies in the range 2 eV to 4 eV are 2.64 eV, 3.38 eV and 3.86 eV.

PROBLEM 5

The energy levels of a hypothetical one electron atom are given by $E_n = -\frac{18}{n^2}$ eV, where $n = 1, 2, 3, \ldots$.

(a) Calculate the four lowest energy levels and construct the energy level diagram.

(b) Find the excitation potential of the stage $n = 2$.

(c) Find the wavelengths (Å) which can be emitted when these atoms in the ground state are bombarded by electrons that have been accelerated through a potential difference of 16.2 V.

(d) Assuming these atoms to be in the ground state, will they absorb radiation having a wavelength of 2000 Å?

(e) Also calculate the photoelectric threshold wavelength of this atom.

SOLUTION

(a) Since, $E_n = -\frac{18}{n^2}$ eV, so we have

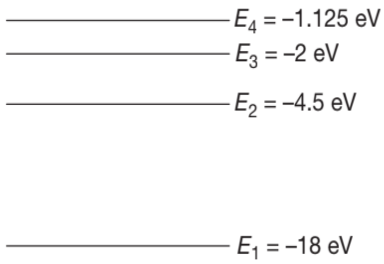
$$E_1 = \frac{-18}{(1)^2} = -18 \text{ eV}$$

$$E_2 = \frac{-18}{(2)^2} = -4.5 \text{ eV}$$

$$E_3 = \frac{-18}{(3)^2} = -2 \text{ eV} \text{ and}$$

$$E_4 = \frac{-18}{(4)^2} = -1.125 \text{ eV}$$

The energy level diagram is shown in figure.



(b) The excitation potential of stage $n = 2$ is

$$E_2 - E_1 = 18 - 4.5 = 13.5 \text{ V}$$

(c) Energy of the electron accelerated by a potential difference of 16.2 V is 16.2 eV. Since we observe that

$$E_4 - E_1 = -1.125 - (-18) = 16.875 \text{ eV} > 16.2 \text{ eV}$$

and $E_3 - E_1 = -2 - (-18) = 16 \text{ eV} < 16.2 \text{ eV}$

With this energy the electron will be able to excite the atom from $n = 1$ to $n = 3$, so we have the possible wavelengths corresponding to the transitions from $3 \to 1$, $3 \to 2$ and $2 \to 1$. Hence

$$\lambda_{32} = \frac{12375}{E_3 - E_2} = \frac{12375}{-2-(-4.5)} = 4950 \text{ Å}$$

$$\lambda_{31} = \frac{12375}{E_3 - E_1} = \frac{12375}{16} = 773 \text{ Å}$$

and $$\lambda_{21} = \frac{12375}{E_2 - E_1} = \frac{12375}{-4.5-(-18)} = 917 \text{ Å}$$

(d) The energy corresponding to $\lambda = 2000$ Å is given by

$$E = \frac{12375}{2000} = 6.1875 \text{ eV}$$

whereas the minimum excitation energy is 13.5 eV ($n = 1$ to $n = 2$).
Hence this is not possible.

(e) Threshold wavelength for photoemission to take place from such an atom is,

$$\lambda_{min} = \frac{12375}{18} = 687.5 \text{ Å}$$

PROBLEM 6

Hydrogen gas in the atomic state is excited to an energy level such that the electrostatic potential energy of H-atom becomes -1.7 eV. Now the photoelectric plate having work function 2.3 eV is exposed to the emission spectra of this gas. Assuming all the transitions to be possible, find the minimum de Broglie wavelength of ejected photo-electrons.

SOLUTION

Given that electrostatic potential energy of H-atom is

$$PE = -1.7 \text{ eV}$$

Since we know that kinetic energy is given by

$$KE = \left|\frac{PE}{2}\right| = \frac{1.7}{2} = 0.85 \text{ eV}$$

So, total energy is $E = -1.7 + 0.85 = -0.85$ eV

$$\text{Since, } E_n = -\frac{13.6}{n^2} = -0.85 \text{ eV}$$

$$\Rightarrow \quad n^2 = \frac{13.6}{0.85} = 16$$

$$\Rightarrow \quad n = 4$$

So, the atom is excited to $n = 4$ state. The maximum energy will be emitted, when electrons will make a transition from $n = 4$ to $n = 1$. The energy emitted for this transition is

$$\Delta E = -0.85 - (-13.6) = 12.75 \text{ eV}$$

When a photon of this energy is incident on a metal plate having work function 2.3 eV, the kinetic energy of fastest electron ejected will be given by

$$K_{max} = \Delta E - \phi_0 = 12.75 - 2.3 = 10.45 \text{ eV}$$

The minimum de Broglie wavelength is given by

$$\lambda_{min} = \frac{h}{p_{max}} = \frac{h}{\sqrt{2mK_{max}}}$$

$$\Rightarrow \quad \lambda_{min} = \frac{6.63 \times 10^{-34}}{\sqrt{2 \times (9.1 \times 10^{-31})(10.45 \times 1.6 \times 10^{-19})}}$$

$$\Rightarrow \quad \lambda_{min} = 3.8 \times 10^{-10} \text{ m} = 3.8 \text{ Å}$$

PROBLEM 7

Two hydrogen like atoms A and B are of different masses and each atom contains equal number of protons and neutrons. The difference in the energies between the first Balmer lines emitted by A and B is $5.667\ \text{eV}$. When the atom A and B, moving with the same velocity, strike a heavy target they rebound back with the same velocity. In the process, atoms B imparts twice momentum to the target than that A imparts. Identify the atoms A and B.

SOLUTION

$$5.667 = 13.6\left(Z_B^2 - Z_A^2\right)\left(\frac{1}{2^2}-\frac{1}{3^2}\right)$$

$$\Rightarrow \quad Z_B^2 - Z_A^2 = 3 \qquad \ldots(1)$$

Applying Law of Conservation of Linear Momentum on atom A and heavy target, we get

$$m_A u = Mv_1 - m_A u$$

$$\Rightarrow \quad 2m_A u = Mv_1$$

Similarly, for atom B and heavy target, we get

$$2m_B u = Mv_2$$

Given $Mv_2 = 2Mv_1$

$$\Rightarrow \quad m_B = 2m_A \qquad \ldots(2)$$

Since, both A and B contain equal number of protons and neutrons, so we have

$$\frac{m_A}{m_B} = \frac{2Z_A}{2Z_B} = \frac{Z_A}{Z_B} \qquad \ldots(3)$$

From these equations we get

$$Z_A = 1 \text{ and } Z_B = 2$$

i.e., A is ${}_1H^2$ and B is ${}_2He^4$ (both having single electron).

PROBLEM 8

Stopping potential of $24\ \text{kV}$, $10\ \text{kV}$, $110\ \text{kV}$ and $115\ \text{kV}$ are measured for photoelectrons emitted from a certain element when it is radiated with monochromatic X-ray. If this element is used as a target in an X-ray tube, what will be the wavelength of K_α line?

SOLUTION

Stopping potentials are $24\ \text{kV}$, $10\ \text{kV}$, $110\ \text{kV}$ and $115\ \text{kV}$

If the electrons are emitted from conduction band then the maximum kinetic energy of photoelectrons would be $115\times10^3\ \text{eV}$.

If they are emitted from next inner shell maximum kinetic energy of photoelectrons would be $110\times10^3\ \text{eV}$ and so on.

For photoelectrons of L shell it would be $100\times10^3\ \text{eV}$ and for K shell it is $24\times10^3\ \text{eV}$. Therefore, difference between energy of L shell and K shell is,

$$\Delta E = E_L - E_K = (100-24)\times10^3\ \text{eV}$$

$$\Rightarrow \quad \Delta E = 76\times10^3\ \text{eV}$$

The wavelength of K_α line (transition of electron from L shell to K shell) is,

$$\lambda_{K_\alpha}(\text{in Å}) = \frac{12375}{\Delta E(\text{in eV})}$$

$$\Rightarrow \quad \lambda_{K_\alpha} = \frac{12375}{76\times10^3} = 0.163\ \text{Å}$$

PROBLEM 9

For a certain hypothetical one-electron atom, the wavelength (in Å) for the spectral lines for transitions originating at $n = p$ and terminating at $n = 1$ are given by

$$\lambda = \frac{1500p^2}{p^2-1} \quad \text{where } p = 2, 3, 4 \ldots..$$

(a) Find the wavelength of the least energetic and the most energetic photons in this series.

(b) Construct an energy level diagram for this element showing the energies of the lowest three levels.

(c) Calculate the ionization potential of this element.

SOLUTION

(a) Since, $\lambda = \dfrac{1500p^2}{p^2-1}$

$$\Rightarrow \quad \lambda = 1500\left(\frac{1}{1-\frac{1}{p^2}}\right)$$

So, $\lambda_{\max}$ corresponds to least energetic photon with $p = 2$

$$\Rightarrow \quad \lambda_{\max} = 1500\left(\frac{1}{1-\frac{1}{4}}\right) = 2000\ \text{Å}$$

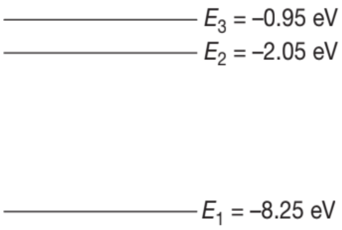
λ_{min} corresponds to most energetic photon with $p \to \infty$

$$\Rightarrow \quad \lambda_{\text{min}} = 1500 \text{ Å}$$

(b) $\lambda_{\infty \to 1} = 1500$ Å

$$\Rightarrow \quad E_\infty - E_1 = \frac{12375}{1500} \text{ eV} = 8.25 \text{ eV}$$

$$\Rightarrow \quad E_1 = -8.25 \text{ eV} \qquad \{\because E_\infty = 0\}$$



Further, $\lambda_{2 \to 1} = 2000$ Å

$$\Rightarrow \quad E_2 - E_1 = \frac{12375}{2000} \text{ eV} = 6.2 \text{ eV}$$

$$\Rightarrow \quad E_2 = -2.05 \text{ eV}$$

Similarly, $\lambda_{31} = 1500\left(\dfrac{1}{1-\dfrac{1}{9}}\right) = 1687.5$ Å

$$\Rightarrow \quad E_3 - E_1 = \frac{12375}{1687.5} \text{ eV} = 7.3 \text{ eV}$$

$$\Rightarrow \quad E_3 = -0.95 \text{ eV}$$

(c) Ionization energy is

$$E_{1 \to \infty} = 8.25 \text{ eV}$$

So, Ionisation Potential equals 8.25 V

PROBLEM 10

Consider an excited hydrogen atom in state n moving with a velocity $v(v \ll c)$. It emits a photon in the direction of its motion and changes its state to a lower state m. Apply momentum and energy conservation principles to calculate the frequency f of the emitted radiation. Compare this with the frequency f_0 emitted if the atom were at rest.

SOLUTION

Applying Conservation of Linear Momentum, we get

$$mv = mv' + \frac{hf}{c} \qquad \ldots(1)$$

Applying Conservation of Energy, we get

$$\frac{1}{2}mv^2 + \Delta E = \frac{1}{2}mv'^2 + hf \qquad \ldots(2)$$

$$\Rightarrow \quad \Delta E = hf - \frac{1}{2}m\left(v^2 - v'^2\right)$$

$$\Rightarrow \quad \Delta E = hf - \frac{1}{2}m(v + v')\left(\frac{hf}{mc}\right)$$

Since, $v \approx v'$, so we have

$$\Delta E = hf - \left(\frac{hf}{2}\right)\left(\frac{2v}{c}\right)$$

$$\Rightarrow \quad \Delta E = hf\left(1 - \frac{v}{c}\right)$$

When atom was at rest, then we have

$$\Delta E \approx hf_0$$

$$\Rightarrow \quad hf_0 = hf\left(1 - \frac{v}{c}\right)$$

$$\Rightarrow \quad f = f_0\left(1 + \frac{v}{c}\right)$$

PROBLEM 11

An electron of a stationary hydrogen atom passes from the fifth energy level to the fundamental state. What velocity did the atom acquire as the result of photon emission? What is the recoil energy? Express your answer in terms of Rydberg constant R mass of hydrogen atom M and universal constants.

SOLUTION

$$E = hcR\left(\frac{1}{1^2} - \frac{1}{5^2}\right) = \frac{24}{25}hcR \qquad \ldots(1)$$

This energy will be shared by photon and the atom.

Thus, $E = hf + E_0 \qquad \ldots(2)$

where $E_0 = \dfrac{p^2}{2M}$ is the atom's recoil energy and p is the momentum due to emission of a photon. Applying the Law of Conservation of Linear Momentum, we get

$$p = p_{Ph} = \frac{hf}{c} \qquad \ldots(3)$$

Solving (1), (2) and (3), we get

$$E_0 = \frac{h^2 f^2}{2Mc^2} \text{ and } hf = \frac{2E}{1 + \sqrt{1 + \dfrac{2E}{Mc^2}}}$$

Since, the transition energy in hydrogen atom is below 13.6 eV and $\frac{2E}{Mc^2} \approx 10^{-8}$, so $\frac{2E}{Mc^2}$ can be neglected.

$$\Rightarrow \quad hf \approx E = \frac{24}{25} hcR$$

So, recoil energy of atom is $\frac{24^2 h^2 R^2}{2 \times 25^2 M} = \frac{h^2 R^2}{2.17 M}$ and the velocity of the atom is $\frac{24hR}{25M}$.

PROBLEM 12

If the wavelength of the nth line of Lyman series is equal to the de-Broglie wavelength of electron in initial orbit of a hydrogen like element $(Z = 11)$. Find the value of n.

SOLUTION

n^{th} line of Lyman series means transition from $(n+1)^{\text{th}}$ state to first state.

$$\frac{1}{\lambda} = RZ^2\left(1 - \frac{1}{(n+1)^2}\right) \qquad \ldots(1)$$

de-Broglie wavelength in $(n+1)^{\text{th}}$ orbit is

$$\lambda = \frac{h}{mv} = \frac{hr}{mvr} = \frac{(2\pi)(hr)}{(n+1)h} = \frac{2\pi r}{(n+1)}$$

$$\Rightarrow \quad \frac{1}{\lambda} = \frac{(n+1)}{2\pi r} \qquad \ldots(2)$$

Equating equations (1) and (2), we get

$$\left(\frac{n+1}{2\pi r}\right) = RZ^2\left(\frac{n(n+2)}{(n+1)^2}\right) \qquad \ldots(3)$$

Since, $r \propto \frac{n^2}{Z}$

$$\Rightarrow \quad r = \frac{(n+1)^2}{11} r_0$$

Substituting in equation (3), we get

$$\frac{11}{2\pi r_0} = \frac{R(11)^2 (n)(n+2)}{(n+1)}$$

$$\Rightarrow \quad (n+1) = (1.09 \times 10^7)(11)(2\pi) \times (0.529 \times 10^{-10})(n^2 + 2n)$$

Solving this equation, we get

$$n = 24$$

PROBLEM 13

A hydrogen-like atom (described by the Bohr model) is observed to emit six wavelengths, orginating from all possible transistions between a group of levels. These levels have energies between -0.85 eV and -0.544 eV (including both these values).

(a) Find the atomic number of the atom.

(b) Calculate the smallest wavelength emitted in these transitions.

(Take $hc = 1240$ eV-nm, ground state energy of hydrogen atom $= -13.6$ eV)

SOLUTION

(a) Total 6 lines are emitted. Therefore,

$$\frac{n(n-1)}{2} = 6$$

$$\Rightarrow \quad n = 4$$

So, transition is taking place between m^{th} energy state and $(m+3)^{\text{th}}$ energy state.

$$E_m = -0.85 \text{ eV}$$

$$\Rightarrow \quad -13.6\left(\frac{z^2}{m^2}\right) = -0.85$$

$$\Rightarrow \quad \frac{z}{m} = 0.25 \qquad \ldots(1)$$

Similarly, $E_{m+3} = -0.544$ eV

$$\Rightarrow \quad -13.6 \frac{z^2}{(m+3)^2} = -0.544$$

$$\Rightarrow \quad \frac{z}{(m+3)} = 0.2 \qquad \ldots(2)$$

Solving equations (1) and (2) for z and m, we get

$$m = 12 \text{ and } z = 3$$

(b) Smallest wavelength corresponds to maximum difference of energies which is obviously $(E_{m+3} - E_m)$.

$$\Rightarrow \quad \Delta E_{\max} = -0.544 - (-0.85) = 0.306 \text{ eV}$$

$$\Rightarrow \quad \lambda_{\min} = \frac{hc}{\Delta E_{\max}} = \frac{1240}{0.306} = 4052.3 \text{ nm}$$

PROBLEM 14

A neutron of kinetic energy 65 eV collides inelastically with a singly ionized helium atom at rest. It is scattered at an angle of $90°$ with respect of its original direction.

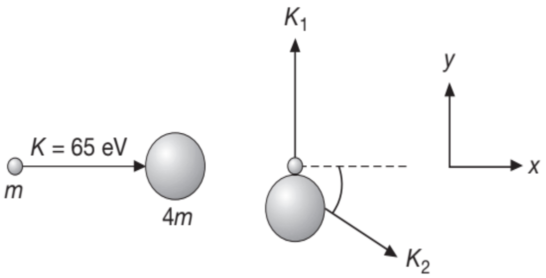
(a) Find the allowed values of the energy of the neutron and that of the atom after the collision.

(b) If the atom gets de-excited subsequently by emitting radiation, find the frequencies of the emitted radiation.

[Given: Mass of He atom $= 4 \times (\text{mass of neutrons})$
Ionization energy of H atom = 13.6 eV]

SOLUTION

(a) Let K_1 and K_2 be the kinetic energies of neutron and helium atom after collision and ΔE be the excitation energy.



Applying Law of Conservation of Linear Momentum along x-direction, we get

$$p_i = p_f$$

$$\Rightarrow \quad \sqrt{2Km} = \sqrt{2(4m)K_2}\cos\theta \qquad \ldots(1)$$

Similarly, Applying Law of Conservation of Linear momentum along y-direction, we get

$$\sqrt{2K_1 m} = \sqrt{2(4m)K_2}\sin\theta \qquad \ldots(2)$$

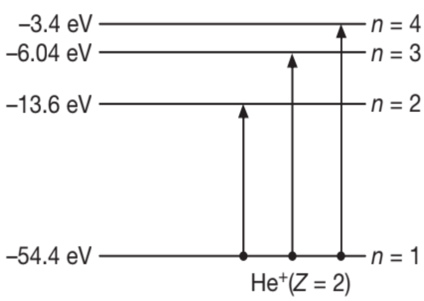
Squaring and adding equations (1) and (2), we get

$$K + K_1 = 4K_2 \qquad \ldots(3)$$

$$\Rightarrow \quad 4K_2 - K_1 = K = 65 \text{ eV} \qquad \ldots(4)$$

Now, during collision, electron can be excited to any higher energy state. Applying Law of Conservation of Energy, we get

$$K = K_1 + K_2 + \Delta E$$



$$\Rightarrow \quad 65 = K_1 + K_2 + \Delta E \qquad \ldots(5)$$

ΔE can have the following values,

$$\Delta E_1 = \{-13.6 - (-54.4)\} \text{ eV} = 40.8 \text{ eV}$$

Substituting in (5), we get

$$K_1 + K_2 = 24.2 \text{ eV} \qquad \ldots(6)$$

Solving (4) and (6), we get

$$K_1 = 6.36 \text{ eV and } K_2 = 17.84 \text{ eV}$$

Similarly, when we put $\Delta E = \Delta E_2$

$$\Rightarrow \quad \Delta E = \{-6.04 - (-54.4)\} \text{ eV}$$

$$\Rightarrow \quad \Delta E = 48.36 \text{ eV}$$

Put in equation (5), we get

$$K_1 + K_2 = 16.64 \text{ eV} \qquad \ldots(7)$$

Solving (4) and (7), we get

$$K_1 = 0.312 \text{ eV and } K_2 = 16.328 \text{ eV}$$

Similarly, when we put

$$\Delta E = \Delta E_3 = \{-3.4 - (-54.4)\} = 51 \text{ eV}$$

Put in equation (5), we get

$$K_1 + K_2 = 14 \text{ eV} \qquad \ldots(8)$$

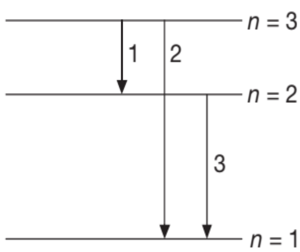
Now, solving (4) and (8), we get

$$K_1 = -1.8 \text{ eV and } K_2 = 15.8 \text{ eV}$$

But since the kinetic energy cannot have the negative values, the electron will not jump to third excited state i.e., $n = 4$.

Therefore, the allowed values of K_1 (KE of neutron) are 6.36 eV and 0.312 eV and of K_2 (KE of the atom) are 17.84 eV and 16.328 eV and the electron can jump upto second excited state only $(n = 3)$.

(b) Possible emission lines are only three as shown in figure. The corresponding frequencies are



$$\nu_1 = \frac{(E_3 - E_2)}{h}$$

$$\Rightarrow \quad \nu_1 = \frac{\{-6.04-(-13.6)\}\times 1.6\times 10^{-19}}{6.63\times 10^{-34}}$$

$$\Rightarrow \quad \nu_1 = 1.82\times 10^{15}\ \text{Hz}$$

$$\nu_2 = \frac{E_3 - E_1}{h}$$

$$\Rightarrow \quad \nu_2 = \frac{\{-6.04-(-54.4)\}\times 1.6\times 10^{-19}}{6.63\times 10^{-34}}$$

$$\Rightarrow \quad \nu_2 = 11.67\times 10^{15}\ \text{Hz and } \nu_3 = \frac{E_2 - E_1}{h}$$

$$\Rightarrow \quad \nu_3 = \frac{\{-13.6-(-54.4)\}\times 1.6\times 10^{-19}}{6.63\times 10^{-34}}$$

$$\Rightarrow \quad \nu_3 = 9.84\times 10^{15}\ \text{Hz}$$

Hence, the frequencies of emitted radiations are 1.82×10^{15} Hz, 11.67×10^{15} Hz and 9.84×10^{15} Hz

PROBLEM 15

The K-absorption edge of an unknown element is 0.171 Å.

(a) Identify the element.

(b) Find the average wavelengths of the K_α, K_β and K_γ lines.

(c) If a 100 eV electron strike the target of this element, what is the minimum wavelength of the X-ray emitted?

Given $hc = 12400$ eVÅ

SOLUTION

From Moseley's law, the wavelength of K series of X-rays is given by taking $b = 1$ in modified in Rydberg's formula. So,

$$\frac{1}{\lambda} = R(Z-1)^2\left(1-\frac{1}{n^2}\right); \text{ for } K \text{ lines where, } n = 2, 3, 4, \ldots$$

(a) For K-absorption edge, substitute $n \to \infty$, in above expression, so we get

$$(Z-1) = \sqrt{\frac{1}{\lambda R}}$$

$$\Rightarrow \quad Z = \sqrt{\frac{1}{(0.171\times 10^{-10})(1.097\times 10^{7})}} + 1$$

$$\Rightarrow \quad Z = 74$$

The element is Tungsten

(b) For K_α line,

$$\frac{1}{\lambda_{K_\alpha}} = R(74-1)^2\left(1-\frac{1}{2^2}\right)$$

$$\Rightarrow \quad \lambda_{K_\alpha} = 0.228\ \text{Å}$$

For K_β line,

$$\frac{1}{\lambda_{K_\beta}} = R(74-1)^2\left(1-\frac{1}{3^2}\right)$$

$$\Rightarrow \quad \lambda_{K_\beta} = 0.192\ \text{Å}$$

For K_γ line,

$$\frac{1}{\lambda_{K_\gamma}} = R(74-1)^2\left(1-\frac{1}{4^2}\right)$$

$$\Rightarrow \quad \lambda_{K_\gamma} = 0.182\ \text{Å}$$

(c) The shortest wavelength corresponding to an electron with kinetic energy 100 eV is given by

$$\lambda_c = \frac{hc}{E} = \frac{12400}{100}\ \text{Å}$$

$$\Rightarrow \quad \lambda_c = 124\ \text{Å}$$