CHAPTER

1 Ray Optics

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Reflection for plane and curved surfaces (i.e. for plane and curved mirrors)
(b) Refraction for plane surfaces (i.e. for glass slab and prism)
(c) Refraction for curved surfaces
(d) Lens
(e) Lens Makers Formula
(f) Human eye
(g) Defects in human eye and optical instruments

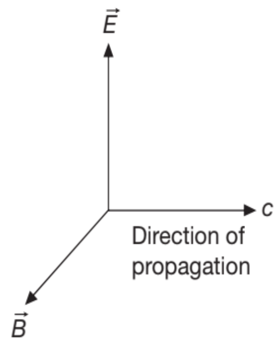
All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

REFLECTION AT PLANE AND CURVED SURFACES

NATURE OF LIGHT: AN INTRODUCTION

Light is a form of energy that makes object visible to our eyes or light is the form of energy that produces in us the sensation of sight. In Seventeenth century **Newton** and **Descartes** believed that light consisted of a stream of particles, called corpuscles. **Huygens** proposed wave theory of light and proposed that light is a disturbance in a medium called **Ether**. This theory could explain the phenomena of interference, diffraction, etc. Thomas Young, through his double slit experiment, measured the wavelength of light.

Maxwell suggested the electromagnetic theory of light. According to this theory, light consists of electric and magnetic fields, in mutually perpendicular directions, and both are perpendicular to the direction of propagation. Heinrich Hertz produced in the laboratory the electromagnetic waves of short wavelengths. He showed that these electromagnetic waves possessed all the properties of light waves.



Light travels in vacuum with a velocity given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ ms}^{-1}$$

where μ_0 and ε_0 are the permeability and permittivity of free space (vacuum).

The magnitudes of electric and magnetic fields are related to the velocity of light by the relation

$$\frac{E}{B} = c$$

In 1905, **Albert Einstein** revived the old corpuscular theory using **Plank's Quantum Hypothesis** and through his photoelectric effect experiment showed that light consists of discrete energy packets, called **photons**. The energy of each photon is

$$E = hf = \frac{hc}{\lambda}$$

So, in view of these developments, light must be regarded to have a dual nature i.e., it exhibits the characteristics of a particle in some situations and that of a wave in other situations. So the question "**Is light a particle or a wave?**" is purely inappropriate to be asked. At present, it is believed that light has **dual nature**, i.e., it has both the characters, wave-like and particle-like.

OPTICS: AN INTRODUCTION

Optics is the study of the properties of light, its propagation through different media and its effects. In most of the situations, the light encounters objects of size much larger than its wavelength. We can assume that light travels in straight lines called **rays**, disregarding its wave nature. This allows us to formulate the rules of optics in the language of **geometry**, as rays of light do not disturb each other on intersection. Such study is called **geometrical** (or **ray**) **optics**. It includes the working of mirrors, lenses, prisms, etc.

When light passes through very narrow slits, or when it passes around very small objects, we have to consider the wave nature of light. This study is called **wave** (or **physical**) **optics**.

DOMAINS OF OPTICS

The study of light can be categorized into three broad domains.

(a) Geometrical Optics (Ray Optics)
(b) Physical Optics (Wave Optics)
(c) Quantum Optics

Please note that these domains are not strictly disjoint as the transitions between them are continuous and not sharp. However for convenience we consider them as distinct. These domains are distinguished as follows.

Geometrical Optics (Ray Optics)

This branch involves the study of propagation of light based on the assumption that light travels in fixed straight line as it passes through a uniform medium and its direction is changed when met by a surface of a different medium or if the optical properties of the medium are non uniform either in time or in space. The ray approximation is valid for the wavelength λ very small compared to the size of the obstacle (d) or the size of the opening through which the ray passes. This approximation $\lambda \ll d$ proves to be very good for the study of mirrors, lenses, prisms and associated optical instruments such as microscope, telescope, cameras etc.

Physical Optics (Wave Optics)

This branch involves the study of propagation of light in the form of a wave and it deals with the phenomenon of interference, diffraction, polarization etc. This nature of light has to be taken when the light passes through very narrow slits or when it goes past very small objects. So this branch works effectively when $\lambda \gg d$.

Quantum Optics

This branch involves the study of propagation of light as a stream of particles called as **Photons**. This concept of light behaving as particles called photons is of utmost importance while studying the origin of spectra, photoelectric effect, concept of radiation pressure, Compton effect etc.

FUNDAMENTAL LAWS OF GEOMETRICAL OPTICS

To a first approximation, we can consider the propagation of light disregarding its wave nature and assuming that light propagates in straight lines called **rays**. This allows us to formulate the laws of optics in the language of **geometry**. Thus, the branch of optics where the wave nature of light is neglected is called **geometrical (or ray) optics**.

Geometrical optics is based on five fundamental laws.

1. **Law of Rectilinear Propagation of Light.** It states that light propagates in straight lines in homogenous media.
2. **Law of Independence of Light Rays.** It states that rays do not disturb each other upon intersection.
3. **The Law of Reversibility of Light.** According to this law, if a ray of light, after suffering a number of reflections and refractions, has its path reversed at any instant, then the ray retraces its path back to the source.
4. **The Laws of Reflection.** The Laws of Reflection govern the bouncing back of the incident ray after striking a surface to the medium from which it was coming.
5. **The Laws of Refraction (discussed later).** The Laws of Refraction govern the bending of light when the light goes from one medium to the other (rarer to denser or denser to rarer) medium.

BASIC TERMS AND DEFINITIONS

Source

A body which emits light is called **source**. The source can be a point one or an extended one. A source is of two types.

(a) **Self luminous:** The source which possess light of its own.
EXAMPLE: sun, electric arc, candle etc.

(b) **Non-luminous:** It is a source of light which does not possess light of its own but acts as source of light by reflecting the light received by it.
EXAMPLE: moon, objects around us, book etc.

Remark(s)

Sources are also classified as isotropic and non-isotropic. Isotropic sources give out light uniformly in all directions whereas non-isotropic sources do not give out light uniformly in all directions.

RAY

The straight line path along which the light travels between two points in a homogeneous medium or in a pair of media is called a Ray. It is represented by an arrow head on a straight line, the arrow head represents the direction of propagation of light. A ray of light will always follow a path along which the time taken is the minimum.

Remark(s)

A single ray cannot be isolated from a source of light.

MEDIUM

Substance through which light propagates or tends to propagate is called a **medium**. It is of following three types.

(a) **Transparent:** It is a medium through which light can be propagated easily.
EXAMPLE: glass, water etc.

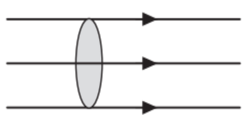
(b) **Translucent:** It is a medium through which light is propagated partially.
EXAMPLE: oil paper, ground glass etc.

(c) **Opaque:** It is a medium through which light cannot be propagated.
EXAMPLE: wood, iron etc.

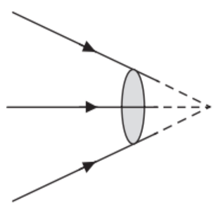
BEAM

A bundle or bunch of rays is called a **beam**. It is of following three types.

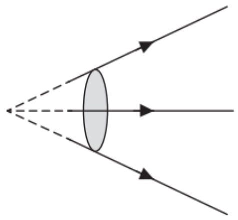
(a) **Parallel beam:** It is a beam in which all the rays constituting the beam move parallel to each other and diameter of beam remains same. A very narrow beam is called a Pencil of Light.



(b) **Convergent beam:** In this case diameter of beam decreases in the direction of ray.



(c) Divergent beam: It is a beam in which all the rays meet at a point when produced backward and the diameter of beam goes on increasing as the rays proceed forward.

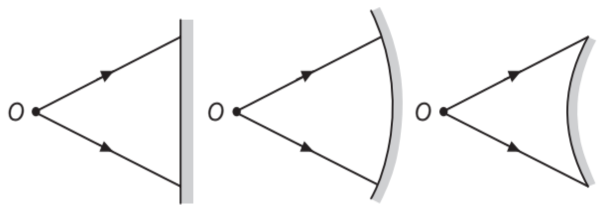


OBJECT(S)

The object for a mirror can be real or virtual. Generally we can define an object as the point where the incident rays intersect (real object) or appear to intersect (virtual object).

Real Object(s)

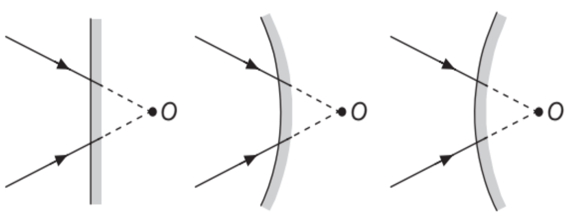
If the rays from a point on an object actually diverge from it and fall on the mirror, the object is said to be real.



In simple language the incident rays are diverging and the point of divergence is the position of the real object. The following diagrams support the arguments given.

Virtual Object(s)

If the rays incident on the mirror appear to converge to a point, then this point is said to be virtual point object for the mirror.



In simple language the incident rays are converging and the point of convergence is the position of the virtual object. The following diagrams support the arguments given.

Conceptual Note(s)

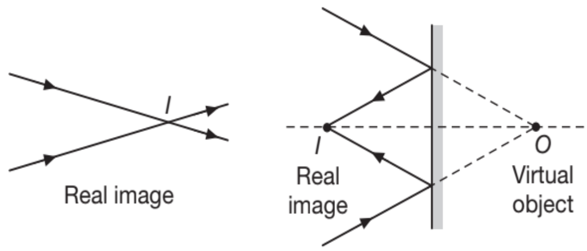
Virtual object cannot be seen by human eye, because for an object or an image to be seen by the eyes, the rays received by the eyes must be diverging.

IMAGE(S)

An **optical image** is a point where reflected or refracted rays of light either intersect or appear to intersect. Thus, the image of an infinite object is actually an assembly of the image points corresponding to various parts or the points of the object. The images formed can again be real or virtual.

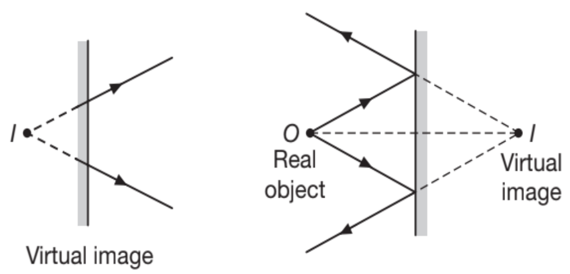
Real Image(s)

If the rays after reflection or refraction actually converge (or meet) at a point then the image is said to be **real** and it can be obtained on a screen.



Virtual Image(s)

However, if the rays do not actually converge but appear to diverge from a point (or appear to meet at a point), then the image so formed is said to be **virtual image.** A virtual image cannot be obtained on a screen.



Conceptual Note(s)

(a) The real images can be obtained on a suitably placed screen, but virtual images cannot be obtained on a screen.

(b) Human eye cannot distinguish between the real image and the virtual image because in both the cases the rays are diverging.

REFLECTION OF LIGHT

When light strikes the surface on an object, some part of the light or the complete light is sent back into the same medium. This phenomenon is called as reflection. The surface, which reflects light, is called mirror. A mirror could be plane or curved.

Conceptual Note(s)

In reflection, the frequency, speed and wavelength remain unchanged, but a phase change may occur depending on the nature of reflecting surface.

The reflection from a denser medium causes an addition phase change of π or a path change of $\frac{\lambda}{2}$ (by Stoke's Law) while reflection from rarer medium does not cause any phase change.

Diffused (irregular) reflection takes place from a rough surface where as **Specular** (regular) reflection takes place from an extraordinarily smooth surface. However, the Laws of Reflection are applicable for both kinds of surfaces.

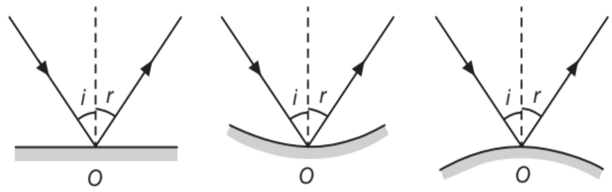
LAWS OF REFLECTION

(a) The incident-ray, the reflected-ray and the normal to the reflecting surface at the point of incidence, all lie in the same plane.

(b) The angle of reflection is equal to the angle of incidence $(i = r)$.

The angle of incidence i is the angle made by the incident ray with the normal.

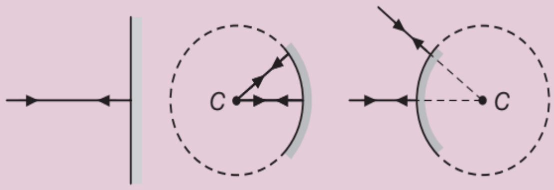
The angle of reflection r is the angle made by the reflected ray with the normal.



Plane surface Concave surface Convex surface

SPECIAL CASES

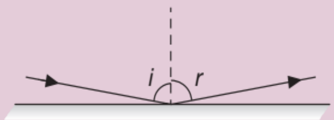
(a) If $i = 0$, then $r = 0$. It means a ray incident normally on a boundary, after reflection it retraces its path.



Plane mirror (a) Concave mirror (b) Convex mirror (c)

(b) The angle made by the incident ray with the plane reflecting surface is called glancing angle. Thus, the glancing angle $= 90° - i$.

(c) **For grazing incidence**, the incident ray grazes the reflecting surface, so $i \to \frac{\pi}{2}$ and hence $r \to \frac{\pi}{2}$ as shown in the figure.



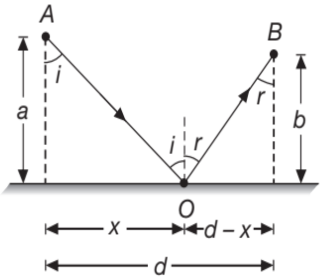
FERMAT'S PRINCIPLE OF LEAST TIME

According to this theorem, the path of a ray of light between any two points is the path along which the time taken is the minimum. This principle is sometimes taken as the definition of a ray of light.

To understand this theorem, let us consider two points A and B in the same medium. Since, we know that between these two points light travels in a straight line, so the time taken by the light to go from A to B must logically be the minimum.

LAWS OF REFLECTION USING FERMAT'S THEOREM

Consider a plane mirror on which light is incident as shown.



Let the incident light start from A, hit the mirror at O and get reflected to point B. Let the points A and B be at perpendicular distances a and b from the mirror and let A and B have a separation d between them as shown in figure. The time taken by the light to go from A to O to B is given by

$$t = t_{A\to O} + t_{O\to B}$$

$$\Rightarrow \quad t = \frac{AO}{c} + \frac{OB}{c}$$

$$\Rightarrow \quad t = \frac{1}{c}\left(\sqrt{a^2+x^2} + \sqrt{b^2+(d-x)^2}\right)$$

Now, according to Fermat's Principle, t is MINIMUM, so

$$\frac{dt}{dx} = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(\sqrt{a^2+x^2}\right) + \frac{d}{dx}\left(\sqrt{b^2+(d-x)^2}\right) = 0$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{2x}{\sqrt{a^2+x^2}}\right) + \frac{1}{2}\left(\frac{2(d-x)(-1)}{\sqrt{b^2+(d-x)^2}}\right) = 0$$

$$\Rightarrow \quad \frac{x}{\sqrt{a^2+x^2}} = \frac{(d-x)}{\sqrt{b^2+(d-x)^2}}$$

From the figure, we observe that

$$\frac{x}{\sqrt{a^2+x^2}} = \sin i \text{ and } \frac{(d-x)}{\sqrt{b^2+(d-x)^2}} = \sin r$$

$$\Rightarrow \quad \sin i = \sin r$$

$$\Rightarrow \quad i = r \qquad \text{\{The Law of Reflection\}}$$

Problem Solving Technique(s)

(a) Basic Problems in Optics: Most of the problems asked in optics expect us to find the position and nature of the final image formed by certain optical systems for a given object. The optical system may be just a mirror, or a lens or a combination of several reflecting and refracting surfaces.

(b) Basic Strategy for Solving the Problems: To handle these kinds of problems, first of all, we identify the sequence in which the reflection and refraction are taking place. The several events of reflection or refraction can be named as Event 1, Event 2 and so on following the sequence in which they occur.
Now, the image of Event 1 would be object for Event 2, image of Event 2 will be object of Event 3 and so on. This way one can proceed to find the final image.

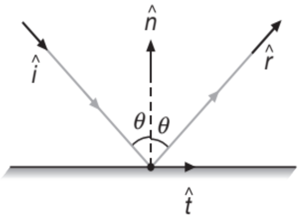
VECTOR FORM OF LAWS OF REFLECTION

Laws of reflection can be redefined with the help of vector algebra by considering unit vectors in the direction of incident rays, reflected rays and normal to the boundary.

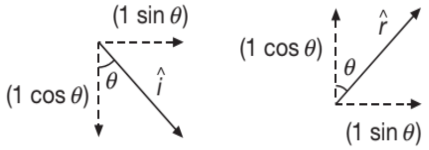
The reflection of a light ray incident on a plane surface is shown in figure. If $\hat{i}$, $\hat{r}$ and $\hat{n}$ are unit vectors along the direction of incident ray, reflected ray and normal to the surface as shown, then first we can write components of $\hat{i}$ and $\hat{r}$ in terms of the unit vectors along the normal and along the surface i.e. tangential to surface. Let $\hat{t}$ be a unit vector tangential to the surface, so we have

$$\hat{i} = (\sin\theta)\hat{t} - (\cos\theta)\hat{n} \qquad \ldots(1)$$

$$\hat{r} = (\sin\theta)\hat{t} + (\cos\theta)\hat{n} \qquad \ldots(2)$$



$$|\hat{i}| = |\hat{r}| = |\hat{n}| = |\hat{t}| = 1$$



Subtracting equation (1) from (2), we get

$$\hat{r} = \hat{i} + (2\cos\theta)\hat{n} \quad \ldots(3)$$

Also, we know that

$$\hat{i}\cdot\hat{n} = |\hat{i}||\hat{n}|\cos(180-\theta) = -\cos\theta \quad \ldots(4)$$

Substituting (4) in (3), we get

$$\hat{r} = \hat{i} - 2(\hat{i}\cdot\hat{n})\hat{n}$$

This equation gives us the Laws of Reflection in vector form.

ILLUSTRATION 1

A ray of light is incident on a plane mirror along a vector $\hat{i}+\hat{j}-\hat{k}$. The normal at the point of incidence is along $\hat{i}+\hat{j}$. Find a unit vector along the reflected ray.

SOLUTION

Reflection of a ray of light is just like an elastic collision of a ball with a horizontal ground. The component of the incident ray along the inside normal gets reversed while the component perpendicular to it remains unchanged. So, the component of incident ray vector $\vec{A} = \hat{i}+\hat{j}-\hat{k}$ parallel to normal, i.e., $\hat{i}+\hat{j}$ gets reversed while perpendicular to it, i.e., $-\hat{k}$ remains unchanged. So, the reflected ray is written as,

$$\vec{R} = -\hat{i}-\hat{j}-\hat{k}$$

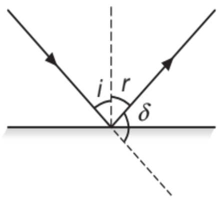
A unit vector along the reflected ray will be,

$$\hat{r} = \frac{\vec{R}}{R} = \frac{-\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$$

$$\Rightarrow \quad \hat{r} = -\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$$

ANGLE OF DEVIATION (δ)

Deviation (δ) is defined as the angle between the initial direction of the incident ray and the final direction of the reflected ray or the emergent ray.

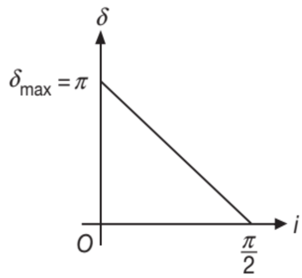


Deviation produced in Reflection is $\delta = 180° - (i+r)$

Since $r = i$

$$\Rightarrow \quad \delta = 180° - 2i$$

The variation of deviation (δ) with the angle of incidence (i) is shown in figure.



Problem Solving Technique(s)

(a) The deviation is maximum for normal incidence i.e., when $i=0$ then, $\delta = \delta_{max} = 180°$.

(b) The deviation is minimum for grazing incidence i.e., when $i \to \frac{\pi}{2}$, then $\delta = \delta_{min} = 0°$.

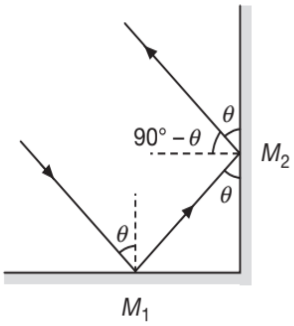
(c) While dealing with the case of multiple reflections suffered by a ray, the net deviation suffered by the incident ray is the algebraic sum of deviation due to each single reflection. So,

$$\delta_{total} = \Sigma\,\delta_{\substack{individual\\reflection}}$$

DO NOT FORGET TO TAKE INTO ACCOUNT THE SENSE OF ROTATION WHILE SUMMING UP THE DEVIATIONS DUE TO SINGLE REFLECTION.

TWO IDENTICAL PERPENDICULAR PLANE MIRRORS

If two plane mirrors are inclined to each other at $90°$, the emergent ray is always antiparallel to the incident ray if it suffers one reflection from each (as shown in figure) whatever be the angle of incidence.



Conceptual Note(s)

The same is found to hold good for three plane mirrors arranged mutually perpendicular to each other thus forming the corner of a cube such that the light incident on this arrangement suffers one reflection from each of the mirrors so as to emerge out anti-parallel to the incident light. This arrangement of three mutually perpendicular plane mirrors forming the corner of a cube is called the CORNER REFLECTOR.

ILLUSTRATION 2

Two plane mirrors are inclined to each other at an angle θ. A ray of light is reflected first at one mirror and then at the other. Find the total deviation suffered by the ray.

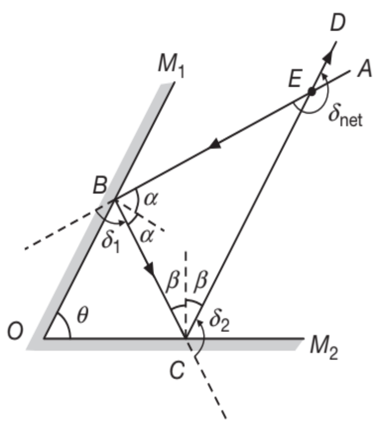
SOLUTION

α be the angle of incidence for mirror M_1

β be the angle of incidence for mirror M_2

δ_1 be the deviation due to mirror M_1 and

δ_2 be the deviation due to mirror M_2



From figure, we observe

$$\delta_1 = \pi - 2\alpha,\ \delta_2 = \pi - 2\beta$$

Also ray is rotated in same sense i.e., anticlockwise, so

$$\delta_{net} = \delta = \text{Total deviation} = \delta_1 + \delta_2$$

$\Rightarrow \quad \delta = 2\pi - 2(\alpha + \beta)$

Now in ΔOBC, $\angle OBC + \angle BCO + \angle COB = 180°$

$\Rightarrow \quad (90° - \alpha) + (90° - \beta) + \theta = 180°$

$\Rightarrow \quad \alpha + \beta = \theta$

$\Rightarrow \quad \delta = 2\pi - 2\theta = 360° - 2\theta$

Alternative Method:

$$\delta = \angle BEC + \angle CEA + \angle AED$$

Now, $\angle BEC = \angle AED$ (vertically opposite angle)

$\Rightarrow \quad \angle BEC = 180° - 2(\alpha + \beta)$

$\Rightarrow \quad \angle BEC = 180° - 2\theta \qquad \{\because \theta = \alpha + \beta\}$

Also, $\angle CEA = 2\alpha + 2\beta \Rightarrow \angle CEA = 2(\alpha + \beta) = 2\theta$

$\Rightarrow \quad \delta = (180° - 2\theta°) + 2\theta + (180° - 2\theta°)$

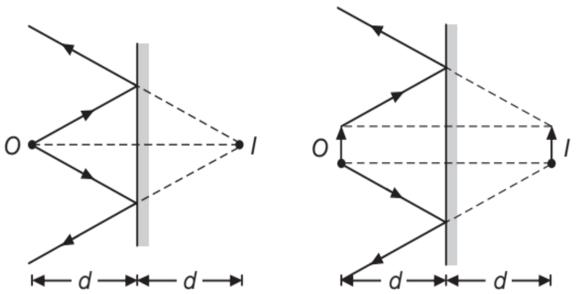
$\Rightarrow \quad \delta = 360° - 2\theta$

Conceptual Note(s)

If two mirrors are inclined at $\angle\theta$ then the ray incident on any one mirror will suffer a total deviation $\delta = 2\pi - 2\theta$ after suffering reflection from both of the mirrors.

REFLECTION FROM A PLANE SURFACE OR PLANE MIRROR

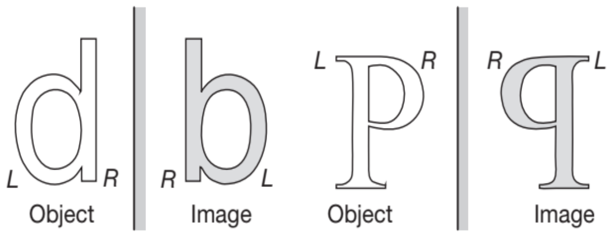
When a real object is placed in front of a plane mirror, the image is always erect, virtual and of same size as the object. It is at same distance behind the mirror as the object is in front of it.



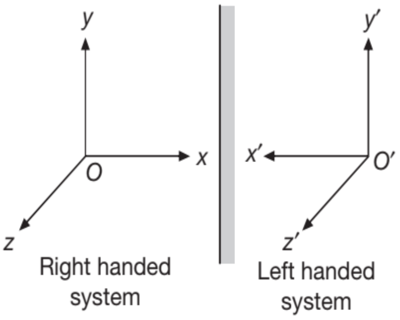
(a) Point object (b) Extended object

LATERAL INVERSION

The image formed by a plane mirror suffers **lateral-inversion**. That is, in the image the left is turned to the right and vice-versa with respect to object. However, the plane mirror does not turn up and down, as shown in figure.

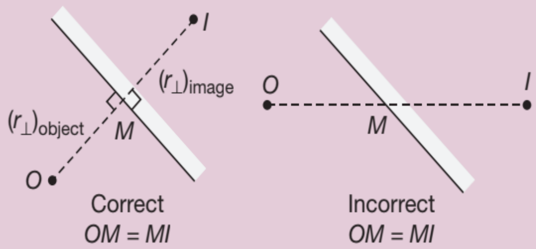


Actually, the plane mirror reverses forward and back in three-dimensions (and not left into right). If we keep a right-handed coordinate system in front of a plane mirror, only the z-axis is reversed. So, a plane mirror changes right-handed co-ordinate system (or screw) to left-handed.



Problem Solving Technique(s)

(a) For finding the location of an image of a point object placed in front of a plane mirror, we must see the perpendicular distance of the object from the mirror.

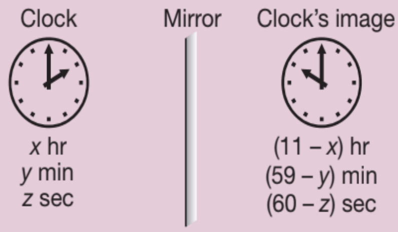


$$\text{i.e.,} \begin{pmatrix}\perp \text{ distance of} \\ \text{O from mirror}\end{pmatrix} = \begin{pmatrix}\perp \text{ distance of} \\ \text{I from mirror}\end{pmatrix}$$

(b) When a wall clock is seen in a mirror then

Image Time = 11 hour 60 minute – Actual Time

$$\Rightarrow \quad t_{image} = 11:60 - t_{actual}$$



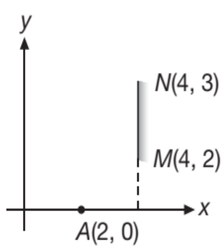
Example 1: If actual time in the clock is 5 : 25, then the image of the clock in the plane mirror will show a time given by

$$t_{image} = 11:60 - 5:25 = 6:35$$

Example 2: If actual time in the clock is 2 hr, 17 min, 25 sec then the image clock will show a time of 9 hr, 42 min, 35 sec.

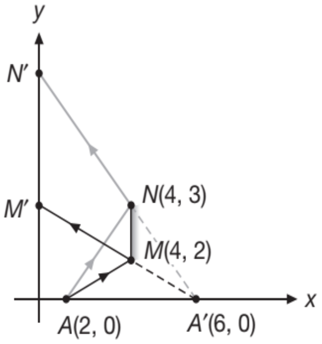
ILLUSTRATION 3

An object is lying at $A(2, 0)$ and MN is a plane mirror, as shown. Find the region on Y-axis in which reflected rays are present.



SOLUTION

The image of point A, in the mirror is at $A'(6, 0)$.



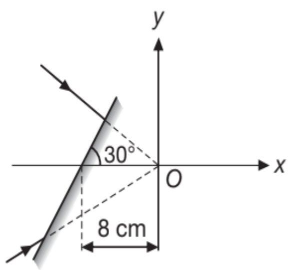
Let us join A' to M and extend it to cut the Y axis at M'. (Ray originating from A which strikes the mirror at M gets reflected as the ray MM' which appears to come from A'). Join $A'N$ and extend to cut Y axis at N' (Ray originating from A which strikes the mirror at N gets reflected as the ray NN' which appears to come from A'). Using Geometry, we get

$$M' = (0,\ 6) \text{ and } N' = (0,\ 9).$$

$M'N'$ is the region on Y axis in which reflected rays are present.

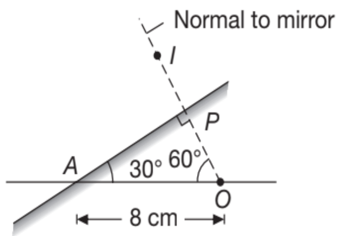
ILLUSTRATION 4

Find the co-ordinates of the location of the image formed for an object kept at origin as shown in figure.



SOLUTION

The first thing we observe is that the object is virtual, because the ray of light is converging on plane mirror. Also, the co-ordinates of object are $(0,\ 0,\ 0)$ and the image co-ordinates are the reflection of object coordinates in the mirror as shown in figure.



The image lies on normal of mirror at I. From ΔAOP, we have

$$\sin(30°) = \frac{PO}{8}$$

$$\Rightarrow \quad PO = 4 \text{ cm}$$

$$\Rightarrow \quad OI = 2(PO) = 8 \text{ cm}$$

So co-ordinates of I are

$$x = -8\cos(60°) = -4 \text{ cm},$$

$$y = 8\sin(60°) = 4\sqrt{3} \text{ cm and}$$

$$z = 0$$

So, the co-ordinates of image are $\left(-4,\ 4\sqrt{3},\ 0\right)$

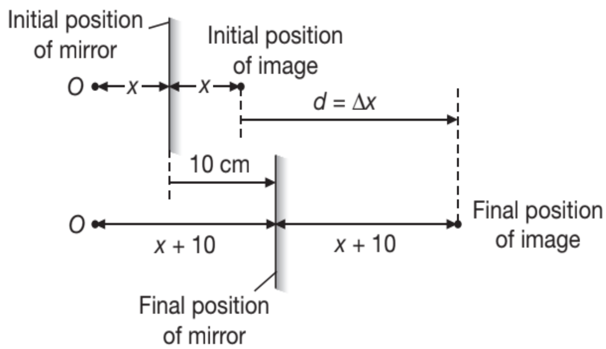
ILLUSTRATION 5

There is a point object and a plane mirror. If the mirror is moved by 10 cm away from the object find the distance which the image will move.

SOLUTION

Since we know that the image distance from the plane mirror is equal to the object distance from the plane mirror.

$$\Rightarrow \quad |x_{im}| = |x_{om}|$$



From figure we observe that

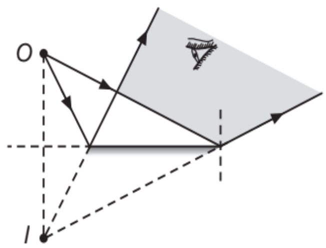
$$2(x+10) = 2x + d$$

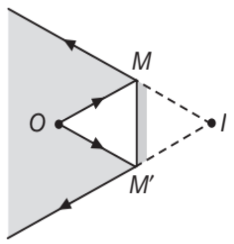
$$\Rightarrow \quad d = \Delta x = 20 \text{ cm}$$

FIELD OF VIEW OF AN OBJECT

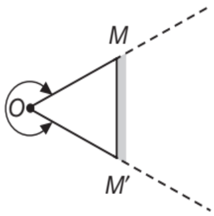
Suppose a point object O is placed in front of a mirror, then a question arises in mind whether this mirror will form the image of this object or not. The answer is yes, it will form. A mirror, irrespective of its size, forms the images of all objects lying in front of it. But every object has its own field of view for a given mirror.

Field of view is the region where diverging rays from object or image are present. If our eyes are present in field of view then only we can see the object or an image as the case may be. Field of view of image is decided by rays which get reflected or refracted from the extremities or the extreme ends of the mirror or a lens and depends on the location of the object in front of mirror or lens.

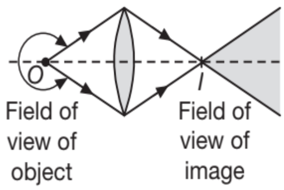




Field of view of image

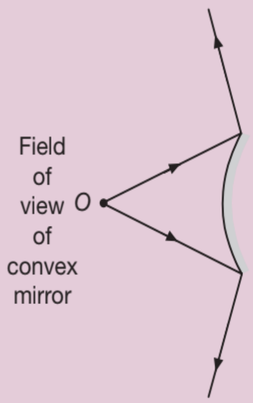


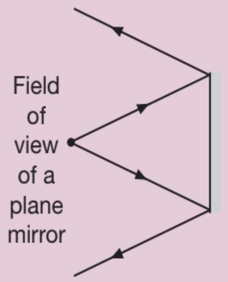
Field of view of object



Conceptual Note(s)

It has been observed that a convex mirror gives a wider field of view than a plane mirror. Therefore, the convex mirrors are used as rear view mirrors in vehicles. Though they make the estimation of distances more difficult but still they are preferred because for a large movement of the object vehicle there is only a small movement of the image.

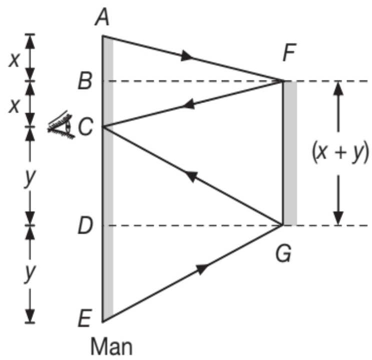




MINIMUM SIZE OF A PLANE MIRROR TO SEE A COMPLETE IMAGE

CASE-1: To find the minimum size of mirror to see a full image we use the fact that light rays from extreme parts of object should reach eye after reflection from mirror. Let us consider following two situations

(a) The minimum size of mirror to see one's full height is $\frac{H}{2}$ where H is the height of man. To see full image mirror is positioned in such a way so that rays from head and foot reach eye after reflection from mirror, as shown in the figure.



(b) A ray starting from head (A) after reflecting from upper end of the mirror (F) reaches the eye at C. Similarly the ray starting from the foot (E) after reflecting from the lower end (G) also reaches the eye at C. in similar triangles ABF and BFC

$AB = BC = x$ (say)

Similarly in triangles CDG and DGE, we have

$CD = DE = y$ (say)

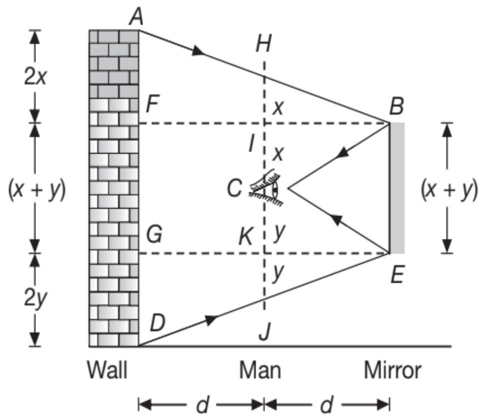
Now, we observe that height of the man is $2(x+y)$ and that the length of mirror is $(x+y)$, i.e., the length of the mirror is half the height of the man. Please note that the mirror can be placed anywhere between the centre line BF (of AC) and DG (of CE).

Conceptual Note(s)

(a) In order to see full image of the man, the mirror is positioned such that the lower edge of mirror is at height half the eye level from the ground.

(b) Minimum size is independent of the distance between man and mirror.

CASE-2: The minimum length of the mirror required to see the full image of a wall behind the man who is standing midway between mirror and the wall is $\frac{H}{3}$, where H is the height of wall. The ray diagram for this situation is shown in figure.



In triangles HBI and IBC let $HI = IC = x$. Now, in triangles HBI and ABF, we have

$$\frac{AF}{HI} = \frac{FB}{BI}$$

$$\Rightarrow \quad \frac{AF}{x} = \frac{2d}{d}$$

$$\Rightarrow \quad AF = 2x$$

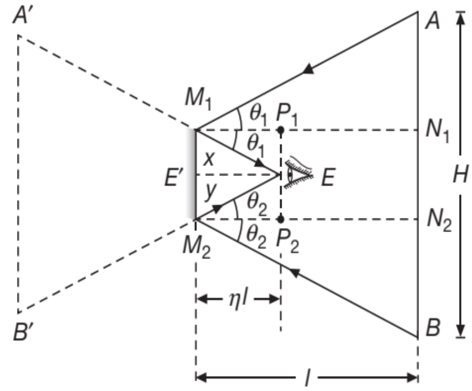
Similarly if, $CK = KJ = y$, then $DG = 2y$. Now, we observe that height of the wall is $3(x+y)$ while that of the mirror is $(x+y)$.

ILLUSTRATION 6

A plane mirror is fitted on a wall and a person gazing at it intends to view the complete image of the rear wall of height H. If η be the fraction of the distance of the person from the mirror, as compared to the length of the room, then calculate the minimum height h of the mirror to do so.

SOLUTION

Let us first draw the ray diagram of the arrangement given in the problem. For this to happen, the rays from the top A and bottom B of wall AB must fall into the eye E of the observer after being reflected from the edges M_1 and M_2 of the mirror. So, in this case height of the mirror will be the minimum.



From similar triangles AM_1N_1 and EM_1P_1

$$\frac{x}{\eta l} = \frac{AN_1}{l}$$

$$\Rightarrow \quad AN_1 = \frac{x}{\eta} \qquad \ldots(1)$$

Similarly, from similar triangle BM_2N_2 and EM_2P_2, we get

$$\frac{y}{\eta l} = \frac{BN_2}{l}$$

$$\Rightarrow \quad BN_2 = \frac{y}{\eta} \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$AN_1 + BN_2 = \frac{(x+y)}{\eta} \qquad \ldots(3)$$

Now, total height of the room is

$$AB = H$$

$$\Rightarrow \quad AN_1 + N_1N_2 + BN_2 = H$$

$$\Rightarrow \quad AN_1 + (M_1E' + E'M_2) + BN_2 = H$$

$$\Rightarrow \quad \frac{x}{\eta} + (x+y) + \frac{y}{\eta} = H$$

Since, height of the mirror is $h = (x+y)$

$$\Rightarrow \quad h = (x+y) = \frac{\eta H}{(\eta+1)}$$

So, height of mirror is $h = \frac{\eta H}{(\eta+1)}$

Note that the height of mirror obtained above is minimum since, light coming from the extreme edges of the room A and B is just able to enter the person's eye after reflection from the mirror.

Conceptual Note(s)

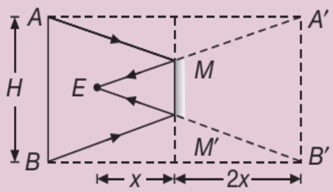
(a) Result is independent of the height or the eye level of the person.

(b) $E'M_2 = EP_2 = y$. But $y + \frac{y}{\eta} = h$, height of eye level above the floor of the room

$$y = \frac{\eta h}{(\eta + 1)}$$

For a person to see the complete image of the rear wall, the lower edge of the mirror, i.e., M_2 should be at a level $\frac{\eta h}{\eta + 1}$ lower than his eye level.

(c) If the person stands at the middle of the room, then $\eta = \frac{1}{2}$.

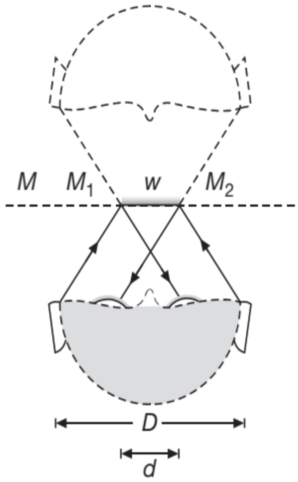


So, minimum height of mirror required is

$$h_{min} = \frac{\left(\frac{1}{2}\right)H}{\left(\frac{1}{2}+1\right)} = \frac{H}{3}$$

REQUIRED MINIMUM WIDTH OF A PLANE MIRROR FOR A PERSON TO SEE THE COMPLETE WIDTH OF HIS FACE

The minimum width of a plane mirror required for a person to see the complete width of his face is $\frac{(D-d)}{2}$ where, D is the width of his face and d is the distance between his two eyes.



$$MM_1 = \frac{1}{2}\left[D - \frac{1}{2}(D-d)\right]$$

$$\Rightarrow \quad MM_1 = \frac{(D+d)}{4} \qquad \ldots(1)$$

and $MM_2 = D - \frac{(D+d)}{4}$

$$\Rightarrow \quad MM_2 = \frac{(3D-d)}{4} \qquad \ldots(2)$$

So, Width of the mirror is $w = M_1M_2$

$$\Rightarrow \quad w = MM_2 - MM_1$$

$$\Rightarrow \quad w = \frac{2D-2d}{4}$$

$$\Rightarrow \quad w = \frac{(D-d)}{2}$$

NUMBER OF IMAGES IN INCLINED MIRRORS

Let θ be the angle between two plane mirrors and n be the number of images formed.

$$\text{Then } n = \begin{cases} \frac{360}{\theta}, & \text{if } \frac{360}{\theta} \text{ is odd} \\ \left(\frac{360}{\theta} - 1\right), & \text{if } \frac{360}{\theta} \text{ is even} \end{cases}$$

Further when $\frac{360}{\theta}$ is odd, then

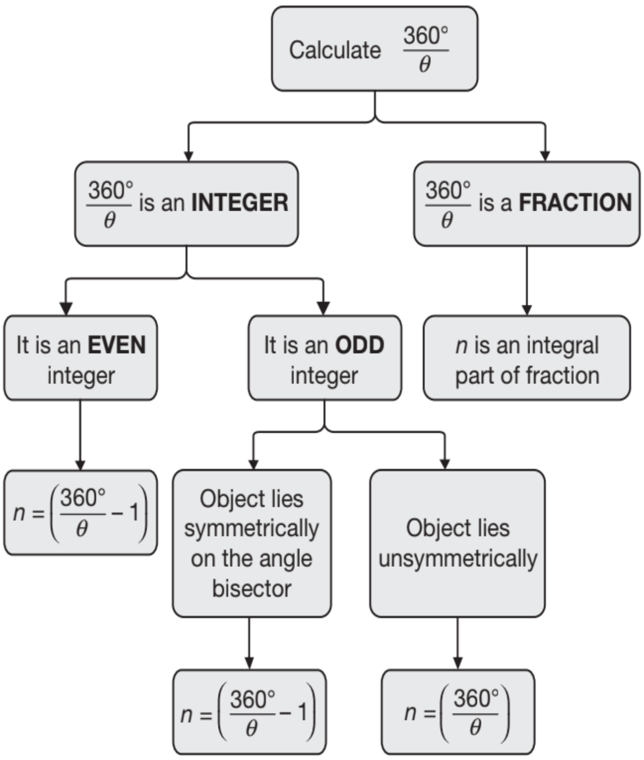
$$n = \begin{cases} \left(\dfrac{360}{\theta} - 1\right), & \text{if object lies symmetrically on the angle bisector of two mirrors} \\ \dfrac{360}{\theta}, & \text{if object lies unsymmetrically} \end{cases}$$

Further if $\dfrac{360}{\theta}$ is a fraction, then the number of images formed will be integral part of the fraction e.g. if $\dfrac{360}{\theta}$ is 4.8, then $n = 4$. Following diagram shows the process to calculate n.

Net deviation produced by two plane mirrors inclined at an angle θ is

$$\delta = 360^\circ - 2\theta$$

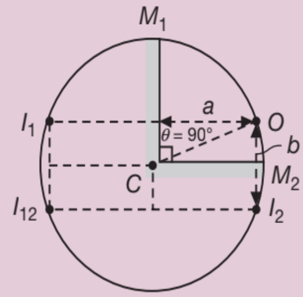
Clearly δ is independent of the angle of incidence of the ray of light.



Conceptual Note(s)

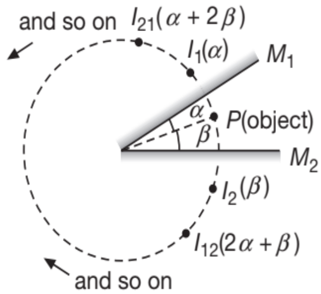
(a) If an object is placed between two parallel mirrors ($\theta = 0^\circ$), the number of images formed will be infinite.

(b) The number of images formed may be different from the number of images seen (which depends on the position of the observer).



LOCATING ALL THE IMAGES FORMED BY TWO PLANE MIRRORS

Consider two plane mirrors M_1 and M_2 inclined at an angle $\theta = \alpha + \beta$ as shown in figure.



Point P is an object kept such that it makes angle α with mirror M_1 and angle β with mirror M_2. Image of object P formed by M_1, denoted by I_1, will be inclined by angle α on the other side of mirror M_1. This angle is written in bracket in the figure besides I_1. Similarly image of object P formed by M_2, denoted by I_2, will be inclined by angle β on the other side of mirror M_2. This angle is written in bracket in the figure besides I_2.

Now I_2 will act as an object for M_1 which is at an angle $(\alpha + 2\beta)$ from M_1. Its image will be formed at an angle $(\alpha + 2\beta)$ on the opposite side of M_1. This image will be denoted as I_{21} and so on. Think when this will process stop.

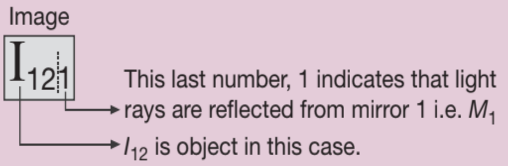
Conceptual Note(s)

(a) The virtual image formed by a plane mirror must not be in front of the mirror or its extension.

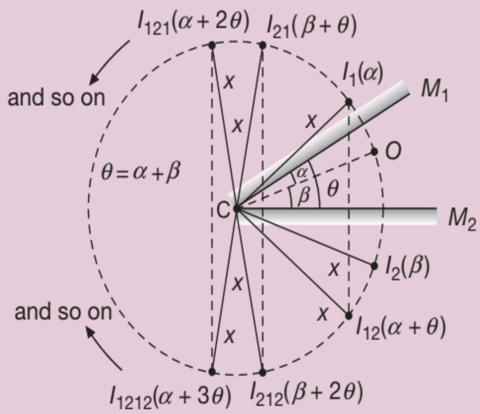
(b) For convenience, we assign symbols to the images formed by mirrors, like

I_1 stands for image of O in M_1
I_{12} stands for image of I_1 in M_2
I_{121} stands for image of I_{12} in M_1
I_{1212} stands for image of I_{121} in M_2

i.e., the last subscript digit in above images tells that reflection is taking place from mirror corresponding to that subscript as shown in the figure.



(c) All the images lie on a circle of radius x where x is the distance between the object O and the point of intersection of the mirrors C.



(d) The angular position of the images formed by mirrors M_1 and M_2 inclined to each other at an angle θ, when an object O is placed between them making an angle α with M_1 and β with M_2 (i.e. $\theta = \alpha + \beta$) can be obtained conveniently by using the following tabular format.

Images formed by mirror M_1 (Angles are measured from the mirror M_1)			Images formed by mirror M_2 (Angles are measured from the mirror M_2)	
I_1	α	$+\theta$	β	I_2
I_{21}	$\beta+\theta$	$+\theta$	$\alpha+\theta$	I_{12}
I_{121}	$\alpha+2\theta$	$+\theta$	$\beta+2\theta$	I_{212}
I_{2121}	$\beta+3\theta$	$+\theta$	$\alpha+3\theta$	I_{1212}
I_{12121}	$\alpha+4\theta$	$+\theta$	$\beta+4\theta$	I_{21212}
Till you get this angle to be less than 180°			Till you get this angle to be less than 180°	

ILLUSTRATION 7

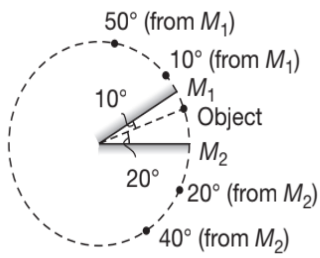
Two mirrors are inclined by an angle 30°. An object is placed making 10° with the mirror M_1. Find the positions of first two images formed by each mirror. Find the total number of images by

(i) counting the images.
(ii) using direct formula.

SOLUTION

We must understand that the angular position of the image in the mirror is same as the angular position of its object in the same mirror. So, first image of O in mirror M_1 is 10° behind mirror M_1. Similarly first image of O in mirror M_2 is 20° behind the mirror M_2.

Now the first image of O formed in mirror M_1 acts as an object for the mirror M_2. This image is at an angular position $(10° + 30°) = 40°$ with respect to the mirror M_2 and hence the second image is located at angular position of 40° from M_2.



Similarly, the first image of O formed in mirror M_2 acts as an object for the mirror M_1. This image is at an angular position $(20° + 30°) = 50°$ with respect to the mirror M_1 and hence the second image is located at angular position of 50° from M_1.

(i) By counting: Let us draw the following table to locate the position of images from the respective mirrors.

Images formed by mirror M_1 (Angles are measured from the mirror M_1)		Images formed by mirror M_2 (Angles are measured from the mirror M_2)
10°	+30°	20°
50°	+30°	40°
70°	+30°	80°
110°	+30°	100°
130°	+30°	140°
170°	+30°	160°
Stop because next angle will be more than 180°		Stop because next angle will be more than 180°

To check whether the final images made by the two mirrors coincide or not, proceed as follows. We shall be adding the last obtained angles (i.e. $170° + 160°$) and the angle between mirrors (i.e. 30°). If this sum comes out to be exactly 360°, then it simply means that the final images formed by the two mirrors coincide. Here last angles made by the mirrors + the angle between the mirrors is $160° + 170 + 30° = 360°$. Therefore in this case the last two images coincide due to which they together will be counted as a one single image and hence we shall be subtracting 1 from the total number of images being obtained.

So, the total number of images formed is the sum of the number of images formed by mirrors M_1 (i.e. 6) and M_2 (also 6) minus 1 (as the last images coincide)

$\Rightarrow \; N = 6 + 6 - 1 = 11.$

(ii) Let's first calculate $\dfrac{360°}{30°} = 12$ (even number)

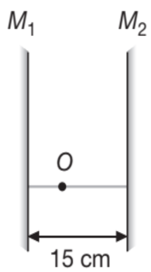
$\Rightarrow$ number of images $= 12 - 1 = 11$

IMAGES FORMED BY TWO PLANE MIRRORS

Since number of images formed is given by $N = \dfrac{360}{\theta} - 1$, so for $\theta = 0°$, $N \to \infty$. This is because when rays after getting reflected from one mirror strike second mirror, the image formed by first mirror will function as an object for second mirror, and this process will continue for every successive reflection.

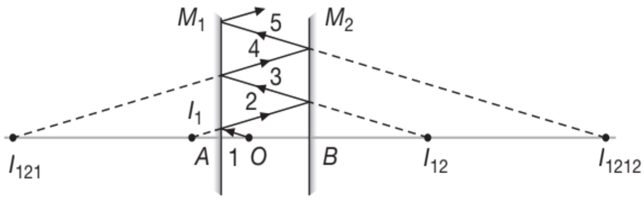
ILLUSTRATION 8

Figure shows a point object O placed between two parallel mirrors separated by 15 cm. O lies at a distance of 5 cm from M_1. Find the distance of images from the two mirrors considering reflection on mirror M_1 first.



SOLUTION

The following ray diagram helps us to understand the formation of images due to subsequent reflections from mirrors M_1 and M_2.



For convenience, we assign symbols to the images formed by mirrors, like

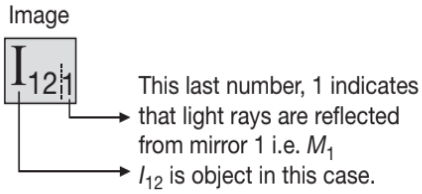
I_1 stands for image of O in M_1

I_{12} stands for image of I_1 in M_2

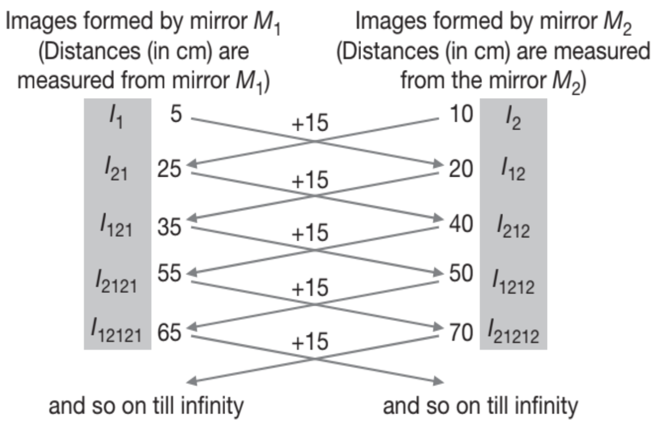
I_{121} stands for image of I_{12} in M_1

I_{1212} stands for image of I_{121} in M_2

i.e., the last subscript digit in above images tells that reflection is taking place from mirror corresponding to that subscript as shown in the figure.

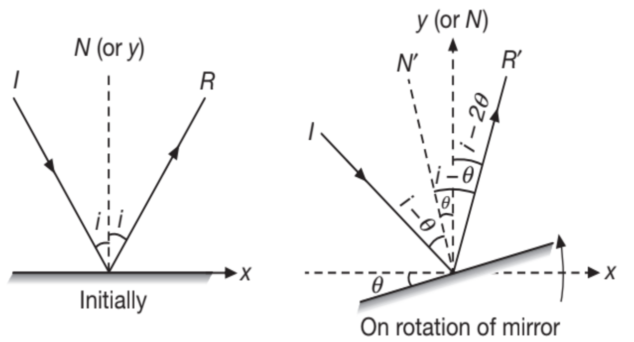


The following figure shows the location of images formed by mirrors M_1 and M_2.



ROTATION OF A PLANE MIRROR

When a mirror is rotated by an angle θ (say anticlockwise), keeping the incident ray fixed, then the reflected ray rotates by 2θ along the same sense, i.e., anticlockwise.



Let I be the incident ray, N the normal and R the reflected ray, then on rotation, I remains as it is, N and R shift to N' and R'.

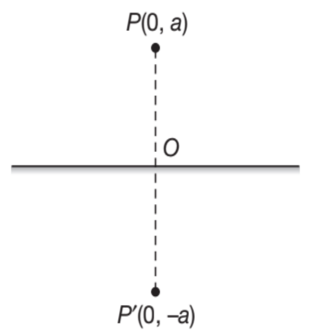
From the two figures we can observe that the reflected ray earlier made an angle i with y-axis while after rotating the mirror it makes the angle $(i-2\theta)$. So, we conclude that the reflected ray has been rotated by an angle 2θ.

Conceptual Note(s)

If a plane mirror rotates with angular velocity ω, then the reflected ray rotates with angular velocity 2ω (excluding rotation of mirror with normal as the axis).

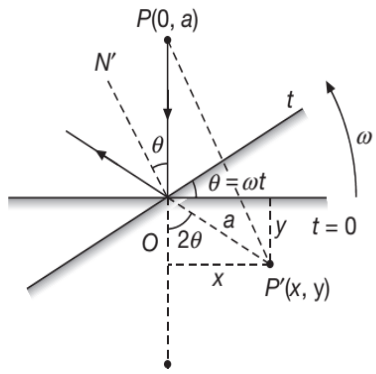
ILLUSTRATION 9

A plane mirror is placed along the xz-plane and an object P is placed at point $(0,\ a)$. The mirror rotates about z-axis with constant angular velocity ω. Calculate the position and velocity of image as function of time $t\left(<\dfrac{\pi}{2\omega}\right)$.



SOLUTION

When the mirror rotates through an angle $\theta=\omega t$, the reflected ray rotates through an angle 2θ as shown. Let the new image be now formed at point $P(x,y)$. Since $OP=OP'=a$. So, we have



$$x=a\sin(2\theta)=a\sin(2\omega t) \text{ and}$$

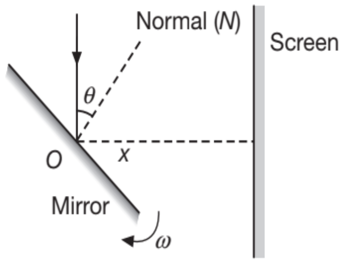
$$y=-a\cos(2\theta)=-a\cos(2\omega t)$$

$$\Rightarrow \quad v_x=\frac{dx}{dt}=2a\omega\cos(2\omega t) \text{ and}$$

$$v_y=\frac{dy}{dt}=2a\omega\sin(2\omega t)$$

ILLUSTRATION 10

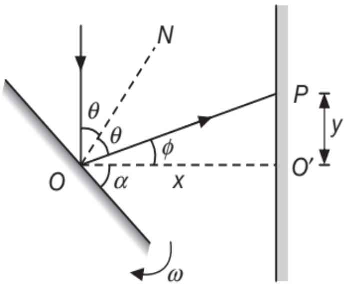
A plane mirror hinged at O is free to rotate in a vertical plane. The point O is at a distance x from a long screen placed in front of the mirror as shown in figure.



A laser beam of light incident vertically downward is reflected by the mirror at O so that a bright spot is formed at the screen. At the instant shown, the angle of incidence is θ and the mirror is rotating clockwise with constant angular velocity ω. Find the speed of the spot at this instant.

SOLUTION

Let P be the bright spot, shown on the screen. Let the distance of point P from O' be y at this instant shown in figure. Then according to the problem we need to calculate $\dfrac{dy}{dt}$



From the figure

$$\theta+\theta+\phi=90^\circ \quad \ldots(1)$$

$$\theta+\phi+\alpha=90^\circ \quad \ldots(2)$$

$\Rightarrow \quad \alpha=\theta$

$\Rightarrow \quad \phi+2\theta=90^\circ$

$\Rightarrow \quad \phi+2\alpha=90^\circ$

$$\Rightarrow \quad \frac{d\phi}{dt}+2\frac{d\alpha}{dt}=0$$

$$\Rightarrow \quad \frac{d\phi}{dt}=-2\frac{d\alpha}{dt}$$

So, the angular speed of the reflected ray is double the angular speed of the mirror.

Since, $y=x\tan\phi$

$$\Rightarrow \quad \frac{dy}{dt}=x\sec^2\phi\frac{d\phi}{dt}$$

Since $\left|\frac{d\phi}{dt}\right|=2\omega$

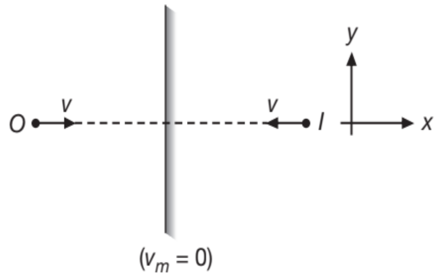
$$\Rightarrow \quad \left|\frac{dy}{dt}\right|=\left(x\sec^2\phi\right)(2\omega)$$

So, the speed of the spot is $\left|\frac{dy}{dt}\right|=2x\omega\sec^2\phi$

VELOCITY OF IMAGE IN A PLANE MIRROR

To understand and interpret the moving images of moving objects in front of plane mirror, we must understand the following cases.

CASE-1: Object moving along the normal to the plane mirror which is at rest. All velocities measured w.r.t. ground frame.



Velocity of object with respect to mirror is

$$\vec{v}_{Om}=v\hat{i}$$

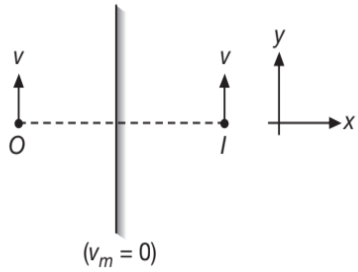
Velocity of image with respect to mirror is

$$\vec{v}_{\text{Im}}=-v\hat{i}$$

Velocity of object with respect to image is

$$\vec{v}_{OI}=\vec{v}_O-\vec{v}_I=(2v)\hat{i}$$

CASE-2: Object moving parallel to the plane of mirror (at rest)



Velocity of object w.r.t. mirror is

$$\vec{v}_{Om}=v\hat{j}$$

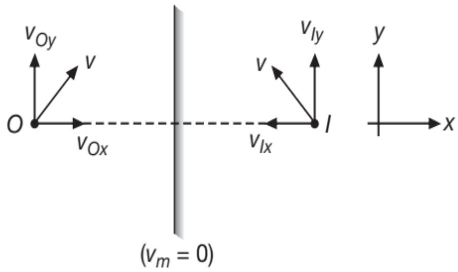
Velocity of image w.r.t. mirror is

$$\vec{v}_{\text{Im}}=v\hat{j}$$

Velocity of object w.r.t. image is

$$\vec{v}_{OI}=\vec{0}$$

CASE-3: Object moving neither along the normal nor along the parallel to the plane mirror (at rest).

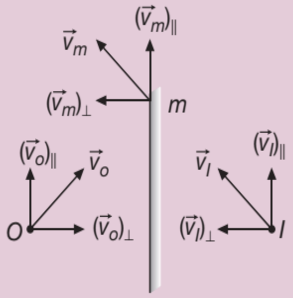


Clearly, we observe this case to be a combination of CASE-1 and CASE-2. So, here

$$(v_{OI})_x = 2v_{Ox} \text{ and } (v_{OI})_y = 0$$

Problem Solving Technique(s)

While solving problems that involve the calculation of image of an object w.r.t. any observer, then



STEP-1: Firstly, calculate the velocity of image w.r.t. mirror keeping in mind that

$$(\vec{v}_{Im})_{\text{along mirror}} = (\vec{v}_{Om})_{\text{along mirror}}$$

$$\Rightarrow (\vec{v}_{Im})_{\parallel} = (\vec{v}_{Om})_{\parallel}$$

Since, both the object and the image approach the mirror with equal and opposite speed, so we have

$$(\vec{v}_{Im})_{\text{normal to mirror}} = -(\vec{v}_{Om})_{\text{normal to mirror}}$$

$$\Rightarrow (\vec{v}_{Im})_{\perp} = -(\vec{v}_{Om})_{\perp}$$

$$\Rightarrow \vec{v}_I - \vec{v}_m = -(\vec{v}_O - \vec{v}_m)$$

$$\Rightarrow \vec{v}_I = 2\vec{v}_m - \vec{v}_O$$

STEP-2: Then the velocity of image w.r.t. mirror is

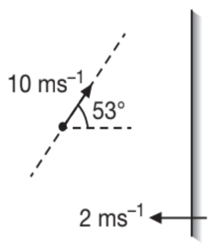
$$\vec{v}_{Im} = (\vec{v}_{Im})_{\parallel} + (\vec{v}_{Im})_{\perp}$$

However, velocity of image w.r.t. any other observer, say A is then given by

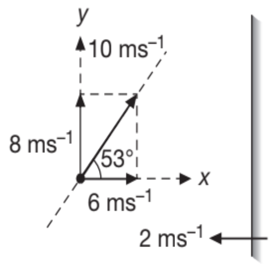
$$\vec{v}_{IA} = \vec{v}_I - \vec{v}_A$$

ILLUSTRATION 11

Find the velocity of image of a moving particle shown in figure.



SOLUTION



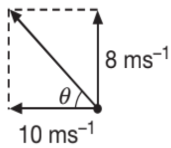
The component of velocity of image perpendicular to mirror is

$$\vec{V}_I = 2\vec{V}_m - \vec{V}_O$$

$$\Rightarrow (\vec{V}_I)_{\perp} = 2(-2) - (6) = -10 \text{ ms}^{-1}$$

For component of velocity of image parallel to the mirror

$$(V_I)_{\parallel} = 8 \text{ ms}^{-1}$$

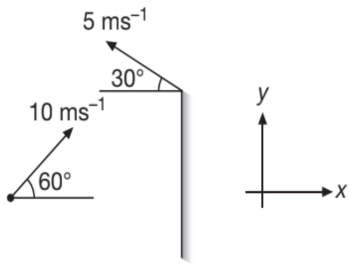


$$\therefore \quad \text{Velocity of image } (V_I) = \sqrt{(V_I)_{\perp}^2 + (V_I)_{\parallel}^2}$$

$$\Rightarrow \quad V_I = \sqrt{100+64} = \sqrt{164} \text{ ms}^{-1} \text{ and } \theta = \tan^{-1}\left(\frac{4}{5}\right)$$

ILLUSTRATION 12

In the situation shown in figure, find the velocity of image.



SOLUTION

Along x direction, applying $v_i = v_m = -(v_0 - v_m)$

$$v_i - (-5\cos 30°) = -(10\cos 60° - (-5\cos 30°))$$

$\Rightarrow \quad v_i = -5(1+\sqrt{3})\ \text{ms}^{-1}$

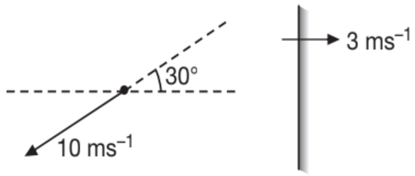
Along y direction $v_0 = v_i$

$\Rightarrow \quad v_i = 10\sin 60° = 5\sqrt{3}\ \text{ms}^{-1}$

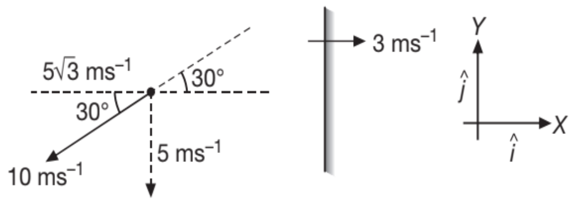
$\Rightarrow \quad$ Velocity of the image $= -5(1+\sqrt{3})\hat{i} + 5\sqrt{3}\hat{j}\ \text{ms}^{-1}$.

ILLUSTRATION 13

A point object is moving with a speed of $10\ \text{ms}^{-1}$ in front of a mirror moving with a speed of $3\ \text{ms}^{-1}$ as shown in figure. Find the velocity of image of the object with respect to mirror, object and ground.



SOLUTION



Velocity of object, $\vec{v}_O = (-5\sqrt{3}\hat{i} - 5\hat{j})\ \text{ms}^{-1}$

Velocity of mirror, $\vec{v}_M = 3\hat{i}\ \text{ms}^{-1}$

For component of velocity perpendicular to mirror, we have

$$(\vec{v}_{IM})_\perp = -(\vec{v}_{OM})_\perp = -(\vec{v}_O - \vec{v}_M)$$

$$\Rightarrow \quad (\vec{v}_{IM})_\perp = -(-5\sqrt{3}\hat{i} - 3\hat{i}) = (5\sqrt{3}+3)\hat{i}\ \text{ms}^{-1}$$

For component of velocity parallel to mirror, we have

$$(\vec{v}_{IM})_\parallel = (\vec{v}_{OM})_\parallel = \vec{v}_O - \vec{v}_M = -5\hat{j} - 0 = -5\hat{j}$$

Since, $(\vec{v}_{IM}) = (\vec{v}_{IM})_\perp + (\vec{v}_{IM})_\parallel$

$$\Rightarrow \quad (\vec{v}_{IM}) = (5\sqrt{3}+3)\hat{i} - 5\hat{j}$$

Also, $\vec{v}_{IM} = \vec{v}_I - \vec{v}_M$

$$\Rightarrow \quad (5\sqrt{3}+3)\hat{i} - 5\hat{j} = \vec{v}_I - (3\hat{i})$$

$$\Rightarrow \quad (5\sqrt{3}+3)\hat{i} + 3\hat{i} - 5\hat{j} = \vec{v}_I$$

$$\Rightarrow \quad \vec{v}_I = \left[(5\sqrt{3}+6)\hat{i} - 5\hat{j}\right]\ \text{ms}^{-1}$$

Further $\vec{v}_{IO} = \vec{v}_I - \vec{v}_O$

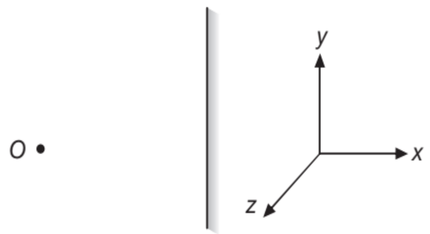
$$\Rightarrow \quad \vec{v}_{IO} = (5\sqrt{3}+6)\hat{i} - 5\hat{j} - (-5\sqrt{3}\hat{i} - 5\hat{j})$$

$$\Rightarrow \quad \vec{v}_{IO} = (5\sqrt{3}+6+5\sqrt{3})\hat{i} + (5-5)\hat{j}$$

$$\Rightarrow \quad \vec{v}_{IO} = (10\sqrt{3}+6)\hat{i}\ \text{ms}^{-1}$$

ILLUSTRATION 14

A plane mirror in y-z plane moves with a velocity $-3\hat{i}$ as shown in figure. An object O starts moving with a velocity $4\hat{i} + \hat{j} - 4\hat{k}$. Find the velocity of the image.



Mirror (M)

SOLUTION

Since the mirror is placed in y-z plane, so the y and z components of the velocity of the image remain the same as that of the object. However, **perpendicular to the mirror**, the velocity of approach of object towards the mirror is always equal and opposite to the velocity of approach of the image towards the mirror, so, we have

$$(\vec{v}_{OM})_x = -(\vec{v}_{IM})_x$$

$$\Rightarrow \quad (\vec{v}_O)_x - (\vec{v}_M)_x = -(\vec{v}_I)_x + (\vec{v}_M)_x$$

$$\Rightarrow \quad (\vec{v}_I)_x = 2(\vec{v}_M)_x - (\vec{v}_O)_x$$

$$\Rightarrow \quad (\vec{v}_I)_x = 2(-3\hat{i}) - 4\hat{i} = -10\hat{i}$$

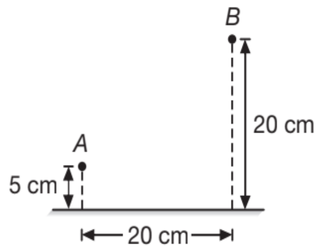
So, $\vec{v}_I = -10\hat{i} + \hat{j} - 4\hat{k}$

Test Your Concepts-I

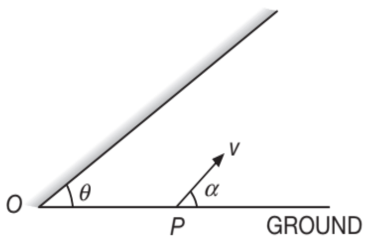
Based on Reflection at Plane Surfaces

(Solutions on page H.3)

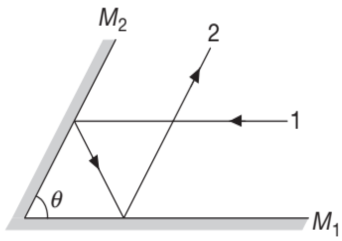
1. A ray of light travelling in the direction $\frac{1}{2}(\hat{i}+\sqrt{3}\hat{j})$ is incident on a plane mirror. After reflection, it travels along the direction $\frac{1}{2}(\hat{i}-\sqrt{3}\hat{j})$. Find the angle of incidence.
2. A ray of light travels from point *A* to a point *B* after being reflected from a plane mirror as shown in figure. From where should it strike the mirror?



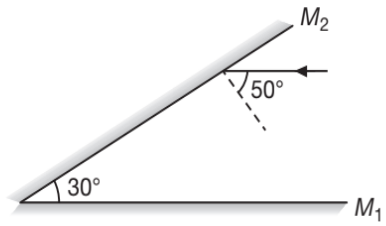
3. A plane mirror is inclined at an angle $\theta = 60°$ with horizontal surface. A particle is projected from point *P* on the ground (see figure) at $t = 0$ with a velocity *v* at an angle α with horizontal. The image of the particle is observed from the frame of the particle projected. Assuming the particle does not collide the mirror. Find the time when image will come momentarily at rest with respect to particle.



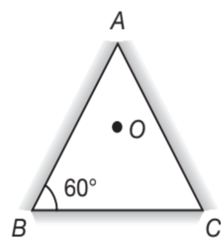
4. Two plane mirrors are inclined to each other such that a ray of light incident on the first mirror and parallel to the second is reflected from the second mirror parallel to the first mirror.
   (a) Find the angle between the two mirrors.
   (b) Also calculate the total deviation produced in the incident ray due to the two reflections.
5. Two plane mirrors M_1 and M_2 are inclined at angle θ as shown in figure. A ray of light 1, which is parallel to M_1 strikes M_2 and after two reflections, the ray 2 becomes parallel to M_2. Find the angle θ.



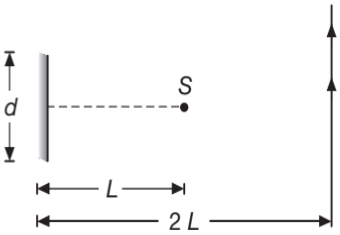
6. Calculate the deviation suffered by an incident ray in the situation shown in figure after it suffers three successive reflections.



7. Two plane mirrors are placed parallel to each other and 40 cm apart. An object is placed 10 cm from one mirror. Find the distance from the object to the respective image for each of the five images that are closest to the object.
8. Find the number of images formed of an object *O* enclosed by three mirrors *AB*, *BC*, *AC* having equal lengths in situation shown in figure.

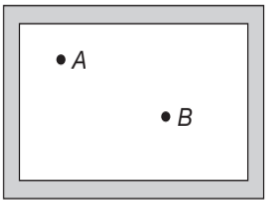


9. A point source of light *S*, placed at a distance *L* in front of the centre of a mirror of width *d*, hangs vertically on a wall. A man walks in front of the mirror along a line parallel to the mirror at a distance 2*L* from it as shown. Find the greatest distance over which he can see the image of the light source in the mirror.

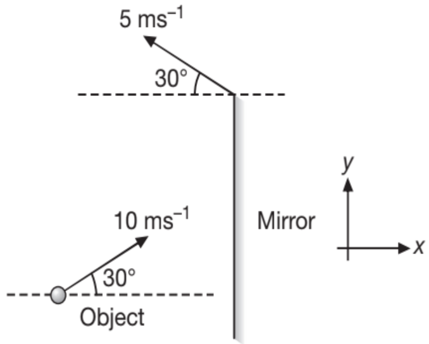


10. Find the smallest size of a looking glass which a man with a face 24 cm × 16 cm should purchase that will enable him to see his whole face completely, if the
(a) man is one eyed.
(b) man is two eyed.
Given that the separation between his eyes is 8 cm.

11. In what direction should *A* beam of light is to be sent from point *A* (shown in figure) contained in a mirror box for it to fall onto point *B* after being reflected once from each of the four walls. If the points *A* and *B* are in one plane perpendicular to the walls of the box (i.e., in the plane of the drawing) then in what direction should the beam be sent from *B* to *A*?



12. The object and the mirror move with velocity shown in figure. Calculate the velocity of the image.



13. A ray of light is incident on an arrangement of two plane mirrors inclined at an angle θ with each other. It suffers two reflections one from each mirror and finally moves in a direction making angle α with the incident ray (α is acute). Find the angle α and show that it is independent of angle of incidence.

14. A ray of light is incident at an angle of 30° with the horizontal. At what angle with horizontal must a plane mirror be placed in its path so that it becomes vertically upwards after reflection?

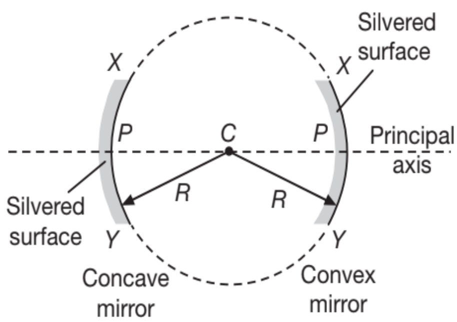
15. Two plane mirrors are inclined to each other at an angle of 70°. A ray is incident on one mirror at an angle θ. The ray reflected from this mirror falls on the second mirror from where it is reflected parallel to the first mirror. Find the value θ.

REFLECTION FROM CURVED SURFACES

A small curved reflecting surface can be considered to be a part of a sphere. Hence, such surfaces are called spherical mirrors. Depending upon the surface silvered, these are of two types—concave and convex, as shown in figure. Some important terms are described below.

(a) **Pole or Vertex:** Centre *P* of the surface of the mirror.
(b) **Centre of Curvature:** Centre C of the sphere.
(c) **Radius of Curvature:** Radius R of the sphere.
(d) **Principal Axis:** Line PC, joining the pole and the centre.
(e) **Linear Aperture:** Distance XY between the extremities of the mirror surface.

Note that since lenses are also made of spherical surfaces, the above terms also apply to lenses, except that the pole is replaced by a new term called as Optical Centre.



Important Terms and Definitions

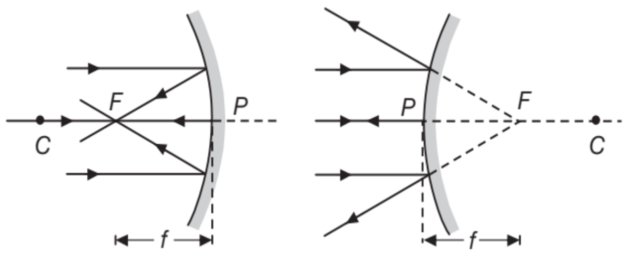
(a) Centre of curvature: It is the centre of the sphere of which the mirror/lens is a part.
(b) Radius of curvature: It is the radius of the sphere of which the mirror/lens is a part.
(c) Pole: It is the geometrical centre of the spherical reflecting surface of which the mirror/lens is a part.
(d) Principal axis (for a spherical mirror): It is the straight line joining the centre of curvature to the pole.
(e) Focus: When a narrow beam of rays of light, parallel to the principal axis and close to it, is incident on the surface of a mirror (lens), the reflected (refracted) beam either converges to a point or appears to diverge from a point on the principal axis. This point is called the focus (F).
(f) Focal length (for a mirror): It is the distance between pole and the principal focus (F).
(g) Real image: If reflected (or refracted) rays converge to a point (i.e. intersect there), then the point is a real image.
(h) Virtual image: If reflected (or refracted) rays appear to diverge from a point, then the point is a virtual image.
(i) Real object: If the incident rays diverge from a point, then the point is a real object.
(j) Virtual object: If incident rays converge and appear to intersect at a point behind the mirror (or lens), then the point is a virtual object.

PARAXIAL RAYS

Paraxial rays are the rays which are either parallel to the principal axis or make small angles with it i.e., these rays are nearly parallel to the principal axis. Our treatment for the spherical mirrors has been restricted to these rays and due to this we shall be considering the **curved mirrors** that have **smaller aperture**. However, for the sake of convenience, comfort and clarity, we shall be drawing the diagrams of larger size.

FOCUS, FOCAL LENGTH AND POWER OF A MIRROR

When a narrow beam of light, parallel to the principal axis and close to it, is incident on the surface of a mirror (lens), the reflected (refracted) beam is found to converge to or appears to diverge from a point on the principal axis. This point is the focus also called Principal Focus in case of mirror(s). The plane passing through the focus and perpendicular to the principal axis is called **focal plane**.



Concave mirror Convex mirror

Focal length (f) is the distance of focus (F) from the pole (P) of the mirror or the optical centre for a lens.

The focal length of a mirror does not change when it is immersed completely in a liquid, i.e. the focal length of the mirror is independent in the medium surrounding it.

Power of a mirror P is defined as the negative reciprocal of the focal length f of the mirror (taken in metre), so

$$P = -\frac{1}{f(\text{in metre})}$$

SIGN CONVENTIONS FOR MIRRORS

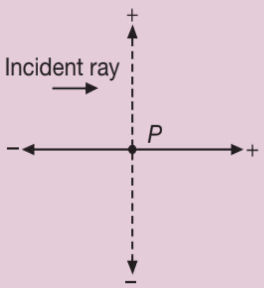
While solving problems, we must follow a set of sign conventions given for convenience. According to this sign convention

(a) Origin is placed at the pole (P).
(b) All distances are to be measured from the pole (P).
(c) Distances measured in the direction of incident rays are taken as positive.
(d) Distances measured in a direction opposite to that of the incident rays are taken as negative.
(e) Distances above the principal axis are taken as positive.
(f) Distances below the principal axis are taken as negative.
(g) This sign convention is used to find the position and nature (virtual or real, erect or inverted) of the image formed by the mirror (or lens).
(h) Object distance is denoted by u, image distance by v, focal length by f and radius of curvature by R.

(i) Note that generally we keep the object to the left of the mirror (or lens), so that the ray of light starting from object must go from left to the right i.e., towards positive direction of x-axis. Now since the distances have to be measured from the pole consequently,
u must always be negative,
v is positive (for a virtual image) and negative (for a real image).
f is positive (for a convex mirror) and negative (for a concave mirror).
For both the mirrors and lenses.
Magnification for a real image is negative i.e., $m_{\text{real}} = \ominus$
Magnification for a virtual image is positive i.e., $m_{\text{virtual}} = \oplus$

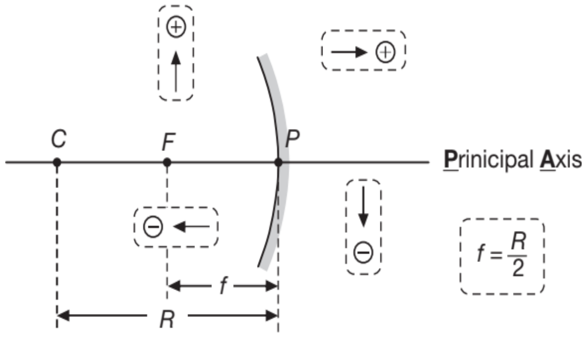
Conceptual Note(s)

The convention that all distances measured along the ray of light are positive and all distances measured opposite to the ray of light are negative matches exactly with the **Cartesian coordinate system**, where we can simply place the origin at the pole P and say that all distances to the left of the pole are negative, all distances to the right of the pole are positive, all distances above the pole are positive and all distances below the pole are negative.

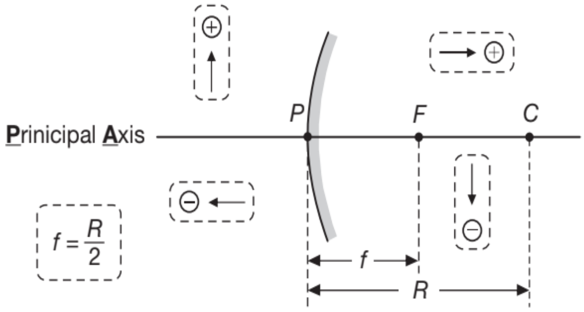


(j) For solving problems in which any of u, v, f (or R) is to be found, we must make sure that no convention should be applied on the quantity to be found. The unknown quantity will automatically take up its sign from which we shall make obvious conclusion.

(k) The diagrams show the application of sign convention to curved mirrors.



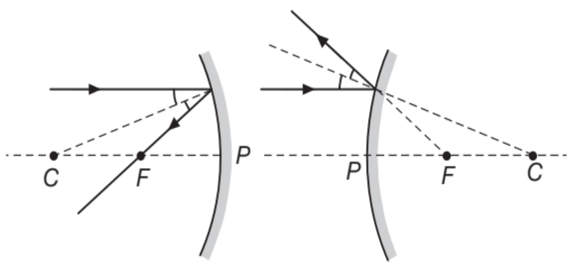
P is pole, **F** is focus and **C** is centre of curvature



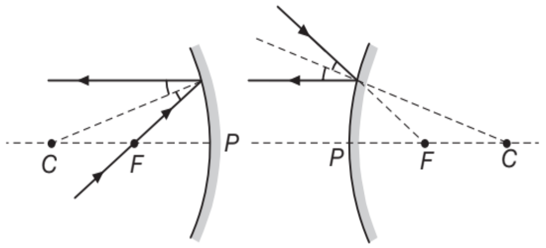
RULES FOR OBTAINING IMAGE BY RAY TRACING

These rules are based on the laws of reflection, i.e. the angle of incidence equals the angle of reflection, $i = r$ and are used to find the location, nature (real or virtual, inverted or erect) and size of the image formed by a spherical mirror. Take any two rays coming from any given point on the object. Find out at which point these rays actually meet (or appear to meet) after reflection from the mirror. This point is the real (or virtual) image. In this way, taking one point after another on the object, the entire image can be constructed.

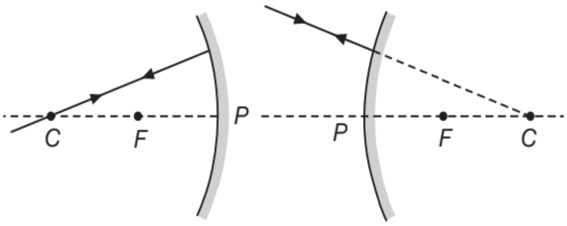
(a) A ray of light **coming parallel to principal axis,** after reflection passes through the focus (in case of concave mirror) or appears to come from the focus (in case of convex mirror).



(b) A ray of light **passing through the focus** (in case of concave mirror) or appearing to pass through the focus (in case convex mirror) is reflected parallel to the principle axis.



(c) A ray of light **passing through the centre** of curvature falls normally on the mirror and is therefore reflected back along the same path i.e., retraces its path.



(d) Incident and reflected rays at the pole of a mirror are symmetrical about the principal axis. (Because for the pole principle axis acts as normal and by Laws of Reflection $i = r$). So by observing the size of erect image in a mirror we can decide the nature of the mirror i.e., whether it is convex, concave or a plane mirror.

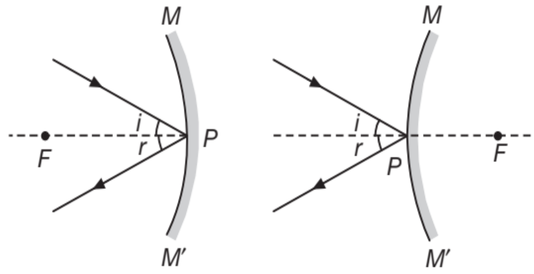
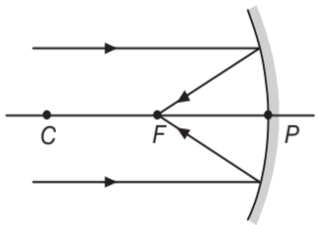
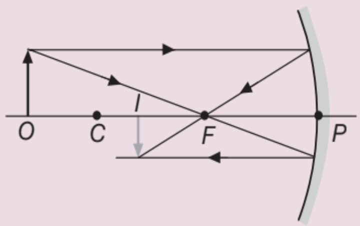
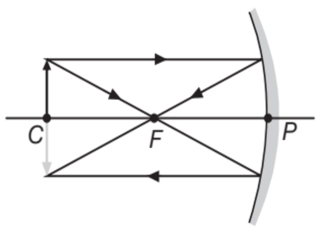


IMAGE FORMATION BY CONCAVE MIRROR

Object Position	Diagram	Position of Image	Nature of Image
At infinity		At the principal focus (F) or in the focal plane	Real, inverted and extremely diminished
Beyond C		Between F and C	Real, inverted and diminished
At C		At C	Real, inverted and of same size as the object

(Continued)

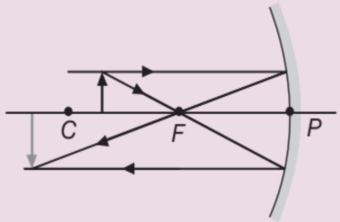
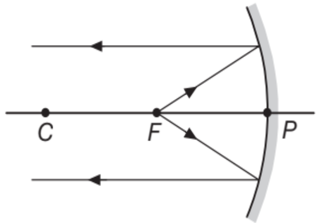
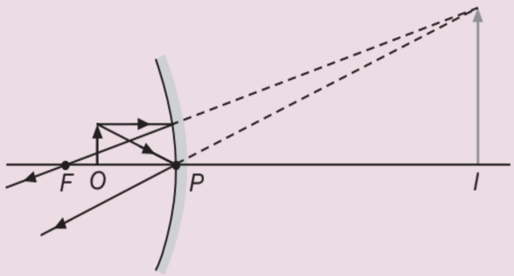
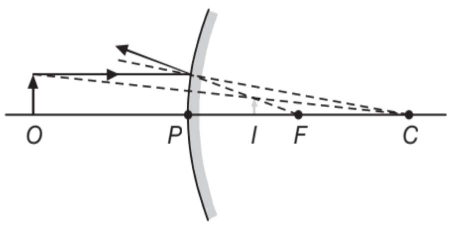
Object Position	Diagram	Position of Image	Nature of Image
Between F and C		Beyond C	Real, inverted and magnified
At F or in the focal plane		At infinity	Real, inverted and highly magnified
Between F and P		Behind the mirror	Virtual, erect and magnified

IMAGE FORMATION BY CONVEX MIRROR

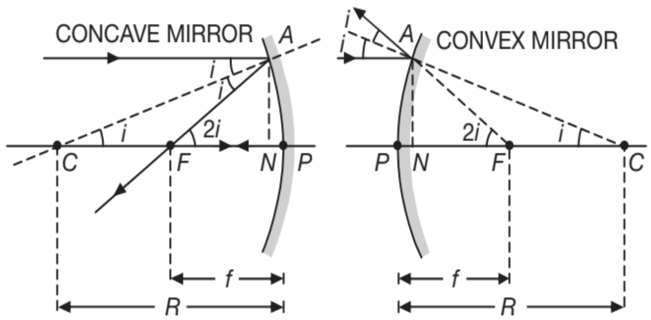
Object Position	Diagram	Position of Image	Nature and Size of Image
For all positions of object		Images formed between the Pole and the focus (F).	Always forms a Virtual, Erect and Diminished Image

RELATION BETWEEN FOCAL LENGTH (f) AND RADIUS OF CURVATURE (R)

A ray parallel to the principal axis passes through the focus (as in concave mirror) or appears to pass through the focus (as in convex mirror). The normal to the mirror(s) at the point of reflection i.e., A must pass through the centre of curvature. In triangle CAN, we have $\tan i = \frac{AN}{NC}$

For paraxial rays and mirrors of small aperture, we have

$$\tan i \cong i = \frac{AN}{NC} \quad \ldots(1)$$



In triangle FAN, we have

$$\tan(2i) = \frac{AN}{NF}$$

Again for paraxial rays and mirror of small aperture, we have

$$\tan(2i) \cong 2i = \frac{AN}{NF} \quad \ldots(2)$$

From (1) and (2), we get

$$2\left(\frac{AN}{NC}\right) = \frac{AN}{NF}$$

$$\Rightarrow \quad \frac{2}{NC} = \frac{1}{NF} \quad \ldots(3)$$

Since, aperture is small, so N coincides with P, so we have

$$NC \cong PC \text{ and } NF \cong PF$$

For convex mirror, we have

$$PC = +R \text{ and } PF = +f$$

$$\Rightarrow \quad f = \frac{R}{2}$$

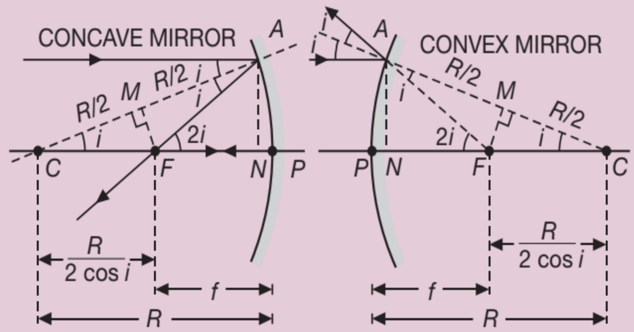
For concave mirror, we have

$$PC = -R \text{ and } PF = -f$$

$$f = \frac{R}{2}$$

So, for a curved mirror of small aperture, focal length is half the radius of curvature.

Conceptual Note(s)



If paraxial rays are not taken into account, then we have

$$f = R - \frac{R}{2\cos i}$$

Since, we see that $CM = MA = \dfrac{R}{2}$

Also, in triangle CFM, we have

$$\cos i = \frac{R/2}{FC}$$

$$\Rightarrow \quad FC = \frac{R}{2\cos i}$$

Since $PF = PC - FC$

$$\Rightarrow \quad f = R - \frac{R}{2\cos i}$$

For paraxial rays $i \to 0$, so $\cos i \to 1$

$$\Rightarrow \quad f = \frac{R}{2}$$

MIRROR FORMULA

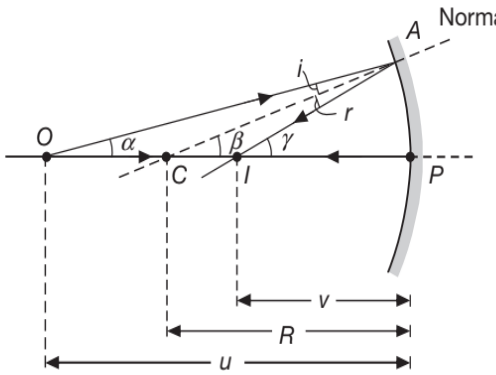
For Concave Mirror

Consider a point object O placed on the principal axis of a concave mirror. A ray of light, incident on the point A at an angle of incidence i on the mirror makes an angle r with the normal as shown in the figure. From the Laws of Reflection we know that $i = r$. Further to find the location of the image let us take another ray along the principal axis so that it hits the mirror normally at the point P to reverse its path and meet the other ray at I. This point of intersection of the two rays happens to be the place where the image is formed.

Since from geometry we know that in a triangle, external angle equals sum of internal opposite angles, so for triangle CAO and triangle CAI, we have

$$\beta = \alpha + i$$

and $\gamma = r + \beta$



Since by Laws of Reflection, we have

$$i = r$$

$\Rightarrow \quad \alpha + \gamma = 2\beta$

Applying paraxial ray approximation, we get

$$\tan\alpha \approx \alpha = \frac{AP}{PO},\ \tan\beta \approx \beta = \frac{AP}{PC} \text{ and } \tan\gamma = \gamma = \frac{AP}{PI}$$

$\Rightarrow \quad \tan\alpha + \tan\gamma = 2\tan\beta$

$\Rightarrow \quad \frac{AP}{PO} + \frac{AP}{PI} = 2\left(\frac{AP}{PC}\right)$

Using sign conventions, we have

$$PO = -u,\ PI = -v \text{ and } PC = -R$$

$\Rightarrow \quad \frac{1}{(-u)} + \frac{1}{(-v)} = \frac{2}{(-R)}$

Since we know that $f = \frac{R}{2}$, so we get

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{R} = \frac{1}{f} \qquad \text{\{Mirror Formula\}}$$

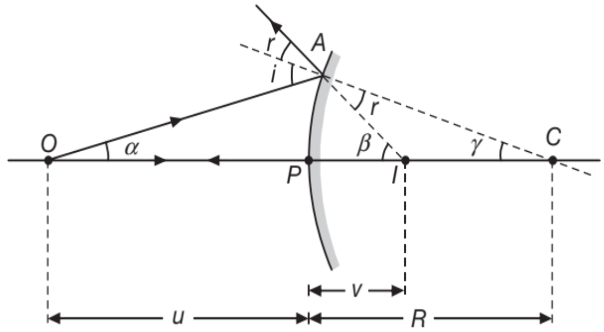
FOR CONVEX MIRROR

Similarly we can drive a formula for a convex mirror.

Since from geometry we know that in a triangle, external angle equals sum of internal opposite angles, so for triangle CAO and triangle CAI, we have

$$i = \alpha + \gamma$$

and $\beta = r + \gamma$



Since $i = r$, so we get

$$-\alpha + \beta = 2\gamma$$

Applying paraxial ray approximation, we get

$$-\tan\alpha + \tan\beta = 2\tan\gamma$$

$\Rightarrow \quad -\frac{AP}{PO} + \frac{AP}{PI} = 2\left(\frac{AP}{PC}\right)$

Using sign conventions, we have

$$PO = -u,\ PI = +v \text{ and } PC = +R$$

$\Rightarrow \quad -\frac{1}{(-u)} + \frac{1}{v} = \frac{2}{R}$

$\Rightarrow \quad \frac{1}{u} + \frac{1}{v} = \frac{2}{R} = \frac{1}{f} \qquad \text{\{Mirror Formula\}}$

Interestingly, the mirror formula is the same irrespective of the mirror used.

NEWTON'S FORMULA

If instead of measuring the object distance and the image distance from the pole, the distances are measured from the focus, then we get a modified mirror formula. This modified mirror formula is called the **Newton's Formula**. Let

x_1 be the distance of object from focus and

x_2 be the distance of image from focus, then

$$u = f + x_1$$

and $v = f + x_2$

According to the mirror formula, we have

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{(f+x_1)}+\frac{1}{(f+x_2)}=\frac{1}{f}$$

$$\Rightarrow \quad (2f+x_1+x_2)f=(f+x_1)(f+x_2)$$

$$\Rightarrow \quad 2f^2+(x_1+x_2)f=f^2+(x_1+x_2)f+x_1x_2$$

$$\Rightarrow \quad x_1x_2=f^2$$

This is known as Newton's formula.
This formula is applicable to real object and real images.

LINEAR MAGNIFICATION OR LATERAL MAGNIFICATION OR TRANSVERSE MAGNIFICATION

To have an idea of the relative size of the image and the object, we define linear magnification also called as lateral magnification as

$$m=\frac{\text{size of the image}}{\text{size of the object}}=\frac{h_2}{h_1}=\frac{h_i}{h_0}$$

For both concave and convex mirrors, it can be shown that

$$m=-\frac{v}{u}$$

Since we know that $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$ so we get

$$m=-\frac{v}{u}=\frac{f}{f-u}=\frac{f-v}{f}$$

For spherical mirrors positive value of m means v and u are having opposite signs i.e. when u is negative v is positive and vice versa.

So for a real object if the image formed is virtual, erect and three times the size of the real object then, we have $m=+3$.

Similarly for a real object if the image formed is real, inverted and one third the size of the real object then $m=-\frac{1}{3}$.

ILLUSTRATION 15

An object is placed at a distance of 15 cm from a concave mirror of focal length 10 cm. Describe the size, nature and position of the image formed.

SOLUTION

The rough figure indicating the pole of the mirror, focus, and the given distances is shown. The sign convention is also given.

Since, $u=-15$ cm (negative since it lies to the left of O)

$f=-10$ cm (negative since it lies to the left of O)

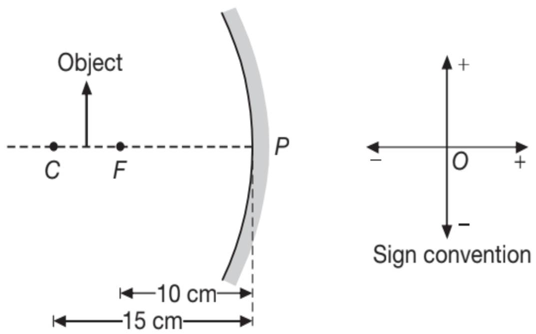
Since we have, from mirror formula that

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{1}{f}-\frac{1}{u}=\frac{1}{(-10)}-\frac{1}{(-15)}=-\frac{1}{10}+\frac{1}{15}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{-15+10}{150}=-\frac{5}{150}$$

$$\Rightarrow \quad v=-\frac{150}{5}=-30 \text{ cm}$$



The negative sign for v shows that the image lies to the left of O.
Now, the magnification is given by

$$m=-\frac{v}{u}=-\frac{-30}{-15}=-2$$

The negative sign for m indicates that the image is inverted, and hence real and is double the size of the object.

Thus, we find that the image is real, inverted, twice the size of the object, and is formed 30 cm in front of the mirror. The ray diagram is shown in figure.

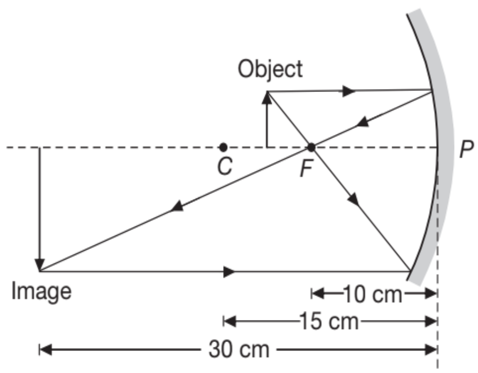


ILLUSTRATION 16

A beam of light converges towards a point O, behind a convex mirror of focal length 20 cm. Find the nature and position of image if the point O is

(a) 10 cm behind the mirror

(b) 30 cm behind the mirror

SOLUTION

(a) Here, in this case the object is virtual. So, for this we have

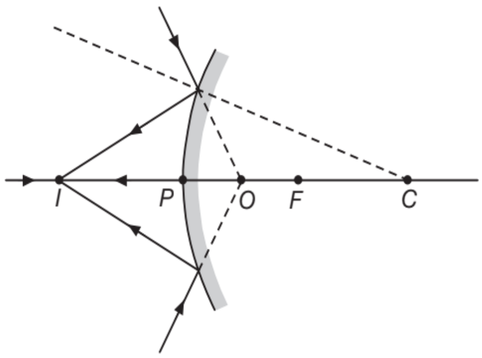
$$u = +10 \text{ cm}, \ f = +20 \text{ cm}$$

Using mirror formula, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}=\frac{1}{20}-\frac{1}{+10}=\frac{1-2}{20}=\frac{-1}{20}$$

$$\Rightarrow \quad v = -20 \text{ cm}$$

Magnification, $m=-\frac{v}{u}=-\frac{(-20)}{10}=2$



Hence, the image formed will be real, erect and enlarged, and at a distance of 20 cm in front of the mirror.

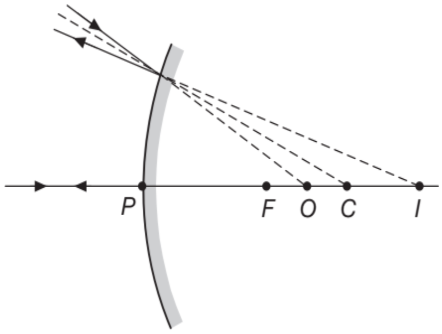
(b) Again, in this case too the object is virtual. So, we have

$$u = +30 \text{ cm}, \ f = +20 \text{ cm}$$

Using mirror formula $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}=\frac{1}{20}-\frac{1}{30}=\frac{3-2}{60}=\frac{1}{60}$$

$$\Rightarrow \quad v = 60 \text{ cm}$$



Magnification, $m=-\frac{v}{u}=-\frac{60}{30}=-2$

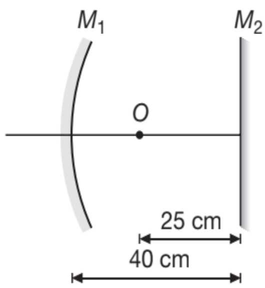
Hence, the image formed will be virtual, inverted and enlarged, and at a distance of 60 cm behind the mirror.

Conceptual Note(s)

Note that for the real objects, a convex mirror always gives virtual and diminished image, but for virtual objects it gives real image if $u < f$ and virtual image if $u > f$.

ILLUSTRATION 17

A concave mirror M_1 of radius of curvature 20 cm and a plane mirror M_2 are placed 40 cm apart as shown. An object O is placed 25 cm in front of the plane mirror. Find the position of final image formed after three successive reflections, assuming that the first reflection takes place from the curved mirror.



SOLUTION

Since $R = -20$ cm, so focal length of the mirror is given by $f=\frac{R}{2}=-10$ cm

For 1st reflection at M_1

$$u_1 = -15 \text{ cm},\ v_1 = ?\ ,\ f = -10 \text{ cm}$$

Since, $\frac{1}{v_1} + \frac{1}{u_1} = \frac{1}{f}$

$\Rightarrow \quad v_1 = -30$ cm

For 2nd reflection at plane mirror M_2

$$u_2 = -(40 - 30) = -10 \text{ cm}$$

Since, for a plane mirror the image distance from the plane mirror behind it is equal to the object distance from the plane mirror.

$\Rightarrow \quad v_2 = 10$ cm

For 3rd reflection at the curved mirror M_2

$$u_3 = -(40 + 10) = -50 \text{ cm},\ v_3 = ?,\ f = -10 \text{ cm}$$

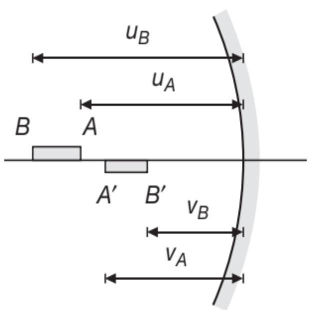
Since, $\frac{1}{v_3} + \frac{1}{u_3} = \frac{1}{f}$

$\Rightarrow \quad v = -12.5$ cm

LONGITUDINAL MAGNIFICATION OR AXIAL MAGNIFICATION

When an object of finite length is placed along the principal axis, then instead of defining the linear magnification we define the axial magnification. Mathematically we define axial magnification, for small objects as

$$m_{\text{axial}} = \frac{\text{Size of image along principal axis}}{\text{Size of object along principal axis}}$$



$$m_{\text{axial}} = -\left(\frac{v_2 - v_1}{u_2 - u_1}\right) = -\frac{\Delta v}{\Delta u}$$

where, u_1, u_2 and v_1, v_2 are the respective object and image distances of the ends of the object from the pole, such that

$$\frac{1}{v_1} + \frac{1}{u_1} = \frac{1}{f} \text{ and } \frac{1}{v_2} + \frac{1}{u_2} = \frac{1}{f}$$

However, if the object has infinitesimal size du and the corresponding image size is dv, then we have

$$m_{\text{axial}} = -\frac{dv}{du}$$

Further since we know that $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ i.e., $v^{-1} + u^{-1} = f^{-1}$. Taking the derivative of this equation with respect to u, we get

$$\frac{d}{du}\left(v^{-1}\right) + \frac{d}{du}\left(u^{-1}\right) = \frac{d}{du}\left(f^{-1}\right)$$

$$\Rightarrow \quad -v^{-2}\frac{dv}{du} - u^{-2} = 0$$

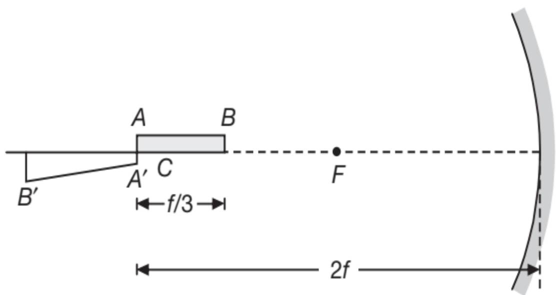
$$\Rightarrow \quad m_{\text{axial}} = \frac{dv}{du} = -\frac{v^2}{u^2} = -\left(\frac{f}{f-u}\right)^2 = -\left(\frac{f-v}{f}\right)^2$$

ILLUSTRATION 18

A thin rod of length $\frac{f}{3}$ is placed along the principal axis of a concave mirror of focal length f such that its image, which is real and elongated, just touches the rod. What is the magnification ?

SOLUTION

According to the problem, the image is real and enlarged, the object must have been placed between C and F. Since one end of the image just touches one end of the object so, this end must lie on C. Let AB be the object and $A'B'$ be its image, such that A and A' both lie at C, as shown in figure.



Now, as the length of the object AB is $\frac{f}{3}$, so the distance of end B of the object from the pole P is

$$u_B = -\left(PA - \frac{f}{3}\right) = -\left(2f - \frac{f}{3}\right) = -\left(\frac{5}{3}\right)f$$

The distance of the image of end B, v_B, is calculated by using the mirror formula,

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v_B} + \frac{1}{-\frac{5}{3}f} = \frac{1}{f}$$

$$\Rightarrow \quad v_B = -\frac{5}{2}f$$

Therefore, the size of the image $A'B'$ is

$$A'B' = |v_B| - |v_A| = \frac{5}{2}f - 2f = \frac{1}{2}f$$

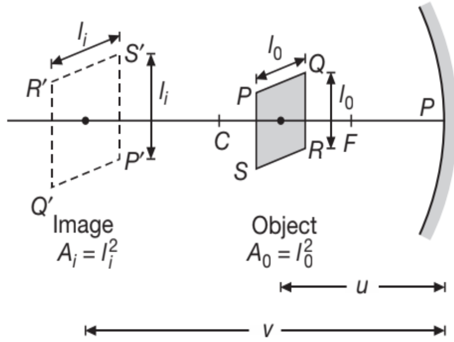
Now, magnification, $m = -\frac{A'B'}{AB} = -\frac{\left(\frac{f}{2}\right)}{\left(\frac{f}{3}\right)} = -\frac{3}{2}$

SUPERFICIAL OR AREAL MAGNIFICATION BY A SPHERICAL MIRROR

Let us find the magnification in area for the image produced by a spherical mirror. There are two cases in which area of image is calculated. In both of these cases, the object size is considered very small compared to the distance of object from the mirror.

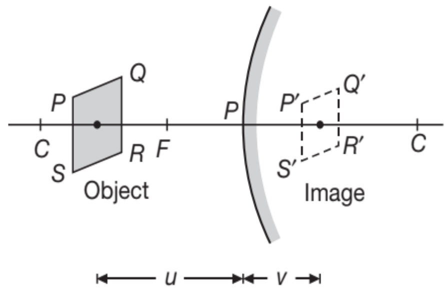
CASE-1: Object is Placed Normal to Principal Axis

Consider a square object $PQRS$ of area $A_0 = l_0^2$ which is placed at a point between F and C of a concave mirror at a distance u from the mirror. The image is produced as $P'Q'R'S'$ at a distance v from the mirror as shown. The image will also be of square shape because both edges of the object are perpendicular to principal axis of mirror. So, for both the edges we use the concept of lateral magnification and hence size of both will be same.



(For concave mirror)

A similar treatment can also be extended to an object placed in front of a convex mirror as shown.



(For convex mirror)

The image edge length l_i can be obtained by using the concept of lateral magnification.

Since $m = \frac{h_i}{h_0} = \frac{l_i}{l_0}$

$$\Rightarrow \quad l_i = ml_0$$

for both the edges. So final image produced for a concave mirror is a magnified square real image and similar to this for convex mirror, the final image will be a diminished square virtual image.
The area of image produced is given by

$$A_i = l_i^2 = m^2 l_0^2 = m^2 A_0$$

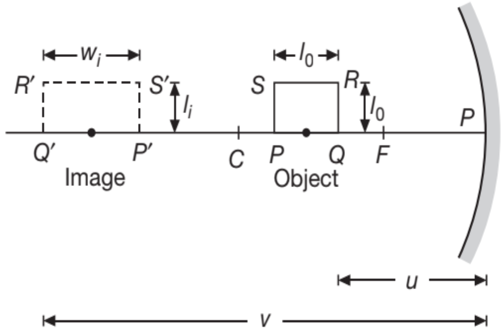
So, in this arrangement we can simply define areal magnification as the ratio of area of image to the area of object and hence

$$m_{\text{areal}} = \frac{\text{Area of image}}{\text{Area of object}} = \frac{A_I}{A_O} = \frac{v^2}{u^2}$$

$$\Rightarrow \quad m_{\text{areal}} = \left(\frac{f}{f-u}\right)^2 = \left(\frac{f-v}{f}\right)^2$$

CASE-2: Object is Placed with One Edge on the Principal Axis

Consider a small square object $PQRS$ of edge $l_0\left(l_0 \ll f\right)$ area $A_0 = l_0^2$, which is placed at a point between F and C of a concave mirror at a distance u from the mirror. The image $A'B'C'D'$ is produced at a distance v from the mirror as shown.



(For concave mirror)

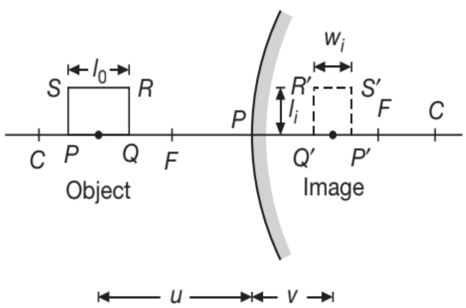
In this case the image produced will be in shape of a rectangle, because for the edges QR and PS of the object (which are perpendicular to the principal axis) we use the concept of lateral magnification and for the edges PQ and RS (which are along the principal axis) we use the concept of longitudinal magnification.

So in this case, the image height l_i and width w_i are given by

$$l_i = ml_0 \text{ and}$$

$$w_i = m^2 l_0$$

where m is the lateral magnification given as $m = -\dfrac{v}{u}$



(For convex mirror)

RELATION BETWEEN OBJECT AND IMAGE VELOCITY FOR CURVED MIRRORS

According to the mirror formula, we have

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

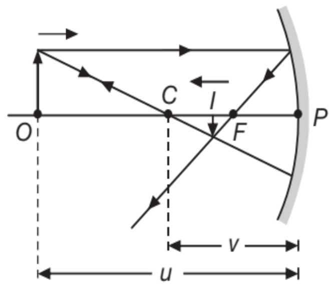
$$\Rightarrow \quad v^{-1}+u^{-1}=f^{-1}=\text{ constant}$$

Differentiating with respect to time, we get

$$-v^{-2}\frac{dv}{dt}-u^{-2}\frac{du}{dt}=0$$

$$\Rightarrow \quad \frac{dv}{dt}=-\left(\frac{v^2}{u^2}\right)\frac{du}{dt} \qquad \ldots(1)$$

Here $\dfrac{du}{dt}$ is the rate at which the object distance u is changing i.e., it is the object speed if the mirror is stationary. Similarly, $\dfrac{dv}{dt}$ is the rate at which v (distance between image and mirror) is changing i.e., it is image speed if the mirror is stationary. So if at a known values of v and u, the object speed is given, we can find the image speed from the above formula.



Let us take the example for a concave mirror.
Suppose the object is moved from infinity towards focus, then since u is decreasing therefore,

$$-\left(\frac{du}{dt}\right)=\text{rate of decrease of } u \qquad \text{\{object speed\}}$$

$$\Rightarrow \quad \left(\frac{dv}{dt}\right)=\text{ rate of increase of } v \qquad \text{\{image speed\}}$$

Further, when the object lies between ∞ and C , then $v<u$,

$$\Rightarrow \quad \left(\frac{dv}{dt}\right)<\left(-\frac{du}{dt}\right) \qquad \text{\{from equation (1)\}}$$

Hence, when the object is moved towards the mirror, its image (which is real) will recede from the mirror with speed less than the speed of object.

When the object is at C, image is also at C

$$\Rightarrow \quad v = u$$

$$\Rightarrow \quad \left(\frac{dv}{dt}\right) = \left(-\frac{du}{dt}\right)$$

Hence, when the object is at C speed of image is equal to the speed of object.

When the object lies between C and F then $v > u$ so, the image speed is more than the object speed.

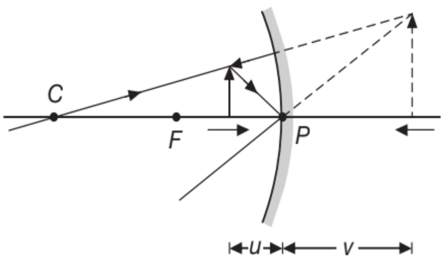
When the object lies between F and P, then the image becomes virtual i.e., u and f are negative while v is positive. So from mirror formula we get,

$$\frac{1}{v} + \frac{1}{(-u)} = \frac{1}{(-f)} \quad \Rightarrow \quad \frac{1}{u} - \frac{1}{v} = \frac{1}{f}$$

$$\Rightarrow \quad -u^{-2}\left(\frac{du}{dt}\right) - v^{-2}\left(\frac{dv}{dt}\right) = 0$$

$$\left(-\frac{dv}{dt}\right) = \left(\frac{v^2}{u^2}\right)\left(-\frac{du}{dt}\right)$$

Now, when u is further decreased, v also decreases to keep $\frac{1}{f}$ constant. So, $-\frac{du}{dt}$ is the rate at which object is approaching towards mirror and $\left(-\frac{dv}{dt}\right)$ is rate at which the image is approaching towards the mirror.



Further in this case we observe that the image is always enlarged i.e., $v > u$. Therefore, image speed is more than the object speed. Thus, the above entire discussion can simply be concluded as follows.

CONCLUSION

(a) When an object is moved from $-\infty$ to F, the image (real) moves from F to $-\infty$ and then when the object is further moved from F to P image (now virtual) moves from $+\infty$ to P.

(b) Therefore for a real image formed by a curved mirror we have

$$\frac{dv}{dt} = -\frac{v^2}{u^2}\left(\frac{du}{dt}\right)$$

and for a virtual image formed by a curved mirror we have

$$\frac{dv}{dt} = +\frac{v^2}{u^2}\left(\frac{du}{dt}\right)$$

(c) When the object is either at centre of curvature C or at pole P, the two speeds are equal. However, when the object is at pole, then due to the small aperture of the mirror, it appears as if the image is being formed by a plane mirror.

ILLUSTRATION 19

An object approaches a convex mirror of focal length 25 cm with speed $10\ \text{ms}^{-1}$. Calculate the velocity of the image when object is 25 cm from the mirror?

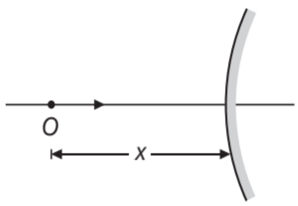
SOLUTION

Let at any instant t, the object be at a distance x from mirror and is moving towards it. Then,

$$u = -x$$

$$f = +25\ \text{cm}$$

Since $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$



Let the image be formed at a distance y from pole of mirror, then

$$\frac{1}{y} = \frac{1}{25} + \frac{1}{x}$$

$$\Rightarrow \quad y = \frac{25x}{25+x}$$

Differentiating both sides w.r.t. time, we get

$$\frac{dy}{dt} = \frac{625}{(25+x)^2}\frac{dx}{dt}$$

Since object is approaching towards mirror, x decreases as t increases

$$\Rightarrow \quad \frac{dx}{dt} = -10 \text{ ms}^{-1}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{625}{(25+25)^2}(-10)$$

$$\Rightarrow \quad \frac{dy}{dt} = -2.5 \text{ ms}^{-1}$$

So velocity of image has a magnitude 2.5 ms^{-1} and negative sign indicates that y will be decreasing as t increases i.e., image is moving towards pole of mirror.

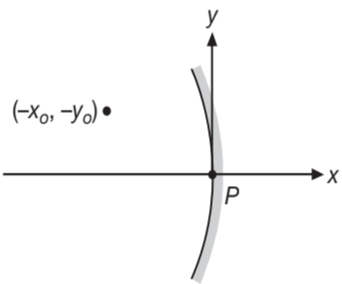
FINDING COORDINATES OF IMAGE OF A POINT

If the coordinates of a point object $(-x_O, -y_O)$ with respect to the coordinate axes shown in figure are known to us and the coordinates of image be (x_i, y_i) then for finding the x-coordinate, we use the mirror formula, according to which

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{x_i} + \frac{1}{x_O} = \frac{1}{f} \qquad \{\because u = -(-x_O) = x_O\}$$

$$\Rightarrow \quad x_i = \frac{fx_O}{x_O - f}$$



For finding the y-coordinate, we apply the concept of magnification (m), according to which, we have

$$m = \frac{y_i}{y_O} = -\frac{v}{u} = -\frac{x_i}{x_O}$$

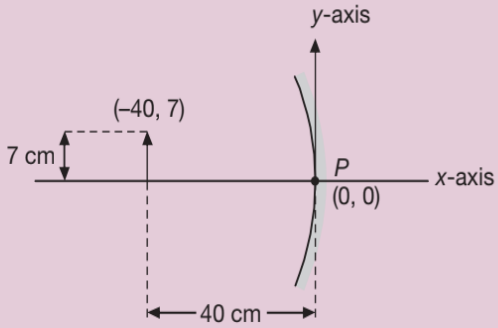
$$\Rightarrow \quad y_i = \frac{fy_O}{f - x_O}$$

EXAMPLE

A point object is placed at (–40, 7) cm in front of a concave mirror of focal length 5 cm having its pole at origin (0, 0). Assuming the principal axis to be along x-axis, find the position of the image formed.

SOLUTION

The situation discussed in the problem is shown in figure.



Since, $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad \frac{1}{v} + \frac{1}{(-40)} = \frac{1}{(-5)}$$

$$\Rightarrow \quad v = -\frac{40}{7} \text{ cm}$$

Since, $m = \frac{h_I}{h_O} = -\frac{v}{u}$

$$\Rightarrow \quad \frac{h_I}{h_O} = -\frac{\left(-\frac{40}{7}\right)}{-40}$$

$$\Rightarrow \quad \frac{h_I}{h_O} = -\frac{1}{7}$$

But $h_O = 7$ cm

$$\Rightarrow \quad h_I = -1 \text{ cm}$$

So, image coordinates are $\left(-\frac{40}{7}, -1\right)$ cm

Problem Solving Technique(s)

(a) Place the object to the left of the mirror (or lens), so that sign convention matches with the familiar sign convention in the coordinate geometry.

(b) Both for concave as well as convex mirrors, use the same mirror formula i.e.

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f} \quad \text{and} \quad m=-\frac{v}{u}$$

(c) Substitute the numerical values of the given quantities with proper sign (+ve or –ve) as per sign convention.

(d) Though the SI unit of distance is metre, it may be more convenient in some problems to take the given distances in cm rather than in m. But then your answer too will be in cm.

(e) Do not give any sign to the quantity to be determined. In your answer, the unknown quantity will be obtained with its proper sign.

In addition to the above hints, if you remember the following facts, it will help you.

(a) Since the object is generally placed to the left of the mirror so, u is negative.

(b) For a concave mirror, f is negative.

(c) For a convex mirror, f is positive.

(d) A real image is formed in front of the mirror, so for a real image v is negative.

(e) A virtual image is formed behind the mirror, so for a virtual image v is positive.

(f) A real image is always inverted, so for a real image h is negative.

(g) A virtual image is always erect, so for a virtual image h is positive.

(h) For the real image of a real object and the virtual image of a virtual object, m is negative.

(i) For the virtual image of a real object and the real image of a virtual object, m is positive.

GRAPH OF $\frac{1}{v}$ VERSUS $\frac{1}{u}$

Let us first take the case of a concave mirror. Here, two cases are possible.

CASE-1: When the Image formed is Real.

When the image is real, i.e., object lies between F and infinity. In such a situation u, v and f are negative. Hence, the mirror formula i.e., $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$ becomes

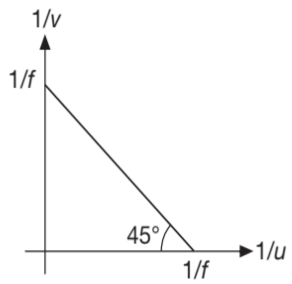
$$-\frac{1}{v}-\frac{1}{u}=-\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v}=-\frac{1}{u}+\frac{1}{f}$$

Comparing with $y=mx+c$, the desired graph will be a straight line with slope -1 and intercept on y-axis is equal to $\frac{1}{f}$.

Do not confuse here, the slope m with magnification.

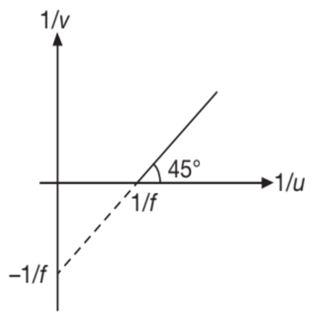


CASE-2: When the Image formed is Virtual.

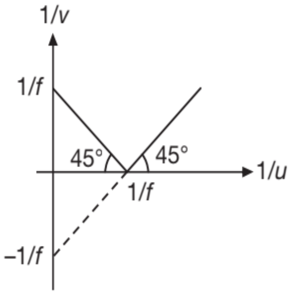
When the image is virtual, i.e., object lies between F and P. Under such situation u and f are negative while v is positive. The mirror formula thus becomes

$$\frac{1}{v}=\frac{1}{u}-\frac{1}{f}$$

Comparing it with $y=mx+c$ the desired graph is a straight line with slope $m=1$ and intercept on y-axis is equal to $-\frac{1}{f}$.



The graph is thus shown in figure. The two graphs can be drawn in one single graph as in figure.



Conceptual Note(s)

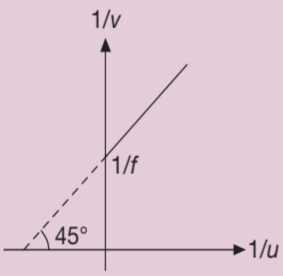
Please note that $\frac{1}{u}$ and $\frac{1}{v}$ are actually the magnitudes of $\frac{1}{u}$ and $\frac{1}{v}$ (i.e., without sign)

For a convex mirror, the image formed is always virtual, i.e., u is always negative while v and f are always positive. Hence, the mirror formula becomes,

$$\frac{1}{v}+\frac{1}{(-u)}=\frac{1}{f}$$

$$\Rightarrow \frac{1}{v}=\frac{1}{u}+\frac{1}{f}$$

Comparing with $y=mx+c$, the desired graph is a straight line of slope m = 1 and intercept on y-axis equal to $\frac{1}{f}$. The graph is thus shown in figure.



GRAPH OF *v* VERSUS *u*

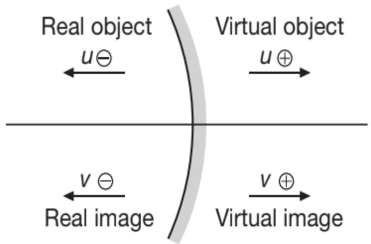
For Concave Mirror

For a spherical mirror, we have $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

$$\Rightarrow \quad v=\frac{fu}{u-f}$$

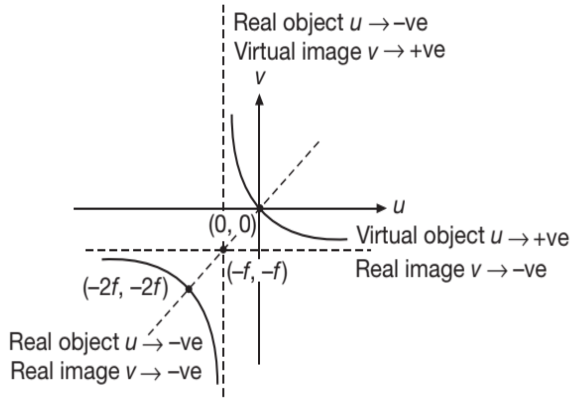
For concave mirror of focal length f, we have $f=-f$

$$\Rightarrow \quad v=-\frac{fu}{u+f} \qquad \ldots(1)$$



Substituting the following values of u in equation (1) to get the corresponding values of v for purpose of plotting the u-v graph.

u	$-\infty$	$-2f$	$-f$	0	$+f$	$+2f$	$+\infty$
v	$-f$	$-2f$	$\pm\infty$	0	$-\frac{f}{2}$	$-\frac{2f}{3}$	$+f$

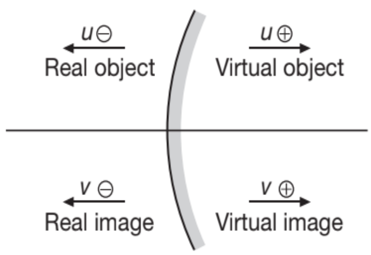


For Convex Mirror

Since $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

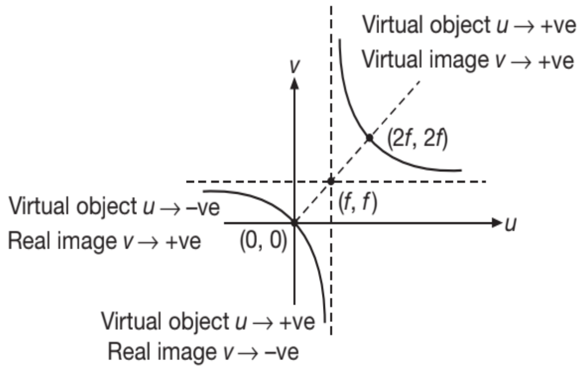
For concave mirror of focal length f, we have $f=+f$

$$\Rightarrow \quad v=\frac{fu}{u-f} \qquad \ldots(1)$$



Substituting the following values of u in equation (1) to get the corresponding values of v for purpose of plotting the u-v graph.

u	$-\infty$	$-2f$	$-f$	0	$+f$	$+2f$	$+\infty$
v	$+f$	$-\dfrac{2f}{3}$	$-\dfrac{f}{2}$	0	$\pm\infty$	$+2f$	$+f$



Problem Solving Technique(s)

(a) As focal-length of a spherical mirror $f = \dfrac{R}{2}$ depends only on the radius of mirror and is independent of wavelength of light and refractive index of medium so the focal length of a spherical mirror in air or water and for red or blue light is same. This is also why the image formed by mirrors do not show chromatic aberration.

(b) In case of spherical mirror if $R \to \infty$ (i.e., it becomes plane), so, $f = \dfrac{R}{2} \to \infty$. The mirror formula $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f}$ reduces to

$$\frac{1}{v} + \frac{1}{u} = 0 \text{ i.e., } v = -u$$

i.e., image is at same distance behind the mirror as the object is in front of it. This in turn verifies the correctness of mirror formula.

(c) Every part of a mirror forms complete image. If some portion of a mirror is obstructed (say covered with black paper), then complete image will be formed but intensity will be reduced.

(d) In case of concave spherical mirrors if a real object is placed at a distance x_1 from the focus and a real image is formed at a distance x_2 from the focus (instead of pole), then

$$u = -(f + x_1) \text{ and } v = -(f + x_2)$$

Since, we know that $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f}$

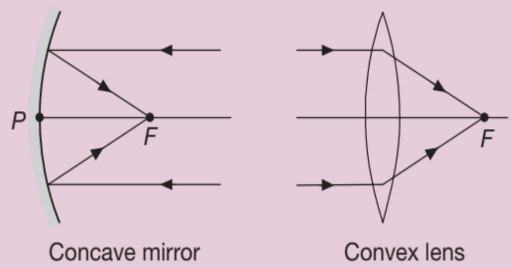
$$\Rightarrow \quad \frac{1}{-(f + x_2)} + \frac{1}{-(f + x_1)} = \frac{1}{-f}$$

$$\Rightarrow \quad x_1 x_2 = f^2$$

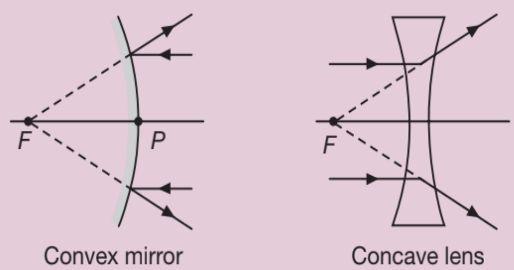
This result is called 'Newton's formula'.

(e) If an object is moved at constant speed towards a concave mirror from infinity to focus, the image will move (slower in the beginning and faster later on) away from the mirror. This is because, during the time the object moves from ∞ to C the image will move from F to C and when the object moves from C to F the image will move from C to ∞. At C the speed of object and image will be equal.

(f) Concave mirror behaves as convex lens (both convergent) while convex mirror behaves as concave lens (both divergent). This is shown in figure.



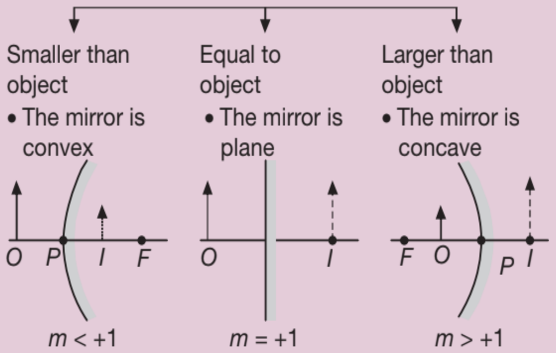
(a) Convergent behaviour



(b) Divergent behaviour

(g) As convex mirror gives erect, virtual and diminished image, field of view is increased. This is why it is used as rear-view mirror in vehicles. Concave mirrors give enlarged erect and virtual image (if object is between F and P) so are used by dentists for examining teeth. Further due to their converging property concave mirrors are also used as reflectors in automobiles head lights and search lights and by ENT surgeons in ophthalmoscope.

(h) For real extended objects, if the image formed by a single mirror is erect it is always virtual and in this situation if the size of the image is



(i) For real extended objects, if the image formed by a single mirror is inverted, it is always real (i.e., m is –ve) and the mirror is concave. In this situation if the size of image is

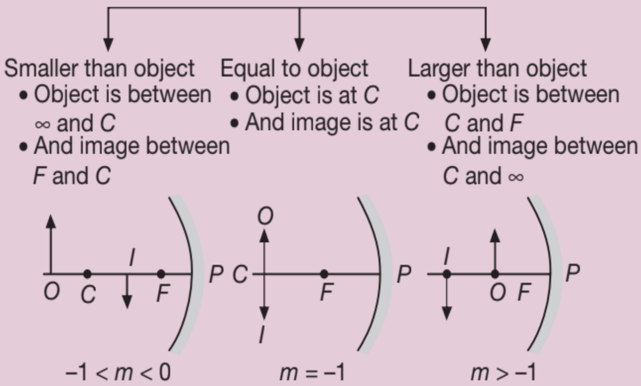


ILLUSTRATION 20

A gun of mass m_1 fires a bullet of mass m_2 with a horizontal speed v_0. The gun is fitted with a concave mirror of focal length f facing towards the receding bullet. Find the speed of separations of the bullet and the image at the instant just after the bullet is fired from the gun.

SOLUTION

Let v_1 be the speed of gun (or mirror) just after the firing of bullet. By Law of Conservation of Linear Momentum, we have

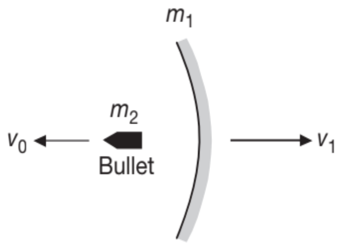
$$m_2 v_0 = m_1 v_1$$

$$\Rightarrow \quad v_1 = \frac{m_2 v_0}{m_1} \quad \ldots(1)$$

Now, $\frac{du}{dt}$ is the rate at which distance between mirror and bullet is increasing, so

$$\frac{du}{dt} = v_1 + v_0 \quad \ldots(2)$$

Since, we know that $\left|\frac{dv}{dt}\right| = \left(\frac{f}{f-u}\right)^2 \left|\frac{du}{dt}\right| \quad \ldots(3)$



Since, at the instant just after the bullet is fired from the gun, the bullet is actually very close to the pole of the mirror, so $u \to 0$ and hence we get at that instant

$$\frac{v^2}{u^2} = \left(\frac{f}{f-u}\right)^2 = m^2 = 1$$

So, from (2) and (3), we get

$$\frac{dv}{dt} = \frac{du}{dt} = v_1 + v_0 \quad \ldots(4)$$

where $\frac{dv}{dt}$ is the rate at which distance between image (of bullet) and mirror is increasing.

Therefore, speed of separation of bullet and image will be,

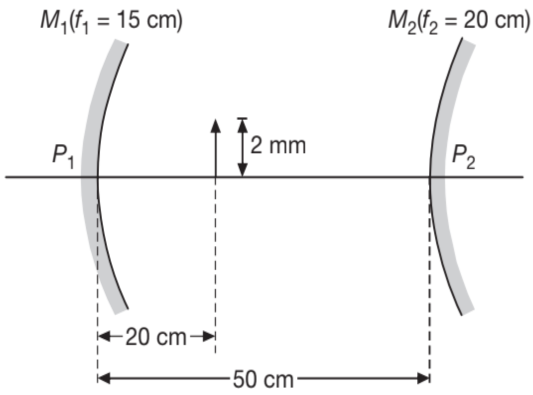
$$v_r = 2(v_1 + v_0)$$

Substituting value of v_1 from equation (1) we get

$$v_r = 2\left(1 + \frac{m_2}{m_1}\right) v_0$$

ILLUSTRATION 21

Find the location, size and the nature of the image of an object of height 2 mm kept between two mirrors (as shown in figure) after two successive reflections, considering the first reflection at the concave mirror and then at the convex mirror.



SOLUTION

As asked in the problem, let us first consider the reflection at mirror M_1. Before executing the mirror formula, we must keep two things in mind.

1. The incident light must go from the object to the mirror and we preferably take it parallel to the principal axis.
2. All distances have to be measured from the pole of the respective mirror for which reflection is being considered.
3. All distances measured along the incident ray are positive and all distances measured opposite to the incident ray are negative.

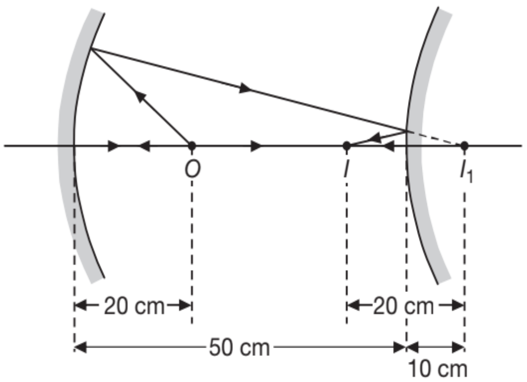


Figure is just representative and not to scale

Now, for reflection at concave mirror M_1, the incident ray from the object goes to left of object and object distance is measured towards right of pole P_1, so

$$u = -20 \text{ cm}$$

Similarly, $f_1 = -15$ cm

Now, according to mirror formula, we have

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v}+\frac{1}{(-20)}=\frac{1}{(-15)}$$

$$\Rightarrow \quad v = -60 \text{ cm}$$

Negative sign with v means that it is formed to right of pole P_1 at a distance of 60 cm from P_1 (10 cm behind M_2).

$$\Rightarrow \quad m_1 = -\frac{v}{u} = -\frac{(-60)}{-20} = -3$$

So, image (I_1) formed is real, inverted and three times size of object i.e., 6 mm.

This image (I_1) formed now acts as object for the convex mirror. Further, this image formed is 10 cm to the left of P_2 and the incident ray from the original object goes to the right for reflection at M_2 to take place, so

$$u = +10 \text{ cm}$$

Similarly, $f = +20$ cm

Applying the mirror formula, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}+\frac{1}{10}=\frac{1}{20}$$

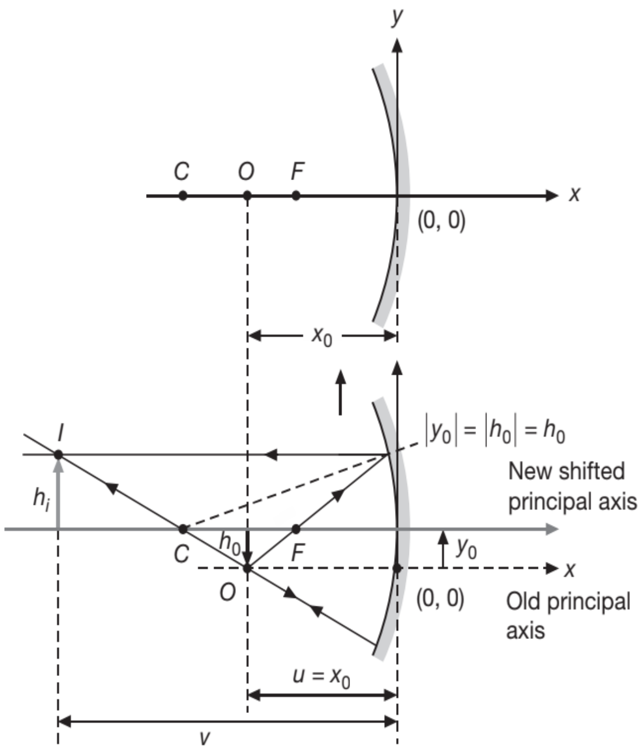
$$\Rightarrow \quad v = -20 \text{ cm}$$

$$\Rightarrow \quad m_2 = -\frac{v}{u} = -\frac{(-20)}{10} = 2$$

So, image (I) formed is virtual, erect and two times the size of object (here I_1). Hence the size of I is 12 mm. So, finally I is formed at 20 cm in front of convex mirror M_2, with size 12 mm, virtual and erect.

EFFECT OF SHIFTING THE PRINCIPAL AXIS OF A SPHERICAL MIRROR

Let a point object O, be placed on the principal axis of a concave mirror as shown. Correspondingly, the image of this point object will also suitably lie on the principal axis. Let the mirror be now displaced by a small distance y_0 (say) perpendicular to the principal axis. Then, obviously the principal axis also shifts along the mirror by a distance y_0.



To find the location of image, we use mirror formula.

So we use $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad \frac{1}{x_i} - \frac{1}{(-x_0)} = \frac{1}{-f}$$

$$\Rightarrow \quad \frac{1}{x_i} = -\left(\frac{1}{x_0} + \frac{1}{f}\right)$$

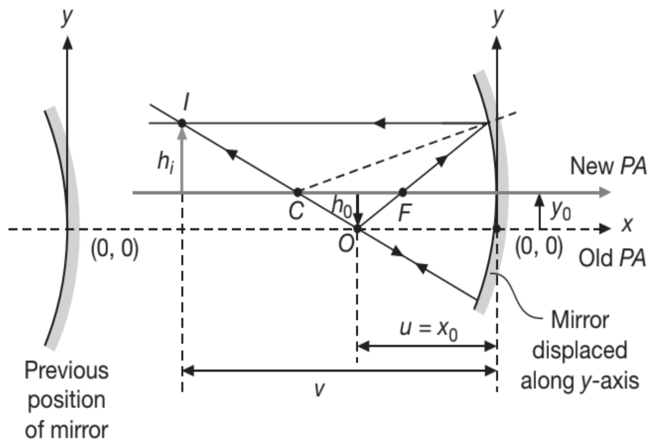
$$\Rightarrow \quad x_i = -\left(\frac{x_0 f}{x_0 + f}\right)$$

However, we must note that the object (O) now lies at a distance y_0 below the new shifted principal axis. To calculate the location of I from this new shifted principal axis, we simply use the concept of magnification, according to which

$$m = \frac{h_i}{h_0} = \left|-\frac{v}{u}\right|$$

$$\Rightarrow \quad h_i = mh_0 = my_0$$

So, after shifting the mirror, the image is formed mh_0 above the new principal axis or formed $mh_0 + h_0 = (m+1)h_0$ above the old principal axis.



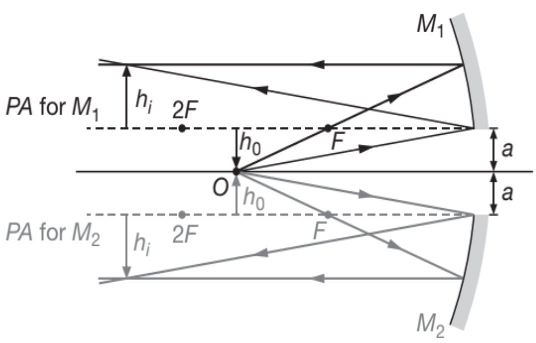
Perpendicular separation between the tip of object and tip of image is $(OI)_\perp = h_i + h_0 = mh_0 + h_0 = (m+1)h_0$

Also, the coordinates of the image I w.r.t. (0, 0) are $\left(-\frac{x_0 f}{x_0 + f},\ (m+1)y_0\right)$

SPLITTING OF A MIRROR

Let a concave mirror of focal length f be cut into two parts M_1 and M_2 at the pole and then each part is displaced perpendicular to the principal axis through A. Due to this the point object O lies at a distance a from the new principal axis for each of the mirrors M_1 and M_2.

CASE-1: When the object O lies between F and $2F$, then the ray diagram for this arrangement is shown in figure.



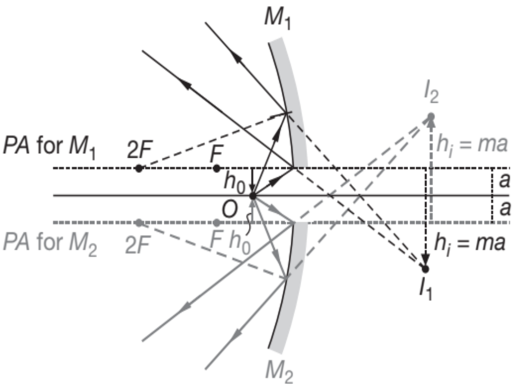
Since the magnification $m = \frac{h_i}{h_o} = \frac{h_i}{a}$

$$\Rightarrow \quad h_i = ma\text{, where } m = \frac{h_i}{h_o} = -\frac{v}{u}$$

The separation I_1I_2 between the tips of images is

$$I_1I_2 = 2h_i + 2a = 2ma + 2a = 2a(m+1)$$

CASE-2: When the object O lies between F and P, then the ray diagram for this arrangement is shown in figure.



Since the magnification $m = \frac{h_i}{h_o} = \frac{h_i}{a}$

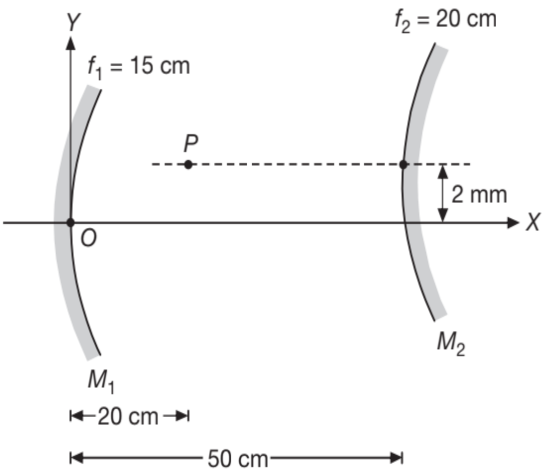
$\Rightarrow \quad h_i = ma$, where $m = \frac{h_i}{h_o} = -\frac{v}{u}$

The separation I_1I_2 between the tips of images is

$$I_1I_2 = 2h_i - 2a = 2ma - 2a = 2a(m-1)$$

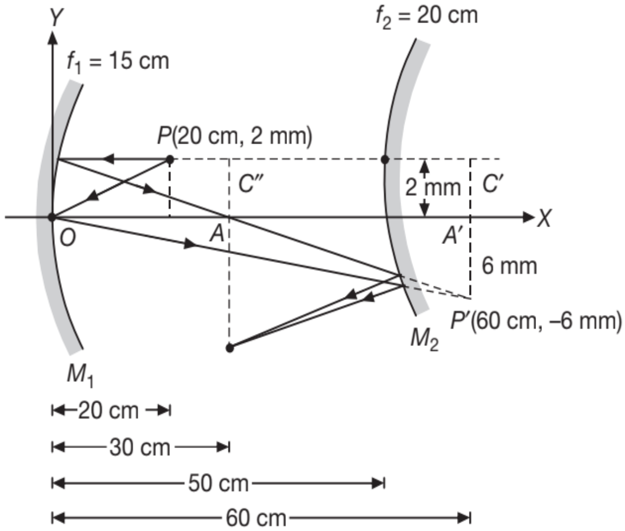
ILLUSTRATION 22

Find the co-ordinates of image of point object P formed after two successive reflections in figure, considering the first reflection at concave mirror and then at convex mirror.



SOLUTION

The ray diagram for the situation is drawn in figure (but not to scale).



For reflection at concave mirror M_1, we have

$$u = -20 \text{ cm}$$

$$f_1 = -15 \text{ cm}$$

Since, $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad \frac{1}{v_1} + \frac{1}{(-20)} = \frac{1}{(-15)}$$

$$\Rightarrow \quad v_1 = -60 \text{ cm}$$

So, magnification $m_1 = -\frac{v_1}{u} = -\frac{-60}{-20} = -3$ (Inverted)

$$\Rightarrow \quad A'P' = m_1(AP) = 3 \times 2 = 6 \text{ mm}$$

For reflection at convex mirror M_2, we have

$$u = +10 \text{ cm}$$

$$f_2 = +20 \text{ cm}$$

Since $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad \frac{1}{v_2} + \frac{1}{10} = \frac{1}{20}$$

$$\Rightarrow \quad v_2 = -20 \text{ cm}$$

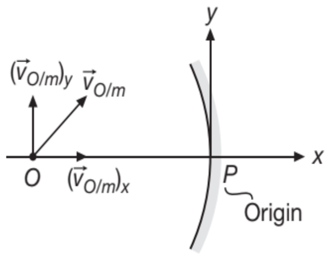
Again, magnification, $m_2 = -\frac{v_2}{u} = -\frac{(-20)}{10} = 2$

$\Rightarrow \quad C''P'' = m_2(C'P') = 2\times 8 = 16$ mm

So, the co-ordinate of image of point object P as measured from the origin O is $(30 \text{ cm}, -14 \text{ mm})$

VELOCITY OF IMAGE IN SPHERICAL MIRROR

Let pole of mirror be origin of co-ordinate system and x-axis be the principal axis of mirror and y-axis is perpendicular to principal axis. Further object is placed such that incident rays travel along +ve x-axis.



From mirror equation, we have

$$\frac{1}{x_{I/m}} + \frac{1}{x_{O/m}} = \frac{1}{f}$$

Differentiating both sides w.r.t. t we get

$$-\frac{1}{x_{I/m}^2}\frac{d}{dt}(x_{I/m}) - \frac{1}{x_{O/m}^2}\frac{d}{dt}(x_{O/m}) = 0$$

$$\Rightarrow \quad \frac{d}{dt}(x_{I/m}) = -\left(\frac{x_{I/m}}{x_{O/m}}\right)^2 \frac{d}{dt}(x_{O/m})$$

$$\Rightarrow \quad (V_{I/m})_x = -m^2 (V_{O/m})_x$$

We know that,

$$m = \frac{f}{f-u} = \frac{\text{Height of Image}}{\text{Height of Object}}$$

$$\Rightarrow \quad y_{I/m} = \left(\frac{f}{f-u}\right)(y_{O/m})$$

Differentiating w.r.t. t we get

$$\frac{d}{dt}(y_{I/m}) = \frac{d}{dt}\left[\left(\frac{f}{f-u}\right)(y_{O/m})\right]$$

$$\Rightarrow \quad (V_{I/m})_y = \left(\frac{f}{f-u}\right)\frac{d}{dt}(y_{O/m}) + (y_{O/m})\frac{d}{dt}\left(\frac{f}{f-u}\right)$$

(Using product rule)

$$\Rightarrow \quad (V_{I/m})_y = \left(\frac{f}{f-u}\right)(V_{O/m})_y + (y_{O/m})\frac{f}{(f-u)^2}\frac{du}{dt}$$

CASE-1: If object is on principal axis, then $y_{O/m} = 0$

$$\Rightarrow \quad (V_{I/m})_y = \left(\frac{f}{f-u}\right)(V_{O/m})_y$$

CASE-2: If object is not on principal axis but moving parallel to principal axis then $(V_{O/m})_y = 0$

$$\Rightarrow \quad V_{I/m} = (V_{O/m})\frac{f}{(f-u)^2}\frac{du}{dt}$$

Note that $\frac{du}{dt}$ is negative if u is decreasing with time and it is taken positive if u is increasing with time.

CASE-3: If object is on principal axis and moving along it then $y_{O/m} = 0$ and $(V_{O/m})_y = 0$

$$\Rightarrow \quad (V_{I/m})_y = 0$$

ILLUSTRATION 23

A thief is driving away on a road in a car with velocity of 20 ms^{-1}. A police jeep is chasing him, which is sighted by thief in his rear view mirror, which is a convex mirror of focal length 10 m. He observes that the image of the jeep is moving towards him with a velocity of 1 cms^{-1}. If the magnification of the mirror for the jeep at that time is $\frac{1}{10}$. Find

(a) The actual speed of the jeep.
(b) The rate at which magnification is changing.

Assume that police jeep is on axis of the mirror.

SOLUTION

(a) The velocity of image with respect to mirror is related to velocity of object with respect to mirror is given as

$$(V_{I/m})_\perp = -m^2 (V_{O/m})$$

$$\Rightarrow \; -1\times10^{-2} = -\frac{1}{10^2}\left(V_{O/m}\right)$$

$$\Rightarrow \; \left(V_{O/m}\right) = +1\text{ ms}^{-1} = +1\hat{i}$$

Velocity of object with respect to ground is given as

$$\vec{V}_{O/G} = \vec{V}_{O/m} + \vec{V}_{m/G}$$

$$\Rightarrow \; \vec{V}_{O/G} = 1 + 20 = \left(+21\text{ ms}^{-1}\right)\hat{i}$$

(b) The magnification produced by the mirror is

$$m = \frac{f}{f-u} = \frac{1}{10}$$

If police jeep is at a distance d behind the thief's car then we can use $u = -d$ so we have

$$\frac{10}{10-(-d)} = \frac{1}{10}$$

$$\Rightarrow \; d = 90\text{ m}$$

Thus distance of image from mirror is

$$v = -mu = -\frac{1}{10}\times -90 = 9\text{ m}$$

Now rate at which magnification is changing is given as

$$\frac{dm}{dt} = \frac{u\dfrac{dv}{dt} - v\dfrac{du}{dt}}{u^2}$$

$$\Rightarrow \; \frac{dm}{dt} = \left[\frac{(-90)\left(-1\times10^{-2}\right) - 9(1)}{90^2}\right]\text{ s}^{-1}$$

$$\Rightarrow \; \frac{dm}{dt} = -\left[\frac{81}{10\times8100}\right] = +1\times10^{-3}\text{ s}^{-1}$$

Test Your Concepts-II

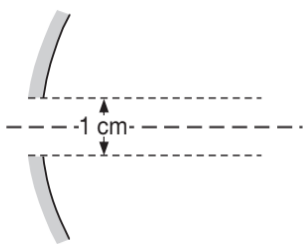
Based on Reflection at Curved Surfaces

(Solutions on page H.6)

1. An object of height 2.5 cm is placed at a 1.5 f from a concave mirror, where *f* is the magnitude of the focal length of the mirror. The height of the object is perpendicular to the principal axis. Find the height of the image. Is the image erect or inverted?

2. A mirror (in a laughing gallery) forms an erect image four times enlarged, of a boy standing 2.5 m away. Is the mirror concave or convex? What is its radius of curvature?

3. A concave mirror forms the real image of a point source lying on the optical axis at a distance of 50 cm from the mirror. The focal length of the mirror is 25 cm. The mirror is cut in two halves and these halves are drawn apart at a distance of 1 cm in a direction perpendicular to the optical axis. How will the images formed by the halves of the mirror be arranged?



4. Find the distance of object from a concave mirror of focal length 10 cm so that image size is four times the size of the object.

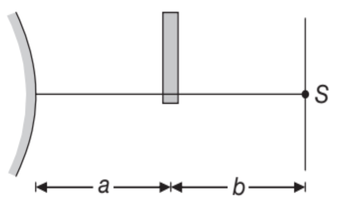
5. A concave mirror has a radius of curvature of 24 cm. How far is an object from the mirror when the image formed is

(a) virtual and 3 times the size of the object.

(b) real and 3 times the size of the object and

(c) real and $\frac{1}{3}$ the size of the object?

6. A thin flat glass plate is placed in front of a convex mirror. At what distance *b* from the plate should a point source of light *S* be placed so that its image produced by the rays reflected from the front surface of the plate coincides with the image formed by the rays reflected from the mirror? The focal length of the mirror is $f = 20$ cm and the distance from the plate to the mirror $a = 5$ cm. How can the coincidence of the images be established by direct observation?

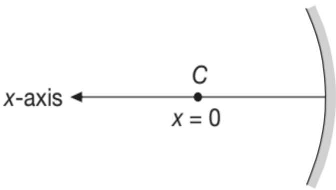


7. A ball swings back and forth in front of a concave mirror. The motion of the ball is described approximately by the equation $x = f\cos(\omega t)$, where f is the focal length of the mirror and x is measured along the axis of mirror. The origin is taken at the centre of curvature of the mirror.

(a) Derive an expression for the distance from the mirror to the image of the swinging ball.

(b) At what point does the ball appear to coincide with its image.

(c) What will be the lateral magnification of the image of the ball at time $t = \frac{T}{2}$, where T is time period of oscillation?



8. An image I is formed of a point object O by a mirror whose principal axis is AB as shown in figure.

(a) State whether it is a convex mirror or a concave mirror.

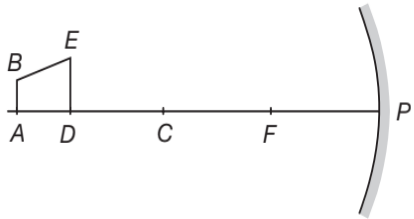
(b) Draw a ray diagram to locate the mirror and its focus. Write down the steps of construction of the ray diagram. Consider the possible two cases:

(i) When distance of I from AB is more than the distance of O from AB and

(ii) When distance of O from AB is more than the distance of I from AB

9. Convex and concave mirrors have the same radii of curvature R. The distance between the mirrors is $2R$. At what point on the common optical axis of the mirrors should a point source of light A be placed for the rays to converge at the point A after being reflected first on the convex and then on the concave mirror?

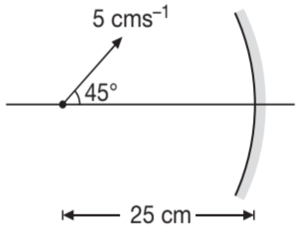
10. An object $ABED$ is placed in front of a concave mirror beyond centre of curvature C as shown in figure. State the shape of the image.



11. An object is 30 cm from a spherical mirror, along the central axis. The absolute value of lateral magnification of an inverted image is $\frac{1}{2}$. Find the focal length of the mirror?

12. A thin rod of length $\frac{f}{3}$ is placed along the principal axis of a concave mirror of focal length f such that its image just touches the rod. Calculate magnification.

13. A point object on the principal axis of a concave mirror of focal length 20 cm is moving at a speed of $5\ \text{cms}^{-1}$ making an angle of 45° with the principal axis as shown in figure.

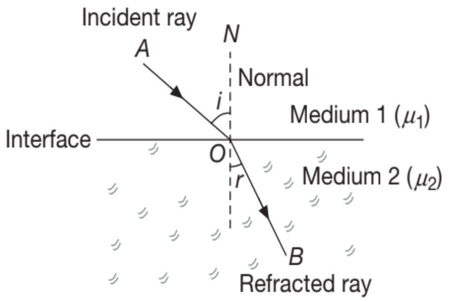


Initially object is located at a distance of 25 cm from the pole of mirror. Calculate the velocity components of image along and normal to principal axis at this instant.

REFRACTION AT PLANE SURFACES

REFRACTION OF LIGHT AT PLANE SURFACES

The phenomenon of the bending of light rays as they travel from one medium to the other is called Refraction. The surface separating two media is called an Interface. In other words, the phenomenon of bending of light rays at the boundary between two media is called refraction.



LAWS OF REFRACTION

(a) The incident ray, the refracted ray and normal at the point of incidence to the surface separating the two media all lie in the same plane.

(b) **Snell's Law**

For two media, **the ratio of sine of angle of incidence i to the sine of the angle of refraction r is constant** (for a beam of particular wavelength). For a given set of media this constant is called the **refractive index** of the medium 2 with respect to medium 1 (represented as ${}^1\mu_2$) i.e.,

$$\frac{\sin i}{\sin r} = \text{constant} = \frac{\mu_2}{\mu_1} = {}^1\mu_2 \quad \text{(SNELL'S LAW)}$$

OR $\mu_1 \sin i = \mu_2 \sin r$

where μ_1 and μ_2 are Absolute Refractive Indices of Medium 1 and 2 respectively and ${}^1\mu_2$ is the refractive index of medium 2 with respect to medium 1. If medium 1 happens to be the vacuum, then the constant is simply called as the **Absolute Refractive Index** of medium 2, expressed as μ_2 or simply μ.

REFRACTIVE INDEX (RI)

The refractive index of a medium is not determined by its density. It is governed by the velocity of light in the medium. The lesser the value of the velocity of light, the more is the refractive index of the medium, and the denser is the medium. A medium having greater refractive index is called denser medium whereas the other medium is called **rarer medium**.

ABSOLUTE REFRACTIVE INDEX

The absolute refractive index of a medium is defined as the ratio of the speed of light in vacuum to the speed of light in the medium,

$$\mu = \frac{\text{speed of light in vacuum}}{\text{speed of light in medium}} = \frac{c}{v} > 1$$

Absolute refractive index is more than one because the speed of light is maximum in vacuum/air.

RELATIVE REFRACTIVE INDEX

The relative refractive index of medium 2 with respect to medium 1 is denoted by ${}^1\mu_2$ and is given by

$${}^1\mu_2 = \frac{\mu_2}{\mu_1} = \frac{\left(\frac{c}{v_2}\right)}{\left(\frac{c}{v_1}\right)} = \frac{v_1}{v_2}$$

The relative refractive index of medium 1 with respect to medium 2 is denoted by ${}^2\mu_1$ and is given by

$${}^2\mu_1 = \frac{\mu_1}{\mu_2} = \frac{\left(\frac{c}{v_1}\right)}{\left(\frac{c}{v_2}\right)} = \frac{v_2}{v_1}$$

Conceptual Note(s)

(a) The velocity of light in air is not much different from that in vacuum. Hence, while defining the refractive index of a medium we often take velocity of light in air rather than that in vacuum.

Medium	Refractive Index (μ)
Water	$\frac{4}{3} = 1.33$
Glass	$\frac{3}{2} = 1.50$

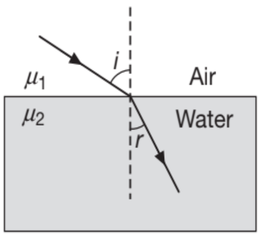
(b) Relative refractive index can be less than one. If we calculate the refractive index of water with respect to glass, then

$$^{g}\mu_w = \frac{\mu_w}{\mu_g} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{3}{2}\right)} = \frac{8}{9} < 1$$

(c) Refractive index is different for different wavelengths for a pair of media, because $\mu_1\lambda_1 = \mu_2\lambda_2$.

BENDING OF A LIGHT RAY

(a) When light passes from rarer to denser medium it bends towards the normal i.e., a light ray passing from air to water bends towards the normal as shown in the figure.



According to Snell's Law

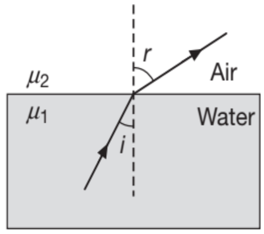
$$\mu_1 \sin i = \mu_2 \sin r$$

$$\Rightarrow \frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} > 1$$

$$\Rightarrow \sin i > \sin r$$

$$\Rightarrow i > r$$

(b) When light passes from denser to rarer medium it bends away from the normal as i.e., a light ray passing from water to air bends away from the normal shown in the figure.



According to Snell's Law

$$\mu_1 \sin i = \mu_2 \sin r$$

$$\Rightarrow \frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} < 1$$

$$\Rightarrow \sin i < \sin r$$

$$\Rightarrow i < r$$

REFRACTION: IMPORTANT POINTS

(a) Whenever light goes from one medium to another, the frequency of light (f) remains unchanged. Since

$$\mu = \frac{c}{v} = \frac{\text{Speed of light in vacuum}}{\text{Speed of light in medium}}$$

$$\Rightarrow \mu = \frac{f\lambda_{\text{air}}}{f\lambda_{\text{medium}}} = \frac{\lambda_{\text{air}}}{\lambda_{\text{medium}}}$$

where λ_{air} and λ_{medium} being wavelengths of light in air and medium respectively.

$$\Rightarrow \frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} = \frac{\frac{c}{v_2}}{\frac{c}{v_1}} = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2}$$

(MODIFIED FORM OF SNELL'S LAW)

From above we conclude that

$$\mu_1\lambda_1 = \mu_2\lambda_2$$

$$\Rightarrow \mu\lambda = \text{constant}$$

Also, we conclude that

$$\mu_1 v_1 = \mu_2 v_2$$

$$\Rightarrow \mu v = \text{constant}$$

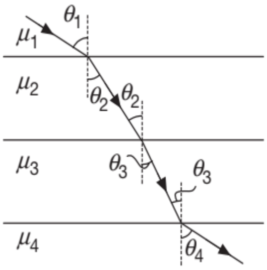
(b) $^{2}\mu_1 \times {}^{1}\mu_2 = 1$

$$\Rightarrow {}^{2}\mu_1 = \frac{1}{{}^{1}\mu_2}$$

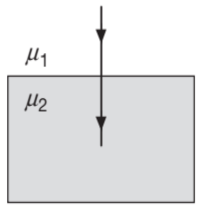
(c) When light propagates through a series of parallel layers of different medium as shown in the figure, then the Snell's Law may be written as

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2 = \mu_3 \sin\theta_3 = \mu_4 \sin\theta_4 = \text{constant}$$

In general, $\mu \sin\theta = \text{constant}$

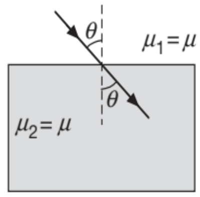


(d) If light is incident normal to a boundary (i.e. $i = 0$), then, it passes undeviated from the boundary as shown in the figure.



$\mu_2 \neq \mu_1$
Condition for no refraction

(e) If the refractive indices of the two media are equal as shown in figure, then also the light ray is not refracted and the boundary between the two media is not visible. This is why a transparent solid is invisible in a liquid of same refractive index.



$\mu_2 = \mu_1 = \mu$
Condition for no refraction

(f) Note that for sound waves,

speed in air, $v_1 = 330 \text{ ms}^{-1}$

speed in water, $v_w = 1500 \text{ ms}^{-1}$

Therefore, the refractive index of water with respect to air, for sound waves is

$$^a\mu_w = \frac{v_a}{v_w} = \frac{330}{1500} = 0.22$$

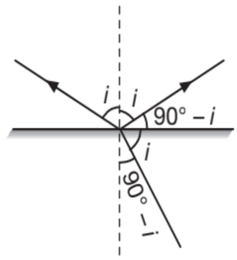
Thus, we find that for the refraction of sound waves, water is rarer than air.

ILLUSTRATION 24

A ray of light falls on a glass plate of refractive index $n = \sqrt{3}$. What is the angle of incidence of the ray if the angle between the reflected and refracted rays is $90°$?

SOLUTION

According to Snell's Law



$$n = \frac{\sin i}{\sin r}$$

Since $i + r = 90°$

$\Rightarrow \quad r = 90 - i$

$$\Rightarrow \quad \sqrt{3} = \frac{\sin i}{\sin(90 - i)} = \tan i$$

$$\Rightarrow \quad i = \tan^{-1}(\sqrt{3}) = 60°$$

ILLUSTRATION 25

A ray of light passes through a medium whose refractive index varies with distance as $n = n_0\left(1 + \frac{x}{2a}\right)$. If ray enters the medium parallel to x-axis, what will be the time taken for ray to travel between $x = 0$ and $x = a$?

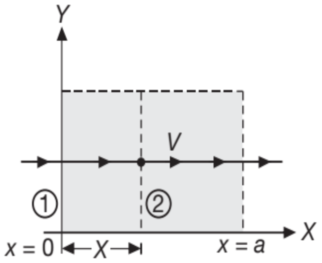
SOLUTION

Since, we know that $\mu = \frac{c}{v}$

$$\Rightarrow \quad v = \frac{c}{\mu}$$

So, if v be the speed at a distance x from y-axis, then

$$v = \frac{c}{n_0\left(1 + \frac{x}{2a}\right)}$$



Since, $v = \frac{dx}{dt}$, so we have

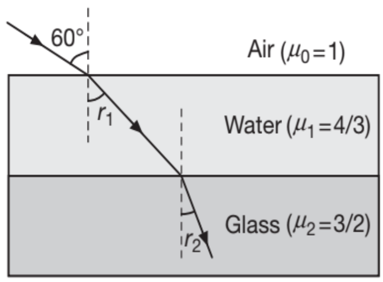
$$\frac{dx}{dt} = \frac{c}{n_0\left(1+\frac{x}{2a}\right)}$$

$$\Rightarrow \int_0^t dt = \frac{n_0}{c}\int_0^a\left(1+\frac{x}{2a}\right)dx$$

$$\Rightarrow t = \frac{5n_0 a}{4c}$$

ILLUSTRATION 26

For the arrangement shown in the figure, a light ray is incident at an angle of $60°$ on the layer of water. Find the angle between this ray and the normal to the glass.



SOLUTION

According to Snell's Law, we have

$$\mu_0 \sin(60°) = \mu_1 \sin r_1 = \mu_2 \sin r_2$$

$$\Rightarrow \sin r_2 = \frac{\mu_0 \sin(60°)}{\mu_2} = \frac{1\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{3}{2}\right)} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow r_2 = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 35°$$

ILLUSTRATION 27

A ray of light goes from air to medium of refractive index μ. If i be the angle of incidence, r be the angle of refraction and δ be the angle of deviation, then prove that $\tan\left(\frac{\delta}{2}\right) = \left(\frac{\mu-1}{\mu+1}\right)\tan\left(\frac{i+r}{2}\right)$.

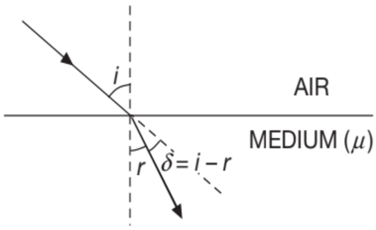
SOLUTION

The angle of deviation δ is given by

$$\delta = i - r \qquad \ldots(1)$$

According to Snell's Law,

$$\mu = \frac{\sin i}{\sin r}$$



Applying componendo and dividendo, we get

$$\frac{\mu-1}{\mu+1} = \frac{\sin i - \sin r}{\sin i + \sin r}$$

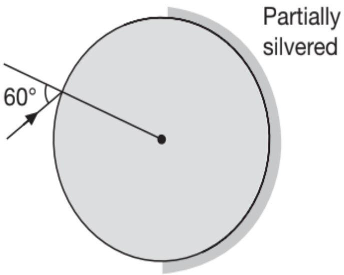
$$\Rightarrow \frac{\mu-1}{\mu+1} = \frac{2\cos\left(\frac{i+r}{2}\right)\sin\left(\frac{i-r}{2}\right)}{2\sin\left(\frac{i+r}{2}\right)\cos\left(\frac{i-r}{2}\right)}$$

$$\Rightarrow \frac{\mu-1}{\mu+1} = \frac{\tan\left(\frac{\delta}{2}\right)}{\tan\left(\frac{i+r}{2}\right)}$$

$$\Rightarrow \tan\left(\frac{\delta}{2}\right) = \left(\frac{\mu-1}{\mu+1}\right)\tan\left(\frac{i+r}{2}\right)$$

ILLUSTRATION 28

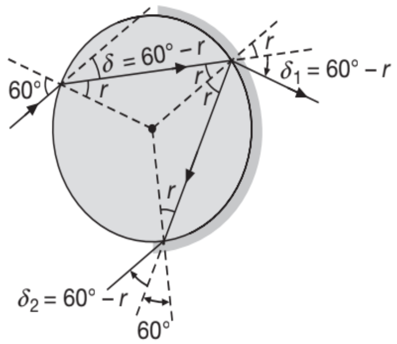
A ray is incident on a glass sphere as shown in figure. The opposite surface of the sphere is partially silvered. If the net deviation of the ray transmitted at the partially silvered surface is one third of the net deviation suffered by the ray reflected at the partially silvered surface (after emerging out of the sphere), find the refractive index of the sphere.



SOLUTION

Since, the distance of all the points lying on the sphere is constant from the centre, all the angles of refraction are same. Here we consider δ_1 is the deviation of the light ray at first refraction, δ_2 is the deviation of the transmitted ray through partially polished surface and δ_3 is the deviation of the light ray emerging out of the sphere at final refraction. Figure shows the ray diagram of the given situation then according to the given condition, we have

$$(60-r)+(60-r)=\frac{1}{3}((60-r)+(60-r)+(180-2r))$$



$$\Rightarrow \quad 120-2r=\frac{1}{3}(300-4r)$$

$$\Rightarrow \quad 360-6r=300-4r$$

$$\Rightarrow \quad 60=2r$$

$$\Rightarrow \quad r=30^\circ$$

Now using Snell's law, we have

$$1\sin(60^\circ)=\mu\sin r$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}=\mu\times\frac{1}{2}$$

$$\Rightarrow \quad \mu=\sqrt{3}$$

LIGHT INCIDENT ON A MEDIUM HAVING VARIABLE REFRACTIVE INDEX

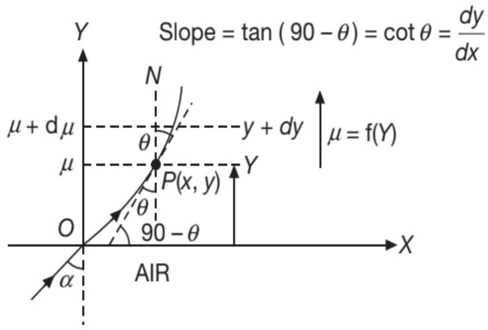
Let us find the mathematical expression for the equation of a ray in the medium when the medium is of variable refractive index. Consider a ray of light to be incident at an angle α at air-medium interface as shown. Now, two cases arise i.e. refractive index is varying either as function of y or function of x.

CASE-1: Refractive index μ varies with y i.e. $\mu=f(y)$

At point $P(x, y)$, let the angle of incidence be θ and refractive index be $f(y)$.

From Snell's Law, we have $\mu\sin\theta=\text{constant}$

$$\Rightarrow \quad 1\times\sin\alpha=f(y)\sin\theta \qquad \ldots(1)$$



Snell's law at $O'A'$ interface.

Slope of curve at A is

$$\frac{dy}{dx}=\tan(90-\theta)$$

$$\Rightarrow \quad \cot\theta=\frac{dy}{dx}$$

So, from equation (1), we get

$$\frac{dy}{dx}=\frac{\sqrt{\mu^2-\sin^2\alpha}}{\sin\alpha}=\frac{\sqrt{[f(y)]^2-\sin^2\alpha}}{\sin\alpha}$$

CASE-2: Refractive index μ varies as function of x i.e. $\mu=f(x)$

According to Snell's law applied at interface $O'A'$, we have $\mu\sin\theta=\text{constant}$

For initial refraction at the air-medium interface,

$$1\times\sin\alpha=\mu_0\sin(90-r)$$

$$\Rightarrow \quad \sin\alpha=\mu_0\cos r$$

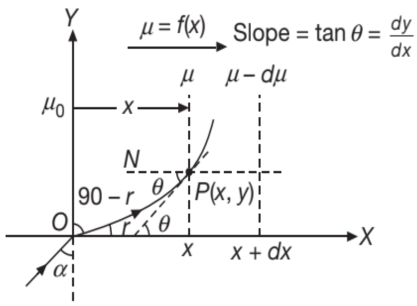
where r is angle of refracting ray at point O with line OX

$$\Rightarrow \quad \cos r = \frac{\sin\alpha}{\mu_0}$$

$$\Rightarrow \quad \sin r = \sqrt{1 - \frac{\sin^2\alpha}{\mu_0^2}}$$

Applying Snell's law at P, we get

$$\mu \sin\theta = \mu_0 \sin r$$



$$\Rightarrow \quad \sin\theta = \frac{\mu_0}{f(x)}\sqrt{1 - \frac{\sin^2\alpha}{\mu_0^2}} \qquad \{\because \mu = f(x)\}$$

$$\Rightarrow \quad \sin\theta = \frac{\sqrt{\mu_0^2 - \sin^2\alpha}}{f(x)}$$

Now slope of tangent at P is

$$\frac{dy}{dx} = \tan\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{\mu_0^2 - \sin^2\alpha}{[f(x)]^2 - \mu_0^2 + \sin^2\alpha}}$$

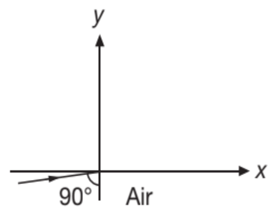
Conceptual Note(s)

(a) For Case-1: When refractive index varies along the y-axis, then normal is taken along the y-axis.

(b) For Case-2: When refractive index varies along the y-axis, then normal is taken along the x-axis.

ILLUSTRATION 29

Find the variation of Refractive index assuming it to be a function of y such that a ray entering origin at grazing incident follows a parabolic path $y = x^2$ as shown in figure.

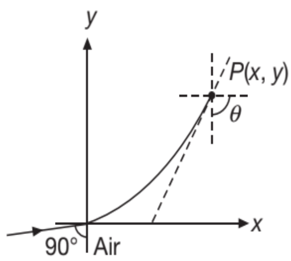


SOLUTION

We draw a tangent at any point (x, y) which makes an angle θ with optical normal parallel to y axis. From the Snell's law, we have

$$1\sin(90) = \mu \sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\mu} \qquad \ldots(1)$$



Slope of tangent is

$$\frac{dy}{dx} = \tan(90 - \theta) = \cot\theta \qquad \ldots(2)$$

The trajectory of the light ray is

$$y = x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x$$

From equation (2), we have

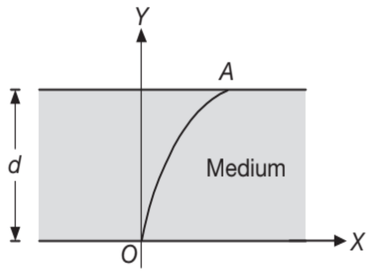
$$\cot\theta = 2x$$

This gives $\mu = \dfrac{1}{\sin\theta} = \text{cosec}\theta$

$$\Rightarrow \quad \mu = \sqrt{1 + \cot^2\theta} = \sqrt{1 + 4x^2} = \sqrt{1 + 4y}$$

ILLUSTRATION 30

A long rectangular slab of transparent medium of thickness d is placed on a table with length parallel to the x-axis and width parallel to the y-axis. A ray of light is travelling along y-axis at origin. The refractive index μ of the medium varies as $\mu = \dfrac{\mu_0}{1-\left(\dfrac{x}{a}\right)}$, where μ_0 and $a(>1)$ are constants. The refractive index of air is 1.



(a) Determine the x-coordinate of the point A, where the ray intersects the upper surface of the slab-air boundary.

(b) Write down the refractive index of the medium at A.

(c) Indicate the subsequent path of the ray in air.

SOLUTION

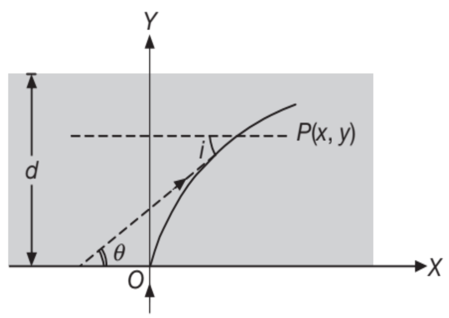
(a) Refractive index is a function of x, i.e., the plane separating the two media is parallel to y-z plane or normal to this plane at any point is parallel to x-axis.

Further refractive index increases as x is increases. So, the ray of light will bend towards normal and the path is shown in figure. Let at the point $P(x,\ y)$ the angle of incidence be i. Then

$$\theta = i$$

$$\Rightarrow \tan\theta = \tan i$$

$$\Rightarrow \frac{dy}{dx} = \tan i \qquad \ldots(1)$$



Applying Snell's Law at O and P, we get

$$\mu_0 \sin i_0 = \mu_P \sin i_P$$

$$\Rightarrow (1)(\sin 90^\circ) = \left(\frac{\mu_0}{1-\dfrac{x}{a}}\right)\sin i$$

$$\Rightarrow \sin i = \left(\frac{1-\dfrac{x}{a}}{\mu_0}\right)$$

$$\Rightarrow \tan i = \frac{\left(1-\dfrac{x}{a}\right)}{\sqrt{\mu_0^2-\left(1-\dfrac{x}{a}\right)^2}} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$dy = \frac{\left(1-\dfrac{x}{a}\right)}{\sqrt{\mu_0^2-\left(1-\dfrac{x}{a}\right)^2}}dx$$

Integrating, we get

$$\int_0^d dy = \int_0^x \frac{1-\dfrac{x}{a}}{\sqrt{\mu_0^2-\left(1-\dfrac{x}{a}\right)^2}}dx$$

$$\Rightarrow x = a\left[1-\sqrt{\mu_0^2-\left(\frac{d}{a}+\sqrt{\mu_0^2-1}\right)^2}\right]$$

(b) At point A, $1-\dfrac{x}{a} = \sqrt{\mu_0^2-\left(\dfrac{d}{a}+\sqrt{\mu_0^2-1}\right)^2}$

$$\Rightarrow \mu = \frac{\mu_0}{\sqrt{\mu_0^2-\left(\dfrac{d}{a}+\sqrt{\mu_0^2-1}\right)^2}} \qquad \left\{\because\ \mu = \frac{\mu_0}{1-\dfrac{x}{a}}\right\}$$

(c) After A, medium is again air. Hence, from Snell's Law, angle of incidence will again become 90° or it will move parallel to y-axis as shown.

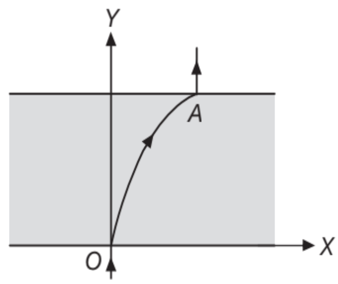
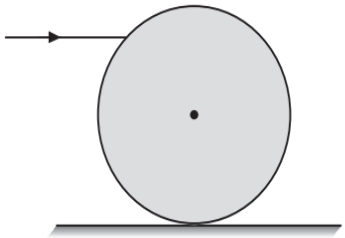


ILLUSTRATION 31

A cylindrical glass rod of radius 0.1 m and refractive index $\sqrt{3}$ lies on a horizontal plane mirror. A horizontal ray of light moving perpendicular to the axis of the rod is incident on it.

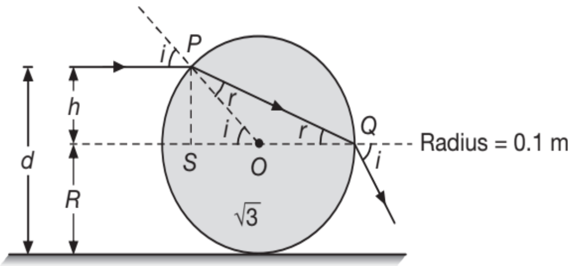
(a) At what height from the mirror should the ray be incident so that it leaves the rod at a height of 0.1 m above the plane mirror?

(b) At what distance a second similar rod, parallel to the first, be placed on the mirror, such that the emergent ray from the second rod is in line with the incident ray on the first rod?



SOLUTION

Let us first draw the ray diagram for the situation.



(a) Since, $PO = OQ$

$\Rightarrow \angle OPQ = \angle OQP = r$ (say)

Also, $i = r + r = 2r$

In ΔPOS, we have

$$h = OP \sin i = 0.1 \sin i$$

$\Rightarrow h = 0.1 \sin 2r$

$\Rightarrow h = 0.2 \sin r \cos r$ ...(1)

Applying Snell's Law at P, we get

$$\sqrt{3} = \frac{\sin i}{\sin r} = \frac{2 \sin r \cos r}{\sin r} = 2 \cos r$$

$\Rightarrow r = 30°$

Substituting in equation (1), we get

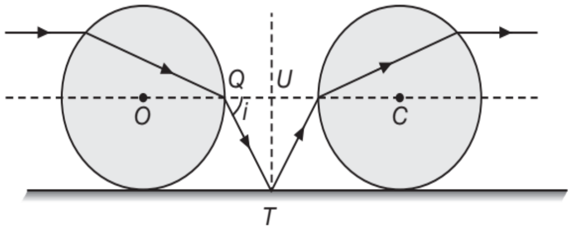
$$h = (0.2)\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 0.086 \text{ m}$$

Hence, height from the mirror is

$$d = h + R = 0.1 + 0.086 = 0.186 \text{ m}$$

(b) Using the principle of reversibility of light, we get

$i = 2r = 60°$



Now, $\dfrac{QU}{TU} = \cot i = \cot(60°) = \dfrac{1}{\sqrt{3}}$

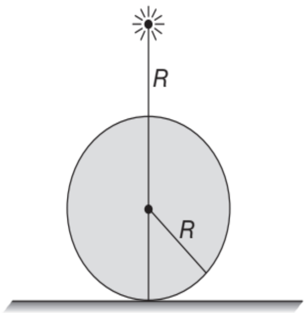
$\Rightarrow QU = \dfrac{TU}{\sqrt{3}} = \dfrac{0.1}{\sqrt{3}}$

So, the desired distance is

$$OC = 2(0.1) + 2\left(\frac{0.1}{\sqrt{3}}\right) = 0.315 \text{ m}$$

ILLUSTRATION 32

An opaque sphere of radius R lies on a horizontal plane. On the perpendicular through the point of contact there is a point source of light a distance R above the sphere.

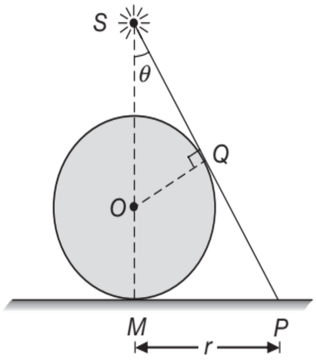


(a) Find the area of the shadow on the plane.

(b) A transparent liquid of refractive index $\sqrt{3}$ is filled above the plane such that the sphere is just covered with the liquid. Show that new area of the shadow.

SOLUTION

(a) The situation is shown in the figure



Since, we observe that

$$\sin\theta = \frac{OQ}{OS} = \frac{R}{2R} = \frac{1}{2}$$

$\Rightarrow \theta = 30^\circ$

Further, radius of shadow is given by

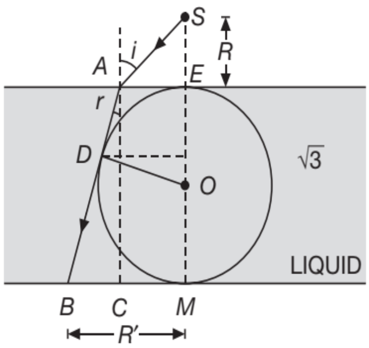
$$r = MP = MS\tan 30^\circ$$

$$\Rightarrow r = (3R)\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3}R$$

So, area of the shadow is

$$A = \pi\left(\sqrt{3}R\right)^2 = 3\pi R^2$$

(b) The situation is again shown in the figure.



$$AB = AD + BD$$

Since, by geometry, we have $AD = AE$ and $BD = BM$

$\Rightarrow AB = AE + BM$

$\Rightarrow AB = R\tan i + R'$

$$\Rightarrow \cos r = \frac{AC}{AB} = \frac{2R}{R' + R\tan i}$$

$$\Rightarrow R' = 2R\sec r - R\tan i \quad \ldots(1)$$

Also, we observe that

$$R' = BC + CM = 2R\tan r + R\tan i \quad \ldots(2)$$

From equations (1) and (2), we get

$$2\sec r - \tan i = 2\tan r + \tan i$$

$\Rightarrow \sec r - \tan r = \tan i$

Using the concepts of trigonometry, we get

$$\tan i = \frac{1 - \sin r}{\cos r} = \frac{\left(\cos\frac{r}{2} - \sin\frac{r}{2}\right)^2}{\cos^2\frac{r}{2} - \sin^2\frac{r}{2}}$$

$$\Rightarrow \tan i = \frac{\cos\left(\frac{r}{2}\right) - \sin\left(\frac{r}{2}\right)}{\cos\left(\frac{r}{2}\right) + \sin\left(\frac{r}{2}\right)} = \frac{1 - \tan\left(\frac{r}{2}\right)}{1 + \tan\left(\frac{r}{2}\right)}$$

$$\Rightarrow \tan i = \tan\left(\frac{\pi}{4} - \frac{r}{2}\right)$$

$$\Rightarrow 2i = \frac{\pi}{2} - \frac{r}{2} \quad \ldots(3)$$

Also, from Snell's Law, we get

$$\sin i = \sqrt{3}\sin r \quad \ldots(4)$$

Solving the above equations, we get

$$R' = \sqrt{2}R$$

So the new area of shadow is

$$A' = \pi R'^2 = 2\pi R^2$$

CONCEPT OF OPTICAL PATH LENGTH (OPL) AND REDUCED THICKNESS

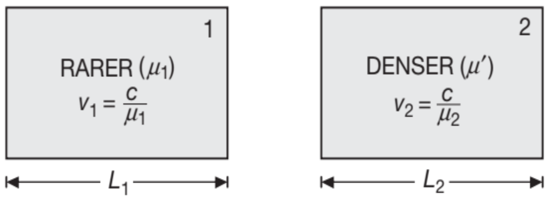
If a distance L separates two buildings, then the measured distance has nothing to do with the medium between the buildings. If this separation is filled with water, then too the distance between the buildings is L. However the time taken by the light to travel between the buildings is different for different media between the buildings. This time difference is due to the interaction of the light with the molecules of the medium which impede (slow down) the light's velocity and this causes the light to take more time to travel the same physical distance for different media.

Due to this, a new concept of distance needs to be introduced that accounts for the delay in the travelling time of the light in water (or a denser medium) in comparison to air (or a rarer medium). This new distance is called the **Optical Path Length (OPL) or Optical Path** and takes into account the slower velocity of light within a denser medium and it is simply the product of the distance with the refractive index i.e.,

$$OPL = \mu L$$

Thus, light passing through a denser medium seems to travel a longer distance than the light propagating in free space/vacuum, during the same time intervals for both the media.

Let me illustrate this thing to you. For that let me take two media, one rarer of length L_1, refractive index μ_1 and other denser of length L_2 and refractive index μ_2, as shown.



Time taken by light to travel a distance L_1 in rarer medium with speed $v_1 = \dfrac{c}{\mu_1}$ is

$$t_1 = \frac{L_1}{v_1} = \frac{L_1}{\left(\dfrac{c}{\mu_1}\right)} = \frac{\mu_1 L_1}{c} \quad \ldots(1)$$

The time taken by light to travel a distance L_2 in denser medium with a speed $v_2 = \dfrac{c}{\mu_2}$ is

$$t_2 = \frac{L_2}{v_2} = \frac{L_2}{\left(\dfrac{c}{\mu_2}\right)} = \frac{\mu_2 L_2}{c} \quad \ldots(2)$$

Now if both times are equal, as said above, then

$$t_1 = t_2$$

$$\Rightarrow \quad \mu_1 L_1 = \mu_2 L_2 \quad \ldots(3)$$

Since, Optical Path Length (OPL) is the distance travelled by light in vacuum/air/rarer medium during the same time it travels a distance L_2 in medium. So, from (3), we get

$$\mu_1 L_1 = \mu_{air} L_{air} = \mu_{medium} L_{medium} = \mu_2 L_2$$

Since, $\mu_{air} = 1$, so, we get

$$OPL = L_{air} = \mu_{medium} L_{medium}$$

{for same time in air and medium}

Since light always travels slower in denser medium, so the OPL (the distance in air corresponding to same time in both) is always longer than the actual thickness L of the medium.

Conceptual Note(s)

Also, note that $\mu_1 l_1$ is OPL in air/rarer medium and $\mu_2 l_2$ is OPL in denser medium. However for standard purposes OPL is the distance travelled by light in vacuum/air to travel a distance L in a medium during the same time in either air or medium.

So, from equation (3), we conclude that for a pair of media,

$$\begin{pmatrix} \text{Optical Path} \\ \text{Length in} \\ \text{Air/Rarer Medium} \end{pmatrix} = \begin{pmatrix} \text{Optical Path} \\ \text{Length in} \\ \text{Denser Medium} \end{pmatrix}$$

$$\Rightarrow \quad \mu_1 L_1 = \mu_2 L_2$$

$$\Rightarrow \quad L_2 = \frac{\mu_1}{\mu_2} L_1$$

Since $\mu_1 < \mu_2$, so we get

$$L_2 < L_1$$

Due to this reason, L_2 is also called the **Reduced Thickness**. So, in general, we get

$$\begin{pmatrix} \text{Reduced} \\ \text{Thickness} \end{pmatrix} = \left(\frac{\mu_{rarer}}{\mu_{denser}}\right) L_{rarer} = \frac{OPL \text{ in air}}{\mu_{denser}}$$

ILLUSTRATION 33

A light beam of wavelength 600 nm in air passes firstly through film 1 of thickness 1 μm and refractive index $n_1 = 1.2$, then through an air film 2 of thickness 1.5 μm and finally through film 3 of thickness 1 μm and refractive index $n_3 = 1.8$.

(a) Which film does the light cross in the least time and what is that least time?

(b) Calculate the total number of wavelengths (at any instant) across all three films together.

SOLUTION

(a) Since, $t_1 = \dfrac{d_1}{v_1} = \dfrac{10^{-6}}{\dfrac{(3\times10^8)}{1.2}} = 4\times10^{-15}$ s $\quad \left\{v = \dfrac{c}{n}\right\}$

Similarly, $t_2 = \dfrac{d_2}{c} = \dfrac{1.5\times10^{-6}}{3\times10^8} = 2\times10^{-15}$ s

and $t_3 = \dfrac{d_3}{v_3} = \dfrac{10^{-6}}{\dfrac{(3\times10^8)}{1.8}} = 6\times10^{-15}$ s

So, $t_{min} = 2\times10^{-15}$ s

(b) The total number of wavelengths in a film of refractive index μ, thickness d is

$$n = \frac{\text{Optical Path Length}}{\text{Wavelength of Light}}$$

$$\Rightarrow \quad n = \frac{\mu d}{\lambda}$$

So, total number of wavelengths, is

$$n = \frac{\mu_1 d_1}{\lambda} + \frac{\mu_2 d_2}{\lambda} + \frac{\mu_3 d_3}{\lambda}$$

$$\Rightarrow \quad n = \frac{1}{\lambda}(\mu_1 d_1 + \mu_2 d_2 + \mu_3 d_3)$$

$$\Rightarrow \quad n = \frac{10^{-6}}{600\times10^{-9}}((1.2)(1)+(1)(1.5)+(1.8)(1))$$

$$\Rightarrow \quad n = \frac{1000}{600}(4.5) = \frac{4500}{600} = 7.5$$

ILLUSTRATION 34

A light ray enters the atmosphere of a planet and descends vertically 20 km to the surface. The index of refraction where the light enters the atmosphere is 1 and it increases linearly to the surface where it has a value 1.005. How long does it take the ray to traverse this path.

SOLUTION

Since variation is linear, so we have

$$\frac{x-0}{\mu-1} = \frac{2\times10^4 - 0}{1.005-1}$$

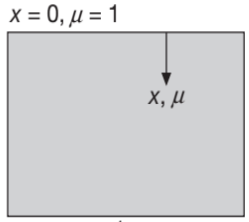
$$\Rightarrow \quad \mu = 1 + \frac{0.005x}{2\times10^4}$$

Now, by definition, we have

$$\mu(x) = \frac{c}{v(x)}$$

$$\Rightarrow \quad v(x) = \frac{c}{\mu(x)}$$

where $\mu(x) = \mu = 1 + \dfrac{0.005x}{2\times10^4}$ and $c = 3\times10^8\ \text{ms}^{-1}$



$$\Rightarrow \quad v(x) = \frac{c}{\mu(x)} = \frac{3\times10^8}{1+\dfrac{0.005}{2\times10^4}x} = \frac{3\times10^8}{1+2.5\times10^{-7}x}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\times10^8}{1+2.5\times10^{-7}x}$$

$$\Rightarrow \quad dt = \frac{1}{3\times10^8}\left[(1+2.5\times10^{-7}x)dx\right]$$

$$\Rightarrow \quad \int_0^t dt = \frac{1}{3\times10^8}\int_0^{2\times10^4}(1+2.5\times10^{-7}x)dx$$

$$\Rightarrow \quad t = \frac{1}{3\times10^8}\left[2\times10^4 + \frac{(2\times10^4)^2(2.5\times10^{-7})}{2}\right]$$

$$\Rightarrow \quad t = 6.68\times10^{-5}\ \text{s}$$

LAWS OF REFRACTION USING FERMAT'S PRINCIPLE

Consider a refracting surface/interface separating medium 1 from medium 2. Let the incident light start from A, in medium 1, hit the surface at O and get refracted to a point B, in medium 2. Let the points A and B be at perpendicular distances a and b from the interface. Further, let A and B be at a separation d as shown in figure. The time taken by the light to go from A to O to B is

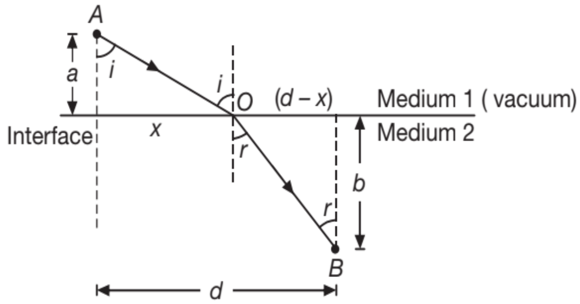
$$t = t_{A\to O} + t_{O\to B}$$

$$\Rightarrow \quad t = \frac{AO}{c} + \frac{OB}{v}$$

$$\Rightarrow \quad t = \frac{\sqrt{a^2+x^2}}{c} + \frac{\sqrt{b^2+(d-x)^2}}{v}$$

Now, according to Fermat's Theorem, t is MINIMUM, so

$$\frac{dt}{dx} = 0$$



$$\Rightarrow \quad \frac{1}{c}\frac{d}{dt}\left(\sqrt{a^2+x^2}\right) + \frac{1}{v}\frac{d}{dt}\left(\sqrt{b^2+(d-x)^2}\right)$$

$$\Rightarrow \quad \frac{1}{2c}\frac{2x}{\sqrt{a^2+x^2}} + \frac{1}{2v}\left(\frac{2(d-x)(-1)}{\sqrt{b^2+(d-x)^2}}\right) = 0$$

$$\Rightarrow \quad \frac{1}{c}\frac{x}{\sqrt{a^2+x^2}} = \frac{1}{v}\frac{(d-x)}{\sqrt{b^2+(d-x)^2}}$$

From the figure, we observe that

$$\frac{x}{\sqrt{a^2+x^2}} = \sin i \text{ and } \frac{d-x}{\sqrt{b^2+(d-x)^2}} = \sin i$$

$$\Rightarrow \quad \frac{1}{c}\sin i = \frac{1}{v}\sin r$$

$$\Rightarrow \quad \frac{\sin i}{\sin r} = \frac{c}{v} = \mu \qquad \text{\{The Law of Refraction\}}$$

VECTOR FORM OF SNELL'S LAW

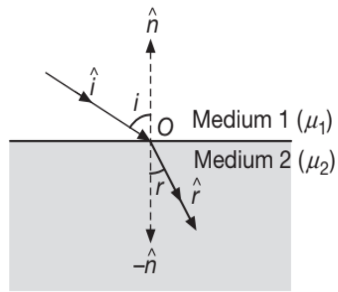
According to Snell's Law, we have

$$\mu_1 \sin i = \mu_2 \sin r$$

Vectorially, Snell's Law can be written as

$$\mu_1\left(\hat{i}\times\hat{n}\right) = \mu_2\left(\hat{r}\times\hat{n}\right)$$

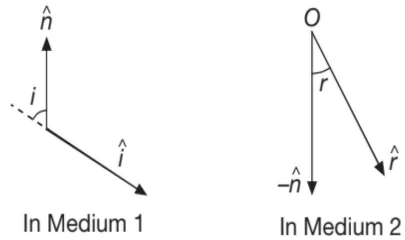
Let $\hat{i}$ be the unit vector along the incident ray, $\hat{r}$ be the unit vector along the refracted ray and $\hat{n}$ be a unit vector along the normal as shown.



Then, $\left|\hat{i}\right| = \left|\hat{n}\right| = \left|\hat{r}\right| = 1$

Using our knowledge of cross product of vectors, we have

$$\hat{i}\times\hat{n} = (1)(1)\sin i, \odot \text{ outwards}$$



and $-\hat{n}\times\hat{r} = \hat{r}\times\hat{n} = (1)(1)\sin r$, $\odot$ outwards

So, in vector form, the Snell's Law can be expressed as

$$\mu_1\left(\hat{i}\times\hat{n}\right) = \mu_2\left(\hat{r}\times\hat{n}\right)$$

REFRACTION THROUGH A COMPOSITE SLAB

Consider the refraction of light ray through a series of media as shown in figure. The ray AB is incident on interface X_1Y_1 at an angle i. The ray is deviated in medium 2 along BC towards the normal. Then it falls on interface X_2Y_2 and is again deviated towards normal along CD. If the last medium is again Medium 1, the ray emerges parallel to the incident ray. Let r_1 and r_2 be angles of refraction in Medium 2 and Medium 3 respectively. Then from Snell's Law,

$$\frac{\sin i}{\sin r_1}=\frac{\mu_2}{\mu_1}={}^1\mu_2 \quad \ldots(1)$$

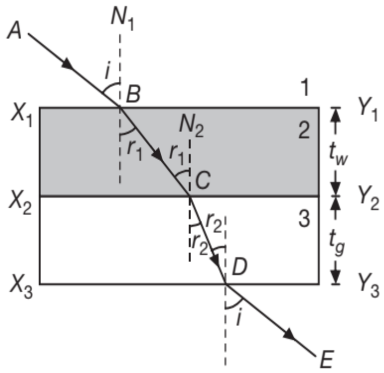
$$\frac{\sin r_1}{\sin r_2}=\frac{\mu_3}{\mu_2}={}^2\mu_3 \quad \ldots(2)$$

$$\frac{\sin r_2}{\sin i}=\frac{\mu_1}{\mu_3}={}^3\mu_1 \quad \ldots(3)$$

μ_1 = refractive index of medium 1

μ_2 = refractive index of medium 2

μ_3 = refractive index of medium 3



Multiplying (1), (2) and (3), we get

$${}^1\mu_2\times{}^2\mu_3\times{}^3\mu_1=1$$

$$\Rightarrow \quad {}^1\mu_2\times{}^2\mu_3={}^1\mu_3$$

In general if a ray passes through a number of composite parallel plate glass slabs, then

$${}^1\mu_2\times{}^2\mu_3\times{}^3\mu_4\times{}^4\mu_5={}^1\mu_5$$

ILLUSTRATION 35

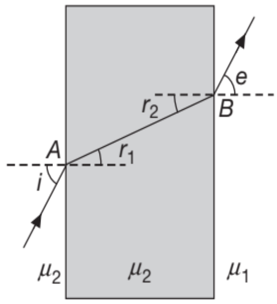
A light beam passes from a parallel plate glass slab of refractive index μ_2 placed in a medium of refractive index μ_1. Show that the emerging beam is parallel to the incident beam.

SOLUTION

Applying Snell's Law at A, we get

$$\mu_1\sin i=\mu_2\sin r_1$$

$$\Rightarrow \quad \frac{\mu_1}{\mu_2}=\frac{\sin r_1}{\sin i} \quad \ldots(1)$$



Applying Snell's Law at B, we get

$$\mu_2\sin r_2=\mu_1\sin e$$

$$\Rightarrow \quad \frac{\mu_1}{\mu_2}=\frac{\sin r_2}{\sin e} \quad \ldots(2)$$

From equation (1) and (2), we get

$$i=e$$

i.e., the emergent ray is parallel to incident ray.

ILLUSTRATION 36

Refractive index of glass with respect to water is $\frac{9}{8}$. Refractive index of glass with respect to air is $\frac{3}{2}$. Find the refractive index of water with respect to air.

SOLUTION

Given, ${}^w\mu_g=\frac{9}{8}$ and ${}^a\mu_g=\frac{3}{2}$

Since, ${}^a\mu_g\times{}^g\mu_w={}^a\mu_w$

$$\Rightarrow \quad {}^a\mu_w=\frac{{}^a\mu_g}{{}^w\mu_g} \qquad \left\{\because\ {}^g\mu_w=\frac{1}{{}^w\mu_g}\right\}$$

$$\Rightarrow \quad {}^a\mu_w=\frac{\left(\frac{3}{2}\right)}{\left(\frac{9}{8}\right)}=\frac{4}{3}$$

LATERAL SHIFT ON PASSING THROUGH A GLASS SLAB

Consider a ray AO incident on the slab at an angle of incidence i through the glass slab $EFGH$ of thickness t. After refraction the ray emerges parallel to the incident ray.

Let PQ be perpendicular dropped from P on incident ray produced as OQ.

The lateral displacement caused by plate,

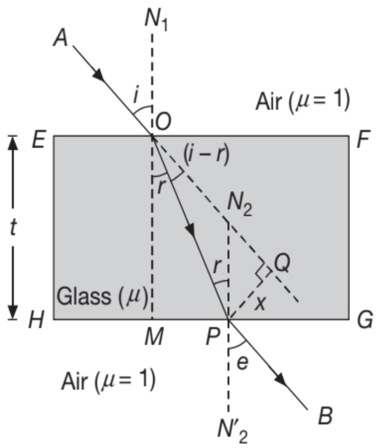
$$x = PQ = OP\sin(i-r)$$

$$x = \frac{OM}{\cos r}\sin(i-r) \qquad \left\{\because OP = \frac{OM}{\cos r}\right\}$$

$$\Rightarrow \quad x = \frac{t}{\cos r}\sin(i-r)$$

$$\Rightarrow \quad x = \frac{t}{\cos r}(\sin i\cos r - \cos i\sin r)$$

$$\Rightarrow \quad x = t(\sin i - \cos i\tan r)$$



Since $\mu = \frac{\sin i}{\sin r}$

$$\Rightarrow \quad \sin r = \frac{\sin i}{\mu}$$

$$\Rightarrow \quad \tan r = \frac{\sin i}{\sqrt{\mu^2 - \sin^2 i}}$$

$$\Rightarrow \quad y = t\sin i\left(1 - \frac{\cos i}{\sqrt{\mu^2 - \sin^2 i}}\right)$$

If i is very small then r is also very small, hence

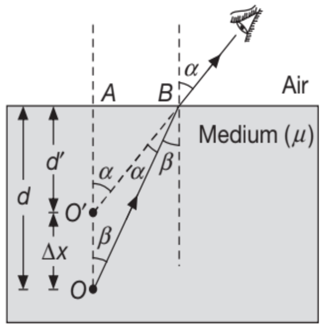
$\sin i \to i$, and $\cos i \to 1$

Then expression for lateral displacement takes the form

$$x = ti\left(1 - \frac{1}{\mu}\right)$$

APPARENT DEPTH

An object O placed in a medium of refractive index μ is observed from air at a small angle α to the normal to the interface (in figure, angle α is shown exaggerated for clarity) i.e., for near normal incidence.



If the object O is at a real depth d from the interface, its apparent depth d' can be calculated. From Δs ABO and ABO',

$$\frac{\tan\alpha}{\tan\beta} = \frac{d}{d'}$$

Since angles α and β are small, so $\sin\alpha \approx \tan\alpha$ and $\sin\beta \approx \tan\beta$.

Therefore, from Snell's Law, we get

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin\alpha}{\sin\beta} \approx \frac{\tan\alpha}{\tan\beta} = \frac{d}{d'}$$

$$\Rightarrow \quad \text{Apparent depth, } d' = \frac{d}{\mu}$$

The **apparent shift** in normal direction (or the **normal shift**) in the position of the object is

$$\Delta x = d - d' = d\left(1 - \frac{1}{\mu}\right)$$

In case the object is seen through n number of slabs with different refractive indices, the total apparent shift is simply the sum of individual shifts, so

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 + ... + \Delta x_n$$

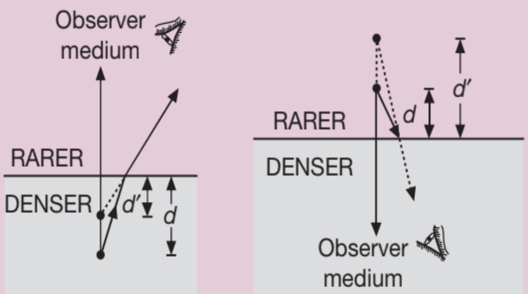
$$\Rightarrow \quad \Delta x = d_1\left(1 - \frac{1}{\mu_1}\right) + d_2\left(1 - \frac{1}{\mu_2}\right) + d_3\left(1 - \frac{1}{\mu_3}\right) + .. + d_n\left(1 - \frac{1}{\mu_n}\right)$$

Conceptual Note(s)

(a) If the medium in which the object is placed is rarer ($\mu < 1$) and it is seen from the denser medium, the apparent shift calculated will be **negative**. It means that the object apparently shifts away from the observer.

If the shift comes out to be positive, the image of the object shifts towards the observer.

(b) At near normal incidence (small angle of incidence i) apparent depth (d') is given by



$$d' = \frac{d}{\mu_{\text{relative}}} \text{ and } v' = \frac{v}{\mu_{\text{relative}}}$$

where

$$\mu_{\text{relative}} = \frac{\text{RI of medium of incidence object}}{\text{RI of medium of refraction observer}}$$

$$= \frac{\mu_{\text{object}}}{\mu_{\text{observer}}}$$

d = distance of object from the interface = real depth.

d' = distance of image from the interface = apparent depth.

v = velocity of object perpendicular to interface relative to surface.

v' = velocity of image perpendicular to interface relative to surface.

ILLUSTRATION 37

A fish is rising vertically to the surface of water in a lake at a uniform speed of 3 ms^{-1}. It observes that a bird is diving vertically towards the water at a uniform speed of 9 ms^{-1}. If the refractive index of water is $\frac{4}{3}$, find the actual speed of dive of the bird.

SOLUTION

Let x be the depth of the fish F below the surface of water, and y be the height of the bird B above the surface at an instant.

To the fish, the bird will appear to be farther away, at an apparent height y' given by

$$y' = \frac{y}{\mu_{\text{relative}}} = \frac{y}{\left(\dfrac{\mu_{\text{object medium}}}{\mu_{\text{observer medium}}}\right)}$$

$$\Rightarrow \quad y' = \frac{y}{\left(\dfrac{\mu_{\text{bird medium}}}{\mu_{\text{fish medium}}}\right)}$$

$$\Rightarrow \quad y' = \frac{y}{\left(\dfrac{1}{\mu}\right)} = \mu y$$

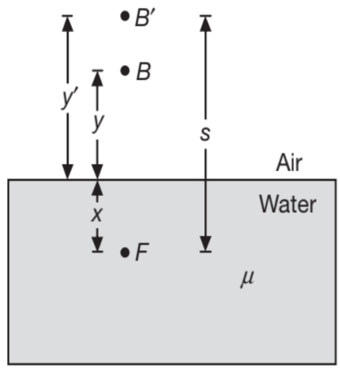
The total apparent distance of the bird from the fish is

$$s = x + y'$$

$$\Rightarrow \quad s = x + \mu y$$

Differentiating w.r.t. time t, we get

$$\frac{ds}{dt} = \frac{dx}{dt} + \mu \frac{dy}{dt}$$



Substituting the values, we get

$$9 = 3 + \left(\frac{4}{3}\right)\frac{dy}{dt}$$

Therefore, the actual speed of dive of the bird is given by

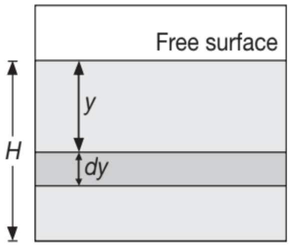
$$\frac{dy}{dt} = (9 - 3)\left(\frac{3}{4}\right) = 4.5 \text{ ms}^{-1}$$

ILLUSTRATION 38

A vessel is filled with a non-homogeneous liquid whose refractive index varies with the depth y from the free surface of liquid as $\mu = \left(1+\frac{y}{H}\right)$. Calculate the apparent depth as seen by an observer from above, if H is the height to which the liquid is filled in the vessel.

SOLUTION

Let us consider a thin layer of liquid of thickness dy at a distance y below the free surface of liquid. The apparent depth of this layer having real depth dy is $dH' = \frac{dy}{\mu}$.



$$\Rightarrow \quad dH' = \frac{dy}{\mu}$$

$$\Rightarrow \quad dH' = \frac{dy}{\left(1+\frac{y}{H}\right)} \qquad \left\{\because \mu = 1+\frac{y}{H}\right\}$$

Total apparent depth is obtained by integrating this expression within appropriate limits. So,

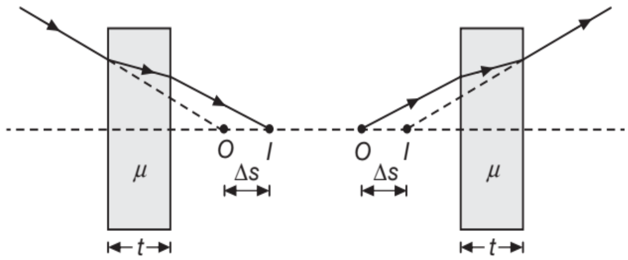
$$H' = \int dH' = \int_0^H \frac{dy}{\left(1+\frac{y}{H}\right)} = H\log_e(H+y)\Big|_0^H$$

$$\Rightarrow \quad H' = H\log_e 2$$

SHIFT OF POINT OF CONVERGENCE OR DIVERGENCE

If a glass slab of thickness t, refractive index μ is placed in the path of a convergent (or divergent) beam of light, the point of convergence (or divergence) gets shifted by

$$\Delta s = t\left(1-\frac{1}{\mu}\right)$$

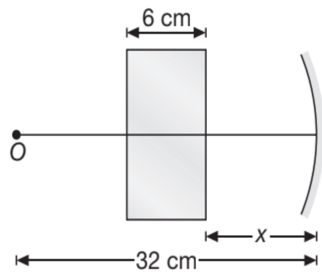


(A) Convergent beam (B) Divergent beam

ILLUSTRATION 39

A point object O is placed in front of a concave mirror of focal length 10 cm. A glass slab of refractive index $\mu = \frac{3}{2}$ and thickness 6 cm is inserted between object and mirror. Find the position of final image when the distance x shown in figure is

(a) 5 cm
(b) 20 cm



SOLUTION

The normal shift produced by a glass slab is given by

$$\Delta x = \left(1-\frac{1}{\mu}\right)t = \left(1-\frac{2}{3}\right)(6) = 2 \text{ cm}$$

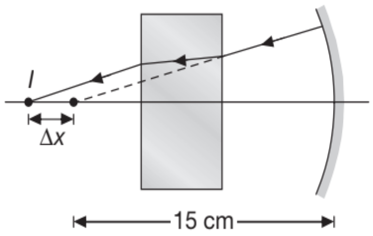
i.e., for the mirror the object is placed at a distance $(32-\Delta x) = 30$ cm from it. Applying mirror formula i.e. $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}-\frac{1}{30}=-\frac{1}{10}$$

$$\Rightarrow \quad v = -15 \text{ cm}$$

(a) When $x = 5$ cm

The light falls on the slab after being reflected from the mirror as shown. But the slab will again shift it by a distance $\Delta x = 2$ cm. Hence, the final real image is formed at a distance $(15+2)=17$ cm from the mirror.



(b) When x = 20 cm

This time too the final image is at a distance 17 cm from the mirror but it is virtual as shown.

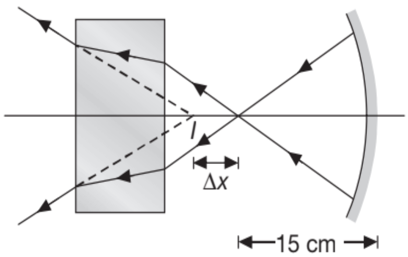
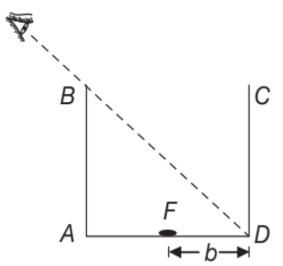


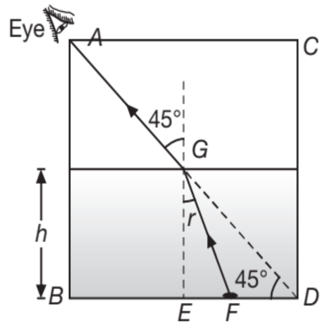
ILLUSTRATION 40

A cubical vessel with non-transparent walls is so located that the eye of an observer does not see its bottom but sees all of the wall CD. To what height should water be poured into the vessel for the observer to see an object F arranged at a distance of $b=10$ cm from corner D? The face of the vessel is $a=40$ cm and refractive index of water is $\frac{4}{3}$.



SOLUTION

Since, the vessel is cubical, $\angle GDE = 45°$ and $GE = ED = h$ (say) then $EF = ED - FD$



But $\tan(45°) = 1 = \dfrac{GE}{ED}$

$\Rightarrow \quad ED = GE = h$

$\Rightarrow \quad EF = ED - FD = h - 10$

Further, $\dfrac{4}{3} = \dfrac{\sin(45°)}{\sin r}$

$\Rightarrow \quad r = 32°$

Now $\dfrac{EF}{GE} = \tan r = \tan(32°)$

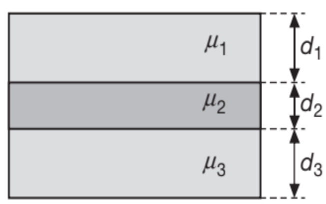
$\Rightarrow \quad \dfrac{h-10}{h} = 0.62$

Solving this, we get

$$h = 26.65 \text{ cm}$$

MULTISLABS

If a number of slabs (or immiscible liquids) of depth d_1, d_2, d_3,.... and refractive index μ_1, μ_2, μ_3, ... are placed one over the other as shown.



Then the real depth is

$$d = d_1 + d_2 + d_3 + ...$$

The apparent depth is given as

$$d' = \frac{d_1}{\mu_1} + \frac{d_2}{\mu_2} + \frac{d_3}{\mu_3} + ...$$

Therefore, for the combination, the effective μ is

$$\mu = \frac{d}{d'} = \frac{d_1 + d_2 + d_3 + \dots}{\left(\frac{d_1}{\mu_1}\right) + \left(\frac{d_2}{\mu_2}\right) + \left(\frac{d_3}{\mu_3}\right) + \dots} = \frac{\Sigma d_i}{\Sigma\left(\frac{d_i}{\mu_i}\right)}$$

If there are only two slabs, of equal thickness, $d_1 = d_2 = d$, $\mu = \frac{d + d}{\left(\frac{d}{\mu_1}\right) + \left(\frac{d}{\mu_2}\right)} = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} =$ harmonic mean of μ_1 and μ_2

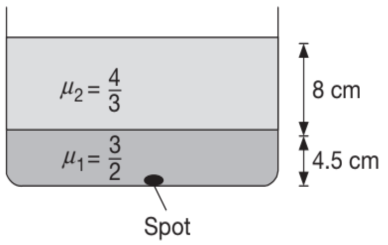
ILLUSTRATION 41

The bottom of a tub has a black spot. A glass slab of thickness 4.5 cm is placed over it and then water is filled to the height of 8 cm above the glass slab. Looking from top, what shall be the apparent depth of the spot below the water surface? Also find the effective refractive index of the combination of glass slab and water layer. (Refractive index of glass is $\frac{3}{2}$ and of water is $\frac{4}{3}$).

SOLUTION

The apparent depth is given as

$$d_a = \frac{d_1}{\mu_1} + \frac{d_2}{\mu_2} = \frac{4.5}{\left(\frac{3}{2}\right)} + \frac{8}{\left(\frac{4}{3}\right)} = 3 + 6 = 9 \text{ cm}$$



The effective refractive index is given as

$$\mu_{\text{effective}} = \frac{\text{Real Depth}}{\text{Apparent Depth}}$$

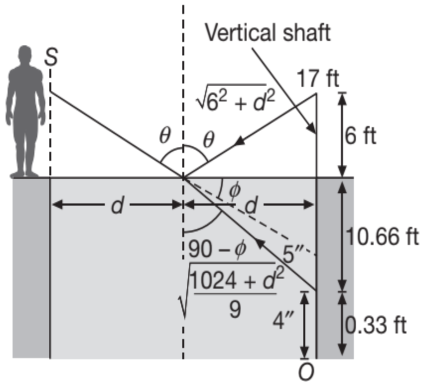
$$\Rightarrow \quad \mu_{\text{effective}} = \frac{d_r}{d_a} = \frac{d_1 + d_2}{d_a} = \frac{4.5 + 8}{9} = \frac{12.5}{9} = 1.39$$

ILLUSTRATION 42

A surveyor on one bank of a canal observes the image of the 4 inch mark and 17 ft mark on a vertical staff, which is partially immersed in the water and held against the bank directly opposite to him. He sees that the reflected and refracted rays come from the same point which is the centre of the canal. If the 17 ft mark and the surveyor's eye are both 6 ft above the water level, estimate the width of the canal (in foot), assuming that the refractive index of the water is $\frac{4}{3}$. Zero mark is at the bottom of the canal.

SOLUTION

Figure below shows the ray diagram of image produced at 5 ft mark for both the marks – One at 4 inch by refraction and other at 17 ft by reflection.



By using Snell's law we have

$$1 \times \sin\theta = \frac{4}{3}\sin(90 - \phi)$$

$$\Rightarrow \quad 1 \times \frac{d}{\sqrt{36 + d^2}} = \frac{4}{3} \times \frac{d}{\sqrt{\left(\frac{32}{3}\right)^2 + d^2}}$$

$$\Rightarrow \quad \sqrt{1024 + 9d^2} = 4\sqrt{36 + d^2}$$

On squaring both sides, we get

$$1024 + 9d^2 = 16(36 + d^2)$$

$$\Rightarrow \quad 1024 + 9d^2 = 576 + 16d^2$$

$$\Rightarrow \quad 7d^2 = 448$$

$$\Rightarrow \quad d^2 = \frac{448}{7} = 64$$

$$\Rightarrow \quad d = 8 \text{ foot}$$

$$\Rightarrow \quad \text{Width of canal is } 2d = 16 \text{ foot}.$$

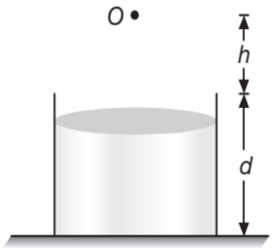
ILLUSTRATION 43

A point source of light is arranged at a height h above the surface of water. Where will the image of this source in the flat mirror-like bottom of a vessel be if the depth of the vessel full of water is d? Refractive index of water is $n = \frac{4}{3}$. Consider only two steps.

SOLUTION

When we consider only two steps, then the ray of light starting from object O first gets refracted and then reflected. Distance of image I_1 formed after refraction from the plane surface is given by

$$x = nh + d = \frac{4}{3}h + d$$



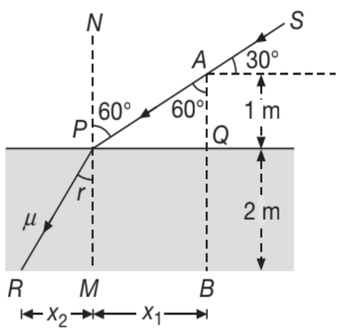
Therefore, distance of image I_2 formed by plane mirror will be $\frac{4}{3}h + d$

ILLUSTRATION 44

In a river 2 m deep, a water level measuring post embedded into the river stands vertically with 1 m of it above the water surface. If the angle of inclination of sun above the horizon is $30°$, calculate the length of the post on the bottom surface of the river. $\left(\mu \text{ for water} = \frac{4}{3}\right)$

SOLUTION

The situation is shown in figure.



The length of shadow of pole AB at the bottom of river is given as

$$L = x_1 + x_2 \quad \ldots(1)$$

From ΔAPQ, we have

$$x_1 = AQ \tan(60°) = \sqrt{3} \text{ meter} \quad \ldots(2)$$

From ΔRPM, we have

$$x_2 = PM \tan r = 2 \tan r \quad \ldots(3)$$

From Snell's law, at point P, we have

$$\frac{\sin 60}{\sin r} = \mu = \frac{4}{3}$$

$$\Rightarrow \quad \sin r = \frac{3}{4}\sin(60°) = \frac{3\sqrt{3}}{8}$$

$$\Rightarrow \quad \cos r = \sqrt{1 - \sin^2 r} = \frac{\sqrt{37}}{8}$$

We substitute the value of r in Equation (3)

$$x_2 = 2 \tan r$$

$$\Rightarrow \quad x_2 = 2 \times \frac{\frac{3\sqrt{3}}{8}}{\frac{\sqrt{37}}{8}} = \frac{6\sqrt{3}}{\sqrt{37}}$$

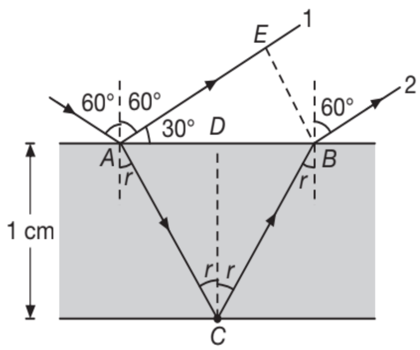
ILLUSTRATION 45

A ray of light falls onto a plane-parallel glass plate 1 cm thick at an angle of $60°$. The refractive index of the glass is $\sqrt{3}$. Some of the light is reflected and the rest, being refracted, passes into the glass is reflected from the bottom of the plate, refracted a second time and emerges back into the air parallel to the first reflected ray. Determine the distance l between the rays.

SOLUTION

From Snell's Law, we have $\sqrt{3} = \frac{\sin(60°)}{\sin r}$

$$\Rightarrow \quad r = 30°$$



Since, $AB = 2(AD) = 2(DC\tan r)$

$$\Rightarrow \quad AB = (2)(1)\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} \text{ cm}$$

So, the distance between rays 1 and 2 is given by

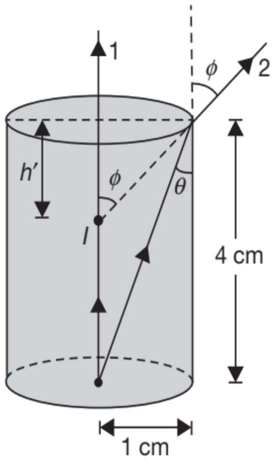
$$BE = AB\sin(30°) = \frac{1}{\sqrt{3}} \text{ cm}$$

ILLUSTRATION 46

A small object is kept at the centre of bottom of a cylindrical beaker of diameter 6 cm and height 4 cm filled completely with water $\left(\mu = \frac{4}{3}\right)$. Consider the light ray from an object leaving the beaker through a corner. If this ray and the ray along the axis of beaker is used to locate the image, find the apparent depth in this case.

SOLUTION

Figure shows the ray diagram of the image formation as described in the given condition.



By using Snell's law, we have

$$\mu_W \sin\theta = 1\sin\phi$$

$$\Rightarrow \quad \frac{4}{3}\times\frac{3}{5} = \sin\phi$$

$$\Rightarrow \quad \phi = 53°$$

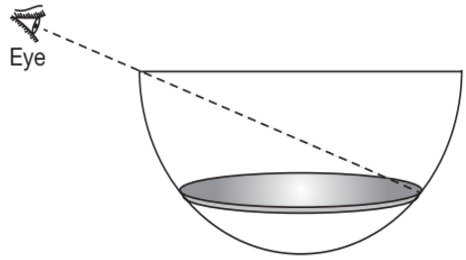
Here we have $\tan\phi = \frac{3}{h'}$

$$\Rightarrow \quad h' = \frac{3}{\frac{4}{3}} = \frac{9}{4} = 2.25 \text{ cm}$$

ILLUSTRATION 47

A circular disc of diameter d lies horizontally inside a metallic hemispherical bowl of radius a. The disc is just visible to an eye looking over the edge. The bowl is now filled with a liquid of refractive index μ. Now, the whole of the disc is just visible to the eye in the same position. Show that

$$d = 2a\left(\frac{\mu^2 - 1}{\mu^2 + 1}\right)$$

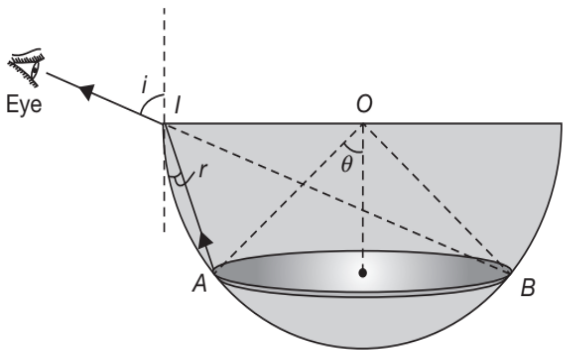


SOLUTION

In the figure, let AB be the disc and O be the centre of the bowl.

Let $\angle AOM = \theta$

Then by symmetry, $\angle AIB = \theta$



Now from geometry of figure, we have

$$\angle AOI = 2\angle IBA$$

$$\Rightarrow \quad \angle IBA = \frac{1}{2}(90° - \theta)$$

$$\Rightarrow \quad \angle r = \angle IBA = \frac{90° - \theta}{2} \quad \ldots(1)$$

Also $\angle i = \frac{90° - \theta}{2} + \theta = \frac{90° + \theta}{2} \quad \ldots(2)$

Now, Snell's Law gives

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin\left(\frac{90° + \theta}{2}\right)}{\sin\left(\frac{90° - \theta}{2}\right)}$$

$$\Rightarrow \quad \left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)^2=\mu^2$$

$$\Rightarrow \quad \frac{1+\sin\theta}{1-\sin\theta}=\mu^2$$

$$\Rightarrow \quad \sin\theta=\frac{\mu^2-1}{\mu^2+1}$$

Since, $\sin\theta=\frac{d}{2a}$

$$\Rightarrow \quad d=2a\left(\frac{\mu^2-1}{\mu^2+1}\right)$$

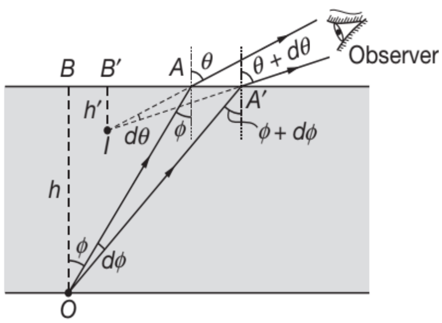
ILLUSTRATION 48

A man standing on the edge of a swimming pool looks at a stone lying at the bottom of the pool. The depth of the swimming pool is h. At what distance from the surface of water is the image of the stone formed if the line of sight makes an angle θ with the normal to the surface?

SOLUTION

Please note that, this problem is not the case of normal viewing (as discussed earlier), in which the apparent depth of the object lying at the bottom of the pool was $h'=\frac{h}{\mu}$.

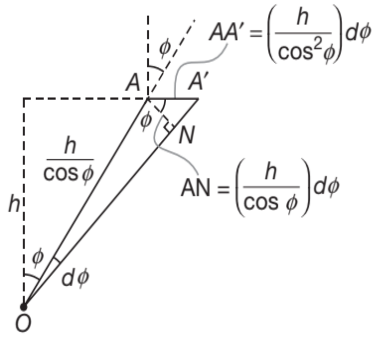
However, in this case, the observer is seeing the stone standing at the edge of the pool and the observer has line of sight angle θ. Let us draw two rays OA and OA' very close to each other. Out of these two rays, one ray is passing through the edge of the pool. The rays incident on the interface in the region AA', after refraction will reach the eye of the observer, so that he sees the object O as shown in figure.



Let us now produce the refracted rays backwards so that their point of intersection (I) is the place where the image of the object O is formed. So $OB=h$ is the real depth and $IB'=h'$ is the apparent depth.

To calculate h', let us consider triangles OAA' and IAA'. From triangle OAA', we calculate AA' and also from triangle IAA' we again calculate AA' (common side to both triangles) and equate both.

From A, drop a perpendicular on OA' at the point N so that AN is an arc of a circular portion of radius $OA=\frac{h}{\cos\phi}$ as shown in figure.



Since $AN=(OA)d\phi=\left(\frac{h}{\cos\phi}\right)d\phi$

In triangle $AA'N$, we have

$$\cos\phi=\frac{AN}{AA'}$$

$$\Rightarrow \quad AA'=\frac{AN}{\cos\phi}=\left(\frac{h}{\cos^2\phi}\right)d\phi \qquad \ldots(1)$$

Similarly, from triangle IAA', we get

$$AA'=\left(\frac{h'}{\cos^2\theta}\right)d\theta \qquad \ldots(2)$$

Equating equations (1) and (2), we get

$$\left(\frac{h}{\cos^2\phi}\right)d\phi=\left(\frac{h'}{\cos^2\theta}\right)d\theta$$

$$\Rightarrow \quad h'=h\left(\frac{\cos^2\theta}{\cos^2\phi}\right)\left(\frac{d\phi}{d\theta}\right) \qquad \ldots(3)$$

Now applying Snell's Law at A, we get

$$\mu=\frac{\sin\theta}{\sin\phi} \qquad \ldots(4)$$

$$\Rightarrow \quad \sin\theta=\mu\sin\phi$$

Differentiating both sides, we get

$$\cos\theta d\theta = \mu\cos\phi d\phi$$

$$\Rightarrow \quad \frac{d\phi}{d\theta} = \frac{\cos\theta}{\mu\cos\phi} \qquad \ldots(5)$$

Put equation (5) in (1), we get

$$h' = \frac{h}{\mu}\left(\frac{\cos^3\theta}{\cos^3\phi}\right) \qquad \ldots(6)$$

Further from equation (4), we get $\sin\phi = \dfrac{\sin\theta}{\mu}$

$$\Rightarrow \quad \cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\frac{\sin^2\theta}{\mu^2}}$$

$$\Rightarrow \quad \cos^3\phi = \left(1-\frac{\sin^2\theta}{\mu^2}\right)^{\frac{3}{2}} = \frac{(\mu^2-\sin^2\theta)^{\frac{3}{2}}}{\mu^3} \qquad \ldots(7)$$

Substituting equation (7) in (6), we get

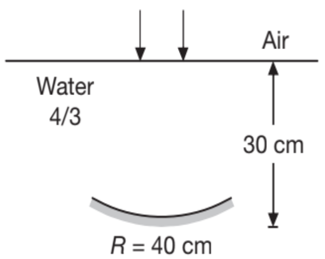
$$h' = \frac{\mu^2 h\cos^3\theta}{(\mu^2-\sin^2\theta)^{\frac{3}{4}}}$$

Test Your Concepts-III

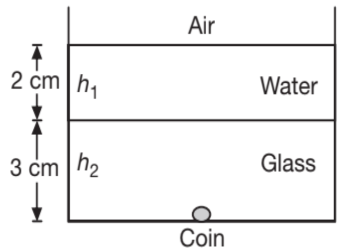
Based on General Refraction

(Solutions on page H.9)

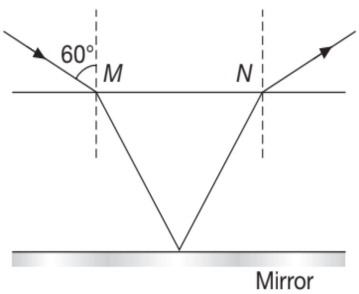
1. An object lies 100 cm inside water. It is viewed from air nearly normally. Find the apparent depth of the object.
2. A concave mirror is placed inside water with its shining surface upwards and principal axis vertical as shown. Rays are incident parallel to the principal axis of concave mirror. Find the position of final image.



3. A small object is placed on the principal axis of a concave spherical mirror of radius 20 cm at a distance of 30 cm. By how much will the position and size of the image alter, when a parallel-sided slab of glass of thickness 6 cm and refractive index 1.5 is introduced between the centre of curvature and the object? The parallel sides are perpendicular to the principal axis.
4. The velocity of light in air is 3×10^8 ms^{-1}. If yellow light of wavelength 6000 Å is passed from air to glass of refractive index 1.5, determine the velocity, the wavelength and the colour of light in glass.
5. A 2 cm thick layer of water covers a 3 cm thick glass slab. A coin is placed at the bottom of the slab and is being observed from the air side along the normal to the surface. Find the apparent position of the coin from the surface.



6. A plate with plane parallel faces having refractive index 1.8 rests on a plane mirror. A light ray is incident on the upper face of the plate at 60°. How far from the entry point will the ray emerge after reflection by the mirror of the plate is 6 cm thick?



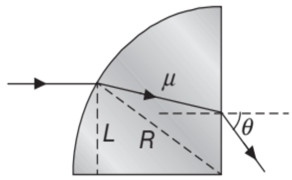
7. A pole 4 m high is driven into the bottom of a lake and happens to be 1 m above the water. Determine the length of the shadow of the pole at the bottom of the lake if the sunrays make an angle of 45° with the water surface. The refractive index of water is $\frac{4}{3}$.

8. A ray of light is refracted through a sphere whose material has a refractive index μ in such a way that it passes through the extremities of two radii which make an angle β with each other. prove that if α is the deviation of the ray caused by its passage through the sphere,

$$\cos\left(\frac{\beta-\alpha}{2}\right)=\mu\cos\left(\frac{\beta}{2}\right)$$

9. A vertical beam of light of cross-sectional radius r is incident symmetrically on the curved surface of a glass hemisphere $\left(\mu=\frac{3}{2}\right)$ of radius $2r$ placed with its base on a horizontal table. Find the radius of the luminous spot formed on the table.

10. A material having an index of refraction μ is surrounded by vacuum and is in the shape of a quarter circle of radius R. A light ray parallel to the base of the material is incident from the left at a distance of L above the base and emerges out of the material at an angle θ. Determine an expression for θ.



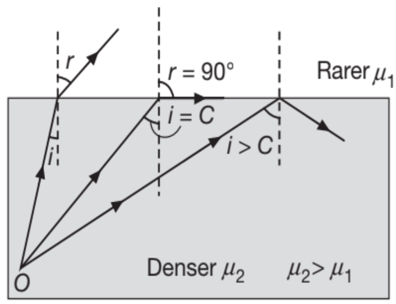
11. How much water should be filled in a container of height 21 cm so that it will appear half filled when viewed along normal to water surface. Take refractive index of water $\mu_W=\frac{4}{3}$.

12. A ray of light is incident on a glass slab at grazing incidence. The refractive index of the material of the slab is given by $\mu=\sqrt{1+y}$. If the thickness of the slab is d, determine the equation of the trajectory of the ray inside the slab and the coordinates of the point where the ray exits from the slab. Take the origin to be at the point of entry of the ray.

TOTAL INTERNAL REFLECTION (TIR)

When a ray of light goes from a denser to a rarer medium, it bends away from the normal. If the angle of incidence in the denser medium is increased the angle of refraction in the rarer medium also increases. At a particular angle of incidence in the denser medium (called as the Critical angle C), the angle of refraction in the rarer medium is 90° (i.e., the refracted ray grazes the interface). **This angle of refraction in the denser medium for which the refracted ray grazes the interface is called the critical angle for the pair of interface**.

Please note that for small angles of incidence, both reflection and refraction occur, however we shall be neglecting the reflection at the interface as most of the light is refracted. However, when $i>C$, no part of light is refracted and the entire light is reflected back to the denser medium itself. This phenomenon is called total internal reflection (TIR) and was first noted by Kepler in 1604.



Images formed by TIR are much brighter than those formed by the mirrors (or lenses). Some loss of intensity always takes place, when light is reflected from a mirror (or refracted through a lens).

CRITICAL ANGLE

According to Snell's Law, we have

$$\frac{\sin i}{\sin r} = {}^{d}\mu_r$$

$$\Rightarrow \quad \frac{\sin C}{\sin 90} = \frac{\mu_{\text{rarer}}}{\mu_{\text{denser}}}$$

$$\Rightarrow \quad \sin C = \frac{\mu_{\text{rarer}}}{\mu_{\text{denser}}} = \frac{1}{\mu_{\text{denser}}}$$

Thus $C = \sin^{-1}\left(\frac{\mu_{\text{rarer}}}{\mu_{\text{denser}}}\right) = \sin^{-1}\left(\frac{1}{\mu_{\text{denser}}}\right)$

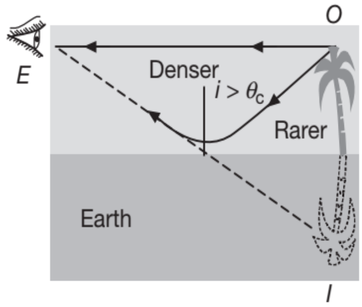
where μ_{denser} is the refractive index of the denser medium w.r.t. the rarer medium. The lesser the value of μ_{denser}, the greater is the critical angle C.

For a given pair of media, since μ depends on the wavelength of light the critical angle also depends on the wavelength. The greater the wavelength, the greater will be the critical angle.

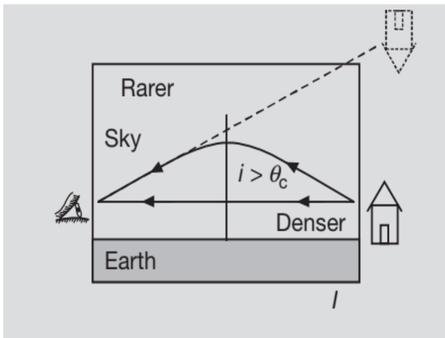
Media Pair	μ_{denser}	Critical angle $C = \sin^{-1}\left(\frac{1}{\mu_{\text{denser}}}\right)$
Water-Air	$\mu_d = \frac{\mu_w}{\mu_a} = \frac{4/3}{1} = \frac{4}{3}$	49°
Glass-Air	$\mu_d = \frac{\mu_g}{\mu_a} = \frac{3/2}{1} = \frac{3}{2}$	42°
Glass-Water	$\mu_d = \frac{\mu_g}{\mu_w} = \frac{3/2}{4/3} = \frac{9}{8}$	63°

EXAMPLES OF TOTAL INTERNAL REFLECTION

(a) Mirage: Mirage is an optical illusion observed in deserts and roads on a hot day. When the air near the ground is hotter (and hence rarer) than the air above, there occurs a continuous decrease of refractive index of air towards the ground.



(b) Looming: Similarly, in extremely cold regions (near polar regions), the refractive index decreases with height. Due to TIR (shown in figure), the image of a hut appears hanging in the air. This is called looming.

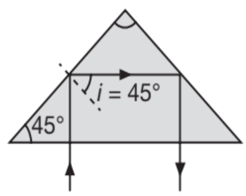


(c) The μ of diamond is 2.5, for which C is only $24°$. Diamonds are cut such that $i > C$, so TIR takes place again and again inside it. The light coming out from few meticulously cut surfaces makes it sparkle.

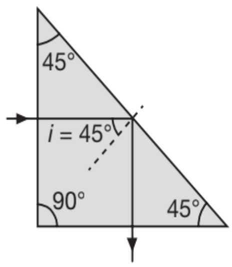
(d) Air bubbles in water shine due to TIR.

(e) The working of an optical fibre is due to multiple TIR inside it.

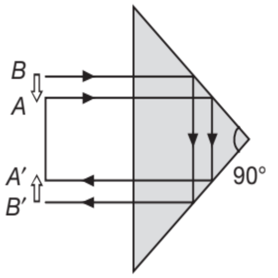
(f) Porro prisms used in periscopes or binoculars bend the ray due to TIR. Some examples are shown in figure.



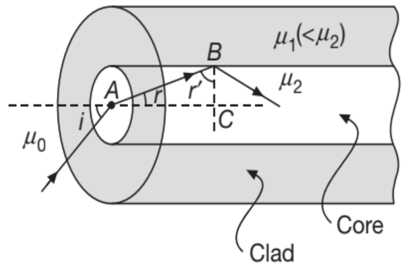
(a) Bending of rays by 180°



(b) Bending of rays by 90°



(c) Erecting of image



The ray of light enters into the core at an angle of incidence i as shown

From Snell's Law

$$\mu_0 \sin i = \mu_2 \sin r \qquad \ldots(1)$$

From ΔABC,

$$r = 90 - r' \qquad \ldots(2)$$

From equation (1) and (2)

$$\mu_0 \sin i = \mu_2 \sin(90 - r')$$

$$\Rightarrow \mu_0 \sin i = \mu_2 \cos r'$$

$$\Rightarrow \mu_0 \sin i = \mu_2 \sqrt{1 - \sin^2 r'}$$

$$\Rightarrow \mu_2^2 (1 - \sin^2 r') = \mu_0^2 \sin^2 i$$

$$\Rightarrow \sin^2 r' = \frac{\mu_2^2 - \mu_0^2 \sin^2 i}{\mu_2^2}$$

$$\Rightarrow \sin r' = \frac{\sqrt{\mu_2^2 - \mu_0^2 \sin^2 i}}{\mu_2}$$

Now for total internal reflection at B

$r' \geq C$, where C is critical angle for a ray coming from core to clad

$$\Rightarrow \sin r' \geq \sin C$$

$$\Rightarrow \sin r' \geq \frac{\mu_1}{\mu_2}$$

$$\Rightarrow \frac{\sqrt{\mu_2^2 - \mu_0^2 \sin^2 i}}{\mu_2} \geq \frac{\mu_1}{\mu_2}$$

$$\Rightarrow \mu_2^2 - \mu_0^2 \sin^2 i \geq \mu_1^2$$

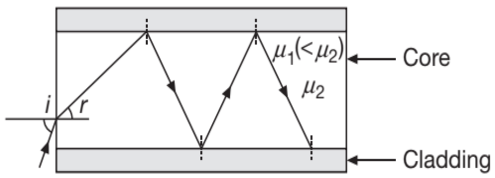
$$\Rightarrow \sin^2 i \leq \frac{\mu_2^2 - \mu_1^2}{\mu_0^2}$$

$$\Rightarrow \sin i \leq \frac{\sqrt{\mu_2^2 - \mu_1^2}}{\mu_0}$$

OPTICAL FIBRE

An optical fibre is a transmission medium to carry the optical signal without any appreciable loss. It is a device based on total internal reflection by which signals can be transmitted from one location to another. The optical fibre works even if it is bent or twisted.

The structure of optical fibre consists of a core surrounded by a cladding.



The core is denser medium of refractive index μ_2 and cladding is relatively rarer medium of refractive index μ_1 such that $\mu_1 < \mu_2$. The light incident at one dent on the interface between the core and cladding at an angle greater than the critical angle is continuously reflected in the core.

It is a thin fibre of plastic or specially coated glass in which light enters at one end and leaves it at the other end suffering a number of total internal reflections with little loss of energy. The optical fibre works even if it is bent or twisted.

The thickness of the fibre is of the order of human hair (10^{-6} m).

ANGLE OF ACCEPTANCE

Maximum angle at which the ray should enter into the core for the transmission through optical fibre is called angle of acceptance (i_{max}). Suppose the surrounding medium has refractive index μ_0 and core and clad have refractive indices μ_2 and μ_1 respectively $(\mu_2 > \mu_1)$.

$$\Rightarrow \quad i \le \sin^{-1}\left(\frac{\sqrt{\mu_2^2-\mu_1^2}}{\mu_0}\right)$$

$$\Rightarrow \quad i_{max} = \sin^{-1}\left(\frac{\sqrt{\mu_2^2-\mu_1^2}}{\mu_0}\right)$$

FIELD OF VISION OF A FISH

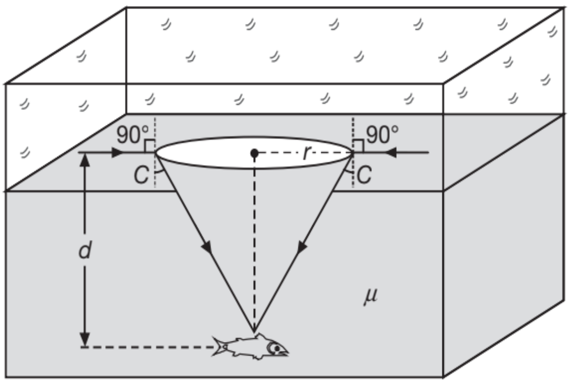
A fish inside a pond does not see the outside world through the entire surface of water. The light from outside can reach the fish only through a circular patch, which forms a cone of half angle equal to the critical angle.

If r is the radius of the circular patch, d is the depth of the fish and μ is the refractive index of water, then

$$r = d\tan C = d\frac{\sin C}{\cos C} = d\frac{\sin C}{\sqrt{1-\sin^2 C}}$$

Since, $\sin C = \frac{1}{\mu}$

$$\Rightarrow \quad r = \frac{d}{\sqrt{\mu^2-1}}$$

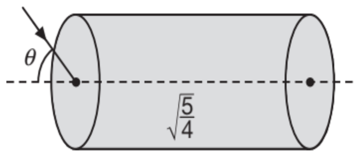


Fish in glass tank

Similarly, if a source of light is kept in a pond, its light will come out only through a circular region. For any incident angle i greater than C, the light will be totally reflected back into the water, making corresponding region on the surface of water appear dark.

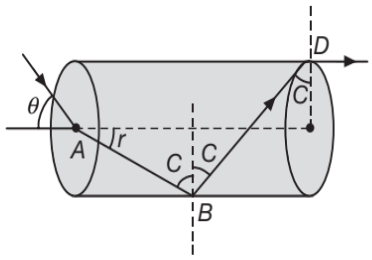
ILLUSTRATION 49

Light is incident making an angle θ with the axis of a transparent cylindrical fiber of refractive index $n = \sqrt{\frac{5}{4}}$ as shown in figure. Determine the maximum value of θ so that the light entering the cylinder does not come out of the curved surface.



SOLUTION

The ray of light is incident at A and it just gets reflected totally at B. Therefore incident angle at B is equal to the critical angle given as $C = \sin^{-1}\left(\frac{1}{n}\right)$



Snell's Law of refraction at A gives

$$\frac{\sin\theta}{\sin r} = n$$

$$\Rightarrow \quad \sin r = \frac{\sin\theta}{n} \qquad \ldots(1)$$

Since $r + C = 90°$

$$\Rightarrow \quad \sin r = \sin(90° - C) = \cos C$$

For a ray not to come through the curved surface,

$$r \le 90 - C$$

$$\Rightarrow \quad \sin r \le \sqrt{1-\sin^2 C} \le \sqrt{1-\frac{1}{n^2}} \qquad \ldots(2)$$

Eliminating $\sin r$ from (1) and (2), we get

$$\frac{\sin\theta}{n} \le \sqrt{1-\frac{1}{n^2}}$$

$$\Rightarrow \quad \sin\theta \le \sqrt{n^2-1}$$

$$\Rightarrow \quad \sin^2\theta \le 1.25 - 1$$

$\Rightarrow \quad \sin^2\theta \le 0.25$

$\Rightarrow \quad \sin\theta \le \frac{1}{2}$

$\Rightarrow \quad \theta \le 30^\circ$

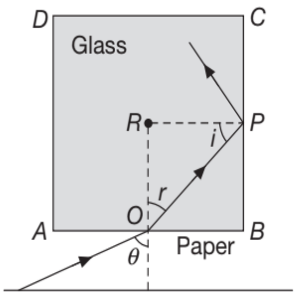
$\Rightarrow \quad \theta_{max} = 30^\circ$

ILLUSTRATION 50

A rectangular block of glass is placed on a printed page lying on a horizontal surface. Find the minimum value of the refractive index of glass for which the letters on the page are not visible from any of the vertical faces of the block.

SOLUTION

Light will not emerge out from the vertical face BC, when



$i >$ Critical Angle (C)

$\Rightarrow \quad \sin i > \sin C$

$\Rightarrow \quad \sin i > \frac{1}{\mu} \qquad \left\{\because \sin C = \frac{1}{\mu}\right\}$

Applying Snell's Law at O, we get

$1\sin\theta = \mu\sin r$

$\Rightarrow \quad \sin\theta = \mu\sin(90^\circ - i) = \mu\cos i$

$\Rightarrow \quad \cos i = \frac{\sin\theta}{\mu}$

$\Rightarrow \quad \sin i = \sqrt{1-\cos^2 i} = \sqrt{1-\frac{\sin^2\theta}{\mu^2}}$

Therefore, the condition for no light to emerge from vertical face BC becomes,

$$\sqrt{\frac{\mu^2 - \sin^2\theta}{\mu^2}} > \frac{1}{\mu}$$

$\Rightarrow \quad \mu^2 > 1 + \sin^2\theta$

$\Rightarrow \quad \mu^2 > 1 + 1 \quad \{\because$ maximum value of θ can be $90^\circ\}$

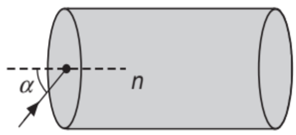
$\Rightarrow \quad \mu > \sqrt{2}$

So, the minimum value of refractive index is

$$\mu_{min} = \sqrt{2}$$

ILLUSTRATION 51

Light is incident at an angle α on one planar end of a transparent cylindrical rod of refractive index n. Determine the least value of n so that the light entering the rod does not emerge from the curved surface of the rod irrespective of the value of α.

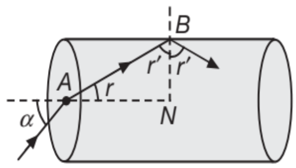


SOLUTION

Since, $\sin C = \frac{1}{n}$

For TIR at B, $(r')_{min} > C$

In triangle ABN, $r' + r + 90^\circ = 180^\circ$



$\Rightarrow \quad r' = 90 - r$

$\Rightarrow \quad (r')_{min} = 90^\circ - (r)_{max}$

and $\quad n = \frac{\sin(i)_{max}}{\sin(r)_{max}} = \frac{\sin 90^\circ}{\sin(r)_{max}} \left(i_{max} = 90^\circ\right)$

Then, $\sin(r)_{max} = \frac{1}{n} = \sin C$

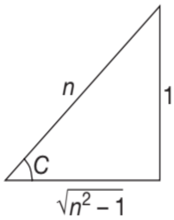
$(r)_{max} = C$

$\Rightarrow \quad (r')_{min} = (90^\circ - C)$

Now, if minimum value of r' i.e., $90^\circ - \theta_c$ is greater than θ_c, then obviously all values of r' will be greater than θ_c i.e., total internal reflection will take place at face AB in all conditions. Therefore, the necessary condition is

$$(r')_{min} > C$$

$$\Rightarrow \quad (90° - C) > C$$



$$\Rightarrow \quad \sin(90° - C) > \sin C$$

$$\Rightarrow \quad \cot C > \sin C$$

$$\Rightarrow \quad \cos C > 1$$

$$\Rightarrow \quad \sqrt{n^2 - 1} > 1$$

$$\Rightarrow \quad n^2 > 2$$

$$\Rightarrow \quad n > \sqrt{2}$$

Therefore, minimum value of n is $\sqrt{2}$

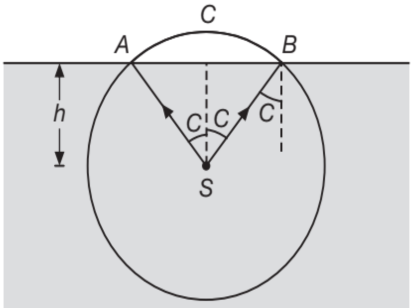
ILLUSTRATION 52

A point source of light is placed at a distance h below the surface of a large and deep lake. Show that the fraction f of light that escapes directly from water surface is independent of h and is given by,

$$f = \frac{\left[1 - \sqrt{1 - \frac{1}{\mu^2}}\right]}{2}.$$

SOLUTION

Due to TIR, light will be reflected back into the water for $i > C$. So only that portion of incident light will escape which passes through the cone of angle $\theta = 2C$.



So, the fraction of light escaping is given by

$$f = \frac{\text{Area of Surface } ACB}{\text{Total Area of Sphere}}$$

$$\Rightarrow \quad f = \frac{2\pi R^2(1 - \cos C)}{4\pi R^2} = \frac{1 - \cos C}{2}$$

Now, as f depends on C, which depends only on μ, hence f is independent of h.

Since, we know that

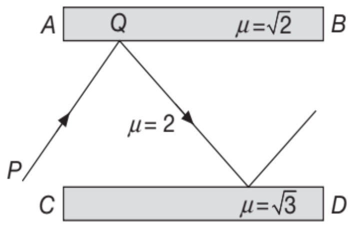
$$\sin C = \frac{1}{\mu}$$

$$\Rightarrow \quad \cos C = \frac{\sqrt{\mu^2 - 1}}{\mu} = \sqrt{1 - \frac{1}{\mu^2}}$$

$$\Rightarrow \quad f = \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{\mu^2}}\right)$$

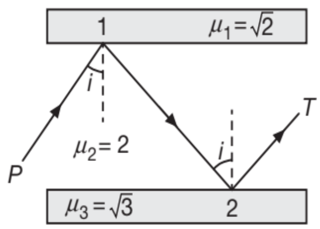
ILLUSTRATION 53

AB and CD are two slabs. The medium between the slabs has refractive index 2. Find the minimum angle of incidence of Q, so that the ray is totally reflected by both the slabs.



SOLUTION

Let the critical angles at 1 and 2 be C_1 and C_2 respectively. Then



$$C_1 = \sin^{-1}\left(\frac{\mu_1}{\mu_2}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

and $$C_2 = \sin^{-1}\left(\frac{\mu_3}{\mu_2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$$

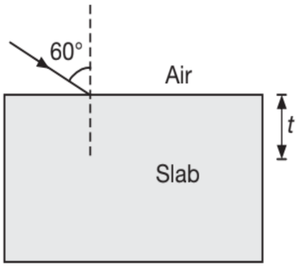
For TIR, $i > C_2$

Therefore, minimum angle of incidence, for total internal reflection to take place on both slabs must be $60°$.

$$i_{min} = 60°$$

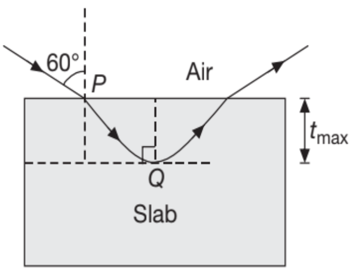
ILLUSTRATION 54

A ray of light enters into a glass slab from air as shown in figure. If refractive of glass slab varies with t, the thickness of the slab measured from the top as $\mu = A - Bt$ where A and B are constants. Find the maximum depth travelled by ray in the slab. Assume thickness of slab to be sufficiently large.



SOLUTION

The path of ray is curved as shown in figure. As it travels successively into denser layers, it bends away from normal and TIR takes place at depth where angle of incidence approaches $\frac{\pi}{2}$.



Applying Snell's Law at interfaces P and Q, we get

$$1\sin(60°) = \mu_B \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = (A - Bt_{max})$$

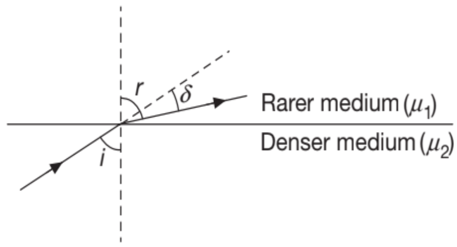
$$\Rightarrow \quad t_{max} = \frac{1}{B}\left(A - \frac{\sqrt{3}}{2}\right)$$

ILLUSTRATION 55

Plot the deviation (δ) versus the angle of incidence (i) graph for a ray travelling from denser to rarer medium.

SOLUTION

CASE-1: When angle of incidence (i) is less than critical angle C i.e., $i < C$



Since, $\delta = \text{Deviation} = r - i$ ...(1)

From Snell's Law, we get

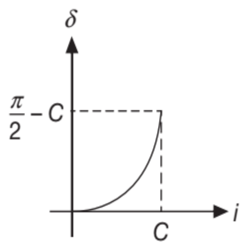
$$\mu_1 \sin i = \mu_2 \sin r$$

$$\Rightarrow \quad r = \sin^{-1}\left(\frac{\sin i}{\frac{1}{\mu_2}}\right)$$

Substituting in equation (1), we get

$$\delta = \sin^{-1}\left(\frac{\sin i}{\frac{1}{\mu_2}}\right) - i$$

This is a non-linear function and graph is given below

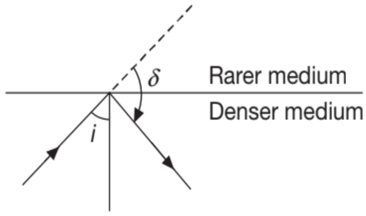


Deviation versus angle of incidence graph when TIR is not taking place.

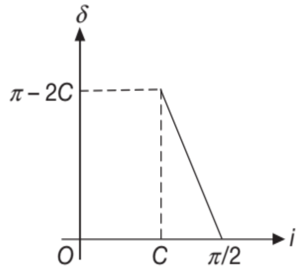
CASE-2: When the angle of incidence i is greater than the critical angle C, i.e., $i > C$

In this case TIR will take place as shown, so deviation is

$$\delta = \pi - 2i \qquad \text{...(2)}$$



This is a linear function and so the graph is given below



Deviation versus angle of incidence graph when TIR is taking place

Conceptual Note(s)

When ray is travelling from rarer to denser medium then deviation is given by

$$\delta = i - \sin^{-1}\left({}^{1}\mu_2 \sin i\right)$$

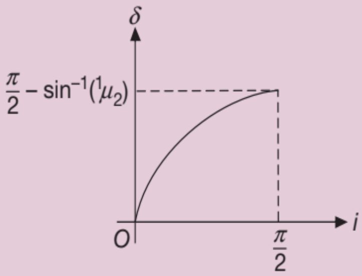
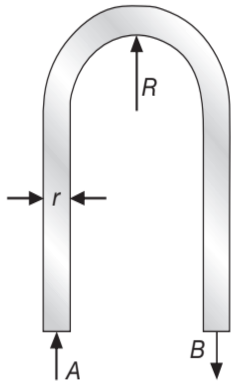


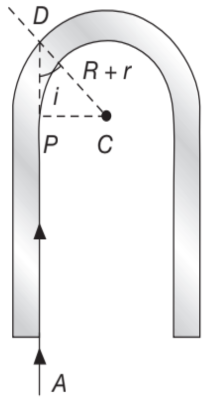
ILLUSTRATION 56

Find the maximum value of $\frac{r}{R}$, so that the beam of light incident normally at the face A of a U shaped glass tube emerges through B as shown in the figure. The refractive index of glass is $\mu = \frac{3}{2}$.



SOLUTION

Incident angle i is least for ray AP and this angle should be greater than the critical angle C



i.e., $i > C$

$\Rightarrow \quad \sin i > \sin C$

$\Rightarrow \quad \frac{R}{R+r} > \frac{1}{\mu}$

$\Rightarrow \quad \frac{R}{R+r} > \frac{2}{3}$

$3R > 2R + 2r$

$\Rightarrow \quad R > 2r$

$\Rightarrow \quad \frac{r}{R} < \frac{1}{2}$

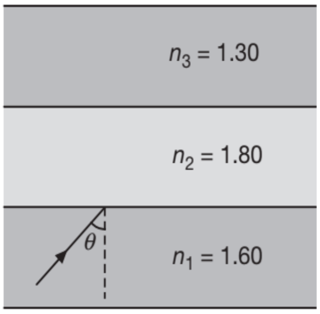
Hence, $\left(\frac{r}{R}\right)_{max} = \frac{1}{2}$

Test Your Concepts-IV

Based on Total Internal Reflection (TIR)

(Solutions on page H.12)

1. Light refracts from medium 1 into a thin layer of medium 2, crosses that layer and then is incident at the critical angle on the interface between media 2 and 3 as shown in figure.



(a) Find the angle θ.

(b) If θ is decreased, will the light be refracted to medium 3?

2. A container contains water upto a height of 20 cm and there is a point source of light at the centre of the bottom of the container. A rubber ring of radius a floats centrally on the water. The ceiling of the room is 2 cm above the water surface.

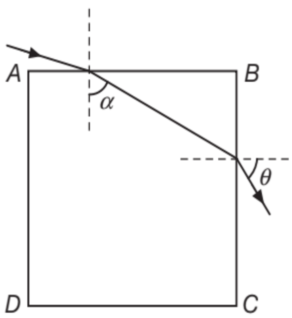
(a) Find the radius of the shadow of the ring formed on the ceiling if $a = 15$ cm.

(b) Find the maximum value of a for which the shadow of the ring is formed on the ceiling.

Refractive index of water $= \dfrac{4}{3}$.

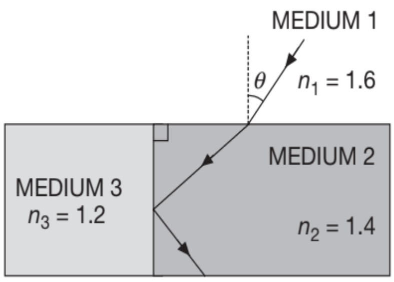
3. $ABCD$ is the plane of glass cube of refractive index μ. A horizontal beam of light enters the face AB at the grazing incidence.

(a) Show that the angle θ which any ray emerging from BC would make with normal to BC is given by $\sin\theta = \cot\alpha$ where α is the critical angle.



(b) What is the greatest value that the refraction index of glass may have if any of the light is to emerge from BC?

4. In figure, light refracts into material 2, crosses that material and is then incident at the critical angle on the interface between materials 2 and 3.

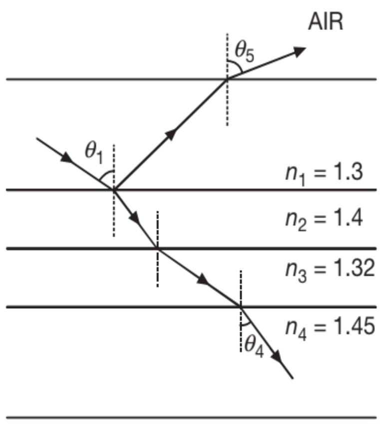


(a) What is angle θ?

(b) If θ is increased, is there refraction of light into material 3?

5. An isotropic point source is placed at a depth h below the water surface. An opaque disc capable of floating on water surface is placed on the surface of water so that the bulb is not visible from the surface. Find the minimum radius of the disc for the bulb not to be visible. Take refractive index of water $= \mu$.

6. In figure, light begins from medium of refractive index $n_1 = 1.3$, undergoes three refractions as it heads downward and a reflection and then a refraction to reach the air. The initial angle $\theta_1 = 30°$. Find the values of the angles



(a) θ_5 and

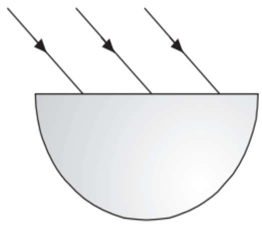
(b) θ_4.

7. Determine the maximum angle θ for which the light ray incident on the end of pipe shown in figure are subject to TIR along the walls of the pipe. Assume that the pipe has an index of refraction of 1.36 and the outside medium is air.

8. A point source of light S is placed at the bottom of a vessel containing a liquid of refractive index $\frac{5}{3}$. A person is viewing the source from above the surface. There is an opaque disc of radius 1 cm floating on the surface. The center of the disc lies vertically above the source S. The liquid from the vessel is gradually drained out through a tap. What is the maximum height of the liquid for which the source cannot at all seen from above.

9. A rectangular glass block is placed on top of a sheet of paper on which there is a small cross. When the paper is soaked in alcohol and a sodium lamp is placed opposite to one vertical of the block the cross can be seen through the opposite vertical face up to a point where the angle of emergence of the light is 30°. If the refractive index of the glass is 1.5, find the refractive index of alcohol. Why can't the black cross be seen through the face when the paper is dry.

10. Rays of light fall on the plane surface of a half cylinder at an angle 45° in the plane perpendicular to the axis (see figure). Refractive index of glass is $\sqrt{2}$. Discuss the condition that the rays do not suffer total internal reflection.

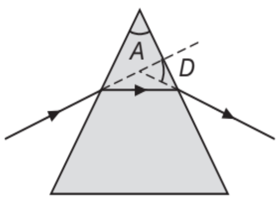


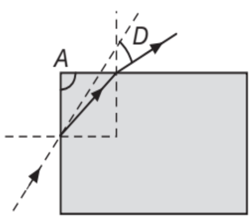
PRISM

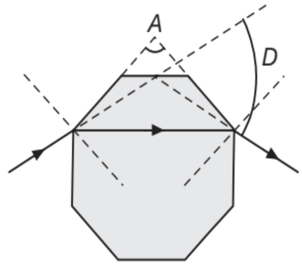
Prism is a transparent medium bounded by any number of surfaces in such a way that the surface on which light is incident and the surface from which light emerges are **plane** and **non-parallel**.

Refracting angle of prism, or simply the **angle of prism** is the angle between the faces on which light is incident and from which light emerges. In all the prisms shown in figure above, angle A is the angle of prism.

Angle of deviation (D) is the angle between the incident ray and the emergent ray. Sometimes the angle of deviation is also denoted by δ.



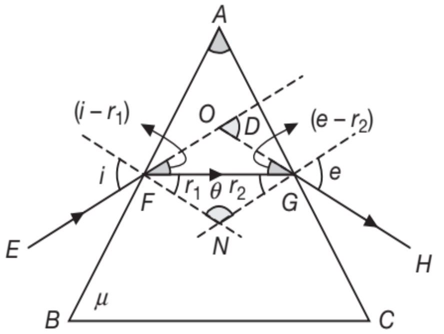




Please note that, for a glass-slab, the angle of prism is zero, and the incident ray emerges parallel to itself, i.e., there is no deviation. If μ of the prism material is same as that of its surroundings, no refraction takes place and light passes through undeviated.

REFRACTION THROUGH A PRISM

Consider a monochromatic ray EF to be incident on the face AB of prism ABC of refracting angle A at angle of incidence i. The ray is refracted along FG, r_1 being angle of refraction. The ray FG is incident on the face AC at angle of incidence r_2 and is refracted in air along GH. Thus GH is the emergent ray and e is the angle of emergence. The angle between incident ray EF and emergent ray GH (produced backwards) is called angle of deviation D.



In triangle OFG,

$$D=(i-r_1)+(e-r_2)$$

$$\Rightarrow \quad D=(i+e)-(r_1+r_2) \qquad \ldots(1)$$

Also in quadrilateral $AFNG$,

$$A+90^\circ+\theta+90^\circ=360^\circ$$

$$\Rightarrow \quad A+\theta=180^\circ \qquad \ldots(2)$$

And in triangle FGN,

$$r_1+r_2+\theta=180^\circ \qquad \ldots(3)$$

Comparing equations (2) and (3), we get

$$A=r_1+r_2 \qquad \ldots(4)$$

From (1), we get

$$D=i+e-A$$

$$\Rightarrow \quad i+e=A+D \qquad \ldots(5)$$

If μ is the refractive index of material of prism, then from Snell's Law

$$\mu=\frac{\sin i}{\sin r_1}=\frac{\sin e}{\sin r_2} \qquad \ldots(6)$$

Since, $D=i+e-A$, where $\sin e=\mu\sin r_2=\mu\sin(A-r_1)$

$$\Rightarrow \quad \sin e=\mu(\sin A\cos r_1-\sin r_1\cos A)\text{, where}$$

$$\sin r_1=\frac{\sin i}{\mu}$$

$$\Rightarrow \quad \sin e=\sin A\sqrt{\mu^2-\sin^2 i}-\sin i\cos A$$

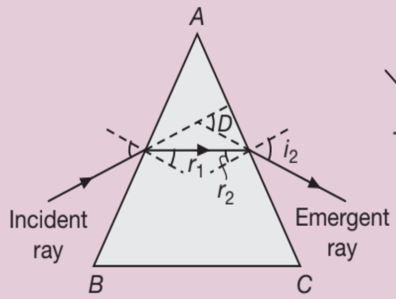
$$\Rightarrow \quad D=i+\sin^{-1}\left(\sin A\sqrt{\mu^2-\sin^2 i}-\sin i\cos A\right)$$

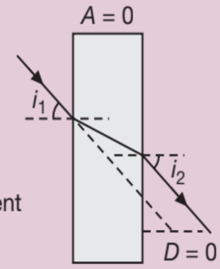
For a prism with small refracting angle, we have

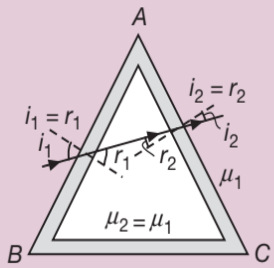
$$D=(\mu-1)A\ .$$

Conceptual Note(s)

(a) **Angle of deviation** (D) means the angle between emergent and incident rays i.e., the angle through which incident ray turns in passing through a prism. It is represented by D and is shown in figure.







(b) If the faces of a prism on which light is incident and from which it emerges becomes parallel (as in figure), angle of prism will be zero and as incident ray will emerge parallel to itself, deviation will also be zero i.e., the prism will act as a slab.

(c) If μ of the material of the prism becomes equal to that of surroundings, no refraction at its faces will take place and light will pass through it undeviated. So, deviation is zero.
i.e., $D = 0$

CONDITION OF NO EMERGENCE

The light entering the prism at surface AB, will not be able to come out from the surface AC, if TIR takes place at this surface. For any angle of incidence, this condition will be satisfied, provided we have at surface AC,

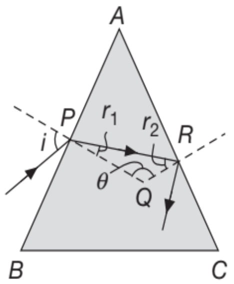
$$(r_2)_{min} > C \quad \ldots(1)$$

Since, $r_1 + r_2 = A$

$$\Rightarrow \quad r_2 = A - r_1$$

So, r_2 is minimum, when r_1 is maximum, because A is constant.

$$\Rightarrow \quad (r_2)_{min} = A - (r_1)_{max} \quad \ldots(2)$$



But $(r_1)_{max}$ is possible when $i = i_{max} = 90°$ i.e., incident ray grazes the interface AB.
Now, applying Snell's Law at AB,

$$1 \times \sin i = \mu \sin r_1$$

$$\Rightarrow \quad \sin(90°) = \mu \sin r_1$$

$$\Rightarrow \quad r_1 = \sin^{-1}\left(\frac{1}{\mu}\right)$$

$$\Rightarrow \quad r_1 = C \quad \ldots(3)$$

From equations (1), (2) and (3), we get

$$A - C > C \quad \ldots(4)$$

Therefore, the condition becomes

$$A > 2C \text{ where } \sin C = \frac{1}{\mu}$$

$$\Rightarrow \quad \sin\left(\frac{A}{2}\right) > \sin C$$

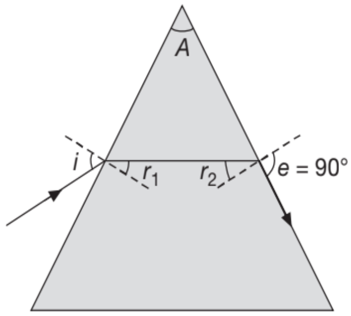
$$\Rightarrow \quad \sin\left(\frac{A}{2}\right) > \frac{1}{\mu}$$

$$\Rightarrow \quad \mu > \text{cosec}\left(\frac{A}{2}\right)$$

Thus, a ray of light will not emerge out of a prism (whatever be the angle of incidence) if $A > 2C$, that is, if $\mu > \text{cosec}\left(\frac{A}{2}\right)$.

CONDITION FOR GRAZING EMERGENCE

A ray can enter a prism in such a way that the angle of emergence, $e = 90°$, as shown in the figure.



We can determine the angle of incidence i for such grazing emergence. We should have

$$r_2 = C$$

Since, for a prism, $r_1 + r_2 = A$

$$\Rightarrow \quad r_1 = A - r_2 = A - C$$

Using Snell's Law,

$$1 \sin i = \mu \sin r_1 = \mu \sin(A - C)$$

$$\Rightarrow \quad \sin i = \mu(\sin A \cos C - \cos A \sin C)$$

$$\Rightarrow \quad \sin i = \mu\left[(\sin A)\sqrt{1 - \sin^2 C} - (\cos A)(\sin C)\right]$$

$$\Rightarrow \quad \sin i = \mu\left[\sin A\sqrt{1 - \frac{1}{\mu^2}} - \cos A\left(\frac{1}{\mu}\right)\right]$$

$$\Rightarrow \quad \sin i = \sin A\sqrt{\mu^2 - 1} - \cos A$$

$$\Rightarrow \quad i = \sin^{-1}\left(\sin A\sqrt{\mu^2 - 1} - \cos A\right)$$

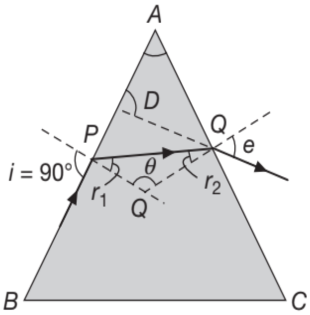
The light will emerge out of the prism only if the angle of incidence i is greater than the above value.

MAXIMUM DEVIATION

The angle of deviation D is maximum when the angle i is maximum, i.e., $i = 90°$.

$$D_{max} = (i + e) - A = (90° + e) - A$$

Under such conditions of grazing incidence, $r_1 = C$



And at the second surface,

$$\mu \sin r_2 = 1 \sin e$$

$$\Rightarrow \quad \sin e = \mu \sin r_2$$

Since $r_1 + r_2 = A$

$$\Rightarrow \quad \sin e = \mu \sin(A - r_1) = \mu \sin(A - C)$$

$$\Rightarrow \quad e = \sin^{-1}(\mu \sin(A - C)) = \sin^{-1}\left[\frac{\sin(A - C)}{\sin C}\right]$$

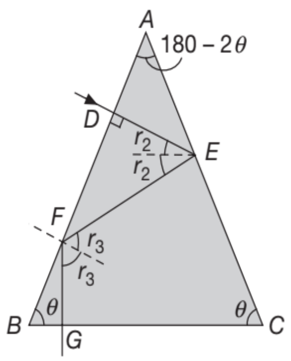
ILLUSTRATION 57

An isosceles glass prism has one of its faces coated with silver. A ray of light is incident normally on the other face (which is equal to the silvered face). The ray of light is reflected twice on the same sized faces and then emerges through the base of the prism perpendicularly. Find angles of prism.

SOLUTION

As the ray is incident normally at the face AB, so

$$r_1 = 0$$



Since, we know that $r_1 + r_2 = A$, so we get

$$r_2 = A = 180° - 2\theta \qquad \ldots(1)$$

Now, $\angle DFE = 180° - 90° - 2r_2$

$$\Rightarrow \quad \angle DFE = 180° - 90° - 360° + 4\theta \quad \{\because r_2 = 180 - 2\theta\}$$

$$\Rightarrow \quad \angle DFE = 4\theta - 270° \qquad \ldots(2)$$

Since, $r_3 = 90° - \angle DFE$ ...(3)

$$\Rightarrow \quad r_3 = 360° - 4\theta$$

Again $\angle BFG = 90° - \theta = 90° - r_3$

$$\Rightarrow \quad r_3 = \theta \qquad \ldots(4)$$

From equations (3) and (4), we get

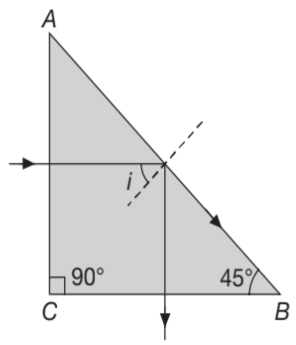
$$5\theta = 360°$$

$$\Rightarrow \quad \theta = 72° \text{ and } 180° - 2\theta = 36°$$

So, the angles of prism are $72°$, $72°$ and $36°$.

ILLUSTRATION 58

A ray of light incident normally on one of the faces of a right angle isosceles prism is found to be totally reflected as shown. What is the minimum value of the refractive index of the material of the prism? When prism is immersed in water $(\mu = 1.33)$ trace the path of the emergent ray for the same incident ray, indicating the values of all the angles.



SOLUTION

For total internal reflection to take place at surface AB, we have

$$i > C$$

$$\Rightarrow \quad \sin i > \sin C$$

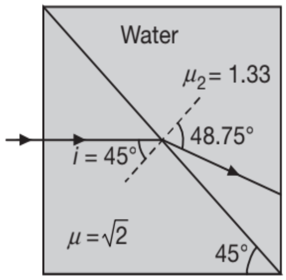
Since, $\sin C = \dfrac{1}{\mu}$

$$\Rightarrow \quad \sin 45° > \left(\frac{1}{\mu}\right)$$

$$\Rightarrow \quad \mu > \sqrt{2}$$

$$\Rightarrow \quad \mu_{min} = \sqrt{2}$$

When the prism is immersed in water, the boundary AB now separates glass from water.



$$\Rightarrow \quad C = \sin^{-1}\left(\frac{\mu_{\text{rarer}}}{\mu_{\text{denser}}}\right) = \sin^{-1}\left(\frac{1.33}{\sqrt{2}}\right)$$

$$\Rightarrow \quad C = 70.12°$$

Since $i = 45°$ and also, we observe that $i < C$

Hence, TIR will not take place.

From Snell's Law, we get

$$\mu_1 \sin i = \mu_2 \sin r$$

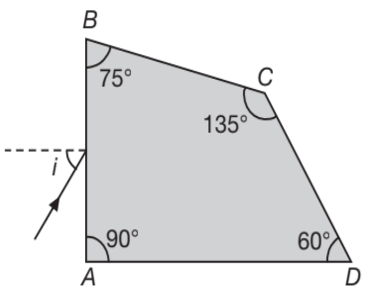
$$\Rightarrow \quad \sqrt{2}\sin(45°) = 1.33 \sin r$$

$$\Rightarrow \quad \sin r = \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)}{1.33} = 0.752$$

$$\Rightarrow \quad r = \sin^{-1}(0.752) = 48.75°$$

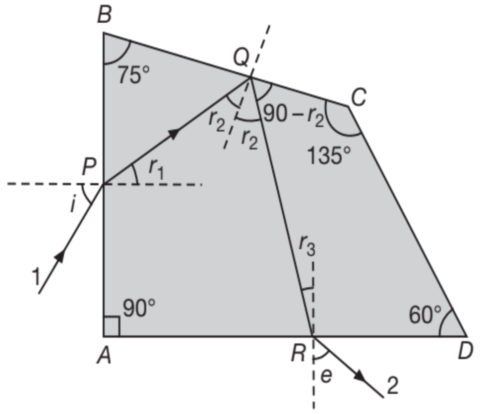
ILLUSTRATION 59

A ray of light is falling on face AB of a tetrahedral of refractive index μ at angle of incidence i. The ray after getting internally reflected on face BC emerges from AD perpendicularly to the incident beam. Find the range of μ and i.

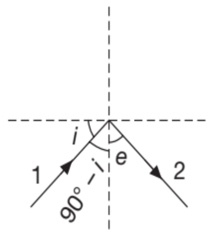


SOLUTION

Since, $r_1 + r_2 = \angle B = 75°$ …(1)



From figure, we observe that $e = i$, because 1 and 2 are perpendicular



(Ray 1) ⊥ (Ray 2)

In quadrilateral $QCDR$, we have

$$(90° - r_2) + (90° + r_3) + 60° + 135° = 360°$$

$$r_3 = 360° - 60° - 135° - (90° - r_2) - 90° \quad \text{…(2)}$$

$$r_3 = r_2 - 15°$$

Further, $\mu = \dfrac{\sin i}{\sin r_1} = \dfrac{\sin e}{\sin r_3}$

$$\Rightarrow \quad r_3 = r_1 \quad \{\text{because } i = e\} \quad \text{…(3)}$$

Solving equations (1), (2) and (3), we get

$$r_2 = 45° \text{ and } r_1 = 30°$$

Now, for TIR (total internal reflection) to take place at the face BC; we have

$$r_2 > C$$

$$\Rightarrow \quad \sin r_2 > \sin C$$

$$\Rightarrow \quad \sin(45°) > \frac{1}{\mu}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} > \frac{1}{\mu} \quad \left\{\because \sin C = \frac{1}{\mu}\right\}$$

$\Rightarrow \quad \mu > \sqrt{2}$

Further, we have $\mu = \dfrac{\sin i}{\sin r_1} = \dfrac{\sin i}{\sin(30^\circ)} = 2\sin i$

Since, $\mu > \sqrt{2}$

$\Rightarrow \quad 2\sin i > \sqrt{2}$

$\Rightarrow \quad \sin i > \dfrac{1}{\sqrt{2}}$

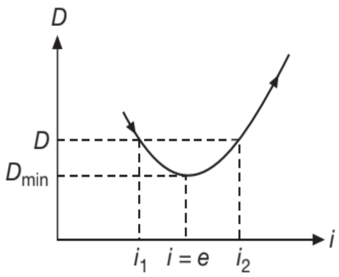
$\Rightarrow \quad i > 45^\circ$

MINIMUM DEVIATION

As discussed and derived already we know that the angle of deviation D is given by

$$D = i + \sin^{-1}\left(\sin A\sqrt{\mu^2 - \sin^2 i} - \sin i\cos A\right)$$

The above function of deviation D, when plotted against i the angle of incidence gives a plot that is unsymmetrical as shown in the figure. It must be observed that for two different angles of incidence, we have the same deviation.



It is found that D is minimum when $i = e$. Thus,

$$D_{\min} = (i + e) - A = 2i - A$$

Using Snell's Law,

$$1\sin i = \mu \sin r_1$$

and $\mu \sin r_2 = 1\sin e = \sin i$

$\Rightarrow \quad \mu \sin r_1 = \mu \sin r_2$

$\Rightarrow \quad r_1 = r_2 = r$ (say)

Since, $r_1 + r_2 = A$

$\Rightarrow \quad r = \dfrac{A}{2}$

$$\Rightarrow \quad \mu = \frac{\sin i}{\sin r_1} = \frac{\sin\left(\dfrac{A + D_{\min}}{2}\right)}{\sin\left(\dfrac{A}{2}\right)}$$

Note that if the prism is equilateral or isosceles, then the ray inside the prism is parallel to its base.

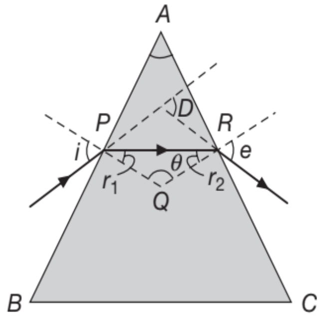


ILLUSTRATION 60

The angle of minimum deviation for a glass prism with refractive index $\sqrt{3}$ equals the refracting angle of the prism. What is the angle of the prism?

SOLUTION

Since we know that

$$\mu = \frac{\sin\left(\dfrac{A + \delta_m}{2}\right)}{\sin\left(\dfrac{A}{2}\right)}$$

Since $\delta_{\text{minimum}} = \delta_{\min} = A$, so we get

$$\sqrt{3} = \frac{\sin A}{\sin\left(\dfrac{A}{2}\right)}$$

$\Rightarrow \quad \sqrt{3} = 2\cos\left(\dfrac{A}{2}\right)$

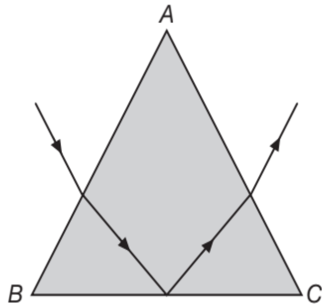
$\Rightarrow \quad \cos\left(\dfrac{A}{2}\right) = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad \dfrac{A}{2} = 30^\circ$

$\Rightarrow \quad A = 60^\circ$

ILLUSTRATION 61

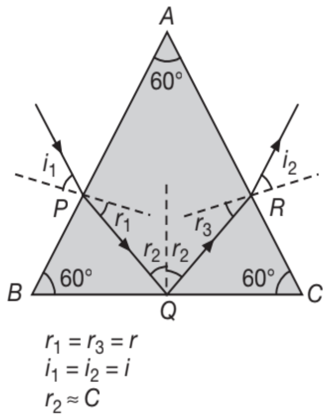
The path of a ray of light passing through an equilateral glass prism ABC is shown in the figure.



The ray of light is incident on face BC at an angle just greater than the critical angle for total internal reflection to take place. The total angle of deviation after the refraction at face AC is $120°$. Calculate the refractive index of the glass.

SOLUTION

The ray diagram is drawn for the sake of convenience.



Since, $r_1 + r_2 = r_2 + r_3 = 60°$

$\Rightarrow \quad r_1 = r_3 = r$ (say)

Similarly by symmetry, we have $i_1 = i_2 = i$ (say)

Also, $r_2 \approx C$

$\Rightarrow \quad r = 60° - C$

Given, $\delta_{\text{Total}} = 120°$

$\Rightarrow \quad \delta_P + \delta_Q + \delta_R = 120°$

$\Rightarrow \quad (i - r) + (180 - 2C) + (i - r) = 120°$

$\Rightarrow \quad 2i - 2(60° - C) + 180° - 2C = 120°$

$\Rightarrow \quad 2i = 60°$

$\Rightarrow \quad i = 30°$

Now, according to Snell's Law, we have

$$\mu = \frac{\sin i_1}{\sin r_1} = \frac{\sin i}{\sin r}$$

But $r_1 + r_2 = 60°$

$\Rightarrow \quad r + C = 60°$

$\Rightarrow \quad r = 60° - C$

$$\Rightarrow \quad \mu = \frac{\sin(30°)}{\sin(60° - C)}$$

$$\Rightarrow \quad \mu = \frac{0.5}{\frac{\sqrt{3}}{2}\cos C - \frac{1}{2}\sin C}$$

But $\sin C = \dfrac{1}{\mu}$

$$\Rightarrow \quad \mu = \frac{0.5}{0.5\left(\sqrt{3}\sqrt{1 - \frac{1}{\mu^2}} - \frac{1}{\mu}\right)}$$

$$\Rightarrow \quad \mu = \frac{\mu}{\sqrt{3}\sqrt{\mu^2 - 1} - 1}$$

$\Rightarrow \quad \sqrt{3}\sqrt{\mu^2 - 1} = 2$

$\Rightarrow \quad 3(\mu^2 - 1) = 4$

$\Rightarrow \quad 3\mu^2 = 7$

$$\Rightarrow \quad \mu = \sqrt{\frac{7}{3}}$$

$\Rightarrow \quad \mu = 1.52$

WHITE LIGHT

White light consists of infinite number of continuous wavelengths (colours) ranging from 4000 Å to 7800 Å. For convenience it is divided into seven colours. Violet, Indigo, Blue, Green, Yellow, Orange, Red called as 'VIBGYOR' pattern.

The Violet having least wavelength (maximum frequency) and Red having maximum wavelength (minimum frequency).

VARIATION OF REFRACTIVE INDEX WITH COLOUR (CAUCHY'S FORMULA)

The refractive index (μ) of a medium varies with wavelength (λ) according to Cauchy's formula

$$\mu = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \ldots$$

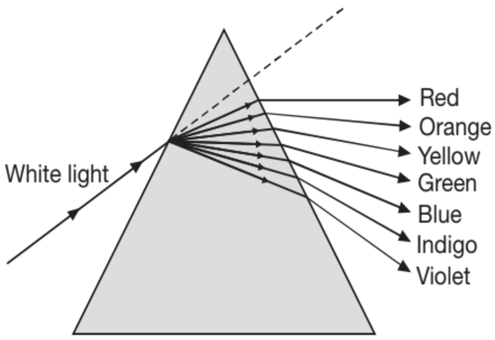
where A, B and C are constants.

From above we observe that refractive index decreases with increase of wavelength. It is maximum for violet and minimum for red colour and due to this variation of the refractive index with the wavelength or the colour, a composite beam of light entering a prism splits into constituent colours.

DISPERSION

It has been observed that when a beam of composite light (consisting of several wavelengths) passes though a prism, it splits into its constituent colours. This phenomenon is called **dispersion**. The band of colours thus obtained on a screen is called the **spectrum**.

If white light is used, seven colours are obtained as shown in the figure. The sequence of colours is VIBGYOR, from bottom to top.



The dispersion of light takes place because the refractive index μ of the medium depends on the wavelength of light as given by **Cauchy's formula**, according to which

$$\mu = A + \frac{B}{\lambda^2}$$

where A and B are constants. The smaller the value of λ, the larger is the value of μ. Thus, μ is maximum for violet colour and minimum for red. The deviation of a ray depends on μ it is larger for higher μ. Hence, violet suffers the maximum deviation and red the minimum.

If light from sodium lamp falls on a prism then it disperses (breaks) into two lines called $D_1\ (5890\,\text{Å})$ and $D_2\ (5896\,\text{Å})$ lines. **Thus we observe that a prism causes deviation as well as dispersion.**

If D_V, D_R and D_Y are the deviations caused by prism for violet, red and mean yellow rays, then for prism with small refracting angle (A), we have

$$\text{Angular Dispersion } D = D_V - D_R = (\mu_V - \mu_R)A$$

DISPERSIVE POWER OF A PRISM

The ratio of angular dispersion to the mean deviation is called dispersive power, so Dispersive Power is

$$\omega = \frac{\text{Angular Dispersion}}{\text{Mean Deviation}} = \frac{D}{D_Y} = \frac{D_V - D_R}{D_Y}$$

where D_Y is the deviation of mean light i.e., yellow light, whose wavelength is considered as mean of all the wavelengths present. Further for a prism of small refracting angle A, we have

$$D = (\mu - 1)A$$

So, we have

$$D_V = (\mu_V - 1)A,\ D_R = (\mu_R - 1)A \text{ and } D_Y = (\mu_Y - 1)A$$

So the dispersive power ω becomes

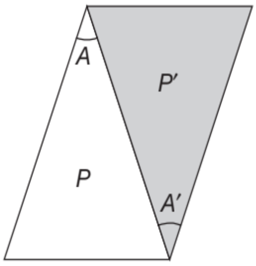
$$\omega = \frac{(\mu_V - \mu_R)A}{(\mu_Y - 1)A} = \frac{(\mu_V - \mu_R)}{\mu_Y - 1} = \frac{d\mu}{\mu - 1}$$

where $d\mu = \mu_V - \mu_R$ and $\mu = \mu_Y$

The dispersive power ω has no units and no dimensions. Its value depends on the material of the prism.

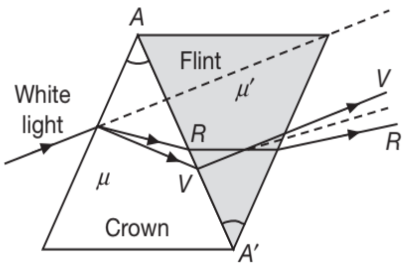
COMBINATION OF TWO PRISMS

From a single prism, it is not possible to get deviation without dispersion, or to get dispersion without deviation. However, two small angled prisms may be combined to produce Dispersion without Deviation or Deviation without Dispersion. The prism placement for both is shown here. The placement remains the same. It is just that we are to decide the relation between their refractive indices such that required condition may be achieved.



Dispersion Without Deviation (Chromatic Combination)

Two prisms can be combined in such a way that the deviation of the mean ray produced by one is equal and opposite to that produced by the other. Such a combination is called a **direct vision prism**.



So, in this arrangement of prisms, the mean deviation (D) caused by one prism is cancelled by the mean deviation (D') caused by the other prism i.e.

$$D - D' = 0$$

$$\Rightarrow \quad (\mu - 1)A - (\mu' - 1)A' = 0$$

$$\text{or} \quad A' = \left(\frac{\mu - 1}{\mu' - 1}\right)A.$$

The net dispersion produced is

$$D_{\text{net}} = (D_V - D_R) - (D'_V - D'_R)$$

$$\Rightarrow \quad D_{\text{net}} = (\mu_V - \mu_R)A - (\mu'_V - \mu'_R)A'$$

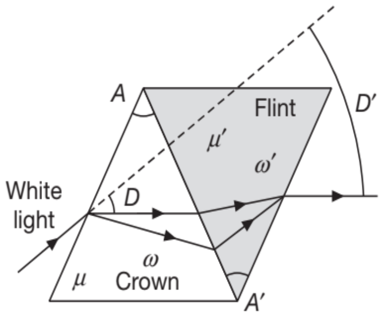
$$\Rightarrow \quad D_{\text{net}} = \left(\frac{\mu_V - \mu_R}{\mu - 1}\right)(\mu - 1)A - \left(\frac{\mu'_V - \mu'_R}{\mu' - 1}\right)(\mu' - 1)A'$$

$$\Rightarrow \quad D_{\text{net}} = \omega D - \omega' D'$$

where ω and ω' are dispersive powers of prisms P and P'.

Deviation Without Dispersion (Achromatic Prism)

It is possible to combine two prisms of different materials in such a way that each cancels the dispersion due to the other. Thus, the net dispersion is zero but a deviation is produced. So, in this arrangement of prisms, the dispersion $(D_V - D_R)$ caused by one prism is cancelled by dispersion $(D'_V - D'_R)$ produced by the other prism.



$$\text{i.e.,} \quad (D_V - D_R) - (D'_V - D'_R) = 0$$

$$\text{or} \quad (\mu_V - \mu_R)A - (\mu'_V - \mu'_R)A' = 0 \qquad \ldots(1)$$

This gives

$$A' = \left(\frac{\mu_V - \mu_R}{\mu'_V - \mu'_R}\right)A$$

Also from (1) we get

$$(\mu_V - \mu_R)A = (\mu'_V - \mu'_R)A'$$

$$\Rightarrow \quad \left(\frac{\mu_V - \mu_R}{\mu - 1}\right)(\mu - 1)A = \left(\frac{\mu'_V - \mu'_R}{\mu - 1}\right)(\mu' - 1)A'$$

$$\Rightarrow \quad \omega D = \omega' D'$$

is the condition for Deviation without Dispersion.
The net mean deviation is

$$D - D' = (\mu - 1)A - (\mu' - 1)A'$$

ILLUSTRATION 62

The refractive indices of the crown glass for blue and red light are 1.51 and 1.49 respectively and those of the flint glass are 1.77 and 1.73 respectively. An isosceles prism of angle $6°$ is made of crown glass. A beam of white light is incident at a small angle on this prism. The other flint glass isosceles prism is combined with the crown glass prism such that there is no deviation of the incident prism.

(a) Determine the angle of the flint glass prism.
(b) Calculate the net dispersion of the combined system.

SOLUTION

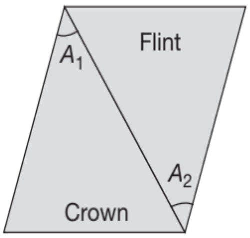
(a) When angle of prism is small and angle of incidence is also small, the deviation is given by $\delta = (\mu - 1)A$

Net deviation by the two prisms is zero, when deviation due to one cancels the deviation due to the other. So,

$$\delta_1 - \delta_2 = 0$$

$$\Rightarrow (\mu_1 - 1)A_1 - (\mu_2 - 1)A_2 = 0 \quad \ldots(1)$$

Here, μ_1 and μ_2 are the refractive indices for crown and flint glasses respectively, where



$$\mu_1 = \frac{1.51 + 1.49}{2} = 1.5 \text{ and } \mu_2 = \frac{1.77 + 1.73}{2} = 1.75$$

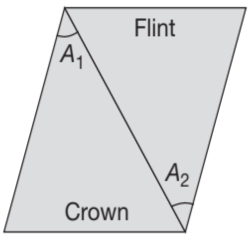
Angle of prism for crown glass is $A_1 = 6°$

Substituting this values in equation (1), we get

$$(1.5 - 1)(6°) - (1.75 - 1)A_2 = 0$$

This gives $A_2 = 4°$

Hence, angle of flint glass prism is $4°$



$\delta_1 + \delta_2 = 0$

(b) Net dispersion due to the two prisms is given by

Net Dispersion $= (\mu_{b_1} - \mu_{r_1})A_1 - (\mu_{b_2} - \mu_{r_2})A_2$

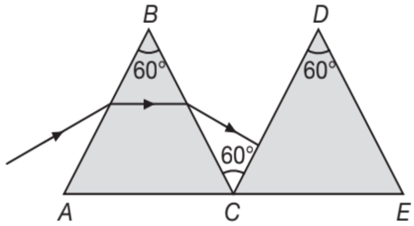
$\Rightarrow$ Net Dispersion $=$

$$(1.51 - 1.49)(6°) - (1.77 - 1.73)(4°) = -0.04°$$

$\Rightarrow$ Net dispersion $= -0.04°$

ILLUSTRATION 63

A ray of light is incident on a prism ABC of refractive index $\sqrt{3}$ as shown in figure.



(a) Find the angle of incidence for which the deviation of light ray by the prism ABC is minimum.
(b) By what angle the second prism must be rotated, so that the final ray suffer net minimum deviation.

SOLUTION

(a) At minimum deviation, we have $r_1 = r_2 = 30°$

According to Snell's Law, we have

$$\mu = \frac{\sin i_1}{\sin r_1}$$

$$\Rightarrow \sqrt{3} = \frac{\sin i_1}{\sin(30°)}$$

$$\Rightarrow \sin i_1 = \frac{\sqrt{3}}{2}$$

$$\Rightarrow i_1 = 60°$$

(b) In the position shown, net deviation suffered by the ray of light should be minimum. Therefore, the second prism should be rotated by $60°$ (anticlockwise).

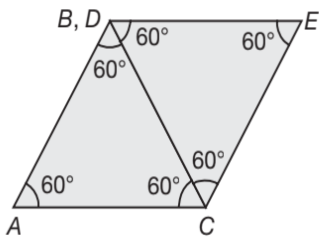


ILLUSTRATION 64

A beam of light enters a glass prism at an angle α and emerges into the air at an angle β. Having passed through the prism, the beam is deflected from the original direction by an angle δ. Find the refracting angle of the prism and the refractive index of the material of the prism.

SOLUTION

$$r_1 + r_2 = A$$

Since $i + e = A + \delta$

$\Rightarrow \quad \delta = \alpha + \beta - A$

Further applying Snell's Law at incident surface and emergent surface, we get

$$\mu = \frac{\sin\alpha}{\sin r_1} \text{ and } \frac{\sin r_2}{\sin\beta} = \frac{1}{\mu}$$

$$\Rightarrow \quad \frac{\sin\alpha}{\sin r_1} = \frac{\sin\beta}{\sin r_2}$$

$$\Rightarrow \quad \frac{\sin\alpha}{\sin(A - r_2)} = \frac{\sin\beta}{\sin r_2}$$

$$\frac{\sin(A - r_2)}{\sin r_2} = \frac{\sin\alpha}{\sin\beta}$$

$$\Rightarrow \quad \frac{\sin A\cos r_2}{\sin r_2} - \frac{\cos A\sin r_2}{\sin r_2} = \frac{\sin\alpha}{\sin\beta}$$

$$\Rightarrow \quad \sin A\cot r_2 = \frac{\sin\alpha}{\sin\beta} + \cos A$$

$$\Rightarrow \quad \cot r_2 = \frac{\sin\alpha}{\sin\beta\sin A} + \cot A$$

Since $\mu = \dfrac{\sin\beta}{\sin r_2} = \sin\beta\,\text{cosec}\, r_2$

$$\Rightarrow \quad \mu = \sin\beta\sqrt{1 + \cot^2 r_2}$$

$$\Rightarrow \quad \mu = \sin\beta\sqrt{1 + \left(\frac{\sin\alpha}{\sin\beta\sin A} + \cot A\right)^2}$$

Since $A = \alpha + \beta - \delta$, so μ is given by

$$\mu = \sin\beta\sqrt{1 + \left(\frac{\sin\alpha}{\sin\beta\sin(\alpha + \beta - \delta)} + \cot(\alpha + \beta - \delta)\right)^2}$$

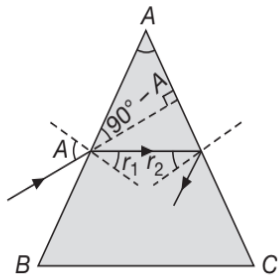
ILLUSTRATION 65

A ray of light is incident upon one face of a prism (angle of prism $< \frac{\pi}{2}$) in a direction perpendicular to the other face. Prove that the ray will fail to emerge from the other face if $\cot A < \cot C - 1$, where C is critical angle for the material of prism.

SOLUTION

From geometry, we observe that the angle of incidence at the face AB is A. Applying Snell's Law at face AB, we get

$$\mu = \frac{\sin A}{\sin r_1} \qquad \ldots(1)$$



If C is the critical angle of the prism, then

$$\mu = \frac{1}{\sin C} \qquad \ldots(2)$$

From (1) and (2), we get

$$\frac{\sin A}{\sin r_1} = \frac{1}{\sin C}$$

$$\Rightarrow \quad \sin r_1 = \sin A\sin C \qquad \ldots(3)$$

The ray does not emerge from the other face AC, when

$$r_2 > C$$

Since, $r_1 + r_2 = A$

$\Rightarrow \quad A - r_1 > C$

$\Rightarrow \quad r_1 < A - C$

$\Rightarrow \quad \sin r_1 < \sin(A - C)$

$\Rightarrow \quad \sin A\sin C < \sin A\cos C - \sin C\cos A$

$\Rightarrow \quad 1 < \cot C - \cot A$

$\Rightarrow \quad \cot A < \cot C - 1$

COLOURS OF OBJECTS AND COLOUR TRIANGLE

The colours of objects are due to a number of phenomena.

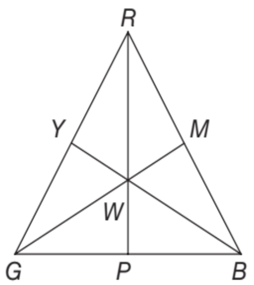
The colours of opaque bodies are due to **Selective Reflection**. For example grass appears green because when white light is incident on grass, it absorbs all colours except green which is reflected. Black appears black because it absorbs all colours falling an it an reflects nothing. Similarly white appears white because it reflects all colours falling on it and absorbs nothing.

The colours of transparent bodies are due to **Selective Transmission**. For example a glass appears blue, because it absorbs all colours except blue, which it transmits.

The colours of sky, rising and setting sun are due to **scattering** while the colours of soap bubble and kerosene oil film are due to **interference**.

COLOUR TRIANGLE

If, Red (R), Green (G) and Blue (B) are primary colours. If P denotes Peacock Blue also called Cyan, M denotes Magenta, Y denotes Yellow and W denotes White, then from colour triangle we observe that



$$R+G+B=W$$
$$R+G=Y$$
$$G+B=P$$
$$R+B=M$$
$$B+Y=W$$
$$R+P=W$$
$$G+M=W$$

RAYLEIGH LAW

According to Lord Rayleigh, intensity (I) of scattered light is inversely proportional to the fourth power of the wavelength λ. So,

$$I \propto \frac{1}{\lambda^4}$$

It can also be concluded that the amplitude (a) of the scattered light is inversely proportional to the square of the wavelength.

So, $a \propto \dfrac{1}{\lambda^2}$ $\qquad \{\because I \propto a^2\}$

COLOUR OF THE SKY

When light from the sun travels through earth's atmosphere, it gets scattered by the large number of molecules of various gases. It is found that the amount of scattering by molecules, called Rayleigh scattering, is inversely proportional to the fourth power of the wavelength. Thus light of shorter wavelength is scattered much more than the light of longer wavelength. Since blue colour has relatively shorter wavelength, it predominates the sky and hence sky appears bluish.

COLOUR OF CLOUDS

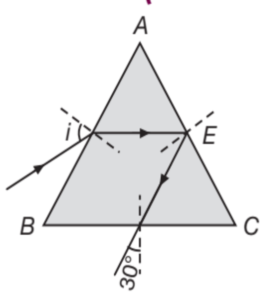
Large particles like water droplets and dust do not have this selective scattering power. They scatter all wavelengths almost equally. Hence clouds appear to the white.

Test Your Concepts-V

Based on Prism

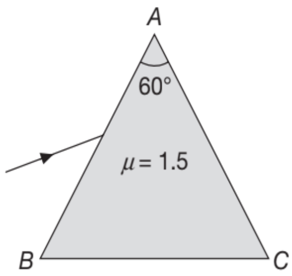
(Solutions on page H.15)

1. The path of a ray undergoing refraction in an equilateral prism is shown in figure. The ray suffers refraction at the face AB and the refracted ray is incident on the face AC at an angle slightly greater than the critical angle and hence, totally reflected. After refraction at the face BC the emergent ray makes an angle of $30°$ with normal at BC at the point of emergence. Find the



(a) corresponding angle of incidence *i*.
(b) refractive index of the prism.

2. In a prism of refractive index $\mu = 1.5$ and refracting angle 60°, the condition for minimum deviation is fulfilled. If face *AC* is polished



(a) Find the net deviation.
(b) If the system is placed in water what will be the net deviation? Refractive index of water $= \frac{4}{3}$

3. A ray of light incident on the face of a prism is refracted and escapes through an adjacent face. What is the maximum permissible angle of refraction of the prism, if it is made of glass with a refractive index of $\mu = 1.5$.

4. In an isosceles prism of angle 45°, it is found that when the angle of incidence is same as the prism angle and the emergent ray grazes the emergent surface.
(a) Find the refractive index of the material of the prism.
(b) For what angle of incidence the angle of deviation will be minimum?

5. A prism of flint glass $\left(\mu_g = \frac{3}{2}\right)$ with an angle of refraction 30° is placed inside water $\left(\mu_w = \frac{4}{3}\right)$.
(a) At what angle should a ray of light fall on the face of the prism so that inside the prism the ray is perpendicular to the bisector of the angle of the prism.
(b) Through what angle will the ray turn after passing through both faces of the prism?

6. Light rays from a source are incident on a glass prism of index of refraction μ and angle of prism α. At near normal incidence, calculate the angle of deviation of the emerging rays.

7. One face of a prism with a refractive angle of 30° is coated with silver. A ray incident on another face at an angle of 45° is refracted and reflected from the silver coated face and retraces its path. What is the refractive index of the prism?

8. A ray of light is incident at an angle of 60° at one face of a prism having refracting angle 30°. The ray emerging out of the prism makes an angle of 30° with the incident ray. Find the angle of emergence and calculate the refractive index of the material of the prism.

9. The index of refraction for violet light in silica flint glass is 1.66 and that for red light is 1.62. Find the angular dispersion of visible light passing through a prism of apex angle 60°, if the angle of incidence is 50°.

10. A light ray is passing through a prism with refracting angle $A = 90°$ and refractive index $\mu = 1.3$. Find the minimum and maximum angle of deviation.

11. A ray of light is incident at an angle of 60° on the face of a prism having refracting angle 30°. The ray emerging out of the prism makes an angle 30° with the incident ray. Find the angle of emergence of the ray.

12. The refracting angle of a glass prism is 30°. A ray is incident onto one of the faces perpendicular to it. Find the angle δ between the incident ray and the ray that leaves the prism. The refractive index of glass is $n = 1.5$.

13. The refractive index of the material of a prism is 1.6 for a certain monochromatic ray. What should be the maximum angle of incidence of this ray on the prism so that no total internal reflection occurs when the ray leaves the prism? The refracting angle of the prism is 45°.

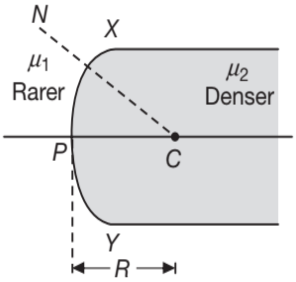
14. A ray of white light falls onto the side surface of an isosceles prism at such an angle that the red ray leaves the prism normally to the second face. Find the deflection of the red and violet rays from the initial direction if the refraction angle of the prism is 45°. The refractive indices of the prism material for red and violet rays are 1.37 and 1.42, respectively.

15. A parallel beam of light falls normally on the first face of a prism of small refracting angle. At the second face it is partly transmitted and partly reflected, the reflected beam striking at the first face again and emerging from it in a direction making an angle of 4° with the reversed direction of the incident beam. The refracted beam is found to have undergone a deviation of 1° from the original direction. Calculate the refractive index of the glass and the angle of the prism.

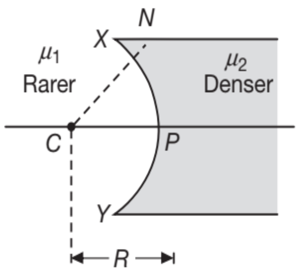
REFRACTION AT CURVED SURFACES AND LENS

SINGLE REFRACTING SURFACE

A spherical surface which separates two media of different refractive index is called a single refracting surface. The convexity or concavity of the surface is decided by looking at it from rarer medium as shown in figure.



Convex Refracting Surface



Concave Refracting Surface

Some Terms Connected with Single Refracting Surface

1. **Pole (*P*):** It is a point which bulges out most (in case of convex surface) or is depressed most (in case of concave surface) as seen from the rarer medium.
2. **Centre of Curvature (*C*):** It is the centre of the sphere of which the surface forms a part.
3. **Radius of Curvature (*R*):** It is the radius of the sphere of which the surface forms a part.
4. **Aperture (*XY*):** The diameter of the refracting surface is called the aperture of the surface.
5. **Principal axis:** The line joining the pole and centre of curvature and extended on either side of the surface is called the principal axis.

SIGN CONVENTIONS

Following sign conventions must be used while dealing with ray diagrams.

(a) All the distances will be measured from the pole of the surface.

(b) The distances measured against the incident ray are taken as negative.

(c) The distances measured along the incident ray are taken as positive.

(d) All transverse measurements done above the principal axis are taken as positive while the ones done below the principal axis are taken as negative.

ASSUMPTIONS

While obtaining some relations, in ray optics, we make some assumptions given below. All those formulae will hold good only if these conditions are satisfied.

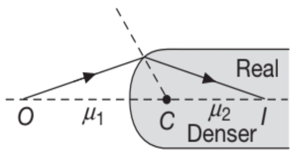
(a) The object/source is considered to be point object/source placed on principal axis.

(b) The aperture of the surface/lens is small.

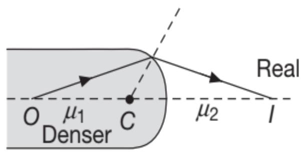
(c) Rays of light make smaller angles with the principal axis i.e., are paraxial in nature.

REFRACTION OF LIGHT AT CURVED SURFACES

For the curved surfaces the same law of refraction are applicable. When a light-ray enters a denser medium, it bends towards the normal. The figures show six situations. The shaded region is denser.

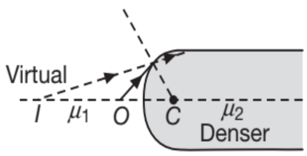


(A)

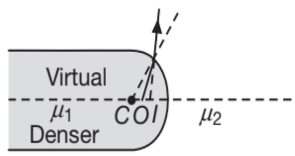


(B)

In Figs. (*A*) and (*B*), the object *O* is kept relatively far from the refracting surface, and the image formed is real.

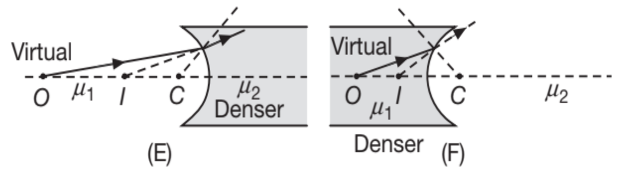


(C)



(D)

In Figs. (*C*) and (*D*), the object is nearer the refracting surface, and the image is virtual.



In Figs. (*E*) and (*F*), the refraction always directs the ray away from the central axis, and hence virtual images are formed.

Note the major difference from the images formed due to reflection from a spherical mirror. Here, real images are formed on the other side of the refracting surface, and virtual images are formed on the same side as the object.

REFRACTION AT CONVEX SURFACE

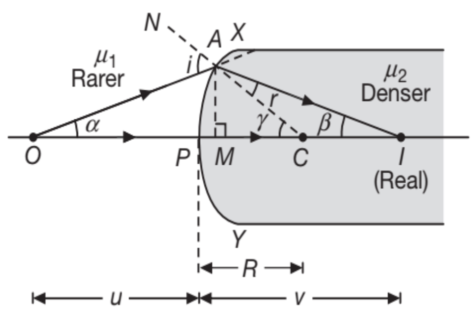
CASE-1: When the object lies in the rarer medium and the image formed is real.

Consider a spherical surface of radius R separating the two media 1 and 2 $(\mu_2 > \mu_1)$. A point object O is placed on the principal axis to the left of the pole P at a considerable distance from it. The incident ray from O falls on point A and is refracted according to

$$\mu_1 \sin i = \mu_2 \sin r \qquad \ldots(1)$$

Since the rays are paraxial, so the angle α is small and hence the angles i and r will also be small. Thus, applying such paraxial approximation, then $\sin i \cong i$ and $\sin r \cong r$, so we get from (1)

$$\mu_1 i = \mu_2 r \qquad \ldots(2)$$



Using the geometrical property that an exterior angle of a triangle is equal to the sum of the two internal opposite angles, we get from triangles AOC and AIC,

$$i = \alpha + \gamma \qquad \ldots(3)$$

and $\gamma = r + \beta$ ...(4)

Substituting the value of i and r from Equations (3) and (4) in Equation (2), we get

$$\mu_1(\alpha + \gamma) = \mu_2(\gamma - \beta)$$

$$\Rightarrow \quad \mu_1\alpha + \mu_2\beta = (\mu_2 - \mu_1)\gamma \qquad \ldots(5)$$

Now, since the aperture of the refracting surface is small, so M and P are very close to each other and hence, we have

$$\alpha \cong \tan\alpha = \frac{AM}{MO} \cong \frac{AM}{PO},$$

$$\beta \cong \tan\beta = \frac{AM}{MI} \cong \frac{AM}{PI} \text{ and}$$

$$\gamma \cong \tan\gamma = \frac{AM}{MC} \cong \frac{AM}{PC}$$

Therefore (5) becomes

$$\Rightarrow \quad \mu_1\left(\frac{AM}{PO}\right) + \mu_2\left(\frac{AM}{PI}\right) = (\mu_2 - \mu_1)\left(\frac{AM}{PC}\right)$$

$$\Rightarrow \quad \frac{\mu_1}{PO} + \frac{\mu_2}{PI} = \frac{\mu_2 - \mu_1}{PC}$$

Since $PO = -u$, $PI = +v$, $PC = +R$ so we get

$$\frac{\mu_1}{-u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$$

If the object is in air, then $\mu_1 = 1$ and $\mu_2 = \mu$, so we get

$$\frac{1}{-u} + \frac{\mu}{v} = \frac{\mu - 1}{R}$$

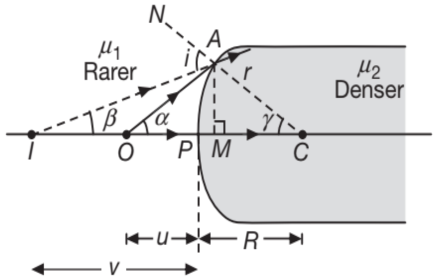
CASE-2: When the object lies in the rarer medium and the image formed is virtual.

Consider a spherical surface of radius R separating the two media 1 and 2 $(\mu_2 - \mu_1)$. A point object O is placed on the principal axis to the left of the pole P. The incident ray from O falls on point A and is refracted according to

$$\mu_1 \sin i = \mu_2 \sin r \qquad \ldots(1)$$

Since the rays are paraxial, so the angle α is small and hence the angles i and r will also be small. Thus, applying such paraxial approximation, then $\sin i \cong i$ and $\sin r \cong r$, so from (1), we have

$$\mu_1 i = \mu_2 r \qquad \ldots(2)$$



Using the geometrical property that an exterior angle of a triangle is equal to the sum of the two internal opposite angles, we get from triangles AOC and AIC,

$$i = \alpha + \gamma \quad \ldots(3)$$

and $r = \beta + \gamma$ ...(4)

Substituting the value of i and r from Equations (3) and (4) in Equation (2), we get

$$\mu_1(\alpha + \gamma) = \mu_2(\beta + \gamma)$$

$$\Rightarrow \mu_1\alpha - \mu_2\beta = (\mu_2 - \mu_1)\gamma \quad \ldots(5)$$

Now, since the aperture of the refracting surface is small, so M and P are very close to each other and hence we have

$$\alpha \cong \tan\alpha = \frac{AM}{MO} \cong \frac{AM}{PO},$$

$$\beta \cong \tan\beta = \frac{AM}{MI} \cong \frac{AM}{PI} \text{ and}$$

$$\gamma \cong \tan\gamma = \frac{AM}{MC} \cong \frac{AM}{PC}$$

$$\Rightarrow \mu_1\left(\frac{AM}{PO}\right) - \mu_2\left(\frac{AM}{PI}\right) = (\mu_2 - \mu_1)\left(\frac{AM}{PC}\right)$$

$$\Rightarrow \frac{\mu_1}{PO} - \frac{\mu_2}{PI} = \frac{\mu_2 - \mu_1}{PC}$$

Since $PO = -u$, $PI = -v$, $PC = +R$ so we get

$$\frac{\mu_1}{-u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$$

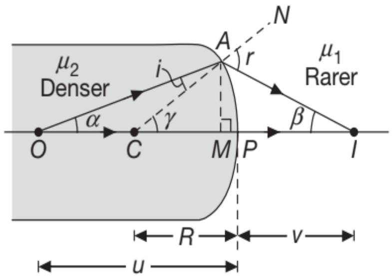
CASE-3: When the object lies in the denser medium and the image formed is real.

Consider a spherical surface of radius R separating the two media 1 and 2 $(\mu_2 > \mu_1)$. A point object O is placed on the principal axis to the left of the pole P. The incident ray from O falls on point A and is refracted according to

$$\mu_2 \sin i = \mu_1 \sin r \quad \ldots(1)$$

Since the rays are paraxial, so the angle α is small and hence the angles i and r will also be small. Thus, applying such paraxial approximation, then $\sin i \cong i$ and $\sin r \cong r$, so from (1), we have

$$\mu_2 i = \mu_1 r \quad \ldots(2)$$



Using the geometrical property that an exterior angle of a triangle is equal to the sum of the two internal opposite angles, we get from triangles AOC and AIC

$$\gamma = \alpha + i \quad \ldots(3)$$

and $r = \beta + \gamma$ ...(4)

Substituting the value of i and r from Equations (3) and (4) in Equation (2), we get

$$\mu_2(\gamma - \alpha) = \mu_1(\beta + \gamma)$$

$$\Rightarrow \mu_2\alpha + \mu_1\beta = (\mu_2 - \mu_1)\gamma \quad \ldots(5)$$

Now, since the aperture of the refracting surface is small, so M and P are very close to each other and hence we have

$$\alpha \cong \tan\alpha = \frac{AM}{MO} \cong \frac{AM}{PO},$$

$$\beta \cong \tan\beta = \frac{AM}{MI} \cong \frac{AM}{PI} \text{ and}$$

$$\gamma \cong \tan\gamma = \frac{AM}{MC} \cong \frac{AM}{PC}$$

$$\Rightarrow \mu_2\left(\frac{AM}{PO}\right) + \mu_1\left(\frac{AM}{PI}\right) = (\mu_2 - \mu_1)\left(\frac{AM}{PC}\right)$$

$$\Rightarrow \frac{\mu_2}{PO} + \frac{\mu_1}{PI} = \frac{\mu_2 - \mu_1}{PC}$$

Since $PO = -u$, $PI = +v$, $PC = -R$ so we get

$$\frac{\mu_2}{-u} + \frac{\mu_1}{v} = \frac{\mu_1 - \mu_2}{R}$$

Simply replace subscript 2 by 1 and 1 by 2 in the formula derived in CASE-1 or CASE-2.

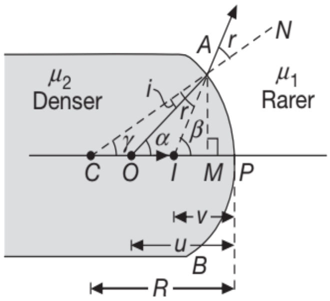
CASE-4: When the object lies in the denser medium and the image formed is virtual.

Consider a spherical surface of radius R separating the two media 1 and 2 $(\mu_2 > \mu_1)$. A point object O is placed on the principal axis to the left of the pole P. The incident ray from O falls on point A and is refracted according to

$$\mu_2 \sin i = \mu_1 \sin r \qquad \ldots(1)$$

Since the rays are paraxial, so the angle α is small and hence the angles i and r will also be small. Thus, applying such paraxial approximation, then $\sin i \cong i$ and $\sin r \cong r$, so from (1), we have

$$\mu_2 i = \mu_1 r \qquad \ldots(2)$$



Using the geometrical property that an exterior angle of a triangle is equal to the sum of the two internal opposite angles, we get from triangles AOC and AIC

$$\alpha = i + \gamma \qquad \ldots(3)$$

and $\beta = r + \gamma \qquad \ldots(4)$

Substituting the value of i and r from Equations (3) and (4) in Equation (2), we get

$$\mu_2(\alpha - \gamma) = \mu_1(\beta - \gamma)$$

$$\Rightarrow \quad \mu_2\alpha - \mu_1\beta = (\mu_2 - \mu_1)\gamma \qquad \ldots(5)$$

Now, since the aperture of the refracting surface is small, so M and P are very close to each other and hence we have

$$\alpha \cong \tan\alpha = \frac{AM}{MO} \cong \frac{AM}{PO},$$

$$\beta \cong \tan\beta = \frac{AM}{MI} \cong \frac{AM}{PI} \text{ and}$$

$$\gamma \cong \tan\gamma = \frac{AM}{MC} \cong \frac{AM}{PC}$$

$$\Rightarrow \quad \mu_2\left(\frac{AM}{PO}\right) - \mu_1\left(\frac{AM}{PI}\right) = (\mu_2 - \mu_1)\left(\frac{AM}{PC}\right)$$

$$\Rightarrow \quad \frac{\mu_2}{PO} - \frac{\mu_1}{PI} = \frac{\mu_2 - \mu_1}{PC}$$

Since $PO = -u$, $PI = -v$, $PC = -R$ so we get

$$\frac{\mu_2}{-u} + \frac{\mu_1}{v} = \frac{\mu_1 - \mu_2}{R}$$

REFRACTION AT CONCAVE SURFACE

For a concave refracting surface the image formed is always virtual irrespective of the placement of the object.

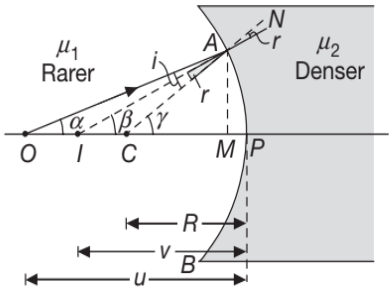
CASE-1: When the object lies in the rarer medium.

Consider a spherical surface of radius R separating the two media 1 and 2 $(\mu_2 > \mu_1)$. A point object O is placed on the principal axis to the left of the pole P. The incident ray from O falls on point A and is refracted according to

$$\mu_1 \sin i = \mu_2 \sin r \qquad \ldots(1)$$

Since the rays are paraxial, so the angle α is small and hence the angles i and r will also be small. Thus, applying such paraxial approximation, then $\sin i \cong i$ and $\sin r \cong r$, so from (1), we have

$$\mu_1 i = \mu_2 r \qquad \ldots(2)$$



Using the geometrical property that an exterior angle of a triangle is equal to the sum of the two internal opposite angles, we get from triangles AOC and AIC

$$\gamma = \alpha + i \qquad \ldots(3)$$

and $\gamma = \beta + r \qquad \ldots(4)$

Substituting the value of i and r from Equations (3) and (4) in Equation (2), we get

$$\mu_1(\gamma - \alpha) = \mu_2(\gamma - \beta)$$

$$\Rightarrow \quad \mu_1\alpha - \mu_2\beta = (\mu_1 - \mu_2)\gamma \qquad \ldots(5)$$

Now, since the aperture of the refracting surface is small, so M and P are very close to each other and hence we have

$$\alpha \cong \tan\alpha = \frac{AM}{MO} \cong \frac{AM}{PO},$$

$$\beta \cong \tan\beta = \frac{AM}{MI} \cong \frac{AM}{PI} \text{ and}$$

$$\gamma \cong \tan\gamma = \frac{AM}{MC} \cong \frac{AM}{PC}$$

$$\Rightarrow \quad \mu_1\left(\frac{AM}{PO}\right) - \mu_2\left(\frac{AM}{PI}\right) = (\mu_1 - \mu_2)\left(\frac{AM}{PC}\right)$$

$$\Rightarrow \quad \frac{\mu_1}{PO} - \frac{\mu_2}{PI} = \frac{\mu_1 - \mu_2}{PC}$$

Since $PO = -u$, $PI = -v$, $PC = -R$ so we get

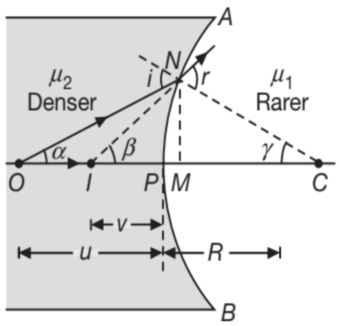
$$\frac{\mu_1}{-u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$$

CASE-2: When the object lies in the denser medium. Consider a spherical surface of radius R separating the two media 1 and 2 $(\mu_2 > \mu_1)$. A point object O is placed on the principal axis to the left of the pole P. The incident ray from O falls on point A and is refracted according to

$$\mu_2 \sin i = \mu_1 \sin r \qquad \ldots(1)$$

Since the rays are paraxial, so the angle α is small and hence the angles i and r will also be small. Thus, applying such paraxial approximation, then $\sin i \cong i$ and $\sin r \cong r$, so from (1), we have

$$\mu_2 i = \mu_1 r \qquad \ldots(2)$$



Using the geometrical property that an exterior angle of a triangle is equal to the sum of the two internal opposite angles, we get from triangles AOC and AIC

$$i = \alpha + \gamma \qquad \ldots(3)$$

$$\text{and} \quad r = \beta + \gamma \qquad \ldots(4)$$

Substituting the value of i and r from Equations (3) and (4) in Equation (2), we get

$$\mu_2(\alpha + \gamma) = \mu_1(\beta + \gamma)$$

$$\Rightarrow \quad \mu_2\alpha - \mu_1\beta = (\mu_1 - \mu_2)\gamma \qquad \ldots(5)$$

Now, since the aperture of the refracting surface is small, so M and P are very close to each other and hence we have

$$\alpha \cong \tan\alpha = \frac{AM}{MO} \cong \frac{AM}{PO},$$

$$\beta \cong \tan\beta = \frac{AM}{MI} \cong \frac{AM}{PI} \text{ and}$$

$$\gamma \cong \tan\gamma = \frac{AM}{MC} \cong \frac{AM}{PC}$$

$$\Rightarrow \quad \mu_2\left(\frac{AM}{PO}\right) - \mu_1\left(\frac{AM}{PI}\right) = (\mu_1 - \mu_2)\left(\frac{AM}{PC}\right)$$

$$\Rightarrow \quad \frac{\mu_2}{PO} - \frac{\mu_1}{PI} = \frac{\mu_1 - \mu_2}{PC}$$

Since $PO = -u$, $PI = -v$, $PC = +R$ so we get

$$\frac{\mu_2}{-u} + \frac{\mu_1}{v} = \frac{\mu_1 - \mu_2}{R}$$

Conceptual Note(s)

(a) For both convex and concave spherical surfaces, the refraction formulae are same, only proper signs of u, v and R are to be used.

(b) For refraction from rarer to denser medium, the refraction formula is

$$\frac{\mu_1}{-u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$$

(c) For refraction from denser to rarer medium, we inter-change μ_1 and μ_2 and obtain the refraction formula,

$$\frac{\mu_2}{-u} + \frac{\mu_1}{v} = \frac{\mu_1 - \mu_2}{R}$$

(d) If the rarer medium is air $(\mu_1 = 1)$ and the denser medium has refractive index μ (i.e., $\mu_2 = \mu$), then

(i) for refraction from air to medium, we have

$$\frac{1}{-u} + \frac{\mu}{v} = \frac{\mu - 1}{R}$$

(ii) for refraction from medium to air, we have

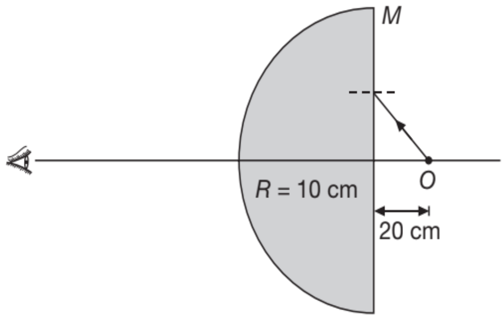
$$\frac{1}{-v}+\frac{\mu}{u}=\frac{\mu-1}{R}$$

(e) The factor $\frac{\mu_2-\mu_1}{R}$ is called power of the spherical refracting surface. It gives a measure of the degree to which the refracting surface can converge or diverge the rays of light passing through it. For air-medium interface, the power is

$$P=\frac{\mu-1}{R}$$

ILLUSTRATION 66

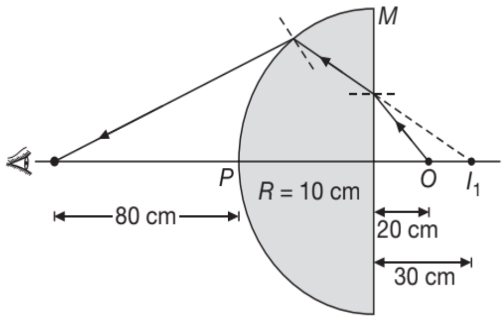
A glass hemisphere M of $\mu=\frac{3}{2}$ and radius 10 cm has a point object O placed at a distance 20 cm behind the flat face which is viewed by an observer from the curved side as shown. Find the location of final image after two refractions as seen by observer.



SOLUTION

After first refraction at flat surface image is produced at a distance given by

$$\mu h=\frac{3}{2}\times 20=30 \text{ cm}$$



For second refraction at spherical surface, for refraction formula we use

$$u=+40 \text{ cm};\ R=+10 \text{ cm};\ \mu_1=\frac{3}{2};\ \mu_2=1$$

Substituting values in refraction formula, we get

$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$$

$$\frac{1}{v}-\frac{3}{2\times 40}=\frac{1-\frac{3}{2}}{10}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{3}{90}-\frac{1}{20}=-\frac{1}{80}$$

$$\Rightarrow \quad v=-80 \text{ cm}$$

Thus final image is seen by observer at a distance 80 cm from the pole P of curved surface and it is a real image.

Conceptual Note(s)

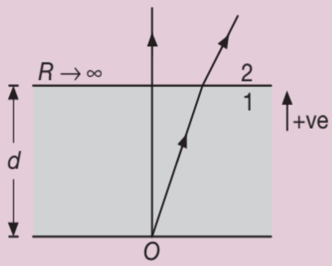
The refraction formula $\frac{\mu_1}{-u}+\frac{\mu_2}{v}=\frac{\mu_2-\mu_1}{R}$ is equally applicable to plane refracting surfaces i.e., surfaces for which $R\to\infty$. Let us derive $\mu=\frac{\text{Real Depth}}{\text{Apparent Depth}}$ using this.

Applying $\frac{\mu_1}{-u}+\frac{\mu_2}{v}=\frac{\mu_2-\mu_1}{R}$

with proper sign conventions and values, we get

$$\frac{\mu}{-(-d)}+\frac{1}{v}=\frac{1-\mu}{\infty}=0$$

$$\Rightarrow \quad v=-\frac{d}{\mu}$$



i.e., image of object O is formed at a distance $\frac{d}{\mu}$ on same side.

So, $d_{app} = \frac{d_{\text{actual}}}{\mu}$

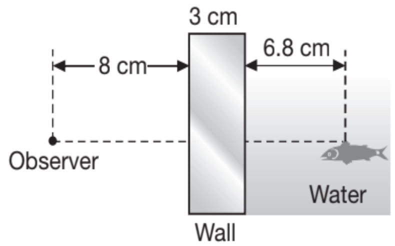
$$\Rightarrow \mu = \frac{\text{Real Depth}}{\text{Apparent Depth}}$$

ILLUSTRATION 67

In figure, a fish watcher watches a fish through a 3 cm thick glass wall of a fish tank. The watcher is in level with the fish; the index of refraction of the glass is $\frac{8}{5}$ and that of the water is $\frac{4}{3}$.

(a) To the fish, how far away does the watcher appear to be?

(b) To the watcher, how far away does the fish appear to be?

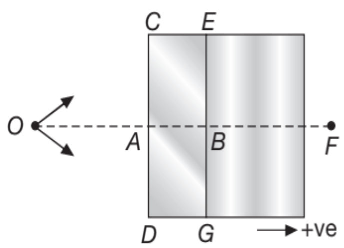


SOLUTION

(a) $OA = 3\text{ cm}$

So, $AI_1 = (n_g)(OA)$

$$\Rightarrow AI_1 = \left(\frac{8}{5}\right)(3) = 4.8\text{ cm}$$



For refraction at $EG(R \to \infty)$, using

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}$$

$$\Rightarrow \frac{\frac{4}{3}}{BI_2} - \frac{\frac{8}{5}}{-(4.8+3)} = 0$$

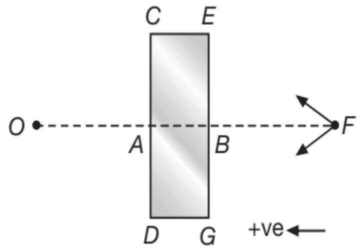
$$\Rightarrow BI_2 = -(7.8)\left(\frac{4}{3}\right)\left(\frac{5}{8}\right) = -6.5\text{ cm}$$

So, $FI_2 = 6.5 + 6.8 = 13.3\text{ cm}$

(b) For face EF, we have

$$\frac{\frac{8}{5}}{BI_1} - \frac{\frac{4}{3}}{-6.8} = 0 \qquad \{\because R \to \infty\}$$

$$\Rightarrow BI_1 = -(6.8)\left(\frac{8}{5}\right)\left(\frac{3}{4}\right) = -8.16\text{ cm}$$



For face CD, we have

$$\frac{1}{AI_2} - \frac{\frac{8}{5}}{-11.16} = 0 \qquad \{\because R \to \infty\}$$

$$\Rightarrow AI_2 = -(11.16)\left(\frac{5}{8}\right) = -6.975\text{ cm}$$

$$\Rightarrow FI_2 = 8 + 6.975$$

$$\Rightarrow FI_2 = 14.975\text{ cm}$$

ILLUSTRATION 68

There are two objects O_1 and O_2 at an identical distance of 20 cm on the two sides of the pole of a spherical concave refracting boundary of radius 60 cm. The indices of refraction of the media on two sides of the boundary are 1 and $\left(\frac{4}{3}\right)$ respectively. Find the location of the object

(a) O_1 when seen from O_2.

(b) O_2 when seen from O_1.

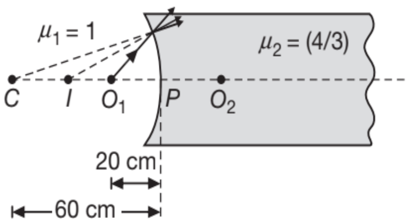
SOLUTION

The formula for refraction from a curved boundary is

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

(a) From the ray diagram drawn, we get

$$u_1 = -20 \text{ cm},\ R = -60 \text{ cm},\ \mu_1 = 1,\ \mu_2 = \frac{4}{3}$$



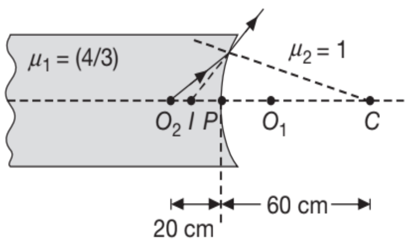
$$\Rightarrow \frac{\left(\frac{4}{3}\right)}{v} - \frac{1}{(-20)} = \frac{\left(\frac{4}{3}\right) - 1}{(-60)}$$

$$\Rightarrow v = -24 \text{ cm}$$

Thus, the object O_1, will appear at a distance of 24 cm from P towards C.

(b) Keeping the object O_2 on the left of the pole P as shown, here, we get

$$u_1 = -20 \text{ cm},\ R = +60 \text{ cm},\ \mu_1 = \frac{4}{3},\ \mu_2 = 1$$



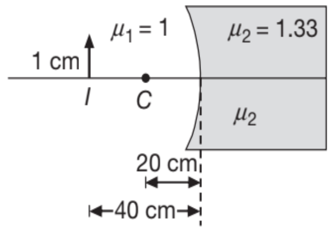
$$\Rightarrow \frac{1}{v} - \frac{\left(\frac{4}{3}\right)}{(-20)} = \frac{1 - \left(\frac{4}{3}\right)}{60}$$

$$\Rightarrow v = -16.36 \text{ cm}$$

Thus, the object O_2 will appear at a distance of 16.36 cm from P towards O_2.

ILLUSTRATION 69

An object of height 1 cm is kept at a distance of 40 cm from a concave spherical surface having radius of curvature $R = 20$ cm, separating air and glass having refractive index $\mu = 1.33$. Find the location, height and the nature of the image formed.



SOLUTION

According to Cartesian sign convention,

$$u = -40 \text{ cm},\ R = -20 \text{ cm}$$

Applying the formula $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$, we get

$$\frac{1.33}{v} - \frac{1}{(-40)} = \frac{1.33 - 1}{(-20)}$$

$$\Rightarrow \quad v = -32 \text{ cm}$$

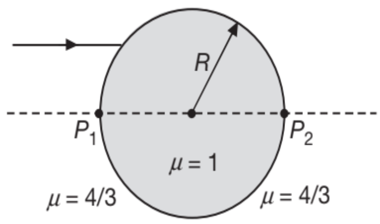
Magnification, $m = \frac{h_2}{h_1} = \frac{\mu_1 v}{\mu_2 u} = \frac{1 \times (-32)}{1.33 \times (-40)} = 0.60$

So, height of image, $h_2 = mh_1 = 0.6 \times 1 = 0.6$ cm

The positive sign of magnification indicates that the image is virtual and erect.

ILLUSTRATION 70

A parallel beam of light travelling in water $\left(\mu = \frac{4}{3}\right)$ is refracted by a spherical air bubble of radius R situated in water.



(a) Find the position of the image due to refraction at the first surface and the position of the final image.

(b) Draw the ray diagram showing the position of the two images.

SOLUTION

(a) Applying the formula for the refraction at the curved boundary i.e.,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

For, refraction at the first surface, the pole is P_1 and we observe that

$$u \to -\infty,\ R = +R,\ \mu_1 = \frac{4}{3} \text{ and } \mu_2 = 1$$

$$\Rightarrow \frac{1}{v} - \frac{\left(\frac{4}{3}\right)}{(-\infty)} = \frac{1-\left(\frac{4}{3}\right)}{(+R)}$$

$$\Rightarrow v = -3R$$

Thus, the first image I_1 is formed at a distance of $3R$ to the left of pole P_1.

This image acts as an object for the refraction at the second surface, with pole P_2. For this refraction, we have

$$u = -(3R + 2R) = -5R,\ R = -R,\ \mu_1 = 1,\ \mu_2 = \frac{4}{3}$$

$$\Rightarrow \frac{\frac{4}{3}}{v} - \frac{1}{-5R} = \frac{\frac{4}{3}-1}{-R}$$

$$\Rightarrow v = -\frac{5}{2}R$$

Thus, the final image I_2 is at a distance $\frac{5R}{2}$ from P_2 towards left.

(b) The ray diagram is shown in figure.

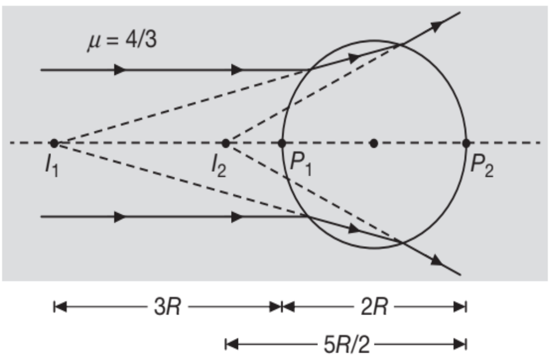
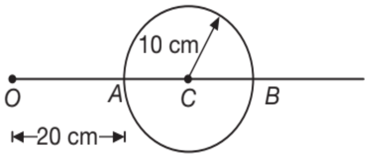


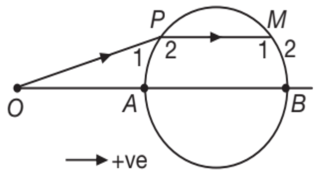
ILLUSTRATION 71

A glass sphere of radius $R = 10$ cm having refractive index $\mu_g = \frac{3}{2}$ is kept inside water. A point object O is placed at 20 cm from A as shown in figure. Find the position and nature of the image when seen from other side of the sphere. Also draw the ray diagram. Given refractive index of water is $\mu_w = \frac{4}{3}$.



SOLUTION

A ray of light starting from O gets refracted twice. The ray of light is travelling in a direction from left to right. Hence, the distances measured in this direction are taken positive. Applying $\frac{\mu_1}{-u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$, twice with appropriate signs at the two refracting surfaces, we get



$$\frac{\left(\frac{4}{3}\right)}{-(-20)} + \frac{\left(\frac{3}{2}\right)}{AI_1} = \frac{\left(\frac{3}{2} - \frac{4}{3}\right)}{10}$$

$$\Rightarrow AI_1 = -30 \text{ cm}$$

Now, the first image I_1, acts as an object for the second surface, so, we have

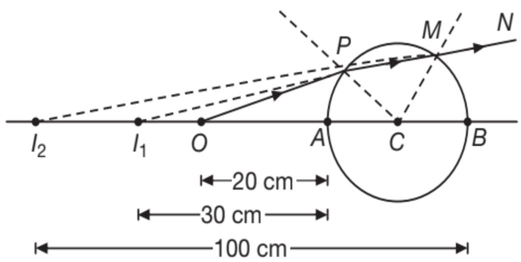
$$BI_1 = u = -(30 + 20) = -50 \text{ cm}$$

Again applying $\frac{\mu_2}{-u} + \frac{\mu_1}{v} = \frac{\mu_1 - \mu_2}{R}$, we get

$$\frac{\left(\frac{3}{2}\right)}{-(-50)} + \frac{\left(\frac{4}{3}\right)}{BI_2} = \frac{\frac{4}{3} - \frac{3}{2}}{-10}$$

$$\Rightarrow BI_2 = -100 \text{ cm}$$

i.e., the final image I_2 is virtual and is formed at a distance 100 cm (towards left) from B. The ray diagram is as shown.



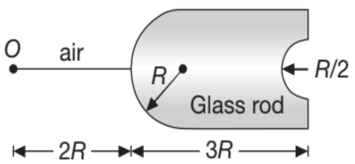
Following points should be kept in mind while drawing the ray diagram.

(i) At P the ray travels from rarer to a denser medium. Hence, it will bend towards normal PC. At M, it travels from a denser to a rarer medium, hence, moves away from the normal MC.

(ii) The ray PM when extended backwards meets the principal axis at I_1 and the ray MN when extended meets the principal axis at I_2.

ILLUSTRATION 72

A glass rod has ends as shown in figure. The refractive index of glass is μ. The object O is at a distance $2R$ from the surface of larger radius of curvature. The distance between apexes of ends is $3R$.



(a) Find the distance of image formed of the point object from right hand vertex.

(b) What is the condition to be satisfied if the image is to be real?

SOLUTION

For refraction at curved surface S_1,

$$\frac{\mu}{v_1} - \frac{1}{(-2R)} = \frac{(\mu-1)}{R}$$

$$\Rightarrow \quad \frac{\mu}{v_1} = \frac{2\mu-3}{2R}$$

$$\Rightarrow \quad v_1 = \frac{2\mu R}{2\mu-3} \qquad \ldots(1)$$

The first image acts as object for refraction at second surface S_2. The origin of our Cartesian coordinate system is now at vertex/pole of surface S_2. Object distance for second refraction is

$$u_2 = -(3R - v_1)$$

$$\Rightarrow \quad u_2 = -\left(3R - \frac{2\mu R}{(2\mu-3)}\right)$$

$$\Rightarrow \quad u_2 = -\left(\frac{4\mu-9}{2\mu-3}\right)R$$

For refraction at curved surface S_2, we have

$$\frac{1}{v_2} - \frac{\mu}{-\left(\frac{4\mu-9}{2\mu-3}\right)R} = \frac{(1-\mu)}{\left(\frac{R}{2}\right)} \qquad \ldots(2)$$

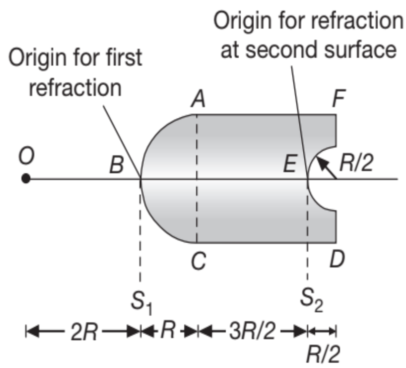
On solving the above expression for v_2, we get

$$v_2 = \frac{(4\mu-9)}{(10\mu-9)(\mu-2)} \qquad \ldots(3)$$

The image will be real if v_2 is positive, i.e.,

$$\frac{(4\mu-9)}{(10\mu-9)(\mu-2)} > 0 \qquad \ldots(4)$$

The equation (4) is satisfied when



CASE-1:

$$(4\mu-9) > 0$$

$$\Rightarrow \quad \mu > \frac{9}{4}$$

and $(10\mu-9)(\mu-2) > 0$

$$\Rightarrow \quad 0.9 < \mu < 2$$

there is no common solution for this condition and hence this is rejected.

CASE-2:

$$4\mu - 9 < 0$$

$$\Rightarrow \quad \mu < \frac{9}{4}$$

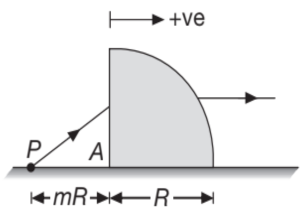
$$(10\mu-9)(\mu-2) < 0$$

$$\Rightarrow \quad \mu > 2 \text{ OR } \mu < 0.9$$

Hence the common result is $2 < \mu < 2.25$

ILLUSTRATION 73

A quarter cylinder of radius R and refractive index 1.5 is placed on a table. A point object P is kept at a distance of mR from it. Find the value of m for which a ray from P will emerge parallel to the table as shown in figure.



SOLUTION

Applying $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$, firstly at the plane surface and then at the curved surface.

For the plane surface, we get

$$\frac{1.5}{AI_1}-\frac{1}{(-mR)}=\frac{1.5-1}{\infty}=0 \qquad \{\because R\to\infty\}$$

$$\Rightarrow \quad AI_1=-(1.5mR)$$

For the curved surface, since the final image is formed at infinity, so we get

$$\frac{1}{\infty}-\frac{1.5}{-(1.5mR+R)}=\frac{1-1.5}{-R}$$

$$\Rightarrow \quad \frac{1.5}{(1.5m+1)R}=\frac{0.5}{R}$$

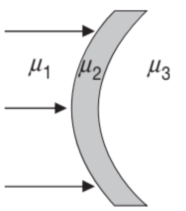
$$\Rightarrow \quad 3=1.5m+1$$

$$\Rightarrow \quad \frac{3}{2}m=2$$

$$\Rightarrow \quad m=\frac{4}{3}$$

ILLUSTRATION 74

In the figure, light is incident on the thin lens as shown. The radius of curvature for both the surface is R. Determine the focal length of this system.

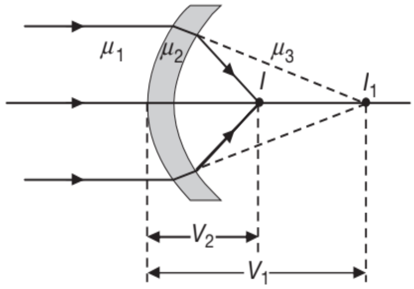


SOLUTION

Since parallel rays after passing through a lens must converge (or appear to converge) at the point. So this point is the place where focus is located and the final image is also formed at the focus.

For refraction at first surface, we get

$$\frac{\mu_2}{v_1}-\frac{\mu_1}{-\infty}=\frac{\mu_2-\mu_1}{+R} \qquad \ldots(1)$$



For refraction at 2nd surface, we get

$$\frac{\mu_3}{v_2}-\frac{\mu_2}{v_1}=\frac{\mu_3-\mu_2}{+R} \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$\frac{\mu_3}{v_2}=\frac{\mu_3-\mu_1}{R}$$

$$\Rightarrow \quad v_2=\frac{\mu_3 R}{\mu_3-\mu_1}$$

Hence, focal length of the given lens system is

$$f=\frac{\mu_3 R}{\mu_3-\mu_1}$$

ILLUSTRATION 75

A parallel beam of light travelling in water having refractive index $\frac{4}{3}$ is refracted by a spherical air bubble of radius 2 mm situated in water. Assuming the light rays to be paraxial.

(i) Find the position of the image due to refraction at the first surface and the position of the final image.

(ii) Draw a ray diagram showing the positions of both the images.

SOLUTION

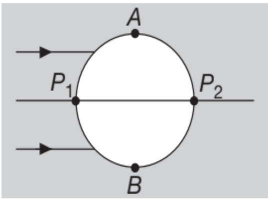
(i) To get the desired result(s), we shall be applying $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$, one by one on two spherical surfaces.

For first refraction at AP_1B, we have

$$\frac{1}{v_1}-\frac{\frac{4}{3}}{\infty}=\frac{1-\frac{4}{3}}{+2}$$

$$\Rightarrow \frac{1}{v_1}=-\frac{1}{6}$$

$$\Rightarrow v_1=-6 \text{ mm}$$



So, the first image will be formed at 6 mm towards left of P_1

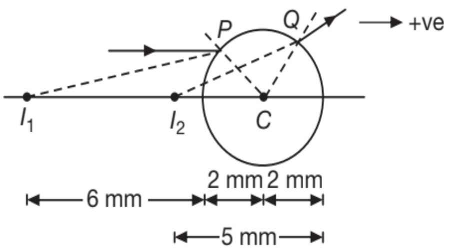
For second refraction at AP_2B, the distance of first image I_1 from P_2 will be $6\text{ mm}+4\text{ mm}=10\text{ mm}$ (towards left). So, we get

$$\frac{\frac{4}{3}}{v_2}-\frac{1}{-10}=\frac{\frac{4}{3}-1}{-2}$$

$$\Rightarrow \frac{4}{3v_2}=-\frac{1}{6}-\frac{1}{10}=-\frac{4}{15}$$

$$\Rightarrow v_2=-5 \text{ mm}$$

(ii) The ray diagram is shown in figure

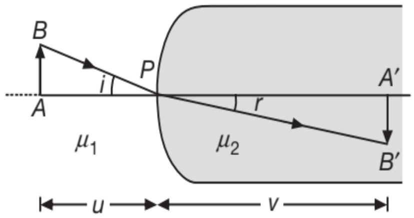


Conceptual Note(s)

(a) At P and Q both normal will pass through C
(b) At P ray of light is travelling from a denser medium (water) to rarer medium (air) therefore, ray of light will bend away from the normal and on extending meet at I_1. Similarly at Q ray of light bends towards the normal.
(c) Both the images I_1 and I_2 are virtual.

LATERAL OR TRANSVERSE MAGNIFICATION

Instead of a point object O let us now, keep an extended object AB at point O such that its image $A'B'$ will be formed at point I. The distance $x(=-u)$ and $y(=v)$ are related by the above formula.



A ray from point B of the object is incident at point P and is refracted, in accordance with Snell's Law, such that

$$\mu_1 \sin i=\mu_2 \sin r$$

$$\Rightarrow \quad \mu_1 i=\mu_2 r \quad \text{\{applying paraxial approximation\}}$$

$$\Rightarrow \quad \frac{r}{i}=\frac{\mu_1}{\mu_2} \quad \ldots(1)$$

Now, in ΔABP and $\Delta A'B'P$, we have

$$AB=u\tan i\cong ui \quad \ldots(2)$$

and $$A'B'=v\tan r\cong vr \quad \ldots(3)$$

The magnification is defined as

$$m=\frac{\text{height of image}}{\text{height of object}}=\frac{A'B'}{AB}$$

Using Equations (1), (2) & (3), we get

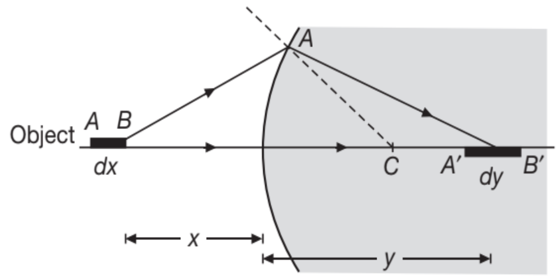
$$m=\frac{A'B'}{AB}=\frac{vr}{ur}=\left(\frac{v}{u}\right)\left(\frac{\mu_1}{\mu_2}\right)$$

$$\Rightarrow \quad m=\frac{v}{u}\frac{\mu_1}{\mu_2}$$

If m is positive, the image is erect and virtual.
If m is negative, the image is inverted and real.

LONGITUDINAL OR AXIAL MAGNIFICATION OF IMAGE

An object of width dx which is placed on principal axis of the refracting surface S at a distance x from the pole P. After refraction its image I is produced at a distance y from the pole and is of width dy as shown in figure.



The distances of object and image from the pole of the surface are related by the refraction formula given as

$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$$

Here we use $u=-x$, $R=+R$ and $v=+y$ which gives

$$\frac{\mu_2}{y}+\frac{\mu_1}{x}=\frac{\mu_2-\mu_1}{R} \qquad \ldots(1)$$

Differentiating the above equation we get

$$-\frac{\mu_2}{y^2}-dy-\frac{\mu_1}{x^2}dx=0 \qquad \ldots(2)$$

From above Equation (2), we get axial magnification as

$$m_{\text{axial}}=\frac{dy}{dx}=-\frac{\mu_1 y^2}{\mu_2 x^2} \qquad \ldots(3)$$

As already discussed that above relation given by Equation (3) is only valid for paraxial rays. Here negative sign shows the lateral inversion of the image.

EFFECT OF MOTION OF OBJECT OR REFRACTING SURFACE ON IMAGE

As already discussed in case of spherical mirrors, for small velocities of the object or refracting surface, the velocity magnification along the principal axis (i.e. $v_{\parallel}$) and perpendicular to the principal axis (i.e. $v_{\perp}$) can be given by the expressions of lateral and longitudinal magnifications.

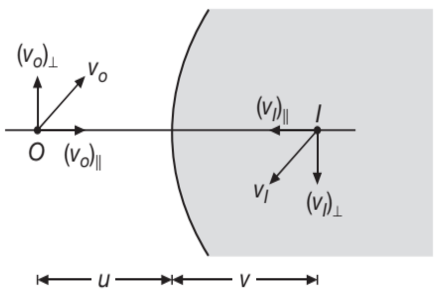


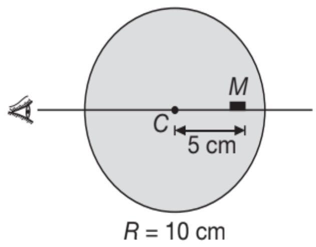
Figure shows an object moving with velocity v_O as shown. Its velocity component parallel to the principal axis with respect to the refracting surface is $(v_O)_{\parallel}$ and in direction perpendicular to the principal axis is $(v_O)_{\perp}$, then the image velocity components can be directly given by

$$(v_I)_{\perp}=m_{\text{lateral}}(v_O)_{\perp}$$

$$(v_I)_{\parallel}=m_{\text{axial}}(v_O)_{\parallel},\ \text{where}\ m_{\text{axial}}=-\frac{\mu_1 v^2}{\mu_2 u^2}$$

ILLUSTRATION 76

Figure shows a small object M of length 1 mm which lies along a diametrical line of a glass sphere of radius 10 cm and $\mu=\frac{3}{2}$ which is viewed by an observer as shown. Find the size of object as seen by the observer.



SOLUTION

For refraction at glass-air interface, we use

$$u=+15\text{ cm},\ R=+10\text{ cm},\ \mu_1=\frac{3}{2}\ \text{and}\ \mu_2=1$$

Substituting values in refraction formula, we get

$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$$

$$\Rightarrow \quad \frac{1}{v}-\frac{\frac{3}{2}}{15}=\frac{1-\frac{3}{2}}{10}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{1}{10}-\frac{1}{20}=\frac{1}{20}$$

$$\Rightarrow \quad v=+20\text{ cm}$$

Axial magnification for refraction is given by

$$m=\frac{l_i}{l_o}=\frac{\mu_1 v^2}{\mu_2 u^2}=\frac{\frac{3}{2}\times(20)^2}{1\times(15)^2}=\frac{1.5\times 400}{225}=\frac{8}{3}$$

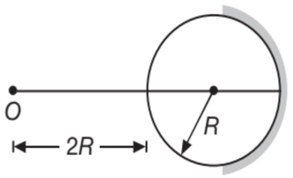
$$\Rightarrow \quad l_i=\frac{8}{3}\times 1=\frac{8}{3}\text{ mm}$$

ILLUSTRATION 77

A solid glass with radius R and an index of refraction 1.5 is silvered over one hemisphere. A small object is located on the axis of the sphere at a distance $2R$ to the left of the vertex of the unsilvered hemisphere. Find the position of final image after all refractions and reflections have taken place.

SOLUTION

The ray of light first gets refracted then reflected and then again refracted. For first refraction and then reflection the ray of light travels from left to right while for the last refraction it travels from right to left. Hence, the sign convention will change accordingly.



First Refraction:

Using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with appropriate sign conventions, we get

$$\frac{1.5}{v_1} - \frac{1}{(-2R)} = \frac{1.5-1}{+R}$$

$$\Rightarrow \quad v_1 \to \infty$$

Second Reflection:

Using $\frac{1}{v} + \frac{1}{u} = \frac{1}{f} = \frac{2}{R}$ with appropriate sign conventions, we get

$$\frac{1}{v_2} + \frac{1}{\infty} = -\frac{2}{R}$$

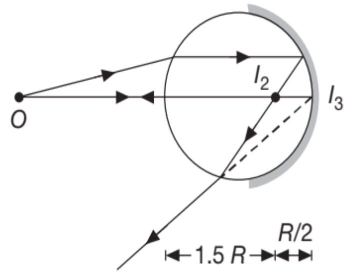
$$\Rightarrow \quad v_2 = -\frac{R}{2}$$

Third Reflection:

Again using $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ with reversed sign convention, we get

$$\frac{1}{v_3} - \frac{1.5}{-1.5R} = \frac{1-1.5}{-R}$$

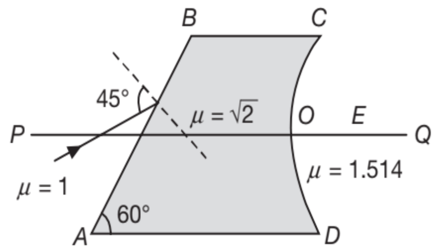
$$\Rightarrow \quad v_3 = -2R$$



i.e., final image is formed at the vertex of the silvered face i.e., at the pole of silvered/curved surface.

ILLUSTRATION 78

Figure shows an irregular block of material of refractive index $\sqrt{2}$. A ray of light strikes the face AB as shown in the figure. After refraction it is incident on a spherical surface CD of radius of curvature 0.4 m and enters a medium of refractive index 1.514 to meet PQ at E. Find the distance OE.



SOLUTION

Applying Snell's Law at face AB, we get

$$(1)\sin 45° = (\sqrt{2})\sin r$$

$$\Rightarrow \quad \sin r = \frac{1}{2}$$

$$\Rightarrow \quad r = 30°$$

i.e., ray becomes parallel to AD inside the block.

Now applying, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ on face CD, we get

$$\frac{1.514}{OE} - \frac{\sqrt{2}}{\infty} = \frac{1.514 - \sqrt{2}}{0.4}$$

Solving this equation, we get

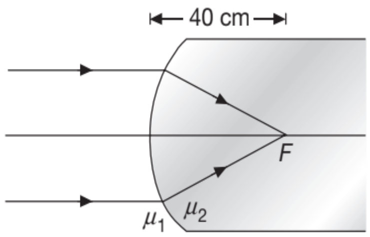
$$OE \approx 6 \text{ m}$$

Test Your Concepts-VI

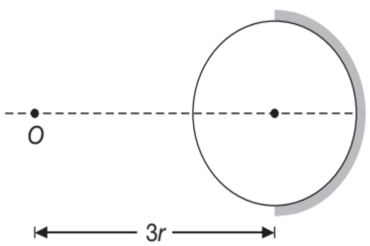
Based on Refraction at Curved Surfaces

(Solutions on page H.18)

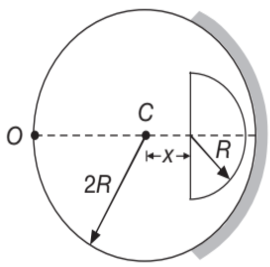
1. A spherical convex surface separates object and image space of refractive index 1 and $\frac{4}{3}$. If radius of curvature of the surface is 10 cm, find its power.



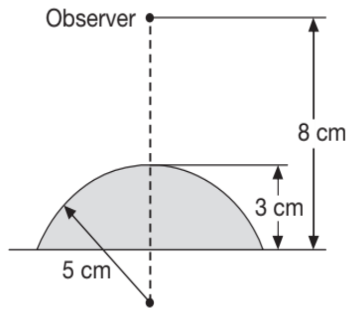
2. A hemispherical portion of the surface of a solid glass sphere of refractive index 1.5 and of radius r is silvered to make the inner side reflecting. An object is placed at the axis of the sphere at a distance $3r$ from the centre of the sphere. The light from the object is refracted at the unsilvered part, then reflected from the silvered part and again refracted at the unsilvered part. Locate the final image formed.



3. Consider the figure shown. A hemispherical cavity of radius R is carved out from a sphere (μ = 1.5) of radius $2R$ such that principal axis of cavity coincides with that of sphere. One side of sphere is silvered as shown. Find the value of x for which the image of an object at O is formed at O itself.



4. A glass sphere has a radius of 5 cm and a refractive index of 1.6. A paperweight is constructed by slicing through the sphere on a plane that is 2 cm from the centre of the sphere and perpendicular to a radius of the sphere that passes through the centre of the circle formed by the intersection of the plane and the sphere. The paperweight is placed on a table and viewed from directly above by an observer who is 8 cm from the table top, as shown in figure. When viewed through the paperweight, how far away does the table top appear to the observer?



5. One end of a long glass rod having refractive index $\mu = 1.5$ is formed into the shape of a convex surface of radius 6 cm. An object is located in air along the axis of the rod, at a distance of 10 cm from the end of the rod.

(a) How far apart are the object and the image formed by the glass rod?

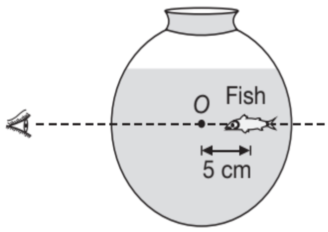
(b) For what range of distances from the end of the rod must the object be located in order to produce a virtual image?

6. An object is at a distance of d = 2.5 cm from the surface of a glass sphere with a radius R = 10 cm. Find the position of the image produced by the sphere. The refractive index of the glass is $\mu = 1.5$.

7. A glass hemisphere of radius 10 cm and refractive index μ = 1.5 is silvered over its curved surface. There is an air bubble in the glass 5 cms from the plane surface along the axis. Find the position of the images of this bubble seen by observer looking along the axis into the flat surface of the hemisphere.

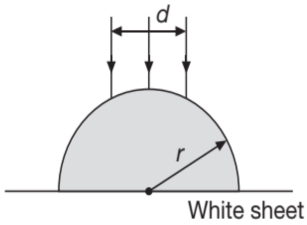
8. A hollow sphere of glass of refractive index μ has a small mark on its interior surface which is observed from a point outside the sphere on the side opposite the centre. The inner cavity is concentric with external surface and the thickness of the glass is everywhere equal to the radius of the inner surface. Prove that the mark will appear nearer than it really is, by a distance $\frac{(\mu-1)R}{(3\mu-1)}$, where R is the radius of the inner surface.

9. Figure shows a fish bowl of radius 10 cm in which along a diametrical line a fish F is moving at speed 2 mms^{-1}. Find the speed of fish as observed by an observer from outside along same line when fish is at a distance 5 cm from the centre of bowl to right of it as shown in figure.

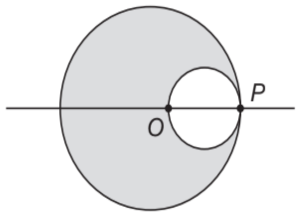


10. A parallel incident beam falls on a solid glass sphere at near normal incidence. Calculate the image distance in terms of refractive index μ of the sphere and its radius R.

11. Figure shows a glass hemisphere placed on a white horizontal sheet. A vertical paraxial light beam of diameter d incident on the curved surface of hemisphere as shown. Find the diameter of the light spot formed on sheet after refraction.



12. A transparent sphere of radius R has a cavity of radius $\frac{R}{2}$ as shown in figure. Find the refractive index of the sphere if a parallel beam of light falling on left surface focuses at point P.



13. A glass sphere ($\mu = 1.5$) with a radius of 15.0 cm has a tiny air bubble 5 cm above its centre. The sphere is viewed looking down along the extended radius containing the bubble. What is the apparent depth of the bubble below the surface of the sphere?

THIN SPHERICAL LENSES

A **lens** is a piece of transparent material with two refracting surfaces, at least one of them being curved. It may have one surface plane.

A **spherical lens** has spherical surfaces as bounds. If the thickness of the lens is small (compared to the radius of curvature of spherical surfaces, the object distance, the image distance, etc.), it is said to be **thin**.

There are two types of lenses:

(a) convex or converging lenses,

(b) concave or diverging lenses.

NAMING CONVENTION FOR LENSES

While naming a lens, the surface with larger radius of curvature is named first. The lens has a nature of the surface that has the smaller radius of curvature.

EXAMPLE:

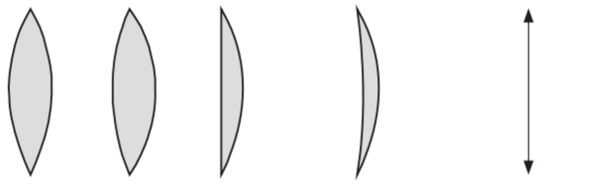
A lens with one surface plane and the other surface convex will be named as Plano-Convex irrespective of its placement and this lens will have converging nature (the same as the nature of the surface having smaller radius of curvature).

Similarly a Convexo-Concave lens will have a diverging nature and Concavo-Convex lens will have a converging nature.

To summarise, we can say that the first name of a bifocal lens is derived from the name of the surface with bigger radius of curvature and the last name of the lens is derived from the nature of the lens.

CONVEX OR CONVERGING LENSES

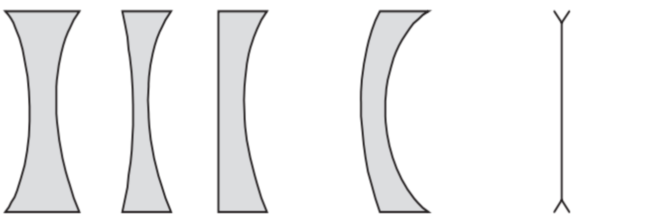
Converging lenses convert a parallel beam of incident rays into a convergent beam. Converging lenses are convex, i.e. such that the thickness at the middle is larger than the thickness of edges. A convex lens is thicker in the centre than at its edges. They include convexo-convex, plano-convex, and cancavo-convex lenses. Sometimes, a converging lens is represented symbolically by a double headed arrow as shown.



Equi-convex | Bi-convex | Plano-convex | Concavo-convex (Convex meniscus) | Symbolic representation

CONCAVE OR DIVERGING LENSES

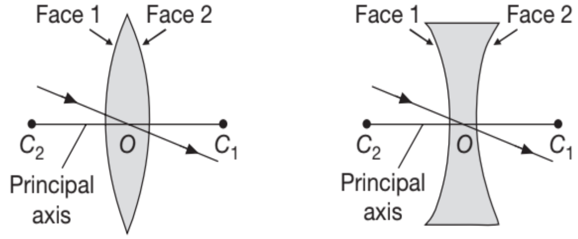
Diverging lenses convert a parallel beam of rays into a divergent beam. Diverging lenses are concave, i.e. such that the thickness at their edges is larger than the thickness at the middle. A concave lens is thinner at the centre. They include concavo-concave, plano-concave and convexo-concave lenses. Sometimes, a diverging lens is represented symbolically by a line with inverted arrows at its two ends.



Equi-concave | Bi-concave | Plano-concave | Convexo-concave (Concave meniscus) | Symbolic representation

OPTICAL CENTRE OF LENS

The central portion of a lens (both convex and concave) behaves as a flat slab. Optical centre O is a point through which any ray passes undeviated.



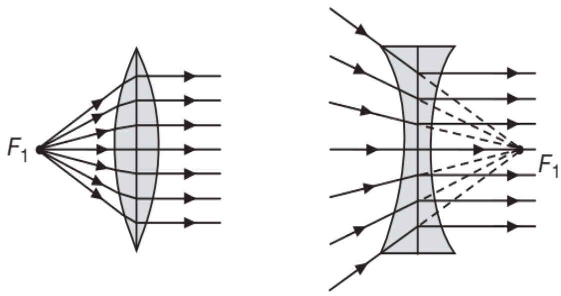
Convex lens | Concave lens

PRINCIPAL AXIS OF A LENS

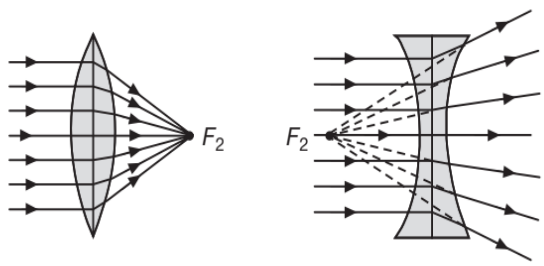
The line joining the centres of curvature C_1C_2 is called the **principal axis** of the lens.

PRINCIPAL FOCUS

A lens has two focal points. The **first focal point** F_1 is a point object on the principal axis for which the image is at infinity.



The **second focal point** F_2 is a point image on the principal axis for which the object is at infinity.

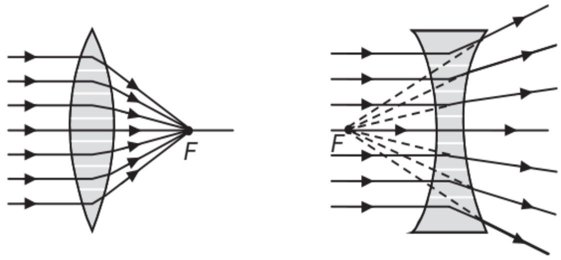


Focal length (f) is the distance between O and the second focus F_2.

Aperture is the effective diameter of the light transmitting area of the lens. The intensity of the image formed by the lens,

$$I \propto (\text{Aperture})^2$$

Converging and diverging action of a lens is due to the fact that a lens may be thought of a combination of small prisms, as shown in figure. A parallel beam of light, when incident on a convex lens, converges to a point called **focus** F.



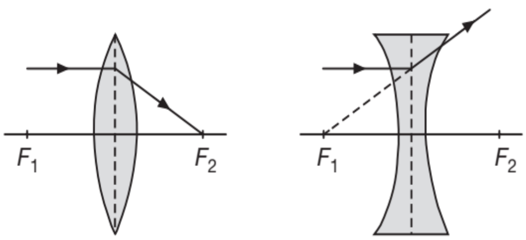
Converging action　　Diverging action

A concave lens diverges a parallel beam of light. It appears to be diverging from a point F, called **focus**.

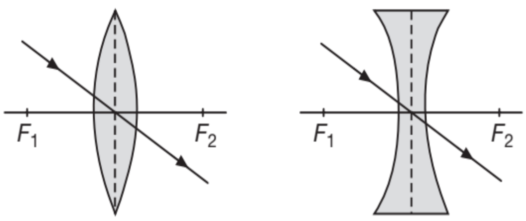
For thin lenses, we need not consider refraction of light at the two surfaces separately. Instead, we say that the light-ray is bent (towards the principal axis in case of convex lens, and away from the principal axis in case of concave lens) when it passes through a thin lens.

RULES FOR OBTAINING IMAGES IN LENSES

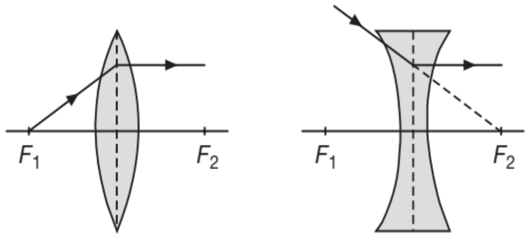
1. A **ray parallel to the principal axis**, after refraction through the lens, converges to the focus (in case of a convex lens) or appears to diverge from the focus (in case of a concave lens).



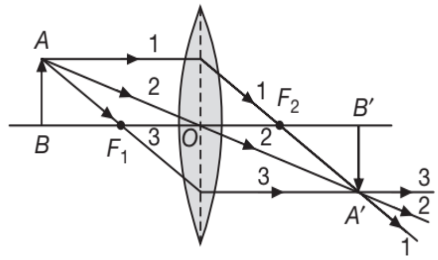
2. A **ray passing through the optical centre** goes through the lens undeviated.



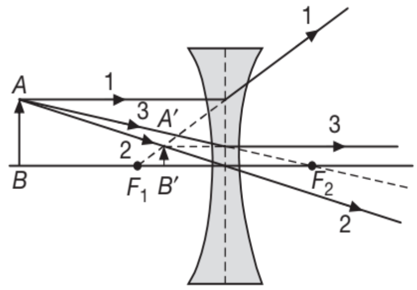
3. A **ray passing through the focus** (in case of a convex lens) or appearing to pass through the focus (in case of a concave lens) is rendered parallel to the principal axis after refraction through the lens.



Any two of the above three rays can be used to obtain the location of the image.



Convex lens



Concave lens

THIN LENS FORMULA FOR A CONVEX LENS

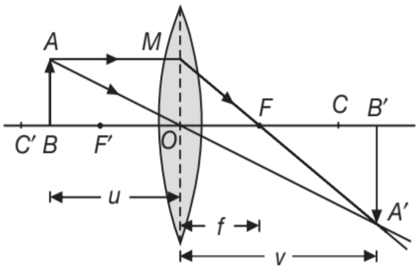
Assumptions used in the derivation of lens formula

(a) The lens used is thin.
(b) The aperture of the lens is small.
(c) The incident and refracted rays make small angles with the principal axis.
(d) The object is a small object placed on the principal axis.

CASE-1: When a real image is formed

Consider an object AB placed perpendicular to the principal axis of a thin convex lens between its F' and C'. A real, inverted and magnified image $A'B'$

is formed beyond C on the other side of the lens, as shown in figure.



Since, $\Delta A'B'O$ and ΔABO are similar,

$$\Rightarrow \frac{A'B'}{AB}=\frac{OB'}{OB} \quad \ldots(1)$$

Also $\Delta A'B'F$ and ΔMOF are similar,

$$\Rightarrow \frac{A'B'}{MO}=\frac{FB'}{OF}$$

Since $MO = AB$,

$$\Rightarrow \frac{A'B'}{AB}=\frac{FB'}{OF} \quad \ldots(2)$$

From (1) and (2), we get

$$\frac{OB'}{OB}=\frac{FB'}{OF}=\frac{OB'-OF}{OF}$$

Using new Cartesian sign convention, we get

Object distance, $OB=-u$

Image distance, $OB'=+v$

Focal length, $OF=+f$

$$\Rightarrow \frac{v}{-u}=\frac{v-f}{f}$$

$$\Rightarrow vf=-uv+uf$$

$$\Rightarrow uv=uf-vf$$

Dividing both sides by uvf, we get

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$$

CASE-2: When a virtual image is formed

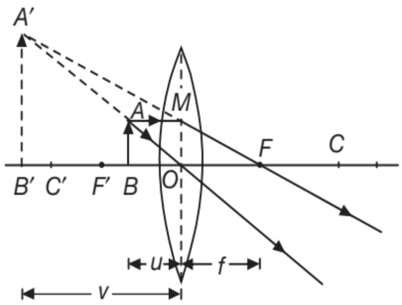
When an object AB is placed between the optical centre O and the focus F of a convex lens, the image $A'B'$ formed by the convex lens is virtual, erect and magnified as shown in figure.

Since, triangles $A'B'O$ and ABO are similar, so we have

$$\frac{A'B'}{AB}=\frac{OB'}{OB} \quad \ldots(1)$$

Also, triangles $A'B'F$ and MOF are similar, so we have

$$\frac{A'B'}{OM}=\frac{B'F}{OF}$$



Since $OM = AB$

$$\Rightarrow \frac{A'B'}{AB}=\frac{B'F}{OF} \quad \ldots(2)$$

From (1) and (2), we get

$$\frac{OB'}{OB}=\frac{B'F}{OF}=\frac{OB'+OF}{OF}$$

Using new Cartesian sign convention,

Object distance $BO=-u$

Image distance $OB'=-v$

Focal length $OF=+f$

$$\Rightarrow \frac{-v}{-u}=\frac{-v+f}{f}$$

$$\Rightarrow -vf=uv-uf$$

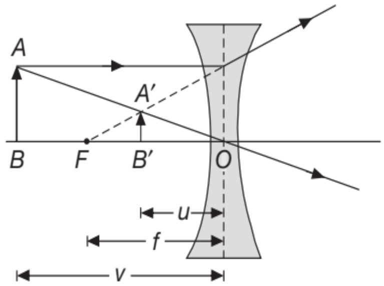
$$\Rightarrow uv=uf-vf$$

Dividing both sides by uvf, we again get

$$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$$

THIN LENS FORMULA FOR A CONCAVE LENS

Let O be the optical centre and F be the principal focus of concave lens of focal length f. AB is an object placed perpendicular to its principal axis. A virtual, erect and diminished image $A'B'$ is formed due to refraction through the lens.



Since, $\Delta A'B'O$ and ΔABO are similar

So, $\dfrac{A'B'}{AB} = \dfrac{OB'}{OB}$ ...(1)

Also, $\Delta A'B'F$ and ΔMOF are similar

So, $\dfrac{A'B'}{OM} = \dfrac{FB'}{OF}$

Since $OM = AB$, therefore

$$\frac{A'B'}{AB} = \frac{FB'}{OF} \qquad ...(2)$$

From (1) and (2), we get

$$\frac{OB'}{OB} = \frac{FB'}{OF} = \frac{OF - OB'}{OF}$$

Using new Cartesian sign convention, we get

$$OB = -u\,,\ OB' = -v\,,\ OF = -f$$

$$\Rightarrow \quad \frac{-v}{-u} = \frac{-f + v}{-f}$$

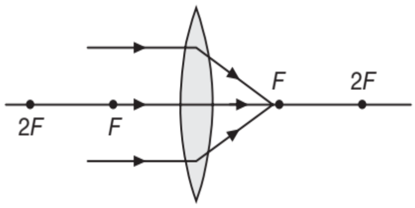
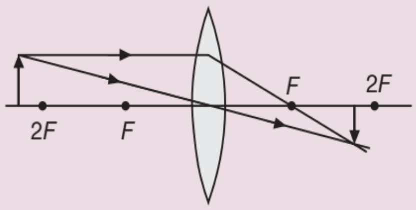
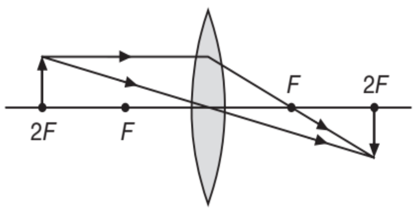
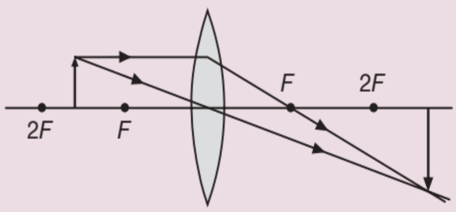
$$\Rightarrow \quad vf = uf - uv$$

$$\Rightarrow \quad uv = uf - vf$$

Dividing both sides by uvf, we again get

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$

IMAGE FORMATION BY CONVEX LENS

OBJECT POSITION	DIAGRAM	POSITION OF IMAGE	NATURE AND SIZE OF IMAGE
At infinity		At the principal Focus (F) or in the focal plane	Real, inverted and extremely diminished
Beyond $2F$		Between F and $2F$	Real, inverted and diminished
At $2F$		At $2F$	Real, inverted and of same size as the object
Between F and $2F$		Beyond $2F$	Real, inverted and magnified

(Continued)

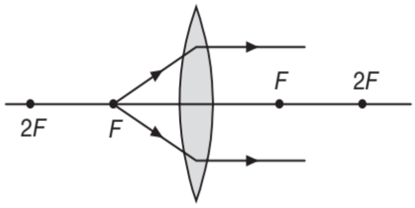
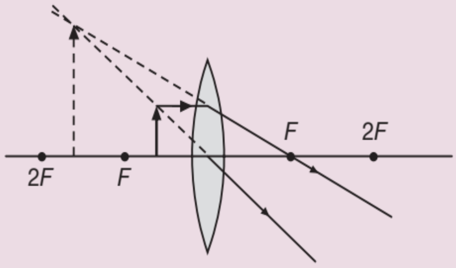
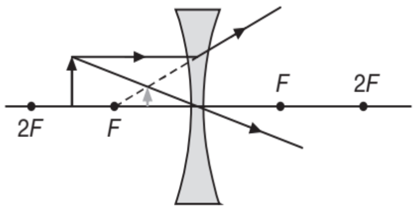
OBJECT POSITION	DIAGRAM	POSITION OF IMAGE	NATURE AND SIZE OF IMAGE
At F		At infinity	Real, inverted and highly magnified
Between F and optical centre		On the same side as the object	Virtual, erect and magnified

IMAGE FORMATION BY A CONCAVE LENS

OBJECT POSITION	DIAGRAM	POSITION OF IMAGE	NATURE AND SIZE OF IMAGE
For all positions of object		Images formed between the optical centre of the lens and the focus (F)	Always forms a Virtual, Erect and Diminished Image

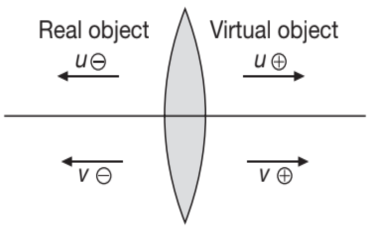
VARIATION CURVES OF IMAGE DISTANCE VS OBJECT DISTANCE FOR A THIN LENS

For Convex Lens

For a lens, we have $\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$

For convex lens of focal length f, we have $f=+f$.

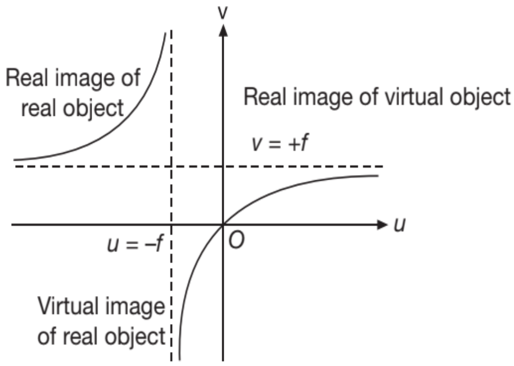
$$\Rightarrow \quad v=\frac{uf}{u+f} \qquad \ldots(1)$$



Substituting the following values of u in equation (1) to get the corresponding values of v for purpose of plotting the u-v graph.

u	$-\infty$	$-2f$	$-f$	$-\frac{f}{2}$	$-\frac{f}{4}$	0	$+\frac{f}{2}$	$+\frac{f}{4}$	$+f$	$+2f$	$+\infty$
v	$+f$	$+2f$	$+\infty$	$-f$	$-\frac{f}{3}$	0	$\frac{f}{3}$	$\frac{f}{5}$	$+\frac{f}{2}$	$+\frac{2f}{3}$	$+f$

The above function can be plotted as shown in figure for a convex lens.

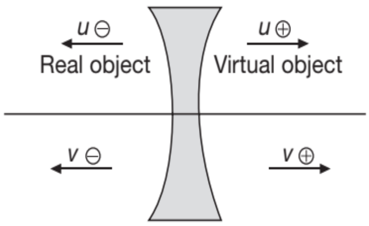


For Concave Lens

For a lens, we have $\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$

For concave lens of focal length *f*, we have $f=-f$.

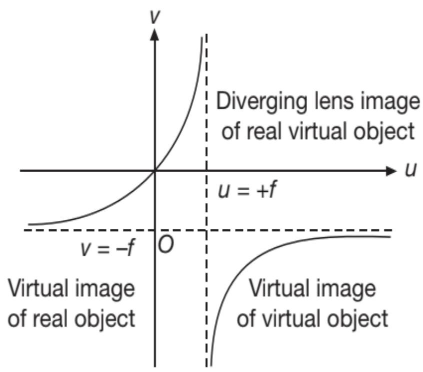
$$\Rightarrow \quad v=\frac{uf}{f-u} \qquad \ldots(1)$$



Substituting the following values of u in equation (1) to get the corresponding values of v for purpose of plotting the u-v graph.

u	$-\infty$	$-2f$	$-f$	$-\frac{f}{2}$	$-\frac{f}{4}$	0	$+\frac{f}{4}$	$+\frac{f}{2}$	$+f$	$+2f$	$+\infty$
v	$-f$	$-\frac{2f}{3}$	$-\frac{f}{2}$	$-\frac{f}{3}$	$-\frac{f}{5}$	0	$\frac{f}{3}$	$+f$	$+\infty$	$-2f$	$-f$

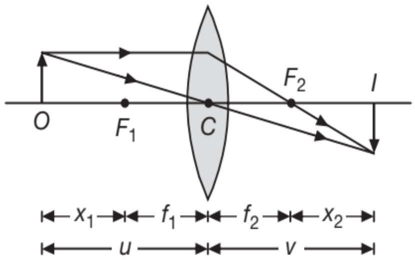
The above function can be plotted as shown in figure for a concave lens.



NEWTON'S FORMULA

If the distance of object and image are not measured from optical centre (C), but from first and second principal foci respectively, and if x_1 is the distance of the object from the first focus x_2 is the distance of the image from the second focus and if f is the focal length of the lens, then we have

$$u=-(f+x_1),\; v=f+x_2$$



According to the lens formula, we have

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

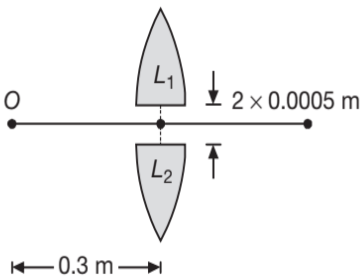
$$\Rightarrow \quad \frac{1}{f+x_2}-\frac{1}{-(f+x_1)}=\frac{1}{f}$$

$$\Rightarrow \quad x_1x_2=f^2$$

This is called Newton's formula.

ILLUSTRATION 79

A point object O is placed at a distance of 0.3 m from a convex lens of focal length 0.2 m. It then cut into two halves each of which is displaced by 0.0005 m as shown in figure. Find the position of the image. If more than one image is formed, find their number and the distance between them.



SOLUTION

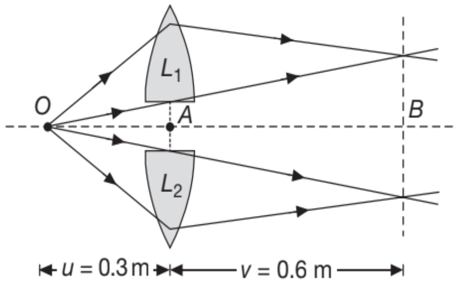
Each part will work as a separate lens and will form its own image. For any part, we have $u=-0.3$ m, $f=+0.2$ m.

Therefore, from lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we have

$$\frac{1}{v}-\frac{1}{0.3}=\frac{1}{0.2}$$

$$\Rightarrow \quad v=0.6\text{ m}$$

So, each part forms a real image of the point object O at 0.6 m from the lens, as shown in figure.



Since the triangles OL_1L_2 and OI_1I_2 are similar. So, we have

$$\frac{I_1I_2}{L_1L_2}=\frac{OB}{OA}=\frac{u+v}{u}$$

$$\Rightarrow \quad \frac{I_1I_2}{L_1L_2}=\frac{0.3+0.6}{0.3}=\frac{0.9}{0.3}=3$$

$$\Rightarrow \quad I_1I_2=3(L_1L_2)=3(2\times 0.0005)=0.003\text{ m}$$

ILLUSTRATION 80

An object is placed 45 cm from a converging lens of focal length 30 cm. A mirror of radius 40 cm is to be placed on the other side of lens so that the object coincides with its image. Find the position of the mirror if it is

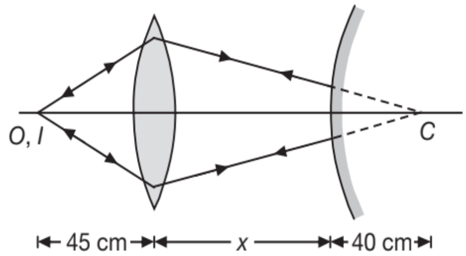
(a) convex?
(b) concave?

SOLUTION

The object and image will coincide only if the light ray retraces its path and it will happen only when the ray strikes the mirror normally. In other words the centre of the curvature of the mirror and the rays incident on the mirror are collinear.

(a) The rays after refraction from lens must be directed towards the centre of curvature of mirror at C. If x is the separation, then for the lens

$$u=-45\text{ cm},\ v=x+40,\ f=30\text{ cm}$$

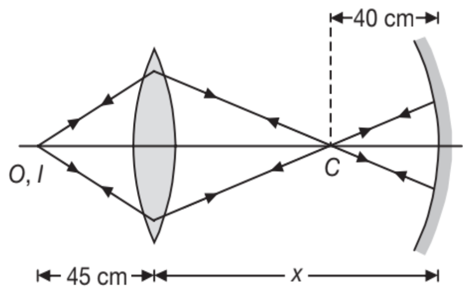


Using lens formula $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$

$$\Rightarrow \quad \frac{1}{x+40}-\frac{1}{-45}=\frac{1}{30}$$

$$\Rightarrow \quad x=\frac{45(30)}{45-30}-40=50\text{ cm}$$

(b) In case of concave mirror, the refracted rays from lens meet at C, the centre of curvature (C) of the mirror.



Using lens formula $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$ where $u=-45$ cm, $v=x-40$, $f=30$ cm, we get

$$\frac{1}{x-40}-\frac{1}{-45}=\frac{1}{30}$$

$$\Rightarrow \quad x-40=\frac{45\times 30}{45-30}$$

$$\Rightarrow \quad x=90+40=130\text{ cm}$$

ILLUSTRATION 81

A lens with a focal length $f=30$ cm placed at a distance of $a=40$ cm from the object produces a sharp image of an object on the screen. A plane parallel plate with thickness of $d=9$ cm is placed between the lens and the object perpendicular to the optical axis of the lens. Through what distance should the screen be shifted for the image of the object to remain distinct? The refractive index of the glass of the plate is $\mu=1.8$.

SOLUTION

In the first case,

$$\frac{1}{v_1}+\frac{1}{40}=\frac{1}{30}$$

$$\Rightarrow \quad v_1=120 \text{ cm}$$

In the second case, shift due to the glass slab is given by

$$\Delta x=\left(1-\frac{1}{\mu}\right)t=\left(1-\frac{1}{1.8}\right)9=4 \text{ cm}$$

i.e., u will now become $(40-4)=36$ cm, so now we have

$$\frac{1}{v_2}+\frac{1}{36}=\frac{1}{30}$$

$$\Rightarrow \quad v_2=180 \text{ cm}$$

Therefore, the screen will have to be shifted 60 cm away from the lens.

ILLUSTRATION 82

Find the distance of an object from a convex lens of focal length 10 cm if the image formed is two times the size of object. Focal length of the lens is 10 cm.

SOLUTION

A convex lens forms both type of images, real as well as virtual. Since, nature of the image is not mentioned here, so we will have to consider both the cases.

CASE-1: When image is real

In this case v is positive and u is negative with $|v|=2|u|$, so when $u=-x$ then $v=2x$ and $f=10$ cm

Substituting in $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{2x}+\frac{1}{x}=\frac{1}{10}$$

$$\Rightarrow \quad \frac{3}{2x}=\frac{1}{10}$$

$$\Rightarrow \quad x=15 \text{ cm}$$

$x=15$ cm, means object lies between F and $2F$.

CASE-2: When image is virtual

In this case v and u both are negative. So when $u=-y$ then $v=-2y$ and $f=10$ cm

Substituting in, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\Rightarrow \quad \frac{1}{-2y}+\frac{1}{y}=\frac{1}{10}$$

$$\Rightarrow \quad \frac{1}{2y}=\frac{1}{10}$$

$$\Rightarrow \quad y=5 \text{ cm}$$

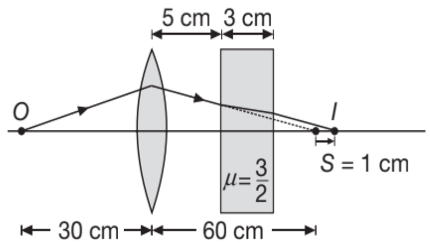
$y=5$ cm, means object lies between F and P.

ILLUSTRATION 83

A convex lens of focal length 20 cm is placed at a distance 5 cm from a glass plate $\left(\mu=\frac{3}{2}\right)$ of thickness 3 cm. An object is placed at a distance 30 cm from lens on the other side of glass plate. Locate the final image produced by this optical setup.

SOLUTION

Figure shows the optical setup described in question and the ray diagram for image formation.



For lens formula to be used in refraction by lens, we use

$$u=-30 \text{ cm}$$

$$f=+20 \text{ cm}$$

$$\Rightarrow \quad \frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v}+\frac{1}{30}=\frac{1}{20}$$

$$\Rightarrow \quad v=\frac{20\times 30}{10}=60 \text{ cm}$$

Shift of image due to refraction by the glass slab is given as

$$S=t\left(1-\frac{1}{\mu}\right)$$

$$\Rightarrow \quad S = 3\left(1-\frac{2}{3}\right) = 1 \text{ cm}$$

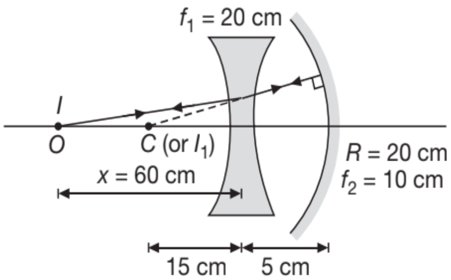
Thus position of final image $= 60+1 = 61$ cm.

ILLUSTRATION 84

A diverging lens of focal length 20 cm is placed coaxially 5 cm toward left of a converging mirror of focal length 10 cm. Where would an object be placed toward left of the lens so that a real image is formed on object itself.

SOLUTION

Due to reflection by a mirror, image of object is formed on itself when reflected rays falls normally on the mirror and retrace the path of incident rays. For this the image produced by the lens must be formed at the centre of curvature of the mirror as shown in ray diagram.



For lens formula, we use

$$u = -x$$
$$f = -20 \text{ cm}$$
$$v = -15 \text{ cm}$$
$$\Rightarrow \quad \frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$
$$\Rightarrow \quad -\frac{1}{15} + \frac{1}{x} = \frac{-1}{20}$$
$$\Rightarrow \quad \frac{1}{x} = \frac{1}{15} - \frac{1}{20} = \frac{4-3}{60} = \frac{1}{60}$$
$$\Rightarrow \quad x = 60 \text{ cm}$$

ILLUSTRATION 85

A thin converging lens of focal length $f = 25.0$ cm forms the image of an object, on the screen, at a distance 5 cm from the lens. The screen is then drawn closer by a distance 18 cm. By what distance should the object be shifted so that its image on the screen is sharp again?

SOLUTION

Since the image is formed on the screen, it is real

Now $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad \frac{1}{u} = \frac{1}{f} - \frac{1}{v} = \frac{v-f}{fv}$$

$$\Rightarrow \quad u = \frac{fv}{v-f} \qquad \ldots(1)$$

In the second case, let the image is formed at $v - \Delta v$. Let the corresponding position of object be $u - \Delta u$. Now

$$u + \Delta u = \frac{f(v-\Delta v)}{(v-\Delta v)-f} \qquad \ldots(2)$$

The shift of the object $u + \Delta u - u = \Delta u$

Subtracting Equation (1) from Equation (2), we get

$$\Delta u = \frac{f(v-\Delta v)}{(v-\Delta v)-f} - \frac{fv}{v-f}$$

$$\Rightarrow \quad \Delta u = \frac{f(v-\Delta v)(v-f) - fv((v-\Delta v)f)}{\{(v-\Delta v)-f\}(v-f)}$$

$$\Rightarrow \quad \Delta u = \frac{f^2 \Delta v}{(v-\Delta v-f)(v-f)}$$

$$\Rightarrow \quad \Delta u = \frac{f^2 \Delta v}{(v-f)^2\left[1-\frac{\Delta v}{(v-f)}\right]}$$

$$\Rightarrow \quad \Delta u = \frac{f^2 \Delta v}{(v-f)^2}\left[1-\frac{\Delta v}{(v-f)}\right]^{-1}$$

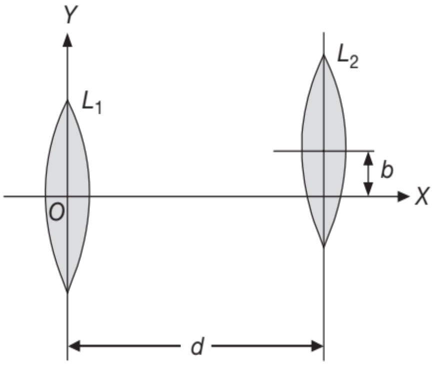
$$\Rightarrow \quad \Delta u \approx \frac{f^2 \Delta v}{(v-f)^2} \qquad \text{(neglecting higher terms)}$$

Substituting the given values, we have

$$\Delta u \approx \frac{(25)^2 \times 18}{(500-25)^2} \approx \frac{(25)^2 \times 18}{(475)^2} \approx 0.5 \text{ mm}$$

ILLUSTRATION 86

Two thin convex lenses of focal lengths f_1 and f_2 are separated by a horizontal distance d (where $d < f_1$, $d < f_2$) and their principal axes are separated by a vertical distance b as shown in the figure. Taking the centre of the first lens (O) as the origin of co-ordinate system and considering a parallel beam of light coming from the left, find the x and y-coordinates of the focal point of this lens system.



SOLUTION

For the refraction through the first lens, we have $u \to \infty$, so

$$v_1 = f_1$$

Since, $d < f_2$, the first image (formed by L_1) lies to the right of second lens L_2, so

$$u_2 = +(f_1 - d)$$

Applying Lens Formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$, we get

$$\frac{1}{v_2} - \frac{1}{(f_1 - d)} = \frac{1}{f_2}$$

$$\Rightarrow \quad v_2 = \frac{f_2(f_1 - d)}{f_1 + f_2 - d}$$

$$\Rightarrow \quad x = v_2 + d = \frac{f_1 f_2 + d(f_1 - d)}{f_1 + f_2 - d}$$

Magnification for second lens is given by

$$m = \frac{v_2}{u_2} = \frac{f_2}{f_1 + f_2 - d}$$

The image due to the second lens is formed below its principal axis and is of the size mb. So, the y coordinate of the focal point system is given by

$$y = b - mb$$

$$\Rightarrow \quad y = b - \frac{f_2 b}{f_1 + f_2 - d}$$

$$\Rightarrow \quad y = \frac{(f_1 - d)b}{f_1 + f_2 - d}$$

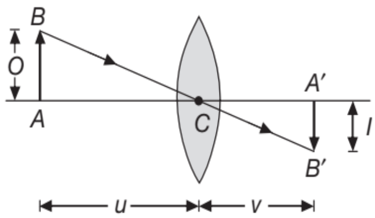
So, the coordinates of the focal point of this system are

$$(x,\ y) = \left[\frac{f_1 f_2 + d(f_1 - d)}{f_1 + f_2 - d},\ \frac{(f_1 - d)b}{f_1 + f_2 - d}\right]$$

LINEAR OR LATERAL OR TRANSVERSE MAGNIFICATION (*m*)

The linear magnification (also called lateral or transverse magnification) m produced by a lens is defined as the ratio of the height of image to the height of the object. So,

$$m = \frac{\text{Height of Image}}{\text{Height of Object}} = \frac{I}{O}$$



Since triangles ABC and $A'B'C$ are similar, so

$$\frac{A'B'}{AB} = \frac{CA'}{CA}$$

Using Conventions, we get

$$A'B' = -I,\ AB = O,\ CA = -u,\ CA' = +v$$

$$\Rightarrow \quad -\frac{I}{O} = \frac{v}{-u}$$

$$\Rightarrow \quad m = \frac{I}{O} = \frac{v}{u}$$

Please note that for both the lens and mirror we have

m_{real} = NEGATIVE i.e. $m_{real} < 0$

$m_{virtual}$ = POSITIVE i.e. $m_{virtual} > 0$

LONGITUDINAL OR AXIAL MAGNIFICATION BY A THIN LENS

Lateral magnification formula for thin lenses gives the image height above the principal axis of mirror and in this section we will discuss about the image width along the principal axis of a thin lens. The relation in object and image width along the principal axis of mirror is called Longitudinal Magnification as given below.

Longitudinal magnification of image is given as

$$m_L = \frac{\text{Width of Image along Principal Axis}}{\text{Width of Object along Principal Axis}}$$

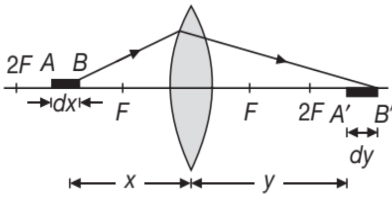


Figure shows image formation of an object located at a distance x from the convex lens of focal length f which produces an image of this object at a distance y which is real inverted and enlarged because object was placed between F and $2F$ points. Here we can see that object edge A was close to C so corresponding image edge A' is also closer to C. If we consider object is of very small width dx and image produced is having a width dy then from lens formula we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

Here by coordinate sign convention we use $u = -x$, $f = +f$ and $v = +y$

$$\frac{1}{f} = \frac{1}{y} - \frac{1}{-x}$$

Differentiating this expression we get

$$0 = -\frac{1}{x^2}dx - \frac{1}{y^2}dy$$

From this relation we can get the Longitudinal Magnification as

$$m_L = \frac{dy}{dx} = -\frac{y^2}{x^2} = -m^2 \qquad \ldots(1)$$

For small width object if image is produced by a thin lens (converging or diverging) then image width can be calculated by using the Equation (1). But if object size is large then this relation cannot be used and in that case we need to calculate the image of both edges of the object along principal axis and take the difference of the image distances obtained.

Conceptual Note(s)

(a) Linear/Transverse/Lateral Magnification produced by a lens is

$$m = \frac{I}{O} = \frac{v}{u} = \frac{f}{f+u} = \frac{f-v}{f}$$

where I is size of image perpendicular to Principal Axis and O is size of object perpendicular to principal Axis.

(b) Axial Magnification: Axial magnification is the ratio of the size of image along the principal axis to the size of the object along the principal axis. So

$$m_{axial} = \frac{\text{Size of Image along Principal Axis}}{\text{Size of Object along Principal Axis}} = \frac{dv}{du}$$

$$\Rightarrow \quad m_{axial} = \frac{dv}{du} = \frac{v^2}{u^2} = m^2$$

(c) Areal Magnification: Areal magnification is the ratio of the area of image to the area of object.

$$m_{areal} = \frac{\text{Area of Image}}{\text{Area of Object}} = \frac{A_I}{A_O}$$

$$m_{areal} = \frac{A_I}{A_o} = \frac{v^2}{u^2} = m^2$$

EFFECT OF MOTION OF OBJECT AND LENS ON IMAGE

When object or lens is in motion the distance between object and lens changes which affects the position and size of image. To find the image velocity and for analysis of image's motion we can differentiate the lens formula and find the rate at which distances between image and lens is changing. If we consider x and y as object and image distance from pole of mirror of focal length f then by lens formula we have

$$\frac{1}{f} = \frac{1}{y} - \frac{1}{x}$$

Differentiating the above relation with respect to time, we get

$$0 = -\frac{1}{x^2}\frac{dx}{dt} - \frac{1}{y^2}\frac{dy}{dt}$$

where $\frac{dx}{dt}$ is the relative velocity of object parallel to principal axis with respect to the lens and $\frac{dy}{dt}$ is the velocity of image parallel to principal axis with respect to lens.

$$\Rightarrow \quad \frac{dx}{dt} = (v_o)_{\|} \text{ and } \frac{dy}{dt} = (v_i)_{\|}$$

$$\Rightarrow \quad 0 = -\frac{1}{x^2}(v_o)_{\|} - \frac{1}{y^2}(v_i)_{\|}$$

$$\Rightarrow \quad (v_i)_{\|} = -\frac{y^2}{x^2}(v_o)_{\|} = -m^2(v_o)_{\|} \qquad \ldots(1)$$

where m is the linear magnification produced by the mirror. The expression of image speed as given in Equation (1) is valid only for the velocity component of the image and object along the principal axis of the lens.

If the object or mirror is in motion along the direction perpendicular to principal axis, then we can directly differentiate the height of object and image above principal axis which are related to each other as

$$h_i = mh_0 \qquad \ldots(2)$$

Differentiating (2) with respect to time we get

$$\frac{dh_i}{dt} = m\frac{dh_0}{dt}$$

where $\frac{dh_i}{dt} = (v_i)_{\perp}$ and $\frac{dh_0}{dt} = (v_o)_{\perp}$

$$\Rightarrow \quad (v_i)_{\perp} = m(v_o)_{\perp}$$

where $\frac{dh_i}{dt} = (v_i)_{\perp}$ and $\frac{dh_0}{dt} = (v_o)_{\perp}$ are the velocity components of image and object respectively in the direction perpendicular to the principal axis.

ILLUSTRATION 87

An object is approaching a thin convex lens of focal length 0.3 m with a speed of $0.01\ \text{ms}^{-1}$. Find the magnitudes of the rates of change of position and lateral magnification of image when the object is at a distance of 0.4 m from the lens.

SOLUTION

Differentiating the lens formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ with respect to time, we get

$$-\frac{1}{v^2}\frac{dv}{dt} + \frac{1}{u^2}\frac{du}{dt} = 0 \qquad \{\because f = \text{constant}\}$$

$$\Rightarrow \quad \left(\frac{dv}{dt}\right) = \left(\frac{v^2}{u^2}\right)\frac{du}{dt} \qquad \ldots(1)$$

Further, substituting proper values in lens formula, we get

$$\frac{1}{v} + \frac{1}{0.4} = \frac{1}{0.3} \qquad \{\because u = -0.4 \text{ m}, f = 0.3 \text{ m}\}$$

$$\Rightarrow \quad v = 1.2 \text{ m}$$

Substituting the values in equation (1), we get
Magnitude of rate of change of position of image is

$$\frac{dv}{dt} = 0.09\ \text{ms}^{-1}$$

Lateral magnification, $m = \frac{v}{u}$

Rate of change of lateral magnification is given by

$$\frac{dm}{dt} = \frac{u\frac{dv}{dt} - v\frac{du}{dt}}{u^2} = \frac{(-0.4)(0.09) - (1.2)(0.01)}{(0.4)^2}$$

$$\Rightarrow \quad \frac{dm}{dt} = -0.3 \text{ per second}$$

$\Rightarrow$ Magnitude of rate of change of lateral magnification is

$$\frac{dm}{dt} = 0.3 \text{ per second}$$

ILLUSTRATION 88

A small pin of size 5 mm is placed along principal axis of a convex lens of focal length 6 cm at a distance 11 cm from the lens. Find the size of image of pin.

SOLUTION

For lens formula, we have

$$u = -11 \text{ cm}$$

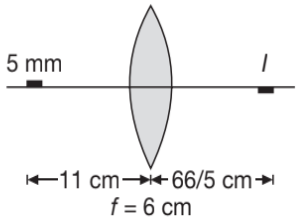
$$f = +6 \text{ cm}$$

$$\Rightarrow \quad \frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v} + \frac{1}{11} = \frac{1}{6}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{1}{6}-\frac{1}{11}=\frac{5}{66}$$

$$\Rightarrow \quad v=\frac{66}{5} \text{ cm}$$



Magnification by lens is given as

$$m=\frac{v}{u}=\frac{66}{5\times 11}=\frac{6}{5}$$

Longitudinal magnification is

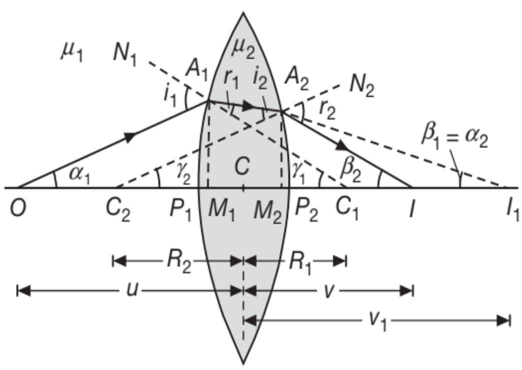
$$m_L=m^2=\frac{36}{25}$$

$$\text{Image size }=\frac{36}{25}\times 5 \text{ mm}$$

$$\Rightarrow \quad \text{Image size }=\frac{36}{5}\text{ mm}=7.2\text{ mm}$$

LENS MAKER'S FORMULA FOR THIN LENS

Consider a thin lens having its optical centre at C and let O be the point object situated on its principal axis as shown in figure. Light starting from O strikes the first surface of the lens at A_1 and heads towards I_1, however, refraction takes place at the second surface, thereby giving a final real image at I.



Consider refraction at the first surface only. Let α_1, β_1 and γ_1 be the angles which the incident ray (OA_1), refracted ray (A_1I_1) and normal (A_1C_1) make with the principal axis.

According the general law of refraction, applied at A_1, we get

$$\mu_1 \sin i_1=\mu_2 \sin r_1$$

Since the angles are small, so $\sin i_1 \cong i_1$ and $\sin r_1 \cong r_1$

$$\Rightarrow \quad \mu_1 i_1=\mu_2 r_1 \quad \ldots(1)$$

In ΔA_1C_1O, we have $i_1=\gamma_1+\alpha_1$

In $\Delta A_1C_1I_1$, we have $\gamma_1=r_1+\beta$

$$\Rightarrow \quad r_1=\gamma_1-\beta_1$$

Substituting for i_1 and r_1 in equation (1), we get

$$\mu_1(\gamma_1+\alpha_1)=\mu_2(\gamma_1-\beta_1)$$

$$\Rightarrow \quad \mu_2\beta_1+\mu_1\alpha_1=(\mu_2-\mu_1)\gamma_1$$

Since the angles are small, so they can be replaced by their tangents.

$$\Rightarrow \quad \mu_2\tan\beta_1+\mu_1\tan\alpha_1=(\mu_2-\mu_1)\tan\gamma_1$$

$$\Rightarrow \quad \mu_2\left(\frac{A_1M_1}{M_1I_1}\right)+\mu_1\left(\frac{A_1M_1}{M_1O}\right)=(\mu_2-\mu_1)\left(\frac{A_1M_1}{M_1C_1}\right)$$

For a thin lens, M_1 lies close to C. Therefore, all the distances measured from M_1 can be replaced by those measured from C. Hence we have

$$\frac{\mu_2}{CI_1}+\frac{\mu_1}{CO}=\frac{\mu_2-\mu_1}{CC_1} \quad \ldots(2)$$

Now consider refraction at the second surface. The ray A_1A_2 which is the refracted ray for first surface becomes the incident ray for second surface. Applying general law of refraction at A_2 (light going from denser to rarer medium), we get

$$\mu_2\sin i_2=\mu_1\sin r_2$$

$$\Rightarrow \quad \mu_2 i_2=\mu_1 r_2 \quad \ldots(3)$$

In $\Delta A_2C_2I_1$ we have $i_2=\gamma_2+\beta_1$

In ΔA_2C_2I, we have $r_2=\gamma_2+\beta_2$

Substituting for i_2 and r_2 in equation (2), we get

$$\mu_2(\gamma_2+\beta_1)=\mu_1(\gamma_2+\beta_2)$$

$$\Rightarrow \quad \mu_1\beta_2-\mu_2\beta_1=(\mu_2-\mu_1)\gamma_2$$

Since angles are small, so replacing the angles by their tangents, we get

$$\mu_1\tan\beta_2-\mu_2\tan\beta_1=(\mu_2-\mu_1)\tan\gamma_2$$

$$\Rightarrow \quad \mu_1\left(\frac{A_2M_2}{M_2I}\right)-\mu_2\left(\frac{A_2M_2}{M_2I_1}\right)=(\mu_2-\mu_1)\left(\frac{A_2M_2}{M_2C_2}\right)$$

Since the lens is thin, M_2 lies close to C, so we get

$$\frac{\mu_1}{CI}-\frac{\mu_2}{CI_1}=\frac{\mu_2-\mu_1}{CC_2} \quad \ldots(4)$$

Adding equations (2) and (4), we get

$$\frac{\mu_2}{CI_1}+\frac{\mu_1}{CO}+\frac{\mu_1}{CI}-\frac{\mu_2}{CI_1}=\frac{\mu_2-\mu_1}{CC_1}+\frac{\mu_2-\mu_1}{CC_2}$$

$$\Rightarrow \quad \mu_1\left(\frac{1}{CO}+\frac{1}{CI}\right)=(\mu_2-\mu_1)\left(\frac{1}{CC_1}+\frac{1}{CC_2}\right)$$

$$\Rightarrow \quad \frac{1}{CO}+\frac{1}{CI}=\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left(\frac{1}{CC_1}+\frac{1}{CC_2}\right)$$

$$\Rightarrow \quad \frac{1}{CO}+\frac{1}{CI}=\left(\frac{\mu_2}{\mu_1}-1\right)\left(\frac{1}{CC_1}+\frac{1}{CC_2}\right)$$

$$\Rightarrow \quad \frac{1}{CO}+\frac{1}{CI}=\left({}^1\mu_2-1\right)\left(\frac{1}{CC_1}+\frac{1}{CC_2}\right)$$

Applying sign convention, we have

$$CO=-u,\ CI=+v,\ CC_1=+R_1,\ CC_2=-R_2$$

$$\Rightarrow \quad \frac{1}{-u}+\frac{1}{v}=\left({}^1\mu_2-1\right)\left(\frac{1}{R_1}+\frac{1}{-R_2}\right)$$

$$\Rightarrow \quad \frac{1}{v}-\frac{1}{u}=\left({}^1\mu_2-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \quad \ldots(5)$$

Since, focal length of a convex lens is defined as the distance of that point from the centre of lens where a beam coming parallel to principal axis comes to focus after refraction through the lens, so when

$u \to \infty$ we have $v = f$

Substituting in equation (5), we get

$$\frac{1}{f}=\left({}^1\mu_2-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \quad \ldots(6)$$

If the first medium is air, then ${}^1\mu_2=\mu$, so we have

$$\frac{1}{f}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

This formula is called Lens Maker's Formula.

Problem Solving Technique(s)

(a) A concave lens forms virtual, erect and diminished image.

(b) A convex lens may form real and virtual images. The real image is inverted, it may be diminished or magnified while virtual image formed by convex lens is erect and enlarged.

(c) Different Media on either side of Lens

If a lens of refractive index μ_2 has different media on either side, the medium of object space has refractive index μ_1 and that of image space has refractive index μ_3, then focal length f of lens is

$$\frac{\mu_3}{f}=\frac{\mu_2-\mu_1}{R_1}+\frac{\mu_3-\mu_2}{R_2}$$

As a special case if we put $\mu_3=\mu_1$, we get the Lens Maker's Formula.

ILLUSTRATION 89

A plano-convex lens has a thickness of **4 cm**. When placed on a horizontal table, with the curved surface in contact with it, the apparent depth of the bottom most point of the lens is found to be **3 cm**. If the lens is inverted such that the plane face is in contact with the table, the apparent depth of the centre of the plane face is found to be $\frac{25}{8}$ cm. Find the focal length of the lens. Assume thickness to be negligible while finding its focal length.

SOLUTION

When placed on a horizontal table with curved surface in contact with it.

In this case refraction of the rays starting from O takes place from a plane surface as shown in Figure 1.1.

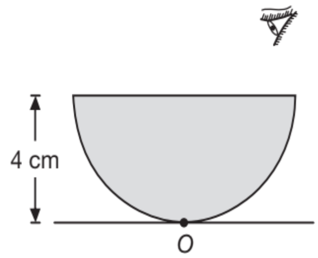


Figure 1.1

So, we can use

$$\text{Apparent Depth} = \frac{\text{Real Depth}}{\mu}$$

$$\Rightarrow \quad 3=\frac{4}{\mu}$$

$$\Rightarrow \quad \mu=\frac{4}{3}$$

When the plane surface is in contact with the horizontal table.

In this case refraction takes place from a spherical surface as shown in Figure 1.2.

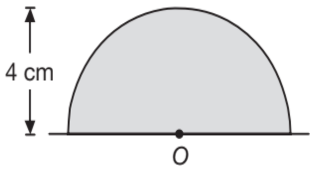


Figure 1.2

Hence, applying

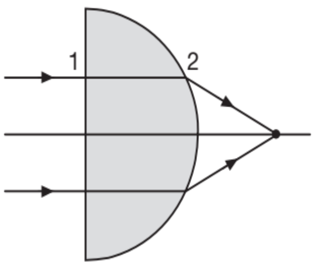
$$\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}\text{, we get}$$

$$\frac{1}{-\frac{25}{8}}-\frac{\frac{4}{3}}{-4}=\frac{1-\frac{4}{3}}{-R}$$

$$\Rightarrow \quad \frac{1}{3R}=\frac{1}{3}-\frac{8}{25}=\frac{1}{75}$$

$$\Rightarrow \quad R=25\text{ cm}$$

To find the focal length, since we know that the parallel rays incident on the lens will converge at the focus of the lens. So using the lens maker formula, we get



$$\frac{1}{f}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\Rightarrow \quad \frac{1}{f}=\left(\frac{4}{3}-1\right)\left(\frac{1}{\infty}-\frac{1}{-25}\right)=\frac{1}{75}$$

$$\Rightarrow \quad f=75\text{ cm}$$

ILLUSTRATION 90

A point source of light is placed inside water and a thin converging lens of refractive index μ_2 is placed just outside the plane surface of water. The image of the source is formed at a distance x from the surface of water. If the lens is now placed just inside water and the image is now formed at a distance x' from the surface of water, show that

$$\frac{1}{x}-\frac{1}{x'}=\left(\frac{\mu_1-1}{\mu_2-1}\right)\frac{1}{f}$$

where f is the focal length of the lens and μ_1 is the refractive index of water.

SOLUTION

Let f and f_w be the focal lengths of the lens when it is outside and inside the water respectively, then

$$\frac{1}{f}=(\mu_2-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \ldots(1)$$

$$\text{and} \quad \frac{1}{f_w}=\left(\frac{\mu_2}{\mu_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \ldots(2)$$

When the lens is in air, let u be the distance of the object from the surface, then we use apparent depth of object to be $\frac{u}{\mu_1}$ from the lens. Using lens formula, we get

$$\frac{1}{x}-\frac{\mu_1}{u}=\frac{1}{f} \qquad \ldots(3)$$

When the lens is in water, then image is formed at a distance x' from lens, so due to refraction from water surface the final image is formed at a distance $\mu_1 x$ from the lens. Again using lens formula, we get

$$\frac{1}{\mu_1 x'}-\frac{1}{u}=\frac{1}{f_w} \qquad \ldots(4)$$

Multiplying equation (4) by μ_1, we get

$$\frac{1}{x'}-\frac{\mu_1}{u}=\frac{\mu_1}{f_W} \qquad \ldots(5)$$

Subtracting equations (5) and (3), we get

$$\frac{1}{x}-\frac{1}{x'}=\frac{1}{f}-\frac{\mu_1}{f_w} \qquad \ldots(6)$$

From equation (2), we have

$$\frac{1}{f_w}=\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

From equation (1), we have

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\frac{1}{f(\mu_2-1)}$$

$$\Rightarrow \quad \frac{1}{f_w}=\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\frac{1}{f(\mu_2-1)}$$

Substituting this value in equation (6), we get

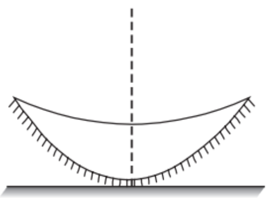
$$\frac{1}{x}-\frac{1}{x'}=\frac{1}{f}-\frac{1}{\mu_1 f}\left(\frac{\mu_2-\mu_1}{\mu_2-1}\right)$$

$$\Rightarrow \quad \frac{1}{x}-\frac{1}{x'}=\frac{1}{f}\left(1-\frac{\mu_2-\mu_1}{\mu_2-1}\right)$$

$$\Rightarrow \quad \frac{1}{x}-\frac{1}{x'}=\frac{1}{f}\left(\frac{\mu_1-1}{\mu_2-1}\right)$$

ILLUSTRATION 91

The convex surface of a thin concavo-convex lens of glass of the refractive index 1.5 has a radius of curvature 20 cm. The concave surface has a radius of curvature 60 cm. The convex side is silvered and placed on a horizontal surface.



(a) Where should a pin be placed on the optic axis such that its image is formed at the same place?

(b) If the concave part is filled with water of refractive index $4/3$, find the distance through which the pin should be moved, so that the image of the pin again coincides with the pin.

SOLUTION

(a) Image of object will coincide with it, if the ray of light after refraction from the concave surface falls normally on concave mirror so formed by silvering the convex surface i.e., image after refraction from concave surface should be formed at centre of curvature of concave mirror or at a distance of 20 cm on same side of the combination. Let x be the distance of pin from the given optical system.

Applying, $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$, we get for

$v=-20$ cm, $u=-x$, $R=-60$ cm

$$\frac{1.5}{-20}-\frac{1}{-x}=\frac{1.5-1}{-60}$$

$$\Rightarrow \quad \frac{1}{x}=\frac{3}{40}-\frac{1}{120}=\frac{8}{120}$$

$$\Rightarrow \quad x=\frac{120}{8}=15 \text{ cm}$$

(b) When the concave part is filled with water, then before striking with the concave surface, the ray is first refracted from a plane surface. So, let x be the distance of pin, then the plane surface will form its image at a distance of $h_{app}=\mu h$ i.e., $\frac{4x}{3}$ from it.

Now, using $\frac{\mu_2}{v}-\frac{\mu_1}{u}=\frac{\mu_2-\mu_1}{R}$ with proper signs, we get

$$\frac{1.5}{-20}-\frac{\frac{4}{3}}{\left(\frac{4}{3}x\right)}=\frac{1.5-\frac{4}{3}}{-60}$$

$$\Rightarrow \quad \frac{1}{x}=\frac{-3}{40}+\frac{1}{360}=\frac{-26}{360}$$

$$\Rightarrow \quad x=\frac{360}{-26}=-13.84 \text{ cm}$$

LENS IMMERSED IN A LIQUID

If a lens (made of glass) of refractive index μ_g is immersed in a liquid of refractive index μ_l, then its focal length in liquid, f_l is given by

$$\frac{1}{f_l}=\left({}^{l}\mu_g-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

If f_a is the focal length of lens of air, then

$$\frac{1}{f_a}=\left({}^{a}\mu_g-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \{\because {}^{a}\mu_g=\mu_g\}$$

$$\Rightarrow \quad f_l=\left[\frac{\mu_g-1}{\frac{\mu_g}{\mu_l}-1}\right]f_a$$

Now three cases arise which are discussed here.

(a) If $\mu_g>\mu_l$, then f_l and f_a are of same sign and $f_l>f_a$.

That is the nature of lens remains unchanged, but its focal length increases and hence power of lens decreases. In other words the convergent lens becomes less convergent and divergent lens becomes less divergent.

(b) If $\mu_g = \mu_l$, then $f_l \to \infty$ and the lens behaves as a simple glass plate.

(c) If $\mu_g < \mu_l$, then f_l and f_a have opposite signs and the nature of lens changes i.e. a convergent lens becomes divergent and vice versa.

ILLUSTRATION 92

A lens has a power of $+5$ dioptre in air. Calculate its power if it is completely immersed in water? Given $\mu_g = \dfrac{3}{2}$ and $\mu_w = \dfrac{4}{3}$.

SOLUTION

Let f_a and f_w be the focal lengths of the lens in air and water respectively, then

$$P_a = \frac{1}{f_a} \quad \text{and} \quad P_w = \frac{\mu_w}{f_w}$$

Since lens has power $+5$ D in air, so

$$f_a = \frac{1}{P} = \frac{1}{5} = 0.2 \text{ m} = 20 \text{ cm}$$

Using Lens Maker's formula, we get

$$\frac{1}{f_a} = (\mu_g - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \ldots(1)$$

Similarly, $$\frac{1}{f_w} = \left(\frac{\mu_g}{\mu_w} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\Rightarrow \quad P_w = \frac{\mu_w}{f_w} = (\mu_g - \mu_w)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \ldots(2)$$

Dividing equation (2) by equation (1), we get,

$$\frac{P_w}{P_a} = \frac{(\mu_g - \mu_w)}{(\mu_g - 1)} = \frac{1}{3}$$

$$\Rightarrow \quad P_w = \frac{P_a}{3} = +\frac{5}{3} \text{ D}$$

DISPLACEMENT METHOD

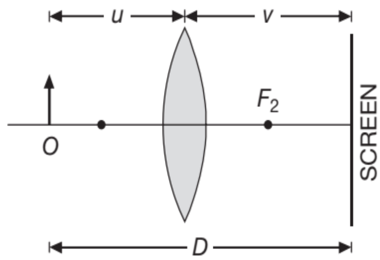
Consider an object and a screen fixed at a distance D apart. Let a lens of focal length f be placed between the object and the screen.

From figure we observe that

$$u + v = D \quad \Rightarrow \quad v = D - u$$

Also from Lens formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$



$$\Rightarrow \quad \frac{1}{D-u} - \frac{1}{-u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{D-u} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad Df = (D-u)u$$

$$\Rightarrow \quad u^2 - Du + Df = 0$$

$$\Rightarrow \quad u = \frac{D \pm \sqrt{D^2 - 4fD}}{2}$$

For u to be mathematically real,

$$D^2 - 4fD \geq 0$$

$$\Rightarrow \quad D \geq 4f$$

Problem Solving Technique(s)

So, if the object and the screen are placed at a distance less than $4f$, then a virtual image will be formed. Hence, for a real image to be formed $D \geq 4f$

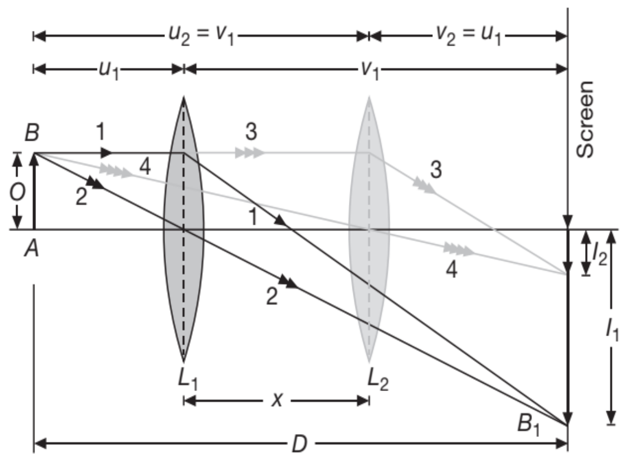
CASE-1: For $D = 4f$

$$u = v = \frac{D}{2}$$

i.e. the lens is placed exactly between the object and the screen.

CASE-2: For $D > 4f$

We get two different position of lens (L_1 and L_2) for which the image of object on the screen is distinct and clear.



L_1 First Position of Lens.

L_2 Second Position of the Same Lens (shown in grey).

Do not develop a misconception that there are two lenses, infact the same lens is displaced through x from position L_1 to L_2.

The object distances for these two positions are given by

$$u_1 = \frac{D - \sqrt{D^2 - 4fD}}{2} \quad \ldots(1)$$

$$u_2 = \frac{D + \sqrt{D^2 - 4fD}}{2} \quad \ldots(2)$$

Since $u + v = D$, so

$$v_1 = \frac{D + \sqrt{D^2 - 4fD}}{2} \quad \ldots(3)$$

$$v_2 = \frac{D - \sqrt{D^2 - 4fD}}{2} \quad \ldots(4)$$

We observe that

$$u_1 = v_2 = u(\text{say}) \quad \ldots(5)$$

$$v_1 = u_2 = v(\text{say}) \quad \ldots(6)$$

Let the lens be displaced through x, then we observe from figure that

$$x = v_1 - u_1 = \sqrt{D^2 - 4fD} \quad \ldots(7)$$

$$\Rightarrow \quad x^2 = D^2 - 4fD$$

$$\Rightarrow \quad \boxed{f = \frac{D^2 - x^2}{4D}} \quad \ldots(8)$$

Using (7) in (1), (2), (3) and (4), we get

$$u_1 = v_2 = \frac{D - x}{2} = u \quad \ldots(9)$$

$$v_1 = u_2 = \frac{D + x}{2} = v \quad \ldots(10)$$

If m_1 is the magnification for the first position of lens i.e. L_1, then

$$m_1 = \frac{I_1}{O} = \frac{v_1}{u_1} = \frac{v}{u} = \frac{D + x}{D - x} \quad \ldots(11)$$

If m_2 is the magnification for the second position of the Lens i.e. L_2, then

$$m_2 = \frac{I_2}{O} = \frac{v_2}{u_2} = \frac{u_1}{v_1} = \frac{u}{v} = \frac{D - x}{D + x} \quad \ldots(12)$$

From (11) and (12), we observe that

$$m_1 m_2 = 1 \quad \ldots(13)$$

So, if magnification for position L_1, is m, then magnification for position L_2 is $\frac{1}{m}$.

Also from (13), we get

$$\frac{I_1}{O}\frac{I_2}{O} = 1$$

$$\Rightarrow \quad O^2 = I_1 I_2$$

$$\Rightarrow \quad O = \sqrt{I_1 I_2}$$

i.e. size of the object (O) is the geometric mean of the sizes of the image for two position of lens L_1 and L_2.

Also,

$$m_1 - m_2 = \frac{D + x}{D - x} - \frac{D - x}{D + x}$$

$$\Rightarrow \quad m_1 - m_2 = \frac{4Dx}{D^2 - x^2}$$

$$\Rightarrow \quad m_1 - m_2 = \frac{x}{\left(\frac{D^2 - x^2}{4D}\right)}$$

$$\Rightarrow \quad m_1 - m_2 = \frac{x}{f}$$

$$\Rightarrow \quad f = \frac{x}{m_1 - m_2}$$

Further if $m_1 = m$, then $m_2 = \frac{1}{m}$

$$\Rightarrow \quad f = \frac{mx}{m^2 - 1}$$

Finally, we observe that

$$\frac{m_1}{m_2} = \left(\frac{D + x}{D - x}\right)^2$$

Problem Solving Technique(s)

Dear Students, you must keep in mind that actually "Displacement Method" is not in the syllabus, but the Examiner generally asks the problems not in its name but by its concept e.g. an examiner's mind may fabricate a problem not having the name Displacement method but then the problem must be having a clue which may state $D > 4f$ or the lens is displaced to get two real images on screen and stuff like that. So, you are advised not to overlook the topic as this is very important (not by name) but by the concept involved.

ILLUSTRATION 93

A thin converging lens of focal length f is moved between a candle and a screen. The distance between the candle and the screen is $D(>4f)$. Show that for two different positions of the lens, two different images can be obtained on the screen. If the ratio of dimensions of the image is β, find the value of $\left(\beta+\frac{1}{\beta}\right)$.

SOLUTION

Let x be the separation between two positions of the lens for which a real image is formed on the screen.

Then, $v+u=D$ ...(1)

and $v-u=x$ ...(2)

Solving we get $u=\frac{D-x}{2}$ and $v=\frac{D+x}{2}$

Now, $m_1=\frac{I_1}{O}=\frac{D+x}{D-x}$

and $m_2=\frac{I_2}{O}=\frac{D-x}{D+x}$

$$\Rightarrow \frac{I_1}{I_2}=\left(\frac{D+x}{D-x}\right)^2=\beta$$

$$\Rightarrow \frac{D+x}{D-x}=\sqrt{\beta}$$

$$\Rightarrow x=\left(\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\right)D$$

Now, $$f=\frac{D^2-x^2}{4D}=\frac{D^2-\left(\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\cdot D\right)^2}{4D}=\frac{D}{2+\sqrt{\beta}+\frac{1}{\sqrt{\beta}}}$$

$$\Rightarrow \sqrt{\beta}+\frac{1}{\sqrt{\beta}}=\frac{D}{f}-2$$

$$\Rightarrow \beta+\frac{1}{\beta}=\left(\frac{D}{f}-2\right)^2-2$$

ILLUSTRATION 94

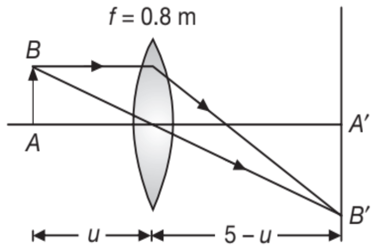
An object is 5 m to the left of a flat screen. A converging lens for which the focal length is $f=0.8$ m is placed between object and screen.

(a) Show that two lens positions exist that form images on the screen and determine how far are these positions from the object?

(b) How do the two images differ from each other?

SOLUTION

(a) Using the lens formula $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get



$$\frac{1}{5-u}-\frac{1}{-u}=\frac{1}{0.8}$$

$$\Rightarrow \frac{1}{5-u}+\frac{1}{u}=\frac{5}{4}$$

$$\Rightarrow \frac{5-u+u}{(5-u)u}=\frac{5}{4}$$

$$\Rightarrow 20=25u-5u^2$$

$$\Rightarrow 5u^2-25u+20=0$$

$$\Rightarrow 5u^2-20u-5u+20=0$$

$$\Rightarrow 5u(u-4)-5(u-4)=0$$

$$\Rightarrow (5u-5)(u-4)=0$$

$$\Rightarrow u=1\text{ m and } u=4\text{ m}$$

Both the values are real, so this means that there exist two positions of lens that form images of object on the screen.

(b) $m=\frac{v}{u}$

$$\Rightarrow m_1=\frac{(5-4)}{(1-4)}=-0.25 \text{ and } m_2=\frac{(5-1)}{(-1)}=-4$$

Hence, both the images are real and inverted, the first has magnification -0.25 and the second -4.

Also, we observe that

$$m_1m_2=(-0.25)(-4)=1$$

ILLUSTRATION 95

For two positions of a converging lens between an object and a screen which are 96 cm apart, two real images are formed. The ratio of the lengths of the two images is 4. Calculate the focal length of the lens.

SOLUTION

Since, $\frac{m_1}{m_2} = 4$

$$\Rightarrow \quad \left(\frac{D+x}{D-x}\right)^2 = 4$$

$$\Rightarrow \quad \frac{D+x}{D-x} = 2$$

Substituting $D = 96$ cm, we get

$$\frac{96+x}{96-x} = 2$$

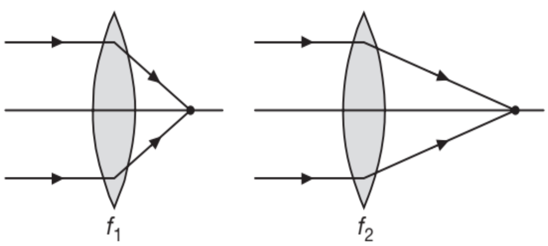
$$\Rightarrow \quad x = 32 \text{ cm}$$

Since, $f = \frac{D^2 - x^2}{4D}$

$$\Rightarrow \quad f = 21.33 \text{ cm}$$

POWER OF A LENS

The power of a lens P is actually the measure of its ability to deviate the incident rays towards axis. The greater the curvature of the two surfaces (i.e., the shorter the focal length f), the greater is the lens action. The shorter the focal length of a lens the more it converges or diverges the light, as shown in figure.



The power of a lens placed in air is actually the reciprocal of the focal length of the lens in metre and is given by

$$P = \frac{1}{f\,(\text{in metre})} = \frac{100}{f\,(\text{in cm})}$$

SI unit of power is dioptre (D).

Power of a lens placed in a medium is defined as

$$P_{\text{med}} = \frac{\mu}{f_{\text{med}}}$$

where μ is the refractive index of the medium and f_{med} is the focal length of the lens in that medium.

As a convention, the power of a converging lens (or **convex lens**) (with focal length positive) is taken to be **positive**. The power of a diverging lens (or **concave lens**) (with focal length negative) is taken to be **negative**.

Also we must note that for a mirror, power is defined as

$$P = -\frac{1}{f\,(\text{in metre})} = -\frac{100}{f\,(\text{in cm})}$$

Thus a convex lens and concave mirror have converging nature and hence they have positive power, whereas the concave lens and convex mirror have diverging nature and hence have negative power.

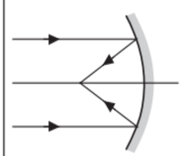
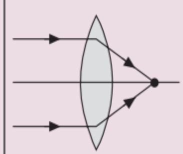
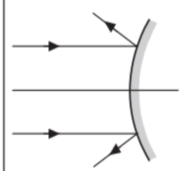
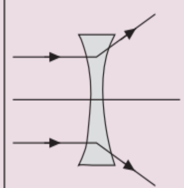
Nature of Lens Mirror	Focal Length (f)	Power $P_{\text{mirror}} = -\frac{1}{f}$, $P_{\text{lens}} = \frac{1}{f}$	Converging/ Diverging	Ray Diagram
Concave mirror	–ve	+ve	Converging	
Convex lens	+ve	+ve	Converging	
Convex mirror	+ve	–ve	Diverging	
Concave lens	–ve	–ve	Diverging	

ILLUSTRATION 96

A convergent lens of power 6 D is combined with a diverging lens of -2 D. Find the power and focal length of the combination.

SOLUTION

Here $P_1 = 6\text{ D}$, $P_2 = -2\text{ D}$

Power of the combination is given by

$$P = P_1 + P_2 = 6 - 2 = 4\text{ D}$$

Since $f = \frac{1}{P}$

$$\Rightarrow \quad f = \frac{1}{4} = 0.25\text{ m} = 25\text{ cm}$$

ILLUSTRATION 97

An object 4 cm high is placed at a distance of 10 cm from a convex lens of focal length 20 cm. Find the position, nature and size of the image. Also find the power of the lens.

SOLUTION

Here, $u = -10\text{ cm}$ (the object assumed to be kept to the left of optical centre)

$f = +20\text{ cm}$ (positive for a convex lens)

$h_1 = +4\text{ cm}$ (object kept above the principal axis)

Using the lens formula, we get

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v} - \frac{1}{-10} = \frac{1}{20}$$

$$\Rightarrow \quad v = -20\text{ cm}$$

That is, the image is 20 cm from the lens, on the same side as the object. Hence, the image is **virtual**. The linear magnification,

$$m = \frac{h_2}{h_1} = \frac{v}{u}$$

So, size of the image is

$$h_2 = h_1\left(\frac{v}{u}\right) = 4 \times \frac{-20}{-10} = 8\text{ cm}$$

The positive sign indicates that the image is **erect** (and **virtual**).

Since, the power of the lens is given by

$$P = \frac{1}{f(\text{in m})} = \frac{1}{+0.2} = +5\text{ D}$$

ILLUSTRATION 98

A converging lens forms a five folds magnified image of an object. The screen is moved towards the object by a distance $d = 0.5\text{ m}$ and the lens is shifted so that the image has the same size as the object. Find the lens power and the initial distance between the object and the screen.

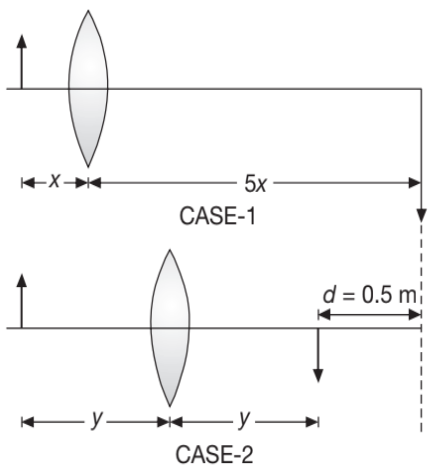
SOLUTION

In the first case image is five times magnified. Hence

$$|v| = 5|u|$$

In the second case image and object are of equal size. Hence

$$|v| = |u|$$



From the two figures, we get

$$6x = 2y + d$$

$$\Rightarrow \quad 6x - 2y = 0.5 \qquad \ldots(1)$$

Using the lens formula for both the cases, we get for

CASE-1,

$$\frac{1}{5x} - \frac{1}{-x} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{6}{5x} = \frac{1}{f} \qquad \ldots(2)$$

CASE-2,

$$\frac{1}{y}-\frac{1}{-y}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{2}{y}=\frac{1}{f} \qquad \ldots(3)$$

Solving these three equations, we get

$$x = 0.1875 \text{ m and } f = 0.15625 \text{ m}$$

Therefore, initial distance between the object and the screen is

$$6x = 1.125 \text{ m}$$

Power of the lens, $P=\frac{1}{f}=\frac{1}{0.15625}\text{ D} = 6.4\text{ D}$

LENSES IN CONTACT

If two or more lenses of focal lengths $f_1, f_2, \ldots$ are placed in contact, then their equivalent focal length f is given by

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}+\ldots=\sum_{i=1}^{n}\frac{1}{f_i}$$

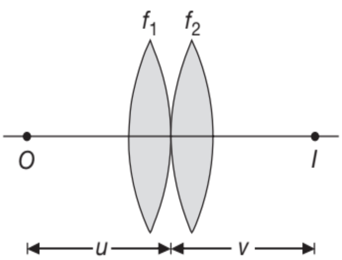
where f_1, f_2,.... are to be substituted with proper signs attached.
The power of combination

$$P = P_1 + P_2 + \ldots = \sum_{i=1}^{n} P_i$$

Here too, P_1, P_2,..... are to be substituted with proper signs attached.
The magnification of the combination is

$$M = m_1 \times m_2 \times \ldots = \prod_{i=1}^{n} m$$

In many optical instruments, the combination of lenses in contact are used so as to improve the performance of the instrument.



Consider two lenses of focal lengths f_1 and f_2 kept in contact. Let a point object O be placed at a distance u from the combination. The first image (say I_1) after refraction from the first lens is formed at a distance v_1 (whatever may be the sign of v_1) from the combination. This image I_1 acts as an object for the second lens and let v be the distance of the final image from the combination. Applying the lens formula $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

For the first lens, $\frac{1}{v_1}-\frac{1}{u}=\frac{1}{f_1}$ ...(1)

and for the second lens, $\frac{1}{v}-\frac{1}{v_1}=\frac{1}{f_2}$ ...(2)

Adding equations, (1) and (2), we get

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f} \text{ (say)}$$

where, f is the equivalent focal length of the combination. Thus,

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

TWO THIN LENSES SEPARATED BY A DISTANCE

If two thin lenses of focal lengths f_1, f_2 are placed at a distance x apart, then equivalent focal length of combination is

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{x}{f_1 f_2}$$

or Power for the combination is

$$P = P_1 + P_2 - xP_1P_2$$

The net magnification of the combination is still remains

$$m = m_1 \times m_2$$

ILLUSTRATION 99

Consider a co-axial system of two thin convex lenses of focal length f each separated by a distance d. Draw ray diagrams for image formation corresponding to an object at infinity placed on the principal axis in the following cases: (a) $d < f$ (b) $d = f$ (c) $f < d < 2f$ (d) $d = 2f$ and (e) $d > 2f$. Indicate the nature of the combination (concave, convex or plane) in each case.

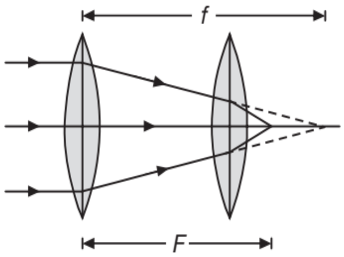
SOLUTION

The formula

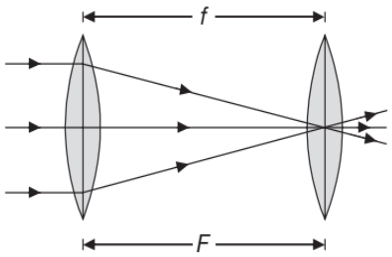
$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}$$

is valid only for small values of d compared to f_1 and f_2. Therefore, we cannot use this formula in the given cases. However, we can draw the ray diagram to decide the nature of the combination.

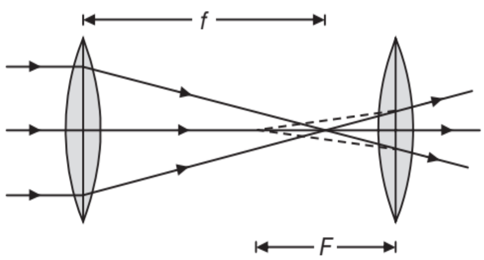
(a) When $d<f$: The ray diagram is shown in figure. The out-coming rays are convergent. Obviously, the combination is a convex lens with $F<f$.



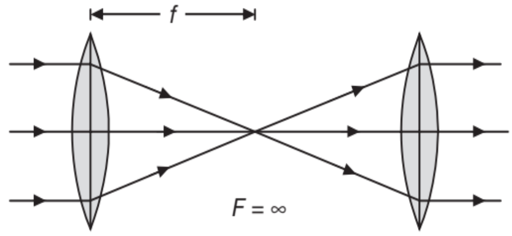
(b) When $d=f$: The incident parallel beam converges to a point and then passes without any more deviation. The combination behaves like a convex lens of $F=f$.



(c) When $f<d<2f$: The incident parallel beam emerges out as a divergent beam. the combination behaves as a divergent or concave lens.



(d) When $d=2f$: The incident parallel beam emerges out as a parallel beam but inverted. The combination behaves as a plane glass slab, which inverts the beam.



(e) When $d>2f$: The incident parallel beam emerges out as a convergent beam. The combination behaves as a convergent or convex lens.

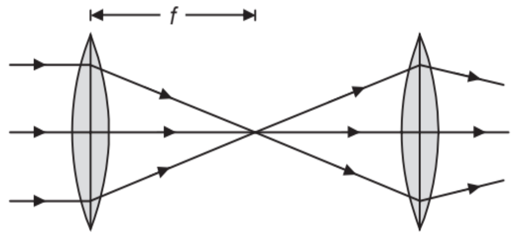


ILLUSTRATION 100

Two equi-convex lenses of focal lengths 30 cm and 70 cm, made of material of refractive index $=1.5$, are held in contact coaxially by a rubber band round their edges. A liquid of refractive index 1.3 is introduced in the space between the lenses filling it completely. Find the position of the image of a luminous point object placed on the axis of the combination lens at a distance of 90 cm from it.

SOLUTION

According to Lens Maker's Formula, we have

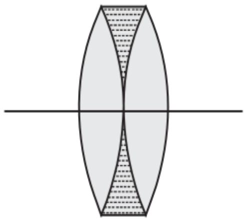
$$\frac{1}{30}=(1.5-1)\left(\frac{1}{R_1}-\frac{1}{-R_1}\right)$$

$\Rightarrow \quad R_1=30\text{ cm}$

Similarly, radius of curvature of the second lens is 70 cm. Since

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3} \quad \ldots(1)$$

Here, $f_1=30\text{ cm}$, $f_2=70\text{ cm}$



Now f_3 is calculated again using the Lens Maker's Formula, so we get

$$\frac{1}{f_3}=(1.3-1)\left(\frac{1}{-30}-\frac{1}{70}\right)$$

$\Rightarrow \quad f_3=-70$ cm

$\Rightarrow \quad F=30$ cm {from equation (1)}

According to Lens formula, applied on the combination of lenses, we have

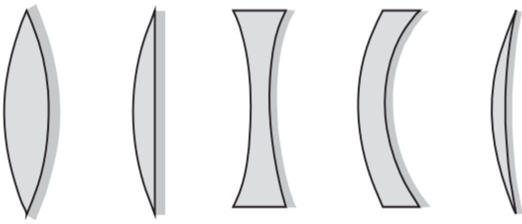
$$\frac{1}{v}-\frac{1}{u}=\frac{1}{F}$$

$$\Rightarrow \quad \frac{1}{v}-\frac{1}{(-90)}=\frac{1}{30}$$

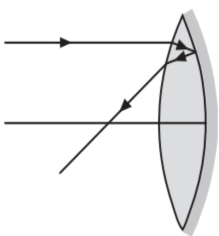
$\Rightarrow \quad v=45$ cm

LENSES WITH ONE SILVERED SURFACE

When one face of a lens is silvered as shown in figure it acts like a lens-mirror combination.



It is obvious from the ray diagram as shown in figure that the incident ray of light is refracted through the lens twice (i.e., once when light is incident on the lens and second time when reflected by the mirror) and reflected from the mirror once.



The combination acts like a mirror whose effective power is given by

$$P_{net}=2P_l+P_m$$

where P_l is the power of the lens and P_m is the power of the mirror.
Since for a mirror we have

$$P_m=-\frac{1}{f_m}$$

and for a lens, we have

$$P_l=\frac{1}{f_l}=(\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

So, the combination acts like a mirror having net focal length given by

$$F_{net}=-\frac{1}{P_{net}}$$

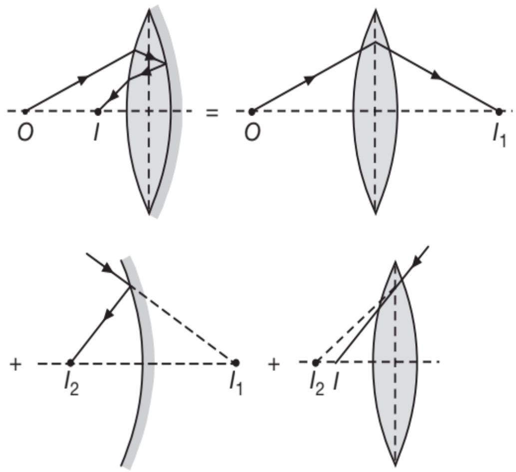
$$\Rightarrow \quad P=-\frac{1}{F}=\frac{2}{f_l}-\frac{1}{f_m}$$

where f_l is focal length of lens and f_m is focal length of spherical mirror formed due to silvering of surface.

To have a fundamental understanding of this we can understand the silvering of lenses using the following arguments.

A ray incident on a lens with its backside silvered will be refracted through the lens twice and will be reflected from the mirror once, as shown.

(a) Light from object O passes through lens to form image I_1.
(b) The image I_1 acts as an object (virtual) for the curved mirror to form image I_2.
(c) The image I_2 acts as an object (virtual) for the lens to form the final image I.



The silvered lens acts like a mirror with equivalent focal length F, given by

$$-\frac{1}{F}=\frac{1}{f_l}-\frac{1}{f_m}+\frac{1}{f_l}=\frac{2}{f_l}-\frac{1}{f_m}$$

where f_l is focal length of lens and f_m is focal length of spherical mirror formed due to silvering of surface.

SIGN CONVENTION

While using the above formula, we make use of the following sign conventions.

(a) f is positive for converging (convex) lens and concave mirror.

(b) f is negative for diverging (concave) lens and convex mirror.

For example, for a plano-convex lens, from Lens Maker's Formula we get

$$\frac{1}{f_l}=(\mu-1)\left(\frac{1}{R}-\frac{1}{\infty}\right)=\frac{\mu-1}{R}$$

$$\Rightarrow \quad f_l=\frac{R}{\mu-1}$$

(a) **when plane surface is silvered, $f_m \to \infty$**

Since we know that

$$-\frac{1}{F}=\frac{2}{f_l}-\frac{1}{\infty}=\frac{2(\mu-1)}{R}$$

$$\Rightarrow F=-\frac{R}{2(\mu-1)}$$

(b) **when convex surface is silvered, then in general we know the relation between radius of curvature and the focal length is given by**

$$f_m=\frac{R}{2}$$

Since we know that

$$-\frac{1}{F}=\frac{2}{f_l}+\frac{2}{R}=\frac{2(\mu-1)}{R}+\frac{2}{R}=\frac{2\mu}{R}$$

$$\Rightarrow F=-\frac{R}{2\mu}$$

ILLUSTRATION 101

The plane surface of a plano-convex lens of focal length 60 cm is silver plated. A point object is placed at a distance 20 cm from the convex face of lens. Find the position and nature of the final image formed.

SOLUTION

Since, $P=2P_l+P_m$

$$\Rightarrow \quad -\frac{1}{F}=\frac{2}{f_l}-\frac{1}{f_m}$$

where, $f_l=+60$ cm and $f_m \to \infty$

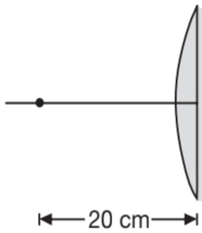
$$\Rightarrow \quad -\frac{1}{F}=\frac{2}{60}-\frac{1}{\infty}=\frac{1}{30}$$

$$\Rightarrow \quad F=-30 \text{ cm}$$

The problem is reduced to a simple case where a point object is placed in front of a concave (converging) mirror of focal length 30 cm.

Using mirror formula i.e.,

$$\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$$



where $u=-20$ cm and $f=-30$ cm

$$\Rightarrow \quad \frac{1}{v}+\frac{1}{-20}=\frac{1}{-30}$$

$$\Rightarrow \quad v=60 \text{ cm}$$

The image is virtual and erect

ILLUSTRATION 102

A concave mirror has the form of a hemisphere with a radius of $R=60$ cm. A thin layer of an unknown transparent liquid is poured into the mirror. The mirror-liquid system forms one real image and another real image is formed by mirror alone of the source in a certain position.

(a) Image produced by combination coincides with the source and that produced by mirror alone is located at a distance of $l=30$ cm from the source away from mirror. Find the refractive index μ of the liquid in this case.

(b) In another case, if the image formed by mirror coincides with the source and that produced by the combination is produced at a distance 30 cm from the source away from mirror, then find the refractive index of the liquid in this case also.

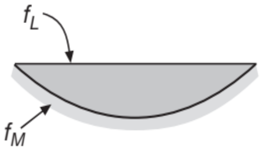
SOLUTION

(a) For concave mirror with unknown liquid, equivalent focal length of the combined mirror is given as

$$\frac{1}{f_{eq}}=\frac{2}{f_L}+\frac{1}{f_M}$$

Where $\frac{1}{f_L} = (\mu - 1)\left(\frac{1}{\infty} - \frac{1}{-60}\right)$

$$\Rightarrow \frac{1}{f_L} = \left(\frac{\mu - 1}{60}\right)$$



and focal length of mirror is

$$f_M = \frac{60}{2} = 30 \text{ cm}$$

Thus equivalent focal length of the combination mirror is given as

$$\frac{1}{f_{eq}} = 2\left(\frac{\mu - 1}{60}\right) + \frac{2}{60} = \frac{2\mu}{60}$$

$$\Rightarrow f_e = \frac{30}{\mu}$$

As image formed by the mirror liquid system coincides with the source, the location of object is at $2f_e$

$$\Rightarrow u = 2f_e = \frac{60}{\mu}$$

According to the given condition, $\frac{60}{\mu} + 30$ is the distance of the image formed by the mirror itself, thus using mirror formula we have

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \frac{1}{v} + \frac{1 \times \mu}{-60} = -\frac{1}{30}$$

$$\Rightarrow v = \frac{60}{\mu - 2} = -\left(\frac{60}{2 - \mu}\right)$$

Image distance from the mirror is $\frac{60}{2 - \mu}$, thus form the already obtained condition we use

$$\frac{60}{\mu} + 30 = \frac{60}{2 - \mu}$$

$$\Rightarrow \mu^2 + 2\mu - 4 = 0$$

$$\Rightarrow \mu = -1 \pm \sqrt{5}$$

Since μ cannot be negative, so

$$\mu = -1 + \sqrt{5}$$

$$\Rightarrow \mu = 2.236 - 1 = 1.236$$

(b) Mirror produces its image on source when the source is located at the centre of curvature thus source position must be at 60 cm from the pole of mirror.

Now we use mirror formula for calculation of image distance for mirror liquid combination

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f_e}$$

$$\Rightarrow \frac{1}{v} + \frac{1}{-60} = -\frac{\mu}{30}$$

$$\Rightarrow v = \frac{60}{1 - 2\mu} = -\left(\frac{60}{2\mu - 1}\right)$$

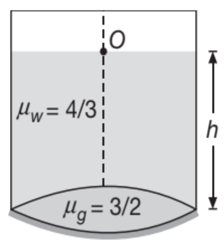
According to the given condition we use

$$\frac{60}{2\mu - 1} + 30 = 60$$

$$\Rightarrow \mu = 1.5$$

ILLUSTRATION 103

Bottom of a glass beaker is made of a thin equi-convex lens having bottom side silver polished as shown in the figure.



Water is filled in the beaker upto a height 4 m. The image of point object, floating at middle point of beaker at the surface of water coincides with it. Find out the radius of curvature of the lens. Given that refractive index of glass is $\frac{3}{2}$ and that of water is $\frac{4}{3}$.

SOLUTION

The silvered lens placed at the bottom of tank behaves like an equivalent mirror and if object is placed at the centre of curvature of the mirror then its image is produced on itself. Here the focal length of the glass lens with respect to water in surrounding is gives as

$$\frac{1}{f_L} = \left(\frac{\frac{3}{2}}{\frac{4}{3}} - 1\right)\left(\frac{1}{R} + \frac{1}{R}\right)$$

$$\Rightarrow \quad \frac{1}{f_L} = \frac{1}{8} \times \frac{2}{R} = \frac{1}{4R}$$

$$\Rightarrow \quad f_2 = 4R$$

Focal length of mirror is $\frac{R}{2}$, so the equivalent focal length of combination is given as

$$\frac{1}{f_{eq}} = \frac{2}{f_L} + \frac{1}{f_M} = \frac{2}{4R} + \frac{2}{R} = \frac{5}{2R}$$

$$\Rightarrow \quad f_{eq} = -\frac{2}{5}R,$$

Thus object is to be placed at $2f_{eq}$ so that its image is produced on itself, thus we have object height given as

$$h = 2 \times \frac{2}{5}R = 4 \text{ m}$$

$$\Rightarrow \quad R = 5 \text{ m}$$

ILLUSTRATION 104

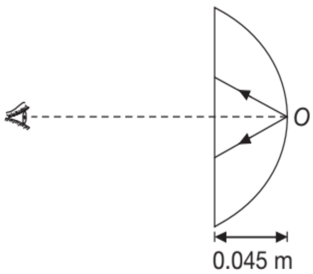
The greatest thickness of a planoconvex lens when viewed normally through the plane surface appears to be 0.03 m and when viewed normally through the curved surface it appears to be 0.036 m. If the actual thickness is 0.045 m, find the

(a) refractive index of the material of the lens.
(b) radius of curvature of lens.
(c) focal length if its plane surface is silvered.
(d) focal length when the curved surface is silvered.

SOLUTION

(a) Since, $\mu = \frac{\text{Real Depth}}{\text{Apparent Depth}}$

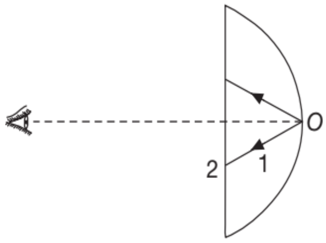
$$\Rightarrow \quad \mu = \frac{d_{\text{actual}}}{d_{\text{app}}} = \frac{0.045}{0.03} = 1.5$$



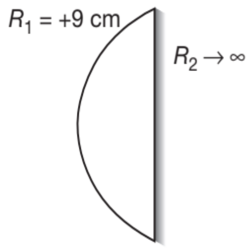
(b) Using, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$, we get

$$\frac{1}{(-0.036)} - \frac{1.5}{(-0.045)} = \frac{1-1.5}{(-R)}$$

$$\Rightarrow \quad R = 0.09 \text{ m} = 9 \text{ cm}$$



(c) If the plane surface is silvered, then



$$\frac{1}{F} = \frac{2}{f_l} + \frac{1}{f_m}$$

But $f_m \to \infty$

$$\Rightarrow \quad \frac{1}{F} = \frac{2}{f_l}$$

where, $\frac{1}{f_l} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{\infty}\right)$

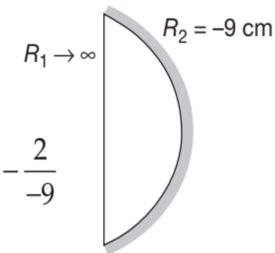
$$\Rightarrow \quad \frac{1}{F} = \frac{2(\mu-1)}{R_1}$$

$$\Rightarrow \quad \frac{1}{F} = \frac{2(1.5-1)}{+9}$$

$$\Rightarrow \quad F = 9 \text{ cm}$$

The nature is given by applying negative sign to the final result. So, this will behave as a concave mirror.

(d) When curved surface is silvered then $R_1 \to \infty$, $R_2 = -9$ cm



$$\frac{1}{F} = \frac{2}{f_l} - \frac{1}{f_m}$$

$$\Rightarrow \quad \frac{1}{F} = 2(\mu-1)\left(\frac{1}{\infty} - \frac{1}{-9}\right) - \frac{2}{-9}$$

$$\Rightarrow \quad \frac{1}{F} = 2\left(\frac{\mu-1}{9}\right) + \frac{2}{9}$$

$$\Rightarrow \quad \frac{1}{F} = \frac{2\mu}{9}$$

$$\Rightarrow \quad \frac{1}{F} = \frac{1.5 \times 2}{+9}$$

$$\Rightarrow \quad f = 3 \text{ cm}$$

Test Your Concepts-VII

Based on Lens Formula

(Solutions on page H.21)

1. The distance between two point sources of light is 24 cm. Find out where would you place a converging lens of focal length 9 cm, so that the images of both the sources are formed at the same point.
2. An object is moved along the principal axis of a convex lens. An image three times the size of the object is obtained when the object is at a distance of 16 cm from the lens and at a distance of 8 cm from the lens. Find the focal length of the lens.
3. The radius of curvature of the convex surface of a plano-convex lens is 10 cm and its focal length is 30 cm. What should be the refractive index of its material?
4. One face of an equi-convex lens ($\mu = 1.5$) of focal length 60 cm is silvered. Does it behave like a concave mirror or convex mirror? Also determine the equivalent focal length of the mirror.
5. A biconvex lens made of glass with a refractive index of $\mu = 1.6$ has a focal length of $f = 10$ cm in air. Calculate the focal length of this lens if it is placed into a transparent medium
   (a) with a refractive index of $\mu_1 = 1.5$
   (b) with a refractive index of $\mu_2 = 1.7$
6. A biconvex thin lens is prepared from glass of refractive index $\frac{3}{2}$. The two bounding surfaces have equal radii of 25 cm each. One of the surfaces is silvered from outside to make it reflecting. Where should an object be placed before this lens so that the image coincides with the object.
7. A converging lens of focal length 5 cm is placed in contact with a diverging lens of focal length 10 cm. Find the combined focal length of the system.
8. A biconvex lens of refractive index 1.5 has a focal length of $f_1 = 10$ cm. One of the lens surfaces having a radius of curvature of $R = 10$ cm is coated with silver. Determine the position of the image if the object is at a distance of $u = 15$ cm from the lens.
9. A convex lens is held 45 cm above the bottom of an empty tank. The image of a point at the bottom of a tank is formed 36 cm above the lens. Now a liquid is poured into the tank to a depth of 40 cm. It is found that the distance of the image of the same point on the bottom of the tank is 48 cm above the lens. Find the refractive index of the liquid.
10. A concave spherical mirror with a radius of curvature of 0.2 m is filled with water. Calculate the focal length of this system? Given that refractive index of water is $\frac{4}{3}$.
11. A convex lens of focal length f_1 is placed in front of a luminous point P so that the distance of the point P from lens is greater than focal length and the image formed is at the shortest possible distance. If now a concave lens of very large focal length f_2 be placed in contact with first, find the shift in the position of the image.
12. At what distance from a biconvex lens of focal length $f = 1$ m should a concave spherical mirror with a radius of curvature of $R = 1$ m be placed for a beam incident on the lens parallel to the major optical axis of the system to leave the lens, remaining parallel to the optical axis, after being reflected from the mirror? Find the image of the object produced by the given optical system.
13. A convex lens of focal length f_1 is placed in front of a luminous point object. The separation between the object and the lens is $3f_1$. A glass slab of thickness t is placed between object and the lens. A real image of the object is formed at the shortest possible distance from the object.
    (a) Find the refractive index of the slab.
    (b) If a concave lens of very large focal length f_2 is placed in contact with the convex lens, find the shifting of the image.
14. An optical system consists of two convergent lenses with focal lengths $f_1 = 20$ cm and $f_2 = 10$ cm. The distance between the lenses is $d = 30$ cm. An object is placed at a distance of $u_1 = 30$ cm from the first lens. At what distance from the second lens will the image be obtained?
15. If r be the radius of curvature of each face of thin converging lens whose one face is silvered and μ is the refractive index of lens material, prove that the lens is equivalent to a concave mirror of focal length $\frac{r}{4\mu - 2}$.

16. Three convergent thin lenses of focal lengths $4a$, a and $4a$ respectively are placed in order along the axis so that the distance between consecutive lenses is $4a$. Prove that this combination simply inverts every small object on the axis without change of magnitude or position.

17. The distance between an object and a divergent lens is m times greater than the focal length of the lens. How many times will the image be smaller than the object?

18. An image I is formed of point object O by a lens whose optic axis is AB as shown in figure.

(a) State whether it is a convex lens or concave?

(b) Draw a ray diagram to locate the lens and its focus.

19. Two thin lenses having focal lengths $f_1 = 7$ cm and $f_2 = 6$ cm are placed at a distance $d = 3$ cm apart. What is the distance of the focus of the system from the second lens? Assume the system to be a centred one.

20. Two glasses with refractive indices of $\mu_1 = 1.5$ and $\mu_2 = 1.7$ are used to make two identical double convex lenses.

(a) Find the ratio between their focal lengths.

(b) How will each of these lenses act on a ray parallel to its optical axis if the lenses are submerged into a transparent liquid with a refractive index of 1.6?

21. A parallel beam of light is incident on a system consisting of three thin lenses with a common optical axis. The focal lengths of the lenses are equal to $f_1 = +10$ cm, $f_2 = -20$ cm and $f_3 = +9$ cm, respectively. The distance between the first and the second lenses is 15 cm and between the second and the third 5 cm. Find the position of the point at which the beam converges when it leaves the system of lenses.

22. Consider a plano-concave lens with one of the radii of curvature r made up of a transparent material whose refractive index varies with intensity (I) of incident light as $\mu = \mu_0 + aI$, where $a > 0$ and $0 < \mu_0 < \dfrac{3}{2}$. Calculate the intensity when the focal length is equal to two times the radius of curvature r.

23. Paraxial rays are incident on surfaces of a thin equi-convex glass lens of refractive index μ and having radius of curvature R. If the final image is formed after n internal reflections, calculate distance of this image from pole of the lens.

24. When the plane surface of a plano-convex lens is silvered it is found that the image of the object pin is formed at the position of the object pin placed at a distance of x_1 from the silvered lens. When the same lens is silvered on the curved surface the image of the object pin is formed at the position of the object pin placed at a distance of x_2 from the silvered lens. Find, in terms of x_1 and x_2, the

(a) focal length of lens

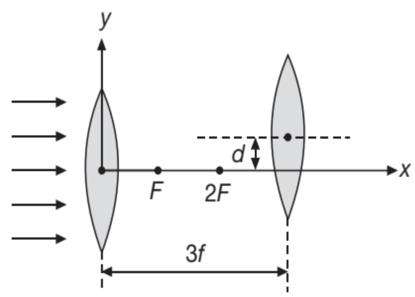
(b) radius of curvature of the curved surface and

(c) index of refraction of the medium of lens.

25. A small fish, 0.4 m below the surface of a lake, is viewed through a simple converging lens of focal length 3 m. The lens is kept at 0.2 m above the water surface such that the fish lies on the optical axis of the lens. Find the position of image of the fish seen by the observer. The refractive index of water is $\dfrac{4}{3}$.

26. Two symmetric double convex lenses A and B have same focal length but the radii of curvature differ so that $R_A = 0.9\,R_B$. If refractive index of A is 1.63, find the refractive index of B.

27. In the figure it is shown, the focal length f of the two thin convex lenses is the same. They are separated by a horizontal distance $3f$ and their optical axes are displaced by a vertical separation $d\,(d \ll f)$, as shown. Taking the origin of coordinates O at the centre of the first lens, find the x and y coordinates of the point where a parallel beam of rays coming from the left finally gets focussed?



DEFECTS OF IMAGES: ABERRATIONS

The simple theory of image formation developed for mirrors and lenses suffers from various approximations. As a result, the actual images formed contain several defects. These defects can be broadly divided in two categories.

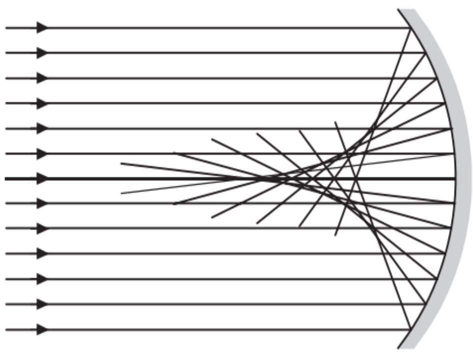
(a) **Monochromatic Aberration:** The defects, which arise when light of a single colour is used, are called **monochromatic aberrations**.

(b) **Chromatic Aberration:** The index of refraction of a transparent medium differs for different wavelengths of the light used. The defects arising from such a variation of the refractive index are termed as **chromatic aberrations**.

MONOCHROMATIC ABERRATIONS

Spherical Aberration

Throughout the discussion of lenses and mirrors with spherical surfaces, it has been assumed that the aperture of the lens or the mirror is small and the light rays of interest make small angles with the principal axis. Only then, it is possible to have a point image of a point object.

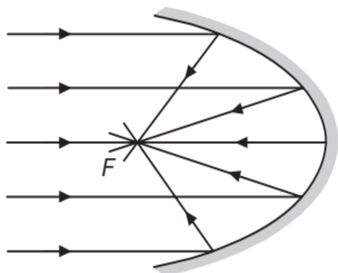


The rays reflect or refract from points at different distances from the principal axis. In general, they meet each other at different points. Thus, the image of a point object is a blurred surface. Such a defect is called **Spherical Aberration**. Figure shows spherical aberration for a concave mirror for an object at infinity. The rays parallel to the principal axis are incident on the spherical surface of the concave mirror. The rays close to the principal axis **(Paraxial Rays)** are focused at the geometrical focus F of the mirror. The rays farthest from the principal axis **(Marginal Rays)** are focused at a point F' somewhat closer to the mirror. The intermediate rays focus at different points between F and F'. Also, the rays reflected from a small portion away from the pole meet at a point off the axis. Thus, a three- dimensional blurred image is formed.

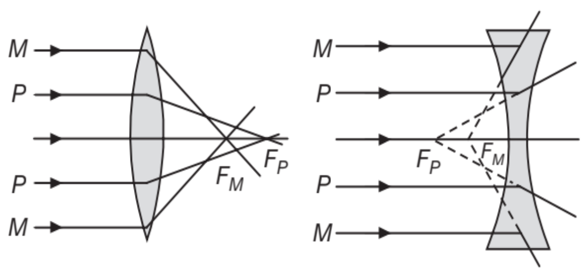
The intersection of this image with the plane of figure is called the **Caustic Curve**.

If a screen is placed perpendicular to the principal axis, a disc image is formed on the screen. As the screen is moved parallel to itself, the disc becomes smallest at one position. This disc is closest to the ideal image and its periphery is called the **Circle of Least Confusion**. The magnitude of spherical aberration may be measured from the distance FF' between the point where the paraxial rays converge and the point where the marginal rays converge.

The parallel rays may be brought to focus at one point if a parabolic mirror is used. Also, if a point source is placed at the focus of a parabolic mirror, the reflected rays will be very nearly parallel. The reflectors used in automobile headlights are made parabolic and the bulb is placed at the focus. The light beam is then nearly parallel and goes up to large distance.



A lens too produces a blurred disc type image of a point object (due to finite aperture of lens). Figure shows the situation for a convex and a concave lens for the rays coming parallel to the principal axis.



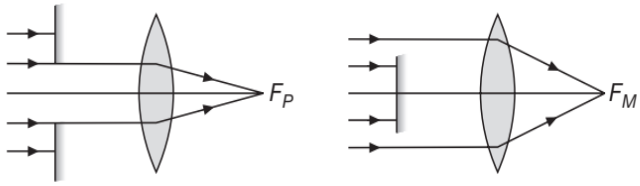
We see from the figure that the marginal rays deviate a bit strongly and hence, they meet at a point different from that given by geometrical optics formulae. Also, in the situation shown, the spherical aberration is opposite for convex and concave lens. The point F_M, where the marginal rays meet, is to the left of the focus for convex lens and is to the right of the focus for the concave lens.

The magnitude of spherical aberration for a lens depends on the radii of curvature and the object distance. For minimum spherical aberration the ratio of radii of curvature of lens is

$$\frac{R_1}{R_2}=\frac{2\mu^2-\mu-4}{\mu(2\mu+1)}$$

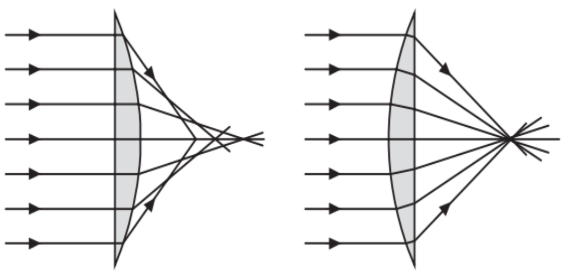
However, it cannot be reduced to zero for a single lens which forms a real image of a real object.

A simple method to reduce spherical aberration is to use a **stop** before and in front of the lens. A stop is an opaque sheet with a small circular opening in it. It only allows a narrow pencil of rays to go through the lens hence reducing the aberration. However, this method reduces the intensity of the image as most of the light is cut off.



(a) (b)

Otherwise, the spherical aberration is less if the total deviation of the rays is distributed over the two surfaces of the lens. Example for this is a planoconvex lens forming the image of a distant object. If the plane surface faces the incident rays, the spherical aberration is much larger than that in the case when the curved surface faces the incident rays. In the former case, the total deviation occurs at a single surface whereas it is distributed at both the surfaces in the latter case.

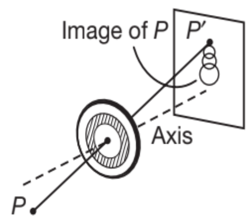


The spherical aberration can also be reduced by using a combination of convex and concave lenses. A suitable combination can reduce the spherical aberration by compensation of positive and negative aberrations. If two thin lenses are separated by a distance d, then condition for minimum spherical aberration is

$$d=f_1-f_2$$

Coma

It has been observed that if a point object is placed on the principal axis of a lens and the image is received on a screen perpendicular to the principal axis, the image has a shape of a disc because of spherical aberration. The basic reason is that the rays passing through different regions of the lens meet the principal axis at different points. If the point object is placed away from the principal axis and the image is received on a screen perpendicular to the axis, the shape of the image is like a comet. This defect is called **Coma**. the lens fails to converge all the rays passing at different distances from the axis at a single point. The paraxial rays form an image of P at P'. The rays passing through the shaded zone forms a circular image on the screen above P'. The rays through outer zones of the lens form bigger circles placed further above P'. The image seen on the screen thus have a comet-like appearance.



Coma can be reduced by properly designing the radii of curvature of lens surfaces. It can also be reduced by using appropriate stops placed at appropriate distance from the lens.

Astigmatism

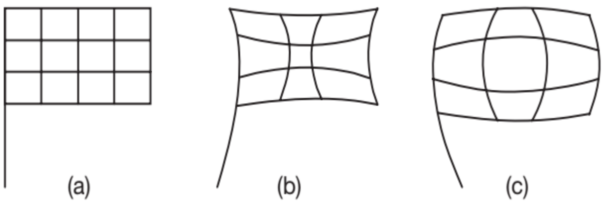
Spherical aberration and coma refer to the spreading of the image of a point object in a plane perpendicular to the principal axis. The image is also spread along the principal axis. Consider a point object placed at a point off the axis of a converging lens. A screen is placed perpendicular to the axis and is moved along the axis. At a certain distance, an approximate line image is focused. If the screen is moved further away, the shape of the image changes but it remains on the screen for quite a distance moved by the screen. The spreading of image along the principal axis is known as **Astigmatism** (you must not confuse this with a defect of vision having the same name).

Curvature

So far we have considered the image formed by a lens on a plane. However, it must be kept in mind that the best image may not be formed along a plane. For a point object placed off the axis, the image is spread both along and perpendicular to the principal axis. The best image is, in general, obtained not on a plane but on a curved surface. This defect is known as **curvature**. It is intrinsically related to astigmatism. The astigmatism or the curvature may be reduced by using proper stops placed at proper locations along the axis.

Distortion

It is the defect arising when extended objects are imaged. Different portions of the object are, in general, at different distances from the axis. The relation between the object distance and the image distance is not linear and hence, the magnification is not the same for all portions of the extended object. Hence a line object is not imaged into a line but into a curve and shown.



Object (a) and its distorted images (b) & (c)

CHROMATIC ABERRATION

The inability of a lens to form the white image of a white object is called **chromatic aberration**. In this case the lens forms coloured images of a white object.

The chromatic aberration arises due to the fact that the focal length of a lens depends upon the refractive index of material of the lens. The lens has different refractive indices for different colours or wavelengths in accordance with Cauchy's formula given by

$$\mu = A + \frac{B}{\lambda^2}$$

Accordingly, the refractive index is maximum for violet $(\lambda = 4000\text{Å})$ and minimum for red $(\lambda = 7800\text{Å})$. Since

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

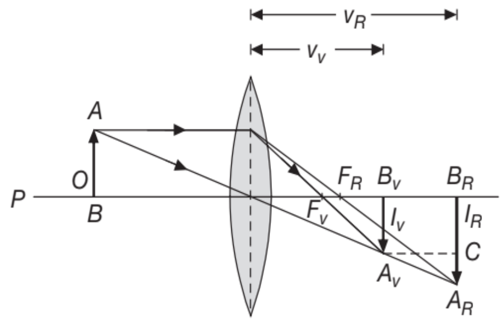
$$\Rightarrow \quad f \propto \frac{1}{\mu - 1}$$

Hence focal length of a lens is maximum for red and minimum for violet

$$\Rightarrow \quad f_{\text{red}} > f_{\text{violet}}.$$

Figure represents the chromatic aberration caused by a lens in the image of an object AB of size O.

F_R and F_V are second principal foci for red and violet colours respectively. The images of object AB are of different sizes and of different colours between $A_V B_V$ and $A_R B_R$. The chromatic aberration is of two types.



Chromatic aberration

Axial or Longitudinal Chromatic Aberration

This is the spread of images along the principal axis and is given by $dv = (v_R - v_V)$ as this spread is very small.

$$v_R - v_V = \frac{\omega v^2}{f}$$

where ω is dispersive power, v is distance of image from lens for mean (yellow) colour and f is mean focal length of lens such that $f = \sqrt{f_V f_R}$

If object is at infinity, then axial chromatic aberration,

$$f_R - f_V = \omega f$$

Lateral Chromatic Aberration

This is the spread of images perpendicular to principal axis and is given by

$$I_R - I_V = \frac{v_R O}{u} - \frac{v_V O}{u} = (v_R - v_V)\frac{O}{u} = \left(\frac{\omega v^2}{f}\right)\frac{O}{u}$$

ACHROMATISM AND ACHROMATIC DOUBLET

The lens system free from chromatic aberration is called **achromatic combination**. This is obtained by using two lenses of different materials and different focal lengths and process is called, **to Achromatise** which satisfies the relation

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$$

$$\Rightarrow \quad \frac{f_1}{f_2} = -\frac{\omega_1}{\omega_2}$$

where ω_1 and ω_2 are dispersive powers of materials of lenses for focal length f_1 and f_2 respectively.

(a) As ω_1 and ω_2 are always positive, therefore f_1/f_2 must be negative. This means the combination must have *one lens convergent and other divergent*.

(b) For the achromatic combination (also called **Achromatic Doublet**) to be convergent, the power of convex lens must be greater or the focal length of convex lens must be smaller than that of concave lens. As dispersive power for crown glass is less than that for flint glass, therefore the convex lens must be made of crown glass while concave lens must be made of flint glass.

Condition for minimum chromatic aberration obtained by two thin lenses of same medium separated by a distance d is

$$d = \frac{f_1 + f_2}{2}$$

ILLUSTRATION 105

An achromatic convergent lens of focal length 150 cm is made by combining flint and crown glass lenses. Calculate the focal lengths of both the lenses and point out which one is divergent, if the ratio of the dispersive powers of flint and crown glasses are 3 : 2.

SOLUTION

For the given combination, we have

$$\frac{1}{150} = \frac{1}{f_1} + \frac{1}{f_2} \qquad \ldots(1)$$

Condition of achromatism is

$$\frac{f_1}{f_2} = -\frac{\omega_1}{\omega_2} = -\frac{2}{3} \qquad \ldots(2)$$

Solving the Equations (1) and (2) we get

$$f_1 = +50 \text{ cm}$$

and $f_2 = -75$ cm

ILLUSTRATION 106

A thin biconvex lens is placed with its principal axis first along a beam of parallel red light and then along a beam of parallel blue light. If the refractive indices of the lens for red and blue light are respectively 1.514 and 1.524 and if the radius of curvature of the faces are 30 cm and 20 cm, calculate the separation of foci for red and blue light. If the focal length for the mean colour (yellow) is 23.1 cm, find the dispersive power of the material of the lens.

SOLUTION

By lens maker's formula, we have

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Here for red light, we use

$$\frac{1}{f_r} = (1.514 - 1)\left(\frac{1}{20} + \frac{1}{30}\right)$$

$$\Rightarrow \quad \frac{1}{f_r} = 0.514 \times \left(\frac{1}{12}\right)$$

$$\Rightarrow \quad f_r = 23.33$$

For blue light, we use

$$\frac{1}{f_b} = (1.524 - 1)\left(\frac{1}{20} + \frac{1}{30}\right)$$

$$\Rightarrow \quad f_b = \frac{12}{0.524} = 22.9 \text{ cm}$$

Separation between the focal points is

$$\Delta f = f_r - f_b = 23.33 - 22.9 = 0.43 \text{ cm}$$

We use $\dfrac{1}{f_b} - \dfrac{1}{f_r} = \left(\dfrac{\mu_b - \mu_r}{\mu - 1}\right)\dfrac{1}{f} = \dfrac{\omega}{f}$

where dispersive power of the lens material is given as

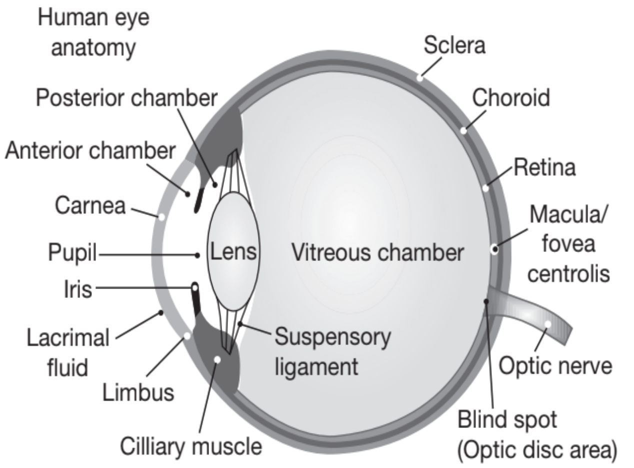
$$\omega = \left(\frac{\mu_b - \mu_r}{\mu - 1}\right)$$

$$\Rightarrow \quad f_r - f_b = \frac{\omega}{f} \times (f_b \times f_r) = \frac{\omega f^2}{f} = \omega f$$

$$\Rightarrow \quad \omega = \frac{\text{separation}}{\text{mean focal length}} = \frac{0.43}{23.1}$$

$$\Rightarrow \quad \omega = 0.019$$

HUMAN EYE



In a number of ways, the human eye works much like a digital camera as discussed.

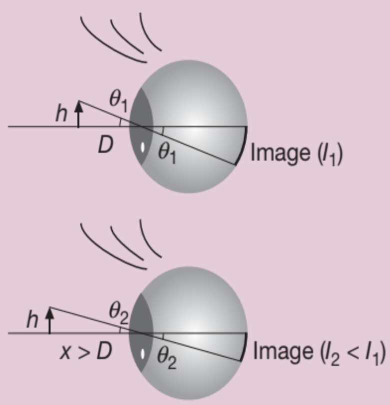
1. Light is focussed primarily by the cornea, the clear front surface of the eye, which acts like a camera lens.
2. The iris of the eye functions like the diaphragm of a camera, controlling the amount of light reaching the back of the eye by automatically adjusting the size of the pupil (aperture).
3. The eye's crystalline lens is located directly behind the pupil and further focusses light. Through a process called accommodation, this lens helps the eye automatically focus on near and approaching objects, like an autofocus camera lens.
4. Light focused by the cornea and crystalline lens (and limited by the iris and pupil) then reaches the retina (a light-sensitive inner lining of the back of the eye). The retina acts like an electronic image sensor of a digital camera, converting optical images into electronic signals. The optic nerve then transmits these signals to the visual cortex. (Cortex is the part of the brain that controls our sense of sight).

Conceptual Note(s)

(a) Human eye lens has a power to adjust its focal length to see the near and far objects. Normally an eye can see objects lying in front of it at distances ranging from 25 cm to infinity (∞). That is a normal eye can see very distant objects clearly but near objects can be seen clearly if they are at a distance greater than equal to 25 cm from it. This ability of the eye to see objects from ∞ to 25 cm by adjusting its focal length is called **the power of accommodation**.

So when the object is brought too closer to the eye, then the focal length cannot be adjusted to form the image on the retina. So, we conclude that there should be a minimum separation between the eye and the object for a clear vision of the object and this separation is called **Least Distance of Distinct Vision** (+D). For a normal eye, D is generally taken to be 25 cm.

(b) If the object is at infinity the eye is least strained i.e. relaxed, however when the object lies at D, then the eye is maximum strained and the visual angle subtended at the eye is maximum.



(c) Persistence of vision is the time interval between two light pulses arriving at the eye which the eye can see distinctly. Persistence of vision of human eye is $\frac{1}{10}$ sec. This simply means that if two light pulses arrive at the eye in a time interval less than $\frac{1}{10}$ sec then these two pulses will be seen by the eye as one single pulse.

(d) Resolving limit or limit of resolution of eye is the minimum angular separation between two objects so that they are just resolved (i.e. can be seen distinctly by eye). For eye, resolving limit (R.L.) is 1 minute i.e. 1′. So,

$$(\text{R.L.}) = 1' = \left(\frac{1}{60}\right)^o = \left(\frac{1}{60}\times\frac{\pi}{180}\right)\text{ radian}$$

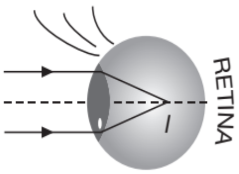
So two objects will not be visible distinctly (as two), when the angle subtended by these two objects at the eye is less than 1′.

DEFECTS OF EYE

A normal eye has nearer point at $D(25\text{ cm})$ called distance of distinct vision and far point at ∞.

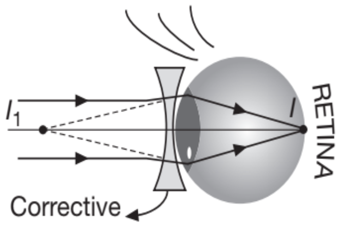
Short-sightedness or Myopia

1. A short-sighted eye can see only nearer objects.
2. It is due to elongation of eye-ball due to which radius of curvature of the lens decreases and hence power P of the lens increases.
3. The image is formed before the retina and it appears as if the separation between the eye lens and retina has increased.



DEFECTIVE EYE
Image is not created on the retina

4. In this case, the far point comes closer to the eye. The far point of a normal eye is generally at infinity but in case of myopia, the person is able to see up to a certain distance and not beyond that. This maximum distance upto which the person can see clearly is called as the **defected far point**.
5. Myopia is corrected by using a concave lens of focal length equal to the far point of defective eye also called as the defected far point.
   It simply means that the concave lens would make the image of an object lying at infinity at the defected far point and then this image will be seen as object by the eye so that the final image is formed at the retina.



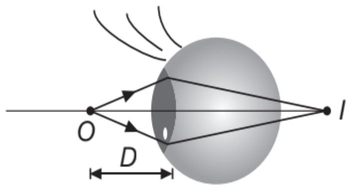
DEFECT CORRECTED

6. If a person can see upto a distance x (defected far point) but wants to see an object placed at distance $y\ (>x)$, then the focal length of the concave lens to be used is

$$f = \frac{xy}{x-y}$$

Long-sightedness or Hypermetropia

1. A long sighted eye can see only farther objects.
2. It is due to contraction of eye-ball due to which radius of curvature of the lens increases and hence power P of the lens decreases.
3. The image is formed behind the retina and it appears as if the separation between the eye lens and retina has decreased.

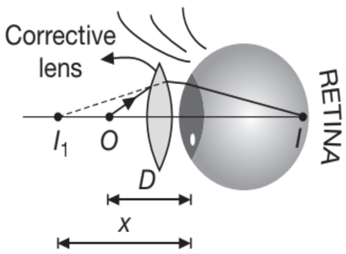


DEFECTIVE EYE
Image is created beyond the retina

4. In this case, the near point moves away from the eye. The near point of a normal eye is generally at $D = 25\text{ cm}$ but in case of hypermetropia it shifts to a distance $> 25\text{ cm}$ and is also called as the **defected near point**.
5. Hypermetropia is corrected by using convex lens. This lens brings the defected nearer point

i.e. the nearer point of defective eye at a distance which equals to the distance of distance of distinct vision $D(=25 \text{ cm})$.

It simply means that for an object lying at the defected near point, the convex lens would make the image of this object at $D=25$ cm and then this image made by the convex lens will act as object for the eye so that the final image is formed at the retina.



DEFECT CORRECTED

6. If a person cannot see before a distance $d(>D)$ but wants to see an object placed at $D=25$ cm, then the focal length of the convex lens to be used is

$$f=\frac{Dd}{d-D}$$

Presbyopia

A presbyopic eye can see objects only within a definite range. This defect is corrected by using bifocal lenses.

Astigmatism

It arises due to distortion in spherical shape in cornea. This defect is corrected by using cylindrical lenses.

ILLUSTRATION 107

The accommodation of eye of a short-sighted man lies between 12 cm and 60 cm. He wears spectacles through which he can see remote objects distinctly. Calculate the minimum distance at which the man can read a book through his spectacles.

SOLUTION

As per the given situation, a man can manage to see objects clearly if placed between 12 cm and 60 cm (accommodation of eye). If v is the distance between eye lens and retina then the focal length of eye lens when an object is placed at 60 cm distance is obtained by using lens formula.

$$\Rightarrow \quad \frac{1}{v}-\frac{1}{(-60)}=\frac{1}{f_e} \quad \ldots(1)$$

If he uses spectacles having lens of focal length f, then he can see far objects clearly. This means, that for far objects, the combination of eye lens and spectacles lens produces the image at retina at distance v from the eye lens. For this combination of the lenses, we use lens formula as

$$\frac{1}{v}-\frac{1}{\infty}=\frac{1}{f_e}+\frac{1}{f} \quad \ldots(2)$$

From equation (1) and (2), we get

$$f=-60 \text{ cm}$$

For the near point of the eye at 12 cm, if the focal length of the eye lens is f_e' then by lens formula, we get

$$\frac{1}{v}-\frac{1}{(-12)}=\frac{1}{f_e'} \quad \ldots(3)$$

The minimum distance D at which the man can read a book through his spectacles (when he places the book at distance D from the eye with spectacles on) is obtained by using the lens formula

$$\frac{1}{v}-\frac{1}{(-D)}=\frac{1}{f_e'}+\frac{1}{-60} \quad \ldots(4)$$

From equation (3) and (4), we get

$$D=15 \text{ cm}$$

OPTICAL INSTRUMENTS

An optical instrument is a device which is constructed by a suitable combination of mirrors, prisms and lenses so that it assists the eye in viewing an object. The principle of working of an optical instrument in based on the laws of reflection and refraction of light. The common types of optical instrument are

(a) Projection instruments: These are used to project on the screen a real, inverted and magnified image of an opaque or transparent object so as to be viewed by a large audience. The object is, however, so fitted that its image is seen in erect form.

An eye, a photographic camera, a projection lantern, an episcope, an epidiascope, an over-head projector, a film projector, etc., are examples of projection instruments.

(b) Microscopes: These are used to see very small objects in magnified form which otherwise cannot be seen distinctly when placed close to the naked eye.

EXAMPLE

A simple microscope and a compound microscope.

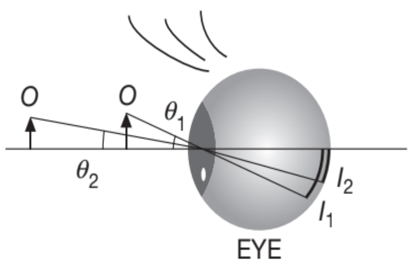
(c) Telescopes: These are used to see astronomical and distant objects in magnified form which, otherwise cannot be seen clearly with the naked eye.

EXAMPLE

An astronomical telescope, a Galilean telescope, a terrestrial telescope, a reflecting telescope, etc.

VISUAL ANGLE

The size of the object as perceived by the eye depends upon the size of image formed at the retina. The size of image formed at the retina is roughly proportional to the angle subtended by the object at the eye called as the visual angle. When the object lies close to the eye its visual angle is large and hence image I_1 formed at the retina is large. When the object is taken far away from the eye its visual angle becomes small and hence the image I_2 formed at the retina is also small.



θ_1 is the visual angle subtended at the eye for near position of object and θ_2 is the visual angle subtended at the eye for the far position of the object. Since $\theta_1 > \theta_2$, so $I_1 > I_2$.

Optical instruments are used for increasing the visual angle artificially so as to increase the size of the image formed at the retina.

MAGNIFYING POWER OR ANGULAR MAGNIFICATION (*M*)

Magnifying power or angular magnification M is the factor by which an image formed on the retina can be enlarged by using a microscope or a telescope.

For Microscope, the magnifying power is the ratio of the visual angle subtended (or formed) by the final image at the eye to the visual angle subtended by the object at the eye (when kept at the distance of distinct vision).

$$M_{\text{microscope}} = \frac{\left(\begin{array}{c}\text{Visual angle subtended by final} \\ \text{image at eye}\end{array}\right)}{\left(\begin{array}{c}\text{Visual angle subtended by the} \\ \text{object at eye when kept at } D\end{array}\right)}$$

For Telescope, the magnifying power is the ratio of the visual angle subtended by the final image at the eye to the visual angle subtended by the object at the eye (when seen from the naked eye).

$$M_{\text{telescope}} = \frac{\left(\begin{array}{c}\text{Visual angle subtended by final} \\ \text{image at eye}\end{array}\right)}{\left(\begin{array}{c}\text{Visual angle subtended by the object} \\ \text{at eye when seen by the naked eye}\end{array}\right)}$$

Conceptual Note(s)

(a) The term linear magnification (m) is different from the term magnifying power (M).

(b) Linear magnification $m = \frac{h_i}{h_o} = \pm\frac{v}{u}$ is the ratio of the height of the image to the height of the object.

(c) Magnifying power (as discussed above) is the ratio of the apparent increase in size of the image seen by the eye.

(d) While m is unitless, the unit of M is X. So if magnifying power of a microscope is 11, then it is written as $11X$.

SIMPLE MICROSCOPE (MAGNIFYING GLASS)

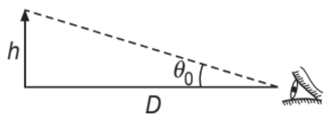
A convex lens of short focal length can be used to see magnified image of a small object and is called **a magnifying glass** or a **simple microscope**.

When a small object is placed between optical centre and focus of a convex lens, its virtual, erect and magnified image is formed on the same side of the lens. The lens is held close to eye and the distance of the object is adjusted, till the image is formed at the least distance of distinct vision from the eye.

When we view an object with naked eyes, the object must be placed somewhere between infinity and the near point. The maximum angle is subtended on the eye when the object is placed at the near point. This angle

$$\theta_0 = \frac{h}{D} \quad \ldots(1)$$

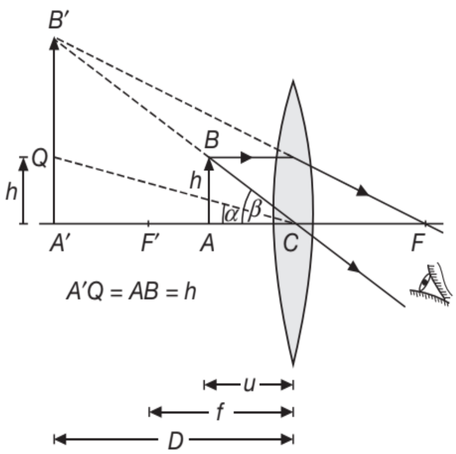
where h is the size of the object and D is the least distance for clear vision.



Magnifying Power (*M*)

CASE-1: When image is formed at *D* i.e. near point adjustment or strained viewing

In this case $M = M_D$ is defined as the ratio of the angle subtended by the image at the eye to the angle subtended by the object (at the eye) seen directly, when both lie at the least distance of distinct vision.



By definition, magnifying power of the simple microscope is given by

$$M_D = \frac{\beta}{\alpha}$$

Since angles α and β are small, therefore, angles α and β can be replaced by their tangents i.e.

$$M_D = \frac{\tan\beta}{\tan\alpha} = \frac{h/u}{h/D} = \frac{D}{u} \quad \ldots(1)$$

If f is focal length of the lens acting as simple microscope, then

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

Since, image is formed at distance of distinct vision, so according to new Cartesian sign convention.

$$u = -u\,,\ v = -D \text{ and } f = +f$$

$$\Rightarrow \quad \frac{1}{(-D)} - \frac{1}{(-u)} = \frac{1}{f}$$

$$\Rightarrow \quad -\frac{1}{D} + \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{D}{u} = 1 + \frac{D}{f} \quad \ldots(2)$$

From equations (1) and (2), we get

$$M_D = \frac{D}{u} = 1 + \frac{D}{f}$$

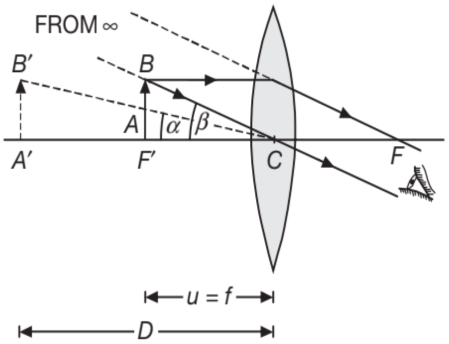
Conceptual Note(s)

(a) From above it follows that lesser is the focal length of the convex lens used as simple microscope, greater is the value of the magnifying power obtained.

(b) Further, the positive value of magnifying power of a simple microscope tells that image formed is erect and hence virtual.

CASE-2: When image is formed at infinity i.e. far point adjustment or relaxed viewing

In this case $M = M_\infty$ is defined as the ratio of the visual angle β subtended (at the eye) by the final image to the visual angle α subtended (at the eye) by the object when kept at D.



Draw a line $A'B' = AB$ and perpendicular to principal axis at a distance $CA' = D$ (least distance of distinct vision) join $B'C$. Then $\angle B'CA' = \alpha \approx \frac{h}{D}$ is the angle formed by object at the eye, when situated at distance D.

The angle formed by the image situated at infinity at the eye is same as the angle formed by the object AB at the eye. Thus, $\angle BCA = \beta \approx \frac{h}{f}$ is the angle formed by the image at the eye.

By definition,

$$M_{\infty} = \frac{\beta}{\alpha} = \frac{\tan\beta}{\tan\alpha} = \frac{h/f}{h/D} = \frac{D}{f}$$

Conceptual Note(s)

It follows that magnifying power of the simple microscope is one less, when the image is formed at infinity. However, the viewing of the image is more comfortable.

USES

(a) Jewellers and watch makers make use of convex lens of short focal length to obtain a magnified view of the fine jewellery work and the small components of the watches.

(b) In science laboratories, a magnifying glass is used to see slides and to read the Vernier scales attached to the instruments.

(c) The use of magnifying glass enables us to place the object close to eye, making it appear bright and yet clearly visible. In position AB, object lies close to the eye. In absence of lens, the object will not be clearly visible.

(d) It is also used by astrologers to read the fate lines of the hand.

(e) Used by Biology students to see slides.

(f) Used by detective department to match finger prints.

ILLUSTRATION 108

A man with normal near point (25 cm) reads a book with small print using a magnifying glass (a thin convex lens) of focal length 5 cm. Find the

(a) closest and farthest distance at which he can read the book when viewing through the magnifying glass.

(b) maximum and minimum magnifying power possible using the above simple microscope.

SOLUTION

(a) For a normal eye, far and near points are ∞ and 25 cm, respectively. So, we have

$$v_{\max} \to -\infty \text{ and } v_{\min} = -25\text{ cm}$$

Using lens formula, $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad u = \frac{f}{\left(\frac{f}{v}\right) - 1}$$

So, u will be minimum, when v is minimum i.e., $v_{\min} = -25\text{ cm}$

$$\Rightarrow \quad (u)_{\min} = \frac{5}{-\left(\frac{5}{25}\right) - 1} = -\frac{25}{6} = -4.17\text{ cm}$$

And u will be maximum, when v is maximum i.e., $v_{\max} \to \infty$

$$\Rightarrow \quad (u)_{\max} = \frac{5}{\left(\frac{5}{\infty}\right) - 1} = -5\text{ cm}$$

(b) Since magnifying power for a lens is

$$m = \frac{v}{u}$$

Magnifying power will be minimum, when u is maximum i.e., $u_{\max} = -5\text{ cm}$

$$\Rightarrow \quad (m)_{\min} = \frac{D}{f} = \frac{-25}{-5} = 5$$

m will be maximum, when u is minimum i.e.,

$$u_{\min} = -\frac{25}{6} = -4.17 \text{ cm}$$

$$\Rightarrow (m)_{\max} = \frac{-25}{-\frac{25}{6}} = 6 \qquad \left\{= 1 + \frac{D}{f}\right\}$$

COMPOUND MICROSCOPE

A compound microscope is used to see extremely small objects. It consists of two lenses. A lens of **short aperture and short focal length** facing the object is called **object lens (or objective lens)** and another lens of **large focal length** and **large aperture** is called **eye lens (or eye piece or ocular)**. The two lenses are placed coaxially at the two ends of a tube. To focus over an object, the distance of the object lens from the object is adjusted with the help of rack and pinion arrangement.

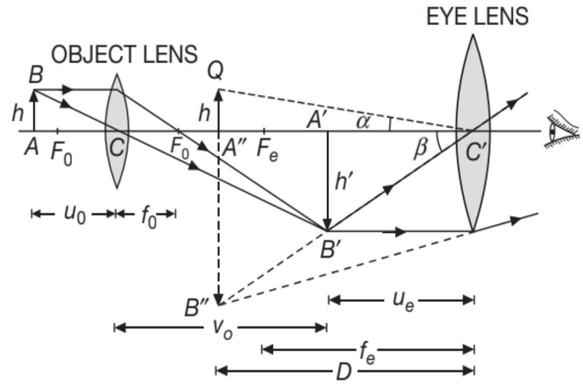
When a small object is placed just outside the focus of the object lens, its real, inverted and magnified image is produced on the other side of the lens between F and $2F$. The image produced by object lens acts as object for the eye lens. The distance of object from the object lens is so adjusted that the final image is formed at the least distance of distinct vision from the eye.

Let AB be an object placed just outside the focus F_0 of the object lens. Its virtual image $A'B'$ is formed on the other side of the lens. The image $A'B'$ lies between focus F_e and optical centre C' of the eye lens and it acts as object for the eye lens. Using the rack and pinion arrangement, the distance between object lens and the object AB is adjusted, till it virtual and magnified image $A''B''$ is formed on the same side at the least distance of distinct vision.

MAGNIFYING POWER (M)

CASE-1: When image is formed at D

In this case $M = M_D$ is defined as the ratio of the visual angle β subtended by the final image at the eye to the visual angle α subtended by the object seen directly, when both are placed at the least distance of distinct vision.



Let $\angle A''C'B'' = \angle A'C'B' = \beta$ be the angle subtended by the final image at the eye. Let us cut $A''Q$ equal to AB and join QC'. Then, $\angle A''C'Q = \alpha$, the angle subtended by the object at the eye, when situated at the least distance of distinct vision. By definition, magnifying power of the compound microscope,

$$M_D = \frac{\beta}{\alpha} = \frac{\tan\beta}{\tan\alpha}$$

Since the angles α and β are small, so they can be replaced by their tangents.

$$M_D = \frac{\tan\beta}{\tan\alpha} = \frac{h'/u_e}{h/D} = \left(\frac{h'}{h}\right)\left(\frac{D}{u_e}\right) \qquad \ldots(1)$$

Since, for the objective lens, we have

$$\frac{h_i}{h_o} = \frac{h'}{h} = \frac{v_o}{u_o} \qquad \ldots(2)$$

For the eye piece, we have $u = -u_e$, $v = v_e = -D$ and $f = +f_e$

$$\text{Since, } \frac{1}{v_e} - \frac{1}{u_e} = \frac{1}{f_e}$$

$$\Rightarrow \frac{1}{-D} + \frac{1}{u_e} = \frac{1}{f_e}$$

$$\Rightarrow \frac{D}{u_e} = 1 + \frac{D}{f_e} \qquad \ldots(3)$$

Substituting (2) and (3) in (1), we get

$$M_D = \frac{v_0}{u_0}\left(1 + \frac{D}{f_e}\right)$$

In general, the focal length if the objective is very small so that $\frac{v_o}{f_o} \gg 1$ and also the first image is formed close to the eye piece so that $v_o \approx L$ (where L is the length of the microscope tube i.e. the separation between the objective and the eye piece).

Since, $\frac{1}{v_o} - \frac{1}{u_o} = \frac{1}{f_o}$

$$\Rightarrow \quad 1 - \frac{v_o}{u_o} = \frac{v_o}{f_o}$$

$$\Rightarrow \quad \frac{v_o}{u_o} = 1 - \frac{v_o}{f_o} \approx -\frac{v_o}{f_o} \approx -\frac{L}{f_o}$$

$$\Rightarrow \quad M_D = -\frac{L}{f_0}\left(1 + \frac{D}{f_e}\right)$$

If we do not take into account the approximation, then the length of the compound microscope tube in this case is denoted by L_D and is given by

$$L_D = v_o + u_e = \frac{u_o f_o}{u_o - f_o} + \frac{f_e D}{f_e + D}$$

Conceptual Note(s)

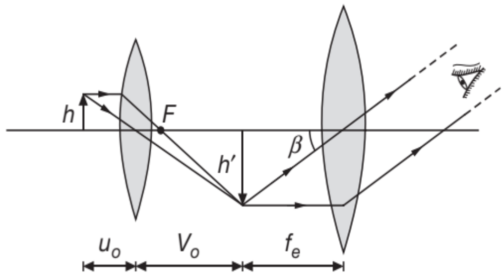
(a) From the above expression, it follows that a compound microscope will have large magnifying power, if both the object lens and the eye lens are of small focal length. In practice, focal length of object lens is smaller than that of eye lens i.e. $f_0 < f_e$. Further the **negative** value of magnifying power of compound microscope tells that final image formed is **inverted**.

(b) In practice, to eliminate chromatic aberration, a combination of two lenses in contact is used. It is called **objective**.

(c) In place of an eye lens, a combination of two lenses at certain distance apart satisfying certain conditions (to minimize chromatic and spherical aberrations) is used. It is called **eye piece**.

CASE-2: When image is formed at infinity

In this case $M = M_\infty$ is defined as the ratio of the visual angle β subtended (at the eye) by the final image to the visual angle α subtended (at the eye) by the object when kept at D.



So, $M_\infty = \frac{\beta}{\alpha} \approx \frac{\tan\beta}{\tan\alpha} = \frac{h'/f_e}{h/D} = \left(\frac{h'}{h}\right)\left(\frac{D}{f_e}\right) = \left(\frac{v_o}{u_o}\right)\left(\frac{D}{f_e}\right)$

and $L_\infty = v_o + f_e$

$$\Rightarrow \quad v_o = L_\infty - f_e$$

Since, we know that $\frac{1}{v_o} - \frac{1}{(-u_o)} = \frac{1}{f_o}$

$$\Rightarrow \quad \frac{v_o}{u_o} = \frac{v_o - f_o}{f_o}$$

Since $M_\infty = \left(\frac{v_o}{u_o}\right)\left(\frac{D}{f_e}\right)$

$$\Rightarrow \quad M_\infty = \left(\frac{v_o}{u_o}\right)\left(\frac{D}{f_e}\right) = \left(\frac{v_o - f_o}{f_o}\right)\left(\frac{D}{f_e}\right)$$

$$\Rightarrow \quad M_\infty = \left(\frac{L_\infty - f_e - f_o}{f_o f_e}\right)D$$

where $L_\infty = v_o + f_e = \frac{u_o f_o}{u_o - f_o} + f_e$

ILLUSTRATION 109

A compound microscope has a magnifying power 30. The focal length of its eye-piece is 5 cm. Assuming the final image to be at the least distance of distinct vision (25 cm), find the magnification produced by the objective.

SOLUTION

For a compound microscope, we have

$$M = m_0 m_e \qquad \ldots(1)$$

Since the final image is formed at least distance of distinct vision, the magnification of eye-piece is

$$m_e = \left(1 + \frac{D}{f_e}\right) = 1 + \frac{25}{5} = 6$$

From equation (1), we get

$$-30 = m_0 \times 6$$

$$\Rightarrow \quad m_0 = -\frac{30}{6} = -5$$

Negative sign implies that image formed by the objective is inverted.

ILLUSTRATION 110

In a compound microscope the objective and eyepiece have focal lengths of 0.95 cm and 5 cm respectively, and are kept at a distance of 20 cm. The final image is formed at a distance of 25 cm from eyepiece. Calculate the position of the object and the total magnification.

SOLUTION

From the lens formula for eyepiece, we use

$$v_e = -25 \text{ cm} \text{ and } f_e = +5 \text{ cm}$$

$$\Rightarrow \quad \frac{1}{u_e} = \frac{1}{v_e} - \frac{1}{f_e} = -\frac{1}{25} - \frac{1}{5} = -\frac{6}{25}$$

$$\Rightarrow \quad u_e = -\left(\frac{25}{6}\right) \text{ cm}$$

For the objective, we use

$$f_0 = 0.95 \text{ cm} \text{ and } v_0 = 20 - \left(\frac{25}{6}\right) = \frac{95}{6} \text{ cm}$$

Using lens formula, we have

$$\frac{1}{v_0} - \frac{1}{u_0} = \frac{1}{f_0}$$

$$\Rightarrow \quad \frac{1}{v_0} - \frac{1}{f_0} = \frac{1}{u_0}$$

$$\Rightarrow \quad \frac{1}{u_0} = \frac{6}{5} - \frac{1}{0.96} = \frac{6-100}{95}$$

$$\Rightarrow \quad \frac{1}{u_0} = -\frac{94}{95}$$

$$\Rightarrow \quad u_0 = \frac{95}{94} \text{ cm}$$

Total Magnification $M = \frac{v_0}{u_0}\left(1 + \frac{D}{f_e}\right)$

$$\Rightarrow \quad M = \frac{\left(\frac{95}{6}\right)}{-\left(\frac{95}{94}\right)}\left(1 + \frac{25}{5}\right) = -94$$

ILLUSTRATION 111

The focal lengths of the objective and eye-lens of a microscope are 1 cm and 5 cm respectively. If the magnifying power for the relaxed eye is 45, then calculate the length of the tube.

SOLUTION

Given that $f_o = 1$ cm, $f_e = 5$ cm, $M_\infty = 45$

Since, we have $M_\infty = \frac{(L_\infty - f_o - f_e)}{f_o f_e}$

$$\Rightarrow \quad 45 = \frac{(L_\infty - 1 - 5) \times 25}{1 \times 5}$$

$$\Rightarrow \quad L_\infty = 15 \text{ cm}$$

ILLUSTRATION 112

The focal length of the objective of a microscope is $f_o = 3$ mm and of the eyepiece $f_e = 5$ cm. An object is placed at a distance of 3.1 mm from the objective. Find the magnification of the microscope for a normal eye, if the final image is produced at a distance 25 cm from the eye (or eyepiece). Also final the separation between the lenses of microscope.

SOLUTION

For the objective, we use

$$u_0 = -0.31 \text{ cm} \text{ and } f_0 = 0.3 \text{ cm}$$

Using lens formula, we have

$$\frac{1}{v_0} - \frac{1}{u_0} = \frac{1}{f_0}$$

$$\Rightarrow \quad \frac{1}{v_0} - \frac{1}{-0.31} = \frac{1}{0.3}$$

$$\Rightarrow \quad v_0 = +9.3 \text{ cm}$$

For the eyepiece, we use

$$v_e = -25 \text{ cm} \text{ and } f_e = +5 \text{ cm}$$

Using lens formula, we have

$$\frac{1}{v_e} - \frac{1}{u_e} = \frac{1}{f_e}$$

$$\Rightarrow \quad \frac{1}{-25} - \frac{1}{u_e} = \frac{1}{5}$$

$$\Rightarrow \quad u_0 = -\frac{25}{6} \text{ cm} = -4.166 \text{ cm}$$

Magnifying power of telescope is given as

$$M=\frac{v_0}{u_0}\left(1+\frac{D}{f_e}\right)=\frac{9.3}{0.31}\left(1+\frac{25}{5}\right)=180$$

Separation between lenses is

$$d=v_0+u_e=9.3+4.166=13.466 \text{ cm}$$

ASTRONOMICAL TELESCOPE (REFRACTING TYPE)

An astronomical telescope is used to see heavenly objects. It produces a virtual and inverted image. As such bodies are round, the inverted image does not affect the observation.

An astronomical refracting telescope consists of two lens systems. The lens system facing the object is called **objective**. It has large aperture and is of **large focal** length (f_0). The other lens system is called **eye-piece**. It has **small aperture** and is of **short focal length** (f_e). The objective and the eye-piece are mounted coaxially in two metallic tubes. The tube holding the eye-piece can be made to slide into the tube holding the objective with the help of rack and pinion arrangement.

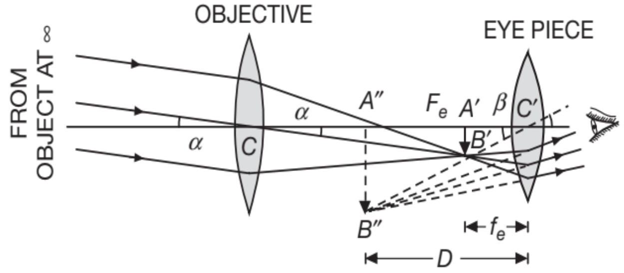
The objective forms the real and inverted image of the distant object in its focal plane. The position of the eye-piece is adjusted, till the final image is formed **at least distance of distinct vision**.

In case, position of the eye-piece is adjusted such that final image is formed at infinity, the telescope is said to be in **normal adjustment**.

Magnifying Power (*M*)

CASE-1: When final image is formed at least distance of distinct vision

When a parallel beam of light rays from the distant object falls on the objective, its real and inverted image $A'B'$ is formed on the other side of the objective. The position of eye-piece is adjusted so that the final image $A''B''$ is formed at least distance of distinct vision. Under such a situation the magnifying power of a telescope is defined as the ratio of the angle subtended at the eye by the image formed at the least distance of distinct vision to angle subtended at the eye by the object lying at infinity, when seen directly.



Again, as the object is at a very large distance, the angle α subtended by it at the objective is practically the same as that subtended by it at the eye. Therefore, if $\angle A''C'B''=\beta$, then

$$M_D=\frac{\beta}{\alpha}$$

Again, as angle α and β are small, they can be replaced by their tangents,

$$\Rightarrow \quad M_D=\frac{\tan\beta}{\tan\alpha}=\frac{CA'}{C'A'}$$

Since, $CA'=f_0$ and $C'A'=u_e$

$$\Rightarrow \quad M_D=\frac{f_0}{u_e}$$

For eye lens, we have

$$\frac{1}{v_e}-\frac{1}{u_e}=\frac{1}{f_e}$$

$$\Rightarrow \quad \frac{1}{u_e}=\frac{1}{v_e}-\frac{1}{f_e}$$

$$\Rightarrow \quad \frac{1}{u_e}=-\frac{1}{f_e}\left(1-\frac{f_e}{v_e}\right)$$

$$\Rightarrow \quad M_D=-\frac{f_0}{f_e}\left(1-\frac{f_e}{v_e}\right)$$

Applying new Cartesian sign conventions we get

$$f_0=+f_0,\ v_e=-D,\ f_e=+f_e$$

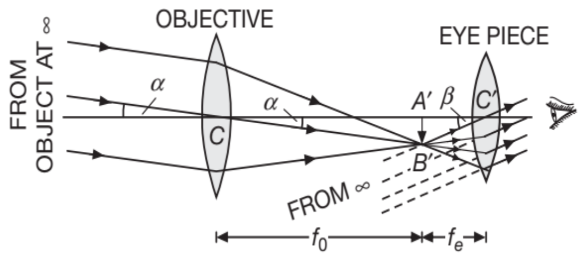
$$\Rightarrow \quad M_D=-\frac{f_0}{f_e}\left(1+\frac{f_e}{D}\right)$$

Therefore, a refracting telescope will have large magnifying power, if the object lens is of large focal length and eye lens is of short focal length. Further, the **negative** value of magnifying power of the telescope tells that the final image formed is **inverted** and **real**. Out of the two adjustments discussed, this adjustment gives a higher magnification, since the factor $\left(1+\frac{f_e}{D}\right)$ is greater than one.

Also, a telescope does not increase the size of object, but it forms an image nearer to the eye, so that the angle of vision is increased and hence it appears to us as if the bigger image of object is formed.

CASE-2: When final image is formed at infinity (Normal adjustment)

When a parallel beam of light rays from the distant object falls on the objective, its real and inverted image $A'B'$ is formed on the other side of the objective. If the position of eye-piece is adjusted, so that the image $A'B'$ lies at its focus, then the final highly magnified image will be formed at infinity. Under such a situation i.e. in normal adjustment the magnifying power of a telescope is defined as the ratio of the angle subtended by the image at the eye as seen through the telescope to the angle subtended by the object seen directly, when both the object and the image lie at infinity. It is also called **angular magnification** of the telescope and is denoted by $M = M_\infty$.



As the object is at a very large distance, the angle subtended by it at the eye is practically the same as that subtended by it at the objective.

Thus, $\angle A'CB' = \alpha$ may be considered as the angle subtended by object at the eye. Let $\angle A'C'B' = \beta$.

Then

$$M_\infty = \frac{\beta}{\alpha}$$

Since the angles α and β are small, $\alpha \approx \tan\alpha$ and $\beta \approx \tan\beta$. Therefore,

$$M_\infty = \frac{\tan\beta}{\tan\alpha} = \frac{CA'}{C'A'}$$

Using new Cartesian sign conventions we get

$$CA' = +f_0$$

{∵ distance of $A'B'$ from object lens is along the incident light}

$$C'A' = -f_e$$

{∵ distance of $A'B'$ from eye lens is against incident light}

$$\Rightarrow \quad M_\infty = \frac{f_0}{f_e}$$

Conceptual Note(s)

(a) It follows that the magnifying power of a telescope in normal adjustment will be large, if objective is of large focal length and the eye-piece is of short focal length.

(b) Further, when telescope is in normal adjustment, the distance between the two lenses is equal to sum of their focal lengths $(f_0 + f_e)$.

(c) Further, the **negative** value of the magnifying power of the telescope tells that final image formed is inverted and real.

ILLUSTRATION 113

The objective of a telescope is a convex lens of focal length 100 cm. Its eye-piece is also a convex lens of focal length 5 cm. Determine the magnifying power of the telescope for normal adjustment.

SOLUTION

For normal adjustment, the magnifying power of a telescope is given by

$$M_\infty = \frac{f_o}{f_e}$$

Here, $f_o = 100$ cm, $f_e = 5$ cm

$$\Rightarrow \quad M_\infty = \frac{100}{5} = 20$$

ILLUSTRATION 114

An astronomical telescope in normal adjustment has a tube length of 93 cm and magnification (angular) of 30. If the eye-piece is to be drawn out by 3 cm to focus a near object, with the final image at infinity, find how far away is the object and the magnification (angular) is this case.

SOLUTION

In the normal adjustment, magnification (angular) produced by telescope is

$$M_\infty = \frac{f_o}{f_e} = 30$$

$$\Rightarrow \quad f_o = 30 f_e$$

and tube length $L_\infty = f_o + f_e = 93$

$$\Rightarrow \quad f_e = \frac{93}{31} = 31 \text{ cm} \text{ and } f_o = 90 \text{ cm}$$

Since $\dfrac{1}{v_e} - \dfrac{1}{u_e} = \dfrac{1}{f_e}$

$$\Rightarrow \quad \frac{1}{\infty} - \frac{1}{-u_e} = \frac{1}{+3}$$

$$\Rightarrow \quad u_e = 3 \text{ cm}$$

Since $v_o = f_o + u_e = 90 + 3 = 93$ cm and $\dfrac{1}{v_o} - \dfrac{1}{u_o} = \dfrac{1}{f_o}$

$$\Rightarrow \quad \frac{1}{+93} - \frac{1}{-u_o} = \frac{1}{+90}$$

$$\Rightarrow \quad \frac{1}{u_o} = \frac{1}{90} - \frac{1}{93} = \frac{3}{90 \times 93}$$

$$\Rightarrow \quad u_e = 30 \times 93 = 2790 \text{ cm} = 27.9 \text{ m}$$

$$\text{Magnification} = \frac{v_o}{u_e} = \frac{93}{3} = 31$$

TERRESTRIAL TELESCOPE

A terrestrial telescope is used to observe objects on earth.

An astronomical telescope is used to view heavenly objects since the inversion of their images does not produce any complication. While viewing earthly objects we would prefer to have their images erect and hence, astronomical telescope is not suitable in such cases. By using an additional convex lens O (of focal length f) in between O_1 and O_2 of an astronomical telescope, we can have the final erect image. The lens O is called **erecting lens**, while the improved version of the telescope is called **Terrestrial Telescope**.

Rays from the distant object get refracted through the objective O_1, giving a real inverted image A_1B_1. The erecting lens O is so adjusted that its distance from A_1B_1 is equal to twice its (erecting lens) focal length. An image A_2B_2 having same size as that of A_1B_1, inverted w.r.t. A_1B_1 and hence erect w.r.t. the object is obtained at a distance $2f$ on other side of O. A_2B_2 acts as an object for lens at O_2 and finally an erect and magnified image is obtained after refraction through O_2. If the distance O_2B_2 is equal to focal length f_e of the eye lens O_2, final image is formed at infinity and the telescope is said to be in **normal adjustment** as in Figure 1.3.

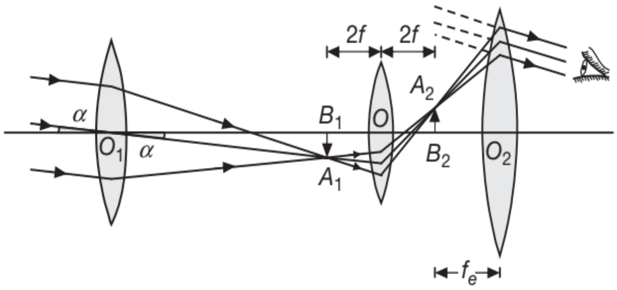


Figure 1.3

If the distance O_2B_2 is less than f_e then corresponding to a certain value of this distance, a virtual and magnified image is obtained at the distance of distinct vision as shown in Figure 1.4.

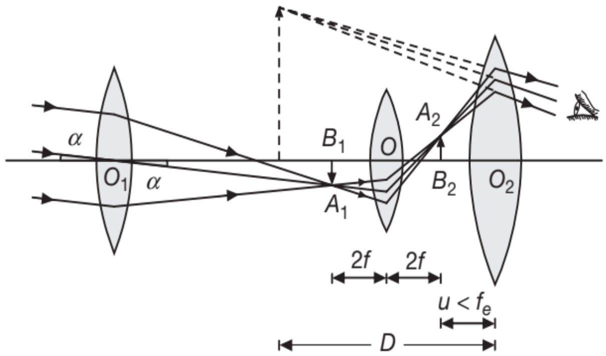
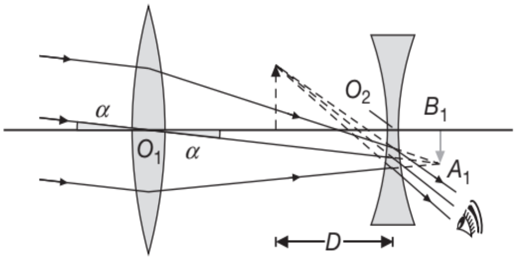


Figure 1.4

Since the sizes of A_2B_2 and A_1B_1 are same, introduction of the erecting lens O has nor produced any change in its magnifying power but, has helped in getting the final image erect only. It may also be noted that the use of erecting lens O results in an increase (equal to four times the focal length of erecting lens) in the length of the tube of telescope.

GALILEO'S TELESCOPE

Instead of using a combination of two, lens O_1 and O_2 for getting an erect image, Galileo used only one concave lenses to get the final erect image.



Parallel beam of incident rays from infinity are focussed by the objective O_1. An inverted image A_1B_1 (shown in grey) would have been formed after refraction through O_1. Before the rays meet at A_1, a concave lens $(\text{at } O_2)$ intercepts them. The beam diverges and the final erect image A_2B_2 is obtained. The distance O_2B_1 is so adjusted that final image is formed at the distance of distinct vision. If O_2B_1 is equal to the focal length f_e of eye lens at O_2 final image is formed at infinity and the telescope is said to be set in normal **adjustment**. In such a case the length of the tube is equal to the difference between the focal lengths of two lenses. The field of view of this telescope is small because of the use of concave lens.
When set in normal adjustment, its magnifying power M is given by

$$M = \frac{\beta}{\alpha} = \frac{\tan\beta}{\tan\alpha} = \frac{\dfrac{A_1B_1}{B_1O_2}}{\dfrac{A_1B_1}{B_1O_1}} = \frac{B_1O_1}{B_1O_2}$$

$$\Rightarrow \quad M = \frac{F}{f} = \frac{\text{Focal length of objective}}{\text{Focal length of eye lens}}$$

LIMIT OF RESOLUTION AND RESOLVING POWER

The *Resolving Power* (RP) of an optical instrument is defined as the reciprocal of smallest angular separation between two neighbouring objects whose images are just distinctly formed by the instrument. The smallest angular separation is called the *Resolving Limit* denoted by RL (also called as **Limit of Resolution**).

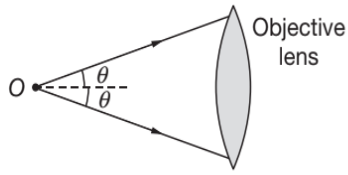
$$\text{Resolving Power} = \frac{1}{\text{Limit of Resolution}}$$

For Microscope

For a microscope, the resolving limit is given by

$$RL = \frac{\lambda}{2\mu\sin\theta}$$

where μ is refractive index of medium between object and objective lens, $\mu\sin\theta$ is called numerical aperture and λ is wavelength of light used to illuminate the object.



The resolving power for the microscope is given by

$$RP = \frac{1}{RL} = \frac{2\mu\sin\theta}{\lambda}$$

Since $RP \propto \frac{1}{\lambda}$, therefore for high resolution of microscope a beam of electrons is used which has wavelength of the order of $1\ \text{Å}$.

For Telescope

If a is aperture or diameter of telescope and λ the wavelength of light, then resolving limit (can also be denoted by $d\theta$ in this case) is

$$d\theta \propto \frac{\lambda}{a}$$

For spherical aperture $d\theta = \dfrac{1.22\lambda}{a}$

Resolving power $\propto \dfrac{a}{\lambda}$

Resolving power of telescope or microscope has no concern with focal lengths of lenses.

PHOTOMETRY

Radiant Flux

It is the radiant energy emitted by a body per second in all directions including all wavelengths. Its unit is watt.

Luminous Flux

The radiant energy emitted by a body per second in the visible region (i.e., between wavelengths from 4000Å to 7800Å) is called luminous flux. Its unit is lumen.

Luminous Intensity (*I*) of a Light Source

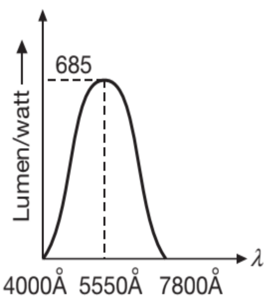
The luminous intensity (I) of a light source in any direction may be defined as the luminous flux (F) per unit solid angle (Ω) in that direction. Its unit is lumen/steradian or candela.

$$\Rightarrow \quad I = \frac{\Delta F}{\Delta \Omega} \qquad \text{(in lumen/steradian or candela)}$$

If the light source is isotropic, then luminous flux is uniform in all directions, so that total luminous flux is given by

$$F = 4\pi I \qquad \text{(since total solid angle for all directions is } 4\pi\text{)}$$

If we plot the ratio luminous flux/radiant flux against wavelength of radiation, we get a graph as shown in figure.



The graph indicates that the ratio luminous flux/radiant flux is maximum for 5550Å i.e. for yellow colour. This indicates that our eye is most sensitive for 5550Å i.e. yellow colour. The maximum value of ratio luminous flux/radiant flux is 685 lumen/watt. Thus when the ratio luminous flux/radiant flux is 685 lumen/watt, the luminous efficiency is said to be 100%.

The tungsten filament bulb converts 2-3% electrical energy into visible light energy, while fluorescent tube converts 8-9% electrical energy into visible light energy.

Illuminance (*E*) of a Surface

It may be defined as the luminous flux falling per unit area on the surface. Its unit is $\text{lumen}\,\text{m}^{-2}$ or lux.

$$E = \frac{\Delta F}{\Delta A}$$

Inverse Square Law Obeyed by Illuminance

When radiant energy falls normally on a surface, the illuminance E of the surface is inversely proportional to the square of distance of surface point from source i.e.

$$E \propto \frac{1}{r^2} \qquad \ldots(1)$$

Lambert's Cosine Law Obeyed by Illuminance

When radiant energy falls obliquely on a surface, the illuminance E of surface is directly proportional to the cosine of angle made by normal to the surface with the direction of incident radiation. So,

$$E \propto \cos\theta \qquad \ldots(2)$$

Combining (1) and (2), we get

$$E = \frac{I\cos\theta}{r^2} \qquad \ldots(3)$$

Total Luminous Energy falling on a surface is given by

$$Q = EAt$$

where E = illuminance of the surface
A = area and
t = exposure time.

The total luminous energy required to be incident on a given type of camera film is constant.

For a box type camera, the time of exposure $\propto \left(\frac{f}{d}\right)^2$, where, d is the diameter of camera lens and f is the focal length of camera lens.

The Luminance (*L*) or Brightness

Luminance of a surface is the luminous flux reflected by unit area of the surface normally.

$\Rightarrow$ Luminance = Illuminance × Reflection Coefficient.

Principle of Photometry

If two sources of light of illuminating power I_1 and I_2 are placed at distances r_1 and r_2 from the screen, then the screen will be equally illuminated due to two sources when

$$\frac{I_1}{r_1^2} = \frac{I_2}{r_2^2}$$

Test Your Concepts-VIII

Based on Aberrations, Human Eye and Optical Instruments

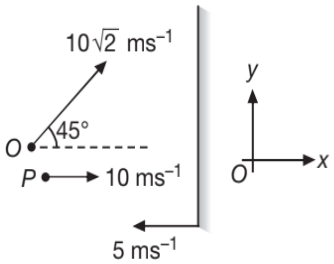
(Solutions on page H.26)

1. For a normal eye, the far point is at infinity and the near point of distinct vision is about 25 cm in front of the eye. The cornea of the eye provides a converging power of about 40 dipotres, and the least converging power of the eye-lens behind cornea is about 20 dioptres. Find the range of accommodation (converging power of eye lens) of the normal eye
2. The focal lengths of a thin convex lens are 1 m and 0.968 m for red and blue colour of light rays respectively. Calculate the chromatic aberration and dispersive power of material of the lens.
3. A short-sighted person cannot see objects situated beyond 2 m from him distinctly. What should be the power of the lens which he should use for seeing distant objects clearly?
4. A projector lens has a focal length 10 cm. It throws an image of a 2 cm × 2 cm slide on a screen 5 metre from the lens. Find
   (i) the size of the picture on the screen and
   (ii) the ratio of illumination of the slide and of the picture on the screen.
5. The eyepiece and objective of a microscope, of focal lengths 0.3 m and 0.4 m respectively, are separated by a distance of 0.2 m. The eyepiece and the objective are to be interchanged such that the angular magnification of the instrument remains same. What is the new separation between the lenses?
6. A compound microscope is used to enlarge an object kept at a distance of 0.03 m from its objective which consists of several convex lenses in contact and has focal length 0.02 m. If a lens of focal length 0.1 m is removed from the objective, find the distance by which the eyepiece of the microscope must be moved to refocus the image.
7. The focal lengths of the objective and eyepiece of a microscope are 4 mm and 25 mm respectively, and the length of the tube is 16 cm. If the final image is formed at infinity and the least distance of distinct vision is 25 cm, then calculate the magnifying power of the microscope.
8. The focal lengths of the objective and the eyepiece of an astronomical telescope are 0.25 m and 0.02 m, respectively. The telescope is adjusted to view an object at a distance of 1.5 m from the objective, the final image being 0.25 m from the eye of the observer. Calculate the tube length of the telescope and the magnification produced by it.
9. An astronomical telescope consisting of two convex lenses of focal length 50 cm and 5 cm is focussed on the moon. What is the distance between the two lenses in this position? If the telescope is then turned towards an object 10 m away, how much would the eye-piece have to be moved to focus on the object without altering the accommodation of the eye? Calculate the magnification (angular) produced by the telescope in the two adjustments.

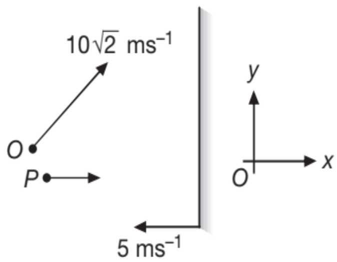
SOLVED PROBLEMS

PROBLEM 1

A plane mirror is moving with a uniform speed of $5\ \text{ms}^{-1}$ along negative x-direction and an observer P is moving with a velocity of $10\ \text{ms}^{-1}$ along $+x$ direction. Calculate the velocity of image of an object O, moving with a velocity of $10\sqrt{2}\ \text{ms}^{-1}$ as shown in the figure, as observed by the observer. Also find its magnitude and direction.



SOLUTION



Let $\vec{v}_O$ be the velocity of the object O, $\vec{v}_P$ be the velocity of the observer P, $\vec{v}_M$ be the velocity of the mirror and $\vec{v}_I$ be the velocity of image (Assume all these velocities w.r.t. ground), then

$$\vec{v}_O = \frac{10\sqrt{2}}{\sqrt{2}}\left(\hat{i}+\hat{j}\right)$$

$$\vec{v}_O = 10\left(\hat{i}+\hat{j}\right)$$

$$\vec{v}_P = 10\hat{i}$$

$$\vec{v}_M = -5\hat{i}$$

$\left(\vec{v}_{IM}\right)_\perp = -\left(\vec{v}_{OM}\right)_\perp$, where the axis perpendicular to the mirror is the x-axis.

$$\Rightarrow \quad \left(\vec{v}_I\right)_x - \left(\vec{v}_M\right)_x = -\left(\vec{v}_O\right)_x + \left(\vec{v}_M\right)_x$$

$$\Rightarrow \quad \left(\vec{v}_I\right)_x = 2\left(\vec{v}_M\right)_x - \left(\vec{v}_O\right)_x$$

$$\Rightarrow \quad \left(\vec{v}_I\right)_x = 2\left(-5\hat{i}\right) - 10\hat{i}$$

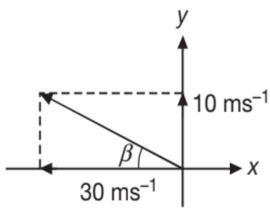
$$\Rightarrow \quad \left(\vec{v}_I\right)_x = -20\hat{i}$$

Further, parallel to the mirror, i.e., along y-axis, we have

$$\left(\vec{v}_I\right)_y = \left(\vec{v}_O\right)_y = 10\hat{j}$$

Since $\vec{v}_I = \left(\vec{v}_I\right)_x + \left(\vec{v}_I\right)_y$

So, absolute velocity of the image is



$$\vec{v}_I = -20\hat{i} + 10\hat{j}$$

Now $\vec{v}_{IP} = \vec{v}_I - \vec{v}_P$

$$\Rightarrow \quad \vec{v}_{IP} = -20\hat{i} + 10\hat{j} - 10\hat{i}$$

$$\Rightarrow \quad \vec{v}_{IP} = -30\hat{i} + 10\hat{j}$$

$$\Rightarrow \quad \left|\vec{v}_{IP}\right| = \sqrt{900+100} = 10\sqrt{10}\ \text{ms}^{-1}$$

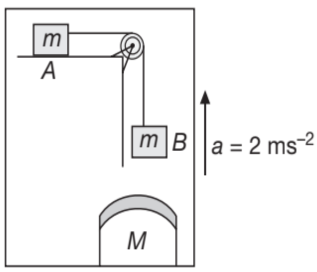
If β is the angle made by $\vec{v}_{IP}$ with $-x$ axis, then

$$\tan\beta = \frac{10}{30}$$

$$\Rightarrow \quad \beta = \tan^{-1}\left(\frac{1}{3}\right), \text{ with } -x \text{ axis}$$

PROBLEM 2

Consider the situation shown in figure. The elevator is going up with an acceleration of $2\ \text{ms}^{-2}$ and the focal length of the mirror is $12\ \text{cm}$. All the surfaces are smooth and the pulley is light. The mass pulley system is released from rest (w.r.t. the elevator) at $t = 0$ when the distance of B from the mirror is $42\ \text{cm}$. Find the distance between the image of the block B and the mirror at $t = 0.2\ \text{s}$. Take $g = 10\ \text{ms}^{-2}$.

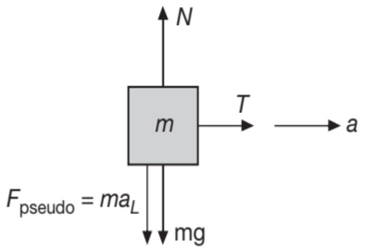


SOLUTION

Let us assume that the acceleration of blocks A and B to be a w.r.t. lift and a_L be the acceleration of lift. Consider block A, as seen from the reference frame attached to the lift (a non-inertial frame), we get

$$N = mg + ma_L \quad \ldots(1)$$

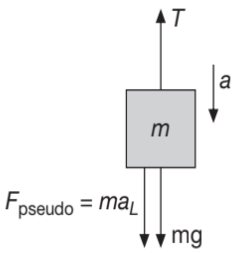
$$T = ma \quad \ldots(2)$$



Free body diagram of *A*

Now, consider block B, as seen from the reference frame attached to the lift, we have

$$mg + ma_L - T = ma \quad \ldots(3)$$



Free body diagram of *B*

On adding equations (2) and (3), we get

$$a = \frac{g + a_L}{2} = \frac{10+2}{2} = 6 \text{ ms}^{-2}$$

So, distance fallen by block (B) is $x = \frac{1}{2}at^2$

$$\Rightarrow \quad x = \frac{1}{2} \times 6 \times (0.2)^2$$

$$\Rightarrow \quad x = 0.12 \text{ m} = 12 \text{ cm}$$

Now, consider reflection at convex mirror, we have

$$u = -(42-12) = -30 \text{ cm}$$

$$f = +12 \text{ cm}$$

Since $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad \frac{1}{v} + \frac{1}{(-30)} = \frac{1}{12}$$

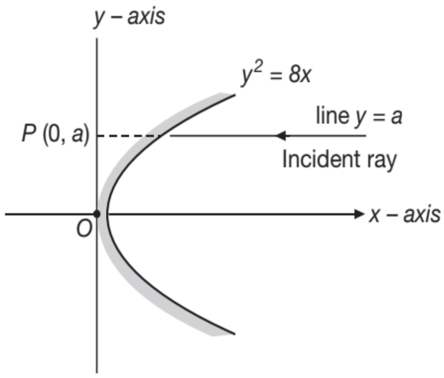
$$\Rightarrow \quad v = \frac{(30)(12)}{40+12} = \frac{360}{42}$$

$$\Rightarrow \quad v = 8.57 \text{ cm}$$

Therefore, the distance between the image of block (B) and mirror is 8.57 cm

PROBLEM 3

Figure shows a parabolic reflector in x-y plane given by $y^2 = 8x$. A ray of light travelling along the line $y = a$ is incident on the reflector. Find where the ray intersects the x-axis after reflection.



SOLUTION

The point at which light ray is incident satisfies $y^2 = 8x$

Since $y = a$

$\Rightarrow \quad x = \frac{a^2}{8}$ i.e., the point at which ray is incident is $P\left(\frac{a^2}{8}, a\right)$

After reflection ray passes through Q.

In ΔPQC, $PQ = QC$ (because sides opposite to equal angles are equal)

In right triangle PNQ

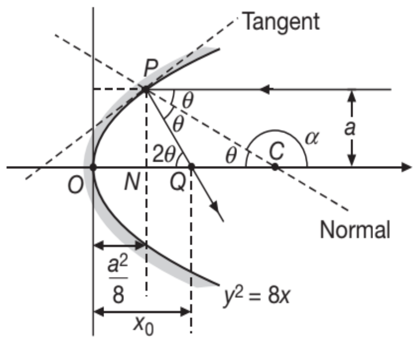
$$\tan(2\theta) = \frac{a}{\left(x_0 - \frac{a^2}{8}\right)} = \frac{8a}{8x_0 - a^2} \quad \ldots(1)$$

Slope of tangent to a curve is $\frac{dy}{dx}$.

Since $y^2 = 8x$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 8$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{y}$$



If m_1 is slope of tangent and m_2 is slope of normal, then $m_1 m_2 = -1$

$$\Rightarrow \quad \text{Slope of normal is } m_2 = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{y}{4}$$

$$\Rightarrow \quad \tan\alpha = -\frac{a}{4} \qquad \{\because y = a\}$$

$$\Rightarrow \quad \tan\theta = \frac{a}{4} \qquad \{\because \theta = 180 - \alpha\} \qquad \ldots(2)$$

From (1), we get

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{8a}{8x_0 - a^2}$$

$$\Rightarrow \quad \frac{2\times\frac{a}{4}}{1-\frac{a^2}{16}} = \frac{8a}{8x_0 - a^2}$$

$$\Rightarrow \quad x_0 = 2$$

Please note that x_0 is independent of a and the rays intersect at the focus of the parabola.

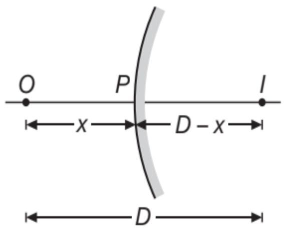
PROBLEM 4

An observer whose least distance of distinct vision is D, views his own face in a convex mirror of radius of curvature R. Prove that the magnification produced cannot exceed $\dfrac{R}{D+\sqrt{D^2+R^2}}$.

SOLUTION

For clear vision, the distance between object and image OI must be more than D. Let object be placed at a distance x from mirror and hence image is at distance $(D-x)$ from mirror.

Since, $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f}$



$$v = +(D-x),\ u = -x \text{ and } f = +\frac{R}{2} (\text{for } x > 0)$$

$$\frac{1}{D-x} - \frac{1}{x} = \frac{2}{R}$$

$$\Rightarrow \quad \frac{x - D + x}{(D-x)(x)} = \frac{2}{R}$$

$$\Rightarrow \quad -2x^2 + 2xD = 2Rx - RD$$

Forming a quadratic in x, we get

$$x = \frac{D - R \pm \sqrt{R^2+D^2}}{2}$$

Discarding the negative value, we get

$$x = \frac{D - R + \sqrt{R^2+D^2}}{2},$$

Magnification produced by the mirror is

$$m = -\frac{v}{u} = -\frac{D-x}{x} = \frac{D+R-\sqrt{R^2+D^2}}{D-R+\sqrt{R^2+D^2}}$$

Rationalising the above equation, we get

$$m = \frac{(D+R) - \left(\sqrt{R^2+D^2}\right)^2}{\left(D-R+\sqrt{R^2+D^2}\right)\left(D+R+\sqrt{R^2+D^2}\right)}$$

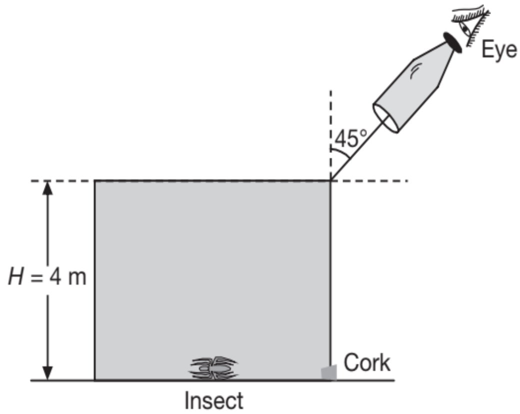
$$\Rightarrow \quad m = \frac{2RD}{2D^2 + 2D\left(\sqrt{R^2+D^2}\right)} = \frac{r}{D+\sqrt{R^2+D^2}}$$

Thus the maximum magnification produced by the mirror can be given as

$$m_{max} = \frac{R}{D+\sqrt{R^2+D^2}}$$

PROBLEM 5

A fixed cylindrical tank of height $H = 4$ m and radius $R = 3$ m is filled up with a liquid. An observer observes through a telescope fitted at the top of the wall of the tank and inclined at $\theta = 45°$ with the vertical. When the tank is completely filled with liquid, he notices an insect, which is at the center of the bottom of the tank. At $t = 0$, he opens a cork of radius $r = 3$ cm at the bottom of tank. The insect moves in such a way that it is visible for a certain time. Determine



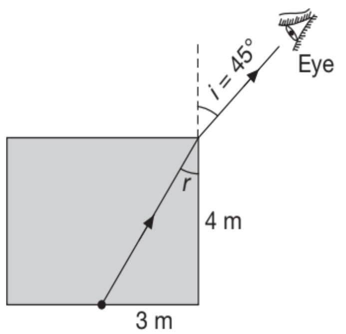
(a) the refractive index of the liquid
(b) the velocity of insect as a function of time.

SOLUTION

(a) At $t = 0$

$$\sin r = \frac{3}{5}$$

$$\mu = \frac{\sin i}{\sin r} = \frac{5}{3\sqrt{2}}$$



(b) Let at time t, insect be at a distance x from centre of the tank. Since,

$$\frac{x_1}{h} = \tan r = \frac{3}{4}$$

$$\Rightarrow \ x_1 = \frac{3}{4}h$$

So, $x = (H - h) + x_1 - 3$

$$\Rightarrow \ x = 4 - h + \frac{3}{4}h - 3$$

$$\Rightarrow \ x = 1 - \frac{h}{4}$$

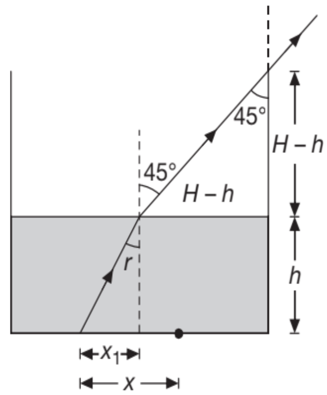
$$\Rightarrow \ \left(\frac{dx}{dt}\right) = \frac{1}{4}\left(-\frac{dh}{dt}\right) \quad \ldots(1)$$

From Equation of Continuity, we have

$$-A_1\left(\frac{dh}{dt}\right) = A_2\sqrt{2gh}$$

$$\Rightarrow \ \left(-\frac{dh}{\sqrt{h}}\right) = \frac{A_2\sqrt{2g}}{A_1}dt$$

$$\Rightarrow \ -\int_H^h \frac{dh}{\sqrt{h}} = \frac{\pi(3\times10^{-2})^2}{\pi(3)^2}\sqrt{2\times9.8}\int_0^t dt$$



Substituting $H = 4$ m, we get

$$h = (2 - 2.21\times10^{-4}t)^2$$

$$\Rightarrow \ -\frac{dh}{dt} = 4.42\times10^{-4}(2 - 2.21\times10^{-4}t)$$

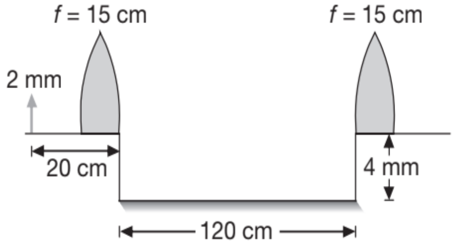
So, speed of insect is

$$v = \frac{dx}{dt} = \frac{1}{4}\left(-\frac{dh}{dt}\right)$$

$$\Rightarrow \ v = 1.1\times10^{-4}(2 - 2.21\times10^{-4}t) \text{ ms}^{-1}$$

PROBLEM 6

A convex lens of focal length 15 cm is split into two halves and the two halves are placed at a separation of 120 cm. Between these two halves of the convex lens, a plane mirror is placed horizontally and at a distance of 4 mm below the principal axis of the lens halves. An object of length 2 mm is placed at a distance of 20 cm from one half lens as shown in figure.



(a) Find the position and size of the final image.

(b) Trace the path of rays forming the image.

SOLUTION

(a) For refraction at first half lens, using lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}-\frac{1}{-20}=\frac{1}{15}$$

$\Rightarrow$ $v = 60$ cm

Magnification, $m=\frac{v}{u}=\frac{60}{-20}=-3$

The image formed by first half lens is shown in Figure 1

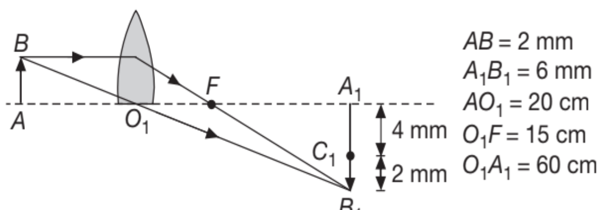


Figure-1

Now, the point B_1 is 6 mm below the principal axis of the lenses. Plane mirror is 4 mm below it. Hence, 4 mm length of A_1B_1 (i.e., A_1C_1) acts as real object for mirror. Mirror forms its virtual image A_2C_2. So, 2 mm length of A_1B_1 (i.e., C_1B_1) acts as virtual object for mirror. Real image C_2B_2 is formed of this part. Image formed by plane mirror is shown in Figure 2.

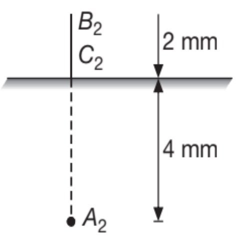


Figure-2

For the second half of the lens, using lens formula $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}-\frac{1}{-60}=\frac{1}{15}$$

$\Rightarrow$ $v = +20$

$$m=\frac{v}{u}=\frac{20}{-60}=-\frac{1}{3}$$

So, length of final image $A_3B_3=\frac{1}{3}A_2B_2=2\text{ mm}$.

However, point B_2 is 2 mm below the optic axis of second half lens. Hence, its image B_3 is formed $\frac{2}{3}$ mm above the principal axis.

Similarly, point A_2 is 8 mm below the principal axis. Hence, its image is $\frac{8}{3}$ mm above it.

Therefore, image is at a distance of 20 cm behind the second half lens and at a distance of $\frac{2}{3}$ mm above the principal axis.

The size of image is 2 mm and is inverted as compared to the given object. Image formed by second half lens is shown in Figure 3.

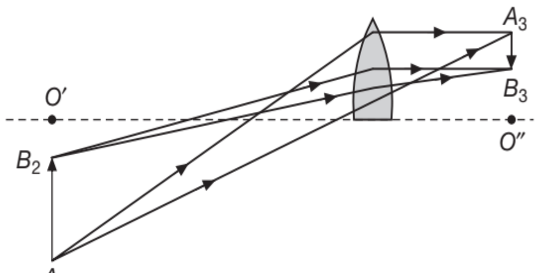


Figure-3

(b) The ray diagram for the final image is shown in Figure 4.

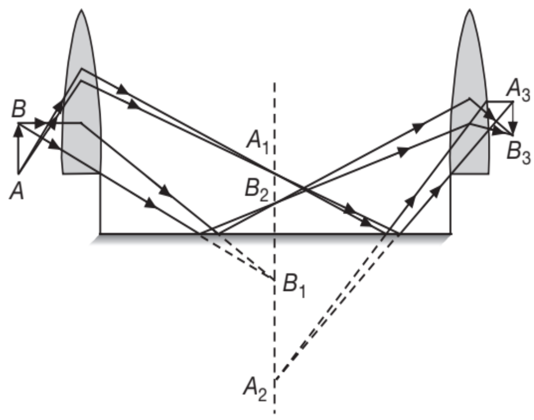
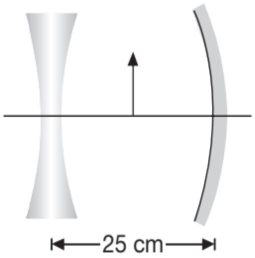


Figure-4

PROBLEM 7

An object lies midway between the lens and a mirror. The mirror's radius of curvature is 20 cm and the lens has a focal length of −16.7 cm. Considering that the rays that leave the object travel first towards the mirror, locate the final image formed by this system. Is this image real or virtual. Is it upright or inverted? What is the overall magnification?



SOLUTION

STEP-1: Image formed by mirror

Using mirror formula $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}=\frac{2}{R}$, we get

$$\frac{1}{v_1}+\frac{1}{-12.5}=\frac{2}{-20}$$

$$\Rightarrow \quad v_1=-50 \text{ cm}$$

$$m_1=-\frac{v}{u}=-\frac{(-50)}{(-12.5)}=-4$$

So, the image formed by the mirror is at a distance of 50 cm from the mirror to the left of it. It is inverted and four times larger.

STEP-2: Image formed by lens

The image formed by mirror acts as an object for the lens and the image formed by the mirror is at a distance of 25 cm to the left of lens. Using the lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$ we get

$$\frac{1}{v_2}-\frac{1}{25}=\frac{1}{-16.7}$$

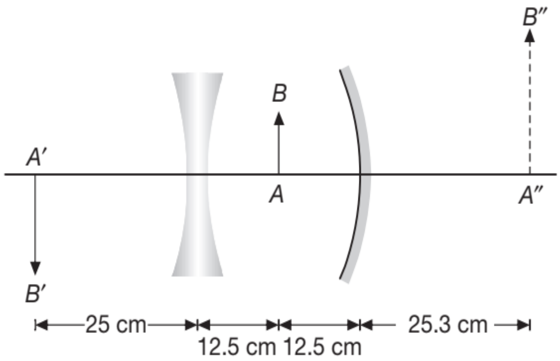
$$\Rightarrow \quad v_2=-50.3 \text{ cm}$$

and $m_2=\frac{v}{u}=\frac{-50.3}{25}=-2.012$

Net magnification is given by

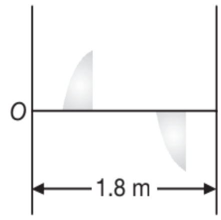
$$m=m_1\times m_2\approx 8$$

Hence, the final image is at a distance 25.3 cm to the right of the mirror, virtual, upright enlarged and approximately 8 times. Positions of the two images are shown in figure.



PROBLEM 8

A thin plano-convex lens of focal length f is split into two halves. One of the halves is shifted along the optical axis as shown in figure. The separation between object and image planes is 1.8 m. The magnification of the image, formed by one of the half lens is 2. Find the focal length of the lens and separation between the two halves. Draw the ray diagram for image formation.



SOLUTION

For both the halves, position of object and image is same, however the only difference is of magnification. Magnification for one of the halves is given as $2(>1)$. This can be for the first one, because for this, $|v|>|u|$. Therefore, magnification, $|m|=\left|\frac{v}{u}\right|>1$. So, for the first half, we have

$$\left|\frac{v}{u}\right|=2$$

$$\Rightarrow \quad |v|=2|u|$$

Let $u=-x$, then $v=+2x$

and $|u|+|v|=1.8$ m

$$\Rightarrow \quad 3x=1.8 \text{ m}$$

$$\Rightarrow \quad x=0.6 \text{ m}$$

Hence, $u = -0.6$ m and $v = +1.2$ m

Using $\frac{1}{f} = \frac{1}{v} - \frac{1}{u} = \frac{1}{1.2} - \frac{1}{-0.6} = \frac{1}{0.4}$

$\Rightarrow \quad f = 0.4$ m

For the second half, we have

$$\frac{1}{f} = \frac{1}{1.2-d} + \frac{1}{-(0.6+d)}$$

$$\Rightarrow \quad \frac{1}{0.4} = \frac{1}{1.2-d} + \frac{1}{(0.6+d)}$$

Solving this, we get $d = 0.6$ m

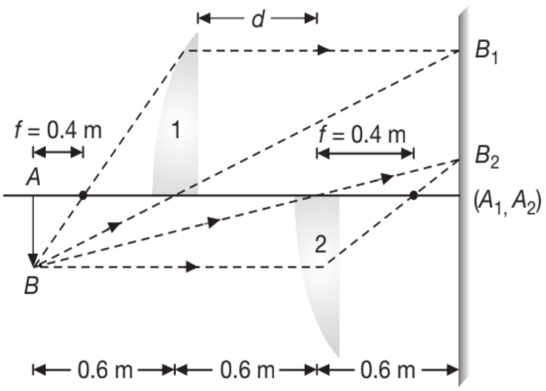
Magnification for the second half will be

$$m_2 = \frac{v}{u} = \frac{0.6}{-(1.2)} = -\frac{1}{2}$$

and magnification for the first half is

$$m_1 = \frac{v}{u} = \frac{1.2}{-(0.6)} = -2$$

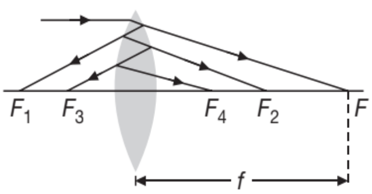
The ray diagram is as follows:



PROBLEM 9

A strong source of light when used with a convex lens produces a number of images of the source owing to feeble internal reflections and refraction called flare spots as shown in figure. These extra images are F_1, F_2,.... If F_n is the position of n^{th} flare spot, then show that

$$\frac{1}{F_n} = \frac{(n+1)\mu - 1}{f(\mu-1)}$$



SOLUTION

Light converges at F_1 after two refractions and one reflection from the lens. So we use

$$\frac{1}{F_1} = \frac{2}{f_e} + \frac{1}{f_m}$$

Where focal length of equivalent independent lens is given do

$$\frac{1}{f_e} = (\mu-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\Rightarrow \quad \frac{1}{f} = (\mu-1)\left(\frac{1}{+R} - \frac{1}{-R}\right) = (\mu-1)\frac{2}{R}$$

$$\Rightarrow \quad R = 2(\mu-1)f$$

$$\Rightarrow \quad \frac{1}{F_1} = \frac{2}{f} + \frac{2}{2(\mu-1)f}$$

$$\Rightarrow \quad \frac{1}{F_1} = \frac{2\mu-1}{(\mu-1)f}$$

For F_2, there are three refractions and two reflections

$$\frac{1}{F_2} = \frac{3}{f_1} + \frac{2}{f_m}$$

$$\Rightarrow \quad \frac{1}{F_2} = \frac{3}{f} + \frac{2}{\frac{R}{2}} = \frac{3}{f} + \frac{4}{R}$$

$$\Rightarrow \quad = \frac{3}{f} + \frac{4}{2(\mu-1)f}$$

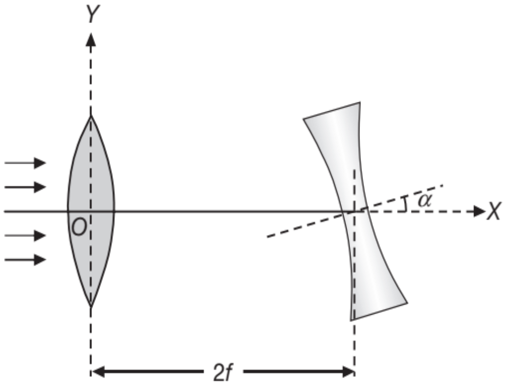
$$\Rightarrow \quad = \frac{3}{f} + \frac{2}{(\mu-1)f}$$

$$\Rightarrow \quad = \frac{3(\mu-1)+2}{(\mu-1)f} = \frac{3\mu-1}{(\mu-1)f}$$

$$\Rightarrow \quad \frac{1}{F_n} = \frac{(n+1)\mu-1}{(\mu-1)f}$$

PROBLEM 10

Two thin lenses of same focal length f are arranged with their principal axes inclined at an angle α as shown in figure. The separation between the optical centers of the lenses is $2f$. A point object lies on the principal axis of the convex lens at a large distance to the left of convex lens.



(a) Find the co-ordinates of the final image formed by the system of lenses taking O as the origin of the co-ordinate axes.

(b) Draw the ray diagram.

SOLUTION

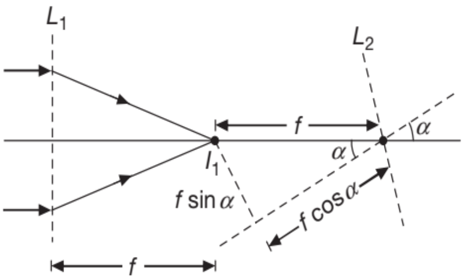
(a) For concave lens (L_2)

$$\frac{1}{v}-\frac{1}{-f\cos\alpha}=\frac{-1}{f}$$

$$\Rightarrow\ v=-\left(\frac{f\cos\alpha}{1+\cos\alpha}\right) \qquad \ldots(1)$$

The magnification is given by

$$m=\frac{v}{u}=\frac{1}{1+\cos\alpha}$$



So, height of I_2 from the principal axis of L_2 is

$$h=(f\sin\alpha)m=\frac{f\sin\alpha}{1+\cos\alpha} \qquad \ldots(2)$$

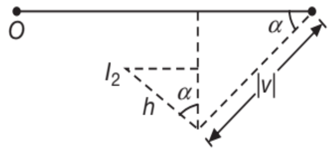
Hence the x-co-ordinate of image I_2 is given by

$$x=2f-|v|\cos\alpha-h\sin\alpha$$

$$\Rightarrow\ x=2f-\frac{f\cos^2\alpha}{1+\cos\alpha}-\frac{f\sin^2\alpha}{1+\cos\alpha}$$

$$\Rightarrow\ x=2f-\frac{f}{1+\cos\alpha}$$

$$\Rightarrow\ x=f\left(\frac{2\cos\alpha+1}{\cos\alpha+1}\right)$$



Similarly, y co-ordinate of image I_2

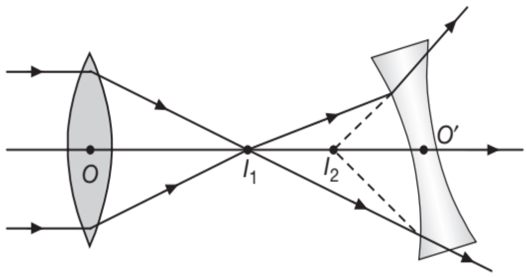
$$y=-(|v|\sin\alpha-h\cos\alpha)$$

On substituting the values of $|v|$ and h from (1) and (2), we get $y=0$.

So, the coordinates of the final image are

$$\left[f\left(\frac{2\cos\alpha+1}{\cos\alpha+1}\right),\ 0\right]$$

(b) Ray diagram is shown in figure

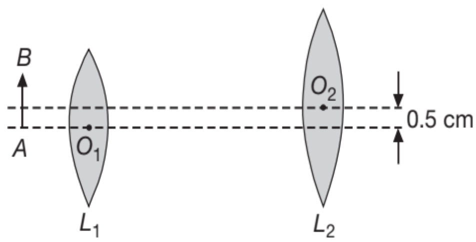


Conceptual Note(s)

The Y-co-ordinate of I_2 is zero is very obvious because a ray of light starting from I_2 and passing through O′ will suffer no deviation. Hence, I_2 must be formed on this line itself i.e., $y=0$.

PROBLEM 11

Two thin lenses $f_1=10$ cm and $f_2=20$ cm are separated by a distance $d=5$ cm. Their optical centres are displaced a distance $\Delta=0.5$ cm. A linear object of size 3 cm placed at 30 cm from the optical centre of left lens. Find the nature position and size of final image.



SOLUTION

STEP-1: Refraction from the first lens L_1

Using the lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v_1}-\frac{1}{-30}=\frac{1}{10}$$

$$\Rightarrow \quad v_1=15 \text{ cm}$$

and $m_1=\frac{v}{u}=\frac{15}{-30}=-\frac{1}{2}$

STEP-2: Refraction from the second lens L_2

Again using the lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v_2}-\frac{1}{(15-5)}=\frac{1}{20}$$

$$\Rightarrow \quad v_2=\frac{20}{3} \text{ cm} \approx 6.67 \text{ cm}$$

and $m_2=\frac{v}{u}=\frac{20/3}{10}=\frac{2}{3}$

About Final Image

Net magnification is given by

$$m=m_1m_2=-\frac{1}{3}$$

i.e., height of the image is $3\times\frac{1}{3}=1$ cm

Since, the net magnification is negative, so the final image is inverted.

Further y-coordinate of a point of the image will be,

$$y_1=my_0-m_2\Delta \quad \ldots(1)$$

with respect to the principal axis of L_1

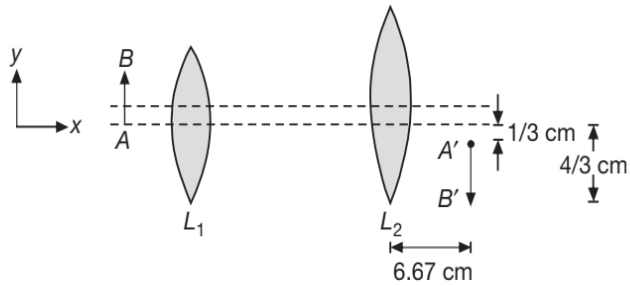
So, y-coordinate of image of A is

$$y_{A'}=\left(-\frac{1}{3}\right)(0)-\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)=-\frac{1}{3} \text{ cm and}$$

y-coordinate of image of B is

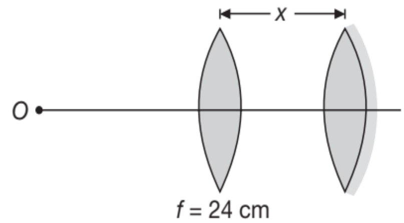
$$y_{B'}=\left(-\frac{1}{3}\right)(3)-\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)=-\frac{4}{3} \text{ cm}$$

Thus final image is as shown in figure.



PROBLEM 12

The radius of curvature of the curved surfaces of an equiconvex lens is 32 cm and its refractive index is $\mu=1.5$. One of its side is silvered and placed 14 cm away from an object as shown in figure. At what distance x should a second convex lens of focal length 24 cm be placed so that the image coincides with the object.



SOLUTION

For the convex lens, we use

$$f=+24 \text{ cm},$$

and $u=-(14-x)$

By refraction formula, we use

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{1}{u}+\frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{1}{24}-\frac{1}{(14-x)}=\frac{14-x-24}{24(14-x)}$$

$$\Rightarrow \quad \frac{1}{v}=\frac{-(x+10)}{24(14-x)}$$

$$\Rightarrow \quad v=-\left[\frac{(336-24x)}{(x+10)}\right]$$

The image will coincide the object if light rays after refraction from un-silvered face fall normally upon silvered face so that these rays will retrace the path of incident rays. This is possible when first surface forms the image at 32 cm from it. Now for the unsilvered surface of the silvered lens, we use

$\mu_2 = 1.5$, $\mu_1 = 1$, $v_1 = -32$ cm,

$u_1 = -(x - v)$ and $R = +32$ cm

By using refraction formula, we have

$$\frac{\mu}{v_1} - \frac{1}{u_1} = \frac{\mu - 1}{R}$$

$$\Rightarrow \frac{1.5}{-32} + \frac{1}{\left[x - \left\{-\frac{336 - 24x}{(x+10)}\right\}\right]} = \frac{1.5 - 1}{32}$$

$$\Rightarrow \frac{(x+10)}{x(x+10) + (336 - 24x)} = \frac{0.5}{32} + \frac{1.5}{32} = \frac{1}{16}$$

$$\Rightarrow x^2 + 10x + 336 - 24x = 16x + 160$$

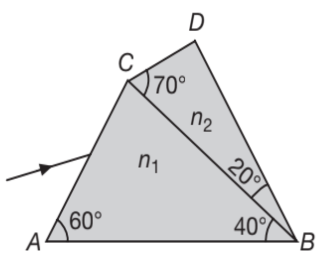
$$\Rightarrow x^2 - 30x + 176 = 0$$

$$\Rightarrow x = 8 \text{ OR } x = 22 \text{ cm}$$

Hence the lens should be placed 8 cm from silvered surface.

PROBLEM 13

A prism of refractive index n_1 and another prism of refractive index n_2 are stuck together with a gap as shown in the figure. The angles of the prism are as shown, n_1 and n_2 depend on λ, the wavelength of light according to the relations given by $n_1 = 1.20 + \frac{10.8 \times 10^4}{\lambda^2}$ and $n_2 = 1.45 + \frac{1.80 \times 10^4}{\lambda^2}$ where λ is in nm.



(a) Calculate the wavelength λ_0 for which rays incident at any angle on the interface BC pass through without bending at that interface.

(b) For light of wavelength λ_0, find the angle of incidence i on the face AC such that the deviation produced by the combination of prisms is minimum.

SOLUTION

Since, $n_1 = 1.20 + \frac{10.8 \times 10^4}{\lambda^2}$ and $n_2 = 1.45 + \frac{1.80 \times 10^4}{\lambda^2}$

where, λ is in nm.

(a) The incident ray will not deviate at BC only if $n_1 = n_2$

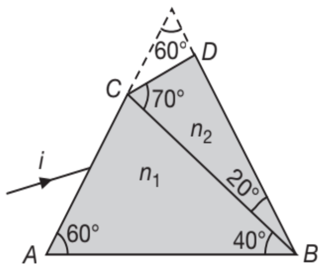
$$\Rightarrow 1.2 + \frac{10.8 \times 10^4}{\lambda_o^2} = 1.45 + \frac{1.80 \times 10^4}{\lambda_o^2} (\lambda = \lambda_0)$$

$$\Rightarrow \frac{9 \times 10^4}{\lambda_o^2} = 0.25$$

$$\Rightarrow \lambda_0 = \frac{3 \times 10^2}{0.5}$$

$$\Rightarrow \lambda_0 = 600 \text{ nm}$$

(b) The given system happens to be a part of an equilateral prism of prism angle 60° as shown in figure.



At minimum deviation, we have

$$r_1 = r_2 = \frac{60°}{2} = 30° = r \quad \text{\{say\}}$$

Since according to Snell's Law, we have

$$n_1 = \frac{\sin i}{\sin r}$$

$$\Rightarrow \sin i = n_1 \sin(30°)$$

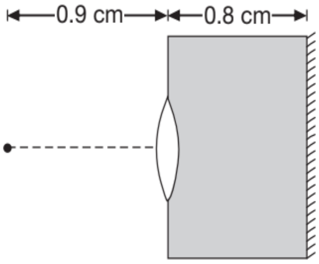
Since, $n_1 = 1.2 + \frac{10.8 \times 10^4}{\lambda_0^2}$, where $\lambda_0 = 600$ nm

$$\Rightarrow \sin i = \left\{1.2 + \frac{10.8 \times 10^4}{(600)^2}\right\}\left(\frac{1}{2}\right) = \frac{1.5}{2} = \frac{3}{4}$$

$$\Rightarrow i = \sin^{-1}\left(\frac{3}{4}\right)$$

PROBLEM 14

A thin equiconvex lens of glass of refractive index $\mu = \frac{3}{2}$ and of focal length 0.3 m in air is sealed into an opening at one end of a tank filled with water $\mu = \frac{4}{3}$. On the opposite side of the lens, a mirror is placed inside the tank on the tank wall perpendicular to the lens axis, as shown in figure. The separation between the lens and the mirror is 0.8 m. A small object is placed outside the tank in front of. Find the position (relative to the lens) of the image of the object formed by the system.



SOLUTION

Applying Lens Maker's Formula

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\Rightarrow \quad \frac{1}{0.3} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{R} - \frac{1}{-R}\right)$$

$$\{\because R_1 = R \text{ and } R_2 = -R\}$$

$$\Rightarrow \quad R = 0.3$$

Now applying, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ at air glass surface, we get

$$\frac{\frac{3}{2}}{v_1} - \frac{1}{-(0.9)} = \frac{\left(\frac{3}{2}\right) - 1}{0.3}$$

$$\Rightarrow \quad v_1 = 2.7 \text{ m}$$

So, the first image I_1 will be formed at 2.7 m from the lens.

This image I_1 will act as the virtual object for glass water surface.

Therefore, applying $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ at glass water surface, we get

$$\frac{\frac{4}{3}}{v_2} - \frac{\frac{3}{2}}{2.7} = \frac{\left(\frac{4}{3}\right) - \left(\frac{3}{2}\right)}{-0.3}$$

$$\Rightarrow \quad v_2 = 1.2 \text{ m}$$

So, the second image I_2 is formed at 1.2 m from the lens or 0.4 m from the plane mirror.

This image I_2 will act as a virtual object for mirror. Therefore, third real image I_3 will be formed at a distance of 0.4 m in front of the mirror after reflection from it. Now this image acts as a real object for water-glass interface. Hence applying, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$, we get

$$\frac{\frac{3}{2}}{v_4} - \frac{\frac{4}{3}}{-(0.8 - 0.4)} = \frac{\left(\frac{3}{2}\right) - \left(\frac{4}{3}\right)}{0.3}$$

$$\Rightarrow \quad v_4 = -0.54 \text{ m}$$

So, the fourth image is formed to the right of the lens at a distance of 0.54 m from it.

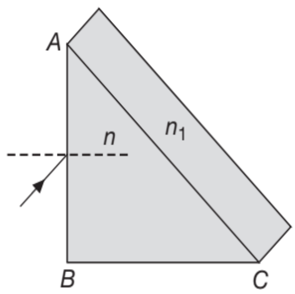
Now finally applying the same formula for glass-air surface, we get

$$\frac{1}{v_5} - \frac{\frac{3}{2}}{-0.54} = \frac{1 - \left(\frac{3}{2}\right)}{-0.3} = -0.9 \text{ m}$$

Hence, the position of final image is 0.9 m relative to the lens (rightwards) i.e., the image is formed 0.1 m behind the mirror.

PROBLEM 15

A right angle prism $(45^\circ - 90^\circ - 45^\circ)$ of refractive index n has a plane of refractive index $n_1\ (n_1 < n)$ cemented to its diagonal face. The assembly is in air. The ray is incident on AB.



(i) Calculate the angle of incidence at AB for which the ray strikes the diagonal face at the critical angle.

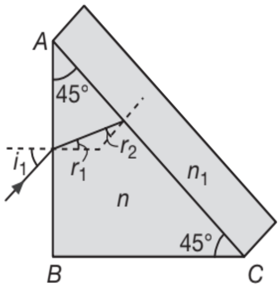
(ii) Assuming $n = 1.352$, calculate the angle of incidence at AB for which the refracted ray passes through the diagonal face undeviated.

SOLUTION

(i) Critical angle C at face AC will be given by

$$C = \sin^{-1}\left(\frac{n_1}{n}\right)$$

$$\Rightarrow \sin C = \frac{n_1}{n}$$



Now, it is given that $r_2 = C$

$$\Rightarrow r_1 = A - r_2 = (45° - C)$$

Applying Snell's Law at face AB, we get

$$n = \frac{\sin i_1}{\sin r_1}$$

$$\Rightarrow \sin i_1 = n \sin r_1$$

$$\Rightarrow i_1 = \sin^{-1}(n \sin r_1)$$

Substituting value of r_1, we get

$$i_1 = \sin^{-1}\{n \sin(45° - C)\}$$

$$\Rightarrow i_1 = \sin^{-1}[n(\sin 45° \cos C - \cos 45° \sin C)]$$

Since $\sin C = \frac{n_1}{n}$

$$\Rightarrow i_1 = \sin^{-1}\left(\frac{n}{\sqrt{2}}\sqrt{1-\sin^2 C} - \sin C\right)$$

$$\Rightarrow i_1 = \sin^{-1}\left[\frac{n}{\sqrt{2}}\left(\sqrt{1-\frac{n_1^2}{n^2}} - \frac{n_1}{n}\right)\right]$$

$$\Rightarrow i_1 = \sin^{-1}\left[\frac{1}{\sqrt{2}}\left(\sqrt{n^2-n_1^2} - n_1\right)\right]$$

Therefore, required angles of incidence (i_1) at face AB for which the ray strikes at AC at critical angle is given by

$$i_1 = \sin^{-1}\left[\frac{1}{\sqrt{2}}\left(\sqrt{n^2-n_1^2} - n_1\right)\right]$$

(ii) The ray will pass undeviated through face AC when

$$n_1 = n \text{ or } r_2 = 0°$$

i.e., ray falls normally on face AC

Since it is given that $n_1 < n$, so the option $n_1 = n$ is ruled out, hence

$$r_2 = 0°$$

$$\Rightarrow r_1 = A - r_2 = 45° - 0° = 45°$$

Now applying Snell's Law at face AB, we get

$$n = \frac{\sin i_1}{\sin r_1}$$

$$\Rightarrow 1.352 = \frac{\sin i_1}{\sin(45°)}$$

$$\Rightarrow \sin i_1 = (1.352)\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \sin i_1 = 0.956$$

$$\Rightarrow i_1 = \sin^{-1}(0.956) \approx 73°$$

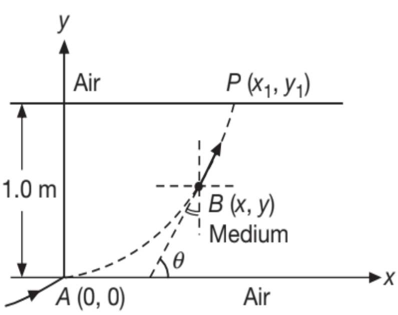
Therefore, required angle of incidence is $i_1 = 73°$.

PROBLEM 16

A ray of light travelling in air is incident at grazing angle (incident angle $= 90°$) on a long rectangular slab of a transparent medium of thickness $t = 1.0$ m. The point of incidence is the origin $A(0, 0)$. The medium has a variable index of refraction $n(y)$ given by

$$n(y) = \left[ky^{3/2} + 1\right]^{1/2} \text{ where } k = 1.0(\text{meter})^{-3/2}$$

The refractive index of air is 1.0



(a) Obtain a relation between the slope of the trajectory of the ray at a point $B(x, y)$ in the medium and the incident angle at the point.

(b) Obtain an equation for the trajectory $y(x)$ of the ray in the medium.

(c) Determine the co-ordinates (x_1, y_1) of the point P, where the ray intersects three upper surface of the slab-air boundary.

(d) Indicate the path of the ray subsequently.

SOLUTION

(a) $i+\theta=90^\circ$, $\theta=90^\circ-i$,

Slope of tangent $=\tan\theta=\tan(90^\circ-i)=\cot i$

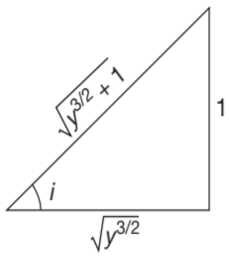
(b) $\tan\theta=\dfrac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx}=\cot i \qquad \ldots(1)$$

Applying Snell's Law at A and B

$$n_A \sin i_A = n_B \sin i_B$$

$n_A=1$ because $y=0$



$\sin i_A=1$ because $i_A=90^\circ$ Grazing incidence

$$n_B=\sqrt{Ky^{3/2}+1}=\sqrt{y^{3/2}+1}$$

because $K=1.0(m)^{-3/2}$

$$\therefore \quad (1)(1)=\sqrt{\left(y^{3/2}+1\right)}\sin i$$

$$\Rightarrow \quad \sin i=\frac{1}{\sqrt{y^{3/2}+1}}$$

$$\Rightarrow \quad \cot i=\sqrt{y^{3/2}} \text{ or } y^{3/4} \qquad \ldots(2)$$

Equating equations (1) and (2), we get

$$\frac{dy}{dx}=y^{3/4} \text{ or } y^{-3/4}dy=dx$$

$$\Rightarrow \quad \int_0^y y^{-3/4}dy=\int_0^x dx \text{ or } 4y^{1/4}=x \qquad \ldots(3)$$

The required equation of trajectory is $4y^{1/4}=x$

$$\Rightarrow \quad y=\frac{x^4}{4^4}=\frac{x^4}{256}$$

(c) At the point of intersection on the upper surface,

$$y=1\text{ m}$$

$$\Rightarrow \quad x=(256)^{1/4}=4\text{ m}$$

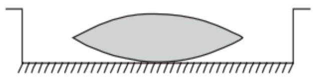
So the co-ordinates are $(4\text{ m}, 1\text{ m})$

(d) As $n_A \sin i_A = n_P \sin i_P$ and as $n_A=n_P=1$

Therefore, $i_P=i_A=90^\circ$ i.e., the ray will emerge parallel to the boundary at P i.e., at grazing emergence.

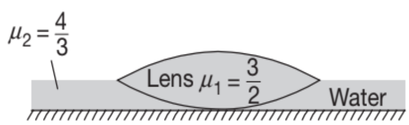
PROBLEM 17

A thin biconvex lens of refractive index $\frac{3}{2}$ is placed on a horizontal plane mirror as shown in the figure. The space between the lens and the mirror is then filled with water of refractive index $\frac{4}{3}$. It is found that when a point object is placed 15 cm above the lens on its principal axis, the object coincides with its own image. On repeating with another liquid, the object and the image again coincide at a distance 25 cm from the lens. Calculate the refractive index of the liquid.



SOLUTION

Let R be the radius of curvature of both the surfaces of the equi-convex lens, then in the first case, the situation is shown in figure.



Let f_1 be the focal length of equi-convex lens of refractive index μ_1 and f_2 be the focal length of plano-concave lens (made of water) of refractive index μ_2. The focal length of the combined lens system is given by

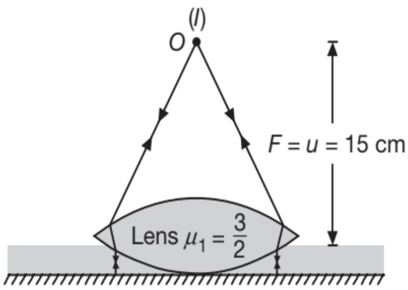
$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

$$\Rightarrow \quad \frac{1}{F}=(\mu_1-1)\left(\frac{1}{R}-\frac{1}{-R}\right)+(\mu_2-1)\left(\frac{1}{-R}-\frac{1}{\infty}\right)$$

$$\Rightarrow \quad \frac{1}{F}=\left(\frac{3}{2}-1\right)\left(\frac{2}{R}\right)+\left(\frac{4}{3}-1\right)\left(-\frac{1}{R}\right)=\frac{1}{R}-\frac{1}{3R}=\frac{2}{3R}$$

$\Rightarrow \quad F = \dfrac{3R}{2}$

Now, image coincides with the object when ray of light retraces its path i.e., it is falls normally on the plane mirror. This is possible only when object lies at centre of curvature of the lens system.



$\Rightarrow \quad F = 15\text{ cm} \qquad \{\because \text{ Distance of object is 15 cm}\}$

$\Rightarrow \quad \dfrac{3R}{2} = 15\text{ cm}$

$\Rightarrow \quad R = 10\text{ cm}$

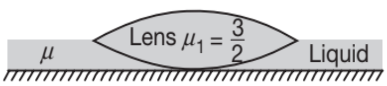
In the second case, let μ be the refractive index of the liquid filled between lens and mirror and let F' be the focal length of new lens system. Then,

$$\frac{1}{F'} = (\mu_1 - 1)\left(\frac{1}{R} - \frac{1}{-R}\right) + (\mu - 1)\left(\frac{1}{-R} - \frac{1}{\infty}\right)$$

$$\Rightarrow \quad \frac{1}{F'} = \left(\frac{3}{2} - 1\right)\left(\frac{2}{R}\right) - \frac{(\mu - 1)}{R} = \frac{1}{R} - \frac{\mu - 1}{R} = \frac{(2 - \mu)}{R}$$

$$\Rightarrow \quad F' = \frac{R}{2 - \mu} = \frac{10}{2 - \mu} \qquad \{\because R = 10\text{ cm}\}$$

Now, the image coincides with the object when it is placed at 25 cm distance.



$\Rightarrow \quad F' = 25$

$\Rightarrow \quad \dfrac{10}{2 - \mu} = 25$

$\Rightarrow \quad 50 - 25\mu = 10$

$\Rightarrow \quad 25\mu = 40$

$\Rightarrow \quad \mu = \dfrac{40}{25} = 1.6$

$\Rightarrow \quad \mu = 1.6$

PROBLEM 18

A telescope has an objective of focal length 50 cm and eyepiece of focal length 5 cm. The distance of distinct vision is 25 cm. The telescope is focussed for distinct vision on a scale 200 cm away from the objective. Calculate

(i) The separation between the objective and eye-piece,

(ii) The magnification produced.

SOLUTION

The situation is shown in figure with ray diagram.

(i) If the separation between the two lenses be x then for lens formula for refraction at objective lens we use

$u_0 = -200\text{ cm}$ and $f_0 = +50\text{ cm}$

From lens formula, we have

$$\frac{1}{v_0} - \frac{1}{u_0} = \frac{1}{f_0}$$

$$\Rightarrow \quad \frac{1}{v_0} = \frac{1}{f_0} + \frac{1}{u_0} = \frac{1}{50} - \frac{1}{200}$$

$$\Rightarrow \quad \frac{1}{v_0} = \frac{4 - 1}{200} = \frac{3}{200}$$

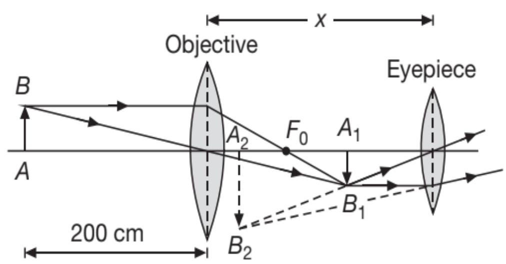
$$\Rightarrow \quad v_0 = +\frac{200}{3}\text{ cm}$$

Thus a real image is formed at a distance of $\dfrac{200}{3}$ from the objective. This image acts as object for the eyepiece. For refraction through eyepiece, we use

$$u_e = -\left(x - \frac{200}{3}\right)$$

$v_e = -25\text{ cm}$ and $f_e = +5\text{ cm}$

$$\Rightarrow \quad -\frac{1}{25} + \frac{1}{\left(x - \frac{200}{3}\right)} = \frac{1}{5}$$



$$\Rightarrow \frac{1}{\left(x-\frac{200}{2}\right)}=\frac{1}{5}+\frac{1}{25}=\frac{6}{25}$$

$$\Rightarrow 6x-400=25$$

$$\Rightarrow 6x=425$$

$$\Rightarrow x=\frac{425}{6}=70.80 \text{ cm}$$

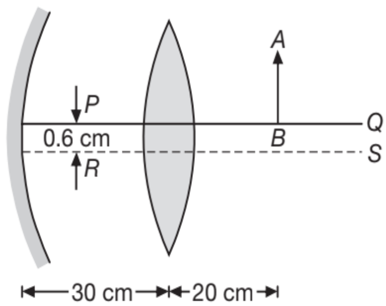
(ii) Magnification of Objective $=\frac{v_0}{u_0}=\frac{200}{3\times 200}=\frac{1}{3}$

Magnification of Eyepiece $=\frac{v_e}{u_e}=\frac{25\times 6}{25}=6$

Total magnification $=\frac{1}{3}\times 6=2$.

PROBLEM 19

A convex lens of focal length 15 cm and a concave mirror of focal length 30 cm are kept with their optic axis PQ and RS parallel but separated in vertical direction by 0.6 cm as shown. The distance between the lens and mirror is 30 cm. An upright object AB of height 1.2 cm is placed on the optic axis PQ of the lens at a distance of 20 cm from the lens. If $A'B'$ is the image after refraction from the lens and the reflection from the mirror, find the distance of $A'B'$ from the pole of the mirror and obtain its magnification. Also locate positions of A' and B' with respect to the optic axis RS.

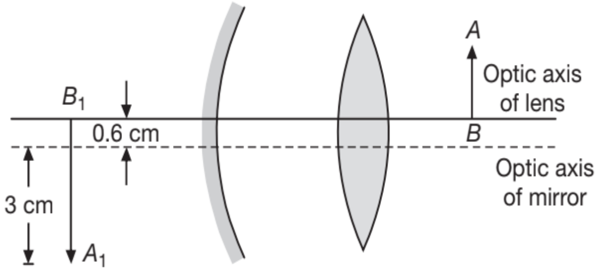


SOLUTION

Rays coming from object AB first refract from the lens and then reflect from the mirror.

For refraction from the lens, we have

$u=-20$ cm, $f=+15$ cm



Applying lens formula, $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}-\frac{1}{(-20)}=\frac{1}{15}$$

$$\Rightarrow v=+60 \text{ cm}$$

and linear magnification is given by

$$m_1=\frac{v}{u}=\frac{+60}{(-20)}=-3$$

So, the first image formed by the lens will be 60 cm from it (or 30 cm from the mirror) towards left and 3 times magnified but inverted. Length of first image A_1B_1 would be $A_1B_1=1.2\times 3=3.6$ cm (inverted).

For reflection from mirror, we have

Image formed by lens (A_1B_1) will behave like a virtual object for the mirror at a distance of 30 cm from it as shown. Therefore $u=+30$ cm, $f=-30$ cm.

Applying mirror formula, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{v}+\frac{1}{30}=-\frac{1}{30}$$

$$\Rightarrow v=-15 \text{ cm}$$

and linear magnification is given by

$$m_2=-\frac{v}{u}=-\frac{(-15)}{+30}=+\frac{1}{2}$$

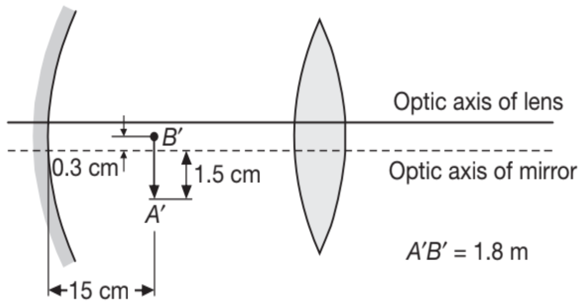
So, the final image $A'B'$ will be located at a distance of 15 cm from the mirror (towards right) and since magnification is $+\frac{1}{2}$, length of final image would be

$$A'B'=3.6\times\frac{1}{2}=1.8 \text{ cm}$$

$$\Rightarrow A'B'=1.8 \text{ cm}$$

Since the point B_1 is 0.6 cm above the optic axis of mirror, therefore, its image B′ would be $(0.6)\left(\frac{1}{2}\right) = 0.3$ cm above optic axis.

Similarly, point A_1 is 3 cm below the optic axis, therefore, its image A' will be $3 \times \frac{1}{2} = 1.5$ cm below the optic axis as shown.



Net magnification of the image is given by

$$m = m_1 \times m_2 = (-3)\left(+\frac{1}{2}\right) = -\frac{3}{2}$$

$$\Rightarrow \quad A'B' = (m)(AB) = \left(-\frac{3}{2}\right)(1.2) = -1.8 \text{ cm}$$

Conceptual Note(s)

If the co-ordinates of the object (X_0, Y_0) are generally known to us with reference to the pole of an optical instrument (whether it is a lens or a mirror), the corresponding co-ordinates of image (X_i, Y_i) are found as follows.

X_i is obtained using $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ (for a mirror)

OR $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ (for a lens)

Here, v is actually X_i and u is X_0 i.e., the above formula can be written as $\frac{1}{X_i} \pm \frac{1}{X_0} = \frac{1}{f}$

Similarly, Y_i is obtained from $m = \frac{I}{O}$

Here, I is Y_i and O is Y_0 i.e., the above formula can be written as $m = \frac{Y_i}{Y_0}$ or $Y_i = mY_0$.

PROBLEM 20

The optical powers of the objective and eyepiece of a microscope are equal to 100 D and 20 D respectively. The microscope magnification is equal to 50 when image is produced at near point of eye. What will be magnification of the microscope be when the distance between the objective and eyepiece is increased by 2 cm ?

SOLUTION

For the eyepiece, we use

$$v_e = -25 \text{ cm and } f_e = 5 \text{ cm}$$

Using lens formula for eyepiece, we have

$$\frac{1}{v_e} - \frac{1}{u_e} = \frac{1}{f_e}$$

$$\Rightarrow \quad \frac{1}{u_e} = \frac{1}{-25} - \frac{1}{5} = -\frac{6}{25} \text{ cm}$$

$$\Rightarrow \quad u_e = -\frac{25}{6} \text{ cm}$$

Magnification of eyepiece is

$$M_e = \frac{v_e}{u_e} = \frac{25}{\left(\frac{25}{6}\right)} = 6$$

The magnification of microscope is given as

$$50 = M_e \times M_0$$

$$\Rightarrow \quad 50 = 6 \times M_0$$

$$\Rightarrow \quad M_0 = \frac{50}{6} = \text{Magnification of objective}$$

The magnifying power of objective is given by

$$M_0 = \frac{v_0}{u_0} = \frac{v_0 - f_0}{f_0}$$

$$\Rightarrow \quad M_0 f_0 = v_0 - f_0$$

$$\Rightarrow \quad v_0 = M_0 f_0 + f_0 = f_0(M_0 + 1)$$

$$\Rightarrow \quad v_0 = f_0\left(\frac{50}{6} + 1\right) = \frac{56}{6} f_0 = \frac{56}{6}$$

When the distance is increased by 2 cm, then new value of v_0 will become

$$v_0' = \frac{56}{6} + 2 = \frac{68}{6}$$

now magnification by objective will be

$$M_0' = \frac{\left(\frac{68}{6}\right)}{1} - 1 = \frac{62}{6}$$

As magnification by eyepiece will remain same, total magnification will now be

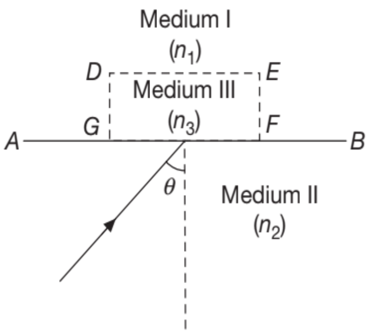
$$M_T = M_0' \times M_e = \frac{62}{6} \times 6 = 62$$

PROBLEM 21

Monochromatic light is incident on a plane interface AB between two media of refractive indices n_1 and n_2 $(n_2 > n_1)$ at an angle of incidence θ as shown in the figure. The angle θ is infinitesimally greater than the critical angle for the two media so that total internal reflection takes place. Now if a transparent slab $DEFG$ of uniform thickness and of refractive index n_3 is introduced on the interface (as shown in the figure), show that for any value of n_3 all light will ultimately be reflected back again into medium II. Consider separately the cases

(a) $n_3 < n_1$ and

(b) $n_3 > n_1$



SOLUTION

At interface AB, θ is infinitesimally greater (slightly greater) than the critical angle for interface, so

$$\theta > \sin^{-1}\left(\frac{n_1}{n_2}\right)$$

(a) When $n_3 < n_1$

$$\Rightarrow n_3 < n_1 < n_2$$

$$\Rightarrow \frac{n_3}{n_2} < \frac{n_1}{n_2}$$

$$\Rightarrow \sin^{-1}\left(\frac{n_3}{n_2}\right) < \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Hence, critical angle for Medium III and Medium II will be less than the critical angle for Medium II and Medium I. So, if TIR is taking place between Medium I and Medium II, then TIR will definitely take place between Medium I and Medium III.

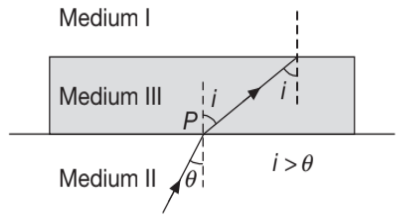
(b) When $n_3 > n_1$, then two further cases may arise.

CASE-1: $n_1 < n_3 < n_2$

In this case there will be no TIR between Medium I and Medium III but TIR will take place between Medium III and Medium II. This is because :

Ray of light first enters from Medium II to Medium III i.e., from denser to rarer. So,

$$i > \theta$$



Applying Snell's Law at P, we get

$$n_2 \sin\theta = n_3 \sin i$$

$$\Rightarrow \sin i = \left(\frac{n_2}{n_3}\right)\sin\theta$$

Since, $\sin\theta$ is slightly greater than $\frac{n_1}{n_2}$, so

$\sin i$ is slightly greater than $\frac{n_2}{n_3} \times \frac{n_1}{n_2} = \frac{n_1}{n_3}$

However, $\frac{n_1}{n_3}$ is nothing but $\sin C$ for Medium I, Medium III interface, so

$\sin i$ is slightly greater than $\sin C$ for Medium I, Medium III interface.

$$\Rightarrow i > (C)_{I,\ III}$$

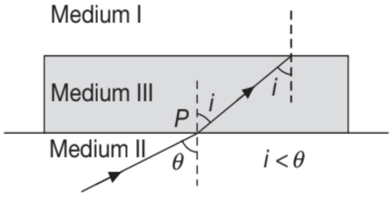
Hence, TIR will now take place on Medium I and Medium III interface and the ray will be reflected back to Medium III.

CASE-2: $n_1 < n_2 < n_3$

This time while moving from Medium II to Medium III, ray of light will bend towards normal. Again applying Snell's Law at P, we get

$$n_2 \sin\theta = n_3 \sin i$$

$$\Rightarrow \sin i = \frac{n_2}{n_3}\sin\theta$$



Since, $\sin\theta$ slightly greater than $\frac{n_1}{n_2}$

So, $\sin i$ will be slightly greater than $\frac{n_2}{n_3}\times\frac{n_1}{n_2}=\frac{n_1}{n_3}$

However $\frac{n_1}{n_3}$ is $\sin C$ for Medium I and Medium III interface, so

$\sin i > \sin C$ for Medium I and Medium III interface.

$\Rightarrow \quad i > (C)_{I,III}$

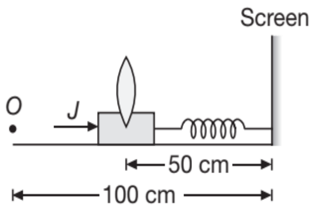
Therefore, TIR will again take place between Medium I and Medium III and the ray will be reflected back.

Conceptual Note(s)

The Cases 1 and 2 for $n_3 > n_1$ can be explained by single equation only. But two cases are deliberately formed for better understanding of refraction, Snell's Law and total internal reflection (TIR).

PROBLEM 22

A point object O is located at a distance of 100 cm from a screen. A lens of focal length 23 cm mounted on a movable frictionless stand is kept between the source and the screen. The stand is attached to a spring of natural length 50 cm and spring constant $800\ \text{Nm}^{-1}$ as shown in figure.



Mass of the stand with lens is 2 kg. How much impulse J should be imparted to the stand so that a real image of the object is formed on the screen after a fixed time gap. Also find this time gap. (Neglect the width of the stand)

SOLUTION

Let the distance of the lens from the object be L when a real image is formed on the screen. Then, we have

$$u=-L,\ v=+(100-L),\ f=+23\text{ cm}$$

Using lens formula $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, we get

$$\frac{1}{100-L}-\frac{1}{-L}=\frac{1}{23}$$

$$\Rightarrow \quad L^2-100L+2300=0$$

Solving, we get

$$L=(50\pm10\sqrt{2})\text{ cm}$$

Since the lens is executing SHM and a real image is formed after a fixed time gap, then this time gap must be such that real image is obtained when the lens passes through two positions at same distance from the mean position and hence separated by a time gap equal to one fourth of the time period of SHM i.e. $\Delta t=\frac{T}{4}$.

So phase difference between the two positions of real image formation must be $\frac{\pi}{2}$, because the two positions are symmetrically located about the mean position and phase difference of any of these positions from origin must be $\frac{\pi}{4}$.

If A is the amplitude of SHM and x be the displacement of the lens executing SHM, then we have

$$10\sqrt{2}\text{ cm}=A\sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad A=20\text{ cm}$$

Velocity of lens, at mean position, in this case is

$$v_0=A\omega=A\sqrt{\frac{K}{m}}$$

Since impulse is equal to the change in momentum of the body, so impulse required to attain this speed is given by

$$J=mv_0=A\sqrt{Km}=8\text{ kgms}^{-1}$$

PROBLEM 23

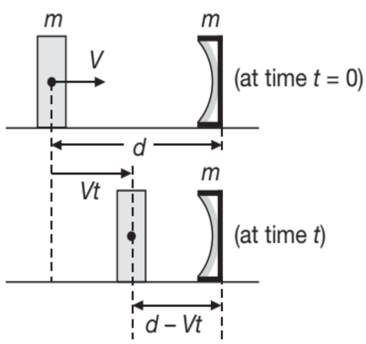
A small block of mass m and a concave mirror of radius R fitted with a stand, lie on a smooth horizontal table with a separation d between them. The mirror together with its stand has a mass m. The block is pushed at $t=0$ towards the mirror so that it starts moving towards the mirror at a constant speed V and collides with it. The collision is perfectly elastic. Find the velocity of the image

(a) at a time $t<\frac{d}{V}$

(b) at a time $t>\frac{d}{V}$

SOLUTION

(a) $t<\frac{d}{V}$



$$u=-(d-Vt)$$

$$f=-\frac{R}{2}$$

We know that $\vec{V}_{I/m}=-m^2\vec{V}_{O/m}$

Here, $m=\frac{f}{f-u}$

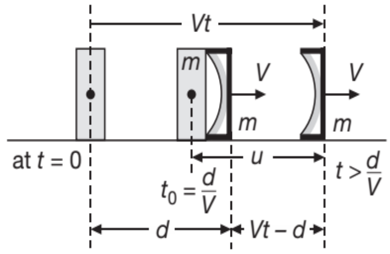
$$\Rightarrow\ m=\frac{-\frac{R}{2}}{-\frac{R}{2}+(d-Vt)}=\frac{-R}{2(d-Vt)-R}$$

So, velocity of image is $v_1=-\left[\frac{-R}{2(d-Vt)-R}\right]^2 v$

$$\Rightarrow\ v_1=\frac{R^2V}{[2(d-Vt)-R]^2}$$

(b) $t>\frac{d}{V}$

Block will collide with mirror assembly after time $t_0=\frac{d}{V}$. Applying Conservation of Linear Momentum, block and mirror assembly will exchange their momentum i.e., block will stop and mirror starts moving with velocity V. So, now



$$u=-(Vt-d)$$

$$\Rightarrow\ u=-V\left(t-\frac{d}{V}\right)$$

Since $m=\frac{f}{f+u}$

$$\Rightarrow\ m=\frac{-\frac{R}{2}}{-\frac{R}{2}+V\left(t-\frac{d}{V}\right)}$$

Also, we know that, $\vec{V}_{I/m}=-m^2\vec{V}_{O/m}$

$$\Rightarrow\ \vec{V}_I-\vec{V}_m=-m^2\left(\vec{V}_{O/m}\right)$$

Let us assume rightward direction as positive, then

$$v_1-V=-m^2(-V)$$

$$\Rightarrow\ v_1=\left(1+m^2\right)V$$

$$\Rightarrow\ v_1=V\left[1+\frac{R^2}{[2(Vt-d)-R^2]}\right]$$