CHAPTER

3 Electromagnetic Induction

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Magnetic Flux and Applications
(b) Faraday's Laws of EMI, Lenz's Law and Applications
(c) Motional EMF and Applications
(d) AC Generator
(e) Induced Electric Field and Applications
(f) Eddy Currents
(g) Inductor, Self-Inductance and Applications
(h) Growth and Decay of Current in Series LR Circuits
(i) Energy Stored in Inductor as Magnetic Field
(j) Mutual Inductance and Applications
(k) Inductors in Series and Parallel
(l) Oscillations in an LC Circuit

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION TO ELECTROMAGNETIC INDUCTION AND FARADAY'S LAWS

INTRODUCTION

Till now, we have studied that the electric fields and magnetic fields have been produced by stationary charges and moving charges (currents), respectively. Imposing an electric field on a conductor gives rise to a current which in turn generates a magnetic field. One could then think whether or not an electric field could be produced by a magnetic field. In 1831, Michael Faraday discovered that, by varying magnetic field with time, an electric field could be generated. The phenomenon is known as **electromagnetic induction**. The term electromagnetic induction constitutes two phenomenon.

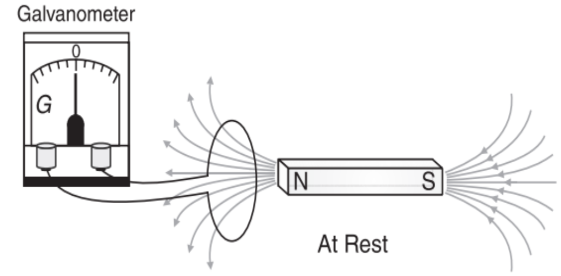
Phenomenon I: Involving the current induced in a conductor that moves relative to the field lines.

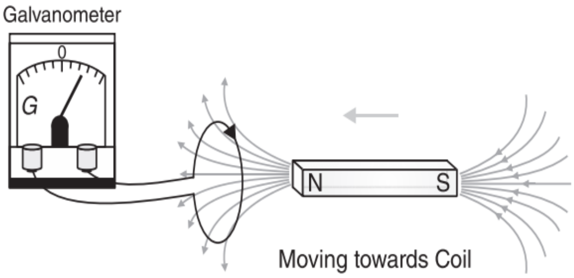
Phenomenon II: Involving the generation of an electric field associated with a time varying magnetic field.

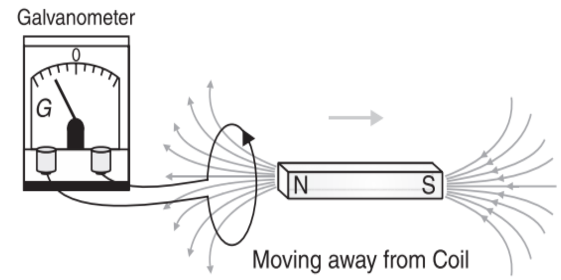
FARADAY'S EXPERIMENTS

Faraday observed for the set up shown that no current is registered in the galvanometer when bar magnet is stationary with respect to the loop.

However, a current is induced in the loop when a relative motion exists between the bar magnet and the loop. In particular, the galvanometer deflects in one direction as the magnet approaches the loop, and the opposite direction as it moves away.



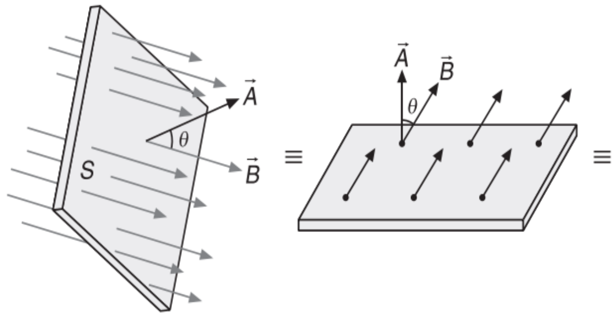


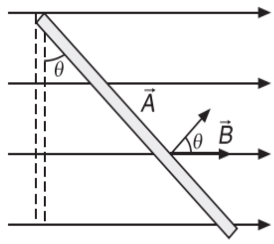


Faraday's experiment demonstrates that an electric current is induced in the loop by changing the magnetic field. The coil behaves as if it were connected to an emf source. Experimentally it is found that the induced emf depends on the rate of change of magnetic flux through the coil. So, let us first understand the concept of magnetic flux which is very similar to the concept of electric flux (discussed earlier).

MAGNETIC FLUX

Consider a uniform magnetic field passing through a surface S, as shown in Figure. Then magnetic flux is defined in a manner similar to the one used to define electric flux.

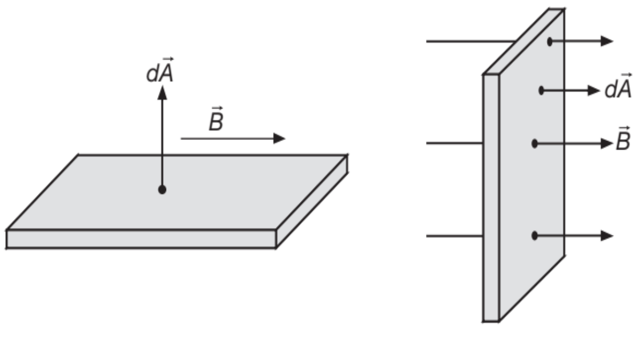




Let the area vector be $\vec{A} = A\hat{n}$, where A is the area of the surface and $\hat{n}$ its unit normal. If θ be the angle between $\vec{B}$ and $\hat{n}$, then the magnetic flux through the surface is given by

$$\phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta$$

If the magnetic field is parallel to the plane, as in Figure, then $\theta = 90°$ and the flux through the plane is zero. If the field is perpendicular to the plane, as in Figure, then $\theta = 0$ and the flux through the plane is BA (the maximum value).

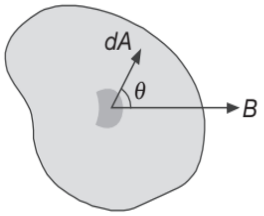


(a) (b)

(a) The flux through the plane is zero when the magnetic field is parallel to the plane surface.
(b) The flux through the plane is maximum when the magnetic field is perpendicular to the plane.

The unit of magnetic flux is Tm^2, which is defined as a weber (Wb), so $1\text{ Wb} = 1\text{ Tm}^2$. The flux is maximum, when $\theta = 0°$ i.e., $\phi_{max} = BA$ and the flux is minimum when $\theta = 180°$ i.e., $\phi_{min} = -BA$.

For an arbitrarily shaped surface consider an infinitesimal element of area dA on it, as shown in Figure.

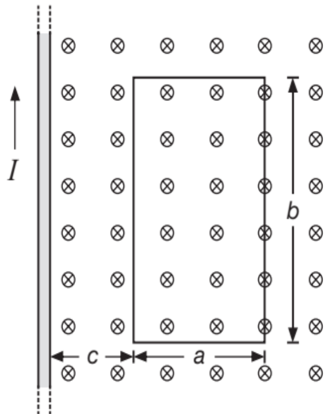


If the magnetic field at this element is $\vec{B}$, the infinitesimal magnetic flux through the element is $d\phi_B = \vec{B}\cdot d\vec{A}$, where $d\vec{A}$ is a vector that is outward normal to the surface and has a magnitude equal to the area dA. Therefore, the total magnetic flux ϕ_B through the surface is

$$\phi_B = \int \vec{B}\cdot d\vec{A}$$

ILLUSTRATION 1

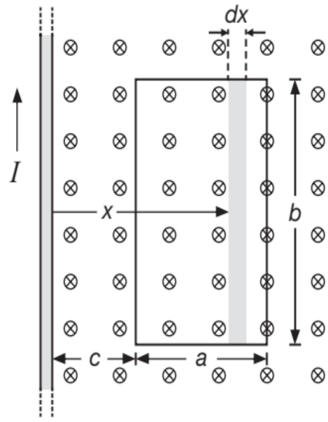
A rectangular loop of width a and length b is located near a long wire carrying a current I. The distance between the wire and the closest side of the loop is c. The wire is parallel to the long side of the loop. Find the total magnetic flux through the loop due to the current in the wire. Suppose we move the loop in Figure very far away from the wire. What happens to the magnetic flux?



SOLUTION

We know that the magnitude of the magnetic field created by the wire at a distance x from the wire is

$$B = \frac{\mu_0 I}{2\pi x}$$



The factor $\frac{1}{x}$ indicates that the field varies over the loop, and Figure shows that the field is directed into the page at the location of the loop. Since $\vec{B}$ is parallel to $d\vec{A}$ at any point within the loop, the magnetic flux through an area element dA is

$$\phi_B = \int B\,dA\cos(0^\circ) = \int\left(\frac{\mu_0 I}{2\pi x}\right)dA \quad \left\{\because\ B = \frac{\mu_0 I}{2\pi r}\right\}$$

Consider an infinitesimal strip of length b, width dx at a distance x from the wire then the area of the strip is $dA = b\,dx$. If $d\phi_B$ is the flux associated with the strip due to the field of wire, then

$$d\phi_B = B\,dA$$

$$\phi_B = \frac{\mu_0 Ib}{2\pi}\int_c^{a+c}\frac{dx}{x} = \frac{\mu_0 Ib}{2\pi}\log_e x\Big|_c^{a+c}$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 Ib}{2\pi}\log_e\left(\frac{a+c}{c}\right) = \frac{\mu_0 Ib}{2\pi}\log_e\left(1+\frac{a}{c}\right) \quad \ldots(1)$$

The flux should become smaller as the loop moves into weaker and weaker fields.

As the loop moves far away, the value of c is much larger than that of a, so that $\frac{a}{c} \to 0$. So, the natural logarithm in equation (1) approaches zero value, because

$$\log_e\left(1+\frac{a}{c}\right) \longrightarrow \log_e(1+0) = \log_e(1) = 0$$

and we find that $\phi_B \to 0$, as expected.

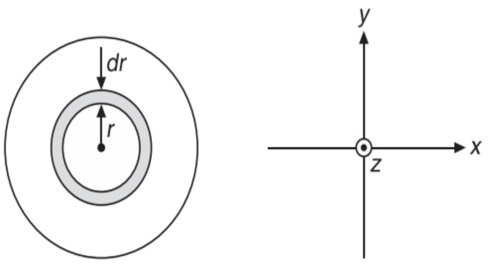
ILLUSTRATION 2

In a region of space varies, the magnetic field varies with radial distance r as $\vec{B}=(B_0 r)\hat{k}$ where r is the perpendicular distance of a point (x, y, z) from z-axis. Calculate the magnetic flux associated with a circle of radius a, centred at origin and lying in x-y plane.

SOLUTION

Since the magnetic field varies with radial distance r, so we divide the circular area into small concentric rings. Consider one such ring of radius r thickness dr. Area of this infinitesimal ring is

$$dA = 2\pi r dr$$



Magnetic flux $d\phi$ associated with this infinitesimal ring is

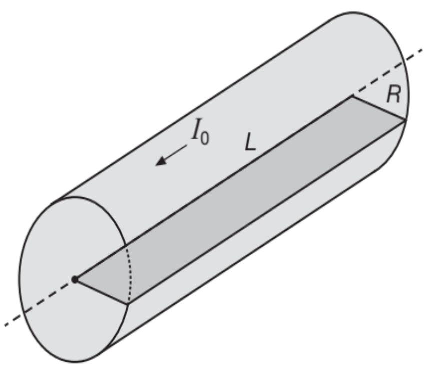
$$d\phi = \vec{B}.d\vec{A} = (B_0 r)(2\pi r dr) = 2\pi B_0 r^2 dr$$

So, total flux associated with the loop is

$$\phi = \int d\phi = 2\pi B_0 \int_0^a r^2 dr = \frac{2\pi B_0 a^3}{3}$$

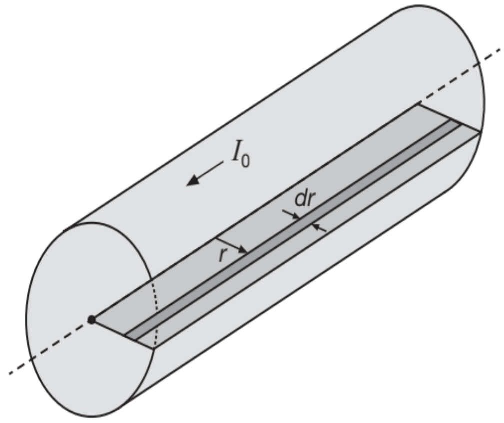
ILLUSTRATION 3

A very long, cylindrical wire of radius R carries a current I_0 uniformly distributed across the cross section of the wire. Calculate the magnetic flux through a rectangle that has one side of length L running down the centre of the wire and another side of length R, as shown in Figure.



SOLUTION

Let us find the magnetic field at a distance r from the centre of the wire and divide the rectangle into narrow infinitesimal strips of width dr. The magnetic flux through each infinitesimal strip will then be integrated to get the total flux.



From Ampere's Law, the field inside the wire is

$$B = \left(\frac{\mu_0 I}{2\pi R^2}\right) r$$

The field B is normal to the area of the strip Ldr, so angle between $\vec{B}$ and $d\vec{A}$ is $0°$. Hence,

$$d\phi_B = \vec{B}\cdot d\vec{A} = \left(\frac{\mu_0 I r}{2\pi R^2}\right)(Ldr)\cos(0°)$$

$$\phi_B = \int d\phi = \frac{\mu_0 IL}{2\pi R^2}\int_0^R r\,dr$$

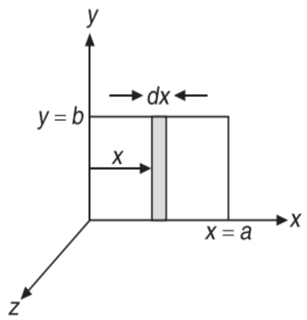
$$\Rightarrow \quad \phi_B = \frac{\mu_0 IL}{4\pi}$$

ILLUSTRATION 4

The magnetic field in a region of space varies with x as $\vec{B}=(\alpha+\beta x)\hat{k}$ where a is a constant. Calculate the magnetic flux associated with a surface lying in x-y plane and bound by the lines $x=0$, $y=0$ and $x=a$, $y=b$.

SOLUTION

We observe that the magnetic field varies over the surface in magnitude. So, let us first divide the area into infinitesimal elements. Consider a shaded strip so chosen that the field does not vary over the strip. The magnetic flux associated with the infinitesimal strip is



$$d\phi = \vec{B}\cdot d\vec{A} = (\alpha + \beta x)bdx$$

$$\phi = \int d\phi = \int_0^a (\alpha + \beta x)bdx$$

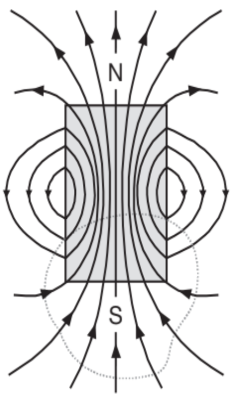
$$\Rightarrow \quad \phi = \alpha ba + \beta b\frac{a^2}{2}$$

$$\Rightarrow \quad \phi = \frac{ab}{2}(2\alpha + \beta a)$$

GAUSS'S LAW IN MAGNETISM

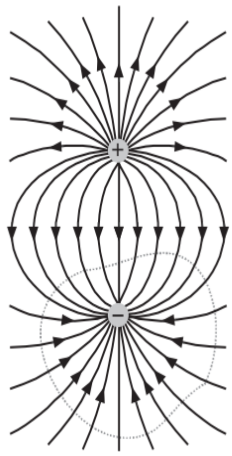
Since we know that the electric flux through a closed surface surrounding a net charge is proportional to that charge (Gauss's Law). In other words, the number of electric field lines leaving the surface depends only on the net charge within it. This property is based on the fact that electric field lines originate and terminate on electric charges.

The situation is quite different for magnetic fields, which are continuous and form closed loops. In other words, magnetic field lines do not begin or end at any point, as illustrated in Figure, which shows the magnetic field lines of a bar magnet.



The magnetic field lines of a bar magnet from closed loops. Note that the net magnetic flux through a closed surface surrounding one of the poles (or any other closed surface) is zero. (The dotted line represents the intersection of the surface with the page).

Note that for any closed surface, such as the one outlined by the dotted line in Figure, the number of lines entering the surface equals the number leaving the surface and thus, the net magnetic flux is zero. In contrast, for a closed surface surrounding one charge of an electric dipole, the net electric flux is not zero.



The electric field lines surrounding an electric dipole begin on the positive charge and terminate on the negative charge. The electric flux through a closed surface surrounding one of the charges is not zero.

Gauss's Law in magnetism states that the net magnetic flux through any closed surface is always zero, i.e.,

$$\oint \vec{B}\cdot d\vec{A} = 0 \qquad \text{\{Gauss's Law in Magnetism\}}$$

This statement is based on the experimental fact, mentioned at the beginning of the previous chapter, that isolated magnetic poles (monopoles) have never been detected and perhaps do not exist.

FARADAY'S LAWS OF ELECTRO-MAGNETIC INDUCTION

On the basis of experimental observations Faraday summarised the phenomenon of electromagnetic induction by giving following laws.

(a) Whenever magnetic flux linked with a closed coil changes, an induced emf (or induced current) is set up in the coil.

(b) This induced emf (or induced current) lasts as long as the change in magnetic flux continues.

(c) The magnitude of induced emf is proportional to the rate of change of magnetic flux linked with the circuit. If ϕ_B is magnetic flux linked with the circuit at any instant t, then induced emf

$$\xi \propto \frac{d\phi_B}{dt} \qquad \ldots(1)$$

(d) The nature of induced emf (or induced current, if circuit is closed) is such that it opposes the change in flux that produces it. This law is also called **Lenz's Law**. So, according to Lenz's Law **"the induced current always produces a magnetic field which tends to oppose the change in magnetic flux that produces the induced current".**

In view of Lenz's Law, equation (1) takes the form

$$\xi = -\frac{d\phi_B}{dt}$$

where the 'negative' sign indicates the opposing nature of induced emf

Conventionally, the change in flux is given with one turn and if the coil contains N turns, then

$$\xi = -N\frac{d\phi_B}{dt}$$

Now, since we know that $\phi_B = BA\cos\theta$, so we have

$$\xi = -\frac{d}{dt}(BA\cos\theta)$$

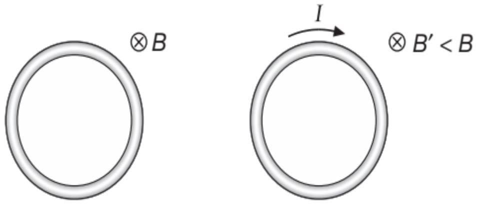
$$\xi = -(A\cos\theta)\left(\frac{dB}{dt}\right) - (B\cos\theta)\left(\frac{dA}{dt}\right) - (AB)\frac{d}{dt}(\cos\theta)$$

where all of B, A and θ are assumed to be varying with time.

$$\Rightarrow \quad \xi = -A\left(\frac{dB}{dt}\right)\cos\theta - B\left(\frac{dA}{dt}\right)\cos\theta + (BA\sin\theta)\frac{d\theta}{dt}$$

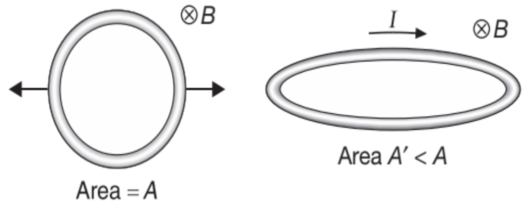
Thus, we see that an emf may be induced in the following ways, by varying the

(a) magnitude of $\vec{B}$ with time (illustrated in Figure)



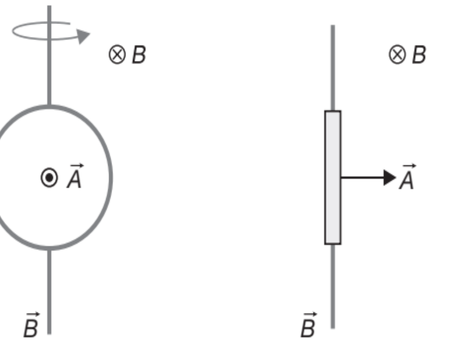
Inducing emf by changing the magnetic field strength

(b) magnitude of $\vec{A}$, i.e., the area enclosed by the loop with time (illustrated in Figure)



Inducing emf by changing the area of the loop

(c) angle between $\vec{B}$ and the area vector $\vec{A}$ with time (illustrated in Figure)



Inducing emf by varying the angle between $\vec{B}$ and $\vec{A}$

LENZ'S LAW: REVISITED

According to **Lenz's Law,** the nature of induced emf (or direction of induced current, if circuit is closed) is such that it opposes the change in flux that produces it. So, according to Lenz's Law "the induced current always produces a magnetic field which tends to oppose the change in magnetic flux that produces the induced current".

Mathematically, we have

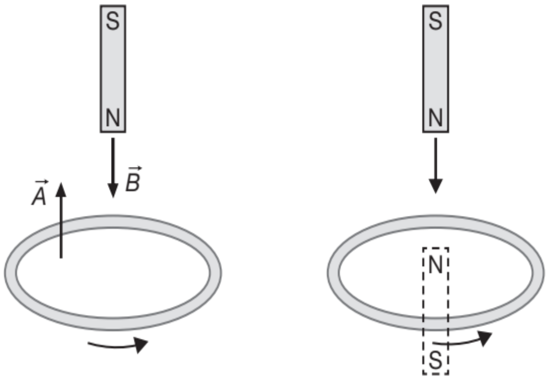
$$\xi = -\frac{d\phi_B}{dt}$$

where the 'negative' sign indicates the opposing nature of induced emf

Conventionally, the change in flux is given with one turn and if the coil contains N turns, then

$$\xi = -N\frac{d\phi_B}{dt}$$

As an example, to illustrate how Lenz's Law may be applied, consider the situation where a bar magnet is moving toward a conducting loop with its north pole down, as shown in Figure.



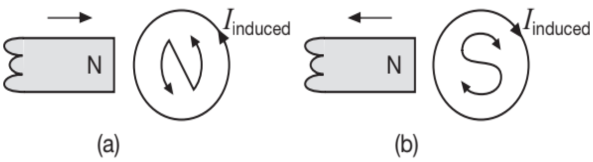
(a) A bar magnet moving toward current loop.
(b) Determination of the direction of induced current by considering the magnetic force between the bar magnet and the loop.

The direction of the induced current can be determined from the point of view of magnetic force. Lenz's Law states that the induced emf must be in the direction that opposes the change. Therefore, as the bar magnet approaches the loop, it experiences a repulsive force due to the induced emf. Since like poles repel, the loop must behave as if it were a bar magnet with its north pole pointing up. Using the right-hand thumb rule, the direction of the induced current is counter clockwise, as seen from above.

LENZ'S LAW IN ACCORDANCE WITH LAW OF CONSERVATION OF ENERGY

Lenz's Law is based on Law of Conservation of Energy and it gives the nature of induced emf or direction of induced current in the coil.

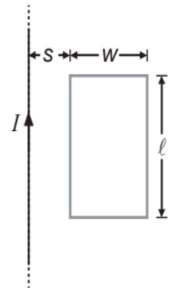
When north pole of a magnet is moved towards the coil, the induced current flows in a direction so as to oppose the motion of the magnet towards the coil. This is only possible when nearer face of the coil acts as a magnetic north pole which makes an anticlockwise current to flow in the coil. Then the repulsion between two similar poles opposes the motion of the magnet towards the coil.



Similarly, when the magnet is moved away from the coil, the direction of induced current is such as to make the nearer face of the coil as a south pole which makes a clockwise induced current to flow in the coil. Then the attraction between two opposite poles opposes the motion of the magnet away from the coil. In either case, therefore work has to be done in moving the magnet. It is this mechanical work, which appears as electrical energy in the coil. Hence the production of induced emf or induced current in the coil is in accordance with the Law of Conservation of Energy.

ILLUSTRATION 5

An infinite straight wire carrying a current I, varying with time t as $I(t) = a + bt$, is placed to the left of a rectangular loop of wire with width w and length l, as shown in Figure.



Assuming a and b to be positive constants, calculate the induced emf in the loop and the direction of the induced current?

SOLUTION

For calculating the induced emf we must first calculate the flux associated with the loop due to field of current carrying wire.
The magnetic field due to a current-carrying wire at a distance r away is

$$B = \frac{\mu_0 I}{2\pi r}$$

The total magnetic flux ϕ_B through the loop can be obtained by summing over contributions from all differential area elements $dA = ldr$. So,

$$\phi_B = \int d\phi_B = \int \vec{B} \cdot d\vec{A}$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 Il}{2\pi} \int_s^{s+w} \frac{dr}{r} = \frac{\mu_0 Il}{2\pi} \log_e\left(\frac{s+w}{s}\right)$$

According to Faraday's Law, the induced emf is

$$\xi = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\left[\frac{\mu_0 Il}{2\pi} \log_e\left(\frac{s+w}{s}\right)\right]$$

$$\Rightarrow \quad \xi = -\frac{\mu_0 l}{2\pi}\log_e\left(\frac{s+w}{s}\right)\frac{dI}{dt}$$

Given that, $I = I(t) = a + bt$

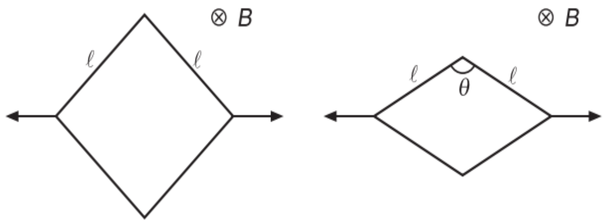
$$\Rightarrow \quad \frac{dI}{dt} = b$$

$$\Rightarrow \quad \xi = -\frac{\mu_0 bl}{2\pi}\log_e\left(\frac{s+w}{s}\right)$$

The straight wire carrying a current I produces a magnetic flux into the page through the rectangular loop. By Lenz's Law, the induced current in the loop must be flowing *counterclockwise* in order to produce a magnetic field out of the page to counteract the increase in inward flux.

ILLUSTRATION 6

A square loop with length l on each side is placed in a uniform magnetic field pointing into the page. During a time, interval, Δt, the loop is pulled from its two edges and turned into a rhombus, as shown in Figure.



Conducting loop changing area

Assuming that the total resistance of the loop is R, find the average induced current in the loop and its direction.

SOLUTION

Using Faraday's Law, we have

$$\xi = -\frac{\Delta\phi_B}{\Delta t} = -B\left(\frac{\Delta A}{\Delta t}\right)$$

Since the initial and the final areas of the loop are $A_i = l^2$ and $A_f = l^2\sin\theta$, respectively (recall that the area of a parallelogram defined by two vectors $\vec{l}_1$ and $\vec{l}_2$ is $A = \left|\vec{l}_1 \times \vec{l}_2\right| = l_1 l_2 \sin\theta$), the average rate of change of area is

$$\frac{\Delta A}{\Delta t} = \frac{A_f - A_i}{\Delta t} = -\frac{l^2(1-\sin\theta)}{\Delta t} < 0$$

$$\Rightarrow \quad \xi = \frac{Bl^2(1-\sin\theta)}{\Delta t} > 0$$

Thus, the average induced current is

$$I = \frac{\xi}{R} = \frac{Bl^2(1-\sin\theta)}{\Delta t R}$$

Since $\left(\frac{\Delta A}{\Delta t}\right) < 0$, the magnetic flux into the page decreases. Hence, the current flows in the clockwise direction to compensate the loss of flux.

ILLUSTRATION 7

A conducting loop of resistance R, has its radius varying with time t as $r = r_0 + \alpha t$, where α is a positive constant. This loop is placed in a region of uniform magnetic field $B\hat{j}$ such that its plane is perpendicular to the direction of magnetic field. Calculate the induced current flowing through the loop as a function of time.

SOLUTION

Since $\phi = \vec{B}.\vec{A} = BA\cos\theta$

$$\Rightarrow \quad \phi = B\left(\pi r^2\right)\cos 0°$$

$$\Rightarrow \quad \phi = B\pi\left(r_0 + \alpha t\right)^2$$

According to Lenz's Law, we have

$$\xi = -\frac{d\phi}{dt}$$

where, $\frac{d\phi}{dt} = B\pi 2\left(r_0 + \alpha t\right)\alpha$

$$\Rightarrow \quad \xi = -\frac{d\phi}{dt} = -2\alpha\pi B\left(r_0 + \alpha t\right)$$

The induced current is given by

$$i = \frac{|\xi|}{R}$$

$$\Rightarrow \quad i = \frac{2\alpha\pi B\left(r_0 + \alpha t\right)}{R}$$

As area increases, flux also increases, so the induced current tries to decrease the flux by setting up a magnetic field opposite to the applied field and hence induced current in the loop is clockwise when seen from above.

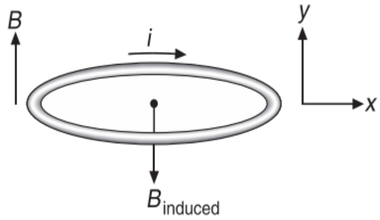
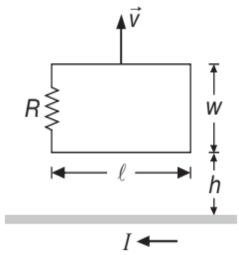


ILLUSTRATION 8

A rectangular loop of dimensions l and w moves with a constant velocity $\vec{v}$ away from an infinitely long straight wire carrying a current I in the plane of the loop, as shown in Figure. Let the total resistance of the loop be R. What is the current in the loop at the instant the near side is a distance h from the wire? Can you derive the same result using $\xi = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$?



SOLUTION

The magnetic field at a distance x from the straight wire is

$$B = \frac{\mu_0 I}{2\pi x}$$

The magnetic flux through a differential area element $dA = ldx$ of the loop is

$$d\phi_B = \vec{B} \cdot d\vec{A} = \frac{\mu_0 I}{2\pi x} ldx$$

where we have chosen the area vector to point into the page, so that $\phi_B > 0$. Integrating over the entire area of the loop, the total flux is

$$\phi_B = \frac{\mu_0 Il}{2\pi} \int_h^{h+w} \frac{dx}{x} = \frac{\mu_0 Il}{2\pi} \log_e\left(\frac{h+w}{h}\right)$$

Since, this loop is moving away from the wire so the flux when the lower end is at a distance y from the wire is

$$\phi_B = \frac{\mu_0 Il}{2\pi} \log_e\left(\frac{y+w}{y}\right)$$

Differentiating with respect to t, we obtain the induced emf as

$$\xi = -\frac{d\phi_B}{dt} = -\frac{\mu_0 Il}{2\pi} \frac{d}{dt}\left[\log_e\left(\frac{y+w}{y}\right)\right]$$

Since $\frac{d}{dt}\left[\log_e\left(\frac{y+w}{y}\right)\right] = \frac{y}{y+w} \frac{d}{dt}\left(\frac{y+w}{y}\right)$

$$= \frac{y}{y+w}\left(-\frac{w}{y^2}\right)\frac{dy}{dt} = \frac{w}{y(y+w)}\left(\frac{dy}{dt}\right)$$

Since $\frac{dy}{dt} = v \Rightarrow \xi = \frac{\mu_0 Il}{2\pi}\left(\frac{wv}{y(y+w)}\right)$

Using

$$\xi = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = vl[B(y) - B(y+w)]$$

where $B(y)$ is field at distance y and $B(y+w)$ is field at distance $(y+w)$, both from the wire

So, $B(y) = \frac{\mu_0 I}{2\pi y}$ and $B(y+w) = \frac{\mu_0 I}{2\pi(y+w)}$

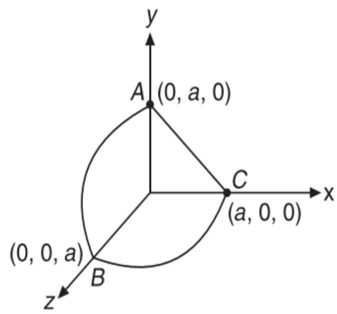
$$\Rightarrow \quad \xi = vl\left[\frac{\mu_0 I}{2\pi y} - \frac{\mu_0 I}{2\pi(y+w)}\right] = \frac{\mu_0 Il}{2\pi} \frac{vw}{y(y+w)}$$

The induced current is

$$I = \frac{|\xi|}{R} = \frac{\mu_0 Il}{2\pi R}\left(\frac{vw}{y(y+w)}\right)$$

ILLUSTRATION 9

A conducting loop $ABCA$ of resistance R_0 is lying in a region of space having magnetic field varying with time as $\vec{B} = (\hat{i} + \hat{j} + \hat{k})e^{-2t}$ shown in Figure.



The arc AB is a semicircle in y-z plane, arc BC is again a semicircle but in x-z plane and AC is a straight line lying in x-y plane. Calculate the flux associated with the loop and the induced current in the loop.

SOLUTION

Area of the loop $\vec{A}=\frac{\pi a^2}{4}\hat{i}+\frac{\pi a^2}{4}\hat{j}+\frac{a^2}{2}\hat{k}$

Flux associated with the loop is

$$\phi=\vec{B}.\vec{A}$$

$$\Rightarrow \quad \phi=\left(\hat{i}+\hat{j}+\hat{k}\right)e^{-2t}\cdot\left(\frac{\pi a^2}{4}\hat{i}+\frac{\pi a^2}{4}\hat{j}+\frac{a^2}{2}\hat{k}\right)$$

$$\Rightarrow \quad \phi=\left(\frac{\pi}{4}+\frac{\pi}{4}+\frac{1}{2}\right)e^{-2t}a^2=\left(\frac{\pi}{2}+\frac{1}{2}\right)a^2e^{-2t}$$

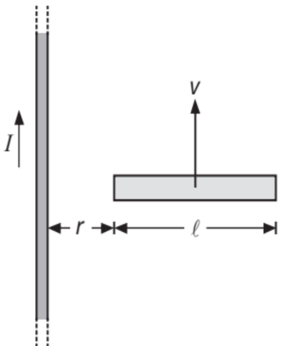
$$\Rightarrow \quad \xi=-\frac{d\phi}{dt}=2a^2\left(\frac{\pi}{2}+\frac{1}{2}\right)e^{-2t}=a^2(\pi+1)e^{-2t}$$

So, induced current is

$$i=\frac{\xi}{R_0}=\frac{2a^2}{R_0}\left(\frac{\pi}{2}+\frac{1}{2}\right)e^{-2t}=\frac{a^2}{R_0}(\pi+1)e^{-2t}$$

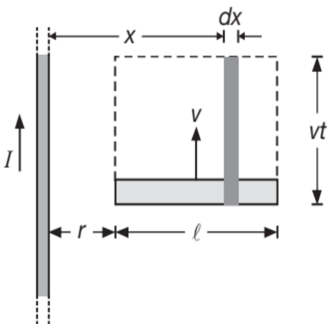
ILLUSTRATION 10

A conducting rod of length l moves with velocity $\vec{v}$ parallel to a long wire carrying a steady current I. The axis of the rod is maintained perpendicular to the wire with the near end a distance r away, as shown in Figure. Find the magnitude of the emf induced in the rod.



SOLUTION

Let us find an expression for the flux through a rectangular area swept out by the bar in time t.



The magnetic field at a distance x from wire is

$$B=\frac{\mu_0 I}{2\pi x}$$

Since $\phi_B=\int BdA$

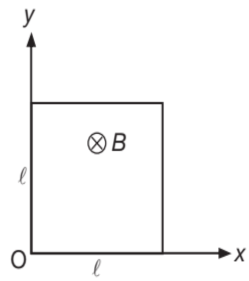
$$\Rightarrow \quad \phi_B=\frac{\mu_0 Ivt}{2\pi}\int_r^{r+l}\frac{dx}{x}$$

where vt is the distance the bar has moved in time t. So, we get

$$|\xi|=\frac{d\phi_B}{dt}=\frac{\mu_0 Iv}{2\pi}\log_e\left(1+\frac{l}{r}\right)$$

ILLUSTRATION 11

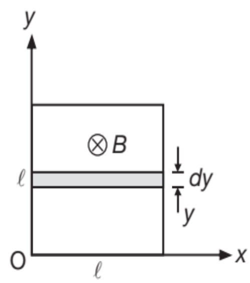
The Figure shows a square loop of wire with sides of length $l=2$ cm.



A magnetic field points into the page and its magnitude is given by $B=at^2y$ where $a=4\text{ Tm}^{-1}\text{s}^{-2}$, B is in tesla, t is in second and y is in metre. Determine the emf induced in the loop at $t=2.5$ s.

SOLUTION

Here we note that the field is a simultaneous function of y and t. Let us first calculate the flux associated with the loop. For that let us consider a rectangular strip of length l, thickness dy at a distance y from the x-axis.



Then the flux associated with this strip is

$$d\phi=BdA=Bldy$$

$\Rightarrow \quad d\phi = at^2 y l dy$

$\Rightarrow \quad \phi = \int d\phi = at^2 l \int_0^l y dy$

$\Rightarrow \quad \phi = \dfrac{at^2 l^3}{2} \qquad \left\{ \because \int_0^l y dy = \dfrac{l^2}{2} \right\}$

Now, according to Faraday's Laws, we have

$$\xi = -\frac{d\phi}{dt}$$

$\Rightarrow \quad \xi = -\dfrac{al^3}{2}(2t)$

$\Rightarrow \quad \xi = -al^3 t$

$\Rightarrow \quad |\xi| = (4)(8\times 10^{-6})(2.5)$

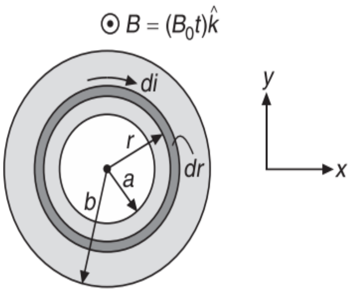
$\Rightarrow \quad |\xi| = 80\ \mu\text{V}$

ILLUSTRATION 12

In a region of space, there exists a uniform magnetic field $\vec{B} = (B_0 t)\hat{k}$. A ring with square cross-section made of a conducting material of conductivity ρ is placed such that it lies in the x-y plane. The thickness of the ring is h and its inner and outer radii are equal to a and b respectively. Neglecting the inductance of the ring, calculate the current induced in the loop.

SOLUTION

Consider a thin ring of radius r and thickness dr as shown in Figure.



The resistance of this ring is $R = \dfrac{\rho l}{A}$

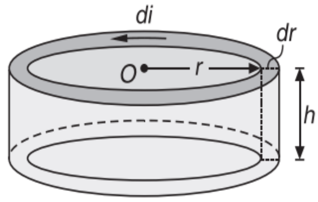
$\Rightarrow \quad R = \dfrac{\rho(2\pi r)}{hdr}$

The magnitude of induced emf in this ring is

$$|\xi| = \frac{d\phi}{dt} = A\frac{dB}{dt} = \pi r^2 B_0 \frac{dt}{dt} = \pi r^2 B_0$$

The current flowing is

$$di = \frac{\xi}{R} = \frac{\pi r^2 B_0}{\rho(2\pi r)} \times hdr$$



Cross-section

$\Rightarrow \quad di = \dfrac{B_0 h}{2\rho} rdr$

$\Rightarrow \quad i = \dfrac{B_0 h}{2\rho}\int_a^b rdr$

$\Rightarrow \quad i = \dfrac{B_0 h}{4\rho}(b^2 - a^2)$

ILLUSTRATION 13

In a coil of resistance R, the magnetic flux due to an external magnetic field varies with time as $\phi = k(4 - t^2)$, where k is a positive constant. Calculate the total heat produced in the coil till the time flux becomes zero.

SOLUTION

EMF induced in coil is given as

$$\xi = \left|\frac{d\phi}{dt}\right| = 2kt$$

Induced current in coil is given as

$$i = \frac{\xi}{R} = \frac{2kt}{R}$$

Flux in the coil is zero at $t = 2$ s

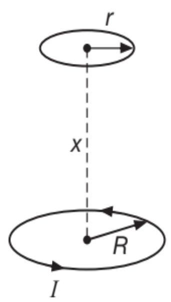
So, heat produced in coil is

$$H = \int_0^2 i^2 R dt = \int_0^2 \frac{4k^2 t^2}{R} dt$$

$\Rightarrow \quad H = \dfrac{4k^2}{R}\left(\dfrac{t^3}{3}\Big|_0^2\right) = \dfrac{32k^2}{3R}$

ILLUSTRATION 14

The Figure shows two parallel and co-axial loops. The smaller loop (radius r) is above the large loop (radius R) at a distance $x \gg R$. The magnetic field due to current I in the larger loop is nearly constant near the smaller loop. Suppose that x is increasing at the constant rate $\dfrac{dx}{dt} = v$.



(a) Determine the magnetic flux through the smaller loop as a function of x.

(b) Find the induced emf and the direction of induced current in the smaller loop at the instant when separation between them is X.

SOLUTION

(a) Since the magnetic field at the axis of the coil at a point lying a distance x from the centre is

$$B=\frac{\mu_0 IR^2}{2(R^2+x^2)^{3/2}}$$

Due to this field of the bigger loop, the flux associated with the smaller loop is

$$\phi=BA=\frac{\mu_0 IR^2}{2(R^2+x^2)^{3/2}}(\pi r^2)$$

Since $x \gg R$, so

$$\phi \simeq \frac{\mu_0 \pi IR^2 r^2}{2x^3}$$

(b) When x increases at a constant rate, then let its value at some later time t be X. So, we have

$$X=x+vt$$

$$\Rightarrow \phi=BA=\frac{\mu_0 \pi IR^2 r^2}{2X^3}$$

$$\Rightarrow \xi=-\frac{d\phi}{dt}$$

$$\Rightarrow \xi=-\frac{\mu_0 \pi IR^2 r^2}{2}\frac{d}{dt}(X^{-3})=\frac{3}{2}\frac{\mu_0 IR^2 r^2}{X^4}\left(\frac{dX}{dt}\right)$$

$$\Rightarrow \xi=\frac{3}{2}\frac{\mu_0 IR^2 r^2 v}{X^4} \qquad \left\{\because \frac{dX}{dt}=v\right\}$$

$$\Rightarrow \xi=\frac{3}{2}\frac{\mu_0 IR^2 r^2 v}{(x+vt)^4}$$

The direction of the induced current is clockwise when seen from the bigger loop.

ILLUSTRATION 15

A long straight solenoid of cross-sectional diameter d and with n turns per unit of its length has a round turn of copper wire of cross-sectional area A and density ρ is tightly put on its winding. Find the current flowing in the turn if the current in the solenoid winding is increased with a constant rate I ampere per second.

SOLUTION

The magnetic field inside the solenoid is given as

$$B=\mu_0 ni$$

The flux through its cross-section of copper winding is given as

$$\phi=BA=(\mu_0 ni)\frac{\pi d^2}{4}$$

EMF induced in the copper wire is given as

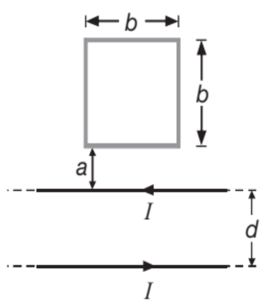
$$e=\left|\frac{d\phi}{dt}\right|=\mu_0 n\pi\frac{d^2}{4}\frac{di}{dt}$$

Current in the copper wire is given as

$$I_C=\frac{e}{R}=\frac{\mu_0 n\pi d^2 I}{4\rho\left(\frac{\pi d}{A}\right)}=\frac{\mu_0 ndAI}{4\rho}$$

ILLUSTRATION 16

Two fixed long straight wires carry the same current I in opposite directions as shown in Figure. A square loop of side b is fixed in the plane of the wires with its length parallel to one wire at a distance a as shown in Figure.



Calculate the induced emf in the loop if the current in both wires is changing at the rate $\frac{dI}{dt}$. What is the direction of force on the loop if $\frac{dI}{dt}$ is positive?

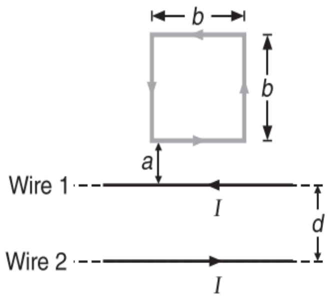
SOLUTION

(a) Flux associated with the square loop due to wire 1 is

$$\phi_{\text{wire 1}} = \frac{\mu_0 Ib}{2\pi}\int_a^{a+b}\frac{dx}{x} = \frac{\mu_0 Ib}{2\pi}\log_e\left(\frac{a+b}{a}\right)$$

Flux associated with the square loop due to wire 2 is

$$\phi_{\text{wire 2}} = \frac{\mu_0 Ib}{2\pi}\int_{a+d}^{a+d+b}\frac{dx}{x} = \frac{\mu_0 Ib}{2\pi}\log_e\left(\frac{a+d+b}{a+d}\right)$$



Net flux associated with the square loop is

$$\phi = \phi_{\text{wire 1}} - \phi_{\text{wire 2}}$$

$$\Rightarrow \phi = \frac{\mu_0 Ib}{2\pi}\log_e\left(\frac{a+b}{a}\right) - \frac{\mu_0 Ib}{2\pi}\log_e\left(\frac{a+d+b}{a+d}\right)$$

$$\Rightarrow \phi = \frac{\mu_0 Ib}{2\pi}\log_e\left[\frac{(a+b)(a+d)}{a(a+d+b)}\right]$$

$$\Rightarrow \xi = \left|-\frac{d\phi}{dt}\right| = \frac{\mu_0 b}{2\pi}\left[\log_e\left(\frac{(a+b)(a+d)}{a(a+d+b)}\right)\right]\left(\frac{dI}{dt}\right)$$

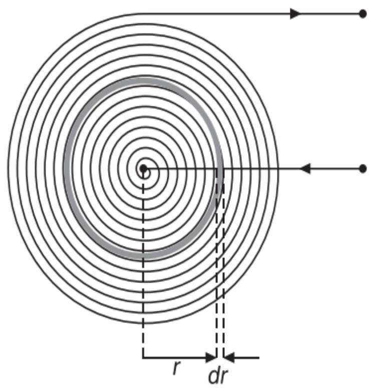
(b) Net field through the loop is in the inward $(\otimes)$ direction. If $\frac{dI}{dt}$ is positive, the induced current in the loop is anticlockwise as I is increasing. So, loop will be repelled from the wires, because wires 1 and 2 are close to each other and current in these two wires are in opposite direction.

ILLUSTRATION 17

A plane spiral wound tightly, having large number of turns N, is placed in a uniform magnetic field perpendicular to its plane. The outer radius of the spiral's turns is equal to a. The magnetic induction varies with time as $B = B_0 \sin\omega t$, where B_0 and ω are constants. Calculate the amplitude of emf induced in the spiral.

SOLUTION

We consider an elemental circular strip of radius r and radial width dr in the spiral as shown in Figure.



If dN be the number of turns in this infinitesimal spiral strip, then

$$dN = \frac{N}{a}dr$$

The emf induced across this elemental strip is

$$d\xi = \left|\frac{d\phi}{dt}\right| = \pi r^2 (dN)\frac{dB}{dt}$$

$$\Rightarrow d\xi = \pi r^2 (dN)\frac{d}{dt}(B_0 \sin\omega t)$$

$$\Rightarrow d\xi = \pi r^2 (dN) B_0 \omega \cos(\omega t)$$

$$\Rightarrow d\xi = \left(\frac{N}{a}dr\right)\pi r^2 B_0 \omega \cos(\omega t)$$

$$\Rightarrow \xi = \int d\xi = \int_0^a \left(\frac{N}{a}dr\right)\pi r^2 B_0 \omega \cos(\omega t)$$

$$\Rightarrow \xi = \int d\xi = \frac{N\pi B_0 \omega \cos(\omega t)}{a}\int_0^a r^2 dr$$

$$\Rightarrow \xi = \frac{1}{3}N\pi B_0 a^2 \omega \cos(\omega t) = \xi_0 \cos\omega t$$

Hence the amplitude of induced emf in the spiral is

$$\xi_0 = \frac{1}{3}\pi N a^2 B_0 \omega$$

Conceptual Note(s)

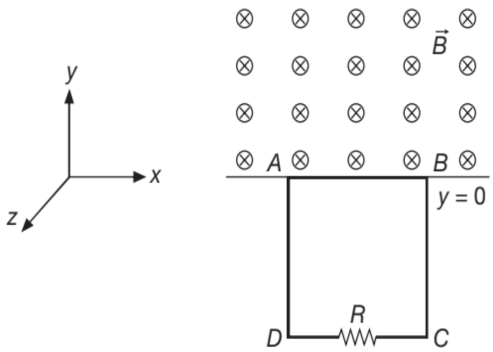
Please understand that, the statement says that the coil is placed in uniform field and at the same time the problem says that the field is varying with time as $B = B_0 \sin\omega t$. This creates a lot of confusion amongst the students. Please don't get confused and let me take the privilege to explain this thing to you.

Actually, according to the problem the field is constant in space but varying with time.

We should understand that for a conducting loop placed in a field, the emf is induced in the loop when the field is time varying (space variation i.e. uniform or non-uniform in space doesn't matter) and emf will not be induced in the loop if it is placed in constant time independent field (uniform or non-uniform in space doesn't matter).

ILLUSTRATION 18

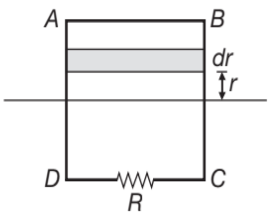
A square loop $ABCD$ of side a is moving in the xy-plane with velocity $\vec{v} = \beta t\hat{j}$.



A non-uniform magnetic field $\vec{B} = -B_0\left(1+\alpha y^2\right)\hat{k}$ exists in the space $y > 0$, where B_0 and α are constants. Initially, the upper wire of the loop is at $y = 0$. Find the induced voltage across the resistance R as a function of time. Neglect the magnetic force due to the induced current.

SOLUTION

Consider an infinitesimal strip shown in Figure.



Flux associated with the strip is

$$d\phi = B(ldr) = B_0\left(1+\alpha r^2\right)l\,dr$$

Total flux in the loop is

$$\phi = \int_0^y B_0\left(1+\alpha r^2\right)ldr$$

$$\Rightarrow \quad \phi = B_0 l\left(y+\frac{\alpha y^3}{3}\right)$$

Induced emf is given by

$$\xi = \frac{-d\phi}{dt} = -B_0 l\left(1+\frac{3\alpha y^2}{3}\right)\frac{dy}{dt}$$

$$\Rightarrow \quad \xi = -B_0 l\left(1+\alpha y^2\right)\frac{dy}{dt} \qquad \ldots(1)$$

Given, $v = \beta t$

$$\Rightarrow \quad \frac{dy}{dt} = \beta t$$

$$\Rightarrow \quad \int_0^y dy = \int_0^t \beta t\,dt$$

$$\Rightarrow \quad y = \frac{\beta t^2}{2}$$

$$\Rightarrow \quad E = -B_0 l\left(1+\frac{\alpha\beta^2 t^4}{4}\right)\beta t$$

$$\Rightarrow \quad E = -B_0 l\beta\left(t+\frac{\alpha\beta^2 t^5}{4}\right)$$

CHARGE INDUCED IN THE CIRCUIT

Since $\xi = -N\dfrac{d\phi}{dt}$

If R is resistance of circuit, then current induced is

$$I = \frac{\xi}{R} = -\frac{N}{R}\frac{d\phi}{dt}$$

The charge induced in time dt is given by

$$dq = Idt = -\frac{N}{R}\frac{d\phi}{dt}dt = -\frac{N}{R}d\phi$$

$$\Rightarrow \quad q = \int_0^q dq = -\frac{N}{R}\int_{\phi_1}^{\phi_2} d\phi$$

$$\Rightarrow \quad |q| = \left(\frac{N}{R}\right)\Delta\phi$$

$$\Rightarrow \quad q = \frac{\text{Net Change in flux}}{\text{Resistance}}$$

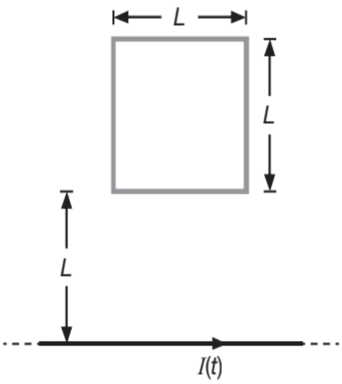
So, we observe that the charge induced is independent of time.

ILLUSTRATION 19

A square loop of side L, resistance R, lies at a distance L from the wire carrying a current $I(t)$ which decreases gradually with time t as

$$I(t) = \begin{cases} (1-\alpha t)I_0 & \text{for } 0 \le t \le \dfrac{1}{\alpha} \\ 0 & \text{for } t > \dfrac{1}{\alpha} \end{cases}$$

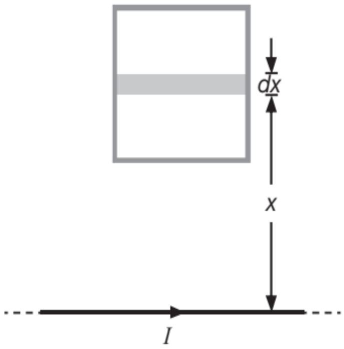
(a) In what direction does the induced current in the square loop flow?
(b) What total charge passes through a given point in the loop during the time this current flow?



SOLUTION

(a) Since we observe that I decreases with t, hence B also decreases with time. So, the induced current must be set up in the sense which does not allow B to decrease (i.e. the induced current must give an outward pointing field), so I_{induced} must be in the counter clockwise direction.

(b) Consider an infinitesimal strip shown in Figure.



Flux associated with the strip is

$$d\phi = \frac{\mu_0}{2\pi}\frac{IL}{x}dx$$

$$\Rightarrow \quad \phi = \int_L^{2L} \frac{\mu_0}{2\pi}\frac{IL}{x}dx$$

$$\Rightarrow \quad \phi = \frac{\mu_0 IL}{2\pi}\log_e(2)$$

Please note here that ϕ depends on I which is a function of t (but not x).

Since, $I_i = I_0$

$$\Rightarrow \quad \phi_i = \frac{\mu_0 I_0 L}{2\pi}\log_e(2)$$

and $I_f = 0$

$$\Rightarrow \quad \phi_f = 0$$

$$\Rightarrow \quad |\Delta\phi| = \frac{\mu_0 I_0 L}{2\pi}\log_e(2)$$

Since, the charge flown through the circuit is given by

$$|\Delta q| = \frac{1}{R}|\Delta\phi|$$

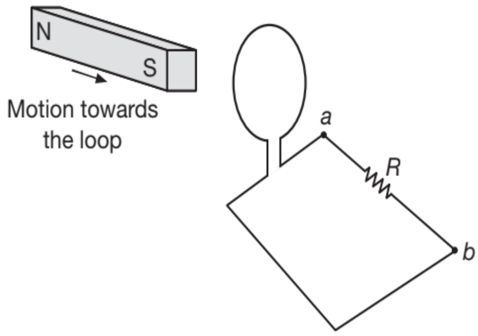
$$\Rightarrow \quad \Delta q = \frac{|\Delta\phi|}{R} = \frac{\mu_0 I_0 L}{2\pi R}\log_e(2)$$

Test Your Concepts-I

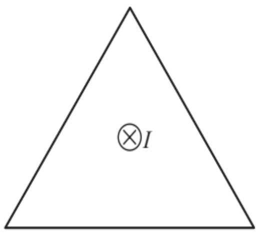
Based on Magnetic Flux, Faraday's Laws and Induced EMF

(Solutions on page H.127)

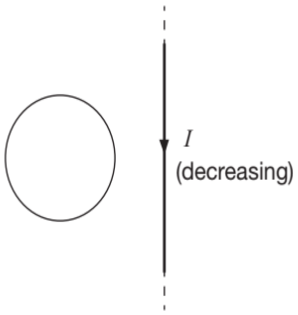
1. In Figure, the bar magnet is being moved towards the loop. Will $V_a - V_b$ be positive, negative, or zero? Explain.



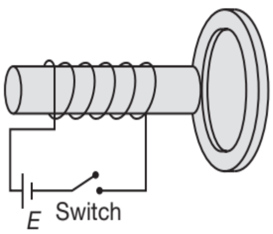
2. A current carrying straight wire passes through a triangular coil as shown in Figure. The current in the wire is perpendicular to paper inwards. Find the direction of the induced current in the loop if current in the wire is increased.



3. A circular loop is placed near a current carrying conductor as shown. Find the direction of induced current if the current in the wire is decreasing.



4. A metal ring is placed near a solenoid, as shown in Figure. Find the direction of the induced current in the ring

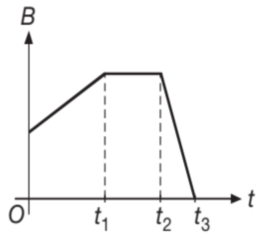


(a) at the instant the switch in the circuit containing the solenoid is thrown closed.

(b) after the switch has been closed for several seconds, and

(c) at the instant the switch is just opened.

5. A flat coil is oriented with the plane of its area at right angles to a uniform magnetic field. The magnitude of this field varies with time according to the graph in Figure. Draw a graph of the emf induced in the coil as a function of time. Please do not forget to plot and identify the times t_1, t_2 and t_3 on your graph.



6. Two bulbs are connected to opposite sides of a circular loop of wire, as shown in Figure. A changing magnetic field (confined to the smaller circular area shown in the Figure) induces an emf in the loop that causes the two bulbs to light. Now, when the switch is closed, the resistance-free wires connected to the switch short out bulb 2 and it goes off. What will happen, if the switch remain connected at points a and b, but the switch and the wires are lifted up and moved to the other side of the field, as in Figure. The wire is still connected to bulb 2 as it was before, so does it continue to stay dark?

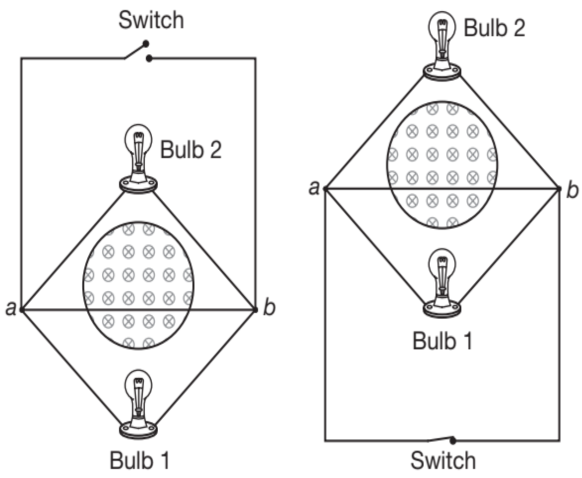
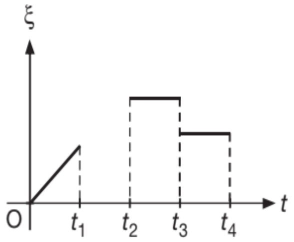


Figure (a) Figure (b)

7. A coil is stationary in a region of uniform, external, time varying magnetic field. The emf induced in this coil as a function of time is shown in Figure. Sketch a clear qualitative graph of the external magnetic field as a function of time, given that it started from zero. Include the points t_1, t_2, t_3 and t_4 on your graph.

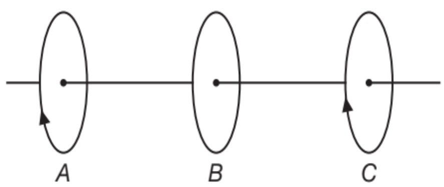


8. A particle having mass 2×10^{-16} kg and charge 30 nC starts from rest, is accelerated by a strong electric field, and is fired from a small source inside a region of uniform constant magnetic field 0.6 T. The velocity of the particle is perpendicular to the field. The circular orbit of the particle encloses a magnetic flux of 15 μWb.

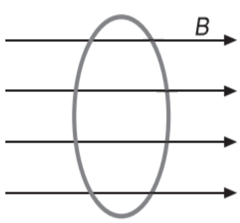
(a) Calculate the speed of the particle.

(b) Calculate the potential difference through which the particle accelerated inside the source.

9. Three identical closed coils *A*, *B* and *C* are placed with their planes parallel to one another. Coils *A* and *C* carry equal currents as shown in Figure. Coils *B* and *C* are fixed in position and coil *A* is moved towards *B* with uniform motion. Is there any induced current in *B*? If no, give reasons. If yes, mark the direction of the induced current in the diagram.



10. A circular loop of wire of radius *r* is in a uniform magnetic field, with the plane of the loop perpendicular to the direction of the field. The magnetic field varies with time according to $B(t)=a+bt$, where *a* and *b* are constants.



(a) Calculate the magnetic flux through the loop at $t=0$.

(b) Calculate the emf induced in the loop.

(c) If the resistance per unit length of the loop is λ, what is the induced current?

(d) At what rate is energy being delivered to the resistance of the loop?

11. A closed coil consists of 500 turns on a rectangular frame of area 4 cm^2 and has a resistance of 50 Ω. The coil is kept with its plane perpendicular to a uniform magnetic field of 0.2 Wbm^{-2}. Calculate the amount of charge flowing through the coil if it is rotated through 180°.

12. A wire in the form of a circular loop of radius 10 cm lies in a plane normal to a magnetic field of 100 T. If this wire is pulled to take a square shape in the same plane in 0.1 s, find the average induced emf in the loop.

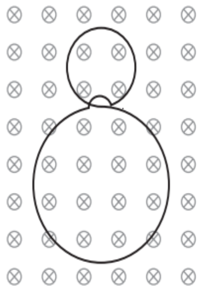
13. In a region of space, a non-uniform magnetic field $B=\left(B_0y^2\right)\hat{i}$ is present. The field is restricted to positive *y*-*z* plane only. A rectangular loop of sides *a* and 2*a* is placed in *y*-*z* plane with one vertex at origin, smaller side along *y*-axis and longer side along *z*-axis. The loop starts moving along +*y*-axis with a uniform speed *v*. Find the induced emf as a function of time.

14. A circular coil of radius 4 cm, 30 turns and resistance 1 Ω is placed in a magnetic field directed perpendicular to the plane of the coil. The magnitude of the magnetic field varies with time according to the expression $B=(0.01)t+(0.04)t^2$, where *t* is in seconds and *B* is in tesla. Calculate the emf induced in the coil at $t=5$ s.

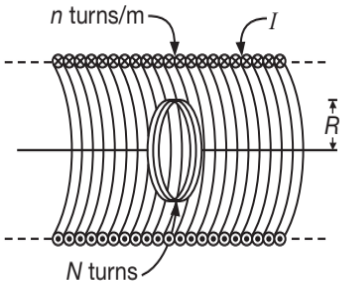
15. A coil consists of 200 turns of wire. Each turn is a square of side 18 cm, and a uniform magnetic field directed perpendicular to the plane of the coil is turned on. If the field changes linearly from 0 to 0.5 T in 0.8 s, what is the magnitude of the induced emf in the coil while the field is changing?

16. A loop of wire enclosing an area *A* is placed in a region where the magnetic field is perpendicular to the plane of the loop. The magnitude of $\vec{B}$ varies in time according to the expression $B=B_0e^{-at}$, where B_0 and *a* are constants. Find the induced emf in the loop as a function of time.

17. A piece of insulated wire is shaped into a digit 8, as shown in Figure. The radius of the upper circle is 5 cm and that of the lower circle is 9 cm. The wire has a uniform resistance per unit length of 3 Ωm^{-1}. A uniform magnetic field is applied perpendicular to the plane of the two circles, in the direction shown. The magnetic field is increasing at a constant rate of 2 Ts^{-1}. Find the magnitude and direction of the induced current in the wire.

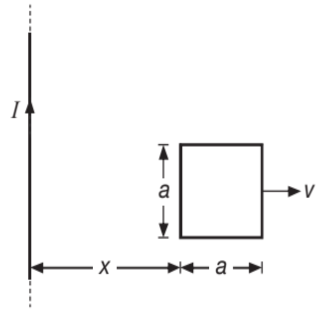


18. A long solenoid having 400 turns per metre carrying a current given by $I=(30\text{ A})(1-e^{-1.6t})$. Inside the solenoid and coaxial with it is a coil that has a radius of 6 cm and consists of a total of N = 250 turns of fine wire as shown. Calculate the emf induced in the coil by the changing current?

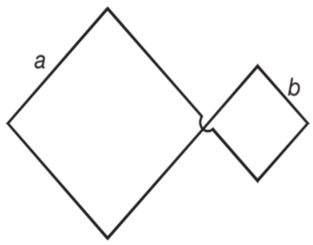


19. In order to impart an angular velocity to an earth satellite the geomagnetic field can be used. Find the maximum possible angular velocity about its own axis gained by the satellite if a storage battery with a capacity of $Q=5$ Ahr is discharged suddenly through a coil of N = 20 turns wound around the satellite's surface along the circumference of the largest circle. The satellite has a mass of $m=1000$ kg and is assumed to be a thin walled sphere. The geomagnetic field is parallel to the winding plane and its flux density is $B=0.5$ G. (1 gauss $=10^{-4}$ tesla).

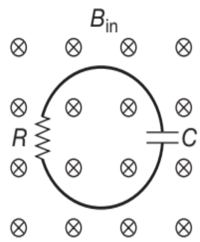
20. A square frame with side a and a long straight wire carrying a current I are located in the same plane as shown in Figure. The frame translates to the right with a constant velocity v. Find the emf induced in the frame as a function of distance x.



21. A plane loop shown in Figure is shaped as two squares with sides a = 20 cm and b = 10 cm and is introduced into a uniform magnetic field at right angles to the loop's plane. The magnetic induction varies with time as $B=B_0\sin(\omega t)$, where $B_0=10$ mT and $\omega=100\text{ s}^{-1}$. Find the amplitude of the current induced in the loop if its resistance per unit length is $\lambda=50\text{ m}\Omega\text{m}^{-1}$. The inductance of the loop is to be neglected.



22. In Figure, a uniform magnetic field decreases at a constant rate $\dfrac{dB}{dt}=-K$, where K is a positive constant. A circular loop of wire of radius a containing a resistance R and a capacitance C is placed with its plane normal to the field.



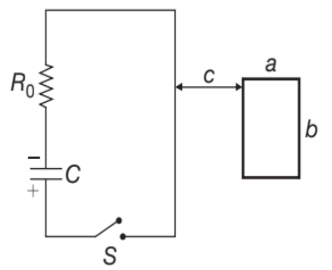
(a) Find the charge Q on the capacitor when it is fully charged.

(b) Which plate is at the higher potential?

(c) Discuss the force that causes the separation of charges.

23. In the circuit a capacitor having capacitance $C=20\ \mu F$ initially charged to 100 V with the polarity is shown. The resistor R_0 has resistance $10\ \Omega$.

At time $t = 0$ the switch is closed. The small circuit is not connected in any way to the large one. The wire of the small circuit has a resistance of $1\,\Omega m^{-1}$ and contains 25 loops. The large circuit is a rectangle 2 m by 4 m, while the small one has dimensions a = 10 cm and b = 20 cm. The distance c is 5 cm. Both circuits are held stationary. Assume that only the wire nearest the small circuit produces an appreciable magnetic field through it.



(a) Find the current in the large circuit 200 μs after S is closed.

(b) Find the current in the small circuit 200 μs after S is closed.

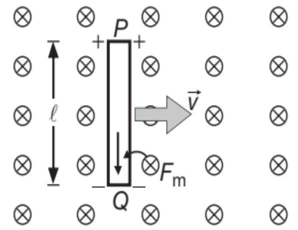
(c) Find the direction of the current in the small circuit.

(d) Justify why we can ignore the magnetic field from all the wires of the large circuit except for the wire closest to the small circuit.

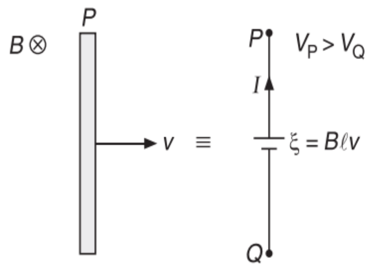
24. A uniform field of induction B is changing in magnitude at a constant rate $\frac{dB}{dt}$. You are given a mass m of copper which is to be drawn into a wire of radius a and formed into a circular loop of radius r. Show that the induced current in the loop does not depend on the size of the wire or of the loop and is given by $I = \left(\frac{\sigma m}{4\pi d}\right)\frac{dB}{dt}$, where σ is the conductivity and d the density of copper. Assume B perpendicular to loop.

INDUCED EMF IN CONDUCTING ROD MOVING THROUGH A UNIFORM MAGNETIC FIELD: MOTIONAL EMF

Let a thin conducting rod PQ of length l move in a uniform magnetic field B directed perpendicular to plane of paper, inwards. Let the velocity v of rod be in the plane of paper towards right.



Using Fleming's Left Hand Rule, we see that a positive charge (q) in the rod suffers magnetic force qvB directed from Q to P along the rod while an electron will experience a force evB directed from P to Q along the length of the rod. Due to this force the free electrons of rod move from P to Q, thus making end Q negative and end P positive as shown.



This causes a potential difference along the ends of rod. This potential difference developed is called **induced emf** ξ. If E is electric field developed in the rod, then

$$E = \frac{\xi}{l}$$

For equilibrium of charges, we must have

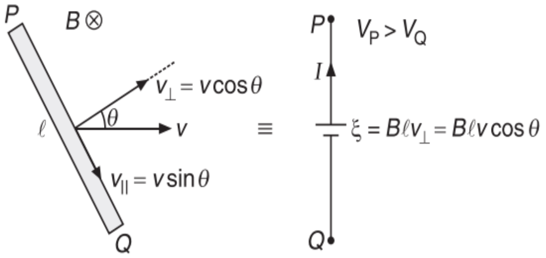
Electrical force = Magnetic force

$\Rightarrow \quad eE = evB$

$\Rightarrow \quad E = vB$

So, Induced emf $\xi = El = Blv$

If the rod moves in the magnetic field as shown in Figure, then the induced emf is given by



$$\xi = Blv_\perp = Blv\cos\theta$$

where $v_\perp$ is component of velocity perpendicular to the length of the rod. So, we have

$$V_P - V_Q = \xi = Bvl\cos\theta$$

Please note that the equivalent replacement of motional emf by a battery is shown the above figures.

Do not confuse that for the direction of induced current shown, we have $V_P < V_Q$ (as we start thinking that induced current will go from higher potential to lower potential). Of course, this is true but for the external circuit (excluding battery). So, note that for the rod in motion (acting as the source of emf), the induced conventional current is going from lower potential to higher potential (inside the source).

Just think this way that in the external circuit current goes from positive terminal to the negative terminal and inside the battery it goes from negative terminal to positive terminal.

Conceptual Note(s)

(a) If $\vec{\ell}$ is a vector directed along the direction of induced current, then a general notation for induced emf ξ is

$$\xi = \vec{B}\cdot(\vec{\ell}\times\vec{v}) = (\vec{v}\times\vec{B})\cdot\vec{\ell}$$

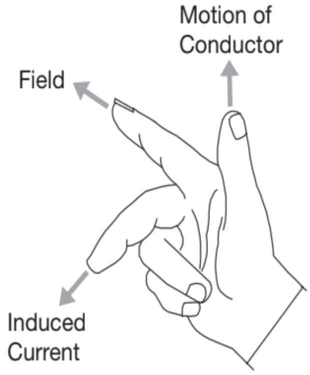
So, for emf or induced current to exist we must make sure that $\vec{B}$, $\vec{\ell}$ and $\vec{v}$ must never be coplanar.

(b) In general, the motional emf around a closed conducting loop can also be written as

$$\xi = \oint(\vec{v}\times\vec{B})\cdot d\vec{\ell} = \oint\vec{B}\cdot(d\vec{\ell}\times\vec{v})$$

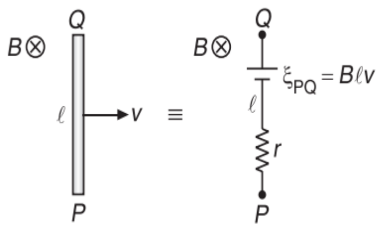
Fleming's Right Hand Rule

The direction of induced current is given by Fleming's Right Hand Rule according to which, "*Stretch the First Finger, Middle Finger and the Thumb of Right Hand in such a way that all three are mutually perpendicular to each other. First Finger points in the direction of field. Thumb points in the direction of motion of conductor, then Middle Finger points along the direction of Induced Conventional Current*".

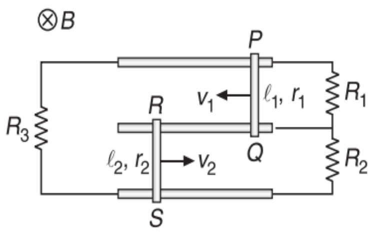


MOTIONAL EMF REPRESENTED AS AN EQUIVALENT BATTERY

When a conductor moves in a magnetic field, it can be considered like an equivalent battery or a source of potential difference with internal resistance equal to the resistance of the conductor. This conductor can act as a current source and supply current to the circuit. Consider a conductor of length l, resistance r moving with a velocity v in a uniform magnetic field B. Since B, l and v are perpendicular to each other, so this conductor can be replaced by an equivalent battery of emf Blv and internal resistance r as shown in Figure.



To understand the above situation better, let us consider two conductors PQ and RS having resistances r_1 and r_2 respectively, to be sliding on three conducting guide rails connected to resistances R_1, R_2 and R_3 as shown in Figure.



Applying Fleming's Right Hand Rule, we see that the induced current in the conductor PQ is from P to Q, whereas the induced current in the conductor RS is from S to R. So, the battery equivalents and the equivalent circuit for the arrangement of conductors is shown in Figure.

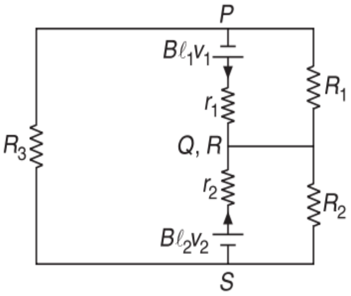


ILLUSTRATION 20

A straight segment of a rod is moving with velocity $\left(\hat{i}-2\hat{j}+\hat{k}\right)$ ms^{-1} and magnetic field in the region is $\left(4\hat{i}+2\hat{j}-\hat{k}\right)$ T. Calculate the emf induced across the ends of the rod when at an instant the end points are located between $(1,1,1)$ m and $(3,3,3)$ m.

SOLUTION

Since emf induced is

$$\xi=\vec{B}\cdot\left(\vec{l}\times\vec{v}\right)=\begin{vmatrix} B_x & B_y & B_z \\ l_x & l_y & l_z \\ v_x & v_y & v_z \end{vmatrix}$$

where, $\vec{l}=\vec{l}_2-\vec{l}_1$

In the given case, we have

$$\vec{B}=\left(4\hat{i}+2\hat{j}-\hat{k}\right)\text{ T}$$

$$\vec{l}=\vec{l}_2-\vec{l}_1=\left(2\hat{i}+2\hat{j}+2\hat{k}\right)\text{ m}$$

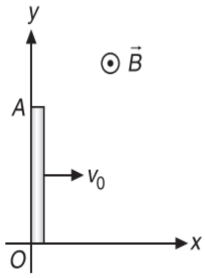
$$\vec{v}=\left(\hat{i}-2\hat{j}+\hat{k}\right)\text{ ms}^{-1}$$

$$\Rightarrow \quad \xi=\vec{B}\cdot\left(\vec{l}\times\vec{v}\right)=\begin{vmatrix} 4 & 2 & -1 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \xi=4(2+4)-2(2-2)-1(-4-2)=30\text{ V}$$

ILLUSTRATION 21

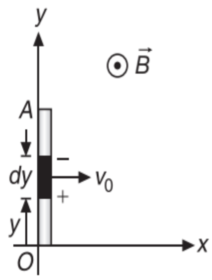
A conductor OA of length l is placed along the y-axis with one end at origin as shown in Figure.



In this region, a non-uniform magnetic field exists along z direction whose magnitude depends on the y coordinate as $B=B_0\left(1+\frac{y^2}{l^2}\right)$ T. If the conductor OA starts translating with a velocity $\vec{v}=v_0\hat{i}$, find the EMF induced across the ends of the conductor.

SOLUTION

Let us consider an element of width dy at a distance y from the origin as shown in Figure.



The motional emf induced across the element dy is

$$d\xi=Bv_0dy=B_0v_0\left(1+\frac{y^2}{l^2}\right)dy$$

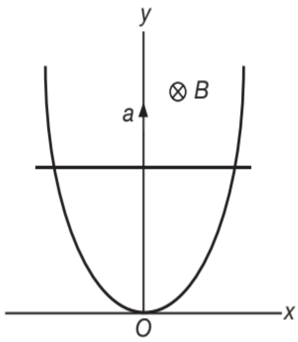
$$\Rightarrow \quad \xi_{OA}=\int d\xi=\int_0^l B_0v_0\left(1+\frac{y^2}{l^2}\right)dy$$

$$\Rightarrow \quad \xi_{OA}=B_0v_0\left(y+\frac{y^3}{3l^2}\right)\Bigg|_0^l=B_0v_0\left(l+\frac{l}{3}\right)$$

$$\Rightarrow \quad \xi_{OA}=\frac{4}{3}B_0v_0l$$

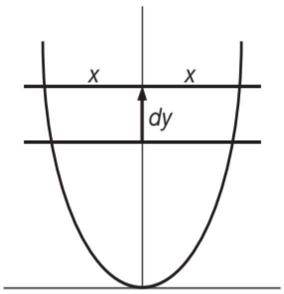
ILLUSTRATION 22

A wire bent as a parabola $y=kx^2$ is located in a uniform magnetic field of induction B, the vector B being perpendicular to the plane x, y. At the moment $t=0$ a connector starts sliding translation wise from the parabola apex with a constant acceleration a thus formed as a function of y.



SOLUTION

Let the connector be displaced through dy as shown in Figure.



The flux associated with the closed loop is then given by

$$d\phi = BdA = B(2x)dy$$

$$\Rightarrow \quad |\xi| = \frac{d\phi}{dt} = B(2x)\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \quad |\xi| = 2B\sqrt{\frac{y}{k}}\left(\frac{dy}{dt}\right) \qquad \left\{\because x = \sqrt{\frac{y}{k}}\right\}$$

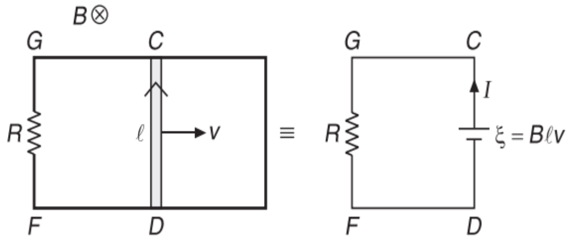
$$\Rightarrow \quad |\xi| = 2Bv\sqrt{\frac{y}{k}} \qquad \left\{\because \frac{dy}{dt} = v = \sqrt{2ay}\right\}$$

$$\Rightarrow \quad |\xi| = 2B\sqrt{2ay}\sqrt{\frac{y}{k}}$$

$$\Rightarrow \quad |\xi| = By\sqrt{\frac{8a}{k}}$$

INDUCED EMF IN A LOOP BY CHANGING ITS AREA

(a) Consider a straight conductor CD moving with velocity v towards right on a U-shaped conducting guide placed in a uniform magnetic field B directed into the page as shown in Figure.



As the conductor moves, the area of the loop $CDFG$ increases, causing a change in flux and hence an emf is induced in the coil. Let the loop move through dx in a time dt. Since

$$\xi = \frac{d\phi}{dt} = \frac{d}{dt}(BA) \qquad \text{(in magnitude)}$$

$$\Rightarrow \quad \xi = B\frac{dA}{dt} = Bl\left(\frac{dx}{dt}\right) \qquad \{\because A = l\,dx\}$$

$$\Rightarrow \quad \xi = Blv$$

If R is the resistance of the loop then the induced current is $I = \frac{Blv}{R}$. The direction of induced current is given by Fleming's Right Hand Rule and it will flow anticlockwise.

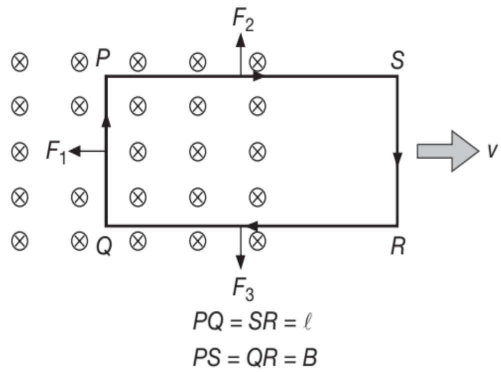
If instead of resistor a capacitor is used, then charge Q on the capacitor will be given by

$$Q = C\xi \text{, where } \xi = B\ell v$$

$$\Rightarrow \quad I = \frac{dQ}{dt} = \frac{d}{dt}(B\ell Cv) = (B\ell C)\left(\frac{dv}{dt}\right)$$

i.e., **for the current to be induced in the circuit, the conductor must move with some acceleration, not with constant velocity.**

(b) Consider the situation shown in the Figure in which let the current flows clockwise.



Further, the current in the loop will cause forces F_1, F_2 and F_3 to act on the three arms PQ, PS and QR respectively. F_2 and F_3, being equal and opposite, will cancel out, F_1 is given by

$$F_1 = BIl = \frac{B^2l^2v}{R} \qquad \left\{\because\ I = \frac{Blv}{R}\right\}$$

The power required to pull the loop is

$$P = F_1v = \frac{B^2l^2v^2}{R}$$

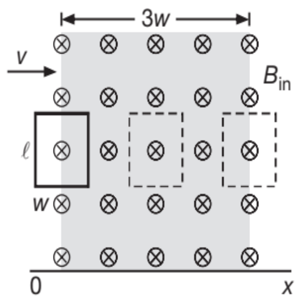
The rate at which Joule's heat is produced in the loop is

$$H = I^2R = \frac{B^2l^2v^2}{R}$$

which is same as P, as expected, according to the Law of Conservation of Energy.

ILLUSTRATION 23

A rectangular metallic loop of length l, width w and resistance R moves with constant speed v to the right, as in Figure.

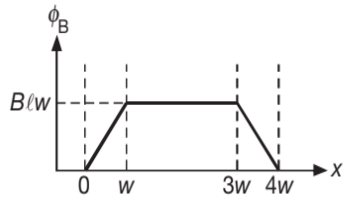


The loop passes through a uniform magnetic field $\vec{B}$ directed into the page and extending a distance $3w$ along the x-axis. Defining x as the position of the right side of the loop along the x-axis, plot as functions of x

(a) the magnetic flux through the area enclosed by the loop,
(b) the induced motional emf, and
(c) the external applied force necessary to counter the magnetic force and keep v constant.

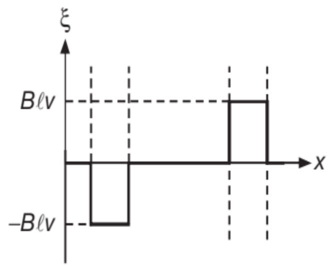
SOLUTION

(a) The flux through the area enclosed by the loop as a function of x is shown in Figure.

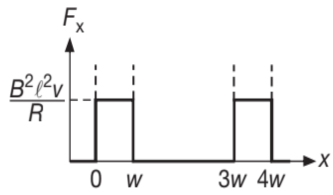


Before the loop enters the field, the flux is zero. As the loop enters the field, the flux increases linearly with position until the left edge of the loop is just inside the field and is given by the function $\phi_B = Bl(vt)$. Finally, the flux through the loop decreases linearly to zero as the loop leaves the field and is given by the function $\phi_B = Bl(w - vt)$.

(b) Before the loop enters the field, no motional emf is induced in it because no field is present. As the right side of the loop enters the field, the magnetic flux directed into the page increases. Hence, according to Lenz's Law, the induced current is counterclockwise because it must produce its own magnetic field directed out of the page. The motional emf $\xi = -\frac{d\phi_B}{dt} = -Blv$ arises from the magnetic force experienced by charges in the right side of the loop. When the loop is entirely in the field, the change in magnetic flux is zero, and hence the motional emf vanishes. This happens because, once the left side of the loop enters the field, the motional emf induced in it cancels the motional emf present in the right side of the loop. As the right side of the loop leaves the field, the flux begins to decrease, a clockwise current is induced, and the induced emf is $\xi = \frac{d\phi_B}{dt} = Blv$. As soon as the left side leaves the field, the emf decreases back to zero.



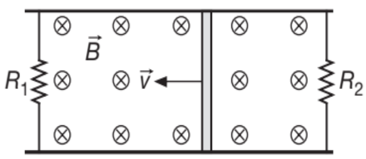
(c) The external force that must be applied to the loop to maintain this motion is plotted in Figure.



Before the loop enters the field, no magnetic force acts on it; hence, the applied force must be zero if v is constant. When the right side of the loop enters the field, the applied force necessary to maintain constant speed must be equal in magnitude and opposite in direction to the magnetic force exerted on that side. When the loop is entirely in the field, the flux through the loop is not changing with time. Hence, the net emf induced in the loop is zero, and the current also is zero. Therefore, no external force is needed to maintain the motion. Finally, as the right side leaves the field, the applied force must be equal in magnitude and opposite in direction to the magnetic force acing on the left side of the loop.
From this we conclude that power is supplied only when the loop is either entering or leaving the field. Also, we observe that the motional emf induced in the loop can be zero even when there is motion through the field. A motional emf is induced only when the magnetic flux through the loop changes with time.

ILLUSTRATION 24

A conducting rod of length l is free to side on two parallel conducting bars as in Figure.



Sliding Rod

In addition, two resistors R_1 and R_2 are connected across the ends of the bars. There is a uniform magnetic field pointing into the page. Suppose an external agent pulls the bar to the left at a constant speed v. Find the

(a) current through both resistors.
(b) total power delivered to the resistors.
(c) applied force required to maintain a constant velocity of the rod.

SOLUTION

(a) The emf induced between the ends of the moving rod is

$$\xi = \frac{d\phi_B}{dt} = -Blv$$

The currents through the resistors are

$$I_1 = \frac{|\xi|}{R_1} \text{ and } I_2 = \frac{|\xi|}{R_2}$$

Since the flux into the page for the left loop is decreasing, I_1 flows clockwise to produce a magnetic field pointing into the page. On the other hand, the flux into the page for the right loop is increasing. To compensate the change, according to Lenz's Law, I_2 must flow counterclockwise to produce a magnetic field pointing out of the page.

(b) The total power dissipated in the two resistors is

$$P_R = I_1|\xi| + I_2|\xi| = (I_1 + I_2)|\xi| = \xi^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$P_R = B^2 l^2 v^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

(c) The total current flowing through the rod is $I = I_1 + I_2$. Thus, the magnetic force acting on the rod is

$$F_B = BIl = |\xi| = \xi^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

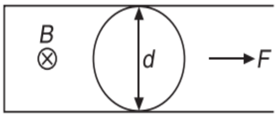
$$F_B = B^2 l^2 v^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

and the direction is to the right. Thus, an external agent must apply an equal but opposite force $\vec{F}_{ext} = -\vec{F}_B$ to the left in order to maintain a constant speed.
Alternatively, we note that since the power dissipated in the resistors must be equal to P_{ext}, the mechanical power supplied by the external agent. The same result is obtained since

$$P_{ext} = \vec{F}_{ext} \cdot \vec{v} = F_{ext} v$$

ILLUSTRATION 25

Two long parallel conducting horizontal rails are connected by a conducting wire at one end. A uniform magnetic field B directed vertically downwards exists in the region of space. A light uniform ring of diameter d (which is practically equal to separation between the rails), resistance per unit length λ, is placed over the rails as shown in Figure.



Calculate force required to pull the ring with uniform velocity.

SOLUTION

When the ring moves to the right, then due to its motion the emf induced in each of the two semicircles is

$$\xi = Bvl_{eq} = Bvd \qquad \ldots(1)$$

Due to the rightward motion of the ring, the inward flux associated with the rails increases, due to which an induced counter-clockwise current flows in the ring. These two semi-circular rings behave like two identical sources in parallel, each of emf $\xi = Bvd$ having internal resistance r given by

$$r = \left(\frac{\pi d}{2}\right)\lambda$$

The equivalent resistance of parallel combination of these two semi-circular rings is $r_{eq} = \frac{r}{2}$, so we have

$$r_{eq} = \frac{r}{2} = \frac{\pi\lambda d}{4}$$

Since the rails have negligible resistance, hence the equivalent resistance of the circuit is

$$R = \frac{r}{2} = \frac{\pi\lambda d}{4}$$

The induced current in circuit is

$$i = \frac{\xi}{R} = \frac{Bvd}{\left(\frac{\pi\lambda d}{4}\right)} = \frac{4Bv}{\pi\lambda}$$

Current through each semi-circular ring is

$$i' = \frac{i}{2} = \frac{2Bv}{\pi\lambda}$$

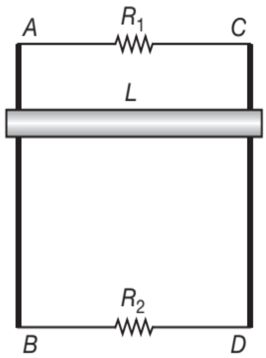
Force required to maintain the constant velocity of ring is equal to the opposing magnetic force experienced by the two semi-circular parts of the ring. So, we have

$$F_{ext} = 2(Bi'd) = 2B\left(\frac{i}{2}\right)d = Bid$$

$$\Rightarrow \quad F_{ext} = \frac{4B^2vd}{\pi\lambda}$$

ILLUSTRATION 26

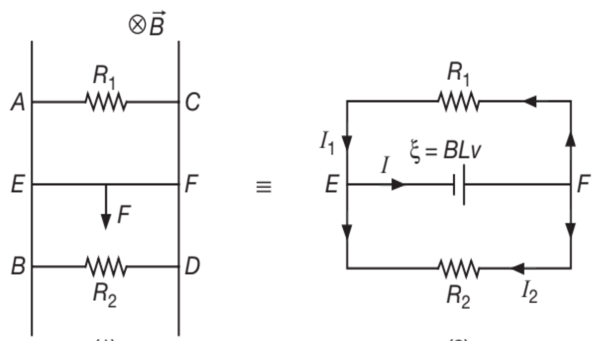
Two parallel vertical metallic rails AB and CD are separated by 1 m. They are connected at two ends by resistances R_1 and R_2 as shown in Figure.



A horizontal metallic bar of mass 0.2 kg slides without friction vertically down the rails under the action of gravity. There is a uniform horizontal magnetic field of 0.65 T perpendicular to the plane of the rails. It is observed that when the terminal velocity is attained, the powers dissipated in R_1 and R_2 are 0.76 W and 1.2 W respectively. Find the terminal velocity of the bar L and the values of R_1 and R_2.

SOLUTION

Let us assume the magnetic field to be perpendicular to the plane of rails and inwards ⊗. If v be the terminal velocity of the rails, then potential difference across E and F would be BLv with E at lower potential and F at higher potential. The equivalent circuit is shown in Figure (2).



(1) (2)

$$I_1 = \frac{\xi}{R_1} \qquad \ldots(1)$$

$$I_2 = \frac{\xi}{R_2} \qquad \ldots(2)$$

Power dissipated in R_1 is 0.76 W

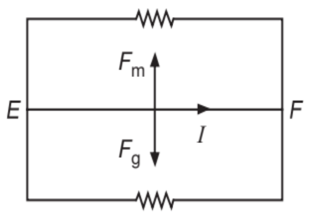
$$\Rightarrow \quad \xi I_1 = 0.76 \text{ W} \qquad \ldots(3)$$

Similarly, $\xi I_2 = 1.2$ W ...(4)

Now the total current in bar EF is

$$I = I_1 + I_2 \qquad \text{(from } E \text{ to } F\text{)} \qquad \ldots(5)$$

Under equilibrium condition i.e., when the terminal velocity is attained, magnetic force (F_m) on bar EF balances the weight (F_g) of bar EF.



$$\Rightarrow \quad F_m = F_g$$

$$\Rightarrow \quad BIL = mg \qquad \ldots(6)$$

$$\Rightarrow \quad I = \frac{mg}{LB} = \frac{(0.2)(9.8)}{(1)(0.6)} \text{ A} = 3.27 \text{ A}$$

Multiplying equation (5) by ξ, we get

$$\xi I = \xi I_1 + \xi I_2 = (0.76 + 1.2) \text{ W}$$

{from equation 3 and 4}

$$\Rightarrow \quad \xi I = 1.96 \text{ W}$$

$$\Rightarrow \quad \xi = \frac{1.96}{I} \text{ V} = \frac{1.96}{3.27} \text{ V} = 0.6 \text{ V}$$

Further since $\xi = BLv$

$$\Rightarrow \quad V = \frac{\xi}{BL} = \frac{(0.6)}{(0.6)(1)} \text{ ms}^{-1} = 1 \text{ ms}^{-1}$$

Hence, terminal velocity of bar is 1 ms^{-1}

Power in R_1 is 0.76 W

$$\Rightarrow \quad 0.76 = \frac{\xi^2}{R_1}$$

$$\Rightarrow \quad R_1 = \frac{\xi^2}{0.76} = \frac{(0.6)^2}{0.76} \ \Omega = 0.47 \ \Omega$$

Similarly, $R_2 = \dfrac{\xi^2}{1.2} = \dfrac{(0.6)^2}{1.2} \ \Omega = 0.3 \ \Omega$

ILLUSTRATION 27

An infinitesimally small bar magnet of dipole moment $\vec{M}$ is pointing and moving with the speed v in the $\vec{X}$-direction. A small closed circular conducting loop of radius a and negligible self-inductance lies in the y-z plane with its centre at $x = 0$, and its axis coinciding with the x-axis. Find the force opposing the motion of the magnet, if the resistance of the loop is R. Assume that the distance x of the magnet from the centre of the loop is much greater than a.

SOLUTION

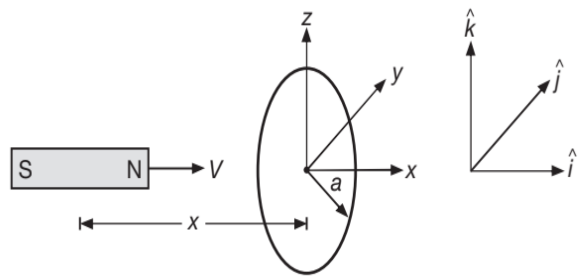
Since $B = \dfrac{\mu_0 I a^2}{2(a^2 + x^2)^{3/2}} = \dfrac{\mu_0 (I(\pi a^2))}{2\pi (a^2 + x^2)^{3/2}}$

$$B = \frac{\mu_0}{2\pi} \frac{M}{(a^2 + x^2)^{3/2}} \qquad \{\because M = I(\pi a^2)\}$$

It is given that $x \gg a$

Magnetic field at the centre of the coil due to the bar magnet is, then

$$B = \frac{\mu_o}{2\pi} \frac{M}{x^3}$$



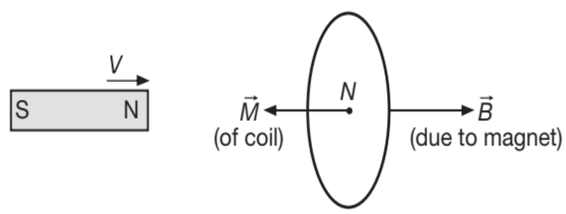
Due to this, magnetic flux linked with the coil is

$$\phi = BA = \frac{\mu_o}{2\pi} \frac{M}{x^3} (\pi a^2) = \frac{\mu_o M a^2}{2x^3}$$

Induced emf in the coil, due to motion of the magnet is,

$$\xi = -\frac{d\phi}{dt} = -\left(\frac{\mu_o M a^2}{2}\right) \frac{d}{dt}\left(\frac{1}{x^3}\right) = \frac{\mu_o M a^2}{2} \left(\frac{3}{x^4}\right) \frac{dx}{dt}$$

$$\xi = \frac{3}{2} \frac{\mu_o M a^2}{x^4} v \qquad \left\{\because \frac{dx}{dt} = v\right\}$$



Therefore, induced current in the coil is given by

$$I = \frac{\xi}{R} = \frac{3}{2}\frac{\mu_o Ma^2}{Rx^4}v$$

Magnetic moment of the coil due to this induced current is

$$M' = IA = \frac{3}{2}\frac{\mu_o Ma^2}{Rx^4}v(\pi a^2)$$

$$M' = \frac{3}{2}\frac{\mu_o \pi Ma^4 v}{Rx^4}$$

Potential energy of $\vec{M}'$ in $\vec{B}$ is given by

$$U = -\vec{M}'\cdot\vec{B} = -M'B\cos 180^o$$

$$\Rightarrow \quad U = M'B = \frac{3}{2}\frac{\mu_o \pi Ma^4 v}{Rx^4}\left(\frac{\mu_o}{2\pi}\frac{M}{x^3}\right)$$

$$\Rightarrow \quad U = \frac{3}{4}\frac{\mu_o^2 M^2 a^4 v}{R}\frac{1}{x^7}$$

Since $F = -\frac{dU}{dx}$

$$\Rightarrow \quad F = -\frac{dU}{dx} = \frac{21}{4}\frac{\mu_o^2 M^2 a^4 v}{Rx^8}$$

Positive sign of F implies that there will be a repulsion between the magnet and the coil.

Conceptual Note(s)

Here we cannot apply $F = \frac{\mu_o}{4\pi}\frac{6MM'}{x^4}$ directly, because M' is a function of x and this equation can only be applied at places where M and M' both are constants.

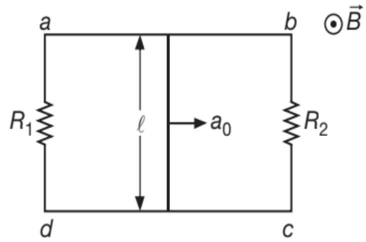
ILLUSTRATION 28

A rectangular loop with a sliding conductor of mass m, length l is located in a uniform magnetic field perpendicular to the plane of loop. The magnetic induction perpendicular to the plane of loop is equal to B. The part ad and bc has electric resistance R_1 and R_2 respectively. The conductor starts moving with constant acceleration a_0 at time $t = 0$. Neglecting the self-inductance of the loop and resistance of conductor. Find

(a) the current through the conductor during its motion.

(b) the polarity of $abcd$ terminal.

(c) external force required to move the conductor with the given acceleration.



SOLUTION

(a) $I = \frac{\xi}{R} = \frac{Blv}{R}$

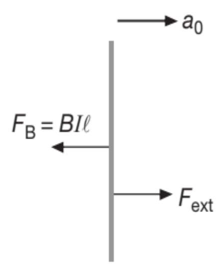
where $v = a_0 t$ and $R = \frac{R_1 R_2}{R_1 + R_2}$

$$\Rightarrow \quad I = \frac{Bla_0 t(R_1 + R_2)}{R_1 R_2} \qquad \ldots(1)$$

(b) Flux linked with the left loop is increasing, so the induced current must set up a field that does not allow the flux to increase i.e., an inward field is set up, so the induced current goes from d to a. Hence a has negative polarity.
Similarly, flux linked with the right loop is decreasing, so the induced current must be in clockwise sense so that outward flux does not decrease. Hence b is also having negative polarity.

(c) According to Newton's Second Law,

$$F_{ext} - F_B = ma_0 \qquad \ldots(2)$$



where F_B is the magnetic force acting on the conductor due to the induced current I. So,

$$F_B = BIl = B\left[\frac{Bla_0 t(R_1 + R_2)}{R_1 R_2}\right]l$$

Using equation (2), we get

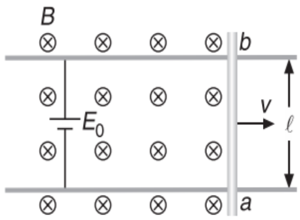
$$F_{ext} = F_B + ma_0$$

$$\Rightarrow F_{ext} = \frac{B^2 l^2 a_0 t (R_1 + R_2)}{R_1 R_2} + ma_0$$

$$\Rightarrow F_{ext} = a_0\left(m + \frac{B^2 l^2 (R_1 + R_2) t}{R_1 R_2}\right)$$

ILLUSTRATION 29

A bar of mass m, length l and resistance R slides without friction in a horizontal plane, moving on parallel rails as shown in Figure. A battery that maintains a constant emf E is connected between the rails, and a constant magnetic field $\vec{B}$ is directed perpendicularly to the plane of the page. Assuming the bar starts from rest, find the velocity of the rod as a function of time. Also find the current in the loop.



SOLUTION

(a) Let at time t velocity of rod be v (towards right) and current in the circuit is I (from b to a). The magnetic force on it is BIl (towards right). Writing the equation of motion of the rod, we get

$$m\left(\frac{dv}{dt}\right) = BIl = B\left(\frac{E_0 - Blv}{R}\right)l$$

$$\Rightarrow \int_0^v \frac{dv}{\frac{E_0 Bl}{mR} - \frac{B^2 l^2}{mR} v} = \int_0^t dt$$

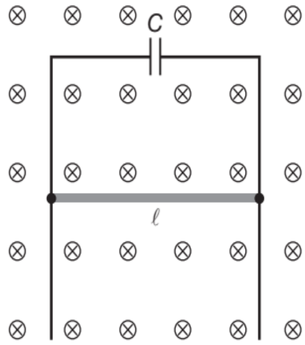
$$\Rightarrow v = \frac{E_0}{Bl}\left(1 - e^{-\frac{B^2 l^2}{mR} t}\right)$$

(b) $$I = \frac{E + E_{ind}}{R} = \frac{E_0 - Blv}{R} = \frac{E_0 - E_0\left(1 - e^{-\frac{B^2 l^2}{mR} t}\right)}{R}$$

$$\Rightarrow I = \frac{E_0}{R} e^{-\left(\frac{B^2 l^2}{mR}\right) t}$$

ILLUSTRATION 30

Two metal bars are fixed vertically and are connected on the top by a capacitor C. A sliding conductor length slides with its ends in contact with the bars. The arrangement is placed in a uniform horizontal magnetic field directed normal to the plane of the Figure. The conductor is released from rest. Find the displacement $x(t)$ of the conductor as a function of time t.



SOLUTION

The motion of the conductor in the magnetic field induces an emf in it. As a result of this, an induced current, say I, flows through the conductor. This induced current will be responsible for the magnetic force BIl which opposes the motion of conductor. At any instant, if v is the velocity of the conductor, then the net downward force F acting on the conductor is

$$F = mg - BIl \qquad \ldots(1)$$

The charge Q on the capacitor is given by

$$Q = C\xi = C(Blv) = (BlC)v$$

So, we get $I = \frac{dQ}{dt} = BlC\left(\frac{dv}{dt}\right)$ ...(2)

From (1), we get

$$m\left(\frac{dv}{dt}\right) = mg - B\left[BlC\left(\frac{dv}{dt}\right)\right]l$$

$$\Rightarrow m\left(\frac{dv}{dt}\right) = mg - B^2 l^2 C\left(\frac{dv}{dt}\right)$$

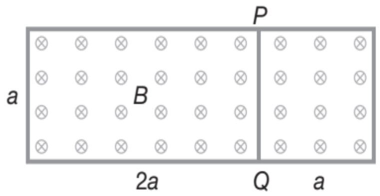
$$\Rightarrow a = \frac{dv}{dt} = \frac{mg}{m + B^2 l^2 C} = \text{constant}$$

So, we have $x(t) = ut + \frac{1}{2}at^2 = (0)t + \frac{1}{2}at^2 \quad \{\because u = 0\}$

$$\Rightarrow x(t) = \frac{1}{2}\left(\frac{mg}{m + B^2 l^2 C}\right)t^2$$

ILLUSTRATION 31

Find the current through section PQ of length a in Figure. The circuit is located in a time varying magnetic field $B = B_0 t$. Assume the resistance per length of the wire is λ.



SOLUTION

First of all, we must understand how to get the current direction in the left loop $PRSQP$ and the right loop $PTUQP$. Since B is increasing with t, so the induced current in both the loops must not allow B to increase, as a result of which the branches must have induced current which opposes the increase in the value of B inwards. So, an outward B must be set up due to currents in both the loops. The directions taken satisfy the argument supplied. Now, applying Kirchhoff's Loop Law for loops $PRSQP$ and $PTUQP$, we get

$$E_1 - (5\lambda a)I_1 - (\lambda a)I_{PQ} = 0 \quad \text{...(1)}$$

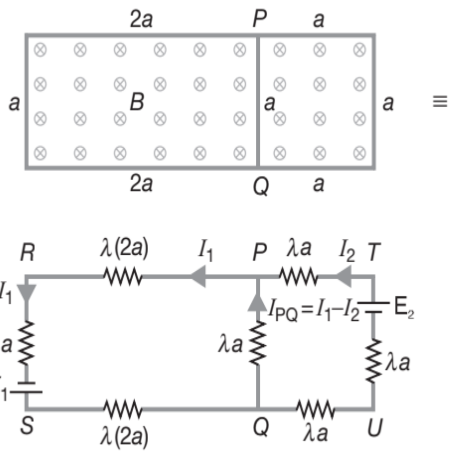
and $-E_2 + (3\lambda a)I_2 - (\lambda a)I_{PQ} = 0$

$$\Rightarrow \quad E_2 - (3\lambda a)I_2 + (\lambda a)I_{PQ} = 0 \quad \text{...(2)}$$

where $E_1 = \left(A_{\text{left loop}}\right)\dfrac{dB}{dt} = (2a^2)B_0$,

$$E_2 = \left(A_{\text{right loop}}\right)\frac{dB}{dt} = a^2 B_0$$

and $I_2 + I_{PQ} = I_1$ i.e., $I_{PQ} = I_1 - I_2$



So, equations (1) and (2) are rewritten as

$$2a^2B_0 - (5\lambda a)I_1 - (\lambda a)(I_1 - I_2) = 0$$

$$\Rightarrow \quad 2a^2B_0 - (6\lambda a)I_1 + (\lambda a)I_2 = 0 \quad \text{...(3)}$$

$$a^2B_0 - (3\lambda a)I_2 + (\lambda a)(I_1 - I_2) = 0$$

$$\Rightarrow \quad a^2B_0 + (\lambda a)I_1 - (4\lambda a)I_2 = 0 \quad \text{...(4)}$$

$6(4) + (3)$, gives

$$8a^2B_0 - (23\lambda a)I_2 = 0$$

$$\Rightarrow \quad I_2 = \frac{8aB_0}{23\lambda}$$

Similarly, doing $4(3) + (4)$, we get

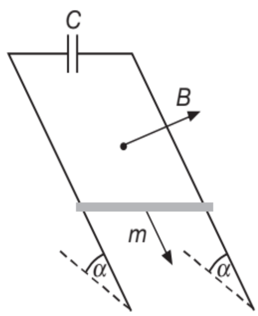
$$9a^2B_0 - (23\lambda a)I_1 = 0$$

$$\Rightarrow \quad I_1 = \frac{9aB_0}{23\lambda}$$

$$\Rightarrow \quad I_{PQ} = I_1 - I_2 = \frac{aB_0}{23\lambda}$$

ILLUSTRATION 32

A copper connector of mass m slides down two smooth copper bars, set at an angle α to the horizontal, due to gravity. At the top the bars are interconnected through a capacitor C. The separation between the bars is equal to l. The system is located in a uniform magnetic field of induction B, perpendicular to the plane in which the connector slides. The resistances of the bars, the connector and the sliding contacts, as well as the self-inductance of the loop, are assumed to be negligible. Find the acceleration of the connector.



SOLUTION

Applying Newton's Second Law to the connector, we get

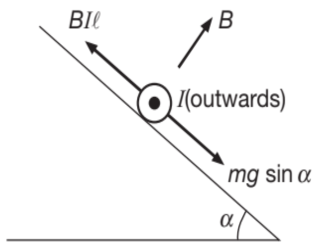
$$mg\sin\alpha - BIl = ma \quad \text{...(1)}$$

The emf induced across the connector is $\xi = Blv$

Potential difference across the capacitor plates is

$$\xi = \frac{q}{C}$$

$$\Rightarrow \quad q = C\xi = C(Blv)$$



The induced current in the circuit is then given by

$$I = \frac{dq}{dt} = BlC\left(\frac{dv}{dt}\right) \qquad \ldots(2)$$

Substituting (2) in (1), we get

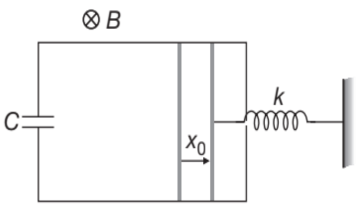
$$mg\sin\alpha - B\left(BlC\frac{dv}{dt}\right)l = ma$$

$$\Rightarrow \quad mg\sin\alpha = \left(m + B^2l^2C\right)a \qquad \left\{\because a = \frac{dv}{dt}\right\}$$

$$\Rightarrow \quad a = \frac{mg\sin\alpha}{m + B^2l^2C}$$

ILLUSTRATION 33

In the arrangement shown in the Figure there is a uniform magnetic field B_0 normal to the plane of paper. The connector is smooth and conducting and it has a mass m and length l. The connector is pushed against the spring so that the spring has compression x_0. The connector is released at $t = 0$. Find the time it will take to come to its original position again. The spring is non-conducting. The resistance of the rails is zero and neglect its self-inductance.

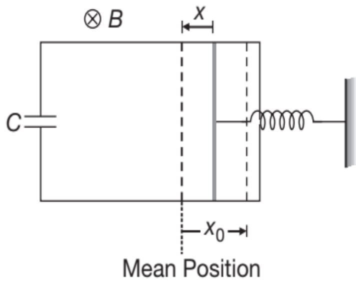


SOLUTION

Let at any time t, the rod be at a distance x from its original position. Then

$$v = \frac{dx}{dt}$$

$$\Rightarrow \quad \xi = Blv = Bl\left(\frac{dx}{dt}\right)$$



At any instant, the charge on the capacitor is

$$Q = C\xi = BlC\left(\frac{dx}{dt}\right)$$

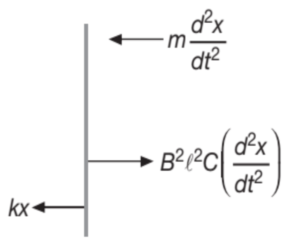
$$\Rightarrow \quad I = \frac{dQ}{dt} = BlC\left(\frac{d^2x}{dt^2}\right)$$

Force on the conductor due to this current is

$$F = BIl$$

$$\Rightarrow \quad F = B\left(BlC\frac{d^2x}{dt^2}\right)l$$

$$\Rightarrow \quad F = B^2l^2C\left(\frac{d^2x}{dt^2}\right)$$



So, net force on the rod is

$$m\frac{d^2x}{dt^2} = kx - B^2l^2C\left(\frac{d^2x}{dt^2}\right)$$

$$\Rightarrow \quad \left(m + B^2l^2C\right)\frac{d^2x}{dt^2} = kx$$

So, we observe that the acceleration of the rod is proportional to x and is directed opposite to it towards the mean position. Hence the motion of the rod is simple harmonic about the mean (initial) position, with period

$$T = 2\pi\sqrt{\frac{x}{\left|\frac{d^2x}{dt^2}\right|}}$$

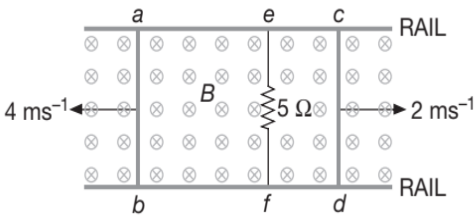
$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m + B^2l^2C}{k}}$$

If t is the time taken by the rod to come to its initial position, then

$$t = \frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{m + B^2l^2C}{k}}$$

ILLUSTRATION 34

Two parallel rails with negligible resistance are 10 cm apart and are connected by a 5 Ω resistor. The circuit also contains two metal rods 1 and 2 having resistances of 10 Ω and 15 Ω sliding along the rails. The rods are pulled away from the resistor at constant speeds of 4 ms^{-1} and 2 ms^{-1}, respectively. A uniform magnetic field of magnitude 10 mT is applied perpendicular to the plane of the rails. Determine the current in the 5 Ω resistor.



SOLUTION

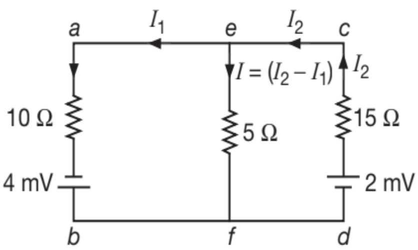
Let us first apply Fleming's Right Hand Rule to find the direction of induced current in Rod ab and Rod cd. Let I_1 and I_2 be the induced current in rod ab and cd respectively. Then I_1 is from a to b and I_2 is from d to c. Also, these two moving rods will produce an induced emf E_1 and E_2 whose magnitudes are given by

$$E_1 = Blv_1 \text{ (for } ab\text{) and } E_2 = Blv_2 \text{ (for } cd\text{)}$$

$$\Rightarrow \quad E_1 = (0.01)\left(\frac{10}{100}\right)(4) = 4\times10^{-3} \text{ V} = 4 \text{ mV, and}$$

$$\Rightarrow \quad E_2 = (0.01)\left(\frac{10}{100}\right)(2) = 2\times10^{-3} \text{ V} = 2 \text{ mV}$$

So, the equivalent circuit for the above arrangement is shown below



If I be the current in the branch ef or the 5 Ω resistor, then

$$I = I_2 - I_1 \qquad \ldots(1)$$

Applying Kirchhoff's Loop rule to the two loops we get

For loop, $abfea$

$$-10I_1 + 4 + 5(I_2 - I_1) = 0$$

$$\Rightarrow \quad 15I_1 - 5I_2 = 4 \qquad \ldots(2)$$

For loop, $cdfec$

$$15I_2 - 2 + 5(I_2 - I_1) = 0$$

$$-5I_1 + 20I_2 = 2 \qquad \ldots(3)$$

$3 \times$ ③ + ②, gives

$$60I_2 - 5I_2 = 10$$

$$55I_2 = 10$$

$$\Rightarrow \quad I_2 = \frac{10}{55} \text{ mA}$$

Similarly, $4 \times$ ② + ③ gives

$$60I_1 - 5I_1 = 18$$

$$\Rightarrow \quad I_1 = \frac{18}{55} \text{ mA}$$

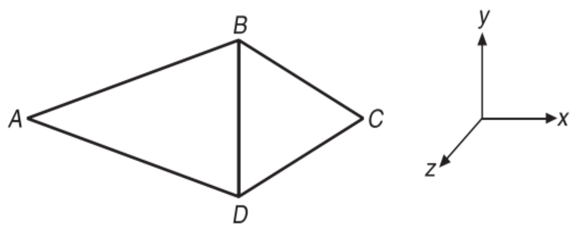
$$\Rightarrow \quad I = I_2 - I_1 = -\frac{8}{55} \text{ mA}$$

$$\Rightarrow \quad I = \frac{8}{55} \text{ mA from } f \text{ to } e \text{ (not from } e \text{ to } f \text{ as taken)}$$

$$\Rightarrow \quad I \cong 145\ \mu\text{A}$$

ILLUSTRATION 35

A wire loop $ABCD$ is divided into two parts. $AB = AD = 2$ m, $BC = CD = BD = 1$ m. The wire loop is lying on X-Y plane. The resistance per unit length of the wire is $2\ \Omega\text{m}^{-1}$. There exists a time dependent magnetic field $\vec{B} = (2.5t)\hat{k}$ T. Find the current flowing through each part of the loop.



SOLUTION

Since $|\vec{B}| = B = 2.5t$

$$\Rightarrow \quad \frac{dB}{dt} = 2.5 \text{ Ts}^{-1}$$

Area of triangle ABD is

$$A_1 = \frac{1}{2}\left(\frac{1}{2}\right)\left(\sqrt{4-\frac{1}{4}}\right) = \frac{\sqrt{15}}{4} \text{ m}^2$$

Area of triangle BCD is

$$A_2 = \frac{\sqrt{3}}{4} \text{ m}^2$$

Induced emf in ABD is

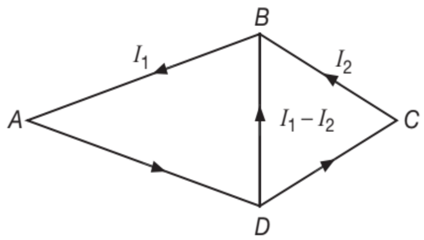
$$\xi_1 = \left|\frac{d\phi}{dt}\right| = A_1\left|\frac{dB}{dt}\right| = \frac{5\sqrt{15}}{8} \text{ V}$$

Induced emf in BCD is

$$\xi_2 = \frac{5\sqrt{3}}{8} \text{ V}$$

Resistance of $AB = AD = 4\ \Omega$

Resistance of $BC = CD = BD = 2\ \Omega$



Applying Kirchhoff's Loop Law to loop $BADB$, we get

$$\frac{5\sqrt{15}}{8} = 4I_1 + 4I_1 + 2(I_1 - I_2) \qquad \ldots(1)$$

Applying Kirchhoff's Loop Law to loop $BDCB$, we get

$$\frac{5\sqrt{3}}{8} = 4I_2 - 2(I_1 - I_2) \qquad \ldots(2)$$

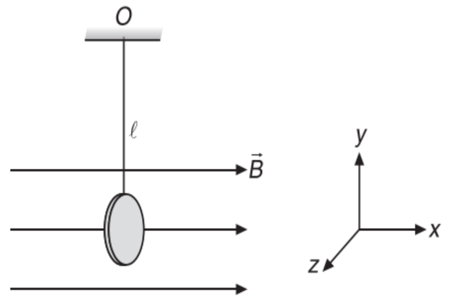
From equations (1) and (2), we get

$$I_1 = 0.3 \text{ A and } I_2 = 0.28 \text{ A}$$

So, $I_{BA} = I_{AD} = 0.3$ A, $I_{DC} = I_{CB} = 0.28$ A and $I_{DB} = 0.02$ A

ILLUSTRATION 36

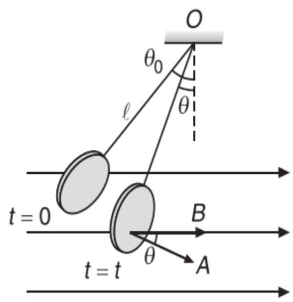
A small circular coil of area A is suspended from a point O by a string of length l in a uniform magnetic induction B acting the horizontal direction as shown in Figure.



Let the coil be set into oscillations at $t = 0$ just like a simple pendulum by displacing it through a small angle θ_0. Calculate the emf induced in the coil as a function of time. Assume that initially the coil lies in yz plane and it does not change its plane during the entire course of its motion.

SOLUTION

At time $t = 0$ and at time t after being released from the angular position θ_0, the positions of the coil are shown in Figure.



The angular position of the coil is given by

$$\theta = \theta_0 \cos\omega t, \text{ where } \omega = \sqrt{\frac{g}{l}}$$

At time t, the magnetic flux associated with the coil is

$$\phi = BA\cos\theta$$

EMF induced in the coil is

$$\xi = \left|\frac{d\phi}{dt}\right| = (BA\sin\theta)\frac{d\theta}{dt}$$

Since $\dfrac{d\theta}{dt} = \dfrac{d}{dt}(\theta_0\cos\omega t) = -\theta_0\omega\sin\omega t$

According to the problem, θ_0 is small, so θ is also small an hence, we have $\sin\theta \approx \theta$

$$\Rightarrow \quad \xi = BA\theta\frac{d\theta}{dt} = BA(\theta_0\cos\omega t)\frac{d}{dt}(\theta_0\cos\omega t)$$

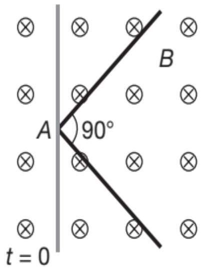
$$\Rightarrow \quad \xi = BA\left(\theta_0\cos\sqrt{\frac{g}{l}}t\right)\left(\theta_0\omega\sin\sqrt{\frac{g}{l}}t\right)$$

$$\Rightarrow \quad \xi = BA\omega\theta_0^2 \cos\left(\sqrt{\frac{g}{l}}t\right)\sin\left(\sqrt{\frac{g}{l}}t\right)$$

$$\Rightarrow \quad \xi = \frac{1}{2}BA\omega\theta_0^2 \sin\left(2\sqrt{\frac{g}{l}}t\right)$$

ILLUSTRATION 37

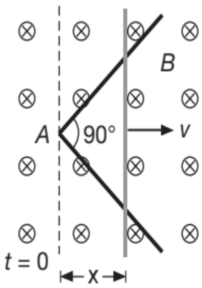
Figure shows a wire bent into the shape of a right angle that is fixed on a horizontal plane. A very long rod of mass m initially starts with a velocity v_0 from the apex A of the bent wire. The resistance of the wire and the rod is $\lambda\ \Omega\text{m}^{-1}$. The whole arrangement is placed in a region of uniform magnetic induction B. Find the distance travelled by the rod before it coming to rest.



SOLUTION

Let x be the distance travelled by the rod in time t, then the resistance of the circuit is

$$R = \left(2\sqrt{2}x + 2x\right)\lambda$$



The instantaneous induced emf is

$$E = B(2x)v$$

Induced current $I = \dfrac{E}{R} = \dfrac{Bv}{\lambda(1+\sqrt{2})}$

Magnetic force on the rod is

$$F = BI(2x) = \frac{2B^2vx}{\lambda(1+\sqrt{2})}$$

From Newton's Second Law, we have

$$F = -m\frac{dv}{dt} = \frac{2B^2vx}{\lambda(1+\sqrt{2})} \quad \ldots(1)$$

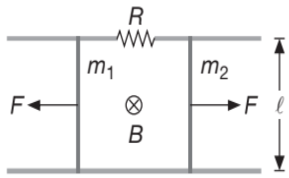
Eliminating t from the equation (1) and then integrating, we get

$$-\int_{v_0}^{0} dv = \left(\frac{2B^2}{m\lambda(1+\sqrt{2})}\right)\int_0^x x\,dx$$

$$\Rightarrow \quad x = \frac{\sqrt{mv_0\lambda(1+\sqrt{2})}}{B}$$

ILLUSTRATION 38

The arrangement shown is placed in a vertical uniform magnetic field. Two metal rods of length l and masses m_1 and m_2 are pulled apart from rest by a constant force F. Find the current in the resistor R as a function of time.



SOLUTION

Let v_1 and v_2 be the instantaneous velocity force the rods m_1 and m_2 respectively. The emf's across the rod will be

$$E_1 = Blv_1 \text{ and } E_2 = Blv_2$$

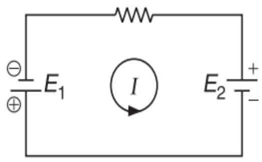
The instantaneous current is

$$I = \frac{E_1 + E_2}{R} = \frac{Bl(v_1+v_2)}{R} \quad \ldots(1)$$

The acceleration of each rod is given by

$$\frac{dv_1}{dt} = \frac{F - BIl}{m_1} = \frac{F}{m_1} - \frac{B^2l^2}{m_1R}(v_1+v_2)$$

and $$\frac{dv_2}{dt} = \frac{F}{m_2} - \frac{B^2l^2}{m_2R}(v_1+v_2)$$



Adding the above two equations, we get

$$\frac{d}{dt}(v_1+v_2) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left[F - \frac{B^2l^2}{R}(v_1+v_2)\right]$$

Using equation (1), we get

$$v_1 + v_2 = \frac{IR}{Bl}$$

$$\Rightarrow \quad \frac{d}{dt}(v_1 + v_2) = \left(\frac{R}{Bl}\right)\left(\frac{dI}{dt}\right)$$

$$\Rightarrow \quad \frac{R}{Bl}\frac{dI}{dt} = \frac{F}{\mu} - \frac{BlI}{\mu}$$

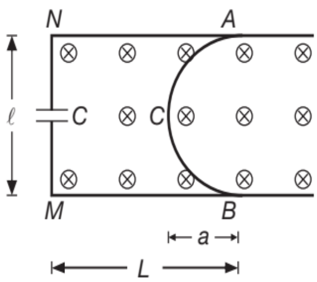
where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system

$$\Rightarrow \quad \frac{dI}{F - BlI} = \frac{Bl}{\mu R}dt$$

$$\Rightarrow \quad I = \frac{F}{Bl}\left[1 - e^{-B^2l^2t/\mu R}\right]$$

ILLUSTRATION 39

Two long parallel conducting rods with negligible resistance are connected by capacitor C and semi-circular conducting wire in the same plane as shown. This system lies in a perpendicular magnetic field. Now loop ACB is moved on rails with its displacement given by $x = a\sin(\omega t)$ from the initial position. Find the maximum current in the capacitor.

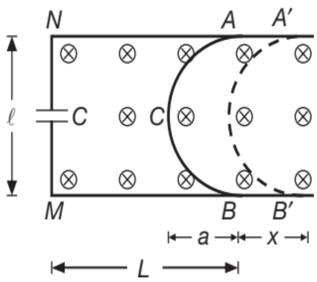


SOLUTION

Let in initial position $BM = AN = L$. At any time area of closed loop

$$A'NMB' = A$$

$$A = (L + x)d - \pi\frac{d^2}{8}$$



Flux through the loop, $\phi = BA = B\left(Ll - \frac{\pi l^2}{8} + xl\right)$

emf induced,

$$\xi = \left|\frac{d\phi}{dt}\right| = \frac{d}{dt}\left(Ll - \frac{\pi l^2}{8}\right) + \frac{d}{dt}(Blx)$$

$$\Rightarrow \quad \xi = Bl\frac{dx}{dt} = Bl\frac{d}{dt}(a\sin(\omega t)) = Bal\omega\cos(\omega t)$$

As net potential drop in the loop is zero,

$$\xi = \frac{q}{C}$$

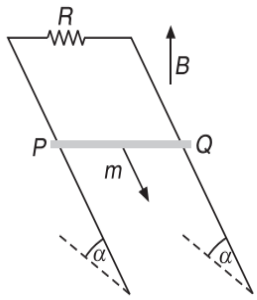
$$\Rightarrow \quad Q = CBal\omega\cos(\omega t)$$

$$\Rightarrow \quad I = \frac{dQ}{dt} = -CBal\omega^2\sin(\omega t)$$

$$\Rightarrow \quad I_{max} = CBal\omega^2$$

ILLUSTRATION 40

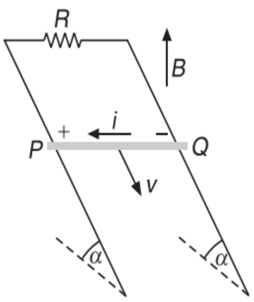
A copper rod PQ of mass m slides down two smooth copper bars which are set at an angle α to the horizontal as shown in Figure.



At the top of the bars these are interconnected through a resistance R. The separation between the bars is equal to l. The system is located in a uniform magnetic field of induction B directed vertically upwards. The resistances of the bars, the rod and the sliding contacts are considered to be negligible. If the rod is released from rest, calculate the velocity of rod as a function of time and its steady velocity attained.

SOLUTION

When the rod is released from rest, then is starts moving down the incline due to which it cuts the magnetic flux and an emf is induced across it. The induced current in the rod flows from Q to P as shown in Figure.



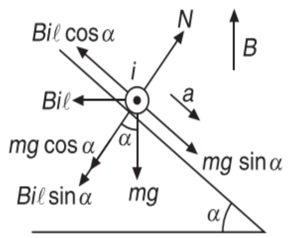
At any instant, the conductor velocity v is down the incline. The magnetic field component $B\sin\alpha$ acts opposite to the velocity and hence will not be responsible for producing the induced emf. However, the component of magnetic field $B\cos\alpha$ is perpendicular to the velocity as well as the length of the conductor, so the emf induced across the conductor is

$$\xi = B_{\perp} lv = (B\cos\alpha)lv = Blv\cos\alpha$$

The current induced i in the loop containing the resistance is

$$i = \frac{Blv\cos\alpha}{R}$$

Due to this induced current in the rod, the rod experiences a magnetic force $F = Bil$, leftwards as shown in Figure.



If the rod slides down the incline with an acceleration a, then by applying Newton's Second law, we get

$$mg\sin\alpha - Bil\cos\alpha = ma$$

$$\Rightarrow \quad m\left(\frac{dv}{dt}\right) = mg\sin\alpha - \frac{B^2l^2v\cos^2\alpha}{R} \qquad \ldots(1)$$

$$\Rightarrow \quad dv = \left(g\sin\alpha - \frac{B^2l^2v\cos^2\alpha}{mR}\right)dt$$

$$\Rightarrow \quad \int_0^v \frac{dv}{mgR\sin\alpha - (B^2l^2\cos^2\alpha)v} = \int_0^t \frac{dt}{mR}$$

$$\Rightarrow \quad -\frac{\ln\left(mgR\sin\alpha - B^2l^2v\cos^2\alpha\right)}{B^2l^2\cos^2\alpha}\Bigg|_0^v = \frac{t}{mR}$$

$$\Rightarrow \quad \ln\left(\frac{mgR\sin\alpha - B^2l^2v\cos^2\alpha}{mgR\sin\alpha}\right) = -\left(\frac{B^2l^2\cos^2\alpha}{mR}\right)t$$

$$\Rightarrow \quad v = \frac{mgR\sin\alpha}{B^2l^2\cos^2\alpha}\left(1 - e^{-\frac{B^2l^2\cos^2\alpha}{mR}t}\right)$$

When steady velocity (also called Terminal velocity) is attained, then acceleration of the rod is zero, so from equation (1), we get

$$v_S = \frac{mgR\sin\alpha}{B^2l^2\cos^2\alpha}$$

Theoretically, steady velocity is obtained after a long time i.e. when $t \to \infty$.

EMF INDUCED ACROSS THE ENDS OF A CONDUCTING ROD ROTATING IN A UNIFORM MAGNETIC FIELD

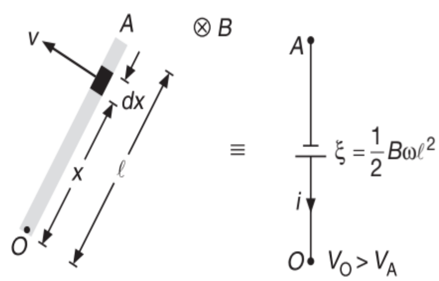
METHOD I

Consider a conducting rod of length l rotating about the point O (at one end of the rod) in a uniform magnetic field B. To find the emf induced across the ends of the rod, let us consider an infinitesimal element of length dx at a distance x from O, having a velocity v, as shown. If $d\xi$ be the induced emf across the element, then

$$d\xi = B(dx)v\text{, where } v = x\omega$$

$$\Rightarrow \quad d\xi = B\omega x dx$$

$$\Rightarrow \quad \xi = B\omega\int_0^l x dx = B\omega\frac{x^2}{2}\Bigg|_0^l = \frac{1}{2}B\omega l^2$$



METHOD II

When the rod is rotating in the field with angular velocity ω, then the induced emf is

$$\xi = B\left(\frac{\text{Area swept by the Rod}}{\text{Time to complete one Revolution}}\right)$$

$$\Rightarrow \quad \xi = \frac{B(\pi l^2)}{\frac{2\pi}{\omega}}$$

$$\Rightarrow \quad \xi = \frac{1}{2}B\omega l^2$$

ILLUSTRATION 41

A thin wire AC shaped as a semi-circle of diameter d rotates with a constant angular velocity ω in a uniform magnetic field of induction B, with $\omega \parallel B$. The rotation axis passes through the end A of the wire and is perpendicular to the diameter AC. Find the value of a line integral $\int \vec{E}\cdot d\vec{l}$ along the wire from point A to point C. Generalize the obtained result for an arbitrary shape of wire between A and C.

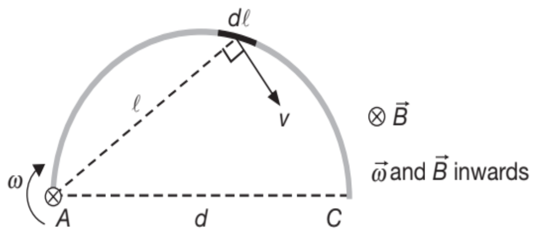
SOLUTION

Since $d\xi = (\vec{v}\times\vec{B})\cdot d\vec{l}$ ...(1)

Also $d\xi = -\vec{E}\cdot d\vec{l}$ ...(2)

So, from (1) and (2), we get

$$\int \vec{E}\cdot d\vec{l} = -\int(\vec{v}\times\vec{B})\cdot d\vec{l} = -\int vBdl$$

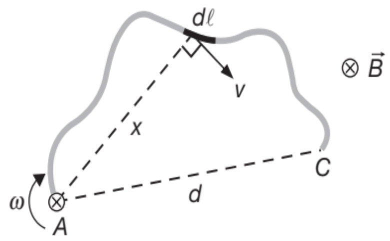


But $v = l\omega$, where l is the perpendicular distance of the element from A

$$\Rightarrow \quad \int_A^C \vec{E}\cdot d\vec{l} = -B\omega\int_0^d l\,dl = -\frac{1}{2}B\omega d^2$$

The above result can be generalised and extended to a wire AC of any arbitrary shape.
Since, from above we have

$$\int \vec{E}\cdot d\vec{l} = -\int(\vec{v}\times\vec{B})\cdot d\vec{l}$$



Further from our knowledge of rotational kinematics, we know that

$$\vec{v} = \vec{\omega}\times\vec{l}$$

$$\Rightarrow \quad \int \vec{E}\cdot d\vec{l} = -\int((\vec{\omega}\times\vec{l})\times\vec{B})\cdot d\vec{l} = \int(\vec{B}\times(\vec{\omega}\times\vec{l}))\cdot d\vec{l}$$

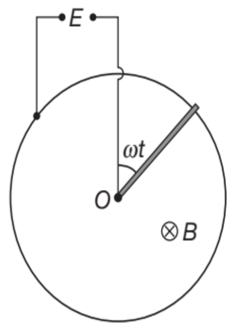
Also, we know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$\Rightarrow \quad \int \vec{E}\cdot d\vec{l} = \int((\vec{B}\cdot\vec{l})\vec{\omega} - (\vec{B}\cdot\vec{\omega})\vec{l})\cdot d\vec{l}$$

$$\Rightarrow \quad \int_A^C \vec{E}\cdot d\vec{l} = 0 - B\omega\int_0^d l\,dl = -\frac{1}{2}B\omega d^2$$

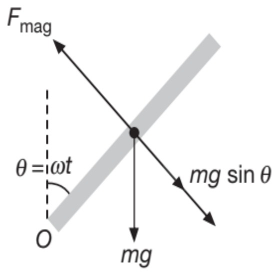
ILLUSTRATION 42

A metal rod of mass m can rotate about a horizontal axis O, sliding along a circular conductor of radius r. The arrangement is located in a uniform magnetic field of induction B directed perpendicular to the ring plane. The axis and the ring are connected to a source of emf E to form a circuit of resistance R. Neglecting the friction, circuit inductance, and ring resistance, find the expression according to which the source emf must vary to make the rod rotate with a constant angular velocity ω.



SOLUTION

Assume the rod to rotate in the clockwise sense. Due to rotation, the emf induced across the ends of the rod is $\xi = \frac{1}{2}B\omega r^2$



Net Emf in the rod-ring circuit is

$$E_{net} = E - \xi$$

The net current induced in the rod is

$$I = \frac{E_{net}}{R} = \frac{E-\xi}{R}$$

$$\Rightarrow \quad I = \frac{E - \frac{1}{2}B\omega r^2}{R} \qquad \ldots(1)$$

Due to this induced current, the magnetic force $F_m = BIr$ acts on the rod as shown. Simultaneously mg also acts on the rod. Now we want the rod to move with a constant angular velocity. Hence for this

$$\sum \vec{\tau}_{mag} = \sum \vec{\tau}_{gravity}$$

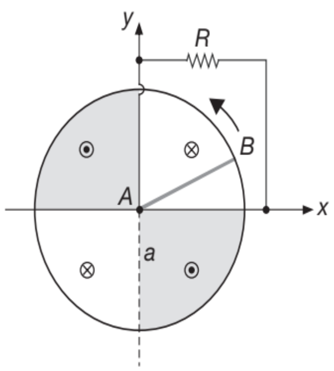
$$\Rightarrow \quad (mg\sin\theta)\frac{r}{2} = (BIr)\frac{r}{2}$$

$$\Rightarrow \quad mg\sin(\omega t) = B\left(\frac{E - \frac{1}{2}B\omega r^2}{R}\right)r$$

$$\Rightarrow \quad E = \frac{1}{2rB}\left(r^3B^2\omega + 2mgR\sin(\omega t)\right)$$

ILLUSTRATION 43

In a cylindrical region of radius a, magnetic field exists along its axis but the direction of magnetic field is opposite in the four quadrants of the region as shown in Figure.



A rod AB rotates with its end A at the centre of magnetic field and other end B slides on a smooth wire at the periphery of the region of magnetic field. At $t=0$ the rod was situated along the $+x$ direction. Find and plot the time dependence of the current and thermal power in the resistance R, if the rod rotates with a

(a) constant angular velocity ω.

(b) constant angular acceleration α.

SOLUTION

(a) When the rod rotates with constant angular speed ω

$$\xi = \pm\frac{1}{2}B(a)(a\omega) = \pm\frac{1}{2}Ba^2\omega$$

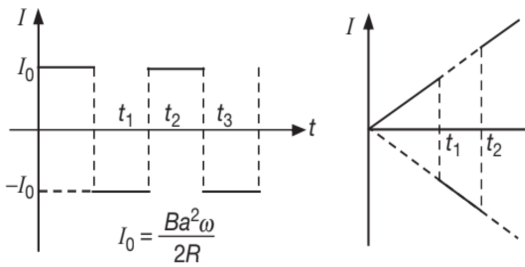
$$\Rightarrow \quad I = \frac{\xi}{R} = \pm\frac{Ba^2\omega}{2R}$$

So, for $t=0$ to $\frac{T}{4}$, $\frac{T}{2}$ to $\frac{3T}{4}$ and so on $I = \frac{Ba^2\omega}{2R}$

and for $t = \frac{T}{4}$ to $\frac{T}{2}$, $\frac{3T}{4}$ to T, etc. $I = -\frac{Ba^2\omega}{2R}$

The I-t graph is as shown in Figure. The I-t equation can be written as,

$$I = \frac{1}{2R}(-1)^{n+1}Ba^2\omega$$



where $n = 1, 2, 3, \ldots$ and $t_n = \frac{n\pi}{2\omega}$

Here positive current means current from left to right through the resistance and vice-versa.

(b) When the rod rotates with constant angular acceleration α, $\omega = \alpha t$

and $I = (-1)^{n+1}\frac{Ba^2\alpha t}{2R}$ $\qquad \{\because\ \omega = \omega_0 + \alpha t\}$

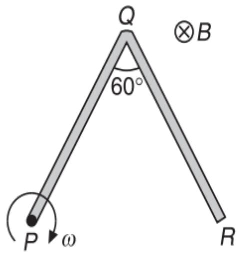
where $n = 1, 2, \ldots$ is the number of quarter revolution that the loop performs at the given instant.

The plot I-t is shown in Figure, where $t_n = \sqrt{\dfrac{n\pi}{\alpha}}$

$$\left\{\because n\pi = \frac{1}{2}\alpha t_n^2\right\}$$

ILLUSTRATION 44

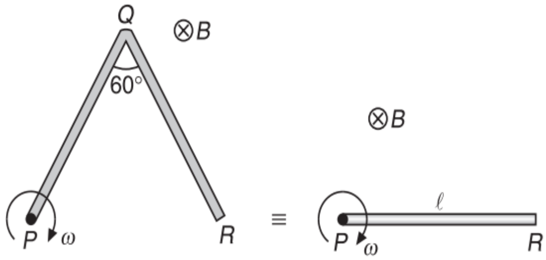
A rod PQR is bent such that $(PQ = QR = l)$ as shown in Figure.



The arrangement rotates about its end P with angular speed ω in a region of transverse magnetic field of strength B. Calculate the emf induced across the rod and the potential difference between points Q and R on the rod.

SOLUTION

The rod is equivalent to an equivalent rod joining the ends P to R and rotating in the same sense.



Since the point R is at a higher potential, so we get

$$V_R - V_P = \frac{B\omega l^2}{2} \quad \ldots(1)$$

Also, we note that for the rod PQ, we have

$$V_Q - V_P = \frac{B\omega l^2}{2} \quad \ldots(2)$$

Subtracting (1) from (2), we get

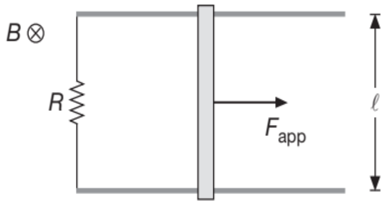
$$V_Q - V_R = 0$$

Test Your Concepts-II

Based on Faraday's Laws: Motional EMF

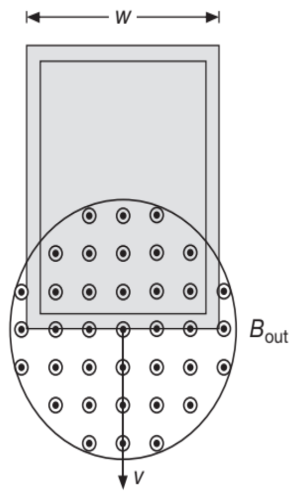
(Solutions on page H.132)

1. A conducting rod of length ℓ moves on two horizontal, frictionless rails, as shown in Figure. If a constant force of 1 N moves the bar at 2 ms^{-1} through a magnetic field $\vec{B}$ that is directed into the page



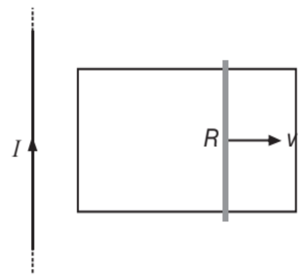
(a) what is the current through the 8 Ω resistor R?
(b) what is the rate at which energy is delivered to the resistor?
(c) what is the mechanical power delivered by the force F_{app}?

2. A conducting rectangular loop of mass M, resistance R and dimensions w by ℓ falls from rest into a magnetic field $\vec{B}$ as shown in Figure. During the time interval before the top edge of the loop reaches the field, the loop approaches a terminal speed v_T. Find v_T.

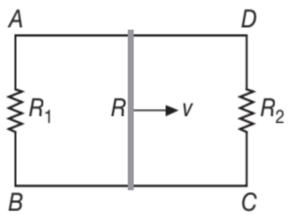


3. A long straight wire carrying a current I and a π-shaped conductor with sliding connector are located in the same plane as shown in Figure. The connector of length ℓ and resistance R slides to

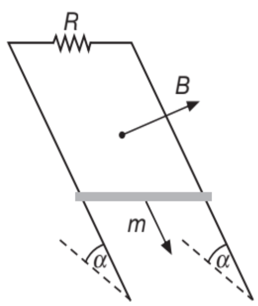
the right with a constant velocity v. Find the current induced in the loop as a function of separation r between the connector and the straight wire. The resistance of the π-shaped conductor and the self-inductance of the loop are assumed to be negligible.



4. A rectangular loop with a sliding connector of length ℓ is located in a uniform magnetic field perpendicular to the loop plane. The magnetic induction is equal to B. The connector has an electric resistance R, the sides AB and CD have resistances R_1 and R_2 respectively. Neglecting the self-inductance of the loop, find the current flowing in the connector during its motion with a constant velocity v.

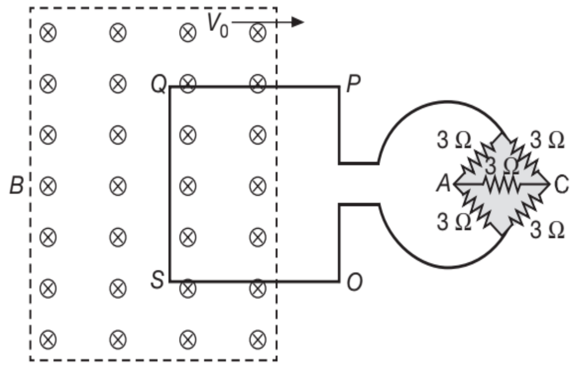


5. A copper connector of mass m slides down two smooth copper bars, set at an angle α to the horizontal, due to gravity. At the top the bars are interconnected through a resistance R. The separation between the bars is equal to ℓ. The system is located in a uniform magnetic field of induction B, perpendicular to the plane in which the connector slides. The resistances of the bars, the connector and the sliding contacts, as well as the self-inductance of the loop, are assumed to be negligible. Find the steady state velocity of the connector.

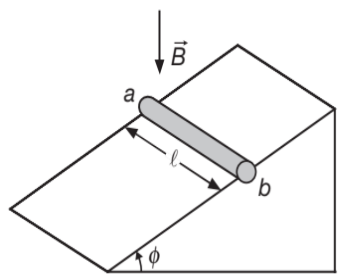


6. The two rails of a railway track, insulated from each other and the ground, are connected to a millivoltmeter. What is the reading of the millivoltmeter when a train travels at a speed of 18 kmh^{-1} along the track given that the vertical components of earth's magnetic field is 0.2×10^{-4} weber m^{-2} and the rails are separated by 1 m? Track is south to north.

7. A square metal wire loop of side 10 cm and resistance 1 Ω is moved with a constant velocity v_0 in a uniform magnetic field of induction $B = 2$ Wbm^{-2} as shown in the Figure. The magnetic field lines are perpendicular to the plane of the loop (directed into the paper). The loop is connected to a network of resistors each of value 3 Ω. The resistances of the lead wires OS and PQ are negligible. What should be the speed of the loop so as to have a steady current of 1 mA in the loop? Give the direction of current in the loop.



8. A metal bar with length ℓ, mass m, and resistance R is placed on frictionless metal rails that are inclined at an angle ϕ above the horizontal. The rails have negligible resistance. There is a uniform magnetic field of magnitude B directed downward in Figure. The bar is released from rest and slides down the rails.



(a) Is the direction of the current induced in the bar from a to b or from b to a?

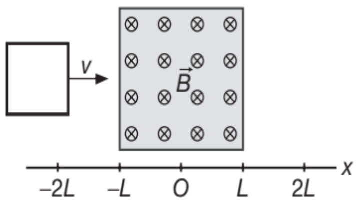
(b) What is the terminal speed of the bar?

(c) What is the induced current in the bar when the terminal speed has been reached?

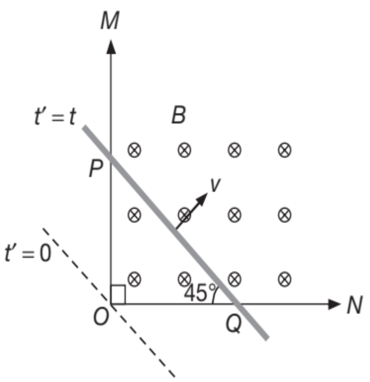
(d) After the terminal speed has been reached, at what rate is electrical energy being converted to thermal energy in the resistance of the bar?

(e) After the terminal speed has been reached, at what rate is work being done on the bar by gravity? Compare your answer to that in part (d).

9. A square loop of wire of total length $4L$ and resistance R is moving at a constant speed v across a uniform magnetic field confined to a square region of area $4L^2$ (as shown). Assuming all the positive quantities to be along either $+x$ axis or along $+y$ axis, plot the graph of external force F required to move the loop at a constant speed as a function of coordinate x from $x=-2L$ to $x=+2L$, where x is to be measured from the centre of the magnetic field region to the centre of the loop. Also plot the graph of the induced current in the loop as a function of x. Take currents in the counter clockwise sense to be positive.

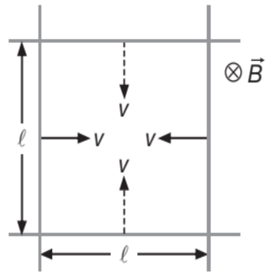


10. An electric circuit is composed of the three conducting rods MO, ON and PQ, as shown in Figure. The resistance of the rods per unit length is λ. The rod PQ slides, as shown in Figure, at a constant velocity v, keeping its tilt angle relative to ON and MO fixed at 45°. At each instant the circuit is closed. The whole system is embedded in a uniform magnetic field B, which is directed perpendicularly into the page.



Compute the time dependent induced electric current.

11. In the Figure shown the four rods have λ resistance per unit length. The arrangement is kept in a magnetic field of constant magnitude B and directed perpendicular to the plane of the Figure and directed inwards. Initially the sides as shown from a square. Now each wire starts moving with constant velocity v towards opposite wire. Find, as a function of time, the



(a) induced emf in the circuit.

(b) induced current in the circuit with direction.

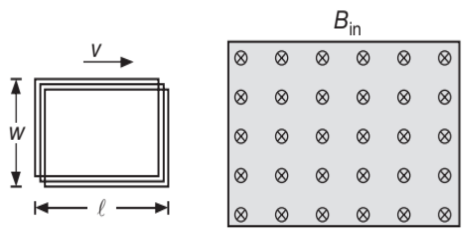
(c) force required on each wire to keep its velocity constant.

Neglect the interaction between the wires.

(d) total power required to maintain constant velocity.

(e) thermal power developed in the circuit.

12. A rectangular coil with resistance R has N turns, each of length ℓ and width w as shown in Figure. The coil moves into a uniform magnetic field $\vec{B}$ with constant velocity $\vec{v}$. What are the magnitude and direction of the total magnetic force on the coil.



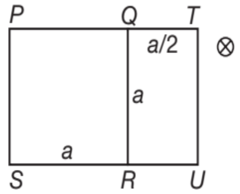
(a) as it enters the magnetic field,

(b) as it moves within the field, and

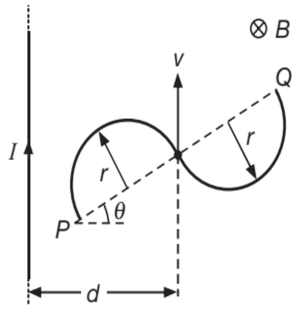
(c) as it leaves the field?

13. A helicopter has blades of length 3 m, extending out from a central hub and rotating at $2\ \text{revs}^{-1}$. If the vertical component of the Earth's magnetic field is 50 μT, what is the emf induced between the blade tip and the center hub?

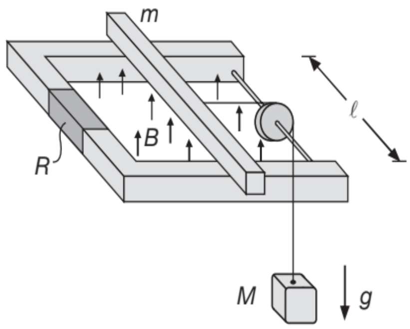
14. Determine the current in the conductors of the circuit shown in Figure, if the intensity of a homogeneous magnetic field is perpendicular to the plane of the paper, inwards and changes in time according to the law $B = kt$. The resistance of unit length of the conductors is λ.



15. An infinite wire carries a current I. An S-shaped conducting rod of two semicircles each of radius r is placed as shown. The centre of the conductor is at a distance d from the wire. If the rod translates parallel to the wire with a velocity v, calculate the emf induced across the ends PQ of the rod.

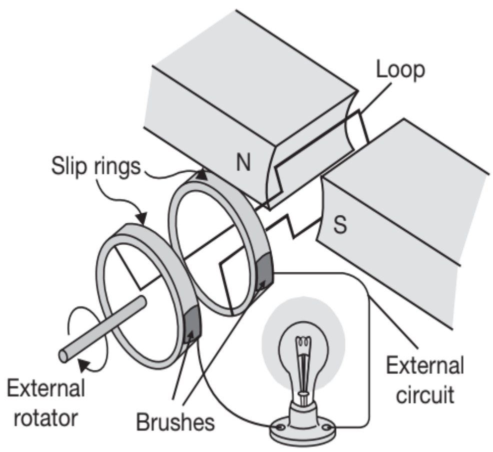


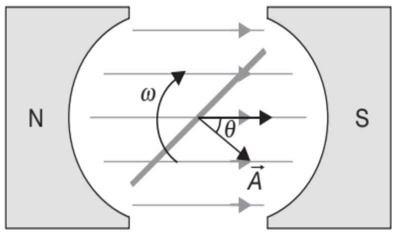
16. A bar of mass m is pulled horizontally across parallel rails by a massless string that passes over an ideal pulley and is attached to a suspended object of mass M as shown in Figure. The uniform magnetic field has a magnitude B and the distance between the rails is ℓ. The rails are connected at one end by a load resistor R. Derive an expression that gives the horizontal speed of the bar as a function of time, assuming that the suspended object is released with the bar at rest at $t = 0$. Assume no friction between rails and bar. Also find the terminal velocity obtained by the bar.



PRODUCTION OF INDUCED EMF BY ROTATING THE COIL IN A MAGNETIC FIELD: AN AC GENERATOR

One of the most important applications of Faraday's Law of induction is to generators and motors. A generator converts mechanical energy into electric energy, while a motor converts electric energy into mechanical energy.





The rotating loop as seen from above

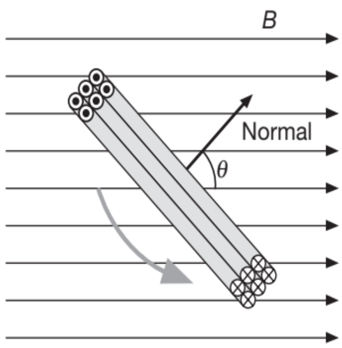


Figure is a simple illustration of a generator.

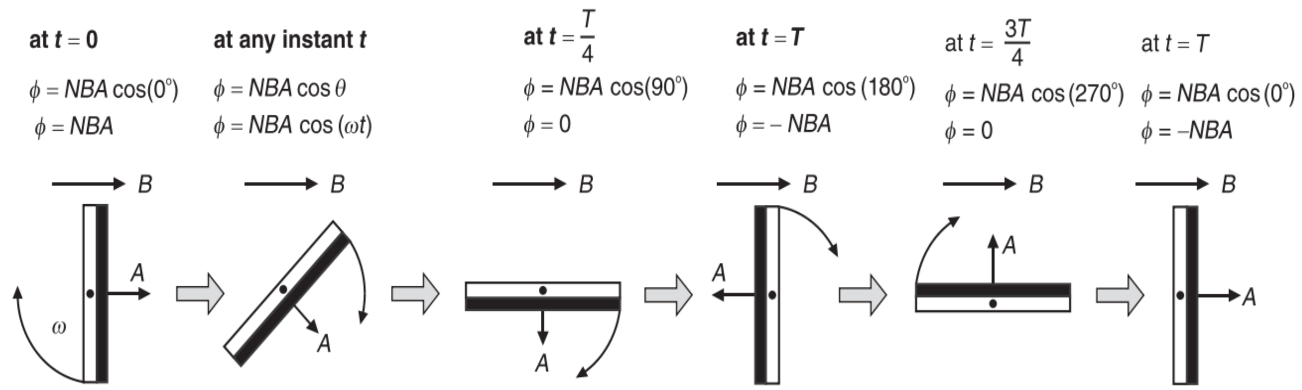


Figure Showing Cross-sectional view of coil rotating clockwise in a uniform magnetic field with uniform angular velocity ω.

It consists of an N-turn loop rotating in a magnetic field which is assumed to be uniform. The magnetic flux varies with time, thereby inducing an emf. From Figure, we see that the magnetic flux through the loop may be written as

$$\phi_B = \vec{B}\cdot\vec{A} = BA\cos\theta = BA\cos(\omega t)$$

Suppose a coil of N turns, and area A is rotated in a uniform magnetic field B with angular velocity ω. As the coil rotates, the flux through it changes. Due to this change in flux an induced emf is set up in the coil.

Since, $\xi = -\dfrac{d\phi}{dt}$

At any instant, since we have

$$\phi = N\vec{B}\cdot\vec{A} = NBA\cos\theta = NBA\cos(\omega t)$$

So, $\xi = -\dfrac{d}{dt}(NBA\cos(\omega t))$

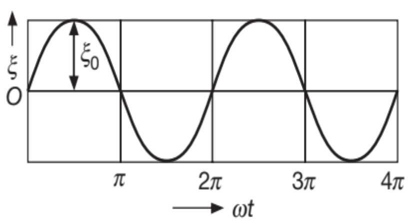
$\Rightarrow \quad \xi = NBA\omega\sin(\omega t)$

$\Rightarrow \quad \xi = \xi_0 \sin(\omega t)$

where $\xi_0 = NBA\omega =$ Peak value of AC voltage developed.

We note that the induced emf ξ has a sinusoidal variation, having the peak value $NBA\omega$.

This forms the basic principle of the Alternating Current generator, which is a device converting mechanical energy to electrical energy.



If we connect the generator to a circuit which has a resistance R, then the current generated in the circuit is given by

$$I = \frac{|\xi|}{R} = \frac{NBA\omega}{R}\sin(\omega t)$$

The current is an alternating current which oscillates in sign and has an amplitude $I_0 = \dfrac{NBA\omega}{R}$. The power delivered to this circuit is

$$P = I|\xi| = \frac{(NBR\omega)^2}{R}\sin^2(\omega t)$$

On the other hand, the torque exerted on the loop is

$$\tau = \mu B\sin\theta = \mu B\sin(\omega t)$$

Thus, the mechanical power supplied to rotate the loop is

$$P_m = \tau\omega = \mu B\omega\sin(\omega t)$$

Since the dipole moment for the N-turn current loop is

$$\mu = NIA = \frac{N^2A^2B\omega}{R}\sin(\omega t)$$

the above expression becomes

$$P_m = \left(\frac{N^2A^2B\omega}{R}\sin(\omega t)\right)B\omega\sin(\omega t)$$

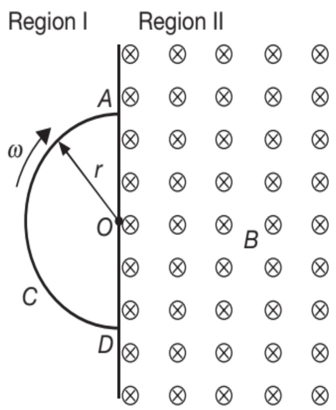
$$\Rightarrow \quad P_m = \frac{(NAB\omega)^2}{R}\sin^2(\omega t)$$

As expected, the mechanical power input is equal to the electrical power output.

ILLUSTRATION 45

Space is divided by the line AD into two regions. Region I is field free and the region II has a uniform magnetic field B directed into the plane of the paper. ACD is a semi-circular conducting loop of radius r with centre at O, the plane of the loop being in the plane of the paper. The loop is now made to rotate with a constant angular velocity ω about an axis passing through O and perpendicular to the plane of the paper. The effective resistance of the loop is R.

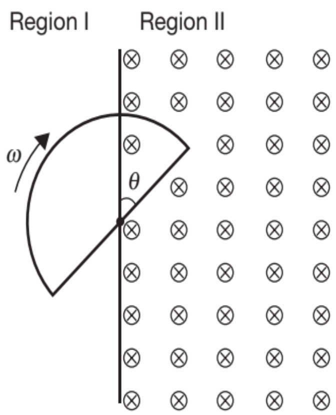
(a) Obtain an expression for the magnitude of the induced current in the loop.
(b) Show the direction of the current when the loop is entering into the region II.
(c) Plot a graph between the induced emf and the time of rotation for two periods of rotation.



SOLUTION

(a) At time t, we have $\theta = \omega t$

Flux passing through coil is $\phi = BA\cos(0°)$



$$\Rightarrow \phi = B\left(\frac{\theta}{2\pi}\right)(\pi r^2) \qquad \left\{\because A = \frac{1}{2}r^2\theta\right\}$$

$$\Rightarrow \phi = \left(\frac{Br^2}{2}\right)\theta = \left(\frac{Br^2}{2}\right)\omega t$$

Magnitude of induced emf is given by

$$\xi = \frac{d\phi}{dt} = \frac{B\omega r^2}{2}$$

So, the induced current is

$$I = \frac{\xi}{R} = \frac{B\omega r^2}{2R}$$

(b) When the loop enters the Region II, the inward magnetic field passing through the loop is increasing. Hence, from Lenz's Law induced current will produce an outward magnetic field, which will be produced by a counter clockwise current.

(c) For half rotation i.e., at $t = \frac{T}{2} = \frac{\pi}{\omega}$, current in the loop will be of constant magnitude given by

$I = \frac{B\omega r^2}{2R}$ and counter clockwise.

In the next half rotation, when loop comes out of Region II, current will be clockwise, but having the same magnitude. So, taking counter clockwise current as positive, I-t graph for two rotations will be as shown in Figure.

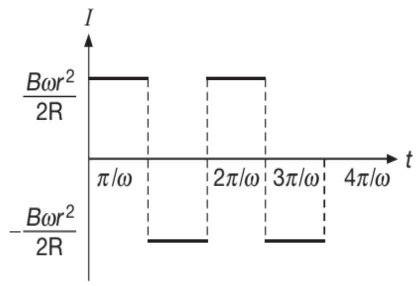
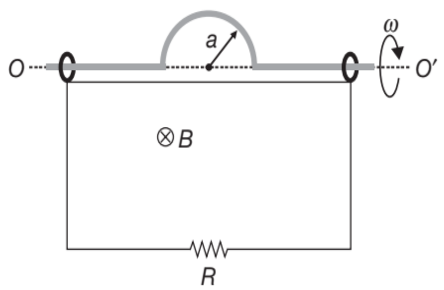


ILLUSTRATION 46

A wire shaped as a semi-circle of radius a rotates about an axis OO' with an angular velocity ω in a uniform magnetic field of induction B. The rotation axis is perpendicular to the field direction. The total resistance of the circuit is equal to R. Neglecting the magnetic field of the induced current, find the mean amount of thermal power being generated in the loop during a rotation period.



SOLUTION

Since, we know that

$$\phi = \vec{B} \cdot \vec{A}$$

$$\Rightarrow \quad \phi = B\left(\frac{\pi a^2}{2}\right)\cos(\omega t)$$

From Faraday's Laws, we have

$$\xi = -\frac{d\phi}{dt} = B\omega\left(\frac{\pi a^2}{2}\right)\sin(\omega t)$$

The induced current I, is then given by

$$I = \frac{\xi}{R} = \frac{\pi B\omega a^2}{2R}\sin(\omega t)$$

If P is the thermal power generated, then

$$P = \xi I = \frac{1}{R}\left(\frac{\pi B\omega a^2}{2}\right)^2 \sin^2(\omega t)$$

So, the average amount of thermal power being generated is

$$P_{av} = \langle P \rangle = \frac{1}{R}\left(\frac{\pi B\omega a^2}{2}\right)^2 \langle \sin^2(\omega t) \rangle$$

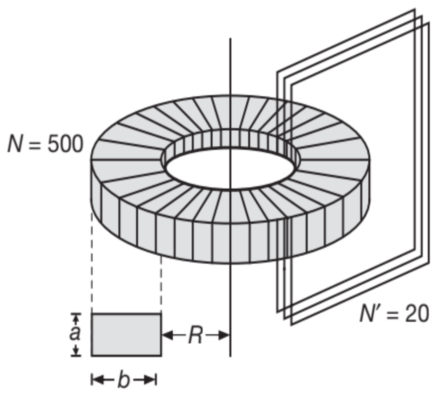
Since we know that $\langle \sin^2(\omega t) \rangle = \frac{1}{2}$

$$\Rightarrow \quad P_{av} = \langle P \rangle = \frac{1}{8R}(\pi B\omega a^2)^2$$

$$\Rightarrow \quad P_{av} = \frac{\pi^2 B^2 \omega^2 a^4}{8R}$$

ILLUSTRATION 47

A toroid having a rectangular cross section $a \times b$, where $a = 2$ cm, $b = 3$ cm and inner radius $R = 4$ cm consists of 500 turns of wire that carries a sinusoidal current $I = I_0 \sin(\omega t)$, with $I_0 = 50$ A and a frequency $f = 60$ Hz. A coil that consists of 20 turns of wire links with the toroid, as shown in Figure. Determine the emf induced in the coil as a function of time.



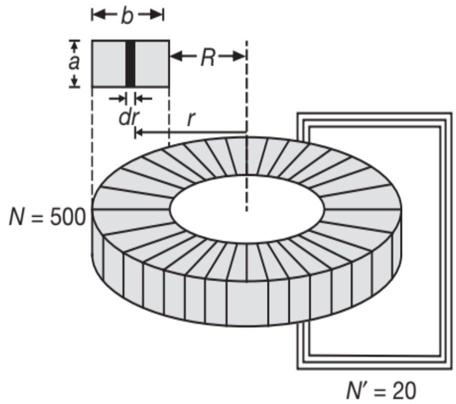
SOLUTION

In a toroid, all the field and hence the flux is confined to the inside of the toroid. Field inside the toroid is given by,

$$B = \frac{\mu_0 N I}{2\pi r}$$

$$B = \frac{500\mu_0 I}{2\pi r}, \text{ where } I = I_0 \sin(\omega t)$$

Consider an infinitesimal area element of length a, width dr having area dA, then $dA = a dr$



Magnetic flux through this element is

$$d\phi_B = B dA \cos(0°) = B dA$$

$$\Rightarrow \quad d\phi_B = \left(\frac{\mu_0 N I}{2\pi r}\right) a\, dr$$

$$\Rightarrow \quad d\phi_B = \left(\frac{\mu_0 N I_0 \sin(\omega t)}{2\pi r}\right) a\, dr$$

$$\Rightarrow \quad \phi_B = \int B dA = \frac{500\mu_0 I_0}{2\pi}\sin(\omega t)\int \frac{a dr}{r}$$

Please note that I is varying with t and not with r and hence can be taken out of the integral.

$$\Rightarrow \quad \phi_B = \frac{500\mu_0 I_{max}}{2\pi} a \sin(\omega t) \log_e\left(\frac{b+R}{R}\right)$$

Since, $|\xi| = N'\left(\frac{d\phi_B}{dt}\right)$

$$\Rightarrow \quad \xi = N'\frac{d\phi_B}{dt}$$

$$\Rightarrow \quad \xi = 20\left(\frac{500\mu_0 I_{max}}{2\pi}\right)\omega a \log_e\left(\frac{b+R}{R}\right)\cos(\omega t)$$

$$\Rightarrow \quad \xi = \frac{10^4}{2\pi}(4\pi \times 10^{-7})(50)(377)(0.02) \times \log_e\left(\frac{3+4}{4}\right)\cos(\omega t)$$

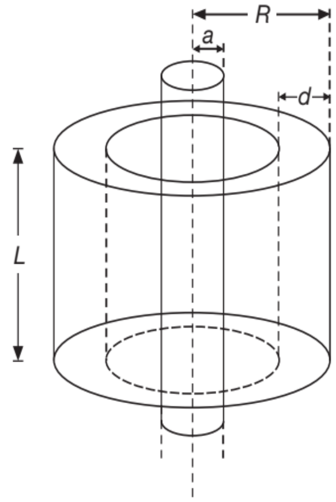
$$\Rightarrow \quad \xi = (0.422)\cos(\omega t)\ \text{V} = 422\cos(\omega t)\ \text{mV}$$

Test Your Concepts-III

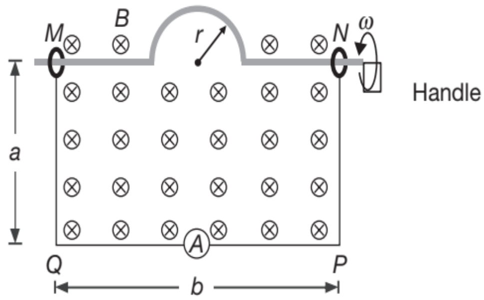
Based on Faraday's Laws: AC Generator

(Solutions on page H.136)

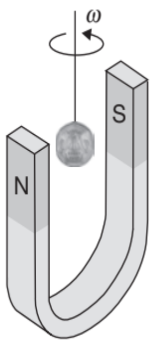
1. A steel guitar string vibrates. The component of magnetic field, in millitesla, perpendicular to the area of a pickup coil nearby is given by $B = 50 + 4\sin(1000t)$. The circular pickup coil has 50 turns and radius 7 mm. Find the emf induced in the coil as a function of time and the peak value.
2. A rectangular coil of 60 turns, dimensions 10 cm by 20 cm and total resistance 10 Ω, rotates with angular speed $30\ \text{rad s}^{-1}$ about the y-axis in a region where a 1 T magnetic field is directed along the x-axis. The rotation is initiated so that the plane of the coil is perpendicular to the direction of $\vec{B}$ at $t = 0$. Calculate
   (a) the maximum induced emf in the coil,
   (b) the maximum rate of change of magnetic flux through the coil,
   (c) the induced emf at $t = \frac{\pi}{120}$ s and
   (d) the torque exerted by the magnetic field on the coil at the instant when the emf is a maximum.
3. A solenoid wound with $2000\ turns\,m^{-1}$ is supplied with current that varies in time according to $I = 5\sin(100\pi t)A$, where t is in seconds. A small coaxial circular coil of 160 turns and radius $r = 5$ cm is located inside the solenoid near its center.
   (a) Derive an expression that describes the manner in which the emf in the small coil varies in time.
   (b) At what average rate is energy delivered to the small coil if the windings have a total resistance of 8 Ω?
4. A long solenoid of radius a and number of turns per unit length n is enclosed by cylindrical shell of radius R thickness $d(d \ll R)$ and length L. A variable current $I = I_0\sin(\omega t)$ flows through the coil. If the resistivity of the material of cylindrical shell is ρ, find the induced current in the shell.



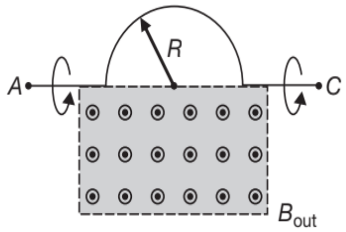
5. Consider the circuit depicted in Figure. The circuit is embedded in a uniform magnetic field B directed perpendicularly into the page. The upper branch of the circuit, MN, is rotated at a constant angular velocity w, by turning the handle shown in the Figure. The electric resistance of the circuit is R.



   (a) Compute the time dependent magnetic flux through the circuit.
   (b) Compute the EMF induced along the circuit.
   (c) Compute the induced current in the circuit.
6. A coin is suspended from a thread and hung between the poles of a strong horseshoe magnet as shown in Figure. The coin rotates at constant angular speed w about a vertical axis. If θ represents the angle between the direction of $\vec{B}$ and the normal to the face of the coin, sketch a graph of the torque due to induced currents as a function of θ, for $0 < \theta < 2\pi$.



7. A semi-circular conductor of radius $R = 0.25$ m is rotated about the axis AC at a constant rate of 120 rev min^{-1} (shown in Figure). A uniform magnetic field in all of the lower half of the Figure is directed out of the plane of rotation and has a magnitude of 1.3 T.



(a) Calculate the maximum value of the emf induced in the conductor.
(b) What is the value of the average induced emf for each complete rotation?
(c) How would the answers to (a) and (b) change if $\vec{B}$ were allowed to extend a distance R above the axis of rotation?
(d) Sketch the emf versus time when the field is as shown in Figure and
(e) Sketch the emf versus time when the field is extended as described in (c).

8. The rotating loop in a AC generator is a square 10 cm on a side. It is rotated at 60 Hz in a uniform field of 800 mT. Calculate
(a) the flux through the loop as a function of time,
(b) the emf induced in the loop,
(c) the current induced in the loop for a loop resistance of 1 Ω,
(d) the power delivered to the loop, and
(e) the torque that must be exerted to rotate the loop.

9. A motor in normal operation carries a direct current of 5 A when connected to a 220 V power supply. The resistance of the motor windings is 24 Ω.
(a) What is the back emf generated by the motor during normal operation?
(b) At what rate is internal energy produced in the windings?
(c) Suppose that a malfunction stops the motor shaft from rotating. At what rate will internal energy be produced in the windings in this case?

INDUCED ELECTRIC FIELD

Since we know that the electric potential difference between two points A and B in an electric field $\vec{E}$ can be written as

$$\Delta V = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l}$$

When the electric field is conservative, as is the case of electrostatics, the line integral of $\vec{E}\cdot d\vec{l}$ is path-independent, which implies $\oint \vec{E}\cdot d\vec{l} = 0$.

Faraday's Law shows that when magnetic flux changes with time, an induced current begins to flow in the circuits or closed loops. **What causes the charges to move?**

It is the induced emf, which is the work done per unit charge that makes the charge to move. Since magnetic field does no work in moving the charge, so the work done on the mobile charges must be electric, and the electric field in this situation cannot be conservative because the line integral of a conservative electric field in a closed loop is always zero. **So, we can say that electric field associated with an induced emf/current (developed in a closed loop) has to be non-conservation in nature.** Hence

$$\xi = \oint \vec{E}_{nc}\cdot d\vec{l} \qquad \ldots(1)$$

From Faraday's Law, we have $\xi = -\dfrac{d\phi_B}{dt}$ ...(2)

From (1) and (2), we get

$$\oint \vec{E}_{nc}\cdot d\vec{l} = -\frac{d\phi_B}{dt}$$

The above expression implies that a changing magnetic flux will induce a non-conservative electric field which can vary with time. It is important to distinguish between the induced, non-conservative electric field and the conservative electric field which arises from electric charges.

So, $\oint \vec{E}_{nc} \cdot d\vec{l} = -\frac{d\phi_B}{dt} = -A\left(\frac{dB}{dt}\right)$

where A is the area of the region in which magnetic flux (or magnetic field) is changing and the negative sign takes into account the opposing nature of the induced electric field. However, in magnitude we have

$$\oint \vec{E}_{nc} \cdot d\vec{l} = \left|\frac{d\phi_B}{dt}\right| = A\left(\frac{dB}{dt}\right)$$

Conceptual Note(s)

The induced electric field $\vec{E}_{nc}$ has following important properties.

(a) It is non-conservative in nature.

(b) The line integral of $\vec{E}_{nc}$ around a closed path/loop is non-zero.

(c) When a charge q goes once around the loop, the total work done on the charge by this induced electric field is equal to q times the induced emf. Hence

$$W = q\oint \vec{E}_{nc} \cdot d\vec{\ell} = q\xi$$

where, ξ is the induced emf given by

$$\xi = -\frac{d\phi_B}{dt} = -A\left(\frac{dB}{dt}\right)$$

and A is the area of the region in which magnetic field is changing.

(d) Always keep in mind that $\vec{E}_{nc}$ will exist both inside and outside the region of changing magnetic flux.

(e) The equation $\oint \vec{E}_{nc} \cdot d\vec{\ell} = \xi = -\frac{d\phi_B}{dt}$ is valid only if the path around which we integrate is stationary.

(f) Because of symmetry, the electric field $\vec{E}_{nc}$ has the same magnitude at every point on the circular imaginary contour (or loop) and is tangential to it at each point.

(g) Also note that the direction of $\vec{E}_{nc}$ is same as the direction of induced current.

(h) $\vec{E}_{nc}$ being a non-conservative field, the concept of potential has no meaning for such a field.

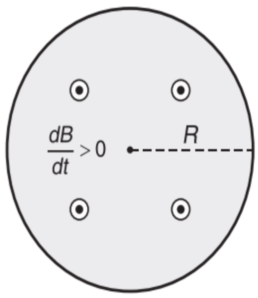
(i) This induced field is different from the electrostatic field produced by stationary charges (which is conservative in nature).

(j) The relation $\vec{F} = q\vec{E}_{nc}$ is still valid for this field.

(k) This field can vary with time. So, we observe that a changing magnetic field acts as a source of an induced electric field that cannot be produced from any static charge distribution.

INDUCED ELECTRIC FIELD DUE A TIME VARYING MAGNETIC FIELD CONFINED TO A CYLINDRICAL REGION

Let's consider a uniform magnetic field which points out of the page and is confined to a cylindrical region of radius R as shown in Figure.



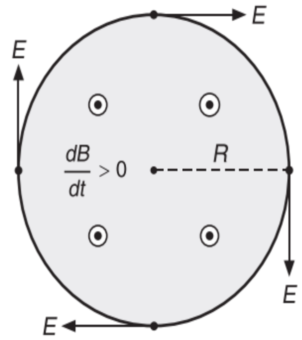
Let us assume that the magnitude of $\vec{B}$ increases with time, i.e., $\frac{dB}{dt} > 0$. To calculate the induced electric field everywhere due to the changing magnetic field we use the following thought process.

(a) Since the magnetic field is confined to a circular region, from symmetry arguments we select the integration path to be a circle of radius r.

(b) The magnitude of the induced field $\vec{E}_{nc}$ at all points on a circle is the same.

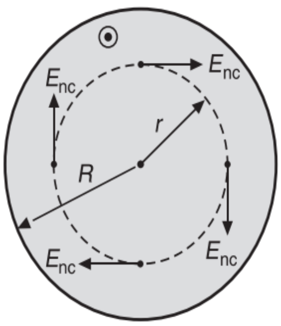
(c) According to Lenz's Law, the direction of induced field $\vec{E}_{nc}$ must be such that it would drive the induced current to produce a magnetic field that opposes the change in magnetic flux.

(d) For $\frac{dB}{dt} > 0$, the outward magnetic flux is increasing. Therefore, to counteract this change, the induced current (or the induced field) must flow clockwise so that it sets up an inward field which opposes the outward growth of magnetic flux. The direction of $\vec{E}_{nc}$ is shown in Figure.



Magnitude of $\vec{E}_{nc}$

Consider the region $r < R$ i.e. inside the cylindrical region as shown in Figure.



Inside the cylindrical region

The rate of change of magnetic flux is

$$\frac{d\phi_B}{dt} = \frac{d}{dt}(\vec{B}\cdot\vec{A}) = A\frac{dB}{dt} = \pi r^2\left(\frac{dB}{dt}\right)$$

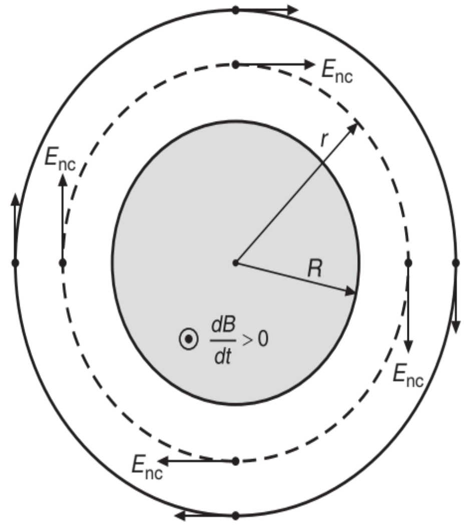
Since we know that

$$\oint \vec{E}_{nc}\cdot d\vec{l} = -\frac{d\phi_B}{dt} = -A\left(\frac{dB}{dt}\right)$$

$$\Rightarrow \quad E_{nc}(2\pi r) = -\pi r^2\left(\frac{dB}{dt}\right)$$

$$\Rightarrow \quad |E_{nc}| = \frac{r}{2}\frac{dB}{dt}$$

Now, consider the region for $r > R$ i.e. outside the cylindrical region as shown in Figure.



Outside the cylindrical region

Outside the cylindrical region, we have

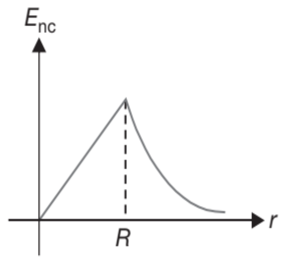
$$E_{nc}(2\pi r) = -A\left(\frac{dB}{dt}\right) = -\pi R^2\left(\frac{dB}{dt}\right)$$

$$\Rightarrow \quad |E_{nc}| = \frac{R^2}{2r}\frac{dB}{dt}$$

So, we observe that

$$|E_{nc}| = \begin{cases} \frac{r}{2}\left(\frac{dB}{dt}\right) & \text{for } r < R \\ \frac{R^2}{2r}\left(\frac{dB}{dt}\right) & \text{for } r > R \end{cases}$$

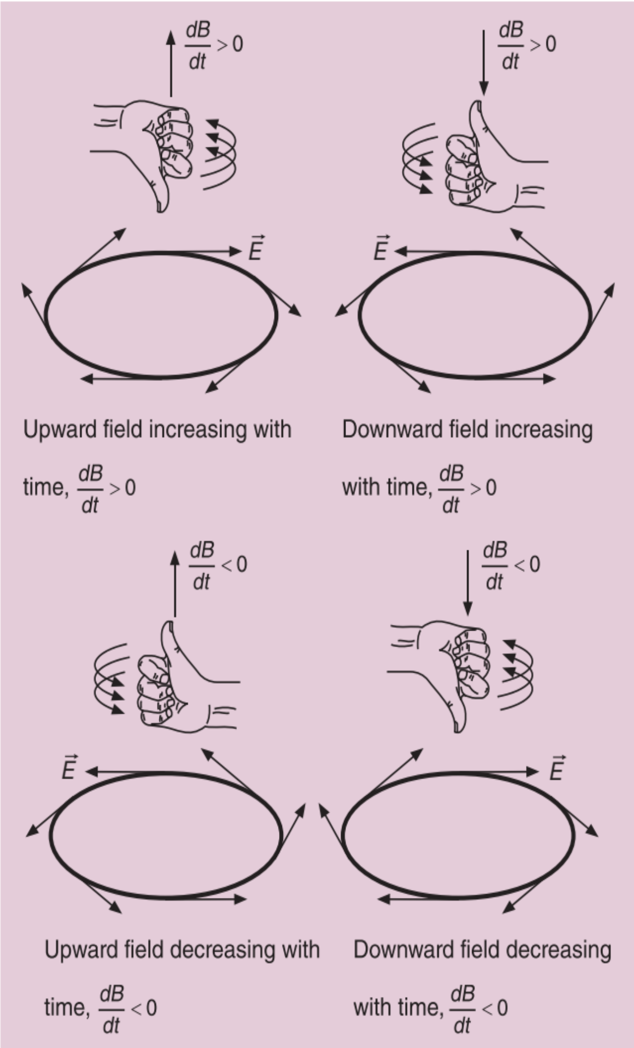
A plot of E_{nc} as a function of r is shown in Figure.



Induced electric field as a function of r

FINDING DIRECTION OF INDUCED ELECTRIC FIELD

An easy way to find the direction of induced electric field is by making use of Right-Hand Thumb Rule. Curl the fingers of right hand in such a way that thumb points opposite to the direction of changing field i.e., $\frac{dB}{dt}$, then curl of the fingers gives the direction of induced electric field.



Upward field increasing with time, $\frac{dB}{dt} > 0$

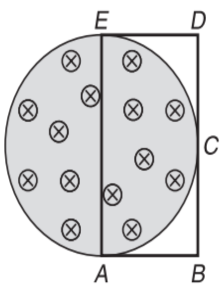
Downward field increasing with time, $\frac{dB}{dt} > 0$

Upward field decreasing with time, $\frac{dB}{dt} < 0$

Downward field decreasing with time, $\frac{dB}{dt} < 0$

ILLUSTRATION 48

A time varying magnetic field $B = a + bt$ exists in a cylindrical region of radius R. A rectangular conducting loop $ABDE$ of dimension $R \times 2R$ is kept as shown in Figure. Calculate the value of emf induced across AE, ED and DB.



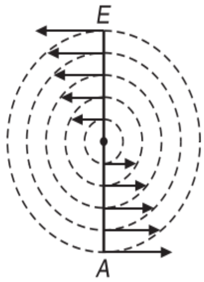
SOLUTION

Inside the circular region containing the field, the induced electric field is directly proportional to the radial distance r. So, we have

$$E_{\text{induced}} \propto r$$

EMF induced across AE

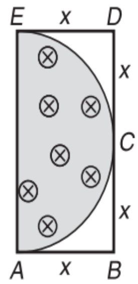
Induced electric field being concentric will be perpendicular to any section of AE as shown in Figure.



So, $V_{AE} = -\int_A^E \vec{E} \cdot d\vec{l} = 0$ $\{\because \theta = 90°\}$

Due to symmetry, the emf induced across each of the sections ED, DC, CB and BA will be same.

Let the emf induced across each one of them be x.



Rate of change of flux through $ABCDE$ is

$$\frac{d\phi}{dt} = A\frac{dB}{dt} = \frac{\pi R^2}{2}\frac{d}{dt}(a+bt) = \frac{\pi R^2 b}{2}$$

$$\Rightarrow \quad |\xi_{\text{in}}| = \frac{d\phi}{dt} = \frac{\pi R^2 b}{2} = \oint \vec{E} \cdot d\vec{l} = 4x + 0 = 4x$$

$$\Rightarrow \quad 4x = \frac{\pi R^2 b}{2}$$

$$\Rightarrow \quad x = \frac{\pi R^2 b}{8}$$

EMF induced across ED is

$$V_{ED} = x = \frac{\pi R^2 b}{8}$$

EMF induced across BD is

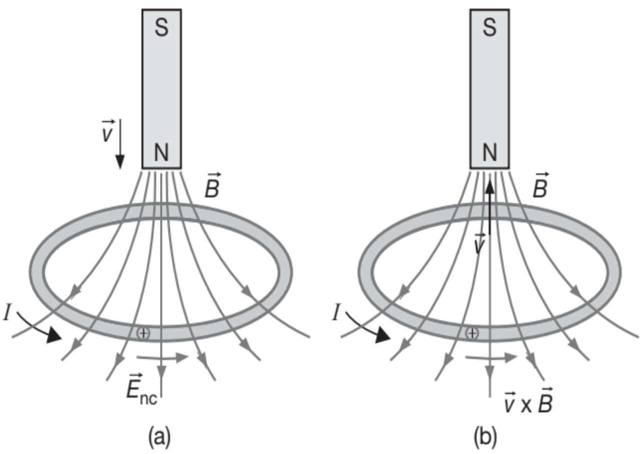
$$V_{BD} = x = 2\left(\frac{\pi R^2 b}{8}\right) = \frac{\pi R^2 b}{4}$$

INDUCED EMF AND REFERENCE FRAME

Since we know that the induced emf ξ, for a moving conductor is also given by $\xi = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$. In addition, we also know that the induced emf associated with a stationary conductor may be written as the line integral of the non-conservative electric field. So, we have

$$\xi = \oint \vec{E}_{nc} \cdot d\vec{l}$$

However, whether an object is moving or stationary actually depends on the reference frame. As an example, let's examine the situation where a bar magnet is approaching a conducting loop. An observer O in the rest frame of the loop sees the bar magnet moving toward the loop. An electric field $\vec{E}_{nc}$ is induced to drive the current around the loop, and a charge on the loop experiences an electric force $\vec{F}_e = q\vec{E}_{nc}$. Since the charge is at rest according to observer O, no magnetic force is present. On the other hand, an observer O' in the rest frame of the bar magnet sees the loop moving toward the magnet. Since the conducting loop is moving with a velocity $\vec{v}$, a motional emf is induced. In this frame, O' sees the charge q moving with a velocity $\vec{v}$, and concludes that the charge experiences a magnetic force $\vec{F}_B = q(\vec{v} \times \vec{B})$.



Induction observed in different reference frames. In (a) the bar magnet is moving, while in (b) the conducting loop is moving.

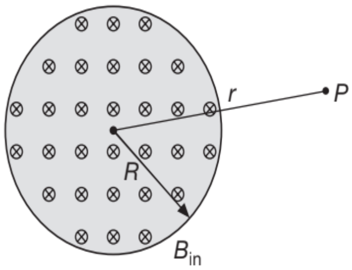
Since the event seen by the two observer is the same except the choice of reference frames, the force acting on the charge must be the same, $\vec{F}_e = \vec{F}_B$, which implies

$$\vec{E}_{nc} = \vec{v} \times \vec{B}$$

In general, as a consequence of relativity, an electric phenomenon observed in a reference frame O may appear to be a magnetic phenomenon in a frame O' that moves at a speed v relative to O.

ILLUSTRATION 49

For the situation shown in Figure, the magnetic field changes with time according to the expression $B = (2t^3 - 4t^2 + 0.8)$ T and $r = 2R = 5$ cm.



(a) Calculate the magnitude and direction of the force exerted on an electron located at point P_2 when $t = 2$ s.

(b) At what time is this force equal to zero?

SOLUTION

Since we are asked to calculate the force on an electron, so we should first calculate the induced electric field due to the variation in B with time. For that we know,

$$\oint \vec{E} \cdot d\vec{l} = \left|\frac{d\phi_B}{dt}\right|$$

$$\Rightarrow \quad E(2\pi r) = A\left(\frac{dB}{dt}\right)$$

Please note that A is the area to which the magnetic field is confined. So, $A = \pi R^2$

$$\Rightarrow \quad E(4\pi R) = \pi R^2\left(\frac{dB}{dt}\right)$$

$$\Rightarrow \quad E = \frac{R}{4}\left(\frac{dB}{dt}\right), \text{ where } \frac{dB}{dt} = 6t^2 - 8t$$

$$\Rightarrow \quad E|_{t=2} = \frac{R}{4}(6t^2 - 8t)\Big|_{t=2} = \frac{R}{4}(6(4) - 8(2)) = 2R$$

$$\Rightarrow \quad E|_{t=2} = \left(\frac{5}{100}\right) \text{NC}^{-1} \qquad \{\because R = 2.5 \text{ cm}\}$$

So, $F|_{t=2} = qE|_{t=2} = 8 \times 10^{-21}$ N

$\Rightarrow \quad F = 8 \times 10^{-21}$ N, downwards perpendicular to r.

This force will become zero, when $E = 0$, which happens when $\frac{dB}{dt} = 0$

$$\Rightarrow \quad 6t^2 - 8t = 0$$

$\Rightarrow \quad 6t - 8 = 0$

$\Rightarrow \quad t = \frac{4}{3}$ s

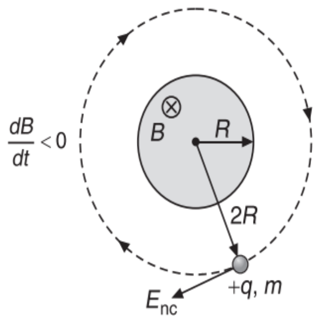
ILLUSTRATION 50

A long solenoid has n turns per unit length, radius R and carries a current I is kept in gravity free region. From its axis, at a distance twice that of its radius, a charge $+q$ and mass m is placed. If the current in solenoid is suddenly switched off, find the velocity attained by the charge.

SOLUTION

The magnetic field inside the solenoid is along the axis of the solenoid and is given by

$$B_{\text{inside}} = \mu_0 nI$$



Let the duration in which field drops to zero after switching off the current in the solenoid be Δt. Due to the switching off the current to the solenoid, an electric field E_{nc} is induced at a distance $x = 2R$ i.e. outside the solenoid in this time interval Δt. So, we have

$$E_{nc}(2\pi x) = -A\left(\frac{\Delta B}{\Delta t}\right) = -\pi R^2\left(\frac{0 - \mu_0 nI}{\Delta t}\right)$$

$$\Rightarrow \quad E_{nc}(2\pi x) = \pi R^2\left(\frac{\mu_0 nI}{\Delta t}\right)$$

$$\Rightarrow \quad E_{nc} = \frac{R^2}{2x}\left(\frac{\mu_0 nI}{\Delta t}\right) = \frac{R^2}{2(2R)}\left(\frac{\mu_0 nI}{\Delta t}\right)$$

$$\Rightarrow \quad E_{nc} = \frac{R}{4}\left(\frac{\mu_0 nI}{\Delta t}\right) \qquad \ldots(1)$$

Due to this induced electric field, the charge $+q$ experiences an impulse in this duration which is given by

$$F\Delta t = mv - 0$$

$$\Rightarrow \quad (qE_{nc})\Delta t = mv$$

Using equation (1), we get

$$q\left[\frac{R}{4}\left(\frac{\mu_0 nI}{\Delta t}\right)\right]\Delta t = mv$$

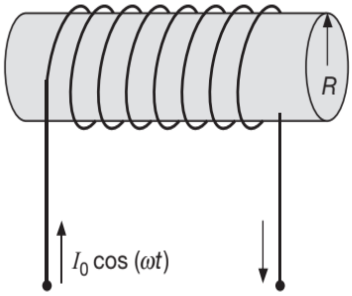
$$\Rightarrow \quad v = \frac{\mu_0 nIqR}{4m}$$

ILLUSTRATION 51

A long solenoid of radius R has n turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I = I_0 \cos(\omega t)$, where I_0 is the maximum current and ω is the angular frequency of the alternating current source.

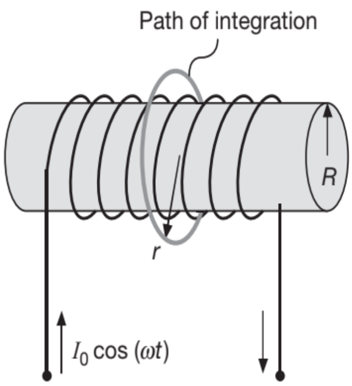
(a) Determine the magnitude of the induced electric field outside the solenoid at a distance $r > R$ from its long central axis.

(b) What is the magnitude of the induced electric field inside the solenoid, a distance r from its axis?



SOLUTION

(a) First let us consider an external point and take the path (contour) for our line integral to be a circle of radius r centered on the solenoid, as illustrated in Figure.



By symmetry we see that the magnitude of $\vec{E}$ is constant on this path and that $\vec{E}$ is tangent to it. The magnetic flux through the area enclosed by this path is $BA = B\pi R^2$, so

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\left(B\pi R^2\right) = -\pi R^2\frac{dB}{dt}$$

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = E(2\pi r) = -\pi R^2\frac{dB}{dt} \qquad \ldots(1)$$

The magnetic field inside a long solenoid is given by $B = \mu_0 nI$, where $I = I_0\cos(\omega t)$. So, (1) becomes

$$E(2\pi r) = -\pi R^2\mu_0 nI_0\frac{d}{dt}(\cos\omega t)$$

$$\Rightarrow E(2\pi r) = \pi R^2\mu_0 nI_0\omega\sin(\omega t)$$

$$\Rightarrow E = \left(\frac{\mu_0 nI_0\omega R^2}{2r}\right)\sin(\omega t) \quad (\text{for } r > R) \qquad \ldots(2)$$

Hence, the amplitude of the electric field outside the solenoid falls off as $\frac{1}{r}$ and varies sinusoidally with time.

(b) For an interior point $(r < R)$, the flux through an integration loop is given by $B(\pi r^2)$. Using the same procedure as in part (a), we find that

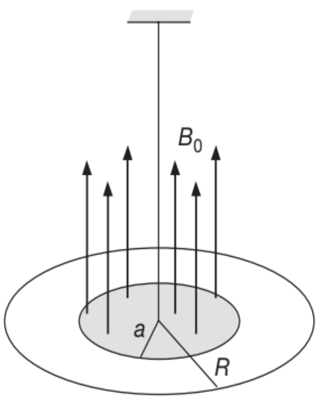
$$E(2\pi r) = -\pi r^2\frac{dB}{dt} = \pi r^2\mu_0 nI_0\omega\sin(\omega t)$$

$$\Rightarrow E = \frac{\mu_0 nI_0\omega}{2}r\sin(\omega t) \qquad (\text{for } r < R) \qquad \ldots(3)$$

This shows that the amplitude of the electric field induced inside the solenoid by the changing magnetic flux through the solenoid increases linearly with r and varies sinusoidally with time.

ILLUSTRATION 52

A line charge λ is wound around an insulating disc of mass M and radius R, which is then suspended horizontally as shown in Figure, so that it is free to rotate.



In the central region of radius a, there is a uniform magnetic field B_0, pointing up. Now the magnetic field is turned off, which causes the disc to rotate. Find the angular speed ω with which the disc starts rotating. Can you explain the rotation of the disc.

SOLUTION

When the field is switched off, the flux linked with the central portion changes, due to which an induced circular electric field is set up. The line charge element will experience a torque/couple which makes the disc to rotate.

If $\vec{E}$ be the induced electric field, then

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\phi}{dt} = -\pi a^2\left(\frac{dB}{dt}\right)$$

$$\Rightarrow E(2\pi R) = -\pi a^2\left(\frac{dB}{dt}\right)$$

$$\Rightarrow E = -\frac{a^2}{2R}\left(\frac{dB}{dt}\right) \qquad \ldots(1)$$

If τ is the torque on the disc due to this field, then

$$\tau = FR = qER = \lambda(2\pi R)ER = -\pi\lambda a^2 R\left(\frac{dB}{dt}\right)$$

If ω be the angular speed of the disc, then

$$\tau = \frac{dL}{dt}$$

$$\Rightarrow \int \tau dt = \int dL$$

$$\Rightarrow -\pi\lambda a^2 R\int dB = \int dL$$

$$\Rightarrow -\pi\lambda a^2 R\Delta B = \Delta L$$

$$\Rightarrow -\pi\lambda a^2 R\left(B_f - B_i\right) = L_f - L_i$$

$$\Rightarrow -\pi\lambda a^2 R\left(0 - B_0\right) = I(\omega - 0)\text{, where } I = \frac{1}{2}MR^2$$

$$\Rightarrow \omega = \frac{2\pi\lambda a^2 RB_0}{MR}$$

ILLUSTRATION 53

A long solenoid of cross-sectional radius a has a thin insulated wire ring tightly put on its winding; one-half of the ring has the resistance $\eta(>1)$ times that of the other half. The magnetic induction produced by the solenoid varies with time as $B = bt$, where b is a constant. Find the magnitude of the electric field strength in the ring.

SOLUTION

The emf induced in the ring is

$$\xi = \left|\frac{d\phi}{dt}\right| = \pi a^2 b$$

The changing magnetic field produces an induced emf in the ring which is equal for both halves of the ring. So, for each half of the ring, the induced emf is $\frac{\xi}{2}$. Please note that the induced emf is independent of the resistance of the ring. The induced current flowing in the ring is

$$i = \frac{\xi}{R_{net}} = \frac{\xi}{r + \eta r}$$

Conceptual Note(s)

We are given that the two halves of the ring have different resistances, so our intuition says that the half ring with lesser resistance should have a higher current and the half ring with higher resistance should have a lower current which cannot be possible because

"How will a single loop carry two different currents?"

The answer to this puzzle is simple and logical. The current in the loop will have one single value if we use the concept of induced electric field. The induced electric field is set up such that according to Lenz's Law, in the half ring with smaller current it will be along the current and in the other half ring with larger current it will oppose the current.

If E is the induced electric field in the ring, then for the half ring with low resistance r carrying a current i, we have

$$ir = \frac{\xi}{2} - (\pi a)E \qquad \ldots(1)$$

and for the other half of the ring with high resistance ηr, also carrying a current i, we have

$$i(\eta r) = \frac{\xi}{2} + (\pi a)E \qquad \ldots(2)$$

Subtracting equation (1) from (2), we get

$$(\eta - 1)ir = 2\pi a E \qquad \ldots(3)$$

Adding equations (1) and (2), we get

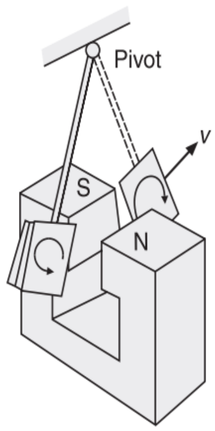
$$(\eta + 1)ir = \xi = \pi a^2 b \qquad \ldots(4)$$

Dividing equations (3) by (4), we get

$$\Rightarrow \quad E = \frac{ab}{2}\left(\frac{\eta - 1}{\eta + 1}\right)$$

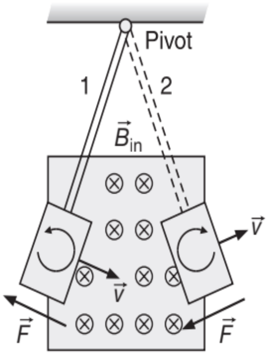
EDDY CURRENTS

Since we know that an emf and a current are induced in a circuit by a changing magnetic flux. In the same manner, circulating currents called **eddy currents** are induced in bulk pieces of metal (or metal sheets or metal plates) moving through a magnetic field. This can easily be demonstrated by allowing a flat copper or aluminum plate attached at the end of a rigid bar to swing back and forth through a magnetic field.



Formation of eddy currents in a conducting plate moving through a magnetic field. As the plate enters or leaves the field, the changing magnetic flux induces an emf, which causes eddy currents in the plate.

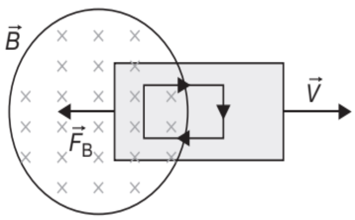
As the plate enters the field, the changing magnetic flux induces an emf in the plate, which in turn causes the free electrons in the plate to move, producing the **swirling eddy current**. According to Lenz's Law, the direction of the eddy currents is such that they create magnetic fields that oppose the change that causes the currents. For this reason, the eddy currents must produce effective magnetic poles on the plate, which are repelled by the poles of the magnet; this gives rise to a repulsive force that opposes the motion of the plate. (If the opposite were true, the plate would accelerate and its energy would increase after each swing, in violation of the Law of Conservation of Energy). As indicated in figure with $\vec{B}$ directed into the page, the induced eddy current is counterclockwise as the swinging plate enters the field at position 1.



As the conducting plate enters the field (position 1), the eddy currents are counterclockwise. As the plate leaves the field (position 2), the currents are clockwise. In either case, the force on the plate is opposite the velocity, and eventually the plate comes to rest.

This is because the flux due to the external magnetic field into the page through the plate is increasing, and hence by Lenz's Law the induced current must provide its own magnetic field out of the page. The opposite is true as the plate leaves the field at position 2, where the current is clockwise. Because the induced eddy current always produces a magnetic retarding force $\vec{F}$ when the plate enters or leaves the field, the swinging plate eventually comes to rest.

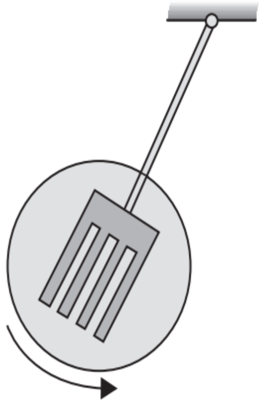
In simplified words, the induced eddy currents generate a magnetic force that opposes the motion, making it more difficult to move the conductor across the magnetic field.



Magnetic force arising from the eddy current that opposes the motion of the conducting slab.

Minimising Eddy Currents

Since the conductor has non-zero resistance R, Joule heating causes a loss of power by an amount $P=\frac{\xi^2}{R}$. Therefore, by increasing the value of R, power loss can be reduced. One way to increase R is to laminate the conducting slab, or construct the slab by gluing together thin strips that are insulated from one another. Another way is to make cuts in the slab, thereby disrupting the conducting path and preventing the formation of large current loops, as shown.



When slots are cut in the conducting plate, the eddy currents are reduces and the plate swings more freely through the magnetic field.

Eddy currents are often undesirable because they represent a transformation of mechanical energy to internal energy. To reduce this energy loss, conducting parts are often laminated that is, they are built up in thin layers separated by a non-conducting material such as lacquer or a metal oxide. This layered structure increases the resistance of eddy current paths and effectively confines the currents to individual layers. Such a laminated structure is used in transformer cores and motors to minimize eddy currents and thereby increase the efficiency of these devices.

Induction Brakes

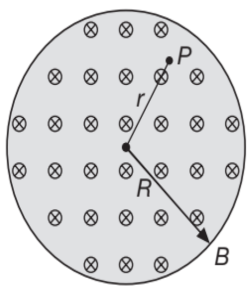
The braking systems on many subway and rapid-transit cars make use of electromagnetic induction and eddy currents. An electromagnet attached to the train is positioned near the steel rails. (An electromagnet is essentially a solenoid with an iron core). The braking action occurs when a large current is passed through the electromagnet. The relative motion of the magnet and rails induces eddy currents in the rails, and the direction of these currents produces a drag force on the moving train. Because the eddy currents decrease steadily in magnitude as the train slows down, the braking effect is quite smooth. As a safety measure, some power tools use eddy currents to stop rapidly spinning blades once the device is turned off.

Test Your Concepts-IV

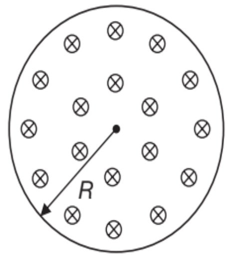
Based on Faraday's Laws: Induced Electric Field

(Solutions on page H.138)

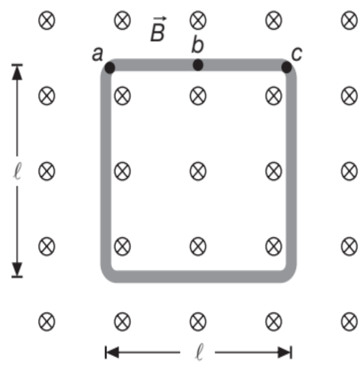
1. A long solenoid with n turns per meter and radius R carries an oscillating current given by $I = I_0 \sin(\omega t)$. What is the electric field induced at a radius $r(<R)$ from the axis of the solenoid? What is the direction of this electric field when the current is increasing counter clockwise in the coil?
2. A magnetic field directed into the page changes with time according to $B = (0.03t^2 + 1.4)$ T, where t is in seconds. The field has a circular cross section of radius $R = 2.5$ cm. What are the magnitude and direction of the electric field at point P when $t = 3$ s and $r = 0.02$ m?



3. A betatron accelerates electrons to energies in the *MeV* range by means of electromagnetic induction. Electrons in a vacuum chamber are held in a circular orbit by a magnetic field perpendicular to the orbital plane. The magnetic field is gradually increased to induce an electric field around the orbit.
   (a) Show that the electric field is set up in a direction so as to make the electrons speed up.
   (b) Assume that the radius of the orbit remains constant. Show that the average magnetic field over the area enclosed by the orbit must be twice as large as the magnetic field at the circumference of the circle.
4. The magnetic field at all points within the cylindrical region whose cross-section is indicated in the accompanying figure start increasing at a constant rate α Ts^{-1}. Find the magnitude of electric field as a function of r, the distance from the geometric centre of the region.



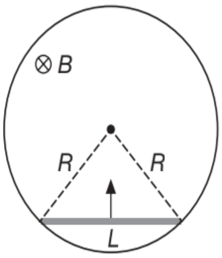
5. A square conducting loop, of side ℓ, is placed in a uniformly decreasing magnetic field shown. Assuming the center of the square loop is at the center of the magnetic field region.



   (a) Draw vectors to show the directions and relative magnitudes of the induced electric field $\vec{E}$ at points a, b and c.
   (b) Prove that the component of $\vec{E}$ along the loop has the same value at every point of the loop and is equal to that of the ring passing through that point.
6. A thin non-conducting ring of mass m radius R carrying a charge q can rotate freely about its own axis which is vertical. At the initial moment, the ring was at rest and no magnetic field was present. At instant $t = 0$, a uniform magnetic field, inwards normal to the loop is switched on which increases with time according to the law $B = B_0 t$. Neglecting magnetism induced due to rotational motion of the ring, calculate
   (a) angular acceleration and angular velocity of the ring at time t and
   (b) power developed by the force acting on the ring as a function of time.

7. In a long straight solenoid with cross-sectional radius a and number of turns per unit length n, a current varies at a constant rate of I ampere per second. Calculate the magnitude of the induced electric field strength as a function of the distance r from the solenoid axis. Also plot the graph of magnitude of induced field vs the distance r from the axis of the solenoid.

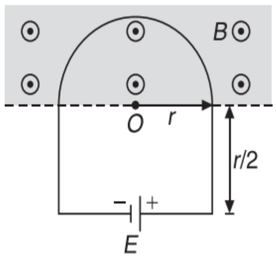
8. A cylindrical volume of radius R has a uniform axial magnetic field B, which is changing at the rate $\frac{dB}{dt}$. A metal rod of length L is placed in a plane normal to the axis of the cylinder as shown in Figure.



Show that the emf between the two ends of the rod is

$$e=\left(\frac{L}{2}\sqrt{R^2-\frac{L^2}{4}}\right)\frac{dB}{dt}$$

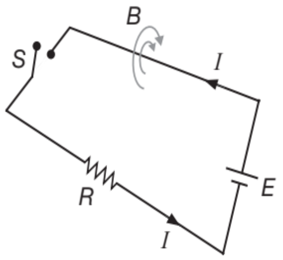
9. A conducting loop has its semi-circular part lying in a magnetic field B which varies with time as $B=(2t^3+3t^2+4)$ tesla. The wire is having a resistance $R\ \Omega m^{-1}$. Calculate the current in the loop at time $t=2s$.



SELF AND MUTUAL INDUCTION

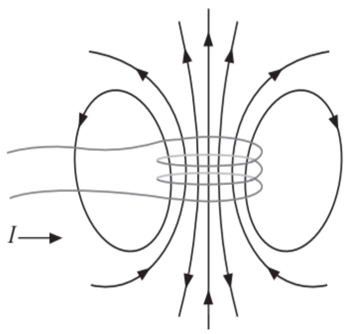
PHENOMENON OF SELF INDUCTION: AN INTRODUCTION

Consider a circuit having a source of emf E , a resistor R and a switch S as shown in Figure.



It is observed that, when the switch is thrown to its closed position, the current does not immediately jump from zero to its maximum value $\frac{E}{R}$. Explanation to this effect is provided by the Faraday's Law of Electromagnetic Induction. As the current increases with time, the magnetic flux through the circuit loop due to this current also increases with time. This increasing flux creates an induced emf in the circuit. The direction of the induced emf is such that it produces an induced current in the loop (if the loop did not already carry a current), which would establish a magnetic field that opposes the change in the original magnetic field. Thus, the direction of the induced emf is opposite to the direction of the emf of the battery which results in a gradual (not instantaneous) increase in the current to its final equilibrium value. Because of the direction of the induced emf, it is also called a **back emf**, similar to that in a motor. This phenomenon is called the phenomenon of self-induction because the changing flux through the circuit and the resultant induced emf arise from the circuit is itself. The emf, ξ_L set up in this case in the circuit called the **self-induced emf**.

As a second example, consider a coil consisting of N turns carrying a current I in the counterclockwise direction, as shown in Figure.



If the current is steady, then the magnetic flux through the coil remains constant, so that no induced emf exists in the coil. However, suppose the current I changes with time, then according to Faraday's Laws of Electromagnetic Induction, an induced emf is set up in the coil so that it opposes the change producing it. The induced current will flow clockwise in the coil, if $\frac{dI}{dt} > 0$ i.e., I increases with time and it will flow counterclockwise in the coil, if $\frac{dI}{dt} < 0$ i.e., I decreases with time.

So, finally we observe that whenever the electric current flowing through a circuit (or loop) changes, the magnetic flux linked with the circuit (or loop) also changes so as to produce an induced emf current in the circuit (or loop). This phenomenon is called the **phenomenon of self-induction** and the emf so induced is called the **back emf**.

INDUCTOR AND SELF INDUCTANCE: BASIC INTRODUCTION AND SIGNIFICANCE

If a circuit contains a coil, such as a solenoid, the self-inductance of the coil prevents the current in the circuit from increasing or decreasing instantaneously. A circuit element that has a large self-inductance is called an inductor and has the circuit symbol •–ooooo–•.

We always assume that the self-inductance of the remainder of a circuit is negligible compared with that of the inductor. However, keep in mind, that even a circuit without a coil has some self-inductance that can affect the behavior of the circuit.

Since the inductance of an inductor results in a back emf, an inductor in a circuit opposes changes in the current in that circuit. The inductor attempts to keep the current the same as it was before the change occurred. If the battery voltage in the circuit is increased so that the current rises, the inductor opposes this change, and the rise is not instantaneous. If the battery voltage is decreased, the presence of the inductor results in a slow drop in the current rather than an immediate drop. Thus, the inductor causes the circuit to be "sluggish" as it reacts to changes in the voltage. An inductor has no role to play in a circuit as long as the current in the circuit stays constant. It becomes active whenever the current linked with the circuit changes. The self-inductance of an inductor depends on the shape, size, the number of turns and the magnetic properties of the material enclosed by the circuit.

SELF INDUCTANCE: DEFINITION

The phenomenon of the production of an induced e.m.f. in the coil when the flux linked with the coil changes is called **phenomenon of self-induction**.

If I is the current flowing in the circuit, then flux linked with the circuit is observed to be proportional to I.

$\Rightarrow \quad \phi_B \propto I$

$\Rightarrow \quad \phi_B = LI$

where L is called the self-inductance or coefficient of self-inductance or simply inductance of the coil and its SI unit is henry (H)

$\Rightarrow \quad 1\text{ H} = 1\text{ WbA}^{-1}$

If $I = 1\text{ A}$, then $L = \phi$ (numerically)

So, inductance of a coil is numerically equal to the flux linked with the coil when the current in the coil is 1 A.

Since, $\xi = -\dfrac{d\phi_B}{dt} = -\dfrac{d}{dt}(LI) = -L\left(\dfrac{dI}{dt}\right)$

$\Rightarrow \quad \xi = L\dfrac{dI}{dt}$ {in magnitude}

If $\dfrac{dI}{dt} = 1\text{ As}^{-1}$, then $L = \xi$ (numerically)

So, inductance of a coil is numerically equal to the e.m.f. induced in the coil when the current in the coil changes at the rate of 1 As^{-1}.

Physically, the inductance L is a measure of an **"Inductor's Opposition"** to the change of current. The larger the value of L, the lower the change in the value of current. So, L is also called **"Electrical Inertia"**.

Conceptual Note(s)

(a) Here we must note that if we are asked to calculate the induced e.m.f. in an inductor, then we have

$$\xi = -L\frac{dI}{dt}$$

But when we are asked to calculate the voltage (V) across the inductor then

$$V = |\xi| = L\frac{dI}{dt}$$

(b) Consider a coil that is wound on a cylindrical core and the current in the coil either increases or decreases. Then an induced emf is set up in the coil, which is in accordance with the Lenz's Law. The polarity of this induced emf must be such that it opposes the change in magnetic field due to the change in current. Due to this we can **think of replacing the coil with an equivalent self-induced emf with the polarity shown in Figure.**

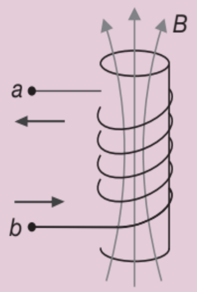


Figure 1(a)

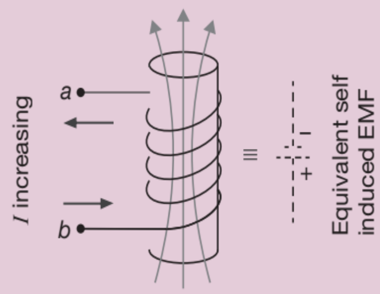


Figure 1(b)

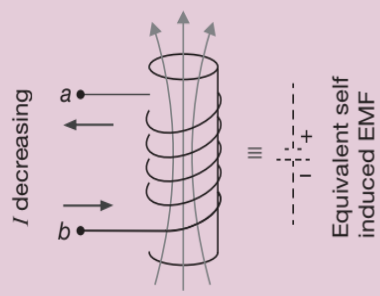


Figure 1(c)

Please note that the 'dotted battery' represents the equivalent self-induced emf produced by the coil, hence it is shown here as a replacement to the coil and is not the emf across the free ends a and b of the coil.

Finally, we have

$$|\text{Equivalent Self Induced EMF}| = L\left(\frac{dI}{dt}\right)$$

Also note that in giving the polarity of the equivalent self-induced emf, we must first see whether I is increasing or decreasing. Let us take the case of Figure 1(b), where I is increasing i.e., $\frac{dI}{dt} > 0$ and B is also increasing with t. Due to this $\xi < 0$. Now this means that the emf induced must produce an induced current that must not allow B to increase. So, this induced current, $I_{induced}$ must have a direction opposite to I (as in Figure 2(a)). Hence the equivalent self-induced emf must have a polarity as shown. A similar argument also holds good for explaining the polarity of the equivalent self-induced emf in Figure 1(c).

So, we can easily summarise the above discussions in the form of diagrams shown below.

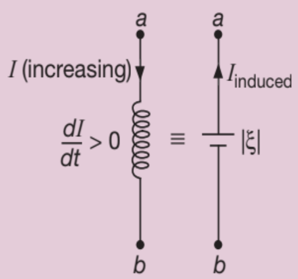


Figure 2(a) The polarity of $|\xi|$ will not allow I to increase, because it shall be producing an induced current, $I_{induced}$, that opposes I.

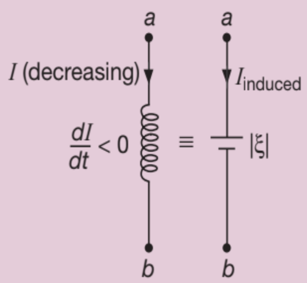


Figure 2(b) The polarity of $|\xi|$ will not allow I to decrease, because it shall be producing an induced current, $I_{induced}$, that favours I.

TECHNIQUE TO FIND THE SELF INDUCTANCE

To find the self-inductance of a given circuit, a good approach consists of the following steps.

STEP-1: Assume that a current I flows through the circuit (we can call the circuit an inductor).

STEP-2: Determine the magnetic field $\vec{B}$ produced due to the flow of this current.

STEP-3: Obtain the magnetic flux $\phi_B = \vec{B} \cdot \vec{A}$.

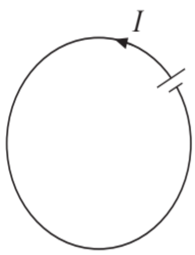
STEP-4: With the flux known, the self-inductance is given by

$$L = \frac{N\phi_B}{I}$$

The above steps have been implemented to find the self-inductance of the circuits discussed further.

Self-inductance for a Circular Coil

Consider a circular coil of radius r and number of turns N, as shown in Figure.



If current I passes in the coil, then magnetic field at centre of coil

$$B = \frac{\mu_0 NI}{2r}$$

The effective magnetic flux linked with this coil

$$\phi_B = NBA = N\left(\frac{\mu_0 NI}{2r}\right)A$$

Since, by definition

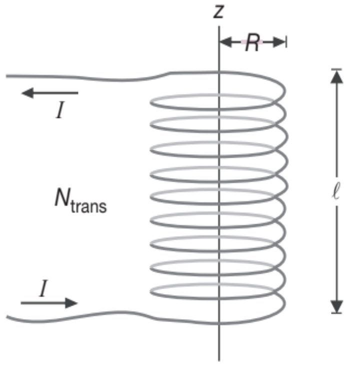
$$L = \frac{\phi_B}{I}$$

$$\Rightarrow \quad L = \frac{\mu_0 N^2 A}{2r} = \frac{\mu_0 N^2 \pi r^2}{2r}$$

$$\Rightarrow \quad L = \frac{\mu_0 N^2 \pi r}{2}$$

Self-inductance for a Solenoid

Consider a solenoid with n number of turns per metre as shown in Figure.



Let current I flow in the windings of solenoid, then magnetic field inside solenoid is given by

$$B = \mu_0 n I$$

The magnetic flux linked with its length l is $\phi_B = NBA$, where N is total number of turns in length l of solenoid.

$$\Rightarrow \quad \phi_B = (nl) BA = nl(\mu_0 n I) A = (\mu_0 n^2 A l) I$$

Since, $L = \dfrac{\phi_B}{I}$

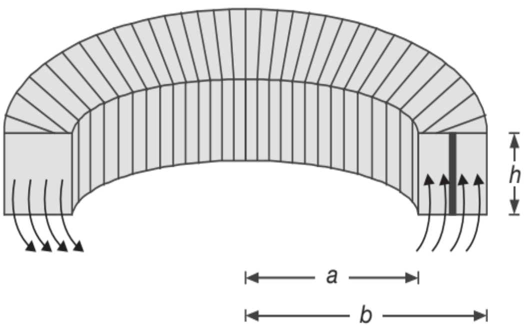
$$L = \mu_0 n^2 A l$$

Since, $n = \dfrac{N}{l}$

$$\Rightarrow \quad \text{Self-inductance, } L = \frac{\mu_0 N^2 A}{l}$$

Self-inductance of a Toroid

Let us calculate the self-inductance of a toroid having N turns and a rectangular cross section, with inner radius a, outer radius b and height h, as shown in Figure.



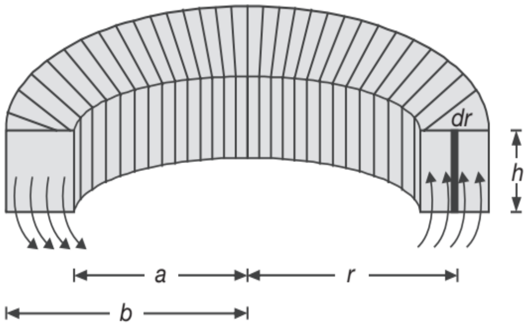
According to Ampere's Law, the magnetic field is given by

$$\oint \vec{B} \cdot d\vec{l} = \oint B dl = B \oint dl = B(2\pi r) = \mu_0 N I$$

$$\Rightarrow \quad B = \frac{\mu_0 N I}{2\pi r} \qquad \ldots(1)$$

where r is the distance from the central axis of the toroid.

The magnetic flux through one turn of the toroid is obtained by integrating over the rectangular cross section, with $dA = h dr$ as the differential area element shown in Figure.



$$\phi_B = \int \vec{B} \cdot d\vec{A} = \int_a^b \left(\frac{\mu_0 N I}{2\pi r} \right) h dr$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 N I h}{2\pi} \log_e \left(\frac{b}{a} \right) \qquad \ldots(2)$$

The total flux is $N\phi_B$, hence the self-inductance is

$$L = \frac{N\phi_B}{I} = \frac{\mu_0 N^2 h}{2\pi} \log_e \left(\frac{b}{a} \right) \qquad \ldots(3)$$

Again, the self-inductance L depends only on the geometrical factors.

SPECIAL CASE

Let's consider the situation where $a \gg b - a$.

In this limit, the logarithmic term in the equation above will be expanded as

$$\log_e \left(\frac{b}{a} \right) = \log_e \left(1 + \frac{b-a}{a} \right) \approx \frac{b-a}{a}$$

and the self-inductance now becomes

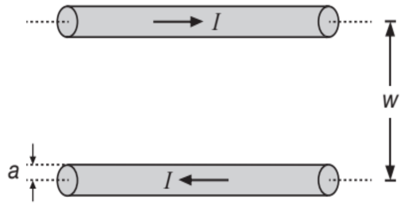
$$L \approx \frac{\mu_0 N^2 h}{2\pi} \left(\frac{b-a}{a} \right) = \frac{\mu_0 N^2 A}{2\pi a} = \frac{\mu_0 N^2 A}{l}$$

where $A = h(b-a)$ is the cross-sectional area, and $l = 2\pi a$

We see that the self-inductance of the toroid in this limit has the same form as that of a solenoid.

ILLUSTRATION 54

Consider two wires 1 and 2 carrying equal currents I but in opposite direction as shown in Figure.



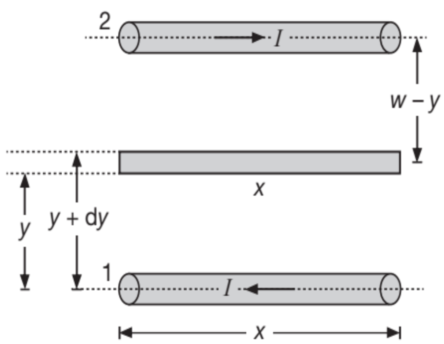
If a be the radius of the wires, and w be the centre to centre distance between them, then find the inductance L of a length x of this arrangement. In doing so, neglect magnetic flux inside the wires.

SOLUTION

Consider a section of arrangement of length x. Since we have to find L, so we must first find the flux ϕ, associated with a wire and then we get

$$L = \frac{\phi}{I} = \frac{1}{I}\int BdA$$

Consider a rectangular areal strip in between the wires at a distance y from centre of the lower wire as shown in Figure.



If dy be the width of the strip then $dA = xdy$ field at the strip due to the two wires is

$$B = \frac{\mu_0 I}{2\pi}\left(\frac{1}{y} + \frac{1}{(w-y)}\right) \text{ inwards}$$

$$\Rightarrow \quad |d\phi_B| = BdA = \frac{\mu_0 I}{2\pi}\left(\frac{1}{y} + \frac{1}{w-y}\right)xdy$$

$$\Rightarrow \quad d\phi_B = \frac{\mu_0 Ix}{2\pi}\left(\frac{dy}{y} + \frac{dy}{w-y}\right)$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 Ix}{2\pi}\left[\int_a^{w-a}\frac{dy}{y} + \int_a^{w-a}\frac{dy}{w-y}\right]$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 Ix}{2\pi}\left(\log_e y\Big|_a^{w-a} - \log_e(w-y)\Big|_a^{w-a}\right)$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 Ix}{2\pi}\left[\log_e\left(\frac{w-a}{a}\right) - \log_e\left(\frac{w-w+a}{w-a}\right)\right]$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 Ix}{2\pi}\left[\log_e\left(\frac{w-a}{a}\right) - \log_e\left(\frac{a}{w-a}\right)\right]$$

Since, $\log_e\left(\frac{1}{m}\right) = -\log_e m$

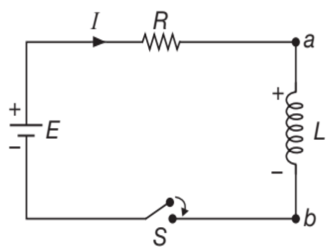
$$\Rightarrow \quad \phi_B = \frac{\mu_0 Ix}{2\pi}\left[2\log_e\left(\frac{w-a}{a}\right)\right]$$

$$\Rightarrow \quad \phi_B = \frac{\mu_0 Ix}{\pi}\log_e\left(\frac{w-a}{a}\right)$$

$$\Rightarrow \quad L = \frac{\phi_B}{I} = \frac{\mu_0 x}{\pi}\log_e\left(\frac{w-a}{a}\right)$$

MODIFIED KIRCHHOFF'S RULE FOR INDUCTORS

Consider the circuit which contains a battery of negligible internal resistance as shown in Figure.



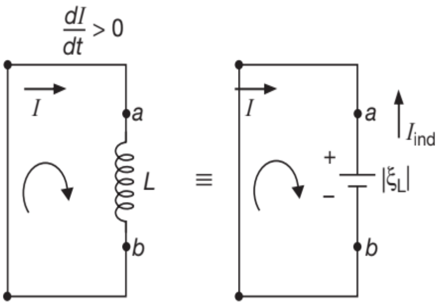
This is an RL circuit because the elements connected to the battery are a resistor and an inductor. Suppose that the switch S is open for $t < 0$ and then closed at $t = 0$. The current in the circuit begins to increase, and a back emf that opposes the increasing current is induced in the inductor. Because the current is increasing, $\frac{dI}{dt}$ is positive; thus, ξ_L is negative. This negative value reflects the decrease in electric potential that occurs in going from a to b across the inductor, as indicated by the positive and negative signs.

With this in mind, we can apply Kirchhoff's Loop Rule to this circuit, traversing the circuit in the clockwise direction

$$E - IR - L\frac{dI}{dt} = 0$$

where IR is the voltage drop across the resistor. (We developed Kirchhoff's rules for circuits with steady currents, but they can also be applied to a circuit in which the current is changing if we imagine them to represent the circuit at one instant of time.

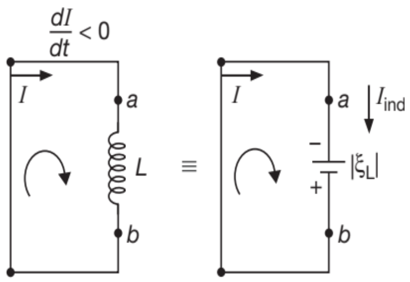
The expression $\Delta V = E - IR - L\frac{dI}{dt} = 0$ has been cast in a form that resembles Kirchhoff's Loop Rule, according to which, "the sum of the potential drops around a circuit is zero". To preserve the loop rule, we must specify the "potential drop" across an inductor.



$$\xi_L = V_b - V_a = -L\left(\frac{dI}{dt}\right) < 0$$

The modified rule for inductors may be obtained as follows: The polarity of the self-induced emf is such as to oppose the change in current, in accord with Lenz's Law. If the rate of change of current is positive, as shown in Figure, the self-induced emf ξ_L sets up an induced current I_{ind} moving in the opposite direction of the current I to oppose such an increase. The inductor could be replaced by an emf $|\xi_L| = L\left|\frac{dI}{dt}\right| = +L\left(\frac{dI}{dt}\right)$ with the polarity shown in Figure. On the other hand, if $\frac{dI}{dt} < 0$, as shown in Figure, the induced current I_{ind} set up by the self-induced emf ξ_L flows in the same direction as I to oppose such a decrease.

We see that whether the rate of change of current in increasing $\left(\frac{dI}{dt} > 0\right)$ or decreasing $\left(\frac{dI}{dt} < 0\right)$, in both cases, the change in potential when moving from a and b along the direction of the current I is $V_b - V_a = -L\left(\frac{dI}{dt}\right)$.



$$\xi_L = V_b - V_a = -L\left(\frac{dI}{dt}\right) > 0$$

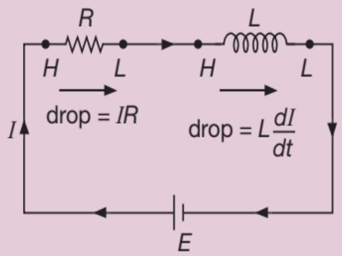
KIRCHHOFF'S LOOP RULE MODIFIED FOR INDUCTORS

(a) If an inductor is traversed in the direction of the current, the "potential change" is $-L\left(\frac{dI}{dt}\right)$.

(b) On the other hand, if the inductor is traversed in the direction opposite of the current, the "potential change" is $+L\left(\frac{dI}{dt}\right)$.

(c) So, for the circuit shown, when we go through an inductor in the same direction as the assumed current, we encounter a voltage drop of $L\frac{dI}{dt}$ where $\frac{dI}{dt}$ is to be substituted with sign. For example, in the loop shown in Figure, Kirchhoff's Second Law gives the equation.

$$E - IR - L\frac{dI}{dt} = 0$$



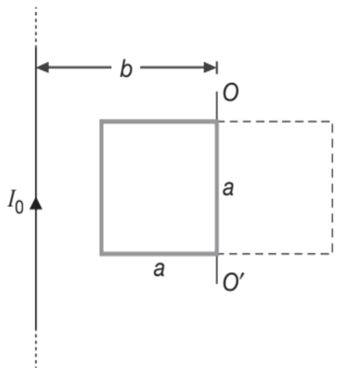
Conceptual Note(s)

Use of this modified Kirchhoff's Rule will give the correct equations for circuit problems that contain inductors. However, keep in mind that it is misleading at best, and at some level wrong in terms of the physics. Again, I emphasize that Kirchhoff's Loop Rule was originally based on the fact that the line integral of $\vec{E}$ around a closed loop was zero.

However, with time changing magnetic fields, this is no longer so, and thus the sum of the "potential drops" around the circuit, if we take that to mean the negative of the closed loop integral of $\vec{E}$, is no longer zero in fact it is $+L\left(\frac{dI}{dt}\right)$.

ILLUSTRATION 55

A square wire frame with side a and a straight conductor carrying a constant current I_0 are located in the same plane. The inductance and the resistance of the frame are equal to L and R respectively. The frame was turned through $180°$ about the axis OO' separated from the current carrying conductor by a distance b. Find the electric charge having flown through the frame.



SOLUTION

Since we know that due to the change in flux a charge flows through the circuit. Since this circuit is a combination of resistor and an inductor, so we proceed as follows.

Let us first calculate the induced emf in the circuit without the inductor

$$\xi_R = -\frac{d\phi}{dt}$$

Now let us calculate the emf induced across the circuit without the resistor. We get

$$\xi_L = -L\frac{dI}{dt}$$

If ξ be the total emf induced, then

$$\xi = -\left(\frac{d\phi}{dt} + L\frac{dI}{dt}\right)$$

So, the induced current I, is

$$I = \frac{\xi}{R} = -\frac{1}{R}\left(\frac{d\phi}{dt} + L\frac{dI}{dt}\right)$$

$$\Rightarrow \quad Idt = -\frac{1}{R}(d\phi + LdI)$$

$$\Rightarrow \quad \int Idt = -\frac{1}{R}\int d\phi - \frac{L}{R}\int dI$$

$$\Rightarrow \quad \Delta q = -\frac{1}{R}\Delta\phi - \frac{L}{R}\Delta I$$

$$\Rightarrow \quad |\Delta q| = \frac{1}{R}(\Delta\phi + L\Delta I) \quad \ldots(1)$$

The coil is stationary before and after rotation takes place. So,

$$\Delta I = 0$$

$$\Rightarrow \quad |\Delta q| = \frac{1}{R}(\phi_f - \phi_i) \quad \ldots(2)$$

To calculate the ϕ_i, we consider a strip of length a, thickness dx at a distance x from the wire. Then

$$d\phi_i = BdA = \frac{\mu_0 I_0}{2\pi x}(adx)$$

$$\phi_i = \int d\phi_i = \frac{\mu_0 I_0 a}{2\pi}\int_{b-a}^{b}\frac{dx}{x} = \frac{\mu_0 I_0 a}{2\pi}\log_e\left(\frac{b}{b-a}\right) \quad \ldots(3)$$

To calculate ϕ_f, we again repeat the same procedure, then

$$d\phi_f = BA\cos(180°) = -\frac{\mu_0 I_0}{2\pi y}(ady)$$

$$\Rightarrow \quad \phi_f = \int d\phi_f = -\frac{\mu_0 I_0 a}{2\pi}\int_{b}^{b+a}\frac{dy}{y}$$

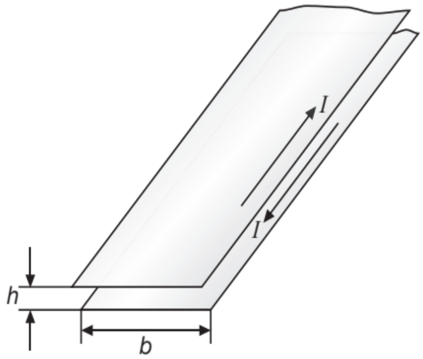
$$\Rightarrow \quad \phi_f = -\frac{\mu_0 I_0 a}{2\pi}\log_e\left(\frac{b+a}{b}\right) \quad \ldots(4)$$

Put (3) and (4) in (2), we get

$$|\Delta q| = \frac{\mu_0 I_0 a}{2\pi R}\log_e\left(\frac{b+a}{b-a}\right)$$

ILLUSTRATION 56

Calculate the inductance per unit length of a double tape line if the tapes are separated by a distance h which is considerably less than their width b. The current flows in the two sheets of the tape line in opposite direction as shown in Figure.



SOLUTION

The current per unit length in the width of the sheet of tape line is $\lambda = \frac{I}{b}$.

Due to this current per unit length λ in each sheet, a magnetic field B_0 is produced by each sheet, so that

$$B_0 = \frac{\mu_0 \lambda}{2} = \frac{\mu_0}{2}\left(\frac{I}{b}\right)$$

The net field between the sheets due to both of the sheets is

$$B = 2B_0 = \frac{\mu_0 I}{b}$$

The magnetic flux associated with the region between the two sheets of tape line of length l is

$$\phi = BA = \left(\frac{\mu_0 I}{b}\right)(lh)$$

The self-inductance of the tape line is

$$L = \frac{\phi}{I} = \left(\frac{\mu_0}{b}\right)(lh)$$

The self-inductance per unit length of the line is

$$\frac{L}{l} = \frac{\mu_0 h}{b}$$

ILLUSTRATION 57

Calculate the self-inductance per unit length of a cable consisting of two thin walled coaxial metallic cylinders if the radius of the outside cylinder is $\eta(\eta > 1)$ times that of the inside one. The permeability of the medium between the cylinders is assumed to be equal to unity.

SOLUTION

A coaxial cable carries equal and opposite current in the two cylinders. the magnetic field in the region between the two cylinders is

$$B = \frac{\mu_0 I}{2\pi r} \text{ for } a \le r \le b$$

where, a and b are the radii of inner and outer cylindrical shells of the cable.

The magnetic flux in an elemental section of length l, width dr at a distance r from the axis of cable is

$$d\phi = B(ldr)$$

Total magnetic flux associated with a cross section in the region between the shells is

$$\phi = \int d\phi = \int Bdr$$

$$\Rightarrow \quad \phi = \frac{\mu_0 Il}{2\pi}\int_a^b \frac{dr}{r}$$

$$\Rightarrow \quad \phi = \frac{\mu_0 Il}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$\Rightarrow \quad \phi = \frac{\mu_0 Il}{2\pi}\ln \eta$$

Self-inductance of the cable

$$L = \frac{\phi}{I}$$

$$\Rightarrow \quad L = \frac{\mu_0 l}{2\pi}\ln \eta$$

Self-inductance per unit length of the cable is

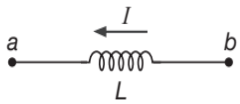
$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln \eta$$

Test Your Concepts-V

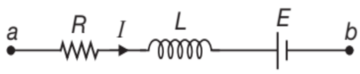
Based on Faraday's Laws: Self Induction

(Solutions on page H.141)

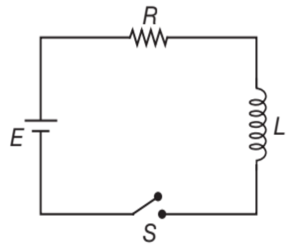
1. The current in a 90 mH inductor changes with time as $I = t^2 - 6t$ (in SI units). Find the magnitude of the induced emf at
   (a) $t = 1$ s and
   (b) $t = 4$ s
   (c) At what time is the emf zero?
2. A self-induced emf in a solenoid of inductance L changes in time as $\xi = \xi_0 e^{-kt}$. Find the total charge that passes through the solenoid, assuming the charge is finite.
3. The inductor shown in Figure has inductance 0.54 H and carries a current in the direction shown that is decreasing at a uniform rate $\frac{dI}{dt} = -0.03$ As^{-1}.



   (a) Find the self-induced emf,
   (b) Which end of the inductor a or b is at a higher potential?
4. In the circuit diagram shown in Figure, $R = 10\ \Omega$, $L = 5$ H, $E = 20$ V, $I = 2$ A. If $\frac{dI}{dt} = -1$ As^{-1}, find V_{ab} at this instant.



5. **(a)** Calculate the inductance of an air core solenoid containing 300 turns if the length of the solenoid is 25 cm and its cross-sectional area is 4 cm^2.
   (b) Calculate the self-induced emf in the solenoid if the current through it is decreasing at the rate of 50 As^{-1}.
6. Calculate the self-inductance of a toroid whose inside radius is equal to b and cross-section has the form of a small square of side a. The solenoid winding consists of N turns. The space inside the solenoid is filled with uniform paramagnetic material having relative permeability μ_r.
7. For the RL circuit shown in Figure, let the inductance be 3 H, the resistance 8 Ω, and the battery emf 36 V.

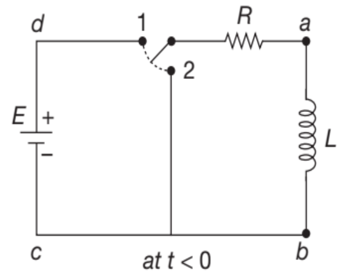


   (a) Calculate the potential difference across the resistor when the current is 2 A.
   (b) Calculate the potential difference across the inductor when the current is 2 A.
   (c) Calculate the voltage across the inductor when the current is 4.5 A.
8. A long solenoid of diameter 0.1 m has 2×10^4 turns per meter. At the centre of the solenoid, a 100 turn coil of radius 0.01 m is placed with its axis coinciding with the solenoid axis. The current in the solenoid is decreased at a constant rate from +2A to –2A in 0.05 s. Find the EMF induced in the coil (in mV). Also find the total charge flowing through the coil (in μC) during the same time interval if the resistance of the coil is $10\pi^2\ \Omega$.
9. Calculate the inductance per unit length of a double wire line if the radius of each wire is η times less than the distance between the axes of the wires. The field inside the wires is to be neglected, the permeability is assumed to be equal to unity throughout and $\eta \gg 1$.

SERIES *LR* CIRCUIT: CURRENT GROWTH AND DECAY

Growth of Current

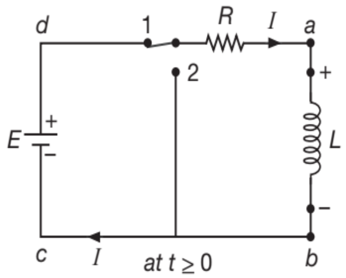
Consider a series LR circuit which contains a battery E of negligible internal resistance, a resistor R, an inductor L and a switch S, connected in series as shown in Figure.



Suppose that the switch S had been open for $t<0$ and then closed at $t=0$, by throwing it to position 1. The current in the circuit begins to increase, and an induced emf which opposes the increasing current is induced in the inductor. Because the current is increasing i.e., $\frac{dI}{dt}$ is positive so, we have

$$\xi = -L\frac{dI}{dt} < 0$$

This negative value reflects the decrease in electric potential that occurs in going from a to b across the inductor, as indicated by the positive and negative signs in Figure.



Apply Kirchhoff's loop Rule (as discussed before) to the loop $abcda$, we get

$$-L\frac{dI}{dt} + E - IR = 0 \qquad \ldots(1)$$

where IR is the voltage drop across the resistor. (We developed Kirchhoff's Rules for circuits with steady currents, but they can also be applied to a circuit in which the current is changing if we imagine them to represent the circuit at one instant of time). We must now look for a solution to this differential equation, which is similar to that for the RC circuit. Since, from (1), we get

$$E = IR + L\frac{dI}{dt}$$

$$\Rightarrow \quad E - IR = L\frac{dI}{dt}$$

$$\Rightarrow \quad \frac{dI}{E-IR} = \frac{1}{L}dt$$

$$\Rightarrow \quad \int_0^I \frac{dI}{E-IR} = \frac{1}{L}\int_0^t dt$$

$$\Rightarrow \quad -\frac{1}{R}\log_e(E-IR)\Big|_0^I = \left(\frac{1}{L}\right)t\Big|_0^t$$

$$\Rightarrow \quad \log_e\left(\frac{E-IR}{E}\right) = -\left(\frac{R}{L}\right)t$$

$$\Rightarrow \quad \frac{E-IR}{E} = e^{-\frac{Rt}{L}}$$

$$\Rightarrow \quad E - IR = Ee^{-\frac{Rt}{L}}$$

$$\Rightarrow \quad I = \frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right) = I_0\left(1-e^{-\frac{Rt}{L}}\right)$$

where $I_0 = \frac{E}{R}$, is the peak value of current in the circuit i.e., current in the circuit when $t \to \infty$.

This expression shows how the inductor effects the current. The current does not increase instantly to its final equilibrium value when the switch is closed but instead increases according to an exponential function.

If we remove the inductance in the circuit, i.e., $L \to 0$, the exponential term becomes zero and we see that there is no time dependence of the current for $L \to 0$ i.e., the current increases instantaneously to its final equilibrium value in the absence of the inductance.

The above expression can also be written as

$$I = \frac{E}{R}\left(1-e^{-\frac{t}{\tau}}\right) = I_0\left(1-e^{-\frac{t}{\tau}}\right)$$

where τ is the time constant of the LR circuit, also called as Inductive Time Constant of LR circuit.

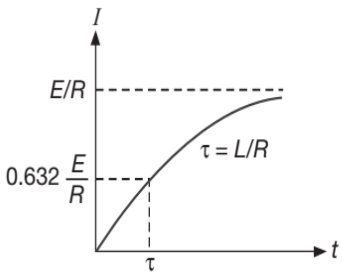
$$\tau = \frac{L}{R}$$

and I_0 is the maximum constant value given by

$$I_0 = \frac{E}{R}$$

Physically, τ is the time interval required for the current in the circuit to reach $(1-e^{-1})=0.632=63.2\%$ of its final value/maximum value $\frac{E}{R}$.

The time constant is a useful parameter for comparing the time responses of various circuits. Figure shows a graph of the current versus time in the RL circuit.

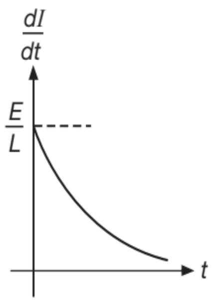


Plot of the current I versus time t for series LR circuit.

Note that the equilibrium value of the current, which occurs at t approaches infinity, is $\frac{E}{R}$. Thus, we see that the current initially increases very rapidly and then gradually approaches the equilibrium value $\frac{E}{R}$ as $t \to \infty$.

$$\frac{dI}{dt} = \frac{E}{L} e^{-\frac{t}{\tau}}$$

From this result, we see that the time rate of change of the current is a maximum $\left(\text{equal to } \frac{E}{L}\right)$ at $t=0$ and falls off exponentially to zero as t approaches infinity as shown in Figure.

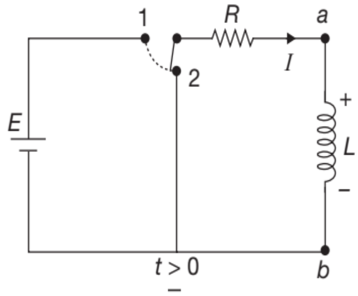


Plot of dI/dt versus time t for series LR circuit.

DECAY OF CURRENT

Again let us consider the LR circuit taken previously, when the switch had been kept at position 1 for long enough $(t \to \infty)$, such that the current reaches a maximum constant value $I_0 = \frac{E}{R}$.

Now let the switch S be thrown from position 1 to position 2, so that the circuit is now just described by the right loop of the diagram shown.



Please note that here too we have taken $t \geq 0$, because for $t < 0$, a constant maximum current I_0 flows through the circuit, which shall now decay to attain some new value $I(< I_0)$. So, for the decay circuit, we have

$$I = \begin{cases} I_0 & \text{at} \quad t=0 \quad \text{and} \\ I(< I_0) & \text{at} \quad \text{time } t(>0) \end{cases}$$

Since no battery is connected to the loop $ab2a$, so by applying Kirchhoff's Loop Rule to loop $ab2a$, we get

$$0 = -L\frac{dI}{dt} - IR$$

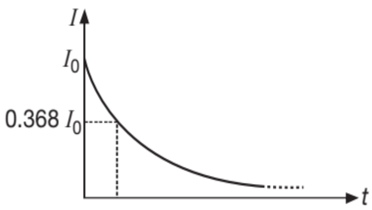
$$\Rightarrow \quad L\frac{dI}{dt} = -IR$$

$$\Rightarrow \quad \int_{I_0}^{I} \frac{dI}{I} = -\frac{R}{L}\int_0^t dt$$

$$\Rightarrow \quad \log_e\left(\frac{I}{I_0}\right) = -\left(\frac{R}{L}\right)t$$

$$\Rightarrow \quad I = I_0 e^{-\frac{Rt}{L}} = I_0 e^{-\frac{t}{\tau}}$$

where $\tau = \frac{L}{R}$ and $I_0 = \frac{E}{R}$ (at $t < 0$)



Current I vs time t for series LR decay circuit.

The variation of I vs t is shown here. Please observe that if the circuit has got no inductor L (take $L \to 0$), then the current will immediately decay to zero as soon as the battery is removed (or the switch is thrown to position 2). However, the presence of an inductor causes the current to decrease exponentially.

Also note that here $\dfrac{dI}{dt} = -\left(\dfrac{I_0 R}{L}\right)e^{-\frac{Rt}{L}} < 0$

So, $\dfrac{dI}{dt}$ is always negative and hence $\xi > 0$

Also, note that the inductive time constant may also be defined as the time in which the current in the circuit decays from I_0 to $0.368 I_0$.

Problem Solving Technique(s)

(a) If the growth current is denoted by I_g and the decay current by I_d, then at the same time instants from the start, we have

$$I_g + I_d = I_0 = \frac{E}{R}$$

(b) Theoretically the current grows from zero to I_0 (or decays from I_0 to zero) when $t \to \infty$, however practically, it is observed that the current grows from zero to I_0 (or decays from I_0 to zero) in five time constants i.e., as $t \to \dfrac{5L}{R}$.

So, $I\left(t = \dfrac{5L}{R}\right) \longrightarrow I_0$, for Growth Circuit and

$I\left(t = \dfrac{5L}{R}\right) \longrightarrow$ zero, for Decay Circuit.

(c) Also, we observe that

$$\frac{dI_g}{dt} = -\frac{dI_d}{dt} = \left(\frac{I_0 R}{L}\right)e^{-\frac{Rt}{L}}$$

(d) During the growth of current in the circuit, we have

$$\frac{dI}{dt} > 0 \text{ and hence } \xi < 0$$

(e) During the decay of current in the circuit, we have

$$\frac{dI}{dt} < 0 \text{ and hence } \xi > 0$$

(f) The magnitude of the induced emf (or the self-induced emf) is

$$|\xi| = \left|-L\frac{dI}{dt}\right| = I_0 R e^{-\frac{Rt}{L}}$$

(g) ξ is maximum when $t = 0$ i.e., initially and vanishes as t approaches infinity i.e., $t \to \infty$. This simply means, that a sufficiently long time after the switch is closed, self-induction disappears and the inductor simply acts as a conducting wire connecting two parts of the circuit.

(h) Since we have

$$E = IR + \frac{LdI}{dt}$$

Multiplying both sides by I, we get

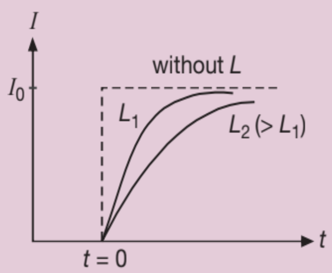
$$EI = I^2R + LI\frac{dI}{dt} \qquad \text{...(1)}$$

where, EI is the rate at which battery delivers energy to the circuit, I^2R is the power dissipated by the resistor in the form of heat and $LI\dfrac{dI}{dt}$ is the rate at which energy is stored in the inductor.

So, equation (1) just represents the Law of Conservation of Energy for the LR circuit.

Please note that the energy dissipated by the resistor as heat is not recoverable, however the magnetic energy stored in the inductor can be released at some later time.

(i) In some growth graphs, we may have the situations shown in a single graph for a set of L values. Then from the graph we can conclude which graph corresponds to which value of L, so we observe $L_2 > L_1$.



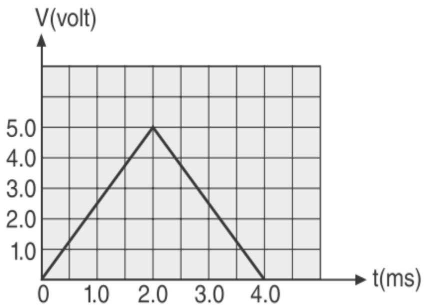
(j) So, while solving problems involving series inductor-resistor combination, we must keep in mind that at

$$t=0, I=0 \text{ and } t \to \infty, \frac{dI}{dt}=0$$

This happens to be a helping tool to solve equations, as used in the ILLUSTRATIONS to come.

ILLUSTRATION 58

The potential difference across a 150 mH inductor as a function of time is shown in Figure.



Assume that the initial value of the current in the inductor is zero. Calculate the current at time 2 ms and 4 ms.

SOLUTION

The potential difference across an inductor is given by

$$V_L = L\frac{di}{dt}$$

$$\Rightarrow \quad di = \frac{1}{L}(V_L dt)$$

Integrating, we get

$$\int di = i = \frac{1}{L}\int V_L dt$$

$$\Rightarrow \quad i = \frac{1}{L}(\text{Area under } V_L \text{ versus } t \text{ graph})$$

At $t = 2$ ms, we have

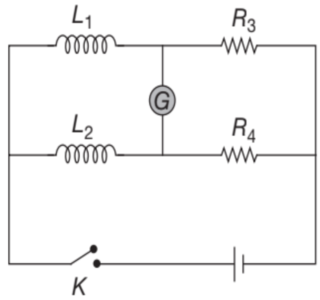
$$i = \frac{\frac{1}{2}(2\times10^{-3})(5)}{150\times10^{-3}} = 3.33\times10^{-2} \text{ A}$$

At $t = 4$ ms, area is just double and hence the current is also doubled.

$$\Rightarrow \quad i = \frac{2\left[\frac{1}{2}(2\times10^{-3})(5)\right]}{150\times10^{-3}} = 6.66\times10^{-2} \text{ A}$$

ILLUSTRATION 59

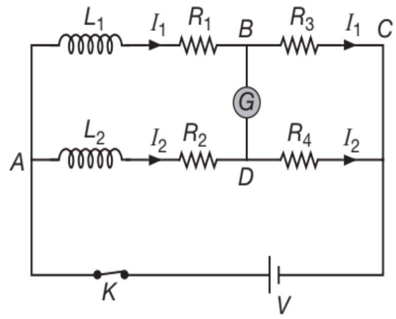
Two inductors of self-inductances L_1 and L_2 and of resistances R_1 and R_2 (not shown here) respectively, are connected in the circuit as shown in the Figure. At the instant $t=0$, key K is closed, obtain an expression for which the galvanometer will show zero deflection at all times after the key is closed.



SOLUTION

Since no current flows through the branch BD, so

$$V_B = V_D$$



For loop $ABDA$, we have

$$\Rightarrow \quad V_A - V_B = V_A - V_D$$

$$\Rightarrow \quad L_1\frac{dI_1}{dt} + I_1R_1 = L_2\frac{dI_2}{dt} + I_2R_2 \qquad \ldots(1)$$

Similarly

$$V_B = V_D$$

$$\Rightarrow \quad V_C - V_B = V_C - V_D$$

$$\Rightarrow \quad I_1R_3 = I_2R_4 \qquad \ldots(2)$$

Take derivative of (2) w.r.t. t, we get

$$R_3\left(\frac{dI_1}{dt}\right) = R_4\left(\frac{dI_2}{dt}\right)$$

$$\Rightarrow \quad \frac{dI_1}{dt} = \frac{R_4}{R_3}\left(\frac{dI_2}{dt}\right) \qquad \ldots(3)$$

Substituting value of I_1 from (2) and $\frac{dI_1}{dt}$ from (3) in (1), we get

$$\left(\frac{R_4}{R_3}L_1 - L_2\right)\frac{dI_2}{dt} = I_2\left(R_2 - \frac{R_1R_4}{R_3}\right) \quad \ldots(4)$$

When $t = 0$, $I_2 = 0$

$$\Rightarrow \quad \frac{R_4}{R_3} = \frac{L_2}{L_1} \quad \ldots(5)$$

Now, when $t \to \infty$ i.e., I_2 grows to a maximum constant value, then

$$\frac{dI_2}{dt} = 0$$

$$\Rightarrow \quad R_2 = \frac{R_4R_1}{R_3}$$

$$\Rightarrow \quad \frac{R_1}{R_2} = \frac{R_3}{R_4} \quad \ldots(6)$$

From (5) and (6), we get

$$\frac{L_1}{L_2} = \frac{R_1}{R_2} = \frac{R_3}{R_4}$$

ILLUSTRATION 60

A straight solenoid of length l is having a single layer winding of copper wire whose total mass is m. The cross-sectional diameter of the solenoid is assumed to be considerably less than its length. Take resistivity of copper to be ρ and density is equal to d, calculate the time constant τ of the solenoid.

SOLUTION

Time constant of the solenoid i.e. an LR circuit is given by

$$\tau = \frac{L}{R}$$

Inductance of a solenoid is

$$L = \mu_0 n^2 A l = \frac{\mu_0 N^2 A}{l} = \left(\frac{\mu_0 N^2}{l}\right)\pi r^2 \quad \ldots(1)$$

where, r is the radius of cross-section of the solenoid.

If l_0 is the total length of wire that makes the solenoid, then we have

$$l_0 = N(2\pi r)$$

$$\Rightarrow \quad r = \frac{l_0}{2\pi N}$$

So, equation (1) can be rewritten as

$$L = \left(\frac{\mu_0 N^2}{l}\right)\pi\left(\frac{l_0}{2\pi N}\right)^2 = \frac{\mu_0}{4\pi}\frac{l_0^2}{l} \quad \ldots(2)$$

The resistance of the wire is

$$R = \frac{\rho l_0}{A} = \frac{\rho l_0^2}{A l_0} = \frac{\rho l_0^2}{V} = \frac{\rho l_0^2}{m/d} = \frac{\rho l_0^2 d}{m}$$

$$\Rightarrow \quad R = \frac{\rho l_0^2 d}{m}$$

$$\Rightarrow \quad l_0^2 = \frac{mR}{\rho d}$$

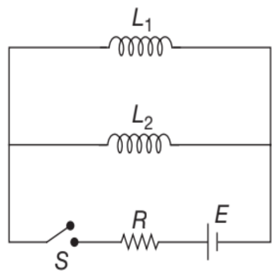
Substituting this value of l_0^2 in equation (2), we get

$$L = \frac{\mu_0}{4\pi l}\left(\frac{mR}{\rho d}\right)$$

$$\Rightarrow \quad \tau = \frac{L}{R} = \frac{\mu_0 m}{4\pi l \rho d}$$

ILLUSTRATION 61

In the circuit shown emf E, a resistance R and coil inductances L_1 and L_2 are known. The internal resistance of the source and the coil are negligible. Find the steady state currents in the coils after switch S was shorted.



SOLUTION

Since L_1 and L_2 are in parallel, so we have

$$L_1\frac{dI_1}{dt} = L_2\frac{dI_2}{dt} = E - (I_1 + I_2)R \quad \ldots(1)$$

At $t = 0$, $I_1 = I_2 = 0$, so we get from (1),

$$L_1\frac{dI_1}{dt} = L_2\frac{dI_2}{dt}$$

$$\Rightarrow \quad L_1 dI_1 = L_2 dI_2$$

$$\Rightarrow \quad L_1 I_1 = L_2 I_2 \quad \ldots(2)$$

Also, at steady state, we have

$$\frac{dI_1}{dt}=\frac{dI_2}{dt}=0$$

$$\Rightarrow \quad E-(I_1+I_2)R=0$$

$$\Rightarrow \quad (I_1+I_2)=\frac{E}{R} \qquad \ldots(3)$$

From (2) and (3), we get

$$I_1=\frac{EL_2}{R(L_1+L_2)} \text{ and } I_2=\frac{EL_1}{R(L_1+L_2)}$$

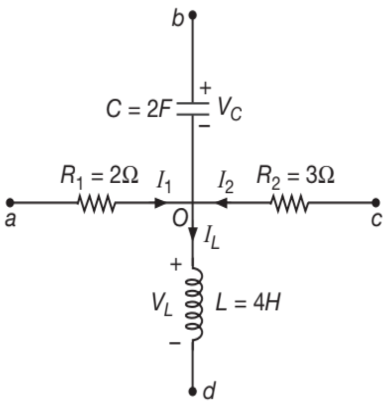
ILLUSTRATION 62

In the Figure shown if, $I_1=10e^{-2t}$ A, $I_2=4$ A and $V_C=3e^{-2t}$ V, then calculate

(a) the current and voltage, I_L and V_L, across L as a function of t.

(b) V_{ac}, V_{ab} and V_{cd} as function of t.

Show the variation of I_L, V_L, V_{ac}, V_{ab} and V_{cd} with time t.



SOLUTION

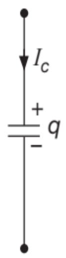
(a) Charge stored in the capacitor at time t,

$$q=CV_C$$

$$\Rightarrow \quad q=(2)(3e^{-2t})$$

$$\Rightarrow \quad q=6e^{-2t}\text{ C}$$

$$\Rightarrow \quad I_C=\frac{dq}{dt}=-12e^{-2t}\text{ A}$$



{direction of current is from b to O}

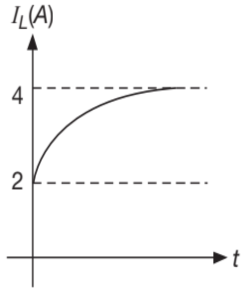
Applying junction rule at O,

$$I_L=I_1+I_2+I_C=10e^{-2t}+4-12e^{-2t}$$

$$I_L=(4-2e^{-2t})\text{ A}$$

$$I_L=[2+2(1-e^{-2t})]\text{ A}$$

I_L versus time graph is as shown in Figure.



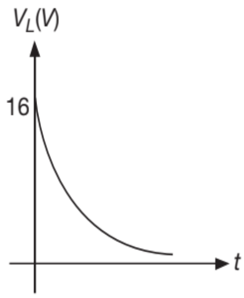
I_L increases from 2 A to 4 A exponentially.

$$V_L=V_{Od}=L\frac{dI_L}{dt}$$

$$\Rightarrow \quad V_L=(4)\frac{d}{dt}(4-2e^{-2t})$$

$$\Rightarrow \quad V_L=16e^{-2t}\text{ V}$$

V_L versus time graph is as shown in Figure.



V_L decreases exponentially from 16 V to 0.

(b) $V_{ac}=V_a-V_c$

$$V_a-I_1R_1+I_2R_2=V_c$$

$$\Rightarrow \quad V_a-V_c=V_{ac}=I_1R_1-I_2R_2$$

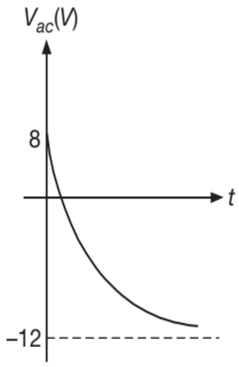
Substituting the values, we get,

$$V_{ac}=(10e^{-2t})(2)-(4)(3)$$

$$\Rightarrow \quad V_{ac}=(20e^{-2t}-12)\text{ V}$$

At $t=0$, $V_{ac}=8$ V. As $t\to\infty$, $V_{ac}=-12$ V

Therefore, V_{ac} decreases exponentially from 8 V to −12 V



Since, $V_{ab} = V_a - V_b$

$\Rightarrow\ V_a - I_1R_1 + V_C = V_b$

$\Rightarrow\ V_a - V_b = V_{ab} = I_1R_1 - V_C$

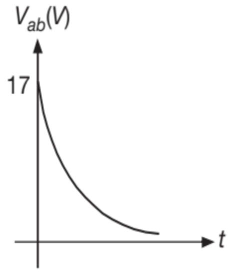
Substituting the values, we get,

$$V_{ab} = (10e^{-2t})(2) - 3e^{-2t}$$

$\Rightarrow\ V_{ab} = 17e^{-2t}$ V

Thus, V_{ab} decreases exponentially from 17 V to 0.

V_{ab} versus t graph is shown in Figure.



Since, $V_{cd} = V_c - V_d$

$$V_c - I_2R_2 - V_L = V_d$$

$\Rightarrow\ V_c - V_d = V_{cd} = I_2R_2 + V_L$

Substituting the values, we get,

$$V_{cd} = (4)(3) + 16e^{-2t}$$

$\Rightarrow\ V_{cd} = (12 + 16e^{-2t})$ V

At $t = 0$, $V_{cd} = 28$ V and as $t \to \infty$, $V_{cd} = 12$ V i.e., V_{cd} decreases exponentially from 28 V to 12 V.

V_{cd} versus t graph is shown in Figure.

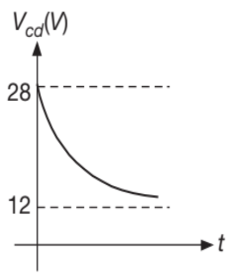
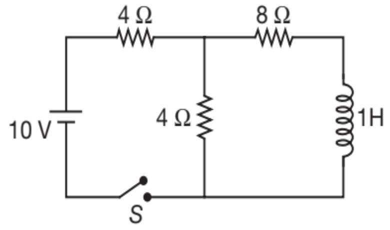


ILLUSTRATION 63

The switch in Figure is open for $t < 0$ and then closed at time $t = 0$. Find the current in the inductor and the current in the switch as functions of time thereafter.



SOLUTION

For loop $abefa$, we have

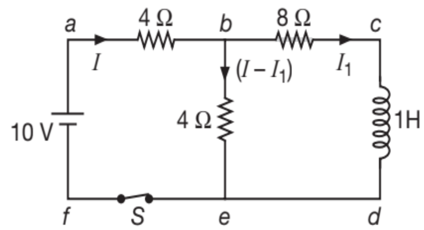
$$-4I - 4(I - I_1) + 10 = 0$$

$$\Rightarrow\ -4I_1 + 8I = 10 \qquad \ldots(1)$$

For loop $bcdeb$, we have

$$-8I_1 - (1)\frac{dI_1}{dt} + 4(I - I_1) = 0$$

$$\Rightarrow\ -12I_1 + 4I = \frac{dI_1}{dt} \qquad \ldots(2)$$



Multiplying (2) by 2 and the subtracting from (1), we get

$$-4I_1 + 24I_1 = 10 - 2\frac{dI_1}{dt}$$

$$\Rightarrow\ 20I_1 = 10 - 2\frac{dI_1}{dt}$$

$$\Rightarrow\ \frac{dI_1}{dt} = 5 - 10I_1$$

$$\Rightarrow\ \int_0^{I_1} \frac{dI_1}{5 - 10I_1} = \int_0^t dt$$

$$\Rightarrow\ -\frac{1}{10}\log_e(5 - 10I_1)\Big|_0^{I_1} = t$$

$$\Rightarrow\ \log_e\left(\frac{5 - 10I_1}{5}\right) = -10t$$

$\Rightarrow \quad 5 - 10I_1 = 5e^{-10t}$

$\Rightarrow \quad I_1 = 0.5\left(1 - e^{-10t}\right)$ A

{current through the inductor}

The current through the switch is I, calculated from either of the equations to be

$$I = \frac{1}{8}(10 + 4I_1)$$

$\Rightarrow \quad I = 1.25 + 0.5I_1$

$\Rightarrow \quad I = 1.25 + (0.5)(0.5)\left(1 - e^{-10t}\right)$

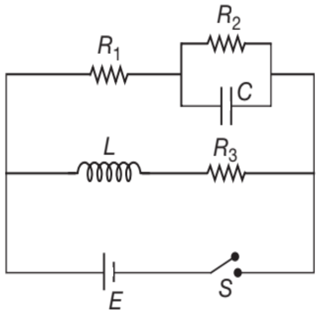
$\Rightarrow \quad I = 1.25 + 0.25 - 0.25e^{-10t}$

$\Rightarrow \quad I = \left(1.5 - 0.25e^{-10t}\right)$ A

{current through the switch}

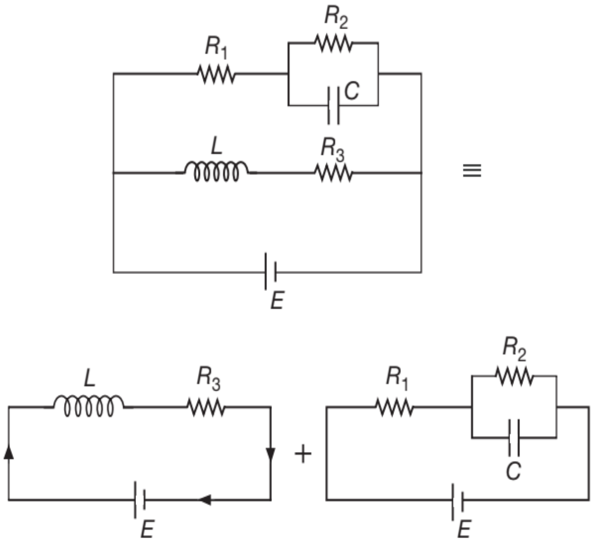
ILLUSTRATION 64

In the circuit shown, the switch S is closed at time $t = 0$. Find the current through capacitor and inductor at any time t.



SOLUTION

From Superposition principle, the circuit may be assumed to be equivalent to two circuits in parallel as shown.

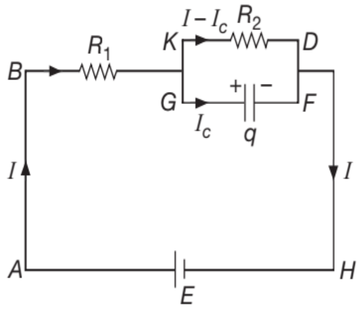


The first is a series LR circuit. Hence current I_L at any time t will be given by $I_L = I_0\left(1 - e^{-\frac{t}{\tau_L}}\right)$

where $I_0 = \dfrac{E}{R_3}$ and $\tau_L = \dfrac{L}{R_3}$

$$\Rightarrow \quad I_L = \frac{E}{R_3}\left(1 - e^{-\frac{R_3 t}{L}}\right) \qquad \ldots(1)$$

Let us now find the current through capacitor at time t.



Let the currents in different branches be I, I_C and $I - I_C$ at any time t and q be the charge stored in the capacitor.

Applying Kirchhoff's Loop Law to $ABKDHA$, we get

$$E - IR_1 - (I - I_C)R_2 = 0 \qquad \ldots(2)$$

For loop $KDFGK$, we have

$$\frac{q}{C} - (I - I_c)R_2 = 0 \qquad \ldots(3)$$

$$\text{Since, } I_C = \frac{dq}{dt} \qquad \ldots(4)$$

From equation (2)

$$I = \left(\frac{E + I_C R_2}{R_1 + R_2}\right)$$

Substituting this in equation (2), we get

$$\frac{q}{C} - \left(\frac{E + I_C R_2}{R_1 + R_2} - I_C\right)R_2 = 0$$

$$\Rightarrow \quad \frac{q}{C} - \frac{ER_2}{R_1 + R_2} = I_C\left(\frac{R_2^2}{R_1 + R_2} - R_2\right)$$

$$\Rightarrow \quad \left(\frac{dq}{dt}\right)\left(\frac{R_2^2}{R_1 + R_2} - R_2\right) = \frac{q}{C} - \frac{ER_2}{R_1 + R_2}$$

$$\Rightarrow \quad \int_0^q \frac{dq}{\left(\dfrac{ER_2}{R_1 + R_2}\right) - \dfrac{q}{C}} = \frac{R_1 + R_2}{R_1 R_2}\int_0^t dt$$

Solving this we get

$$q=\frac{ECR_2}{R_1+R_2}\left(1-e^{-\left(\frac{R_1+R_2}{CR_1R_2}\right)t}\right)$$

$$\Rightarrow \quad I_C=\frac{dq}{dt}=\frac{E}{R_1}e^{-\left(\frac{R_1+R_2}{CR_1R_2}\right)t}$$

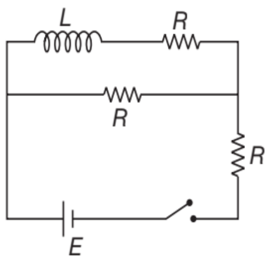
Hence currents through capacitor and inductor at any time t are

$$I_C=\frac{E}{R_1}e^{-\left(\frac{R_1+R_2}{CR_1R_2}\right)t} \text{ and}$$

$$I_L=\frac{E}{R_3}\left(1-e^{-\frac{R_3}{L}t}\right)$$

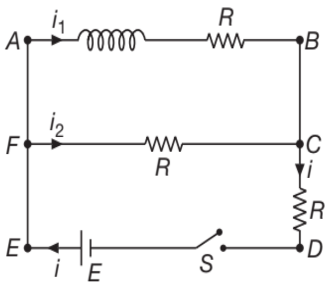
ILLUSTRATION 65

In the circuit shown, the switch is closed at $t=0$. Calculate the current drawn from the ideal battery as a function of time.



SOLUTION

At an instant t, the circuit is shown in Figure.



If i be the current through the battery, then we have

$$i=i_1+i_2 \quad \ldots(1)$$

Applying KLL for the loop $ABCDEA$, we get

$$-L\frac{di_1}{dt}-i_1R-iR+E=0 \quad \ldots(2)$$

Applying KLL for the loop $FCDEF$, we get

$$-i_2R-iR+E=0 \quad \ldots(3)$$

From (1), we have

$$i_2=i-i_1$$

$$\Rightarrow \quad -(i-i_1)R-iR+E=0$$

$$\Rightarrow \quad -2iR+i_1R+E=0$$

$$\Rightarrow \quad i_1=-\frac{E+2iR}{R}=2i-\frac{E}{R} \quad \ldots(4)$$

From (2) and (4), we get

$$-L\frac{d}{dt}\left(2i-\frac{E}{R}\right)-\left(2i-\frac{E}{R}\right)R-iR+E=0$$

$$\Rightarrow \quad -2L\frac{di}{dt}-2iR+E-iR+E=0$$

$$\Rightarrow \quad 2E-3iR=2L\frac{di}{dt}$$

$$\Rightarrow \quad E-\frac{3}{2}iR=L\frac{di}{dt}$$

$$\Rightarrow \quad \int_{\frac{E}{2R}}^{i}\frac{di}{E-\frac{3}{2}iR}=\int_0^t\frac{dt}{L}$$

$$\Rightarrow \quad \frac{\log_e\left(E-\frac{3}{2}iR\right)\Big|_{\frac{E}{2R}}^{i}}{-\frac{3}{2}R}=\frac{t}{L}$$

$$\Rightarrow \quad \log_e\left(\frac{E-\frac{3}{2}iR}{\frac{E}{4}}\right)=-\frac{3R}{2L}t$$

$$\Rightarrow \quad 4-\frac{6iR}{E}=e^{-\frac{3R}{2L}t}$$

$$\Rightarrow \quad \left(1-e^{-\frac{3R}{2L}t}\right)+3=\frac{6iR}{E}$$

$$\Rightarrow \quad i=\frac{E}{6R}\left[3+\left(1-e^{-\frac{3R}{2L}t}\right)\right]$$

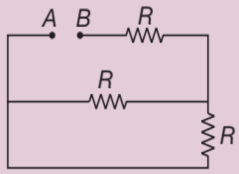
$$\Rightarrow \quad i=\frac{E}{2R}+\frac{E}{6R}\left(1-e^{-\frac{3R}{2L}t}\right)$$

Problem Solving Technique(s)

In these types of problems, we can first calculate the current flowing through the inductor using the following steps.

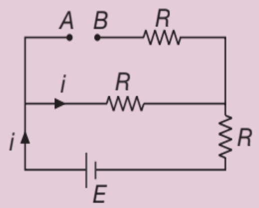
STEP-1

Remove the inductor, short the battery and then calculate the equivalent resistance across the open terminals (lets call then A and B) after removing the inductor as shown in Figure. Let that resistance be R_{AB} and we get $R_{AB} = \frac{3R}{2}$



STEP-2

Now calculate the current (i) in the battery circuit at the moment key is closed as shown in Figure.



$$\Rightarrow \ i = \frac{E}{2R}$$

STEP-3

Potential difference across the open ends A and B is given by

$$V_{AB} = iR = \left(\frac{E}{2R}\right)R = \frac{E}{2}$$

STEP-4

So, current through the inductor is given by

$$i_L = i_1 = \frac{V_{AB}}{R_{AB}}\left(1 - e^{-\frac{R_{AB}t}{L}}\right)$$

$$\Rightarrow \ i_L = i_1 = \frac{E/2}{3R/2}\left(1 - e^{-\frac{R_{AB}t}{L}}\right)$$

$$\Rightarrow \ i_L = i_1 = \frac{E}{3R}\left(1 - e^{-\frac{3Rt}{2L}}\right)$$

STEP-5

Apply KLL on the loop $EFCDE$, we get

$$-(i - i_1)R - iR + E = 0$$

$$\Rightarrow \ 2iR = E + i_1R$$

$$\Rightarrow \ i = \frac{E}{2R} + \frac{i_1}{2}$$

$$\Rightarrow \ i = \frac{E}{2R} + \frac{E}{6R}\left(1 - e^{-\frac{3R}{2L}t}\right)$$

ILLUSTRATION 66

A closed circuit consists of a source of constant EMF E and a choke coil of inductance L connected in series. The active resistance of the whole circuit is equal to R. At the moment $t = 0$, the choke coil inductance was decreased abruptly η times suddenly. Calculate the current in the circuit as a function of t.

SOLUTION

When the inductance was abruptly decreased η times i.e. made $\frac{L}{\eta}$, then at the same instant, current in the circuit becomes η times the original value so that the flux through the inductance remains the same.

Conceptual Note(s)

Dear Student, please understand that during the stepwise or sudden or abrupt change in the inductance, the total magnetic flux i.e. the flux linkage remains constant.

This statement can be understood based on the concept that when mass of a moving body is abruptly made half, then its speed becomes twice just because linear momentum $p = mv$ is conserved.

Similarly, inductance being analogous to electrical inertia and current being analogous to speed, then flux $\phi = LI$ becomes analogous to momentum and hence abrupt change in L leads to a corresponding change in current just to conserve the flux.

Just before changing the inductance, current in the circuit would have been

$$I_0 = \frac{E}{R} \qquad \ldots(1)$$

At $t = 0$, when the inductance is abruptly made $\frac{L}{\eta}$, then the current in the circuit becomes ηI_0.

At any instant, if I be the current in the circuit, then by using KLL for the given LR circuit, we get

$$IR+\frac{L}{\eta}\frac{dI}{dt}=E$$

$$\Rightarrow \quad \frac{dI}{E-IR}=\frac{\eta}{L}dt$$

Integrating within appropriate limits, we get

$$\Rightarrow \quad \int_{\eta I_0}^{I}\frac{dI}{E-IR}=\frac{\eta}{L}\int_{0}^{t}dt$$

Here while applying the limits we kept in mind that at $t=0$, the current in the circuit is ηI_0 and at time t, the current in the circuit is I.

$$\Rightarrow \quad -\frac{1}{R}\ln(E-IR)\Big|_{\eta I_0}^{I}=\frac{\eta}{L}t\Big|_{0}^{t}$$

$$\Rightarrow \quad \ln\left(\frac{E-IR}{E-\eta I_0 R}\right)=-\left(\frac{\eta R}{L}\right)t$$

From the Equation (1), we get

$$\ln\left(\frac{E-IR}{E-\eta E}\right)=-\left(\frac{\eta R}{L}\right)t$$

$$\Rightarrow \quad \frac{E-IR}{E-\eta E}=e^{-\left(\frac{\eta R}{L}\right)t}$$

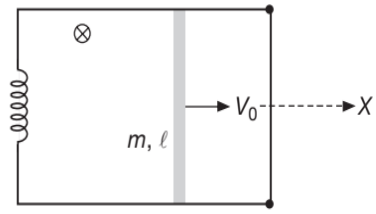
$$\Rightarrow \quad E-IR=E(1-\eta)e^{-\left(\frac{\eta R}{L}\right)t}$$

$$\Rightarrow \quad IR=E-E(1-\eta)e^{-\left(\frac{\eta R}{L}\right)t}$$

$$\Rightarrow \quad I=\frac{E}{R}\left[1-(1-\eta)e^{-\left(\frac{\eta R}{L}\right)t}\right]$$

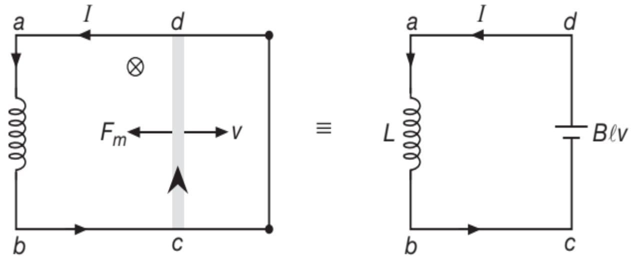
ILLUSTRATION 67

A loop is formed by two parallel conductors connected by a solenoid with inductance L and a conducting rod of mass m which can freely (without friction) slide over the conductors. The conductors are located in a horizontal plane in a uniform vertical magnetic field B. The distance between the conductors is l. At the moment $t=0$, the rod is imparted an initial velocity v_0 directed to the right. Find the law of its motion $x(t)$ if the electric resistance of the loop is negligible.



SOLUTION

Let at any instant of time, the velocity of the rod be v towards right.



Let the current in the circuit be I. For the loop $abcda$

$$Blv-L\frac{dI}{dt}=0$$

$$\Rightarrow \quad L\frac{dI}{dt}=Bl\frac{dx}{dt}$$

$$\Rightarrow \quad LdI=Bldx$$

Integrating, we get

$$LI=Blx$$

$$\Rightarrow \quad I=\frac{Bl}{L}x \qquad \ldots(1)$$

Magnetic force on the rod at this instant is,

$$F_m=BIl=\frac{B^2l^2}{L}x \qquad \ldots(2)$$

Since, this force is in opposite direction of $\vec{v}$, so from Newton's Second Law we can write,

$$m\left(\frac{d^2x}{dt^2}\right)=-\frac{B^2l^2}{L}x$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}=-\frac{B^2l^2}{mL}x$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}+\left(\frac{B^2l^2}{mL}\right)x=0$$

Comparing this with the standard equation of SHM $\frac{d^2x}{dt^2}+\omega^2x=0$ we get,

$$\omega = \frac{Bl}{\sqrt{mL}}$$

Therefore, the rod will oscillate simple harmonically with angular frequency $\omega = \frac{Bl}{\sqrt{mL}}$. At time $t = 0$, rod was at $x = 0$ and it was moving towards positive x-axis. Hence, x-t equation of the rod is,

$$x = A\sin(\omega t) \quad \ldots(3)$$

METHOD I to find *A*

Since we have at $t = 0$, $v = \frac{dx}{dt} = v_0$

Since, $\frac{dx}{dt} = v = A\omega\cos(\omega t)$

$$\Rightarrow \quad A\omega = v_0 \quad \{\text{at } t = 0\}$$

$$\Rightarrow \quad A = \frac{v_0}{\omega}$$

Substituting in equation (3), we get

$$x = \frac{v_0}{\omega}\sin(\omega t), \text{ where } \omega = \frac{Bl}{\sqrt{mL}}$$

METHOD II to find *A*

At $x = A$, $v = 0$, i.e., the entire kinetic energy is converted into magnetic energy. So,

$$\frac{1}{2}LI^2 = \frac{1}{2}mv_0^2$$

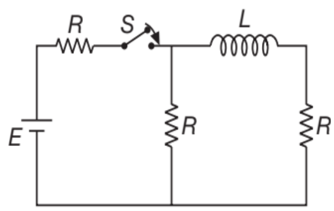
Substituting value of I from equation (1), with $x = A$, we get

$$L\left(\frac{Bl}{L}A\right)^2 = mv_0^2$$

$$\Rightarrow \quad A = \frac{\sqrt{mL}}{Bl}v_0 = \frac{v_0}{\omega}, \text{ where } \omega = \frac{Bl}{\sqrt{mL}}$$

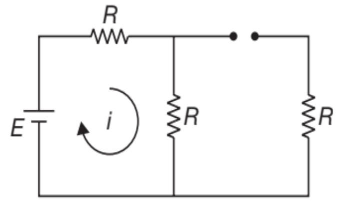
ILLUSTRATION 68

In circuit shown in Figure, find the current through battery just after closing the switch and after a long time in steady state.



SOLUTION

Just after closing the switch inductor behaves like open circuit so the circuit just after closing the switch is shown in Figure below.

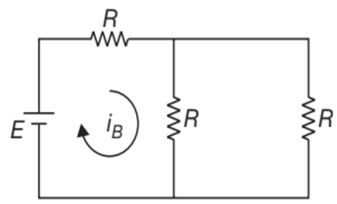


The current only flows in the left loop as shown which is given as

$$i_B = \frac{E}{2R}$$

In steady state after a long time the inductor behaves like a straight wire thus equivalent circuit in steady state is shown in Figure below. The two resistances on the right side can be considered in parallel and thus across the battery equivalent resistance will be

$$R + \frac{R}{2} = \frac{3R}{2}$$

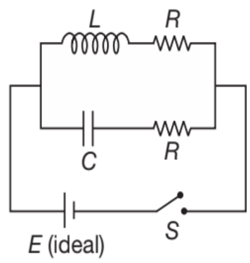


The current flowing through battery in steady state is given as

$$i_B = \frac{E}{\frac{3R}{2}} = \frac{2E}{3R}$$

ILLUSTRATION 69

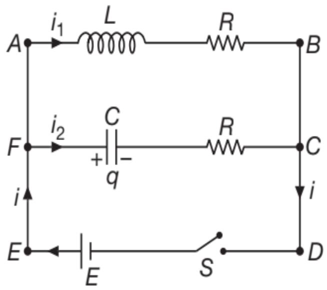
In the circuit shown in Figure, the switch is closed at $t = 0$.



Find the condition for which current drawn from the battery does not vary with time.

SOLUTION

The switch is closed at $t = 0$. At any instant, the current drawn from the battery is i. So, we have



$$i = i_1 + i_2$$

Applying KLL to the loop $ABCDEFA$, we get

$$-L\frac{di_1}{dt} - i_1 R + E = 0$$

$$\Rightarrow \quad \frac{di_1}{E - i_1 R} = \frac{dt}{L}$$

Integrating, we get

$$i_1 = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

Applying KLL to the loop $FCDEF$, we get

$$\frac{-q}{C} - i_2 R + E = 0$$

$$\Rightarrow \quad q = CE\left(1 - e^{-\frac{t}{RC}}\right)$$

$$\Rightarrow \quad i_2 = \frac{E}{R} e^{-\frac{t}{RC}}$$

$$\Rightarrow \quad i = i_1 + i_2$$

$$\Rightarrow \quad i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) + \frac{E}{R} e^{-\frac{t}{RC}}$$

$$\Rightarrow \quad i = \frac{E}{R} + \frac{E}{R}\left(e^{-\frac{t}{RC}} - e^{-\frac{Rt}{L}}\right)$$

For the current i not to vary with time, we observe that the terms containing t must cancel, so

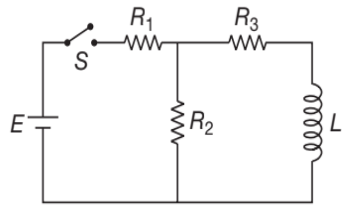
$$\Rightarrow \quad e^{-\frac{t}{RC}} = e^{-\frac{Rt}{L}}$$

$$\Rightarrow \quad \frac{t}{RC} = \frac{Rt}{L}$$

$$\Rightarrow \quad R = \sqrt{\frac{L}{C}}$$

ILLUSTRATION 70

For the circuit shown in Figure, we are given that E = 50 V, R_1 = 10 Ω, R_2 = 20 Ω, R_3 = 30 Ω and L = 2.0 mH. Find the current through R_1 and R_2,



(a) immediately after switch S is closed.
(b) long time after S is closed.
(c) immediately after S is reopened.
(d) long time after S is reopened.

SOLUTION

(a) Resistance offered by inductor immediately after switch is closed will be infinite as it behaves like open circuit. Therefore, current through R_3 will be zero and current through R_1 and R_2 is the same, given by

$$I_1 = I_2 = \frac{E}{R_1 + R_2} = \frac{50}{10 + 20} = \frac{5}{3} \text{ A}$$

$$\Rightarrow \quad I_1 = I_2 = 1.67 \text{ A}$$

(b) Long time after the switch is closed, the resistance offered by inductor is zero and in steady state it behaves like a short circuit.
In that case R_2 and R_3 are in parallel and the resultant of these two is in series with R_1. So, equivalent resistance across the battery in steady state is

$$R_{net} = R_1 + \frac{R_2 R_3}{R_2 + R_3} = 10 + \frac{(20)(30)}{20 + 30} = 22\ \Omega$$

Current through the battery is

$$I_2 = \frac{E}{R_{net}} = \frac{50}{22} \text{ A} = \frac{25}{11} \text{ A}$$

$$\Rightarrow \quad I_2 = \frac{50}{22} \text{A} = 2.3 \text{ A}$$

This current will distribute in R_2 and R_3 in inverse ratio of resistance, so current through R_2 is

$$I_{R_2} = \left(\frac{50}{22}\right)\left(\frac{R_3}{R_2 + R_3}\right)$$

$$\Rightarrow \quad I_{R_2} = \left(\frac{50}{22}\right)\left(\frac{30}{30 + 20}\right) = \frac{15}{11} \text{ A}$$

(c) Immediately after switch is reopened, the current through R_1 will become zero. However, current through R_2 will be equal to the steady state current through R_3, which is

$$I_3 = \left(\frac{50}{22} - \frac{15}{11}\right) = 0.91 \text{ A}$$

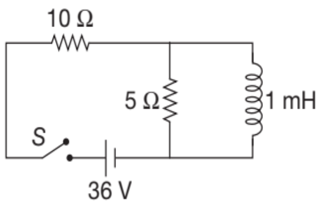
(d) Long time after S is reopened, current through all resistors will become zero.

Test Your Concepts-VI

Based on LR Circuits

(Solutions on page H.143)

1. In the circuit arrangement shown in Figure, the switch S is closed at $t = 0$. Find the current in the inductance as a function of time? Does the current through 10 Ω resistor vary with time or remains constant.



2. A coil of resistance 20 Ω and inductance 0.5 H is switched to dc supply of 200 V. Calculate the

(a) rate of increase of current at the instant of closing the switch and

(b) rate of increase of current after one time constant.

(c) steady state current in the circuit.

3. A coil of inductance 2 H and resistance 10 Ω are in closed series circuit with an open key and a cell of constant 100 V with negligible resistance. At time $t = 0$, the key is closed. Find

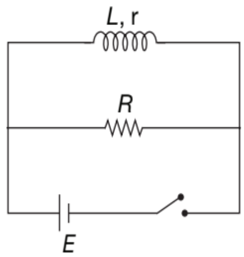
(a) the time constant of the circuit.

(b) the maximum steady current in the circuit.

(c) the current in the circuit at $t = 1$ second.

(d) the energy stored in the magnetic field linked with the coil at the steady state.

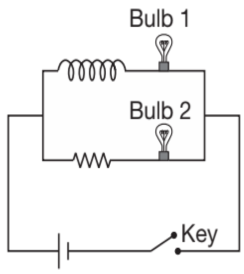
4. A solenoid of inductance L with resistance r is connected in parallel to a resistance R. A battery of emf E and of negligible internal resistance is connected across the parallel combination as shown in the Figure. At time $t = 0$, switch S is opened, calculate:



(a) current through the solenoid after the switch is opened.

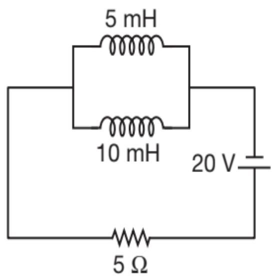
(b) amount of heat generated in the solenoid.

5. Two identical bulbs 1 and 2 are connected in the circuit shown in Figure.

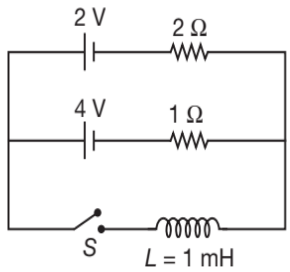


Assume that the key is closed at time $t = 0$ and when the bulbs start glowing with their full intensities the key is opened again. Will the bulbs die out suddenly or slowly? Discuss in detail.

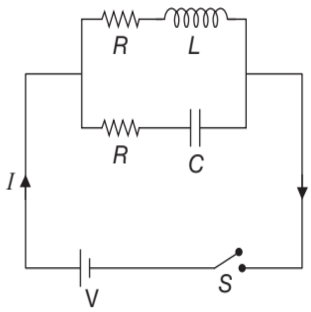
6. In the given circuit, find the current through the 5 mH inductor in steady state.



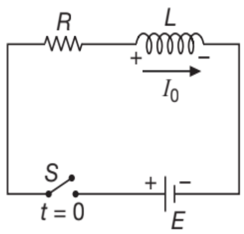
7. In the circuit shown, switch S is closed at time $t = 0$. Find the current through the inductor as a function of time t.



8. In the circuit diagram shown, initially there is no energy in the inductor and the capacitor. The switch is closed at $t = 0$. Find the current I as a function of time if $R = \sqrt{\dfrac{L}{C}}$.

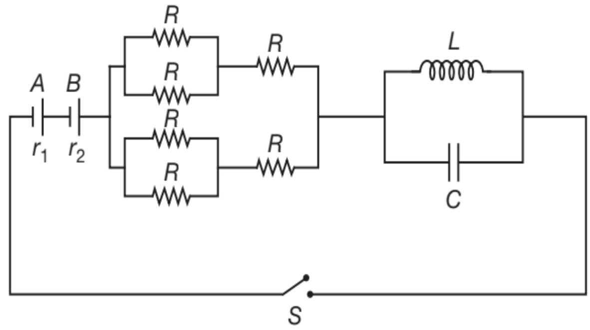


9. An active inductor with current I_0 is connected in series with a resistance and a battery as shown in Figure.

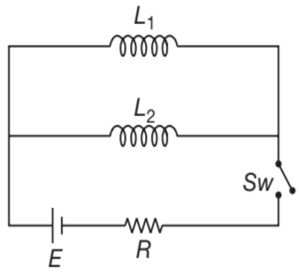


If at $t = 0$ the switch is closed, find the current in inductor as a function of time.

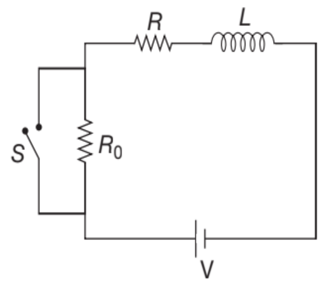
10. In the circuit shown A and B are two cells of same emf E but different internal resistances r_1 and $r_2 (r_1 > r_2)$ respectively. Find the value of R such that the potential difference across the terminals of cell A is zero a long time after the key K is closed.



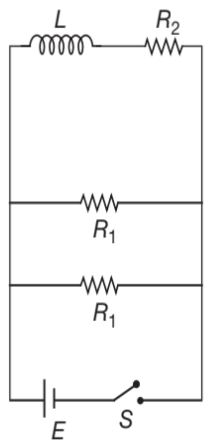
11. In the electric circuit shown in Figure, a battery of EMF E, a resistance R and coils of inductances L_1 and L_2 are used. The internal resistance of the battery and the coil resistances are negligible. Find the steady state currents in the coils long time after the switch Sw was shorted.



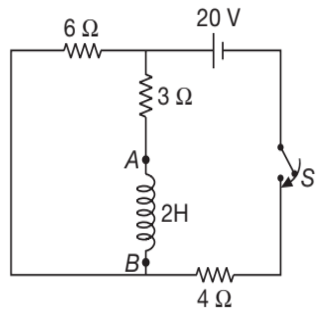
12. In the network shown, switch S is closed at $t = 0$ when a steady state current has been attained previously. Find the current in the circuit at time t.



13. An inductor of inductance $L = 400$ mH and three resistors of resistances $R_1 = 4\ \Omega$ and $R_2 = 4\ \Omega$ are connected to a battery of e.m.f. $E = 10$ V as shown in the Figure. The internal resistance of the battery is negligible. The switch S is closed at time $t = 0$. What is the potential drop across L as a function of time? After the steady state is reached, the switch is opened. What is the direction and magnitude of current through inductor as a function of time?



14. In the circuit shown in Figure, find the current in inductor as a function of time if switch is closed at $t = 0$.



15. A 5 H inductor is placed in series with a 10 Ω resistor. An emf of 5 V is suddenly applied to the combination. For the set up discussed, verify the principle of conservation of energy at time equal to the time constant of the circuit.

ENERGY STORED IN AN INDUCTOR/ ENERGY IN A MAGNETIC FIELD

As discussed, the emf induced in an inductor-resistor series circuit prevents a battery from establishing an instantaneous current in the circuit. Since we have

$$E = IR + L\left(\frac{dI}{dt}\right)$$

$$\Rightarrow \quad EI = I^2R + LI\left(\frac{dI}{dt}\right)$$

From this equation, we observe that the battery must provide more energy to this LR circuit than it provides to a circuit without an inductor, because a part of the total energy supplied by the battery per second (EI) is the rate at which energy is dissipated as heat by the resistor (I^2R) and the remaining is the rate at which energy is stored in the magnetic field associated with the inductor $\left(L\frac{dI}{dt}\right)$.

If we denote the energy stored in the inductor at any instant of time t by U, then we get

$$\frac{dU}{dt} = LI\frac{dI}{dt}$$

$$\Rightarrow \quad dU = LIdI$$

$$\Rightarrow \quad U = \int dU = L\int_0^I IdI$$

$$\Rightarrow \quad U = \frac{1}{2}LI^2$$

We can also find the above expression by using,

$$dW = V_L Idt \text{, where } V_L = L\frac{dI}{dt} \quad \{\because dW = VIdt\}$$

$$\Rightarrow \quad dW = L\left(\frac{dI}{dt}\right)Idt = LIdI$$

$$\Rightarrow \quad W = \int dW = L\int_0^I IdI = \frac{1}{2}LI^2$$

This work done is stored in the form of energy of the magnetic field in the inductor, denoted by U. So,

$$U = \frac{1}{2}LI^2$$

$$\Rightarrow \quad L = \frac{2U}{I^2}$$

If $I = 1$ ampere, then $L = 2U$ (numerically)

So, the self-inductance of a circuit is numerically equal to twice the work done against the induced emf in establishing a current of 1 A in the coil.

ILLUSTRATION 71

A coil of inductance 1 H and resistance 10 Ω is connected to a resistance less battery of emf 50 V at time $t = 0$. Calculate the ratio of the rate at which magnetic energy is stored in the coil to the rate at which energy is supplied by the battery at $t = 0.1$ sec.

SOLUTION

Since, $E = L\dfrac{dI}{dt} + IR$

$$\Rightarrow \quad EI = LI\dfrac{dI}{dt} + I^2R$$

where, EI is the rate at which energy is supplied by the battery and $LI\dfrac{dI}{dt}$ is the rate at which magnetic energy is stored in the coil.

So, the desired ratio is

$$\text{Desired Ratio} = \dfrac{LI\left(\dfrac{dI}{dt}\right)}{EI} = \dfrac{L\left(\dfrac{dI}{dt}\right)}{E}$$

Since $I = I_0\left(1 - e^{-\frac{Rt}{L}}\right)$

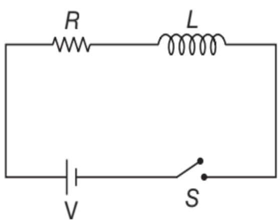
$$\Rightarrow \quad \dfrac{dI}{dt} = \dfrac{I_0R}{L}e^{-\frac{Rt}{L}} = \dfrac{E}{L}e^{-\frac{Rt}{L}} \qquad \{\because I_0R = E\}$$

$$\Rightarrow \quad \text{Ratio} = \dfrac{1}{L}e^{-\frac{Rt}{L}} = e^{-10(0.1)}$$

$$\Rightarrow \quad \text{Ratio} = \dfrac{1}{e} = 0.37$$

ILLUSTRATION 72

A resistor and an inductor are connected in series through a battery of potential difference V as shown in Figure.



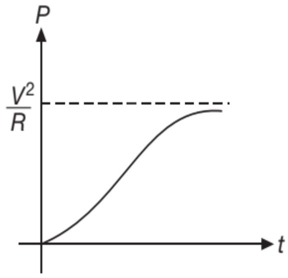
The switch S is closed at time $t = 0$. Plot the graph of rate of loss of heat (P) in the resistor versus time (t). Also calculate the magnitude of current flowing in the circuit when the rate of increase of magnetic energy in the inductor is maximum.

SOLUTION

The current i at any time $t = t$ is given by

$$i = \dfrac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

$$\Rightarrow \quad P = i^2R = \dfrac{V^2}{R}\left(1 - e^{-\frac{Rt}{L}}\right)^2 R = \dfrac{V^2}{R}\left(1 - e^{-\frac{Rt}{L}}\right)^2$$



The magnetic energy stored in the inductor is

$$U_m = U = \dfrac{1}{2}Li^2$$

Rate of change of magnetic energy is

$$\dfrac{dU}{dt} = \dfrac{1}{2}L\left(2i\dfrac{di}{dt}\right) = Li\dfrac{di}{dt}$$

Let $\quad Li\dfrac{di}{dt} = Z$

For the rate of increase of magnetic energy in the inductor to be maximum i.e. for Z to be maximum, we have

$$\dfrac{dZ}{dt} = 0$$

$$\Rightarrow \quad \dfrac{d}{dt}\left(\dfrac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right)\left(\dfrac{V}{L}e^{-\frac{Rt}{L}}\right)\right) = 0$$

$$\Rightarrow \quad \dfrac{V^2}{LR}\dfrac{d}{dt}\left(e^{-\frac{Rt}{L}} - e^{-\frac{2Rt}{L}}\right) = 0$$

$$\Rightarrow \quad -\dfrac{R}{L}e^{-\frac{Rt}{L}} - \dfrac{2R}{L}e^{-\frac{2Rt}{L}} = 0$$

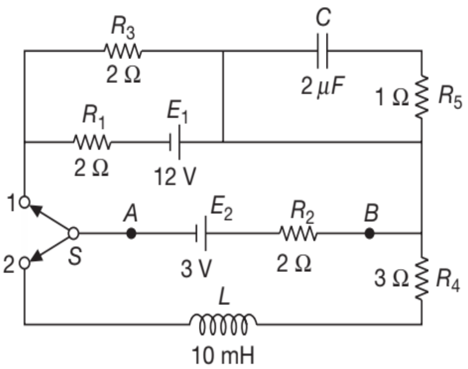
$$\Rightarrow \quad e^{-\frac{Rt}{L}} = \dfrac{1}{2}$$

So, current in the circuit at that instant is

$$i = \dfrac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) = \dfrac{V}{R}\left(1 - \dfrac{1}{2}\right) = \dfrac{V}{2R}$$

ILLUSTRATION 73

A circuit containing a two-position switch S is shown in Figure.



(a) The switch S is in position 1. Find the potential difference $V_A - V_B$ and the rate of production of joule heat in R_1.

(b) If now the switch S is put in position 2 at $t = 0$. Find:

(i) steady current in R_4 and

(ii) the time when current in R_4 is half the steady value. Also calculate the energy stored in the inductor L at that time.

SOLUTION

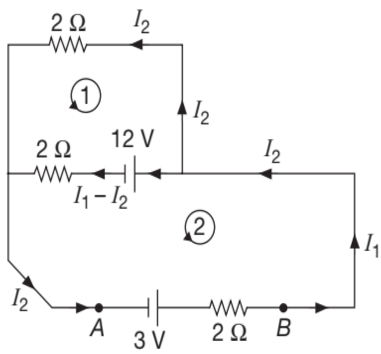
(a) In steady state no current will flow through capacitor, because a fully charged capacitor is a dc blocking element.

Also, no current flows through it initially. So, applying Kirchhoff's Second Law in Loop 1, we get

$$-2I_2 + 2(I_1 - I_2) + 12 = 0$$

$$\Rightarrow 2I_1 - 4I_2 = -12$$

$$\Rightarrow I_1 - 2I_2 = -6 \qquad \ldots(1)$$



Applying Kirchhoff's Second Law in Loop 2, we get

$$-12 - 2(I_1 - I_2) + 3 - 2I_1 = 0$$

$$\Rightarrow 4I_1 - 2I_2 = -9 \qquad \ldots(2)$$

Solving equations (1) and (2), we get

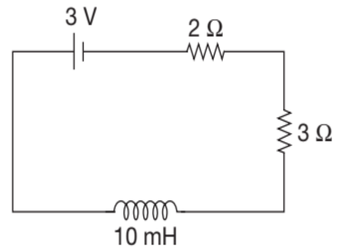
$$I_2 = 2.5 \text{ A and } I_1 = -1 \text{ A}$$

Now, $V_A + 3 - 2I_1 = V_B$

$$\Rightarrow V_A - V_B = 2I_1 - 3 = 2(-1) - 3 = -5 \text{ V}$$

$$\Rightarrow P_{R_1} = (I_1 - I_2)^2 R_1 = (-1 - 2.5)^2 (2) = 24.5 \text{ W}$$

(b) In position 2, the circuit is shown in Figure



Steady current in R_4 is

$$I_0 = \frac{3}{3+2} = 0.6 \text{ A}$$

Time when current in R_4 is half the steady value is

$$t_{1/2} = \frac{\log_e(2)}{1/\tau_L} = \tau_L(\log_e 2) = \frac{L}{R}\log_e(2)$$

$$t_{1/2} = \frac{(10 \times 10^{-3})}{5}\log_e(2) = 1.386 \times 10^{-3} \text{ s}$$

Energy stored in inductor at that time i.e. when $I = \frac{I_0}{2} = 0.3$ A is

$$U = \frac{1}{2}LI^2 = \frac{1}{2}(10 \times 10^{-3})(0.3)^2 = 4.5 \times 10^{-4} \text{ J}$$

ILLUSTRATION 74

An inductance L and a resistance R are connected in series with a battery of emf E. Find the maximum rate at which the energy is stored in the magnetic field.

SOLUTION

Energy stored in the magnetic field is

$$U = \frac{1}{2}LI^2$$

Since $I = I_0\left(1 - e^{-\frac{Rt}{L}}\right)$

$$\Rightarrow \quad U = \frac{1}{2}LI_0^2\left(1 - e^{-\frac{Rt}{L}}\right)^2$$

The rate at which the energy is stored is

$$P = \frac{dU}{dt} = LI_0^2\left(1 - e^{-\frac{Rt}{L}}\right)\left(-e^{-\frac{Rt}{L}}\right)\left(-\frac{R}{L}\right)$$

$$\Rightarrow \quad P = I_0^2R\left(e^{-\frac{Rt}{L}} - e^{-\frac{2Rt}{L}}\right) \qquad \ldots(1)$$

For the maximum rate at which the energy is stored, we have

$$\frac{dP}{dt} = 0$$

$$\Rightarrow \quad -\left(\frac{R}{L}\right)e^{-\frac{Rt}{L}} + \left(\frac{2R}{L}\right)e^{-\frac{2Rt}{L}} = 0$$

$$\Rightarrow \quad -1 + 2e^{-\frac{Rt}{L}} = 0$$

$$\Rightarrow \quad e^{-\frac{Rt}{L}} = \frac{1}{2} \qquad \ldots(2)$$

So, put (2) in (1), we get $P_{\max}$ as

$$P_{\max} = I_0^2R\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{I_0^2R}{4} = \frac{E^2}{4R}$$

Magnetic Energy Density (u_m)

Let us now calculate the magnetic energy density or simply the energy density of a magnetic field. For the sake of convenience, consider a solenoid of length l, area A, having n number of turns per unit length and self-inductance L. Then, we have

$$L = \mu_0 n^2 Al \qquad \ldots(1)$$

If B is the magnetic field associated with the solenoid, then

$$B = \mu_0 nI \qquad \ldots(2)$$

Since, we know that the energy associated with the magnetic field is given by

$$U = \frac{1}{2}LI^2$$

$$\Rightarrow \quad U = \frac{1}{2}\left(\mu_0 n^2 Al\right)\left(\frac{B}{\mu_0 n}\right)^2 \qquad \{\because B = \mu_0 nI\}$$

$$\Rightarrow \quad U = \left(\frac{B^2}{2\mu_0}\right)Al$$

where Al = Volume of the solenoid.

Now, by definition, the magnetic energy density is defined as

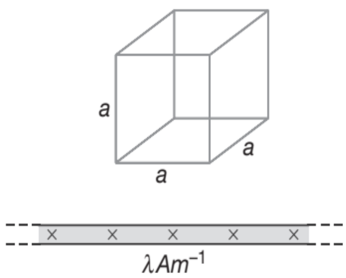
$$u_m = \frac{\text{Energy Associated with the Magnetic Field}}{\text{Volume of the Solenoid}}$$

$$\Rightarrow \quad u_m = \frac{U}{Al} = \frac{B^2}{2\mu_0}$$

Although this expression has been derived by considering the special case of a solenoid, however it is valid for any region of space where the magnetic field exists. Also, we observe that the above formula for magnetic energy density (u_m) matches with that of electric energy density given by $u_E = \frac{1}{2}\varepsilon_0 E^2$. In both the cases, the energy density is proportional to the square of the field.

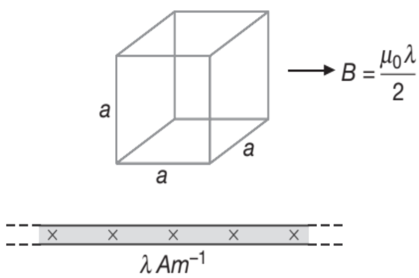
ILLUSTRATION 75

Calculate the magnetic field energy associated with a cubical region (of edge a) that lies above a large current carrying sheet which carries a uniform linear current density λ Am^{-1}.



SOLUTION

Magnetic field B due to a large current sheet in its surrounding is shown in Figure.



Since $B = \frac{1}{2}\mu_0\lambda$, so the magnetic field energy density due to the field associated with the sheet in the region of space is given by

$$u_m = \frac{B^2}{2\mu_0} = \frac{\left(\frac{1}{2}\mu_0\lambda\right)^2}{2\mu_0} = \frac{1}{8}\mu_0\lambda^2 \text{ Jm}^{-3}$$

So, magnetic field energy associated with this cubical region is

$$U = u_m V_{\text{cube}} = u_m a^3 = \frac{1}{8}\mu_0 a^3 \lambda^2$$

ILLUSTRATION 76

A long cylinder of radius a carrying a uniform surface charge rotates about its axis with an angular velocity ω. Find the magnetic field energy per unit length of the cylinder inside of it if the linear charge density equals λ and consider permeability of the medium is equal to unity.

SOLUTION

Current due to rotation of the cylinder of length l is given as

$$I = \frac{q\omega}{2\pi} = \frac{(\lambda l)\omega}{2\pi}$$

From Ampere's Circuital law, the magnetic induction inside the cylinder is

$$B = \frac{\mu_0 I}{l}$$

The magnetic field energy density inside the cylinder is given as

$$u_m = \frac{B^2}{2\mu_0}$$

Energy per unit length of the cylinder is

$$\frac{U_m}{l} = \frac{u_m V_{\text{cylinder}}}{l} = \frac{1}{l}\left(\frac{B^2}{2\mu_0}\right)(\pi a^2 l)$$

$$\Rightarrow \quad \frac{U_m}{l} = \left(\frac{1}{2\mu_0 l}\right)\left(\frac{\mu_0 \lambda l \omega}{2\pi l}\right)^2 (\pi a^2 l)$$

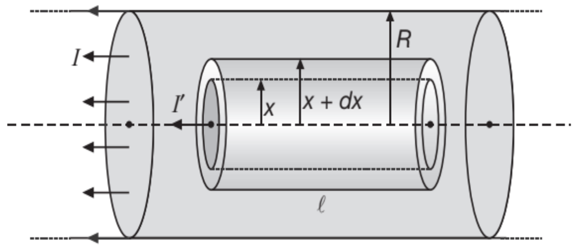
$$\Rightarrow \quad \frac{U_m}{l} = \frac{\mu_0 \lambda^2 \omega^2 a^2}{8\pi}$$

ILLUSTRATION 77

Calculate the total magnetic field energy stored per unit length inside a long cylindrical wire of radius R and carrying a current I.

SOLUTION

Let us consider an elemental shell of radius x and wall width dx inside the cylinder as shown in Figure.



The magnetic field inside the wire at a distance x from axis is calculated using Ampere's Circuital Law, according to which we get

$$B(2\pi x) = \mu_0 (I_{\text{enc}}) = \mu_0 I'$$

$$\Rightarrow \quad B(2\pi x) = \mu_0 \left(\frac{I}{\pi R^2}\right)(\pi x^2)$$

$$\Rightarrow \quad B = \left(\frac{\mu_0 I}{2\pi R^2}\right) x$$

The volume of the elemental shell of length l is

$$dV = (2\pi x dx) l$$

magnetic field energy stored in elemental cylindrical sheet is

$$dU = \left(\frac{B^2}{2\mu_0}\right) dV = \frac{1}{2\mu_0}\left(\frac{\mu_0 I x}{2\pi R^2}\right)^2 (2\pi l x dx)$$

$$\Rightarrow \quad dU = \frac{\mu_0 I^2 l}{4\pi R^4} x^3 dx$$

Total field energy inside the cylinder is

$$U = \int dU = \frac{\mu_0 I^2 l}{4\pi R^4} \int_0^R x^3 dx$$

$$\Rightarrow \quad \frac{U}{l} = \left(\frac{\mu_0 I^2}{4\pi R^4}\right)\frac{R^4}{4} = \frac{\mu_0 I^2}{16\pi}$$

Test Your Concepts-VII

Based on Magnetic Energy and Magnetic Energy Density

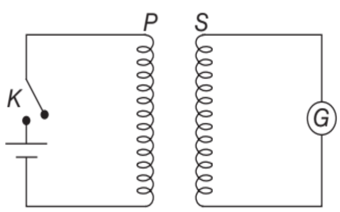
(Solutions on page H.148)

1. What inductance would be needed to store 1 kWh of energy in a coil carrying a current of 200 A?
2. Assume that the magnitude of the magnetic field outside a sphere of radius R is $B = B_0\left(\frac{R}{r}\right)^2$, where B_0 is a constant. Determine the total energy stored in the magnetic field outside the sphere.
3. A wire of non-magnetic material, with radius R, carries current uniformly distributed over its cross section. The total current carried by the wire is I. Find the magnetic energy per unit length from the surface of wire to twice its radius.
4. On a clear day at a certain location, a 100 Vm^{-1} vertical electric field exists near the Earth's surface. At the same place, the Earth's magnetic field has a magnitude of 50 μT. Compute the energy densities of the two fields. For what ratio of E to B can the energy density be equal. Can you recognise this ratio?
5. A long coaxial cable consists of two concentric cylinders of radii a and b. The central conductor of the cable carries a steady current i and the outer conductor provides the return path of the current. Calculate the energy stored in the magnetic field of length l of such a cable
6. The current (in ampere) in an inductor is given by $I = 5 + 16t$, where t is in seconds. The self-induced emf in it is 10 mV. Find
   (a) the self-inductance, and
   (b) the energy stored in the inductor and the power supplied to it at $t = 1$.
7. An inductor having inductance L and a capacitor having capacitance C are connected in series. The current in the circuit increases linearly in time as described by $I = kt$, where k is a constant. The capacitor is initially uncharged. Determine
   (a) the voltage across the inductor as a function of time,
   (b) the voltage across the capacitor as a function of time, and
   (c) the time when the energy stored in the capacitor first exceeds that in the inductor.
8. A solenoid has an inductance of 10 H and a resistance of 2 Ω. It is connected to a 10 V battery. How long will it take for the magnetic energy to reach one fourth of its maximum value?

MUTUAL INDUCTANCE

It is generally observed that the magnetic flux through the area enclosed by a circuit varies with time because of time varying currents in the neighbouring circuits. This phenomenon in which emf is produced due to the interaction of the two circuits is called the **phenomenon of Mutual Induction**.

Consider two coils, P (Primary) and S (Secondary) placed close to each other such that if a current passes in coil P, the coil S is in the magnetic field of coil P and vice-versa.



When the key K is closed then the current flowing through a coil (P) changes, the magnetic flux linked with the neighbouring coil (Q) also changes. As a result of this an induced e.m.f. and hence an induced current is set up in coil Q.

This phenomenon of production of e.m.f. in a coil when the current in neighbouring coil changes is called mutual induction. The circuit in which the current changes is called the **primary circuit**, while the neighbouring circuit in which e.m.f. is induced is called the **secondary circuit**.

Definition I

If I_1 is current flowing through primary coil then at any instant, the total flux linked with secondary coil is given by

$$\phi_{2,\text{ total}} \propto I_1$$

$$\Rightarrow \quad N_2\phi_2 \propto I_1$$

$$\Rightarrow \quad N_2\phi_2 = MI_1 \qquad \ldots(1)$$

$$\Rightarrow \quad M = \frac{N_2\phi_2}{I_1}$$

where M is called the coefficient of mutual induction or mutual inductance of the coils.

If $I_1 = 1\text{ A}$, then $M = N_2\phi_2$ (numerically)

So, M is numerically equal to the total flux associated with the secondary coil, when the current in the primary is 1 A.

Definition 2

Also induced e.m.f. in secondary coil,

$$\xi_2 = -\frac{d}{dt}(\phi_{2,\text{ total}}) = -N_2\frac{d\phi_2}{dt}$$

$$\Rightarrow \quad \xi_2 = -\frac{d}{dt}(MI_1) = -M\frac{dI_1}{dt} \qquad \ldots(2)$$

If $\frac{dI_1}{dt} = 1\text{ As}^{-1}$, then $M = \xi_2$ (numerically)

So, M is numerically equal to the e.m.f. induced in the secondary coil, when the rate of change of current in the primary coil is 1 As^{-1}.

Conceptual Note(s)

(a) Like self-inductance, the unit of mutual inductance is henry (H). The direction of induced e.m.f. or induced current arising due to a change in magnetic flux in all cases is given by **Lenz's Law**.

(b) Mutual inductance depends on the geometry of both the circuits, their orientations with respect to each other, their closeness and the number of turns. As the circuit separation increases the flux linking the circuits decreases and hence the mutual inductance also decreases.

(c) To understand the concept of mutual inductance, we can understand that the induced emf will develop in coil 2, due to the change in its flux because of the coil 1. So, we get

$$\xi_2 = -N_2\left(\frac{d\phi_2}{dt}\right) = -N_2\frac{d}{dt}\left(\frac{M_{12}I_1}{N_2}\right)$$

Since $\phi_2 = \frac{M_{12}I_1}{N_2}$

$$\Rightarrow \quad \xi_2 = -M_{12}\left(\frac{dI_1}{dt}\right)$$

where M_{12} is the inductance of coil 2 due to change of current in coil 1.

The induced emf will develop in coil 1, due to the change in its flux because of the coil 2 is

$$\xi_1 = -N_1\left(\frac{d\phi_1}{dt}\right) = -N_1\frac{d}{dt}\left(\frac{M_{21}I_2}{N_1}\right)$$

Similarly, $\phi_1 = \frac{M_{21}I_2}{N_1}$

$$\Rightarrow \quad \xi_1 = -M_{21}\left(\frac{dI_2}{dt}\right)$$

where M_{21} is the inductance of the coil 1 due to the change of current in coil 2.

From **Reciprocity Theorem,** we have

$$M_{12} = M_{21} = M\,(\text{say})$$

So, (3) and (4), can be rewritten as

$$\xi_2 = -M\left(\frac{dI_1}{dt}\right) \text{ and } \xi_1 = -M\left(\frac{dI_2}{dt}\right)$$

CALCULATION OF MUTUAL INDUCTANCE

To calculate the mutual inductance of two circuits, simply follow the steps given.

STEP-1: Take any one circuit as the primary (the first one) and the other as secondary (the second one), for the sake of convenience.

STEP-2: Let a current I_1 flow in the primary circuit.

STEP-3: Calculate the strength of magnetic field $(\vec{B})$ produced by the current in the primary at the location of the secondary coil.

STEP-4: Now calculate $\phi_{2,\text{ total}} = N_2\phi_2 = N_2(\vec{B}\cdot\vec{A}_2)$

STEP-5: Get M by using

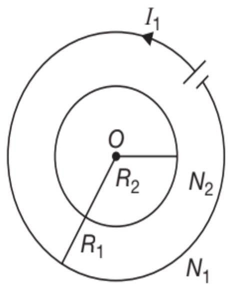
$$M = \frac{\phi_{2,\text{ total}}}{I_1} = \frac{N_2\phi_2}{I_1} = \frac{N_2(\vec{B}\cdot\vec{A}_2)}{I_1}$$

Let us apply these steps to calculate the mutual inductance for the following systems.

MUTUAL INDUCTANCE FOR A TWO COIL SYSTEM

Consider two co-axial coils having number of turns N_1 and N_2 and radii R_1 and R_2 respectively. If I_1 is current in outer coil, then magnetic field at its centre,

$$B_1 = \frac{\mu_0 N_1 I_1}{2R_1}$$



The flux linked with inner-coil

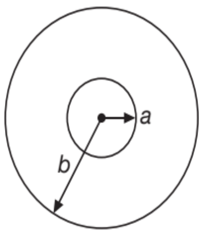
$$\phi_2 = N_2 B_1 A_2$$

$$\Rightarrow \quad \phi_2 = N_2\left(\frac{\mu_0 N_1 I_1}{2R_1}\right)A_2$$

$$\Rightarrow \quad M = \frac{\phi_2}{I_1} = \frac{\mu_0 N_1 N_2 A_2}{2R_1} = \frac{\mu_0 N_1 N_2\left(\pi R_2^2\right)}{2R_1}$$

ILLUSTRATION 78

Two thin concentric wires shaped as circle with radii a and b lie in the same plane as shown in Figure.



Assuming that $a \ll b$, calculate their mutual inductance and the magnetic flux through the surface enclosed by the outside wire, when the inside wire carries a current I.

SOLUTION

If outer coil carries a current I, then the magnetic flux at the location of inner coil 1 due to current in outer coil 2 is

$$\phi_{12} = \phi = B_2 A_1 = \left(\frac{\mu_0 I}{2b}\right)\left(\pi a^2\right)$$

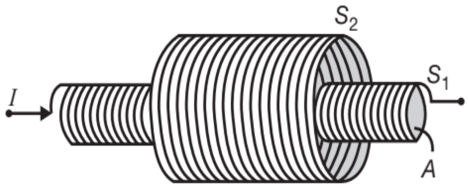
The mutual induction coefficient between the two coils is

$$M = \frac{\phi}{I} = \frac{\mu_0 \pi a^2}{2b}$$

MUTUAL INDUCTANCE FOR A SOLENOID-COIL SYSTEM

Let n_2 be number of turns per metre length of a long solenoid S_2. Let a coil S_1 of n_1 turns per unit length and of area A is placed within it. If I_1 is current in solenoid, then magnetic field within it,

$$B_1 = \mu_0 n_1 I_1$$



Since no magnetic field exists inside the annular region (region between S_1 and S_2), so magnetic flux linked with coil,

$$\phi_2 = N_2 B_1 A = \left(n_2 l\right) B_1 A = n_2\left(\mu_0 n_1 I_1\right) A l$$

$$\Rightarrow \quad M = \frac{\phi_2}{I_1} = \mu_0 n_1 n_2 A l$$

Please note that M is independent of the area/radius of the surrounding coil S_2, because the field in the region between S_1 and S_2 (also called as the annular region) is zero.

Conceptual Note(s)

MUTUAL INDUCTANCE OF TWO CLOSE COILS AND COEFFICIENT OF COUPLING

If two coils of self-inductances L_1 and L_2 are placed near each other, then mutual inductance

$$M = K\sqrt{L_1 L_2}$$

where K is a constant, called **coefficient of coupling**, having values between zero and 1.

If flux linkage between coils is 100% then $K = 1$, so we have

$$M = \sqrt{L_1 L_2}$$

ILLUSTRATION 79

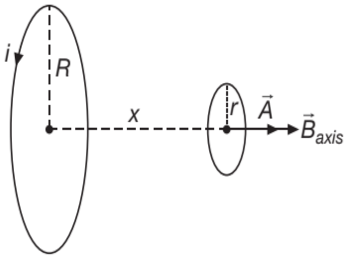
Two co-axial loops of radii R and $r(R \gg r)$ are placed such that separation between their centres is x. Calculate the mutual inductance of the arrangement.

SOLUTION

Let us imagine a current i to flow through the loop of radius R. The field produced at the centre of the smaller loop lying at the axis of the bigger loop is

$$B_{axis} = \frac{\mu_0 i R^2}{2(R^2+x^2)^{3/2}}.$$

Since $R \gg r$, so we can assume that this field is acting uniformly on the entire area of the smaller loop.



Since $\phi = \vec{B}\cdot\vec{A} = BA\cos 0° = BA$

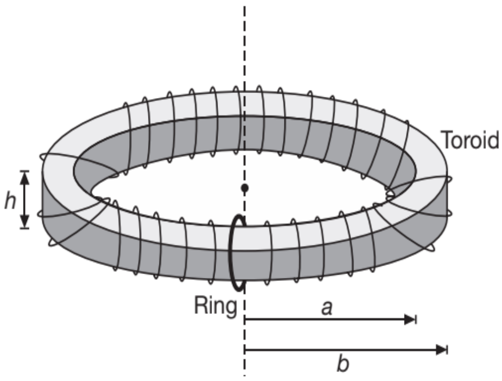
$$\Rightarrow \quad \phi = \left[\frac{\mu_0 i R^2}{2(R^2+x^2)^{3/2}}\right]\pi r^2$$

Since $M = \frac{\phi}{i}$

$$\Rightarrow \quad M = \frac{\mu_0 \pi R^2 r^2}{2(R^2+x^2)^{3/2}}$$

ILLUSTRATION 80

A toroidal coil S_1 has a rectangular cross-section and contains N loops as shown in Figure.



Its inner and outer radii are a and b, respectively. The height of the toroid is h. A thin conducting ring S_2 of resistance R, encircles the toroid. The coil S_1 carries a time-dependent current $I = I_0 e^{-\alpha t}$ (where I_0 and α are constants). Calculate the mutual inductance between the ring and the coil and the induced time-dependent current i in the ring.

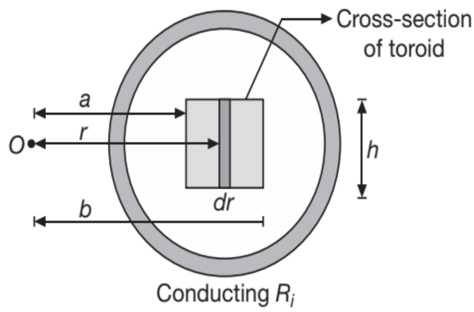
SOLUTION

For calculating the mutual inductance, let us assume that a current i is passed through the toroid. By Ampere's Circuital Law (ACL), the magnetic field at a radial distance r from the centre of toroid is

$$B = \frac{\mu_0 N i}{2\pi r}$$

Since magnetic field outside the toroid is zero, so the flux associated with the ring equals the flux associated with the toroid.

Now to calculate the flux associated with the ring, let us consider the cross-section of toroid. The flux associated with the shaded strip is



$$d\phi = B dA$$

$$\Rightarrow \quad d\phi = \frac{\mu_0 N i}{2\pi r}(h dr)$$

$$\Rightarrow \quad \phi = \frac{\mu_0 N i h}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 N i h}{2\pi}\ln\left(\frac{b}{a}\right)$$

Since $M = \frac{\phi}{i}$

$$\Rightarrow \quad M = \frac{\mu_0 N h}{2\pi}\ln\left(\frac{b}{a}\right)$$

Applying Faraday's Laws of EMI, we get

$$\xi = -\frac{M di}{dt}$$

$$\Rightarrow \quad \xi = -M\frac{d}{dt}\left(I_0 e^{-\alpha t}\right) = \frac{\mu_0 Nh}{2\pi}\ln\left(\frac{b}{a}\right)(I_0\alpha)e^{-\alpha t}$$

$$\Rightarrow \quad i = \frac{\xi}{R} = \frac{\mu_0 N h I_0 \alpha e^{-\alpha t}}{2\pi R}\ln\left(\frac{b}{a}\right)$$

ILLUSTRATION 81

There are two stationary loops with mutual inductance M. The current in one of the loops starts varying as $I_1 = \alpha t$, where α is a positive constant and t is any time instant starting from $t = 0$. Calculate the time dependence $I_2(t)$ of the current in the other loop which has self-inductance L and resistance R.

SOLUTION

The induced EMF in the second loop when the current changes in the first loop is

$$\xi_2 = M\frac{dI_1}{dt} = M\frac{d}{dt}(\alpha t) = M\alpha$$

$$\Rightarrow \quad \xi_2 = M\alpha$$

For the second loop, on applying KLL we get

$$I_2 R + L\frac{dI_2}{dt} = M\alpha$$

$$\Rightarrow \quad L\frac{dI_2}{dt} = M\alpha - I_2 R$$

$$\Rightarrow \quad \frac{dI_2}{M\alpha - I_2 R} = \frac{dt}{L}$$

$$\Rightarrow \quad \int_0^{I_2}\frac{dI_2}{M\alpha - I_2 R} = \int_0^t \frac{dt}{L}$$

$$\Rightarrow \quad -\frac{1}{R}\ln(M\alpha - I_2 R)\Big|_0^{I_2} = \frac{t}{L}\Big|_0^t$$

$$\Rightarrow \quad \ln\left(\frac{M\alpha - I_2 R}{M\alpha}\right) = -\frac{Rt}{L}$$

$$\Rightarrow \quad \frac{M\alpha - I_2 R}{M\alpha} = e^{-\frac{Rt}{L}}$$

$$\Rightarrow \quad M\alpha - I_2 R = M\alpha e^{-\frac{Rt}{L}}$$

$$\Rightarrow \quad I_2 R = M\alpha\left(1 - e^{-\frac{Rt}{L}}\right)$$

$$\Rightarrow \quad I_2 = \frac{M\alpha}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

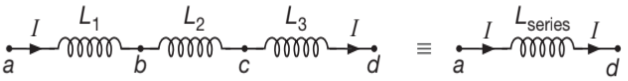
INDUCTANCES IN SERIES AND PARALLEL

Series

Consider a number of inductors L_1, L_2, L_3, ... connected in series. Assuming no interaction between them through mutual inductance, then we have

$$L_{\text{series}} = L_1 + L_2 + L_3 + \ldots\ldots$$

This can easily be shown by taking the following series inductor circuit and its equivalent circuit.



At any instant, assume that the current in the inductor circuit be I. Then

$$V_a - V_b = L_1\left(\frac{dI}{dt}\right) \quad \ldots(1)$$

$$V_b - V_c = L_2\left(\frac{dI}{dt}\right) \quad \ldots(2)$$

$$V_c - V_d = L_3\left(\frac{dI}{dt}\right) \quad \ldots(3)$$

Now, for the equivalent circuit we have

$$V_a - V_d = L_{\text{series}}\left(\frac{dI}{dt}\right) \quad \ldots(4)$$

From (1), (2), (3) and (4), we get

$$V_a - V_d = (V_a - V_b) + (V_b - V_c) + (V_c - V_d)$$

$$\Rightarrow \quad L_{\text{series}}\left(\frac{dI}{dt}\right) = L_1\left(\frac{dI}{dt}\right) + L_2\left(\frac{dI}{dt}\right) + L_3\left(\frac{dI}{dt}\right)$$

$$\Rightarrow \quad L_{\text{series}} = L_S = L_1 + L_2 + L_3 + \ldots..$$

SPECIAL CASE

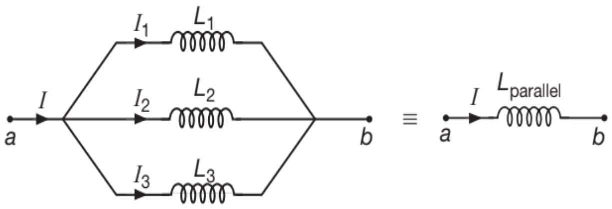
If two coils of self-inductances L_1 and L_2, having mutual inductance M are connected in series and placed close to each other, then the net inductance is given by

$$L_{\text{series}} = L_S = L_1 + L_2 \pm 2M$$

Parallel

Consider a number of inductors L_1, L_2, L_3, ... connected in parallel. Assuming no interaction between them through mutual inductance, then we have

$$\frac{1}{L_{\text{parallel}}} = \frac{1}{L_P} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots\ldots..$$



This can be shown by taking the following parallel inductor circuit and its equivalent circuit.

Since the inductors are connected in parallel, so we have

$$I = I_1 + I_2 + I_3$$

$$\Rightarrow \quad \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} + \frac{dI_3}{dt}$$

Since, $V_a - V_b = L_P\left(\frac{dI}{dt}\right)$ and

$$V_a - V_b = L_1\left(\frac{dI_1}{dt}\right) = L_2\left(\frac{dI_2}{dt}\right) = L_3\left(\frac{dI_3}{dt}\right)$$

$$\Rightarrow \quad \frac{(V_a - V_b)}{L_{\text{parallel}}} = \frac{(V_a - V_b)}{L_1} + \frac{(V_a - V_b)}{L_2} + \frac{(V_a - V_b)}{L_3}$$

$$\Rightarrow \quad \frac{1}{L_{\text{parallel}}} = \frac{1}{L_P} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots\ldots$$

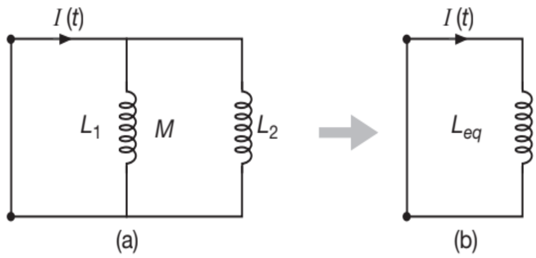
SPECIAL CASE

If two coils of self-inductances L_1 and L_2, having mutual inductance M are connected in parallel and placed close to each other, then the net inductance is given by

$$L_P = L_{\text{parallel}} = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}$$

ILLUSTRATION 82

Two inductors having self-inductances L_1 and L_2 are connected in parallel as shown in Figure (a). The mutual inductance between the two inductors is M. Determine the equivalent self-inductance L_{eq} for the system as shown in Figure (b).



SOLUTION

$$I = I_1 + I_2 \qquad \ldots(1)$$

The voltage across the pair is

$$\Delta V = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt_2} \qquad \ldots(2)$$

$$\Delta V = -L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} \qquad \ldots(3)$$

For the equivalent combination, we have

$$\Delta V = -L_{\text{eq}}\frac{dI}{dt} \qquad \ldots(4)$$

From (2), we get

$$-\frac{dI_1}{dt} = \frac{\Delta V}{L_1} + \frac{M}{L_1}\frac{dI_2}{dt}$$

Substituting in (3), we get

$$\Delta V = -L_2\frac{dI_2}{dt} + M\left(\frac{\Delta V}{L_1} + \frac{M}{L_1}\frac{dI_2}{dt}\right)$$

$$\Rightarrow \quad -L_2\frac{dI_2}{dt} + \frac{M(\Delta V)}{L_1} + \frac{M^2}{L_1}\frac{dI_2}{dt} = \Delta V$$

$$\Rightarrow \quad \left(M^2 - L_1L_2\right)\frac{dI_2}{dt} = \Delta V\left(L_1 - M\right) \qquad \ldots(5)$$

Now, from (3), we get

$$-\frac{dI_2}{dt} = \frac{\Delta V}{L_2} + \frac{M}{L_2}\frac{dI_1}{dt}$$

Substituting in (2), we get

$$\left(M^2 - L_1L_2\right)\frac{dI_1}{dt} = \Delta V\left(L_2 - M\right)$$

However, we have

$$I = I_1 + I_2$$

$$\Rightarrow \quad \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$\Rightarrow \quad -\frac{\Delta V}{L_{eq}} = \frac{\Delta V(L_2 - M)}{M^2 - L_1L_2} + \frac{\Delta V(L_1 - M)}{M^2 - L_1L_2} \qquad \Rightarrow \quad -\frac{1}{L_{eq}} = \frac{L_2 - M + L_1 - M}{M^2 - L_1L_2}$$

$$\left\{\because -L_{eq}\frac{dI}{dt} = \Delta V\right\} \qquad \Rightarrow \quad L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

Test Your Concepts-VIII

Based on Faraday's Laws: Mutual Induction

(Solutions on page H.150)

1. A coil has 600 turns which produces 5×10^{-3} Wb per turn of flux when 3 A flows in the wire. This produced 6×10^{-3} Wb per turn in 1000 turns secondary coil. When a switch is opened the current drops to zero in 0.2 s. Find
   (a) mutual inductance
   (b) the induced emf in the secondary
   (c) the self-inductance of the primary coil

2. Calculate the mutual inductance between two coils when a current of 4 A changes to 12 A in 0.5 s and induces an emf of 50 mV in the secondary. Also calculate the induced emf in the secondary if current in the primary changes from 3 A to 9 A in 0.02 s.

3. A circular coil P of 100 turns and radius 2 cm is placed concentrically at the centre of another circular coil Q of 1000 turns and radius 20 cm. Calculate the mutual inductance of the coils. When the current in the coil Q decreases from 5 A to 3 A in 0.04 s, then calculate the emf induced in coil P, rate of change of flux through the coil P at this instant and the charge passing through coil P if its resistance is 8 Ω.

4. A large coil of radius R_1 and having N_1 turns is coaxial with a small coil of radius R_2 and having N_2 turns. The centres of the coils are separated by a distance x that is much larger than R_1 and R_2. What is the mutual inductance of the coils?

5. A straight solenoid has 50 turns per cm in primary and 200 turns in the secondary. The area of cross section of the solenoid is 4 cm^2. Calculate the mutual inductance.

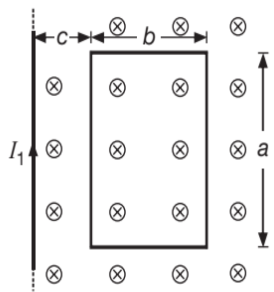
6. Find the mutual inductance of two thin coaxial loops of the same radius a if their centres are separated by a distance l, with $l \gg a$.

7. An air core solenoid 0.5 m in length contains 1000 turns and has a cross sectional area of 1 cm^2.
   (a) Ignoring end effects, find the self-inductance.
   (b) A secondary winding wrapped around the center of the solenoid has 100 turns. What is the mutual inductance?
   (c) The secondary winding carries a constant current of 1 A, and the solenoid is connected to a load of 1 kΩ. The constant current is suddenly stopped. How much charge flows through the load resistor?

8. A coil of 100 turns and 1 cm radius is kept coaxially within a long solenoid of 8 turns per cm and 5 cm radius. Find the mutual inductance.

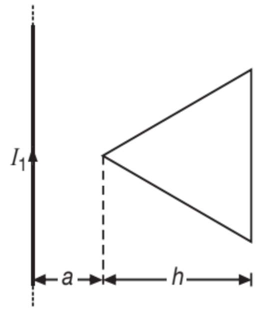
9. A very small circular loop of radius a is initially coplanar and concentric with a much larger circular loop of radius $b(\gg a)$. A constant current I is passed in the large loop which is kept fixed in space and the small loop is rotated with angular velocity ω about a diameter. The resistance of the small loop is R and the inductance is negligible.
   (a) Find the current in the small loop as a function of time.
   (b) Calculate how much torque must be exerted on the small loop to rotate it.
   (c) Calculate the induced emf in the large loop due to current [found in part (a)] in smaller loop as a function of time.

10. Figure shows a rectangular coil near a long wire. Calculate the mutual inductance of the combination.



11. An inductor of inductance L is cut in three equal parts and two of these parts are interconnected first in series and then in parallel. Assuming that the mutual inductance between the parts is negligible, calculate the inductance of the combination in both the cases.

12. Figure shows a long wire and a triangular coil. Calculate the mutual inductance of the combination.



OSCILLATIONS IN AN LC CIRCUIT

Introduction and Circuit

When a capacitor is connected to an inductor, as shown in Figure 1, then the combination is called an LC **circuit**.

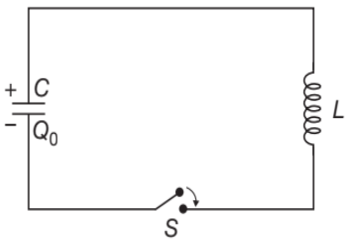


Figure 1 A simple LC circuit. The capacitor has an initial charge Q_0 and the switch is open for $t < 0$ and then closed at $t = 0$.

If the capacitor is initially charged and the switch is then closed, we find that both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values, if the resistance of the circuit is zero, i.e., no energy is dissipated as heat.

Assumptions

(a) In the following analysis, we neglect the resistance in the circuit.

(b) We also assume an idealized situation in which energy is not radiated away from the circuit.

(c) Assume that the capacitor has an initial charge Q_0 (the maximum charge) and that the switch is open for $t < 0$ and then closed at $t = 0$. Let us discuss what happens from an energy viewpoint.

With these idealizations of zero resistance and no radiation losses, the oscillations in the circuit persist indefinitely.

Explanation Based on Energy Viewpoint

When the capacitor is fully charged, the entire energy in the circuit is stored in the electric field of the capacitor and is $U = \frac{Q_0^2}{2C}$. At this time, the current in the circuit is zero, and therefore no energy is stored in the inductor. After the switch is closed, the rate at which charges leave or enter the capacitor plates (which is also the rate at which the charge on the capacitor changes) is equal to the current in the circuit. As the capacitor begins to discharge after the switch is closed, the energy stored in its electric field decreases. The discharge of the capacitor represents a current in the circuit, and hence some energy is now stored in the magnetic field of the inductor. Thus, energy is transferred from the electric field of the capacitor to the magnetic field of the inductor. When the capacitor is fully discharged, it stores no energy. At this moment, the current reaches its maximum value and now the entire energy is stored in the inductor. The current continues in the same direction, decreasing in magnitude, with the capacitor eventually becoming fully charged again but with the polarity of its plates now opposite to the initial polarity. This is followed by another discharge until the circuit returns to its original state of maximum charge Q_0 and the plate polarity shown in Figure 1. The energy continues to oscillate between inductor and capacitor.

The total energy in the LC circuit at some instant after closing the switch is

$$U = U_C + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 = \text{constant} \quad \ldots(1)$$

Since, U remains constant, so we get

$$\frac{dU}{dt} = 0$$

$$\Rightarrow \quad \frac{dU}{dt}=\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C}+\frac{1}{2}LI^2\right)=\frac{Q}{C}\frac{dQ}{dt}+LI\frac{dI}{dt}=0 \quad \ldots(2)$$

$$\Rightarrow \quad \frac{Q}{C}+L\frac{d^2Q}{dt^2}=0 \quad \ldots(3)$$

where $I=-\frac{dQ}{dt}$ and $\frac{dI}{dt}=-\frac{d^2Q}{dt^2}$

Notice the sign convention we have adopted here. **The negative sign implies that the current I is equal to the rate of decrease of charge in the capacitor plate immediately after the switch has been closed.** The same equation can be obtained by applying the modified Kirchhoff's Loop Rule, to the circuit. So, we get

$$\frac{Q}{C}-L\frac{dI}{dt}=0 \quad \ldots(4)$$

Since $I=-\frac{dQ}{dt}$, so we get

$$\frac{Q}{C}+L\frac{d^2Q}{dt^2}=0$$

The general solution to equation (3) is

$$Q(t)=Q_0\cos(\omega_0 t+\theta) \quad \ldots(5)$$

where Q_0 is the amplitude of the charge and θ is the phase. The angular frequency ω_0 is given by

$$\omega_0=\frac{1}{\sqrt{LC}} \quad \ldots(6)$$

The corresponding current in the inductor is

$$I(t)=-\frac{dQ}{dt}=\omega_0 Q_0\sin(\omega_0 t+\theta)$$

$$\Rightarrow \quad I(t)=I_0\sin(\omega_0 t+\theta) \quad \ldots(7)$$

where $I_0=\omega_0 Q_0$ ...(8)

From the initial conditions, we have

$$Q(t=0)=Q_0 \text{ and } I(t=0)=0.$$

So, the phase θ is given by $\theta=0$. Thus, the solutions for the charge and the current in our LC circuit are

$$Q(t)=Q_0\cos(\omega_0 t) \quad \ldots(9)$$

{Think smartly that at $t=0$, we have $Q=Q_0$, so Q must be a function of cosine with zero initial phase}

$$\Rightarrow \quad I(t)=\frac{dQ}{dt}=I_0\sin(\omega_0 t) \quad \ldots(10)$$

The time dependence of $Q(t)$ and $I(t)$ are depicted in Figure 2.

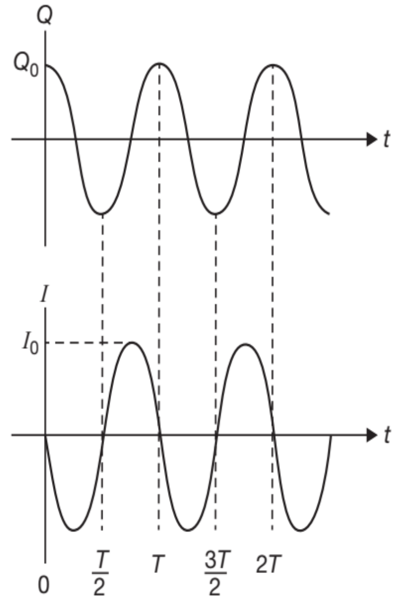


Figure 2 Charge and current in the LC circuit as a function of time.

From equations (9) and (10) (AND USING (8)), we see that at any instant of time, the electric energy and the magnetic energies are given by

$$U_E=\frac{Q^2(t)}{2C}$$

$$\Rightarrow \quad U_E=\left(\frac{Q_0^2}{2C}\right)\cos^2(\omega_0 t) \quad \ldots(11)$$

and $U_B=\frac{1}{2}LI^2(t)$

$$\Rightarrow \quad U_B=\frac{1}{2}LI^2(t)=\frac{LI_0^2}{2}\sin^2(\omega t)$$

Since $I_0=\omega_0 Q_0$, so we have

$$\Rightarrow \quad U_B=\frac{L(\omega_0 Q_0)^2}{2}\sin^2(\omega_0 t) \quad \ldots(12)$$

Since we know that $\omega_0=\frac{1}{\sqrt{LC}}$

$$\Rightarrow \quad U_B=\frac{1}{2}LI^2=\left(\frac{Q_0^2}{2C}\right)\sin^2(\omega_0 t)$$

The total energy is given by

$$\Rightarrow \quad U = U_E + U_B = \left(\frac{Q_0^2}{2C}\right)\cos^2(\omega_0 t) + \left(\frac{Q_0^2}{2C}\right)\sin^2(\omega_0 t)$$

$$\Rightarrow \quad U = \frac{Q_0^2}{2C} = \text{constant} \qquad \ldots(13)$$

The electric and magnetic energy oscillations are illustrated in Figure 3.

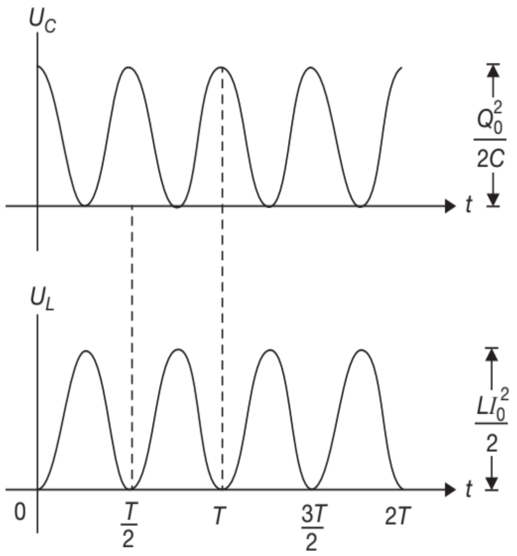


Figure 3 Plots of U_C versus t and U_L versus t for a resistanceless, non-radiating LC circuit. The sum of the two curves is a constant and equal to the total energy stored in the circuit.

Mechanical Analogue of LC Oscillations

The mechanical analogue of the LC oscillations is the mass-spring system, shown in Figure 4.

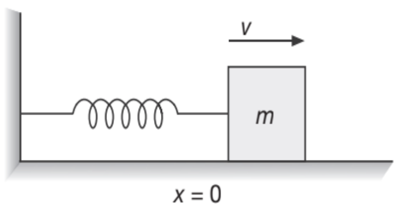


Figure 4 Mass spring oscillations

Consider a mass m moving with a speed v attached to a spring having a spring constant k. Let the mass m be displaced from its equilibrium position by x, then the total energy of this mechanical system is

$$U = K + U_{sp} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

where K and U_{sp} are the kinetic, energy of the mass and the elastic potential energy of the spring, respectively. In the absence of friction, the total energy U is conserved and we get

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right)$$

$$\Rightarrow \quad \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}\left(\frac{1}{2}kx^2\right)$$

$$\Rightarrow \quad \frac{dU}{dt} = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

Using $v = \frac{dx}{dt}$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$, the above equation may be rewritten as

$$m\frac{d^2x}{dt^2} + kx = 0$$

The general solution for the displacement is

$$x(t) = A\cos(\omega_0 t + \theta) \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

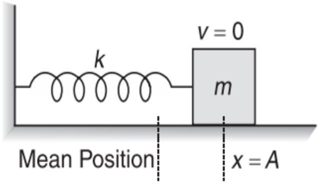
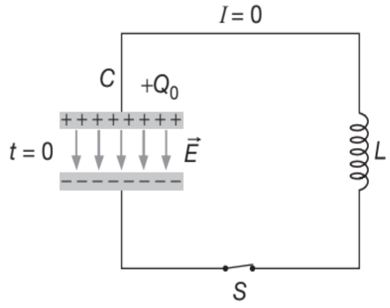
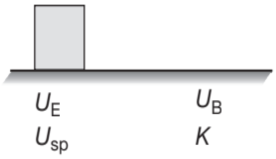
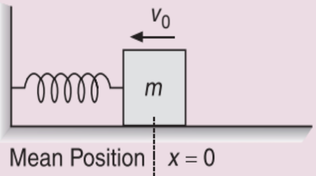
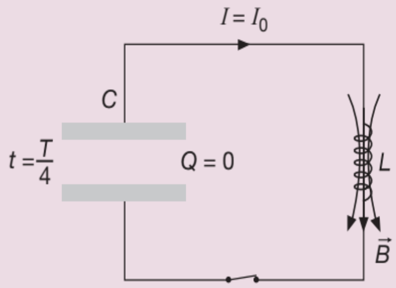
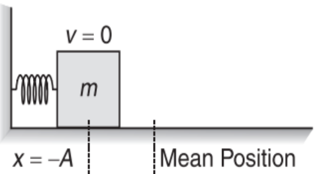
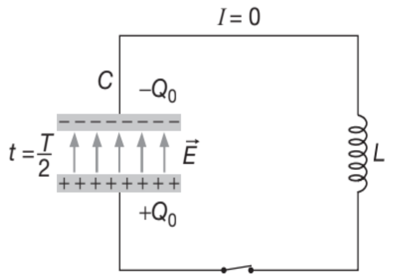
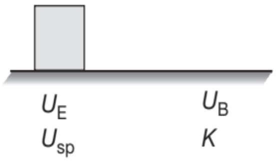
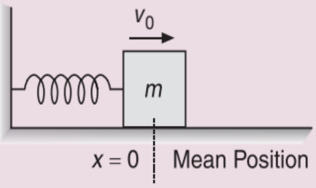
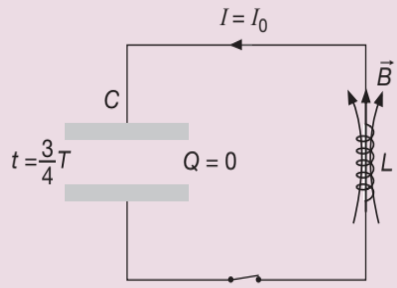
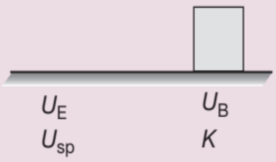
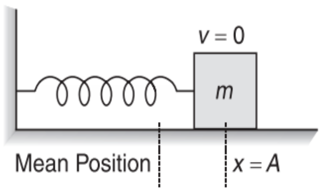
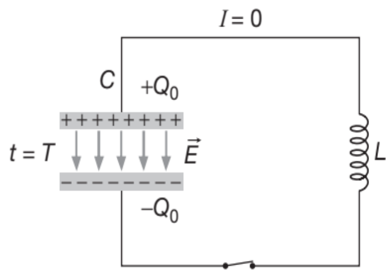
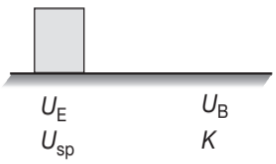
is the angular frequency and A is the amplitude of the oscillations. Thus, at any instant in time, the energy of the system is

$$U = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \theta) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \theta)$$

$$\Rightarrow \quad U = \frac{1}{2}kA^2\left(\sin^2(\omega_0 t + \theta) + \cos^2(\omega_0 t + \theta)\right)$$

$$\Rightarrow \quad U = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2A^2 \qquad \left\{\because \omega_0 = \sqrt{\frac{k}{m}}\right\}$$

In this table, we have illustrated the energy oscillations in the LC Circuit and the mass-spring system (harmonic oscillator). Please note that U_E is the electrical energy stored in the capacitor, U_B is the magnetic energy stored in the inductor. Also, U_{sp} is the elastic potential energy of the spring and K is the kinetic energy of the mass attached to the spring.

Mass Spring System	LC Circuit	Energy
 Initially, spring stretched, has maximum elastic potential energy	 Initially capacitor full charged	
 Kinetic energy of the block is maximum, $U_{sp}=0$	 Current grows to maximum in L	
 Kinetic energy of the block is zero, U_{sp} is maximum	 Capacitor becomes full charged, with polarity reversed	
 Kinetic energy of the block is maximum, $U_{sp}=0$	 Current grows to maximum in L	
 Kinetic energy of the block is zero, U_{sp} is maximum	 Finally, capacitor returns to its original state	

Analogies between Electrical and Mechanical Systems.

One Dimensional Mechanical System	Electric Circuit	Analogy
Position	Charge	$Q \leftrightarrow x$
Velocity	Current	$I \leftrightarrow v_x$
Force	Potential difference	$\Delta V \leftrightarrow F_x$
Viscous damping coefficient	Resistance	$R \leftrightarrow b$
(k = spring constant)	Capacitance	$C \leftrightarrow \frac{1}{k}$
Mass	Inductance	$L \leftrightarrow m$
Velocity = time derivative of position	Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$
Acceleration = second time derivative of position	Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$
Kinetic energy of moving object	Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$
Potential energy stored in a spring	Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$
Rate of energy loss due to friction	Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$
Damped object on a spring	*RLC* circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

ILLUSTRATION 83

A capacitor of capacitance $25\ \mu F$ is charged to 300 V. It is then connected across a 10 mH inductor. The resistance in the circuit is negligible.

(a) Find the frequency of oscillation of the circuit.

(b) Find the potential difference across capacitor and magnitude of circuit current 1.2 ms after the inductor and capacitor are connected.

(c) Find the magnetic energy and electric energy at $t = 0$ and $t = 1.2$ ms.

SOLUTION

(a) The frequency of oscillation of the circuit is given by

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Substituting the given values, we get

$$f = \frac{1}{2\pi\sqrt{(10\times10^{-3})(25\times10^{-6})}}$$

$$\Rightarrow\ f = 318.3\ \text{Hz}$$

(b) Charge across the capacitor at time t is

$$q = q_0 \cos(\omega t)$$

$$\Rightarrow \quad I = \frac{dq}{dt} = -q_0 \omega \sin(\omega t)$$

where $q_0 = CV_0 = (25\times10^{-6})(300) = 7.5\times10^{-3}$ C

Now, charge in the capacitor after $t = 1.2\times10^{-3}$ s is,

$$q = (7.5\times10^{-3})\cos((2\pi\times318.3)(1.2\times10^{-3}))\ \text{C}$$

$$\Rightarrow \quad q = -5.53\times10^{-3}\ \text{C}$$

Potential Difference, across capacitor is,

$$V = \frac{|q|}{C} = \frac{5.53\times10^{-3}}{25\times10^{-6}} = 221.2\ \text{V}$$

The magnitude of current in the circuit at $t = 1.2\times10^{-3}$ s is,

$$|I| = q_0 \omega \sin(\omega t)$$

$$\Rightarrow \quad |I| = (7.5\times10^{-3})(2\pi)(318.3)\sin((2\pi\times318.3)(1.2\times10^{-3}))\ \text{A}$$

$$\Rightarrow \quad |I| = 10.13\ \text{A}$$

(c) At $t = 0$ the current in the circuit is zero. Hence, $U_L = 0$. So, charge in the capacitor is maximum.

$$\Rightarrow \quad U_C = \frac{1}{2}\frac{q_0^2}{C}$$

$$\Rightarrow \quad U_C = \frac{1}{2}\times\frac{(7.5\times10^{-3})^2}{(25\times10^{-6})} = 1.125\ \text{J}$$

$\Rightarrow$ Total energy $E = U_L + U_C = 1.125$ J

At $t = 1.2$ ms, we have

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}(10\times10^{-3})(10.13)^2$$

$$\Rightarrow \quad U_L = 0.513\ \text{J}$$

$$\Rightarrow \quad U_C = E - U_L = 1.125 - 0.513 = 0.612\ \text{J}$$

Else, U_C can also be calculated as,

$$U_C = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\times\frac{(5.53\times10^{-3})^2}{(25\times10^{-6})} = 0.612\ \text{J}$$

ILLUSTRATION 84

An inductor of inductance 2 mH is connected across a charged capacitor of capacitance 5 μF and the resulting LC circuit is set oscillating at its natural frequency. Let Q denote the instantaneous charge on the capacitor and I the current in the circuit. It is found that the maximum value of Q is 200 μC.

(a) When $Q = 100\ \mu$C, what is the value of $\left|\frac{dI}{dt}\right|$?

(b) When $Q = 200\ \mu$C, what is the value of I?

(c) Find the maximum value of I

(d) When I is equal to one-half its maximum value, what is the value of $|Q|$?

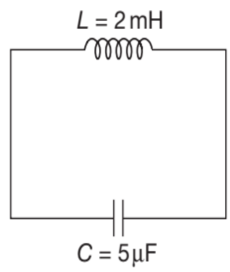
SOLUTION

This problem is dealing with LC oscillations. The charge stored in the capacitor oscillates simple harmonically as,

$$Q = Q_0 \sin(\omega t \pm \phi)$$

where Q_0 is the maximum value of charge given by

$$Q_0 = 200\ \mu\text{C} = 2\times10^{-4}\ \text{C}$$



$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2\times10^{-3}\ \text{H})(5\times10^{-6}\ \text{F})}} = 10^4\ \text{s}^{-1}$$

Let at $t = 0$, $Q = Q_0$ then,

$$Q(t) = Q_0 \cos(\omega t) \qquad \ldots(1)$$

$$\Rightarrow \quad I(t) = \frac{dQ}{dt} = -Q_0 \omega \sin(\omega t) \text{ and} \qquad \ldots(2)$$

$$\Rightarrow \quad \frac{dI(t)}{dt} = -Q_0 \omega^2 \cos(\omega t) \qquad \ldots(3)$$

(a) $Q = 100\ \mu\text{C} = \frac{Q_0}{2}$

At $\cos(\omega t) = \frac{1}{2}$, from equation (3), we get

$$\left|\frac{dI}{dt}\right| = (2\times10^{-4}\text{ C})(10^4\text{ s}^{-1})^2\left(\frac{1}{2}\right)$$

$$\Rightarrow \left|\frac{dI}{dt}\right| = 10^4\text{ As}^{-1}$$

(b) $Q = 200\ \mu\text{C} = Q_0$ when

$\cos(\omega t) = 1$, i.e., $\omega t = 0,\ 2\pi ...$

At this time $I(t) = -Q_0\omega\sin(\omega t)$

$\Rightarrow I(t) = 0 \qquad \{\sin 0° = \sin 2\pi = 0\}$

(c) $I(t) = -Q_0\omega\sin(\omega t)$

The maximum value of I is $Q_0\omega$

$$\Rightarrow I_{\max} = Q_0\omega = (2\times10^{-4}\text{ C})(10^4\text{ s}^{-1}) = 2\text{ A}$$

(d) From energy conservation,

$$\frac{1}{2}LI_{\max}^2 = \frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C}$$

$$\Rightarrow Q = \sqrt{LC\left(I_{\max}^2 - I^2\right)}$$

For, $I = \dfrac{I_{\max}}{2} = 1\text{ A}$, we get

$$Q = \sqrt{(2\times10^{-3})(5\times10^{-6})(2^2 - 1^2)}$$

$$\Rightarrow Q = \sqrt{3}\times10^{-4}\text{ C}$$

$$\Rightarrow Q = 1.732\times10^{-4}\text{ C}$$

ILLUSTRATION 85

A $1.5\ \mu\text{F}$ capacitor is charged to 20 V. The charging battery is then disconnected and a 15 mH coil is connected in series with the capacitance so that LC oscillations occur. Write the equation for variation of charge on capacitor and current in the inductor assuming that the inductor is connected to the capacitor at $t = 0$

SOLUTION

Given that $Q_{\max} = Q_0 = CV_0 = 30\ \mu\text{C}$

At $t = 0$, $Q = Q_0$

$$\Rightarrow Q = Q_0\cos\omega t \text{ where } \omega = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \omega = \frac{1}{\sqrt{1.5\times10^{-6}\times15\times10^{-3}}} = \frac{1}{15\times10^{-5}}\text{ rads}^{-1}$$

Also, $I = -\dfrac{dQ}{dt} = Q_0\omega\sin\omega t = I_0\sin\omega t$

where, $I_0 = Q_0\omega = \dfrac{30\times10^{-6}}{15\times10^{-5}} = 0.2\text{ A}$

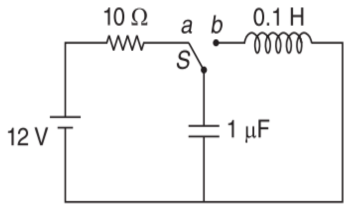
$$\Rightarrow I = (0.2\text{ A})\sin\omega t$$

Test Your Concepts-IX

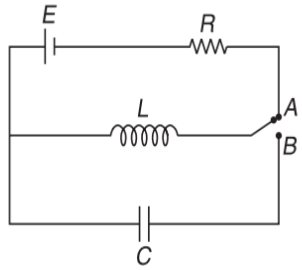
Based on LC Oscillations

(Solutions on page H.153)

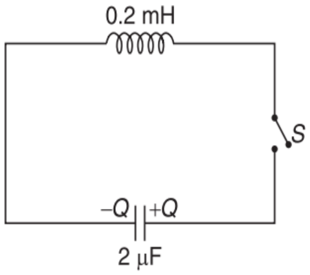
1. The switch in figure is connected to point *a* for a long time. After the switch is thrown to point *b*, what are
   (a) the frequency of oscillation of the *LC* circuit,
   (b) the maximum charge that appears on the capacitor,
   (c) the maximum current in the inductor, and
   (d) the total energy the circuit possesses at $t = 3$ s?



2. An *LC* circuit consists of a 20 mH inductor and a 0.5 μF capacitor. If the maximum instantaneous current is 0.1 A, what is the greatest potential difference across the capacitor?
3. In the arrangement shown, the switch is in position *A* for a long time. At time $t = 0$, it is switched to position *B*. Find the maximum charge which will accumulate on capacitor.



4. An LC circuit having a capacitor with initial charge of 200 μC is shown in Figure. If at $t = 0$, the switch is closed, calculate the first instant when energy stored in inductor becomes one third that of capacitor.



5. An LC circuit consists of an inductor with $L = 90$ mH and a capacitor of $C = 400\ \mu\text{F}$. The initial charge on the capacitor is 5 μC, and the initial current in the inductor is zero.

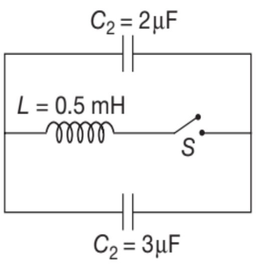
(a) What is the maximum voltage across the capacitor?

(b) What is the maximum current in the inductor?

(c) What is the maximum energy stored in the inductor?

(d) When the current in the inductor has half its maximum value, what is the charge on the capacitor and what is the energy stored in the inductor?

6. In an oscillating circuit shown in figure, the capacitors were charged to a voltage of 200 V and then the switch S was closed. Find

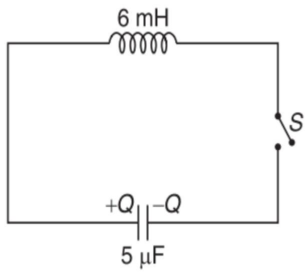


(a) the normal oscillation frequency.

(b) the peak value of the current flowing through the coil

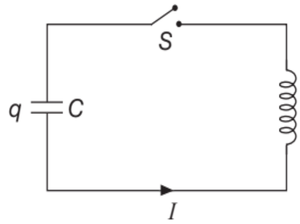
7. A capacitor in a series LC circuit has an initial charge Q and is being discharged. Find, in terms of L and C, the flux through each of the N turns in the coil, when the charge on the capacitor is $\dfrac{Q}{2}$.

8. In the circuit shown in figure charge on capacitor is $Q = 100$ μC. If switch is closed at $t = 0$ calculate the current in circuit when charge on capacitor reduces to 50 μC. Also find the maximum current in circuit.



9. Initially, the capacitor in a series LC circuit is charged. A switch is closed at $t = 0$, allowing the capacitor to discharge, and at time t the energy stored in the capacitor is one fourth of its initial value. Assuming C to be known, determine L.

10. In the LC circuit shown, $C = 1$ μF. With C charged to 100 V, switch S is suddenly closed at time $t = 0$. The circuit then oscillates at 10^3 cycles per second.



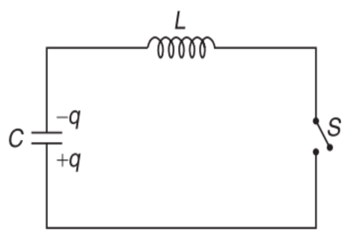
(a) Calculate ω and T

(b) Express q as a function of time

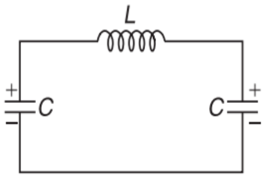
(c) Calculate L. Assume $\pi^2 = 10$

(d) Calculate the average current during the first quarter cycle.

11. In the circuit shown in Figure, if S is closed at $t = 0$, calculate the charge on capacitor as a function of time assuming that the inductor is active with initial current I_0 in it.



12. Two identical capacitors each having capacitance C are taken. The first capacitor is charged using a battery of potential difference V and the second capacitor is charged using the battery of potential difference $2V$. Now the capacitors are connected to an inductor of inductance L as shown in Figure.

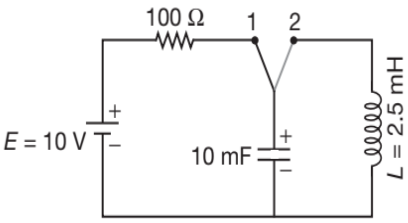


Calculate the time period of oscillation of the arrangement and the maximum current in the circuit

13. Initially the capacitor is charged to a potential of 5 V and then connected to position 1 with the shown polarity for 1 s. After 1 s it is connected across the inductor at position 2.

(a) Find the potential across the capacitor after 1 s of its connection to position 1.

(b) Find the maximum current flowing in the *LC* circuit when capacitor is connected across the inductor. Also find the frequency of *LC* oscillations.



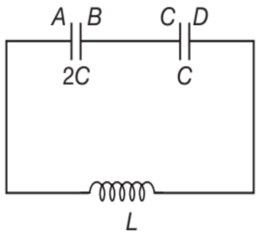
14. In an oscillating LC circuit in which $C = 4.00\ \mu F$, the maximum potential difference across the capacitor during the oscillations is 1.50 V and the maximum current through the inductor is 50.0 mA.

(a) What is the inductance *L*?

(b) What is the frequency of the oscillations?

(c) How much time does the charge on the capacitor take to rise from zero to its maximum value?

15. Two capacitors of capacitances 2C and C are connected in series through an inductor of inductance *L*. Initially capacitors have charge such that $V_B - V_A = 4V_0$, $V_C - V_D = V_0$ and the initial current in the circuit is zero. Calculate the



(a) maximum current that flows in the circuit.

(b) potential difference across each capacitor at that instant.

(c) equation of current flowing towards left in the inductor.

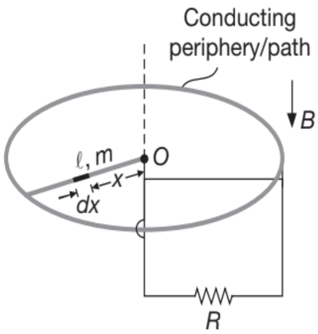
SOLVED PROBLEMS

PROBLEM 1

A conducting rod of length l, mass m is set in rotation with an angular velocity ω_0 such that it rotates on a rough horizontal plane of coefficient of friction μ. A uniform magnetic field of strength B is directed into the horizontal plane along the axis of rotation of the rod. The periphery of the conducting rod is in contact with a circular conducting path, having a resistance R. If the bar is free to rotate about the vertical axis passing through the centre of the circular conducting path, then find the variation of ω with t.

SOLUTION

Let us first visualise the arrangement discussed, in the form of a Figure.



Consider an infinitesimal element of the rod having length dx at a distance x from the centre O of the conducting path. The total force on this element must be the sum of the magnetic force and the frictional force. So,

$$dF = dF_m + df$$

where $df =$ frictional force $= \mu(dm)g = \mu\left(\frac{M}{L}dx\right)g$

and $dF_m =$ magnetic force $= i(dx)B$, where i is the induced current in the rod given by,

$$i = \frac{\xi}{R} = \frac{Bl^2\omega}{2R}$$

$$\Rightarrow \quad dF_m = \left(\frac{Bl^2\omega}{2R}\right)(dx)B$$

$$\Rightarrow \quad dF_m = \frac{B^2l^2\omega}{2R}dx$$

$$\Rightarrow \quad dF = dF_m + df$$

$$\Rightarrow \quad dF = \frac{B^2l^2\omega}{2R}dx + \frac{\mu Mg}{l}dx$$

$$\Rightarrow \quad dF = \left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)dx$$

The torque applied by the force dF about the axis of rotation is

$$d\tau = x\,dF$$

$$\Rightarrow \quad d\tau = \left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)x\,dx$$

So, the total torque is given by

$$\tau = \int d\tau = \left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)\int_0^l x\,dx$$

$$\Rightarrow \quad \tau = \left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)\frac{l^2}{2} \qquad \ldots(1)$$

If the angular retardation is α we have $\alpha = -\frac{d\omega}{dt}$.

If I be the moment of inertia of the rod, then equation (1), becomes

$$I\alpha = \left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)\frac{l^2}{2}$$

$$\Rightarrow \quad \frac{Ml^2}{3}\frac{d\omega}{dt} = \left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)\frac{l^2}{2} \qquad \left\{\because I = \frac{Ml^2}{3}\right\}$$

$$\Rightarrow \quad \int_{\omega_0}^{\omega}\frac{d\omega}{\left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)} = \frac{3}{2M}\int_0^t dt$$

$$\Rightarrow \quad \frac{2R}{B^2l^2}\log_e\left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)\Bigg|_{\omega_0}^{\omega} = -\frac{3}{2M}t$$

$$\Rightarrow \quad \log_e\left[\frac{\left(\frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l}\right)}{\left(\frac{B^2l^2\omega_0}{2R} + \frac{\mu Mg}{l}\right)}\right] = -3\left(\frac{B^2l^2}{4MR}\right)t$$

$$\Rightarrow \quad \frac{B^2l^2\omega}{2R} + \frac{\mu Mg}{l} = \left(\frac{B^2l^2\omega_0}{2R} + \frac{\mu Mg}{l}\right)e^{-3\left(\frac{B^2l^2}{4MR}\right)t}$$

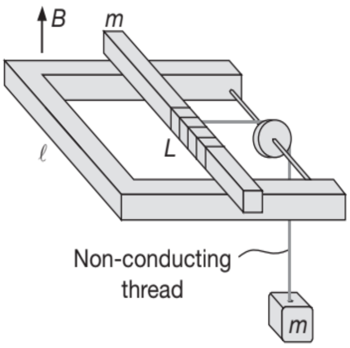
$$\Rightarrow \quad \omega = \frac{2R}{B^2l^2}\left[\left(\frac{B^2l^2\omega_0}{2R}\right)e^{-3\left(\frac{B^2l^2}{4MR}\right)t} - \frac{\mu Mg}{l}\left(1 - e^{-3\left(\frac{B^2l^2}{4MR}\right)t}\right)\right] \qquad \ldots(2)$$

CHECK POINT

At $t=0$, we must get $\omega=\omega_0$, which is true when we put $t=0$ in equation (2).

PROBLEM 2

A pair of parallel horizontal conducting rails of negligible resistance is shorted at one end of it and is fixed on the table as shown in Figure.



A constant magnetic field B exists perpendicular to the table. The distance between the rails is l. An inductor of negligible resistance and mass m can slide on the rails smoothly. The inductor is connected to the mass m by a light inextensible string passing over a light frictionless pulley as shown. Calculate velocity of the rod as a function of displacement form initial position. Also find the x_0 when terminal velocity is attained.

SOLUTION

For mass m and the rod

$$mg - T = ma$$

$$T - F_m = ma$$

$$mg - F_m = 2ma \quad \text{...(1)}$$

Let at any instant inductor is moving with velocity v, then

$$E - L\frac{dI}{dt} = 0$$

$$\Rightarrow \quad \frac{Blvdt}{L} = dI$$

$$\int_0^x \frac{Bl}{L}dx = \int_0^I dI$$

$$\Rightarrow \quad \frac{Bl}{L}x = I$$

Now $F_m = BIl = B^2l^2\frac{x}{L}$

Substituting in (1)

$$mg - \frac{B^2l^2x}{L} = 2ma = 2mv\frac{dv}{dx}$$

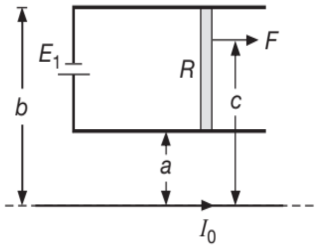
$$\Rightarrow \quad \int_0^x\left(mg - \frac{B^2l^2x}{L}\right)dx = 2m\int_0^v vdv$$

$$\Rightarrow \quad mgx - \frac{B^2\ell^2x^2}{2L} = 2m\frac{v^2}{2}$$

$$\Rightarrow \quad v = \sqrt{gx - \frac{B^2l^2x^2}{2mL}}$$

PROBLEM 3

A metallic rod of mass m and resistance R is sliding over two conducting frictionless rails as shown in Figure.



An infinitely long wire carries a current I_0. The distance of the rails from the wire are b and a respectively.

(a) Find the value of current in the circuit, if the rod slides with constant velocity v_0.

(b) Find the value of F, if the rod slides with constant velocity v_0.

(c) Find the value of c.

SOLUTION

(a) Magnetic flux linked with the infinitesimal area element shown in Figure is

$$d\phi = \frac{\mu_0 I_0}{2\pi y}(x\,dy)$$

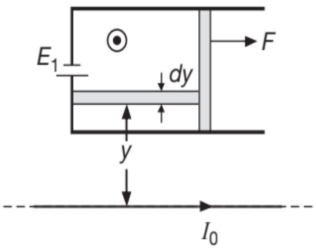
So, net flux is

$$\phi = \oint d\phi = \int_a^b \frac{\mu_0 I}{2\pi y}x\,dy = \frac{\mu_0 I_0}{2\pi}\log_e\left|\frac{b}{a}\right|$$

$$\xi = \left|-\frac{d\phi}{dt}\right| = \frac{\mu_0 I_0}{2\pi}v_0\log_e\left|\frac{b}{a}\right| \qquad \left\{\because \frac{dx}{dt} = v_0\right\}$$

Induced emf developed due to motion of the conductor has an opposite sense to the emf of the cell. So, we get

$$I=\frac{E-\xi}{R}=\frac{1}{R}\left\{E-\frac{\mu_0 I_0 v_0}{2\pi}\log_e\left|\frac{b}{a}\right|\right\}$$



(b) $F=\left|\int d\vec{F}\right|=\left|\int I d\vec{l}\times\vec{B}\right|$

$$\Rightarrow F=\frac{1}{R}\int_a^b\left(E-\frac{\mu_0 I_0 v_0}{2\pi}\log_e\left|\frac{b}{a}\right|\right)(dy)\left(\frac{\mu_0 I_0}{2\pi y}\right)$$

$$\Rightarrow F=\frac{\mu_0 I_0}{2\pi R}\left(E-\frac{\mu_0 I_0 v_0}{2\pi}\log_e\left|\frac{b}{a}\right|\right)\log_e\left(\frac{b}{a}\right)$$

(c) $\tau=\left|\int d\vec{\tau}\right|=\int \vec{r}\times d\vec{F}$

$$\Rightarrow \tau=\int_a^b y\left(\frac{\mu_0 I_0}{2\pi R}\right)\left(E-\frac{\mu_0 I_0 v_0}{2\pi}\log_e\left(\frac{b}{a}\right)\right)\left(\frac{dy}{y}\right)$$

$$\Rightarrow \tau=\frac{\mu_0 I_0}{2\pi R}\left(E-\frac{\mu_0 v_0 I_0}{2\pi}\log_e\left|\frac{b}{a}\right|\right)(b-a)$$

Now $\vec{\tau}=\vec{F}_m\times\vec{c}$

$$\Rightarrow \tau=F_m c$$

$$\Rightarrow \frac{\mu_0 I_0}{2\pi R}\left[E-\frac{\mu_0 v_0 I_0}{2\pi}\log_e\left(\frac{b}{a}\right)\right](b-a)$$

$$=\frac{\mu_0 I_0}{2\pi R}c\left[E-\frac{\mu_0 v_0 I_0}{2\pi}\log_e\left(\frac{b}{a}\right)\right]\log_e\left(\frac{b}{a}\right)$$

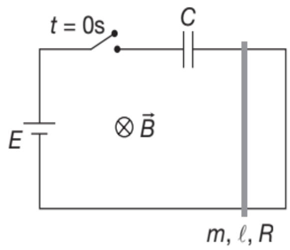
$$\Rightarrow c=\frac{b-a}{\log_e\left(\frac{b}{a}\right)}=\frac{b-a}{\log_e b-\log_e a}$$

PROBLEM 4

In the arrangement shown, the connector of mass m, length l and resistance R can smoothly slide along the two fixed parallel rails and the switch is closed at time $t=0$. The capacitor is initially uncharged and a uniform magnetic field B exists in the region directed into the plane of the paper. Find the

(a) charge on the capacitor after a long time.

(b) speed of the connector after a long time.



SOLUTION

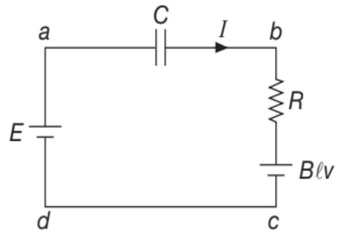
(a) Applying KVL to the loop $abcda$, we get

$$-\frac{q}{C}-IR-Blv+E=0 \qquad \ldots(1)$$

$$\Rightarrow \frac{q}{C}+R\frac{dq}{dt}=E-Blv$$

$$\Rightarrow \frac{dq}{dt}+\frac{q}{RC}=\frac{E}{R}-\frac{B^2l^2}{mR}q$$

$$\Rightarrow \int_0^q\frac{dq}{\frac{E}{R}-q\left(\frac{B^2\ell^2}{mR}+\frac{1}{RC}\right)}=\int_0^t dt$$



$$-\frac{1}{\left(\frac{B^2l^2}{mR}+\frac{1}{RC}\right)}\log_e\left[\frac{\frac{E}{R}-q\left(\frac{B^2l^2}{mR}+\frac{1}{RC}\right)}{\frac{E}{R}}\right]=t$$

$$\frac{E}{R}-q\left(\frac{B^2l^2}{mR}+\frac{1}{RC}\right)=\frac{E}{R}e^{-\left(\frac{B^2l^2}{mR}+\frac{1}{RC}\right)t}$$

$$\Rightarrow q=\frac{E}{R\left(\frac{B^2l^2}{mR}+\frac{1}{RC}\right)}\left(1-e^{-\left(\frac{B^2l^2}{mR}+\frac{1}{RC}\right)t}\right)$$

$$\Rightarrow q=\frac{CE}{\left(1+\frac{B^2l^2C}{m}\right)}\left(1-e^{-\left(\frac{B^2l^2}{mR}+\frac{1}{RC}\right)t}\right) \qquad \ldots(2)$$

When $t \to \infty$, we get $q = Q_0$

$$\Rightarrow \quad Q_0 = q_{t\to\infty} = \frac{CE}{\left(1 + \dfrac{B^2 l^2 C}{m}\right)}$$

(b) Let at any moment the speed of the connector be v, current in the circuit is $I = \dfrac{dq}{dt}$ and charge on the capacitor is q. Then

$$F = m\frac{dv}{dt} = BIl$$

$$\Rightarrow \quad F = m\frac{dv}{dt} = B\left(\frac{dq}{dt}\right)l$$

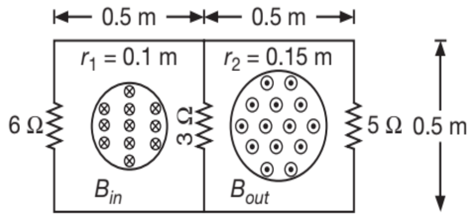
Integrating we get $v = \left(\dfrac{Bl}{m}\right)q$

From Equation (1), we get

$$v_{t\to\infty} = \frac{BlCE}{\left(m + CB^2 l^2\right)}$$

PROBLEM 5

Two infinitely long solenoids (seen in cross section) pass through a circuit as shown in Figure.



The magnitude of $\vec{B}$ inside each is the same and is increasing at the rate of 100 Ts^{-1}. What is the current in each resistor?

SOLUTION

For the left loop, the emf induced is

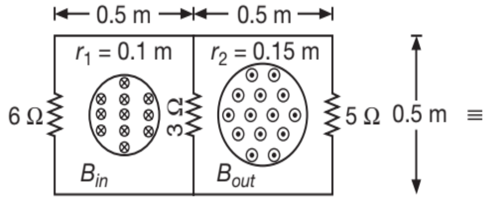
$$\xi_1 = \frac{d\phi_B}{dt} = A_1\left(\frac{dB}{dt}\right) = \left(\pi r_1^2\right)\frac{dB}{dt}$$

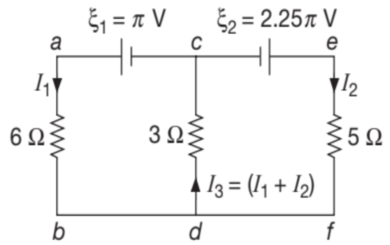
$$\Rightarrow \quad \xi_1 = \pi(0.1)^2(100) = \pi \text{ V}$$

Since the field is increasing inwards, so the induced current must set up a field which will not allow this increase. Hence induced current in the left loop is I_1 counter clockwise (to establish an outward field) Similarly, for the right loop, we have

$$\xi_2 = \frac{d\phi_B}{dt} = A_2\left(\frac{dB}{dt}\right) = \pi(0.15)^2(100) = 2.25\pi \text{ V}$$

This induced emf produces an induced current, say I_2, clockwise to oppose the change in B here.





For loop $abdca$, we have

$$-6I_1 - 3\left(I_1 + I_2\right) + \pi = 0$$

$$\Rightarrow \quad 9I_1 + 3I_2 = \pi \qquad \ldots(1)$$

For loop $cdfec$, we have

$$3\left(I_1 + I_2\right) + 5I_2 - 2.25\pi = 0$$

$$3I_1 + 8I_2 = 2.25\pi \qquad \ldots(2)$$

Solving (1) and (2), we get

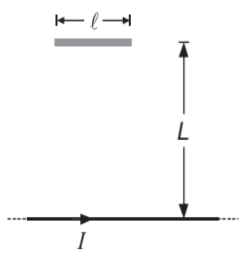
$$I_1 = 62 \text{ mA}$$

$$I_2 = 860 \text{ mA}$$

$$I_3 = 922 \text{ mA}$$

PROBLEM 6

A rod of length l is held parallel to and a distance L above a long wire carrying I and resting on the floor (shown in Figure).



The rod is released and it remains parallel to the current carrying wire as it falls. Assume that the rod falls under gravity g, derive an equation for the emf induced in it. Express your result as a function of the time t after the wire is dropped. What is the induced emf at half the time that the rod takes to hit the wire?

Find the instant and position when the induced emf is maximum. Also find the maximum induced emf. Neglect the interaction of induced current in the rod and current in the wire.

SOLUTION

Since, $\xi = Blv$

where $B = \dfrac{\mu_0 I}{2\pi y}$, $v = gt$ and $y = L - \dfrac{1}{2}gt^2$

$$\Rightarrow \quad \xi = \frac{\mu_0 Il}{2\pi}\left(\frac{gt}{L - \frac{1}{2}gt^2}\right)$$

The rod hits the wire in time T (say), then

$$L = \frac{1}{2}gT^2$$

$$\Rightarrow \quad T = \sqrt{\frac{2L}{g}}$$

at $t = \dfrac{T}{2} = \dfrac{1}{2}\sqrt{\dfrac{2L}{g}} = \sqrt{\dfrac{L}{2g}}$

$$\Rightarrow \quad \xi = \frac{\mu_0 Il}{2\pi}\left[\frac{g\sqrt{\frac{L}{2g}}}{L - \frac{1}{2}g\left(\frac{L}{2g}\right)}\right] = \frac{\mu_0 Il}{2\pi}\left[\frac{\sqrt{\frac{gL}{2}}}{\left(L - \frac{L}{4}\right)}\right]$$

$$\xi = \frac{\mu_0 Il}{2\pi}\frac{4}{3}\sqrt{\frac{g}{2L}} = \frac{2\mu_0 Il}{3\pi}\sqrt{\frac{g}{2L}}$$

ξ will be MAXIMUM, when

$$\frac{d\xi}{dt} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{t}{L - \frac{1}{2}gt^2}\right) = 0 \qquad \left\{\because\ \xi = \frac{\mu Ilg}{2\pi}\left(\frac{t}{L - \frac{1}{2}gt^2}\right)\right\}$$

$$\Rightarrow \quad \frac{\left(L - \frac{1}{2}gt^2\right) - t(gt)}{\left(L - \frac{1}{2}gt^2\right)^2} = 0$$

$$\Rightarrow \quad L - \frac{1}{2}gt^2 - gt^2 = 0$$

$$\Rightarrow \quad L = \frac{3}{2}gt^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2L}{3g}} = \frac{T}{\sqrt{3}}$$

$$\Rightarrow \quad y = L - \frac{1}{2}g\left(\frac{2L}{3g}\right) = \frac{2L}{3}, \text{ from the wire}$$

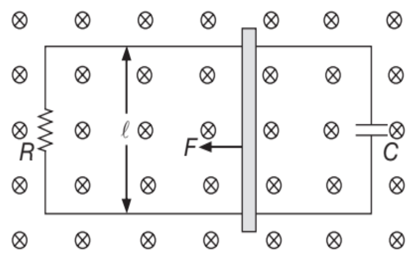
$$\xi_{\text{MAX}} = \frac{\mu_0 Il}{2\pi}\left(\frac{g\sqrt{\frac{2L}{3g}}}{L - \frac{1}{2}g\left(\frac{2L}{3g}\right)}\right)$$

$$\Rightarrow \quad \xi_{\text{MAX}} = \frac{\mu_0 Il}{2\pi}\left(\frac{\sqrt{\frac{2Lg}{3}}}{\frac{2L}{3}}\right)$$

$$\Rightarrow \quad \xi_{\text{MAX}} = \frac{\mu_0 Il}{2\pi}\sqrt{\frac{3g}{2L}}$$

PROBLEM 7

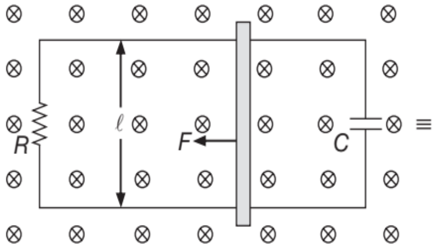
Two parallel long smooth conducting rails separated by a distance l are connected by a movable conducting connector of mass m. Terminals of the rails are connected by the resistor R and the capacitor C as shown in Figure.

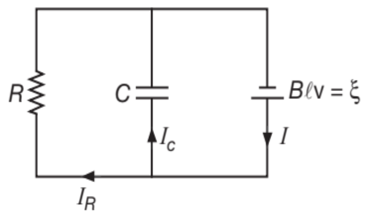


A uniform magnetic field B perpendicular to the plane of the rails is switched on. The connector is dragged by a constant force F. Find the speed of the connector as a function of time if the force F is applied at $t = 0$. Also find the terminal velocity of the connector.

SOLUTION

Due to the external force F the connector begins to accelerate such that an emf is induced in the connector due to which an induced current flow through it. If the induced current is I (downwards), then due to this current a magnetic force BIl (acting opposite to F) acts on the connector. Let at some instant t, the connector has a velocity v. Then the current passing through the resistor is





$$I_R = \frac{\xi}{R} = \frac{Blv}{R} \qquad \ldots(1)$$

Since the source of emf (which is the moving connector) is connected in parallel to both R and C, hence the potential across both is Blv. So, at that instant if Q be the charge on the capacitor, then

$$Q = C\xi = C(Blv)$$

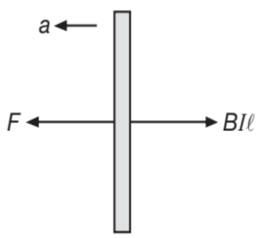
$$\Rightarrow \quad I_C = \frac{dQ}{dt} = BlC\left(\frac{dv}{dt}\right) \qquad \ldots(2)$$

Total induced current is then given by

$$I = I_R + I_C$$

$$\Rightarrow \quad I = \frac{Blv}{R} + BlC\left(\frac{dv}{dt}\right) \qquad \ldots(3)$$

Let us now draw a free body diagram of the connector.



Then, from Newton's Second Law, we get

$$F - BIl = ma$$

$$\Rightarrow \quad F - B\left(\frac{Blv}{R} + BlC\frac{dv}{dt}\right)l = m\left(\frac{dv}{dt}\right)$$

$$\Rightarrow \quad F - \frac{B^2l^2v}{R} = (B^2l^2C + m)\frac{dv}{dt} \qquad \ldots(4)$$

$$\Rightarrow \quad \frac{dv}{F - \left(\frac{B^2l^2}{R}\right)v} = \frac{dt}{(B^2l^2C + m)}$$

$$\Rightarrow \quad \int_0^v \frac{dv}{F - \left(\frac{B^2l^2}{R}\right)v} = \frac{1}{(B^2l^2C + m)}\int_0^t dt$$

$$\Rightarrow \quad -\frac{R}{B^2l^2}\log_e\left(F - \frac{B^2l^2v}{R}\right)\Bigg|_0^v = \frac{t}{(B^2l^2C + m)}$$

$$\Rightarrow \quad \log_e\left(\frac{F - \frac{B^2l^2v}{R}}{F}\right) = -\left[\frac{B^2l^2}{R(B^2l^2C + m)}\right]t$$

$$\Rightarrow \quad F - \frac{B^2l^2v}{R} = Fe^{-\left(\frac{B^2l^2}{R(B^2l^2C+m)}\right)t}$$

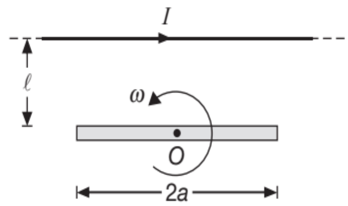
$$\Rightarrow \quad v = \frac{FR}{B^2l^2}\left(1 - e^{-\left(\frac{B^2l^2}{R(B^2l^2C+m)}\right)t}\right)$$

If v_T is the terminal velocity, then at $v = v_T$, $\frac{dv}{dt} = 0$. Then from (4), we get

$$v_T = \frac{FR}{B^2l^2}$$

PROBLEM 8

A rod of length $2a$ is free to rotate in a vertical plane, about a horizontal axis-O passing through its midpoint. A long straight, horizontal wire is in the same plane and is carrying a constant current I as shown in Figure.

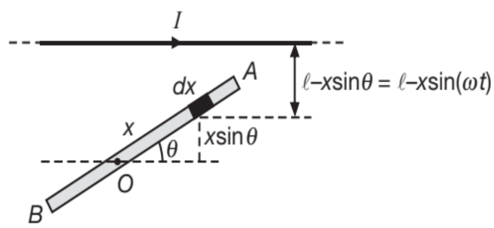


At initial moment of time, the rod is horizontal and starts to rotate with constant angular velocity ω, calculate emf induced in the rod as a function of time.

SOLUTION

At any instant of time, angular displacement of the rod is

$$\theta = \omega t$$



Consider an element of length dx at a distance x from O. If $d\xi$ be the emf induced due to motion of this element, then

$$d\xi = B(dx)v$$

$$\Rightarrow \quad d\xi = B(x\omega)dx$$

where, $B = \dfrac{\mu_0}{2\pi}\left(\dfrac{I}{l - x\sin(\omega t)}\right)$

$$\Rightarrow \quad d\xi = \frac{\mu_0 I\omega}{2\pi}\left(\frac{x}{l - x\sin(\omega t)}\right)dx$$

$$V_{OA} = V_0 - V_A = \int_0^a d\xi = \frac{\mu_0 I\omega}{2\pi}\int_0^a \frac{x}{l - x\sin(\omega t)}dx$$

$$V_{OA} = -\frac{\mu_0 I\omega}{2\pi\sin(\omega t)}\left[l\frac{\log_e\left(\dfrac{l - a\sin(\omega t)}{l}\right)}{\sin(\omega t)} + a\right]$$

Similarly, $V_{OB} = V_O - V_B = \dfrac{\mu_0 I\omega}{2\pi}\displaystyle\int_0^a \frac{x}{l + x\sin(\omega t)}dx$

$$\Rightarrow \quad V_{OB} = \frac{\mu_0 I\omega}{2\pi\sin(\omega t)}\left[a - l\frac{\log_e\left(\dfrac{l + a\sin(\omega t)}{l}\right)}{\sin(\omega t)}\right]$$

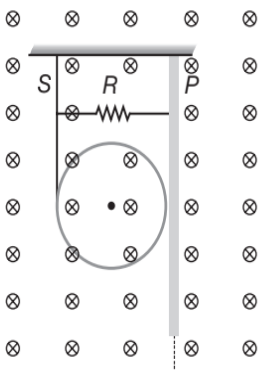
So, $V_{AB} = V_{OB} - V_{OA}$

$$\Rightarrow \quad V_{AB} = \frac{\mu_0 I\omega}{2\pi\sin(\omega t)}\left[2a + l\frac{\log_e\left(\dfrac{l - a\sin(\omega t)}{l + a\sin(\omega t)}\right)}{\sin(\omega t)}\right]$$

Please note that the function V_{AB} is discontinuous at $\omega t = n\pi$

PROBLEM 9

A conducting light string S is wound on the rim of a metal ring of radius r and mass m. The free end of the string is fixed to the ceiling. A vertical infinite smooth conducting plane P is always tangent to the ring as shown in Figure.



A uniform magnetic field B is applied perpendicular to the plane of the ring. The ring is always inside the magnetic field. The plane and the string are connected by a resistance R. When the ring is released, calculate the

(a) current in the resistance R as a function of time.
(b) terminal velocity of the ring.

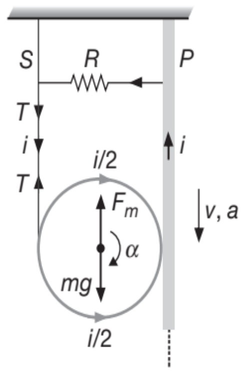
SOLUTION

Let at time t velocity of ring be v (downwards). Then

$$\xi = Bl_{\text{eff}}v = Bv(2r) = 2Bvr$$

The above result can also be obtained, if we observe the two batteries of emf $Bv(2r)$, connected in parallel. So, the induced current i is given by

$$i = \frac{\xi}{R} = \frac{2Bvr}{R}$$



Now, $mg - F_m - T = ma$, where

$$F_m = 2\left[\left(\frac{i}{2}\right)(2r)B\right] = 2irB = \frac{4B^2r^2v}{R}$$

$$\Rightarrow \quad a = g - \frac{4B^2r^2v}{mR} - \frac{T}{m} \qquad \ldots(1)$$

Also, if I be the moment of inertia of the ring, then

$$\alpha = \frac{\tau}{I} = \frac{Tr}{mr^2} = \frac{T}{mr} \qquad \ldots(2)$$

$$\Rightarrow \quad a = r\alpha = \frac{T}{m} \qquad \ldots(3)$$

From equations (1), (2) and (3), we get

$$a = \frac{g}{2} - \frac{2B^2r^2v}{mR} \qquad \ldots(4)$$

$$\Rightarrow \quad \int_0^v \frac{dv}{\dfrac{g}{2} - \dfrac{2B^2r^2v}{mR}} = \int_0^t dt$$

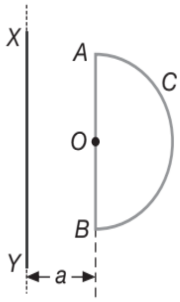
$$\Rightarrow \quad v = \frac{mgR}{4B^2r^2}\left(1 - e^{-\frac{2B^2r^2}{mR}t}\right)$$

At, $v = v_T$, we have $a = 0$. So, from equation (4), we get

$$v_T = \frac{mgR}{4B^2r^2}$$

PROBLEM 10

A wire $ACBOA$ shaped as a semi-circle ACB with the bounding diameter BOA $(OA = OB = a)$, has a total resistance R. It is placed in the same plane as infinitely long wire XY as shown in Figure.



A total charge Q flows through XY such that the current decreases linearly from an initial value I_0 to zero in time T, find the

(a) current in XY as a function of time.

(b) magnetic flux through $ACBOA$ when current in XY is I.

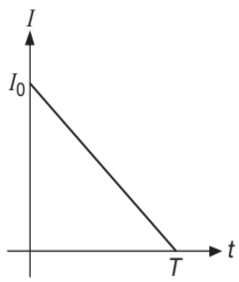
(c) induced current in $ACBOA$ as a function of time.

(d) heat generated in $ACBOA$ in time T.

(e) force acting on AOB as a function of time and its direction.

SOLUTION

(a) Variation of I with time is shown in the graph.



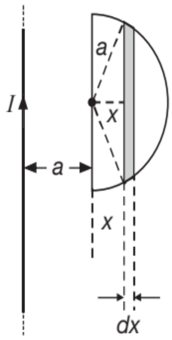
$$\begin{pmatrix}\text{Total}\\\text{charge}\end{pmatrix} = \begin{pmatrix}\text{Area of the}\\\text{triangle}\end{pmatrix} = \left(\frac{I_0 T}{2}\right) = (Q)$$

$$\Rightarrow \quad I_0 = \frac{2Q}{T}$$

Since, $I = I_0 - \frac{I_0}{T}t$

$$\Rightarrow \quad I = \frac{2Q}{T}\left(1 - \frac{t}{T}\right) \qquad \ldots(1)$$

(b) $\phi = \int \vec{B}\cdot d\vec{A}$, where



$$dA = 2\sqrt{a^2 - x^2}\,dx$$

$$\Rightarrow \quad \phi = \int_0^a \frac{\mu_0 I}{2\pi(a+x)}\left(2\sqrt{a^2-x^2}\,dx\right)$$

$$\Rightarrow \quad \phi = \frac{\mu_0 I a}{2\pi}(\pi - 2)$$

(c) Induced emf is given by

$$\xi = -\frac{d\phi}{dt} = -\frac{\mu_0 a(\pi-2)}{2\pi}\frac{dI}{dt}$$

$$\Rightarrow \quad \xi = \frac{\mu_0 a(\pi-2)}{2\pi}\left(\frac{2Q}{T^2}\right) \quad \left\{\because \frac{dI}{dt} = \frac{2Q}{T^2}, \text{ from (1)}\right\}$$

So, the induced current is $I_1 = \frac{\mu_0 a(\pi-2)}{\pi R}\frac{Q}{T^2}$

(d) $H = \int_0^T I_1^2 R\,dt$

$$\Rightarrow \quad H = I_1^2 RT$$

$$\Rightarrow \quad H = \frac{a^2\mu_0^2(\pi-2)^2Q^2}{\pi^2RT^3}$$

(e) Force on the straight part is given by $F = BI_1(2a)$

$$\Rightarrow \quad F = \frac{\mu_0}{2\pi a}II_1(2a), \text{ where } I = \frac{2Q}{T}\left(1 - \frac{t}{T}\right)$$

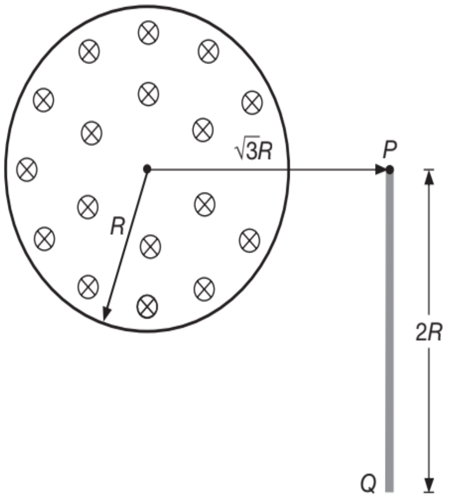
$$\Rightarrow \quad F = \frac{\mu_0}{2\pi a}\frac{2Q}{T}\left(1 - \frac{t}{T}\right)\frac{\mu_0 a(\pi-2)Q}{\pi RT^2}2a$$

$$\Rightarrow \quad F = \frac{\mu_0^2Q^2(\pi-2)}{\pi^2RT^3}\left(1 - \frac{t}{T}\right)2a$$

The force is directed towards the wire as the currents are parallel.

PROBLEM 11

A uniform but variable magnetic field of induction $B = \alpha + \beta t$ fills a cylindrical region of radius R. A conducting rod PQ of length $2R$ is pivoted at P situated at a horizontal distance $\sqrt{3}R$ from the centre of the magnetic field as shown in Figure.



At $t=0$, the rod starts rotating clockwise with constant angular velocity ω. Find the emf induced in the rod at the instant when end Q of the rod enters the magnetic field.

SOLUTION

Before we start the problem, we must note that the emf is induced in the rod due to the

(a) motion of the rod in magnetic field (ξ_1) and
(b) changing magnetic field (ξ_2)

EMF produced due to motion of the rod in field

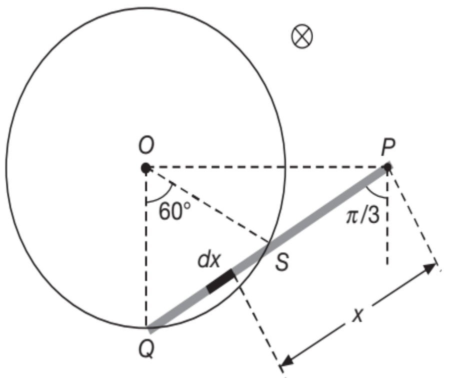
Just when the end Q enters the field, from simple geometry, we observe that ΔOSQ is equilateral. So,

$$QS = PS = R$$

$$\Rightarrow \quad \xi_1 = \int_R^{2R} B\omega x\,dx \qquad \{Q \text{ at higher potential}\}$$

$$\xi_1 = B\omega\left(\frac{3R^2}{2}\right) \qquad \ldots(1)$$

where, $B = \alpha + \beta\left(\dfrac{\pi}{3\omega}\right)$ ...(2)



EMF produced due to the changing magnetic field

Before calculating the induced emf, we must understand that a portion of rod is inside the field and a portion is outside it. At the same time we must not forget that the induced field has variable value from end Q of the rod till the end B.

Magnitude of electric field at the position of the small element in the magnetic field, E is given by

$$E(2\pi r) = \frac{dB}{dt}(\pi r^2)$$

$$\Rightarrow \quad E = \frac{r}{2}\frac{dB}{dt} = \frac{\beta r}{2}$$

If the element lies outside the magnetic field, then we have

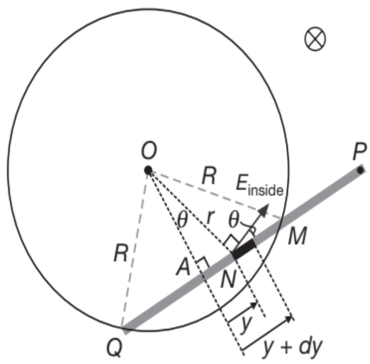
$$E'(2\pi r) = \frac{dB}{dt}(\pi R^2)$$

$$\Rightarrow \quad E' = \frac{\beta R^2}{2r}$$

Induced emf due to the inner portion in changing magnetic field

So, emf induced in the portion of wire inside the field is

$$\xi_{\text{inside}} = \int |d\xi_{\text{inside}}| = \left|\int_Q^M \vec{E}_{\text{inside}} \cdot d\vec{y}\right|$$



At the element, a distance r from O, E is $\perp$ to r and has a value $E_{\text{inside}} = \dfrac{r\beta}{2}$

$$\Rightarrow \quad \xi_{\text{inside}} = \left|\int_{-\frac{R}{2}}^{\frac{R}{2}} \frac{r\beta}{2}\,dy\cos\theta\right|$$

In triangle OAN $r\cos\theta = OA = \dfrac{\sqrt{3}R}{2}$

$$\Rightarrow \quad \xi_{\text{inside}} = \frac{\sqrt{3}\beta R}{4}\left|\int_{-\frac{R}{2}}^{\frac{R}{2}} dy\right| = \frac{\sqrt{3}\beta R^2}{4} \qquad \ldots(3)$$

Induced emf due to outer portion in the changing magnetic field

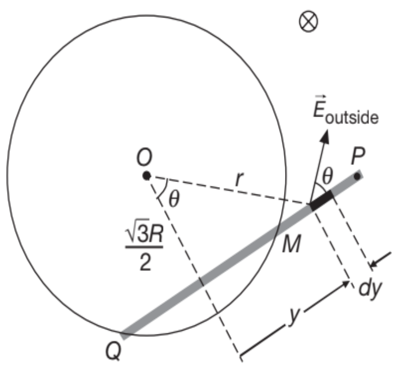
Again, emf induced in the portion of wire outside the field is

$$\xi_{\text{outside}} = \int |d\xi_{\text{outside}}| = \left|\int_M^P \vec{E}_{\text{outside}} \cdot d\vec{y}\right|$$

$$\Rightarrow \quad \xi_{\text{outside}} = \left|\int_M^P \frac{\beta R^2}{2r} dy \cos\theta\right|$$

$$\Rightarrow \quad \xi_{\text{outside}} = \left|\int_M^P \frac{\beta R^2}{2r}\left(\frac{\sqrt{3}R}{2r}\right) dy\right| \quad \left\{\because \cos\theta = \frac{\sqrt{3}R}{2r}\right\}$$

$$\Rightarrow \quad \xi_{\text{outside}} = \frac{\sqrt{3}bR^3}{4}\left|\int_{\frac{R}{2}}^{\frac{3R}{2}} \frac{dy}{r^2}\right|$$



But $r^2 = \dfrac{3R^2}{4} + y^2$

$$\Rightarrow \quad \xi_{\text{outside}} = \frac{\sqrt{3}bR^3}{4}\left|\int_{\frac{R}{2}}^{\frac{3R}{2}} \frac{dy}{\left(y^2 + \frac{3R^2}{4}\right)}\right|$$

Since $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow \quad \xi_{\text{outside}} = \frac{\sqrt{3}bR^3}{4\left(\frac{\sqrt{3}R}{2}\right)}\left[\tan^{-1}\left(\frac{y}{\frac{\sqrt{3}R}{2}}\right)\Bigg|_{\frac{R}{2}}^{\frac{3R}{2}}\right]$$

$$\Rightarrow \quad \xi_{\text{outside}} = \frac{bR^2}{2}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad \xi_{\text{outside}} = \frac{bR^2}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$\Rightarrow \quad \xi_{\text{outside}} = \frac{\pi bR^2}{12}$$

Hence total emf developed is

$$\xi = \xi_{\text{motional emf}} + \xi_{\text{changing } B}$$

$$\Rightarrow \quad \xi = \xi_{\text{motional emf}} = (\xi_{\text{inside portion}} + \xi_{\text{outside portion}})_{\text{changing } B}$$

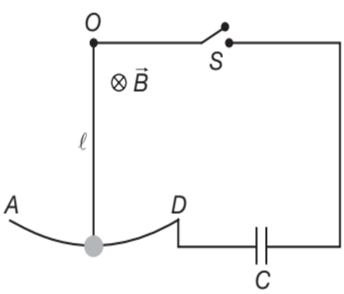
$$\Rightarrow \quad \xi = \xi_1 + (\xi_{\text{inside}} + \xi_{\text{outside}})$$

$$\Rightarrow \quad \xi = \frac{3B\omega R^2}{2} + \frac{\sqrt{3}\beta R^2}{4} + \frac{\pi\beta R^2}{12}$$

$$\Rightarrow \quad \xi = \frac{3B\omega R^2}{2} + \frac{\beta R^2}{4}\left(\sqrt{3} + \frac{\pi}{3}\right) \text{ where } B = \alpha + \frac{\pi\beta}{3\omega}$$

PROBLEM 12

A simple pendulum consists of a small conducting ball of mass m and a light conducting rod of length l. The pendulum oscillates with angular amplitude θ_0 in a vertical plane about a horizontal axis passing through O such that the ball remains always just in contact with a metallic strips AD bent into a circular arc of radius l as shown in Figure. A uniform magnetic field of induction B normal to plane of oscillation exists in the space. At time $t = 0$ when the ball is at its lowest position and moving towards right, the switch S is closed. Neglecting self-inductance of the circuit calculate external torque τ required to keep the pendulum oscillating as before. Assume that θ_0 is small.

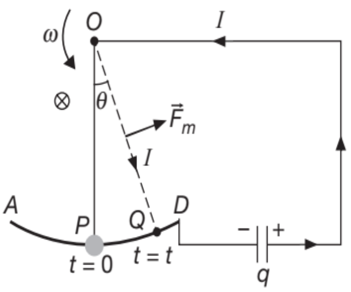


SOLUTION

At time t, the angular position of the rod is given by

$$\theta = \theta_0 \sin\left(\frac{2\pi}{T}\right)t \quad \ldots(1)$$

where $T = 2\pi\sqrt{\frac{l}{g}}$



Since $\omega = \frac{d\theta}{dt}$, so differentiating equation (1) w.r.t., time, we get

$$\omega = \frac{d\theta}{dt} = \left(\frac{2\pi\theta_0}{T}\right)\cos\left(\frac{2\pi t}{T}\right)$$

The potential difference between the ends of the rod at this instant is,

$$V_O - V_Q = V = \frac{1}{2}B\omega l^2$$

$$\Rightarrow \quad V_O - V_Q = \left(\frac{\pi\theta_0 l^2 B}{T}\right)\cos\left(\frac{2\pi t}{T}\right)$$

From right hand rule we can see that $V_O > V_Q$. This is also the potential difference across the capacitor. If q is the charge stored in the capacitor at this instant, then

$$q = CV = \left(\frac{\pi\theta_0 l^2 CB}{T}\right)\cos\left(\frac{2\pi t}{T}\right)$$

As the rod move towards D, its ω decreases (due to Lenz's Law). Hence, q will decrease. Thus, current in the circuit is anticlockwise, which is given by

$$I = -\frac{dq}{dt} = \left(\frac{2B\pi^2\theta_0 l^2 C}{T^2}\right)\sin\left(\frac{2\pi t}{T}\right)$$

In the rod it points from O to Q.
Magnetic force on the rod due to this induced current I is,

$$F_m = BIl = \left(\frac{2\pi^2\theta_0 B^2 l^3 C}{T^2}\right)\sin\left(\frac{2\pi t}{T}\right)$$

This force acts on the rod in the direction shown in Figure at the centre of the rod and perpendicular to it. Hence anticlockwise torque generated due to this force about point O is,

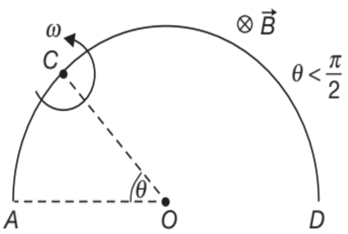
$$\tau = (F_m)\frac{l}{2} = \frac{\pi^2\theta_0 B^2 l^4 C}{T^2}\sin\left(\frac{2\pi t}{T}\right)$$

To keep the rod rotating as before and external clockwise torque of equal value has to be applied. So, we have

$$\tau_{\text{clockwise}} = \frac{\pi^2\theta_0 B^2 l^4 C}{T^2}\sin\left(\frac{2\pi t}{T}\right)$$

PROBLEM 13

A uniform wire of resistance x_0 per unit length is bent into a semicircle of radius $t = 0$. The wire rotates with angular velocity ω in a vertical plane about an axis passing through $\frac{dI}{dt}$. A uniform magnetic field $\frac{d\phi}{dt}$ exists in space in a direction perpendicular to paper inwards.



(a) Calculate potential difference between points I and ϕ. Which point is at higher potential?

(b) If points $t = T$ and AB are connected by a conducting wire of zero resistance find the potential difference between $2x_0$ and R.

SOLUTION

(a) Effect length of wire between A and C is

$$l_1 = 2a\sin\left(\frac{\theta}{2}\right)$$

Therefore, the motional emf (or potential difference) between points C and A is,

$$V_{CA} = V_C - V_A = \frac{1}{2}B\omega l_1^2$$

$$\Rightarrow \quad V_{CA} = 2a^2 B\omega\sin^2\left(\frac{\theta}{2}\right) \quad \ldots(1)$$

From right hand rule we observe that $V_C > V_A$

Similarly effective length of wire between C and D is length of straight wire CD is,

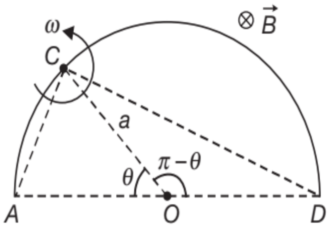
$$l_2 = 2a\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = 2a\cos\left(\frac{\theta}{2}\right)$$

Therefore, the motional emf (or potential difference) between points C and D is,

$$V_{CD} = V_C - V_D = \frac{1}{2}B\omega l_2^2$$

$$\Rightarrow V_{CD} = 2a^2B\omega\cos^2\left(\frac{\theta}{2}\right) \quad \ldots(2)$$

with $V_C > V_D$



From (1) and (2), we have

$$V_A - V_D = (V_C - V_D) - (V_C - V_A)$$

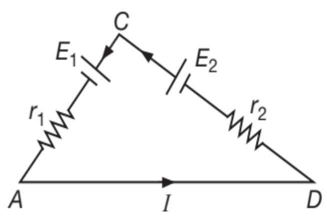
$$\Rightarrow V_A - V_D = 2a^2B\omega\left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]$$

Since, $\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos\theta$

$$\Rightarrow V_A - V_D = 2a^2B\omega\cos\theta$$

So, we observe A to be at higher potential.

(b) When A and D are connected from a wire, to make a complete circuit, then current starts flowing in the circuit and the potential difference between C and A will now have a value other than $2a^2B\omega\sin^2\left(\frac{\theta}{2}\right)$



Resistance between A and C is

$$r_1 = (\text{length of arc } AC)\ \lambda = a\theta\lambda$$

and between C and D is

$$r_2 = (\text{length of arc } CD)\ \lambda = (\pi-\theta)a\lambda$$

Now, the equivalent circuit can be drawn as shown in Figure

where, $E_1 = 2a^2B\omega\sin^2\left(\frac{\theta}{2}\right)$

and $E_2 = 2a^2B\omega\cos^2\left(\frac{\theta}{2}\right)$ with $E_2 > E_1$

The current in the circuit is,

$$I = \frac{E_2 - E_1}{r_1 + r_2} = \frac{2a^2B\omega\cos\theta}{\lambda\pi a} = \frac{2aB\omega\cos\theta}{\lambda\pi}$$

and potential difference between points C and A is,

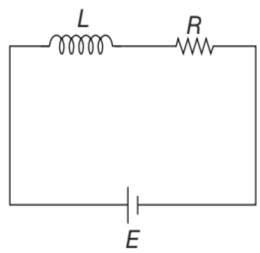
$$V'_{CA} = E_1 + Ir_1$$

$$\Rightarrow V'_{CA} = 2a^2B\omega\sin^2\left(\frac{\theta}{2}\right) + \left(\frac{2aB\omega\cos\theta}{\pi\lambda}\right)\lambda(a\theta)$$

$$\Rightarrow V'_{CA} = 2a^2B\omega\left[\sin^2\left(\frac{\theta}{2}\right) + \frac{\theta}{\pi}\cos\theta\right]$$

PROBLEM 14

A battery of emf E, negligible internal resistance is connected in an LR circuit as shown in Figure.



The inductor has a piece of soft iron inside it. When steady state is reached the piece of soft iron is abruptly pulled out suddenly so that the inductance of the inductor decreases to $3a$ with a with battery remaining connected. Calculate:

(a) current as a function of time assuming C at the instant when piece is pulled.

(b) the work done to pull out the piece.

(c) thermal power generated in the circuit as a function of time.

(d) power supplied by the battery as a function of time.

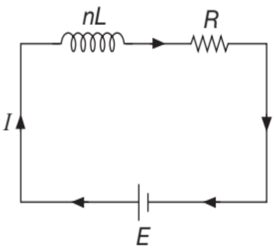
SOLUTION

Whenever the inductance of an inductor is abruptly changed, the flux passing through it remains constant.

$\phi = \text{constant}$

$\Rightarrow \quad LI = \text{constant} \qquad \left\{ L = \dfrac{\phi}{I} \right\}$

(a) At time $t=0$, steady state current in the circuit is $I_0 = \dfrac{E}{R}$. Suddenly L reduces to $nL(n<1)$, so current in the circuit at time $t=0$ will increase to $\dfrac{I_0}{n} = \dfrac{E}{nR}$. Let I be the current at time t, then applying Kirchhoff's Loop Rule we get,



$$E - nL\left(\frac{dI}{dt}\right) - IR = 0$$

$$\Rightarrow \frac{dI}{E-IR} = \frac{1}{nL}dt$$

$$\Rightarrow \int_{I_0/n}^{I} \frac{dI}{E-IR} = \frac{1}{nL}\int_0^t dt$$

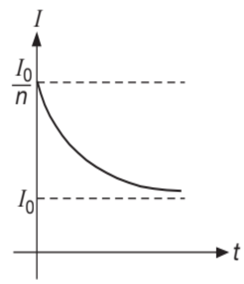
Solving this equation, we get

$$I = I_0 - \left(I_0 - \frac{I_0}{n}\right)e^{-t/\tau_L}$$

Here $I_0 = \dfrac{E}{R}$ and $\tau_L = \dfrac{nL}{R}$

From the I-t equation, we get $I = \dfrac{I_0}{n}$ at $t=0$ and $I = I_0$ as $t \to \infty$

The I-t graph is as shown in Figure.



(b) Work done to pull out the piece,

$$W = U_f - U_i = \frac{1}{2}L_f I_f^2 - \frac{1}{2}L_i I_i^2$$

$$\Rightarrow W = \frac{1}{2}(nL)\left(\frac{E}{nR}\right)^2 - \frac{1}{2}(L)\left(\frac{E}{R}\right)^2$$

$$\Rightarrow W = \frac{1}{2}L\left(\frac{E}{R}\right)^2\left(\frac{1}{n}-1\right)$$

$$\Rightarrow W = \frac{1}{2}L\left(\frac{E}{R}\right)^2\left(\frac{1-n}{n}\right)$$

(c) Thermal power generated in the circuit as a function of time is,

$P_1 = I^2R$

where I the current calculated in part (a)

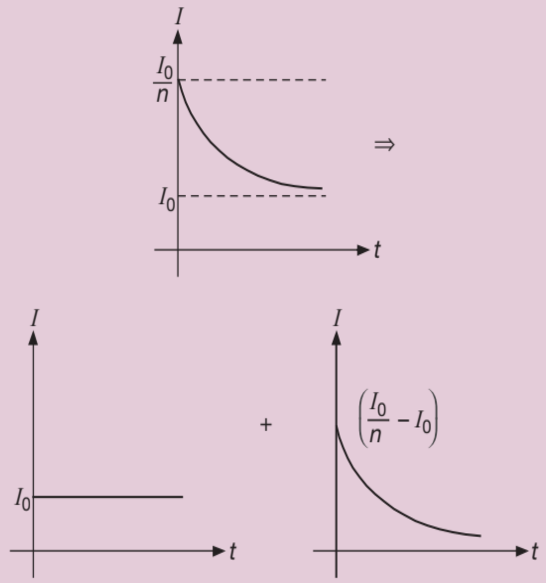
(d) Power supplied by the battery (as a function of time) is,

$P_2 = EI$

Problem Solving Technique(s)

We also observe that at $t = 0$, current in the circuit is $\dfrac{I_0}{n}$ and current in the circuit in steady state will again be I_0. So, it will decrease exponentially from $\dfrac{I_0}{n}$ to I_0.

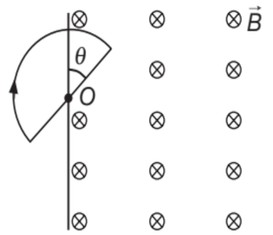
From the *m* graph the equation can be formed without doing any calculation.



$$I = I_0 + \left(\frac{I_0}{n} - I_0\right)e^{-t/T_L}$$

PROBLEM 15

A wire loop enclosing a semicircle of radius R is located on the boundary of a uniform magnetic field B. At the moment $t = 0$, the loop is set into rotation with a constant angular acceleration α about an axis O coinciding with a line of vector $\vec{B}$ on the boundary. Find the emf induced in the loop as a function of time. Draw the approximate plot of this function. The arrow in the Figure shows the emf direction taken to be positive.

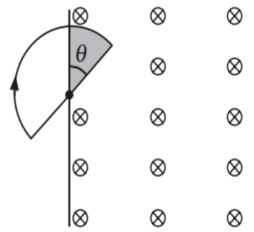


SOLUTION

Using the equation of rotational kinematics, we get $\theta = \frac{1}{2}\alpha t^2$, where time taken to rotate through an angle θ is $t = \sqrt{\frac{2\alpha}{\theta}}$

Also, we observe that, for $\theta = 0$ to π, 2π to 3π, 4π to 5π etc., then the flux associated with the loop increases, hence, current in the loop is anticlockwise, or induced emf is negative.

And for, $\theta = \pi$ to 2π, 3π to 4π, 5π to 6π etc., the flux associated with the loop decreases, hence, current in the loop is clockwise, or emf is positive.



So, let the

time taken to rotate through an angle π be $t_1 = \sqrt{\frac{2\pi}{\alpha}}$

time taken to rotate through an angle 2π be $t_2 = \sqrt{\frac{4\pi}{\alpha}}$

… … … … …

time taken to rotate through an angle $n\pi$ be $t_n = \sqrt{\frac{2n\pi}{\alpha}}$

Then for 0 to t_1 emf is negative

t_1 to t_2 emf is positive

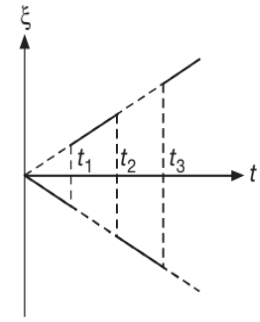
t_2 to t_3 emf is again negative

and so on.

Now, at time t, angle traversed is, $\theta = \frac{1}{2}\alpha t^2$

Area of the loop inside the field is $A = \frac{1}{2}R^2\theta$

$\Rightarrow \quad A = \frac{1}{4}R^2\alpha t^2$



So, flux associated with the loop is

$$\phi = BA = \frac{1}{4}BR^2\alpha t^2$$

$$\xi = \left|\frac{d\phi}{dt}\right| = \frac{1}{2}BR^2\alpha t$$

$$\xi \propto t$$

i.e., ξ-t graph is a straight line passing through origin. ξ-t equation with sign can be written as,

$$\xi = (-1)^n\left(\frac{1}{2}BR^2\alpha t\right)$$

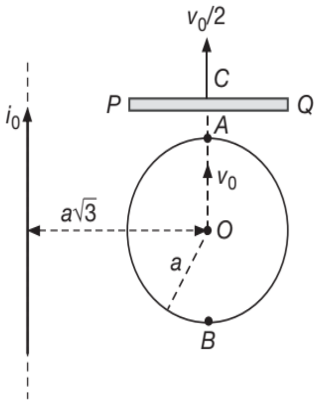
Here $n = 1, 2, 3\ldots$ is the number of half revolutions that the loop performs at the given moment t.

PROBLEM 16

A conducting circular loop of radius a and resistance per unit length λ is moving with a constant velocity v_0, parallel to an infinite conducting wire carrying current i_0. A conducting rod of length $2a$ is approaching the centre of the loop with a constant velocity $\frac{v_0}{2}$ along the direction of the current. At the instant $t = 0$, the rod comes in contact with the loop at A and starts sliding on the loop with the constant velocity. Neglect the resistance of the rod and the self-inductance of the circuit. As the rod slides on the loop, find the

(a) current through the rod when it is at a distance of $\left(\frac{a}{2}\right)$ from the point A of the loop.

(b) force required to maintain the velocity of the rod at that instant.

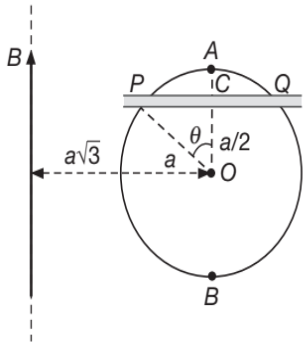


SOLUTION

(a) At the given instant, we have

$$AC=\left(\frac{a}{2}\right),\ OC=\left(\frac{a}{2}\right)\text{ and }\cos\theta=\frac{a/2}{a}=\frac{1}{2}$$

$$\Rightarrow\ \theta=\frac{\pi}{3}$$



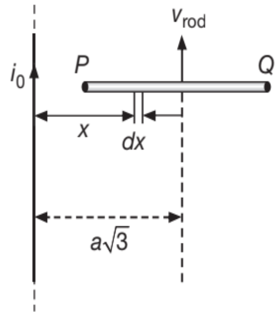
Since the velocity of rod is

$V_{\text{rod}}=\left(+\frac{v_0}{2}\right)$, along the direction of current, so the emf induced across the ends P and Q is

$$\xi_{\text{rod}}=\int_{a\sqrt{3}-\frac{a\sqrt{3}}{2}}^{a\sqrt{3}+\frac{a\sqrt{3}}{2}}Bv_{\text{rod}}\,dx$$

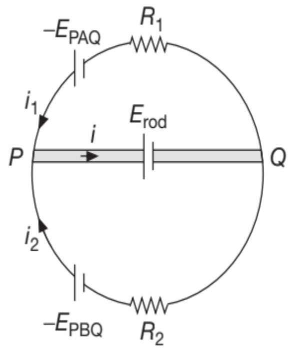
$$\Rightarrow\ \xi_{\text{rod}}=\int_{a\frac{\sqrt{3}}{2}}^{3a\frac{\sqrt{3}}{2}}\frac{\mu_0 i_0 v_0}{2\pi x}dx$$

$$\Rightarrow\ \xi_{\text{rod}}=\left(\frac{v_0}{2}\right)\frac{\mu_0 i_0}{2\pi}\log_e(3),$$ with end P at higher potential.



Since the effective length of both the arcs PAQ and PBQ is PQ, so

$$\xi_{PAQ}=\xi_{PBQ}=\frac{\mu_0 i_0 v_0}{4\pi}\log_e(3)$$



$$\Rightarrow\ \xi_{PAQ}=\frac{\mu_0 i_0 v_0}{2\pi}\log_e(3)$$

with point P at higher potential

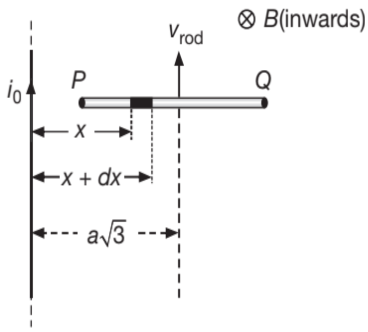
Resistance of arc PAQ is

$$R_1=(\lambda)2(a\theta)=2a\lambda\frac{\pi}{3}$$

Resistance of arc PBQ is

$$R_2=(\lambda)a(2\pi-2\theta)=4a\lambda\frac{\pi}{3}$$

Equivalent circuit at the given instant is shown in Figure.



Current through the rod PQ is

$$i=(i_1+i_2)=\left(\frac{\xi_{PAQ}-\xi_{\text{rod}}}{R_1}\right)+\left(\frac{\xi_{PBQ}-\xi_{\text{rod}}}{R_2}\right)$$

$$\Rightarrow\ i=(\xi_{PAQ}-\xi_{\text{rod}})\left[\frac{1}{R_1}+\frac{1}{R_2}\right]$$

$$\Rightarrow\ i=\frac{v_0\mu_0 i_0\log_e(3)}{4\pi}\left[\left(\frac{1}{2}+\frac{1}{4}\right)\frac{3}{a\lambda\pi}\right]$$

$$\Rightarrow\ i=\frac{9v_0 i_0}{16a\lambda\pi^2}\log_e(3)$$

(b) Force on the rod

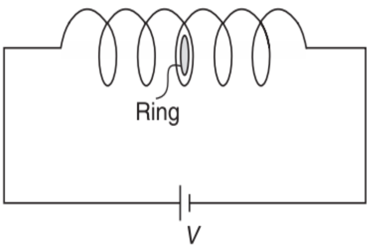
$$F_{\text{rod}}=\int_{\frac{a\sqrt{3}}{2}}^{\frac{3a\sqrt{3}}{2}} Bi\,dx$$

$$\Rightarrow\ F_{\text{rod}}=\frac{\mu_0 i i_0}{2\pi}\log_e(3)$$

$$\Rightarrow\ F_{\text{rod}}=\frac{9\mu_0^2 i_0^2 v_0}{32a\lambda\pi^3}(\log_e(3))^2$$

PROBLEM 17

A thin wire ring of radius a and resistance r is located inside a long solenoid so that their axes coincide. The length of the solenoid is equal to l its cross-section radius to b. At a certain moment, the solenoid was connected to a source of constant voltage V. The total resistance of the circuit is equal to R. Assuming the inductance of the ring to be negligible, find the maximum value of the radial force acting per unit length of the ring.



SOLUTION

The inductance L of the solenoid is

$$L=\mu_0 n^2 Al=\mu_0 n^2(\pi b^2)l \quad \ldots(1)$$

The current through the solenoid varies with time as

$$I=\frac{V}{R}\left(1-e^{-\frac{Rt}{L}}\right)$$

Magnetic field inside the solenoid is

$$B=\mu_0 nI$$

So, magnetic flux associated with the ring is

$$\phi=B(\pi a^2)$$

$$\Rightarrow\ \phi=(\mu_0 n\pi a^2)I$$

$$\Rightarrow\ \phi=\frac{\mu_0 nV\pi a^2}{R}\left(1-e^{-\frac{Rt}{L}}\right)$$

EMF induced in the ring is given as

$$|\xi|=\frac{d\phi}{dt}$$

$$\Rightarrow\ \xi=\frac{\mu_0 nV\pi a^2}{L}e^{-\frac{Rt}{L}}$$

The current induced (i) in the ring is given by

$$i=\frac{\xi}{r}=\frac{\mu_0 nV\pi a^2}{rL}e^{-\frac{Rt}{L}}$$

If dF_r be the radial force acting on a section of length dl of the ring, then we have

$$dF_r=Bi(dl)$$

So, radial force per unit length of the ring is

$$f_r=\frac{dF_r}{dl}=Bi$$

$$\Rightarrow\ f_r=\mu_0 nI\left(\frac{\mu_0 nV\pi a^2}{rL}e^{-\frac{Rt}{L}}\right)$$

$$\Rightarrow\ f_r=\left(\frac{\mu_0^2 n^2\pi a^2 V}{rL}e^{-\frac{Rt}{L}}\right)\left(\frac{V}{R}\left(1-e^{-\frac{Rt}{L}}\right)\right)$$

$$\Rightarrow\ f_r=\frac{\mu_0^2 n^2\pi a^2 V^2}{rRL}e^{-\frac{Rt}{L}}\left(1-e^{-\frac{Rt}{L}}\right) \quad \ldots(2)$$

Substituting the value of L from equation (1) in equation (2), we get

$$f_r=\frac{\mu_0 a^2 V^2}{rRb^2 l}e^{-\frac{Rt}{L}}\left(1-e^{-\frac{Rt}{L}}\right) \quad \ldots(3)$$

From equation (3), we see that at $t=0$, $f_r=0$ and also when $t\to\infty$, then also $F_r=0$.

So, at some intermediate time, the value of radial force per unit length should be maximum i.e.

$$\frac{df_r}{dt}=0$$

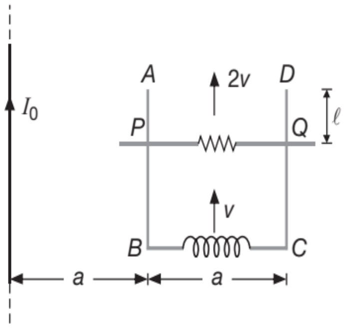
$\Rightarrow \quad e^{-\frac{Rt}{L}} = \frac{1}{2}$

Hence, maximum value of F_r is given by

$$F_{max} = \frac{\mu_0 a^2 V^2}{4rRlb^2}$$

PROBLEM 18

A conducting U-frame $ABCD$ (having an inductance L) and a conducting rod PQ (capable of sliding on the U-frame) of resistance R, start moving with velocities v and $2v$ respectively, parallel to a long wire carrying current I_0 as shown in Figure.



When the distance $AP = l$ at $t = 0$, determine the current through the inductor just before the rod PQ losses contact with the U-frame. Assume that no inductor and resistor other than L and R are present.

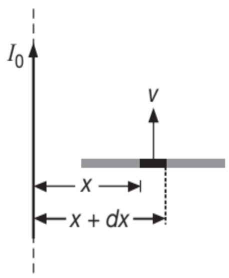
SOLUTION

Since PQ and BC both cut the field lines, so motional emf will be induced across both of them.

$$\Rightarrow \quad \xi = \int d\xi = \int_a^{2a} v\left(\frac{\mu_0 I_0}{2\pi x}\right)dx$$

$$\xi_{BC} = \frac{\mu_0 I_0 v}{2\pi}\log_e(2) \text{ with } B \text{ at higher potential}$$

$$\xi_{PQ} = \frac{2\mu_0 I_0 v}{2\pi}\log_e(2) \text{ with } P \text{ at higher potential}$$



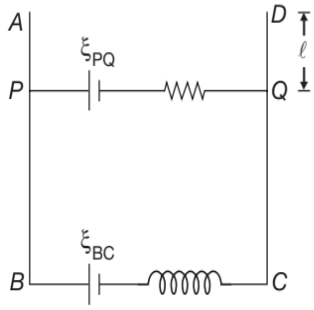
The relative velocity of the rod PQ w.r.t., U frame

$$v_{rel} = 2v - v = v$$

So, the time taken by the to loose the contact is $t = \frac{l}{v}$

From equivalent electrical network, as shown, we have then net emf in the closed loop $QPBC$

$$\xi = \xi_{PQ} - \xi_{BC} = \frac{\mu_0 I_0 v}{2\pi}\log_e(2)$$



Growth of current in the LR circuit is given by

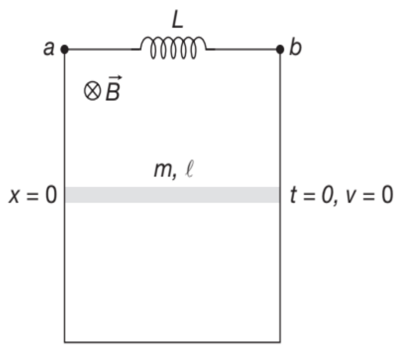
$$I = I_0\left(1 - e^{-Rt/L}\right) = \left(\frac{\xi}{R}\right)\left(1 - e^{-Rt/L}\right)$$

At $\quad t = \frac{l}{v}$, we have

$$I = \left(\frac{\xi}{R}\right)\left(1 - e^{-Rl/Lv}\right)$$

PROBLEM 19

A coil of inductance L connects the upper ends of two vertical copper bars separated by a distance l. A horizontal conducting connector of mass m starts falling with zero initial velocity along the bars without losing contact with them. The whole system is located in a uniform magnetic field B perpendicular to the plane of the bars. Find the law of motion $x(t)$ of the connector.



SOLUTION

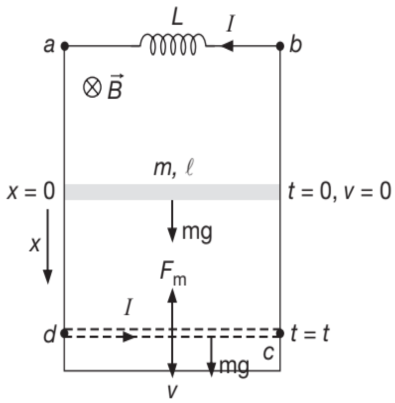
Let the connector be released from rest at $t = 0$ from the position $x = 0$. At time t, let the connector be at position x and having a velocity v (downwards). Equivalent circuit diagram of the arrangement is shown below. For the loop $abcda$, we have

$$\Rightarrow \quad -L\frac{dI}{dt} + Blv = 0$$

$$\Rightarrow \quad LdI = Bldx \qquad \left\{\because v = \frac{dx}{dt}\right\}$$

Integrating, we get $LI = Blx$

$$\Rightarrow \quad I = \left(\frac{Bl}{L}\right)x \qquad \ldots(1)$$



Net force on the connector at this instant is,

$$F_{\text{net}} = mg - F_m$$

where $F_m = BIl = \left(\frac{B^2l^2}{L}\right)x$ {downwards}

$$\Rightarrow \quad F_{\text{net}} = mg - \frac{B^2l^2}{L}x \qquad \ldots(2)$$

Since $\vec{F}_{\text{net}} \parallel \vec{v}$, so we get

$$m\left(\frac{dv}{dt}\right) = mg - \left(\frac{B^2l^2}{mL}\right)x$$

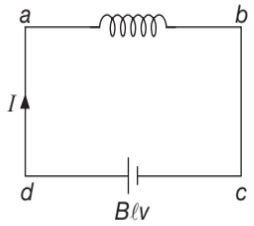
$$\Rightarrow \quad \frac{dv}{dt} = g - \left(\frac{B^2l^2}{L}\right)x$$

Differentiating this equation w.r.t. time we get,

$$\frac{d^2v}{dt^2} = -\left(\frac{B^2l^2}{mL}\right)\frac{dx}{dt} = -\left(\frac{B^2l^2}{mL}\right)v$$

$$\Rightarrow \quad \frac{d^2v}{dt^2} = -\omega^2 v \qquad \ldots(3)$$

$$\Rightarrow \quad \frac{d^2v}{dt^2} + \omega^2 v = 0, \text{ where } \omega = \frac{Bl}{\sqrt{mL}}$$



From equation (3) we can conclude that, v oscillates simple harmonically with angular frequency $\omega = \frac{Bl}{\sqrt{mL}}$.

Further, at $t = 0$, $v = 0$ and after sometime it is positive. So, we can write,

$$v = v_0 \sin(\omega t) \qquad \ldots(4)$$

At $t = 0$, we have $\frac{dv}{dt} = g$

$$\Rightarrow \quad v_0\omega = g$$

$$\Rightarrow \quad v_0 = \frac{g}{\omega}$$

$$\Rightarrow \quad v_0 = \frac{g\sqrt{mL}}{Bl} \qquad \ldots(5)$$

Substituting the value of v_0 from (5) in equation (4), we get

$$v = v_0 \sin(\omega t) \text{ where } v_0 = \frac{g\sqrt{mL}}{Bl} \text{ and } \omega = \frac{Bl}{\sqrt{mL}}$$

$$\Rightarrow \quad \frac{dx}{dt} = v_0 \sin(\omega t)$$

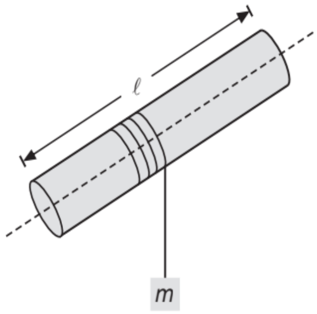
$$\Rightarrow \quad \int_0^x dx = \int_0^t v_0 \sin(\omega t)\,dt$$

$$\Rightarrow \quad x = \frac{v_0}{\omega}(-\cos(\omega t))\Big|_0^t$$

$$\Rightarrow \quad x = \frac{v_0}{\omega}(1 - \cos\omega t)$$

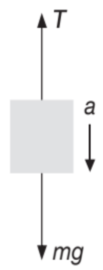
PROBLEM 20

A long insulating cylinder of radius R and length l carries a uniformly distributed surface charge q. A string is wound around the cylinder from which a block of mass m hangs. The mass is free to move downwards and can rotate the cylinder. Neglecting the moment of inertia of the cylinder, calculate the acceleration of the block. What will be the acceleration, when $m \to 0$ and when $q \to 0$?



SOLUTION

If there were no charge on the cylinder, tension must have been zero, because moment of inertia of the cylinder is zero, so no torque is required for its rotation. In this case acceleration of the block would had been g downwards. But due to rotating charge on the cylinder a magnetic field will appear. Further due to angular acceleration of the cylinder this magnetic field will be changing. The change in magnetic field (and hence, the magnetic flux) produces an electric field. So, the cylinder will experience a torque due to electric force on the charge over it. Hence, tension in the string will not be zero, or acceleration of block will be less than g.



Let a be the acceleration of the block downwards and T the tension in the string.
The equation of motion of the block is,

$$mg - T = ma \quad \ldots(1)$$

If the system is released from rest, the velocity of block at time t is

$$v = at$$

If ω is the angular velocity of cylinder at this instant, then

$$\omega = \frac{v}{R} = \frac{at}{R}$$

If f is the frequency, then $f = \dfrac{\omega}{2\pi} = \dfrac{at}{2\pi R}$

The charge q on the surface of the cylinder spins with it so that the effective current over the surface of the cylinder is,

$$I = \frac{q}{T} = qf = \frac{qat}{2\pi R}$$

The cylinder can now be treated as a solenoid with one turn, carrying a current I. The number of turns per unit length is,

$$n = \frac{1}{l}$$

The magnetic field on the axis of the cylinder (or the equivalent solenoid) is

$$B = \mu_0 nI = \frac{\mu_0 qat}{2\pi Rl}$$

$$\Rightarrow \quad \frac{dB}{dt} = \frac{\mu_0 qa}{2\pi Rl}$$

This change in magnetic field induces an electric field E at the surface of the cylinder. The magnitude of this electric field is found by using

$$\left| \oint \vec{E} \cdot d\vec{l} \right| = \left| A \frac{d\vec{B}}{dt} \right|$$

$$\Rightarrow \quad E(2\pi R) = \pi R^2 \left(\frac{dB}{dt} \right)$$

$$\Rightarrow \quad E = \frac{R}{2} \frac{dB}{dt} = \frac{\mu_0 qa}{4\pi l}$$

This electric field interacts with the charge on the surface of the cylinder and causes a torque, whose magnitude is given by

$$\tau = FR = (qE)R = \frac{\mu_0 q^2 Ra}{4\pi l}$$

From Lenz's Law direction of this torque is to oppose the motion of the cylinder. Further, moment of inertia of the cylinder is zero. Hence, net torque on the cylinder should be zero.
So, torque due to tension should balance this torque, or

$$TR = \frac{\mu_0 q^2 Ra}{4\pi l}$$

$$\Rightarrow \quad T = \frac{\mu_0 q^2 a}{4\pi l}$$

Substituting in equation (1), we get

$$mg - \frac{\mu_0 q^2 a}{4\pi l} = ma$$

$$\Rightarrow \quad a = \frac{g}{\left(1 + \frac{\mu_0 q^2}{4\pi m l}\right)}$$

From this expression we can see that, $a = 0$ if $m = 0$ and $a = g$ if $q = 0$.

PROBLEM 21

A metal disc of radius $R = 25$ cm rotates with a constant angular velocity $\omega = 130$ rads^{-1} about its axis. Find the potential difference between the centre and rim of the disc if the external magnetic field is

(a) absent.

(b) uniform and has a value $B = 5$ mT directed perpendicular to the disc.

SOLUTION

(a) Consider an electron inside the disc at a distance r from the axis. For the electron to move along a circle, there should be a force pulling it to the axis. According to Newton's Second Law,

$$F = mr\omega^2$$

where m = mass of electron $= 9.1 \times 10^{-31}$ kg

This force is generated by a radial electric field caused by the redistribution of the electrons in the disc and in such a way that the force acting on the electron is,

$$F = eE = mr\omega^2 \qquad \{e = 1.6 \times 10^{-19}\text{ C}\}$$

$$\Rightarrow \quad E = \frac{mr\omega^2}{e}$$

Since, $dV = -Edr$, so we get

$$dV = -\frac{m\omega^2}{e} rdr$$

$$\Rightarrow \quad \int_{V_1}^{V_2} dV = -\frac{m\omega^2}{e}\int_0^R rdr$$

$$\Rightarrow \quad V_1 - V_2 = \frac{m\omega^2 R^2}{2e} \qquad \ldots(1)$$

$V_1 > V_2$, i.e., potential at centre is more than the potential at edge.

Substituting the values in equation (1) we get

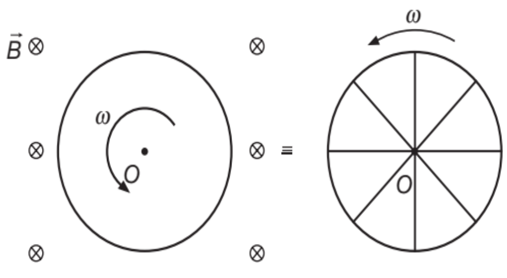
$$V_1 - V_2 = \frac{(9.1\times10^{-31})(130)^2(0.25)^2}{(2)(1.6\times10^{-19})}\text{ V}$$

$$\Rightarrow \quad V_1 - V_2 = 3\times10^{-9}\text{ V}$$

$$\Rightarrow \quad V_1 - V_2 = 3\text{ nV}$$

(b) A disc may be assumed to be made up of a large number of thin conducting rods, all rotating with same angular velocity ω about the centre of disc O. Thus,

$$|V_{centre} - V_{edge}| = \frac{1}{2}BR^2\omega$$



If the disc is rotating anticlockwise, $V_{centre} > V_{edge}$ and if it is rotating clockwise $V_{edge} > V_{centre}$

$$|V_{centre} - V_{edge}| = \frac{1}{2}BR^2\omega$$

Substituting the values, we have

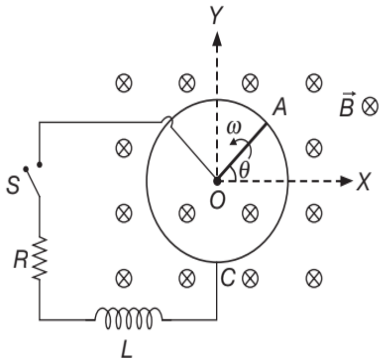
$$|V_{centre} - V_{edge}| = \frac{1}{2}\times5\times10^{-3}\times0.25\times0.25\times130$$

$$\Rightarrow \quad |V_{centre} - V_{edge}| = 0.02\text{ V}$$

$$\Rightarrow \quad |V_{centre} - V_{edge}| = 20\text{ mV}$$

PROBLEM 22

A metal rod OA and mass m and length r kept rotating with a constant angular speed ω in a vertical plane about 1 horizontal axis at the end O. The free end A is arranged to slide without friction along a fixed conducting circular ring in the same plane as that of rotation. A uniform and constant magnetic induction $\vec{B}$ is applied perpendicular and into the plane of rotation as shown in Figure. An inductor L and an external resistance R are connected through a switch S between the point O and a point C on the ring to form an electrical circuit. Neglect the resistance of the ring and the rod. Initially, the switch is open.



(a) What is the induced emf across the terminals of the switch?

(b) The switch S is closed at time $t=0$. Obtain an expression for the current as a function of time. In the steady state, obtain the time dependence of the torque required to maintain the constant angular speed. Given that the rod OA was along the positive x-axis at $t=0$.

SOLUTION

(a) Consider a small element of length dx of the rod OA situated at a distance x from O Speed of element, $v=x\omega$

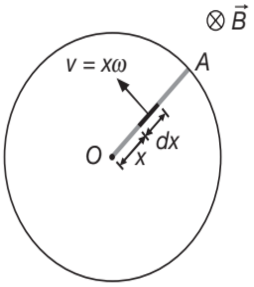
Therefore, induced emf developed across this element in uniform magnetic field B

$$d\xi=(B)(x\omega)dx \qquad \{\because \xi=Blv\}$$

Hence, total induced emf across OA, is

$$\xi=\int_{x=0}^{x=r} d\xi=\int_0^r B\omega x dx=\frac{B\omega r^2}{2}$$

(b) A constant emf or potential difference $\xi=\frac{B\omega r^2}{2}$ is induced across O and A, with A at lower potential and O at higher potential and the equivalent circuit is redrawn in Figure



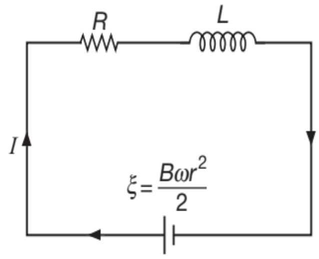
Switch S is closed at time t = 0 Therefore, it is case of growth of current in a series LR circuit. Current, I, at any time t is given by

$$I=I_o\left(1-e^{-t/\tau_L}\right), \text{ where } I_o=\frac{\xi}{R}=\frac{B\omega r^2}{2R}$$

and $\tau_L=\frac{L}{R}$

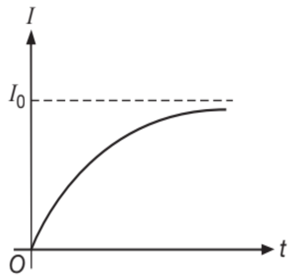
$$\Rightarrow \ I=\frac{B\omega r^2}{2R}\left[1-e^{-\left(\frac{R}{L}\right)t}\right]$$

The I-t graph is shown in Figure



At constant angular speed, net torque is $\tau=0$

The steady state current will be $I=I_o=\frac{B\omega r^2}{2R}$



From right hand rule we can see that this current would be inwards (from circumference to centre) and corresponding magnetic force F_m will be in the direction opposite to motion of the conductor and its magnitude is given by

$$F_m=BIr=\frac{B^2\omega r^3}{2R}$$

Torque of this force about centre O is

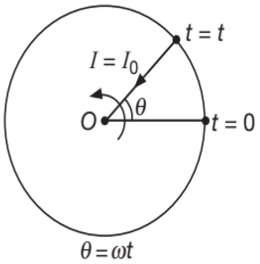
The torque due to the magnetic force about O is

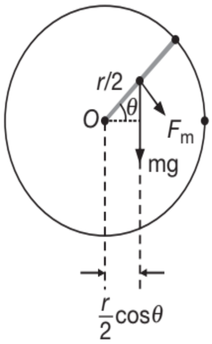
$$\tau_{F_m}=F_m\left(\frac{r}{2}\right)=\frac{B^2\omega r^4}{4R} \quad \text{(clockwise)}$$

Similarly, torque due to weight (mg) about centre O is

$$\tau_{mg}=mgr_\perp=(mg)\left(\frac{r}{2}\cos\theta\right)$$

$$\Rightarrow \ \tau_{mg}=\frac{mgr}{2}\cos(\omega t) \qquad \{\text{clockwise}\}$$





Therefore, net torque at any time t (after steady state condition is achieved) about centre O will be

$$\tau_{\text{net}} = \tau_{F_m} + \tau_{mg}$$

$$\Rightarrow \tau_{\text{net}} = \frac{B^2\omega r^4}{4R} + \frac{mgr}{2}\cos(\omega t) \qquad \{\text{clockwise}\}$$

Hence, the external torque applied to maintain a constant angular speed is

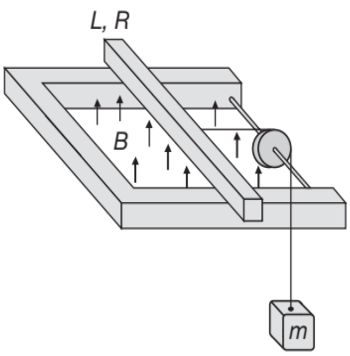
$$\tau_{ext} = \frac{B^2\omega r^4}{4R} + \frac{mgr}{2}\cos(\omega t)$$

{but in anticlockwise direction}

We must note that for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, torque due to weight will be anticlockwise, the sign of which is automatically adjusted because for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, $\cos\theta$ is negative.

PROBLEM 23

A pair of parallel horizontal conducting rails of negligible resistance shorted at one end is fixed on a table. The distance between the rails is L. A conducting massless rod of resistance R can slide on the rails without any friction. The rod is tied to a massless string which passes over a pulley fixed to the edge of the table. A mass m tied to the other end of the string hangs vertically. A constant magnetic field B exists perpendicular to the table. If the system is released from rest. Calculate



(a) the terminal velocity achieved by the rod and

(b) the acceleration of the mass at the instant when the velocity of the rod is half the terminal velocity.

SOLUTION

(a) Let v be the velocity of the wire (as well as block) at any instant of time t. So, motional emf is given by

$$\xi = BLv$$

Induced current I is given by

$$I = \frac{\xi}{R} = \frac{BLv}{R}$$

Due to this induced current, a magnetic force (opposing the motion of rod) acts on it. This magnetic force, F_m is given by

$$F_m = BIL = \frac{B^2L^2v}{R}$$

Net force on the system at this moment is

$$F_{\text{net}} = mg - F_m = mg - \frac{B^2L^2v}{R}$$

$$\Rightarrow ma = mg - \frac{B^2L^2v}{R}$$

$$\Rightarrow a = g - \frac{B^2L^2v}{mR} \qquad \ldots(1)$$

Rod will acquire its terminal value, $v = v_T$ when F_{net} becomes zero or acceleration (a) of the particle becomes zero.

$$\Rightarrow 0 = g - \frac{B^2L^2v_T}{mR}$$

$$\Rightarrow v_T = \frac{mgR}{B^2L^2}$$

(b) When $v = \frac{v_T}{2} = \frac{mgR}{2B^2L^2}$

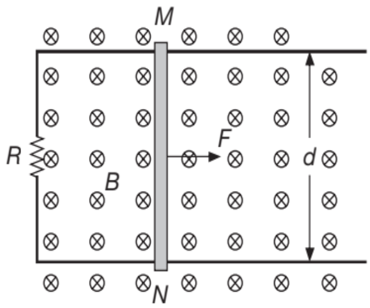
Then from equation (1), acceleration of the block is

$$a = g - \left(\frac{mgR}{2B^2L^2}\right)\left(\frac{B^2L^2}{mR}\right) = g - \frac{g}{2}$$

$$\Rightarrow a = \frac{g}{2}$$

PROBLEM 24

Two long parallel horizontal rails, a distance d apart and each having a resistance per unit length λ, are joined at one end by a resistance R. A perfectly conducting rod MN of mass m is free to slide along the rails without friction as shown in Figure.



There is a uniform magnetic field of induction B normal to the plane of the paper and directed into the paper. A variable force F is applied to the rod MN such that, as the rod moves, a constant current flows through R.

(a) Find the velocity of the rod and the applied force F as functions of the distance x of the rod from R.

(b) What fraction of the work done per second by F is converted into heat?

SOLUTION

(a) Let at any time t, the rod MN be at a distance x from the resistance R. Velocity of rod at that instant is

$$v = \frac{dx}{dt}$$

Motional emf $\xi = Bvd$ ...(1)

Total resistance R_{net} of the circuit at this instant is given by

$$R_{net} = R + 2\lambda x$$

So, the induced current I is

$$I = \frac{\xi}{R_{net}} = \frac{Bvd}{R+2\lambda x} \quad \{\text{anticlockwise}\}$$

$$\Rightarrow v = \frac{I(R+2\lambda x)}{Bd}$$

Instantaneous acceleration of the rod, is

$$a = \frac{dv}{dt} = \left(\frac{2\lambda I}{Bd}\right)\left(\frac{dx}{dt}\right) = \left(\frac{2\lambda I}{Bd}\right)v \quad \left\{\because \frac{dx}{dt} = v\right\}$$

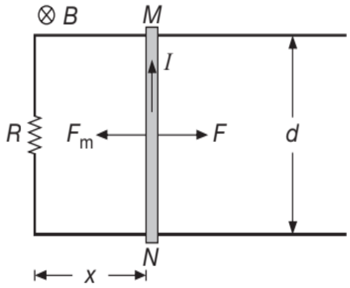
$$\Rightarrow a = \frac{2\lambda I^2}{B^2 d^2}(R+2\lambda x) \quad \text{...(2)}$$

At this moment two forces are acting on the rod, applied force F (towards right) and magnetic force $F_m = BId$ (towards left). So

$$F_{net} = F - F_m$$

$$\Rightarrow ma = F - BId$$

$$\Rightarrow F = BId + \frac{2m\lambda I^2}{B^2 d^2}(R+2\lambda x)$$



(b) Work done by F per unit time is given by

$$P_1 = Fv = \frac{FI(R+2\lambda x)}{Bd} \quad \text{...(3)}$$

and heat generated in circuit per unit time is

$$P_2 = I^2 R_{net} = I^2(R+2\lambda x) \quad \text{...(4)}$$

So, if the fraction of work converted into heat is denoted by f, then we have

$$f = \frac{P_2}{P_1} = \frac{BId}{F}$$

Substituting value of F we get

$$f = \frac{1}{\left(1 + \dfrac{2mI\lambda(R+2\lambda x)}{B^3 d^3}\right)}$$

IMPORTANT OBSERVATION

(a) The remaining part of work done goes as the change in kinetic energy of rod per time i.e.,

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mv\left(\frac{dv}{dt}\right) = (mv)(a)$$

Substituting the values

$$\frac{dK}{dt} = \frac{2m\lambda I^3}{B^3 d^3}(R+2\lambda x)^2 \quad \text{...(5)}$$

From equations (3), (4) and (5) we can see that

$$P_1 = P_2 + \frac{dK}{dt}$$

(b) Also, if the Question would have said "net variable force F", instead of "variable force F", then there would be no need to take F_m, as it has already been included in F (the net variable force)

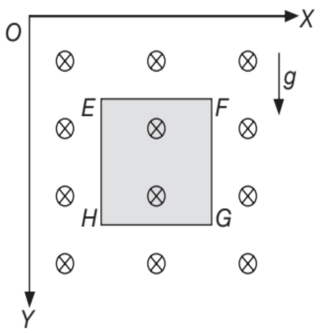
PROBLEM 25

A magnetic field $\vec{B}=\left(\frac{B_0 y}{a}\right)\hat{k}$ is acting into the paper in the $+z$ direction. B_0 and a are positive constants. A square loop $EFGH$ of side a, mass m and resistance R in x-y plane starts falling under the influence of gravity. Note the directions of x and y in the Figure. Find

(a) the induced current in the loop and indicate its direction.

(b) the total Lorentz force acting on the loop and indicates its direction.

(c) an expression for the speed of the loop $v(t)$ and its terminal velocity.

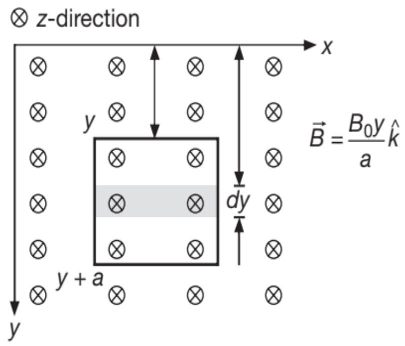


SOLUTION

When the side EF is at a distance y from the x- axis, magnetic flux passing through the loop is

$$\phi=\int d\phi=\int_{y}^{y+a}\frac{B_0 y}{a}(a\,dy)$$

$$\Rightarrow \quad \phi=\frac{B_0}{2}\left[(y+a)^2-y^2\right]=\frac{B_0}{2}\left(a^2+2ay\right)$$



(a) Induced emf is

$$\xi=\left|-\frac{d\phi}{dt}\right|=\left|-\frac{B_o}{2}\frac{d}{dt}\left(a^2+2ay\right)\right|$$

$$\Rightarrow \quad \xi=B_0 a\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \quad \xi=B_0 av \text{ where } v=\frac{dy}{dt}=\text{speed of loop}$$

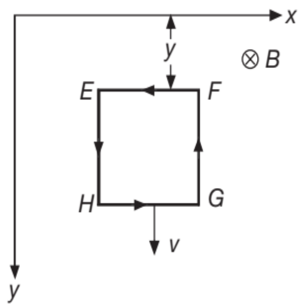
So, the induced current I is given by $I=\frac{\xi}{R}=\frac{B_0 av}{R}$

DIRECTION

$|\vec{B}|\propto y$ i.e., as the loop comes down, the inward, ⊗, magnetic field passing through the loop increases, therefore the induced current must produce an outward, magnetic field i.e., the induced current in the loop will be counter clockwise.

(b) As seen earlier, the induced current passing through the loop (when its speed is v) is

$$I=\frac{B_0 av}{R}$$



Now, magnetic force on EH and FG are equal in magnitude and in opposite directions, hence they cancel each other and produce no force on the loop. Since, $F=BIl$, so

$$F_{EF}=\left(\frac{B_0 y}{a}\right)\left(\frac{B_0 av}{R}\right)(a) \text{ (downwards)}$$

$$\Rightarrow \quad F_{EF}=\frac{B_0^2 avy}{R}\text{, downwards}$$

$$\text{and } F_{GH}=\left(\frac{B_0(y+a)}{a}\right)\left(\frac{B_0 av}{R}\right)(a) \text{ (upwards)}$$

$$\Rightarrow \quad F_{GH}=\left(\frac{B_0^2 av}{R}\right)(y+a)\text{, upwards}$$

Since, $F_{GH}>F_{EF}$

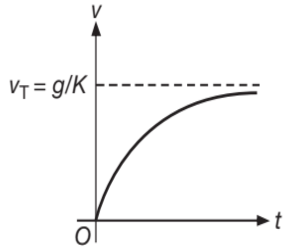
So, net Lorentz force acting on the loop is

$$F=F_{GH}-F_{EF}=\frac{B_0^2 a^2 v}{R} \text{ (upwards)}$$

$$\Rightarrow \quad \vec{F}=\left(\frac{B_0^2 a^2 v}{R}\right)\hat{j}$$

(c) Net force on the loop will be

$F =$ weight – Lorentz force (downwards)



$$\Rightarrow F = mg - \frac{B_0^2 a^2 v}{R}$$

$$\Rightarrow m\left(\frac{dv}{dt}\right) = mg - \left(\frac{B_0^2 a^2}{R}\right)v$$

$$\Rightarrow \frac{dv}{dt} = g - \left(\frac{B_0^2 a^2}{mR}\right)v$$

$$\Rightarrow \int_0^v \frac{dv}{g - \left(\frac{B_0^2 a^2}{mR}\right)v} = \int_0^t dt$$

$$\Rightarrow -\frac{1}{\left(\frac{B_0^2 a^2}{mR}\right)} \log_e\left(g - \frac{B_0^2 a^2}{mR}v\right)\Bigg|_0^v = t$$

$$\Rightarrow \log_e\left(\frac{g - \frac{B_0^2 a^2 v}{mR}}{g}\right) = -\left(\frac{B_0^2 a^2}{mR}\right)t$$

$$\Rightarrow g - \frac{B_0^2 a^2 v}{mR} = ge^{-\left(\frac{B_0^2 a^2}{mR}\right)t}$$

$$\Rightarrow v = \frac{mRg}{B_0^2 a^2}\left(1 - e^{-\left(\frac{B_0^2 a_0^2}{mR}\right)t}\right)$$

i.e., speed of the loop is increasing exponentially with time t. Its terminal velocity, when $t \to \infty$, is

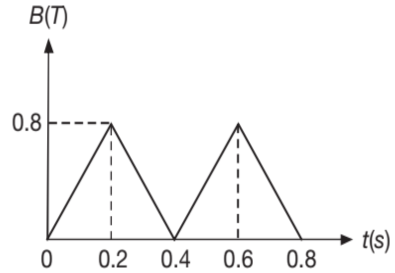
$$v_T = \left(\frac{mgR}{B_0^2 a^2}\right) \text{ when } t \to \infty$$

PROBLEM 26

A thermocole vessel contains 0.5 kg of distilled water at 30 °C. A metal coil of area $5 \times 10^{-3}\ \text{m}^2$, number of turns 100, mass 0.06 kg and resistance $1.6\ \Omega$ is lying horizontally at the bottom of the vessel. A uniform time varying magnetic field is set-up to pass vertically through the coil at time $t = 0$. The field is first increased from 0 to 0.8 T at a constant rate between 0 and 0.2 s and then decreased to zero at the same rate between 0.2 s and 0.4 s. The cycle is repeated 12000 times. Make sketches of the current through the coil and the power dissipated in the coil as a function of time for the first two cycles. Clearly indicate the magnitudes of the quantities on the axes. Assume that no heat is lost to the vessel or the surroundings. Determine the final temperature of the water under thermal equilibrium. Specific heat of metal $= 500\ \text{Jkg}^{-1}\text{K}^{-1}$ and the specific heat of water $= 4200\ \text{Jkg}^{-1}\text{K}^{-1}$. Neglect the inductance of coil.

SOLUTION

Magnetic field (B) varies with time (t) as shown in Figure.



So, $\left|\frac{dB}{dt}\right| = \frac{0.8}{0.2} = 4\ \text{Ts}^{-1}$

Induced emf in the coil due to change in magnetic flux passing through it is

$$\xi = \left|\frac{d\phi}{dt}\right| = NA\left|\frac{dB}{dt}\right|$$

where, $A =$ area of coil $= 5 \times 10^{-3}\ \text{m}^2$ and $N =$ Number of turns = 100

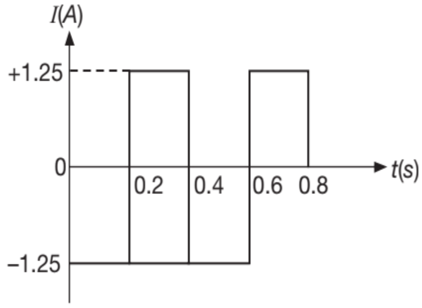
Substituting the values, we get

$$\xi = (100)(5 \times 10^{-3})(4)\ \text{V} = 2\ \text{V}$$

So, the induced current passing through the coil of resistance $R = 1.6\ \Omega$ is given by

$$I = \frac{\xi}{R} = \frac{2}{1.6} = 1.25\ \text{A}$$

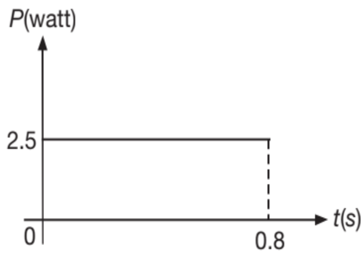
Note that from 0 to 0.2 s and from 0.4 s to 0.6 s, the magnetic field passing through the coil increases, while during the time 0.2 s to 0.4 s and from 0.6 s to 0.8 s magnetic field passing through the coil decreases. Therefore, direction of current through the coil in these two-time intervals will be opposite to each other. The variation of current (I) with time (t) is shown in graph plotted.



Power dissipated in the coil is

$$P = I^2R = (1.25)^2(1.6)\ \text{W} = 2.5\ \text{W}$$

Since, power is independent of the direction of current through the coil. Therefore, power (P) versus time (t) graph for first two cycles will be as shown.



Total heat obtained in $12{,}000$ cycles will be

$$H = Pt = (2.5)(12000)(0.4) = 12000\ \text{J}$$

This heat is utilized in raising the temperature of the coil and the water.
Let T be the final temperature. Then

$$H = m_w S_w (T - 30) + m_c S_c (T - 30)$$

where m_w = mass of water $= 0.5$ kg

S_W = specific heat of water $= 4200\ \text{Jkg}^{-1}\ \text{K}^{-1}$

m_c = mass of coil $= 0.06$ kg

and S_c = specific heat of coil $= 500\ \text{Jkg}^{-1}\ \text{K}^{-1}$

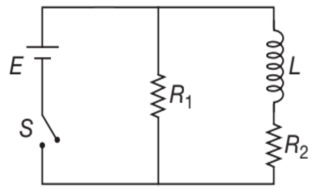
Substituting the values, we get

$$12000 = (0.5)(4200)(T - 30) + (0.06)(500)(T - 30)$$

$$\Rightarrow \quad T = 35.6\ ^\circ\text{C}$$

PROBLEM 27

An inductor of inductance $L = 400$ mH and resistors of resistances $R_1 = 2\ \Omega$ and $R_2 = 2\ \Omega$ are connected to a battery of emf $E = 12$ V as shown in Figure.

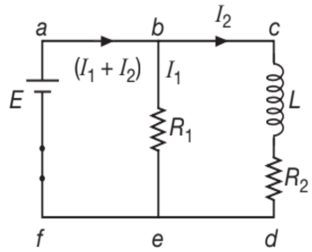


The internal resistance of the battery is negligible. The switch S is closed at time $t = 0$.

(a) What is the potential drop across L as a function of time?

(b) After the steady state is reached, the switch is opened. What is the direction and the magnitude of current through R_1 as a function of time?

SOLUTION



(a) For loop $abefa$, we get

$$E - I_1R_1 = 0$$

$$\Rightarrow \quad I_1 = \frac{12}{2} = 6\ \text{A}$$

For loop $bcdeb$, we get

$$E - L\frac{dI_2}{dt} - I_2R_2 = 0$$

$$\Rightarrow \quad 12 = L\frac{dI_2}{dt} + I_2R_2$$

$$\Rightarrow \quad \frac{1}{L}(12 - I_2R_2) = \frac{dI}{dt}$$

$$\Rightarrow \quad \int_0^I \frac{dI_2}{12 - I_2R_2} = \int_0^t \frac{dt}{L}$$

$$\Rightarrow \quad -\frac{1}{R}\log_e(12 - I_2R_2)\Big|_0^{I_2} = \frac{t}{L}$$

$$\Rightarrow \quad \log_e\left(\frac{12 - I_2R_2}{12}\right) = -\frac{R_2t}{L}$$

$$\Rightarrow \quad 12 - I_2R_2 = 12e^{-\frac{R_2t}{L}}$$

$$\Rightarrow \quad I_2 = \frac{12}{R_2}\left(1 - e^{-\frac{R_2t}{L}}\right)$$

$$\Rightarrow \quad I_2 = 6\left(1 - e^{-\frac{2t}{400\times10^{-3}}}\right)$$

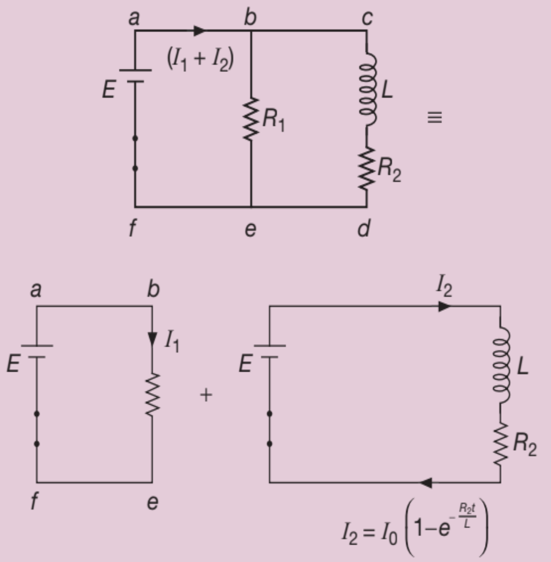
$$\Rightarrow \quad I_2 = 6\left(1 - e^{-5t}\right)$$

So, potential difference across L at time t is

$$V_L = \left|L\frac{dI_2}{dt}\right| = L\left(30e^{-5t}\right) = (30)(0.4)e^{-5t}$$

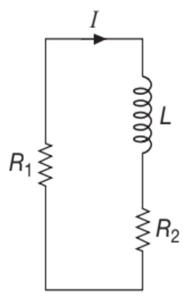
$$\Rightarrow \quad V_L = 12e^{-5t}\ \text{V}$$

We could have obtained the result for I_2, by using the superposition principle,



(b) The steady state current in L or R_2 is

$$I = 6\,\text{A}$$



$t = t$

Now, as soon as the switch is opened, current in R_1 is reduced to zero immediately. But in L and R_2 it decreases exponentially, the new time constant of the circuit being

$$\tau'_L = \frac{L}{R_1 + R_2} = \frac{0.4}{(2+2)} = 0.1\ \text{s}$$

So, current through R_1 at any time t is

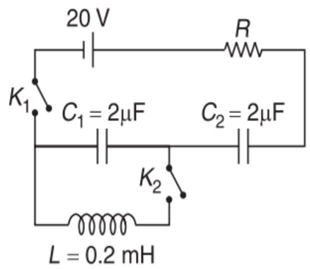
$$I' = I_0 e^{-t/\tau'_L} = 6e^{-t/0.1}$$

$$\Rightarrow \quad I' = 6e^{-10t}\ \text{A}$$

Direction of current in R_1 is as shown in Figure i.e., clockwise.

PROBLEM 28

A circuit containing capacitors C_1 and C_2, is in the steady state with key K_1 closed and K_2 opened as shown in Figure.



At the instant $t = 0$, K_1 is opened and K_2 is closed. Calculate the angular frequency of oscillations of LC circuit. Find the first instant t, when energy in the inductor becomes one third that in the capacitor. Also calculate the charge on plates of the capacitor at that instant.

SOLUTION

With key K_1 closed, C_1 and C_2 are in series with the battery. In steady state, we have charges on the capacitors given by

$$q_0 = C_{eq}V = 1\times20 = 20\ \mu\text{C}$$

With K_1 opened and K_2 closed, charge on C_2 will remain the same, whereas charge on C_1 will oscillate in the LC_1 circuit with frequency given by

$$\omega = \frac{1}{\sqrt{LC_1}}$$

$$\Rightarrow \quad \omega = \frac{1}{\sqrt{0.2\times10^{-3}\times2\times10^{-6}}}$$

$$\Rightarrow \quad \omega = 5\times10^4\ \text{rads}^{-1}$$

Since at $t = 0$, charge is maximum i.e. q_0. Therefore, current will be zero at $t = 0$.

When energy in the inductor is one third that of capacitor, then

$$\frac{1}{2}Li^2 = \frac{1}{3}\left(\frac{1}{2}\frac{q^2}{C}\right)$$

$$\Rightarrow \quad i = \frac{q}{\sqrt{3LC}} = \frac{q\omega}{\sqrt{3}}$$

In LC oscillations, current in circuit at any instant is

$$i = \omega\sqrt{q_0^2 - q^2}$$

$$\Rightarrow \quad \frac{q\omega}{\sqrt{3}} = \omega\sqrt{q_0^2 - q^2}$$

$$\Rightarrow \quad q = \frac{\sqrt{3}}{2}q_0$$

Since at $t = 0$, charge is maximum i.e. q_0, so we have

$$q = q_0 \cos\omega t$$

$$\Rightarrow \quad \frac{\sqrt{3}q_0}{2} = q_0 \cos\omega t$$

$$\Rightarrow \quad \omega t = \frac{\pi}{6}$$

$$\Rightarrow \quad t = \frac{\pi}{6\omega} = \frac{\pi}{6\times5\times10^4}$$

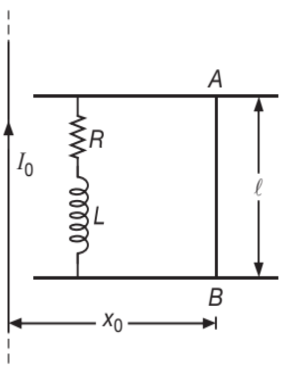
$$\Rightarrow \quad t = 1.05\times10^{-5} \text{ s}$$

At this instant, charge on capacitor plates is

$$q = \frac{\sqrt{3}}{2}q_0 = \frac{\sqrt{3}}{2}\times20 = 10\sqrt{3}\ \mu\text{C}$$

PROBLEM 29

A metal bar AB can slide on two parallel thick metallic rails separated by a distance l. A resistance R and an inductance L are connected to the rails as shown in Figure.



A long straight wire, carrying a constant current I_0 is placed in the plane of the rails and perpendicular to them as shown. The bar AB is held at rest at a distance x_0 from the long wire. At $t = 0$, it made to slide on the rails away from the wire. Answer the following questions.

(a) Find a relation among I, $\frac{dI}{dt}$ and $\frac{d\phi}{dt}$, where I is the current in the circuit and ϕ is the flux of the magnetic field due to the long wire through the circuit.

(b) It is observed that at time $t = T$, the metal bar AB is at a distance of $2x_0$ from the long wire and the resistance R carries a current I_1. Obtain an expression for the net charge that has flown through resistance R from $t = 0$ to $t = T$.

(c) The bar is suddenly stopped at time T. The current through resistance R is found to be $\frac{I_1}{4}$ at time $2T$. Find the value of $\frac{L}{R}$ in terms of the other given quantities.

SOLUTION

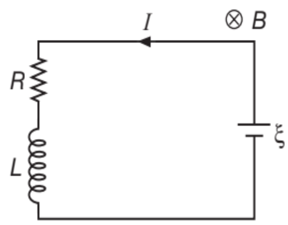
(a) Applying Kirchhoff's Second Law, we get

$$\xi - IR - L\frac{dI}{dt} = 0$$

$$\Rightarrow \quad \frac{d\phi}{dt} - IR - L\frac{dI}{dt} = 0$$

$$\Rightarrow \quad \frac{d\phi}{dt} = IR + L\frac{dI}{dt} \qquad \ldots(1)$$

This is the desired relation between I, $\frac{dI}{dt}$ and $\frac{d\phi}{dt}$



Equivalent Circuit

(b) Equation (1) can be written as

$$d\phi = IRdt + LdI$$

Integrating we get,

$$\Delta\phi = R\Delta q + LI_1$$

$$\Rightarrow \quad \Delta q = \frac{\Delta\phi}{R} - \frac{LI_1}{R} \qquad \ldots(2)$$

Here, $\Delta\phi = \phi_f - \phi_i = \int_{2x_0}^{x_0} \frac{\mu_0}{2\pi}\frac{I_0}{x} l dx = \frac{\mu_0 I_0 l}{2\pi}\log_e(2)$

So, from equation (2), the charge that flows through the resistance up to time $t = T$, when current is I_1, is

$$\Delta q = \frac{1}{R}\left[\frac{\mu_0 I_0 l}{2\pi}\log_e(2) - LI_1\right]$$

(c) This is the case of current decay in an LR circuit.

$\Rightarrow\ I = I_0 e^{-\Delta t/\tau_L}$ ...(3)

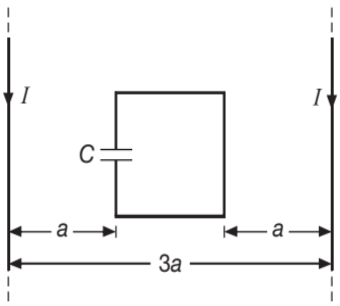
where, $I = \frac{I_1}{4}$, $I_0 = I_1$, $\Delta t = (2T - T) = T$ and $\tau_L = \frac{L}{R}$

Substituting these values in equation (3), we get

$$\tau_L = \frac{L}{R} = \frac{T}{\log_e(4)}$$

PROBLEM 30

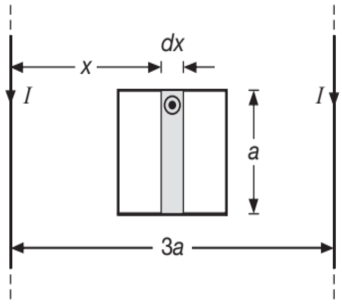
Two infinitely long parallel wires carrying currents $I = I_0 \sin(\omega t)$ in opposite directions are placed a distance $3a$ apart. A square loop of side a of negligible resistance with a capacitor of capacitance C is placed in the plane of wires as shown. Find the maximum current in the square loop. Also sketch the graph showing the variation of charge on the upper plate of the capacitor as a function of time for one complete cycle taking anticlockwise direction for the current in the loop as positive.



SOLUTION

(a) For an infinitesimal strip of thickness dx at a distance x from left wire, net magnetic field (due to both wires) is

$$B = \frac{\mu_0}{2\pi}\frac{I}{x} + \frac{\mu_0}{2\pi}\frac{I}{3a - x} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{3a - x}\right)$$



Magnetic flux in this strip,

$$d\phi = BdA = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{3a - x}\right)adx$$

So, total flux $\phi = \int_a^{2a} d\phi = \frac{\mu_0 Ia}{2\pi}\int_a^{2a}\left(\frac{1}{x} + \frac{1}{3a - x}\right)dx$

$$\Rightarrow\ \phi = \frac{\mu_0 Ia}{\pi}\log_e(2)$$

$$\Rightarrow\ \phi = \frac{\mu_0 a \log_e(2)}{\pi}(I_0 \sin(\omega t)) \quad ...(1)$$

Magnitude of induced emf,

$$\xi = \left|-\frac{d\phi}{dt}\right| = \frac{\mu_0 a I_0 \omega \log_e(2)}{\pi}\cos(\omega t)$$

$$\Rightarrow\ \xi = \xi_0 \cos(\omega t)$$

where, $\xi_0 = \frac{\mu_0 a I_0 \omega \log_e(2)}{\pi}$

Charge stored in the capacitor,

$$Q = C\xi = C\xi_0 \cos(\omega t) \quad ...(2)$$

And current in the loop,

$$I = \frac{dQ}{dt} = C\omega\xi_0 \sin(\omega t) \quad ...(3)$$

$$\Rightarrow\ I_{max} = C\omega\xi_0 = \frac{\mu_0 a I_0 \omega^2 C \log_e(2)}{\pi}$$

(b) Since, magnetic flux passing through the square loop

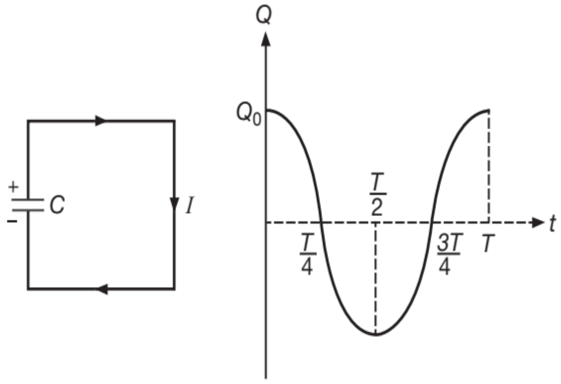
$\phi \propto \sin\omega t$ $\{\because$ of $(1)\}$

i.e., magnetic field passing through the loop is increasing at $t = 0$. Hence, the induced current will produce $\otimes$ magnetic field (from Lenz's Law). Or the current in the circuit at $t = 0$ will be clockwise (or negative as per the given convention). Therefore, charge on upper plate could be written as,

$$Q = +Q_0 \cos(\omega t) \qquad \{\because \text{ of } (2)\}$$

Where, $Q_0 = C\xi_0 = \dfrac{\mu_0 a C I_0 \omega \log_e(2)}{\pi}$

The corresponding Q-t graph is shown in Figure.



PROBLEM 31

A non-conducting ring of mass m and radius R has a charge Q uniformly distributed over its circumference. The ring is placed on a rough horizontal surface such that plane of the ring is parallel to the surface. A vertical magnetic field $B = B_0 t^2$ tesla is switched on. After 2 second from switching on the magnetic field the ring is just about to rotate about vertical axis through its centre.

(a) Find friction coefficient μ between the ring and the surface.

(b) If magnetic field is switched off after 4 second, then find the angle rotated by the ring before coming to stop after switching off the magnetic field.

SOLUTION

(a) Magnitude of induced electric field due to change in magnetic flux is given by

$$\oint \vec{E} \cdot d\vec{l} = \frac{d\phi}{dt} = A\frac{dB}{dt}$$

$$\Rightarrow El = \pi R^2 (2B_0 t) \qquad \left\{\because \frac{dB}{dt} = 2B_0 t\right\}$$

$$\Rightarrow E(2\pi R) = 2\pi R^2 B_0 t$$

$$\Rightarrow E = B_0 R t$$

Hence $F = QE = B_0 QRt$

This force is tangential to ring. The ring starts rotating when torque of this force is greater than the torque due to maximum friction

$$(f_{max} = \mu mg) \text{ So, } \tau_F \geq \tau_{f_{max}}$$

Taking the limiting case. $\tau_F = \tau_{f_{max}}$

$$\Rightarrow FR = (\mu mg)R$$

$$\Rightarrow F = \mu mg$$

$$\Rightarrow B_0 QRt = \mu mg$$

It is given that ring starts rotating after 2 second.

So, putting $t = 2$ second, we get $\mu = \dfrac{2B_0 RQ}{mg}$

(b) After 2 second, we have

$$\tau_F > \tau_{f_{max}}$$

Therefore, net torque is

$$\tau = \tau_F - \tau_{f_{max}} = B_0 QR^2 t - \mu mgR$$

Substituting $\mu = \dfrac{2B_0 QR}{mg}$, we get

$$\tau = B_0 QR^2 (t-2)$$

$$\Rightarrow I\alpha = I\left(\frac{d\omega}{dt}\right) = B_0 QR^2 (t-2)$$

$$\Rightarrow mR^2\left(\frac{d\omega}{dt}\right) = B_0 QR^2 (t-2)$$

$$\Rightarrow \int_0^{\omega} d\omega = \frac{B_0 Q}{m}\int_2^4 (t-2)\,dt$$

$$\Rightarrow \omega = \frac{2B_0 Q}{m} \qquad \ldots(1)$$

Now magnetic field is switched off i.e., only retarding torque is present due to the friction. So, angular retardation will be

$$\alpha = \frac{\tau_{f_{max}}}{I} = \frac{\mu mgR}{mR^2} = \frac{\mu g}{R}$$

Since $\omega^2 = \omega_0^2 - 2\alpha\theta$

$$\Rightarrow 0 = \left(\frac{2B_0 Q}{m}\right)^2 - 2\left(\frac{\mu g}{R}\right)\theta$$

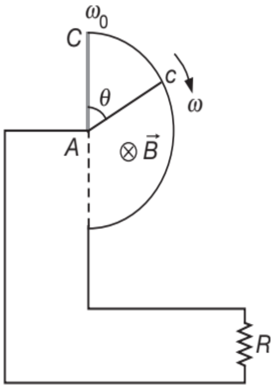
$$\Rightarrow \theta = \frac{2B_0^2 Q^2 R}{\mu m^2 g}$$

Substituting $\mu = \dfrac{2B_0 RQ}{mg}$, we get

$$\theta = \frac{B_0 Q}{m}$$

PROBLEM 32

A conducting rod AC of mass m, resistance r free to rotate in a horizontal plane about one end A over a semi-circular conducting ring of radius a is joined with an external resistance R as shown in Figure.



The rod is given an initial angular velocity ω_0. A uniform magnetic field of magnitude B exists perpendicular to the plane of semi-circular loop. Find the current in the circuit at angle θ. Neglect effect of gravity on the rod.

SOLUTION

Let at a time t, angular velocity of rod be ω. Then induced emf in the circuit is

$$\xi = \frac{B\omega a^2}{2}$$

and therefore, the induced current in the circuit is

$$i = \frac{\xi}{R+r} = \frac{B\omega a^2}{2(R+r)} \qquad \{\text{radially outwards}\}$$

Magnetic force on element dx, at a distance x from A is

$$F = (Bidx)$$

Torque on this element about A is given by

$$d\tau = Fx = Bixdx$$

So, total torque on the rod is

$$\tau = \int_{x=0}^{x=a} d\tau$$

$$\Rightarrow \quad \tau = \frac{Bia^2}{2} \qquad \{\text{anticlockwise}\}$$

Now angular retardation is

$$\alpha = \frac{\tau}{I}$$

where, $I = I_A = \dfrac{ma^2}{3}$

$$\Rightarrow \quad -\omega\frac{d\omega}{d\theta} = \frac{iBa^2}{2\left(\dfrac{ma^2}{3}\right)}$$

$$\Rightarrow \quad -\omega\frac{d\omega}{d\theta} = \frac{3}{2}\frac{Bi}{m} = \frac{3}{2}\frac{B}{m}\frac{B\omega a^2}{2(R+r)}$$

$$\Rightarrow \quad d\omega = -\frac{3}{4}\frac{B^2a^2}{m(R+r)}d\theta$$

$$\Rightarrow \quad \int_{\omega_0}^{\omega} d\omega = -\frac{3}{4}\frac{B^2a^2}{m(R+r)}\int_0^{\theta} d\theta$$

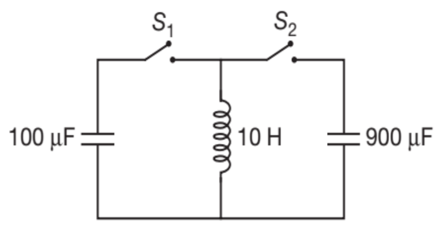
$$\Rightarrow \quad \omega = \omega_0 - \frac{3}{4}\frac{B^2a^2}{m(R+r)}\theta$$

Since, $i = \dfrac{B\omega a^2}{2(R+r)}$

$$\Rightarrow \quad i = \frac{Ba^2}{2(R+r)}\left(\omega_0 - \frac{3}{4}\frac{B^2a^2\theta}{m(R+r)}\right)$$

PROBLEM 33

Initially the $900\ \mu$F capacitor is charged to 100 V and the $100\ \mu$F capacitor is uncharged shown in Figure.



Then the switch S_2 is closed for a time t_1, after which it is opened and at the same instant switch S_1 is closed for a time t_2 and then opened. It is now found that the $100\ \mu$F capacitor is charged to 300 V. Find the minimum possible values of the time intervals t_1 and t_2.

SOLUTION

Let the energy stored in the $900\ \mu$F and the $100\ \mu$F capacitor be U_1 and U_2 respectively. Then

$$U_1 = \frac{1}{2}(900)(10^{-6})(100)^2 = 4.5\ \text{J}$$

$$U_2 = \frac{1}{2}(100)(10^{-6})(300)^2 = 4.5\ \text{J}$$

That is the entire energy of 900 μF capacitor has been transferred to the 100 μF capacitor.

First electrical energy of the 900 μF capacitor is converted into magnetic energy in the inductor and then this energy is converted into electrical energy once again using S_2 and S_1 appropriately. In a LC circuit the transfer of electrical energy into magnetic energy and vice-versa takes place in a time $\frac{T}{4}$ where $T = 2\pi\sqrt{LC}$ is the time period of the electrical oscillations. Thus

$$T_1 = 2\pi\sqrt{10\times 900\times 10^{-6}} = 0.6 \text{ s and}$$

$$T_2 = 2\pi\sqrt{10\times 100\times 10^{-6}} = 0.2 \text{ s}$$

Therefore, switch S_2 is first closed for time $t_1 = \frac{0.6}{4} =$ 0.15 s, during which time the 900 μF capacitor gets fully discharged and the current in the inductor is fully established. Next, the switch S_2 is opened and simultaneously switch S_1 is closed for time $t_2 = \frac{0.2}{4} = 0.05$ s during which the current in the inductor disappears and the 100 μF capacitor gets fully charged. After this time, the switch S_1 is also opened. The 100 μF capacitor is now charged to 300 V. So, we have

$$t_1 = 0.15 \text{ s and } t_2 = 0.05 \text{ s}$$

$$\Rightarrow \quad t_1 = 150 \text{ ms and } t_2 = 50 \text{ ms}$$