CHAPTER

4 Alternating Currents

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Introduction to AC, Average Value and RMS Value of AC
(b) AC Through Resistor, Inductor, Capacitor, Their Combinations and Applications
(c) Resonance in LCR Circuit
(d) Average Power Consumed in an AC Circuit
(e) Half Power Frequencies, Band Width and Sharpness of Resonance
(f) Parallel LCR Circuit
(g) Transformer and Applications

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

ALTERNATING CURRENT: AN INTRODUCTION

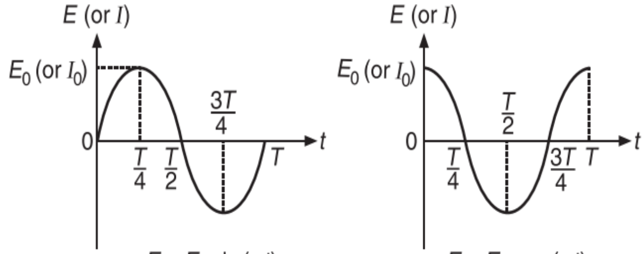
As studied earlier, we know that a changing magnetic flux can induce an emf and hence a current in a closed circuit. Also, we have seen that when a coil rotates in the presence of a magnetic field the induced emf varies sinusoidally with time leading to an alternating current (AC) and provides a source of AC power. The symbol for an AC voltage is

An alternating current change in magnitude and direction periodically and is abbreviated as AC (alternating current).

The alternating e.m.f. E at any instant may be expressed as

$$E = E_0 \sin(\omega t) \qquad \textbf{OR} \qquad E = E_0 \cos(\omega t)$$

where ω is angular frequency of alternating e.m.f. and E_0 is the peak value or amplitude of alternating e.m.f.



$E = E_0 \sin(\omega t)$
$I = I_0 \sin(\omega t)$

$E = E_0 \cos(\omega t)$
$I = I_0 \cos(\omega t)$

The frequency of alternating e.m.f.,

$$f = \frac{\omega}{2\pi} = \frac{1}{T}$$

where T is the time period.

Similarly, the alternating current in the circuit is given by

$$I = I_0 \sin(\omega t) \qquad \textbf{OR} \qquad I = I_0 \cos(\omega t)$$

where I_0 is the peak value of current and other symbols have the same notations.

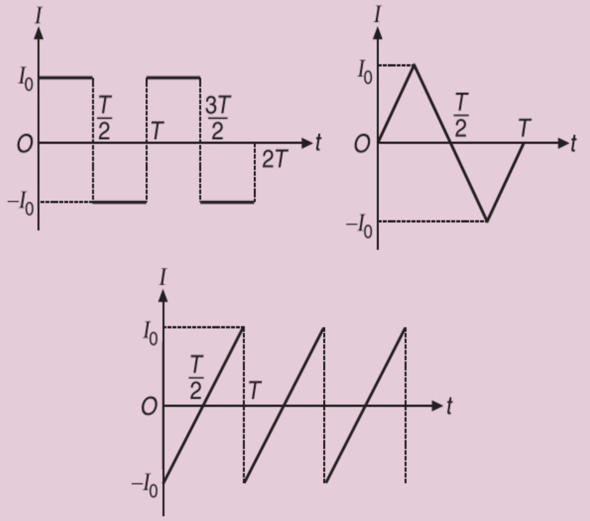
Alternating current in circuits fed by an alternating source of emf may be controlled by inductance L, resistance R and capacitance C. Due to presence of elements L and C, the current is not necessarily in phase with the applied emf. Hence alternating current, in general, may be expressed as

$$I = I_0 \sin(\omega t + \phi)$$

where ϕ is the phase angle (in radian) which may be positive, zero or negative depending on the values of reactive components R, L and C.

Conceptual Note(s)

An alternating current change its direction of flow periodically. For a half cycle, it flows in one direction and for next half cycle, it flows in opposite direction. The following graphs can also be a good representation of alternating currents



MEAN VALUE OF AC

Since current represents the rate of flow of charge, so average current is defined as charge flown per unit time interval and can be written as

$$I_{mean} = I_{av} = \langle I \rangle = \frac{\Delta q}{\Delta t}$$

where, Δq represents the charge flown in a time interval Δt. If the current is varying, then we have

$$I = \frac{dq}{dt}$$

$$\Rightarrow \quad dq = Idt$$

So, the charge flown through the circuit in a time T is obtained by integrating the above expression.

$$\Rightarrow \quad \Delta q = \int_0^T Idt$$

$$\Rightarrow \quad I_{mean} = I_{av} = \langle I \rangle = \frac{\Delta q}{T} = \frac{\int_0^T Idt}{T}$$

For an alternating current, mean value during one complete cycle is zero as there is a reversal in the direction of current after every half cycle. Therefore, we shall be finding the mean value of ac for the half cycle.

AVERAGE VALUE OR MEAN VALUE OF AN A.C. [I_{av} OR E_{av} OR $\langle I \rangle$ OR $\langle E \rangle$]

Physically, the mean value of an ac for one half cycle is defined as the steady current which when passes through the circuit makes the same amount of charge to pass through it as is done by an ac for the same time. The average value of AC over full cycle is zero since there are equal positive and negative half cycles. Mathematically, if $I = I_0 \sin(\omega t)$ or $I = I_0 \cos(\omega t)$, then for both the sine and cosine functions we have

$$\langle I \rangle = (I_{av})_{\text{full cycle}} = \frac{\int_0^T I\,dt}{\int_0^T dt} = 0$$

The average value of AC over half cycle $\left(t = 0 \text{ to } \frac{T}{2}\right)$ of sine function is given by

$$\langle I \rangle = (I_{av})_{\text{half cycle}} = \frac{\int_0^{T/2} Idt}{\int_0^{T/2} dt} = \frac{2I_0}{\pi}$$

Similarly, the average values of alternating voltage are given by

$$\langle E \rangle = (E_{av})_{\text{full cycle}} = 0$$

$$\langle E \rangle = (E_{av})_{\text{half cycle}} = \frac{2E_0}{\pi}$$

ROOT MEAN SQUARE OR VIRTUAL VALUE OF A.C. (R.M.S VALUE)

Physically, the r.m.s. value of alternating current is defined as the direct current which produces the same heating effect in a given resistor as is produced by an alternating current flowing through the resistor for the same time.

Due to this reason the r.m.s. value of current is also known as **effective or apparent or virtual value of current**,

$$\Rightarrow \quad I_{\text{effective}} = I_{\text{virtual}} = \frac{I_0}{\sqrt{2}}$$

Similarly, the r.m.s. value of alternating voltage is called the **effective or virtual value of alternating voltage (or e.m.f.).** So,

$$E_{\text{virtual}} = E_{\text{rms}} = \frac{E_0}{\sqrt{2}}$$

Mathematically, the root mean square value of AC is defined by the expression given by

$$I_V = I_{\text{rms}} = \sqrt{\langle I^2 \rangle} = \sqrt{\frac{\int_0^T I^2 dt}{\int_0^T dt}}$$

For $I = I_0 \sin \omega t$ or $I = I_0 \cos \omega t$, we have

$$\langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle = \frac{1}{2}$$

$$\Rightarrow \quad I_V = I_{\text{rms}} = \sqrt{\langle I^2 \rangle} = \frac{I_0}{\sqrt{2}}$$

Similarly, $E_V = E_{\text{rms}} = \dfrac{E_0}{\sqrt{2}}$

RMS represents the squared mean of currents for a given time interval.

Remark(s)

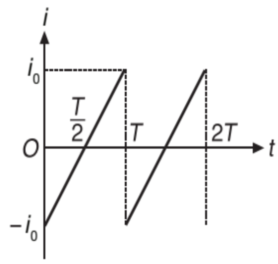
(a) Alternating currents show only heating effect, hence meters used for AC are based on heating effect and are called **hot wire meters**. The deflection in hot wire meters is proportional to mean square value of current but the scale is so calibrated that its reading is proportional to the square root of deflection. Hence these meters read directly the r.m.s values. As direct current also shows the same heating effect as produced by r.m.s value, therefore hot wire (i.e., AC) meters can read both AC and DC while dc meters based on magnetic effects of current can only read d.c.

(b) We must remember that all labelled values or the designated values of voltages and currents are the virtual values.

EXAMPLE: If we are given a 50 Hz, 220 V ac then 220 V is the virtual voltage and to get the peak value we must multiply it by $\sqrt{2}$.

ILLUSTRATION 1

Calculate the mean and rms values (for half cycle) for the current for which the graph here shows its variation with time?



SOLUTION

From the graph, current as function of time is

$$i = \left(\frac{2i_0}{T}\right)t - i_0$$

$$\Rightarrow \quad I_{\text{mean}} = \frac{\int_0^{T/2} i\,dt}{\frac{T}{2}} = \frac{\int_0^{T/2} \left[\left(\frac{2i_0}{T}\right)t - i_0\right]dt}{\frac{T}{2}} = -\frac{i_0}{2}$$

Negative sign appears as we have computed the mean for first half cycle. If we had calculated the mean for the second half cycle, then it would have been $\dfrac{i_0}{2}$. Since we know that

$$i_{\text{rms}}^2 = \langle i^2 \rangle = \frac{\int_0^{\frac{T}{2}} i^2 dt}{\frac{T}{2}}$$

$$\Rightarrow \quad i_{\text{rms}}^2 = \frac{\int_0^{\frac{T}{2}} i^2 dt}{\frac{T}{2}} = \frac{\int_0^{\frac{T}{2}} \left[\left(\frac{2i_0}{T}\right)t - i_0\right]^2 dt}{\frac{T}{2}} = \frac{i_0^2}{3}$$

$$\Rightarrow \quad i_{\text{rms}} = \frac{i_0}{\sqrt{3}}$$

ILLUSTRATION 2

An AC voltage is given as $e = e_1 \sin \omega t + e_2 \cos \omega t$. Calculate the RMS value of this voltage.

SOLUTION

The given AC voltage can be written as

$$e = e_1 \sin \omega t + e_2 \cos \omega t \qquad \ldots(1)$$

Let $e_1 = e_0 \cos\theta \qquad \ldots(2)$

$$e_2 = e_0 \sin\theta \qquad \ldots(3)$$

Substituting these values in equation (1), we get

$$e = e_0 (\sin \omega t \cos\theta + \cos \omega t \sin\theta)$$

$$\Rightarrow \quad e = e_0 \sin(\omega t + \theta)$$

Squaring and adding equations (2) and (3), we get

$$e_1^2 + e_2^2 = e_0^2$$

$$\Rightarrow \quad e_0 = \sqrt{e_1^2 + e_2^2}$$

$$\Rightarrow \quad e_{\text{rms}} = \frac{e_0}{\sqrt{2}} = \sqrt{\frac{e_1^2 + e_2^2}{2}}$$

We can also calculate the RMS value of the given time function of voltage by using the formula for RMS value given by

$$e_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T e^2 dt}, \text{ where } T = \frac{2\pi}{\omega}$$

ILLUSTRATION 3

In a wire, direct current i_1 and an AC current $i_2 = i_0 \sin \omega t$ are superposed. Find the RMS value of current in wire.

SOLUTION

Total current in wire is given by

$$i = i_1 + i_0 \sin \omega t$$

Since $i_{\text{rms}} = \sqrt{\langle i^2 \rangle}$

where, $i^2 = (i_1 + i_0 \sin \omega t)^2$

$$\Rightarrow \quad i^2 = i_1^2 + i_0^2 \sin^2 \omega t + 2i_1 i_0 \sin \omega t$$

$$\Rightarrow \quad \langle i^2 \rangle = \langle i_1^2 + i_0^2 \sin^2 \omega t + 2i_1 i_0 \sin \omega t \rangle$$

$$\Rightarrow \quad \langle i^2 \rangle = i_1^2 + i_0^2 \langle \sin^2 \omega t \rangle + 2i_1 i_0 \langle \sin \omega t \rangle$$

Since we know that $\langle \sin^2 \omega t \rangle = \frac{1}{2}$ and $\langle \sin \omega t \rangle = 0$

$$\Rightarrow \quad \langle i^2 \rangle = i_1^2 + i_0^2 \left(\frac{1}{2}\right)$$

$$\Rightarrow \quad i_{\text{rms}} = \sqrt{\langle i^2 \rangle} = \sqrt{i_1^2 + \frac{i_0^2}{2}}$$

ILLUSTRATION 4

For a given AC flowing through a specific branch of a circuit, the variation of current as a function of time is given by

$$i = \begin{cases} i_0 \sin^2 \omega t & \text{for} \quad 0 \le \omega t < \pi \\ i_0 \sin \omega t & \text{for} \quad \pi \le \omega t < 2\pi \end{cases}$$

Calculate the average current per cycle for this AC.

SOLUTION

Average value of the given AC current is

$$\Rightarrow \quad i_{\text{av}} = \frac{1}{T}\left[\int_0^{\frac{T}{2}} (i_0 \sin^2 \omega t) dt + \int_{\frac{T}{2}}^{T} (i_0 \sin \omega t) dt\right]$$

$$\Rightarrow \quad i_{\text{av}} = \frac{i_0}{T}\left(\int_0^{\frac{T}{2}} \left(\frac{1 - \cos 2\omega t}{2}\right) dt + \int_{\frac{T}{2}}^{T} \sin \omega t dt\right)$$

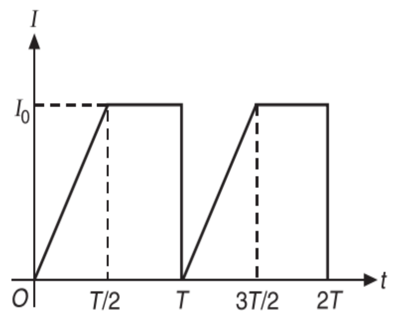
$$\Rightarrow \quad i_{av} = \frac{i_0}{T}\left(\frac{1}{2}\left(t - \frac{\sin 2\omega t}{2\omega}\right)\Bigg|_0^{\frac{T}{2}} + \left(-\frac{\cos \omega t}{\omega}\right)\Bigg|_{\frac{T}{2}}^{T}\right)$$

$$\Rightarrow \quad i_{av} = \frac{i_0}{T}\left[\frac{1}{2}\left(\frac{T}{2}\right) - \frac{2}{\omega}\right] = \frac{i_0}{T}\left[\frac{T}{4} - \frac{2}{2\pi/T}\right]$$

$$\Rightarrow \quad i_{av} = i_0\left(\frac{1}{4} - \frac{1}{\pi}\right)$$

ILLUSTRATION 5

Calculate the average value of the AC per cycle for which the time variation of current is shown in Figure.



SOLUTION

Since $I_{av} = \frac{\text{Charge Flown}}{\text{Time Duration}} = \frac{\Delta q}{\Delta t}$

The charge flown Δq in one cycle i.e. from $t = 0$ to $t = T$ is the area under the current-time graph. So, we have

$$\Rightarrow \quad \Delta q = \frac{1}{2} I_0 \left(\frac{T}{2}\right) + I_0\left(T - \frac{T}{2}\right)$$

$$\Rightarrow \quad \Delta q = \frac{3}{4} I_0 T$$

$$\Rightarrow \quad I_{av} = \frac{\Delta q}{\Delta t} = \frac{\Delta q}{T} = \frac{3}{4} I_0$$

ILLUSTRATION 6

If the current in an ac circuit is represented by the equation, $I = 5\sin\left(300t - \frac{\pi}{4}\right)$, where, t is in second and I in ampere. Calculate the

(a) peak and r.m.s. value of current.
(b) frequency of ac.
(c) average current for the half cycle.

SOLUTION

(a) Since, for an ac input we have

$$I = I_0 \sin(\omega t \pm \phi)$$

So, the peak value $I_0 = 5$ A

and $I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{5}{\sqrt{2}} = 3.535$ A

(b) Angular frequency $\omega = 300 \text{ rads}^{-1}$

$$\Rightarrow \quad f = \frac{\omega}{2\pi} = \frac{300}{2\pi} \simeq 47.75 \text{ Hz}$$

(c) $I_{av} = \left(\frac{2}{\pi}\right) I_0 = \left(\frac{2}{\pi}\right)(5) = 3.18$ A

IMPEDANCE, REACTANCE AND ADMITTANCE

Impedance (Z)

In alternating current circuit, the ratio of e.m.f. applied and consequent current produced is called the **impedance** and is denoted by Z i.e.,

$$Z = \frac{E}{I} = \frac{E_0}{I_0} = \frac{E_{rms}}{I_{rms}}$$

Physically, impedance of ac circuit is the opposition by the circuit to the flow of ac through it.

Reactance (X)

The opposition offered by pure inductance or capacitance to the flow of ac in an ac circuit is called **reactance** and is denoted by X. In other words, when there is no ohmic resistance in the circuit, the reactance is equal to impedance.

The reactance due to inductance alone is called **inductive reactance** and is denoted by X_L while the reactance due to capacitance alone is called the **capacitive reactance** and is denoted by X_C.

Z, X_L and X_C all have same SI unit i.e. ohm (Ω)

Admittance (Y)

The reciprocal of impedance is called the **admittance** and is denoted by Y i.e.,

$$Y = \frac{1}{Z}$$

Its SI unit is ohm^{-1} (i.e. Ω^{-1}), sometimes written as mho.

PHASOR DIAGRAMS AND PHASORS

The concept of phasor diagrams is introduced when resultant of scalar quantities having a constant phase difference (such as voltage, current) is to be found. Actually, the concept of phasors fits well at those places where the quantities are scalars and have a constant phase difference between them, because on superposition they behave like vectors e.g., ac currents, ac voltages, amplitudes of superimposing waves etc. So, phasor quantities are those quantities which possess magnitude along with a phase angle. In simple words we can say that a **phasor** is a rotating vector having the following properties.

(a) **Length**, which corresponds to the amplitude.
(b) **Angular speed**, with which the vector rotates counter clockwise.
(c) **Projection**, of the vector along the vertical axis corresponds to the value of alternating current or voltage at that time.

EXAMPLE

Alternating Current and Voltages. If an alternating voltage is given by $V = 100\sin\left(\omega t + \frac{\pi}{6}\right)$ and the current given by $I = 10\cos(\omega t)$, then before concluding anything, we must first convert both to sine or to cosine. Using the property $\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right)$, we conclude that voltage **lags behind the current** by a phase angle of

$$\phi = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

Alternatively, it can be said that the **current leads the voltage** by a phase angle of

$$\phi = \frac{\pi}{3}$$

Problem Solving Technique(s)

While drawing phasor diagram for a pure element (e.g. a resistance, a capacitance or an inductance) either of the current or voltage can be plotted along x-axis

But when phasor diagrams for a combination of elements is drawn then the quantity which remains constant for the combination must be plotted along x-axis So, we observe that

(a) in series circuits current has to be plotted along x-axis.

(b) in parallel circuits voltage has to be plotted along x-axis and the phasor diagram is then completed by plotting the variation of other quantity for a particular element.

IMPEDANCES AND PHASES OF AC CIRCUITS CONTAINING DIFFERENT ELEMENTS

As already pointed out that in an ac circuit the current and applied emfs are not necessarily in same phase. The applied e.m.f. (E) and current produced (I) may be expressed as

$$E = E_0 \sin \omega t$$

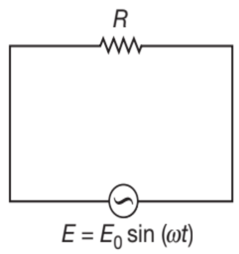
and $I = I_0 \sin(\omega t + \phi)$ with $I_0 = \frac{E_0}{Z}$

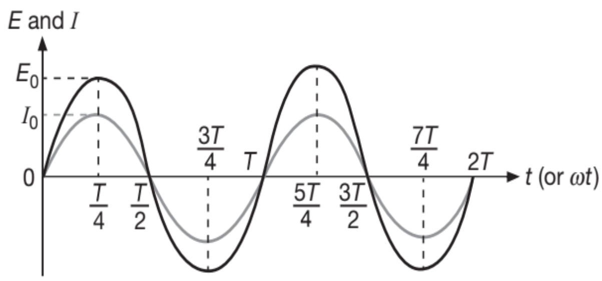
where E_0 and I_0 are peak values of alternating e.m.f. and current.

A.C. THROUGH A PURE RESISTOR

Consider a circuit, fed by an alternating e.m.f. $E = E_0 \sin(\omega t)$, containing pure resistance R, then current is given by

$$I = \frac{E}{R} = \frac{E_0 \sin(\omega t)}{R} = I_0 \sin(\omega t), \text{ where } I_0 = \frac{E_0}{R}$$

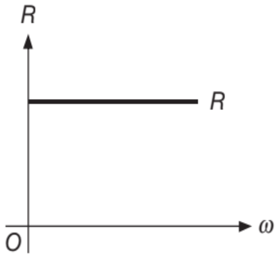




Phasor Diagram

Comparing this with standard equation $E = E_0 \sin(\omega t)$, we observe that Impedance of circuit, $Z = R$ and phase lead of current over e.m.f., $\phi = 0$ i.e. current and e.m.f. both are in the same phase, the phasor diagram for which is also shown.

Graph of R vs ω is also shown here.

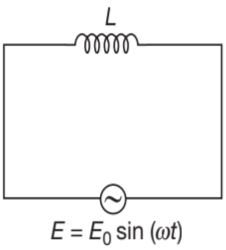


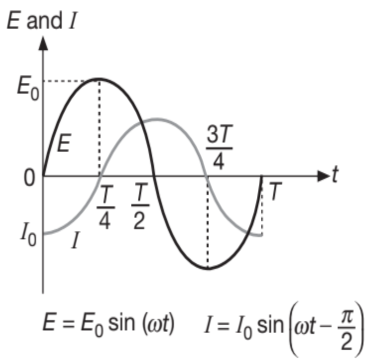
AC THROUGH A PURE INDUCTOR

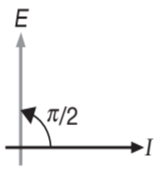
Let an alternating e.m.f. $E = E_0 \sin(\omega t)$ be applied across a pure inductance L, then at any instant we have

$$L\frac{dI}{dt} = E_0 \sin(\omega t)$$

or $$dI = \frac{E_0}{L}\sin(\omega t)\,dt$$







Phasor Diagram

Integrating, we get

$$I = \int \frac{E_0}{L}\sin(\omega t)\,dt = -\frac{E_0}{\omega L}\cos(\omega t)$$

$$\Rightarrow \quad I = -\frac{E_0}{\omega L}\sin\left(\frac{\pi}{2} - \omega t\right)$$

$$\Rightarrow \quad I = \frac{E_0}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad I = I_0 \sin\left(\omega t - \frac{\pi}{2}\right)$$

where $I_0 = \dfrac{E_0}{L\omega} = \dfrac{E_0}{X_L}$

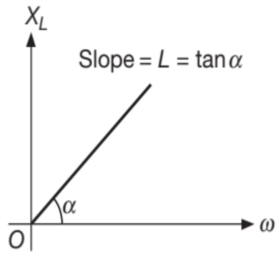
where X_L = Inductive Reactance = $L\omega$

In a purely inductive circuit the current lags behind the applied voltage by $\dfrac{\pi}{2}$ or the voltage leads the current by a phase angle of $\dfrac{\pi}{2}$.

Also, we note that since

$$X_L = L\omega = 2\pi f L$$

Graph of X_L vs ω is shown in figure.



Problem Solving Technique(s)

(a) For a dc $f = 0$

$\Rightarrow \quad X_L = 0$

i.e. an inductor offers zero resistance path to the flow of dc.

(b) If an oscillating voltage of a given amplitude V_0 is applied across an inductor the resulting current will have a smaller amplitude I_0 for larger value of ω. Since, X_L is proportional to frequency, a high frequency voltage applied to the inductor gives only a small current while a lower frequency voltage of the same amplitude gives rise to a larger current. Inductors are used in some circuit applications, such as power supplies and radio interference filters to block high frequencies while permitting lower frequencies to pass through. A circuit device that uses an inductor for this purpose is called a **Low Pass Filter**.

AC THROUGH A PURE CAPACITOR

Let a circuit contain pure capacitance and the applied alternating e.m.f. be $E = E_0 \sin(\omega t)$.

The current in circuit is given by

$$I=\frac{dq}{dt}=\frac{d}{dt}(CE)=\frac{d}{dt}(CE_0\sin\omega t)$$

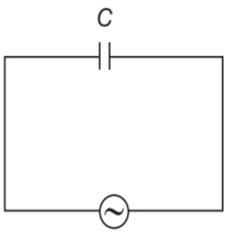
$$\Rightarrow \quad I=\omega CE_0\cos\omega t$$

$$\Rightarrow \quad I=\frac{E_0}{1/\omega C}\sin\left(\frac{\pi}{2}+\omega t\right)=\frac{E_0}{X_C}\sin\left(\omega t+\frac{\pi}{2}\right)$$

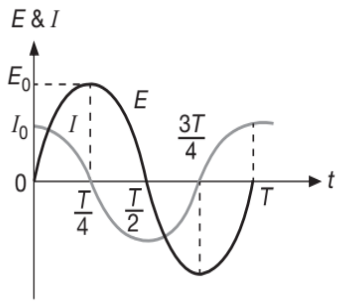
$$\Rightarrow \quad I=I_0\sin\left(\omega t+\frac{\pi}{2}\right)$$

where $I_0=\dfrac{E_0}{1/\omega C}=\dfrac{E_0}{X_C}$

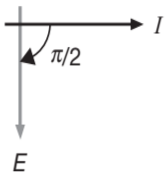
where $X_C=\dfrac{1}{C\omega}=$ Capacitance Reactance



$E = E_0 \sin(\omega t)$



$E = E_0\sin(\omega t)$ $\quad I = I_0\sin\left(\omega t - \frac{\pi}{2}\right)$



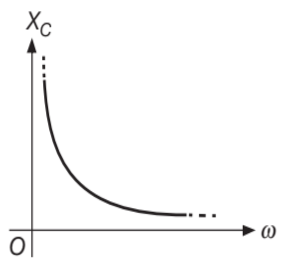
Phasor Diagram

In a purely capacitive circuit the current leads the applied e.m.f. by a phase angle $\dfrac{\pi}{2}$ or the voltage lags behind the current by a phase angle of $\dfrac{\pi}{2}$.

Also, we note that since

$$X_C=\frac{1}{C\omega}=\frac{1}{2\pi fC}$$

Graph of X_C vs ω is shown in figure.



Problem Solving Technique(s)

(a) For a dc circuit $f=0$

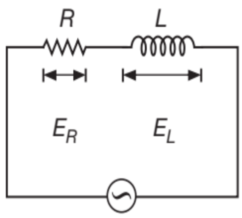
$\Rightarrow \quad X_C\to\infty$

i.e. a capacitor offers an infinite resistance path to the flow of dc i.e. it bypasses all the dc components of supply and hence is called a **dc blocking element.**

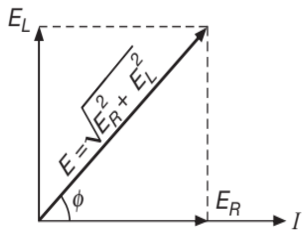
(b) The capacitive reactance of a capacitor is inversely proportional both to the capacitance C and to angular frequency ω. The greater the capacitance and the higher the frequency, the smaller is the capacitive reactance X_C. Capacitors tend to pass high frequency current and to block low frequency current, just the opposite of inductors. A device that passes signals of high frequency is called a **High Pass Filter**.

AC THROUGH A NON-IDEAL INDUCTOR OR SERIES *LR* CIRCUIT

Consider a circuit containing resistance R and inductance L in series.



$E = E_0\sin\omega t$



Phasor Diagram

Let I be the current flowing in the circuit and $E_R(=IR)$ the potential difference across resistance and $E_L(=IX_L)$ the potential difference across inductance.

The current I and the potential difference E_R are always in phase, but the potential difference E_L across inductance leads the current I by an angle $\dfrac{\pi}{2}$.

Therefore, resultant voltage is given by

$$E=\sqrt{\left(E_R^2+E_L^2\right)}=\sqrt{(IR)^2+(IX_L)^2}$$

$$\Rightarrow \quad \frac{E}{I}=\sqrt{R^2+L^2\omega^2}$$

So, impedance of RL circuit,

$$Z=\frac{E}{I}=\frac{E_v}{I_v}=\sqrt{R^2+L^2\omega^2}$$

From phasor diagram we observe that net voltage E leads the current by ϕ. Further,

$$\tan\phi = \frac{E_L}{E_R} = \frac{IX_L}{IR} = \frac{X_L}{R}$$

$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}\left(\frac{L\omega}{R}\right)$$

ILLUSTRATION 7

An 220 V AC voltage at a frequency of 40 cycles/s is applied to a circuit containing a pure inductance of 0.01 H and a pure resistance of $6\ \Omega$ in series. Calculate

(a) The current supplied by source
(b) The potential difference across the resistance
(c) The potential difference across the inductance
(d) The time lag between maxima of current and voltage in circuit

SOLUTION

The impedance of RL series circuit is given as

$$Z = \sqrt{R^2 + \omega^2 L^2}$$

$$\Rightarrow \quad Z = \sqrt{R^2 + (2\pi f L)^2} = \sqrt{R^2 + 4\pi^2 f^2 L^2}$$

$$\Rightarrow \quad Z = \sqrt{(6)^2 + 4(3.14)^2(40)^2(0.01)^2}$$

$$\Rightarrow \quad Z \approx 6.5\ \Omega$$

(a) Current supplied by ac source is

$$I_{\text{rms}} = \frac{E_{\text{rms}}}{Z} = \frac{220}{6.5} = 33.84\text{ mA}$$

(b) The potential difference across the resistance is

$$E_R = I_{\text{rms}} R = 33.83 \times 6 = 202.98\text{ V}$$

(c) Potential difference across inductance is

$$E_L = I_{\text{rms}} X_L = I_{\text{rms}}(L\omega)$$

$$\Rightarrow \quad E_L = 33.83 \times (2 \times 3.14 \times 40 \times 0.01)$$

$$\Rightarrow \quad E_L = 96.83\text{ V}$$

(d) Phase angle between current and voltage is

$$\phi = \tan^{-1}\left(\frac{L\omega}{R}\right)$$

$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{2\times 3.14\times 40\times 0.01}{6}\right)$$

$$\Rightarrow \quad \phi = \tan^{-1}(0.419) = 22.73°$$

$$\Rightarrow \quad \phi = 22.73 \times \frac{\pi}{180}\text{ radian} \approx 0.4\text{ radian}$$

Since $\phi = \omega t$, so the time lag corresponding to the above phase angle is given by

$$t = \frac{\phi}{\omega} = \frac{0.4}{80\pi} \approx 1.6\text{ ms}$$

ILLUSTRATION 8

A resistance and inductance are connected in series across a voltage, $V = 283\sin(314t)$. The current is found to be $I = 4\sin\left(314t - \frac{\pi}{4}\right)$. Find the values of the inductance and resistance.

SOLUTION

In series LR circuit, the current lags the voltage by an angle, ϕ given by

$$\phi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}\left(\frac{L\omega}{R}\right)$$

Since $\phi = \frac{\pi}{4}$

$$\Rightarrow \quad X_L = L\omega = R \qquad \{\because \omega = 314\text{ rads}^{-1}\}$$

$$\Rightarrow \quad 314L = R \qquad \ldots(1)$$

Further, since $V_0 = I_0|Z|$

$$\Rightarrow \quad 283 = 4\sqrt{R^2 + X_L^2}$$

$$\Rightarrow \quad R^2 + (\omega L)^2 = \left(\frac{283}{4}\right)^2 = 5005.56$$

$$\Rightarrow \quad 2R^2 = 5005.56 \qquad \{\because \omega L = R\}$$

$$\Rightarrow \quad R \approx 50\ \Omega$$

and from equation (1), we get $L = 0.16$ H

ILLUSTRATION 9

An electric lamp which runs at 40 V and consumes 10 A current is connected to 100 V, 50 Hz AC main supply. Calculate the inductance of the required choke for lamp to glow at full brightness.

SOLUTION

Resistance of the lamp is given as

$$R = \frac{V}{I} = \frac{40}{10} = 4\ \Omega$$

Current in the circuit is

$$I=\frac{E}{\sqrt{R^2+(\omega L)^2}}$$

$$\Rightarrow \quad 10=\frac{100}{\sqrt{(4)^2+(2\times 3.14\times 50\times L)^2}}$$

Solving, we get

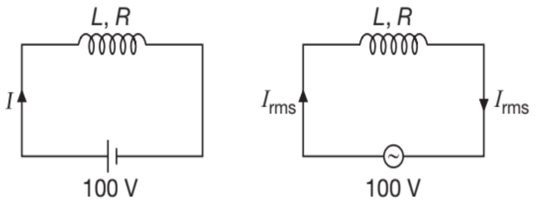
$$L\approx 0.029\text{ H}$$

ILLUSTRATION 10

When 100 V dc is applied across a solenoid, a steady current of 1 A flows in it. When 100 V ac is applied across the same solenoid, current drops to 0.5 A. If the frequency of ac source is $\frac{150\sqrt{3}}{\pi}$ Hz, find the resistance and inductance of the solenoid.

SOLUTION

Steady current through solenoid with dc source is given by



$$I=\frac{E}{R}=\frac{100}{R}=1$$

$$\Rightarrow \quad R=100\ \Omega$$

When ac is applied, the current through the solenoid is given by

$$I_{rms}=\frac{E_{rms}}{Z}=\frac{100}{\sqrt{R^2+X_L^2}}=0.5$$

$$\Rightarrow \quad \sqrt{R^2+X_L^2}=200$$

$$\Rightarrow \quad X_L^2=40000-10000=30000$$

$$\Rightarrow \quad X_L=100\sqrt{3}\ \Omega$$

Inductive reactance is given by

$$X_L=L\omega$$

$$\Rightarrow \quad L=\frac{X_L}{\omega}=\frac{100\sqrt{3}}{2\pi}\frac{\pi}{150\sqrt{3}}=\frac{1}{3}\text{ H}$$

ILLUSTRATION 11

A $130\sqrt{2}$ V, 50 Hz ac source is applied across a series LR circuit having $L=\frac{175}{11}$ mH and $R=12\ \Omega$. Calculate the impedance of the circuit and the phase difference between voltage and current in the circuit.

SOLUTION

For a series LR circuit, impedance is given by

$$Z=\sqrt{R^2+X_L^2}$$

Inductive reactance X_L is given by

$$X_L=L\omega=2\pi fL$$

$$\Rightarrow \quad X_L=2\left(\frac{22}{7}\right)(50)\left(\frac{175}{11}\times 10^{-3}\right)\Omega$$

$$\Rightarrow \quad X_L=5000\times 10^{-3}\ \Omega=5\ \Omega$$

$$\Rightarrow \quad Z=\sqrt{(12)^2+(5)^2}=\sqrt{169}=13\ \Omega$$

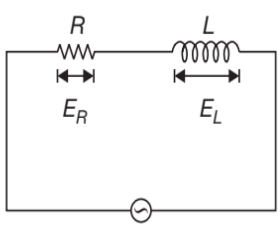
Since we know that in a series LR circuit, the voltage leads the current by a phase angle of

$$\phi=\tan^{-1}\left(\frac{X_L}{R}\right)=\tan^{-1}\left(\frac{L\omega}{R}\right)$$

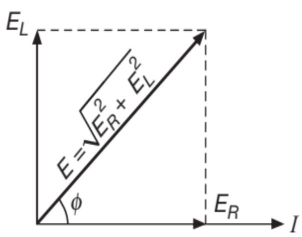
$$\Rightarrow \quad \phi=\tan^{-1}\left(\frac{5}{12}\right)$$

AC THROUGH A NON-IDEAL CAPACITOR OR SERIES *CR* CIRCUIT

Let a circuit contain resistance R and capacitance C in series.



$E=E_0\sin\omega t$



Phasor Diagram

Let I be the current flowing in the circuit, $E_R\,(=I_R)$ the potential difference across resistance and $E_C\,(=IX_C)$ the potential difference across capacitance.

The potential difference E_R and current I are in same phase and the potential difference E_C lags behind the current I (and hence E_R) by phase angle $\frac{\pi}{2}$.

The resultant e.m.f is given by

$$E=\sqrt{E_R^2+E_C^2}=\sqrt{(IR)^2+(IX_C)^2}$$

$\Rightarrow$ Impedance, $Z=\frac{E}{I}=\frac{E_V}{I_V}=\sqrt{R^2+\frac{1}{C^2\omega^2}}$

The current leads the applied e.m.f by phase angle ϕ given by

$$\tan\phi=\frac{E_C}{E_R}=\frac{IX_C}{IR}=\frac{X_C}{R}$$

or $\phi=\tan^{-1}\left(\frac{X_C}{R}\right)=\tan^{-1}\left(\frac{1}{RC\omega}\right)$

ILLUSTRATION 12

A $100\ \mu F$ capacitor in series with a $40\ \Omega$ resistor is connected to a $110\ V,\ 60\ Hz$ supply. Calculate the maximum current in the circuit. Also find the time lag between current maxima and voltage maxima.

SOLUTION

The peak current in a series RC circuit is given by

$$I_0=\frac{E_0}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}$$

Given that $E_{rms}=110\ V$, so peak voltage is given by

$$E_0=\sqrt{2}E_{rms}=1.414\times110=155.5\ V$$

$$\Rightarrow\quad I_0=\frac{155.5}{\sqrt{(40)^2+\left(\frac{1}{376.8\times10^{-4}}\right)^2}}$$

$$\Rightarrow\quad I_0=3.24\ A$$

In RC circuit, the voltage lags behind the current by phase angle ϕ, where

$$\tan\phi=\frac{X_C}{R}=\frac{1}{RC\omega}=\frac{1}{(376.8\times10^{-4})(40)}$$

$$\Rightarrow\quad \tan\phi=0.6635$$

$$\Rightarrow\quad \phi=\tan^{-1}(0.6635)=33.56^\circ$$

The time lag is given by $\phi=\omega t$

$\Rightarrow$ $t=\frac{\phi}{\omega}$, where $\phi=33.56\times\frac{\pi}{180}$ radian and $\omega=2\pi f=376.8\ \text{rad s}^{-1}$

$$\Rightarrow\quad t=1.55\times10^{-3}\ s$$

ILLUSTRATION 13

In a circuit, a resistance of $40\ \Omega$ and capacitor of capacitance $\frac{1250}{9\pi}\ \mu F$ are connected in series across a $500\ V,\ 120\ Hz$ ac source. Find the effective current in circuit and phase difference between current and voltage source.

SOLUTION

Impedance Z of the series RC circuit is given by

$$Z=\sqrt{R^2+X_C^2}$$

where $X_C=\frac{1}{C\omega}$ is the capacitive reactance.

$$\Rightarrow\quad X_C=\frac{1}{C\omega}=\frac{1}{2\pi fC}$$

$$\Rightarrow\quad X_C=\frac{9\pi\times10^6}{2\pi(120)(1250)}\ \Omega$$

$$\Rightarrow\quad X_C=\frac{3\times10^6}{10^5}=30\ \Omega$$

Circuit impedance is given by

$$Z=\sqrt{(40)^2+(30)^2}=50\ \Omega$$

Current in the circuit is

$$I_{rms}=\frac{E_{rms}}{Z}=\frac{500}{50}=10\ A$$

Phase difference between current and voltage source is given by

$$\phi=\tan^{-1}\left(\frac{X_C}{R}\right)=\tan^{-1}\left(\frac{30}{40}\right)$$

$$\Rightarrow\quad \phi=\tan^{-1}\left(\frac{3}{4}\right)=37^\circ$$

Since in RC circuit, the current leads voltage, so in this case, current leads voltage by a phase angle of 37°.

ILLUSTRATION 14

An ac source of angular frequency ω is fed across resistor R and a capacitor C in series. The current registered is i. If now the frequency of the source is changed to $\frac{\omega}{3}$ but maintaining the same voltage, the current in the circuit is found to be halved. Calculate the ratio of reactance to resistance at the original frequency.

SOLUTION

At angular frequency ω, the current in RC circuit is given as

$$I_{\text{rms}} = \frac{E_{\text{rms}}}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \quad \ldots(1)$$

When frequency is changed to $\frac{\omega}{3}$, the current is halved so we have

$$\frac{I_{\text{rms}}}{2} = \frac{E_{\text{rms}}}{\sqrt{R^2 + \frac{1}{(\omega/3)^2 C^2}}}$$

$$\Rightarrow \quad \frac{I_{\text{rms}}}{2} = \frac{E_{\text{rms}}}{\sqrt{R^2 + \frac{9}{C^2\omega^2}}} \quad \ldots(2)$$

From equations (1) and (2), we get

$$\frac{1}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} = \frac{2}{\sqrt{R^2 + \frac{9}{C^2\omega^2}}}$$

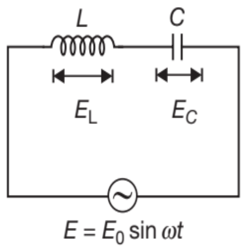
$$\Rightarrow \quad 3R^2 = \frac{5}{C^2\omega^2}$$

$$\Rightarrow \quad 3R^2 = 5X_C^2$$

$$\Rightarrow \quad \frac{X_C}{R} = \sqrt{\frac{3}{5}}$$

AC THROUGH A SERIES *LC* CIRCUIT

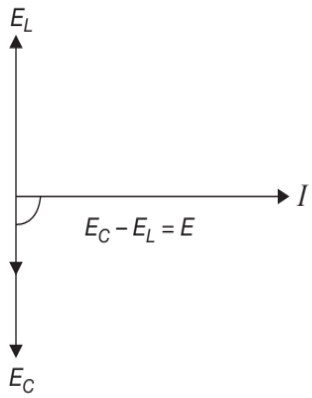
Let a circuit contain inductance L and capacitance C in series.



Let I be the current flowing in circuit, $L = IX_L$ the potential difference across inductance L and $E_C = IX_C$ the potential difference across capacitance C.

The potential difference E_C lags behind the current by angle $\frac{\pi}{2}$ and the potential difference E_C leads the current by angle $\frac{\pi}{2}$.

So, resultant applied e.m.f, $E = E_C \sim E_L = IX_C \sim IX_L$



So, impedance of circuit is

$$Z = \frac{E}{I} = X_C - X_L = \left(\frac{1}{\omega C} - \omega L\right)$$

The leading of current over applied e.m.f is $\pm\frac{\pi}{2}$

In case $X_C = X_L$ then $Z = 0$

$$\Rightarrow \quad \frac{1}{\omega_0 C} = \omega_0 L$$

$$\Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

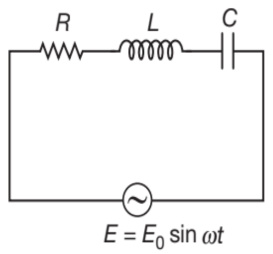
where ω_0 is the resonant angular frequency.

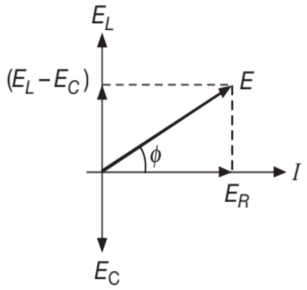
$$\Rightarrow \quad \text{Frequency } f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

This frequency is called the **resonant frequency**.

AC THROUGH A SERIES *LCR* CIRCUIT

Let a circuit contain a resistance R, inductance L and capacitance C in series.





Let I be the current flowing in circuit E_R, E_L and E_C the respective potential differences across resistance R, inductance L and capacitance C. The potential difference E_R is in phase with current I. The potential difference E_C lags behind the current by angle $\frac{\pi}{2}$. The potential difference V_L leads the current by angle $\frac{\pi}{2}$ as shown in the phasor diagram.
So, Resultant e.m.f is

$$E = \sqrt{E_R^2 + (E_L - E_C)^2}$$

$$\Rightarrow \quad E = \sqrt{(IR)^2 + (IX_L - IX_C)^2}$$

$$\Rightarrow \quad Z = \frac{E}{I} = \frac{E_V}{I_V} = \sqrt{R^2 + (X_L - X_C)^2}$$

The phase lead of e.m.f over current is

$$\tan\phi = \frac{E_L - E_C}{E_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R}$$

$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

(a) If $X_C > X_L$ i.e. $\omega_0 > \omega$ the value of ϕ is negative i.e., current leads the applied e.m.f by a phase angle ϕ.

(b) If $X_C < X_L$ i.e. $\omega_0 < \omega$ the value of ϕ is positive i.e., current lags behind the applied e.m.f by a phase angle ϕ.

(c) If $X_C = X_L$ i.e. $\omega_0 = \omega$ the value of ϕ is zero i.e., current and e.m.f are in same phase. This is called the case of **resonance** and **resonant frequency** is given by condition $X_C = X_L$

$$\Rightarrow \quad \frac{1}{\omega_0 C} = \omega_0 L$$

$$\Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \quad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

Problem Solving Technique(s)

At resonance

(a) $X_L = X_C$

(b) $\omega_0 = \frac{1}{\sqrt{LC}}$

(c) E and I both are in the same phase.

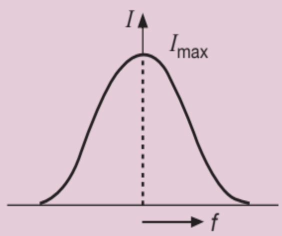
(d) Impedance is minimum and $Z_{min} = R$

(e) Current in the circuit is maximum and $I_{max} = \frac{E}{R}$

(f) Resistance is the only active element in the *LCR* series circuit.

(g) Power factor $(\cos\phi)$ is maximum i.e. Power factor $= 1$

(h) The variation of current with the frequency of the applied voltage is shown in the figure.



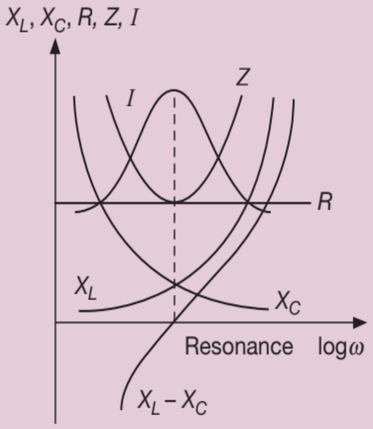
If the applied voltage consists of a number of frequency components, the current will be large for the component having frequency f_0. This resonant behaviour of the *LCR* circuit is used in radio tuning.
The tuning circuit of a radio receiver contains an *LCR* circuit, usually having a variable capacitor (also called **Variac**). It is varied till the resonant frequency of the circuit is equal to the particular frequency from some radio station. Then the current corresponding to this signal is maximum and the receiver responds to it.

(i) In *L-C-R*, circuit whenever voltage across various elements is asked, find r.m.s. values unless stated in the question for the peak or instantaneous value.

The r.m.s. values are, $V_R = I_{rms}R$, $V_L = I_{rms}X_L$ and $V_C = I_{rms}X_C$

The peak values can be obtained by multiplying the r.m.s. values by $\sqrt{2}$. The instantaneous values across different elements is rarely asked.

(j) Response curves of series circuit: The impedence of an *LCR* circuit depends on the frequency. The dependence is shown in figure. The frequency is taken on logarithmic scale because of its wide range. From the figure we can see that at resonance.



(i) $X_L = X_C$ or $\omega = \dfrac{1}{\sqrt{LC}}$

(ii) $Z = Z_{min} = R$ and

(iii) I is maximum.

Here by Z we mean the modulus of Z and I means I_{rms}.

(k) Acceptor Circuit

If the frequency of the ac supply can be varied (e.g., in radio or television signal) then in series *LCR* circuit, at a frequency $f = \dfrac{1}{2\pi\sqrt{LC}}$ maximum current flows in the circuit and have a maximum P.D. across its inductance (or capacitance). This is the method by which a radio or television set is tuned at a particular frequency. The circuit is known as **acceptor circuit**.

ILLUSTRATION 15

A series *LCR* circuit containing a resistance of 120 Ω has angular frequency 4×10^5 rads^{-1}. At resonance the voltages across resistance and inductance are 60 V and 40 V respectively. Find the values of *L* and *C*. At what angular frequency the current in the circuit lags the voltage by $\dfrac{\pi}{4}$.

SOLUTION

At resonance, $X_L - X_C = 0$

$$\Rightarrow \quad Z = R = 120\ \Omega \qquad \left\{\because Z^2 = (X_L - X_C)^2 + R^2\right\}$$

$$\Rightarrow \quad I_{rms} = \frac{(V_R)_{rms}}{R} = \frac{60}{120} = \frac{1}{2}\ \text{A}$$

$$(V_L)_{rms} = I_{rms}X_L = I_{rms}(L\omega)$$

$$\Rightarrow \quad L = \frac{(V_L)_{rms}}{I_{rms}\omega} = \frac{40}{(4 \times 10^5)\left(\frac{1}{2}\right)} = 2 \times 10^{-4}\ \text{H}$$

$$\Rightarrow \quad L = 0.2\ \text{mH}$$

The resonant frequency is given by,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \quad C = \frac{1}{\omega_0^2 L}$$

Substituting the values, we have

$$C = \frac{1}{(4 \times 10^5)^2(2 \times 10^{-4})} = \frac{1}{32 \times 10^6}$$

$$\Rightarrow \quad C = \frac{1}{32} \times 10^{-6} = 3.125 \times 10^{-8}\ \text{F} = 31.25\ \text{nF}$$

For the current to lag behind the voltage by $\dfrac{\pi}{4}$, we have

$$\tan\left(\frac{\pi}{4}\right) = \frac{\omega L - \dfrac{1}{\omega C}}{R}$$

Substituting the values of *L*, *C*, *R* and $\tan\left(\dfrac{\pi}{4}\right)$, we get

$$(1)(120) = 0.2\omega - \frac{1}{\omega\left(\frac{10^{-6}}{32}\right)}$$

$\Rightarrow$ $$\omega^2 - 6\times10^5\omega - 16\times10^{10} = 0$$

$\Rightarrow$ $$\omega = \frac{6\times10^5 \pm \sqrt{36\times10^{10} + 64\times10^{10}}}{2}$$

$\Rightarrow$ $$\omega = \frac{6\times10^5 + 10\times10^5}{2} = 8\times10^5 \text{ rads}^{-1}$$

$\Rightarrow$ $$\omega = 8\times10^5 \text{ rads}^{-1}$$

ILLUSTRATION 16

A 200 km long telegraph line wire has a capacitance of 0.014 μFkm^{-1}. If it carries an alternating current of 50 kHz, what should be the value of an inductance required to be connected in series so that impedance is minimum.

SOLUTION

The total capacitance of telegraph line wire is

$$C = 0.014\times200 = 2.8\ \mu\text{F} = 2.8\times10^{-6}\text{ F}$$

For minimum impedance, we have $X_L = X_C$

$\Rightarrow$ $$L\omega = \frac{1}{C\omega}$$

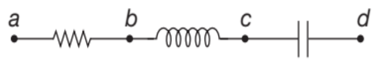
$\Rightarrow$ $$L = \frac{1}{\omega^2 C} = \frac{1}{(2\pi f)^2 C} = \frac{1}{4\pi^2 f^2 C}$$

$\Rightarrow$ $$L = \frac{1}{4\times(3.14)^2\times(50\times10^3)^2\times(2.8\times10^{-6})}\text{ H}$$

$\Rightarrow$ $$L = 3.6\times10^{-6} = 3.6\ \mu\text{H}$$

ILLUSTRATION 17

An AC source with $V_{max} = 205$ V and $f = 50$ Hz is connected between points a and d having a resistance of $R = 40\ \Omega$, inductance of $L = 185$ mH and a capacitance of $C = 65\ \mu$F as shown in figure. Calculate the maximum voltages between points (a) a and b, (b) b and c, (c) c and d, and (d) b and d. A power supply with $\Delta V_{rms} = 120$ V is connected between points a and d in figure. At what frequency will it deliver a power of 250 W?



SOLUTION

Let us first calculate X_L and X_C. Since

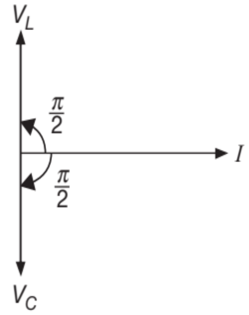
$$X_L = L\omega = (2\pi)(50)(185\times10^{-3}) = 58\ \Omega$$

$$X_C = \frac{1}{C\omega} = \frac{1}{(2\pi)(50)(65\times10^{-6})} = 49\ \Omega$$

Since R, L and $C$are connected in series, so we have

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$\Rightarrow$ $$Z = \sqrt{1600+81} = 41\ \Omega$$



Now $I_{max} = \frac{\Delta V_{max}}{Z} = \frac{205}{41} = 5$ A

(a) $(V_{ab})_{max} = I_{max}R = (5)(40) = 200\text{ V} = (V_R)_{max}$

(b) $(V_{bc})_{max} = I_{max}X_L = (5)(58) = 290\text{ V} = (V_L)_{max}$

(c) $(V_{cd})_{max} = I_{max}X_C = (5)(49) = 245\text{ V} = (V_C)_{max}$

(d) $(V_{bd})_{max} = (V_L)_{max} - (V_C)_{max} = (V_{bc})_{max} - (V_{cd})_{max}$

$\Rightarrow$ $(V_{bd})_{max} = 45$ V

Also note that, for this arrangement we have

$$V_{max} = \sqrt{(V_L - V_C)^2_{max} + (V_R)^2_{max}} = 205\text{ V}$$

ILLUSTRATION 18

A resistance R and inductance L and a capacitor C are all connected in series with an AC supply. The resistance is of 16 Ω and for a given frequency, the inductive reactance is 24 Ω and the capacitive reactance is 12 Ω. If the current in the circuit is 5 A, calculate the

(a) potential difference across R, L and C.
(b) impedance of the circuit.
(c) ac supply voltage.
(d) phase angle between current and voltage.

SOLUTION

(a) Potential difference across resistance is

$E_R = IR = 5\times16 = 80\text{ V}$

Potential difference across inductor is

$E_L = IX_L = 5\times24 = 120\text{ V}$

Potential difference across capacitor is

$E_C = IX_C = 5\times12 = 60\text{ V}$

(b) The impedance of circuit is

$$Z=\sqrt{R^2+\left(L\omega-\frac{1}{C\omega}\right)^2}$$

$$\Rightarrow\ Z=\sqrt{(16)^2+(24-12)^2}=20\ \Omega$$

(c) The AC supply voltage is

$E = IZ = 5\times20 = 100\text{ V}$

(d) Phase angle between current and voltage is given as

$$\phi=\tan^{-1}\left(\frac{X_L-X_C}{R}\right)=\tan^{-1}\left(\frac{24-12}{16}\right)$$

$$\Rightarrow\ \phi=\tan^{-1}(0.75)=37^\circ$$

ILLUSTRATION 19

A series circuit consists of a resistance of $15\ \Omega$, an inductance of 0.08 H and a capacitor of capacitance $30\ \mu\text{F}$. The applied voltage has a frequency of 500 rads^{-1}. Find whether the current leads or lags the applied voltage and also the angle of lead or lag.

SOLUTION

The inductive reactance of the circuit is

$$X_L = L\omega = 500\times0.08 = 40\ \Omega$$

The capacitive reactance of the circuit is

$$X_C=\frac{1}{C\omega}=\frac{1}{(30\times10^{-6})(500)}=66.7\ \Omega$$

Since $X_C > X_L$, so the circuit is capacitive in nature and hence the current leads the voltage by a phase angle ϕ given by

$$\tan\phi=\frac{X_C-X_L}{R}=\frac{66.7-40}{15}=1.78$$

$$\Rightarrow\ \phi=60.65^\circ$$

Hence current leads voltage by a phase angle of 60.65°.

ILLUSTRATION 20

An inductor coil, a capacitor and an AC source of 24 V are connected in series. When the frequency of the source is varied, a maximum current of 6 A flows through the circuit. If the inductor coil is connected to a battery of emf 12 V and internal resistance $4\ \Omega$, what will be steady state current through battery.

SOLUTION

Let R be the resistance of the inductor coil, then at resonance, current in the circuit is maximum.

$$\Rightarrow\ I_{\text{max}}=\frac{E_V}{Z_{\text{min}}}=\frac{E_V}{R}=\frac{24}{R}=6\text{ A}$$

$$\Rightarrow\ R=4\ \Omega$$

When connected across a DC source, the steady current in circuit is

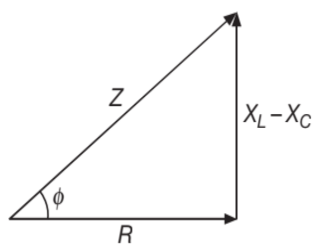
$$I=\frac{E}{R+r}=\frac{12}{4+R}=\frac{12}{8}=1.5\text{ A}$$

IMPEDANCE: REVISITED

We have already seen that the inductive reactance $X_L(=\omega L)$ and capacitance reactance $X_C\left(=\frac{1}{C\omega}\right)$ play the role of an effective resistance in the purely inductive and capacitive circuits, respectively. In the series *RLC* circuit, the effective resistance is the impedance Z, given by

$$Z=\sqrt{R^2+(X_L-X_C)^2}$$

The relationship between Z, X_L and X_C is best represented by the diagram shown.



The *SI* unit of impedance is ohm (Ω). In terms of Z, the current may be rewritten as

$$I(t)=\frac{V_0}{Z}\sin(\omega t-\phi)$$

Notice that the impedance Z also depends on the angular frequency ω, as do X_L and X_C. Using $\tan\phi = \left(\frac{X_L - X_C}{R}\right)$ and $Z^2 = R^2 + (X_L - X_C)^2$, we can easily get the limits for simple circuit (with only one element) as provided in the table.

Table 1 Simple-circuit limits of the series *RLC* circuit

Simple Circuit	R	L	C	$X_L = \omega L$	$X_C = \frac{1}{C\omega}$	$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$	$Z = \sqrt{R^2 + (X_L - X_C)^2}$
Pure resistance	R	0	∞	0	0	0	R
Pure inductance	0	L	∞	X_L	0	$\frac{\pi}{2}$	X_L
Pure capacitance	0	0	C	0	X_C	$-\frac{\pi}{2}$	X_C

Problem Solving Technique(s)

(a) Please note that

$$I_0 = \frac{V_0}{|Z|},\ I_{rms} = \frac{V_{rms}}{|Z|}.$$

(b) The complex impedance for an inductor is

$X_L = j(L\omega)$, where $j = \sqrt{-1}$

(c) The complex impedance of a capacitor is

$$X_C = \frac{1}{jC\omega} = -\frac{j}{C\omega}, \text{ where } j = \sqrt{-1}$$

Please note that here we are using the complex number symbol as $j = \sqrt{-1}$, because i can be misled to be current. Also note that $\frac{1}{j} = \frac{j}{j^2} = -j$.

(d) If we have any complex number written as

$z = x + jy$, where $j = \sqrt{-1}$

Then the real part of this number is x and the imaginary part is y. The magnitude of the number is

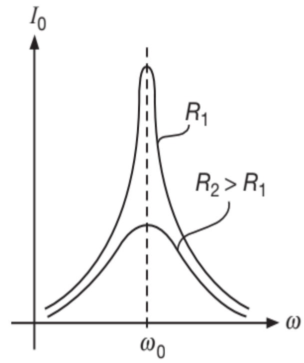
$$|z| = \sqrt{x^2 + y^2}$$

(e) When any complex number is multiplied by $j = \sqrt{-1}$, then that number is rotated counter clockwise through an angle of $90°$.

QUALITY FACTOR OR Q-FACTOR

The Q-factor of an LCR series circuit is defined as the ratio of the voltage across inductor or capacitor at resonance to the voltage across the resistor. So,

$$Q = \frac{E_L}{E_R} = \frac{E_C}{E_R}$$



The amplitude of the current as a function of ω in the driven RLC circuit.

$$\Rightarrow \quad Q = \frac{IX_L}{IR} = \frac{IX_C}{IR}$$

$$\Rightarrow \quad Q = \frac{\omega_0 L}{R} = \frac{1}{RC\omega_0}$$

Since $\omega_0 = \dfrac{1}{\sqrt{LC}}$

$$\Rightarrow \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Note that, we get the same result from both equalities. Quality factor is an indicator of the sharpness of the current peak. Higher the value of the quality factor Q, sharper is the current peak.

ILLUSTRATION 21

Prove that in a series LCR circuit, the frequencies at which the current amplitude falls to $\dfrac{1}{\sqrt{2}}$ of the current at resonance are separated by an interval equal to $\dfrac{R}{2\pi L}$.

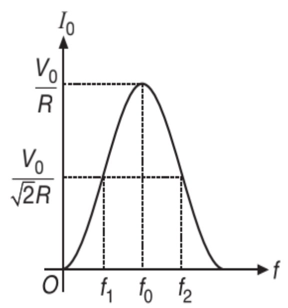
SOLUTION

At resonance, we have

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$\Rightarrow \quad I_0 = \frac{V_0}{R} \qquad \{\text{as } Z = R\}$$

According to the problem, we have $I = \dfrac{I_0}{\sqrt{2}}$



$$\Rightarrow \quad \frac{1}{\sqrt{2}}\frac{V_0}{R} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$$

$$\Rightarrow \quad 2R^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2$$

$$\Rightarrow \quad \omega L - \frac{1}{\omega C} = \pm R$$

If f_1 and f_2 are two corresponding frequencies, then

$$2\pi f_1 L - \frac{1}{2\pi f_1 C} = R \qquad \ldots(1)$$

$$2\pi f_2 L - \frac{1}{2\pi f_2 C} = -R \qquad \ldots(2)$$

Dividing (1) by f_2, (2) by f_1 and solving, we get

$$(f_1 - f_2) = \frac{R}{2\pi L}$$

AVERAGE POWER CONSUMED IN AN AC CIRCUIT

The power is defined as the rate at which work is being done in the circuit.

In an ac circuit, the current and e.m.f are not necessarily in the same phase. Consider

$$E = E_0 \sin(\omega t + \phi) \text{ and } I = I_0 \sin(\omega t)$$

If dW is the work done in time dt, then

$$dW = EIdt$$

$$\Rightarrow \quad dW = E_0 I_0 \sin(\omega t + \phi)\sin(\omega t)dt$$

$$\Rightarrow \quad dW = \frac{E_0 I_0}{2}\left[2\sin(\omega t + \phi)\sin(\omega t)\right]dt$$

From Trigonometry

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\Rightarrow \quad dW = \frac{E_0 I_0}{2}\left[\cos(\omega t + \phi - \omega t) - \cos(2\omega t + \phi)\right]dt$$

So, work done in one complete cycle is

$$W = \int dW = \frac{E_0 I_0}{2}\left[\int_0^T \cos\phi dt - \int_0^T \cos(2\omega t + \phi)dt\right]$$

Since, $\displaystyle\int_0^T \cos(2\omega t + \phi)dt = 0$

$$\Rightarrow \quad W = \left(\frac{E_0 I_0}{2}\cos\phi\right)T - 0$$

$$\Rightarrow \quad P_{av} = \frac{W}{T} = \frac{E_0 I_0}{2}\cos\phi$$

$$\Rightarrow \quad P_{av} = \frac{E_0 I_0}{2}\cos\phi = \left(\frac{E_0}{\sqrt{2}}\right)\left(\frac{I_0}{\sqrt{2}}\right)\cos\phi \qquad \ldots(1)$$

Since, $E_V = \frac{E_0}{\sqrt{2}}$, $I_V = \frac{I_0}{\sqrt{2}}$ and from phasor diagrams of various cases already discussed we know that

$$\cos\phi = \frac{\text{Resistance }(R)}{\text{Impedance }(Z)}$$

$\cos\phi$ is called the **Power Factor (PF) of an AC circuit**. So, we must keep in mind that power factor

$$PF = \cos\phi = \frac{R}{Z}$$

So, equation (1) can be re-written as

$$P_{av} = E_V I_V \cos\phi \text{, where } I_V = \frac{E_V}{Z}$$

$$\Rightarrow \quad P_{av} = (I_V Z) I_V \left(\frac{R}{Z}\right) = I_V^2 R$$

This result is completely justified as we are aware that there is no power consumption across an ideal inductor and an ideal capacitor because in both, the power factor is zero. Hence the power is consumed only across the resistor in the circuit.

Also $P_{av} = \frac{E_V^2}{Z}\cos\phi = I_V^2 Z\cos\phi \qquad \left\{\because Z = \frac{E_V}{I_V}\right\}$

CASE-1:

If $R = 0$, $\cos\phi = 0$ and $P_{av} = 0$ i.e., in resistance less circuit the power consumed is zero. Such a circuit is called the **wattless circuit** and the current flowing is called the **wattless current**.

CASE-2:

If $R = Z$ (in purely resistive circuit and LCR circuit at resonance), the power factor $\cos\phi = R/Z = 1$ and the power loss is maximum. Therefore, the use of resistance is avoided in AC circuits.

HALF POWER FREQUENCIES (HPF), BAND WIDTH AND SHARPNESS OF RESONANCE

Since the average power is given by

$$P_{av} = \langle P\rangle = E_{rms} I_{rms}\cos\phi = E_V I_V \cos\phi \qquad \ldots(1)$$

This average power consumed in the circuit varies with the frequency, because the impedance Z depends upon the frequency.

We want to find the frequencies where the power is half the maximum power. These frequencies are called the Half Power Frequencies.

Equation (1) can be re-written as

$$\langle P\rangle = \left(\frac{E_0}{\sqrt{2}}\right)\left(\frac{I_0}{\sqrt{2}}\right)\cos\phi = \frac{E_0 I_0}{2}\cos\phi \qquad \ldots(2)$$

Further we know that $I_0 = \frac{E_0}{Z}$ and $\cos\phi = \frac{R}{Z}$

So, from equation (2), we get

$$\langle P\rangle = \frac{E_0}{2}\left(\frac{E_0}{Z}\right)\left(\frac{R}{Z}\right) = \left(\frac{E_0^2 R}{2}\right)\frac{1}{Z^2}$$

where, $Z^2 = R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2$

$$\Rightarrow \quad \langle P\rangle = \frac{E_0^2 R}{2\left[R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right]} \qquad \ldots(3)$$

We can clearly see from equation (3), that the average power is dependent on the frequency fed across the circuit.

From equation (3), we also conclude that

(a) $\langle P\rangle = 0$, when $\omega \to 0$ or when $\omega \to \infty$ i.e. $\langle P\rangle$ is zero for extremely small and extremely large frequencies.

(b) $\langle P\rangle$ is maximum at resonance i.e. when

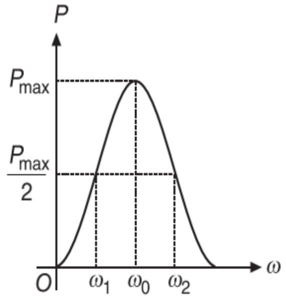
$$L\omega_0 = \frac{1}{C\omega_0}$$

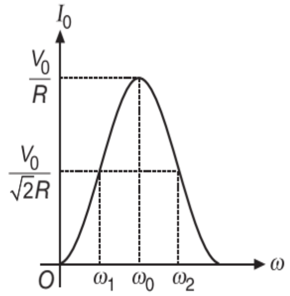
where, $\omega_0 = \frac{1}{\sqrt{LC}}$ is the resonant frequency.

This maximum value of average power i.e. $\langle P\rangle_{max}$ is given by

$$\langle P\rangle_{max} = \frac{E_0^2}{2R} \qquad \ldots(4)$$

The variation of average power with frequency is shown in Figure.





The frequencies at which the average power consumed in the circuit is half of the maximum average power (i.e. the power at resonance) are called Half Power Frequencies (HPF). So, at half power frequencies we have

$$\langle P \rangle = \frac{\langle P \rangle_{max}}{2}$$

Using equations (3) and (4), we get

$$\frac{E_0^2 R}{2\left[R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right]} = \frac{1}{2}\left(\frac{E_0^2}{2R}\right)$$

$$\Rightarrow \quad R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2 = 2R^2$$

$$\Rightarrow \quad \left(L\omega - \frac{1}{C\omega}\right)^2 = R^2$$

$$\Rightarrow \quad L\omega - \frac{1}{C\omega} = \pm R$$

From this equation, we conclude that

EITHER, $L\omega - \frac{1}{C\omega} = +R$ ...(5)

OR, $L\omega - \frac{1}{C\omega} = -R$ ...(6)

To calculate these frequencies, we proceed using equations (5) and (6). From equation (5), we have

$$L\omega - \frac{1}{C\omega} = +R$$

Multiplying both sides by $\frac{\omega}{L}$, we get

$$\omega^2 - \frac{1}{LC} = \frac{R\omega}{L}$$

Rearranging this equation, we get

$$\omega^2 - \frac{R\omega}{L} - \frac{1}{LC} = 0$$

Solving this quadratic equation, we get

$$\omega = \frac{1}{2}\left(\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}\right)$$

Rejecting the negative value of frequency ω, we get

$$\omega = \frac{1}{2}\left(\frac{R}{L} + \sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}\right)$$

$$\Rightarrow \quad \omega = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \qquad ...(7)$$

Now from equation (6), we have

$$L\omega - \frac{1}{C\omega} = -R$$

$$\Rightarrow \quad \omega^2 + \frac{R\omega}{L} - \frac{1}{LC} = 0$$

Solving this quadratic equation, we get

$$\omega = \frac{1}{2}\left(-\frac{R}{L} + \sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}\right)$$

$$\Rightarrow \quad \omega = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \qquad ...(8)$$

From equations (7) and (8), we observe that one frequency is higher and the other frequency is lower that the resonant frequency. So, they are denoted by

$$\omega_{Lower} = \omega_L = \omega_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \text{ and}$$

$$\omega_{Higher} = \omega_H = \omega_2 = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

The circuit is capacitive in nature for $\omega < \omega_0$ and inductive in nature for $\omega > \omega_0$. For ω_L, the circuit is capacitive in nature and for ω_H, the circuit is inductive in nature.
Also, we see that

$$\omega_L \omega_H = \frac{1}{LC} = \omega_0^2$$

which simply makes us conclude that

$$\omega_L < \omega_0 < \omega_H$$

and ω_0 is the geometric mean of ω_L and ω_H.
The difference of these two frequencies is called the band-width given by

$$\Delta\omega = \omega_H - \omega_L = \frac{R}{L}$$

The quality factor Q is defined as the ratio of the resonant frequency ω_0 to the band width $\Delta\omega$, so we have

$$Q = \frac{\text{Resonant Frequency}}{\text{Band Width}} = \frac{\omega_0}{\Delta\omega}$$

$$\Rightarrow \quad Q = \frac{1/\sqrt{LC}}{L/R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

The larger the value of Q, the smaller is the value of bandwidth and sharper is the resonance. If sharpness of the circuit is small, then its selectivity is also less. If quality factor is large, then R is small, C is small and L is large, so the circuit is more selective.

ILLUSTRATION 22

A DC of $2\ \text{A}$ and an AC of peak value $2\ \text{A}$ flow through a resistance of $2\ \Omega$ and $1\ \Omega$ respectively. Calculate the ratio of heat produced in the two resistances in the same time interval.

SOLUTION

In time t heat produced by a DC current i_{dc} is given by

$$H_1 = i_{dc}^2 Rt$$

$$\Rightarrow \quad H_1 = (2)^2(2)t = 8t$$

RMS value of AC is given by

$$i_{\text{rms}} = \frac{i_0}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}\ \text{A}$$

Heat produced by the rms current in time t is given by

$$H_2 = i_{\text{rms}}^2 Rt$$

$$\Rightarrow \quad H_2 = \left(\sqrt{2}\right)^2(1)t = 2t$$

$$\Rightarrow \quad \frac{H_1}{H_2} = \frac{8t}{2t} = 4$$

ILLUSTRATION 23

A $\frac{2.2}{\pi}\ \text{H}$, $220\ \Omega$ coil is applied across a $220\ \text{V}$, $50\ \text{Hz}$ ac. Calculate wattless current in the circuit.

SOLUTION

For a series LR circuit, impedance is given by

$$Z = \sqrt{R^2 + X_L^2}$$

Inductive reactance is given by

$$X_L = \omega L$$

$$\Rightarrow \quad X_L = 2\pi(50) \times \frac{2.2}{\pi}$$

$$\Rightarrow \quad X_L = 220\ \Omega$$

Since, $X_L = R = 220\ \Omega$

$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}(1) = 45^\circ$$

Circuit in the circuit is

$$I_{\text{rms}} = \frac{E_{\text{rms}}}{Z} = \frac{220}{\sqrt{(220)^2 + (220)^2}} = \frac{1}{\sqrt{2}}\ \text{A}$$

Wattless current in an ac circuit is given by

$$I_{WL} = I_V \sin\phi = I_{\text{rms}} \cos\phi$$

$$\Rightarrow \quad I_{WL} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} = 0.5\ \text{A}$$

ILLUSTRATION 24

A series LCR circuit has $L = 10\ \text{mH}$, $R = 3\ \Omega$ and $C = 1\ \mu\text{F}$ connected in series to an ac source of $E = 15\cos(\omega t)$ volt. Calculate the current amplitude and the average power dissipated per cycle at a frequency which is 10% lower than the resonant frequency.

SOLUTION

At resonance, the frequency of circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \quad \omega_0 = \sqrt{\frac{1}{(10\times10^{-3})(1\times10^{-6})}} = 10^4\ \text{rads}^{-1}$$

A frequency which is 10% less than the resonant frequency will be

$$\omega = 10^4 - 10^4 \times \frac{10}{100} = 9\times10^3\ \text{rads}^{-1}$$

At this frequency, the inductive reactance is

$$X_L = L\omega = 9\times10^3 \times (10\times10^{-3}) = 90\ \Omega$$

and the capacitive reactance is

$$X_C = \frac{1}{C\omega} = \frac{1}{(1\times10^{-6})(9\times10^3)} = 111.1\ \Omega$$

Since, $Z=\sqrt{R^2+(X_L-X_C)^2}$

$$\Rightarrow \quad Z=\sqrt{3^2+(90-111.1)^2}=21.32\ \Omega$$

Current amplitude is

$$I_0=\frac{E_0}{Z}=\frac{15}{21.32}=0.704\text{ A}$$

Average power consumed in the circuit is

$$P_{av}=E_V I_V\cos\phi=I_V^2R=\frac{I_0^2R}{2}$$

$$\Rightarrow \quad P=\frac{(0.704)^2\,3}{2}=0.744\text{ W}$$

ILLUSTRATION 25

In an AC circuit, the voltage applied is $E=5\sin(\omega t)$ and a current $I=3\cos(\omega t)$ flows in circuit. Calculate the average power dissipated in the circuit.

SOLUTION

Given that

$$E=5\sin(\omega t)\text{ and}$$

$$I=3\cos(\omega t)=3\sin\left(\omega t+\frac{\pi}{2}\right)$$

From above we conclude that, the phase difference between voltage and current is $\frac{\pi}{2}$, the current leads voltage by $\phi=\frac{\pi}{2}$ and hence the circuit is purely capacitive in nature. Since, $P_{av}=E_V I_V\cos\phi$

$$\Rightarrow \quad P_{av}=E_V I_V\cos\left(\frac{\pi}{2}\right)=0$$

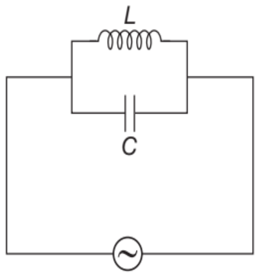
So, the average power dissipated in circuit will be zero.

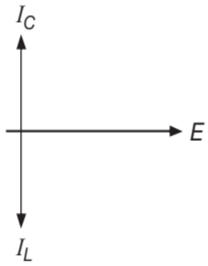
CHOKE COIL

Choke coil is a coil having high inductance and negligible resistance. It is used to control current in AC circuit and is used in fluorescent tubes. The power loss in a circuit containing choke coil is least.

PARALLEL RESONANT CIRCUIT: REJECTOR CIRCUIT

A parallel resonant circuit consists of an inductance L and a capacitance C in parallel as shown in figure.





The condition of resonance is again that the current and applied e.m.f must be in same phase.

i.e. $\quad I_C=I_L$

$$\Rightarrow \quad \frac{V}{X_C}=\frac{V}{X_L}$$

$$\Rightarrow \quad X_L=X_C$$

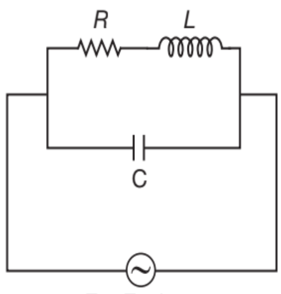
$$\Rightarrow \quad \omega_0=\frac{1}{\sqrt{LC}}$$

The frequency of parallel resonant circuit at resonance is

angular frequency $\omega_0=\dfrac{1}{\sqrt{LC}}$

and linear frequency $f_0=\dfrac{1}{2\pi\sqrt{LC}}$

At resonance the current in the circuit is minimum and impedance is infinite.



If inductance has also a resistance R as shown. The condition of resonant angular frequency is,

$$\omega_0=\sqrt{\frac{1}{LC}-\frac{R^2}{L^2}}$$

So, resonant frequency

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

The impedance at resonance,

$$Z = \frac{R^2 + \omega_0^2 L^2}{R} = \frac{L}{RC}$$

In parallel resonant circuit, the impedance is maximum (or admittance is minimum) and the current is minimum, when $\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$

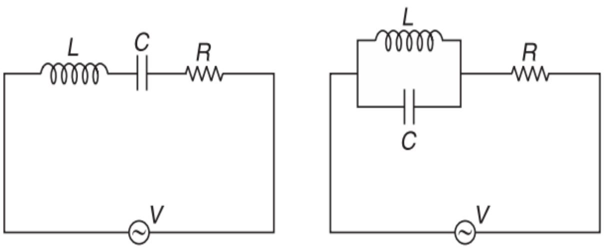
If $R \to 0$, then $f_0 = \frac{1}{2\pi\sqrt{LC}}$ and $Z \to \infty$

REJECTOR CIRCUIT

We observe that at $\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$, the admittance in the circuit is the minimum or the impedance is maximum or the current is the minimum. Thus, the parallel circuit does not allow this particular frequency from the source to pass in the circuit and due to this reason, the circuit with such frequency (f_0 or ω_0) is called a REJECTOR CIRCUIT.

ILLUSTRATION 26

An AC source is connected to two circuits as shown in figure. Obtain the current through resistance R at resonance in both the circuits.



(a) (b)

SOLUTION

The first circuit is a series LCR circuit. The impedance for this circuit is

$$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

At resonance, we have

$$L\omega = \frac{1}{C\omega}$$

$$\Rightarrow \quad Z = R$$

The current in the circuit at resonance is given as

$$I = \frac{V}{Z} = \frac{V}{R}$$

In the second circuit, the inductor and the capacitor are connected in parallel and hence the potential difference across each will be the same.

At resonance, we have

$$X_L = X_C$$

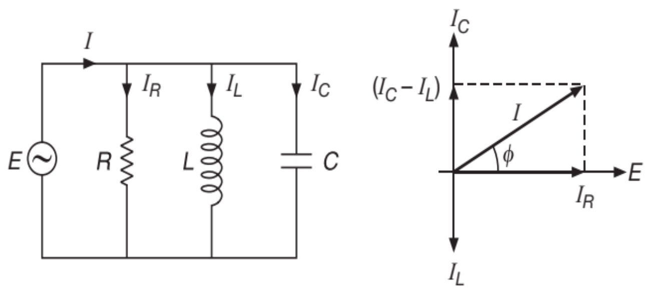
$$\Rightarrow \quad L\omega = \frac{1}{C\omega}$$

So, the current in both inductor and capacitor will be equal in magnitude. Also, we know that the phase difference between currents through the inductor L and the capacitor C will be 180° or the currents in them will be out of phase. So, we conclude that the two currents will be equal in magnitude but 180° out of phase and hence the current through the resistor R will be zero in this circuit.

THE *LCR* PARALLEL CIRCUIT

In this case the voltage E across each element is the same. The currents are related as

$$I = \sqrt{I_R^2 + (I_C - I_L)^2}$$



$$\Rightarrow \quad I = \sqrt{\left(\frac{E}{R}\right)^2 + \left(C\omega E - \frac{E}{L\omega}\right)^2}$$

$$I = E\sqrt{\frac{1}{R^2} + \left(C\omega - \frac{1}{L\omega}\right)^2}$$

$$\Rightarrow \quad Z = \frac{E}{I} = \frac{1}{\sqrt{\frac{1}{R^2} + \left(C\omega - \frac{1}{L\omega}\right)^2}}$$

So, we observe that at resonance, i.e. when $L\omega = \dfrac{1}{C\omega}$ then the current in circuit is minimum.

Problem Solving Technique(s)

In this chapter, we have seen how phasors provide a powerful tool for analysing the AC circuits. Below are some important techniques that help in problem solving.

(a) Keep in mind the phase relationships for simple circuits.

(i) For a resistor, the voltage and the phase are always in phase.

(ii) For an inductor, the current lags the voltage by $90°$.

(iii) For a capacitor, the current leads to voltage by $90°$.

(b) When circuit elements are connected in series, the instantaneous current is the same for all elements, and the instantaneous voltages across the elements are out of phase.

On the other hand, when circuit elements are connected in parallel, the instantaneous voltage is the same for all elements, and the instantaneous currents across the elements are out of phase.

(c) For series connection, draw a phasor diagram for the voltages. The amplitudes of the voltage drop across all the circuit elements involved should be represented with phasors. In figure the phasor diagram for a series RLC circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

From figure (a), we see that $V_L > V_C$ in the inductive case and $\vec{V}$ leads $\vec{I}$ by a phase ϕ. On the other hand, in the capacitive case shown in figure (b), $V_C > V_L$ and $\vec{I}$ leads $\vec{V}$ by a phase ϕ.

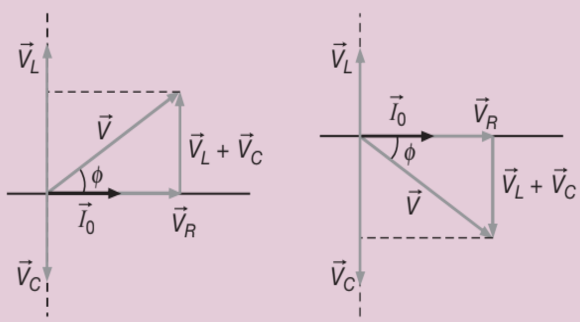


FIGURE (a) FIGURE (b)

Phasor diagram for the series RLC circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

(d) When $V_L = V_C$, or $\phi = 0$, the circuit is at resonance. The corresponding resonant frequency is $\omega_0 = \dfrac{1}{\sqrt{LC}}$, and the power delivered to the resistor is a maximum.

(e) For parallel connection, draw a phasor diagram for the currents. The amplitudes of the currents across all the circuit elements involved should be represented with phasors. In figure the phasor diagram for a parallel RLC circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

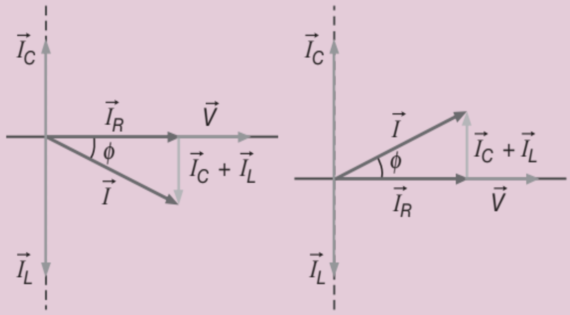


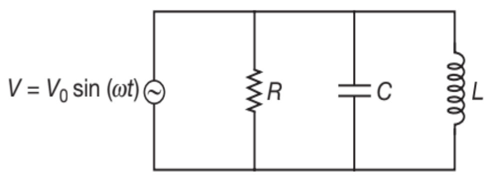
FIGURE (a) FIGURE (b)

Phasor diagram for the paralled RLC circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From figure (a), we see that $I_L > I_C$ in the inductive case and $\vec{V}$ leads $\vec{I}$ by a phase ϕ. On the other hand, in the capacitive case shown in figure (b), $I_C > I_L$ and $\vec{I}$ leads $\vec{V}$ by a phase ϕ.

ILLUSTRATION 27

For the circuit shown in figure, find the instantaneous current through each element. Also find the total instantaneous current through the source, and find expressions for phase angle of this current and the impedance of the circuit.



SOLUTION

The three current equations are,

$$V = I_R R,\ V = L\frac{dI_L}{dt} \text{ and } \frac{dV}{dt} = \frac{1}{C} I_C \qquad \ldots(1)$$

The steady state solutions of equation (1) are,

$$I_R = \frac{V_0}{R}\sin(\omega t) \equiv (I_0)_R \sin(\omega t)$$

$$I_L = -\frac{V_0}{\omega L}\cos(\omega t) = -\frac{V_0}{X_L}\cos(\omega t)$$

$$\Rightarrow \quad I_L = -(I_0)_L \cos(\omega t)$$

and $I_C = V_0 \omega C \cos(\omega t) = \dfrac{V_0}{X_C}\cos(\omega t)$

$$\Rightarrow \quad I_C = (I_0)_C \cos(\omega t)$$

where, $X_L = L\omega$ and $X_C = \dfrac{1}{C\omega}$

For the total current we have,

$$I = I_R + I_L + I_C$$

$$I = V_0\left[\frac{1}{R}\sin(\omega t) + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)\cos(\omega t)\right] \quad \ldots(2)$$

Let $\dfrac{V_0}{R} = I_0 \cos\phi$ and $V_0\left(\dfrac{1}{X_L} - \dfrac{1}{X_C}\right) = I_0 \sin\phi$

then equation (2), becomes

$$I = I_0 \sin(\omega t + \phi)$$

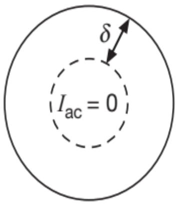
where $I_0 = \dfrac{V_0}{Z} = V_0\sqrt{\dfrac{1}{R^2} + \left(\dfrac{1}{X_L} - \dfrac{1}{X_C}\right)^2}$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_L} - \frac{1}{X_C}\right)^2}$$

and $\tan\phi = \dfrac{\left(\dfrac{1}{X_C} - \dfrac{1}{X_L}\right)}{\left(\dfrac{1}{R}\right)} = R\left(C\omega - \dfrac{1}{L\omega}\right)$

SKIN EFFECT

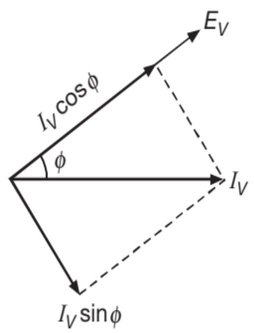
A direct current flows uniformly throughout the cross-section of the conductor. An alternating current, on the other hand, flows mainly along the surface of the conductor. This effect is known as **skin effect**.



The reason is that when alternating current flows through a conductor, the flux changes in the inner part of the conductor are higher. Therefore, the inductance of the inner part is higher than that of the outer part. Higher the frequency of alternating current, more is the skin effect. The depth upto which ac current flows through a wire is called **skin depth** (δ).

WATTLESS CURRENT ($I_V \sin\phi$)

Since $P_{av} = E_V(I_V \cos\phi)$



i.e. the component $I_V \cos\phi$ contributes to the power whereas $I_V \sin\phi$ does not contribute to the power and hence is called the **Wattless current**.

ILLUSTRATION 28

An LCR series circuit with $100\ \Omega$ resistance is connected to an ac source of 200 V and angular frequency 300 rads^{-1}. When only the capacitance is removed, the current lags behind the voltage by $60°$. When only the inductance is removed, the current leads the voltage by $60°$. Calculate the current and the power dissipated in the LCR circuit.

SOLUTION

When capacitance is removed,

$$\tan\phi = \frac{X_L}{R}$$

$$\Rightarrow \quad \tan 60° = \frac{X_L}{R}$$

$$\Rightarrow \quad X_L = \sqrt{3}R \quad \ldots(1)$$

When inductance is removed, $\tan\phi = \dfrac{X_C}{R}$

$$\Rightarrow \quad \tan 60° = \frac{X_C}{R}$$

$$\Rightarrow \quad X_C = \sqrt{3}R \quad \ldots(2)$$

From equations (1) and (2) we get

$$X_L = X_C$$

So, the LCR circuit is in resonance and hence

$$Z = R$$

$$\Rightarrow \quad I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{200}{100} = 2 \text{ A}$$

$$\Rightarrow \quad \langle P \rangle = V_{\text{rms}} I_{\text{rms}} \cos\phi$$

At resonance current and voltage are in phase i.e., $\phi = 0°$

$$\Rightarrow \quad \langle P \rangle = (200)(2)(1) = 400 \text{ W}$$

ILLUSTRATION 29

A 750 Hz, 20 V source is connected to a resistance of $100\ \Omega$, an inductance of 180 mH and a capacitance of 10 μF all in series. Calculate the time in which the resistance having a thermal capacity $2\ \text{J}\ °\text{C}^{-1}$, will get heated by $10\ °\text{C}$.

SOLUTION

The impedance of the circuit,

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\Rightarrow \quad Z = \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}$$

Now, $X_L = 2\pi fL = 2 \times 3.14 \times 750 \times 0.18 \approx 848\ \Omega$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2 \times 3.14 \times 750 \times 10^{-5}} \approx 21\ \Omega$$

$$\Rightarrow \quad Z = \sqrt{(100)^2 + (848 - 21)^2} = \sqrt{(100)^2 + (827)^2}$$

$$\Rightarrow \quad Z = 833\ \Omega$$

In case of an ac,

$$\langle P \rangle = V_{\text{rms}} I_{\text{rms}} \cos\phi$$

$$\Rightarrow \quad \langle P \rangle = (V_{\text{rms}})\left(\frac{V_{\text{rms}}}{Z}\right)\left(\frac{R}{Z}\right) = \left(\frac{V_{\text{rms}}}{Z}\right)^2 R$$

$$\Rightarrow \quad \langle P \rangle = \left(\frac{20}{833}\right)^2 \times 100 = 0.058 \text{ Js}^{-1}$$

Now, $\langle P \rangle \times t = (mc)\Delta T$

where, $mc =$ heat capacity of wire $= 2\ \text{J}(°\text{C})^{-1}$.

$$\Rightarrow \quad t = \frac{(mc)\Delta T}{\langle P \rangle} = \frac{2 \times 10}{0.058} = 345 \text{ s}$$

ILLUSTRATION 30

A resistor R, inductor L, and capacitor C are connected in series to an AC source of rms voltage V and variable frequency. Find the energy that is delivered to the circuit during one period if the operating frequency is twice the resonance frequency.

SOLUTION

Since, at resonance, the angular frequency ω_0 is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Now, when $\omega = 2\omega_0 = \frac{2}{\sqrt{LC}}$, then

$$X_L = L\omega = \frac{2L}{\sqrt{LC}} = 2\sqrt{\frac{L}{C}} \text{ and}$$

$$X_C = \frac{1}{C\omega} = \frac{\sqrt{LC}}{2C} = \frac{1}{2}\sqrt{\frac{L}{C}}$$

Since by definition, we have

$$Z^2 = (X_L - X_C)^2 + R^2$$

$$\Rightarrow \quad Z^2 = R^2 + 2.25\left(\frac{L}{C}\right)$$

Further, we know that

$$P = \left(\frac{V^2}{Z}\right)\cos\phi = \left(\frac{V^2}{Z}\right)\left(\frac{R}{Z}\right) = \frac{V^2 R}{Z^2}$$

$$\Rightarrow \quad P = \frac{V^2 R}{R^2 + 2.25\left(\frac{L}{C}\right)}$$

So, energy delivered to the circuit in one cycle is

$$E = PT = P\left(\frac{2\pi}{\omega}\right) = \left(\frac{2\pi RCV^2}{R^2C + 2.25L}\right)\frac{\sqrt{LC}}{2}$$

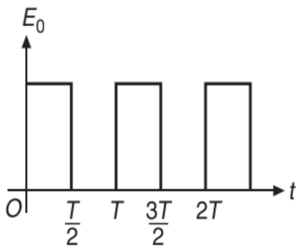
$$\Rightarrow \quad E = \frac{4\pi RCV^2\sqrt{LC}}{4R^2C + 9L}$$

Test Your Concepts-I

Based on AC

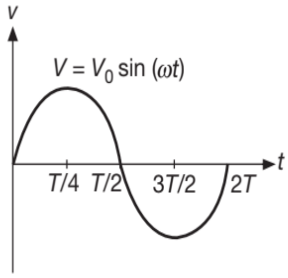
(Solutions on page H.227)

1. Show that average heat produced during a cycle of ac is same as produced by dc with $I = I_{rms}$.
2. Calculate the reading which will be given by a hot-wire voltmeter if it is connected across the terminals of a generator whose voltage waveform is represented by $V = 200\sin(\omega t) + 100\sin(3\omega t) + 50\sin(5\omega t)$.
3. The current flowing through a resistor R varies as $i = kt$, where k is constant. Calculate the rms current for first 2 seconds.
4. Calculate the RMS value of the voltage (for a cycle) whose time variation is shown in Figure.



5. An AC voltage is given by $V = V_0 + V_1 \cos\omega t$. Calculate the rms value for one cycle.
6. A $100\ \Omega$ resistance is connected in series with a 4 H inductor. The voltage across the resistor is, $V_R = (2\ V)\sin\left((10^3\ \text{rads}^{-1})t\right)$
   (a) Find the expression of circuit current
   (b) Find the inductive reactance
   (c) Derive an expression for the voltage across the inductor.
7. A 200 V, 50 Hz AC is connected to a circuit of resistance 1Ω and inductance 0.01 H. What is the phase difference between the current and the emf in the circuit? Also find the virtual current in the circuit.
8. A sinusoidal voltage of frequency 60 Hz and peak value 150 V is applied to a series LR circuit, where $R = 20\ \Omega$ and $L = 40$ mH. Calculate the values of
   (a) T, ω, X_L, Z and ϕ
   (b) current amplitude, $(V_R)_{max}$ and $(V_L)_{max}$.
9. A choke coil is needed to operate an arc lamp at 160 V (r.m.s.) and 50 Hz. The lamp has an effective resistance of $5\ \Omega$ when running at 10 A (r.m.s.). Calculate the inductance of the choke coil. If the same arc lamp is to be operated on 160 V (dc), what additional resistance is required? Compare the power loses in both cases.
10. A $300\ \Omega$ resistor is connected in series with a 0.8 H inductor. The voltage across the resistor as a function of time is
    $$V_R = (2.5\ V)\cos\left[(950\ \text{rads}^{-1})t\right].$$
    (a) Derive an expression for the circuit current.
    (b) Determine the inductive reactance of the inductor.
    (c) Derive an expression for the voltage V_L across the inductor.
11. A $300\ \Omega$ resistor, a 250 mH inductor, and a $8\ \mu F$ capacitor are in series with an ac source with voltage amplitude 120 V and angular frequency 400 rads^{-1}.
    (a) What is the current amplitude?
    (b) What is the phase angle of the source voltage with respect to the current? Does the source voltage lag or lead the current?
    (c) What are the voltage amplitudes across the resistor, inductor, and capacitor?
12. **(a)** What is the reactance of a 2 H inductor at a frequency of 50 Hz?
    (b) What is the inductance of an inductor whose reactance is $2\ \Omega$ at 50 Hz?
    (c) What is the reactance of a $2\ \mu$F capacitor at a frequency of 50 Hz?
    (d) What is the capacitance of a capacitor whose reactance is $2\ \Omega$ at 50 Hz?
13. An ac circuit consists of a $220\ \Omega$ resistance and a 0.7 H choke. Find the power absorbed from 220 V and 50 Hz source connected in this circuit if the resistance and choke are joined in
    (a) series
    (b) parallel
14. In an LR series circuit, a sinusoidal voltage $V = V_0 \sin(\omega t)$ is applied. It is given that $L = 35$ mH, $R = 11\Omega$, $V_{rms} = 220$ V, $\frac{\omega}{2\pi} = 50$ Hz and $\pi = \frac{22}{7}$. Find the amplitude of current in the steady state and obtain the phase difference between the

current and the voltage. Also plot the variation of current for one cycle on the given graph.



15. In an RLC series circuit that includes a source of alternating current operating at fixed frequency and voltage, the resistance R is equal to the inductive reactance. If the plate separation of the capacitor is reduced to half of its original value, the current in the circuit doubles. Find the initial capacitive reactance in terms of R.

16. A voltage $V = 100\sin(\omega t)$ (in SI units) is applied across a series combination of a 2 H inductor, a $10\ \mu\text{F}$ capacitor, and a $10\ \Omega$ resistor.
- **(a)** Determine the angular frequency ω_0 at which the power delivered to the resistor is a maximum.
- **(b)** Calculate the power delivered at that frequency.
- **(c)** Determine the two angular frequencies ω_1 and ω_2 at which the power is half the maximum value.

17. Voltage and current for a circuit with two elements in series are expressed as follows:

$$V(t) = 170\sin\left(6280t + \frac{\pi}{3}\right) \text{ volt}$$

$$I(t) = 8.5\sin\left(6280t + \frac{\pi}{2}\right) \text{ ampere}$$

- **(a)** Determine the frequency in Hz.
- **(b)** Determine the power factor stating its nature.
- **(c)** What are the values of the elements?

18. A capacitor of capacitance 250 pF is connected in parallel with a choke coil having inductance of $1.6\times10^{-2}\text{ H}$ and resistance $20\ \Omega$. Calculate the
- **(a)** resonance frequency and
- **(b)** circuit impedance at resonance

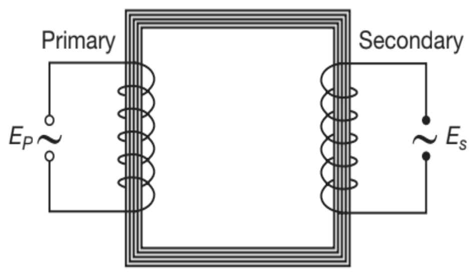
19. A radar transmitter contains an LC circuit oscillating at 1000 MHz.
- **(a)** What capacitance will resonate with a one-turn loop of inductance 400 pH at this frequency?
- **(b)** If the capacitor has square parallel plates separated by 1 mm of air, what should the edge length of the plates be?
- **(c)** What is the common reactance of the loop and capacitor at resonance? Take $\pi^2 = 10$.

20. Two LC circuits have same resonant frequency f_0. When connected in series (assume all the elements to be in series), find the new resonant frequency.

TRANSFORMER

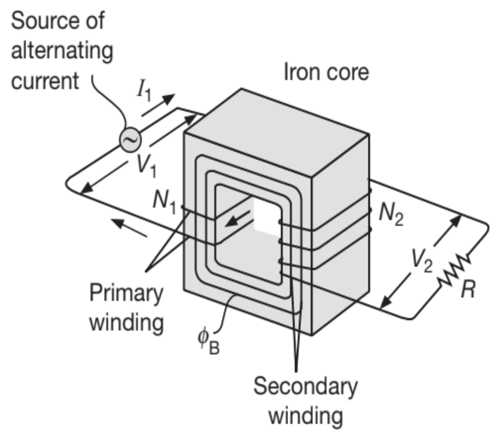
A transformer is a device for converting high voltage at low current to low voltage at high current and vice-versa.

It consists of two coils wound on a soft iron core. The primary coil is connected to an a.c. source. The secondary coil is connected to the load which may be a resistor or any other electrical device.

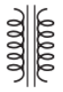


Construction

Figure shows an idealized transformer. The key components of the transformer are two coils or windings, electrically insulated from each other but wound on the same core.



The core is typically made of a material, such as iron, with a very large relative permeability K_m. This keeps the magnetic field lines due to a current in one winding almost completely within the core. Hence almost all of these field lines pass through the other winding, maximizing the mutual inductance of the two windings. The winding to which power is supplied is called the primary; the winding from which power is delivered is called the secondary. The circuit symbol for a transformer with an iron core, such as those used in power distribution systems, is shown here



Types

There are two types of transformers.

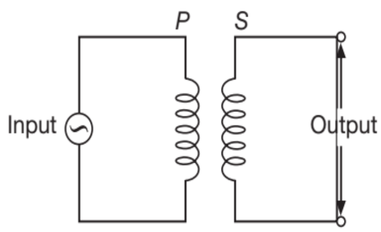
Step Up Transformer

It converts low voltage at high current into high voltage at low current.

Step Down Transformer

It converts high voltage at low current into low voltage at high current.

The principle of transformer is based on mutual induction and a transformer always works on AC. The input is applied across primary terminals and output across secondary terminals. If the resistance of primary coil is zero then voltage across the primary is equal to the applied voltage.



$P \rightarrow$ Primary Coil
$S \rightarrow$ Secondary Coil

The ratio of number of turns in secondary and primary is called the **turn ratio.** So,

$$\text{Turn Ratio } N = \frac{N_S}{N_P}$$

If E_P and E_S are alternating voltages, I_P and I_S the alternating currents, ϕ_P and ϕ_S be the magnetic flux across primary and secondary terminals respectively, then

For an Ideal Transformer (Efficiency = 1) we have

$$\eta = 1 = \frac{\text{Output Power}}{\text{Input Power}} = \frac{\xi_S I_S}{\xi_P I_P}$$

$$\Rightarrow \quad \frac{\xi_S}{\xi_P} = \frac{I_P}{I_S}$$

Also, we observe that

$$\frac{\xi_S}{\xi_P} = \frac{\phi_S}{\phi_P} = \frac{N_S}{N_P}$$

So, for an Ideal Transformer

$$\frac{\xi_S}{\xi_P} = \frac{\phi_S}{\phi_P} = \frac{N_S}{N_P} = \frac{I_P}{I_S} \quad \ldots(1)$$

Since $\xi_S I_S = \xi_P I_P$

The value of the load resistance (across which output is to be taken), determines the value of current in the secondary coil. So,

$$I_S = \frac{\xi_S}{R_L} \quad \ldots(2)$$

Also, the current in the primary is

$$I_P = \frac{\xi_P}{R_{eq}} \quad \ldots(3)$$

From (2) and (3), we get

$$\frac{I_P}{I_S} = \frac{\xi_P}{\xi_S}\frac{R_L}{R_{eq}}$$

From (1), we get

$$\frac{N_S}{N_P} = \frac{N_P}{N_S}\left(\frac{R_L}{R_{eq}}\right)$$

$$\Rightarrow \quad R_{eq} = \left(\frac{N_P}{N_S}\right)^2 R_L$$

From this analysis, we observe that a transformer may be used to match resistances between the primary circuit and the load. In this way, maximum power transfer can be achieved between a given power source and the load resistance. In other words, we can say that a transformer 'transforms' not only voltages and currents, but resistances as well.

For a non-ideal transformer (with efficiency $\eta < 1$)

we have $\eta = \dfrac{\xi_S I_S}{\xi_P I_P}$

$$\frac{\xi_S}{\xi_P} = \eta \frac{I_P}{I_S}$$

So, we get

$$\frac{\xi_S}{\xi_P}=\frac{N_S}{N_P}=\frac{\phi_S}{\phi_P}=\frac{\eta I_P}{I_S}$$

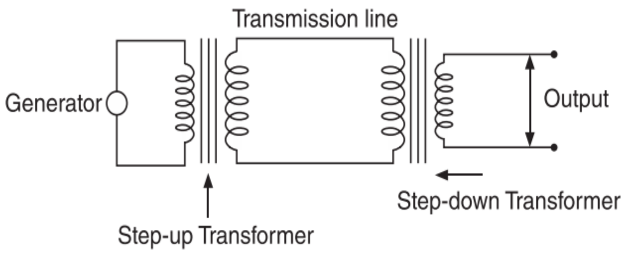
In actual transformers, there is some power loss. The main sources of power loss are:

(a) I^2R loss due to Joule heat in copper windings.
(b) Heating produced due to Eddy currents in the iron core. This is reduced by using laminated core.
(c) Hysteresis loss due to repeated magnetisation of the iron core.
(d) Loss due to flux leakage.

When all the losses are minimized, the efficiency of the transformer becomes very high (90-99%).

LONG DISTANCE TRANSMISSION OF ELECTRIC POWER

One of the great advantages of AC over DC for electric power distribution is that it is much easier to step voltage levels up and down with AC than with DC. For long-distance power transmission it is desirable to use as high a voltage and as small a current as possible; this reduces I^2R losses in the transmission lines, and smaller wires can be used, saving on material costs. To minimize the I^2R loss during long distance transmission of electric power, the voltage is first stepped up to a high value using a transformer.



Since $P=VI$ is constant, as V is increased I decreases and hence I^2R loss becomes small. By increasing V to a very high value, this power loss can be made negligible.

INDUCTION COIL

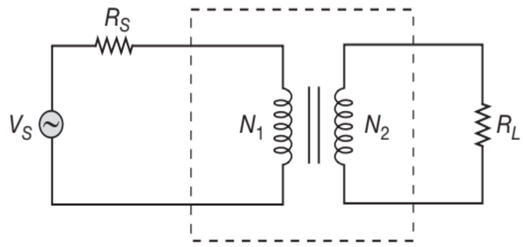
Induction coil is based on mutual induction and is used to produce a larger e.m.f. from a source of low e.m.f.

It has two coils, Primary and Secondary. When current in primary coil changes, the magnetic flux linked with the secondary also changes thus setting up an induced e.m.f. in the secondary. During self-induction of primary the rate of growth of current is slow but when the circuit breaks rate of change of current $\left|\frac{dI}{dt}\right|$ in the secondary is very large and so induced e.m.f. in the secondary is also very large.

Induction coil can be used to produce an e.m.f. of the order of 50,000 V from a 12 V battery. Induction coil is often used to operate a discharge tube.

ILLUSTRATION 31

In the transformer shown in figure, the load resistor is $50\,\Omega$. The turns ratio $N_1:N_2$ is $10:2$ and the source voltage is 200 V (r.m.s.). If a voltmeter across the load measures 25 V (r.m.s.), what is the source resistance R_S?



SOLUTION

The r.m.s. voltage across the primary of the transformer is given by

$$V_P=V_1=\left(\frac{N_1}{N_2}\right)V_2$$

So, the source voltage, if current in the primary is I_1 is

$$V_s=I_1R_s+V_P$$

$$\Rightarrow \quad V_s=I_1R_s+\left(\frac{N_1}{N_2}\right)V_2$$

The current in the secondary coil is I_2, then

$$I_1=\left(\frac{N_2}{N_1}\right)I_2=\left(\frac{N_2}{N_1}\right)\left(\frac{V_2}{R_L}\right)$$

$$\Rightarrow \quad V_s=\left(\frac{N_2}{N_1}\right)\left(\frac{V_2}{R_L}\right)R_s+\left(\frac{N_1}{N_2}\right)V_2$$

$$\Rightarrow \quad V_s=V_2\left[\left(\frac{N_2}{N_1}\right)\left(\frac{R_s}{R_L}\right)+\frac{N_1}{N_2}\right]$$

$$R_s = \frac{N_1 R_L}{N_2 V_2}\left(V_s - V_2\left(\frac{N_1}{N_2}\right)\right)$$

Since $\frac{N_1}{N_2} = \frac{10}{2}$, $V_s = 200$ V, $R_L = 50\ \Omega$, $V_2 = 25$ V, so

$$R_s = \left(\frac{10}{2}\right)\left(\frac{50}{25}\right)\left(200 - (25)\left(\frac{10}{2}\right)\right)$$

$$\Rightarrow \quad R_s = (10)(200 - 125)$$

$$\Rightarrow \quad R_s = 750\ \Omega$$

ILLUSTRATION 32

An ideal transformer has 50 turns in its primary winding and 25 turns in its secondary winding. If the current in the secondary winding in 4 A, calculate the current in primary winding if a 200 V AC is applied across it.

SOLUTION

Since the ratio of turns in primary to secondary winding is 2:1, so it is a step-down transformer for which, we have

$$\frac{\xi_s}{\xi_p} = \frac{N_s}{N_p}$$

$$\Rightarrow \quad \xi_s = \left(\frac{N_s}{N_p}\right)\xi_p$$

$$\Rightarrow \quad \xi_s = \frac{25}{50} \times 200 = 100 \text{ V}$$

For an ideal transformer, the power equation is

$$\xi_s i_s = \xi_p i_p$$

$$\Rightarrow \quad i_p = \frac{\xi_s i_s}{\xi_p} = \frac{100 \times 4}{200} = 2 \text{ A}$$

ILLUSTRATION 33

An ideal power transformer is used to step up an alternating voltage of 220 V to 4.4 kV for transmitting 0.6 kW of power. If the primary coil has 1000 turns, calculate the number of turns in the secondary coil and the current rating of the secondary coil.

SOLUTION

Since, we know that

$$N_s = \left(\frac{\xi_s}{\xi_p}\right)N_p$$

$$\Rightarrow \quad N_s = \left(\frac{4.4 \times 1000}{220}\right) \times 1000 = 20000 \text{ turns}$$

Power supplied at the primary coil is

$$P_p = i_p \xi_p = 6.6 \times 10^3 \text{ W}$$

$$\Rightarrow \quad i_p = \frac{6.6 \times 10^3}{220} = 30 \text{ A}$$

For an ideal transformer we use

$$\frac{i_s}{i_p} = \frac{N_p}{N_s} = \left(\frac{1000}{20,000}\right) = \frac{1}{20}$$

$$\Rightarrow \quad i_s = \frac{i_p}{20} = 1.5 \text{ A}$$

ILLUSTRATION 34

A transformer has secondary turns to primary turns ratio of 4. If a 200 V AC is applied across its primary and it carries a current of 1 A, then calculate the current in circuit connected to secondary coil if transformer is 80% efficient.

SOLUTION

Power supplied at primary coil is

$$P_1 = \xi_1 i_1 = (200)(1) = 200 \text{ W}$$

Voltage across secondary coil is

$$\xi_2 = \left(\frac{N_2}{N_1}\right)\xi_1 = 200 \times 4 = 800 \text{ V}$$

Power available at the secondary coil is

$$P_2 = \left(\frac{80}{100}\right)200 \text{ W} = 160 \text{ W}$$

$$\Rightarrow \quad P_2 = \xi_2 i_2 = 160 \text{ W}$$

$$\Rightarrow \quad i_2 = \frac{160}{800} = 0.2 \text{ A}$$

ILLUSTRATION 35

In a step-down transformer having primary to secondary turn ratio $20:1$, the input voltage applied is 250 V and output current is 8 A. Assuming 100% efficiency, calculate the voltage across secondary coil, current in primary coil and power output.

SOLUTION

Since we know that

$$\frac{\xi_S}{\xi_P}=\frac{N_S}{N_P}$$

$$\Rightarrow \quad \xi_S=\left(\frac{N_S}{N_P}\right)\xi_P$$

where, $\frac{N_S}{N_P}=\frac{1}{20}$, $\xi_P=250\text{ V}$ and $i_S=8\text{ A}$

$$\Rightarrow \quad \xi_S=\frac{1}{20}(250)=12.5\text{ V}$$

For 100% efficiency, we have

$$\frac{i_P}{i_S}=\frac{N_S}{N_P}$$

$$\Rightarrow \quad i_P=\left(\frac{N_S}{N_P}\right)i_S$$

$$\Rightarrow \quad i_p=\frac{1}{20}\times 8=0.4\text{ A}$$

Power output of transformer is

$$P_{\text{output}}=\xi_S i_S$$

$$\Rightarrow \quad P_{\text{output}}=12.5\times 8=100\text{ W}$$

ILLUSTRATION 36

A $2000\text{ V}-200\text{ V}$, 20 kVA transformer has 66 turns in the secondary. Calculate the primary and secondary full-load current when power losses in the transformer are ignored.

SOLUTION

Primary and secondary coil voltage are

$$\xi_P=2000\text{ V} \text{ and } \xi_S=200\text{ V}$$

Power of the transformer at primary coil is 20 kVA, so we have

$$P_P=\xi_P i_P=20\times 10^3\text{ W}$$

$$\Rightarrow \quad i_P=\frac{20\times 10^3}{2000}=10\text{ A}$$

Since power losses in the transformer are ignored, so power of the transformer at secondary coil is also 20 kVA and hence we have

$$P_S=\xi_S i_S=20\times 10^3\text{ W}$$

$$\Rightarrow \quad i_S=\frac{20\times 10^3}{200}=100\text{ A}$$

ILLUSTRATION 37

A transformer is used to light a 140 W, 24 V lamp from 240 V ac mains. If the current in the mains cable is 0.7 A, calculate the efficiency of transformer.

SOLUTION

After reading the problem, we can understand that the input voltage is 240 V and the output voltage is, hence we have

$$\xi_P=240\text{ V} \text{ and } \xi_S=24\text{ V}$$

Resistance of lamp is

$$R=\frac{V^2}{P}=\frac{(24)^2}{140}=\frac{144}{35}\ \Omega$$

So, current in the secondary coil is

$$i_S=\frac{\xi_S}{R}=\frac{24}{144}\times 35=\frac{35}{6}\text{ A}$$

Power supplied at the input coil i.e. the primary coil is

$$P_P=\xi_P i_P=240\times 0.7=168\text{ W}$$

Power output at the secondary coil is

$$P_S=\xi_S i_S=24\times\frac{35}{6}=140\text{ W}$$

So, efficiency of the transformer is

$$\eta=\frac{\xi_S i_S}{\xi_P i_P}\times 100\%=\frac{148}{168}\times 100\%$$

$$\Rightarrow \quad \eta=83.33\%$$

Test Your Concepts-II

Based on Transformer

(Solutions on page H.231)

1. A transformer has 200 turns in primary coil and 600 turns in secondary coil. If a 220 V DC is applied across primary coil, what will be the voltage across secondary coil.
2. The secondary voltage of an ignition transformer in a furnace is 10 kV. When the primary operates at rms voltage of 120 V, the primary impedance is $24\ \Omega$ and the transformer is 90% efficient. Calculate the
   (a) turns ratio.
   (b) current in the secondary.
   (c) impedance in the secondary.
3. In a transformer there are 10000 turns in primary coil and 25000 turns in secondary coil. An AC of voltage $E = 50\sin(100\pi t)$ is applied across the primary coil, calculate the peak voltage output across secondary coil in ideal conditions.
4. A transmission line that has a resistance per unit length of $5\times10^{-4}\ \Omega\text{m}^{-1}$ is to be used to transmit 5 MW over 600 km. The output voltage of the generator is 4.5 kV.
   (a) What is the line loss if a transformer is used to step up the voltage to 500 kV?
   (b) What fraction of the input power is lost to the line under these circumstances?
   (c) What difficulties will be encountered in attempting to transmit the 5 MW at the generator voltage of 4.5 kV?

SOLVED PROBLEMS

PROBLEM 1

A series LCR circuit with inductance 0.12 H, capacitance 480 nF and resistance $23\ \Omega$ is connected to a 230 V variable frequency ac supply.

(a) Calculate the ac source frequency for which current in the circuit is maximum. Calculate the maximum current amplitude.
(b) Calculate the source frequency for which average power consumed by the circuit is maximum. Calculate the value of maximum power.
(c) For which angular frequencies of the source, the power transferred to the circuit is half of the power at resonant frequency?
(d) Calculate the Q factor of the circuit.

SOLUTION

(a) The current in the circuit is I and is given by

$$I=\frac{E}{\sqrt{R^2+\left(L\omega-\frac{1}{C\omega}\right)^2}}$$

Current in circuit is maximum at resonance i.e. when $X_L=X_C$. If ω_0 be the resonant angular frequency, then we have

$$\omega_0=\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{(0.12)(480\times10^{-9})}}$$

$$\Rightarrow\ \omega_0=4167\text{ rads}^{-1}$$

The resonant frequency f_0 is

$$f_0=\frac{\omega_0}{2\pi}=\frac{4167}{2\times3.14}=663.5\text{ Hz}$$

The maximum current amplitude at resonance is given

$$I_0=\frac{E_0}{R}=\frac{\sqrt{2}E_{\text{rms}}}{R}=\frac{\sqrt{2}\times230}{23}=14.14\text{ A}$$

(b) The average power consumed by the circuit is

$$P=E_{\text{rms}}I_{\text{rms}}\cos\phi$$

At resonance, power factor is maximum, so

$$\cos\phi=1$$

$$\Rightarrow\ P_{\max}=E_{\text{rms}}I_{\text{rms}}=\frac{E_{\text{rms}}^2}{R}$$

$$\Rightarrow\ P_{\max}=\frac{(230)^2}{23}=2300\text{ W}$$

(c) The half power frequencies are the frequencies at which the power in the circuit is half the maximum power. These are given by

$$\omega_{hp}=\omega_0\pm\frac{R}{2L}=4167\pm\left(\frac{23}{2\times0.12}\right)$$

$$\Rightarrow\ \omega_{hp}=4263\text{ rads}^{-1}\text{ and }4071\text{ rads}^{-1}$$

(d) The quality factor of the circuit is given as

$$Q=\frac{L\omega_0}{R}=\frac{(0.12)(4167)}{23}=21.74$$

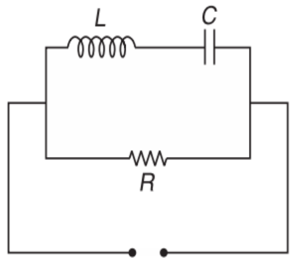
PROBLEM 2

A box contains an inductor, capacitor and a resistor. When 250 V DC voltage is applied to the terminals of the box, a current of 1 A flows in the circuit. When a 250 V, 2250 rads^{-1} AC source is applied across the box, a current of 1.25 A flows through it. It is observed that the current rises with frequency and becomes maximum at 4500 rads^{-1}. Calculate the values of inductance, capacitance and the resistance in the box. Also draw the circuit diagram.

SOLUTION

Let L be the inductance, C be the capacitance and R be the resistance in the circuit. When a dc voltage is applied to the circuit containing L, C and R, then the capacitor acts as a dc blocking element and if all L, C and R are in series then no current should flow in the circuit.

However, we observe a current to flow through the circuit which makes us conclude that the resistor R must have been in parallel to the series combination of L and C as shown in figure.



So, for the dc source, we have

$$R=\frac{E_{\text{dc}}}{I_{\text{dc}}}=\frac{250}{1}=250\ \Omega$$

The impedance Z of the circuit at $2250\ \text{rads}^{-1}$ is

$$\frac{1}{Z}=\sqrt{\frac{1}{R^2}+\frac{1}{\left(L\omega-\frac{1}{C\omega}\right)^2}}$$

Circuit impedance can also be directly given as

$$Z=\frac{V}{i}=\frac{250}{1.25}=200\ \Omega$$

$$\Rightarrow\quad \frac{1}{(200)^2}=\frac{1}{R^2}+\frac{1}{\left(L\omega-\frac{1}{C\omega}\right)^2}$$

$$\Rightarrow\quad \frac{1}{\left(L\omega-\frac{1}{C\omega}\right)^2}=\frac{1}{(200)^2}-\frac{1}{(250)^2}=\frac{9}{(10)^6}$$

$$\Rightarrow\quad L\omega-\frac{1}{C\omega}=\frac{1000}{3}$$

$$\Rightarrow\quad 2250L-\frac{1}{2250C}=\frac{1000}{3} \qquad \ldots(1)$$

At resonance, the frequency is $\omega_0=4500\ \text{rads}^{-1}$

$$\Rightarrow\quad \omega_0^2=\frac{1}{LC}$$

$$\Rightarrow\quad 4500\times4500=\frac{1}{LC}$$

$$\Rightarrow\quad 2250L=\frac{1}{9000C} \qquad \ldots(2)$$

Substituting equation (2) in equation (1), we get

$$\frac{1}{9000C}-\frac{1}{2250C}=\frac{1000}{3}$$

$$\Rightarrow\quad C=10^{-6}\ \text{F}=1\ \mu\text{F}$$

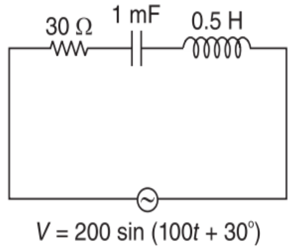
From equation (2), we get

$$2250L=\frac{1}{9000\times10^{-6}}$$

$$\Rightarrow\quad L=\frac{1}{2250\times9000\times10^{-6}}\approx0.05\ \text{H}$$

PROBLEM 3

An LCR circuit is fed with an ac voltage given by $V=200\sin(100t+30°)$ volt as shown in Figure.



Find the current in the circuit, voltage across resistor, capacitor and inductor as a function of time. Also, calculate the average power consumed in the circuit.

SOLUTION

From the voltage equation given in the problem, we conclude that $\omega=100\ \text{rads}^{-1}$ and $V_0=200\ \text{V}$.

The inductive reactance is

$$X_L=\omega L=50\ \Omega$$

The capacitive reactance is

$$X_C=\frac{1}{\omega C}=\frac{1}{100\times10^{-3}}=10\ \Omega$$

The impedance of the circuit is

$$Z=\sqrt{R^2+(X_L-X_C)^2}$$

$$\Rightarrow\quad Z=\sqrt{(30)^2+(50-10)^2}=50\ \Omega$$

Maximum current flowing through the circuit is

$$I_0=\frac{V_0}{Z}=\frac{200}{50}=4\ \text{A}$$

From the calculations done, we see that $X_L>X_C$, therefore the circuit is inductive in nature and hence voltage leads the current by a phase difference ϕ where,

$$\cos\phi=\frac{R}{Z}=\frac{30}{50}=\frac{3}{5}$$

$$\Rightarrow\quad \phi=53°$$

So, we conclude that current in the circuit lags behind the voltage by a phase angle of $\phi=53°$.

The current function is given by

$$I=4\sin(100t+30°-53°)$$

$$\Rightarrow\quad I=4\sin(100t-23°)$$

For Resistor

Maximum value of potential drop across the resistor i.e. $(V_R)_{max}$ is given by

$$(V_R)_{\max} = I_0R = 4\times 30 = 120 \text{ volt}$$

Since, V_R and I are in same phase, so we have

$$V_R = 120\sin(100t - 23°)$$

For Capacitor

Maximum value of potential drop across the capacitor i.e. $(V_C)_{\max}$ is given by

$$(V_C)_{\max} = I_0X_C = 4\times 10 = 40 \text{ volt}$$

Since voltage across the capacitor lags behind the current by a phase angle of $90°$, so we have

$$V_C = 40\sin(100t - 23° - 90°)$$

$$\Rightarrow \quad V_C = 40\sin(100t - 113°)$$

For Inductor

Maximum value of potential drop across the inductor i.e. $(V_L)_{\max}$ is given by

$$(V_L)_{\max} = I_0X_L = 4\times 50 = 200 \text{ volt},$$

Since voltage across the inductor leads the current by a phase angle of $90°$, so we have

Therefore,

$$V_L = 200\sin(100t - 23° + 90°)$$

$$\Rightarrow \quad V_C = 200\sin(100t - 67°)$$

Please note that at any instant in the circuit, we have

$$V = V_R + V_L + V_C$$

Average power is consumed in an AC circuit only across a resistance and this power is given by

$$\langle P\rangle = V_{\text{rms}}I_{\text{rms}}\cos\phi = I_{\text{rms}}^2R = \frac{I_0^2R}{2}$$

$$\Rightarrow \quad \langle P\rangle = \frac{I_0^2R}{2} = \frac{(4)^2(30)}{2} = 240 \text{ W}$$

PROBLEM 4

An AC circuit draws a power of 550 W when connected across a 220 V, 50 Hz ac source. If the power factor of the circuit is 0.8 and the current lags the voltage. For making the power factor of the circuit to be the maximum, calculate the capacitance required to be connected in series with it.

SOLUTION

Since the current lags behind the voltage, so the circuit must be having a resistance and an inductance. Since the average power consumed by the circuit is

$$P_{\text{av}} = E_{\text{rms}}I_{\text{rms}}\cos\phi$$

$$\Rightarrow \quad P = \left(\frac{E_{\text{rms}}^2}{Z}\right)\cos\phi$$

$$\Rightarrow \quad Z = \frac{E_{\text{rms}}^2\cos\phi}{P}$$

$$\Rightarrow \quad Z = \frac{(220)^2\times 0.8}{550} = 70.4\ \Omega$$

Power factor of circuit is given to be 0.8, so we have

$$\cos\phi = \frac{R}{Z}$$

$$\Rightarrow \quad R = Z\cos\phi$$

$$\Rightarrow \quad R = 70.4\times 0.8 = 56.32\ \Omega$$

Also, the initial circuit impedance is

$$Z^2 = R^2 + (\omega L)^2$$

$$\Rightarrow \quad L\omega = \sqrt{Z^2 - R^2}$$

$$\Rightarrow \quad L\omega = \sqrt{(70.4)^2 - (56.32)^2} = 42.24\ \Omega$$

Now, when the capacitor is connected in the circuit for making the power factor to be maximum i.e. $\cos\phi = 1$, then the circuit obeys the condition of resonance i.e. $X_L = X_C$

$$\Rightarrow \quad L\omega = \frac{1}{C\omega}$$

$$\Rightarrow \quad C = \frac{1}{\omega(L\omega)} = \frac{1}{2\pi f(L\omega)}$$

$$\Rightarrow \quad C = \frac{1}{(2\times 3.14\times 50)(42.24)}$$

$$\Rightarrow \quad C \approx 75\times 10^{-6} \text{ F} = 75\ \mu\text{F}$$

PROBLEM 5

A series circuit consists of a resistance, inductance and capacitance. The applied voltage and the current at any instant are given by

$$E = 141.4\cos(3000t - 10°)$$

and $\quad I = 5\cos(3000t - 55°)$

The inductance is 0.01 H. Calculate the values of the resistance and capacitance.

SOLUTION

The phase difference between current and voltage is given as

$$\phi = -10° - (-55°) = 45°$$

This makes us conclude that the voltage is leading the current by a phase angle of $\phi = 45°$

For a series LCR circuit, we have

$$\tan\phi = \frac{L\omega - \frac{1}{C\omega}}{R}$$

$$\Rightarrow \quad \tan 45° = \frac{L\omega - \frac{1}{C\omega}}{R} = 1$$

$$\Rightarrow \quad L\omega - \frac{1}{C\omega} = R \qquad \ldots(1)$$

The impedance Z of the circuit is given by

$$Z = \frac{E_0}{I_0} = \frac{141.4}{5} = 28.28\ \Omega$$

Also, we know that

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\Rightarrow \quad Z = \sqrt{R^2 + R^2} = 1.414R$$

$$\Rightarrow \quad 1.414R = 28.28\ \Omega$$

$$\Rightarrow \quad R = 20\ \Omega \qquad \ldots(2)$$

From equation (1), we get

$$L\omega - \frac{1}{C\omega} = 20$$

$$\Rightarrow \quad (0.01)(3000) - \frac{1}{3000C} = 20$$

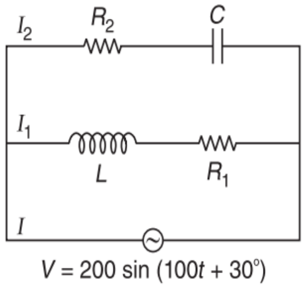
$$\Rightarrow \quad \frac{1}{3000C} = 30 - 20 = 10$$

$$\Rightarrow \quad C = \frac{1}{(3000)(10)} = 33.33 \times 10^{-6}\ \text{F}$$

$$\Rightarrow \quad C = 33.33\ \mu\text{F}$$

PROBLEM 6

Two resistors, a capacitor and an inductor are joined to form a circuit shown in Figure.



An ac source voltage given by $V = 200\sin(100t + 30°)$ is applied to the circuit. If $R_1 = 30\ \Omega$, $R_2 = 40\ \Omega$, $L = 0.4$ H and $C = \frac{1}{3}$ mF, then find I, I_1, I_2, voltage across resistor R_1, voltage across resistor R_2, voltage across inductor and voltage across capacitor as function of time. Also, calculate the average power consumed in the circuit.

SOLUTION

In this problem, let us consider the circuit containing inductor and resistor R_1 (thinking that R_2 and C being absent) and then reconsider the same circuit but now containing R_2 and capacitor (thinking L and R_1 to be absent) as shown in Figures 4.1 and 4.2.

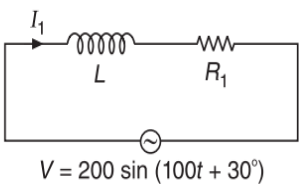


Figure 4.1

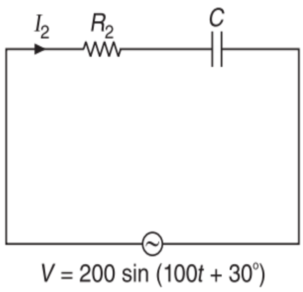


Figure 4.2

Then by Principle of Superposition as already discussed in Current Electricity, we know that

$$I = I_1 + I_2$$

Let us now calculate I_1 (i.e. current in the circuit when R_2 and C are absent).
From the voltage equation given in the problem, we conclude that $\omega = 100$ rads^{-1} and $V_0 = 200$ V.

FOR CIRCUIT IN FIGURE 4.1

Inductive reactance of this circuit is

$$X_L = \omega L = 100 \times 0.4 = 40\ \Omega$$

Since $R_1 = 30\ \Omega$, so the impedance for this circuit is

$$Z_1 = \sqrt{R_1^2 + X_L^2} = \sqrt{(30)^2 + (40)^2} = 50\ \Omega$$

Maximum value of current in this first circuit is

$$(I_1)_{\max} = \frac{V_0}{Z_1} = \frac{200}{50} = 4\text{ A}$$

Since, this first circuit only contains L and R_1, so in this inductive natured circuit, the current will lag behind the voltage by an angle ϕ_1 given by

$$\cos\phi_1 = \frac{R_1}{Z_1} = \frac{30}{50} = \frac{3}{5}$$

$\Rightarrow \quad \phi_1 = 53^\circ$

So, for the applied voltage

$$V = 200\sin(100t + 30^\circ)\text{ volt}$$

the current in this circuit should lag behind the applied voltage by 53° and hence is given by

$\Rightarrow \quad I_1 = 4\sin(100t + 30^\circ - 53^\circ)$

$\Rightarrow \quad I_1 = 4\sin(100t - 23^\circ)$

Maximum value of potential drop across the resistor i.e. $(V_{R_1})_{\max}$ is given by

$$(V_{R_1})_{\max} = (I_1)_{\max} R_1 = (4)(30) = 120\text{ V}$$

However, current in the resistor is in phase with the voltage, so we have

$$V_{R_1} = 120\sin(100t - 23^\circ)$$

Maximum value of potential drop across the inductor i.e. $(V_L)_{\max}$ is given by

$$(V_L)_{\max} = (I_1)_{\max} X_L = (4)(40) = 160\text{ V}$$

However, voltage in an inductor leads the current by 90°, so we have

$\Rightarrow \quad V_L = 160\sin(100t - 23^\circ + 90^\circ)$

$\Rightarrow \quad V_L = 160\sin(100t + 67^\circ)$

In this circuit, power will be consumed only across R_1. If $\langle P_1 \rangle$ is the average power consumed in this circuit, then we have

$$\langle P \rangle = V_{\text{rms}} I_{\text{rms}} \cos\phi = I_{\text{rms}}^2 R = \frac{I_0^2 R}{2}$$

$$\Rightarrow \quad \langle P_1 \rangle = \frac{I_0^2 R_1}{2} = \frac{(4)^2(30)}{2} = 240\text{ W}$$

FOR CIRCUIT IN FIGURE 4.2

Now, let us calculate I_2 (i.e. current in the circuit when R_1 and L are absent).

Capacitive reactance of the circuit is

$$X_C = \frac{1}{\omega C} = \frac{1}{100 \times \frac{1}{3} \times 10^{-3}} = 30\ \Omega$$

Since $R_2 = 40\ \Omega$, so the impedance of this second circuit is

$$Z_2 = \sqrt{R_2^2 + X_C^2} = \sqrt{(40)^2 + (30)^2} = 50\ \Omega$$

Maximum value of current in this second circuit is

$$(I_2)_{\max} = \frac{V_0}{Z_2} = \frac{200}{50} = 4\text{ A}$$

Since, this second circuit only contains R_2 and C, so in this capacitive natured circuit, current leads the voltage by a phase angle ϕ_2 given by

$$\cos\phi_2 = \frac{R_2}{Z_2} = \frac{40}{50} = \frac{4}{5}$$

$\Rightarrow \quad \phi_2 = 37^\circ$

So, for the applied voltage

$$V = 200\sin(100t + 30^\circ)\text{ volt}$$

the current in this circuit should lead the applied voltage by 37° and hence is given by

$\Rightarrow \quad I_2 = 4\sin(100t + 30^\circ + 37^\circ)$

$\Rightarrow \quad I_2 = 4\sin(100t + 67^\circ)$

Maximum value of potential drop across the resistor R_2 is $(V_{R_2})_{\max}$, where

$$(V_{R_2})_{\max} = (I_2)_{\max} R_2 = (4)(40) = 160\text{ V}$$

Since, current in the resistor is in phase with the voltage, so we get

$$V_{R_2} = 160\sin(100t + 67^\circ)$$

Maximum value of potential drop across the capacitor i.e. $(V_C)_{\max}$ is given by

$$(V_C)_{\max} = (I_2)_{\max} X_C = (4)(30) = 120\text{ V}$$

Since, voltage in a capacitor lags behind the current by 90°, so we have

$$V_C = 120\sin(100t + 67° - 90°)$$

$$\Rightarrow \quad V_C = 120\sin(100t - 23°)$$

In this circuit, power will be consumed only across R_2. If $\langle P_2 \rangle$ is the average power consumed in this circuit, then we have

$$\langle P \rangle = V_{rms} I_{rms} \cos\phi = I_{rms}^2 R = \frac{I_0^2 R}{2}$$

$$\Rightarrow \quad \langle P_2 \rangle = \frac{I_0^2 R_2}{2} = \frac{(4)^2 (40)}{2} = 320 \text{ W}$$

Total average power consumed in the circuit is

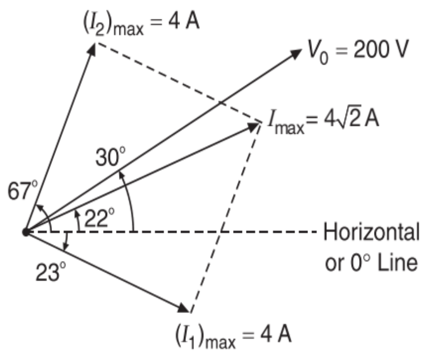
$$\langle P \rangle = \langle P_1 \rangle + \langle P_2 \rangle = 240 + 320 = 560 \text{ W}$$

The current I is given by

$$I = I_1 + I_2$$

$$\Rightarrow \quad I = 4\sin(100t - 23°) + 4\sin(100t + 67°)$$

The current amplitude can be found by drawing the phasor shown in Figure.



Resultant of 4 A and 4 A at 90° is $4\sqrt{2}$ A at 45° from both currents or at 22° with the horizontal i.e. the $100t$ line.

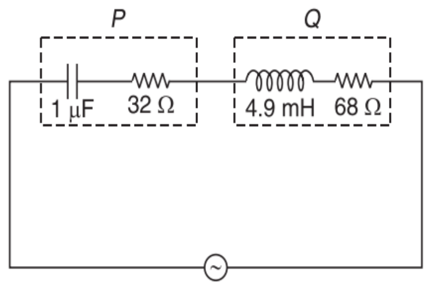
$$I = 4\sqrt{2}\sin(100t + 22°)$$

PROBLEM 7

A box P and a coil Q are connected in series with a 100 V ac source of variable frequency. The box P contains a $32\,\Omega$, $1\,\mu$F capacitor and the coil Q is a $68\,\Omega$, 4.9 mH inductor. The frequency is adjusted so that the maximum current flows both through P and Q. Calculate the impedance of P and Q at this frequency. Also find the voltage across P and Q respectively.

SOLUTION

The situation described in the problem is shown in Figure.



The current flowing in a series LCR circuit is

$$I_{rms} = \frac{E_{rms}}{Z} = \frac{E_{rms}}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

where $R = R_P + R_Q = 100\,\Omega$

The maximum current flows through P and Q when the circuit is in resonance i.e. when $X_L = X_C$

$$\Rightarrow \quad L\omega = \frac{1}{C\omega}$$

$$\Rightarrow \quad \omega^2 = \frac{1}{LC}$$

$$\Rightarrow \quad \omega = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \quad \omega = \frac{1}{\sqrt{(4.9\times 10^{-3})(10^{-6})}} = \frac{10^5}{7} \text{ rads}^{-1}$$

The current flowing in the circuit is

$$I_{max} = \frac{E_{rms}}{R_P + R_Q} = \frac{100}{32+68} = \frac{100}{100} = 1 \text{ A}$$

The impedance of P is

$$Z_P = \sqrt{R_P^2 + X_C^2} = \sqrt{R_1^2 + \left(\frac{1}{C\omega}\right)^2}$$

$$\Rightarrow \quad Z_P = \sqrt{(32)^2 + \left(\frac{7}{10^5} \times \frac{1}{10^{-6}}\right)^2}$$

$$\Rightarrow \quad Z_p = \sqrt{(32)^2 + (70)^2}$$

$$\Rightarrow \quad Z_p = \sqrt{1024 + 4900} = \sqrt{5924} \approx 77\,\Omega$$

The impedance of Q is

$$Z_Q = \sqrt{R_Q^2 + L^2\omega^2}$$

$$\Rightarrow \quad Z_Q = \sqrt{(68)^2 + \left(4.9\times10^{-3}\times\frac{10^5}{7}\right)^2}$$

$$\Rightarrow \quad Z_Q = \sqrt{(68)^2+(70)^2} = \sqrt{9524} \approx 98\ \Omega$$

Voltage across P is

$$V_P = I_{\text{rms}} Z_P = 1\times76 = 76\ \text{V}$$

Voltage across Q is given as

$$V_Q = I_{\text{rms}} Z_Q = 1\times98 = 98\ \text{V}$$

PROBLEM 8

A $10\ \Omega$ resistor is joined in series with a 0.5 H inductor. What capacitance should be connected in series with the combination to obtain the maximum current? Find the maximum current. Assuming that the current is being supplied by a 200 V, 50 Hz ac source, calculate the potential difference across the resistor, inductor and capacitor.

SOLUTION

The current in the circuit is maximum at resonance i.e. when

$$L\omega = \frac{1}{C\omega}$$

$$\Rightarrow \quad C = \frac{1}{\omega^2 L}$$

$$\Rightarrow \quad C = \frac{1}{(2\pi f)^2 L} = \frac{1}{(2\times3.14\times50)^2\times0.5}$$

$$\Rightarrow \quad C = 20.24\times10^{-6}\ \text{F}$$

At resonance, impedance is minimum, circuit is resistive in nature and current is maximum, so

$$Z_{\text{min}} = R = 10\ \Omega$$

$$\Rightarrow \quad I_{\text{max}} = \frac{E}{R} = \frac{200}{10} = 20\ \text{A}$$

Potential difference across resistance

$$E_R = IR = 20\times10 = 200\ \text{V}$$

Potential difference across inductor

$$E_L = I(L\omega) = I(2\pi fL)$$

$$\Rightarrow \quad E_L = 20[2\pi(50)(0.5)] = 3142\ \text{V}$$

Potential difference across capacitor

$$E_C = IX_C = \frac{I}{C\omega} = IL\omega = 3142\ \text{V}$$

PROBLEM 9

A 20 V, 5 W lamp is applied across a 200 V, 50 Hz AC source.

(a) Calculate the capacitance to be connected in series with the lamp to run the lamp at its peak brightness.
(b) Calculate the inductance to be connected in series with the lamp to run the lamp at its peak brightness.
(c) How much additional resistance can be connected in series with the lamp (as a replacement to inductor or a capacitor) so that the lamp can run on its peak brightness?
(d) Which of the above arrangements will be more economical and why?

SOLUTION

The current required by the lamp (a resistor) to run at its peak brightness is

$$I_{\text{rms}} = \frac{P}{E_{\text{rms}}} = \frac{5}{20} = 0.25\ \text{A}$$

The resistance of the lamp is

$$R = \frac{E_{\text{rms}}}{I_{\text{rms}}} = \frac{20}{0.25} = 80\ \Omega$$

(a) When a capacitor C is connected in series with the lamp, then its impedance is

$$Z = \sqrt{R^2 + \left(\frac{1}{C\omega}\right)^2}$$

The current through the circuit is

$$I_{\text{rms}} = \frac{E_{\text{rms}}}{Z} = \frac{200}{\sqrt{R^2+\left(\frac{1}{C\omega}\right)^2}} = 0.25\ \text{A}$$

$$\Rightarrow \quad \frac{200}{\sqrt{(80)^2 + \frac{1}{4\pi^2(50)^2C^2}}} = 0.25$$

$$\Rightarrow \quad C = 4.0\times10^{-6}\ \text{F} = 4.0\ \mu\text{F}$$

(b) When an inductance L is connected in series with the lamp, then the impedance is

$$Z = \sqrt{R^2 + L^2\omega^2}$$

The current through the circuit is

$$I_{\text{rms}} = \frac{E_{\text{rms}}}{Z} = \frac{200}{\sqrt{R^2 + L^2\omega^2}} = 0.25 \text{ A}$$

$$\Rightarrow \frac{200}{\sqrt{(80)^2 + 4\pi^2(50)^2 L^2}} = 0.25$$

$$\Rightarrow L = 2.53 \text{ H}$$

(c) When an additional resistance r replaces the inductor or capacitor and it is connected in series with the lamp of resistance, then the current through the lamp resistor circuit is

$$I_{\text{rms}} = \frac{200}{R+r} = 0.25 \text{ A}$$

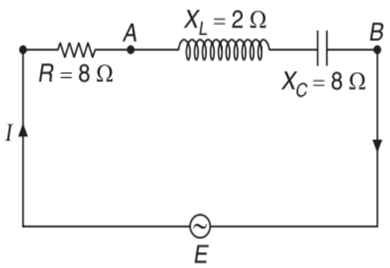
$$\Rightarrow \frac{200}{80+r} = 0.25$$

$$\Rightarrow r = 720\ \Omega$$

(d) It will be more economical to use inductor or capacitor in series with the lamp to run it, because the reactive components of circuit (i.e. inductor or capacitor) do not consume any power, whereas there would be dissipation of power when resistance is inserted in series with the lamp.

PROBLEM 10

An ac source $E = 10\cos(100\pi t)$ is applied across the circuit shown in Figure.



Calculate the circuit impedance. When the potential difference across the points A and B i.e. V_{AB} is half of the voltage of source, then calculate V_{AB}.

SOLUTION

Circuit impedance is given as

$$\Rightarrow \quad Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(8)^2 + (6)^2}$$

$$\Rightarrow \quad Z = 10\ \Omega$$

Since $X_L < X_C$, so current will lead the voltage by a phase angle

$$\phi = \tan^{-1}\left(\frac{6}{8}\right) = 37^\circ$$

So, at any instant, current in the circuit is given by

$$I = I_0 \cos(100\pi t + 37^\circ)$$

where, $I_0 = \dfrac{E_0}{Z} = 1$ A

$$\Rightarrow \quad I = \cos(100\pi t + 37^\circ) \qquad \text{...(1)}$$

Since between the points A and B, only capacitor and inductor are connected and $X_C > X_L$, so V_{AB} must lag behind the current by 90° which is possible only when we have potential difference across points is given as

$$V_{AB} = 6\cos(100\pi t - 53^\circ) \qquad \text{...(2)}$$

Please note that from equations (1) and (2), we see that I is leading V_{AB} by a phase angle of 90°.
Now, according to the problem, we have

$$V_{AB} = \frac{1}{2}E$$

$$\Rightarrow \quad 6\cos(100\pi t - 53^\circ) = 5\cos 100\pi t \qquad \text{...(3)}$$

$$\Rightarrow \quad 6\left[\frac{3}{5}\cos(100\pi t) + \frac{4}{5}\sin(100\pi t)\right] = 5\cos(100\pi t)$$

$$\Rightarrow \quad \frac{24}{5}\sin(100\pi t) = \frac{7}{5}\cos(100\pi t)$$

$$\Rightarrow \quad \tan(100\pi t) = \frac{7}{24}$$

Since we know that, $1 + \tan^2\theta = \sec^2\theta = \dfrac{1}{\cos^2\theta}$

$$\Rightarrow \quad \cos(100\pi t) = \frac{24}{25}$$

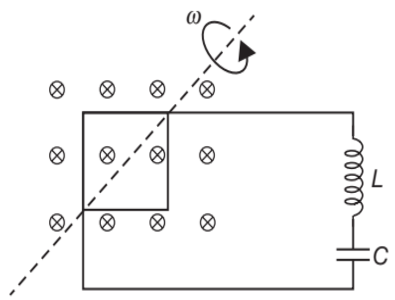
From equation (3), we get

$$V_{AB} = 5\cos(100\pi t) = 5\left(\frac{24}{25}\right) \text{ V}$$

$$\Rightarrow \quad V_{AB} = \frac{24}{5} \text{ V}$$

PROBLEM 11

In the given arrangement the square loop of area $10\ \text{cm}^2$ rotates with an angular velocity ω about its diagonal. The loop is connected to a inductance of $L = 100\ \text{mH}$. I and a capacitance of 10 mF in series. The lead wires have a net resistance of $10\ \Omega$. Given that $B = 0.1\ \text{T}$ and $\omega = 63\ \text{rads}^{-1}$, find the



(a) rms current in the circuit.
(b) energy dissipated in 50 sec.
(c) if the current is in phase with voltage, what should be the frequency of rotation of the coil.

SOLUTION

Since, $\phi = BA\cos(\omega t)$

$$\Rightarrow \quad \xi = -\frac{d\phi}{dt} = BA\omega\sin(\omega t)$$

(a) The current in the circuit is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z}, \text{ where } V_{\text{rms}} = \frac{\xi_0}{\sqrt{2}} = \frac{BA\omega}{\sqrt{2}}$$

$$\Rightarrow I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{\dfrac{BA\omega}{\sqrt{2}}}{\sqrt{\left(\omega L - \dfrac{1}{\omega C}\right)^2 + R^2}}$$

$$\Rightarrow I_{\text{rms}} = \frac{(0.1)(10^{-3})(63)/\sqrt{2}}{\sqrt{\left(63\times 0.1 - \dfrac{1}{63\times 0.01}\right)^2 + (10)^2}}$$

$$\Rightarrow I_{\text{rms}} = 4\times 10^{-4}\ \text{A}$$

(b) The energy dissipated in time interval t is

$$E = P_{av}t = V_{\text{rms}}I_{\text{rms}}t\cos\phi$$

$$\Rightarrow E = \left(\frac{B^2A^2\omega^2}{2Z}\right)\left(\frac{R}{Z}\right)(50)$$

$$\Rightarrow E = \frac{(0.1)^2(10^{-3})^2(63)^2(10)(50)}{2\left[\left(63\times 0.1 - \dfrac{1}{63\times 0.01}\right)^2 + (10)^2\right]}$$

$$\Rightarrow E = 8.12\times 10^{-5}\ \text{J}$$

(c) Since the current is in phase with the voltage which happens only at resonance, so we have

$$X_L = X_C$$

$$\Rightarrow L\omega_0 = \frac{1}{C\omega_0}$$

$$\Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

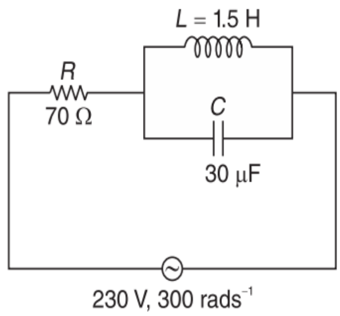
$$\Rightarrow \omega_0 = \frac{1}{\sqrt{(0.1)(0.01)}} = 31.6\ \text{rads}^{-1}$$

PROBLEM 12

A circuit with $R = 70\ \Omega$ in series with a parallel combination of $L = 1.5\ \text{H}$ and $C = 30\ \mu\text{F}$ is driven by a $230\ \text{V}$ ac supply of angular frequency $300\ \text{rads}^{-1}$. Calculate the impedance of the circuit. Find the RMS value of current. Calculate the current amplitude in the L and C arms of the circuit. How will the circuit behave when $\omega = \dfrac{1}{\sqrt{LC}}$?

SOLUTION

The situation described in the problem is shown in Figure.



In the given circuit, the inductor and capacitor are connected in parallel. Let Z' be their complex ac impedance. Then

$$\frac{1}{Z'} = \frac{1}{Z_L} + \frac{1}{Z_C}, \text{ where}$$

$$Z_L = j(L\omega) \text{ and } Z_C = -\frac{j}{C\omega} = \frac{1}{jC\omega}$$

$$\Rightarrow \quad \frac{1}{Z'} = \frac{1}{jL\omega} + \frac{1}{\frac{1}{j\omega C}} = \frac{1}{j\omega L} + j\omega C$$

$$\Rightarrow \quad \frac{1}{Z'} = \frac{1-\omega^2 LC}{jL\omega} \qquad \{\because j^2 = -1\}$$

$$\Rightarrow \quad Z' = \frac{jL\omega}{1-\omega^2 LC}$$

The total complex ac impedance of the circuit is given by

$$Z_{net} = Z = R + Z'$$

$$\Rightarrow \quad Z = R + \frac{j\omega L}{1-\omega^2 LC}$$

$$\Rightarrow \quad |Z| = \sqrt{R^2 + \left(\frac{\omega L}{1-\omega^2 LC}\right)^2}$$

Substituting these values and solving, we get

$$|Z| = 163.3\ \Omega$$

$$\Rightarrow \quad I_{rms} = \frac{E_{rms}}{Z} = \frac{230\ \text{V}}{163.3\ \Omega} = 1.41\ \text{A}$$

Let I_L and I_C be the rms values of current in L and C respectively.

$$I_L = \left(\frac{Z_C}{Z_L + Z_C}\right) I_{rms} \text{ and } I_C = \left(\frac{Z_L}{Z_L + Z_C}\right) I_{rms}$$

where, $Z_L = jL\omega$ and $Z_C = \frac{1}{jC\omega} = -jC\omega$

$$\Rightarrow \quad I_L = \left(\frac{1}{I-\omega^2 LC}\right) I_{rms},\ I_C = \left(\frac{\omega^2 LC}{\omega^2 LC - 1}\right) I_{rms}$$

Substituting the values and solving, we get

$$I_L = 0.462\ \text{A} \text{ and } I_C = 1.87\ \text{A}$$

So, the corresponding current amplitudes are

$$(I_L)_0 = \sqrt{2} \times 0.462 = 1.414 \times 0.462 = 0.653\ \text{A}$$

and $(I_C)_0 = \sqrt{2} \times 1.87 = 1.414 \times 1.87 = 2.64\ \text{A}$

When $\omega = \frac{1}{\sqrt{LC}}$, then $\omega^2 LC = 1$ and hence

$$Z = R + \frac{j\omega L}{1-\omega^2 LC} \to \infty$$

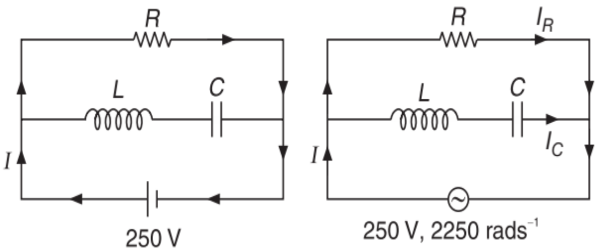
So, the current flowing in the circuit is zero.

PROBLEM 13

A box contains an inductor of inductance L, a capacitor of capacitance C and a resistor of resistance R. When a $250\ \text{V}$ dc is applied to the terminals of the box, a current of $1\ \text{A}$ flows in the circuit. When a $250\ \text{V}$, $2250\ \text{rads}^{-1}$ ac source is connected, a current of $1.25\ \text{A}$ flows. It is observed that the current rises with frequency and becomes maximum at $4500\ \text{rads}^{-1}$. Find the values of L, C and R. Also draw the circuit diagram.

SOLUTION

Since, the capacitor is a dc blocking element, but still the current in the circuit for a dc source is not zero so, all the elements must not be series. Further in case of ac, current rises with frequency and has a maximum value at $4500\ \text{rads}^{-1}$, L and C should be in series. The circuit diagram should therefore, be as shown in Figure.



CASE I: When dc is applied   CASE II: When ac is applied

CASE-1:

When dc is applied, we have

$$R = \frac{V}{I} = \frac{250}{1} = 250\ \Omega$$

CASE-2:

When AC is applied, then for $\omega = 2250\ \text{rads}^{-1}$, let the applied voltage be $V = V_0 \sin(\omega t)$. Then,

$$I_R = \frac{V_0}{R}\sin(\omega t)$$

Since $V_{rms} = 250\ \text{volt}$, so $V_0 = 250\sqrt{2}\ \text{volt}$

$$I_R = \frac{250\sqrt{2}}{250}\sin(\omega t) = \sqrt{2}\sin(\omega t)$$

and $I_L = I_C = \dfrac{V_0}{X_C \sim X_L}\sin\left(\omega t \pm \dfrac{\pi}{2}\right)$

$$\Rightarrow \quad I_L = \left(\frac{V_0}{X_C \sim X_L}\right)\cos(\omega t)$$

where, the sign '~' indicates positive difference between X_L and X_C

So, the total current through the circuit at any instant is

$$I = I_R + I_C = \frac{V_0}{R}\sin(\omega t) + \frac{V_0}{X_C \sim X_L}\cos(\omega t)$$

$$\Rightarrow \quad I = I_0 \sin(\omega t + \phi)$$

where $I_0 = V_0\sqrt{\dfrac{1}{R^2} + \dfrac{1}{(X_C \sim X_L)^2}}$

$$\Rightarrow \quad I_{\text{rms}} = V_{\text{rms}}\sqrt{\frac{1}{R^2} + \frac{1}{(X_C \sim X_L)^2}}$$

Given that, $V_{\text{rms}} = 250$ volt, $I_{\text{rms}} = 1.25$ A

$$\Rightarrow \quad \frac{1}{R^2} + \frac{1}{(X_C \sim X_L)^2} = \left(\frac{1.25}{250}\right)^2$$

$$\Rightarrow \quad \frac{1}{(X_C \sim X_L)^2} = \left(\frac{1.25}{250}\right)^2 - \left(\frac{1}{250}\right)^2$$

$$\Rightarrow \quad X_C \sim X_L = \frac{1000}{3}\,\Omega$$

$$\Rightarrow \quad \frac{1}{\omega C} \sim \omega L = \frac{1000}{3}\,\Omega \qquad \ldots(1)$$

Current in the circuit will be maximum at

$$X_L = X_C$$

$$\Rightarrow \quad L\omega_0 = \frac{1}{C\omega_0}$$

$$\Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \quad LC = \frac{1}{\omega_0^2} = \frac{1}{(4500)^2} \qquad \ldots(2)$$

Solving equations (1) and (2) with $\omega = 2250\ \text{rads}^{-1}$, we get $C = 10^{-6}\ \text{F} = 1\ \mu\text{F}$ and $L = 0.049$ H

$$\Rightarrow \quad C = 1\ \mu\text{F},\ L = 49\ \text{mH and } R = 250\ \Omega$$

PROBLEM 14

A solenoid with inductance $L = 7$ mH and active resistance $R = 44\ \Omega$ is first connected to a source of direct voltage V_0 and then to a source of sinusoidal voltage with effective value $V = V_0$. At what frequency of the oscillator will the power consumed by the solenoid be $\eta = 5.0$ times less than in the former case?

SOLUTION

For the case of direct current flowing through the circuit, we have

$$P_{\text{dc}} = \frac{V_0^2}{R}$$

When connected across an ac source of effective voltage V_0 (i.e. $V_{\text{rms}} = V_0$), the impedance of this series LR circuit is

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2 L^2}$$

$$\Rightarrow \quad I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\Rightarrow \quad P_{\text{ac}} = I_{\text{rms}}^2 R = \frac{V_0^2 R}{R^2 + \omega^2 L^2}$$

According to problem, the power consumed by the solenoid is $\eta = 5.0$ times less than the power consumed when solenoid is connected across dc power. So, we have

$$P_{\text{dc}} = \eta P_{\text{ac}}, \text{ where } \eta = 5$$

$$\Rightarrow \quad \frac{V_0^2}{R} = \eta \frac{V_0^2 R}{R^2\left(1 + \left(\dfrac{\omega L}{R}\right)^2\right)}$$

$$\Rightarrow \quad 1 + \left(\frac{\omega L}{R}\right)^2 = \eta$$

$$\Rightarrow \quad \frac{\omega L}{R} = \sqrt{\eta^2 - 1}$$

$$\Rightarrow \quad \omega = \frac{R}{L}\sqrt{\eta^2 - 1}$$

$$\Rightarrow \quad 2\pi f = \frac{R}{L}\sqrt{\eta^2 - 1}$$

$$\Rightarrow \quad f = \frac{R}{2\pi L}\sqrt{\eta^2 - 1} = 2\ \text{kHz}$$