CHAPTER 1

Magnetic Effects of Current

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

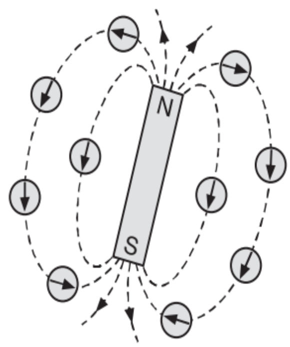
(a) Force on charge particle moving in magnetic field
(b) Lorentz force
(c) Cyclotron
(d) Path of a charged particle in combined effect of electric and magnetic field
(e) Biot Savart's Law
(f) Magnetic field due to a moving charge
(g) Application of Biot Savart's Law and Solenoid
(h) Ampere's Circuital Law (ACL) and Application
(i) Magnetic force on a current carrying conductor placed in magnetic field
(j) Magnetic pressure and magnetic field energy
(k) Magnetic dipole
(l) Gyromagnetic ratio (GMR)
(m) Torque on a current loop placed in uniform field
(n) Magnetic dipole moment
(o) Work done to change orientation of current carrying coil in magnetic field
(p) Moving coil galvanometer
(q) Magnetic force between two straight parallel current carrying wires

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION TO MAGNETIC FIELD AND MOTION OF CHARGED PARTICLES IN MAGNETIC FIELD

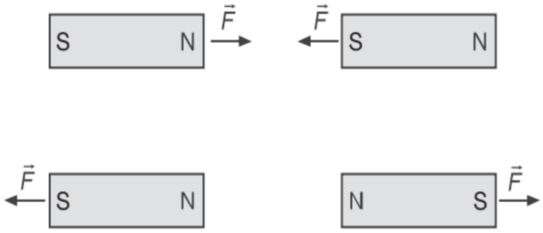
INTRODUCTION

We are aware of the fact that a charged object produces an electric filed $\vec{E}$ at all points in space. In a similar manner, a bar magnet is a source of a magnetic field $\vec{B}$. This can be readily demonstrated by moving a compass near the magnet. The compass needle always aligns itself along the direction of the magnetic field produced by the magnet, as shown in Figure.



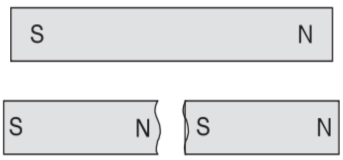
Magnetic field produced by a bar magnet

The bar magnet consists of two poles, which are designated as the north (N) and the south (S). Magnetic fields are strongest at the poles. The magnetic field lines leave from the north pole and enter the south pole. When holding two bar magnets close to each other, the like poles repel each other while the opposite or unlike poles attract.



Magnets attracting and repelling

Unlike electric charges which can be isolated, the two magnetic poles always exist in a pair. Whenever you break a bar magnet, two new bar magnets are obtained, each with a north pole and a south pole as shown.



Magnetic monopoles do not exist in isolation

In other words, "magnetic monopoles do not exist in isolation", although they are of theoretical interest.

How do we define the magnetic field $\vec{B}$?

In the case of an electric field $\vec{E}$, we have already seen that the field is defined as the force per unit charge, given by

$$\vec{E}=\frac{\vec{F}_e}{q}$$

However, due to the absence of magnetic monopoles, $\vec{B}$ must be defined in a different way.

FORCE ON A CHARGED PARTICLE IN MAGNETIC FIELD AND THE DEFINITION OF A MAGNETIC FIELD (*B*)

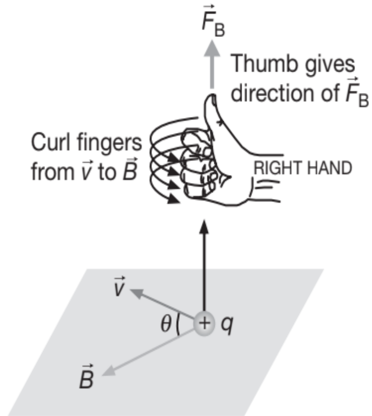
To define the magnetic field at a point, consider a particle of charge q moving with a velocity $\vec{v}$. Experimentally, we have the following observations.

(a) The magnitude of the magnetic force $\vec{F}_B$ or $\vec{F}_m$ exerted on the charged particle is proportional to both q and v.

(b) The magnitude and direction of $\vec{F}$ depends on $\vec{B}$ and $\vec{v}$.

(c) The magnetic force $\vec{F}$ vanishes when $\vec{v}$ is parallel to $\vec{B}$. However, when $\vec{v}$ makes an angle θ with $\vec{B}$, the direction of $\vec{F}$ is normal to the plane containing $\vec{v}$ as well as $\vec{B}$ and the magnitude of $\vec{F}_B$ is proportional to $\sin\theta$.

(d) When the sign of the charge of the particle is changed from positive to negative (or vice versa), the direction of the magnetic force also reverses.



From the above observations we conclude that this magnetic force must be expressed vectorially as

$$\vec{F}=q\left(\vec{v}\times\vec{B}\right) \quad \ldots(1)$$

The above expression is regarded as the working definition of the magnetic field at a point in space. The magnitude of $\vec{F}_B$ is given by

$$F=|q|vB\sin\theta \quad \ldots(2)$$

$$\Rightarrow \quad B=\frac{F}{qv\sin\theta}$$

If $q=+1$ coulomb, $v=1\text{ ms}^{-1}$, $\theta=90^\circ$ then

$$B=F\left(\text{numerically}\right)$$

Hence, magnetic field is numerically equal to the force experienced by a charged particle of +1 coulomb when it enters a magnetic field with a velocity of 1 ms^{-1} at right angles to the field.

The SI unit of magnetic field is the tesla (T).

$$1\text{ tesla}=1\text{ T}=1\frac{\text{newton}}{\left(\text{coulomb}\dfrac{\text{meter}}{\text{second}}\right)}$$

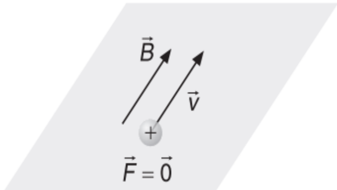
$$\Rightarrow \quad 1\text{ tesla}=1\frac{\text{N}}{\text{Cms}^{-1}}=1\frac{\text{N}}{\text{Am}}$$

Another commonly used non-SI unit for $\vec{B}$ is the gauss (G), where $1\,T = 10^{-4}\,G$. Instruments used to measure magnetic field are sometimes called as Gaussmeters.

The magnetic force has several important properties:

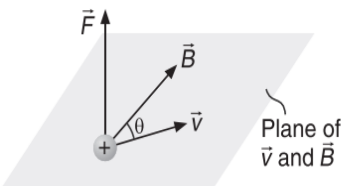
(a) Only a moving charge experiences a magnetic force. There is no magnetic force on a charge at rest $(v = 0)$ in a magnetic field.

(b) There is no force on a charge moving parallel $(\theta = 0°)$ or antiparallel $(\theta = 180°)$ to a magnetic field.



There is no force on the charge moving parallel to $\vec{B}$

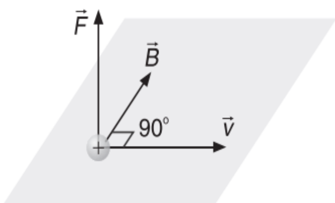
(c) When there is a force, the force is perpendicular to both $\vec{v}$ and $\vec{B}$.



The magnetic force is perpendicular to $\vec{v}$ and $\vec{B}$. Its magnitude is $qvB\sin\theta$

(d) The force on a negative charge is in the direction opposite to $\vec{v} \times \vec{B}$.

(e) For a charge moving perpendicular to $\vec{B}$ $(\theta = 90°)$, the magnitude of the magnetic force is maximum given by $F_{max} = |q|vB$.



The magnetic force is maximum when the charge moves perpendicular to $\vec{B}$.

DIRECTION OF $\vec{F}$ (THE MAGNETIC FORCE)

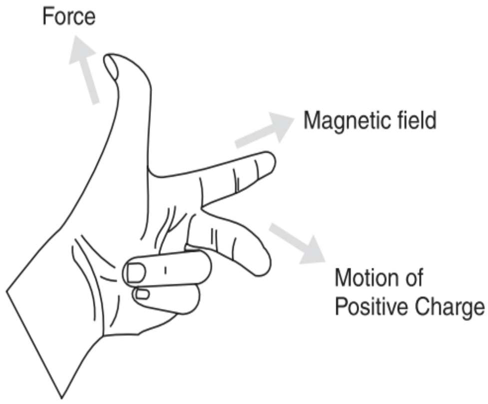
The direction of the magnetic force $\vec{F}$, can be found by using either of the two methods

(a) Fleming's Left Hand Rule

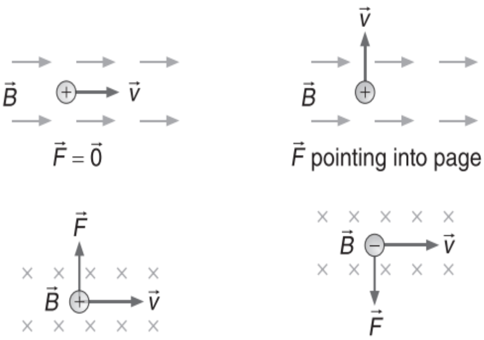
(b) Right Hand Thumb Rule (also used to find the direction of cross product of two vectors)

Fleming's Left Hand Rule

Stretch the first finger, the middle finger, and the thumb of left hand in such a way that they are mutually perpendicular to each other.



First finger points in the direction of Field, middle finger points in the direction of motion of positive charge (or the velocity of positive charge) then thumb gives the direction of force experienced by the positive charge.



Magnetic Forces on a Moving Charge

Right Hand Thumb Rule

To use this method for finding the direction of magnetic force $(\vec{F})$ simply follow the steps given.

STEP-1:

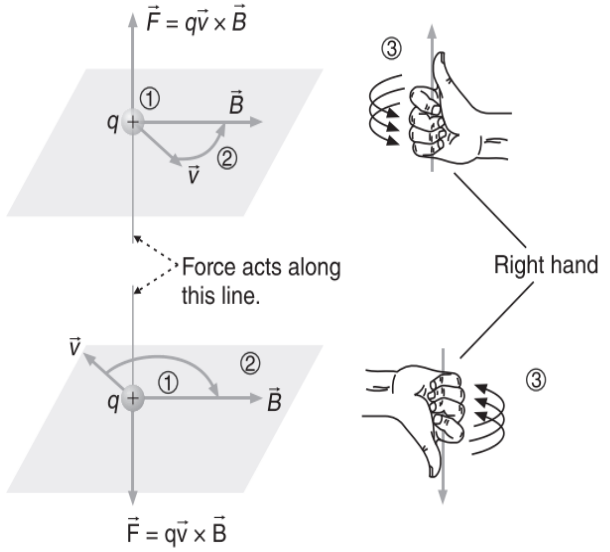
Place $\vec{v}$ (of the positive charge) and $\vec{B}$ tail to tail.

STEP-2:

Curl fingers of right hand from $\vec{v}$ to $\vec{B}$ (in the $\vec{v}$-$\vec{B}$ plane), as shown.

STEP-3:

The magnetic force $(\vec{F})$ acts along the line normal to the $\vec{v}$-$\vec{B}$ plane i.e., when we curl the fingers of the right hand from $\vec{v}$ to $\vec{B}$ in the $\vec{v}$-$\vec{B}$ plane, then direction of thumb gives the direction of the magnetic force as shown in Figure.



STEP-4:

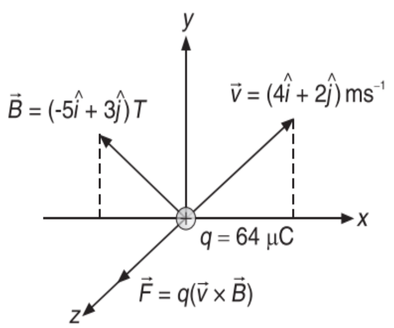
If the charge is negative, then the direction of the magnetic force is opposite to that given by the Right-Hand Rule. So, just find the direction of force as if charge were positive and then just reverse it.

ILLUSTRATION 1

A 64 μC charge travelling with velocity $\vec{v} = (4\hat{i} + 2\hat{j})\ \text{ms}^{-1}$ enters a region where there is a magnetic field $\vec{B} = (-5\hat{i} + 3\hat{j})\ \text{T}$. Find the magnetic force on the charge.

SOLUTION

A charge q moving with velocity $\vec{v}$ entering a magnetic field $\vec{B}$ is shown in Figure.



Because both $\vec{v}$ and $\vec{B}$ lie in the xy-plane, we feel that F should be along the z-axis. We use the determinant method to compute the cross product indicated by equation.

$$\vec{F} = q(\vec{v} \times \vec{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & 0 \\ -5 & 3 & 0 \end{vmatrix}$$

$$\Rightarrow \quad \vec{F} = (64 \times 10^{-6})(12 + 10)\hat{k}$$

$$\Rightarrow \quad \vec{F} = 1.41 \times 10^{-3}\hat{k}\ \text{N}$$

Problem Solving Technique(s)

(a) Since $\vec{F} = q(\vec{v} \times \vec{B})$, so from the property of cross product of two vectors we conclude that $\vec{F}$ is perpendicular to $\vec{v}$ as well as $\vec{B}$.

$$\Rightarrow \quad \vec{F} \perp \vec{v} \quad \text{and} \quad \vec{F} \perp \vec{B}$$

$$\Rightarrow \quad \vec{F} \cdot \vec{v} = 0 \quad \text{and} \quad \vec{F} \cdot \vec{B} = 0$$

If $\vec{a}$ is the acceleration of the particle then $\vec{a} \cdot \vec{v} = 0$ and $\vec{a} \cdot \vec{B} = 0$.

(b) From above property we can also conclude that $\vec{F}$ is always normal to the plane containing $\vec{v}$ and $\vec{B}$.

(c) Since $\vec{F} = q(\vec{v} \times \vec{B})$, so we observe that apart from other factors $\vec{F}$ depends upon sign of q also. Mathematically

$$\vec{F} = \begin{cases} \text{parallel to } \vec{v} \times \vec{B}, \text{ if } q \text{ is positive,} \\ \text{antiparallel to } \vec{v} \times \vec{B} \text{ if } q \text{ is negative or} \\ \text{parallel to } \vec{B} \times \vec{v}, \text{ if } q \text{ is negative} \end{cases}$$

(d) Since we have noted that $\vec{F}$ is perpendicular to $\vec{v}$ as well as $\vec{B}$, so the displacement of a charged particle in a field will always be perpendicular to the force. Due to this, we have

$$dW = \vec{F} \cdot d\vec{\ell} = q(\vec{v} \times \vec{B}) \cdot d\vec{\ell}$$

$$\Rightarrow \quad dW = q(\vec{v} \times \vec{B}) \cdot (\vec{v}dt) = 0$$

This makes us conclude that the magnetic force does no work to displace a particle in a magnetic field (both varying and static) Further since, according to Work - Energy Theorem, we have

Work Done = Change in KE

$$\Rightarrow \quad 0 = \Delta K$$

$$\Rightarrow \quad K_f = K_i$$

$$\Rightarrow \quad |\vec{v}_{\text{final}}| = |\vec{v}_{\text{initial}}| \text{ (in magnitude only)}$$

$$\Rightarrow \quad v_x^2 + v_y^2 + v_z^2 = u_x^2 + u_y^2 + u_z^2$$

i.e., a magnetic force is just capable of changing the direction of velocity of the charged particle without changing its magnitude.

Hence, keep in mind that a **magnetic force is incapable of speeding up or slowing down a charge particle**. However, it can just change the direction of the velocity of the particle (without any change in its magnitude).

(e) $F = qvB\sin\theta$ has a maximum value qvB when $\theta = 90°$ i.e., whenever a charged particle enters at right angles to the field, then it will experience a maximum magnetic force. So, when, $\theta = \frac{\pi}{2}$

$$F_{max} = qvB$$

(f) By convention, the magnetic field perpendicular to and directed out of the page (i.e., a field that has its tips of arrows coming towards you) is represented by $\odot$ and the magnetic field perpendicular to and directed inside the page (i.e., a field that has its tips of arrows directed away from you) is represented by $\otimes$, as shown.

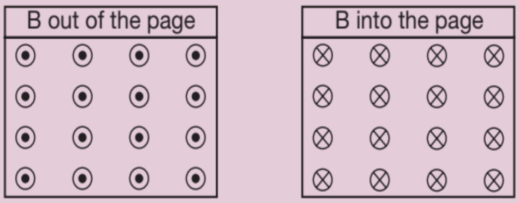


ILLUSTRATION 2

A charged particle moving in a magnetic field $\vec{B} = (\hat{i} - \hat{j})$ T has an acceleration of $(2\hat{i} + \alpha\hat{j})$ ms^{-2} at an instant. Calculate the value of α.

SOLUTION

Since, $\vec{F} = q(\vec{v} \times \vec{B})$

$$\Rightarrow \quad \vec{a} = \frac{q}{m}(\vec{v} \times \vec{B})$$

$$\Rightarrow \quad \vec{a} \perp \vec{B}$$

$$\Rightarrow \quad \vec{a} \cdot \vec{B} = 0$$

$$\Rightarrow \quad (2\hat{i} + \alpha\hat{j})(\hat{i} - \hat{j}) = 0$$

$$\Rightarrow \quad 2 - \alpha = 0$$

$$\Rightarrow \quad \alpha = 2$$

ILLUSTRATION 3

A particle having charge $q = 10\ \mu\text{C}$ moves in uniform magnetic field with velocity $v_1 = 10^6$ ms^{-1} at angle 45° with x-axis in the x-y plane and experiences a force $F_1 = 5\sqrt{2}$ mN along the negative z-axis. When the same particle moves with velocity $v_2 = 10^6$ ms^{-1} along the z-axis it experiences a force F_2 in y direction. Find the magnitude and direction of the magnetic field. Also find the magnitude of the force F_2.

SOLUTION

Since we know that $\vec{F} = q(\vec{v} \times \vec{B})$

For the first case, we get

$$-(5\sqrt{2} \times 10^{-3})\hat{k} = (10^{-5})\left(\frac{10^6}{\sqrt{2}}\right)(\hat{i} + \hat{j}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\Rightarrow \quad -(5\sqrt{2} \times 10^{-3})\hat{k} = \frac{10}{\sqrt{2}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ B_x & B_y & B_z \end{vmatrix}$$

$$\Rightarrow \quad -(5\sqrt{2} \times 10^{-3})\hat{k} = \frac{10}{\sqrt{2}}\left[B_z\hat{i} - B_z\hat{j} + (B_y - B_x)\hat{k}\right]$$

$$\Rightarrow \quad B_z = 0 \text{ and } B_y - B_x = -10^{-3}\ \text{T} \qquad \text{...(1)}$$

For the second case, we have

$$F_2\hat{j} = (10^{-5})(10^6)\hat{k} \times (B_x\hat{i} + B_y\hat{j})$$

$$\Rightarrow \quad F_2\hat{j} = 10\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ B_x & B_y & 0 \end{vmatrix} = 10B_x\hat{j} - 10B_y\hat{i} \qquad \text{...(2)}$$

From (2), we get

$$B_y = 0$$

$$\Rightarrow \quad B_x = 10^{-3}\ \text{T and } F_2 = 10B_x = 10^{-2}\ \text{N}$$

$$\Rightarrow \quad \vec{B} = (10^{-3}\ \text{T})\hat{i} \text{ and } F_2 = 10^{-2}\ \text{N}$$

ILLUSTRATION 4

A particle of charge $q > 0$ is moving at speed v in the $+z$-direction through a region of uniform magnetic field $\vec{B}$. The magnetic force on the particle is $\vec{F} = F_0(3\hat{i} + 4\hat{j})$, where F_0 is a positive constant.

(a) Calculate the components B_x, B_y, and B_z, or at least as many as possible from the information given.

(b) If $|\vec{B}| = \frac{6F_0}{qv}$, calculate the left-out component(s) of $\vec{B}$, if any.

SOLUTION

(a) $\vec{v} = v\hat{k}$ and $\vec{F} = F_0(3\hat{i} + 4\hat{j})$

Let $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Since, $\vec{F} = q(\vec{v} \times \vec{B})$

$$\Rightarrow \vec{F} = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & v \\ B_x & B_y & B_z \end{vmatrix}$$

$$\Rightarrow \vec{F} = q\left[\hat{i}(-vB_y) - \hat{j}(-vB_x) + \hat{k}(0-0)\right]$$

Since, $\vec{F} = F_0(3\hat{i} + 4\hat{j})$

$$\Rightarrow F_0(3\hat{i} + 4\hat{j}) = -(qvB_y)\hat{i} + (qvB_x)\hat{j}$$

$$\Rightarrow B_y = -\frac{3F_0}{qv} \text{ and } B_x = \frac{4F_0}{qv}$$

However, nothing can be said about B_z with the information provided

(b) $|\vec{B}| = \frac{6F_0}{qv}$

$$\Rightarrow B_x^2 + B_y^2 + B_z^2 = \frac{36F_0^2}{q^2v^2}$$

$$\Rightarrow \frac{16F_0^2}{qv^2} + \frac{9F_0^2}{q^2v^2} + B_z^2 = \frac{36F_0^2}{q^2v^2}$$

$$\Rightarrow B_z = \pm\sqrt{11}\left(\frac{F_0}{qv}\right)$$

ILLUSTRATION 5

A group of particles is travelling in a magnetic field of unknown magnitude and direction. It is observed that a proton moving at 1.5 kms^{-1} in the $+x$ direction experiences a force of 2.25×10^{-16} N in the $+y$ direction, and an electron moving at 4.75 kms^{-1} in the $-z$ direction experiences a force of 8.5×10^{-16} N. Calculate the magnitude and direction of the magnetic field.

SOLUTION

Let $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Since the proton moving along $+x$ direction experiences a force in the $+y$ direction and also we know that this force must be perpendicular to $\vec{v}$ as well as $\vec{B}$. Hence, for the proton the magnetic force given by

$$\vec{F} = +e\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & 0 & 0 \\ B_x & B_y & B_z \end{vmatrix}$$

$$\Rightarrow \vec{F} = +e\left[\hat{i}(0-0) - \hat{j}(v_xB_z - 0) + \hat{k}(v_xB_y - 0)\right]$$

$$\Rightarrow \vec{F} = +ev_x(-B_z\hat{j} + B_y\hat{k}) \quad \ldots(1)$$

Since $\vec{F} = 2.25 \times 10^{-16}$ N along $+y$ direction

$$\Rightarrow \vec{F} = (2.25 \times 10^{-16})\hat{j} \text{ N} \quad \ldots(2)$$

On equating (1) and (2), we observe that

$$B_y = 0 \text{ and } -ev_xB_z = 2.25 \times 10^{-16}$$

$$\Rightarrow B_z = \frac{2.25 \times 10^{-16}}{1.6 \times 10^{-19} \times 1.5 \times 10^3} = 0.938 \text{ T} = 938 \text{ mT}$$

Further, for the electron we have $\vec{v} = -v_z\hat{k}$ and $|\vec{F}| = 8.5 \times 10^{-16}$ N i.e., here we have just been provided with F without direction i.e., only with magnitude.

Since $\vec{F} = q(\vec{v} \times \vec{B})$

$$\Rightarrow \vec{F} = -e\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & -v_z \\ B_x & 0 & B_z \end{vmatrix}$$

$$\Rightarrow \vec{F} = -e\left[\hat{i}(0-0) - \hat{j}(0 + v_zB_x) + \hat{k}(0-0)\right]$$

$$\Rightarrow \vec{F} = +e(v_zB_x)\hat{j} \text{ where } v_z = 4.75 \times 10^3 \text{ kms}^{-1}$$

Since $|\vec{F}| = 8.5 \times 10^{-16}$ N

$$\Rightarrow ev_z|B_x| = 8.5 \times 10^{-16}$$

$$\Rightarrow |B_x| = \frac{8.5 \times 10^{-16}}{(1.6 \times 10^{-19})(4.75 \times 10^3)} = 1.12 \text{ T}$$

$$\Rightarrow B_x = \pm 1.12 \text{ T}$$

$$\Rightarrow \vec{B} = (\pm 1.12\hat{i} + 0.938\hat{k}) \text{ T}$$

$$\Rightarrow |\vec{B}| = \sqrt{(\pm 1.12)^2 + (0.938)^2} = 1.46 \text{ T}$$

Direction of $\vec{B}$

Let $\vec{B}$ make an angle θ with $+x$-axis, then

$$\tan\theta = \frac{B_z}{B_x} = \frac{-0.938}{\pm 1.12}$$

$$\Rightarrow \quad \theta = \pm 40^\circ$$

i.e., $\vec{B}$ is in the xz plane and makes an angle of 40° with $+x$-axis either in anticlockwise sense or in clockwise sense.

ILLUSTRATION 6

A particle of mass 0.195 g carries a charge of -2.5×10^{-8} C. The particle is given an initial horizontal velocity that is due north and has magnitude 4×10^4 ms^{-1}. Calculate the magnitude and direction of the minimum magnetic field that keeps the particle moving in the earth's gravitational field in the same horizontal, northward direction?

SOLUTION

For the net force to be zero, the magnetic force (F_m) must balance the weight (W) of the particle
Now, for equilibrium we have

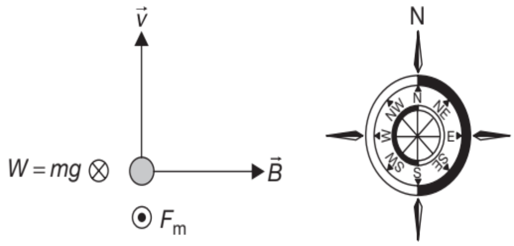
$$F_m = W$$

$$\Rightarrow \quad |q|vB\sin\theta = W$$

$$\Rightarrow \quad B = \frac{W}{|q|v\sin\theta} = \frac{mg}{|q|v\sin\theta}$$

Since, we are asked to calculate the minimum B, so we must have $\sin\theta = MAX = 1$, i.e., $\theta = 90^\circ$ i.e., $\vec{v}\perp\vec{B}$. Hence

$$B_{\min} = \frac{mg}{|q|v} = \frac{(0.195\times10^{-3})(9.8)}{(2.5\times10^{-8})(4\times10^4)} = 1.91\text{ T}$$



Since the velocity of the particle is due North and the weight of the particle $W(=mg)$ is acting towards the centre of the earth, which happens to be a direction denoted by ⊗ shown in figure. Now for the negative charge that moves due North, the magnetic force that balances W must be acting in the outward direction ⊙, just opposite to W so as to balance it. Please note that the direction of the force on negative charge will be opposite to that obtained by Fleming's Left Hand Rule (specified for positive charge). So, be careful while finding the direction of $\vec{F}_m$. Hence to conclude we get to have B acting towards the East.

Test Your Concepts-I

Based on Force and Fleming's Left Hand Rule

(Solutions on page H.3)

1. Can you find the dimensional formula for the ratio $\frac{E}{B}$, where E denotes the electric field and B denotes the magnetic field?
2. If $\vec{F}$ denotes the magnetic force acting on a charge q entering a field $\vec{B}$ with a velocity $\vec{v}$, then find
   (a) $\vec{v}\cdot(\vec{v}\times\vec{B})$
   (b) $\vec{B}\cdot(\vec{v}\times\vec{B})$
   (c) $\vec{v}\times(\vec{v}\times\vec{B})$
3. A charged particle is projected in a magnetic field $\vec{B} = (3\hat{i}+2\hat{j}-10\hat{k})\times10^{-2}$ T. The acceleration of the particle is found to be, $\vec{a} = (x\hat{i}+4\hat{j}+2\hat{k})$ ms^{-2}. Find the value of x.
4. A particle with mass 1.81×10^{-3} kg and a charge of 1.22×10^{-8} C has, at a given instant, a velocity $\vec{v} = (3\times10^4 \text{ ms}^{-1})\hat{j}$. What are the magnitude and direction of the particle's acceleration produced by a uniform magnetic field $\vec{B} = (1.63\text{ T})\hat{i} + (0.98\text{ T})\hat{j}$?
5. A particle with charge -5.6 nC is moving in a uniform magnetic field $\vec{B} = -(1.25\text{ T})\hat{k}$. The magnetic force on the particle is measured to be

$$\vec{F} = -(3.4\times10^{-7}\text{ N})\hat{i} + (7.4\times10^{-7}\text{ N})\hat{j}$$

(a) Calculate all the components of the velocity of the particle that you can calculate from this information.

(b) Are there components of the velocity that are not determined by the measurement of the force? Explain.

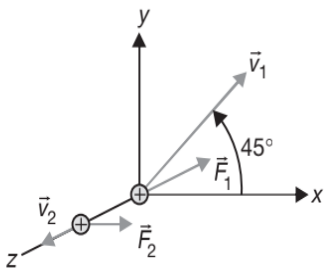
(c) Calculate the scalar product $\vec{v}\cdot\vec{F}$. What is the angle between $\vec{v}$ and $\vec{F}$?

6. To hit a target from several meters away with a charged coin having a mass of 5 g and a charge of $+2500\ \mu C$, a coin is given an initial velocity of $12.8\ ms^{-1}$. A downward, uniform electric field with field strength $27.5\ NC^{-1}$ exists throughout the region. Assuming that you aim directly at the target and fire the coin horizontally, what magnitude and direction of uniform magnetic field are needed in the region for the coin to hit the target?

7. A charged particle of charge $1\ mC$ and mass $2\ g$ is moving with a speed of $5\ ms^{-1}$ in a uniform magnetic field of $0.5\ T$. Calculate the maximum acceleration of the charged particle.

8. When a proton has a velocity $\vec{v}=(2\hat{i}+3\hat{j})\times10^6\ ms^{-1}$ it experiences a force $\vec{F}=-(1.28\times10^{-13}\hat{k})\ N$. When its velocity is along the z-axis, it experiences a force along the x-axis. What is the magnetic field?

9. When a particle of charge $q>0$ moves with a velocity of $\vec{v}_1$ at $45°$ from the $+x$-axis in the xy-plane, a uniform magnetic field exerts a force $\vec{F}_1$ along the $-z$-axis. When the same particle moves with a velocity $\vec{v}_2$ with the same magnitude as $\vec{v}_1$ but along the $+z$-axis, a force $\vec{F}_2$ of magnitude F_2 is exerted on it along the $+x$-axis.



(a) What are the magnitude (in terms of q, v_1 and F_2) and direction of the magnetic field?

(b) What is the magnitude of $\vec{F}_1$ in terms of F_2?

10. A charged particle of specific charge (i.e. charge unit mass) $0.2\ Ckg^{-1}$ has velocity $(2\hat{i}-3\hat{j})\ ms^{-1}$ at an instant in a uniform magnetic field $(5\hat{i}+2\hat{j})\ T$. Calculate the acceleration of the particle at this instant.

11. A proton moving in a uniform magnetic field with a velocity $\vec{v}_1=10^6\hat{i}\ ms^{-1}$ experiences a force $\vec{F}_1=1.2\times10^{-16}\hat{k}\ N$. A second proton moving in the same field with a velocity $\vec{v}_2=2\times10^6\hat{j}\ ms^{-1}$ experiences a force $\vec{F}_2=-4.16\times10^{-16}\hat{k}\ N$. Calculate the magnetic field $\vec{B}$? Give your answer as a magnitude and an angle measured CCW from the $+x$-axis.

12. A particle with charge $9.45\times10^{-8}\ C$ is moving in a region where there is a uniform magnetic field of $0.45\ T$ in the $+x$-direction. At a particular instant of time the velocity of the particle has components

$$v_x=-1.68\times10^4\ ms^{-1},$$

$$v_y=-3.11\times10^4\ ms^{-1}\text{ and}$$

$$v_z=5.85\times10^4\ ms^{-1}$$

What are the components of the force on the particle at this time?

13. A small ball having mass m, charge q is suspended from a rigid support by means of a light inextensible string of length l. It is made to revolve on a horizontal circular path in a uniform magnetic field B directed vertically upwards. If the time period of revolution of the ball is T_0 and the thread is always tight, then calculate the radius of circular path on which the ball moves.

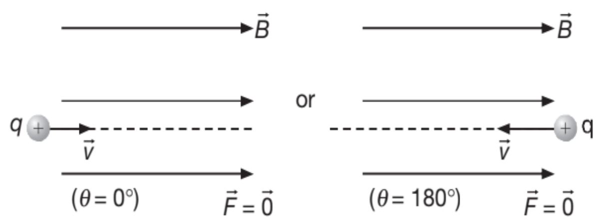
PATH OF A CHARGED PARTICLE IN UNIFORM MAGNETIC FIELD

When a charged particle moves in a magnetic field, it is acted upon by a magnetic force given by $|\vec{F}| = qvB\sin\theta$ and the motion is determined by Newton's Laws.

The path of a charged particle in uniform magnetic field depends on the angle θ (the angle between $\vec{v}$ and $\vec{B}$). Based on the different values of θ, following three cases are possible.

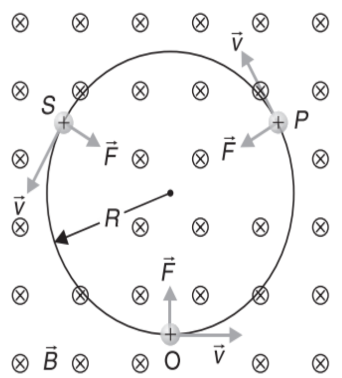
CASE-1: When θ is 0° or 180° (= π radian)

Since $|\vec{F}| = qvB\sin\theta$, so when θ is either 0° or 180° i.e. the charged particle enters the field with a velocity either parallel to it $(\theta = 0°)$ or anti parallel to the field $(\theta = 180°)$, then we have $|\vec{F}| = 0$. Hence, path of the charged particle is a straight line (undeviated) when it enters parallel or antiparallel to the magnetic field.



CASE-2: When the charged particle moves perpendicular to the field, i.e., $\theta = 90°$ or $\frac{\pi}{2}$ radian

Consider a positive point charge q of mass m at **point** O, moving with a velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$ directed into the plane of the page as shown in Figure.



The vectors $\vec{v}$ and $\vec{B}$ are perpendicular to each other initially as well as at later instants of time. The magnetic force $\vec{F} = q(\vec{v} \times \vec{B})$ will have a magnitude $F = qvB\sin 90° = qvB$ and a direction as shown in the figure. We observe that the force is always perpendicular to $\vec{v}$ and hence cannot change the magnitude of the velocity, but can change its direction only.

To understand this concept differently, we know that the magnetic force has got no component parallel to the motion of the particle (it always possesses component perpendicular to the particle's motion) and hence cannot do any work on the particle. This is true even if the magnetic field is non-uniform. So, always keep in mind that the motion of a charged particle under the action of a magnetic field alone is always a motion with constant speed.

So, we see that in the situation shown, the magnitudes of both $\vec{F}$ and $\vec{v}$ are constant. At points P and S, the directions of force and velocity have changed as shown, but their magnitudes are still the same. The particle therefore moves under the influence of a constant-magnitude force which is always at right angles to the velocity of the particle and we see that the particle's path is a circle of radius R, traced out with constant speed v. The centripetal acceleration of the particle is $\frac{v^2}{R}$ and the only force acting on the particle is the magnetic force, so from Newton's Second Law, we get

$$F = |q|vB = \frac{mv^2}{R}$$

$$\Rightarrow \quad R = \frac{mv}{|q|B} \text{ (radius of a circular orbit in a magnetic field)}$$

Since $p = mv$ = momentum of the particle, so we can also write $R = \frac{mv}{qB} = \frac{p}{qB}$.

If the charge q is negative, the particle traces the circle in clockwise sense.

The angular speed ω of the particle can be found by using $v = R\omega$. So, we get

$$\omega = \frac{v}{R} = v\frac{|q|B}{mv} = \frac{|q|B}{m}$$

The number of revolutions per unit time is $f = \frac{\omega}{2\pi}$.

This frequency f is independent of the radius R of the path. It is called the **cyclotron frequency**.

If the charged particle has a kinetic energy K, then we have

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

$\Rightarrow \quad p = \sqrt{2mK}$

$$\Rightarrow \quad R = \frac{mv}{|q|B} = \frac{p}{|q|B} = \frac{\sqrt{2mK}}{|q|B}$$

Also, there is a possibility that a charged particle is first accelerated through a potential V to gain a velocity v.

Then, $\frac{1}{2}mv^2 = |q|V$

$$\Rightarrow \quad v = \sqrt{\frac{2|q|V}{m}}$$

$$\Rightarrow \quad R = \frac{mv}{|q|B} = \frac{m}{|q|B}\sqrt{\frac{2|q|V}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{|q|}} = \frac{\sqrt{2|q|mV}}{|q|B}$$

So, finally, we observe that the radius of the particle moving in a circle is given by

$$R = \frac{mv}{|q|B} = \frac{p}{|q|B} = \frac{\sqrt{2mK}}{|q|B} = \frac{\sqrt{2|q|mV}}{|q|B} = \frac{1}{B}\sqrt{\frac{2mV}{|q|}}$$

where

$p =$ momentum of the particle $= mv$

$K =$ kinetic energy of the particle $= \frac{1}{2}mv^2$

$V =$ electrostatic potential through which the charge is accelerated to gain a speed v.

If T be the period of revolution of the particle, then we have

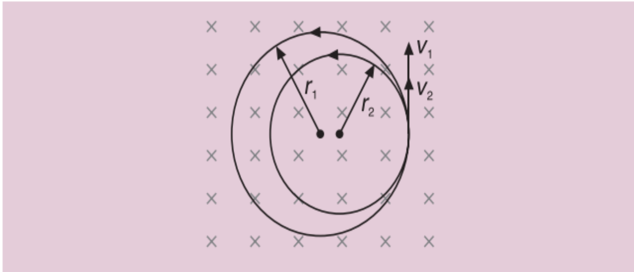
$$T = \frac{2\pi R}{v}$$

$$\Rightarrow \quad T = \frac{2\pi}{v}\left(\frac{mv}{|q|B}\right) = \frac{2\pi m}{|q|B} = \frac{1}{f}$$

$$\Rightarrow \quad T = \frac{2\pi m}{|q|B} = \frac{1}{f}$$

So, we observe that T (the period of revolution), f (the frequency of revolution) and ω (the angular frequency or the angular velocity) are the independent of v.

So, if two identical charge particles are given different speeds from a same point in the same direction, then they will return to the initial point simultaneously.



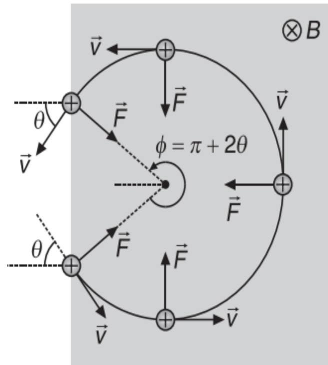
Also, the kinetic energy of the particle is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{qBR}{m}\right)^2 = \frac{q^2B^2R^2}{2m}$$

CHARGED PARTICLE ENTERING INTO MAGNETIC FIELD REGION FROM OUTSIDE

When a charged particle enters a region of uniform magnetic field, such that the particle moves in a circular arc, but cannot complete a full circle.

CASE-1: For a positively charged particle



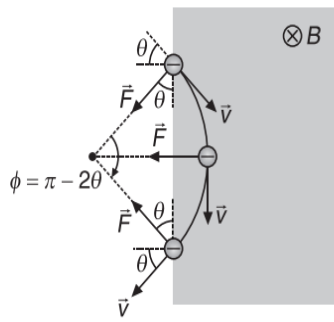
Time for which the charged particle stays inside the field is given by

$\phi = \omega t$, where $\phi = \pi + 2\theta$

$$\Rightarrow \quad t = \frac{\phi}{\omega}, \text{ where } \omega = \frac{qB}{m}$$

$$\Rightarrow \quad t = \left(\frac{m}{qB}\right)\phi = \frac{m(\pi + 2\theta)}{qB}$$

CASE-2: For a negatively charged particle



Time for which the charged particle stays inside the field is given by

$$\phi = \omega t \text{, where } \phi = \pi - 2\theta$$

$$\Rightarrow \quad t = \frac{\phi}{\omega}, \text{ where } \omega = \frac{qB}{m}$$

$$\Rightarrow \quad t = \left(\frac{m}{qB}\right)\phi = \frac{m(\pi - 2\theta)}{qB}$$

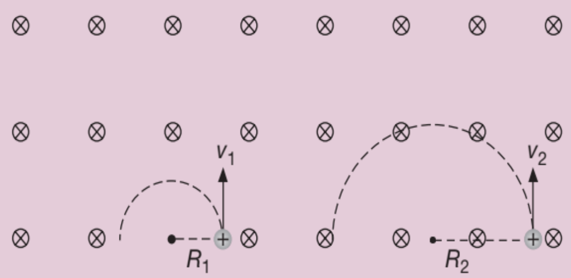
Problem Solving Technique(s)

(a) The above results can also be expressed in terms of the specific charge $\left(\frac{q}{m}\right)$, sometimes denoted by α. So, we get $R = \frac{v}{\alpha B}$, $T = \frac{2\pi}{\alpha B}$, $\omega = \alpha B$, $f = \frac{\alpha B}{2\pi}$

(b) As discussed, T, f and ω are independent of the velocity v, however R (radius of curvature) is directly proportional to v. So, for two identical charged particles (having same q and m) entering at right angles to the field $\vec{B}$ with different speeds v_1 and v_2, where $v_2 > v_1$, we get

$$T_1 = T_2,\ f_1 = f_2 \text{ and } \omega_1 = \omega_2$$

However, $R_2 > R_1$ (as shown)



(c) For a particle moving with velocity at right angles to the field, the plane of the circle is normal to the magnetic field. If the magnetic field is along z-direction, the circular path is in x-y plane. In other words, the plane of the circle is actually the plane that contains both $\vec{F}$ and $\vec{v}$ which of course happens to be normal to $\vec{B}$.

Also note that F acts towards the centre of the circle.

(d) The speed of the particle does not change in magnetic field (as discussed already).

Hence, if v_0 be the speed of the particle, then velocity of particle at any instant of time will be given by

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \text{ where } v_x^2 + v_y^2 = v_0^2$$

If $\vec{u}$ be the initial velocity of the particle, then

$$\vec{u} = u_x\hat{i} + u_y\hat{j}$$

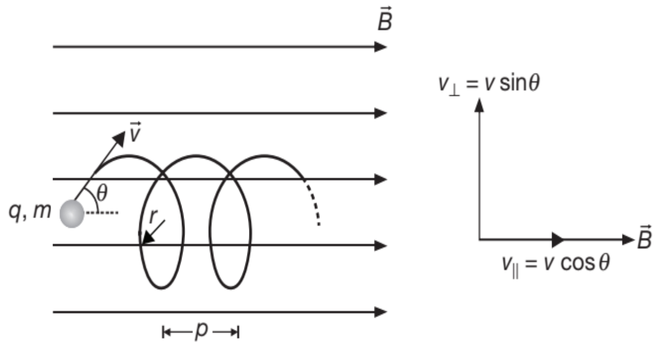
However, $|\vec{v}| = |\vec{u}| = v_0$

$$\Rightarrow \quad u_x^2 + u_y^2 = v_x^2 + v_y^2 = v_0^2$$

This strategy will help us solve problems easily.

CASE-3: When the charged particle moves with velocity not perpendicular to the field i.e. $\theta \neq 0°$, $90°$, $180°$

If the direction of the initial velocity of the charged particle is not perpendicular to the field, then the velocity can be resolved into two components, one along $\vec{B}$ and other perpendicular to $\vec{B}$. Now since the magnetic force has got no component parallel to the field (remember that $\vec{F} \perp \vec{B}$) so the component of velocity parallel to the field remains constant.



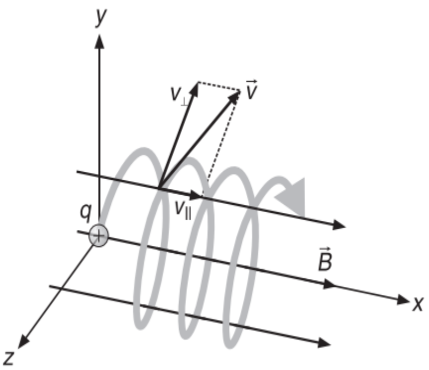
If $v_{\parallel}$ be the component of velocity parallel to the field and $v_{\perp}$ be the component of velocity perpendicular to the field. Since $\vec{B}$ cannot influence the component of velocity parallel to it, so we have

$$v_{\parallel} = v\cos\theta = \text{constant}$$

Also, we have

$$v_{\perp} = v\sin\theta$$

This component of velocity $(v_{\perp})$ perpendicular to field gives a circular path and the component of velocity $(v_{\parallel})$ parallel to the field $(v_{\parallel})$ gives a straight line path. The resultant path is a helical path called **helix** as shown in Figure.



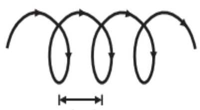
The radius of this helical path is,

$$r = \frac{mv_\perp}{|q|B} = \frac{mv\sin\theta}{|q|B}$$

Time period and frequency do not depend on velocity and so they are given by,

$$T = \frac{2\pi m}{|q|B} \text{ and } f = \frac{|q|B}{2\pi m}$$

Pitch (p) of the helical path is defined as the horizontal forward distance travelled by the particle in one complete cycle as shown in Figure.



So, mathematically pitch is given by

$$p = v_{||}T$$

$$\Rightarrow \quad p = (v\cos\theta)\left(\frac{2\pi m}{|q|B}\right)$$

$$\Rightarrow \quad p = \frac{2\pi mv\cos\theta}{|q|B}$$

Problem Solving Technique(s)

(a) The plane of the circle of the helix is normal to the magnetic field.

(b) The axis of the helix is parallel to magnetic field.

(c) The particle while moving in helical path in magnetic field touches the line passing through the starting point parallel to the magnetic field after every pitch.

For example, a charged particle is projected from origin in a magnetic field (along x-direction) at angle θ from the x-axis as shown. As the velocity vector $\vec{v}$ makes an angle θ with $\vec{B}$ its path is a helix. The plane of the circle of the helix is y-z (perpendicular to magnetic field) and axis of the helix is parallel to x-axis. The particle while moving in helical path touches the x-axis after every pitch, i.e., it will touch the x-axis at a distance.

$$x = np \text{ where } n = 0, 1, 2 \ldots$$

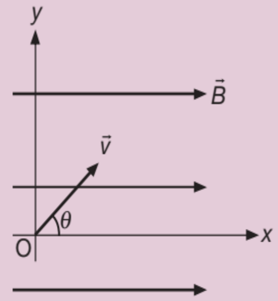
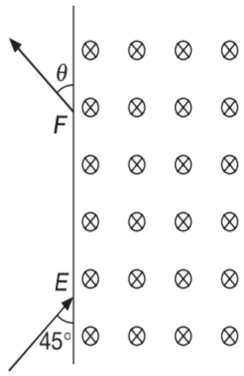


ILLUSTRATION 7

A particle of mass $m = 1.6\times10^{-27}$ kg and charge $q = 1.6\times10^{-19}$ C enters a region of uniform magnetic field of strength 1 T along the direction shown in figure. The speed of the particle is 10^7 ms^{-1}.



(a) The magnetic field is directed along the inward normal to the plane of the paper. The particle leaves the region of the field at the point F. Find the distance EF and the angle θ.

(b) If the direction of the field is along the outward normal to the plane of the paper, find the time spent by the particle in the region of the magnetic field after entering it at E.

SOLUTION

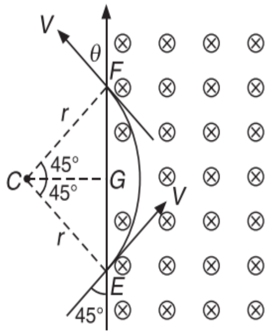
Inside a magnetic field speed of charged particle does not change. Further, velocity is perpendicular to magnetic field in both the cases hence path of the particle in the magnetic field will be circular.

Centre of circle can be obtained by drawing perpendicular to velocity (or tangent to the circular path) at

E and F. Radius and angular speed of circular path would be

$$r = \frac{mv}{qB} \quad \text{and} \quad \omega = \frac{qB}{m}$$

(a) From the figure we observe that



(a)

$$\angle CFG = 90° - \theta$$

and $\angle CEG = 90° - 45° = 45°$

Since, $CF = CE$

$\Rightarrow \angle CFG = \angle CEG$

$\Rightarrow 90° - \theta = 45°$

$\Rightarrow \theta = 45°$

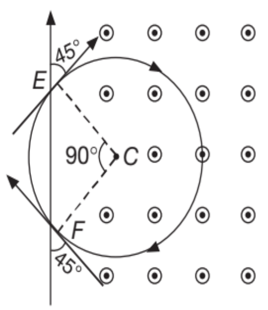
Further, $FG = GE = R\cos 45°$

$$\Rightarrow EF = 2FG = 2r\cos 45° = \frac{2mv\cos 45°}{qB}$$

$$EF = \frac{2(1.6\times10^{-27})(10^7)\left(\frac{1}{\sqrt{2}}\right)}{(1.6\times10^{-19})(1)} = 0.14 \text{ m}$$

In this case, the particle completes $\frac{1}{4}$th of circle in the magnetic field.

(b) In this case, the particle will complete $\frac{3}{4}$th of circle in the magnetic field.



(b)

Hence, the time spent in the magnetic field is

$$t = \frac{3}{4}(\text{time period of circular motion})$$

$$t = \frac{3}{4}\left(\frac{2\pi m}{qB}\right) = \frac{3\pi m}{2qB}$$

$$\Rightarrow t = \frac{(3\pi)(1.6\times10^{-27})}{(2)(1.6\times10^{-19})(1)}$$

$$\Rightarrow t = 4.712\times10^{-8} \text{ s} \cong 47 \text{ ns}$$

ILLUSTRATION 8

A beam of protons with a velocity 4×10^5 ms^{-1} enters a uniform magnetic field of 0.3 T at an angle of $60°$ to the magnetic field. Find the radius of the helical path taken by the proton beam. Also find the pitch of the helix.

SOLUTION

$$r = \frac{mv\sin\theta}{qB} = \frac{(1.67\times10^{-27})(4\times10^5)(\sin 60°)}{(1.6\times10^{-19})(0.3)}$$

$$\Rightarrow r = 1.2\times10^{-2} \text{ m} = 1.2 \text{ cm}$$

Since, $p = \left(\frac{2\pi m}{qB}\right)(v\cos\theta)$

$$\Rightarrow p = \frac{(2\pi)(1.67\times10^{-27})(4\times10^5)(\cos 60°)}{(1.6\times10^{-19})(0.3)}$$

$$\Rightarrow p = 4.37\times10^{-2} \text{ m} = 4.37 \text{ cm}$$

Problem Solving Technique(s)

MOTION OF CHARGED PARTICLE

STEP-1

In analysing the motion of a charged particle in electric and magnetic fields, you will apply Newton's Second Law of motion, with the net force given by

$$\Sigma\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

Often other forces such as gravity can be neglected.

STEP-2

Read and analyse the problem carefully. The use of components is the most efficient approach. Select a coordinate system and then express all vector quantities (including $\vec{E}$, $\vec{B}$, $\vec{v}$, $\vec{F}$, and $\vec{a}$) in terms of their components in this system.

STEP-3
If the particle moves perpendicular to a uniform magnetic field, the trajectory is a circle whose radius and angular speed are given by

$$R=\frac{mv}{|q|B} \text{ and } \omega=\frac{|q|B}{m}$$

STEP-4
If our calculation involves a more complex trajectory then use $\Sigma\vec{F}=m\vec{a}$ in component form $\Sigma F_x=ma_x$, and so forth. This approach is particularly useful when both electric and magnetic fields are present.

STEP-5
If $\vec{u}=u_x\hat{i}+u_y\hat{j}$ be the initial velocity of the particle and $\vec{v}=v_x\hat{i}+v_y\hat{j}$ be the final velocity of the particle in a magnetic field, then

$$|\vec{v}|=|\vec{u}|$$

$$\Rightarrow\quad v_x^2+v_y^2=u_x^2+u_y^2$$

ILLUSTRATION 9

A particle having mass m and charge q, enters a uniform magnetic field $\vec{B}=-B_0\hat{k}$ with velocity $\vec{v}=v_0\hat{i}$ from the origin. Find the time dependence of velocity and position of the particle.

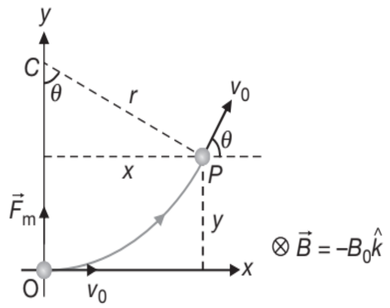
SOLUTION

In such type of problems, we should follow the steps given.

STEP-1:
First of all see the angle between $\vec{v}$ and $\vec{B}$. Because it only decides the path of the particle. Here, the angle is $90°$. Therefore, the path is a circle.

STEP-2:
If it is a circle, see the plane of the circle (normal to the magnetic field). Here the plane is x-y.

STEP-3:
Apply Fleming's Left Hand Rule to find the sense of rotation of the particle, which happens to be counter clockwise as shown in Figure.



$OC = CP =$ radius of circle

STEP-4:

Now find the deviation and radius of the particle,

$$\theta=\omega t=\left(\frac{qB_0}{m}\right)t \text{ and } r=\frac{mv_0}{qB_0}$$

STEP-5:
Now from the figure, find $\vec{v}(t)$ and $\vec{r}(t)$.
Since velocity of the particle at any time t is,

$$\vec{v}(t)=v_x\hat{i}+v_y\hat{j}=v_0\cos\theta\hat{i}+v_0\sin\theta\hat{j}$$

$$\Rightarrow\quad \vec{v}(t)=v_0\cos(\omega t)\hat{i}+v_0\sin(\omega t)\hat{j}\text{, where}$$

$$\omega=\frac{qB_0}{m}$$

Position of particle at time t is,

$$\vec{r}(t)=x\hat{i}+y\hat{j}=r\sin\theta\hat{i}+(r-r\cos\theta)\hat{j}$$

$$\vec{r}(t)=r\left((\sin\theta)\hat{i}+(1-\cos\theta)\hat{j}\right)$$

Substituting the values of r and θ, we have

$$\vec{r}(t)=\frac{mv_0}{qB_0}\left(\sin\left(\frac{qB_0t}{m}\right)\hat{i}+\left(1-\cos\left(\frac{qB_0t}{m}\right)\right)\hat{j}\right)$$

ILLUSTRATION 10

A particle of charge q and mass m is projected from origin with velocity $\vec{v}=v_0\left(\hat{i}-\hat{k}\right)$ in a uniform magnetic field $\vec{B}=-B_0\hat{k}$. Find the velocity and position of the particle as a function of time t.

SOLUTION

Let us first calculate the angle between $\vec{v}$ and $\vec{B}$. If θ is the angle between $\vec{v}$ and $\vec{B}$, then

$$\theta=\cos^{-1}\left(\frac{\vec{v}\cdot\vec{B}}{|\vec{v}||\vec{B}|}\right)=\cos^{-1}\left(\frac{B_0v_0}{\sqrt{2}v_0B_0}\right)$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

Since this happens to be an angle other than $0°$, $90°$ and $180°$, hence, the path followed by the particle is a helix. The axis of the helix is along z-axis (parallel to $\vec{B}$) and plane of the circle of helix is the xy plane (normal to $\vec{B}$). So, in x-y plane, the velocity components and x and y co-ordinates are same as the ones calculated in **ILLUSTRATION 9**. The only change is in the values along z-axis. Velocity component in this direction will remain unchanged while the z-coordinate of particle at time t would be $v_z t$.

Velocity of particle at time t is,

$$\vec{v}(t) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\vec{v}(t) = v_0\cos\left(\frac{qB_0t}{m}\right)\hat{i} + v_0\sin\left(\frac{qB_0t}{m}\right)\hat{j} - v_0\hat{k}$$

v_x and v_y have been calculated in the similar manner as done in **ILLUSTRATION 9.**

The position of the particle at time t would be,

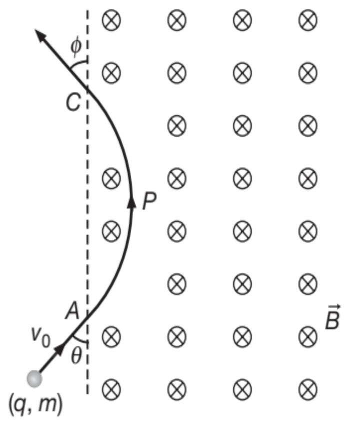
$$\vec{r}(t) = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where, $z = v_z t = -v_0 t$ and x and y are same as calculated in **ILLUSTRATION 9**. Hence,

$$\vec{r} = \frac{mv_0}{qB_0}\left[\sin\left(\frac{qB_0t}{m}\right)\hat{i} + \left(1-\cos\left(\frac{qB_0t}{m}\right)\right)\hat{j}\right] - v_0t\hat{k}$$

ILLUSTRATION 11

A charged particle of mass m, charge q enters a uniform magnetic field $\vec{B}$ with speed v_0 at angle θ as shown in figure.

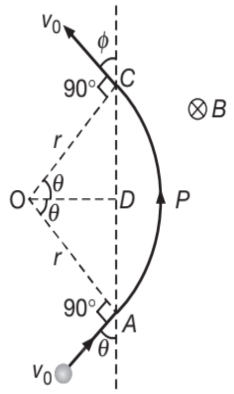


Calculate the

(a) angle ϕ at which it leaves the magnetic field.
(b) time spent by the particle in magnetic field and
(c) separation AC.

SOLUTION

(a) Here, velocity of the particle is in the plane of paper while the magnetic field is perpendicular to the paper inwards. i.e., angle between $\vec{v}$ and $\vec{B}$ is $90°$.



So, the path is a circle, having radius $r = \frac{mv_0}{qB}$

If, O is the centre of the circle, then in ΔAOC,

$\angle OCD = \angle OAD$

$\Rightarrow 90° - \phi = 90° - \theta$

$\Rightarrow \phi = \theta$

(b) $\angle COD = \angle DOA = \theta$

$\{\because \angle OCD = \angle OAD = 90° - \theta\}$

$\Rightarrow \angle AOC = 2\theta$

Length of arc $APC = r(2\theta) = \left(\frac{2mv_0}{qB}\right)\theta$

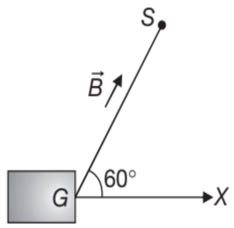
$$\Rightarrow t_{APC} = \frac{APC}{v_0} = \frac{2m\theta}{qB}$$

(c) Separation $AC = 2(AD) = 2(r\sin\theta)$

$$AC = \left(\frac{2mv_0}{qB}\right)\sin\theta$$

ILLUSTRATION 12

An electron gun G emits electrons of energy 2 keV travelling in the positive X-direction. The electrons are required to hit the spot S where $GS = 0.1$ m, and the line GS makes an angle of $60°$ with the x-axis as shown in figure. A uniform magnetic field $\vec{B}$ parallel to GS exists in the region outside the electron gun. Find the minimum value of B needed to make the electrons hit S.



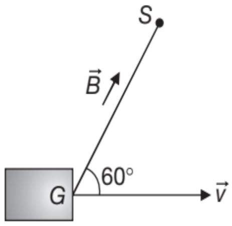
SOLUTION

Kinetic energy of electron, $K=\frac{1}{2}mv^2=2\text{ keV}$

So, the speed of electrons hitting the spot S is

$$v=\sqrt{\frac{2K}{m}}=\sqrt{\frac{2\times2\times1.6\times10^{-16}}{9.1\times10^{-31}}}\text{ ms}^{-1}$$

$$\Rightarrow\quad v=2.65\times10^{7}\text{ ms}^{-1}$$



Since, the velocity $(\vec{v})$ of the electron makes an angle of $\theta=60^\circ$ with the magnetic field $\vec{B}$, the path will be a Helix of pitch p.

So, the particle will hit S if we have

$GS=np$, where $n=1,2,3,4,\ldots$

and $p=$ pitch of helix $=\frac{2\pi mv\cos\theta}{qB}$

However, for B to be minimum we must have $n=1$

$$\Rightarrow\quad GS=p=\frac{2\pi m}{qB}v\cos\theta$$

$$\Rightarrow\quad B=B_{\text{min}}=\frac{2\pi mv\cos\theta}{q(GS)}$$

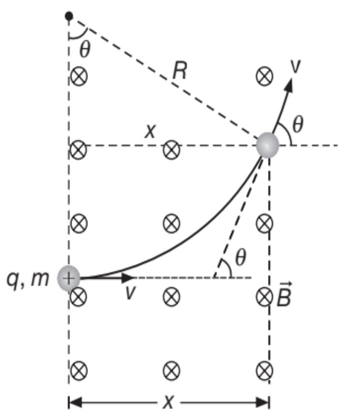
Substituting the values, we get

$$B_{\text{min}}=\frac{(2\pi)(9.1\times10^{-31})(2.65\times10^{7})\left(\frac{1}{2}\right)}{(1.6\times10^{-19})(0.1)}\text{ T}$$

$$\Rightarrow\quad B_{\text{min}}=4.73\times10^{-3}\text{ T}$$

DEVIATION OF A CHARGED PARTICLE IN MAGNETIC FIELD

Consider a charged particle of mass m, charge q entering a uniform magnetic field $\vec{B}$ at right angles with speed v as shown in Figure.



Let the magnetic field extend in a region of space upto a length x. The path of the particle is a circle of radius R, where

$$R=\frac{mv}{|q|B}$$

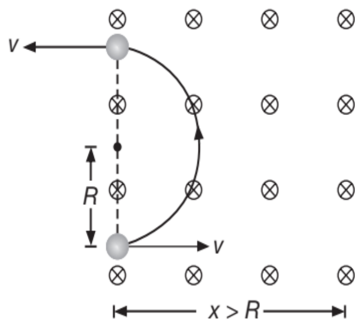
The speed of the particle in magnetic field does not change. But it gets deviated in the magnetic field. The angular deviation θ can be found and expressed as a function of time t or in terms of the length of the magnetic field space.

(a) After time t, deviation will be,

$$\theta=\omega t=\left(\frac{|q|B}{m}\right)t\text{ as }\omega=\frac{|q|B}{m}$$

(b) In terms of the length of the magnetic field (i.e., when the particle leaves the magnetic field) the deviation will be,

$$\theta=\begin{cases}\sin^{-1}\left(\frac{x}{R}\right) & \text{for } x\le R\\ \pi(\text{or }180^\circ) & \text{for } x>R\end{cases}$$



Please note that the deviation, when x is slightly less than R is $\frac{\pi}{2}$. Also note that the deviation suffered by a charged particle is given by

$$\theta=\sin^{-1}\left(\frac{x}{R}\right)$$

$$\Rightarrow\quad \sin\theta=\frac{x}{R}\text{ where}$$

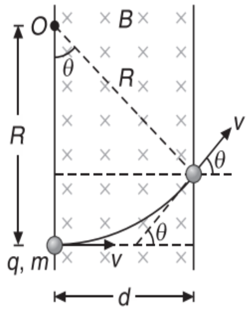
$$R=\frac{mv}{|q|B}=\frac{p}{|q|B}=\frac{\sqrt{2mE}}{|q|B}=\frac{\sqrt{2m|q|V}}{|q|B}$$

ILLUSTRATION 13

An α-particle is accelerated by a potential difference of 10^4 V. Find the change in its direction of motion, if it enters normally in a region of thickness 0.1 m having transverse magnetic induction of 0.1 tesla. Given: mass of α-particle 6.4×10^{-27} kg.

SOLUTION

The situation is shown in figure.



When a charged particle with charge q is accelerated through a potential difference V volt, then it gains a speed given by

$$\frac{1}{2}mv^2 = qV$$

$$\Rightarrow \quad v = \sqrt{\frac{2qV}{m}} \qquad \ldots(1)$$

When α-particle enters the magnetic field, it moves in a circle of radius R as shown in figure.

Since $R = \dfrac{mv}{qB}$

$$\Rightarrow \quad R = \frac{1}{B}\sqrt{\frac{2mV}{q}}$$

Given that $A = 0.1$ m, $B = 0.1$ T $V = 10^4$ volt

Also, $q_\alpha = 2e = 2\times1.6\times10^{-19} = 32\times10^{-19}$ C

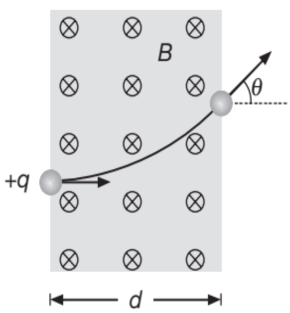
and $m_\alpha = 6.4\times10^{-27}$ kg

$$\Rightarrow \quad \sin\theta = (0.1)(0.1)\sqrt{\frac{3.2\times10^{-19}}{2\times6.4\times10^{-27}\times10^4}}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

ILLUSTRATION 14

A particle of mass m and charge q is projected into a region having a perpendicular uniform magnetic field B.

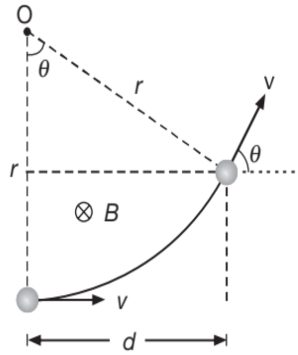


Find the angle of deviation θ of the particle as it comes out of the magnetic field if width d of the region is

(a) $\dfrac{mv}{2qB}$ (b) $\dfrac{mv}{qB}$ (c) $\dfrac{2mv}{qB}$

SOLUTION

(a) The radius of the circular orbit is given by $r = \dfrac{mv}{qB}$.

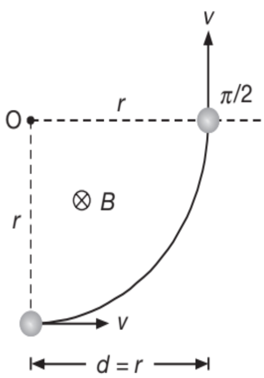


So, the angle of deviation for $d = \dfrac{mv}{2qB} = \dfrac{r}{2}$ is

$$\sin\theta = \frac{d}{r} = \frac{1}{2}$$

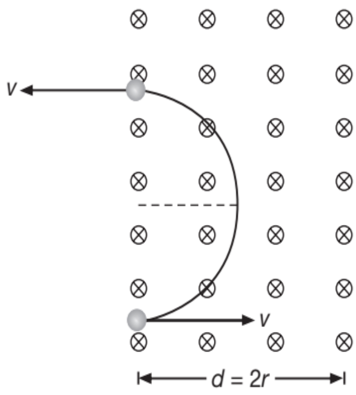
$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} = 30^\circ$$

(b) When $d = \dfrac{mv}{qB} = r$, the charged particle deviates through an angle of $\dfrac{\pi}{2}$ as shown in Figure.



So, we have $\theta = \dfrac{\pi}{2}$

(c) When $d = 2\left(\dfrac{mv}{qB}\right) = 2r$, the charged particle completes one semi-circle and deviates through π, as shown in Figure.



So, we get $\theta = \pi$

ILLUSTRATION 15

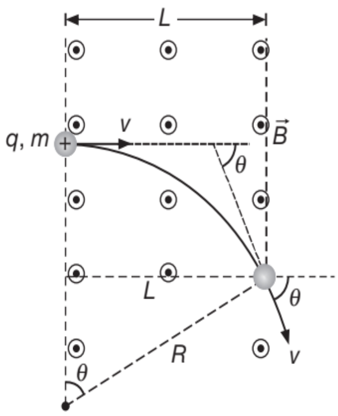
The region between $x = 0$ and $x = L$ is filled with uniform steady magnetic field $B_0\hat{k}$. A particle of mass m, positive charge q and velocity $v_0\hat{i}$ travels along x-axis and enters the region of the magnetic field. Neglecting gravity throughout the question, find the

(a) value of L if the particle emerges from the region of magnetic field with its final velocity at an angle $30°$ to its initial velocity.

(b) final velocity of the particle and the time spent by it in the magnetic field, if the magnetic field now expands upto 2.1 L.

SOLUTION

The situation is shown in Figure.



(a) For $\theta = 30°$

Since, we know that $\sin\theta = \dfrac{L}{R}$

where, $R = \dfrac{mv_0}{qB_0}$

$$\Rightarrow \sin(30°) = \frac{L}{\dfrac{mv_0}{qB_0}}$$

$$\Rightarrow \frac{1}{2} = \frac{qB_0L}{mv_0}$$

$$\Rightarrow L = \frac{mv_0}{2qB_0}$$

(b) Since, we know that $\sin\theta = \dfrac{L}{R}$

$$\Rightarrow \sin(30°) = \frac{L}{R}$$

$$\Rightarrow \frac{1}{2} = \frac{L}{R}$$

$$\Rightarrow L = \frac{R}{2}$$

Now when $L' = 2.1L = \dfrac{2.1R}{2} = 1.05R$

$$\Rightarrow L' > R$$

Therefore, deviation of the particle is $\theta = 180°$, so that

$$\vec{v}_f = -v_0\hat{i} = \vec{v}_B \text{ and } t_{AB} = \frac{T}{2} = \frac{\pi m}{qB_0}$$

ILLUSTRATION 16

A proton is at rest at the plane vertical boundary of a region containing a uniform vertical magnetic field B. An alpha particle moving horizontally makes a head-on elastic collision with the proton. Immediately after the collision, both particles enter the magnetic field, moving perpendicular to the direction of the field. The radius of the proton's trajectory is R. Find the radius of the alpha particle's trajectory.

SOLUTION

Let u represent the original speed of the alpha particle. Let v_α and v_p represent the particles speeds after the collision. By Law of Conservation of Momentum, we have $4m_pu = 4m_pv_\alpha + m_pv_p$ and the relative velocity equation gives

$$u - 0 = v_p - v_\alpha \qquad \{\because e = 1\}$$

Eliminating u,

$$4v_p - 4v_\alpha = 4v_\alpha + v_p$$

$$\Rightarrow 3v_p = 8v_\alpha$$

$$\Rightarrow v_\alpha = \frac{3}{8}v_p$$

For the proton's motion in the magnetic field,

$$\sum F = ma$$

$$\Rightarrow \quad ev_pB\sin(90°) = \frac{m_pv_p^2}{R}$$

$$\Rightarrow \quad \frac{eBR}{m_p} = v_p$$

For the alpha particle,

$$q_\alpha v_\alpha B\sin 90° = \frac{m_\alpha v_\alpha^2}{r_\alpha}$$

$$\Rightarrow \quad (2e)v_\alpha B = \frac{(4m_p)v_\alpha^2}{r_\alpha}$$

$$\Rightarrow \quad r_\alpha = \frac{2m_pv_\alpha}{eB}$$

$$\Rightarrow \quad r_\alpha = \frac{2m_p}{eB}\frac{3}{8}v_p = \frac{2m_p}{eB}\frac{3}{8}\frac{eBR}{m_p} = \frac{3}{4}R$$

ILLUSTRATION 17

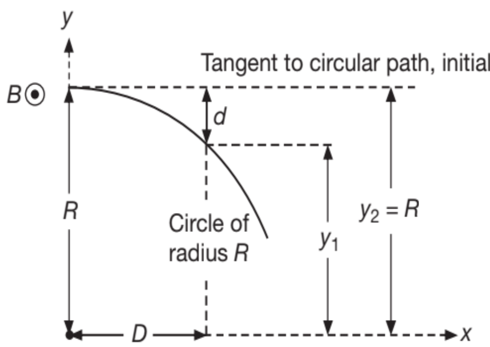
In the electron gun of a TV picture tube the electrons (charge $-e$, mass m) are accelerated by a voltage V. After leaving the electron gun, the electron beam travels a distance D to the screen through a region having transverse magnetic field of magnitude B and no electric field. Sketch the path of the electron beam in the tube. Show that the approximate deflection of the beam due to this magnetic field is

$$d = \frac{BD^2}{2}\sqrt{\frac{e}{2mV}}$$

Calculate the value of d if $V = 750$ V, $D = 50$ cm and $B = 5\times10^{-5}$ T. Is this deflection significant?

SOLUTION

The path is sketched in figure. It is circular with radius $R = \frac{mv}{qB}$. Here we must note that $R \gg D$



Since the motion is circular, therefore we have $x^2 + y^2 = R^2$. So, at $x = D$ we get $y_1 = \sqrt{R^2 - D^2}$ (path of deflected particle).

$y_2 = R$ (equation for tangent to the circle, path of undeflected particle).

$$d = y_2 - y_1 = R - \sqrt{R^2 - D^2}$$

$$\Rightarrow \quad d = R - R\sqrt{1 - \frac{D^2}{R^2}}$$

$$\Rightarrow \quad d = R\left[1 - \sqrt{1 - \frac{D^2}{R^2}}\right]$$

When $R \gg D$ we get,

$$d \approx R\left[1 - \left(1 - \frac{1}{2}\frac{D^2}{R^2}\right)\right] = \frac{D^2}{2R}$$

For a particle moving in a magnetic field,

$$R = \frac{mv}{qB}$$

But $\frac{1}{2}mv^2 = qV$

$$\Rightarrow \quad R = \frac{1}{B}\sqrt{\frac{2mV}{q}}$$

Thus, the deflection d is given by

$$d \approx \frac{D^2B}{2}\sqrt{\frac{q}{2mV}} = \frac{D^2B}{2}\sqrt{\frac{e}{2mV}}$$

$$\Rightarrow \quad d = \frac{(0.5)^2(5\times10^{-5})}{2}\sqrt{\frac{1.6\times10^{-19}}{2(9.11\times10^{-31})(750)}}$$

$$\Rightarrow \quad d = 0.067 \text{ m} = 6.7 \text{ cm}$$

Since we see that $d \approx 13\%$ of D, which is fairly significant.

Also, here we observe that

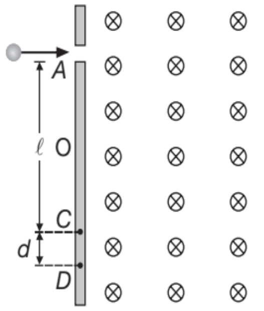
$$R = \frac{1}{B}\sqrt{\frac{2mV}{e}} = \frac{D^2}{2d} = \left(\frac{D}{2d}\right)D = 3.7D$$

$$\Rightarrow \quad \left(\frac{R}{D}\right)^2 = 14$$

So, the approximation made i.e., $R \gg D$ is valid.

ILLUSTRATION 18

A beam of equally charged particles after being accelerated through a voltage V enters into a magnetic field B as shown in the figure. It is found that all the particles hit the plate between C and D. Find the ratio between the masses of the heaviest and lightest particles of the beam.



SOLUTION

When accelerated through a potential V, the particles gain a velocity v such that

$$\frac{1}{2}mv^2 = qV$$

$$\Rightarrow \quad v = \sqrt{\frac{2qV}{m}}$$

When the particles enter the field, they follow a circular track $(\because \vec{v} \perp \vec{B})$ of radius r, where

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{q}}$$

$$\Rightarrow \quad r \propto \sqrt{m}$$

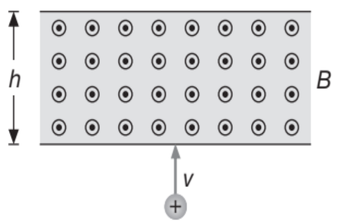
$$\Rightarrow \quad m \propto r^2$$

i.e., the heavier particles will hit the point D and have a radius $(d+l)$ whereas the light particles will hit the point C and have a radius l.

$$\Rightarrow \quad \frac{m_{\text{Heavy}}}{m_{\text{Light}}} = \left(\frac{d+l}{l}\right)^2 = \left(1+\frac{d}{l}\right)^2$$

ILLUSTRATION 19

As shown in figure, a particle of mass m having positive charge q is initially travelling with velocity $v\hat{j}$. At the origin of coordinates it enters a region between $y=0$ and $y=h$ containing a uniform magnetic field $B\hat{k}$ directed perpendicularly out of the page.



(a) What is the critical value of velocity, $v = v_c$, such that the particle just reaches $y = h$? Describe the path of the particle under this condition and predict its final velocity.

(b) Specify the path that the particle takes and its final velocity, if v is less than the critical value.

(c) Specify the path that the particle takes and its final velocity if v is greater than the critical value.

SOLUTION

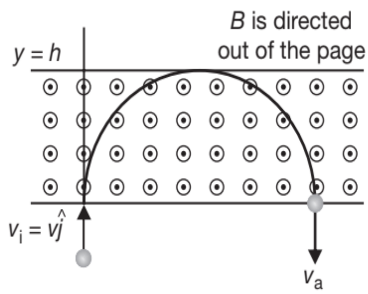
When the particle is in the field, then it follows a circular path of radius r.

Since $qvB = \dfrac{mv^2}{r}$

$$\Rightarrow \quad r = \frac{mv}{qB}$$

(a) For the particle just to reach h, we have

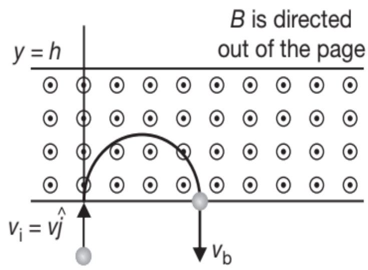
$$r = h = \frac{mv_c}{qB}$$



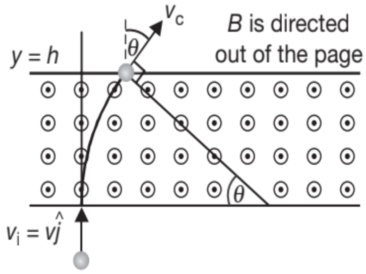
$$\Rightarrow \quad v_c = \frac{qBh}{m},$$

So, the particle will cross the band of field. It will move in a full semicircle of radius h, leaving the field at $(2h, 0, 0)$ with velocity $v_f = -v_c\hat{j}$

(b) When $v < v_c$ i.e., $v < \dfrac{qBh}{m}$, the particle will move in a smaller semicircle of radius $r = \dfrac{mv}{qB} < h$. It will leave the field at $(2r, 0, 0)$ with velocity $v_f = -v\hat{j}$



(c) When $v > \dfrac{qBh}{m}$, the particle moves in a circular arc of radius $r = \dfrac{mv}{qB} > h$, centred at $(r, 0, 0)$. The arc subtends an angle given by $\theta = \sin^{-1}\left(\dfrac{h}{r}\right)$.



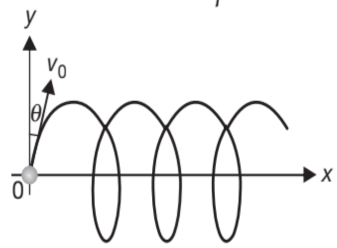
It will leave the field at the point with coordinates $[r(1-\cos\theta),\ h,\ 0]$ with velocity $v_f = v\sin\theta\hat{i} + v\cos\theta\hat{j}$

ILLUSTRATION 20

A particle of mass m and change q is lying at the origin in a uniform magnetic field B directed along x-axis. At time $t=0$, it is given a velocity v_0 at an angle θ with the y-axis in the xy-plane. Find the coordinates of the particle after one revolution.

SOLUTION

After one revolution the particle will have a forward motion equal to the pitch of the helix, so its x component equals pitch $p = \dfrac{2\pi m v_0 \cos(90-\theta)}{qB}$



$$\Rightarrow \quad x = \frac{2\pi m v_0 \sin\theta}{qB}$$

However, y and z coordinates still continue to be zero. Hence, after one revolution, the coordinates are

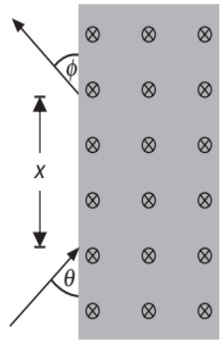
$$(x,\ y,\ z) \equiv \left(\frac{2\pi m v_0 \sin\theta}{qB},\ 0,\ 0\right)$$

Test Your Concepts-II

Based on Charged Particle in a Magnetic Field

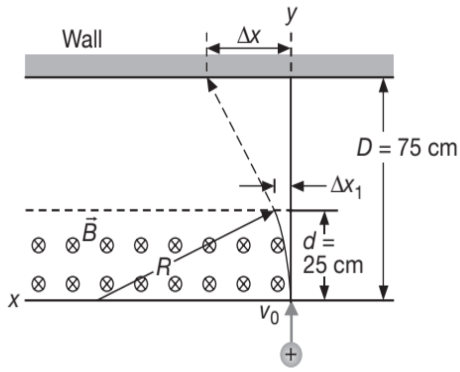
(Solutions on page H.5)

1. In a region of space, a beam of charged particles travels in a straight line. What can be the conclusion made by you about the magnetic field in the region. Assume that no electric field exists in the region.
2. In a region of space where only magnetic field is present, an electron beam projected along positive x-axis deflects along the positive y-axis. What is the direction of the field?
3. Can a charged particle be accelerated by a magnetic field? Can its speed be increased?
4. An electron and a proton are projected with same velocity perpendicular to a magnetic field.
   (a) Which particle will describe the smaller circle?
   (b) Which particle will have greater frequency?
5. A proton moving in the plane of the page has a kinetic energy of 6 MeV. A magnetic field of magnitude $B = 1$ T is directed into the page. The proton enters the magnetic field with its velocity vector at an angle $\theta = 45°$ to the linear boundary of the field as shown in figure.



   (a) Find x, the distance from the point of entry to where the proton will leave the field.
   (b) Determine ϕ, the angle between the boundary and the proton's velocity vector as it leaves the field.
6. A particle with charge $2.15\ \mu C$ and mass 3.2×10^{-11} kg is initially travelling in the $+y$-direction with a speed $v_0 = 1.45\times10^5\ ms^{-1}$ and then enters a region containing a uniform magnetic field that is directed inwards as shown. The magnitude of the field is 0.42 T. The region extends

a distance of 25 cm along the initial direction of travel; 75 cm from the point of entry into the magnetic field region is a wall. The length of the field free region is thus 50 cm. When the charged particle enters the magnetic field, it follows a curved path whose radius of curvature is R. It then leaves the magnetic field after a time t_1, having been deflected a distance Δx_1. The particle then travels in the field free region and strikes the wall after undergoing a total deflection Δx.



(a) Determine the radius R of the curved part of the path.

(b) Determine t_1, the time the particle spends in the magnetic field.

(c) Determine Δx_1, the horizontal deflection at the point of exit from the field.

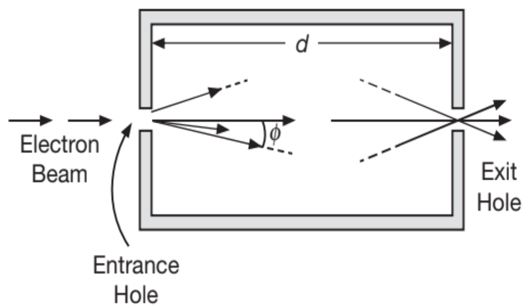
(d) Determine Δx, the total horizontal deflection.

7. A proton of charge e and mass m enters a uniform magnetic field $\vec{B} = B\hat{i}$ with an initial velocity $\vec{v} = v_x\hat{i} + v_y\hat{j}$. Find an expression in unit-vector notation for its velocity at time t.

8. A singly charged ion of mass m is accelerated from rest by a potential difference ΔV. It is then deflected by a uniform magnetic field (perpendicular to the ion's velocity) into a semicircle of radius R. Now a doubly charged ion of mass m' is accelerated through the same potential difference and deflected by the same magnetic field into a semicircle of radius $R' = 2R$. What is the ratio of the masses of the ions?

9. A negatively charged particle (of mass m and charge q) is thrown with speed v along x-axis in a region of uniform, magnetic field B directed along positive z-axis. If thickness of the region is $d\left(<\dfrac{mv}{qB}\right)$, calculate the angular deviation suffered by the particle as it comes out of the magnetic field.

10. Electrons in a beam are accelerated from rest through a potential difference ΔV. The beam enters an experimental chamber through a small hole, as shown in figure. The electron velocity vectors lie within a narrow cone of half angle ϕ oriented along the beam axis. Calculate the magnitude of a uniform magnetic field directed parallel to the axis to focus the beam, so that all of the electrons can pass through a small exit hole on the opposite side of the chamber after travelling a length d in the chamber.

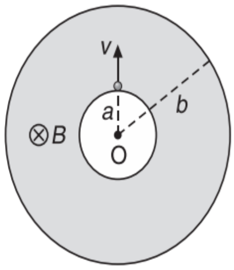


11. A proton (charge $+e$, mass m_p), a deuteron (charge $+e$, mass $2m_p$) and an alpha particle (charge $+2e$, mass $4m_p$) are accelerated through a common potential difference ΔV. Each of the particles enters a uniform magnetic field $\vec{B}$, with its velocity in a direction perpendicular to $\vec{B}$. The proton moves in a circular path of radius r_p. Calculate the radii of the circular orbits for the deuteron, r_d and the alpha particle, r_α, in terms of r_p.

12. One electron collides elastically with a second electron initially at rest. After the collision, the radii of their trajectories are 1 cm and 2.4 cm. The trajectories are perpendicular to a uniform magnetic field of magnitude 0.044 T. Calculate the energy (in keV) of the incident electron.

13. A particle of mass m and charge q is accelerated by a potential difference V and made to enter a magnetic field region at an angle θ with the field. At the same moment another particle of same mass and charge is projected in the direction of the field from the same point. Magnetic field induction is B. What should be the speed of second particle so that both the particles meet again and again after regular minimum interval of time? Also find the time interval after which they meet and the distance travelled by the second particle during that interval.

14. A uniform magnetic field B exists in the annular space enclosed between two cylindrical shells of the inner radius a and outer radius of b as shown in Figure.



An electron is projected from the surface of inner cylindrical shell perpendicular to it with some initial velocity. The magnetic field is directed along the common axis of the cylindrical shells. Calculate the maximum initial velocity with which this electron should be projected so that it will not hit the inner surface of the outer shell.

15. An electron moves in a circular path perpendicular to a constant magnetic field of magnitude 1 mT. The angular momentum of the electron about the centre of the circle is 4×10^{-25} Js. Find the

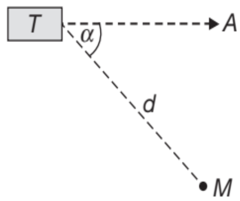
(a) the radius of the circular path and

(b) the speed of the electron.

16. A charged particle having mass m and charge q is accelerated by a potential difference V. It flies through a region having uniform transverse magnetic field B. The field occupies a region of space d. Find the time interval for which it remains inside the magnetic field.

17. A beam of protons with a velocity of 2×10^5 ms^{-1} enters a uniform magnetic field of 0.3 T. The velocity makes an angle of 60° with the magnetic field. Calculate the radius of helical path followed by the proton beam and pitch of the helix.

18. An electron gun emits electrons accelerated by potential difference of 1000 V along direction TA as depicted in figure. We want the electrons leaving T to strike target M in the direction making an angle $\alpha = 60°$ with direction TA at a distance d from T. Given $d = 5$ cm.



(a) Calculate the uniform magnetic field, which is applied normal to the plane ATM so that the electrons leaving T strike the target M.

(b) Also calculate the minimum value of B applied parallel to TM so that the electrons leaving T strike the target M.

LORENTZ FORCE

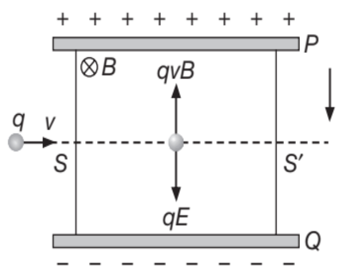
When a particle is subjected to both electric and magnetic fields in the same region, the total force on it is called the **Lorentz force**.

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

We know that the path of particle in a uniform magnetic field is a helix and the path in a uniform electric field is *parabola*. When a particle is subject to both electric and magnetic fields, the motion is in general quite complex. However, the special cases in which the fields are either *parallel* to *perpendicular* to each other are simple to analyse.

CROSSED ELECTRIC AND MAGNETIC FIELDS: VELOCITY SELECTOR

Suppose both the uniform electric and the magnetic fields are applied in the region between two parallel plates P and Q as shown in Figure.



If $\vec{F}=\vec{0}$, then $\vec{E}+\vec{v}\times\vec{B}=\vec{0}$

$\Rightarrow \quad \vec{E}=-\vec{v}\times\vec{B}$

$\Rightarrow \quad \vec{E}=\vec{B}\times\vec{v}$

i.e. $\vec{E}$ must be perpendicular to both $\vec{B}$ and $\vec{v}$.

A positive charge q enters through the slit S perpendicular to both $\vec{E}$ and $\vec{B}$. It is acted upon by forces qE downwards and qvB upwards. If the two forces are equal in magnitude the particle will pass through the region undeflected. So, for equilibrium, we have

$$F_m = F_e$$

$$\Rightarrow \quad qvB = qE$$

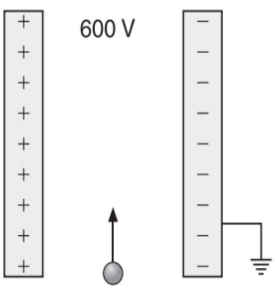
$$\Rightarrow \quad v=\frac{E}{B}$$

This situation is obtained when all three $(v,\ E \text{ and } B)$ are mutually perpendicular

Only those particles will pass out of the slit S' which satisfy the above relation. This device is used to select charged particles of a particular speed out of a beam having particles which move at different speeds and hence, it is called a **velocity selector**.

ILLUSTRATION 21

A potential difference of 600 V is applied across the plates of a parallel plate condenser. The separation between the plates is 3 mm. An electron projected vertically, parallel to the plates, with a velocity of $2\times10^{6}\ \text{ms}^{-1}$ moves undeflected between the plates. Find the magnitude and direction of the magnetic field in the region between the condenser plates. (Neglect the edge effects). (Charge of the electron $=1.6\times10^{-19}$ C).



SOLUTION

Electron passes undeviated, so

$$|\vec{F}_e| = |\vec{F}_m|$$

$$\Rightarrow \quad eE = eBv$$

Since, $E=\dfrac{V}{d}$

$$\Rightarrow \quad B=\frac{E}{v}=\frac{V/d}{v}$$

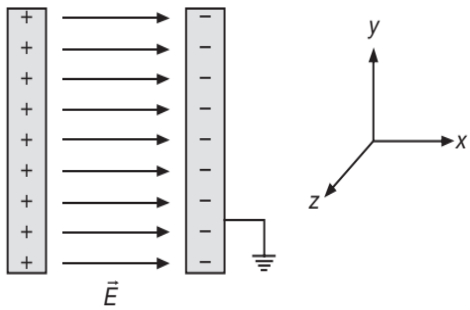
$$\Rightarrow \quad B=\frac{V}{vd}$$

Substituting the values, we have

$$\Rightarrow \quad B=\frac{600}{3\times10^{-3}\times2\times10^{6}}=0.1\ \text{T}$$

Further, for the electron to pass undeviated, direction of $\vec{F}_e$ should be opposite of $\vec{F}_m$ and both must have same magnitude.

Since, $\vec{E}$ is in positive x-direction as shown in Figure, so the electrostatic force on the electron will be opposite to $\vec{E}$ i.e. along negative x-direction and hence the magnetic force $\vec{F}_m = q(\vec{v}\times\vec{B})$ must be directed along positive x-direction.



Therefore $\vec{v}\times\vec{B}$ should be in positive x-direction and hence the magnetic field $\vec{B}$ should be in negative z-direction (or perpendicular) to paper inwards, because velocity of electron is in positive y-direction.

ILLUSTRATION 22

A particle of mass 1×10^{-26} kg and charge $+1.6\times10^{-19}$ C travelling with a velocity $1.28\times10^{6}\ \text{ms}^{-1}$ in the $+X$ direction enters a region in which a uniform electric field E and a uniform magnetic field of induction B are present such that $E_x=E_y=0$, $E_z=-102.4\ \text{kVm}^{-1}$ and $B_x=B_z=0$, $B_y=8\times10^{-2}\ \text{Wbm}^{-2}$. The particle enters this region at the origin at time $t=0$. Determine the location (x, y and z co-ordinates) of the particle at $t=5\times10^{-6}$ s. If the electric field is switched off at this instant (with the magnetic field still present), what will be the position of the particle at $t=7.45\times10^{-6}$ s?

SOLUTION

$$\vec{F}_e = q\vec{E} = \left(1.6\times10^{-19}\right)\left(-102.4\times10^{3}\right)\hat{k}$$

$$\Rightarrow \quad \vec{F}_e = -\left(1.6384\times10^{-14}\right)\hat{k}\ \text{N}$$

$$\vec{F}_m = q\left(\vec{v}\times\vec{B}\right)$$

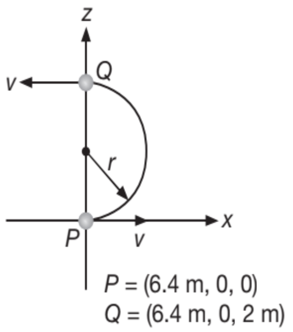
$$\Rightarrow \quad \vec{F}_m = \left(1.6\times10^{-19}\right)\left(1.28\times10^{6}\hat{i}\times8\times10^{-2}\hat{j}\right)$$

$$\Rightarrow \quad \vec{F}_m = \left(1.6384\times10^{-14}\right)\hat{k}\ \text{N}$$

Since, $\vec{F}_e + \vec{F}_m = \vec{0}$, so net force on the charged particle is zero and hence the particle will move undeviated. In time $t = 5\times10^{-6}$ s, the x-coordinate of particle will become,

$$x = v_x t = \left(1.28\times10^{6}\right)\left(5\times10^{-6}\right) = 6.4\ \text{m}$$

The y and z-coordinates will be zero.



At $x = 5\times10^{-6}$ s, electric field is switched-off. Only magnetic field is left which is perpendicular to its velocity. Hence, path of the particle will now become circular. Plane of circle will be normal to magnetic field i.e., x-z plane. Radius and angular velocity of circular path is given by

$$r = \frac{mv}{qB} = \frac{\left(10^{-26}\right)\left(1.28\times10^{6}\right)}{\left(1.6\times10^{-19}\right)\left(8\times10^{-2}\right)} = 1\ \text{m}$$

$$\omega = \frac{qB}{m} = \frac{\left(1.6\times10^{-19}\right)\left(8\times10^{-2}\right)}{\left(10^{-26}\right)} = 1.28\times10^{6}\ \text{rads}^{-1}$$

In the remaining time, i.e.,

$$(7.45-5)\times10^{-6} = 2.45\times10^{-6}\ \text{s}$$

Angle rotated by particle, in this time is

$$\theta = \omega t = \left(1.28\times10^{6}\right)\left(2.45\times10^{-6}\right)$$

$$\Rightarrow \quad \theta = 3.14\ \text{rad} \approx 180^\circ$$

The z-coordinate of particle is then given by

$$z = 2r = 2\ \text{m}$$

and y-coordinate will be zero.

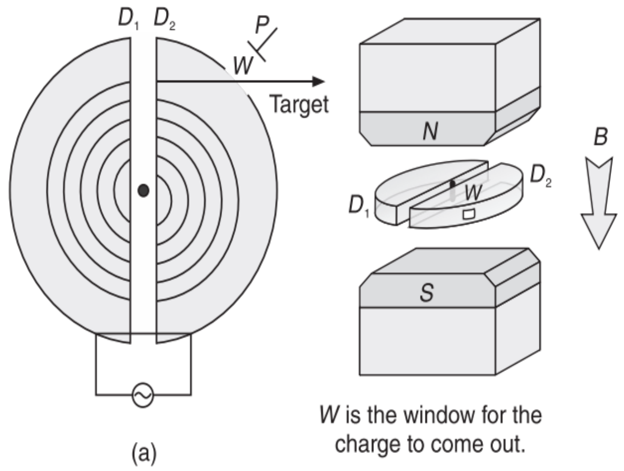
Position of particle at

$t = 5\times10^{-6}$ s is $P \equiv (6.4\ \text{m}, 0, 0)$ and at

$t = 7.45\times10^{-6}$ s is $Q \equiv (6.4\ \text{m}, 0, 2\ \text{m})$

THE CYCLOTRON

It is a device to accelerate charged particles to high speeds. The alternating potential difference source accelerates the particles in the gap between the Dees D_1 and D_2 and the magnetic field makes them move in circular orbits. The frequency of the alternating potential difference is made equal to the cyclotron frequency $\dfrac{qB}{2\pi m}$ so that the polarity of the dees is reversed exactly when the particle comes back in the gap. It then gets accelerated towards the other dee. This process continues till the particle is taken out through a gap with the help of a deflector.



(a) (b)

If r is the radius of the chamber then the speed of the particle circulating at this radius is

$$v = \frac{qBr}{m}$$

Therefore, the kinetic energy of the particle is

$$K = \frac{1}{2}mv^2 = \frac{q^2B^2r^2}{2m}$$

PATH OF A CHARGED PARTICLE IN BOTH ELECTRIC AND MAGNETIC FIELD

When the particle is exposed to simultaneous electric and magnetic field, then the force $\vec{F}$ acting on the particle is

$$\vec{F} = \vec{F}_e + \vec{F}_m$$

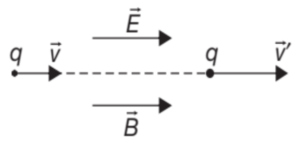
$$\Rightarrow \quad \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

CASE-1: When $\vec{v}$, $\vec{E}$ and $\vec{B}$ are Collinear

As the particle is moving parallel or antiparallel to the magnetic field, the magnetic force on it will be zero. The electric force $\vec{F}_e$ will produce an acceleration,

$$\vec{a} = \frac{\vec{F}_e}{m} = \frac{q\vec{E}}{m}$$

The particle follows a straight-line path with change in speed. So, in this situation speed, velocity, momentum and kinetic energy all will change, without change in direction, of motion as shown in figure.



$\vec{v}$, $\vec{E}$ and $\vec{B}$ are collinear

At any instant, the velocity of the particle is $v = at = \left(\frac{qE}{m}\right)t$, displacement is $s = \frac{1}{2}at^2 = \left(\frac{qE}{2m}\right)t^2$.

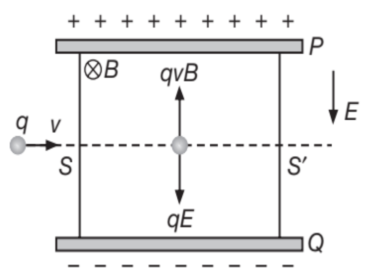
CASE-2: $\vec{v}$, $\vec{E}$ and $\vec{B}$ are Mutually Perpendicular (just as the case of velocity selector)

In this situation we consider a specific case when $\vec{E}$ and $\vec{B}$ are such that

$$\vec{F} = \vec{F}_e + \vec{F}_m = \vec{0}$$

$$\Rightarrow \quad \vec{a} = \left(\frac{\vec{F}}{m}\right) = \vec{0}$$

The particle will pass through the field with same velocity as shown in Figure.



$\vec{v}$, $\vec{E}$ and $\vec{B}$ are mutually perpendicular

So, in this situation, we have

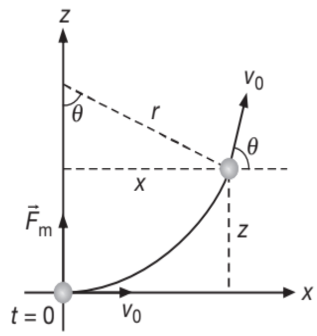
$$F_e = F_m$$

$$\Rightarrow \quad qE = qvB$$

$$\Rightarrow \quad v = \frac{E}{B}$$

CASE-3: When $\vec{E} \parallel \vec{B}$ and particle velocity is perpendicular to both of these fields.

Consider a particle of charge q and mass m released from the origin with velocity $\vec{v} = v_0\hat{i}$ into a region of uniform electric and magnetic fields parallel to y-axis. i.e., $\vec{E} = E_0\hat{j}$ and $\vec{B} = B_0\hat{j}$ as shown in Figure.



The electric field accelerates the particle in y-direction, i.e., y component of velocity goes on increasing with acceleration,

$$a_y = \frac{F_y}{m} = \frac{F_e}{m} = \frac{qE_0}{m} \qquad \ldots(1)$$

The magnetic field rotates the particle in a circle in x-z plane (perpendicular to magnetic field). The resultant path of the particle is a helix with increasing pitch. The axis of the plane is parallel to y-axis. Velocity of the particle at time t would be

$$\vec{v}(t) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

where, $v_y = a_y t = \left(\frac{qE_0}{m}\right)t$

Since, $v_x^2 + v_z^2 = \text{constant} = v_0^2$ and $\theta = \omega t = \left(\frac{qB}{m}\right)t$

$$\Rightarrow \quad v_x = v_0\cos\theta = v_0\cos\left(\frac{qBt}{m}\right) \text{ and}$$

$$v_z = v_0\sin\theta = v_0\sin\left(\frac{qBt}{m}\right)$$

$$\Rightarrow \quad \vec{v}(t) = v_0\cos\left(\frac{qBt}{m}\right)\hat{i} + \left(\frac{qE_0}{m}t\right)\hat{j} + v_0\sin\left(\frac{qBt}{m}\right)\hat{k}$$

Similarly, position vector of particle at time t can be given by

$$\vec{r}(t)=x\hat{i}+y\hat{j}+z\hat{k}$$

where $x=r\sin\theta=\left(\frac{mv_0}{qB}\right)\sin\left(\frac{qBt}{m}\right)$

$$y=\frac{1}{2}a_y t^2=\frac{1}{2}\left(\frac{qE_0}{m}\right)t^2$$

and $z=r(1-\cos\theta)=\left(\frac{mv_0}{qB}\right)\left[\left\{1-\cos\left(\frac{qBt}{m}\right)\right\}\right]$

$$\Rightarrow \quad \vec{r}(t)=\left(\frac{mv_0}{qB}\right)\sin\left(\frac{qBt}{m}\right)\hat{i}+\frac{1}{2}\left(\frac{qE_0}{m}\right)t^2\hat{j}+\left(\frac{mv_0}{qB}\right)\left[\left\{1-\cos\left(\frac{qBt}{m}\right)\right\}\right]\hat{k}$$

CASE-4: When $\vec{E}\perp\vec{B}$ and the particle is released at rest from origin

Consider a particle of charge q and mass m emitted at origin with zero initial velocity into a region of uniform electric and magnetic fields. The field $\vec{E}$ is acting along x-axis and field $\vec{B}$ along y-axis i.e.,

$$\vec{E}=E_0\hat{i} \quad \text{and} \quad \vec{B}=B_o\hat{j}$$

Electric field will provide the particle an acceleration (and therefore a velocity component) in x-direction and the magnetic field will rotate the particle in x-z plane (perpendicular to $\vec{B}$). Hence, at any instant of time its velocity (and hence, position) will have only x and z components. Let at time t its velocity be,

$$\vec{v}=v_x\hat{i}+v_z\hat{k}$$

Net force on it at this instant is

$$\vec{F}=\vec{F}_e+\vec{F}_m=q\vec{E}+q(\vec{v}\times\vec{B})$$

$$\Rightarrow \quad \vec{F}=q\left[E_0\hat{i}+(v_x\hat{i}+v_z\hat{k})\times(B_0\hat{j})\right]$$

$$\Rightarrow \quad \vec{F}=q(E_0-v_zB_0)\hat{i}+qv_xB_0\hat{k}$$

From Newton's Second Law, we have $\vec{F}=m\vec{a}$

$$\Rightarrow \quad \vec{a}=\frac{\vec{F}}{m}=a_x\hat{i}+a_z\hat{k}$$

where, $a_x=\frac{q}{m}(E_0-v_zB_0)$ ...(1)

and $a_z=\frac{q}{m}v_xB_0$ ...(2)

Differentiating Equation (1), w.r.t. time, we have,

$$\frac{da_x}{dt}=\frac{d^2v_x}{dt^2}=-\frac{qB_0}{m}\left(\frac{dv_z}{dt}\right)$$

Since, $\frac{dv_z}{dt}=a_z=\frac{qB_0}{m}v_x$

$$\Rightarrow \quad \frac{d^2v_x}{dt^2}=-\left(\frac{qB_0}{m}\right)^2 v_x \qquad \text{...(3)}$$

Comparing this equation with the differential equation of SHM $\left(\frac{d^2y}{dt^2}=-\omega^2 y\right)$, we get

$$\omega=\frac{qB_0}{m}$$

and the general solution of Equation (3) is,

$$v_x=A\sin(\omega t+\phi) \qquad \text{...(4)}$$

At time $t=0$, $v_x=0$, hence, $\phi=0$

Again,

$$\frac{dv_x}{dt}=A\omega\cos(\omega t) \qquad \{\text{as } \phi=0\}$$

From equation (1),

$$a_x=\frac{qE_0}{m} \text{ at } t=0, \text{ as } v_z=0 \text{ at } t=0$$

$$\Rightarrow \quad A\omega=\frac{qE_0}{m}$$

$$\Rightarrow \quad A=\frac{qE_0}{m\omega}$$

Substituting $\omega=\frac{qB_0}{m}$, we get $A=\frac{qE_0}{m\left(\frac{qB_0}{m}\right)}=\frac{E_0}{B_0}$

Therefore, equation (4) becomes,

$$v_x=\frac{E_0}{B_0}\sin(\omega t) \text{ where } \omega=\frac{qB_0}{m} \qquad \text{...(5)}$$

Now substituting value of v_x in equation (2), we get

$$a_z=\frac{dv_z}{dt}=\frac{qE_0}{m}\sin(\omega t)$$

$$\Rightarrow \quad \int_0^{v_z} dv_z=\frac{qE_0}{m}\int_0^t \sin(\omega t)\,dt$$

$$\Rightarrow \quad v_z=\frac{qE_0}{m\omega}(1-\cos(\omega t))$$

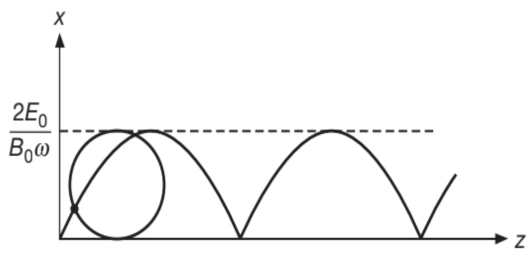
Again substituting $\omega = \dfrac{qB_0}{m}$, we get

$$v_z = \frac{E_0}{B_0}\left(1 - \cos(\omega t)\right) \qquad \ldots(6)$$

On integrating equations for v_x and v_z, from (5) and (6), and knowing that at $t = 0$, $x = 0$ and $z = 0$, we get

$$x = \frac{E_0}{B_0\omega}(1 - \cos\omega t) \text{ and } z = \frac{E_0}{B_0\omega}(\omega t - \sin\omega t)$$

These equations are the equations for a cycloid which is defined as the path generated by the point on the circumference of a wheel rolling on a ground.



In the present case, radius of the rolling wheel is $\dfrac{E_0}{B_0\omega}$.

The maximum displacement along x-direction is $\dfrac{2E_0}{B_0\omega}$.

The x-displacement becomes zero at $t = 0$, $\dfrac{2\pi}{\omega}$, $\dfrac{4\pi}{\omega}$, etc.

Problem Solving Technique(s)

MOTION OF CHARGED PARTICLE IN COMBINED ELECTRIC AND MAGNETIC FIELDS

STEP-1

In analysing the motion of a charged particle in electric and magnetic fields, you will apply Newton's Second Law of motion, with the net force given by

$$\Sigma\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Often other forces such as gravity can be neglected.

STEP-2

Read and analyse the problem carefully. The use of components is the most efficient approach. Select a coordinate system and then express all vector quantities (including $\vec{E}$, $\vec{B}$, $\vec{v}$, $\vec{F}$, and $\vec{a}$) in terms of their components in this system.

STEP-3

If the particle moves perpendicular to a uniform magnetic field, the trajectory is a circle whose radius and angular speed are given by

$$R = \frac{mv}{|q|B} \text{ and } \omega = \frac{|q|B}{m}$$

STEP-4

If our calculation involves a more complex trajectory then use $\Sigma\vec{F} = m\vec{a}$ in component form $\Sigma F_x = ma_x$, and so forth. This approach is particularly useful when both electric and magnetic fields are present.

STEP-5

If $\vec{u} = u_x\hat{i} + u_y\hat{j}$ be the initial velocity of the particle and $\vec{v} = v_x\hat{i} + v_y\hat{j}$ be the final velocity of the particle in a magnetic field, then

$$|\vec{v}| = |\vec{u}|$$

$$\Rightarrow \; v_x^2 + v_y^2 = u_x^2 + u_y^2$$

ILLUSTRATION 23

A particle of mass m and charge q is moving in a region where uniform, constant electric and magnetic fields $\vec{E}$ and $\vec{B}$ are present. $\vec{E}$ and $\vec{B}$ are parallel to each other. At time $t = 0$, the velocity $\vec{v}_0$ of the particle is perpendicular to $\vec{E}$ (Assume that its speed is always $\ll c$, the speed of light in vacuum). Find the velocity $\vec{v}$ of the particle at time t. You must express your answer in terms of t, q, m the vectors $\vec{v}_0$, $\vec{E}$, $\vec{B}$ and their magnitudes v_0, E and B.

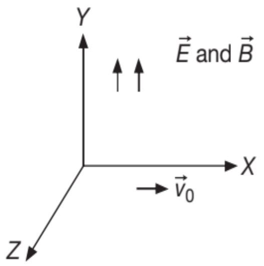
SOLUTION

Let us first find the unit vectors denoting directions of $\vec{E}$, $\vec{B}$, $\vec{v}_0$ and $\vec{F}$. So, we have

$$\hat{j} = \frac{\vec{E}}{E} \quad \text{or} \quad \hat{j} = \frac{\vec{B}}{B}, \; \hat{i} = \frac{\vec{v}_0}{v_0}$$

$$\text{and} \quad \hat{k} = \frac{\vec{v}_0 \times \vec{B}}{|\vec{v}_0 \times \vec{B}|} = \frac{\vec{v}_0 \times \vec{B}}{v_0 B} \qquad \{\because \vec{v}_0 \perp \vec{B}\}$$

Force due to electric field will be along Y-axis. Magnetic force, acting along z-axis, will not affect the motion of charged particle in the direction of electric field (or Y-axis).

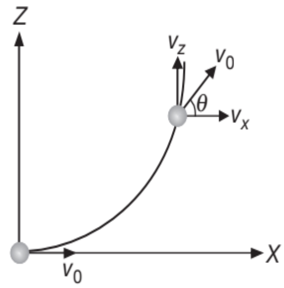


So, $a_y = \dfrac{F_e}{m} = \dfrac{qE}{m} = \text{constant}$

$$\Rightarrow \quad v_y = a_y t = \left(\frac{qE}{m}\right)t \qquad \ldots(1)$$

The charged particle under the action of magnetic field describes a circle in x-z plane (perpendicular to $\vec{B}$) with period and angular frequency given by

$$T = \frac{2\pi m}{qB} \quad \text{or} \quad \omega = \frac{2\pi}{T} = \frac{qB}{m}$$



Initially (i.e., at $t = 0$) velocity was along X-axis. Therefore, magnetic force $(\vec{F}_m)$ is given by $\vec{F}_m = q(\vec{v}_0 \times \vec{B})$ and is directed along positive Z-axis. Let v_0 make an angle θ with X-axis at time t, then $\theta = \omega t$. So, we get

$$v_x = v_0 \cos(\omega t) = v_0 \cos\left(\frac{qB}{m}t\right) \text{ and} \qquad \ldots(2)$$

$$v_z = v_0 \sin(\omega t) = v_0 \sin\left(\frac{qB}{m}t\right) \qquad \ldots(3)$$

Since, $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

So, from equations (1), (2) and (3), we get

$$\Rightarrow \quad v = v_0 \cos\left(\frac{qBt}{m}\right)\left(\frac{\vec{v}_0}{v_0}\right) + \frac{qEt}{m}\left(\frac{\vec{E}}{E}\right) + v_0 \sin\left(\frac{qBt}{m}\right)\left(\frac{\vec{v}_0 \times \vec{B}}{v_0 B}\right)$$

$$\Rightarrow \quad \vec{v} = \cos\left(\frac{qBt}{m}\right)(\vec{v}_0) + \left(\frac{qt}{m}\right)(\vec{E}) + \sin\left(\frac{qBt}{m}\right)\left(\frac{\vec{v}_0 \times \vec{B}}{B}\right)$$

The path of the particle will be a helix of increasing pitch. The axis of the helix is parallel to Y-axis.

ILLUSTRATION 24

A particle with specific charge s leaves the origin in the direction of x-axis with an initial velocity v_0. Uniform electric and magnetic fields with strength E and B are directed along the y-axis. Find the

(a) y-coordinate of the particle when it crosses the y-axis for nth time.

(b) angle α between the particle's velocity vector and the y-axis at that moment.

SOLUTION

(a) The path of the particle will be a helix of increasing pitch. The axis of the helix is parallel to y-axis (parallel to $\vec{E}$) and plane of circle of the helix is the xz plane (perpendicular to $\vec{B}$). The particle will cross the y-axis after time,

$$t = nT = n\left(\frac{2\pi m}{qB}\right) = \frac{2\pi n}{sB}$$

The y-coordinate of particle at this instant is,

$$y = \frac{1}{2}a_y t^2 \quad \text{where, } a_y = \frac{E_y}{m} = sE$$

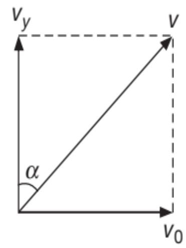
$$\Rightarrow \quad y = \frac{1}{2}(sE)\left(\frac{2\pi n}{sB}\right)^2$$

$$\Rightarrow \quad y = \frac{2\pi^2 n^2 E}{sB^2}$$

(b) At this moment, the y component of its velocity is,

$$v_y = a_y t = (sE)\left(\frac{2\pi n}{sB}\right) = 2\pi n\left(\frac{E}{B}\right)$$

The angle α between the particle's velocity vector and the y-axis at this moment as shown in Figure.



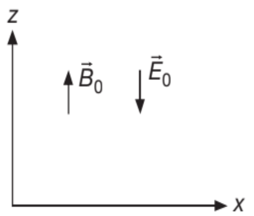
$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{v_0}{v_y}\right)$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{v_0 B}{2\pi n E}\right)$$

Please note that v_0 is the resultant of the x and z components of the velocity of the particle and $\sqrt{v_x^2 + v_z^2} = v_0$

ILLUSTRATION 25

In a certain region uniform electric field $\vec{E}=-E_0\hat{k}$ and magnetic field $\vec{B}=B_0\hat{k}$ are present. At time $t=0$ a particle of mass m and charge q is given a velocity $\vec{v}=v_0\left(\hat{j}+\hat{k}\right)$. Find the minimum speed of the particle and the time when happens so.



SOLUTION

$$\vec{E}=-E_0\hat{k} \text{ and } \vec{B}=B_0\hat{k}$$

At $t=0$, velocity of the particle is $\vec{v}=v_0\hat{j}+v_0\hat{k}$

The electric field will not change the x component of velocity because it is acting along $-z$ direction.

The magnetic force, $\vec{F}_m$ is given by

$$\vec{F}_m=q\left(\vec{v}\times\vec{B}\right)=q\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & v_0 & v_0\\ 0 & 0 & B_0\end{vmatrix}$$

$$\Rightarrow \quad \vec{F}_m=\left(qv_0B_0\right)\hat{i}$$

Since $\vec{F}_m$ is acting along $+x$ direction, so it will have no effect or change in the $\vec{v}$ (in the yz plane).

Further since $\vec{E}=-E_0\hat{k}$, so the electric force is going to change the velocity of the particle in yz plane.

$$\Rightarrow \quad \vec{v}=\vec{u}+\vec{a}t$$

$$\Rightarrow \quad \vec{v}=\left(v_0\hat{j}+v_0\hat{k}\right)-\left(\frac{qE_0}{m}\right)t\hat{k}$$

$$\Rightarrow \quad \vec{v}=v_0\hat{j}+\left(v_0-\frac{qE_0t}{m}\right)\hat{k}$$

$$\Rightarrow \quad |\vec{v}|=\sqrt{v_0^2+\left(v_0-\frac{qE_0t}{m}\right)^2}$$

For $|\vec{v}|$ to be MINIMUM, we have

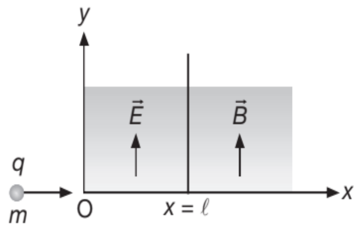
$$v_0-\frac{qE_0t}{m}=0$$

$$\Rightarrow \quad t=\frac{mv_0}{qE_0} \text{ and } v_{MIN}=|\vec{v}|=\sqrt{v_0^2+0^2}$$

$$\Rightarrow \quad v_{MIN}=v_0 \text{ at } t=\frac{mv_0}{qE_0}$$

ILLUSTRATION 26

A positively charged particle, having charge q, is accelerated by a potential difference V. This particle moving along the x-axis enters a region where an electric field E exists. The direction of the electric field is along positive y-axis. The electric field exists in the region bounded by the lines $x=0$ and $x=l$. Beyond the line $x=l$ (i.e., in the region $x\geq l$) there exists a magnetic field of strength B, directed along the positive y-axis. Find the



(a) distance of the point from x-axis where the particle meets the line $x=l$.

(b) pitch of the helix formed after the particle enters the region $x\geq l$.

Mass of the particle is m.

SOLUTION

Since the particle is accelerated through a potential V, so

$$\frac{1}{2}mv^2=qV$$

$$\Rightarrow \quad v=\sqrt{\frac{2qV}{m}}$$

(a) In region from $0<x<\ell$, electric field E is present along y-axis. So, acceleration of particle is

$$a_y=\frac{qE}{m}$$

Under the influence of this acceleration a_y and initial velocity v (along x-axis) the particle will follow a parabolic path so that it hits the line $x=\ell$ at a distance y from x-axis.

IN REGION OF E

For motion along x-axis

$$v_x=u_x+a_xt$$

$$\Rightarrow \quad v_x=u_x=v \qquad \{\because a_x=0\}$$

and $l=u_xt=vt$ ...(1)

For motion along y-axis

$$y = u_y t + \frac{1}{2} a_y t^2$$

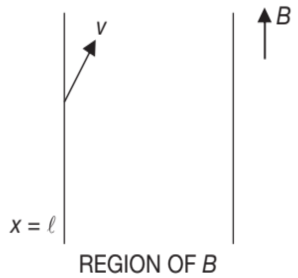
$$\Rightarrow\ y = 0 + \frac{1}{2}\left(\frac{qE}{m}\right)\left(\frac{l}{v}\right)^2$$

$$\Rightarrow\ y = 0 + \frac{1}{2}\left(\frac{qE}{m}\right)\frac{l^2}{\left(\frac{2qV}{m}\right)}$$

$$\Rightarrow\ y = \frac{El^2}{4V}$$

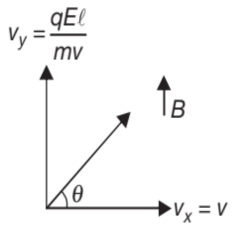
(b) Just when the particle is entering the "Region of B" beyond $x = l$, then it has a velocity $\vec{v}$ having two components v_x and v_y given by

$$v_x = v \text{ and } v_y = \left(\frac{qE}{m}\right)t = \frac{qEl}{mv}$$



Since $v_x \perp B$, so we shall get a helix whose pitch p is given by

$$p = v_y T$$



Please note that here we must understand that the pitch of the helix is due to the component of velocity parallel to the field and in this part of the question v_y happens to be parallel to B. So,

$$p = \frac{2\pi m(v_y)}{qB}$$

$$\Rightarrow\ p = \left(\frac{2\pi m}{qB}\right)\left(\frac{qEl}{mv}\right)$$

$$\Rightarrow\ p = \frac{2\pi El}{B\sqrt{\frac{2qV}{m}}} = \frac{\pi El}{B}\sqrt{\frac{2m}{qV}}$$

Test Your Concepts-III

Based on Charged Particle in Magnetic and Electric Field

(Solutions on page H.9)

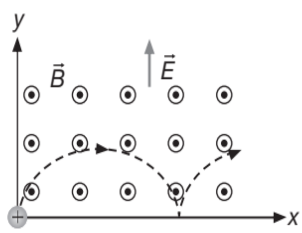
1. A positive charge $q = 3.2 \times 10^{-19}$ C moves with a velocity $\vec{v} = (2\hat{i} + 3\hat{j} - \hat{k})\ \text{ms}^{-1}$ through a region where both a uniform magnetic field and a uniform electric field exist.

(a) Calculate the total force on the moving charge (in unit-vector notation), taking $\vec{B} = (2\hat{i} + 4\hat{j} + \hat{k})$ T and $\vec{E} = (4\hat{i} - \hat{j} - 2\hat{k})\ \text{Vm}^{-1}$.

(b) What angle does the force vector make with the positive x-axis?

2. A velocity selector consists of electric and magnetic fields described by the expressions $\vec{E} = E\hat{k}$ and $\vec{B} = B\hat{j}$, with $B = 15$ mT. Find the value of E such that a 750 eV electron moving along the positive x-axis is undeflected.

3. A particle with mass m, charge q starts from rest at the origin shown in figure. There is a uniform electric field $\vec{E}$ in the +y-direction and a uniform magnetic field $\vec{B}$ directed out of the page. With our knowledge of Physics, we conclude that the path is a cycloid whose radius of curvature at the top points is twice the y-coordinate at that level.



(a) Explain why the path has this general shape and why it is repetitive.

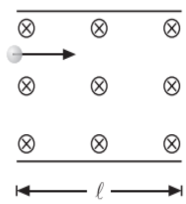
(b) Prove that the speed at any point is equal to $\sqrt{\dfrac{2qyE}{m}}$.

(c) Applying Newton's Second Law at the top point and assuming that the radius of curvature at the top point equals $2y$, prove that the speed at this point is $\dfrac{2E}{B}$.

4. A particle of mass m and charge q starts moving from the origin under the action of an electric field $\vec{E} = E_0\hat{i}$ and magnetic field $\vec{B} = B_0\hat{k}$. If its velocity at $(x_0, 0, 0)$ is $(4\hat{i} + 3\hat{j})$. Then calculate x_0.

5. A particle of mass m and charge q starts moving from the origin under the action of an electric field $\vec{E} = E\hat{i}$ and magnetic field $\vec{B} = B\hat{i}$ with a velocity $\vec{v} = v_0\hat{j}$. Calculate the time t after which the speed of the particle becomes $2v_0$.

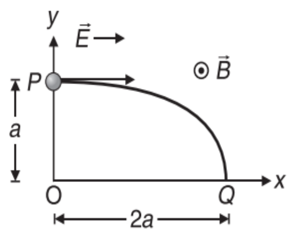
6. An electron moves straight inside a charged parallel plate capacitor of uniform surface charge density σ. The space between the plates is filled with constant magnetic field of induction $\vec{B}$. Calculate the time for which the motion of electron in the capacitor is along a straight line.



7. A charge q of mass m moving with velocity $v_0\hat{i}$ enters a region of magnetic field $B_0\hat{i}$ and electric field $E_0\hat{i}$. If $\dfrac{E_0}{B_0} = v_0$, then discuss the motion of charged particle.

8. A charge q of mass m moving with velocity $v_0\hat{i} + v_0\hat{j}$ enters a region of magnetic field $B_0\hat{j}$ and electric field $E_0\hat{j}$. Calculate the pitch of the charged particle for this case.

9. A particle of mass m, charge q is moving under the influence of a uniform electric field $E\hat{i}$ and a uniform magnetic field $B\hat{k}$. It follows a trajectory from P to Q as shown. The velocities at P and Q are $v\hat{i}$ and $-2v\hat{j}$. Calculate E, rate of work done by the electric field and magnetic field at the points P and Q.

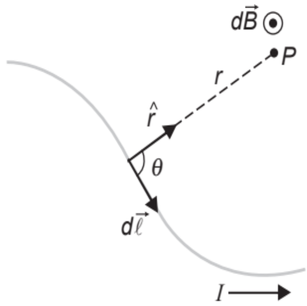


10. A charged particle moving with a constant velocity passes through a region of space without any change in its velocity. If E and B represent the electric and magnetic fields respectively that co-exist in this region of space, then what conclusions can you make about the magnitudes of E and B in this region?

SOURCES OF MAGNETIC FIELD, BIOT SAVART'S LAW AND AMPERE'S CIRCUITAL LAW

BIOT SAVART'S LAW (BSL)

Currents which arise due to the motion of charges are the source of magnetic fields. When charges move in a conducting wire, they produce a current I. The magnetic field at any point P due to the current can be calculated by adding up the magnetic field contributions, $d\vec{B}$, from small segments of the wire, $d\vec{l}$.



Magnetic field $d\vec{B}$ at point P due to a current carrying element $I\,d\vec{\ell}$

The current segment can be thought of as a vector quantity having a magnitude of the length of the segment and pointing in the direction of the current flow. The infinitesimal current source can then be written as $Id\vec{l}$ and is also called as the **Current Element**.

Let r denote as the distance from the current source to the field point P, and $\hat{r}$ the corresponding unit vector (from the current element to the field point P). On the basis of experiments, Jean Biot and Felix Savart arrived at a mathematical expression that gives the magnetic field at some point in space in terms of the current that produces the field. That expression is based on the following observations for the magnetic field $d\vec{B}$ at a point P associated with a length element $d\vec{l}$ of a wire carrying a steady current I (or the current element $Id\vec{l}$).

(a) The vector $d\vec{B}$ is perpendicular to both $d\vec{l}$ (which points in the direction of the current) and the unit vector $\hat{r}$ directed from $d\vec{l}$ to P.

(b) The magnitude of $d\vec{B}$ is inversely proportional to r^2, where r is the distance from $d\vec{l}$ to P.

(c) The magnitude of $d\vec{B}$ is proportional to the current and to the magnitude (dl) of the length element $d\vec{l}$.

(d) The magnitude of $d\vec{B}$ is proportional to sine of the angle between the current element $(Id\vec{l})$ and $\hat{r}$.

The Biot Savart Law gives an expression for the magnetic field contribution, $d\vec{B}$, from the current source, $Id\vec{l}$.

So, mathematically, we write Biot Savart's Law as

$$d\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{Id\vec{l}\times\hat{r}}{r^2}\right) \quad \ldots(1)$$

where μ_0 is a constant called the permeability of free space, having value

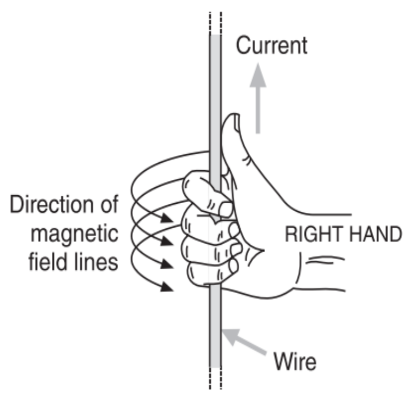
$$\mu_0 = 4\pi\times10^{-7}\ \text{TmA}^{-1} \quad \ldots(2)$$

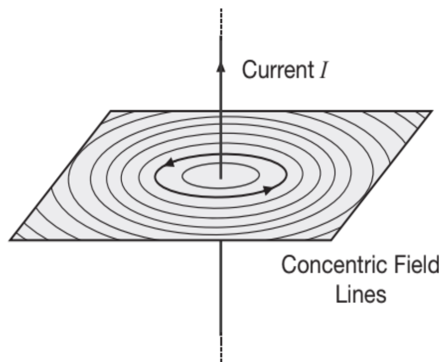
Magnitude of $d\vec{B}$ is given by

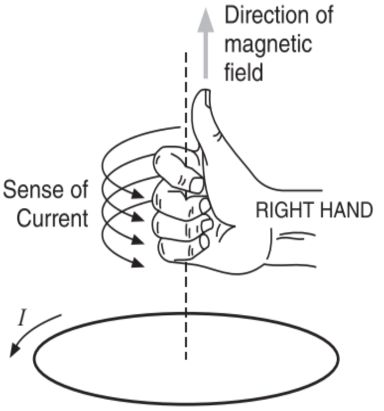
$$|d\vec{B}| = \frac{\mu_0}{4\pi}\left(\frac{Idl\sin\theta}{r^2}\right)$$

The direction of $d\vec{B}$ at the field point P is found by using the RIGHT HAND THUMB RULE, according to which

'Curl the fingers of Right Hand in such a way that thumb points in the direction of current, then the curl of the fingers gives the direction of magnetic field and vice-versa i.e. if we curl the fingers in the sense of current (clock wise or counter clockwise), then direction of the thumb gives direction of the magnetic field.'







Notice that the expression is remarkably similar to the Coulomb's Law for the electric field due to a charge element dq, given by

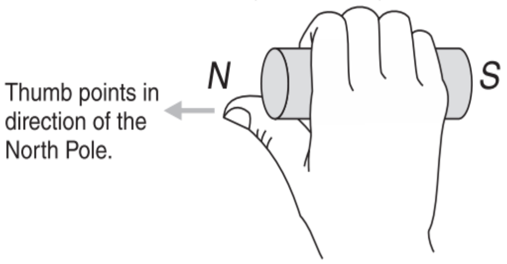
$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}$$

Adding up these contributions to find the magnetic field at the field point P requires integrating over the entire current source. So, we get

$$\vec{B} = \int_{\text{wire}} d\vec{B} = \frac{\mu_0 I}{4\pi}\int_{\text{wire}} \frac{d\vec{l}\times\hat{r}}{r^2} \quad \ldots(3)$$

The integral is a vector integral, which means that the expressions for $\vec{B}$ is actually obtained by calculating three integrals (one for each component) of $\vec{B}$. The vector nature of this integral appears in the cross product $Id\vec{l}\times\hat{r}$. Understanding how to evaluate this cross product and then perform the integral will be the key to use the Biot-Savart Law efficiently.

With the help of Right Hand Thumb Rule we can also find the polarity (North and South) of the current carrying conductor as shown.



Problem Solving Technique(s)

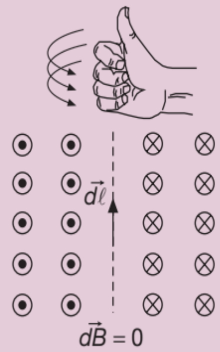
Following points are worth noting regarding the Biot Savart Law.

(a) Magnitude of $d\vec{B}$ is given by,

$$|d\vec{B}| = \frac{\mu_0}{4\pi}\frac{Id\ell\sin\theta}{r^2}$$

$|d\vec{B}|$ is zero at $\theta = 0°$ or $180°$ and maximum at $\theta = 90°$.

(b) For finding the direction of $d\vec{B}$, either of the following methods can be used.



(i) Since $d\vec{B} \parallel (d\vec{\ell}\times\hat{r})$. So, $d\vec{B}$ is along $d\vec{\ell}\times\vec{r}$ i.e., $d\vec{B}\perp d\vec{\ell}$ and $d\vec{B}\perp\hat{r}$.

(ii) If $d\vec{\ell}$ is in the plane of paper. $d\vec{B} = 0$ at all points lying on the straight line that passes through $d\vec{\ell}$. The magnetic field to the right of this line is in ⊗ direction and to the left of this line is in ⊙ direction (simply use RIGHT HAND THUMB RULE)

MAGNETIC FIELD OF A MOVING POINT CHARGE

A point charge q, at rest, in the observer's inertial frame produces an electric field. If this charge moves relative to the observer's inertial system, then it also it produces a magnetic field. The magnitude of the magnetic field produced is proportional to the speed of the charge relative to the observer (provided speed $v < c$, the speed of light). An observer travelling with a moving charge (with the same v) detects no magnetic field.

The magnetic field vector $\vec{B}$ at point P at position vector $\vec{r}$ from the charge q moving with a velocity $\vec{v}$ is found by modifying the Biot Savart's Law.

MAGNETIC FIELD DUE TO UNIFORMLY MOVING CHARGE

Biot Savart's Law gives the magnetic field produced by a current or current element $Id\vec{l}$. Since $I = \frac{dq}{dt}$, we can write,

$$Id\vec{l} = \frac{dq}{dt}d\vec{l} = dq\frac{d\vec{l}}{dt} = \vec{v}dq$$

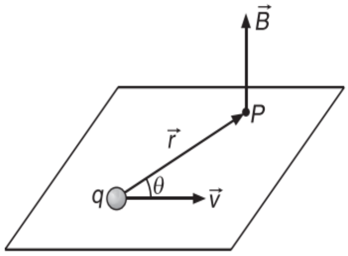
Since $d\vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{Id\vec{l}\times\vec{r}}{r^3}$

If a single charge q is moving with a velocity $\vec{v}$, it creates a magnetic field given by

$$\vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{q(\vec{v}\times\vec{r})}{r^3} = \frac{\mu_0}{4\pi}\frac{q(\vec{v}\times\hat{r})}{r^2}$$

If θ is the angle between $\vec{v}$ and $\vec{r}$ (or $\hat{r}$), then we get

$$|\vec{B}| = \frac{\mu_0}{4\pi}\left(\frac{qv\sin\theta}{r^2}\right)$$



For the expression $\vec{B} = \frac{\mu_0}{4\pi}\frac{q(\vec{v}\times\vec{r})}{r^3}$ following points must be kept in mind.

(a) Direction of $\vec{B}$ is along $\vec{v}\times\vec{r}$ if q is positive and opposite to $\vec{v}\times\vec{r}$ if q is negative.

(b) $|\vec{B}|$ is zero at $\theta = 0°$ and $\theta = 180°$.

(c) $|\vec{B}|$ is and maximum at $\theta = 90°$.

(d) $|\vec{B}|$ is inversely proportional to r^2 and not r^3. i.e., $|\vec{B}| \propto \frac{1}{r^2}$.

(e) If a charge q_1 is moving with velocity $\vec{v}_1$ and another charge q_2 is moving with velocity $\vec{v}_2$ at position vector $\vec{r}$ relative to q_1, then force on q_2 is given by

$$\vec{F} = q_2(\vec{v}_2\times\vec{B})$$

Since we know that $\vec{B} = \frac{\mu_0}{4\pi}\frac{q(\vec{v}\times\vec{r})}{r^3}$, so we get

$$\vec{F} = q_2\left[(\vec{v}_2)\times\left(\frac{\mu_0}{4\pi}\frac{q_1}{r^3}\right)(\vec{v}_1\times\vec{r})\right]$$

$$\Rightarrow \quad \vec{F} = \frac{\mu_0}{4\pi}\frac{q_1q_2}{r^3}[(\vec{v}_2)\times(\vec{v}_1\times\vec{r})]$$

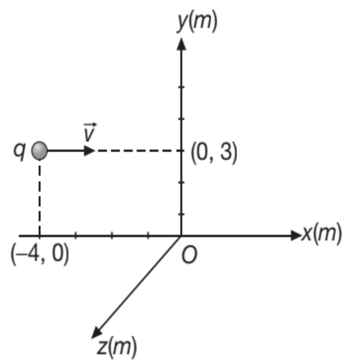
Since, $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

$$\Rightarrow \quad \vec{F} = \frac{\mu_0}{4\pi}\frac{q_1q_2}{r^3}[(\vec{v}_2\cdot\vec{r})\vec{v}_1 - (\vec{v}_2\cdot\vec{v}_1)\vec{r}]$$

This corresponds to Coulomb's electrical force between the charges q_1 and q_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$ respectively relative to an observer at rest.

ILLUSTRATION 27

A point charge of magnitude $q = 4.5$ nC is moving with speed $v = 3.6\times10^7$ ms^{-1} parallel to the x-axis along the line $y = 3$ m. Calculate the magnetic field at the origin produced by this charge when the charge is at the point $P(-4, 3)$ m as shown in Figure.

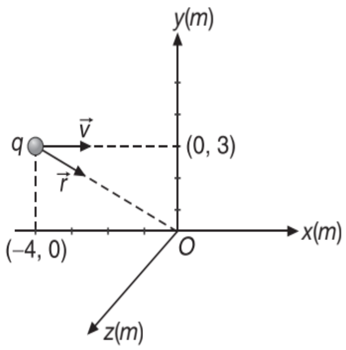


SOLUTION

The magnetic field is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2},$$

where $\vec{v} = v\hat{i}$ and $\vec{r}$ is a vector drawn from the charge to the point at which field is to be calculated.



$\Rightarrow \quad \vec{r} = (4\hat{i} - 3\hat{j})$ m

$\Rightarrow \quad r = \sqrt{4^2 + 3^2}$ m $= 5$ m

Unit vector in the direction of $\vec{r}$ is

$$\hat{r} = \frac{\vec{r}}{r} = \frac{4\hat{i} - 3\hat{j}}{5} = 0.8\hat{i} - 0.6\hat{j}$$

$$\Rightarrow \quad \vec{v} \times \hat{r} = (v\hat{i}) \times (0.8\hat{i} - 0.6\hat{j}) = -0.6v\hat{k}$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi}\left(\frac{q\vec{v} \times \hat{r}}{r^2}\right) = \frac{\mu_0}{4\pi}\frac{q(-0.6v\hat{k})}{r^2}$$

$$\Rightarrow \quad \vec{B} = -(10^{-7})\frac{(4.5\times 10^{-9})(0.6)(3.6\times 10^{7})}{(5)^2}\hat{k}$$

$$\Rightarrow \quad \vec{B} = (-3.89\times 10^{-10}\text{ T})\hat{k}$$

ILLUSTRATION 28

A negative point charge $q = -7.2$ mC is moving in a reference frame. When the point charge is at the origin, the magnetic field it produces at the point $(25, 0, 0)$ cm is $\vec{B} = (6\ \mu\text{T})\hat{j}$, and its speed is 800 kms^{-1}.

(a) Find the x, y and z components of the velocity $\vec{v}_0$ of the charge.

(b) At this same instant, what is the magnitude of the magnetic field that the charge produces at the point $(0, 25, 0)$ cm?

SOLUTION

(a) Since $\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{q(\vec{v}_0 \times \hat{r})}{r^2}$, where

$\hat{r} = \hat{i}$, $r = 0.25$ m and $\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j} + v_{0z}\hat{k}$

$$\Rightarrow \quad \vec{v}_0 \times \hat{r} = v_{0z}\hat{j} - v_{0y}\hat{k}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q}{r^2}(v_{0z}\hat{j} - v_{0y}\hat{k}) = (6\times 10^{-6}\text{ T})\hat{j}$$

$$\Rightarrow \quad v_{0y} = 0$$

$$\Rightarrow \quad \frac{\mu_0}{4\pi}\frac{q}{r^2}v_{0z} = 6\times 10^{-6}\text{ T}$$

$$\Rightarrow \quad v_{0z} = \frac{4\pi(6\times 10^{-6}\text{ T})(0.25\text{ m})^2}{\mu_0(-7.2\times 10^{-3}\text{ C})} = -521\text{ ms}^{-1}$$

Since $v_0^2 = v_{0x}^2 + v_{0y}^2 + v_{0z}^2$

$$v_{0x} = \pm\sqrt{v_0^2 - v_{0y}^2 - v_{0z}^2}$$

$$\Rightarrow \quad v_{0x} = \pm\sqrt{(800\text{ ms}^{-1})^2 - (-521\text{ ms}^{-1})^2}$$

$$\Rightarrow \quad v_{0x} = \pm 607\text{ ms}^{-1}$$

The sign of v_{0x} isn't determined

(b) Now $\vec{r} = \hat{j}$ and

$r = 0.25$ m

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q(\vec{v}_0 \times \hat{r})}{r^2} = \frac{\mu_0}{4\pi}\frac{q}{r^2}(v_{0x}\hat{k} - v_{0z}\hat{i})$$

$$\Rightarrow \quad |\vec{B}| = B = \frac{\mu_0}{4\pi}\frac{|q|}{r^2}\sqrt{v_{0x}^2 + v_{0z}^2} = \frac{\mu_0}{4\pi}\frac{|q|}{r^2}v_0$$

$$\Rightarrow \quad |\vec{B}| = \frac{\mu_0}{4\pi}\frac{(7.2\times 10^{-3}\text{ C})}{(0.25\text{ m})^2}(800\text{ ms}^{-1})$$

$$\Rightarrow \quad |\vec{B}| = 9.2\times 10^{-6}\text{ T} = 9.2\ \mu\text{T}$$

The magnetic field in part (b) doesn't depend on the sign of v_{0x}.

ILLUSTRATION 29

Point charges Q_1 and Q_2 are constrained to move along the x and y-axes, respectively, with the same uniform speed v. At time $t = 0$, both the charges are at the origin. Calculate the Lorentz force $\vec{F}$ acting on Q_2 due to the magnetic field of Q_1 at time t.

SOLUTION

The expression for the magnetic field $\vec{B}$ at a point P at position vector $\vec{r}$ from the charge Q moving with a velocity $\vec{v}$ is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{Q(\vec{v} \times \vec{r})}{r^3}$$

Since both charges start from the origin at $t = 0$, so the position vector of the charge Q_1 is $\vec{r}_1 = \vec{v}_1 t$ and the position vector of the charge Q_2 is $\vec{r}_2 = \vec{v}_2 t$. The vector displacement $\vec{r}$ of Q_2 w.r.t. Q_1 is

$$\vec{r}_{12} = (\vec{v}_2 - \vec{v}_1)t$$

The magnetic field at the position of Q_2 due to the charge Q_1 is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{Q_1(\vec{v}_1 \times \vec{r}_{12})}{r^3} = \frac{\mu_0}{4\pi}\frac{Q_1\vec{v}_1 \times (\vec{v}_2 - \vec{v}_1)t}{t^3|\vec{v}_2 - \vec{v}_1|^3}$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi}\frac{Q_1\vec{v}_1 \times (\vec{v}_2 - \vec{v}_1)}{t^2|\vec{v}_2 - \vec{v}_1|^3}$$

The force on a charge Q_2 moving with at velocity $\vec{v}_2$ in a magnetic field $\vec{B}$ is

$$\vec{F}_{21} = Q_2(\vec{v}_2 \times \vec{B})$$

$$\Rightarrow \quad \vec{F}_{21} = \left(\frac{\mu_0 Q_1 Q_2}{4\pi t^2|\vec{v}_2 - \vec{v}_1|^3}\right)\left[\vec{v}_2 \times \{\vec{v}_1 \times (\vec{v}_2 - \vec{v}_1)\}\right]$$

Since $\vec{v}_1 \times \vec{v}_1 = 0$, so we get

$$\vec{F}_{21} = \left(\frac{\mu_0 Q_1 Q_2}{4\pi t^2|\vec{v}_2 - \vec{v}_1|^3}\right)\left[\vec{v}_2 \times (\vec{v}_1 \times \vec{v}_2)\right] \quad \ldots(1)$$

Taking $\vec{v}_1 = v\hat{i}$ and $\vec{v}_2 = v\hat{j}$, we get

$$\vec{v}_1 \times \vec{v}_2 = v^2\hat{k}$$

$$\vec{v}_2 \times (\vec{v}_1 \times \vec{v}_2) = v^3\hat{i} \text{ and}$$

$$|\vec{v}_1 - \vec{v}_2| = \sqrt{2}v$$

$$\Rightarrow \quad |\vec{v}_1 - \vec{v}_2|^3 = (2^{3/2})v^3$$

From equation (1), the magnetic force is given by

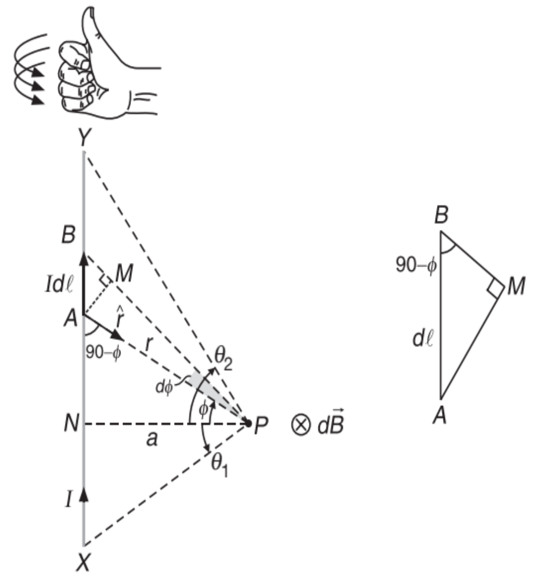
$$\vec{F}_{21} = \frac{\mu_0}{4\pi}\left(\frac{Q_1 Q_2}{2\sqrt{2}t^2}\right)\hat{i}$$

$$\Rightarrow \quad F_{21} = \frac{\mu_0 Q_1 Q_2}{8\pi t^2\sqrt{2}}, \text{ parallel to the } x\text{-axis.}$$

MAGNETIC FIELD AROUND A THIN, STRAIGHT CURRENT CARRYING CONDUCTOR

METHOD I

Consider a current carrying conductor XY carrying current I from X to Y. Let us find the magnetic field at the point P (at perpendicular distance a) due to the conductor. Let the angles subtended by the conductor at the point P be θ_1 and θ_2 as shown in Figure.



According to Biot Savart's Law, the field at P due to an infinitesimal current element $Id\vec{l}$ is given by

$$dB = |d\vec{B}| = \frac{\mu_0}{4\pi}\frac{Id\ell\sin(90-\phi)}{r^2}$$

$$\{\because \angle PAN = 90 - \phi\}$$

$$\Rightarrow \quad dB = \frac{\mu_0}{4\pi}\left(\frac{Idl\cos\phi}{r^2}\right), \text{ into the page} \quad \ldots(1)$$

Since Idl is very small, so

$$\angle PAN \cong \angle PBN \cong (90 - \phi)$$

Now, in triangle BAM, we have

$$\sin(90-\phi) = \frac{AM}{AB} = \frac{AM}{dl}$$

$$\Rightarrow \quad AM = dl\cos\phi$$

Substituting in (1), we get

$$\Rightarrow \quad dB = \frac{\mu_0}{4\pi}\frac{I(AM)}{r^2}$$

But $AM = rd\phi$

$$\Rightarrow \quad dB = \frac{\mu_0}{4\pi}\frac{I(rd\phi)}{r^2} = \frac{\mu_0}{4\pi}\left(\frac{I}{r}\right)d\phi \quad \ldots(2)$$

However, we observe that in triangle APN, $\cos\phi = \frac{a}{r}$

$$\Rightarrow \quad r = \frac{a}{\cos\phi} \quad \ldots(3)$$

Substituting, (3) in (2), we get

$$dB = \frac{\mu_0}{4\pi}\left(\frac{I}{a}\right)\cos\phi\, d\phi \quad \ldots(4)$$

Total field is obtained by integrating (4), over the complete wire. Hence

$$B=\int dB=\frac{\mu_0}{4\pi}\left(\frac{I}{a}\right)\int_{-\theta_1}^{\theta_2}\cos\phi\, d\phi=\frac{\mu_0 I}{4\pi a}\left(\sin\phi\Big|_{-\theta_1}^{\theta_2}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0 I}{4\pi a}\left[\sin\theta_2-\sin(-\theta_1)\right]=\frac{\mu_0 I}{4\pi a}(\sin\theta_2+\sin\theta_1)$$

So finally, we get

$$B=\frac{\mu_0 I}{4\pi a}(\sin\theta_1+\sin\theta_2) \qquad \text{(inwards)}$$

WORD OF ADVICE

Please note that, $AB\neq rd\phi$ (or $d\ell\neq rd\phi$). However, $AM=rd\phi$.

MISCONCEPTION REMOVAL

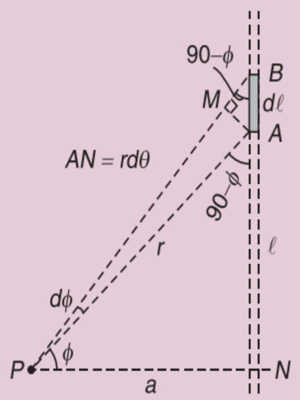
While attempting this problem, we may think that $d\ell=rd\phi$, but that would be a wrong step. From the magnified diagram of the element shown $PA=r$ and $PM=r$. So, its AM that equals $rd\theta$ and not AB. Since AB is extremely small, so $\angle NAP=\angle NBP=90-\theta$. Now, we observe the triangle ABM, then

$$\sin(90-\phi)=\frac{AM}{AB}=\frac{rd\phi}{d\ell}$$

$$\Rightarrow \quad d\ell=\frac{rd\phi}{\cos\phi}$$

$$\Rightarrow \quad d\ell=\frac{ad\phi}{\cos^2\phi}$$

$$\Rightarrow \quad d\ell=a\sec^2\phi\, d\phi$$

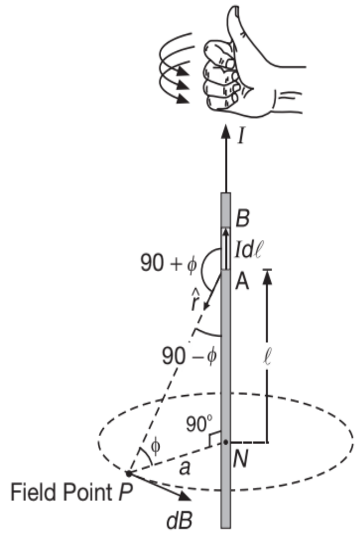


Otherwise we would have got $d\ell=a\sec\phi\, d\phi$ which definitely would not fetch us correct result for the magnetic field B.

METHOD II

According to Biot Savart's Law, $d\vec{B}$ due to infinitesimal element Idy is given by

$$dB=\frac{\mu_0}{4\pi}\frac{Idl\sin(90+\phi)}{r^2}$$



$$\Rightarrow \quad dB=\frac{\mu_0}{4\pi}\frac{I(dl\cos\phi)}{r^2} \qquad \ldots(1)$$

In triangle APN, we have

$$\tan\phi=\frac{l}{a}$$

$$\Rightarrow \quad l=a\tan\phi$$

$$\Rightarrow \quad dl=a\sec^2\phi\, d\phi \qquad \ldots(2)$$

Substitute (2) in (1), we get

$$dB=\frac{\mu_0}{4\pi}\frac{I(a\sec^2\phi\, d\phi)\cos\phi}{r^2}$$

$$\Rightarrow \quad dB=\frac{\mu_0}{4\pi}\frac{Ia\sec\phi\, d\phi}{r^2}$$

But $a\sec\phi=r$ (see from triangle APN)

$$\Rightarrow \quad dB=\frac{\mu_0}{4\pi}\left(\frac{Id\phi}{r}\right)=\frac{\mu_0}{4\pi}\left(\frac{I\cos\phi\, d\phi}{a}\right) \quad \left\{\because \frac{a}{r}=\cos\phi\right\}$$

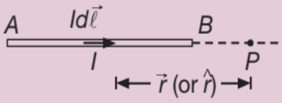
$$\Rightarrow \quad B=\int dB=\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\cos\phi\, d\phi=\frac{\mu_0 I}{4\pi a}\left(\sin\phi\Big|_{-\theta_1}^{\theta_2}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0 I}{4\pi a}\left[\sin\theta_2-\sin(-\theta_1)\right]$$

$$\Rightarrow \quad B=\frac{\mu_0 I}{4\pi a}(\sin\theta_1+\sin\theta_2) \qquad \text{(inwards)}$$

Problem Solving Technique(s)

(a) If the point P is along the length of the wire, as shown, then $d\vec{l}$ and $\vec{r}$ or $\hat{r}$ will either be parallel or antiparallel, thus giving $\theta = 0°$ or $\theta = 180°$.



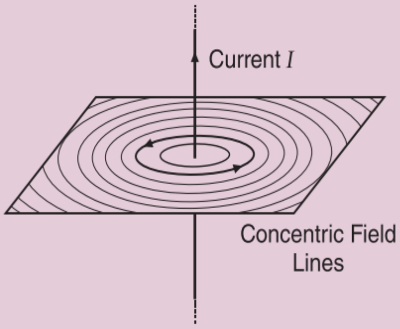
Hence, we get $d\vec{l} \times \hat{r} = \vec{0}$ or $d\vec{l} \times \vec{r} = \vec{0}$

So, at a point P along the length of wire, we have

$$\vec{B} = \vec{0}$$

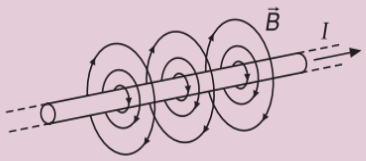
(b) Also note that for the points along the length of the wire (but not on it) the field is always zero.

(c) The field is always perpendicular to the plane containing the wire and the point. So, in a plane perpendicular to the wire the lines of force are concentric circles.

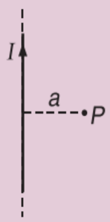


(d) For wire of infinite length, we have $\theta_1 \to \frac{\pi}{2}$ and $\theta_2 \to \frac{\pi}{2}$

$$B = \frac{\mu_0 I}{4\pi a}\left[\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right] = \frac{\mu_0 I}{2\pi a}$$

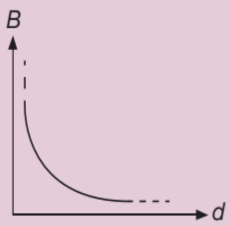


Magnetic field lines due to an infinite wire carrying current I



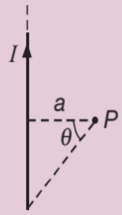
(e) For points at a perpendicular distance d from the wire, field B varies inversely with distance, so

$$B \propto \frac{1}{d} \qquad \left\{\text{and not } \frac{1}{d^2}\right\}$$

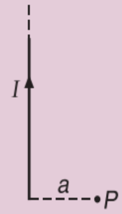


(f) For a semi-infinite wire, we have

$$B = \frac{\mu_0 I}{4\pi a}(1 + \sin\theta)$$



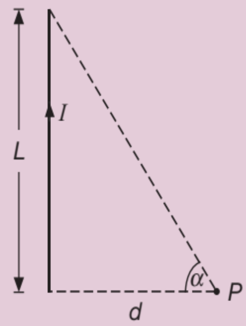
(g) For semi-infinite wire, the field at point P that lies on the perpendicular passing through the finite end of the wire is



$$B = \frac{\mu_0 I}{4\pi a}(1 + \sin\theta) = \frac{\mu_0 I}{4\pi a}$$

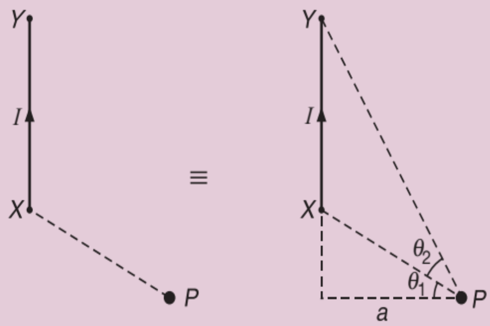
$$\Rightarrow \quad B = \frac{\mu_0 I}{4\pi a}$$

(h) If the wire is of finite length L and the point is near its one end, then $\theta_1 = \alpha$ and $\theta_2 = 0°$. Hence,



$$B = \frac{\mu_0 I}{4\pi d}\sin\alpha \text{ with } \sin\alpha = \frac{L}{\sqrt{(L^2 + d^2)}}$$

(i) Consider the arrangement shown in Figure.

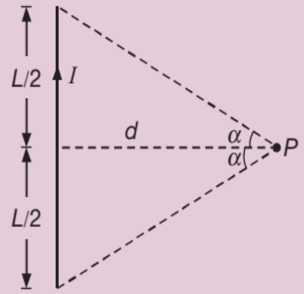


To find field at P due to XY | Do this construction

The field due to current carrying wire XY at point P is given by

$$B=\frac{\mu_0 I}{4\pi a}(\sin\theta_2-\sin\theta_1)$$

(j) If the wire is of finite length L and the point lies at its perpendicular bisector, then $\theta_1=\theta_2=\alpha$ and hence we get



$$B=\frac{\mu_0 I}{2\pi d}\sin\alpha$$

where $\sin\alpha=\dfrac{\frac{L}{2}}{\sqrt{\frac{L^2}{4}+d^2}}=\dfrac{L}{\sqrt{(L^2+4d^2)}}$

where L is length of the wire.

ILLUSTRATION 30

An infinite wire passing through origin along the direction $\hat{i}+\hat{j}+\hat{k}$ carries a current I. Calculate the magnetic field due to the wire at point $(1, 0, 0)$ m.

SOLUTION

From the concept of three-dimensional geometry, we have

$$\vec{r}_\perp=\frac{\left|\overrightarrow{P_0P_1}\times\overrightarrow{\text{Directing Vector}}\right|}{\left|\overrightarrow{\text{Directing Vector}}\right|}$$

The directing vector $(\overrightarrow{DV})$ of current is $(\hat{i}+\hat{j}+\hat{k})$ and $\vec{P}_0=(0, 0, 0)$, $\vec{P}_1=(1, 0, 0)$

$$\Rightarrow \quad \overrightarrow{P_0P_1}=\hat{i}$$

$$\Rightarrow \quad \overrightarrow{P_0P_1}\times\overrightarrow{DV}=\begin{vmatrix}\overset{\oplus}{\hat{i}} & \overset{\ominus}{\hat{j}} & \overset{\oplus}{\hat{k}}\\ 1 & 0 & 0\\ 1 & 1 & 1\end{vmatrix}$$

$$\Rightarrow \quad \overrightarrow{P_0P_1}\times\overrightarrow{DV}=\hat{i}(0-0)-\hat{j}(1-0)+\hat{k}(1-0)$$

$$\Rightarrow \quad \overrightarrow{P_0P_1}\times\overrightarrow{DV}=-\hat{j}+\hat{k}$$

$$\Rightarrow \quad \left|\overrightarrow{P_0P_1}\times\overrightarrow{DV}\right|=\sqrt{2}$$

Also, $\left|\overrightarrow{DV}\right|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

$$\Rightarrow \quad \vec{r}_\perp=\frac{\left|\overrightarrow{P_0P_1}\times\overrightarrow{DV}\right|}{\left|\overrightarrow{DV}\right|}=\frac{\sqrt{2}}{\sqrt{3}}\text{ m}$$

Since $B=\dfrac{\mu_0 I}{2\pi r_\perp}$

$$\Rightarrow \quad B=\frac{\mu_0 I}{2\pi\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}\text{ T}$$

$$\Rightarrow \quad B=\frac{\sqrt{3}\mu_0 I}{2\sqrt{2}\pi}\text{ T}$$

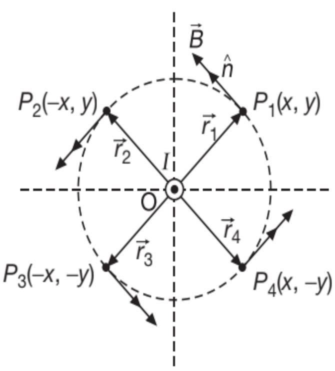
ILLUSTRATION 31

A long straight wire along the z-axis carries a current I in the positive z direction. On the $z=0$ plane, the magnetic field vector is $\vec{B}_1$ at the point $P_1(x, y)$, is $\vec{B}_2$ at the point $P_2(-x, y)$, is $\vec{B}_3$ at the point $P_3(-x, -y)$ and is $\vec{B}_4$ at the point $P_4(x, -y)$. Calculate $\vec{B}_1$, $\vec{B}_2$, $\vec{B}_3$ and $\vec{B}_4$.

SOLUTION.

In magnitude, the field at point $P(x, y)$ is

$$B=\frac{\mu_0 I}{2\pi r}$$



However, vectorially the field is given by

$$\vec{B}=\left(\frac{\mu_0 I}{2\pi r}\right)\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to $Id\vec{l}$ as well as $\vec{r}$.

For the point $P_1(x,y)$, we have

$$\vec{r}_1 = x\hat{i} + y\hat{j}$$

A unit vector perpendicular to $Id\vec{l}$ as well as $\vec{r}_1$ is given by

$$\hat{n}_1 = \frac{y\hat{i} - x\hat{j}}{\sqrt{x^2+y^2}}$$

Because only then, we have $\vec{r}_1 \cdot \hat{n}_1 = 0$

$$\Rightarrow \quad \vec{B}_1 = \left(\frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}\right)\left(\frac{y\hat{i}-x\hat{j}}{\sqrt{x^2+y^2}}\right)$$

$$\Rightarrow \quad \vec{B}_1 = \frac{\mu_0 I}{2\pi}\left(\frac{y\hat{i}-x\hat{j}}{x^2+y^2}\right)$$

Similarly, at the point $P_2(-x,y)$, we have

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi}\left(\frac{-y\hat{i}-x\hat{j}}{x^2+y^2}\right)$$

Similarly, at the point $P_3(-x,-y)$, we have

$$\vec{B}_3 = \frac{\mu_0 I}{2\pi}\left(\frac{y\hat{i}-x\hat{j}}{x^2+y^2}\right)$$

Similarly, at the point $P_4(x,-y)$, we have

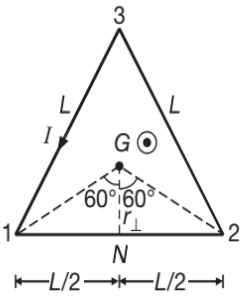
$$\vec{B}_4 = \frac{\mu_0 I}{2\pi}\left(\frac{y\hat{i}+x\hat{j}}{x^2+y^2}\right)$$

ILLUSTRATION 32

Calculate the magnetic field at the centroid of an equilateral current carrying wire frame of length $3L$, carrying a current I.

SOLUTION

Let us find the field due to a single wire '12' first.



$$B_{12} = \frac{\mu_0 I}{4\pi(r_\perp)}(\sin\theta + \sin\theta)$$

where $\theta = 60° = \frac{\pi}{3}$ radian and in triangle $G1N$, we have

$$\tan 60° = \frac{N1}{GN} = \frac{\frac{L}{2}}{r_\perp}$$

$$\Rightarrow \quad r_\perp = \frac{L}{2\sqrt{3}}$$

$$\Rightarrow \quad B_{12} = \frac{\mu_0 I}{4\pi\left(\frac{L}{2\sqrt{3}}\right)}(\sin 60 + \sin 60) = \frac{3\mu_0 I}{2\pi L}, \odot$$

Each side will contribute the field of the same strength in the same direction. Hence, the total field is given by

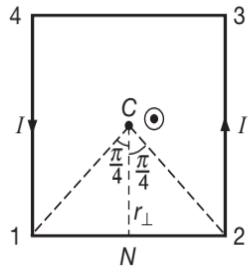
$$B_{\text{total}} = B = 3B_{12} = \frac{9\mu_0 I}{2\pi L}, \odot$$

ILLUSTRATION 33

Calculate the magnetic field at the centre of a current carrying squared wire frame of length $4L$, carrying a current I.

SOLUTION

Let us find the field due to a single wire '12' first.



$$B_{12} = \frac{\mu_0 I}{4\pi(r_\perp)}\left[\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \quad B_{12} = \frac{\mu_0 I}{4\pi\left(\frac{L}{2}\right)}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

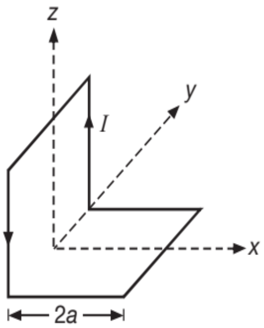
$$\Rightarrow \quad B_{12} = \frac{2\sqrt{2}\mu_0 I}{4\pi L}, \odot$$

Since each side will contribute the field of the same strength in the same direction. Hence, the total field is given by

$$B_{\text{total}} = 4B_{12} = \frac{8\sqrt{2}\mu_0 I}{4\pi L}, \odot$$

ILLUSTRATION 34

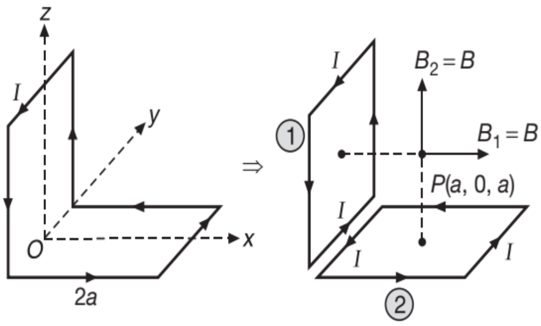
A non-planar loop of conducting wire carrying a current I is placed as shown in Figure.



Each of the straight sections of the loop is of length $2a$. Find the direction of the magnetic field due to this loop at the point $P(a,\ 0,\ a)$.

SOLUTION

Consider the bigger loop to be made up of two loops 1 and 2 as shown in Figure.



Magnetic field due to loop 1 and 2 at point P has same value say B. So,

$$\vec{B}_P = B\hat{i} + B\hat{k} = B(\hat{i} + \hat{k})$$

So, magnetic field points along the direction $\hat{n}$ given by

$$\hat{n} = \frac{\vec{B}_P}{|\vec{B}_P|} = \frac{B(\hat{i}+\hat{k})}{\sqrt{2}B} = \frac{\hat{i}+\hat{k}}{\sqrt{2}}$$

ILLUSTRATION 35

Calculate the magnetic field at the centre of a regular n sided current carrying regular polygon wire frame of length nL. Assume that the current in the wire frame is I. Also interpret the obtained result for $n \to \infty$.

SOLUTION

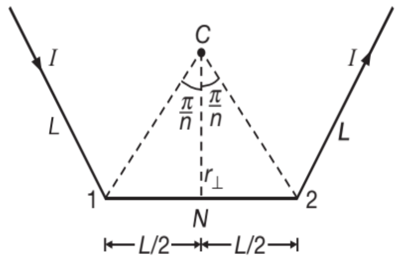
Let us find the field due to a single wire '12' first.

$$B_{12} = \frac{\mu_0 I}{4\pi(r_\perp)}\left[\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right] \qquad \ldots(1)$$

In triangle $CN1$, we have

$$\tan\left(\frac{\pi}{n}\right) = \frac{N1}{CN} = \frac{\frac{L}{2}}{r_\perp}$$

$$\Rightarrow \quad r_\perp = \frac{L}{2\tan\left(\frac{\pi}{n}\right)} \qquad \ldots(2)$$



Substituting, the value of $r_\perp$ from (2) in (1), we get

$$B_{12} = \frac{\mu_0 I\left[2\tan\left(\frac{\pi}{n}\right)\right]}{4\pi L}\left[2\sin\left(\frac{\pi}{n}\right)\right]$$

$$\Rightarrow \quad B_{12} = \frac{\mu_0 I}{4\pi L}\left[4\tan\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right], \odot$$

Each side will contribute the field of the same strength in the same direction. Hence, the total magnetic field at the centre C of the polygon is given by

$$B_{total} = B = n(B_{12}), \odot$$

$$\Rightarrow \quad B_{total} = n\left(\frac{\mu_0 I}{4\pi L}\right)\left\{4\tan\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right\}, \odot$$

Now, when $n \to \infty$ then the polygon approaches the circular shape. In that case, we have

$$2\pi r = nL$$

$$\Rightarrow \quad r = \frac{nL}{2\pi}$$

$$\Rightarrow \quad B = \lim_{n\to\infty} B_{total} = \frac{4\mu_0 I}{n(4\pi L)}\left(\lim_{n\to\infty} n^2 \tan\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right)$$

$$\Rightarrow \quad B = \frac{\mu_0 I \pi^2}{\pi(nL)}\lim_{n\to\infty}\left[\left(\frac{\tan\left(\frac{\pi}{n}\right)}{\left(\frac{\pi}{n}\right)}\right)\left(\frac{\sin\left(\frac{\pi}{n}\right)}{\left(\frac{\pi}{n}\right)}\right)\right]$$

Since, $\lim\limits_{n\to\infty}\frac{\tan\left(\frac{\pi}{n}\right)}{\left(\frac{\pi}{n}\right)} = 1$ and $\lim\limits_{n\to\infty}\frac{\sin\left(\frac{\pi}{n}\right)}{\left(\frac{\pi}{n}\right)} = 1$

$$\Rightarrow \quad B = \frac{\mu_0 I\pi}{nL}(1)(1) = \frac{\mu_0 I}{2\left(\frac{nL}{2\pi}\right)} = \frac{\mu_0 I}{2r}$$

$$\Rightarrow \quad B = \lim_{n\to\infty} B_{\text{total}} = \frac{\mu_0 I}{2r} = B_{\text{circle}}$$

So, in this problem, by discussing the limiting case condition, we have calculated the magnetic field at the centre of a circular loop (having infinite sides), which is given by

$$B_{\text{circle}} = \frac{\mu_0 I}{2r}$$

ILLUSTRATION 36

Find the magnetic induction at the centre of a rectangular wire frame whose diagonal is equal to $d = 16$ cm and the angle between the diagonals is equal to $\varphi = 30°$, carrying a current of $I = 5$ A.

Take $\tan(15°) = 0.27$ and $\sqrt{3} = 1.73$.

SOLUTION

In such type of questions where we have been provided with the current value but not the direction, we can assume the direction of current ourselves and then express our results in accordance with the direction taken by us. Let the current flow in the frame from A to C to D to E to A (or in the counter clockwise sense). Then let us find the field due to AC at O. The field due to the wire AC subtending angles $60°$ and $60°$ at O is given by

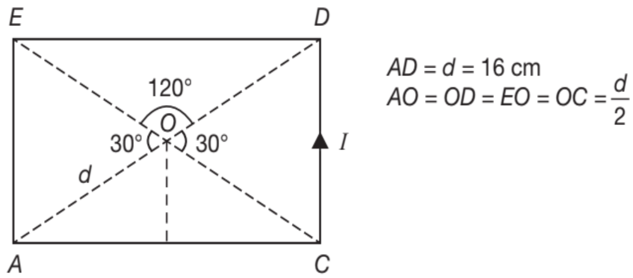
$$B_{AC} = \frac{\mu_0 I}{2\pi(r_\perp)}(\sin 60 + \sin 60)$$

where $r_\perp = \frac{d}{2}\cos 60° = \frac{d}{4}$

$$\Rightarrow \quad B_{AC} = \frac{\mu_0 I}{2\pi\left(\frac{d}{4}\right)}(2\sin 60) = \frac{\mu_0 I}{2\pi\left(\frac{d}{4}\right)}\sqrt{3}, \odot$$

$$\Rightarrow \quad B_{AC} = \frac{2\sqrt{3}\mu_0 I}{\pi d}, \odot$$

Similarly, $B_{DE} = \frac{2\sqrt{3}\mu_0 I}{\pi d}, \odot$



Let us calculate field due to CD at O. Since CD subtends angles $15°$ and $15°$ at O and lies at $r_\perp = \frac{d}{2}\cos(15)$.

So, we get

$$B_{CD} = \frac{\mu_0 I}{2\pi\left(\frac{d}{2}\cos 15\right)}(\sin 15 + \sin 15)$$

$$\Rightarrow \quad B_{CD} = \frac{2\mu_0 I}{\pi d}\tan(15), \odot$$

Similarly, $B_{EA} = \frac{2\mu_0 I}{\pi d}\tan(15), \odot$

So, $B_{\text{total}} = B_{AC} + B_{CD} + B_{DE} + B_{EA}, \odot$

$$\Rightarrow \quad B_{\text{total}} = \frac{4\sqrt{3}\mu_0 I}{\pi d} + \frac{4\mu_0 I}{\pi d}\tan(15), \odot$$

$$\Rightarrow \quad B_{\text{total}} = \frac{4\mu_0 I}{\pi d}\left(\sqrt{3} + \tan 15\right), \odot$$

Since $I = 5$ A, $d = 16$ cm $= 0.16$ m and

$\tan(15) = 0.27$

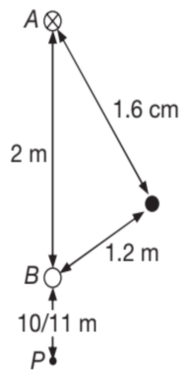
$$\Rightarrow \quad B_{\text{total}} = \frac{4(4\pi \times 10^{-7})(5)}{(\pi)(0.16)}(1.73 + 0.27)$$

$$\Rightarrow \quad B_{\text{total}} = \frac{(16 \times 10^{-7})(5)(100)}{16}(2)$$

$$\Rightarrow \quad B_{\text{total}} = 10^{-4}\ \text{T} = 0.1\ \text{mT}$$

ILLUSTRATION 37

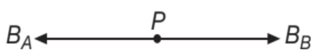
Two long straight parallel wires are 2 m apart, perpendicular to the plane of the paper. The wire A carries a current of 9.6 A, directed into the plane of the paper as shown in Figure.



The wire B carries a current such that the magnetic field of induction at the point P, at a distance of $\frac{10}{11}$ m from the wire B, is zero. Calculate the magnitude and direction of the current in B. Also calculate the magnitude of the magnetic field induction at the point S.

SOLUTION

At the point P, the field due to the wires A and B is shown in Figure.



For the net magnetic field at P to be zero, the direction of current at B should be perpendicular to paper outwards. Let current in the wire B be I_B. Then,

$$\frac{\mu_0}{2\pi}\frac{I_A}{\left(2+\frac{10}{11}\right)}=\frac{\mu_0}{2\pi}\frac{I_B}{\left(\frac{10}{11}\right)}$$

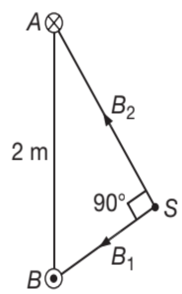
$$\Rightarrow \quad \frac{I_B}{I_A}=\frac{10}{32}$$

$$\Rightarrow \quad I_B=\frac{10}{32}\times I_A=\frac{10}{32}\times 9.6=3\text{ A}$$

From the data provided in the question, we observe that

$$(AS)^2+(BS)^2=(AB)^2$$

$$\Rightarrow \quad \angle ASB=90^\circ$$



At S magnetic field due to I_A is B_1 given by

$$B_1=\frac{\mu_0}{2\pi}\frac{I_A}{(1.6)}=\frac{\left(2\times10^{-7}\right)(9.6)}{1.6}=12\times10^{-7}\text{ T}$$

and magnetic field due to I_B is B_2 given by

$$B_2=\frac{\mu_0}{2\pi}\frac{I_B}{(1.2)}=\frac{\left(2\times10^{-7}\right)(3)}{1.2}=5\times10^{-7}\text{ T}$$

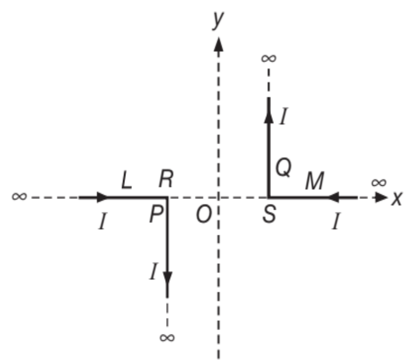
Since, B_1 and B_2 are mutually perpendicular. Net magnetic field at S is

$$B=\sqrt{B_1^2+B_2^2}=\sqrt{\left(12\times10^{-7}\right)^2+\left(5\times10^{-7}\right)^2}$$

$$\Rightarrow \quad B=13\times10^{-7}\text{ T}$$

ILLUSTRATION 38

A pair of stationary and infinitely long bent wires are placed in the xy plane as shown in figure. The wires carry currents of $I=10$ A each as shown. The segments L and M are along the x-axis. The segments P and Q are parallel to the y-axis such that $OS=OR=0.02$ m. Find the magnitude and direction of the magnetic induction at the origin O.



SOLUTION

Magnetic field at O due to L and M is zero. Due to P magnetic field at O is

$$B_1=\frac{1}{2}\left(\frac{\mu_0}{2\pi}\frac{I}{OR}\right)=\frac{\left(10^{-7}\right)(10)}{0.02}\text{ T}=5\times10^{-5}\text{ T}$$

(perpendicular to paper outwards)

Similarly, field at O due to Q would be,

$$B_2=\frac{1}{2}\left(\frac{\mu_0}{2\pi}\frac{I}{OS}\right)=\frac{\left(10^{-7}\right)(10)}{0.02}\text{ T}=5\times10^{-5}\text{ T}$$

(perpendicular to paper outwards)

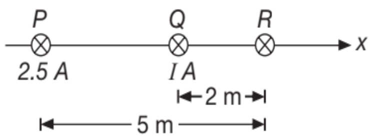
Since, both the fields are in same direction, net field will be sum of these two. Hence

$$B_{net} = B_1 + B_2 = 10^{-4}\text{ T}$$

Direction of field is perpendicular to the paper outwards.

ILLUSTRATION 39

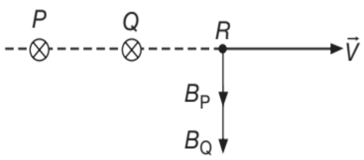
Two long parallel wires carrying currents 2.5 A and I (ampere) in the same direction (directed into the plane of the paper) are held at P and Q respectively such that they are perpendicular to the plane of paper. The points P and Q are located at a distance of 5 m and 2 m respectively from a collinear point R (shown in figure).



An electron moving with a velocity of 4×10^5 ms^{-1} along the positive x-direction experiences a force of magnitude 3.2×10^{-20} N at the point R. Find the value of I.
Find all the positions at which a third long parallel wire carrying a current of magnitude 2.5 A may be placed, so that the magnetic induction at R is zero.

SOLUTION

Magnetic field at R due to both the wires P and Q will be downwards as shown in Figure.



Therefore, net field at R will be sum of these two.

$$B = B_P + B_Q = \frac{\mu_0}{2\pi}\frac{I_P}{5} + \frac{\mu_0}{2\pi}\frac{I_Q}{2} = \frac{\mu_o}{2\pi}\left(\frac{2.5}{5}+\frac{I}{2}\right)$$

$$B = \frac{\mu_0}{4\pi}(I+1) = 10^{-7}(I+1)$$

Net force on the electron will be

$$F_m = qvB\sin 90^\circ = qvB$$

$$\Rightarrow \quad 3.2\times10^{-20} = 1.6\times10^{-19}\left(4\times10^5\right)\left(10^{-7}\right)(I+1)$$

$$\Rightarrow \quad I+1=5$$

$$\Rightarrow \quad I = 4\text{ A}$$

Net field at R due to wires P and Q is

$$B = 10^{-7}(I+1)\text{ T} = 5\times10^{-7}\text{ T} \qquad \{\because I = 4\text{ A}\}$$

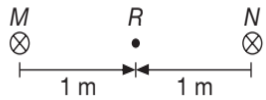
Magnetic field due to third wire carrying a current of 2.5 A should also be 5×10^{-7} T but in upward direction so, that net field at R becomes zero. Let distance of this wire from R be r. Then,

$$\frac{\mu_0}{2\pi}\frac{2.5}{r} = 5\times10^{-7}$$

$$\Rightarrow \quad \frac{\left(2\times10^{-7}\right)(2.5)}{r} = 5\times10^{-7}\text{ m}$$

$$\Rightarrow \quad r = 1\text{ m}$$

So, the third wire can be placed at M or N as shown in figure.



If it is placed at M, then current in it should be outwards and if placed at N, then current be inwards.

ILLUSTRATION 40

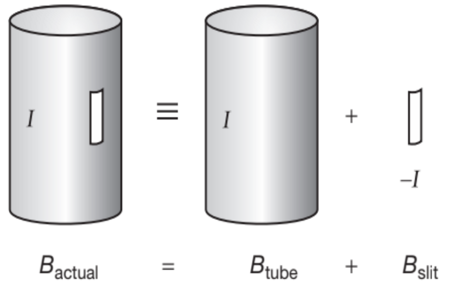
A current I flows along a lengthy thin-walled tube of radius R with longitudinal slit of width w. Find the induction of the magnetic field inside the tube under the condition $w \ll R$.

SOLUTION

The magnetic field due to a lengthy thin walled tube at any internal point is zero. When a longitudinal slit of width w is removed, then a net field will exist inside. The tube with a slit can be visualised as equivalent to a full tube carrying a current I and a slit carrying current $-I$ (i.e. current equal in magnitude and opposite in direction). So, the net field inside the tube is due to the slit carrying current I in opposite direction.
Since

$$B_{tube} = 0$$

$$\Rightarrow \quad B_{actual} = B_{slit} = \frac{\mu_0 I_{slit}}{2\pi R}$$



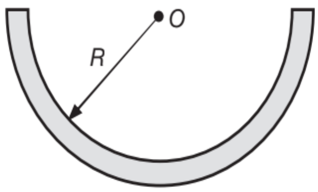
(Since width of slit is very small, so field due to slit is equal to that of a thin wire).
Further

$$I_{slit} = \left(\frac{I}{2\pi R}\right)w$$

$$\Rightarrow \quad B_{actual} = \frac{\mu_0\left(\frac{I}{2\pi R}\right)w}{2\pi R} = \frac{\mu_0 I w}{4\pi^2 R^2}$$

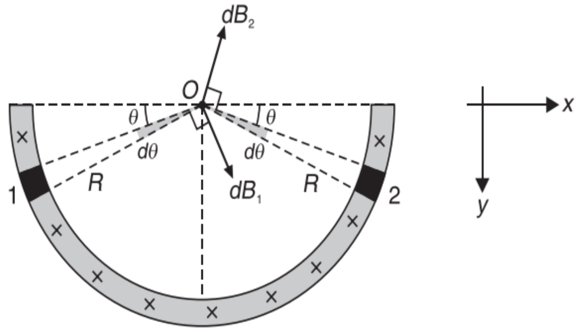
ILLUSTRATION 41

A uniform current I flows in a long straight semi-cylindrical shaped wire with cross-section having the form of a thin half-ring of radius R. Find the induction of the magnetic field at the point O.



SOLUTION

Please note that the diagram drawn shows the cross-sectional view of a straight wire. Consider a long wire (cross-sectional view in black), which is a part of the semi-cylindrical wire of radius R. Let this current carrying infinitesimal wire carry a current dI and be inclined to the x-axis at angle θ and subtends angle $d\theta$ at the centre. For the sake of convenient evaluation, we consider another wire element 2 (mirror image of 1). Due to symmetry of location of point O, both will give equal fields at O. (Please note that 1 and 2 have been attached to the fields just to identify which field is due to which element). Also note that while drawing dB never forget that $d\vec{B} \perp \vec{r}$ (or here $\vec{R}$).



Field due to an infinite wire carrying a current dI at distance R is

$$dB_1 = dB_2 = dB = \frac{\mu_0 (dI)}{2\pi R}$$

where $dI = \frac{I}{(\pi R)}(R d\theta)$

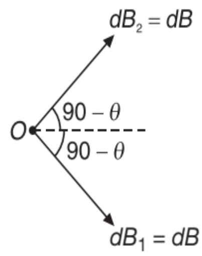
$$\Rightarrow \quad dI = \left(\frac{I}{\pi}\right)d\theta$$

Now, on resolution of dB we observe the components $dB\cos\theta$ cancel. So,

$$B = B_{net} = \int\left(\begin{matrix}\text{contribution due to}\\ \text{a single element}\end{matrix}\right)$$

$$\Rightarrow \quad B = \int dB \sin\theta$$

$$\Rightarrow \quad B = \int \frac{\mu_0 dI}{2\pi R}\sin\theta$$



$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi^2 R}\int_0^{\pi} \sin\theta d\theta$$

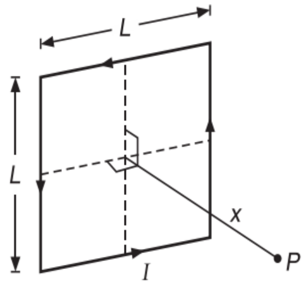
$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi^2 R}\left(-\cos\theta|_0^{\pi}\right)$$

$$\Rightarrow \quad B = -\frac{\mu_0 I}{2\pi^2 R}(\cos\pi - \cos 0)$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{\pi^2 R}$$

ILLUSTRATION 42

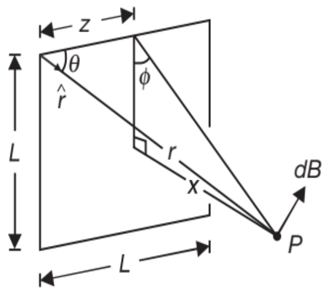
A wire is formed into the shape of a square loop of edge length L carrying a current I. Calculate the magnetic field at point P on the axis of the loop at a distance x from the centre of the loop.



SOLUTION

By symmetry of the arrangement, the magnitude of the net magnetic field at point P is $B = 8B_{0x}$ where B_0 is the contribution to the field due to current in an edge length equal to $\frac{L}{2}$. In order to calculate B_0, we use the Biot-Savart Law and consider the plane of the square to be the yz plane with point P on the x-axis. The contribution to the magnetic field at point P due to a current element of length dz and located a distance z along the axis is given by

$$\vec{B}_0 = \frac{\mu_0 I}{4\pi}\int \frac{d\ell \times \hat{r}}{r^2}$$



From the figure we see that

$$r = \sqrt{x^2 + \left(\frac{L^2}{4}\right) + z^2}$$

and $$|d\ell \times \hat{r}| = dz \sin\theta = dz\sqrt{\frac{\left(\frac{L^2}{4}\right) + x^2}{\left(\frac{L^2}{4}\right) + x^2 + z^2}}$$

By symmetry all components of the field $\vec{B}$ at P cancel except the components along x-axis (perpendicular to the plane of the square) and hence we have

$$B_{0x} = B_0 \cos\phi \text{ where } \cos\phi = \frac{\frac{L}{2}}{\sqrt{\frac{L^2}{4} + x^2}}$$

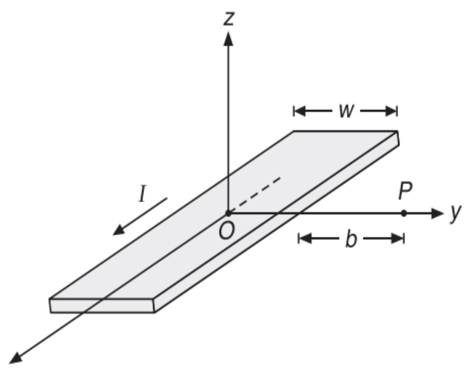
Therefore, $$B_{0x} = \frac{\mu_0 I}{4\pi}\int_0^{\frac{L}{2}} \frac{\sin\theta \cos\phi\, dz}{r^2} \text{ and } B = 8B_{0x}$$

Using the expressions given above for $\sin\theta$, $\cos\phi$ and r, we get

$$B = 8B_{0x} = \frac{\mu_0 I L^2}{2\pi\left(x^2 + \frac{L^2}{4}\right)\left(\sqrt{x^2 + \frac{L^2}{2}}\right)}$$

ILLUSTRATION 43

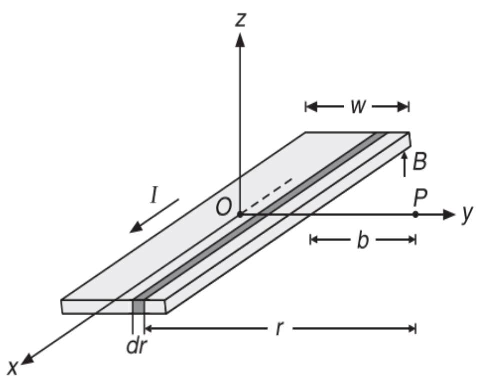
A very long, thin strip of metal of width w carries a current I along its length as shown. Find the magnetic field at the point P in the diagram. The point P is in the plane of the strip at distance b away from it.



SOLUTION

Consider a longitudinal infinitesimal element of the strip of width dr at a distance r from P is shown. The contribution to the field at point P due to the current di in the element dr is

$$dB = \frac{\mu_0 di}{2\pi r}, \text{ upwards towards } z\text{-axis}$$



where $di = I\left(\frac{dr}{w}\right)$

$$\Rightarrow \quad \vec{B} = \int d\vec{B} = \int_b^{b+w} \left(\frac{\mu_0 I dr}{2\pi w r} \right) \hat{k}$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0 I}{2\pi w} \log_e \left(1 + \frac{w}{b} \right) \hat{k}$$

Conceptual Note(s)

If the point is at a large distance from the strip i.e., $b \gg w$

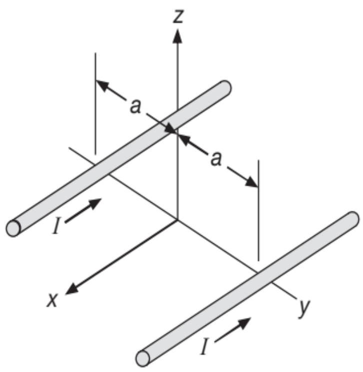
$$\log_e \left[1 + \frac{w}{b} \right] = \frac{w}{b} - \frac{w^2}{2b^2} + \ldots\ldots = \frac{w}{b}$$

$$\Rightarrow \quad B = \frac{\mu_0}{4\pi} \frac{2I}{w} \times \frac{w}{b} = \frac{\mu_0 I}{2\pi a}$$

So, we observe that for a distant point, the strip just behaves like a current carrying wire which justifies the above result obtained by approximation.

ILLUSTRATION 44

In figure, both currents in the infinitely long wires are in the negative x-direction.

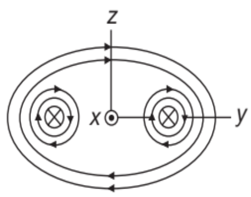


(a) Draw the magnetic field pattern in the yz plane.

(b) Calculate the distance d along the z-axis where the magnetic field is maximum.

SOLUTION

(a)

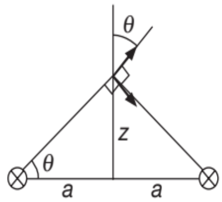


Currents are into the paper

(b) At a point on the z-axis, the contribution from each wire has magnitude $B = \frac{\mu_0 I}{2\pi\sqrt{a^2 + z^2}}$ and is perpendicular to the line from this point to the wire as shown in figure. Combining fields, the vertical components cancel while the horizontal components add, thus giving

$$B_y = 2\left(\frac{\mu_0 I}{2\pi\sqrt{a^2 + z^2}} \sin\theta \right)$$

$$\Rightarrow \quad B_y = \frac{\mu_0 I}{\pi\sqrt{a^2 + z^2}} \left(\frac{z}{\sqrt{a^2 + z^2}} \right) = \frac{\mu_0 I z}{\pi(a^2 + z^2)}$$



At a distance z above the plane of the conductors

The condition for a maximum is

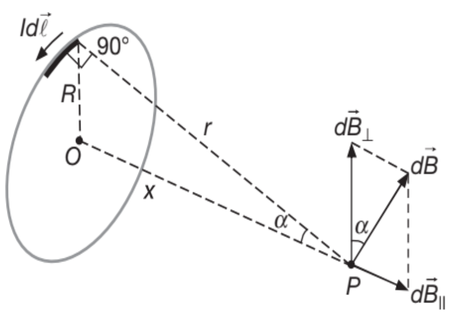
$$\frac{dB_y}{dz} = \frac{-\mu_0 I z(2z)}{\pi(a^2 + z^2)^2} + \frac{\mu_0 I}{\pi(a^2 + z^2)} = 0$$

$$\Rightarrow \quad \frac{\mu_0 I}{\pi} \left(\frac{a^2 - z^2}{a^2 + z^2} \right) = 0$$

Thus, along the z-axis, the field is a maximum at $d = a$

FIELD AT AN AXIAL POINT OF A CURRENT CARRYING CIRCULAR LOOP

Consider a current carrying loop of radius R carrying a current I. Let us find the magnetic field at a point P on the axis of the loop at distance x from its centre as shown in Figure.



For this let us consider a current element $Id\vec{l}$ on circumference of the loop at distance r from P. Please note that r is same for all the current elements taken on the circumference of the loop.

Here, angle θ between the element $d\vec{l}$ and $\vec{r}$ is $\frac{\pi}{2}$ everywhere and r is same for all elements, so we have from Biot Savart's Law

$$dB = \frac{\mu_0}{4\pi}\frac{I(d\vec{l}\times\hat{r})}{r^2} \text{ where } d\vec{l}\times\hat{r} = dl\sin\theta = dl$$

$$\Rightarrow \quad dB = \left(\frac{\mu_0}{4\pi}\right)\frac{Idl\sin(90^\circ)}{r^2} = \left(\frac{\mu_0}{4\pi}\right)\frac{Idl}{r^2}$$

{Biot Savart's Law}

The field $d\vec{B}$ has components both along and perpendicular to the axis of the loop. Also, we observe $d\vec{B}$ to be perpendicular to $Id\vec{l}$ and $\vec{r}$ both. However, if we consider the contributions of the current elements that are diametrically opposite, we see that their components normal to the axis $d\vec{B}_\perp$ will cancel.

But, the component of $d\vec{B}$ along the axis is

$$dB_{\|} = dB\sin\alpha = \left(\frac{\mu_0}{4\pi}\right)\frac{Idl}{r^2}\left(\frac{R}{r}\right) = \frac{\mu_0 IRdl}{4\pi r^3}$$

The total field is given by the integral of this expression over all elements, so

$$B = B_{\|} = B_{\text{axis}} = \int dB_{\|}$$

$$\Rightarrow \quad B = \left(\frac{\mu_0}{4\pi}\right)\frac{IR}{r^3}\int_0^{2\pi R} dl = \left(\frac{\mu_0}{4\pi}\right)\frac{I2\pi R^2}{r^3}$$

Since, $r^2 = R^2 + x^2$, and if the coil has N turns, we get

$$B = \left(\frac{\mu_0}{4\pi}\right)\frac{2\pi R^2 IN}{(R^2+x^2)^{3/2}} = \frac{\mu_0 NIR^2}{2(R^2+x^2)^{3/2}}$$

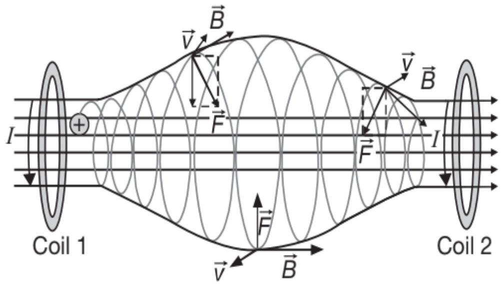
$$\Rightarrow \quad B = \frac{\mu_0 NIR^2}{2(R^2+x^2)^{3/2}}, \text{ along the axis of the loop}$$

At the centre of the coil, we have $x = 0$, so

$$B_{\text{centre}} = \frac{\mu_0 NI}{2R}$$

CONCEPT OF MAGNETIC BOTTLE

Motion of a charged particle in a non-uniform magnetic field is more complex. A field produced by two circular coils separated by some distance is shown in Figure.

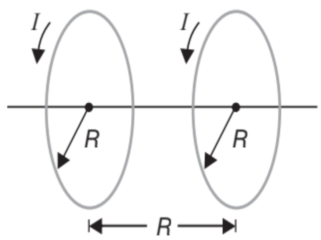


A magnetic bottle. Particles near either end of the region ecperience a magnetic force toward the center of the region.

Particles near either coil experience a magnetic force toward the centre of the region. Particles with appropriate speeds spiral repeatedly from one end of the region to the other and back. Since charged particles can be trapped in such a magnetic field so, it is called a **magnetic bottle**. This technique is used to confine very hot plasmas with temperatures of the order of 10^6 K. In a similar way the earth's non-uniform magnetic field traps charged particles coming from the sun in doughnut-shaped regions around the earth. These regions are called the **Van Allen Radiation Belts**.

ILLUSTRATION 45

Two circular coils of radius R, each with N turns, are perpendicular to a common axis. The centres of the coils are a distance R apart. Each coil carries a steady current I in the same direction, as shown in figure.



Show that the magnetic field on the axis at a distance x from the centre of one coil is

$$B = \frac{N\mu_0 IR^2}{2}\left[\frac{1}{(R^2+x^2)^{3/2}} + \frac{1}{(2R^2+x^2-2Rx)^{3/2}}\right]$$

Also show that $\frac{dB}{dx}$ and $\frac{d^2B}{dx^2}$ are both zero at the point midway between the coils.

SOLUTION

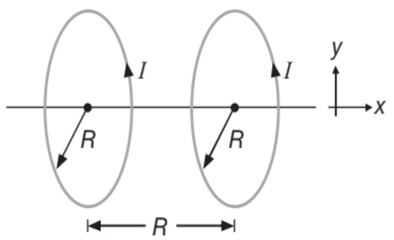
Let us use $B=\frac{\mu_0 NIR^2}{2\left(x^2+R^2\right)^{3/2}}$ twice, for both the coils.

So, we get

$$B=B_{x_1}+B_{x_2}$$

$$B=\frac{N\mu_0 IR^2}{2}\left[\frac{1}{\left(x^2+R^2\right)^{3/2}}+\frac{1}{\left[(R-x)^2+R^2\right]^{3/2}}\right]$$

$$\Rightarrow \quad B=\frac{N\mu_0 IR^2}{2}\left[\frac{1}{\left(x^2+R^2\right)^{3/2}}+\frac{1}{\left(2R^2+x^2-2xR\right)^{3/2}}\right]$$



$$\frac{dB}{dx}=\frac{N\mu_0 IR^2}{2}\left[-\frac{3}{2}(2x)\left(x^2+R^2\right)^{-5/2}-\frac{3}{2}\left(2R^2+x^2-2xR\right)^{-5/2}(2x-2R)\right]$$

Substituting $x=\frac{R}{2}$ in this equation, we get

$$\frac{dB}{dx}=0$$

Further $\frac{d^2B}{dx^2}=\frac{d}{dx}\left(\frac{dB}{dx}\right)$, so we have

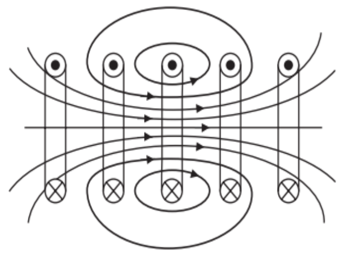
$$\Rightarrow \quad \frac{d^2B}{dx^2}=-\frac{3N\mu_0 IR^2}{2}\left[\left(x^2+R^2\right)^{-\frac{5}{2}}-5x^2\left(x^2+R^2\right)^{-\frac{7}{2}}+\left(2R^2+x^2-2xR\right)^{-\frac{5}{2}}\left(-5(x-R)^2\left(2R^2+x^2-2xR\right)^{-\frac{7}{2}}\right)\right]$$

Substituting $x=\frac{R}{2}$, we get $\frac{d^2B}{dx^2}=0$

This means the magnetic field in the region midway between the coils is uniform. Coils in this configuration are called Helmholtz coils.

SOLENOID (AN INTRODUCTION)

Let us consider the field lines due to a current carrying coil having five turns as shown in Figure.



The field lines for five loops

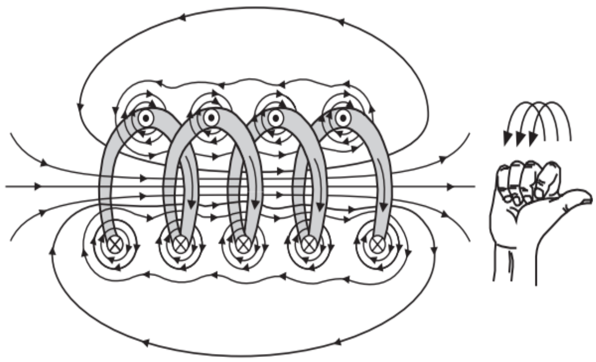
From the figure we observe that

(a) the field lines are always closed loops.
(b) very close to each wire the lines are circular (not shown).
(c) inside the coil the contributions from the different turns reinforce each other, so the field is strong.
(d) near the axis it is fairly uniform.
(e) outside the coil, the contributions from the various current elements tend to cancel, so the field is much weaker and the field outside the coil resembles that of a bar magnet so that one end of the coil acts as a north pole, whereas the other end acts as a south pole.

When the turns are tightly packed and their number becomes very large, the device is called a **solenoid**. The field within a long solenoid is quite uniform and strong, whereas it is essentially zero outside, as shown in figure. Let us now calculate the field strength along the axis of a long solenoid.

Field Along the Axis of a Solenoid

The name solenoid was first given by Ampere to a wire carrying a current I wound in a closely spaced spiral over a hollow cylindrical non-conducting core as shown in Figure.



If n is the number of turns per unit length of the coil, where each turn carries a current I uniformly wound round a cylinder (not shown) of radius R, then the number of turns in length dx are $n(dx)$. Thus, the magnetic field at the axial point O due to this element dx is,

$$dB = \frac{\mu_0}{2}\frac{I(ndx)R^2}{(R^2+x^2)^{3/2}}, \text{ along axis of solenoid}$$

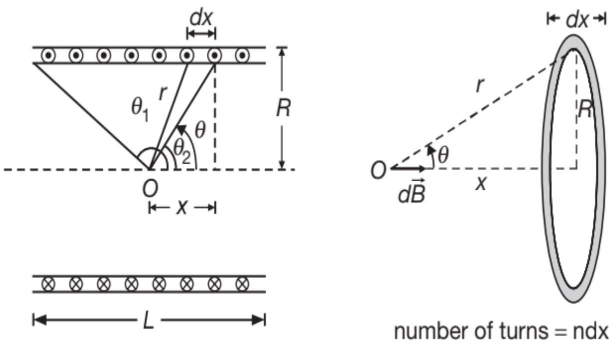
Also, we observe that

$$x = R\cot\theta$$

$$\Rightarrow \quad dx = -R(\text{cosec}^2\,\theta)d\theta$$

$$\Rightarrow \quad dB = -\frac{1}{2}\mu_0(nI\sin\theta)d\theta$$

Total field B due to the entire solenoid is, given by integrating the above expression within appropriate limits of θ as shown in Figure.



$$\Rightarrow \quad B = \frac{1}{2}\mu_0 nI\int_{\theta_1}^{\theta_2}(-\sin\theta)d\theta$$

$$\Rightarrow \quad B = \frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1)$$

CASE-1:
If the solenoid is very long $(L \gg R)$ and the point O is chosen at the middle, i.e., if $\theta_1 = 180°$ and $\theta_2 = 0°$, then, we get

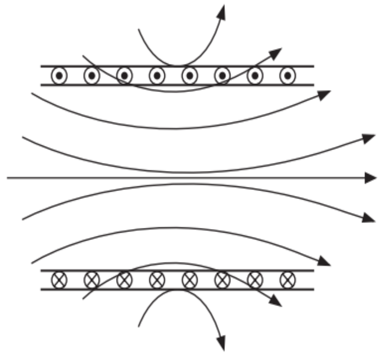
$$B(\text{centre}) = \mu_0 nI \qquad \{\text{For } L \gg R\}$$

CASE-2:
If the solenoid is very long $(L \gg R)$ and the point O is taken at the ends of the solenoid, then

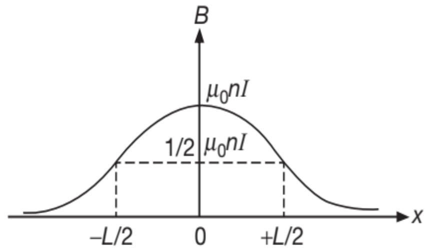
$$\theta_2 = 0°,\ \theta_1 = 90°$$

$$\Rightarrow \quad B(\text{end}) = \frac{1}{2}\mu_0 nI \qquad \{\text{For } L \gg R\}$$

Thus, the field at the end of a solenoid is just one half at the centre. The field lines are as shown in Figure.

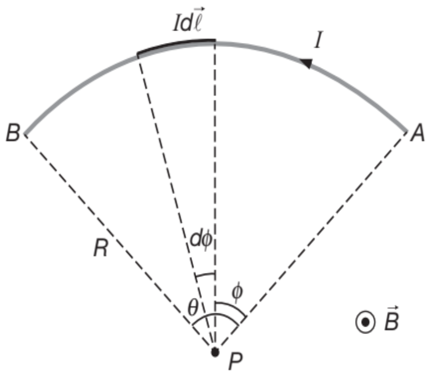


The field variation with distance along the axis of a solenoid is as shown in Figure.



FIELD DUE TO A CIRCULAR CURRENT CARRYING SEGMENT AT ITS CENTRE

Consider a circular segment AB of radius R as shown in Figure.



Let us find the magnetic field at the point P at its centre. We observe that

(a) each element is at the same distance from the centre, i.e., $r = R = \text{constant}$,

(b) the angle between element $d\vec{l}$ and $\vec{r}$ is always $\frac{\pi}{2}$

The contribution of each element to $\vec{B}$ is in the same direction (i.e., out of the page if the current is anticlockwise and into the page if clockwise). From Biot Savart's Law, we have

$$|\vec{B}| = \frac{\mu_0}{4\pi}\int \frac{I(d\vec{l}\times\hat{r})}{r^2} = \frac{\mu_0}{4\pi}\int_A^B \frac{Idl}{R^2}$$

But $dl = R\,d\phi$,

$$\Rightarrow \quad B = \frac{\mu_0}{4\pi}\int_0^{\theta} \frac{IRd\phi}{R^2}$$

$$\Rightarrow \quad B = \left(\frac{\mu_0 I}{4\pi R}\right)\theta$$

(a) Please note that, when we write $B = \left(\frac{\mu_0 I}{4\pi R}\right)\theta$, then the angle θ is in radian.

(b) If the loop is a quarter circle, $\left(\text{i.e., } \theta = \frac{\pi}{2}\right)$, then

$$B = \frac{\mu_0 I}{8R}$$

(c) If the loop is semi-circular (i.e., $\theta = \pi$), $B = \frac{\mu_0 I}{4R}$

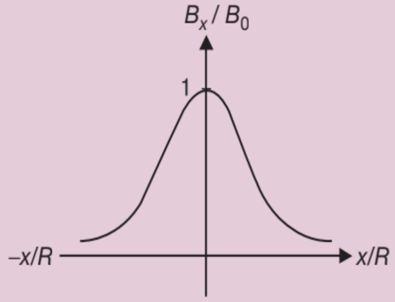
(d) If the loop is a three quarter circle $\left(\text{i.e., } \theta = \frac{3\pi}{2}\right)$, then

$$B = \frac{3\mu_0 I}{8R}$$

(e) If the loop is a full circle with N turns (i.e., $\theta = 2\pi N$), then

$$B = \frac{\mu_0 NI}{2R}$$

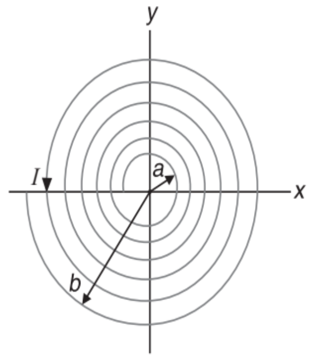
(f) The behavior of $\frac{B_x}{B_0}$ where $B_0 = B_{centre} = \frac{\mu_0 I}{2R}$ is the magnetic field strength at $x = 0$, as a function of $\frac{x}{R}$ is shown in Figure.



The ratio of the magnetic field, B_x / B_0, as a function of x/R

ILLUSTRATION 46

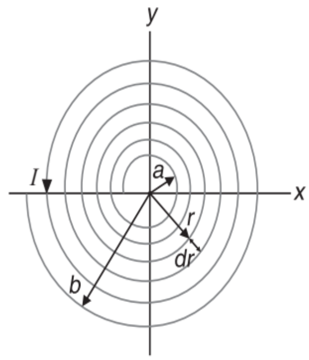
A long insulated copper wire is closely wound as a spiral of N turns. The spiral has inner radius a and outer radius b. The spiral lies in the x-y plane and a steady current I flows through the wire. Calculate the z component of the magnetic field at the centre of the spiral.



SOLUTION

We observe that the radial width $(b-a)$ of the coil is having N turns. So, number of turns per unit radial width of the coil is $n = \frac{N}{b-a}$.

Consider a circular coil of radius r, radial thickness dr as shown in Figure.



If dN be the number of turns in this radial thickness dr of the coil, then we have

$$dN = \frac{Ndr}{b-a}$$

If dB be the magnetic field (at the centre) due to this circular elemental coil of radius r having number of turns dN, then we have

$$dB = \frac{\mu_0 (dN) I}{2r}$$

$$\Rightarrow \quad dB = \frac{\mu_0 NI\,dr}{2(b-a)r}$$

To get the total magnetic field at the centre due to the coil integrating from $r=a$ to $r=b$, we get

$$B=\int_a^b dB=\int_a^b \frac{\mu_0 NI\,dr}{2(b-a)r}=\frac{\mu_0 NI}{2(b-a)}\int_a^b \frac{dr}{r}$$

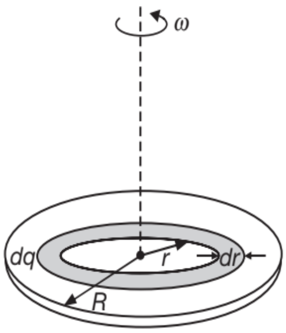
$$\Rightarrow \quad B=\frac{\mu_0 NI}{2(b-a)}\log_e r\Big|_a^b=\frac{\mu_0 NI}{2(b-a)}\log_e\left(\frac{b}{a}\right)$$

ILLUSTRATION 47

Calculate the magnetic field at the centre of a charged disc of radius R having a uniform charge Q, surface charge density σ rotating with angular velocity ω. Also calculate B in terms of σ.

SOLUTION

Consider a concentric infinitesimal ring of radius r, thickness dr, charge dq rotating with angular velocity ω as shown in Figure.



This rotating infinitesimal charge dq gives a current dI given by

$$dI=\frac{dq}{\pi}=\frac{\left(\frac{Q}{\pi R^2}\right)(2\pi r dr)}{\left(\frac{2\pi}{\omega}\right)}$$

$$\Rightarrow \quad dI=\left(\frac{Q\omega}{\pi R^2}\right)r\,dr$$

Since, we know that $dB=\dfrac{\mu_0(dI)}{2r}$

$$\Rightarrow \quad dB=\frac{\mu_0}{2r}\left(\frac{Q\omega}{\pi R^2}\right)r\,dr$$

$$\Rightarrow \quad dB=\frac{\mu_0 Q\omega}{2\pi R^2}dr$$

$$\Rightarrow \quad B=\frac{\mu_0 Q\omega}{2\pi R^2}\int_0^R dr$$

$$\Rightarrow \quad B=\frac{\mu_0 Q\omega}{2\pi R}$$

Also $Q=\sigma(\pi R^2)$

$$\Rightarrow \quad B=\frac{\mu_0\sigma(\pi R^2)\omega}{2\pi R}$$

$$\Rightarrow \quad B=\frac{\mu_0\sigma R\omega}{2}$$

So, finally, we get

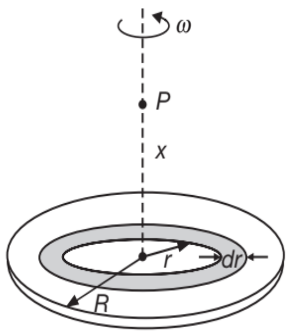
$$B=\frac{\mu_0 Q\omega}{2\pi R}=\frac{\mu_0\sigma R\omega}{2}$$

ILLUSTRATION 48

A disc of radius R rotates with angular velocity ω about an axis perpendicular to its surface passing through centre. Assuming the surface charge density σ varies with r as $\sigma=\alpha r^2$, where r is the distance from its centre, find the magnetic induction on the axis of rotation at a point at distance x from the centre.

SOLUTION

Consider an infinitesimal concentric ring of radius r, thickness dr, having a charge dq on it as shown in Figure.



Then, $dq=(2\pi r dr)(\sigma)$

$$\Rightarrow \quad dq=(2\pi r dr)(\alpha r^2)$$

$$\Rightarrow \quad dq=2\pi\alpha r^2 dr$$

If dI be the infinitesimal current due to rotation of the element of thickness dr, then

$$dI = \frac{dQ}{T} = \frac{2\pi\alpha r^3 dr}{2\pi/\omega}$$

$$\Rightarrow \quad dI = \alpha\omega r^3 dr$$

Magnetic induction at a distance x is given by

$$dB = \frac{\mu_0 (dI) r^2}{2(r^2 + x^2)^{3/2}}$$

$$\Rightarrow \quad dB = \frac{\mu_0 \alpha\omega r^5 dr}{2(r^2 + x^2)^{3/2}}$$

Total magnetic induction is obtained by integrating dB from zero to R. So, we get

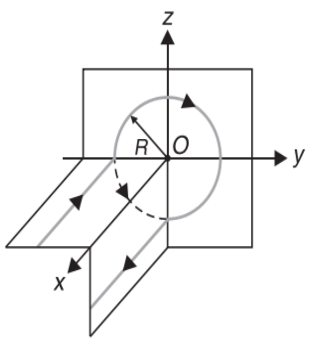
$$B = \int dB = \frac{\mu_0 \alpha\omega}{2} \int_0^R \frac{r^5 dr}{(r^2 + x^2)^{3/2}}$$

Integrating the above expression, we get

$$B = \frac{\mu_0 \alpha\omega}{2}\left[\frac{(R^2 + x^2)^{3/2}}{3} - 2x^2 (R^2 + x^2)^{1/2} - \frac{x^4}{(R^2 + x^2)^{1/2}} + \frac{8}{3}x^3\right]$$

ILLUSTRATION 49

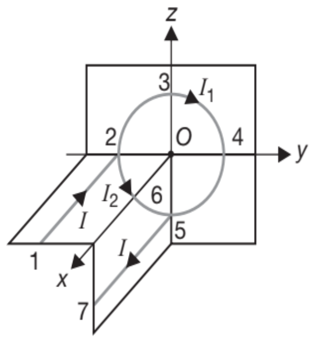
If the wire carrying a current I has the shape shown in figure, then calculate the magnetic field at the point O (both in magnitude and direction). The radius of the curved part of the wire is R, the linear parts of the wire are very long. Also assume the wires to be having uniform resistance.



SOLUTION

If we assume resistance of the quarter part '265' to be $R_2 = R$, then the resistance of three quarters '2345' will be $R_1 = 3R$. So, current passing through the two branches will have the ratio given by

$$\frac{I_1}{I_2} = \frac{R_2}{R_1} = \frac{R}{3R}$$



$$\Rightarrow \quad \frac{I_1}{I_2} = \frac{1}{3} \qquad \ldots(1)$$

Also $I_1 + I_2 = \mathrm{I}$ ...(2)

So, from (1) and (2), we get

$$I_1 = \frac{I}{4} \text{ (from 2 to 3 to 4 to 5)}$$

and $I_2 = \frac{3I}{4}$ (from 2 to 6 to 5)

So, $\vec{B}_{12} = \frac{\mu_0 I}{4\pi R}(-\hat{k})$

$$\vec{B}_{\text{arc } 2345} = \frac{3\mu_0 I_1}{8R}(-\hat{i}) = \frac{3\mu_0 I}{32R}(-\hat{i})$$

$$\vec{B}_{57} = \frac{\mu_0 I}{4\pi R}(-\hat{j}) \text{ and}$$

$$\vec{B}_{\text{arc } 265} = \frac{\mu_0 I_2}{8R}(\hat{i}) = \frac{3\mu_0 I}{32R}(\hat{i})$$

$$\Rightarrow \quad \vec{B} = \vec{B}_{12} + \vec{B}_{\text{arc } 2345} + \vec{B}_{\text{arc } 265} + \vec{B}_{57}$$

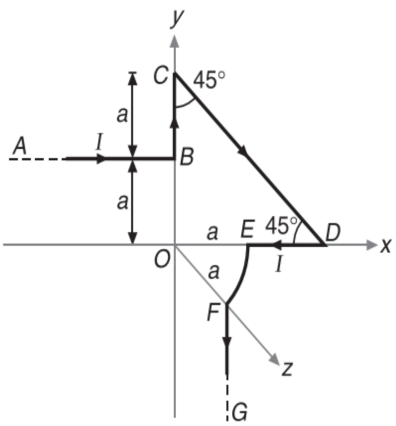
$$\Rightarrow \quad \vec{B} = -\left(\frac{\mu_0 I}{4\pi R}\right)\left[\hat{k} + \left(\frac{3\pi}{8}\right)\hat{i} - \left(\frac{3\pi}{8}\right)\hat{i} + \hat{j}\right]$$

$$\Rightarrow \quad \vec{B} = -\left(\frac{\mu_0 I}{4\pi R}\right)(\hat{j} + \hat{k})$$

$$B = |\vec{B}| = \frac{\sqrt{2}\mu_0 I}{4\pi R}$$

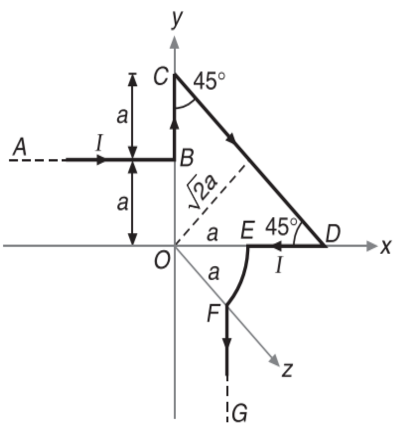
ILLUSTRATION 50

Calculate the magnetic field vector at the origin O due to the current carrying wire configuration shown in figure.



SOLUTION

Due to the wire segments BC and DE no magnetic field will be produced at O because O lies on the extended portions of BC and DE.



For the remaining wire segments the magnetic fields are given by

$$\vec{B}_{AB} = \frac{\mu_0 I}{4\pi a}(-\hat{k})$$

$$\vec{B}_{CD} = \frac{\mu_0 I}{4\pi(\sqrt{2}a)}\sqrt{2}(-\hat{k})$$

$$\vec{B}_{EF} = \frac{\mu_0 I}{8a}(-\hat{j})$$

$$\vec{B}_{FG} = \left(\frac{\mu_0 I}{4\pi a}\right)\hat{i}$$

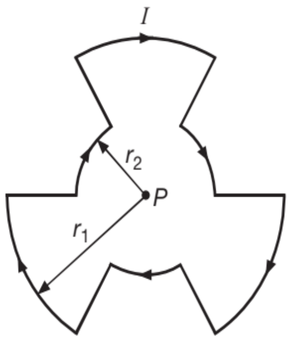
The resultant magnetic field at O due to all the wire segments is

$$\vec{B}_O = \vec{B}_{AB} + \vec{B}_{CD} + \vec{B}_{EF} + \vec{B}_{FG}$$

$$\Rightarrow \quad \vec{B}_0 = \frac{\mu_0 I}{4\pi a}\left(\hat{i} - \frac{\pi}{2}\hat{j} - 2\hat{k}\right)$$

ILLUSTRATION 51

A current I flows around a closed path in the horizontal plane of the circuit shown in figure. The path consists of six arcs with alternating radii r_1 and r_2 connected by radial segments. Each segment of arc subtends an angle of 60° at the common centre P, with $\frac{r_2}{r_1} = \frac{2}{3}$. This current path produces a magnetic field $\vec{B}$ at P. If the path is modified so that the ratio $\frac{r_2}{r_1} = \frac{1}{3}$, by what factor must I be multiplied in order that the field at P remain the same?



SOLUTION

Since $B_{\text{arc}} = \left(\frac{\mu_0 I}{4\pi r}\right)\theta$, where θ is in radian

When $\frac{r_2}{r_1} = \frac{2}{3}$, then the field at P is B given by

$$B = 3\left(\frac{\mu_0 I}{4\pi r_1}\right)\left(\frac{\pi}{3}\right) + 3\left(\frac{\mu_0 I}{4\pi r_2}\right)\left(\frac{\pi}{3}\right), \otimes$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{4}\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \otimes$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{4r_1}\left(1 + \frac{r_1}{r_2}\right) = \frac{\mu_0 I}{4r_1}\left(1 + \frac{3}{2}\right)$$

$$\Rightarrow \quad B = \frac{5\mu_0 I}{8r_1}, \otimes \quad \ldots(1)$$

Now, when $\frac{r_2}{r_1} = \frac{1}{3}$, let the current be kI so that B remains the same.

$$B = 3\left[\frac{\mu_0(kI)}{4\pi r_1}\right]\left(\frac{\pi}{3}\right) + 3\left[\frac{\mu_0(kI)}{4\pi r_2}\right]\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0(kI)}{4}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0(kI)}{4r_1}\left(1+\frac{r_1}{r_2}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0(kI)}{4r_1}(1+3)$$

$$\Rightarrow \quad B=\frac{\mu_0(kI)}{r_1}, \otimes \qquad \ldots(2)$$

From (1) and (2), we get

$$\frac{5\mu_0 I}{8r_1}=\frac{\mu_0(kI)}{r_1}$$

$$\Rightarrow \quad k=\frac{5}{8}$$

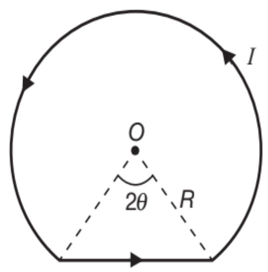
Please note that in both the cases the ratio $\frac{r_2}{r_1}=\frac{2}{3}$ and then the same ratio becomes $\frac{1}{3}$. So, the problem just wants to convey that the radius r_2 is now halved keeping r_1 constant.

Test Your Concepts-IV

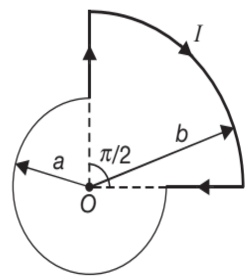
Based on Biot Savart's Law and Applications

(Solutions on page H.11)

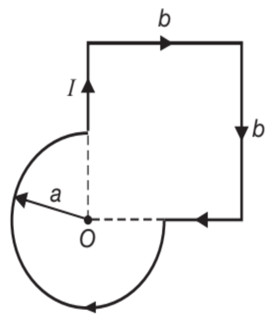
1. A current $I=1\,\text{A}$ circulates in a round thin-wire loop of radius $R=100$ mm. Find the magnetic induction
   (a) at the centre of the loop.
   (b) at the point lying on the axis of the loop at a distance $x=100$ mm from its centre.
2. A current I flows along a thin wire shaped as a regular polygon with n sides which can be inscribed into a circle of radius R. Find the magnetic induction at the centre of the polygon. Analyse the obtained expression at $n\to\infty$.
3. Two straight infinitely long and thin parallel wires are spaced 0.1 m apart and carry a current of 10 A each. Calculate the magnetic field at a point lying at a distance 0.1 m from both wires when the currents in the wires are in the same direction and when the currents in the wires are in the opposite direction.
4. A current I flows along a thin wire shaped as shown in figure. The radius of a curved part of the wire is equal to R, the angle 2θ. Find the magnetic induction of the field at the point O.



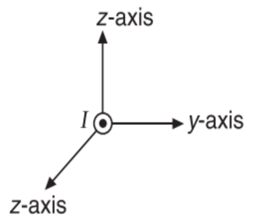
5. Find the magnetic induction of the field at the point O of a loop with current I, whose shape is shown in figure.



6. Find the magnetic induction of the field at the point O of a loop with current I, whose shape is shown in figure.

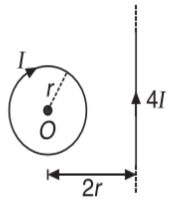


7. A long current carrying conductor carrying a current I lies along x-axis such that its one end is at origin, and the other end lies on positive x-axis at a large distance from origin. The current through the conductor is along positive x-direction as shown in Figure.

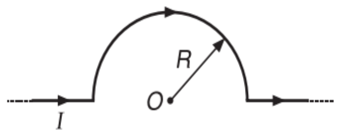


Calculate the magnetic field vector at points whose co-ordinates are $(0, a, 0)$, $(0, a, a)$ and $(-a, a, 0)$.

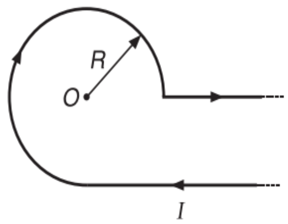
8. For the arrangement shown, calculate the magnetic field at point O.



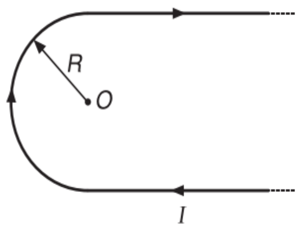
9. Calculate the magnetic field at the point O shown in figure. The radius of the curved part of the wire is R, the linear parts are assumed to be very long.



10. Find the magnetic field at the point O due to the current carrying wire having the shape shown in figure.

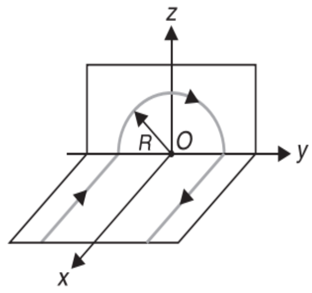


11. In the arrangement shown, calculate the magnetic flux density i.e., magnetic field at the point O.

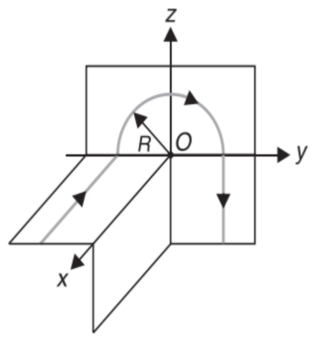


12. A very long wire carrying a current I is bent at right angles. Find the magnetic induction at a point lying on a perpendicular to the wire, drawn through the point of bending, at a distance ℓ from it.

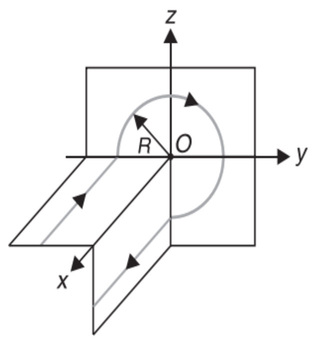
13. Calculate the magnetic field vector at the point O if the wire carrying a current I has the shape shown in figure. Also calculate its magnitude.



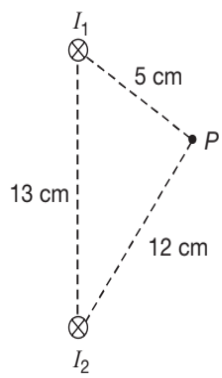
14. Calculate the magnetic field $\vec{B}$ (both in magnitude and direction) at the point O for the current carrying wire arrangement shown in figure.



15. In the current carrying wire arrangement shown, calculate the magnetic flux density i.e., magnetic field at the point O (both in magnitude and direction).

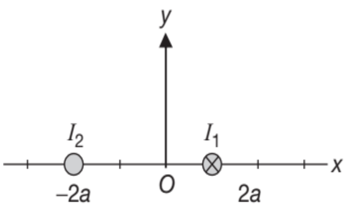


16. Two long, parallel conductors carry currents $I_1 = 3$ A and $I_2 = 3$ A, both directed into the page in figure. Determine the magnitude and direction of the resultant magnetic field at P.



17. In Niels Bohr's 1913 model of the hydrogen atom, an electron circles the proton at a distance of 5.29×10^{-11} m with a speed of 2.19×10^{6} ms^{-1}. Compute the magnitude of the magnetic field that this motion produces at the location of the proton.

18. Two very long, straight, parallel wires carry currents that are directed perpendicular to the page, as in figure. Wire 1 carries a current I_1 into the page (in the $-x$ direction) and passes through the x-axis at $x = +a$. Wire 2 passes through the x axis at $x = -2a$ and carries an unknown current I_2. The total magnetic field at the origin due to the current carrying wires has the magnitude $\frac{2\mu_0 I_1}{2\pi a}$. The current I_2 can have either of two possible values.



(a) Find the value of I_2 with the smaller magnitude, stating it in terms of I_1 and giving its direction.

(b) Find the other possible value of I_2.

19. A very long wire 1 carries a current of 30 A to the left along the x axis. A second very long wire 2 carries current 50 A to the right along the line $y = \frac{2}{3}$ m, $z = 0$

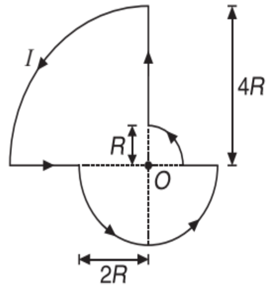
(a) Where in the plane of the two wires is the total magnetic field equal to zero?

(b) A particle with a charge of $-2\ \mu C$ is moving with a velocity of $150\hat{i}$ Mms^{-1} along the line $y = \frac{1}{3}$ m, $z = 0$. Calculate the vector magnetic force acting on the particle.

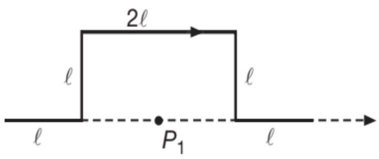
20. A non-conducting ring of radius R is uniformly charged with a total positive charge Q. The ring rotates at a constant angular speed w about an axis through its centre and perpendicular to its plane. Find the magnitude of the magnetic field on the axis of the ring at a point P, a distance $\frac{R}{2}$ from its centre.

21. A long straight wire lies on a horizontal table and carries a current I. In a vacuum, a proton moves parallel to the wire (opposite the current) with a constant speed of v at a distance d above the wire. Determine the value of d. You may ignore the magnetic field due to the Earth.

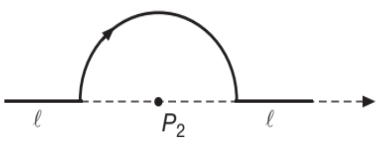
22. Calculate the magnetic field at the centre O of the current carrying loop shown in Figure.



23. A wire is bent into the shape shown in figure (a), and the magnetic field is measured at P_1 when the current in the wire is I. The same wire is then formed into the shape shown in figure (b), and the magnetic field measured at point P_2 when the current is again I. If the total length of wire is the same in each case, what is the ratio of $\frac{B_1}{B_2}$?

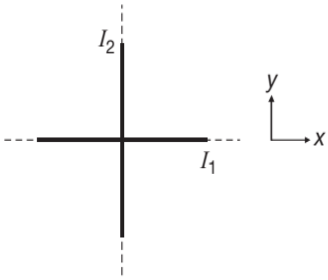


(a)

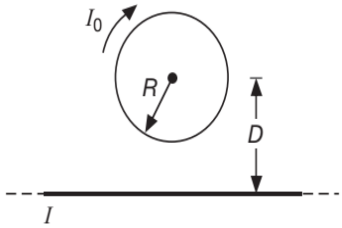


(b)

24. Two mutually perpendicular long conductors carrying currents I_1 and I_2 lie in one plane. Find the locus of points at which the magnetic induction is zero.



25. A circular loop has radius R and carries current I_0 in a clockwise direction. The centre of the loop is a distance D above a long, straight wire. What are the magnitude and direction of the current I in the wire if the magnetic field at the centre of the loop is zero?



AMPERE'S CIRCUITAL LAW (ACL)

Just as we use Gauss' Law to find the electric field at a point due to a charge distribution, similarly we shall be taking help of Ampere's Circuital Law (ACL) to find the magnetic field at a point due to a current carrying conductor.

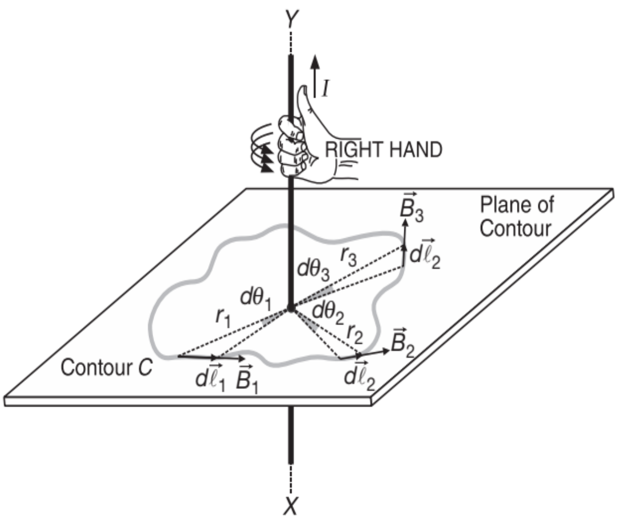
According to this Law, *"the line integral of the resultant magnetic field* $\left(\text{i.e., } \oint \vec{B}\cdot d\vec{l}\right)$ *along a closed contour* ***(also called Amperian Loop)*** *is* μ_0 *times the total current threading through the contour"*.

Mathematically, we have

$$\oint \vec{B}\cdot d\vec{l} = \mu_0(\Sigma I) = \mu_0 I_{\text{net}}$$

Proof of Ampere's Circuital Law (ACL)

To prove the result obtained, let us consider a current carrying long wire XY as shown in Figure.



Let us draw a random contour C around the wire XY. Also, we have shown the plane which contains the contour C. Let us now randomly consider three infinitesimal elements $d\vec{l}_1$, $d\vec{l}_2$ and $d\vec{l}_3$ along the length of the contour (as shown). Let these elements be at distances r_1, r_2 and r_3 from the wire and they subtend angles $d\theta_1$, $d\theta_2$ and $d\theta_3$ at the wire. If $\vec{B}_1$, $\vec{B}_2$ and $\vec{B}_3$ be the magnetic fields due to the wire at these elements, then we observe that angle between $d\vec{l}_1$ and $\vec{B}_2$, between $d\vec{l}_2$ and $\vec{B}_2$, between $d\vec{l}_3$ and $\vec{B}_3$ is zero. Let us now calculate the integral $\oint \vec{B}\cdot d\vec{l}$.

From concepts of integral calculus, for a closed loop, we have

$$\oint \vec{B}\cdot d\vec{l} = \vec{B}_1\cdot d\vec{l}_1 + \vec{B}_2\cdot d\vec{l}_2 + \vec{B}_3\cdot d\vec{l}_3 + \ldots$$

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = B_1 dl_1 \cos(0^\circ) + B_2 dl_2 \cos(0^\circ) + \ldots$$

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = B_1 dl_1 + B_2 dl_2 + B_3 dl_3 + \ldots$$

Since $B_1 = \dfrac{\mu_0 I}{2\pi r_1}$, $B_2 = \dfrac{\mu_0 I}{2\pi r_2}$, $B_3 = \dfrac{\mu_0 I}{2\pi r_3}$ and so on

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\left(\frac{dl_1}{r_1} + \frac{dl_2}{r_2} + \frac{dl_3}{r_3} + \ldots\right)_{\text{in the closed loop}}$$

However, $\dfrac{dl_1}{r_1} = d\theta_1$, $\dfrac{dl_2}{r_2} = d\theta_2$, $\dfrac{dl_3}{r_3} = d\theta_3$ and so on

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}(d\theta_1 + d\theta_2 + d\theta_3 + \ldots)_{\text{in the closed loop}}$$

$$\Rightarrow \quad \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\oint d\theta$$

Since, for a closed loop, we have $\oint d\theta = 2\pi$

$$\Rightarrow \quad \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}(2\pi)$$

$$\Rightarrow \quad \oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

Conceptual Note(s)

(a) Generally, the **Amperian Loop** or the **Contour** is selected in a way such that at each point taken on the loop, either the magnetic field

(i) $\vec{B}$ is normal to the loop (because then $\oint \vec{B}\cdot d\vec{\ell} = 0$, since $\theta = 90°$)

(ii) $\vec{B}$ vanishes everywhere on the loop (because then too $\oint \vec{B}\cdot d\vec{\ell} = 0$)

(iii) $\vec{B}$ is tangential to the loop and has a non-zero constant magnitude all over the loop (because then

$$\oint \vec{B}\cdot d\vec{\ell} = \oint B(d\ell)\cos(0°) = B\oint d\ell,$$

which becomes really easy to evaluate just by knowing the shape of the contour.

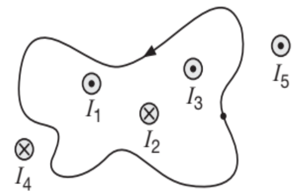
(b) The line integral is independent of the shape of the path and the placement of the current carrying wire in it.

(c) The statement $\oint \vec{B}\cdot d\vec{\ell} = 0$ does not necessarily imply that $B = 0$ everywhere along the path. However, we can conclude that no net current is threaded through the contour or the loop.

Sign Convention to be Followed for Currents while Applying ACL

*Curl the fingers of the Right Hand in the sense of contour (Clockwise or Counter-Clockwise), then **the currents along the direction of the thumb are taken as positive** and **the currents opposite to the direction of thumb are taken as negative**.*

In the arrangement shown,



we observe that the currents I_1 and I_3 are along the direction of thumb when the fingers of the right hand are curled in the sense of contour i.e. counter clockwise, whereas I_2 points opposite to the direction of thumb. Also, the currents I_4 and I_5 are outside the contour, so according to Ampere's Circuital Law, we have

$$\Rightarrow \quad \oint \vec{B}\cdot d\vec{l} = \mu_0 \sum I = \mu_0 (I_1 + I_3 - I_2)$$

AMPERE'S OBJECTION(S) TO BIOT SAVART'S LAW

Ampere had several objections to the work of Jean Biot and Felix Savart.

(a) He felt that their experiments were not precise enough for them to claim certainty in the $\sin\theta$ factor.

(b) He was uncomfortable with their use of "current elements", since isolated current elements do not exist as they are always part of a complete circuit.

Ampere's Line of Action

Consequently, he pursued his own line of experimental and theoretical research and obtained a different relation, now called Ampere's Law, between a current and the magnetic field it produces.

Although Ampere's Law could have been derived from the Biot-Savart's expression for $d\vec{B}$, but Ampere preferred not to do so. Instead, he made it possible by considering the field due to an infinite straight wire. He knew that the field lines due to an infinite straight wire are concentric circles and the magnitude of the field at a distance r from the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

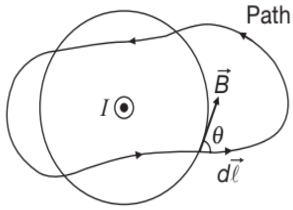
So, he had rewritten the above expression as

$$B(2\pi r) = \mu_0 I$$

and interpreted it as follows.

"$2\pi r$ is the length of a circular path around the wire, B is the component of the magnetic field tangential to the path, and I is the current through the area bounded by the path. Ampere generalized this result to paths and wires of any shape".

Figure shows a current coming out of the page and an arbitrary closed path around it.



A current is coming out of the page. According to Ampere's Law, around any closed path, enclosing the current I,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

For an infinitesimal displacement $d\vec{l}$ along the path, the product of dl and the component of $\vec{B}$ along $d\vec{l}$ is

$$dl(B\cos\theta) = \vec{B} \cdot d\vec{l}$$

According to Ampere's Law, the sum (integral) of this product around a closed path is given by

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

where I is the net current flowing through the surface enclosed by the path.

The sense (clockwise or counter clockwise) in which the integral is to be evaluated is given by a right-hand rule, according to which when the thumb of the right hand points along the current, the curled fingers indicate the positive sense along the path.

Conceptual Note(s)

(a) The equation, $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$ is valid only for steady currents and nonmagnetic materials, such as Cu.

(b) The enclosed current may not have to flow in a wire. It may be a beam of charged particles which also constitutes a current.

(c) In order to use Ampere's Law to determine the magnetic field, it is necessary for the geometry of the current flow to possess sufficient symmetry so that the integral can be evaluated easily. One needs to know the field pattern and then to make a suitable choice for the path of integration.

(d) Ampere's Law in magnetism is analogous to Gauss's Law in electrostatics. In order to apply them, the system must posses certain symmetry. In the case of an infinite wire, the system possesses cylindrical symmetry and Ampere's Law can be readily applied. However, when the length of the wire is finite, Biot-Savart's Law must be used instead.

Biot-Savart's Law	$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{r}}{r^2}$	General current source e.g. finite wire
Ampere's Law	$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$	Current source has certain symmetry e.g. infinite wire (cylindrical)

(e) Ampere's Law is applicable to the following current configurations:

(i) Infinitely long straight current carrying wire.

(ii) Infinitely large sheet of thickness *b* with a current density *J*.

(iii) Infinite solenoid.

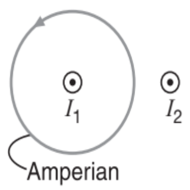
(iv) Toroid.

We shall examine all four configurations in detail.

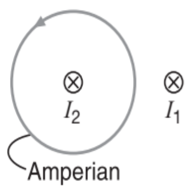
ILLUSTRATION 52

In the following arrangements shown, calculate $\oint \vec{B} \cdot d\vec{l}$ corresponding to the respective Amperian loops for which the direction in which the closed integral has to be performed is indicated by the arrows.

(a)

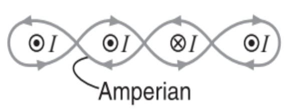


(b)



(c)

(d)



SOLUTION

To solve this problem, we must understand that when we curl the fingers of the Right Hand in the sense of contour (Clockwise or Counter-Clockwise), then the currents along the direction of the thumb are taken as positive and the currents opposite to the direction of thumb are taken as negative.

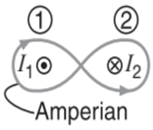
(a) The Amperian loop is traversed counter clockwise and hence the thumb will point outwards along I_1, so

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_1$$

(b) The Amperian loop is traversed counter clockwise and hence the thumb will point outwards opposite to I_2, so

$$\oint \vec{B}\cdot d\vec{l} = \mu_0(-I_2) = -\mu_0 I_2$$

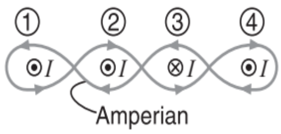
(c) The left side portion 1 of the Amperian loop is traversed clockwise, whereas the right-side portion 2 is traversed counter clockwise as shown in figure.



So, by applying the sign convention, we get that both I_1 and I_2 are negative, so we have

$$\oint \vec{B}\cdot d\vec{l} = \mu_0(-I_1 - I_2) = -\mu_0(I_1 + I_2)$$

(d) The portion 1 of the Amperian loop is traversed counter clockwise, portion 2 clockwise, portion 3 counter clockwise and portion 4 clockwise as shown in figure.

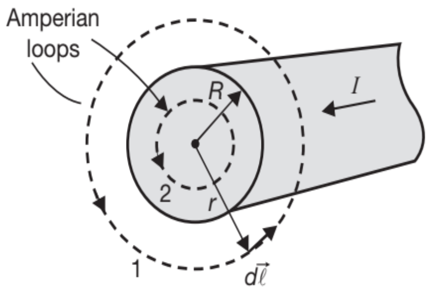


So, by applying the sign convention, we see that current for the loop 1 is to be taken as positive, for loop 2 to be taken as negative, for loop 3 to be taken as negative and for loop 4 to be taken as negative. So, we get

$$\oint \vec{B}\cdot d\vec{l} = \mu_0(I - I - I - I) = -3\mu_0 I$$

MAGNETIC FIELD DUE TO A THICK CURRENT CARRYING WIRE

Consider a long straight wire of radius R carrying a current I of uniform current density, as shown in Figure.



Let us find the magnetic field everywhere i.e., outside, at the surface and inside the wire.

CASE-1: Outside

Outside the wire where $r > R$, the Amperian loop or the contour (circle 1) completely encircles the current, and hence $I_{enc} = I$.

Applying Ampere's Law, we get

$$\oint \vec{B}\cdot d\vec{l} = B\oint dl = B(2\pi r) = \mu_0 I$$

$$\Rightarrow \quad B_{outside} = \frac{\mu_0 I}{2\pi r}$$

CASE-2: At the Surface

Substituting $r = R$ in the above expression, we get

$$B_{surface} = \frac{\mu_0 I}{2\pi R}$$

CASE-3: Inside

Inside the wire where $r < R$, the amount of current encircled by the Amperian loop or the contour (circle 2) is proportional to the area enclosed, i.e.,

$$I_{encl} = \left(\frac{\pi r^2}{\pi R^2}\right)I = \left(\frac{r^2}{R^2}\right)I$$

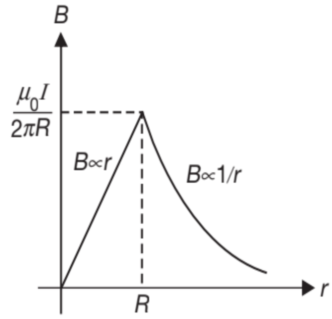
According to Ampere's Circuital Law, we have

$$\oint \vec{B}\cdot d\vec{l} = B(2\pi r) = \mu_0 I\left(\frac{\pi r^2}{\pi R^2}\right)$$

$$\Rightarrow \quad B_{inside} = \left(\frac{\mu_0 I}{2\pi R^2}\right)r$$

We see that the magnetic field is zero at the centre of the wire and increases linearly with r until $r = R$.

Outside the wire, the field falls off as $\frac{1}{r}$. The qualitative behaviour of the field is depicted as shown in Figure.



$$\Rightarrow \quad B = \begin{cases} \dfrac{\mu_0 I}{2\pi r} & r \geq R \text{ (outside and at the surface)} \\ \left(\dfrac{\mu_0 I}{2\pi R^2}\right) r & r < R \text{ (inside)} \end{cases}$$

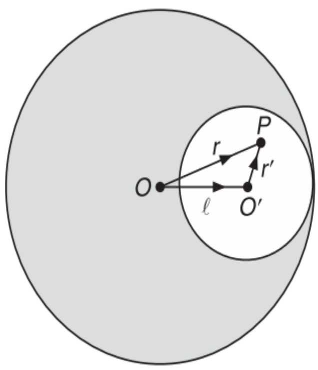
and $B_{\text{centre}} = 0$

ILLUSTRATION 53

Inside a homogeneous long straight current carrying wire of circular cross-section, there is a cylindrical cavity whose axis is parallel to the conductor axis and is displaced relative to it by a distance ℓ. A direct current of density J flows along the wire. Calculate the magnetic field inside the cavity.

SOLUTION

The arrangement for which the magnetic field $(\vec{B})$ has to be calculated at the point P inside the cavity is shown in Figure.



The magnetic field at a point P is

$$\vec{B}_P = \vec{B}_{\substack{\text{Full}\\\text{Cylinder}}} - \vec{B}_{\substack{\text{Cavity}\\\text{Cylinder}}} \qquad \ldots(1)$$

where $\vec{B}_0$ is the magnetic field at P due to the full conductor (without cavity) and $\vec{B}'$ is the magnetic field at P due to current flowing through the part of the conductor which has been removed in order to create the cavity.

According to Ampere's Circuital Law, the magnetic field at a distance r from the axis of solid cylinder (inside the solid cylinder) is given by

$$B(2\pi r) = \mu_0 \pi r^2 J$$

$$\Rightarrow \quad B = \frac{1}{2}\mu_0 J r \text{ (in magnitude)}$$

Vectorially, we have

$$\vec{B} = \frac{\mu_0}{2}(\vec{J} \times \vec{r})$$

The magnetic field due to the full cylinder is given by

$$\vec{B}_{\substack{\text{Full}\\\text{Cylinder}}} = \frac{\mu_0}{2}(\vec{J} \times \vec{r})$$

The magnetic field due to the cylindrical cavity is given by

$$\vec{B}_{\substack{\text{Cavity}\\\text{Cylinder}}} = \frac{\mu_0}{2}(\vec{J} \times \vec{r}')$$

Since $\vec{B}_P = \vec{B}_{\substack{\text{Full}\\\text{Cylinder}}} - \vec{B}_{\substack{\text{Cavity}\\\text{Cylinder}}}$

$$\Rightarrow \quad \vec{B}_P = \frac{\mu_0}{2}(\vec{J} \times \vec{r}) - \frac{\mu_0}{2}(\vec{J} \times \vec{r}')$$

$$\Rightarrow \quad \vec{B}_P = \frac{\mu_0}{2}\vec{J} \times (\vec{r} - \vec{r}') \qquad \ldots(2)$$

In triangle $OO'P$, we have

$$\vec{l} + \vec{r}' = \vec{r}$$

$$\Rightarrow \quad \vec{r} - \vec{r}' = \vec{l}$$

So, equation (2) becomes

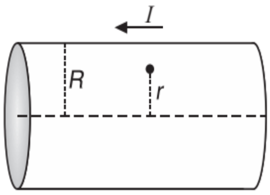
$$\vec{B}_P = \frac{\mu_0}{2}(\vec{J} \times \vec{l})$$

In magnitude, we have

$$B_P = \frac{\mu_0 J l}{2}$$

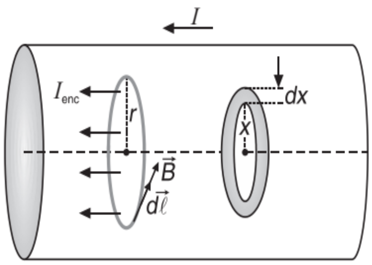
ILLUSTRATION 54

A long cylindrical conductor of radius R carries a current I as shown in figure. The current density J varies across the cross-section with the radial distance r as $J = kr^2$, where k is a constant. Find an expression for the magnetic field B at a distance $r(<R)$ from the axis of the cylindrical conductor.



SOLUTION

Consider an Amperian loop of radius r as shown in figure.



The distribution of current shows that the field will be same everywhere on the Amperian loop of radius r. If I_{enc} be the current threading through the Amperian loop, then by Ampere's Circuital Law, we have

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\Rightarrow \quad \oint \vec{B} \cdot d\vec{l} = B(2\pi r) = \mu_0 I_{enc}$$

$$\Rightarrow \quad B = \frac{\mu_0 I_{enc}}{2\pi r} \qquad \ldots(1)$$

Now, to find the current I_{enc} threading through the Amperian loop, consider a ring of radius x and thickness dx having area $dA = 2\pi x dx$. Since the current density J is defined as

$$J = \frac{dI}{dA}$$

$$\Rightarrow \quad dI = JdA, \text{ where } J = kx^2$$

$$\Rightarrow \quad I_{enc} = \int dI = \int JdA = \int_0^r kx^2 (2\pi x dx)$$

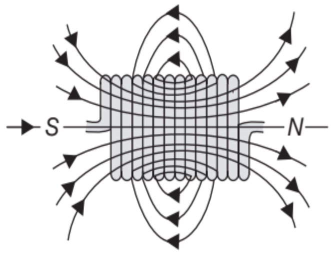
$$\Rightarrow \quad I_{enc} = 2\pi k \int_0^r x^3 dx = 2\pi k \left(\frac{x^4}{4} \bigg|_0^r \right) = \frac{2\pi k r^4}{4}$$

Substituting this value of I_{enc} in equation (1), we get

$$B = \frac{\mu_0 k r^3}{4}$$

SOLENOID (REVISITED)

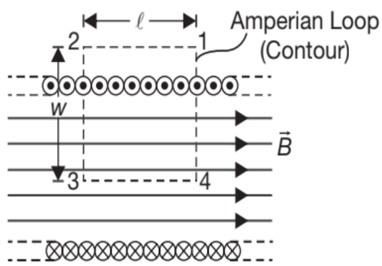
As studied and discussed earlier, a solenoid is a long coil of wire tightly wound in the helical form. The magnetic field lines of a solenoid carrying a steady current I is shown in Figure.



We see that if the turns are closely spaced, the resulting magnetic field inside the solenoid becomes fairly uniform, provided that the length of the solenoid is much greater than its diameter.

For an ideal solenoid, which is infinitely long with turns tightly packed, the magnetic field inside the solenoid is uniform and parallel to the axis, and vanishes outside the solenoid.

We shall be using Ampere's Law to calculate the magnetic field strength inside an ideal solenoid. The cross-sectional view of an ideal solenoid is shown in Figure.



Amperian loop for calculating the magnetic field of an ideal solenoid.

To compute $\vec{B}$, we consider a rectangular path of length l and width w and traverse the path in a counterclockwise manner. The line integral of $\vec{B}$ along this loop, 12341, is

$$\oint \vec{B}\cdot d\vec{l} = \int_1^2 \vec{B}\cdot d\vec{l} + \int_2^3 \vec{B}\cdot d\vec{l} + \int_3^4 \vec{B}\cdot d\vec{l} + \int_4^1 \vec{B}\cdot d\vec{l}$$

In the above, the contributions along sides 23 and 41 are zero because $\vec{B}$ is perpendicular to $d\vec{l}$. In addition, $\vec{B} = \vec{0}$ alongside 12 because the magnetic field is zero outside and non-zero only inside the solenoid. On the other hand, the total current enclosed by the contour (Amperian loop) is $I_{enc} = NI$, where N is the total number of turns. Applying Ampere's Law, we get

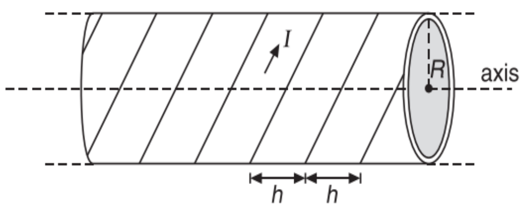
$$\oint \vec{B}\cdot d\vec{l} = Bl = \mu_0 (NI)$$

$$\Rightarrow \quad B = \frac{\mu_0 NI}{l} = \mu_0 nI$$

where $n = \frac{N}{l}$ represents the number of turns per unit length.

ILLUSTRATION 55

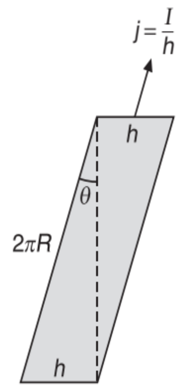
A thin conducting strip of width h is tightly wound in the shape of a very long cylindrical coil with cross-sectional radius R to make a single layer straight solenoid as shown in figure.



A direct current I flows through the strip. Find the magnetic induction inside and outside the solenoid as a function of the distance r from its axis.

SOLUTION

Let us consider one loop, then open it and draw it as shown in Figure.



(one loop completely opened)

Figure (a)

Since the current is flowing in a strip of width h so, the current per unit strip width is given by

$$j = \frac{I}{h}$$

This current per unit length (also called linear current density) can be resolved in two components j_x and j_y as shown in Figure.

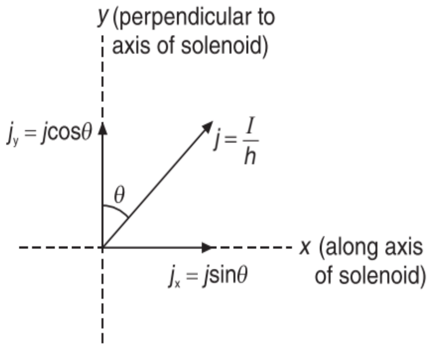


Figure (b)

Outside this type of solenoid, the magnetic field will be due to the component of current along the axis of solenoid as shown in Figure (c).

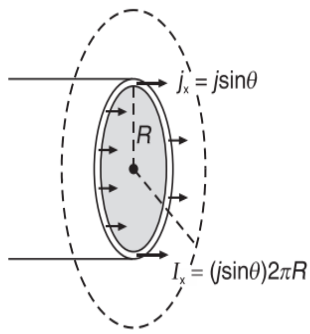


Figure (c)

For current density $j_x = j\sin\theta$ along the length of solenoid, by applying Amperes Circuital Law, we get

$$B_{\text{out}}(2\pi r) = \mu_0 I_x$$

where $I_x = j_x(2\pi R)$

$$\Rightarrow \quad I_x = j\sin\theta(2\pi R)$$

From Figure (a), we have

$$\sin\theta = \frac{h}{2\pi R}$$

$$\Rightarrow \quad I_x = j\left(\frac{h}{2\pi R}\right)(2\pi R) = jh$$

$$\Rightarrow \quad B_{\text{out}} = \frac{\mu_0 I_x}{2\pi r} = \frac{\mu_0 jh}{2\pi r} = \frac{\mu_0 I}{2\pi r}$$

Inside this type of solenoid, the magnetic field will be due to the component of current perpendicular to the axis of the solenoid. Due to component of current density perpendicular to length of solenoid, we have

$B = \mu_0 n I_y$, where

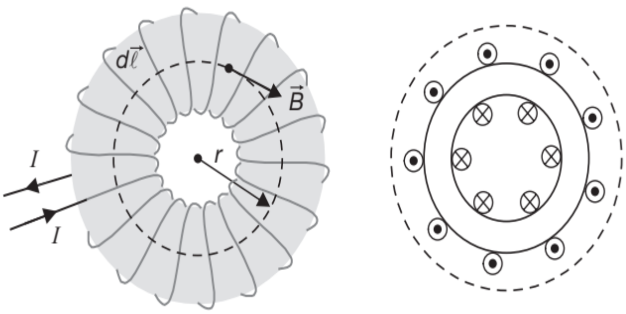
$$n = \frac{N}{Nh} = \frac{1}{h} \text{ and } I_y = (j\cos\theta)h = I\cos\theta$$

$$\Rightarrow \quad B_{\text{inside}} = \mu_0\left(\frac{1}{h}\right)(I\cos\theta)$$

$$\Rightarrow \quad B_{\text{inside}} = \frac{\mu_0 I}{h}\sqrt{1-\left(\frac{h}{2\pi R}\right)^2}$$

MAGNETIC FIELD DUE TO A TOROID

Consider a toroid which consists of N turns, as shown in Figure.



A toroid with N turns

Let us find the magnetic field due to the toroid. To calculate field, due to a toroid, we must evaluate $\oint \vec{B}\cdot d\vec{l}$ over the circle of radius r. Due to symmetry we see that the magnitude of the field is constant on this circle and tangent to it. So,

$$\oint \vec{B}\cdot d\vec{l} = Bl = B(2\pi r)$$

According to Ampere's Circuital Law, we have

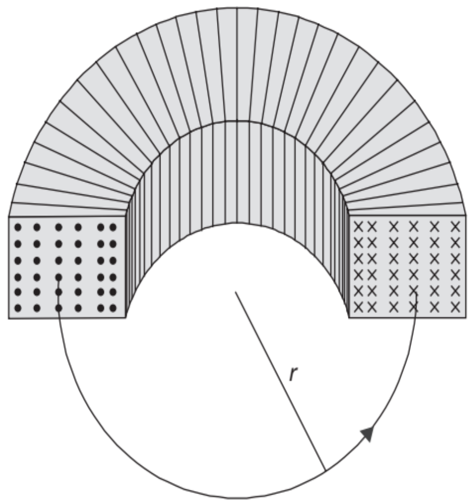
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{net}}$$

Since, the circular closed path surrounds N loops of the wire, each of which carries a current I, so we have $I_{\text{net}} = NI$

$$\Rightarrow \quad \oint \vec{B}\cdot d\vec{l} = \mu_0(I_{\text{net}}) = \mu_0 NI$$

$$\Rightarrow \quad B(2\pi r) = \mu_0 NI$$

$$\Rightarrow \quad B = \frac{\mu_0 NI}{2\pi r}$$



Within a toroid, the field is constant along a given circle of radius r

This result shows that $B \propto \frac{1}{r}$ and hence is non-uniform in the region occupied by toroid.

However, if r is very large compared with the cross-sectional radius of the toroid, then the field is approximately uniform inside the toroid and is given by

$$B = \mu_0 nI$$

In this case, we observe that the number of turns per unit length of toroid is

$$n = \frac{N}{2\pi r}$$

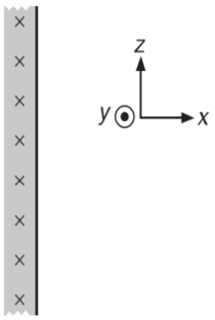
However, If the toroid has inner radius r_1 and an outer radius r_2, then number of turns per unit length of toroid is

$$n = \frac{N}{2\pi r_{av}} = \frac{N}{2\pi\left(\frac{r_1 + r_2}{2}\right)}$$

For an ideal toroid, in which turns are closely spaced, the external magnetic field is zero. This is because the net current passing through any circular path lying outside the toroid is zero. Therefore, from Ampere's Law we find that $B = 0$, in the regions exterior to the toroid.

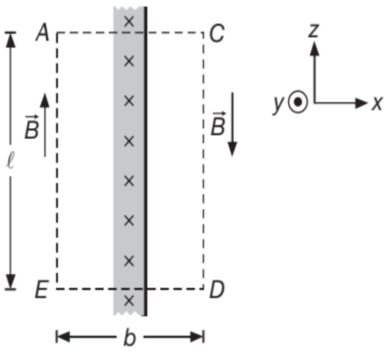
ILLUSTRATION 56

A thin infinitely large sheet lying in the yz plane carries a current of linear current density λ. The current is in the negative y-direction and λ represents the current per unit length measured along the z-axis. Find the magnetic field near the sheet.



SOLUTION

To evaluate the line integral in Ampere's Law, let us consider a rectangular path of length l, width b through the sheet as shown in Figure.



From symmetry we can argue that the magnetic field is constant over the sides of length l because sheet is infinitely long so it should not vary from point to point. The two sides of length b do not contribute to the line integral because the component of $\vec{B}$ along the direction of these paths is zero. The net current passing through the plane of the rectangle is λl. Applying Ampere's Law, we get

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (I_{\text{net}})$$

$$\Rightarrow \int_A^C \vec{B} \cdot d\vec{l} + \int_C^D \vec{B} \cdot d\vec{l} + \int_D^E \vec{B} \cdot d\vec{l} + \int_E^A \vec{B} \cdot d\vec{l} = \mu_0 (\lambda l)$$

$$\Rightarrow 0 + Bl + 0 + Bl = \mu_0 \lambda l$$

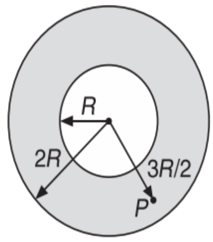
$$\Rightarrow B = \frac{\mu_0 \lambda}{2} = \text{constant}$$

The result shows that the magnetic field is independent of distance from the current sheet.

This is exactly similar to the case where the electric field due to an infinite sheet of charge does not depend on the distance from the sheet.

ILLUSTRATION 57

Figure shows the cross sectional view of a hollow cylindrical conductor with inner radius R and outer radius $2R$ carrying a uniformly distributed current I along it's axis.



Calculate the magnetic induction at point P at a distance $\frac{3R}{2}$ from the axis of the cylinder.

SOLUTION

Using Ampere's Circuital Law, the current inside the Amperian Loop passing through the point P is

$$I' = I\left(\frac{(3R/2)^2 - R^2}{(2R)^2 - R^2}\right) = \frac{5}{12} I$$

Since $\oint \vec{B} \cdot d\vec{l} = \mu_0 I'$

$$\Rightarrow B(2\pi r) = \mu_0 I', \text{ where } r = \frac{3R}{2}$$

$$\Rightarrow \quad 2\pi B\left(\frac{3R}{2}\right)=\mu_0\left(\frac{5}{12}I\right)$$

$$\Rightarrow \quad B=\frac{5\mu_0 I}{36\pi R}$$

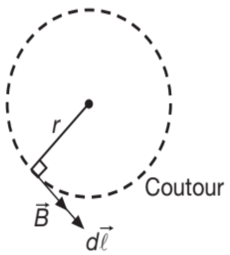
ILLUSTRATION 58

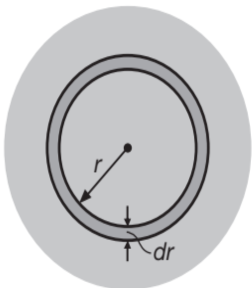
Find the current density as a function of distance r from the axis of a radially symmetrical parallel stream of electrons if the magnetic induction inside the stream varies as $B=br^{\alpha}$, where b and α are positive constants.

SOLUTION

According to Ampere's Circuital Law, we have

$$\int \vec{B}\cdot d\vec{l}=\mu_0\left(I_{\text{total}}\right)$$





Stream of Electrons

Since, $I_{\text{total}}=\int \vec{j}\cdot d\vec{A}=\int j dA$, where $dA=2\pi r dr$

$$\Rightarrow \quad B(2\pi r)=\mu_0\int j\,dA$$

$$\Rightarrow \quad B(2\pi r)=\mu_0\int j(2\pi r\,dr)$$

But $B=br^{\alpha}$

$$\Rightarrow \quad 2\pi b r^{\alpha+1}=\mu_0\int j(2\pi r dr)$$

Differentiating both sides, we get

$$\Rightarrow \quad 2\pi b(\alpha+1)r^{\alpha}dr=\mu_0\left(j2\pi r dr\right)$$

$$\Rightarrow \quad j=\frac{b(\alpha+1)}{\mu_0}r^{\alpha-1}$$

ILLUSTRATION 59

An infinitely long hollow conducting cylinder with inner radius $\frac{R}{2}$ and outer radius R carries a uniform current density along its length. Calculate the magnitude of the magnetic field, $|\vec{B}|$ as a function of the radial distance r from the axis of the cylinder. Also plot $|\vec{B}|$ vs r graph.

SOLUTION

Let r be the distance of a point from centre, then

For $r\le\frac{R}{2}$, using Ampere's circuital law,

$$\oint \vec{B}\cdot dl$$

$$\Rightarrow \quad Bl=\mu_0\left(I_{\text{in}}\right)$$

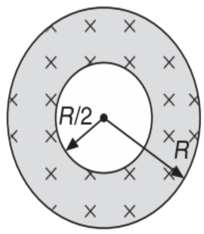
$$\Rightarrow \quad B(2\pi r)=\mu_0\left(I_{\text{in}}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0}{2\pi}\frac{I_{\text{in}}}{r} \qquad \ldots(1)$$

Since, $I_{\text{in}}=0$

$$\Rightarrow \quad B=0$$

If J be the current per unit area, then for $\frac{R}{2}\le r\le R$, we have



$$I_{\text{in}}=\left[\pi r^2-\pi\left(\frac{R}{2}\right)^2\right]J$$

Substituting in equation (1), we have

$$B=\frac{\mu_0}{2\pi}\frac{\left(\pi r^2-\pi\frac{R^2}{4}\right)J}{r}$$

$$\Rightarrow \quad B=\frac{\mu_0 J}{2r}\left(r^2-\frac{R^2}{4}\right)$$

At $r=\frac{R}{2}$, $B=0$

At $r=R$, $B=\frac{3\mu_0 JR}{8}$

For $r\ge R$, we have

$$I_{\text{in}}=I_{\text{Total}}=I(\text{say})$$

Therefore, substituting in equation (1), we have

$$B = \frac{\mu_0 I}{2\pi r}$$

$$\Rightarrow \quad B \propto \frac{1}{r}$$

The plot of $|\vec{B}|$ vs r graph is shown in Figure.

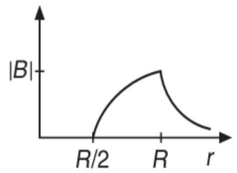


ILLUSTRATION 60

A long, straight, solid cylinder of radius a, oriented with its axis in the z-direction, carries a current whose current density is $\vec{J}$. The current density, although symmetrical about the cylinder axis, is not constant but varies with the radial distance r from the axis of the cylinder according to the relation given by

$$\vec{J} = \begin{cases} \dfrac{2I_0}{\pi a^2}\left(1-\left(\dfrac{r}{a}\right)^2\right)\hat{k} & \text{, for } r \le a \\ 0 & \text{, for } r \ge a \end{cases}$$

where I_0 is a constant having SI unit of current.

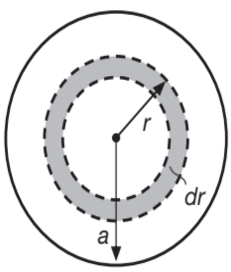
Show that the total current passing through the entire cross-section of the wire is I_0. Also, calculate the current I contained in a circular cross section of radius $r \le a$ and centered at the cylinder axis.

Using Ampere's Law, calculate the magnitude of the magnetic field $\vec{B}$ in the region $r \ge a$ and $r \le a$.

SOLUTION

Use the current density J to find dI through a concentric ring and integrate over the appropriate cross section to find the current through that cross-section. Then use Ampere's Law to find $\vec{B}$ at the specified distance from the centre of the wire.

Let us divide the cross section of the cylinder into thin concentric rings of radius r and width dr, as shown in Figure.



The current through each ring is $dI = JdA = J(2\pi r dr)$

$$dI = JdA = \frac{2I_0}{\pi a^2}\left[1-\left(\frac{r}{a}\right)^2\right]2\pi r dr$$

$$\Rightarrow \quad dI = \frac{4I_0}{a^2}\left[1-\left(\frac{r}{a}\right)^2\right]rdr$$

$$\Rightarrow \quad dI = \frac{4I_0}{a^2}\left[1-\left(\frac{r}{a}\right)^2\right]rdr \qquad \ldots(1)$$

The total current I is obtained by integrating dI over the entire cross-section of the conductor i.e. from $r = 0$ to $r = a$. So, we get

$$I = \int_0^a dI = \left(\frac{4I_0}{a^2}\right)\iint_0^a\left(1-\frac{r^2}{a^2}\right)rdr$$

$$\Rightarrow \quad I = \left(\frac{4I_0}{a^2}\right)\left(\frac{r^2}{2}-\frac{1}{4}\frac{r^4}{a^2}\right)\Bigg|_0^a = \left(\frac{4I_0}{a^2}\right)\left(\frac{a^2}{2}-\frac{1}{4}\frac{a^4}{a^2}\right)$$

$$\Rightarrow \quad I = \left(\frac{4I_0}{a^2}\right)\left(\frac{a^2}{4}\right) = I_0$$

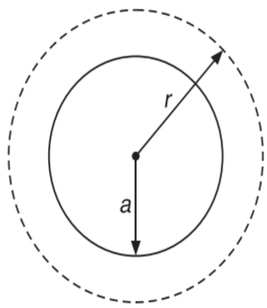
The current I contained in a circular cross section of radius $r \le a$ and centered at the cylinder axis can simply be calculated by integrating equation (1), from $r = 0$ to $r = r$.

$$\Rightarrow \quad I = \int dI = \frac{4I_0}{a^2}\int_0^r\left[1-\left(\frac{r}{a}\right)^2\right]rdr$$

$$\Rightarrow \quad I = \frac{4I_0}{a^2}\left(\frac{r^2}{2}-\frac{1}{4}\frac{r^4}{a^2}\right)\Bigg|_0^r = \frac{4I_0}{a^2}\left(\frac{r^2}{2}-\frac{r^4}{4a^2}\right)$$

$$\Rightarrow \quad I = \frac{I_0 r^2}{a^2}\left(2-\frac{r^2}{a^2}\right) \qquad \ldots(2)$$

To find the magnitude of the magnetic field outside the wire i.e. for $r > a$, let us consider a circular Amperian loop of radius $r > a$, as shown in Figure.



By Ampere's Law, we have

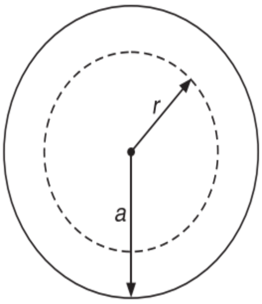
$$\oint \vec{B}\cdot d\vec{l} = B(2\pi r) = \mu_0 I_{enc}$$

Since, $I_{encl} = I_0$, because the path encloses the entire cylinder. So, we get

$$B(2\pi r) = \mu_0 I_0$$

$$\Rightarrow \quad B = \frac{\mu_0 I_0}{2\pi r}$$

To find the magnitude of the magnetic field outside the wire i.e. for $r < a$, let us consider a circular Amperian loop of radius $r < a$, as shown in Figure.



By Ampere's Law, we have

$$\oint \vec{B}\cdot d\vec{l} = B(2\pi r) = \mu_0 I_{enc}$$

From equation (2), for $r < a$, we get

$$I_{encl} = I = \frac{I_0 r^2}{a^2}\left(2 - \frac{r^2}{a^2}\right)$$

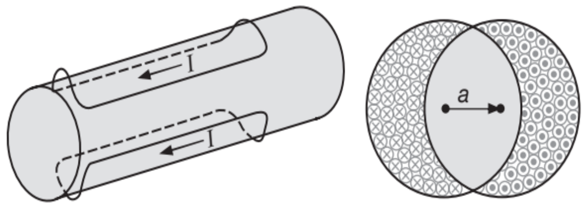
$$\Rightarrow \quad B(2\pi r) = \mu_0 I$$

$$\Rightarrow \quad B(2\pi r) = \mu_0 \frac{I_0 r^2}{a^2}\left(2 - \frac{r^2}{a^2}\right)$$

$$\Rightarrow \quad B = \frac{\mu_0 I_0 r}{2\pi a^2}\left(2 - \frac{r^2}{a^2}\right)$$

ILLUSTRATION 61

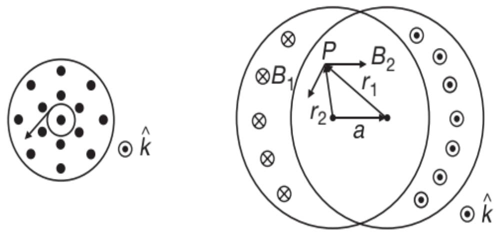
A long solenoid produces a uniform magnetic field directed along the axis of a cylindrical region. However, to produce a uniform magnetic field directed parallel to a diameter of a cylindrical region we can use the saddle coils illustrated in figure (a). The loops are wrapped over a somewhat flattened tube. Assume the straight sections of wire are very long. The end view of the tube, in figure (b) shows how the windings are applied. The overall current distribution is the superposition of two overlapping circular cylinders of uniformly distributed current, one towards the reader and one away from the reader. The current density J is the same for each cylinder. The position of the axis of one cylinder is described by a position vector a relative to the other cylinder. Prove that the magnetic field inside the hollow tube is $\frac{\mu_0 J a}{2}$ downward.



(a) (b)

SOLUTION

Here the use of vector methods will further simplify the calculations done. Let us consider first a solid cylindrical rod of radius R carrying current toward you, uniformly distributed over its cross-sectional area. To find the field at distance r from its centre we consider a circular loop of radius r.



$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{inside}$$

$$B(2\pi r) = \left(\mu_0 \pi r^2\right) J$$

$$\Rightarrow \quad B = \frac{\mu_0 J r}{2}$$

$$\Rightarrow \quad \vec{B} = \left(\frac{\mu_0 J}{2}\right)\left(\hat{k}\times\vec{r}\right)$$

Now the total field at P inside the saddle coils is the field due to a solid rod carrying current toward you, centred at the head of vector $\vec{a}$, plus the field of a solid rod centred at the tail of vector $\vec{a}$ carrying current away from you.

$$\vec{B}_1 + \vec{B}_2 = \frac{\mu_0 J}{2}\hat{k}\times\vec{r}_1 + \frac{\mu_0 J}{2}\left(-\hat{k}\right)\times\vec{r}_2$$

Now note $\vec{a}+\vec{r}_1=\vec{r}_2$

$$\Rightarrow \quad \vec{B}=\vec{B}_1+\vec{B}_2=\frac{\mu_0 J}{2}\hat{k}\times\vec{r}_1-\frac{\mu_0 J}{2}\hat{k}\times(\vec{a}+\vec{r}_1)$$

$$\Rightarrow \quad \vec{B}=\frac{\mu_0 J}{2}(\vec{a}\times\hat{k})=\frac{\mu_0 Ja}{2} \text{ down in the diagram.}$$

ILLUSTRATION 62

A current I flows around a closed loop. It produces a field B_x directed along the axis of the loop. Integrate B_x from $-\infty$ to $+\infty$, i.e., calculate $\int_{-\infty}^{+\infty} B_x dx$. Explain the significance of your result.

SOLUTION

Since, $B_x=\dfrac{\mu_0 Ia^2}{2(x^2+a^2)^{3/2}}$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} B_x dx=\int_{-\infty}^{\infty}\frac{\mu_0 Ia^2}{2(x^2+a^2)^{3/2}}dx=\frac{\mu_0 Ia^2}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^{3/2}}$$

Substitute $x=a\tan\theta$, we get

$$\int_{-\infty}^{\infty} B_x dx=\frac{\mu_0 Ia^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\sec^2\theta d\theta}{a^3\sec^3\theta}=\frac{\mu_0 Ia^2}{2a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} B_x dx=\frac{\mu_0 I}{2}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} B_x dx=\frac{\mu_0 I}{2}\left[\sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} B_x dx=\frac{\mu_0 I}{2}(2)$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} B_x dx=\mu_0 I$$

This is just what Ampere's Law tells us to expect if we imagine the loop runs along the x-axis closing itself at infinity. So, we get

$$\oint \vec{B}\cdot d\vec{\ell}=\mu_0 I$$

Test Your Concepts-V

Based on Ampere's Circuital Law and Applications

(Solutions on page H.18)

1. A long, straight, solid cylinder, oriented with its axis in the z-direction, carries a current whose current density is $\vec{J}$. The current density, although symmetrical about the cylinder axis, is not constant and varies according to the relationship

$$\vec{J}=\begin{cases}\left(\dfrac{b}{r}\right)e^{\frac{(r-a)}{\delta}}\hat{k} & \text{for } r\le a\\ 0 & \text{for } r\ge a\end{cases}$$

where the radius of the cylinder is a, r is the radial distance from the cylinder axis, b is a constant and δ is a constant.

(a) Find the total current I_0, passing through the entire cross section of the wire in terms of b, δ and a.

(b) Use Ampere's Law to derive an expression for the magnetic field $\vec{B}$, in terms of I_0, in the region $r\ge a$.

(c) Obtain an expression for the current I, in terms of I_0, contained in a circular cross section of radius $r\le a$ and centred at the cylinder axis.

(d) Use Ampere's Law to derive an expression for the magnetic field $\vec{B}$ in the region $r\le a$.

2. A conductor is made in the form of a hollow cylinder with inner and outer radii a and b, respectively. It carries a current I uniformly distributed over its cross section. Find the magnitude of the magnetic field in the regions

(a) $r<a$

(b) $a<r<b$

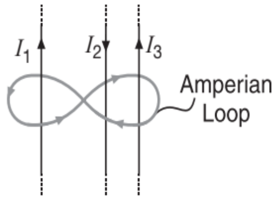
(c) $r>b$

3. A cylinder of radius $R = 0.5$ cm is actually made from a packed bundle of 100 long, straight, insulated wires.

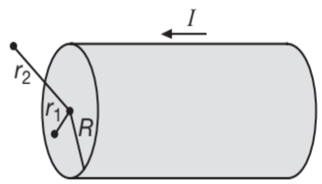
(a) If each wire carries 2 A, calculate the magnitude and direction of the magnetic force per unit length acting on a wire located 0.2 cm from the centre of the bundle.

(b) Consider a wire on the outer edge of the bundle. Will it experience a force greater or smaller than the value calculated in part (a)?

4. Write the equation for Ampere's circuital law for the Amperian loop shown in figure. The arrow marks show the sense in which the Amperian loop is taken.



5. A long cylindrical conductor of radius R carries a current I as shown in figure. The current density $\vec{J}$, however, is not uniform over the cross section of the conductor but is a function of the radius according to $J = br$, where b is a constant. Find an expression for the magnetic field B



(a) at a distance $r_1 < R$ and

(b) at a distance $r_2 > R$, measured from the axis.

6. A very large parallel-plate capacitor caries charge with uniform charge per unit area $+\sigma$ on the upper plate and $-\sigma$ on the lower plate. The plates are horizontal and both move horizontally with speed v to the right.

(a) Calculate the magnetic field between the plates.

(b) Calculate the magnetic field close to the plates but outside of the capacitor.

(c) Calculate the magnitude and direction of the magnetic force per unit area on the upper plate.

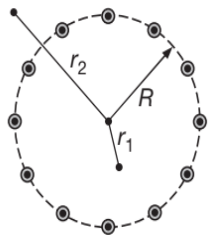
(d) At some extrapolated speed v, the magnetic force on a plate balances the electric force on the plate. Calculate v.

7. Consider the cross section of a non-conducting cylinder that has N wires parallel to the axis of the cylinder and uniformly spaced around the curved surface as shown in figure. The cylinder has a radius R and the current in each conductor is I and directed out of the plane of the figure. Assuming N to be a very large number and the radius of each wire to be small compared with the radius of the cylinder, find an expression for the magnetic field $\vec{B}$.

(a) at $r_1 < R$ and

(b) at $r_2 > R$

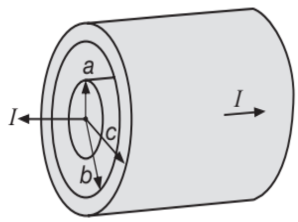
(c) Calculate B when $N = 100$, $R = 5$ cm, $I = 10$ A and $r_2 = 15$ cm.



8. (a) For the coaxial cable shown, derive an expression for the magnitude of the magnetic field at points inside the central solid conductor $(r < a)$.

(b) For this coaxial cable derive an expression for the field within the tube $(b < r < c)$.

Interpret the results for $r = a$, $r = b$ and $r = c$.



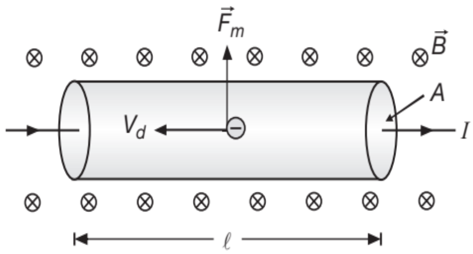
9. A column of electric current when passing through a plasma (ionized gas) makes filaments of current within the column which are magnetically attracted to one another. They can crowd together to yield a very great current density and a very strong magnetic field in a small region. Sometimes the current can be cut off momentarily by this pinch effect. The pinch effect can be demonstrated by making an empty aluminium can carry a large current parallel to its axis. Let R represent the radius of the can and I the upward current, uniformly distributed over its curved wall. Determine the magnetic field just inside the wall and just outside the wall. Also calculate the pressure on the wall.

10. Find the magnetic pressure which the lateral surface of a long straight solenoid with n turns per unit length experiences when a current I flows through it.

MAGNETIC FORCE ON CURRENT CARRYING CONDUCTORS, MAGNETIC MOMENT AND TORQUE

MAGNETIC FORCE ON A CURRENT CARRYING CONDUCTOR

Since a charged particle in motion experiences a magnetic force in a magnetic field. Hence a current carrying wire also experiences a force when placed in a magnetic field because of the fact that the current is due to the motion of collection of many charged particles. Hence, the resultant force exerted by the field on a current carrying wire is the vector sum of the individual forces exerted on all the charged particles constituting the current. The force exerted on the particles is transmitted to the wire when the particles collide with the atoms making up the wire. Consider a conducting wire of length l, area of cross-section A carrying a current I to be placed in a magnetic field $\vec{B}$ as shown in Figure.



Since the free electrons in the wire drift with a speed v_d opposite to the direction of current, so the magnetic force exerted on an electron is given by $\vec{f}$, where

$$\vec{f} = -e\left(\vec{v}_d \times \vec{B}\right)$$

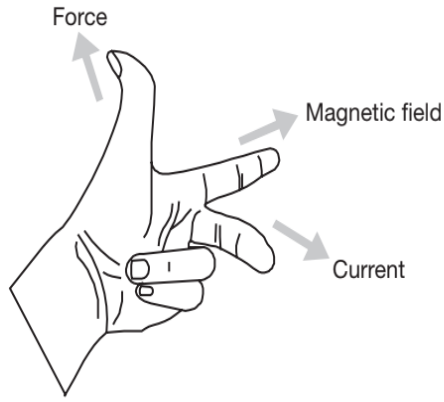
If n be the number of free electrons per unit volume of the wire, then total number of electrons in the wire having volume Al are, $N = n(Al)$. If $\vec{F}$ is the total force on the wire, then

$$\vec{F} = N\vec{f} = -e(nAl)\left(\vec{v}_d \times \vec{B}\right)$$

Since we know that $neAv_d = I$ and if we denote the length l along the direction of the current by $\vec{l}$, then the above equation becomes

$$\vec{F} = I\left(\vec{l} \times \vec{B}\right) \quad \ldots(1)$$

The direction of $\vec{F}$ is given by Fleming's Left Hand Rule as discussed already. According to this rule, stretch the first finger, the middle finger and the thumb of the left hand in such a way that they are mutually perpendicular to each other such that the first finger points in the direction of field $(\vec{B})$, the middle finger points in the direction of conventional current (I), then the thumb gives the direction of magnetic force $(\vec{F})$.



For the above expression, following points are worth noting

(a) Here $\vec{l}$ is a vector that points in the direction of the current I and has a magnitude equal to the length.

(b) Magnitude of $\vec{F}$ is $F = BIl\sin\theta$, where θ is the angle between $\vec{l}$ and $\vec{B}$. F_m is zero for $\theta = 0°$ or $180°$ and maximum for $\theta = 90°$.

(c) The expression $\vec{F} = I\left(\vec{l} \times \vec{B}\right)$ applies only to a straight segment of wire in a uniform magnetic field.

(d) For the magnetic force on an arbitrarily shaped wire segment, we consider the magnetic force exerted on a small segment of vector length $d\vec{l}$, then

$$d\vec{F} = I\left(d\vec{l} \times \vec{B}\right) \quad \ldots(2)$$

The total force $\vec{F}_m$ acting on the wire, as shown earlier is calculated by integrating equation (2) over the length of the wire. So, we get

$$\vec{F} = I\int_P^R \left(d\vec{l} \times \vec{B}\right)$$

Now let us consider two special cases involving $\vec{F} = I\int d\vec{l} \times \vec{B}$. *In both the cases we have taken the magnetic field to be constant in magnitude and direction.*

CASE-1:

A curved wire PQR (as shown in FIGURE 1) carries a current I and is located in a uniform magnetic field $\vec{B}$.

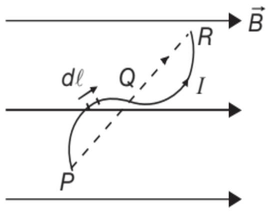


Figure 1

Since the field is uniform, so we can take $\vec{B}$ outside the integral to get

$$\vec{F} = I\left(\int_P^R d\vec{l}\right) \times \vec{B}$$

But the quantity $\int_P^R d\vec{l}$ represents the vector sum of all length elements from P to R. From the polygon law of vector addition this sum equals the vector $\vec{l}_{\text{effective}}$ directed from P to R. Hence, we can rewrite the expression as

$$\vec{F} = I\left(\vec{l}_{\text{effective}} \times \vec{B}\right)$$

$$\Rightarrow \quad \vec{F}_{PQR} = \vec{F}_{PR} = I\left(P\vec{R} \times \vec{B}\right), \text{ in a uniform field.}$$

CASE-2:

For an arbitrarily shaped closed loop carrying a current I placed in a uniform magnetic field (as shown in FIGURE 2).

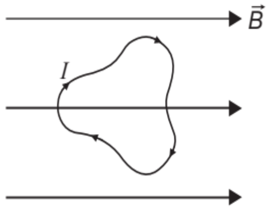


Figure 2

Here too, we again get the force acting on the loop as

$$\vec{F} = I\left(\int d\vec{l} \times \vec{B}\right)$$

However, this time we must take the vector sum of the length elements $d\vec{l}$ over the entire loop, so

$$\vec{F} = I\left(\oint d\vec{l}\right) \times \vec{B}$$

Now since, the set of length elements forms a closed polygon, hence the vector sum $\oint d\vec{l}$ must be zero.

Thus, we get

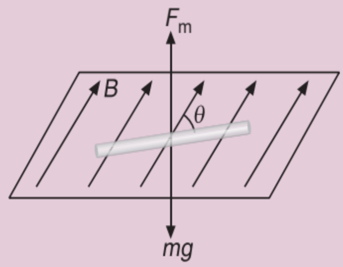
$$\vec{F} = 0$$

Hence the net magnetic force acting on any closed current loop in a uniform magnetic field is always zero.

Conceptual Note(s)

(a) Conductor to Appear Weightless (or Floating In Air)

A current carrying conductor is lying in a horizontal plane such that it is making an angle θ with the direction of magnetic field B, which also lies in the horizontal plane as shown in Figure.



Magnetic field B is inwards making an angle θ with the length of rod

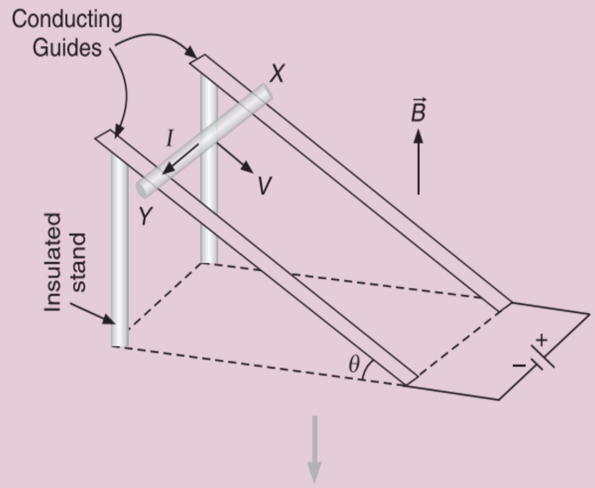
For the conductor to appear weightless, we have

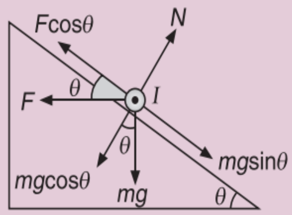
$$mg = BIL\sin\theta$$

$$\Rightarrow \quad I = \frac{mg}{BL\sin\theta}$$

(b) Sliding of conducting rod with constant velocity on inclined rails

When a conducting rod slides on conducting rails. In the following situation conducting rod XY slides with constant velocity (or is in equilibrium), when





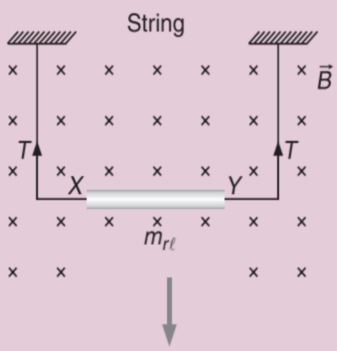
$$F\cos\theta = mg\sin\theta$$

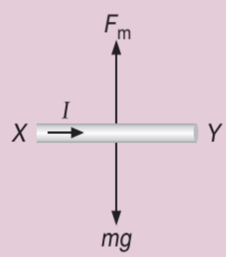
$$\Rightarrow \quad BIL\cos\theta = mg\sin\theta$$

$$\Rightarrow \quad B = \frac{mg}{IL}\tan\theta$$

(c) Tension less strings

In the following figure the value and direction of current through the conductor XY so that strings become tensionless i.e. when weight of the conductor XY is balanced by the magnetic force (F_m).





Hence direction of current is from $X \to Y$ and in balanced condition

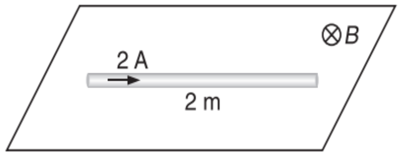
$$F_m = mg$$

$$\Rightarrow \quad BIL = mg$$

$$\Rightarrow \quad I = \frac{mg}{BL}$$

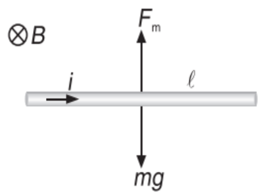
ILLUSTRATION 63

A straight wire having mass 400 g, length 2 m carrying a current of 2 A is suspended in mid air with the help of a uniform horizontal field B as shown in Figure. Calculate the value of the field for this to happen.



SOLUTION

Since the weight mg of the wire always acts vertically downwards, so for the weight to be balanced the magnetic force F_m on the wire must act upwards as shown in Figure.



So, we have

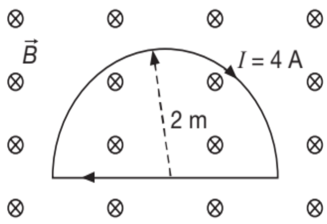
$$Bil\sin\theta = mg, \text{ where } \theta = 90°$$

$$\Rightarrow \quad B = \frac{0.4 \times 9.8}{2 \times 2 \times \sin 90°}$$

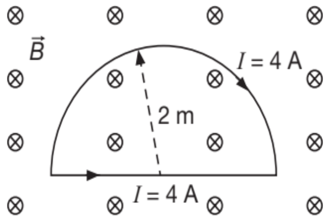
$$\Rightarrow \quad B = 0.98\ \text{T}$$

ILLUSTRATION 64

In the arrangement shown, a semi-circular wire loop is placed in a uniform magnetic field $B = 2$ T. The plane of the loop is perpendicular to the magnetic field. A current $I = 4$ A flows in the loop in the directions shown.



(a)



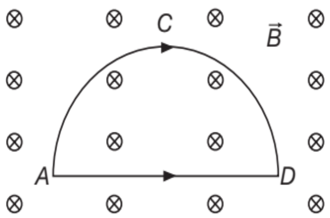
(b)

Find the magnitude of the magnetic force in both the cases (a) and (b). The radius of the loop is 2 m.

SOLUTION

Since in FIGURE (a), the arrangement forms a closed loop and the current completes the loop. Hence the, net force on the loop in uniform field is zero.

In FIGURE (b) although the arrangement forms a closed loop, but the current does not complete the loop. Hence, net force is not zero.



Since $\vec{F}_{ACD} = \vec{F}_{AD}$ and

$$\vec{F}_{\text{loop}} = \vec{F}_{ACD} + \vec{F}_{AD} = 2\vec{F}_{AD} \qquad \left\{\because\ \vec{F}_{ACD} = \vec{F}_{AD}\right\}$$

$$\Rightarrow \quad \left|\vec{F}_{\text{loop}}\right| = F = 2\left|\vec{F}_{AD}\right|$$

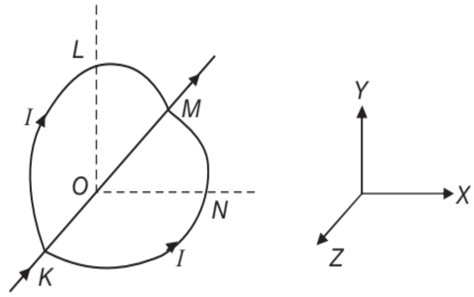
$$\Rightarrow \quad F = 2BIl_{\text{eff}}\sin\theta \qquad \left\{\text{where } l_{\text{eff}} = 2r = 4\text{ m}\right\}$$

$$\Rightarrow \quad F = 4BIr\sin(90^\circ)$$

$$\Rightarrow \quad F = (4)(2)(4)(2) = 64\text{ N}$$

ILLUSTRATION 65

A circular loop of radius R is bent along a diameter and given a shape as shown in figure. One of the semi-circles (KNM) lies in the X-Z plane and the other one (KLM) in the Y-Z plane with their centres at origin. Current I is flowing through each of the semicircles as shown in figure.



(a) A particle of charge q is released at the origin with a velocity $\vec{v} = -v_0\hat{i}$. Find the instantaneous force $\vec{F}$ on the particle. Assume that space is gravity free.

(b) If an external uniform magnetic field $B_0\hat{j}$ is applied, determine the force $\vec{F}_1$ and $\vec{F}_2$ on the semicircles KLM and KNM due to the field and the net force $\vec{F}$ on the loop.

SOLUTION

(a) Magnetic field $(\vec{B})$ at the origin is

$$\vec{B} = \begin{pmatrix}\text{Magnetic} \\ \text{field due to} \\ \text{semicircle } KLM\end{pmatrix} + \begin{pmatrix}\text{Magnetic field} \\ \text{due to other} \\ \text{semicircle } KNM\end{pmatrix}$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0 I}{4R}(-\hat{i}) + \frac{\mu_0 I}{4R}(\hat{j})$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0 I}{4R}(-\hat{i} + \hat{j})$$

Since, magnetic force acting on the particle is given by $\vec{F} = q(\vec{v} \times \vec{B})$

$$\Rightarrow \quad \vec{F} = q\left[(-v_0\hat{i}) \times (-\hat{i} + \hat{j})\right]\frac{\mu_0 I}{4R}$$

$$\Rightarrow \quad \vec{F} = -\left(\frac{\mu_0 q v_0 I}{4R}\right)\hat{k}$$

(b) $\vec{F}_{KLM} = \vec{F}_{KNM} = \vec{F}_{KM}$ and $\vec{F}_{KM} = BI(2R)\hat{i} = 2BIR\hat{i}$

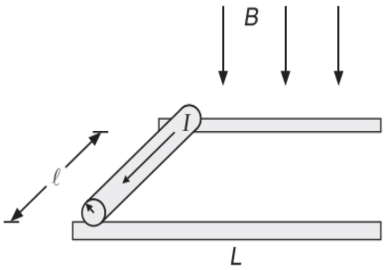
$$\Rightarrow \quad \vec{F}_1 = \vec{F}_2 = (2BIR)\hat{i}$$

Hence total force on the loop is $\vec{F} = \vec{F}_1 + \vec{F}_2$

$$\Rightarrow \quad \vec{F} = (4BIR)\hat{i}$$

ILLUSTRATION 66

A rod of mass m and radius R rests on two parallel rails that are a distance l apart and have a length L. The rod carries a current I in the direction shown in Figure.

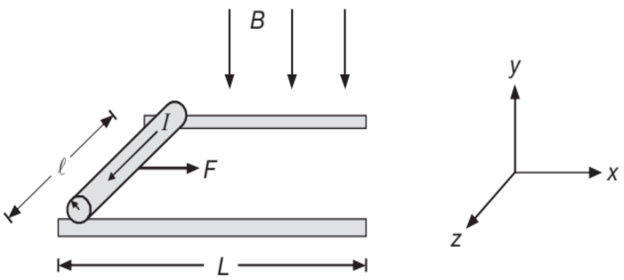


The rod rolls along the rails without slipping. A uniform magnetic field B is directed perpendicular to the rod and the rails. If it starts from rest, what is the speed of the rod as it leaves the rails?

SOLUTION

The magnetic force acting on the rod is given by

$$\vec{F} = I(\vec{l} \times \vec{B}) = Il(\hat{k}) \times B(-\hat{j}) = (BIl)\hat{i}$$



From Work Energy Theorem we have

$$(K_{\text{trans}} + K_{\text{rot}})_{\text{initial}} + \Delta E = (K_{\text{trans}} + K_{\text{rot}})_{\text{final}}$$

$$\Rightarrow \quad 0 + 0 + Fs\cos\theta = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2, \text{ where } I = \frac{1}{2}mR^2$$

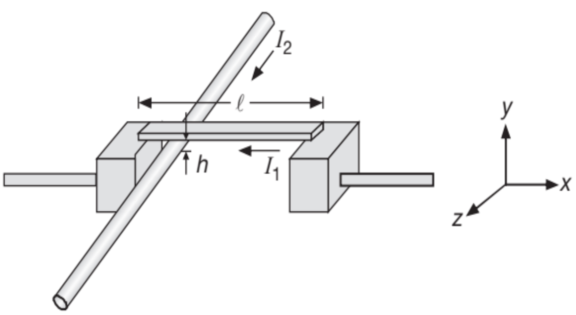
$$\Rightarrow \quad (BIl)L\cos 0° = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2$$

$$\Rightarrow \quad BIlL = \frac{3}{4}mv^2$$

$$\Rightarrow \quad v = \sqrt{\frac{4BIlL}{3m}}$$

ILLUSTRATION 67

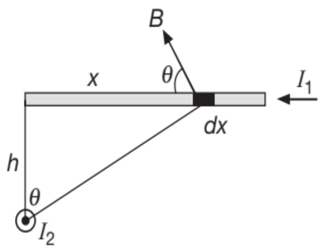
A thin copper bar of length l is supported horizontally by two (non-magnetic) contacts. The bar carries current I_1 in the $-x$ direction, as shown in Figure.



At a distance h below one end of the bar, a long straight wire carries a current I_2 in the z-direction. Determine the magnetic force exerted on the bar.

SOLUTION

At a point at distance x from the left end of the bar, current I_2 creates magnetic field $\vec{B} = \dfrac{\mu_0 I_2}{2\pi\sqrt{h^2 + x^2}}$ to the left and above the horizontal at angle θ where $\tan\theta = \dfrac{x}{h}$ as shown in Figure.



This field exerts force on an element of the rod of length dx

$$d\vec{F} = I_1(d\vec{l} \times \vec{B}) = I_1 \frac{\mu_0 I_2 dx}{2\pi\sqrt{h^2 + x^2}} \sin\theta$$

$$\Rightarrow \quad d\vec{F} = \frac{\mu_0 I_1 I_2 dx}{2\pi\sqrt{h^2 + x^2}} \frac{x}{\sqrt{h^2 + x^2}}, \text{ into the page}$$

$$\Rightarrow \quad d\vec{F} = \frac{\mu_0 I_1 I_2 x dx}{2\pi(h^2 + x^2)}(-\hat{k})$$

The net force is the sum of the forces on all of the elements of the bar. So

$$\vec{F} = \int_{x=0}^{l} \frac{\mu_0 I_1 I_2 x dx}{2\pi(h^2 + x^2)}(-\hat{k})$$

$$\Rightarrow \quad \vec{F} = \frac{\mu_0 I_1 I_2(-\hat{k})}{4\pi} \int_0^l \frac{2x\,dx}{h^2 + x^2}$$

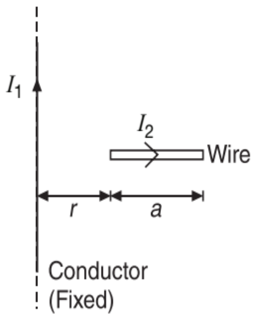
$$\Rightarrow \quad \vec{F} = \frac{\mu_0 I_1 I_2(-\hat{k})}{4\pi} \log_e(h^2 + x^2)\Big|_0^l$$

$$\Rightarrow \quad \vec{F} = \frac{\mu_0 I_1 I_2(-\hat{k})}{4\pi}\left[\log_e(h^2 + l^2) - \log_e h^2\right]$$

$$\Rightarrow \quad \vec{F} = \left(\frac{\mu_0 I_1 I_2}{4\pi}\right)\log_e\left(\frac{h^2 + l^2}{h^2}\right)(-\hat{k})$$

ILLUSTRATION 68

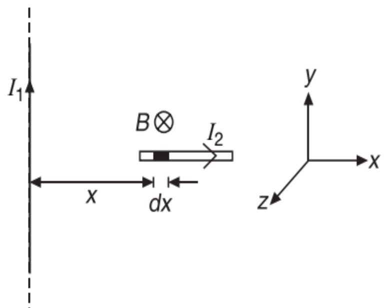
A straight current carrying wire of length a carrying a current I_2 is placed near a very long straight fixed conductor carrying a current I_1 as shown in Figure. Calculate the magnetic force acting on the wire due to the conductor.



SOLUTION

Dear Student, here we cannot directly use the formula $\vec{F}=I(\vec{l}\times\vec{B})$ to calculate force acting on the wire due to the conductor, because the magnetic field generated by the very long conductor is non-uniform and varies from one end of the wire to the other end.

So, to calculate the force, we have to consider an infinitesimal element of length dx at a distance x from the conductor as shown in Figure.



If $d\vec{F}$ be the force on this element due to the field of the conductor, then we have

$$d\vec{F}=I(d\vec{l}\times\vec{B})$$

where, $d\vec{l}=(dx)\hat{i}$ and $\vec{B}=\dfrac{\mu_0 I_1}{2\pi x}(-\hat{k})$

$$\Rightarrow \quad d\vec{F}=I(d\vec{l}\times\vec{B})=\left(\frac{\mu_0 I_1 I_2 dx}{2\pi x}\right)(\hat{i}\times(-\hat{k}))$$

$$\Rightarrow \quad d\vec{F}=\left(\frac{\mu_0 I_1 I_2 dx}{2\pi x}\right)\hat{j}$$

Integrating, we get the net force on the wire to be

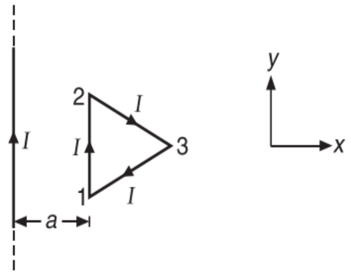
$$\vec{F}=\int d\vec{F}=\left(\frac{\mu_0 I_1 I_2}{2\pi}\right)\hat{j}\int_r^{r+a}\frac{dx}{x}$$

$$\vec{F}=\left(\frac{\mu_0 I_1 I_2}{2\pi}\right)\hat{j}\int_r^{r+a}\frac{dx}{x}$$

$$\Rightarrow \quad F=\frac{\mu_0 I_1 I_2}{2\pi}(\ln x)\Big|_r^{r+a}=\frac{\mu_0 I_1 I_2}{2\pi}\ln\left(\frac{r+a}{r}\right)$$

ILLUSTRATION 69

An equilateral triangular frame with side a carrying a current I is placed at a distance a from an infinitely long straight wire carrying a current I as shown in the figure. One side of the frame is parallel to the wire. The whole system lies in the x-y plane. Find the magnetic force $\vec{F}$ acting on the frame.



SOLUTION

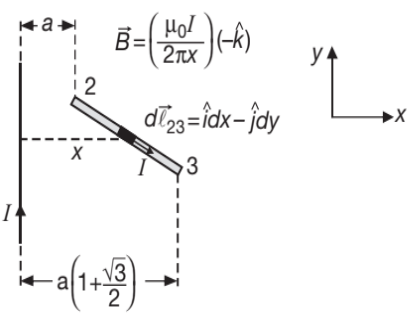
The net force on the frame is

$$\vec{F}=\vec{F}_{12}+\vec{F}_{23}+\vec{F}_{31}$$

Since, we known that $\dfrac{F}{l}=\dfrac{\mu_0 I I_2}{2\pi r}$

$$\Rightarrow \quad \vec{F}_{12}=\left(\frac{\mu_0 I^2 a}{2\pi a}\right)(-\hat{i})=-\left(\frac{\mu_0 I^2}{2\pi}\right)\hat{i} \qquad \dots(1)$$

To calculate $\vec{F}_{23}$ and $\vec{F}_{31}$, let us draw a figure.



Consider an infinitesimal length element $d\vec{l}_{23}$ on 23, then $d\vec{l}_{23}=\hat{i}dx-\hat{j}dy$. If $d\vec{F}_{23}$ is the force due to the field on this wire, then we have

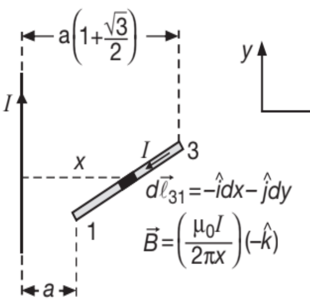
$$d\vec{F}_{23}=I(d\vec{l}_{23}\times\vec{B}), \text{ where } \vec{B}=\left(\frac{\mu_0 I}{2\pi x}\right)(-\hat{k})$$

$$\Rightarrow \quad d\vec{F}_{23} = I\left(\hat{i}dx - \hat{j}dy\right)\times\left(\frac{\mu_0 I}{2\pi x}\right)\left(-\hat{k}\right)$$

$$\Rightarrow \quad d\vec{F}_{23} = \frac{\mu_0 I^2}{2\pi}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & -dy & 0 \\ 0 & 0 & -\frac{1}{x} \end{vmatrix}$$

$$\Rightarrow \quad d\vec{F}_{23} = \frac{\mu_0 I^2}{2\pi}\left[\hat{i}\left(\frac{dy}{x}\right)+\hat{j}\left(\frac{dx}{x}\right)\right] \qquad \ldots(2)$$

Similarly, we shall now find $\vec{F}_{31}$. So, again consider an element $d\vec{l}_{31}$ on 31, then $d\vec{l}_{31} = -\left(\hat{i}dx + \hat{j}dy\right)$ as shown in figure.



If $d\vec{F}_{31}$ is the force due to the field on this wire, then we have

$$d\vec{F}_{31} = I\left(d\vec{l}_{31}\times\vec{B}\right), \text{ where } \vec{B} = \left(\frac{\mu_0 I}{2\pi x}\right)\left(-\hat{k}\right)$$

$$\Rightarrow \quad d\vec{F}_{31} = I\left(-\hat{i}dx - \hat{j}dy\right)\times\left(\frac{\mu_0 I}{2\pi x}\right)\left(-\hat{k}\right)$$

$$\Rightarrow \quad d\vec{F}_{31} = \frac{\mu_0 I^2}{2\pi}\left(\hat{i}dx + \hat{j}dy\right)\times\left(\frac{1}{x}\hat{k}\right)$$

$$\Rightarrow \quad d\vec{F}_{31} = \frac{\mu_0 I^2}{2\pi}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & 0 \\ 0 & 0 & \frac{1}{x} \end{vmatrix}$$

$$\Rightarrow \quad d\vec{F}_{31} = \frac{\mu_0 I^2}{2\pi}\left[\hat{i}\left(\frac{dy}{x}\right)-\hat{j}\left(\frac{dx}{x}\right)\right] \qquad \ldots(3)$$

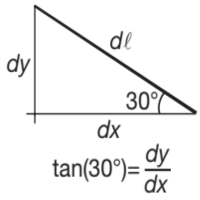
The resultant infinitesimal force on conductor 23 and 31 is given by

$$d\vec{F}_{321} = d\vec{F}_{23} + d\vec{F}_{31}$$

From (2) and (3), we get

$$d\vec{F}_{321} = \frac{\mu_0 I^2}{2\pi}\left[\hat{i}\left(\frac{2dy}{x}\right)\right]$$

To find the relation between dx and dy, we make use of the following figure.



Since, $dy = \frac{dx}{\sqrt{3}}$

$$\Rightarrow \quad d\vec{F}_{321} = \frac{\mu_0 I^2}{2\pi}\left[\hat{i}\left(\frac{2dx}{\sqrt{3}x}\right)\right]$$

Integrating, we get

$$\vec{F}_{321} = \frac{\mu_0 I^2}{\sqrt{3}\pi}\left[\int_a^{a+\frac{\sqrt{3}}{2}a}\frac{dx}{x}\right]\hat{i}$$

$$\Rightarrow \quad \vec{F}_{321} = \frac{\mu_0 I^2}{\sqrt{3}\pi}\left(\log_e x\Big|_a^{a+\frac{\sqrt{3}}{2}a}\right)\hat{i}$$

$$\Rightarrow \quad \vec{F}_{321} = \frac{\mu_0 I^2}{\sqrt{3}\pi}\left[\log_e\left(a+\frac{\sqrt{3}}{2}a\right)-\log_e a\right]\hat{i}$$

$$\Rightarrow \quad \vec{F}_{321} = \frac{\mu_0 I^2}{\sqrt{3}\pi}\left[\log_e\left(1+\frac{\sqrt{3}}{2}\right)\right]\hat{i}$$

So, from equations (1), (2) and (3), we get

$$\vec{F} = \vec{F}_{12} + \vec{F}_{23} + \vec{F}_{31} = \vec{F}_{12} + \vec{F}_{321}$$

$$\Rightarrow \quad \vec{F} = \left(\frac{\mu_0 I^2}{2\pi}\right)\hat{i} + \frac{\mu_0 I^2}{\sqrt{3}\pi}\left[\log_e\left(1+\frac{\sqrt{3}}{2}\right)\right]\hat{i}$$

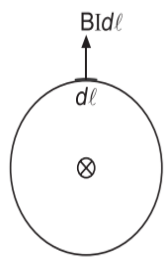
$$\Rightarrow \quad \vec{F} = \frac{\mu_0 I^2}{\pi}\left[-\frac{1}{2}+\frac{1}{\sqrt{3}}\log_e\left(1+\frac{\sqrt{3}}{2}\right)\right]\hat{i}$$

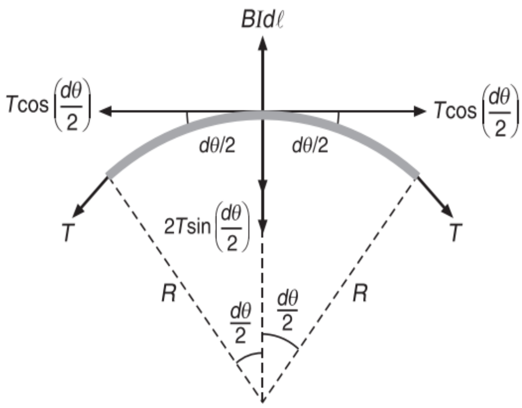
ILLUSTRATION 70

A coil carrying a current I is placed in a uniform magnetic field so that its axis coincides with the field direction. The single-layer winding of the coil is made of wire with diameter d, radius of turns is equal to R. At what value of the induction of the external magnetic field can the coil winding be ruptured? Assume the breaking stress of the wire to be σ_0.

SOLUTION

Each element of length dl experiences a force $BIdl$. This causes a tension T in the wire. Consider an infinitesimal arc element of length dl, subtending an angle $d\theta$ at the centre as shown in Figure.





For equilibrium, we have

$$2T\sin\left(\frac{d\theta}{2}\right)=BIdl$$

Since $d\theta$ is very small, so we get

$$\sin\left(\frac{d\theta}{2}\right)\simeq\frac{d\theta}{2}$$

$\Rightarrow \quad Td\theta = BIdl$

$\Rightarrow \quad Td\theta = BI(Rd\theta)$

$\Rightarrow \quad T = BIR$

The wire experiences a stress σ, which is

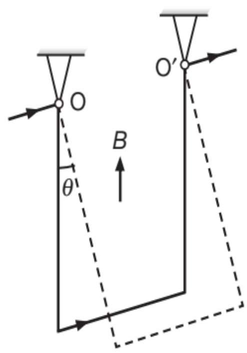
$$\sigma=\frac{\text{Force}}{\text{Area}}=\frac{BIR}{\frac{\pi d^2}{4}}$$

This must equal the breaking stress σ_0 for the winding to rupture. Thus,

$$B_{max}=\frac{\pi d^2\sigma_0}{4IR}$$

ILLUSTRATION 71

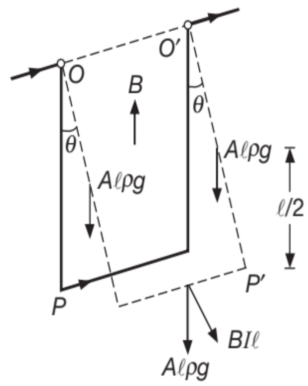
A wire with cross-sectional area A, density ρ bent to make three sides of a square can turn about a horizontal axis OO' as shown in Figure.



The wire is located in uniform vertical magnetic field. Find the magnetic induction if on passing a current I through the wire the latter deflects by an angle θ.

SOLUTION

The magnetic forces on the sides OP and $O'P'$ are directed along the same line, in opposite directions and have equal values, hence the net force as well as the net torque of these forces about the axis OO' is zero. The magnetic force on the segment PP' and the corresponding moment of this force about the axis OO' is effective and is deflecting in nature.



In equilibrium, shown by the dotted position, the deflecting torque must be equal to the restoring torque, developed due to the weight of the shape.

Let, the length of each side be l and ρ be the density of the material then,

$$BIl(l\cos\theta)=(Al\rho)g\left(\frac{l}{2}\sin\theta\right)+$$
$$(Al\rho)g\left(\frac{l}{2}\sin\theta\right)+(Al\rho)g(\sin\theta)$$

$\Rightarrow \quad Il^2B\cos\theta = 2A\rho gl^2\sin\theta$

$\Rightarrow \quad B=\dfrac{2A\rho g}{I}\tan\theta$

ILLUSTRATION 72

A conducting wire of length l is placed on a rough horizontal surface, where a uniform horizontal magnetic field B perpendicular to the length of the wire exists. Least values of the forces required to move the rod when a current I is established in the rod are observed to be F_1 and $F_2(<F_1)$ for the two possible directions of the current through the rod respectively. Find the weight of the rod and the coefficient of friction between the rod and the surface.

SOLUTION

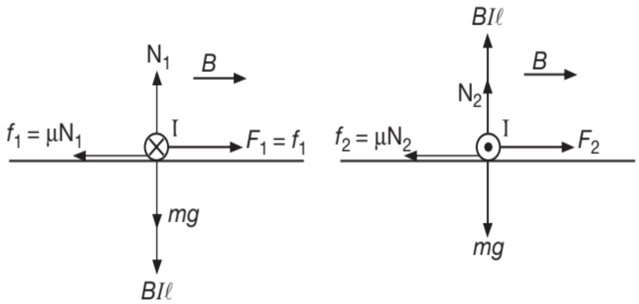
Changing the direction of current in the wire, we can change the normal reaction on the wire by the surface. In one case magnetic force on the wire will be in upward direction while in the other case it will be in the downward direction. Hence normal reaction is given by

$$N = mg \pm BIl$$

$$\Rightarrow \quad f_{(\text{friction limiting})} = \mu(mg \pm BIl) \text{ as } F_1 > F_2$$

$$\Rightarrow \quad F_1 = f_1 = \mu(mg + BIl) \quad \ldots(1)$$

$$\text{and} \quad F_2 = f_2 = \mu(mg - BIl) \quad \ldots(2)$$



From equations (1) and (2),

$$\frac{F_1}{F_2} = \frac{mg + BIl}{mg - BIl}$$

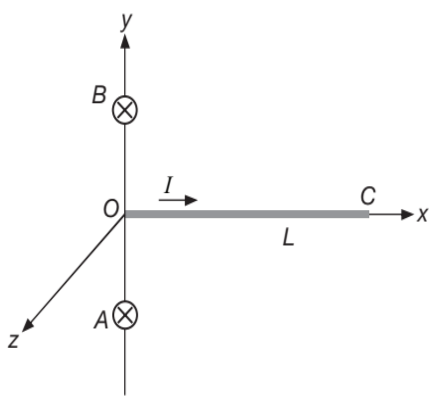
$$\Rightarrow \quad mg = BIl\left(\frac{F_1 + F_2}{F_1 - F_2}\right) \quad \ldots(3)$$

From equations (1) and (3), we get

$$\mu = \frac{F_1 - F_2}{2BIl}$$

ILLUSTRATION 73

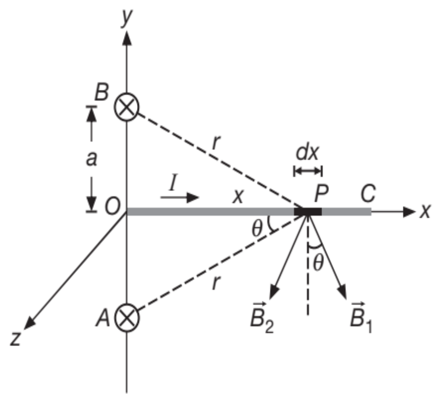
A straight segment OC (of length L) of a circuit carrying a current I is placed along the x-axis. Two infinitely long straight wires A and B, each extending from $z = -\infty$ to $+\infty$, are fixed at $y = -a$ and $y = +a$ respectively, as shown in the Figure.



If the wires A and B each carry a current I into the plane of the paper, obtain the expression for the force acting on the segment OC. What will be the force on OC if the current in the wire B is reversed?

SOLUTION

The current I at A into the plane of the paper produces magnetic field B_1 at $P(x, 0, 0)$ in a direction perpendicular to AP, as shown in Figure.



The magnetic field B_2 which acts perpendicular to BP is due to the current I at B into the plane of paper. The total magnetic field at P due to the currents at A and B is obtained by the vector addition of the fields B_1 and B_2. On resolution, we observe that the x-components of B_1 and B_2 cancel each other whereas y-components add up and thus the net field of B_1 and B_2 would point towards $-y$ direction. Since

$$B = B_1 + B_2, \text{ where } B_1 = \frac{\mu_0 I}{2\pi r} \text{ and } B_2 = \frac{\mu_0 I}{2\pi r}$$

$$\Rightarrow \quad \vec{B} = \frac{2\mu_0 I}{2\pi r}\cos\theta(-\hat{j}) \quad \ldots(1)$$

where θ is the angle which AP or BP makes with OP, (i.e., x-axis) and r is the length of AP or BP. If x is the length OP, then

$$r = \sqrt{x^2 + a^2} \text{ and } \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + a^2}}$$

Substituting these in equation (1), we get

$$\vec{B} = \left(\frac{\mu_0 I}{\pi}\right)\left(\frac{x}{x^2 + a^2}\right)(-\hat{j})$$

The force dF acting on the current element $Idx\hat{i}$ due to the field B is given as

$$d\vec{F} = Id\vec{l} \times \vec{B}$$

$$\Rightarrow \quad d\vec{F} = Idx\hat{i} \times \left(-\hat{j}\right)\left(\frac{\mu_0 I}{\pi}\right)\left(\frac{x}{x^2+a^2}\right)$$

$$\Rightarrow \quad d\vec{F} = \frac{\mu_0 I^2}{\pi}\left(-\hat{k}\right)\frac{xdx}{x^2+a^2}$$

Total force acting on the wire OC is obtaining by integrating the above expression within the limits from $x=0$ to $x=L$. So, we get

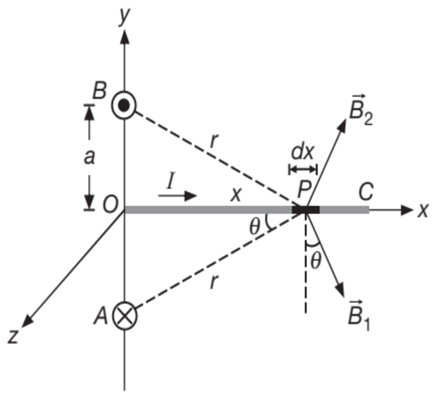
$$\vec{F} = \frac{\mu_0 I^2}{\pi}\left(-\hat{k}\right)\int_0^L \frac{x\,dx}{x^2+a^2}$$

$$\Rightarrow \quad \vec{F} = \frac{\mu_0 I^2}{2\pi}\left(-\hat{k}\right)\log_e\left(\frac{x^2+a^2}{a^2}\right)$$

So, we get

$$F = \frac{\mu_0 I^2}{2\pi}\log_e\left(\frac{x^2+a^2}{a^2}\right), \text{ along } -z \text{ axis}$$

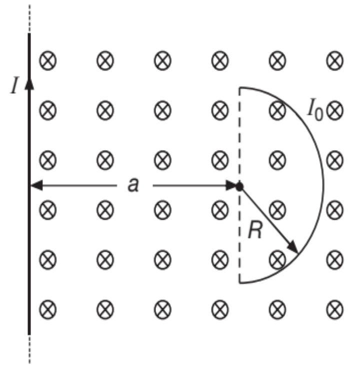
If the current in wire B is reversed, then the magnetic field B_2 at P due to the current I at B would act in the direction as shown in Figure.



On resolution, the resultant field of $\vec{B}_1$ and $\vec{B}_2$ will be along $+x$-axis i.e., along $Id\vec{l}$. So, force due to this arrangement of A and B on the conductor OC is zero.

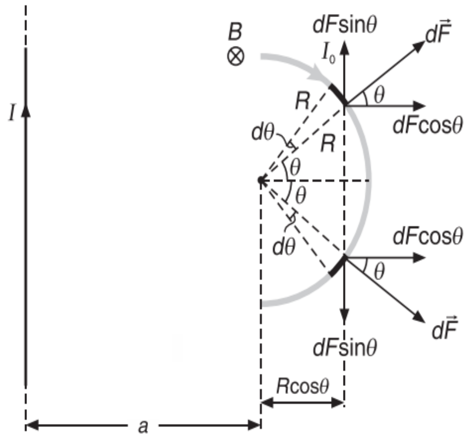
ILLUSTRATION 74

A semi-circular conductor of radius R carrying a current I_0 is coplanar with a very long straight conductor carrying a current I. The distance of the centre of the straight conductor is a from the straight conductor. Find the magnetic interaction force between the conductors for the position shown in Figure.



SOLUTION

According to Biot-Savart's Law the magnetic induction due to very long straight current I is directed normally into the plane of the figure. The magnitude of the magnetic induction due to straight current depends upon the perpendicular distance from it. Here the elements of semi-circular current are at different perpendicular distances from the straight current. Let us take an element of the circular current at an angle θ from horizontal as shown in the Figure.



Taken elemental wire experiences the Ampere force in accordance with the formula.

$d\vec{F} = I\left(d\vec{l} \times \vec{B}\right)$, which gives

$d\vec{F} = I_0 R d\theta \dfrac{\mu_0}{2\pi}\dfrac{I}{\left(a+R\cos\theta\right)}$, directed away from the centre

From the symmetry of the problem, all sine components cancel, hence the net Ampere force $\vec{F}$ is directed towards right and is given by

$$F_{\text{net}} = \int dF\cos\theta$$

$$\Rightarrow \quad F_{net} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mu_0}{2\pi} I I_0 R \frac{\cos\theta d\theta}{(a+R\cos\theta)}$$

$$\Rightarrow \quad F_{net} = \frac{\mu_0}{2\pi} I I_0 R\left[2\int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{(a+R\cos\theta)}\right]$$

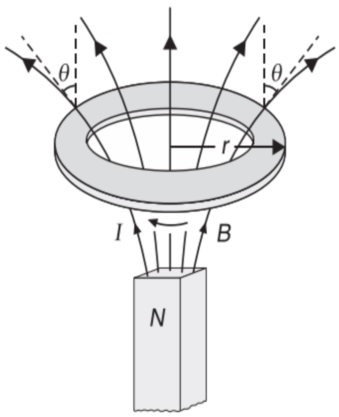
$$\Rightarrow \quad F_{net} = \frac{\mu_0}{\pi} I I_0 R\int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{(a+R\cos\theta)}$$

$$\Rightarrow \quad F_{net} = \frac{\mu_0 I I_0}{\pi}\left(\frac{\pi}{2} - \frac{2a}{\sqrt{a^2-R^2}}\tan^{-1}\sqrt{\frac{a-R}{a+R}}\right)$$

Please note that if the current I_0 were in a circular loop having its centre at O, then the two halves experience forces in the opposite directions. The left half experiences the Ampere force $F_{\text{Left Half}}$ (say) towards left and the right half experiences a net force $F_{\text{Right Half}}$ towards right and so the net Ampere force on the circular current loop becomes $(F_{\text{Left Half}} - F_{\text{Right Half}})$, directed towards left.

ILLUSTRATION 75

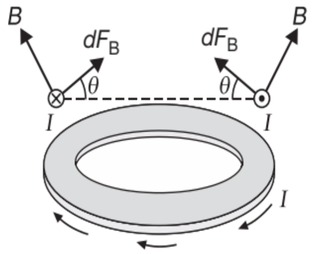
A non-uniform magnetic field exerts a net force on a magnetic dipole. A strong magnet is placed under a horizontal conducting ring of radius r that carries current I as shown in Figure.



If the magnetic field $\vec{B}$ makes an angle θ with the vertical at the ring's location, calculate the magnitude and direction of the resultant force on the ring?

SOLUTION

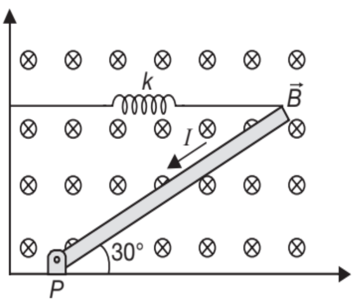
The magnetic force on each bit of ring is $I(d\vec{\ell} \times \vec{B}) = BI(d\ell)$ radially inward and upward, at angle θ above the radial line as shown in Figure.



The radially inward components tend to squeeze the ring but all cancel out as forces. The upward components $Id\ell B\sin\theta$ all add to give $F = I(2\pi r)B\sin\theta$, upwards.

ILLUSTRATION 76

A thin, uniform rod with negligible mass and length ℓ is attached to the floor by a frictionless hinge at point P. A horizontal spring with force constant k connects the other end of the rod to a vertical wall. The rod is in a uniform magnetic field B directed into the plane of the figure. There is current I in the rod, in the direction shown in Figure.



(a) Calculate the torque due to the magnetic force on the rod, for an axis at P. Is it correct to take the total magnetic force to act at the centre of gravity of the rod when calculating the torque? Explain.

(b) When the rod is in equilibrium and makes an angle of $30°$ with the floor, is the spring stretched or compressed?

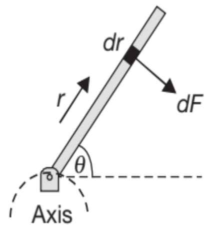
(c) How much energy is stored in the spring when the rod is in equilibrium?

SOLUTION

Let us calculate the force and then the torque on each small section of the rod and then integrating to find the total magnetic torque. At equilibrium the torques from the spring force and from the magnetic force cancel. If the spring stretches by x, then we have

$$U=\frac{1}{2}kx^2$$

which gives the energy stored in the spring.



Let us now divide the rod into infinitesimal sections of length dr, as shown in figure.

(a) The magnetic force on this section of the rod is $dF=IBdr$ and is perpendicular to the rod. The torque $d\tau$ due to the force on this section is $d\tau=rdF=IBrdr$. The total torque is given by

$$\tau=\int d\tau=IB\int_0^{\ell} rdr=\frac{1}{2}I\ell^2B\text{, clockwise.}$$

(b) F produces a clockwise torque so the spring force must produce a counter clockwise torque. The spring force must be to the left so as to keep the spring stretched.

(c) Since $\sum\tau=0$, axis at hinge, counter clockwise torques positive

$$\Rightarrow\ (kx)\ell\sin(30°)-\frac{1}{2}I\ell^2B=0$$

$$\Rightarrow\ x=\frac{BI\ell}{k}$$

$$\Rightarrow\ U=\frac{1}{2}kx^2=\frac{1}{2}k\left(\frac{BI\ell}{k}\right)^2=\frac{B^2I^2\ell^2}{2k}$$

The total magnetic force $F=BI\ell$ acts at the centre of the rod. Also note that, we didn't include the torque due to gravity since the rod had negligible mass.

MAGNETIC PRESSURE AND MAGNETIC FIELD ENERGY

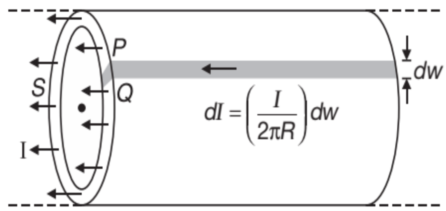
Since we know that a charged metal surface always experiences an electrostatic pressure on it in the outward direction due to its surface charge density (as already studied in Electrostatics).

Now let us consider the case of a hollow current carrying cylindrical conductor. For the case of a current carrying hollow conductor, the current flows at its surface. This current flowing on the surface can be thought to be flowing in parallel strips each carrying current in the same direction, due to which these strips must attract each other and hence the conductor surface experiences an inward magnetic pressure.

To calculate this magnetic pressure on the surface, let us consider a long straight hollow cylindrical shell carrying a current I. Since we know that at every point inside this hollow conductor, the magnetic field is zero, whereas just outside the conductor, the magnetic field is

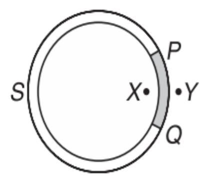
$$B=\frac{\mu_0 I}{2\pi R} \quad \ldots(1)$$

Figure shows the front view and the cross sectional view of the hollow cylindrical conductor which shows an infinitesimal wire PQ of width dw and the remainder conductor shell PSQ.



Front View

Let us draw the cross-sectional view of the circular section and then consider two points X and Y just inside and outside the cylindrical shell as shown in the Figure.



Cross-sectional View

At both these points magnetic field must be the vector sum of the fields due to the infinitesimal wire and the remainder cylindrical shell. At point X, the magnetic field is zero and at point Y, magnetic field is given by $B=\frac{\mu_0 I}{2\pi R}$.

If B_1 and B_2 be the magnetic fields at points X and Y due to the infinitesimal wire PQ (of width dw) and the remainder conductor shell PSQ, then we have

$$B_X = B_1 - B_2 = 0 \quad \text{...(2)}$$

$$B_Y = B_1 + B_2 = B = \frac{\mu_0 I}{2\pi R} \quad \text{...(3)}$$

From equations (2) and (3), we have

$$B_1 = B_2 = \frac{B}{2} \quad \text{...(4)}$$

This is exactly similar to the case studied in electrostatics. So, we see that, the elemental part PQ of the shell contributes exactly half of the magnetic field just outside the shell due to the entire current flowing in it.

If we consider a length l of the elemental wire PQ, then force experienced by it due to the remaining section PSQ is given by

$$dF = B_2 (dI) l, \text{ where } dI = \left(\frac{I}{2\pi R}\right) dw$$

$$\Rightarrow \quad dF = B_2 \left(\frac{I dw}{2\pi R}\right) l$$

$$\Rightarrow \quad dF = \left(\frac{\mu_0 I}{4\pi R}\right)\left(\frac{I dw}{2\pi R}\right) l$$

The area of the elemental wire strip is $dA = l dw$, so, the inward magnetic pressure P_m on the elemental strip is given by

$$P_m = \frac{dF}{dA} = \frac{\mu_0 I^2}{8\pi^2 R^2} = \frac{1}{2\mu_0}\left(\frac{\mu_0^2 I^2}{4\pi^2 R^2}\right)$$

$$\Rightarrow \quad P_m = \frac{B^2}{2\mu_0} \quad \text{...(5)}$$

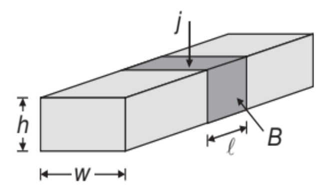
This expression in equation (5) represents the magnetic pressure due to a surface current and this is also equal to the magnetic energy density i.e. energy stored per unit volume by the magnetic field of the current carrying conductor.

Test Your Concepts-VI

Based on Force on Current Carrying Conductor

(Solutions on page H.21)

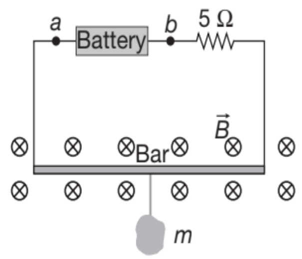
1. Liquid sodium, an excellent thermal conductor which melts at 99°C is used in some nuclear reactors to cool the reactor core. The liquid sodium is moved through the pipes of the reactor through Electromagnetic pumps. These pumps work on the concept that a moving charge in a magnetic field experiences a force. Consider a liquid metal to be in an electrically insulated pipe that has a rectangular cross-section of width w and height h. A uniform magnetic field $\vec{B}$ acts perpendicular to the section of pipe having a length ℓ as shown. Also, an electric field directed perpendicular to the pipe and to the magnetic field produces a current density j in the liquid.



(a) Calculate force that acts on the liquid in the arrangement shown.

(b) Calculate pressure increase, that the section of the liquid of length ℓ experiences in the magnetic field.

2. The circuit shown in figure is used to make a magnetic balance to weigh objects. The mass m to be measured is hung from the center of the bar that is in a uniform magnetic field of 1.5 T, directed into the plane of the figure. The battery voltage can be adjusted to vary the current in the circuit. The horizontal bar is 60 cm long and is made of extremely light weight material. It is connected to the battery by thin vertical wires that can support no appreciable tension. All the weight of the suspended mass m is supported by the magnetic force on the bar. A resistor with $R = 5\ \Omega$ is in series with the bar and the resistance of the rest of the circuit is much less than this.

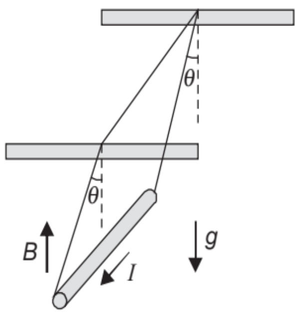


(a) Which point, a or b, should be the positive terminal of the battery?

(b) If the maximum terminal voltage of the battery is 175 V, calculate the greatest mass *m* that this instrument can measure.

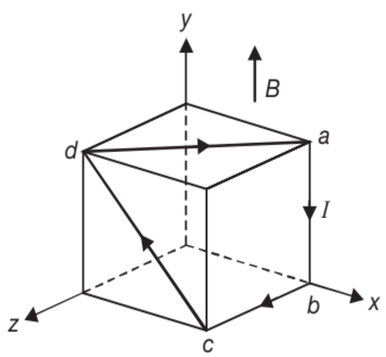
3. A 0.2 kg metal rod carrying a current of 10 A glides on two horizontal rails 0.5 m apart. What vertical magnetic field is required to keep the rod moving at a constant speed if the coefficient of kinetic friction between the rod and rails is 0.1?

4. A metal rod having a mass per unit length λ carries a current *I*. The rod hangs from two vertical wires in a uniform vertical magnetic field as shown in figure. The wires make an angle θ with the vertical when in equilibrium. Determine the magnitude of the magnetic field.

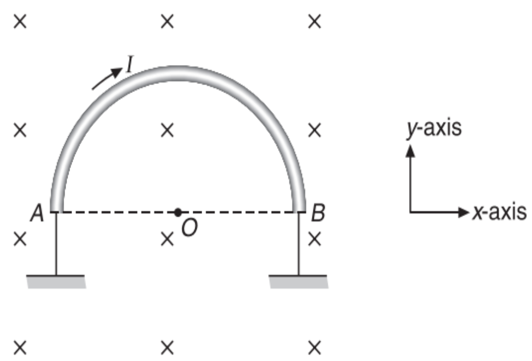


5. A wire having a mass per unit length of 0.5 gcm^{-1} carries a 2 A current horizontally to the south. What are the direction and magnitude of the minimum magnetic field needed to lift this wire vertically upward?

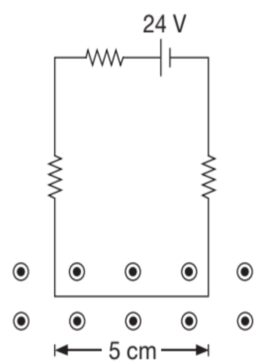
6. In the figure shown, the cube has each side of length ℓ. Four straight segments of wires *ab*, *bc*, *cd* and *da* forming a closed loop carrying a current *I*, in the direction shown are taken. A uniform magnetic field of magnitude *B* is existing in the positive *y*-direction. Determine the magnitude and direction of the magnetic force on each segment.



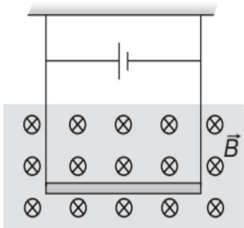
7. The top view of a conducting rod *AB*, bent to form a semicircle of radius *R*, carrying current *I*, placed on a smooth horizontal table is shown in Figure. It is subjected to a uniform magnetic field *B* perpendicular to its plane. The rod is held at rest in its position by two light strings attached to its ends. Calculate the magnetic force acting on the conductor *AB* and the tension in the strings.



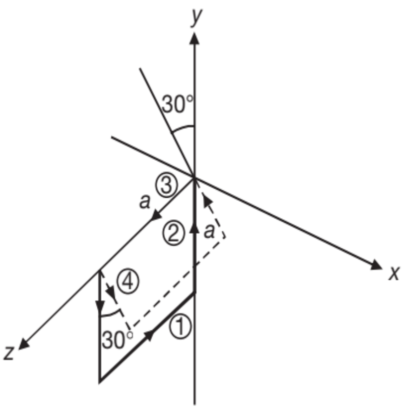
8. The circuit in figure consists of wires at the top and bottom and identical metal springs in the left and right sides. The upper portion of the circuit is fixed. The wire at the bottom has a mass of 10 g and is 5 cm long. The springs stretch 0.5 cm under the weight of the wire and the circuit has a total resistance of 12 Ω. When a magnetic field is turned on, directed out of the page, the springs stretch an additional 0.3 cm. What is the magnitude of the magnetic field?



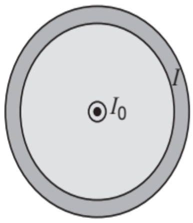
9. A conductor suspended by two flexible wires as shown in figure has a mass per unit length of 0.04 kgm^{-1}. What current must exist in the conductor in order for the tension in the supporting wires to be zero when the magnetic field is 3.6 T into the page? Also find the required direction for the current?



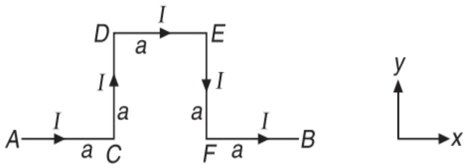
10. Consider a uniform square frame as shown in figure. Each side of the frame is of length a. The longitudinal mass density of the frame is λ. The frame carries an electric current I and is free to move around its upper side (side 3 in figure). Compute the magnetic field B in the $+y$ direction, required to hold the frame titled 30° away from the vertical axis.



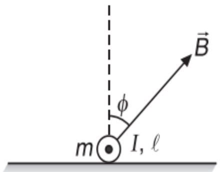
11. A long thin walled hollow cylinder of radius r is carrying a current I. A very long current carrying straight conductor is passing through the axis of the hollow cylinder, carrying a current I_0. Find the tension per unit length developed in the hollow cylinder due to the interaction of the straight current carrying conductor.



12. A uniform magnetic field $\vec{B} = B_0\hat{k}$ exists in a region. A current carrying wire $ACDEFB$ is placed in x-y plane as shown in Figure. Calculate the force acting on the wire AB, if each section of the wire is having length a.



13. A conducting rod of mass m, length l carrying a current I is subjected to a magnetic field of induction B as shown in Figure.



If the coefficients of friction between the conducting rod and rail is μ, find the value of I for which the rod starts sliding.

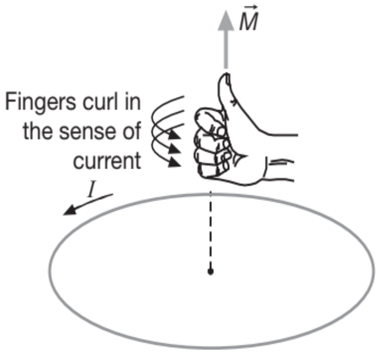
MAGNETIC DIPOLE

Just like a bar magnet , every current carrying loop behaves like a magnetic dipole and has two poles, South (S) and North (N) in which the magnetic field lines emanate from the north pole and after forming a closed path terminate at the south pole. Each magnetic dipole has some magnetic moment $(\vec{M})$. The magnitude of the dipole moment $\vec{M}$ is

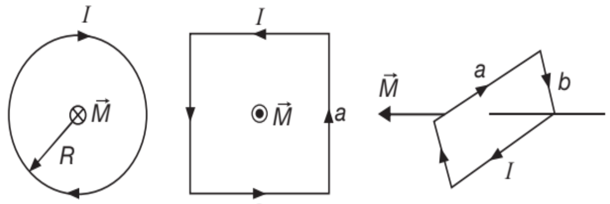
$$\vec{M} = N(I\vec{A})$$

where, N is the number of turns in the loop, I is current in the loop and $\vec{A}$ is area of cross-section of the loop. To find the direction of $\vec{M}$, we can make use of any of the following methods.

The vector $\vec{M}$ is along the normal to the plane of the loop and its orientation (up or down along the normal) is given by the right-hand rule.



For getting the direction of $\vec{M}$, just curl the fingers of your right hand around the perimeter of the loop in the sense of current as shown and extend your thumb so that it is perpendicular to the plane of the loop. The thumb direction gives the direction of $\vec{M}$.



(a) $M = I(\pi R^2)$ (b) $M = I(a^2)$ (c) $M = I(ab)$

Remark(s)

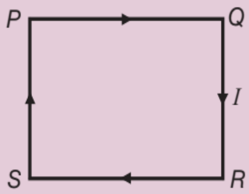
For calculating $\vec{M}$, let us discuss two more methods.

METHOD 1

This method is useful for calculating $\vec{M}$ for a rectangular or square loop.

The magnetic moment $(\vec{M})$ of the rectangular loop shown in figure is,

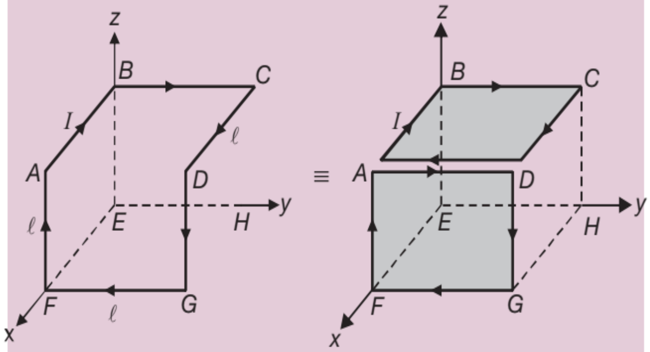
$$\vec{M} = I\left(\overrightarrow{PQ} \times \overrightarrow{QR}\right) = I\left(\overrightarrow{QR} \times \overrightarrow{RS}\right) = I\left(\overrightarrow{RS} \times \overrightarrow{SP}\right) = I\left(\overrightarrow{SP} \times \overrightarrow{PQ}\right)$$



Here the cross product of any two consecutive sides (taken in order) gives the area as well as the correct direction of $\vec{M}$ also.

METHOD 2

Sometimes a current carrying loop may not lie in a single plane. However, by assuming two equal and opposite currents in one branch (which obviously makes net current in that branch to be zero) two (or more) closed loops can be completed in different planes. Now the net magnetic moment of the given loop is the vector sum of individual loops.



(a) (b)

For example, in figure (a), A loop carries a current I in the directions shown. To find the magnetic moment we assume two equal and opposite currents in wire AD and get two complete loops in two different planes (xy and yz). So,

$$\vec{M}_{ABCDA} = -I\ell^2\hat{k} \text{ and } \vec{M}_{ADGFA} = -I\ell^2\hat{i}$$

$$\Rightarrow \vec{M}_{net} = \vec{M}_{ABCDA} + \vec{M}_{ADGFA} = -I\ell^2\left(\hat{i} + \hat{k}\right)$$

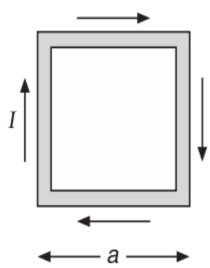
ILLUSTRATION 77

A given length of a constant current carrying straight wire is moulded into a square, equilateral triangle and a circular loop, each of one turn. Which loop has the maximum magnetic moment?

SOLUTION

Let L be the length of the wire to be bent in loops of various shapes and let I be the current in each loop and A_{sqaure}, $A_{triangle}$ and A_{circle} be the area of the square loop, triangular loop and circular loop . Then

For Square



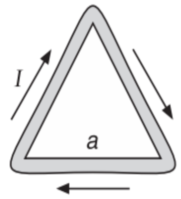
$$L = 4a$$

$$\Rightarrow a = \frac{L}{4}$$

$$\Rightarrow A_{square} = a^2 = \left(\frac{L}{4}\right)^2 = \frac{L^2}{16}$$

$$\Rightarrow M_{square} = IA_{square} = Ia^2 = \frac{IL^2}{16}$$

For Triangle



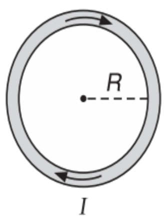
$$L = 3a$$

$$\Rightarrow \quad a = \frac{L}{3}$$

$$\Rightarrow \quad A_{\text{triangle}} = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\left(\frac{L}{3}\right)^2 = \frac{\sqrt{3}L^2}{36}$$

$$\Rightarrow \quad M_{\text{triangle}} = IA_{\text{triangle}} = \frac{\sqrt{3}IL^2}{36}$$

For Circle



$$L = 2\pi R$$

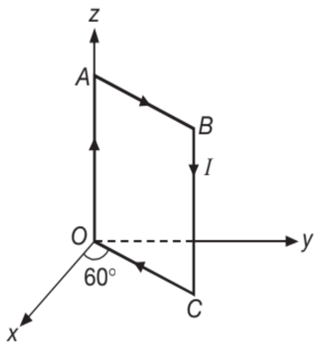
$$\Rightarrow \quad R = \frac{L}{2\pi}$$

$$\Rightarrow \quad A_{\text{circle}} = \pi R^2 = \pi\left(\frac{L^2}{4\pi^2}\right) = \frac{L^2}{4\pi}$$

$$\Rightarrow \quad M_{\text{circle}} = IA_{\text{circle}} = \frac{IL^2}{4\pi} = \text{MAXIMUM}$$

ILLUSTRATION 78

A square loop $OABCO$ of side l carries a current I and is placed as shown in Figure. Calculate the magnetic moment of the loop.



SOLUTION

As discussed, the magnetic moment of the loop is given by

$$\vec{M} = I(\overrightarrow{BC} \times \overrightarrow{CO})$$

where, $\overrightarrow{BC} = -l\hat{k}$

and $\overrightarrow{CO} = -l\cos 60°\hat{i} - l\sin 60°\hat{j} = -\frac{l}{2}\hat{i} - \frac{\sqrt{3}l}{2}\hat{j}$

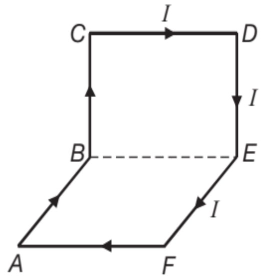
$$\Rightarrow \quad \vec{M} = I\left[(-l\hat{k}) \times \left(-\frac{l}{2}\hat{i} - \frac{\sqrt{3}l}{2}\hat{j}\right)\right]$$

$$\Rightarrow \quad \vec{M} = \frac{Il^2}{2}\left(-\sqrt{3}\hat{i} + \hat{j}\right)$$

$$\Rightarrow \quad |\vec{M}| = Il^2$$

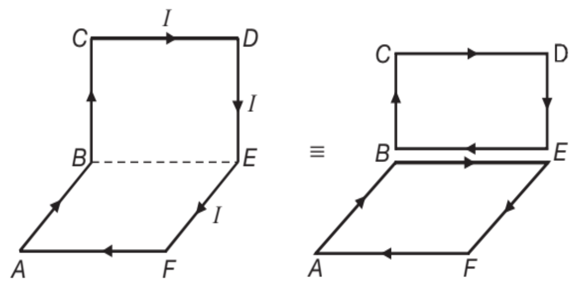
ILLUSTRATION 79

Find the magnitude of magnetic moment of the current carrying loop $ABCDEFA$. Each side of the loop is 10 cm long and current in the loop is $I = 2$ A.



SOLUTION

By assuming two equal and opposite currents in BE, two current carrying loops ($ABEFA$ and $BCDEB$) are formed as shown in Figure.



Their magnetic moments are equal in magnitude but perpendicular to each other. Hence,

$$M_{\text{net}} = \sqrt{M^2 + M^2} = \sqrt{2}M$$

where $M = IA = (2)(0.1)(0.1) = 0.02 \text{ Am}^2$

$$\Rightarrow \quad M_{\text{net}} = (\sqrt{2})(0.02) \text{ Am}^2$$

$$\Rightarrow \quad M_{\text{net}} = 0.028 \text{ Am}^2$$

GYROMAGNETIC RATIO (GMR)

Whenever a non-conducting uniformly charged body is rotated with some angular speed then the ratio of magnetic moment and angular momentum is constant which is equal to $\frac{q}{2m}$, where q is the charge and m the mass of the body. This ratio is called as the GYROMAGNETIC RATIO.

EXAMPLE

In case of a ring, of mass m, radius R and charge q distributed on its circumference, the angular momentum L of the ring about the said axis is given by

$$L = I\omega = \left(mR^2\right)\omega \qquad \ldots(1)$$

where, I is the moment of Inertia of the ring given by

$$I = mR^2$$

Magnetic moment is given by

$$M = iA = (qf)\left(\pi R^2\right)$$

where i is the current due to the circulating charge

$$\Rightarrow \quad i = \frac{q}{T} = qf$$

where f is the frequency given by $f = \dfrac{\omega}{2\pi}$

The magnetic moment of the loop is given by

$$M = iA = (q)\left(\frac{\omega}{2\pi}\right)\left(\pi R^2\right) = q\frac{\omega R^2}{2} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$\frac{M}{L} = \frac{q}{2m}$$

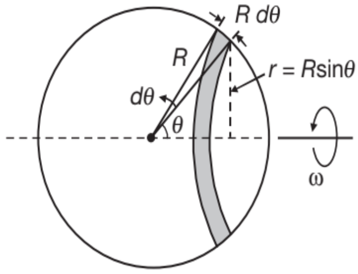
Although this expression is derived for simple case of a ring, it holds good for other bodies also. For example, for a disc or a sphere.

ILLUSTRATION 80

A spherical shell of radius R, having charge Q rotates with angular velocity ω about an axis passing through its centre. Calculate the magnetic induction at the centre of the shell. Find the magnetic moment of the shell in terms of Q and express both the results in terms of σ. Also calculate the ratio of the magnetic moment $|\vec{M}|$ to the angular momentum of the rotating shell $|\vec{L}|$, called as GYROMAGNETIC RATIO. Can you conclude something significant from this ratio? Explain.

SOLUTION

The shell can be assumed to be made up of a number of co-axial rings. Consider one such ring of radius r, azimuth angle θ, thickness dr subtending an angle $d\theta$ at the centre, as shown in Figure.



If dq be the charge on the infinitesimal element, then

$$dq = \sigma dA \text{ where } dA = (2\pi r)dr,\ r = R\sin\theta$$

and $dr = Rd\theta$

$$\Rightarrow \quad dq = \frac{Q}{4\pi R^2}2\pi(R\sin\theta)(Rd\theta)$$

$$\Rightarrow \quad dq = \left(\frac{Q}{2}\right)\sin\theta d\theta$$

Now, this charge dq circulates about the axis shown, so the equivalent current produced by the element is

$$dI = \frac{dq}{\left(\frac{2\pi}{\omega}\right)} = \left(\frac{Q\omega}{4\pi}\right)(\sin\theta d\theta)$$

Since the magnetic field at the axis of the current carrying loop is given by

$$B_{axis} = \frac{\mu_0 I(\text{radius of ring})^2}{2\left(a^2+x^2\right)^{3/2}}, \text{ where}$$

$$\left(a^2+x^2\right)^{3/2} = R^3$$

So, the magnetic field produced due to an infinitesimal current carrying loop at the axis is

$$dB = \frac{\mu_0(dI)r^2}{2R^3}$$

$$\Rightarrow \quad dB = \frac{\mu_0 Q\omega(R\sin\theta)^2(\sin\theta d\theta)}{8\pi R^3}$$

$$\Rightarrow \quad dB = \frac{\mu_0 Q\omega}{8\pi R}\int_0^{\pi}\sin^3\theta d\theta$$

Since, $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

$$\Rightarrow \quad \sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin(3\theta)\right]$$

$$\Rightarrow \quad B = \int dB = \frac{\mu_0 Q\omega}{8\pi R}\left[\frac{1}{4}\left(\int_0^{\pi}3\sin\theta d\theta - \int_0^{\pi}\sin(3\theta)d\theta\right)\right]$$

$$\Rightarrow \quad B=\int dB=\frac{\mu_0 Q\omega}{32\pi R}\left[3\left(-\cos\theta\Big|_0^{\pi}\right)+\frac{\cos(3\theta)}{3}\Big|_0^{\pi}\right]$$

$$\Rightarrow \quad B=\frac{\mu_0 Q\omega}{32\pi R}\left[3(2)+\frac{1}{3}(-2)\right]=\frac{\mu_0 Q\omega}{16\pi R}\left(3-\frac{1}{3}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0 Q\omega}{16\pi R}\left(\frac{8}{3}\right)$$

$$\Rightarrow \quad B=\frac{\mu_0 Q\omega}{6\pi R} \qquad \{\text{in terms of } Q,\ \omega \text{ and } R\}$$

Now, since $Q=(4\pi R^2)\sigma$

$$\Rightarrow \quad B=\frac{\mu_0(4\pi R^2)\sigma\omega}{6\pi R}$$

$$\Rightarrow \quad B=\frac{2}{3}\mu_0\sigma R\omega \qquad \{\text{in terms of } \sigma,\ \omega \text{ and } R\}$$

Now to calculate the magnetic moment, we know that

$$dM=\begin{pmatrix}\text{Current in}\\ \text{the element}\end{pmatrix}\begin{pmatrix}\text{Area of the loop}\\ \text{formed by element}\end{pmatrix}$$

$$\Rightarrow \quad dM=(dI)(\pi r^2)$$

$$\Rightarrow \quad dM=\left(\frac{Q\omega}{4\pi}\right)(\sin\theta d\theta)(\pi R^2\sin^2\theta)$$

$$\Rightarrow \quad M=\int dM=\frac{Q\omega R^2}{4}\int_0^{\pi}\sin^3\theta d\theta$$

$$\Rightarrow \quad M=\frac{Q\omega R^2}{4}\left[\frac{1}{4}\int_0^{\pi}3\sin\theta d\theta-\int_0^{\pi}\sin(3\theta)d\theta\right]$$

$$\Rightarrow \quad M=\frac{Q\omega R^2}{16}\left[3(2)+\frac{1}{3}(-2)\right] \qquad \{\text{as solved earlier}\}$$

$$\Rightarrow \quad M=\frac{Q\omega R^2}{8}\left(\frac{8}{3}\right)=\frac{1}{3}Q\omega R^2$$

$$\{\text{in terms of } Q,\ \omega \text{ and } R\}$$

$$\Rightarrow \quad M=\frac{1}{3}(Q\omega R^2)$$

Since, $Q=(4\pi R^2)\sigma$

$$\Rightarrow \quad M=\frac{4}{3}(\pi\sigma R^4\omega) \qquad \{\text{in terms of } \sigma,\ \omega \text{ and } R\}$$

Let us also calculate the GYROMAGNETIC RATIO, denoted by γ, i.e., the ratio of magnetic moment to the angular momentum. So, we get

Gyromagnetic Ratio, $\gamma=\dfrac{|\vec{M}|}{|\vec{L}|}=\dfrac{\frac{1}{3}Q\omega R^2}{\frac{2}{3}mR^2}=\dfrac{Q}{2m}$

where m = mass of shell, Q = charge on shell.

Please note that, for a charged body of charge Q, mass m rotating about axis of symmetry will possess both angular momentum $(\vec{L})$ and the magnetic moment $(\vec{M})$, on account of its rotation. It can be proved (or also has been shown), that as long as the charge and mass are distributed uniformly on the body, its GYROMAGNETIC RATIO (γ) is always given by

$$\gamma=\frac{|\vec{M}|}{|\vec{L}|}=\frac{Q}{2m}$$

So, we could have calculated the magnetic moment from the expression of Gyromagnetic Ratio, according to which

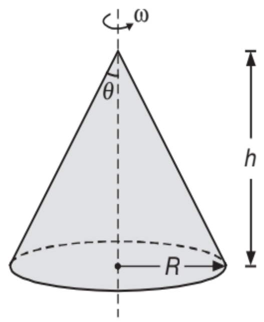
$$\gamma=\frac{|\vec{M}|}{|\vec{L}|}=\frac{Q}{2m}$$

$$\Rightarrow \quad \frac{|\vec{M}|}{\left(\frac{2}{3}mR^2\right)\omega}=\frac{Q}{2m}$$

$$\Rightarrow \quad |\vec{M}|=\frac{1}{3}QR^2\omega$$

ILLUSTRATION 81

A charge Q is uniformly distributed over the slant surface of a thin walled right circular cone of semi-vertex angle θ and height h as shown in Figure.

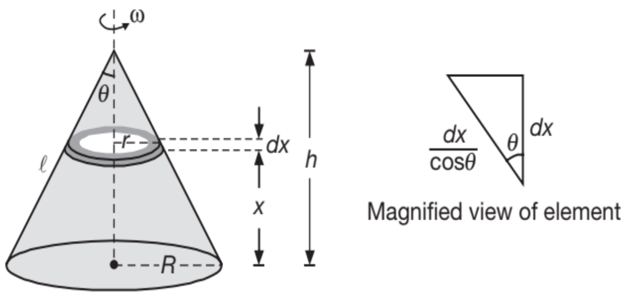


The cone is uniformly rotated about its axis at angular velocity ω. Calculate the magnetic dipole moment associated with the cone.

SOLUTION

$$M = IA = \left(\frac{\text{Charge Circulating}}{\text{Period of Revolution}}\right)(\text{Area of loop})$$

Let us consider a small, current carrying loop element, having thickness dx at a distance x from the base of the cone as shown in Figure.



If dM is the magnetic moment of this small loop having charge dq, radius r and rotating with angular velocity ω then we have

$$dM = iA = \left[\frac{dq}{\left(\frac{2\pi}{\omega}\right)}\right]A = \frac{\omega}{2\pi}(dq)A$$

where A is area of the loop or the area circulated by the charge in the loop. Also, we know that

$$dq = \sigma dA \text{, where } \sigma = \frac{Q}{\pi Rl},\ dA = (2\pi r)\left(\frac{dx}{\cos\theta}\right)$$

and l is the slant height of the cone given by $l = \dfrac{h}{\cos\theta}$

$$\Rightarrow \quad dM = \left(\frac{\omega}{2\pi}\right)\left(\frac{Q}{\pi Rl}\right)(2\pi r)\left(\frac{dx}{\cos\theta}\right)(\pi r^2) \quad \ldots(1)$$

Also, from figure we observe that

$$\tan\theta = \frac{R}{h} = \frac{r}{h-x}$$

$$\Rightarrow \quad r = \left(\frac{h-x}{h}\right)R$$

So, from (1), we get

$$dM = \left(\frac{\omega}{2\pi}\right)\left(\frac{Q\cos\theta}{\pi Rh}\right)\left(\frac{2\pi^2R^3}{\cos\theta}\right)\left(\frac{h-x}{h}\right)^3 dx$$

$$\Rightarrow \quad M = \int dM = \frac{\omega QR^2}{h^4}\int_0^h (h-x)^3\,dx$$

$$\Rightarrow \quad M = \frac{\omega QR^2}{h^4}\left(\frac{(h-x)^4}{-4}\right)\Bigg|_0^h$$

$$\Rightarrow \quad M = \frac{\omega QR^2}{h^4}\left(0 - \frac{h^4}{4}\right)$$

$$\Rightarrow \quad M = \frac{1}{4}Q\omega R^2 = \frac{1}{4}Q\omega h^2\tan^2\theta$$

Since we know that the Gyromagnetic ratio for such an arrangement is given by

$$\gamma = \frac{M}{L} = \frac{Q}{2m},$$

where M is to be calculated and $L = I\omega$. For a thin walled cone, $I = \frac{1}{2}mR^2$. So, we get

$$M = \frac{I\omega Q}{2m} = \frac{\left(\frac{1}{2}mR^2\right)\omega Q}{2m} = \frac{1}{4}Q\omega R^2$$

$$\Rightarrow \quad M = \frac{1}{4}Q\omega h^2\tan^2\theta$$

{we could have by passed the process of integration}

However, for the solid cone, $I = \frac{3}{10}MR^2$. So, our answer would be (if asked)

$$M = \frac{I\omega Q}{2m} = \frac{\left(\frac{3}{10}mR^2\right)\omega Q}{2m}$$

$$\Rightarrow \quad M = \frac{3}{20}Q\omega R^2 = \frac{3}{20}Q\omega h^2\tan^2\theta$$

ILLUSTRATION 82

A solid cylinder has length L and inner and outer radii R_1 and R_2 respectively. The cylinder carries a uniform charge density ρ and is rotating angular velocity ω about its axis. Calculate the magnetic moment of the cylinder. Assume the above cylinder to be solid of radius R, length L and let it carry a uniform charge density $+\rho$ from $r = 0$ to $r(<R)$ and an equal charge density of opposite sign $-\rho$, from $r(<R)$ to R. If the cylinder is rotating uniformly about its axis such that its magnetic moment is zero, then calculate r. Assume that the mass is distributed uniformly on each cylinder in both the cases.

SOLUTION

Since we know that the Gyro-magnetic Ratio

$$\frac{|\vec{M}|}{|\vec{L}|}=\frac{q}{2m} \quad \ldots(1)$$

If m be the mass of the cylinder, then

$$L=\frac{1}{2}m\left(R_1^2+R_2^2\right)\omega \text{ and}$$

$$q=\rho V=\rho\pi\left(R_2^2-R_1^2\right)L$$

From equation (1), we get

$$|\vec{M}|=\frac{q}{2m}|\vec{L}|$$

$$\Rightarrow \quad |\vec{M}|=\frac{1}{2m}\left[\frac{1}{2}m\left(R_1^2+R_2^2\right)\omega\right]\left[\rho\pi\left(R_2^2-R_1^2\right)L\right]$$

$$\Rightarrow \quad |\vec{M}|=\frac{1}{4}\pi\rho L\omega\left(R_2^4-R_1^4\right)$$

Proceeding in a similar way, the magnetic moment of positively charged section of cylinder i.e. from $r=0$ to $r(<R)$ is

$$M_+=\frac{1}{4}\pi\rho\omega Lr^4$$

Similarly, magnetic moment of negatively charged section i.e. from $r(<R)$ to R is

$$M_-=-\frac{1}{4}\pi\rho\omega L\left(R^4-r^4\right)$$

The net magnetic moment should be zero, hence we have

$$M_+ + M_- = 0$$

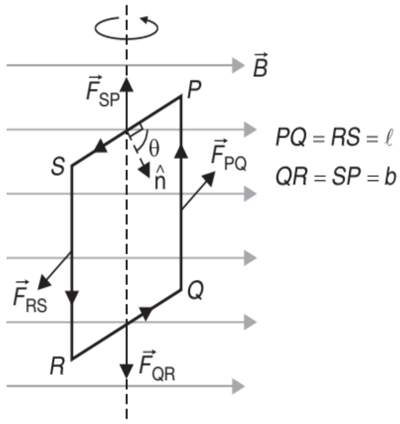
$$\Rightarrow \quad \frac{1}{4}\pi\rho\omega Lr^4=\frac{1}{4}\pi\rho\omega L\left(R^4-r^4\right)$$

$$\Rightarrow \quad 2r^4=R^4$$

$$\Rightarrow \quad r=\left(2^{-1/4}\right)R$$

TORQUE ON A CURRENT LOOP IN UNIFORM MAGNETIC FIELD: REVISITED

Consider a rectangular current carrying coil $PQRS$ having N turns and area A, placed in a uniform field B, in such a way that the normal $(\hat{n})$ to the coil makes an angle θ with the direction of B as shown in Figure.



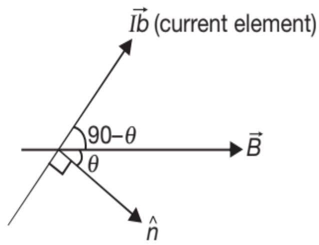
The coil experiences no net force and a torque given by

$$\tau=NBIA\sin\theta$$

This can be derived by simply calculating and drawing the forces on the current carrying conductors PQ, QR, RS and SP. The direction of the forces acting on the wires has been shown in the figure and direction has been found by using the Fleming's Left Hand Rule. Let $\vec{F}_{PQ}$, $\vec{F}_{QR}$, $\vec{F}_{RS}$ and $\vec{F}_{SP}$ the forces acting on the respective wires PQ, QR, RS and SP. Then

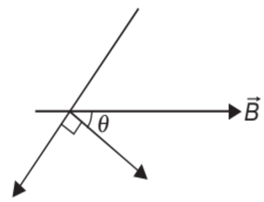
$$\left|\vec{F}_{PQ}\right|=BIl\sin 90^\circ=BIl\text{, inwards}$$

$$\left|\vec{F}_{QR}\right|=BIb\sin(90-\theta)=BIb\cos\theta\text{, downwards}$$



$$\left|\vec{F}_{RS}\right|=BIl\sin 90^\circ=BIl\text{, outwards}$$

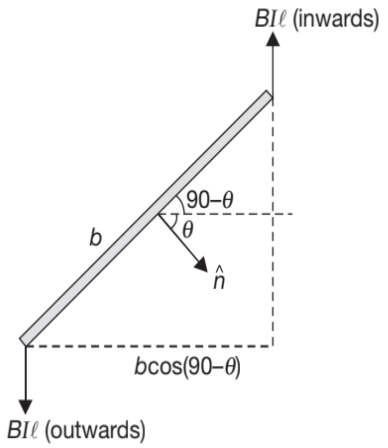
$$\left|\vec{F}_{SP}\right|=BIb\sin(90+\theta)=BIb\cos\theta\text{, upwards}$$



We observe that, these forces must be acting through the centres of the respective wires. Also, from the diagram we observe that $\vec{F}_{QR}$ and $\vec{F}_{SP}$ cancel (due to the same magnitude and having the same line of action). Further, we see that $F_{PQ}\left(=BIl,\text{ inwards}\right)$ and $F_{RS}\left(=BIl,\text{ outwards}\right)$ have different lines of action and hence will make the loop experience a torque so that the loop rotates about the dotted axis. To calculate this torque let us take the overhead view of the loop and draw it. Since we know that

$$\tau = Fr_{\perp}$$

$$\Rightarrow \quad \tau = (\text{Either force})\begin{pmatrix}\text{Perpendicular distance}\\ \text{between the points of}\\ \text{application of forces}\end{pmatrix}$$



$$\Rightarrow \quad \tau = (BIl)(b\cos(90-\theta))$$

$$\Rightarrow \quad \tau = BI(lb)\sin\theta$$

$$\Rightarrow \quad \tau = B(IA)\sin\theta$$

Since we know that $IA = M$, where M is the Magnetic Dipole Moment, so we have

$$\tau = BM\sin\theta$$

If the loop has N turns, then we get

$$\tau = NBIA\sin\theta$$

So, we observe that

(a) τ is zero when $\theta = 0$, i.e., when the plane of the coil is perpendicular to the field.

(b) τ is maximum when $\theta = 90°$, i.e., the plane of the coil is parallel to the field.

$$\Rightarrow \quad \tau_{\max} = NBIA$$

The above expression is valid for coils of all shapes.

The quantity $I\vec{A}$ is a vector and is called the **Magnetic Moment of the loop**. So,

$$\vec{M} = I\vec{A}$$

Vectorially $\vec{\tau} = \vec{M}\times\vec{B}$

In magnitude, $\tau = MB\sin\theta$

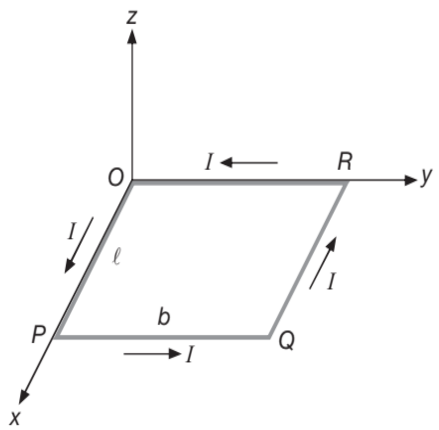
where θ is the angle between magnetic moment and magnetic field OR θ is the angle between normal to the coil and the magnetic field.

Problem Solving Technique(s)

Direction of $\vec{M}$ is found by using Right Hand Thumb Rule according to which "curl the fingers of right hand in the direction of circulation of conventional current, then the thumb gives the direction of $\vec{M}$".

MAGNETIC DIPOLE IN UNIFORM MAGNETIC FIELD: REVISITED USING VECTOR METHOD

Let us consider a rectangular current carrying loop $OPQRO$ of length l, width b placed in xy plane in a uniform magnetic field $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$.



Let us find the net force and torque on the loop.

FORCE ON THE LOOP:

Net force on the loop is,

$$\vec{F} = \vec{F}_{OP} + \vec{F}_{PQ} + \vec{F}_{QR} + \vec{F}_{RO}$$

Since $\vec{F} = I\left(\vec{l}\times\vec{B}\right)$

$$\Rightarrow \quad \vec{F} = I\left[\left(\overrightarrow{OP}\times\vec{B}\right)+\left(\overrightarrow{PQ}\times\vec{B}\right)+\left(\overrightarrow{QR}\times\vec{B}\right)+\left(\overrightarrow{RO}\times\vec{B}\right)\right]$$

$$\Rightarrow \quad \vec{F} = I\left[\left(\overrightarrow{OP}+\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RO}\right)\times\vec{B}\right] = \vec{0} \text{ (null vector)}$$

So, net force on the loop is $\vec{F} = \vec{0}$

$$\left\{\because \ \overrightarrow{OP}+\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RO} \text{ forms a null vector}\right\}$$

TORQUE ON THE LOOP:

The current carrying loop $OPQR$ can be considered to be made up of four current carrying wires OP, PQ, QR and RO. Using $\vec{F}=I(\vec{l}\times\vec{B})$, we get

$$\vec{F}_{OP}=I(\overrightarrow{OP}\times\vec{B})=I\left[(l\hat{i})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})\right]$$

$$\Rightarrow \vec{F}_{OP}=Il(B_y\hat{k}-B_z\hat{j})$$

$$\vec{F}_{PQ}=I(\overrightarrow{PQ}\times\vec{B})=I\left[(b\hat{j})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})\right]$$

$$\Rightarrow \vec{F}_{PQ}=Ib(-B_x\hat{k}+B_z\hat{i})$$

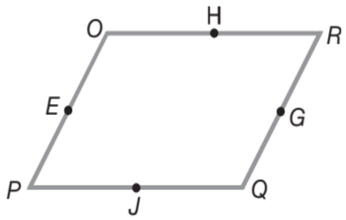
$$\vec{F}_{QR}=I(\overrightarrow{QR}\times\vec{B})=I\left[(-l\hat{i})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})\right]$$

$$\Rightarrow \vec{F}_{QR}=Il\left[-B_y\hat{k}+B_z\hat{j}\right]$$

$$\vec{F}_{RO}=I(\overrightarrow{RO}\times\vec{B})=I\left[(-b\hat{j})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})\right]$$

$$\Rightarrow \vec{F}_{RO}=Ib\left[B_x\hat{k}-B_z\hat{i}\right]$$

All these forces are acting at the centre of the wires. For example, $\vec{F}_{OP}$ will act at the centre of OP. When the forces are in equilibrium then the net torque about any point remains the same. Let us calculate the torque due to the forces about O. Let E, J, G and H are the mid-point of OP, PQ, QR and RO respectively.



Using $\vec{\tau}=\vec{r}\times\vec{F}$, we get

$$\vec{\tau}_O=(\overrightarrow{OE}\times\vec{F}_{OP})+(\overrightarrow{OJ}\times\vec{F}_{PQ})+(\overrightarrow{OG}\times\vec{F}_{QR})+(\overrightarrow{OH}\times\vec{F}_{RO})$$

$$\Rightarrow \vec{\tau}_O=\left[\left(\frac{l}{2}\hat{i}\right)\times\left(Il(B_y\hat{k}-B_z\hat{j})\right)\right]+\left[\left(l\hat{i}+\frac{b}{2}\hat{j}\right)\times\left(Ib(-B_x\hat{k}+B_z\hat{i})\right)\right]+\left[\left(\frac{l}{2}\hat{i}+b\hat{j}\right)\times\left(Il(-B_y\hat{k}+B_z\hat{j})\right)\right]+\left[\left(\frac{b}{2}\hat{j}\right)\times\left(Ib(B_x\hat{k}-B_z\hat{i})\right)\right]$$

$$\Rightarrow \vec{\tau}_O=(Ilb)B_x\hat{j}-(Ilb)B_y\hat{i}$$

$$\Rightarrow \vec{\tau}_O=IA(B_x\hat{j}-B_y\hat{i})$$ where area of the loop is $A=lb$ and $\vec{A}=(lb)\hat{k}$

$$\Rightarrow \vec{\tau}_O=M(B_x\hat{j}-B_y\hat{i})$$ where $M=IA$ and $\vec{M}=I\vec{A}$

This expression obtained can be rewritten as

$$\vec{\tau}_0=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & M\\ B_x & B_y & B_z\end{vmatrix}=(M\hat{k})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$

$$\Rightarrow \vec{\tau}_0=\vec{M}\times\vec{B}$$

The above equation for the torque is very similar to that of an electric dipole in an electric field. The similarity between electric and magnetic dipole extends even further as illustrated in the table shown.

S. No.	Similarity	Electric Dipole	Magnetic Dipole
1.	Magnitude	$\lvert\vec{p}\rvert=q(2d)$	$\lvert\vec{M}\rvert=NIA$
2.	Direction	from $-q$ to $+q$	from south (S) to north (N)
3.	Net force in uniform field	zero	zero
4.	Torque	$\vec{\tau}=\vec{p}\times\vec{E}$	$\vec{\tau}=\vec{M}\times\vec{B}$
5.	Potential energy	$U=-\vec{p}\cdot\vec{E}$	$U=-\vec{M}\cdot\vec{B}$
6.	Work done in rotating the dipole from θ_1 to θ_2	$W=pE(\cos\theta_1-\cos\theta_2)$	$W=MB(\cos\theta_1-\cos\theta_2)$
7.	Field along axis at farther points (Axial Line)	$\vec{E}=\frac{1}{4\pi\varepsilon_0}\left(\frac{2\vec{p}}{r^3}\right)$	$\vec{B}=\frac{\mu_0}{4\pi}\left(\frac{2\vec{M}}{r^3}\right)$
8.	Field perpendicular to axis at farther points (Equitorial Line)	$\vec{E}=-\frac{1}{4\pi\varepsilon_0}\left(\frac{\vec{p}}{r^3}\right)$	$\vec{B}=-\frac{\mu_0}{4\pi}\left(\frac{\vec{M}}{r^3}\right)$
9.	At far points	$\lvert\vec{E}_{axial}\rvert=2\lvert\vec{E}_{equitorial}\rvert$	$\lvert\vec{E}_{axial}\rvert=2\lvert\vec{E}_{equitorial}\rvert$

Remark(s)

(a) The expressions for the magnetic dipole can be obtained from the expressions for the electric dipole by replacing $\vec{p}$ by $\vec{M}$ and ε_0 by $\frac{1}{\mu_0}$. Here μ_0 is called the permeability of free space. It is related with ε_0 and speed of light c as, $c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$ and it has the value, $\mu_0=4\pi\times10^{-7}\ \text{TmA}^{-1}$

(b) Dimensions of $\frac{1}{\sqrt{\varepsilon_0\mu_0}}$ are that of speed or $[LT^{-1}]$

Hence, $\left[\frac{1}{\sqrt{\varepsilon_0\mu_0}}\right]=[LT^{-1}]$

Problem Solving Technique(s)

(a) Note that although $\vec{\tau}=\vec{M}\times\vec{B}$ has been derived for a rectangular loop, it comes out to be true for any shape of loop.

(b) Magnitude of $\vec{\tau}$ is $MB\sin\theta$ or $NB(IA)\sin\theta$. Here θ is the angle between $\vec{M}$ and $\vec{B}$. Torque is zero when $\theta=0^\circ$ or 180° and it is maximum at $\theta=90^\circ$.

Also, we can say that θ is the angle between the normal to the coil and the magnetic field.

(c) If the loop is free to rotate in a magnetic field the axis of rotation becomes an axis parallel to $\vec{\tau}$ passing through the centre of mass of the loop.

ILLUSTRATION 83

A current loop with magnetic dipole moment $\vec{M}$ is placed in a uniform magnetic field $\vec{B}$, with its moment making angle θ with the field. With the arbitrary choice of $U=0$ for $\theta=90^\circ$, prove that the potential energy of the dipole field system is $U=-\vec{M}\cdot\vec{B}$. Also calculate the maximum and minimum values of U, giving its significance.

SOLUTION

Since we have been given to take the arbitrary choice of $U=0$ at $\theta=90^\circ$, so we shall proceed as follows. The torque due to the field on the current loop is

$$\vec{\tau}=\vec{M}\times\vec{B}$$

This torque of magnitude $MB\sin\theta$, tends to rotate the dipole of decreasing θ. Let this torque turn the dipole through an infinitesimal angle $d\theta$, then

$$dW=dU=\tau d\theta$$

$$\Rightarrow \int_0^U dU=\int_{\frac{\pi}{2}}^{\theta} MB\sin\theta d\theta$$

$$\Rightarrow U=MB\left(-\cos\theta\Big|_{\frac{\pi}{2}}^{\theta}\right)$$

$$\Rightarrow U=-MB\left(\cos\theta-\cos\frac{\pi}{2}\right)$$

$$\Rightarrow U=-MB\cos\theta$$

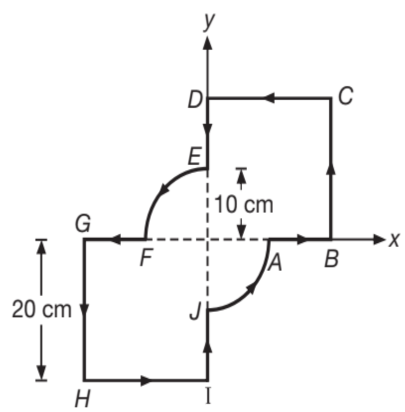
$$\Rightarrow U=-\vec{M}\cdot\vec{B}$$

Now, U is minimum when $\theta=0^\circ$ i.e., $\vec{M}\parallel\vec{B}$ i.e., for a stable system we must have $U=U_{\text{min}}=-MB$.

Further, U is maximum when $\theta=\pi$ i.e., $\vec{M}$ and $\vec{B}$ are antiparallel i.e., for an unstable system we have $U=U_{\text{max}}=MB$.

ILLUSTRATION 84

A current $I=5$ A flows through a thin wire as shown in Figure.



(a) Find the magnetic field produced by the current at point O in the figure,

(b) If there exists an external magnetic field $\vec{B}=(14\hat{i}+14\hat{j})$ T, calculate the torque acting on the wire.

Take $\pi=\frac{22}{7}$

SOLUTION

(a) Let 10 cm = a

$$B_0 = 2(B_{ABCDE} + B_{EF})$$

$$\Rightarrow B_0 = 2\left[2\frac{\mu_0 I}{4\pi 2a}\left(\cos\frac{\pi}{4}+\cos\frac{\pi}{2}\right)+\frac{\mu_0 I}{4\pi a}\frac{\pi}{2}\right]$$

$$\Rightarrow B_0 = \frac{\mu_0 I}{4\pi a}\left[\frac{2}{\sqrt{2}}+\pi\right] = \frac{10^{-7}\times 5}{0.1}\left[\sqrt{2}+3.14\right]$$

$$\Rightarrow B_0 = 22.78\ \mu\text{T in the direction } \hat{k}$$

(b) $\vec{M} = I\vec{A} = 5\left(8a^2+\frac{\pi a^2}{2}\right)\hat{k} = 5a^2\left(8+\frac{\pi}{2}\right)\hat{k}$

$$\vec{\tau} = \vec{M}\times\vec{B} = 5a^2\left(8+\frac{\pi}{2}\right)\hat{k}\times 14\left(\hat{i}+\hat{j}\right)\text{ Nm}$$

$$\Rightarrow \vec{\tau} = (5)(14)\left(8+\frac{\pi}{2}\right)\hat{k}\times\left(\hat{i}+\hat{j}\right)\text{ Nm}$$

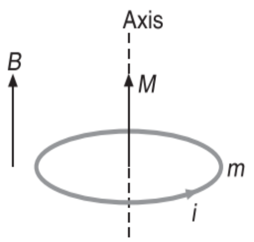
$$\Rightarrow \vec{\tau} = 5a^2(112+22)\left(\hat{j}-\hat{i}\right)\text{ Nm}$$

$$\Rightarrow \vec{\tau} = 5\times 0.01\times 134\left(\hat{j}-\hat{i}\right)\text{ Nm}$$

$$\Rightarrow \vec{\tau} = 6.7\left(\hat{j}-\hat{i}\right)\text{ Nm}$$

ILLUSTRATION 85

A small circular coil of mass m consists of N turns of fine wire and carries a current i. The coil is located in a uniform magnetic field B with the coil axis originally parallel to the direction of field as shown in Figure.



Calculate the period of small angle oscillations of the coil axis around its equilibrium position, assuming friction force mechanical force to be absent. Now, if the coil is rotated through an angle of θ from its equilibrium position and then released, calculate the angular speed of the coil when it passes through the equilibrium position.

SOLUTION

Magnetic moment of coil is given by

$$M = N(iA) = N\left(\pi R^2\right)$$

Let the coil be displaced by a small angle θ, then torque on the coil is

$$\tau = -MB\sin\theta$$

The negative sign indicates that this torque is restoring in nature. Since θ is small, so $\sin\theta \approx \theta$, hence

$$\Rightarrow I\alpha = -MB\theta\text{, where } I = \frac{1}{2}mR^2$$

$$\Rightarrow \left(\frac{1}{2}mR^2\right)\alpha = -Ni\pi R^2 B\theta$$

$$\Rightarrow \alpha = -\left(\frac{2Ni\pi B}{m}\right)\theta$$

Comparing with $\alpha = -\omega^2\theta$ we get

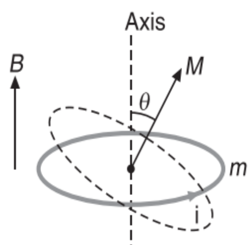
$$\omega = \sqrt{\frac{2NI\pi B}{m}}$$

Time period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2Ni\pi B}}$$

Applying conservation of energy

$$(\text{KE}+\text{PE})_i = (\text{KE}+\text{PE})_f$$



$$\Rightarrow 0 - MB\cos\theta = \frac{1}{2}I\omega^2 - MB\cos 0°$$

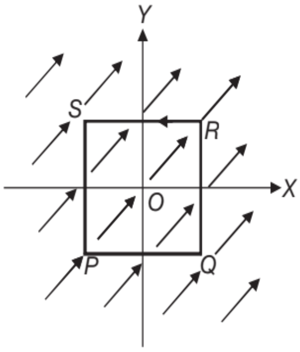
$$\Rightarrow \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 = MB(1-\cos\theta)$$

$$\Rightarrow \frac{1}{4}mR^2\omega^2 = Ni\pi R^2 B(1-\cos\theta)$$

$$\Rightarrow \omega = \sqrt{\frac{4Ni\pi B(1-\cos\theta)}{m}}$$

ILLUSTRATION 86

A uniform constant magnetic field $\vec{B}$ is directed at an angle of $45°$ to the X-axis in X-Y plane. $PQRS$ is a rigid square wire frame carrying a steady current I_0, with its centre at the origin O. At time $t = 0$, the frame is at rest in the position shown in the figure with its sides parallel to X and Y axes. Each side of the frame is of mass M and length L.



(a) What is the torque $\vec{\tau}$ about O acting on the frame due to the magnetic field?

(b) Find the angle by which the frame rotates under the action of this torque in a short interval of time Δt, and the axis about which this rotation occurs (Δt is so short that any variation in the torque during this interval may be neglected).

Given: the moment of inertia of the frame about an axis through its centre perpendicular to its plane is $\frac{4}{3}ML^2$.

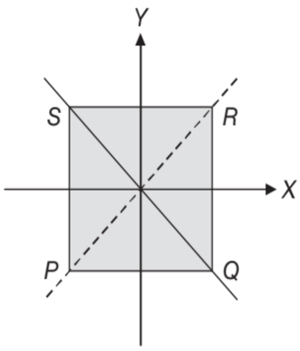
SOLUTION

Magnetic moment of the loop is given by

$$\vec{M} = (IA)\hat{k} = (I_0L^2)\hat{k}$$

Magnetic field is given by

$$\vec{B} = (B\cos 45°)\hat{i} + (B\sin 45°)\hat{j} = \frac{B}{\sqrt{2}}(\hat{i}+\hat{j})$$



(a) Torque acting on the loop,

$$\vec{\tau} = \vec{M}\times\vec{B} = (I_0L^2\hat{k})\times\left[\frac{B}{\sqrt{2}}(\hat{i}+\hat{j})\right]$$

$$\Rightarrow \tau = \frac{I_0L^2B}{\sqrt{2}}(\hat{j}-\hat{i})$$

$$\Rightarrow |\vec{\tau}| = I_0L^2B$$

(b) Since, torque is in $(\hat{j}-\hat{i})$ direction or parallel to QS. Therefore, the loop will rotate about an axis passing through Q and S with an angular acceleration α, given by

$$\alpha = \frac{|\vec{\tau}|}{I_{QS}}$$

where I_{QS} is moment of inertia of loop about the axis QS. ***Please do not confuse moment of inertia with the current.***

From theorem of perpendicular axis, we get

$$I_{QS} + I_{PR} = I_{ZZ}$$

However, since $I_{QS} = I_{PR}$

$$\Rightarrow 2I_{QS} = I_{ZZ} = \frac{4}{3}ML^2$$

$$\Rightarrow I_{QS} = \frac{2}{3}ML^2$$

$$\Rightarrow \alpha = \frac{|\vec{\tau}|}{I_{QS}} = \frac{I_0L^2B}{\frac{2}{3}ML^2} = \frac{3}{2}\frac{I_0B}{M}$$

Since Δt is given very small, so we can assume τ and α almost constant because θ will not change appreciably. Hence, we can use,

$$\Delta\theta = \frac{1}{2}\alpha(\Delta t)^2$$

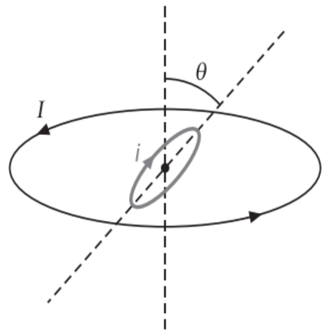
So, the angle by which the frame rotates in time Δt is

$$\Delta\theta = \frac{1}{2}\alpha(\Delta t)^2$$

$$\Rightarrow \Delta\theta = \frac{3}{4}\frac{I_0B}{M}(\Delta t)^2$$

ILLUSTRATION 87

A large horizontal coil of radius R is carrying a current I. Another small coil of radius $r(\ll R)$ carrying a current i and N turns is placed at the centre of the large coil with its plane inclined at an angle θ with the axis of the large coil as shown in Figure. Calculate the torque experienced by the smaller coil.



SOLUTION

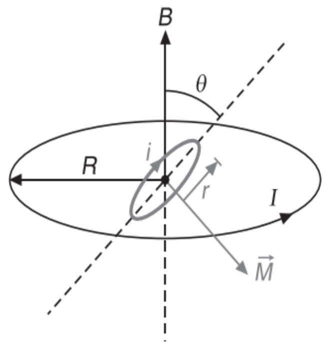
Magnetic moment of smaller coil is given by

$$M = Ni\left(\pi r^2\right) = Ni\pi r^2$$

Magnetic induction due to large coil at its centre is

$$B = \frac{\mu_0 I}{2R}$$

Angle between magnetic induction at centre of large coil and magnetic moment of smaller coil is $\frac{\pi}{2} + \theta$ as shown in Figure.



Torque on smaller coil due to the magnetic induction of larger coil at its centre is given by

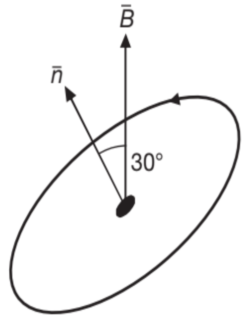
$$\tau = MB\sin\left(\frac{\pi}{2} + \theta\right)$$

$$\Rightarrow \quad \tau = Ni\pi r^2\left(\frac{\mu_0 I}{2R}\right)\cos\theta$$

$$\Rightarrow \quad \tau = \frac{\mu_0 i I N \pi r^2}{2R}\cos\theta$$

ILLUSTRATION 88

An electron in the ground state of hydrogen atom is revolving in anticlockwise direction in a circular orbit of radius R (shown in figure).



(a) Obtain an expression for the orbital magnetic moment of the electron.

(b) The atom is placed in a uniform magnetic induction $\vec{B}$ such that the plane normal of the electron orbit makes an angle of 30° with the magnetic induction. Find the torque experienced by the orbiting electron.

SOLUTION

(a) In ground state $(n = 1)$ according to Bohr's Theory

$$mvR = \frac{h}{2\pi}$$

$$\Rightarrow \quad v = \frac{h}{2\pi m R}$$

Now, time period,

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{h/2\pi m R} = \frac{4\pi^2 m R^2}{h}$$

Magnetic moment $M = IA$, where

$$I = \frac{\text{charge}}{\text{time period}} = \frac{e}{\frac{4\pi^2 m R^2}{h}} = \frac{eh}{4\pi^2 m R^2}$$

and $A = \pi R^2$

$$\Rightarrow \quad M = \left(\pi R^2\right)\left(\frac{eh}{4\pi^2 m R^2}\right)$$

$$\Rightarrow \quad m = \frac{eh}{4\pi m}$$

Direction of magnetic moment $\vec{M}$ is perpendicular to the plane of orbit.

(b) Since, $\vec{\tau} = \vec{M} \times \vec{B}$

$$\Rightarrow \quad \vec{\tau} = MB\sin\theta$$

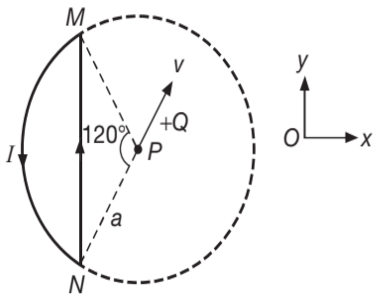
where $\theta = 30°$ is the angle between $\vec{M}$ and $\vec{B}$.

$$\Rightarrow \tau = \left(\frac{eh}{4\pi m}\right)(B)\sin 30^\circ = \frac{ehB}{8\pi m}$$

The direction of $\vec{\tau}$ is perpendicular to both $\vec{M}$ and $\vec{B}$.

ILLUSTRATION 89

A wire loop carrying a current I is placed in the x-y plane as shown in Figure.

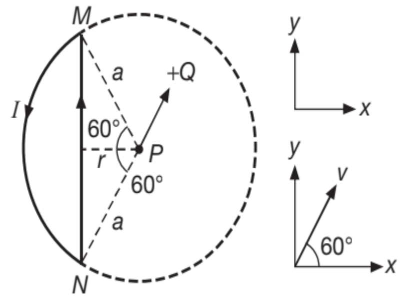


(a) If a particle with charge $+Q$ and mass m is placed at the centre P and given a velocity $\vec{V}$ along NP (shown in figure), find its instantaneous acceleration.

(b) If an external uniform magnetic induction field $\vec{B} = B\hat{i}$ is applied, find the force and the torque acting on the loop due to this field.

SOLUTION

(a) Magnetic field at P due to arc of circle (shown in Figure) subtending an angle of 120° at centre would be



$$B_1 = \frac{1}{3}(\text{Field Due to a Circle})$$

$$\Rightarrow B_1 = \frac{1}{3}\left(\frac{\mu_0 I}{2a}\right) = \frac{\mu_0 I}{6a} \text{ (outwards)}$$

$$\Rightarrow B_1 = \frac{0.16\mu_0 I}{a} \text{ (outwards)}$$

$$\Rightarrow \vec{B}_1 = \left(\frac{0.16\mu_0 I}{a}\right)\hat{k}$$

Magnetic field due to straight wire NM at P is

$$B_2 = \frac{\mu_0 I}{4\pi r_\perp}(\sin 60^\circ + \sin 60^\circ)$$

Here, $r_\perp = r = a\cos 60^\circ$

$$\Rightarrow B_2 = \frac{\mu_0 I}{4\pi(a\cos 60^\circ)}(2\sin 60^\circ)$$

$$\Rightarrow B_2 = \frac{\mu_0 I}{2\pi a}\tan 60^\circ = \frac{0.27\mu_0 I}{a}\text{(inwards)}$$

$$\Rightarrow \vec{B}_2 = -\frac{0.27\mu_0 I}{a}\hat{k}$$

So, $\vec{B}_{net} = \vec{B}_1 + \vec{B}_2 = -\left(\frac{0.11\mu_0 I}{a}\right)\hat{k}$

The velocity of particle can be written as,

$$\vec{v} = (v\cos 60^\circ)\hat{i} + (v\sin 60^\circ)\hat{j} = \frac{v}{2}\left(\hat{i} + \sqrt{3}\hat{j}\right)$$

Magnetic force on the moving charge is given by

$$\vec{F}_m = Q\left(\vec{v}\times\vec{B}\right) = \frac{0.11\mu_0 IQv}{2a}\hat{j} - \frac{0.11\sqrt{3}\mu_0 IQv}{2a}\hat{i}$$

Instantaneous acceleration $\vec{a}$ is given by

$$\vec{a} = \frac{\vec{F}_m}{m} = \frac{0.11\mu_0 IQv}{2am}\left(\hat{j} - \sqrt{3}\,\hat{i}\right)$$

(b) In uniform magnetic field, force on a current loop is zero. Further, magnetic dipole moment of the loop will be,

$$\vec{M} = (IA)\hat{k}$$

where A is the area of the loop given by

$$A = \frac{1}{3}\left(\pi a^2\right) - \frac{1}{2}(2a\sin 60^\circ)(a\cos 60^\circ)$$

$$\Rightarrow A = \frac{\pi a^2}{3} - \frac{a^2}{2}\sin 120^\circ = 0.61a^2$$

$$\Rightarrow \vec{M} = \left(0.61\, Ia^2\right)\hat{k}$$

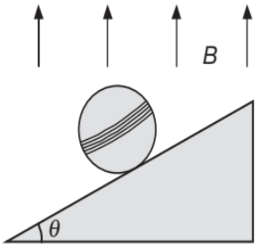
Given, $\vec{B} = B\hat{i}$

$$\Rightarrow \vec{\tau} = \vec{M}\times\vec{B} = \left(0.61\, Ia^2 B\right)\hat{j}$$

ILLUSTRATION 90

A non-conducting sphere has mass 220 g and radius 20 cm. A flat compact coil of wire with 10 turns is wrapped tightly around it, with each turn concentric with the sphere. The sphere is placed on an inclined plane that slopes downward to the left, making an angle θ with the horizontal, so that the coil is parallel to the inclined plane. A uniform magnetic field of 0.35 T vertically upward exists in the region of the sphere. What current in the coil will enable the sphere to rest in equilibrium on the inclined plane? Show that the result does not depend on the value of θ.

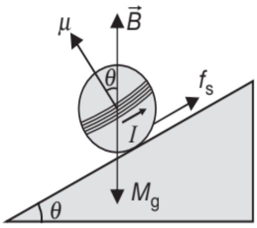
$\left(\text{Take } g = 10\text{ ms}^{-2} \text{ and } \pi = \dfrac{22}{7}\right)$



SOLUTION

For Translational Equilibrium, we have

$$f_s - Mg\sin\theta = 0 \quad \ldots(1)$$



For Rotational Equilibrium, if torques are taken about the centre of the sphere, the magnetic field produces a clockwise torque of magnitude $\mu B\sin\theta$, and the frictional force a counter clockwise torque of magnitude $f_s R$, where R is the radius of the sphere. Thus

$$f_s R - \mu B\sin\theta = 0 \quad \ldots(2)$$

For avoiding confusion, we have taken the magnetic moment here to be $\vec{\mu}$ instead of M.
From (1), we have $f_s = Mg\sin\theta$. Substituting this in (2) and cancelling out $\sin\theta$, we get

$$\mu B = MgR \quad \ldots(3)$$

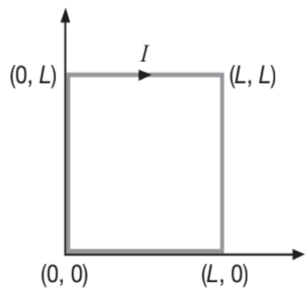
where $\mu = NI(\pi R^2)$. So, we get

$$I = \frac{Mg}{\pi NBR} = \frac{(0.22\text{ kg})(10\text{ ms}^{-2})}{\pi(10)(0.35\text{ T})(0.2\text{ m})} = 1\text{ A}$$

The current must be counter clockwise as seen from above.

ILLUSTRATION 91

Figure shows a square loop of wire that lies in the xy-plane. The loop has corners at $(0, 0)$, $(0, L)$, $(L, 0)$ and (L, L) and carries a constant current I in the clockwise direction. The magnetic field is given by $\vec{B} = \left(\dfrac{B_0 y}{L}\right)\hat{i} + \left(\dfrac{B_0 x}{L}\right)\hat{j}$, where B_0 is a positive constant.

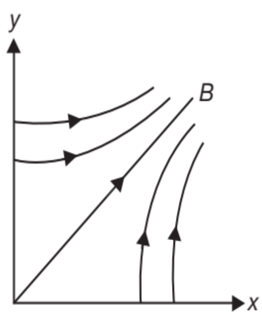


(a) Draw the magnetic field lines in the xy- plane.
(b) Find the magnitude and direction of the magnetic force exerted on each side of the loop.
(c) If the loop is free to rotate about the x-axis, find the magnitude and direction of the magnetic torque on the loop.
(d) If the loop is free to rotate about the y-axis, find the magnitude and direction of the magnetic torque on the loop.
(e) According to you, is the expression $\vec{\tau} = \vec{\mu} \times \vec{B}$, an appropriate description of the torque on this loop? If yes, then why and if no, then why not?

SOLUTION

Apply $d\vec{F} = I(d\vec{\ell} \times \vec{B})$ to each side of the loop $\vec{\tau} = \vec{r} \times \vec{F}$. For each side of the loop, $d\vec{\ell}$ is parallel to that side of the loop and is in the direction of I

(a) The magnetic field lines in the xy-plane are sketched in figure



(b) Side 1, that runs from $(0,\ 0)$ to $(0,\ L)$

$$\vec{F}=\int_0^L I\left(d\vec{\ell}\times\vec{B}\right)=I\int_0^L \frac{B_0 y\,dy}{L}\left(-\hat{k}\right)=-\frac{1}{2}B_0 LI\hat{k}$$

Side 2, that runs from $(0,\ L)$ to $(L,\ L)$

$$\vec{F}=\int_0^L I\left(d\vec{\ell}\times\vec{B}\right)=I\int_0^L \frac{B_0 x dx}{L}\hat{k}=\frac{1}{2}IB_0 L\hat{k}$$

Side 3, that runs from $(L,\ L)$ to $(L,\ 0)$

$$\vec{F}=\int_0^L I\left(d\vec{\ell}\times\vec{B}\right)=I\int_0^L \frac{B_0 y dy}{L}\hat{k}=+\frac{1}{2}IB_0 L\hat{k}$$

Side 4, that runs from $(L,\ 0)$ to $(0,\ 0)$

$$\vec{F}=\int_0^L I\left(d\vec{\ell}\times\vec{B}\right)=I\int_0^L \frac{B_0 x dx}{L}\left(-\hat{k}\right)=-\frac{1}{2}IB_0 L\hat{k}$$

(c) When free to rotate about the x-axis, the torques due to the forces on sides 1 and 3 cancel and the torque due to the forces on side 4 is zero. For side 2, we have $\vec{r}=L\hat{j}$. Therefore,

$$\vec{\tau}=\vec{r}\times\vec{F}=\frac{IB_0L^2}{2}\hat{i}=\frac{1}{2}IAB_0\hat{i}$$

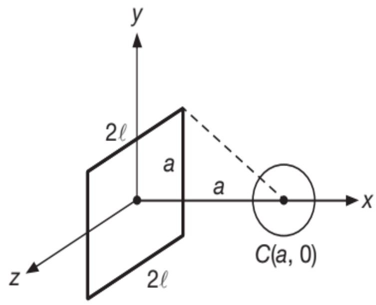
(d) When free to rotate about the y-axis, the torques due to the forces on sides 2 and 4 cancel and the torque due to the forces on side 1 is zero. For side 3, we have $\vec{r}=L\hat{i}$. Therefore,

$$\vec{\tau}=\vec{r}\times\vec{F}=\frac{IB_0L^2}{2}\hat{j}=-\frac{1}{2}IAB_0\hat{j}$$

(e) The equation for the torque $\vec{\tau}=\vec{\mu}\times\vec{B}$ is not appropriate, since the magnetic field is not constant.

ILLUSTRATION 92

A very small circular loop of radius r and carrying a current I_1 is placed in the x-y plane with its centre on x-axis at the point $C(a,0)$. A square loop of side length 2ℓ carrying a current I_2 is fixed in the y-z plane with the centre of the loop at the origin. Calculate the torque exerted by the square loop on the circular loop.



SOLUTION

As derived earlier, B at the axis of a square loop of side L is given by

$$B_{\text{axis of square loop}}=\frac{\mu_0 IL^2}{2\pi\left(x^2+\frac{L^2}{4}\right)\left(x^2+\frac{L^2}{2}\right)^{\frac{1}{2}}}$$

where x is the distance of the point at the axis from the centre of the loop.

However in this problem, we have the loop of side 2ℓ and the point at a distance $x=a$ from its centre, where the field has to be found. So,

$$B_{\substack{\text{square loop}\\ \text{at the axis}}}=\frac{\mu_0 I_2(2\ell)^2}{2\pi\left(x^2+\frac{(2\ell)^2}{4}\right)\left(x^2+\frac{(2\ell)^2}{2}\right)^{\frac{1}{2}}}$$

$$\Rightarrow\quad B_{\substack{\text{Square loop}\\ \text{at the axis}}}=\frac{4\mu_0 I_2\ell^2}{2\pi\left(x^2+\ell^2\right)\sqrt{x^2+2\ell^2}}$$

So, torque due to field of square loop on the circular loop is

$$\tau=MB\sin 90 \qquad \{\because\ \vec{M}\perp\vec{B}\}$$

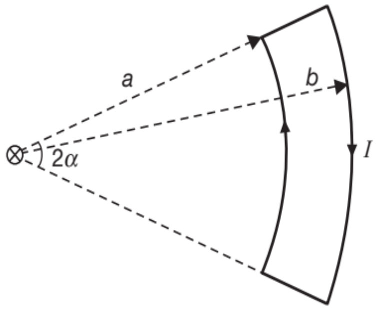
Since, $M=I_1\left(\pi r^2\right)$

$$\Rightarrow\quad \tau=I_1\left(\pi r^2\right)\frac{4\mu_0 I_2\ell^2}{2\pi\left(x^2+\ell^2\right)\sqrt{x^2+2\ell^2}}$$

$$\Rightarrow\quad \tau=\frac{2\mu_0 I_1 I_2\ell^2 r^2}{\left(x^2+\ell^2\right)\sqrt{x^2+2\ell^2}}$$

ILLUSTRATION 93

A loop with current I is in the field of a long straight wire with current I_0. The plane of the loop is perpendicular to the straight wire. Find the moment of Ampere's force acting on this loop.



SOLUTION

Consider a rectangular element of width dr and length $rd\theta$ as shown. If $d\vec{A}$ is the area of the infinitesimal element then

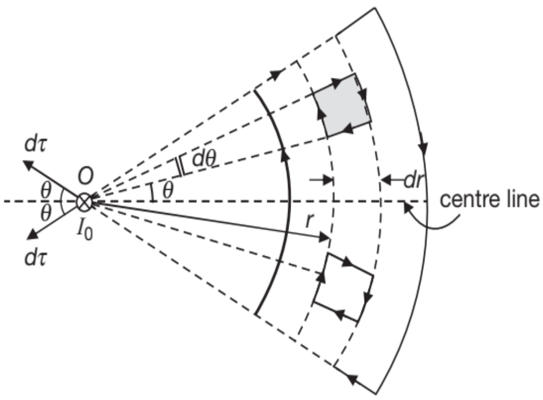
$$d\vec{A} = (dr)(rd\theta) = rdrd\theta$$

$$d\vec{A} = (rd\theta dr) \quad \{\text{inwards}\}$$

$$d\vec{M} = Id\vec{A} = I(rd\theta dr) \quad \{\text{inwards}\}$$

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I_0}{r} \quad \{\text{tangential clockwise}\}$$

$$\Rightarrow \quad d\tau = |d\vec{M} \times \vec{B}| = \frac{\mu_0 I I_0 d\theta\, dr}{2\pi} \quad \{\text{towards centre}\}$$



$$\Rightarrow \quad \tau = \int_{-\alpha}^{\alpha}\int_{a}^{b} d\tau \cos\theta$$

$$\Rightarrow \quad \tau = \frac{\mu_0 I I_0}{2\pi}\int_{-\alpha}^{\alpha}\int_{a}^{b} \cos\theta\, d\theta\, dr$$

$$\Rightarrow \quad \tau = \frac{\mu_0 I I_0 (b-a)\sin\alpha}{\pi}$$

INTERACTION ENERGY FOR A CURRENT CARRYING LOOP IN MAGNETIC FIELD

As already discussed in the chapter of electrostatics (while studying an electric dipole), the interaction potential energy of an electrical dipole with dipole moment p placed in an external uniform electric field E is

$$U = -\vec{p}\cdot\vec{E} = -pE\cos\theta$$

Similarly, for a magnetic dipole with dipole moment M placed in a magnetic field with magnetic induction B with θ as the angle between magnetic moment and magnetic induction, its interaction potential energy is given by

$$U = -\vec{M}\cdot\vec{B} = -MB\cos\theta$$

Since $M = IA$, so the above expression for potential energy can also be written as

$$U = -(IA)B\cos\theta$$

Also, we must know that just like electric flux ϕ_E is defined as $\phi_E = EA\cos\theta$, the magnetic flux ϕ_B associated with the coil is

$$\phi_B = BA\cos\theta$$

$$\Rightarrow \quad U = -I\phi_B$$

Dear Student, we shall be discussing the Magnetic Flux in detail in Electromagnetic Induction.

WORK DONE TO CHANGE ORIENTATION OF A CURRENT CARRYING COIL IN MAGNETIC FIELD

Consider a current carrying coil to be rotated in a uniform magnetic field, such that angle between magnetic induction and magnetic moment of coil changes from θ_1 to θ_2. The potential energy of the coil in magnetic field for its initial and final state are given by

$$U_i = -MB\cos\theta_1 \text{ and}$$

$$U_f = -MB\cos\theta_2$$

Work done to change the orientation of the coil in this process is

$$W = U_f - U_i$$

$$\Rightarrow \quad W = (-MB\cos\theta_2) - (-MB\cos\theta_1)$$

$$\Rightarrow \quad W = -MB(\cos\theta_2 - \cos\theta_1) \quad \ldots(1)$$

Also, we can think that to change the orientation of coil externally, we need to apply an external torque against the magnetic torque such that this external torque has magnitude equal to the magnetic torque. So, work done in changing the orientation in this case is given by

$$W = \int dW = \int \tau d\theta$$

$$\Rightarrow \quad W = \int_{\theta_1}^{\theta_2} MB\sin\theta d\theta$$

$$\Rightarrow \quad W = MB(-\cos\theta)\Big|_{\theta_1}^{\theta_2}$$

$$\Rightarrow \quad W = -MB(\cos\theta_2 - \cos\theta_1) \quad \text{...(2)}$$

So, we see that equations (1) and (2) are identical.
Since $M = IA$, so the above expression for work done can also be written as

$$W = -(IA)B(\cos\theta_2 - \cos\theta_1)$$

$$\Rightarrow \quad W = -I(BA\cos\theta_2 - BA\cos\theta_1)$$

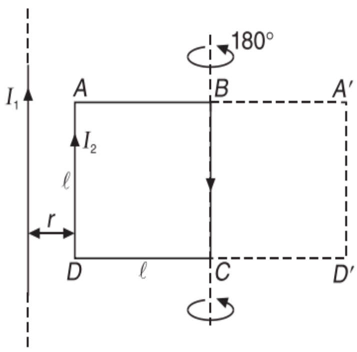
Since $BA\cos\theta_1 = \phi_1 = \phi_{\text{initial}}$ and $BA\cos\theta_2 = \phi_2 = \phi_{\text{final}}$

$$\Rightarrow \quad W = -I(\phi_{\text{final}} - \phi_{\text{initial}})$$

$$\Rightarrow \quad W = -I\Delta\phi$$

ILLUSTRATION 94

A square coil of edge l carrying a current I_2 is placed near to a long straight wire carrying current I_1. Calculate the work required to rotate the coil $ABCD$ about the axis along the edge BC through 180° to the dotted position $A'BCD'$ as shown in Figure.

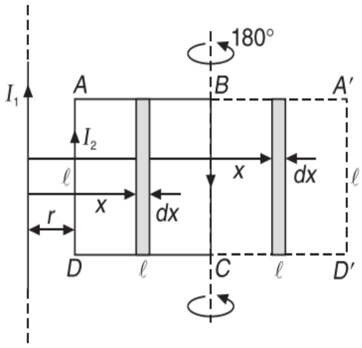


SOLUTION

Work done in the process of changing the orientation of a coil in the magnetic field of the infinite current carrying wire is given by

$$W = -I_2(\phi_f - \phi_i) \quad \text{...(1)}$$

where ϕ_f and ϕ_i are the flux associated with the square coil in the final and the initial position.
To calculate the magnetic flux associated with the square loop let us consider an infinitesimal strip element of length l, width dx at a distance x in the coil as shown in Figure.



The magnetic field B at the strip due to the infinite wire is

$$B = \frac{\mu_0 I_1}{2\pi x}$$

The magnetic flux associated with this infinitesimal strip is

$$d\phi = BdA$$

where $dA = ldx$ is the area of the strip.

$$\Rightarrow \quad d\phi = \left(\frac{\mu_0 I_1}{2\pi x}\right) ldx$$

Total magnetic flux associated with the square loop for the initial position of the loop is obtained by integrating $d\phi$. So, we get

$$\phi_i = \int BdA = \int_r^{r+l} \left(\frac{\mu_0 I_1}{2\pi x}\right) ldx = \left(\frac{\mu_0 I_1 l}{2\pi}\right) \int_r^{r+l} \frac{dx}{x}$$

$$\Rightarrow \quad \phi_i = \frac{\mu_0 I_1 l}{2\pi} (\ln x)\Big|_r^{r+l} = \frac{\mu_0 I_1 l}{2\pi}[\ln(r+l) - \ln r]$$

$$\Rightarrow \quad \phi_i = \frac{\mu_0 I_1 l}{2\pi} \ln\left(\frac{r+l}{r}\right)$$

Similarly, the final flux associated with the coil is

$$\phi_f = \int_{r+l}^{r+2l} \frac{\mu_0 I_1}{2\pi x} ldx = \frac{\mu_0 I_1 l}{2\pi}(\ln x)\Big|_{r+l}^{r+2l}$$

$$\Rightarrow \quad \phi_f = \frac{\mu_0 I_1 l}{2\pi} \ln\left(\frac{r+2l}{r+l}\right)$$

So, work done in this process is calculated by using equation (1), according to which

$$W = -I_2(\phi_f - \phi_i)$$

$$\Rightarrow \quad W = -I_2\left(\frac{\mu_0 I_1 l}{2\pi}\right)\left[\ln\left(\frac{r+2l}{r+l}\right) - \ln\left(\frac{r+l}{r}\right)\right]$$

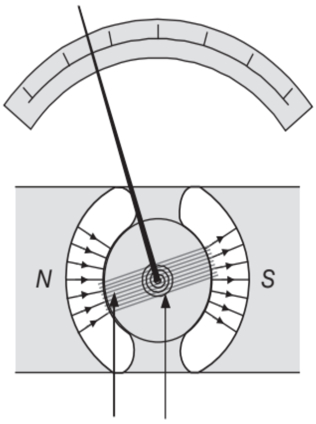
$$\Rightarrow \quad W = I_2\left(\frac{\mu_0 I_1 l}{2\pi}\right)\left[\ln\left(\frac{r+l}{r}\right) - \ln\left(\frac{r+2l}{r+l}\right)\right]$$

$$\Rightarrow \quad W = I_2\left(\frac{\mu_0 I_1 l}{2\pi}\right)\left[\ln\left(\frac{r+l}{r}\right) + \ln\left(\frac{r+l}{r+2l}\right)\right]$$

$$\Rightarrow \quad W = \frac{\mu_0 I_1 I_2 l}{2\pi}\ln\left(\frac{(r+l)^2}{r(r+2l)}\right)$$

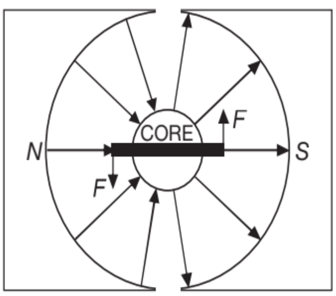
MOVING COIL GALVANOMETER (D'ARSOVNAL GALVANOMETER): RADIAL FIELD

In a moving coil galvanometer, the coil is suspended between the pole pieces of a strong horse-shoe magnet as shown in Figure.



Structure of a moving-coil galvanometer

The pole pieces are made cylindrical and a soft iron cylindrical core is placed within the coil without touching it. This makes the field radial. In a radial field, the plane of the coil always remains parallel to the field.



$F = nBIl$

Therefore $\theta = 90°$ and the deflecting torque always has the value

$$\tau_{def} = NBIA$$

As the coil deflects, a restoring torque is set up in the suspension fibre. If α is the angle of twist, the restoring torque is

$$\tau_{rest} = C\alpha$$

where C is the torsional constant of the fibre.
When the coil is in equilibrium.

$$NBIA = C\alpha$$

$$\Rightarrow \quad I = \frac{C}{NBA}\alpha$$

$$\Rightarrow \quad I = K\alpha,$$

where $K = \frac{C}{NBA}$ is the galvanometer constant. This linear relationship between I and α makes the moving coil galvanometer useful for current measurement and detection.

Current Sensitivity

The current sensitivity of a moving coil galvanometer is defined as

$$S_I = \frac{\alpha}{I} = \frac{NBA}{C}$$

Thus in order to increase the sensitivity of a moving coil galvanometer, N, B and A should be increased and C should be decreased.

Voltage Sensitivity (S_V)

The twist angle per unit voltage is called Voltage Sensitivity. So,

$$S_V = \frac{\alpha}{V} = \frac{\alpha}{IR} = \frac{S_I}{R} = \frac{NBA}{RC}$$

ILLUSTRATION 95

A moving coil galvanometer experiences torque ki where i is current. If N coils of area A each and moment of inertia I is kept in magnetic field B.

(a) Find k in terms of given parameters,

(b) If for current i deflection is $\frac{\pi}{2}$, find out torsional constant of spring.

(c) If a charge Q is passed suddenly through the galvanometer. Find out maximum angle of deflection.

SOLUTION

(a) The torque acting on the coil of moving coil galvanometer is

$$\tau = MB = ki \left(\theta = 90^o\right)$$

$$\Rightarrow k = \frac{MB}{i} = \frac{(NiA)B}{i} = NBA$$

(b) If K is torsional constant of the spring of galvanometer, then we know that

$$\tau = K\theta = NBiA$$

$$\Rightarrow k = \frac{2NBiA}{\pi}$$

(c) $\tau = NBiA$

$$\Rightarrow \int_0^t \tau dt = NBA\int_0^t i\,dt$$

$$\Rightarrow I\omega = (NBA)Q$$

$$\Rightarrow \omega = \frac{NBAQ}{I} \quad \ldots(1)$$

At maximum deflection whole, kinetic energy (rotational) will be converted into potential energy of spring. If θ_{max} is the maximum deflection, then by Law of Conservation of Energy, we have

$$\frac{1}{2}I\omega^2 = \frac{1}{2}k\theta_{max}^2$$

Substituting the values, we get

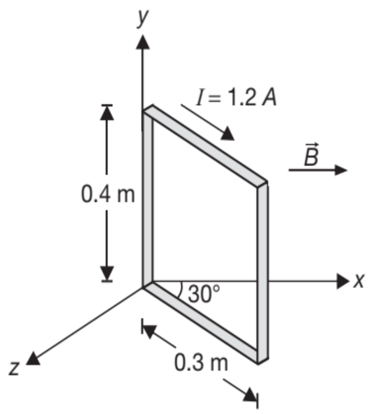
$$\theta_{max} = Q\sqrt{\frac{\pi NBA}{2I}}$$

Test Your Concepts-VII

Based on Magnetic Moment and Torque

(Solutions on page H.24)

1. Consider an electron orbiting a proton and maintained in a fixed circular path of radius $R = 0.53$ Å by the Coulomb force. Treating the orbiting charge as a current loop, calculate the resulting torque when the system is in a magnetic field of 0.4 T directed perpendicular to the magnetic moment of the electron. Take $m_e = 9\times10^{-31}$ kg and $e = 1.6\times10^{-19}$ C.
2. If θ represents the angle between magnetic moment $\vec{M}$ of a current carrying loop and magnetic field $\vec{B}$ in the region in which the loop is placed. Assuming the reference of potential energy to be at $\theta = 0°$, calculate the work done in rotating the loop from $\theta_1 = 60°$ to $\theta_2 = 120°$ and the potential energy of the loop at $\theta = 45°$.
3. A current of 17 mA is maintained in a single circular loop of 2 m circumference. A magnetic field of 0.8 T is directed parallel to the plane of the loop.
   (a) Calculate the magnetic moment of the loop.
   (b) What is the magnitude of the torque exerted by the magnetic field on the loop?
4. A rectangular coil carrying a current of 1.2 A consists of $N = 100$ closely wrapped turns and has dimensions $a = 0.4$ m and $b = 0.3$ m. The coil is hinged along the y-axis, and its plane makes an angle $\theta = 30°$ with the x-axis. The coil is placed in a field of 0.8 T directed along x-axis. Calculate the torque on the coil and the sense of rotation.



5. A long piece of wire with a mass of 0.1 kg and a total length of 4 m is used to make a square coil with a side of 0.1 m. The coil is hinged along a horizontal side, carries a 3.4 A current, and is placed in a vertical magnetic field with a magnitude of 0.01 T.
   (a) Determine the angle that the plane of the coil makes with the vertical when the coil is in equilibrium.
   (b) Find the torque acting on the coil due to the magnetic force at equilibrium.

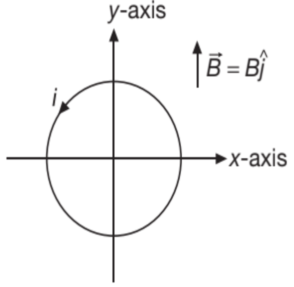
6. A 40 cm length of wire carries a current of 20 A. It is bent into a loop and placed with its normal perpendicular to a magnetic field with a magnitude of 0.52 T. What is the torque on the loop if it is bent into

(a) an equilateral triangle?
(b) a square
(c) a circle?
(d) Which torque is greatest?

7. A wire is formed into a circle having a diameter of 10 cm and placed in a uniform magnetic field of 3 mT. The wire carries a current of 5 A. Find

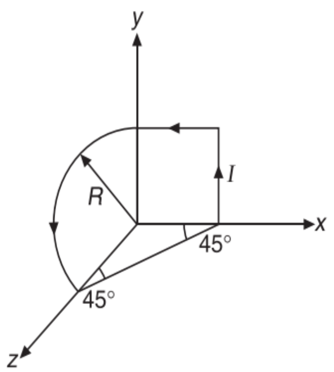
(a) the maximum torque on the wire and
(b) the range of potential energies of the wire-field system for different orientations of the circle.

8. A wire ring of mass m has a radius R carries a current i. It is placed in x-y plane and is free to rotate about a diameter parallel to x-axis as shown in Figure.



A uniform magnetic field B is switched on along the y-axis. Calculate the angular speed acquired by the ring as it rotates through 90°.

9. Calculate the magnetic moment of the current carrying loop shown in figure.

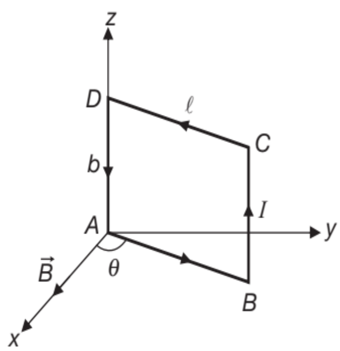


10. A rectangular galvanometer coil of area 5.0×10^{-4} m^2 having 60 turns is pivoted about one of its vertical sides. The coil is in a radial horizontal magnetic field having a magnitude of 9×10^{-3} T. Calculate the torsional constant of the string in Nmrad^{-1} connected to the coil if a current of 0.20 mA produces an angular deflection of 18°.

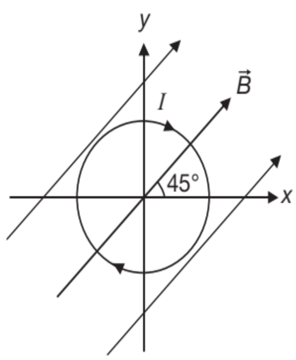
11. Find an expression for the magnetic dipole moment and magnetic field induction at the center of a Bohr's hypothetical hydrogen atom in the nth orbit of the electron in terms of universal constants.

12. Consider a rectangular frame $ABCD$ whose sides are ℓ and b meters long. The frame holds N coils, each of which carries a current I. The frame is embedded in a uniform magnetic field $\vec{B}$ which forms an angle θ with the frame plane.

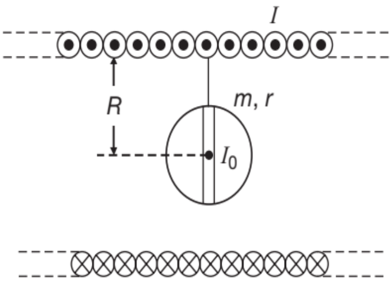
(a) Find the forces which act on the sides of the frame.
(b) Compute the torque about the axis DA.



13. A circular loop of radius $R = 20$ cm is placed in a uniform magnetic field $\vec{B} = 2$ T in x-y plane as shown in figure. The loop carries a current $I = 1$A in the direction shown in figure. Find the magnitude and direction of torque acting on the loop.

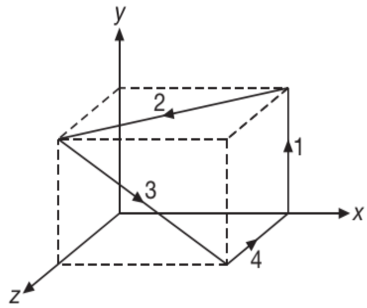


14. A small sphere of mass m and radius r is suspended by a silk thread inside a current carrying solenoid. The solenoid carries a current I and has n numbers of turns per unit length. The sphere is wrapped by a single turn of coil carrying a current I_0 as shown in the diagram. Find the



(a) maximum torque experienced by the sphere
(b) time period of small oscillation.

15. A wire carrying a current of 10 A is bent to pass through various sides of a cube of side 10 cm as shown in Figure.

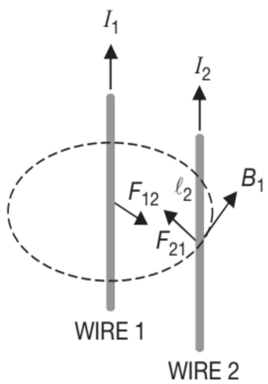


A magnetic field $\vec{B}=(2\hat{i}-3\hat{j}+\hat{k})$ T is present in the region. Calculate the net force on the loop, the magnetic moment of the loop and torque on the loop formed by the wires.

MAGNETIC FORCE BETWEEN TWO PARALLEL CURRENT CARRYING WIRES

CASE-1: Wires Carrying Currents In The Same Direction

Consider two long straight wires 1 and 2 kept parallel to each other at perpendicular separation r, carrying currents I_1 and I_2 respectively, in the same direction as shown in Figure.



The equal and opposite forces exerted by two parallel current carrying wires. The force is attractive in nature.

The wire 2 (carrying current I_2) lies in the magnetic field of wire 1 $\left(B_1=\dfrac{\mu_0 I_1}{2\pi r}, \otimes\right)$ as a result of which the wire 2 experiences a magnetic force. The magnetic force on a length l_2 of wire 2 exerted by the field B_1 of the wire 1 (at wire 2) is given by

$$\vec{F}_{21}=I_2\left(\vec{l}_2\times\vec{B}_1\right) \qquad \{\text{towards wire 1}\}$$

where $\vec{l}_2$ is in the direction of I_2.

Since $\vec{l}_2\perp\vec{B}_1$ (inwards), so we get

$$\left|\vec{F}_{21}\right|=F_{21}=I_2 l_2 B_1\sin 90=I_2 l_2 B_1$$

Since, we have $B_1=\dfrac{\mu_0 I_1}{2\pi r}$, $\otimes$, so we get

$$\Rightarrow \quad F_{21}=I_2 l_2\left(\frac{\mu_0 I_1}{2\pi r}\right) \qquad \ldots(1)$$

The direction of $\vec{F}_{21}$ (towards wire 1) is found by using Fleming's Left Hand Rule.

Similarly, $\vec{F}_{12}$ on a length l_1 of the wire is

$$\vec{F}_{12}=I_1\left(\vec{l}_1\times\vec{B}_2\right) \qquad \{\text{Towards Wire 2}\}$$

Since, we have $B_2=\dfrac{\mu_0 I_2}{2\pi r}$, $\odot$, so we get

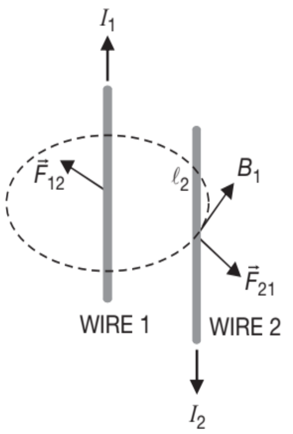
$$\Rightarrow \quad \left|\vec{F}_{12}\right|=F_{12}=I_1 l_1\left(\frac{\mu_0 I_2}{2\pi r}\right) \qquad \ldots(2)$$

From equation (1) and (2), we conclude that the force per unit length on either of the wire is equal

$$\frac{F_{21}}{l_2}=\frac{F_{12}}{l_1}=\frac{\mu_0 I_1 I_2}{2\pi r}$$

CASE-2: Wires Carrying Current In The Opposite Direction

Consider two long straight wires 1 and 2 kept parallel to each other at perpendicular separation r, carrying currents I_1 and I_2 respectively, in the opposite direction as shown in Figure.



The equal and opposite forces exerted by two parallel current carrying wires. Here the force is repulsive in nature.

The wire 2 (carrying current I_2) lies in the magnetic field of wire 1 $\left(B_1 = \dfrac{\mu_0 I_1}{2\pi r}, \otimes\right)$ as a result of which the wire 2 experiences a magnetic force. The magnetic force on a length l_2 of wire 2 exerted by the field B_1 of the wire 1 (at wire 2) is given by

$$\vec{F}_{21} = I_2\left(\vec{l}_2 \times \vec{B}_1\right) \qquad \{\text{away from wire 1}\}$$

where $\vec{l}_2$ is in the direction of I_2.

Since $\vec{l}_2 \perp \vec{B}_1$ (inwards), so we get

$$\left|\vec{F}_{21}\right| = F_{21} = I_2 B_1 l_2 \sin 90° = I_2 B_1 l_2$$

Also, $B_1 = \dfrac{\mu_0 I_1}{2\pi r}, \otimes$, so we get

$$\Rightarrow \quad F_{21} = I_2\left(\frac{\mu_0 I_1}{2\pi r}\right) l_2 \qquad \ldots(1)$$

The direction of $\vec{F}_{21}$ (towards wire 1) is found by using Fleming's Left Hand Rule.

Similarly, $\vec{F}_{12}$ on a length l_1 of the wire is

$$\vec{F}_{12} = I_1\left(\vec{l}_1 \times \vec{B}_2\right) \qquad \{\text{away from wire 2}\}$$

Since, we have $B_2 = \dfrac{\mu_0 I_2}{2\pi r}, \otimes$, so we get

$$\Rightarrow \quad \left|\vec{F}_{12}\right| = F_{12} = I_1\left(\frac{\mu_0 I_2}{2\pi r}\right) l_1 \qquad \ldots(2)$$

From equation (1) and (2), we conclude that the force per unit length on either of the wire is equal

$$\frac{F_{21}}{l_2} = \frac{F_{12}}{l_1} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

Also, we see that the two current carrying wires carrying current in the opposite direction repel each other with a force per unit length given by

$$\frac{F_1}{l_1} = \frac{F_2}{l_2} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

Definition of One Ampere

Since we know that

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

If $I_1 = I_2 = I$ (say), $r = 1$ metre and $\dfrac{F}{l} = 2\times 10^{-7}\ \text{Nm}^{-1}$, then we get

$$2\times 10^{-7} = 2\times 10^{-7} I^2 \quad \{\because \mu_0 = 4\pi\times 10^{-7}\ \text{TmA}^{-1}\}$$

$$\Rightarrow \quad I^2 = 1$$

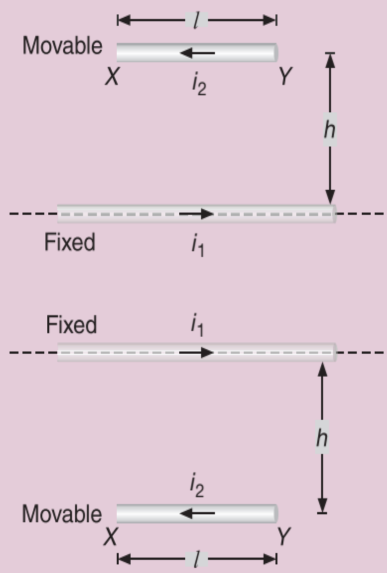
$$\Rightarrow \quad I = 1\ \text{A}$$

So, the current in the wire is one ampere when two identical current carrying wires placed at a separation of one metre, attract (or repel) each other with a force of 2×10^{-7} newton per metre of their length.

Problem Solving Technique(s)

(a) Equilibrium of a current carrying conductor

When a finite length current carrying wire is kept parallel to another infinite length current carrying wire, it can suspend freely in air as shown in Figure



In both the situations for equilibrium of XY, we have

$$\begin{pmatrix}\text{Weight of}\\ \text{Conductor}\end{pmatrix}=\begin{pmatrix}\text{Upward Magnetic}\\ \text{Force}\end{pmatrix}$$

$$\Rightarrow\ mg=\left(\frac{\mu_0 I_1 I_2}{2\pi h}\right)\ell$$

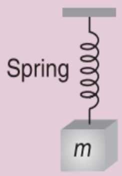
(b) In the first case if wire XY is slightly displaced from its equilibrium position, then it executes SHM and its time period is given by

$$T=2\pi\sqrt{\frac{h}{g}}$$

(c) If direction of current in movable wire is reversed then its instantaneous acceleration produced is $2g$ downwards.

(d) Current carrying spring

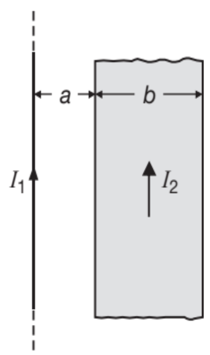
When current is passed through a spring, then it will contract, because the current flow in all the segments of the spring is in the same direction.



So, due to the current flow in the spring the spring contracts and the weight is lifted up.

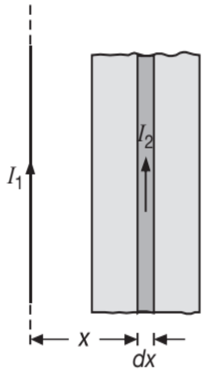
ILLUSTRATION 96

Two long thin parallel conductors of the shape shown in figure carry direct currents I_1 and I_2. The separation between the conductors is a, the width of the right hand conductor is equal to b as shown in Figure. With both conductors lying in one plane, find the magnetic interaction force between them reduced to a unit of their length.



SOLUTION

Consider an infinitesimal thin strip of "unit length" carrying a current dI as shown in Figure.



Then infinitesimal current dI flowing through this element is

$$dI=\left(\frac{I_2}{b}\right)dx$$

The field due to wire at the strip is

$$B=\frac{\mu_0 I_1}{2\pi x}$$

The force per unit length between the wire and the infinitesimal elemental wire is

$$dF=\frac{\mu_0 I_1 (dI)}{2\pi x}$$

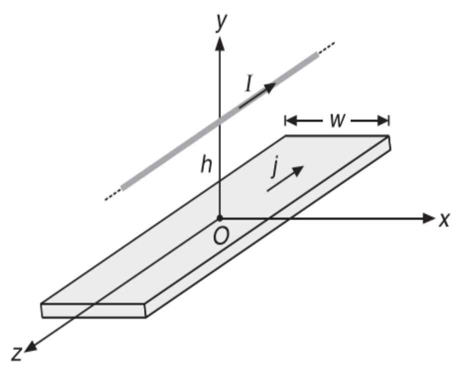
$$\Rightarrow\ dF=\frac{\mu_0 I_1 I_2}{2\pi b}\frac{dx}{x}$$

$$\Rightarrow\ F=\int dF=\frac{\mu_0 I_1 I_2}{2\pi b}\int_a^{a+b}\frac{dx}{x}$$

$$\Rightarrow\ F=\frac{\mu_0 I_1 I_2}{2\pi}\log_e\left(\frac{a+b}{a}\right)$$

ILLUSTRATION 97

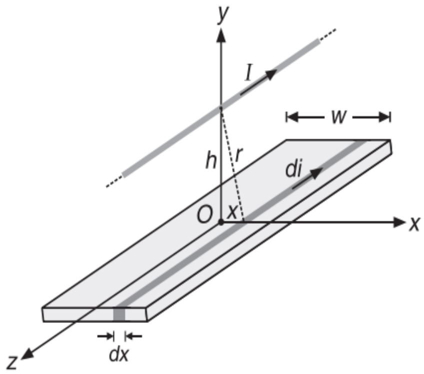
A very long straight conductor carries a current I is placed parallel to a current strip of width w having a uniform current per unit width j flowing throughout its width. The conductor is at a height h above the strip as shown in Figure.



Calculate the force per unit length on the conductor due to the strip. Also discuss the obtained result when the width w of the current sheet approaches infinity.

SOLUTION

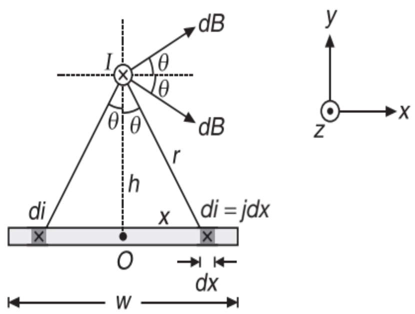
Let us consider an infinitesimal segment of sheet of thickness dx at a distance x to the right of origin as shown in Figure.



If di be the current flowing through this infinitesimal strip, then $di = jdx$. The force of attraction between the infinitesimal strip and the conductor of section length L is

$$dF = (dB)IL = \frac{\mu_0 (jdx)}{2\pi r} IL, \text{ where } r = \sqrt{h^2 + x^2}$$

Let us consider another infinitesimal segment of sheet of thickness dx at a distance x to the left of origin as shown in Figure.



Due to symmetry, the x-components of the forces i.e. $dF\sin\theta$ and $dF\sin\theta$ cancel. So, we get

$$F_{net} = \int dF\cos\theta = \int \frac{\mu_0 IL(jdx)}{2\pi r}\cos\theta,$$

where $\cos\theta = \frac{h}{\sqrt{h^2 + x^2}}$

Integrating over the width of the strip i.e. from $x = -\frac{w}{2}$ to $x = \frac{w}{2}$, we get

$$\Rightarrow \quad F_{net} = \int_{-\frac{w}{2}}^{\frac{w}{2}} \left(\frac{\mu_0 IjLdx}{2\pi\sqrt{h^2 + x^2}}\right)\left(\frac{h}{\sqrt{h^2 + x^2}}\right)$$

$$\Rightarrow \quad F_{net} = \frac{\mu_0 IjLh}{2\pi} \int_{-\frac{w}{2}}^{\frac{w}{2}} \frac{dx}{h^2 + x^2}$$

Since $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow \quad F_{net} = \frac{\mu_0 IjL}{\pi}\tan^{-1}\left(\frac{w}{2h}\right)$$

So the force of attraction per unit length of the conductor is given by

$$\frac{F_{net}}{L} = \frac{\mu_0 Ij}{\pi}\tan^{-1}\left(\frac{w}{2h}\right) \quad \ldots(1)$$

This force is attractive in nature as the currents in wire as well as the sheet are in the same direction.

When $w \to \infty$, then $\tan^{-1}\left(\frac{w}{2h}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

So, from equation (1), we get

$$\frac{F}{L} = \left(\frac{\mu_0 Ij}{\pi}\right)\left(\frac{\pi}{2}\right) = \frac{\mu_0 Ij}{2}$$

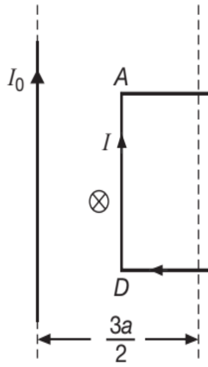
ILLUSTRATION 98

A square frame carrying a current I is located in the same plane as a long straight wire carrying a current I_0. The frame side has a length a. The axis of the frame passing the midpoints of opposite sides is parallel to the wire and is separated from it by the distance which is $\eta = 1.5$ times greater than the side of the frame. Find

(a) the ampere force acting on the frame.
(b) the mechanical work to be performed in order to turn the frame through 180° about its axis, with currents maintained constant.

SOLUTION

The force per unit length on the different sections of the square wire frame due to the straight current carrying wire is given by



$$\vec{F}_{AB}=\left(\frac{\mu_0 I I_0}{2\pi a}\right)a \quad \{\text{attractive}\}$$

$$\vec{F}_{BC}=\frac{\mu_0 I I_0}{2\pi(2a)}a \quad \{\text{repulsive}\}$$

On the sections AB and CD of the square frame, the magnetic force due to the wire has same magnitude but opposite direction, hence the net force on the square wire frame is given by

$$\Rightarrow \quad F_{\text{net}}=\frac{\mu_0 I I_0}{2\pi}\left(1-\frac{1}{2}\right) \quad \{\text{attractive}\}$$

$$\Rightarrow \quad F_{\text{net}}=\frac{\mu_0 I I_0}{4\pi} \quad \{\text{attractive}\}$$

Since, $W=U_f-U_i$

Ley us now find the initial and the final magnetic potential energy. For calculating this let us consider a very small strip element of length a, thickness dx i.e. area $dA=adx$, such that magnetic moment dM due to this small elemental strip is given by

$$dM=I(dA)=I(adx)$$

$$\Rightarrow \quad dU_i=-(dM)B\cos(0^\circ)$$

$$\Rightarrow \quad dU_i=-I(adx)\frac{\mu_0 I_0}{2\pi x}$$

$$\Rightarrow \quad U_i=-\frac{\mu_0 I I_0 a}{2\pi}\int_a^{2a}\frac{dx}{x}$$

$$\Rightarrow \quad U_i=-\frac{\mu_0 I I_0 a}{2\pi}\log_e(2)$$

Similarly, we get

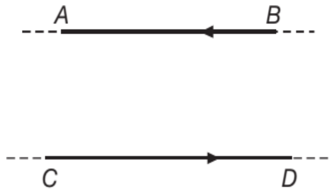
$$U_f=\int dU_f=-\int(dM)B\cos(180^\circ)=(dM)B$$

$$\Rightarrow \quad U_f=\frac{\mu_0(I I_0)a}{2\pi}\log_e(2)$$

$$\Rightarrow \quad W=|U_f-U_i|=\frac{\mu_0(I I_0)a}{\pi}\log_e(2)$$

ILLUSTRATION 99

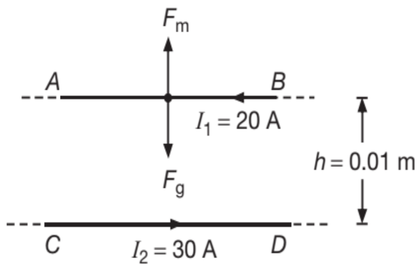
A long horizontal wire AB, which is free to move in a vertical plane and carries a steady current of 20 A, is in equilibrium at a height of 0.01 m over another parallel long wire CD which is fixed in a horizontal plane and carries a steady current of 30 A, as shown in Figure.



Show that when AB is slightly depressed, it executes simple harmonic motion. Find the period of oscillations.

SOLUTION

Let λ be the mass per unit length of wire AB. At a height h above the wire CD, magnetic force per unit length on wire AB will be acting upwards as shown in Figure.



The magnitude of this magnetic fore is given by

$$F_m=\frac{\mu_0}{2\pi}\frac{I_1 I_2}{h}\text{, upwards} \quad \ldots(1)$$

Also, weight per unit length of wire AB is

$$F_g=\left(\frac{m}{l}\right)g=\lambda g\text{, downwards}$$

If the wire is in equilibrium, say at $x=d$, then the magnetic force must balance the weight of the wire. So,

$$F_m=F_g$$

$$\Rightarrow \quad \frac{\mu_0}{2\pi}\frac{I_1 I_2}{h}=\lambda g \quad \ldots(2)$$

When AB is depressed through x, then h decreases therefore, F_m will increase, while F_g remains the same due to which the net restoring force F acting on a portion of length l of wire AB is given by

$$F = \frac{\mu_0 I_1 I_2 l}{2\pi(h-x)} - \lambda l g = \frac{\mu_0 I_1 I_2 l}{2\pi h}\left(1-\frac{x}{h}\right)^{-1} - \lambda l g$$

$$\Rightarrow \quad F = \frac{\mu_0 I_1 I_2 l}{2\pi h}\left(1+\frac{x}{h}\right) - \lambda l g = \frac{\mu_0 I_1 I_2 l}{2\pi h^2}x \qquad \{\because \text{ of } (2)\}$$

$$\Rightarrow \quad m\ddot{x} = -\left(\frac{\mu_0 I_1 I_2 l}{2\pi h^2}\right)x$$

Negative sign indicates the restoring nature of the force.

$$\Rightarrow \quad m\ddot{x} = -\left(\frac{\lambda g l}{h}\right)x \qquad \{\because \text{ of } (2)\}$$

However, we have $\lambda l = m$

$$\Rightarrow \quad \ddot{x} + \frac{g}{h}x = 0$$

Comparing with standard equation of SHM, i.e.

$$\ddot{x} + \omega^2 x = 0$$

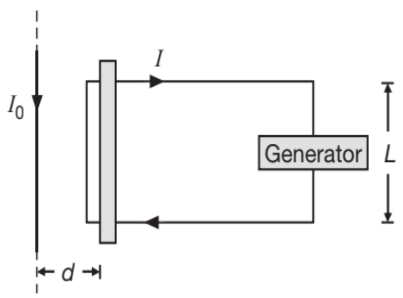
$$\Rightarrow \quad \omega = \sqrt{\frac{g}{h}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{0.01}{9.8}}$$

$$\Rightarrow \quad T = 0.2 \text{ s}$$

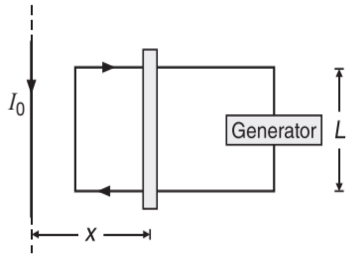
ILLUSTRATION 100

A bar of mass m slides on a smooth horizontal conducting rail. A constant current I is maintained in the rod using a constant current generator as shown. There is a long straight conductor carrying current I_0 parallel to the rod. Find the speed of the bar after it has travelled a distance D.



SOLUTION

Consider an instant when the sliding bar is at a distance x from the straight long conductor as shown in Figure.



The magnetic force acting on the sliding bar at this instant is given by

$$F = \frac{\mu_0 I I_0 L}{2\pi x}$$

Since $F = ma = m\left(v\frac{dv}{dx}\right)$

$$\Rightarrow \quad mv\frac{dv}{dx} = \frac{\mu_0 I I_0 L}{2\pi x}$$

$$\Rightarrow \quad v\,dv = \frac{\mu_0 I I_0 L}{2m}\frac{dx}{x}$$

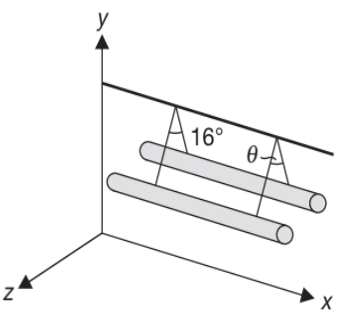
$$\Rightarrow \quad \int_0^v v\,dv = \frac{\mu_0 I I_0 L}{2\pi m}\int_d^{d+D}\frac{dx}{x}$$

$$\Rightarrow \quad \frac{v^2}{2} = \frac{\mu_0 I I_0 L}{2\pi m}\log_e\left(1+\frac{D}{d}\right)$$

$$\Rightarrow \quad v = \sqrt{\frac{\mu_0 I I_0 L \log_e\left(1+\frac{D}{d}\right)}{\pi m}}$$

ILLUSTRATION 101

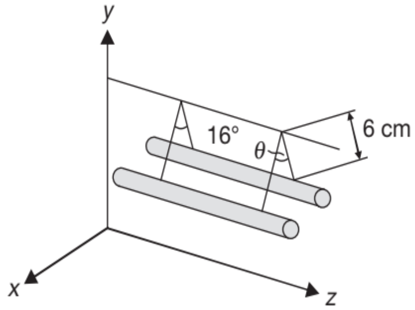
Two long, parallel wires, each having a mass per unit length of 40 gm^{-1}, are supported in a horizontal plane by strings 6 cm long. When both wires carry the same current I, the wires repel each other so that the angle θ between the supporting strings is $16°$. Find the magnitude and the directions of the currents.



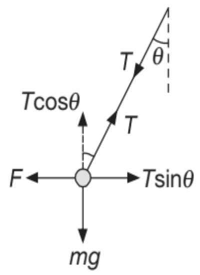
SOLUTION

The separation between the wires is

$$r = 2(6\text{ cm})\sin 8° = 1.67\text{ cm}$$



Since the wires repel so, the currents are in the opposite directions.



Cross Sectional View

The magnetic force F acts horizontally, so we get

$$T\cos\theta = mg \quad \ldots(1)$$

$$T\sin\theta = F = \frac{\mu_0 I^2 \ell}{2\pi r}$$

$$\Rightarrow \tan\theta = \frac{\mu_0 I^2 \ell}{(2\pi r)mg}$$

$$\Rightarrow \tan(8°) = \frac{\mu_0 I^2 \ell}{2\pi rmg}$$

$$\Rightarrow I^2 = \frac{mg2\pi r}{\ell\mu_0}\tan(8°)$$

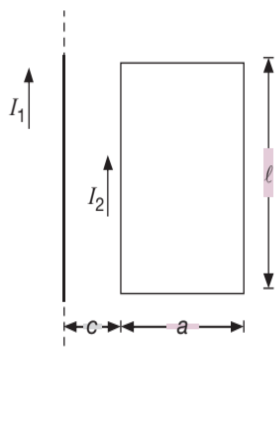
$$\Rightarrow I = 67.8\text{ A}$$

Test Your Concepts-VIII

Based on Force between Current Carrying Conductors

(Solutions on page H.27)

1. In figure, the current in the long, straight wire is $I_1 = 5$ A and the wire lies in the plane of the rectangular loop, which carries the current $I_2 = 10$ A. The dimensions are $c = 0.1$ m, $a = 0.15$ m, and $\ell = 0.45$ m. Find the magnitude and direction of the net force exerted on the loop by the magnetic field created by the wire.

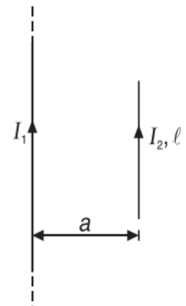


2. Two long, parallel conductors, separated by 10 cm, carry currents in the same direction. The first wire carries current $I_1 = 5$ A and the second carries $I_2 = 8$ A.
- **(a)** What is the magnitude of the magnetic field created by I_1 at the location of I_2?
- **(b)** What is the force per unit length exerted by I_1 on I_2?
- **(c)** What is the magnitude of the magnetic field created by I_2 at the location of I_1?
- **(d)** What is the force per length exerted by I_2 on I_1?

3. Two long, parallel wires are attracted to each other by a force per unit length of 320 μNm^{-1} when they are separated by a vertical distance of 0.5 m. The current in the upper wire is 20 A to the right. Determine the location of the line in the plane of the two wires along which the total magnetic field is zero.

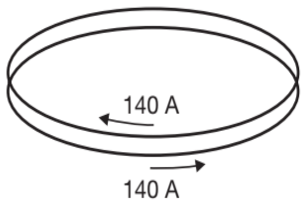
4. Three long wires (wire 1, wire 2, and wire 3) hang vertically. The distance between wire 1 and wire 2 is 20 cm. On the left, wire 1 carries an upward current of 1.5 A. To the right, wire 2 carries a downward current of 4 A. Wire 3 is located such that when it carries a certain current, each wire experiences no net force. Find the position of wire 3. Also find the magnitude and direction of the current in wire 3.

5. A fixed long straight conductor carrying current I_1 is placed at a distance a from another conductor of length l carrying current I_2 as shown in Figure.



Calculate the minimum work required to be done to increase the separation between the conductors through a distance b.

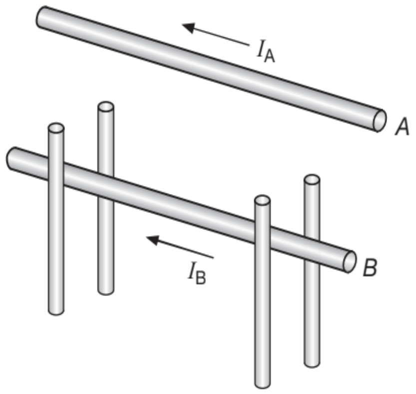
6. Two circular current carrying loops, each of radius 10 cm, are parallel, coaxial, and almost in contact, 1 mm apart. The top loop carries a clockwise current of 140 A. The bottom loop carries a counter clockwise current of 140 A.



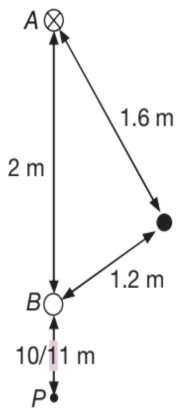
(a) Calculate the magnetic force exerted by the bottom loop on the top loop.

(b) The upper loop has a mass of 0.021 kg. Calculate its acceleration, assuming that the only forces acting on it are the magnetic force and the gravitational force. $\left(\text{Take } \pi = \frac{22}{7} \text{ and } g = 10 \text{ ms}^{-2}\right)$

7. Two long, parallel conductors carry currents in the same direction as shown. Conductor A carries a current of 150 A and is held firmly in position. Conductor B carries a current I_B and is allowed to slide freely up and down (parallel to A) between a set of non-conducting guides. If the mass per unit length of conductor B is 0.1 gcm^{-1}, what value of current I_B will result in equilibrium when the distance between the two conductors is 2.5 cm?



8. Two long straight parallel wires are 2 m apart, perpendicular to the plane of the paper. The wire A carries a current of 9.6 A, directed into the plane of the paper as shown in Figure.



The wire B carries a current such that the magnetic field of induction at the point P, at a distance of $\frac{10}{11}$ m from the wire B, is zero. Calculate the force per unit length on the wire B.

SOLVED PROBLEMS

PROBLEM 1

A particle having charge $-q$ and mass $m = 2.58 \times 10^{-15}$ kg is travelling through a region containing a uniform magnetic field $\vec{B} = -(0.12\text{ T})\hat{k}$. At a particular instant of time the velocity of the particle is $\vec{v} = (1.05\times10^6 \text{ ms}^{-1})(-3\hat{i}+4\hat{j}+12\hat{k})$ and the force $\vec{F}$ on the particle has a magnitude of 1.25 N.

(a) Calculate the charge q.

(b) Calculate the acceleration $\vec{a}$ of the particle.

(c) Explain why the path of the particle is a helix, and determine the radius of curvature R of the circular component of the helical path.

(d) Determine the cyclotron frequency of the particle.

(e) Although helical motion is not periodic in the full sense of the word, the x and y-coordinates do vary in a periodic way. If the coordinates of the particle at $t = 0$ are $(x, y, z) = (R, 0, 0)$, determine its coordinates at a time $t = 2T$, where T is the period of the motion in the xy-plane.

SOLUTION

(a) Given that,

$$\vec{B} = -(0.12\text{ T})\hat{k},$$

$$\vec{v} = (1.05\times10^6 \text{ ms}^{-1})(-3\hat{i}+4\hat{j}+12\hat{k}),$$

$$F = 1.25\text{ N}$$

Since $\vec{F} = q(\vec{v}\times\vec{B})$

$$\Rightarrow \vec{F} = q(1.05\times10^6)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 4 & 12\\ 0 & 0 & -0.12\end{vmatrix}$$

$$\Rightarrow \vec{F} = -q(1.26\times10^5)(4\hat{i}+3\hat{j}) \quad \ldots(1)$$

The magnitude of the vector $(4\hat{i}+3\hat{j})$ is

$$\sqrt{3^2+4^2} = 5$$

$$\Rightarrow F = -q(1.26\times10^5 \text{ NC}^{-1})(5)$$

$$\Rightarrow q = -\frac{F}{5(1.26\times10^5 \text{ NC}^{-1})}$$

$$\Rightarrow q = -\frac{1.25\text{ N}}{5(1.26\times10^5 \text{ NC}^{-1})}$$

$$\Rightarrow q = -1.98\times10^{-6}\text{ C} = -1.98\ \mu\text{C}$$

(b) Since, $\Sigma\vec{F} = m\vec{a}$, so $\vec{a} = \dfrac{\vec{F}}{m}$.

From equation (1), we have

$$\vec{F} = -q(1.26\times10^5)(4\hat{i}+3\hat{j})$$

$$\Rightarrow \vec{F} = -(-1.98\times10^{-6}\text{ C})(1.26\times10^5)(4\hat{i}+3\hat{j})$$

$$\Rightarrow \vec{F} = +0.25(4\hat{i}+3\hat{j})$$

$$\Rightarrow \vec{a} = \frac{\vec{F}}{m} = \left(\frac{0.25}{2.58\times10^{-15}}\right)(4\hat{i}+3\hat{j})$$

$$\Rightarrow \vec{a} = (9.69\times10^{13})(4\hat{i}+3\hat{j})\text{ ms}^{-2}$$

(c) Since, $\vec{F}$ is in the xy-plane, so in the z-direction the particle moves with constant speed $12.6\times10^6 \text{ ms}^{-1}$. In the xy-plane the force $\vec{F}$ causes the particle to move in a circle, with $\vec{F}$ directed in towards the centre of the circle.

$$\Sigma\vec{F} = m\vec{a}$$

$$\Rightarrow F = \frac{mv^2}{R}$$

$$\Rightarrow R = \frac{mv^2}{F}, \text{ where } v^2 = v_x^2 + v_y^2$$

$$\Rightarrow v^2 = (-3.15\times10^6)^2 + (4.2\times10^6)^2$$

$$\Rightarrow v^2 = 2.756\times10^{13}\text{ m}^2\text{s}^{-2}$$

$$\Rightarrow v = 5.25\times10^6 \text{ ms}^{-1}$$

From equation (1), we have

$$F = \sqrt{F_x^2+F_y^2} = (0.25)\sqrt{4^2+3^2} = 1.25\text{ N}$$

$$\Rightarrow R = \frac{mv^2}{F} = \frac{(2.58\times10^{-15})(2.756\times10^{13})}{1.25}$$

$$\Rightarrow R = 0.0569\text{ m} = 5.69\text{ cm}$$

(d) Since the cyclotron frequency is $f = \dfrac{qB}{2\pi m}$

$$\Rightarrow f = 1.47\times10^7\text{ Hz and}$$

$$\Rightarrow \omega = 2\pi f = 9.23\times10^7\text{ rads}^{-1}$$

(e) Compare t to the period T of the circular motion in the xy-plane to find the x and y coordinates at this t. In the z-direction the particle moves with constant speed, so $z = z_0 + v_z t$.

The period of the motion in the xy-plane is given by

$$T = \frac{1}{f} = \frac{1}{1.47\times10^{7}\ \text{Hz}} = 6.8\times10^{-8}\ \text{s}$$

For $t = 2T$, the particle returns to the same x and y coordinates. The motion along the z-direction is the motion with constant velocity v_z given by

$$v_z = (1.05\times10^6)(12) = 12.6\times10^6\ \text{ms}^{-1}$$

$$\Rightarrow\ z = z_0 + v_z t$$

$$\Rightarrow\ z = 0 + (12.6\times10^6\ \text{ms}^{-1})(2)(6.8\times10^{-8}\ \text{s})$$

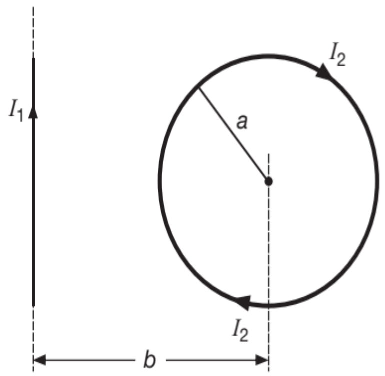
$$\Rightarrow\ z = +1.71\ \text{m}$$

So, the coordinates at $t = 2T$ are $x = R,\ y = 0,\ z = +1.71$ m i.e., $(R,\ 0,\ 1.71)$ metre

Please note that the circular motion is in the plane perpendicular to $\vec{B}$. The radius of this motion gets smaller when B increases and it gets larger when v increases. There is no magnetic force in the direction of $\vec{B}$ so the particle moves with constant velocity in that direction. The superposition of circular motion in the xy-plane and constant speed motion in the z-direction is a helical path.

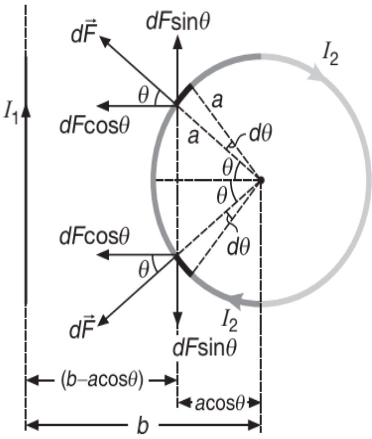
PROBLEM 2

A circular loop of radius a carrying a current I_2 is placed near an infinitely long straight wire carrying a current I_1 as shown. The wire and the loop are coplanar. The distance of the centre of the loop from the wire is b. Find the force of interaction between the wire and the loop.



SOLUTION

For this type of problem, let us divide the circular loop in two equal halves and then proceed. Figure here shows the one half of the loop. We observe that on taking the symmetrical locations of the infinitesimal elements, the sine components cancel. So, the net force on this left half is attractive and equals



$$F_{\text{left half}} = \int dF\cos\theta \qquad \text{\{towards the wire\}}$$

where, $dF = B_1 I_2 (dl)$

Since $B_1 = \dfrac{\mu_0 I_1}{2\pi r_\perp}$, $r_\perp = b - a\cos\theta$ and $dl = ad\theta$

$$\Rightarrow\ dF = B_1 I_2 (dl) = \frac{\mu_0 I_1 I_2 (ad\theta)}{2\pi(b - a\cos\theta)}$$

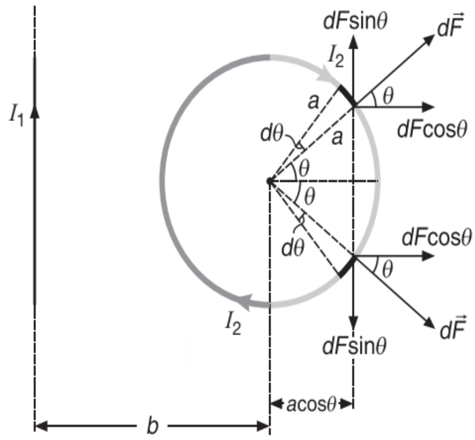
$$\Rightarrow\ F_{\text{left half}} = F_1 = \frac{\mu_0 I_1 I_2 a}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{b - a\cos\theta}$$

$$\Rightarrow\ F_1 = \frac{\mu_0 I_1 I_2 a}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{b - a\cos\theta}$$

$$\Rightarrow\ F_{\text{left half}} = F_1 = \frac{\mu_0 I_1 I_2 a}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{b - a\cos\theta} \quad \ldots(1)$$

{towards the wire}

Similarly, we get for the other half (right half)



$$F_{\text{right half}} = \int dF'\cos\theta \qquad \{\text{away from the wire}\}$$

where $dF' = B_1' I_2 dl$

$$B_1' = \frac{\mu_0 I_1}{2\pi r_\perp'},\ r_\perp' = b + a\cos\theta \text{ and } dl = ad\theta$$

$$\Rightarrow dF' = B_1' I_2 dl = \frac{\mu_0 I_1 I_2 (ad\theta)}{2\pi(b + a\cos\theta)}$$

$$\Rightarrow F_{\text{right half}} = F_2 = \frac{\mu_0 I_1 I_2 a}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{(b + a\cos\theta)}$$

$$\Rightarrow F_2 = \frac{\mu_0 I_1 I_2 a}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{b + a\cos\theta}$$

$$\Rightarrow F_{\text{right half}} = F_2 = \frac{\mu_0 I_1 I_2 a}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{(b + a\cos\theta)} \qquad \ldots(2)$$

{away from wire}

Please note that $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ when $f(x) = f(-x)$, has been used to simplify the integrals (1) and (2).

So, we have

$$\vec{F}_{\text{loop}} = \vec{F}_{\text{left half}} + \vec{F}_{\text{right half}}$$

$$\Rightarrow F_{\text{loop}} = F_1 - F_2 \qquad \{\text{towards the wire}\}$$

where F_1 and F_2 are to be calculated from (1) and (2).

Now $$\int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{b - a\cos\theta} = \frac{1}{a}\left(\frac{2b}{\sqrt{b^2 - a^2}}\tan^{-1}\left(\sqrt{\frac{b+a}{b-a}}\right) - \frac{\pi}{2}\right)$$

and $$\int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{b + a\cos\theta} = \frac{1}{a}\left(\frac{\pi}{2} - \frac{2b}{\sqrt{b^2 - a^2}}\tan^{-1}\left(\sqrt{\frac{b-a}{b+a}}\right)\right)$$

$$\Rightarrow F_1 = \frac{\mu_0 I_1 I_2}{\pi}\left(\frac{2b}{\sqrt{b^2 - a^2}}\tan^{-1}\left(\sqrt{\frac{b+a}{b-a}}\right) - \frac{\pi}{2}\right)$$

and $$F_2 = \frac{\mu_0 I_1 I_2}{\pi}\left(\frac{\pi}{2} - \frac{2b}{\sqrt{b^2 - a^2}}\tan^{-1}\left(\sqrt{\frac{b-a}{b+a}}\right)\right)$$

$$\Rightarrow F = F_1 - F_2$$

$$\Rightarrow F = \frac{\mu_0 I_1 I_2}{\pi}\left[\frac{2b}{\sqrt{b^2 - a^2}}\left(\tan^{-1}\sqrt{\frac{b+a}{b-a}} + \tan^{-1}\sqrt{\frac{b-a}{b+a}}\right) - \pi\right]$$

But we know that

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}$$

$$\Rightarrow F = F_1 - F_2 = \frac{\mu_0 I_1 I_2}{\pi}\left(\frac{2b}{\sqrt{b^2 - a^2}}\frac{\pi}{2} - \pi\right)$$

$$\Rightarrow F = \mu_0 I I_2\left(\frac{b}{\sqrt{b^2 - a^2}} - 1\right) \qquad \{\text{towards the wire}\}$$

PROBLEM 3

A parallel plate capacitor with area of each plate equal to A and the separation between them to d is put into a stream of conducting liquid with resistivity ρ. The liquid moves parallel to the plates with a constant velocity v. The whole system is located in a uniform magnetic field of induction B, parallel to the plates and perpendicular to the stream direction. The capacitor plates are interconnected by means of an external resistance R.

(a) What amount of power is generated in that resistance?

(b) At what value of R is the generated power the highest?

(c) What is this highest power equal to?

SOLUTION

Resistance of the liquid in between the plates is $\frac{\rho d}{A}$

Voltage applied across the plates is $V = Ed$

For constant velocity of the liquid to be maintained under the simultaneous influence of electric and magnetic field, we must have

$$v = \frac{E}{B}$$

$$\Rightarrow V = (vB)d \qquad \ldots(1)$$

If I be the current through the plates then

$$I = \frac{V}{R_{\text{total}}} = \frac{vBd}{R + \frac{\rho d}{A}}$$

(a) So, power generated in the external resistance R is

$$P = I^2 R = \frac{v^2 B^2 d^2 R}{\left(R + \frac{\rho d}{A}\right)^2}$$

(b) $\frac{dP}{dR} = 0$, for P to be MAXIMUM

$$\Rightarrow \frac{d}{dR}\left\{R\left(R + \frac{\rho d}{A}\right)^{-2}\right\} = 0$$

$$\Rightarrow \left(R + \frac{\rho d}{A}\right)^{-2} - 2R\left(R + \frac{\rho d}{A}\right)^{-3} = 0$$

$$\Rightarrow 1 - \frac{2R}{R + \frac{\rho d}{A}} = 0$$

$$\Rightarrow 2R = R + \frac{\rho d}{A}$$

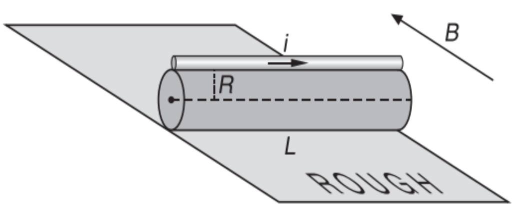
$$\Rightarrow R = \frac{\rho d}{A}$$

(c) $P_{\text{MAX}} = \frac{v^2 B^2 d^2 R}{4R^2} = \frac{v^2 B^2 d^2}{4R}$ where $R = \frac{\rho d}{A}$

$$\Rightarrow P_{\text{MAX}} = \frac{v^2 B^2 d^2}{4\left(\frac{\rho d}{A}\right)} = \frac{v^2 B^2 A d}{4\rho}$$

PROBLEM 4

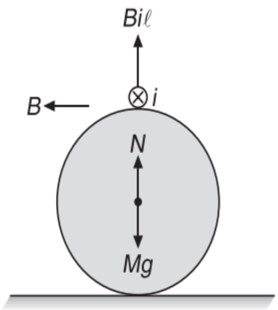
A wooden cylinder of mass M, length L and radius R is placed on a rough horizontal surface. A massless current carrying wire of length L is fixed on its curved surface parallel to its length. There exists a uniform magnetic field B along horizontal as shown in Figure.



Find the various forces acting on the cylinder and draw the free body diagram of cylinder. Assuming that the wooden cylinder does not slip on the ground, find the time period of resulting motion when it is slightly disturbed from the equilibrium position.

SOLUTION

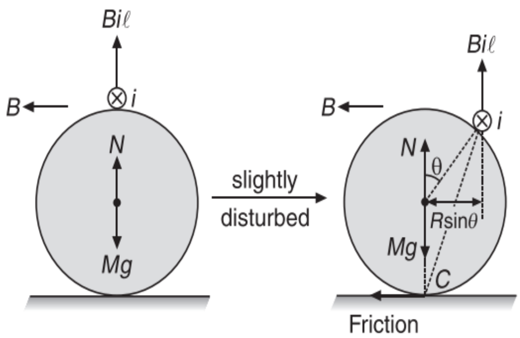
The forces acting on the cylinder are shown in figure.



For equilibrium, we have

$$Mg = N + Bil$$

$$\Rightarrow \quad N = Mg - Bil$$



When there is no slipping at point of contact, C is instantaneous centre of rotation.

Torque about the point C is

$$\tau = (Bil)(R\sin\theta)$$

For small θ, $\sin\theta \approx \theta$

$$\Rightarrow \quad \tau = BilR\theta$$

Since, $\alpha = \frac{\tau}{I}$, where $I = I_{cm} + MR^2$

$$\Rightarrow \quad \alpha = \frac{BilR\theta}{\left(\frac{1}{2}MR^2 + MR^2\right)}$$

$$\Rightarrow \quad \alpha = \frac{2Bil}{3MR}\theta$$

Also, we see that, the angular displacement θ is clockwise, whereas the angular acceleration $\alpha = \ddot{\theta}$ is counter-clockwise, so

$$\Rightarrow \quad \alpha = -\frac{2Bil}{3MR}\theta$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{2Bil}{3MR}\theta$$

$$\Rightarrow \quad \ddot{\theta} + \left(\frac{2Bil}{3MR}\right)\theta = 0$$

Comparing with standard equation of SHM i.e., $\ddot{\theta} + \omega^2\theta = 0$, we get

$$\omega^2 = \frac{2Bil}{3MR}$$

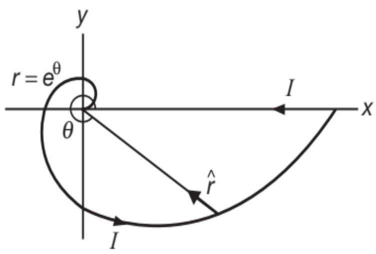
$$\Rightarrow \quad \omega = \sqrt{\frac{2Bil}{3MR}}$$

Since $T = \frac{2\pi}{\omega}$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{3MR}{2Bil}}$$

PROBLEM 5

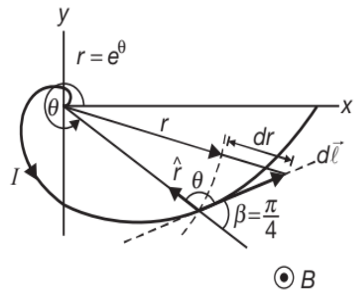
A wire carrying a current I is bent into the shape of an exponential spiral, $r = e^{\theta}$, from $\theta = 0$ to $\theta = 2\pi$ as shown in figure. To complete a loop, the ends of the spiral are connected by a straight wire along the x-axis. Find the magnitude and direction of $\vec{B}$ at the origin.



SOLUTION

Before we start with the problem, we must know and keep in mind that the angle β between a radial line and its tangent line at any point on the curve $r = f(\theta)$ are related to the function as

$$\tan\beta = \frac{r}{\frac{dr}{d\theta}}$$



Thus in this case, we have $r = e^{\theta}$ and so we get $\tan\beta = 1$ and $\beta = \frac{\pi}{4}$. Therefore, the angle between $d\vec{l}$ and $\hat{r}$ is $(\pi - \beta) = \frac{3\pi}{4}$. Also

$$d\vec{l} = \frac{dr}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}dr$$

From Biot-Savart's Law, we know that there is no contribution from the straight portion of the wire since $d\vec{l} \times \vec{r} = 0$. For the field of the spiral, we have

$$dB = \frac{\mu_0 I}{4\pi}\frac{(d\vec{l} \times \hat{r})}{r^2}$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{4\pi}\int_{\theta=0}^{2\pi}\frac{|d\vec{l}|\sin\theta|\hat{r}|}{r^2}$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{4\pi}\int_{\theta=0}^{2\pi}\sqrt{2}dr\left[\sin\left(\frac{3\pi}{4}\right)\right]\frac{1}{r^2}$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{4\pi}\int_{\theta=0}^{2\pi} r^{-2}dr = -\frac{\mu_0 I}{4\pi}\left(r^{-1}\right)\Big|_{\theta=0}^{2\pi}$$

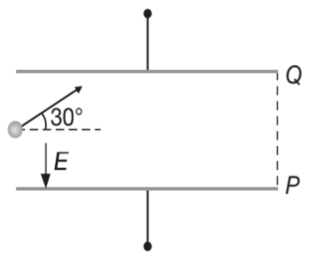
Substitute $r = e^{\theta}$ we get

$$B = -\frac{\mu_0 I}{4\pi}\left(e^{-\theta}\right)\Big|_0^{2\pi} = -\frac{\mu_0 I}{4\pi}\left(e^{-2\pi} - e^0\right)$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{4\pi}\left(1 - e^{-2\pi}\right) \text{ out of the page.}$$

PROBLEM 6

A charged particle having charge 10^{-6} C and mass 10^{-10} kg is fired from the middle of the plate making an angle 30° with plane of the plate. Length of the plate is 0.17 m and it is separated by 0.1 m. An electric field $E = 10^{-3}$ NC^{-1} is present between the plates and just outside the plates a magnetic field is present. Find the velocity of projection of charged particle and magnitude of the magnetic field perpendicular to the plane of the figure, if it has to graze the plate at Q and P parallel to the surface of the plate. (Neglect gravity).



SOLUTION

The acceleration of the particle is given by

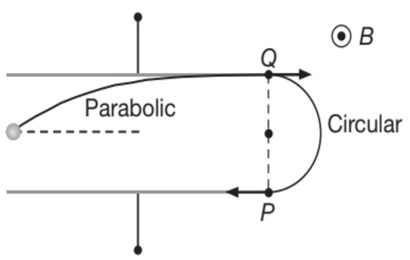
$$a = \frac{qE}{m} = \frac{(10^{-6})(10^{-3})}{(10^{-10})} = 10 \text{ ms}^{-2}\text{, downwards}$$

Path of the particle is a projectile. The particle will graze at Q when length of the plates equals half the range i.e. $l = \frac{R}{2}$

$$\Rightarrow \quad l = \frac{R}{2} = \frac{u^2 \sin 2\theta}{2g}$$

$$\Rightarrow \quad 0.17 = \frac{u^2 \sin(2\theta)}{2g} = \frac{u^2 \sin 60°}{20}$$

$$\Rightarrow \quad u = 1.98 \text{ ms}^{-1}$$



Speed of particle at Q is,

$$v = u\cos\theta = 1.98\cos 30° = 1.7 \text{ ms}^{-1}$$

It will graze at P if PQ is twice the radius of circular path followed by the particle in magnetic field i.e.

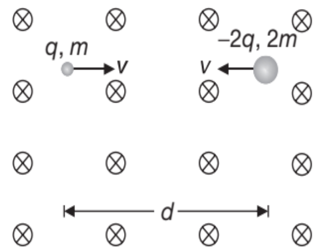
$$PQ = 2R = 2\left(\frac{mv}{qB}\right)$$

$$\Rightarrow \quad 0.1 = 2\left(\frac{mv}{qB}\right)$$

$$\Rightarrow \quad B = \frac{2\times 10^{-10}\times 1.7}{0.1\times 10^{-6}} = 3.4\times 10^{-3} \text{ T} = 3.4 \text{ mT}$$

PROBLEM 7

A charged particle $+q$ of mass m is placed at a distance d from another charged particle $-2q$ of mass 2 m in a uniform magnetic field B as shown in figure. If the particles are projected towards each other with equal speeds v.



(a) Find the maximum value of projection speed v_m so that the two particles do not collide.

(b) Find the time after which collision occurs between the particles if projection speed equals $2v_m$.

(c) Assuming the collision to be perfectly inelastic find the radius of particle in subsequent motion. (Neglect the electric force between the charges).

SOLUTION

(a) The particles will move in circular paths, as velocity vector is perpendicular to magnetic field. Time period of both the particles is same $\left(T = \frac{2\pi m}{qB}\right)$. So, for collision not to take place, we must have

$$r_1 + r_2 < d$$

$$\Rightarrow \quad \frac{mv}{qB} + \frac{2mv}{2qB} < d$$

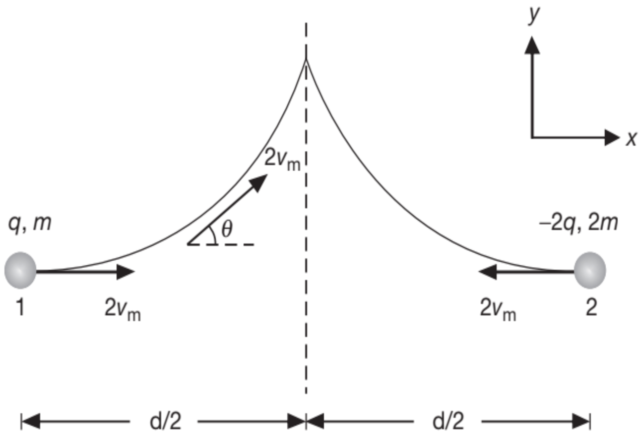
$$\Rightarrow \quad v < \frac{qBd}{2m}$$

Therefore, maximum speed should be $\frac{qBd}{2m}$

$$\Rightarrow \quad v_{\max} = v_m = \frac{qBd}{2m}$$

(b) From symmetry, we observe that they collide at $\frac{d}{2}$.

$$\Rightarrow \theta = \omega t = \left(\frac{qB}{m}\right)t$$



x-component of velocity at time t is,

$$v_x = 2v_m \cos\theta = 2v_m \cos\left(\frac{qB}{m}\right)t$$

$$\Rightarrow \frac{d}{2} = \int_0^t v_x dt = 2v_m \int_0^t \cos\left(\frac{qB}{m}t\right)dt$$

$$\Rightarrow \frac{d}{2} = \frac{2mv_m}{qB}\left[\sin\left(\frac{qB}{m}t\right)\right]_0^t$$

$$\Rightarrow \frac{d}{2} = \frac{2mv_m}{qB}\sin\left(\frac{qB}{m}t\right)$$

Substituting, $v_{max} = v_m = \frac{qBd}{2m}$, we get

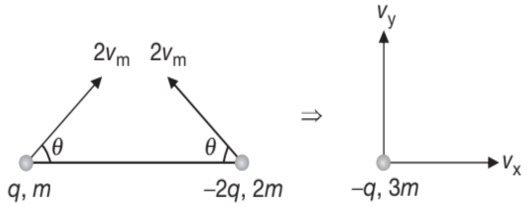
$$\sin\left(\frac{qB}{m}t\right) = \frac{1}{2}$$

$$\Rightarrow \frac{qBt}{m} = \frac{\pi}{6}$$

$$\Rightarrow t = \frac{m\pi}{6qB}$$

(c) At the time of collision

$$\theta = \omega t = \left(\frac{qB}{m}\right)\left(\frac{m\pi}{6qB}\right) = \frac{\pi}{6} = 30^\circ$$



After the collision

$$v_x = \frac{2mv_m \cos 30^\circ - 4mv_m \cos 30^\circ}{3m} = -\frac{v_m}{\sqrt{3}}$$

$$v_y = \frac{2mv_m \sin 30^\circ + 4mv_m \sin 30^\circ}{3m} = v_m$$

So, speed of combined mass is

$$v = \sqrt{v_x^2 + v_y^2} = \frac{2}{\sqrt{3}}v_m$$

$$\Rightarrow R = \frac{(3m)(v)}{qB} = \frac{(3m)\left(\frac{2}{\sqrt{3}}v_m\right)}{qB} = \frac{2\sqrt{3}mv_m}{qB}$$

Substituting, $v_m = \frac{qBd}{2m}$, we get

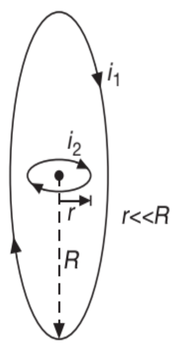
$$R = \sqrt{3}\,d$$

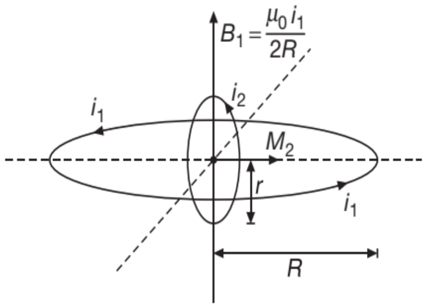
PROBLEM 8

A coil of radius R, mass m_1 carries a current i_1. Another concentric coil of radius $r(\ll R)$, mass m_2 carries a current i_2. The planes of two coils are mutually perpendicular and both the coils are free to rotate about common diameter. Calculate the maximum kinetic energy attained by the two coils, when both are released from rest.

SOLUTION

Let us first draw the arrangement described for a clear visualisation of the problem.





The magnetic field due to coil of bigger radius R is

$$B_1 = \frac{\mu_0 i_1}{2R}$$

This field exerts a torque on smaller coil due to which an equal and opposite torque is exerted by the shorter coil on the larger coil.

Since no external torque is acting on the system, so the angular momentum remains conserved. Also, the total mechanical energy of the system remains conserved.

Now, when the system is released, the magnetic dipole starts rotating to attain equilibrium position and the magnetic energy is converted into kinetic energy of both the coils.

The kinetic energy is maximum when it reaches the equilibrium position or the kinetic energies are maximum when both the coils become coplanar. The two coils rotate due to their mutual interaction only and if one coil rotates clockwise, then the other coil rotates anticlockwise.

Let I_1 and I_2 be the moment of inertia of the respective coils about the axis of rotation. Let ω_1 and ω_2 be the angular velocities of larger and smaller coils when they become coplanar, then by Law of Conservation of Angular Momentum, we have

$$I_1\omega_1 = I_2\omega_2 \qquad \ldots(1)$$

By Law of Conservation of Energy, we have

$$(U+K)_{\text{initial}} = (U+K)_{\text{final}}$$

where, $U = -MB\cos\theta$ and $K = \frac{1}{2}I\omega^2$

$$\Rightarrow \quad \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 = U \qquad \ldots(2)$$

From equations (1) and (2), we get

$$\frac{1}{2}I_1\left(\frac{I_2\omega_2}{I_1}\right)^2 + \frac{1}{2}I_2\omega_2^2 = U$$

$$\Rightarrow \quad \frac{1}{2}I_2\omega_2^2\left(\frac{I_2}{I_1}+1\right) = U$$

$$\Rightarrow \quad \frac{1}{2}I_2\omega_2^2 = K_2 = \left(\frac{I_1}{I_1+I_2}\right)U$$

Since, $K_1 + K_2 = U$

$$\Rightarrow \quad \frac{1}{2}I_1\omega_1^2 = K_1 = \left(\frac{I_2}{I_1+I_2}\right)U$$

Also, we have $U = \left(\pi r^2 i_2\right)\left(\frac{\mu_0 i_1}{2R}\right)$

So, maximum kinetic energy of smaller coil is

$$K_2 = \left(\frac{\mu_0 \pi r^2 i_1 i_2}{2R}\right)\left(\frac{\frac{1}{2}m_1R^2}{\frac{1}{2}m_1R^2 + \frac{1}{2}m_2r^2}\right)$$

$$\Rightarrow \quad K_2 = \frac{\mu_0 \pi r^2 i_1 i_2 m_1 R}{2\left(m_1R^2 + m_2r^2\right)}$$

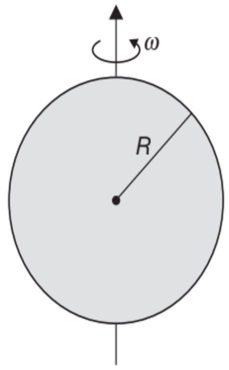
Also, $K_1 = \frac{1}{2}I_1\omega_1^2 = \left(\frac{I_2}{I_1+I_2}\right)U$

$$\Rightarrow \quad K_1 = \left(\frac{\mu_0 \pi r^2 i_1 i_2}{2R}\right)\left(\frac{\frac{1}{2}m_2r^2}{\frac{1}{2}m_1R^2 + \frac{1}{2}m_2r^2}\right)$$

$$\Rightarrow \quad K_1 = \frac{\mu_0 \pi r^4 i_1 i_2 m_2}{2R\left(m_1R^2 + m_2r^2\right)}$$

PROBLEM 9

A sphere of radius R has a uniform volume charge density ρ. Determine the magnetic field at the centre of the sphere when it rotates as a rigid object with angular speed ω about an axis through its centre. Also find the magnetic dipole moment of the sphere under the situation mentioned. Also express both the results in terms of the charge Q on the sphere. Now if the given sphere is completely embedded in an external magnetic field $\vec{B} = B_0\hat{i}$. Repeat your calculation for the cases when $\vec{B} = B_0\hat{j}$ and $\vec{B} = B_0\hat{k}$.

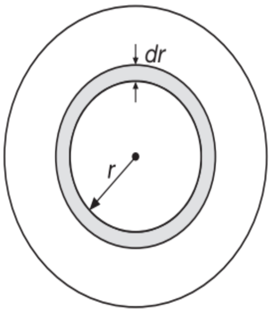


SOLUTION

The sphere can be assumed to be made up of a large number of concentric shells having radii between zero and R. Consider one such shell of radius r thickness dr having a charge dq. Then field contribution of such a shell at the centre is

$$dB=\left(\frac{\mu_0\omega}{6\pi r}\right)dq$$

{as derived earlier in the case of a shell}



where $dq=\left(4\pi r^2 dr\right)\rho$

$$\Rightarrow \quad dB=\left(\frac{\mu_0\omega}{6\pi r}\right)\left(4\pi r^2 dr\right)\rho$$

$$\Rightarrow \quad dB=\frac{2\mu_0\omega\rho}{3}r\,dr$$

$$\Rightarrow \quad B=\int dB=\frac{2\mu_0\omega\rho}{3}\int_0^R r\,dr$$

$$\Rightarrow \quad B=\left(\frac{2\mu_0\omega\rho}{3}\right)\left(\frac{R^2}{2}\right)$$

$$\Rightarrow \quad B=\frac{1}{3}\left(\mu_0\omega\rho R^2\right) \qquad \{\text{in terms of } \omega,\ \rho \text{ and } R\}$$

Since, we have

$$\rho=\frac{Q}{\frac{4}{3}\pi R^3}$$

$$\Rightarrow \quad B=\frac{1}{3}\mu_0\omega\left(\frac{Q}{\frac{4}{3}\pi R^3}\right)R^2$$

$$\Rightarrow \quad B=\frac{\mu_0\omega Q}{4\pi R} \qquad \{\text{in terms of } \omega,\ Q \text{ and } R\}$$

For calculating the magnetic moment, we can directly take the help of Gyromagnetic Ratio, which is given by

$$\gamma=\frac{|\vec{M}|}{|\vec{L}|}=\frac{Q}{2m}\text{, where } Q=\text{charge on sphere,}$$

$m=$ mass of sphere

$$\Rightarrow \quad \gamma=\frac{|\vec{M}|}{\left(\frac{2}{5}mR^2\right)\omega}=\frac{Q}{2m}$$

$$\Rightarrow \quad |\vec{M}|=\left(\frac{2}{5}mR^2\right)\omega\left(\frac{Q}{2m}\right)$$

$$\Rightarrow \quad |\vec{M}|=\frac{1}{5}Q\omega R^2=\frac{4\pi\rho\omega R^5}{15} \qquad \left\{\because\ \rho=\frac{Q}{\frac{4}{3}\pi R^3}\right\}$$

So, please note that the GYROMAGNETIC RATIO (γ) is a strong tool in your hands that will always help you to calculate the magnetic moment of a body of mass m, charge Q rotating with angular velocity ω.

Now since $\vec{\tau}=\vec{M}\times\vec{B}$ where $\vec{M}=\left(\frac{1}{5}Q\omega R^2\right)\hat{j}$ and $\vec{B}=B_0\hat{i}$

$$\Rightarrow \quad \vec{\tau}=\left(\frac{1}{5}QB_0\omega R^2\right)\left(-\hat{k}\right) \qquad \{\because\ \hat{j}\times\hat{i}=-\hat{k}\}$$

Similarly, for the other two cases, we get

$$\vec{\tau}=\vec{0} \text{ and } \vec{\tau}=\left(\frac{1}{5}QB_0\omega R^2\right)\hat{i}$$

PROBLEM 10

A very long straight conductor of circular cross-section with radius R has current density $J(r)$ directed into the plane of paper (along negative z direction) and varying with radial distance r as

$$J(r)=\begin{cases}\text{zero} & \text{for } r<\frac{R}{2}\\ \rho_0\frac{r^2}{R^2} & \text{for } \frac{R}{2}\le r\le R\end{cases}$$

where ρ_0 is a positive constant. Assuming the origin to lie on the axis of the very long straight conductor, consider a point $P(0,a,0)$ outside the conductor i.e. $a > R$. If two infinitely long thin conducting wires each carrying current I_0 in the same direction and parallel to the axis of the conductor are placed at $(a,0,0)$ and $(-a,0,0)$ such that the magnetic field at P is zero, then calculate the current I_0 in the thin wires and the direction of current with respect to the very long straight conductor.

SOLUTION

Current through the very long straight conductor is given by

$$I = \int_0^R J dA \text{, where } dA = 2\pi r dr$$

$$\Rightarrow \quad I = \int_0^{\frac{R}{2}} 2\pi r J(r) dr + \int_{\frac{R}{2}}^{R} 2\pi r J(r) dr$$

$$\Rightarrow \quad I = \int_0^{\frac{R}{2}} 2\pi r (0) dr + \int_{\frac{R}{2}}^{R} 2\pi r \rho_0 \frac{r^2}{R^2} dr$$

$$\Rightarrow \quad I = 0 + \frac{2\pi\rho_0}{R^2}\int_{\frac{R}{2}}^{R} r^3 dr = \frac{2\pi\rho}{R^2}\left(\frac{r^4}{4}\right)\Bigg|_{\frac{R}{2}}^{R}$$

$$\Rightarrow \quad I = \frac{2\pi\rho_0}{4R^2}\left(R^4 - \frac{R^4}{16}\right) = \frac{15}{32}\pi\rho_0 R^2$$

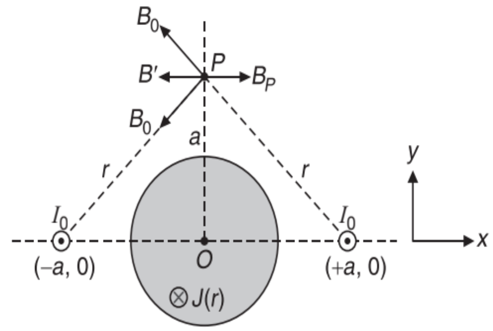
For calculating the magnetic field at P due to the very long straight conductor, let us consider an Amperian loop of radius a having centre at the axis of the conductor, then for all points on the Amperian loop, the magnetic field B has the same value. Applying Ampere's Circuital Law, we get

$$B(2\pi a) = \mu_0 I$$

$$\Rightarrow \quad B(2\pi a) = \mu_0\left(\frac{15}{32}\pi\rho_0 R^2\right)$$

$$\Rightarrow \quad B = B_P = \frac{15}{64}\frac{\mu_0\rho_0 R^2}{a}\text{, along } +x \text{ direction}$$

The visualisation to the situation given in the problem is shown in Figure.



At P, field due to the conductor is directed along positive x direction. Field due to wires A_1 and A_2 must cancel B, which is possible only when the current I_0 in each wire is outwards along positive z axis. Field at P due to wire A_1 is

$$\left|\vec{B}_1\right| = B_0 = \frac{\mu_0 I_0}{2\pi r} = \frac{\mu_0 I_0}{2\pi\left(a\sqrt{2}\right)}$$

Field at P due to wire A_2 is

$$\left|\vec{B}_2\right| = B_0 = \frac{\mu_0 I_0}{2\pi r} = \frac{\mu_0 I_0}{2\pi\left(a\sqrt{2}\right)}$$

Resultant of these two fields B_1 and B_2 (i.e. B') is along negative x direction (i.e. opposite to B_P). So, we have

$$B' = 2B_0\cos(45°) = 2\left(\frac{\mu_0 I_0}{2\pi\left(a\sqrt{2}\right)}\right)\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad B' = \frac{\mu_0 I_0}{2\pi a}\text{, along } -x \text{ direction}$$

Since B' is acting opposite to B_P, so net field at P will be zero when we have both B' and B_P equal in magnitude.

$$\Rightarrow \quad B' = B_P$$

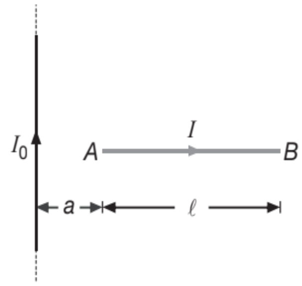
$$\Rightarrow \quad \frac{\mu_0 I_0}{2\pi a} = \frac{15}{64}\frac{\mu_0\rho_0 R^2}{a}$$

$$\Rightarrow \quad I_0 = \frac{15}{32}\pi\rho_0 R^2$$

PROBLEM 11

A finite conductor AB carrying current I is placed near a fixed very long current carrying wire I_0 as shown in the figure. Find the point of application and magnitude of the net ampere force on the conductor AB.

What happens to the conductor AB if it is free to move? (Neglect gravitational field).



SOLUTION

Consider an element of rod AB of length dx at a distance x from the wire. The field at this element is

$$B=\frac{\mu_0 I_0}{2\pi x}$$

$$|d\vec{F}|=BIdx=\frac{\mu_0 I_0 I}{2\pi}\frac{dx}{x}$$

$$\Rightarrow \quad F=\int dF=\frac{\mu_0 I I_0}{2\pi}\int_a^{a+l}\frac{dx}{x}$$

$$\Rightarrow \quad F=\frac{\mu_0 I I_0}{2\pi}\log_e\left(\frac{a+l}{a}\right) \qquad \{\text{in the direction of } I_0\}$$

For finding the point of application of force we shall make use of the following mathematical formula, according to which

$$r=\frac{\int x\,dF}{F} \quad \text{where } F=\frac{\mu_0 I I_0}{2\pi}\log_e\left(\frac{a+l}{a}\right)$$

Consider an element of length dx at a distance x from wire I_0.

Then $dF=B(x)Idx$ where $B(x)=\dfrac{\mu_0 I_0}{2\pi x}$

$$\Rightarrow \quad r=\frac{\int_a^{a+l} x\left(\frac{\mu_0 I_0}{2\pi x}\right)Idx}{F}$$

$$\Rightarrow \quad r=\frac{\left(\frac{\mu_0 I I_0}{2\pi}\right)l}{\left(\frac{\mu_0 I I_0}{2\pi}\right)\log_e\left(\frac{a+l}{a}\right)}$$

$$\Rightarrow \quad r=\frac{l}{\log_e\left(\frac{a+l}{a}\right)}$$

Please note that the formula used $r=\dfrac{\int xdF}{F}$ is just another way of expressing the "Law of Conservation of Moments of force".

PROBLEM 12

A solid sphere of radius R carries a uniform charge density $+\rho$ from $r=0$ to $\dfrac{R}{2}$ and an equal charge density of opposite sign $-\rho$, from $\dfrac{R}{2}$ to R. If the sphere rotates about its diameter with some uniform angular velocity ω, then calculate the magnetic moment of the sphere. Assume that the mass is distributed uniformly on the sphere.

SOLUTION

The solid sphere can be visualized to be made up of a number of infinitesimal concentric spherical shells. Consider one such shell of charge dq, thickness dr, radius r having magnetic moment dM, then due to the rotation of the sphere with uniform angular speed ω, the magnetic moment is given by

$$dM=\frac{1}{3}(dq)\omega r^2$$

where, $dQ=\rho dV=\rho\left(4\pi r^2 dr\right)$

$$\Rightarrow \quad dM=\frac{1}{3}(dq)\omega r^2=\frac{1}{3}\left(4\pi r^2\rho dr\right)\omega r^2$$

Magnetic moment of solid portion between $r=0$ to $r=\dfrac{R}{2}$ is obtained by integrating the above expression. So, we have

$$M_1=\int dM=\int_0^{R/2}\frac{1}{3}\left(4\pi r^2\rho dr\right)\omega r^2$$

$$\Rightarrow \quad M_1=\frac{4}{3}\pi\rho\omega\int_0^{R/2} r^4 dr=\frac{4}{3}\pi\rho\omega\left(\frac{r^5}{5}\right)\Bigg|_0^{\frac{R}{2}}$$

$$\Rightarrow \quad M_1=\frac{4}{15}\pi\rho\omega\left(\frac{R^5}{32}\right)=\frac{1}{120}\pi\rho\omega R^5$$

Similarly, the magnetic moment of solid portion of charge density $-\rho$ from $r=\frac{R}{2}$ to $r=R$ is

$$M_2=\int d\mu=-\int_{R/2}^{R}\frac{1}{3}\left(4\pi r^2\rho dr\right)\omega r^2$$

$$\Rightarrow \quad \mu_2=-\frac{4}{3}\pi\rho\omega\left(\frac{r^5}{5}\right)\Bigg|_{\frac{R}{2}}^{R}$$

$$\Rightarrow \quad M_2=-\frac{4}{15}\pi\rho\omega R^5\left(1-\frac{1}{32}\right)$$

$$\Rightarrow \quad M_2=-\frac{31}{120}\pi\rho\omega R^5$$

So, net magnetic moment is given by

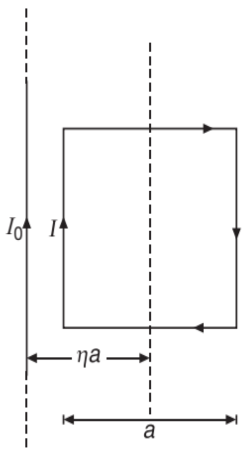
$$M_{\text{net}}=M_1+M_2$$

$$\Rightarrow \quad M_{\text{net}}=-\frac{1}{4}\pi\rho\omega R^5$$

Negative sign implies that the net magnetic moment is directed opposite to the direction of ω.

PROBLEM 13

A square frame carrying a current I is located in the same plane as a long straight wire carrying a current I_0. The frame side has a length a. The axis of the frame passing through the midpoints of the opposite sides is parallel to the wire and is separated from it by the distance which is η times greater than the side of the frame as shown in Figure.



Calculate the force acting on the frame. Also calculate the mechanical work to be performed in order to turn the frame through 180° about its axis, assuming the currents to be constant throughout the process of turning the frame.

SOLUTION

Force of attraction between wires carrying currents in the same direction is

$$F_1=\frac{\mu_0 II_0}{2\pi\left(\eta a-\frac{a}{2}\right)}=\frac{\mu_0 II_0}{\pi(2\eta-1)a}$$

Similarly, force of repulsion between wires carrying currents in the opposite direction is

$$F_2=\frac{\mu_0 II_0}{2\pi\left(\eta a+\frac{a}{2}\right)}=\frac{\mu_0 II_0}{\pi(2\eta+1)a}$$

Net force of attraction between the square frame and the long straight wire is

$$F=F_1-F_2$$

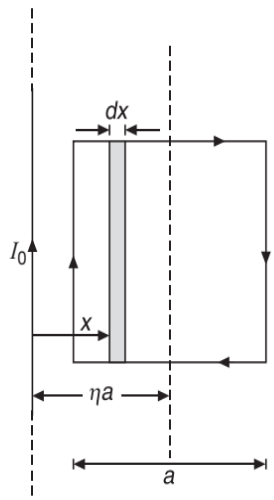
$$\Rightarrow \quad F=\frac{\mu_0 II_0}{\pi(2\eta-1)a}-\frac{\mu_0 II_0}{\pi(2\eta+1)a}$$

$$\Rightarrow \quad F=\frac{2\mu_0 II_0}{\pi\left(4\eta^2-1\right)}$$

Since the work to be done in turning the frame through 180° is given by

$$W=-I\Delta\phi=-I(-\phi-\phi)=2I\phi \qquad \ldots(1)$$

where ϕ is the magnetic flux associated with the coil due to the magnetic field of the wire placed near it. To calculate the magnetic flux associated with the square loop let us consider an infinitesimal strip element of length a, width dx at a distance x in the coil as shown in Figure.



The magnetic field B at the strip due to the infinite wire is

$$B = \frac{\mu_0 I_0}{2\pi x}$$

The magnetic flux associated with this infinitesimal strip is

$$d\phi = BdA$$

where $dA = adx$ is the area of the strip.

$$\Rightarrow \quad d\phi = \left(\frac{\mu_0 I_0}{2\pi x}\right) adx$$

Total magnetic flux associated with the square loop is obtained by integrating $d\phi$. So, we get

$$\phi = \int d\phi = \int_{\eta a - \frac{a}{2}}^{\eta a + \frac{a}{2}} \left(\frac{\mu_0 I_0}{2\pi x}\right) adx$$

$$\Rightarrow \quad \phi = \frac{\mu_0 I_0 a}{2\pi} \int_{\eta a - \frac{a}{2}}^{\eta a + \frac{a}{2}} \frac{dx}{x}$$

$$\Rightarrow \quad \phi = \frac{\mu_0 I_0 a}{2\pi} (\ln x)\Big|_{\eta a - \frac{a}{2}}^{\eta a + \frac{a}{2}}$$

$$\Rightarrow \quad \phi = \frac{\mu_0 I_0 a}{2\pi} \left[\ln\left(\eta a + \frac{a}{2}\right) - \ln\left(\eta a - \frac{a}{2}\right)\right]$$

$$\Rightarrow \quad \phi = \frac{\mu_0 I_0 a}{2\pi} \ln\left(\frac{2\eta + 1}{2\eta - 1}\right)$$

From equation (1), the work done to rotate the coil is given by

$$W = 2I\phi$$

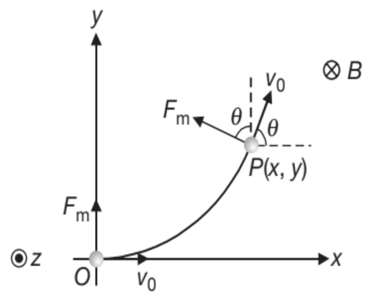
$$\Rightarrow \quad W = \frac{\mu_0 I I_0 a}{\pi} \ln\left(\frac{2\eta + 1}{2\eta - 1}\right)$$

PROBLEM 14

A particle of charge q and mass m is projected from the origin with velocity $\vec{v} = v_0 \hat{i}$ in a nonuniform magnetic field $\vec{B} = -(B_0 x)\hat{k}$. Here v_0 and B_0 are positive constants of proper dimensions. Find the maximum positive x coordinate of the particle during its motion.

SOLUTION

Magnetic field is along negative z-direction. So, in the coordinate axes shown in figure, it is perpendicular to paper inwards. $(\otimes)$ magnetic force on the particle at origin is along positive y-direction. So, it will rotate in x-y plane as shown. The path is not a perfect circle as the magnetic field is nonuniform. Speed of the particle in magnetic field remains constant.



Magnetic force is always perpendicular to velocity. Let at point $P(x, y)$ its velocity vector makes an angle θ with positive x-axis. Then magnetic force $\vec{F}_m$ will be at angle θ with positive y-direction. So,

$$a_y = \left(\frac{F_m}{m}\right)\cos\theta$$

$$\Rightarrow \quad \frac{dv_y}{dt} = \frac{(B_0 x)(qv_0 \cos\theta)}{m} \qquad \{\because F_m = Bqv_0 \sin 90^\circ\}$$

$$\Rightarrow \quad \left(\frac{dv_y}{dx}\right)\left(\frac{dx}{dt}\right) = \left(\frac{B_0 q x}{m}\right)(v_0 \cos\theta) \qquad \ldots(1)$$

where $\frac{dx}{dt} = v_x = v_0 \cos\theta$

From (1), we get

$$\frac{dv_y}{dx} = \left(\frac{B_0 q}{m}\right)x$$

$$\Rightarrow \quad \int_0^{v_0} dv_y = \left(\frac{B_0 q}{m}\right)\int_0^{x_{max}} x dx$$

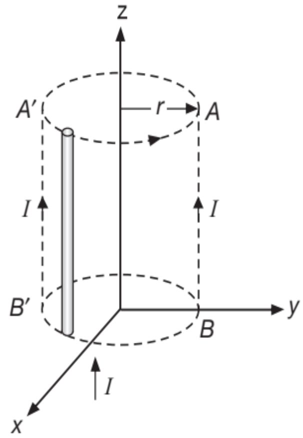
$$\Rightarrow \quad v_0 = \left(\frac{B_0 q}{m}\right)\left(\frac{x_{max}^2}{2}\right)$$

$$\Rightarrow \quad x_{max} = \sqrt{\frac{2mv_0}{B_0 q}}$$

Note that here, when the x displacement is maximum, then the velocity is along positive y direction.

PROBLEM 15

Find the work and power required to move the conductor of length l shown in the figure through one full turn in the clockwise direction at a rotational frequency of n revolutions per second if the magnetic field is of magnitude B_0 everywhere and points radially outwards from Z-axis. The figure shows the surface traced by the wire AB.



SOLUTION

Since we know that

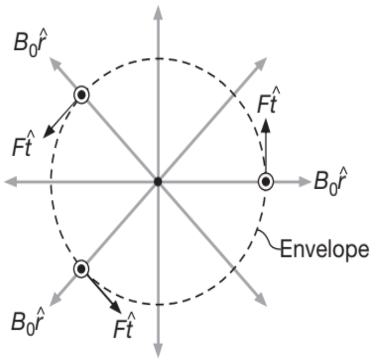
$$\vec{F} = I(\vec{l} \times \vec{B})$$

where $I\vec{l} = (Il)\hat{k}$ and $\vec{B} = B_0\hat{r}$

where $\hat{r}$ is a unit vector directed radially outwards.

$$\Rightarrow \quad \vec{F} = (IlB_0)\hat{t}$$

where we have $\hat{k} \times \hat{r} = \hat{t}$, with $\hat{t}$ directed towards the tangential direction.



View From the Top
$\hat{r}$ = Unit Radial Vector
$\hat{t}$ = Unit Tangential Vector

If dW is the work done by the external force to turn the conductor through angle $d\theta$ in the clockwise sense, then

$$dW = \vec{F} \cdot d\vec{x}$$

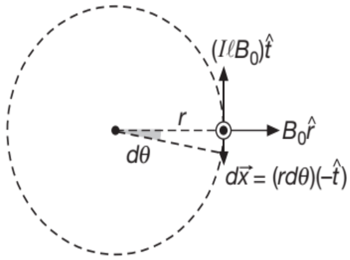
$$\Rightarrow \quad dW = (IlB_0)\hat{t} \cdot rd\theta(-\hat{t}) \qquad \{\because d\vec{x} = rd\theta(-\hat{t})\}$$

$$\Rightarrow \quad W = \int dW = -IlB_0 r \int_0^{2\pi} d\theta$$

$$\Rightarrow \quad W = -IlB_0(2\pi r) = -(2\pi r)(B_0 Il)$$

The negative sign shows that the field does work. Also, work done a round a closed loop is non-zero thus showing that the force doing this work is non-conservative in nature.

$$\text{Power} = \frac{\text{Work Done}}{\text{Time of one Revolution}} = \frac{W}{\left(\dfrac{2\pi}{\omega}\right)}$$



Since $\omega = n$ revolution per second

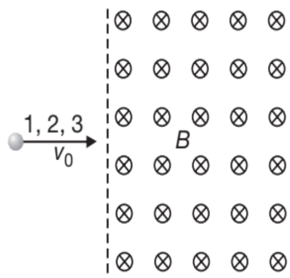
$$\Rightarrow \quad \omega = 2\pi n \text{ rads}^{-1}$$

$$\Rightarrow \quad T = \frac{2\pi}{\omega} = \left(\frac{1}{n}\right) \text{ second}$$

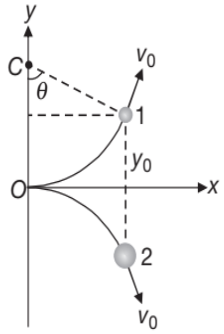
$$\Rightarrow \quad P = -\frac{2\pi r(B_0 Il)}{\left(\dfrac{1}{n}\right)} = -2\pi r n B_0 Il$$

PROBLEM 16

Three charged particles 1, 2 and 3 of mass m, $2m$ and $3m$ having charges q, $-2q$ and $6q$ are projected into a magnetic field B as shown in figure. Find the distance between the particles 1, 2 and 1, 3 after time t, if at $t = 0$ they enter into the region of magnetic field. Ignore electric and magnetic effects due to charges on themselves.



SOLUTION



Distance between 1 and 2:

Since, $r_1 = \dfrac{mv_0}{qB}$

$$r_2 = \frac{2mv_0}{2qB} = \frac{mv_0}{qB}$$

Also, $\omega_1 = \dfrac{qB}{m}$ and $\omega_2 = \dfrac{2qB}{2m} = \dfrac{qB}{m}$

$$\Rightarrow \quad S_{12} = 2y_0 = 2r_1(1-\cos\theta) = 2r_1(1-\cos\omega_1 t)$$

$$\Rightarrow \quad S_{12} = \frac{2mv_0}{qB}\left(1-\cos\left(\frac{qBt}{m}\right)\right)$$

Distance between 1 and 3:

Since, $r_3 = \dfrac{3mv_0}{6qB} = \dfrac{mv_0}{2qB}$ and $\omega_3 = \dfrac{6qB}{3m} = \dfrac{2qB}{m}$

$$\Rightarrow \quad S_{13} = \sqrt{(x_1-x_3)^2+(y_1-y_3)^2}$$

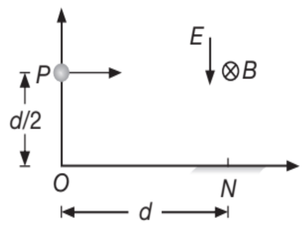
$$\Rightarrow \quad S_{13} = \sqrt{\begin{array}{l}\left[r_1\sin(\omega_1 t)-r_3\sin(\omega_3 t)\right]^2+\\ \left[r_1(1-\cos(\omega_1 t))-r_3(1-\cos(\omega_3 t))\right]^2\end{array}}$$

Substituting the values, we get

$$S_{13} = \frac{mv_0}{qB}\sqrt{\begin{array}{l}\left\{\sin\left(\frac{qBt}{m}\right)-\frac{1}{4}\sin\left(\frac{2qBt}{m}\right)\right\}^2+\\ \left\{\frac{3}{4}+\cos\left(\frac{2qBt}{m}\right)-\cos\left(\frac{qBt}{m}\right)\right\}^2\end{array}}$$

PROBLEM 17

A particle of mass m and charge q enters a region of electric field $\vec{E}$ as shown in the figure with some velocity at point P. At the moment the particle collides elastically with smooth surface at N, the electric field $\vec{E}$ is switched off and a magnetic field $\vec{B}$ perpendicular to the plane of paper automatically switched on. If the particle hits the surface at point O, then prove that $B = 2\sqrt{\dfrac{mE}{qd}}$.



SOLUTION

If t is the time taken to reach point N, then

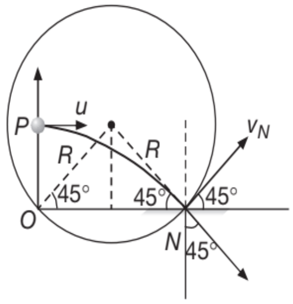
$$\frac{d}{2} = \frac{1}{2}\left(\frac{qE}{m}\right)\times t^2$$

$$\Rightarrow \quad t = \sqrt{\left(\frac{m}{qE}\right)d}$$

If u is the velocity while entering the field, then

$$d = u \times t$$

$$\Rightarrow \quad u = \frac{d}{t} = \sqrt{\frac{qEd}{m}}$$



also if $v_{N\perp}$ is the velocity of the particle in the direction of $\vec{E}$ at N, then

$$v_{N\perp} = \sqrt{2\times\left(\frac{qE}{m}\right)\times\frac{d}{2}} = \sqrt{\frac{qEd}{m}}$$

So, net velocity at N is

$$v_N = \sqrt{v_{N\perp}^2+u^2} = \sqrt{\frac{2qEd}{m}}$$

Since $\tan\theta = \dfrac{v_{N\perp}}{u} = 1$

$$\Rightarrow \quad \theta = 45°$$

Hence, the collision is elastic and the particle is reflected back with the same speed v_N

The charge q now moves with a speed v_N in magnetic field $\vec{B}$. Radius of its path

$$R = \frac{mv_N}{qB} = \left(\sqrt{\frac{2mEd}{q}}\right)\times\frac{1}{B} \qquad \ldots(1)$$

From figure $R = \dfrac{d}{2}\times\sqrt{2} = \dfrac{d}{\sqrt{2}}$

Putting in (1), $\dfrac{d}{\sqrt{2}} = \dfrac{1}{B}\sqrt{\dfrac{2mEd}{q}}$

$$\Rightarrow \quad B = 2\sqrt{\frac{mE}{qd}}$$

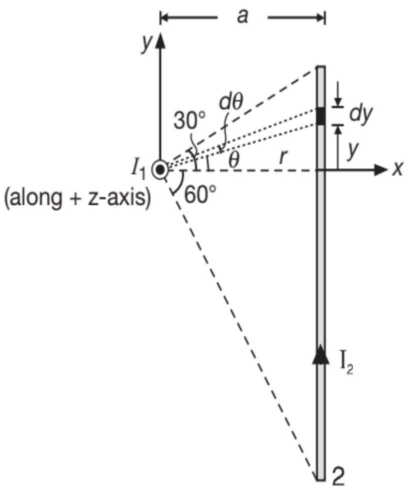
PROBLEM 18

An infinite wire 1, placed along z-axis, has current I_1 in positive z-direction. Another wire 2 is placed parallel to y-axis and carries a current I_2 in positive y-direction. The ends of the wire 2 subtend $+30°$ and $-60°$ at the origin with positive x-direction. The wire 2 is at a distance a from the origin. Find net force on the wire 2.

SOLUTION

The arrangement is shown here.
Please note that the direction of I_1 and the $+z$-axis, both are outwards towards the reader, denoted by $\odot$. Let us consider an infinitesimal element of length dy at a distance y from x-axis. If $\vec{B}$ be the field due to the wire 1 at the element is

$$B_1 = \frac{\mu_0 I_1}{2\pi r} = \frac{\mu_0 I_1}{2\pi\sqrt{a^2+y^2}}$$



The force on this element due to the field of wire 1 is

$$d\vec{F} = I(d\vec{l} \times \vec{B})$$

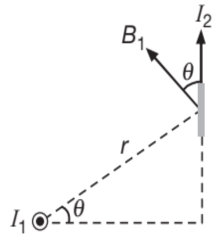
$$\Rightarrow \quad d\vec{F} = I_2(d\vec{y} \times \vec{B}_1)$$

$$\Rightarrow \quad |d\vec{F}| = dF = (I_2 dy)\left(\frac{\mu_0 I_1}{2\pi\sqrt{a^2+y^2}}\right)\sin\theta$$

$$\Rightarrow \quad dF = \frac{\mu_0 I_1 I_2}{2\pi}\left(\frac{\sin\theta}{\sqrt{a^2+y^2}}\right)dy$$

{along positive z-direction, using Right Hand Thumb Rule}

But $\sin\theta = \dfrac{y}{\sqrt{a^2+y^2}}$



$$\Rightarrow \quad dF = \frac{\mu_0 I_1 I_2}{2\pi}\left(\frac{y\,dy}{a^2+y^2}\right)$$

$$F = \frac{\mu_0 I_1 I_2}{2\pi}\int_{-\sqrt{3}a}^{\frac{a}{\sqrt{3}}} \frac{y\,dy}{a^2+y^2}$$

Since, $\displaystyle\int \frac{y\,dy}{a^2+y^2} = \frac{1}{2}\log_e(a^2+y^2)$

$$\Rightarrow \quad F = \frac{\mu_0 I_1 I_2}{4\pi}\left[\log_e(a^2+y^2)\Big|_{-\sqrt{3}a}^{\frac{a}{\sqrt{3}}}\right]$$

$$\Rightarrow \quad F = \frac{\mu_0 I_1 I_2}{4\pi}\log_e\left(\frac{a^2+\frac{a^2}{3}}{a^2+3a^2}\right)$$

$$\Rightarrow \quad F = \frac{\mu_0 I_1 I_2}{4\pi}\log_e\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad F = -\frac{\mu_0 I_1 I_2}{4\pi}\log_e(3)$$

{Along negative z-direction}

In this particular problem, we must observe one thing that though the magnetic force on the infinitesimal element is along $+z$ axis, but the net force on the complete wire 2 is along $-z$ axis. This is quite obvious because the lower current carrying portion has a length more than the upper portion and so the extra unsymmetrical portion contributes to the net force along negative z-direction.

PROBLEM 19

A charged particle of mass m and charge q is projected on a rough horizontal X-Y plane with z-axis in vertically upward direction. Both electric and magnetic fields are acting in the region and given by $\vec{E} = -E_0\hat{k}$ and $\vec{B} = -B_0\hat{k}$ respectively. The particle enters into the field at $(a_0, 0, 0)$ with velocity $\vec{v} = v_0\hat{j}$. The particle starts moving into a circular path on the plane. If coefficient of friction between particle and the plane is μ. Then calculate the

(a) time when the particle will come to rest
(b) time when the particle will hit the centre
(c) distance travelled by the particle when it comes to rest.

SOLUTION

(a) $N = mg + qE_0$ ...(1)

$$qvB_0 = \frac{mv^2}{R} \quad ...(2)$$

and $-m\dfrac{dv}{dt} = \mu N$ ...(3)

From equation (2)

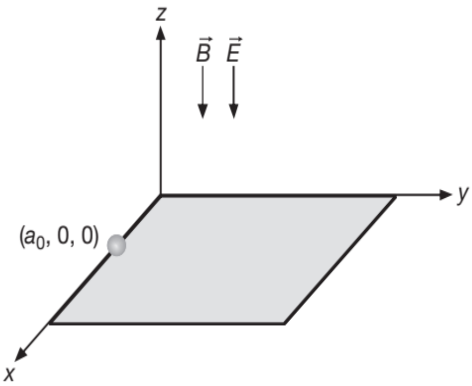
$$R = \frac{mv}{qB_0} \quad ...(4)$$

From equation (1) and (3)

$$-m\frac{dv}{dt} = \mu(mg + qE_0)$$

$$\Rightarrow -m\int_{v_0}^{0} dv = -\mu(mg + qE_0)\int_0^t dt$$

$$t = \frac{mv_0}{\mu(mg + qE_0)}$$



(b) From equation (4)

$$dR = \frac{m}{qB_0}dv = -\frac{\mu(mg + qE_0)dt}{qB_0}$$

$$\Rightarrow \int_{R_i}^{0} dR = \frac{-\mu(mg + qE_0)}{qB_0}\int_0^t dt$$

$$\Rightarrow t = \frac{qB_0R_i}{\mu(mg + qE_0)}$$

Here, $R_i = \dfrac{mv_0}{qB_0}$

$$\Rightarrow t = \frac{mv_0}{\mu(mg + qE_0)}$$

We observe that both the times are equal.

(c) $-mv\dfrac{dv}{dt} = \mu(mg + qE_0)$

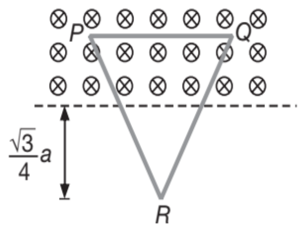
$$\Rightarrow -m\int_{v_0}^{0} v\,dv = \mu(mg + qE_0)\int_0^t dl$$

$$\Rightarrow \frac{mv_0^2}{2} = \mu(mg + qE_0)l$$

$$\Rightarrow l = \frac{mv_0^2}{2\mu(mg + qE_0)}$$

PROBLEM 20

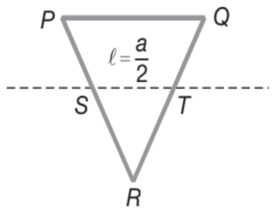
An equilateral triangular frame PQR of mass M and side a is at rest is under the influence of horizontal magnetic field B and gravitational field as shown in the figure.



(a) Find the magnitude and direction of current in the frame so that the frame remains at rest.
(b) If the frame is slightly displaced in its plane perpendicular to PQ, show that its motion is simple harmonic and find its period of oscillation. (Neglect EMF induced due to motion of the loop)

SOLUTION

(a) The current I in the loop should be clockwise.



Now for the equilibrium of the loop

$$Mg = I(l)B$$

$$\Rightarrow \; Mg = I \times \frac{a}{2} \times B$$

$$\Rightarrow \; I = \frac{2Mg}{aB}$$

(b) Now if the loop is displaced downward by distance x

Restoring force

$$F = -[I(S'T')B - Mg]$$

$$\Rightarrow \; F = -I\left[\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{4}a + x\right)\right]B + Mg$$

$$\Rightarrow \; F = -\frac{IaB}{2} - \frac{2IB}{\sqrt{3}}x + Mg$$

$$\Rightarrow \; F = -\left(\frac{2IB}{\sqrt{3}}\right)x$$

$$\Rightarrow \; ma = -\left(\frac{2IB}{\sqrt{3}}\right)x$$

$$\Rightarrow \; m\ddot{x} + \left(\frac{2IB}{\sqrt{3}}\right)x = 0$$

Hence motion is SHM comparing with standard equation of SHM, we get $\ddot{x} + \omega^2 x = 0$

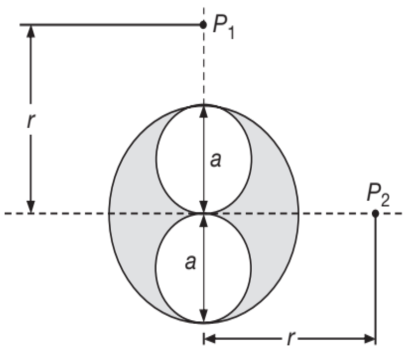
$$\Rightarrow \; \ddot{x} + \left(\frac{2IB}{\sqrt{3}M}\right)x = 0$$

$$\Rightarrow \; \omega^2 = \frac{2IB}{\sqrt{3}M}$$

$$\Rightarrow \; T = 2\pi\sqrt{\frac{\sqrt{3}M}{2IB}} = \pi\sqrt{\frac{\sqrt{3}a}{g}}$$

PROBLEM 21

A long cylindrical conductor of radius a has two cylindrical cavities of diameter a through its entire length, as shown in figure.



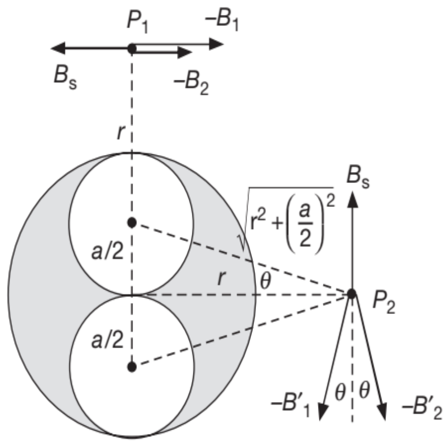
A current I is directed out of the page and is uniform through a cross section of the conductor. Find the magnitude and direction of the magnetic field in terms of μ_0, I, r and a at points P_1 and P_2.

SOLUTION

Note that the current I exists in the conductor with a current density $J = \frac{I}{A}$, where

$$A = \pi\left(a^2 - \frac{a^2}{4} - \frac{a^2}{4}\right) = \frac{\pi a^2}{2}$$

$$\Rightarrow \quad J = \frac{2I}{\pi a^2}$$



To find the field at either point P_1 or P_2 we follow the following steps.

STEP-1:

Find B_s which would exist if the conductor were solid, using Ampere's Law.

STEP-2:

Find B_1 and B_2 that would be due to the conductors of radius $\frac{a}{2}$ that could occupy the void where the holes exist.

STEP-3:

Then use the superposition principle and subtract the field that would be due to the part of the conductor where the holes exist from the field of the solid conductor.

At point P_1

$$B_s = \frac{\mu_0 J(\pi a^2)}{2\pi r}$$

$$B_1 = \frac{\mu_0 J\pi\left(\frac{a}{2}\right)^2}{2\pi\left(r-\frac{a}{2}\right)} \text{ and}$$

$$B_2 = \frac{\mu_0 J\pi\left(\frac{a}{2}\right)^2}{2\pi\left(r+\frac{a}{2}\right)}$$

$$\Rightarrow \quad B = B_s - B_1 - B_2$$

$$\Rightarrow \quad B = \frac{\mu J\pi a^2}{2\pi}\left[\frac{1}{r}-\frac{1}{4\left(r-\frac{a}{2}\right)}-\frac{1}{4\left(r+\frac{a}{2}\right)}\right]$$

$$\Rightarrow \quad B = \frac{\mu_0(2I)}{2\pi}\left[\frac{4r^2-a^2-2r^2}{4r\left(r^2-\frac{a^2}{4}\right)}\right]$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{\pi r}\left[\frac{2r^2-a^2}{4r^2-a^2}\right], \text{ directed to the left}$$

At point P_2

$$B_s = \frac{\mu_0 J(\pi a^2)}{2\pi r}$$

$$\text{and } B_1' = B_2' = \frac{\mu_0 J\pi\left(\frac{a}{2}\right)^2}{2\pi\sqrt{r^2+\left(\frac{a}{2}\right)^2}}$$

The horizontal components of B_1' and B_2' cancel while their vertical components add.

$$\Rightarrow \quad B = B_s - B_1'\cos\theta - B_2'\cos\theta$$

$$\Rightarrow \quad B = \frac{\mu_0 J(\pi a^2)}{2\pi r} - 2\left(\frac{\frac{\mu_0 J\pi a^2}{4}}{2\pi\sqrt{r^2+\left(\frac{a^2}{4}\right)}}\right)\frac{r}{\sqrt{r^2+\left(\frac{a^2}{4}\right)}}$$

$$\Rightarrow \quad B = \frac{\mu_0 J(\pi a^2)}{2\pi r}\left[1-\frac{r^2}{2\left(r^2+\left(\frac{a^2}{4}\right)\right)}\right]$$

$$\Rightarrow \quad B = \frac{\mu_0(2I)}{2\pi r}\left[1-\frac{2r^2}{4r^2+a^2}\right]$$

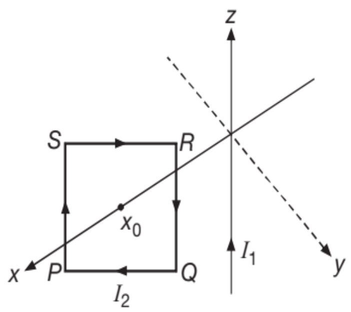
$$\Rightarrow \quad B = \frac{\mu_0 I}{\pi r}\left[\frac{2r^2+a^2}{4r^2+a^2}\right]$$

{directed toward the top of the page}

PROBLEM 22

Current I_1 flows along the positive z-direction in a long thin wire. Another current I_2, flows in a rectangularly shaped wire whose centre lies at $(x_0, 0, 0)$ and whose vertices are located at the points $P(x_0 + d, -a, -b)$, $Q(x_0 - d, a, -b)$, $R(x_0 - d, a, +b)$ and $S(x_0 + d, -a, +b)$ respectively as shown in the figure. Assume that $a, b, d \ll x_0$.

(a) Find the magnetic dipole moment vector of the rectangular wire.
(b) Find the force exerted on the rectangular coil by the magnetic field generated by I_1.



SOLUTION

(a) The magnetic moment of the rectangular coil is

$$\vec{M} = I_2\vec{A}$$

where $\vec{A} = \overrightarrow{QP}\times\overrightarrow{PS}$

here $\overrightarrow{QP} = 2d\hat{i} - 2a\hat{j}$ and $\overrightarrow{PS} = 2b\hat{k}$

$$\Rightarrow \quad \vec{A} = 2\left(d\hat{i} - a\hat{j}\right)\times 2b\hat{k}$$

$$\Rightarrow \quad M = -4bI_2\left(d\hat{j} + a\hat{i}\right)$$

(b) The magnetic field caused by I_1 at the centre $(x_0, 0, 0)$ of rectangular coil is

$$\vec{B} = \left(\frac{\mu_0}{2\pi}\frac{I_1}{x_0}\right)\hat{j}$$

The potential energy of magnetic moment $\vec{M}$ in the field $\vec{B}$

$$U = -\vec{M}\cdot\vec{B}$$

$$\Rightarrow \quad U = (4bI_2I_1)\left(\frac{\mu_0}{2\pi}\frac{d}{x_0}\right) = \frac{2\mu_0 bdI_1I_2}{\pi x_0}$$

$$\text{Since, } \vec{F} = -\left(\frac{\partial U}{\partial x}\right)\hat{i}$$

$$\Rightarrow \quad \vec{F} = \left(\frac{2\mu_0 bdI_1I_2}{\pi x_0^2}\right)\hat{i}$$

PROBLEM 23

A particle having mass m and charge q is released from the origin in a region in which electric field and magnetic field are given by $\vec{B}=-B_0\hat{j}$ and $\vec{E}=E_0\hat{k}$. Find the components of the velocity and the speed of the particle as a function of its z-co-ordinate.

SOLUTION

Initially, when the particle is released, then magnetic force will do nothing, but the electrostatic force will make the particle gain some velocity. Let the velocity of the particle at time t be $\vec{v}=v_x\hat{i}+v_y\hat{j}+v_z\hat{k}$. Then according to the Lorentz Force Formula, we have

$\vec{F}=\vec{F}_m+\vec{F}_e$, where

$$\vec{F}_m=q(\vec{v}\times\vec{B})=q\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ v_x & v_y & v_z\\ 0 & -B_0 & 0\end{vmatrix}$$

$$\Rightarrow \vec{F}_m=q\left[\hat{i}(v_zB_0)+\hat{k}(-v_xB_0)\right]$$

$$\Rightarrow \vec{F}_m=qB_0\left(v_z\hat{i}-v_x\hat{k}\right) \text{ and } \vec{F}_e=q\vec{E}=qE_0\hat{k}$$

$$\Rightarrow \vec{F}=q\left[B_0v_z\hat{i}+(E_0-B_0v_x)\hat{k}\right]$$

$$\Rightarrow ma_x=qB_0v_z,\ ma_y=0,\ ma_z=q(E_0-B_0v_x)$$

$$\Rightarrow a_x=\left(\frac{qB_0}{m}\right)v_z$$

$$\Rightarrow \frac{dv_x}{dt}=\left(\frac{qB_0}{m}\right)v_z$$

$$\Rightarrow \frac{dv_x}{dt}\frac{dz}{dz}=\left(\frac{qB_0}{m}\right)v_z$$

$$\Rightarrow \left(\frac{dv_x}{dz}\right)\left(\frac{dz}{dt}\right)=\left(\frac{qB_0}{m}\right)v_z$$

Since $v_z=\frac{dz}{dt}$, so we get

$$\frac{dv_x}{dz}=\frac{qB_0}{m}$$

$$\Rightarrow \int_0^{v_x}dv_x=\frac{qB_0}{m}\int_0^z dz$$

$$\Rightarrow v_x=\left(\frac{qB_0}{m}\right)z$$

$$v_y=0$$

and $a_z=\frac{q}{m}(E_0-B_0v_x)=\frac{qE_0}{m}-\left(\frac{qB_0}{m}\right)v_x$

But $v_x=\left(\frac{qB_0}{m}\right)z$

$$\Rightarrow a_z=\frac{dv_z}{dt}=\frac{qE_0}{m}-\left(\frac{qB_0}{m}\right)^2 z$$

$$\Rightarrow \frac{dv_z}{dt}\frac{dz}{dz}=\frac{qE_0}{m}-\left(\frac{qB_0}{m}\right)^2 z$$

$$\Rightarrow v_zdv_z=\left(\frac{qE_0}{m}\right)dz-\left(\frac{qB_0}{m}\right)^2 zdz \qquad \left\{\because v_z=\frac{dz}{dt}\right\}$$

$$\Rightarrow \int_0^{v_z}v_zdz=\frac{qE_0}{m}\int_0^z dz-\left(\frac{qB_0}{m}\right)^2\int_0^z z\,dz$$

$$\Rightarrow \frac{v_z^2}{2}=\left(\frac{qE_0}{m}\right)z-\left(\frac{qB_0}{m}\right)^2\frac{z^2}{2}$$

$$\Rightarrow v_z=\sqrt{\left(\frac{2qE_0}{m}\right)z-\left(\frac{qB_0}{m}\right)^2 z^2}$$

$$\Rightarrow v_x=\frac{qB_0z}{m},\ v_y=0 \text{ and}$$

$$v_z=\sqrt{\left(\frac{2qE_0}{m}\right)z-\left(\frac{qB_0}{m}\right)^2 z^2}$$

The speed of the particle is

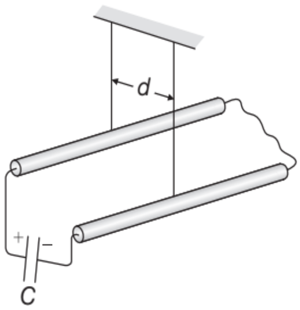
$$v=|\vec{v}|=\sqrt{v_x^2+v_y^2+v_z^2}$$

$$\Rightarrow v=\sqrt{\left(\frac{qB_0}{m}\right)^2 z^2+0+\left(\frac{2qE_0}{m}\right)z-\left(\frac{qB_0}{m}\right)^2 z^2}$$

$$\Rightarrow v=\sqrt{\left(\frac{2qE_0}{m}\right)z}$$

PROBLEM 24

Two long, straight conducting wires with linear mass density λ are suspended from cords so that they are each horizontal, parallel to each other, and a distance d apart. The back ends of the wires are connected to each other by a slack, low resistance connecting wire. A charged capacitor (capacitance C) is now added to the system; the positive plate of the capacitor (initial charge $+Q_0$) is connected to the front end of one of the wires, and the negative plate of the capacitor (initial charge $-Q_0$) is connected to the front end of the other wire. Both of these connections are also made by slack, low resistance wires. When the connection is made, the wires are pushed aside by the repulsive force between the wires, and each wire has an initial horizontal velocity of magnitude v_0. Assume that the time constant for the capacitor to discharge is negligible compared to the time it takes for any appreciable displacement in the position of the wires to occur.



(a) Show that the initial speed v_0 of either wire is given by

$$v_0 = \frac{\mu_0 Q_0^2}{4\pi\lambda RCd}$$

where R is the total resistance of the circuit.

(b) To what height h will each wire rise as a result of the circuit connection?

SOLUTION

(a) METHOD I:

The impulsive force, per unit length, due to (antiparallel) discharging currents is equal to the change in momentum. So,

$$\lambda v = \int_0^\infty F dt = \int_0^\infty \frac{\mu_0 I^2}{2\pi d} dt$$

where $I =$ discharging current

Now $q = Q_0 e^{-t/RC}$

$$\Rightarrow |I| = \frac{Q_0}{RC} e^{-t/RC}$$

$$\Rightarrow \lambda v = \left| \int_0^\infty \frac{\mu_0}{2\pi d}\left(\frac{Q_0^2}{R^2C^2} e^{-\frac{2t}{RC}} \right) dt \right| = \frac{\mu_0 Q_0^2}{4\pi RCd}$$

$$\Rightarrow v = \frac{\mu_0 Q_0^2}{4\pi\lambda RCd}$$

METHOD II:

The current carrying wires repel each other magnetically, causing them to accelerate horizontally. Since gravity is vertical, it plays no initial role

The magnetic force per unit length is $\frac{F}{L} = \frac{\mu_0}{2\pi}\frac{I^2}{d}$

and the acceleration obeys the equation $\left(\frac{F}{L}\right) = \left(\frac{m}{L}\right) a$

Also, the r.m.s. current over a short discharge time is $\frac{I_0}{\sqrt{2}}$

The force per unit length between the wires is given by

$$\frac{F}{L} = \frac{\mu_0}{2\pi}\frac{I^2}{d} = \frac{\mu_0}{2\pi d}\left(\frac{I_0}{\sqrt{2}}\right)^2$$

$$\Rightarrow \frac{F}{L} = \frac{\mu_0}{4\pi d}\left(\frac{V}{R}\right)^2 = \frac{\mu_0}{4\pi d}\left(\frac{Q_0}{RC}\right)^2$$

According to Newton's Second Law, we have

$$\frac{F}{L} = \left(\frac{m}{L}\right) a = \lambda a = \frac{\mu_0}{4\pi d}\left(\frac{Q_0}{RC}\right)^2$$

$$\Rightarrow a = \frac{\mu_0 Q_0^2}{4\pi\lambda R^2C^2 d}$$

From the kinematics equation, we have

$$v_x = v_{0x} + a_x t$$

$$\Rightarrow v_0 = at = aRC = \frac{\mu_0 Q_0^2}{4\pi\lambda RCd}$$

(b) By Law of Conservation of Energy, we have

$$\frac{1}{2} m v_0^2 = mgh$$

$$\Rightarrow h = \frac{v_0^2}{2g} = \frac{\left(\frac{\mu_0 Q_0^2}{4\pi\lambda RCd}\right)^2}{2g} = \frac{1}{2g}\left(\frac{\mu_0 Q_0^2}{4\pi\lambda RCd}\right)^2$$

Please note that, once the wires have swung apart, we would have to consider gravity in applying Newton's Second Law.

PROBLEM 25

A capacitor of capacitance C is connected to a battery of EMF E for a long time and then disconnected. The charged capacitor is then connected across a long solenoid having n turns per meter in its closely packed winding on its core. After connections it is found that the voltage across the capacitor drops to $\frac{E}{\eta}$ in a time Δt. In this period estimate the average magnetic induction at the centre of solenoid.

SOLUTION

Charge (q) on capacitor after time Δt is

$$q = \frac{CE}{\eta}$$

Charge flown (Δq) through the solenoid in this duration is

$$\Delta q = CE - \frac{CE}{\eta} = CE\left(1 - \frac{1}{\eta}\right)$$

Average current (i_{av}) in solenoid in this period is given by

$$i_{av} = \frac{\Delta q}{\Delta t} = \frac{CE}{\Delta t}\left(1 - \frac{1}{\eta}\right)$$

So, average magnetic field at the centre of solenoid is

$$B_{av} = \mu_0 n i_{av}$$

$$\Rightarrow \quad B_{av} = \frac{\mu_0 nCE}{\Delta t}\left(1 - \frac{1}{\eta}\right)$$

PROBLEM 26

A positively charged particle 1 having charge 1 C and mass 40 g, is revolving along a circle of radius 40 cm with velocity 5 ms^{-1} in a uniform magnetic field with centre of circle at origin O of a three dimensional system. At $t = 0$, the particle was at $(0, 0.4\text{ m}, 0)$ and velocity was directed along positive x-direction. Another particle 2 having charge 1 C and mass 10 g moving uniformly parallel to z-direction with velocity 40 ms^{-1} collides with the revolving particle at $t = 0$ and sticks to it. Neglecting gravitational force and coulomb force, calculate x, y, z coordinates of the combined particle at $t = \frac{\pi}{40}$ s.

SOLUTION

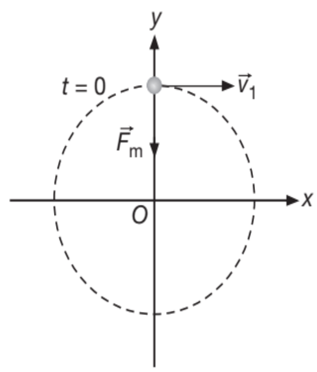
At time $t = 0$ particle 1 is at $(0,\ 0.4\text{ m},\ 0)$ and its velocity is along positive x-direction and magnetic force $\vec{F}_m$ is towards origin or along negative y-direction as shown. Hence according to Fleming's left hand rule, the magnetic field $\vec{B}$ should be along positive z-direction. Since, the radius of the circle, r, is given by

$$r = \frac{m_1 v_1}{B q_1}$$

$$\Rightarrow \quad B = \frac{m_1 v_1}{r q_1} = \frac{(0.04)(5)}{(0.4)(1)} = 0.5\text{ T}$$

Now by Law of Conservation of Linear Momentum, we get

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}$$



$$\Rightarrow \quad (0.04)5\hat{i} + (0.01)(40\hat{k}) = (0.04 + 0.01)\vec{v}$$

$$\Rightarrow \quad 0.05\vec{v} = (0.04)5\hat{i} + (0.01)(40)\hat{k}$$

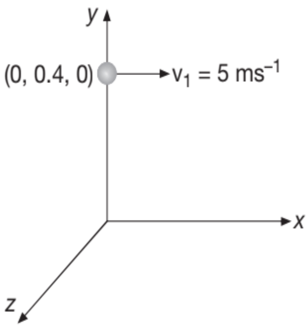
$$\Rightarrow \quad \vec{v} = 4\hat{i} + 8\hat{k}$$

Due to v_x, the combined particle tries to move clockwise along a circular path. However, v_z makes it move uniformly along z-axis so that its path becomes helical.

Also, we observe that this calculated velocity of the combined mass makes some angle $(\neq 0°,\ 180° \text{ or } 90°)$ with the magnetic field. Hence the path of the combined mass is helical, say of radius R, given by

$$R = \frac{(m_1 + m_2)v_x}{(q_1 + q_2)B}$$

$$\Rightarrow \quad R = \frac{(0.05)(4)}{(2)(0.5)} = 0.2\text{ m}$$



Time period of revolution is

$$T = \frac{2\pi(m_1 + m_2)}{(q_1 + q_2)B}$$

$$\Rightarrow \quad T = \frac{2\pi(0.05)}{(2)(0.5)} = \frac{\pi}{10} \text{ s}$$

The time at which the coordinates are to be calculated is

$$t = \frac{\pi}{40} \text{ s} = \frac{T}{4}$$

During this time the combined mass will rotate through quarter circle (a part of Helix) or through an angle $\theta = \omega t = \frac{\pi}{2}$ in x-y plane.

Hence now, the radius of this circle will become 0.2 m

So, $x = R = 0.2$ m

$$y = r - R = 0.2 \text{ m} \text{ and}$$

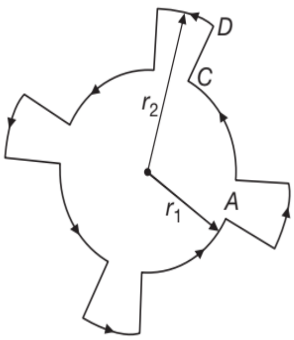
$$z = v_z t = (8)\left(\frac{\pi}{40}\right) = 0.628 \text{ m}$$

Hence, the position of combined mass at $t = \frac{\pi}{40}$ s is

$$(x,\ y,\ z) \equiv (0.2\ ,0.2\ ,0.628\ ) \text{ m}$$

PROBLEM 27

A current of 10 A flows around a closed path in a circuit which is in the horizontal plane as shown in the figure. The circuit consists of eight alternating arcs of radii $r_1 = 0.08$ m and $r_2 = 0.12$ m. Each subtends the same angle at the centre.



(a) Find the magnetic field produced by this circuit at the centre.

(b) An infinitely long straight wire carrying a current of 10 A is passing through the centre of the above circuit vertically with the direction of the current being into the plane of the circuit. What is the force acting on the wire at the centre due to the current in the circuit? What is the force acting on the arc AC and the straight segment CD due to the current at the centre?

SOLUTION

(a) Given $I = 10$ A, $r_1 = 0.08$ m and $r_2 = 0.12$ m. The straight portions, like CD, etc. will produce zero magnetic field at the centre. The remaining eight arcs will produce the magnetic field at the centre in the same direction, i.e., perpendicular to the paper outwards or vertically upwards and its magnitude is

$$B = B_{\text{inner arcs}} + B_{\text{outer arcs}}$$

$$\Rightarrow \quad B = \frac{1}{2}\left(\frac{\mu_0 I}{2r_1}\right) + \frac{1}{2}\left(\frac{\mu_0 I}{2r_2}\right)$$

$$\Rightarrow \quad B = \left(\frac{\mu_0}{4\pi}\right)(\pi I)\left(\frac{r_1 + r_2}{r_1 r_2}\right)$$

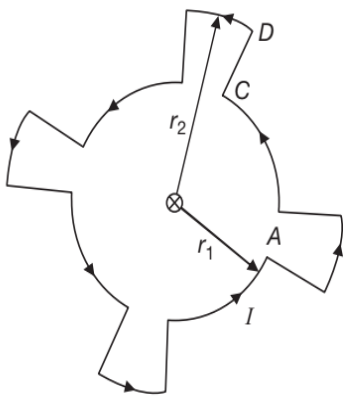
Substituting the values, we have

$$B = \frac{(10^{-7})(3.14)(10)(0.08 + 0.12)}{(0.08 \times 0.12)} \text{ T}$$

$$\Rightarrow \quad B = 6.54 \times 10^{-5} \text{ T}$$

{Vertically upward or outward normal to the paper}

(b) **Force on AC:** Force on circular portions of the circuit, i.e., AC etc. due to the wire at the centre will be zero because magnetic field due to the central wire at these arcs will be tangential ($\theta = 180°$) as shown.



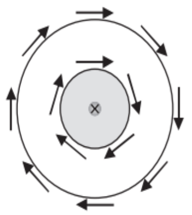
Force on CD: Current in central wire is also 10 A Magnetic field at P due to central wire,

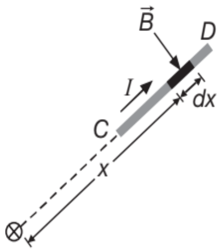
$$B = \frac{\mu_0}{2\pi}\left(\frac{I}{x}\right)$$

Consider an infinitesimal element on CD at a distance x from wire. Let this element have a length dx. Then magnetic force on element dx due to this magnetic field is dF given by

$$dF = (I)\left(\frac{\mu_0}{2\pi}\frac{I}{x}\right)dx \qquad \{\because F = BIl\sin 90°\}$$

$$\Rightarrow \quad dF = \left(\frac{\mu_0 I^2}{2\pi}\right)\frac{dx}{x}$$





So, the net force on CD is given by

$$F = \int_{x=r_1}^{x=r_2} dF = \frac{\mu_0 I^2}{2\pi}\int_{0.08}^{0.12}\frac{dx}{x}$$

$$\Rightarrow \quad F = \frac{\mu_0}{2\pi} I^2 \log_e\left(\frac{3}{2}\right)$$

Substituting the values,

$$F = \left(2\times 10^{-7}\right)(10)^2 \log_e(1.5)$$

$$\Rightarrow \quad F = 8.1\times 10^{-6}\ \text{N (inwards)}$$

Force on wire at the centre: Net magnetic field at the centre due to circuit is in vertical direction and current in the wire in centre is also in vertical direction. Therefore, net force on the wire at the centre will be zero. $(\theta = 180°)$. So, we have the force acting

(i) on the wire at the centre is zero.

(ii) on arc $AC = 0$

(iii) on segment CD is 8.1×10^{-6} N (inwards)

PROBLEM 28

A slightly divergent beam of charged particles accelerated by a potential difference V propagates from a point A along the axis of a solenoid. The beam is brought into focus at a distance l from the point A at two successive values of magnetic induction B_1 and B_2. Find the specific charge $\frac{q}{m}$ of the particles.

SOLUTION

The velocity of the particles accelerated by a potential V is given by

$$\frac{1}{2}mv^2 = Vq$$

$$\Rightarrow \quad v = \sqrt{\frac{2Vq}{m}}$$

Since the charged particles are slightly divergent so, they will follow a helical path.

If θ be the small angle made by a particle with B, then $\cos\theta \approx 1$.

So, pitch of the particle is

$$p = v_{\parallel} \times T$$

$$\Rightarrow \quad p = (v\cos\theta)\left(\frac{2\pi m}{qB}\right) = \frac{2\pi v m}{qB}$$

Particles are focussed at a distance l, if l contains integral number of pitches i.e.,

$$l = np$$

$$\Rightarrow \quad p = \frac{l}{n} = l,\ \frac{l}{2},\ \frac{l}{3}, \ldots\ldots$$

For two consecutive focussings (as B increases, p decreases), we get

$$l = \frac{2\pi mv}{qB_1} \quad \text{and} \quad \frac{l}{2} = \frac{2\pi mv}{qB_2}$$

$$\Rightarrow \quad B_1 = \frac{2\pi mv}{ql} \quad \text{and} \quad B_2 = \frac{4\pi mv}{ql}$$

$$\Rightarrow \quad B_2 - B_1 = \frac{2\pi mv}{ql}$$

$$\Rightarrow \quad B_2 - B_1 = \frac{2\pi m}{ql}\sqrt{\frac{2Vq}{m}}$$

$$\Rightarrow \quad \frac{q}{m} = \frac{8\pi^2 V}{l^2 (B_2 - B_1)^2}$$

PROBLEM 29

Three infinitely long thin wires, each carrying current I in the same direction, are in the X-Y plane of a gravity free space. The central wire is along the y-axis while the other two are along $x=\pm d$.

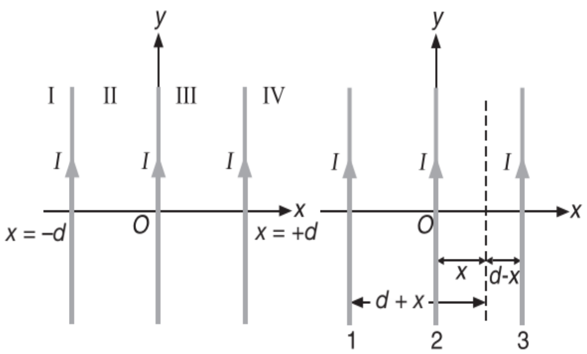
(a) Find the locus of the points for which the magnetic field B is zero.

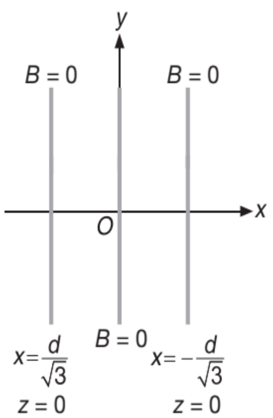
(b) If the central wire is displaced along the z-direction by a small amount and released, show that it will execute simple harmonic motion. If the linear density of the wires is λ, find the frequency of oscillation. '

SOLUTION

(a) Magnetic field will be zero on the y-axis, i.e., $x=0=z$.

Magnetic field cannot be zero in REGION I and REGION IV because in REGION I magnetic field will be along positive z-direction ⊙ due to all the three wires, while in REGION IV magnetic field will be along negative z-axis ⊗ due to all the three wires. It can be zero only in REGIONS II and III.





Let magnetic field is zero on line $z=0$ and $x=x$ (shown as dotted). The magnetic field on this line due to wires 1 and 2 will be along negative z-axis and due to wire 3 along positive z-axis. Thus,

$$B_1+B_2-B_3=0$$

$$\Rightarrow \frac{\mu_0}{2\pi}\frac{I}{(d+x)}+\frac{\mu_0 I}{2\pi x}=\frac{\mu_0}{2\pi}\frac{I}{(d-x)}$$

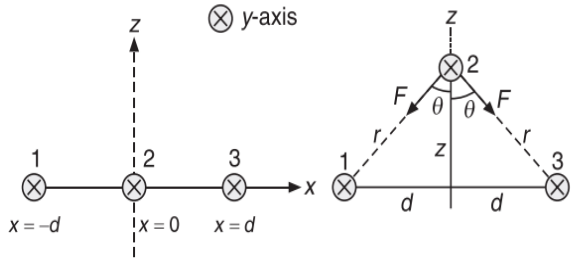
$$\Rightarrow \frac{1}{d+x}+\frac{1}{x}=\frac{1}{d-x}$$

Solving, we get, $x=\pm\dfrac{d}{\sqrt{3}}$

Hence, there will be two lines, where the magnetic field will be zero i.e.,

$$x=\frac{d}{\sqrt{3}} \text{ and } x=-\frac{d}{\sqrt{3}} \qquad \{z=0\}$$

(b) For the sake of convenience and understanding the problem easily, let us change our coordinate axes system as shown.



Three wires 1, 2 and 3 are shown in figure. Let the wire 2 be displaced slightly towards the z-axis, then force of attraction per unit length between wires (1 and 2) and (2 and 3) is given by

$$F=\frac{\mu_0}{2\pi}\frac{I^2}{r}$$

The components of F along x-axis will be cancelled out. Net resultant force will be towards negative z-axis (or mean position) and is given by

$$F_{net}=2F\cos\theta=2\left(\frac{\mu_0}{2\pi}\frac{I^2}{r}\right)\left(\frac{z}{r}\right)$$

$$\Rightarrow F_{net}=\frac{\mu_0}{\pi}\frac{I^2}{(z^2+d^2)}z$$

For small displacements, we have $z\ll d$, then

$$z^2+d^2\approx d^2$$

$$\Rightarrow F_{net}=-\left(\frac{\mu_0}{\pi}\frac{I^2}{d^2}\right)z$$

Negative sign implies that F_{net} is restoring in nature. i.e., $F_{net} \propto -z$

Hence the wire will oscillate simple harmonically. Let a be the acceleration of wire in this position and λ the mass per unit length of wire then

$$F_{net} = \lambda a = -\left(\frac{\mu_0}{\pi}\frac{I^2}{d^2}\right)z$$

$$\Rightarrow \quad a = -\left(\frac{\mu_0 I^2}{\pi\lambda d^2}\right)z$$

Comparing with the standard equation of SHM i.e., $\ddot{z} + \omega^2 z = 0$

we get

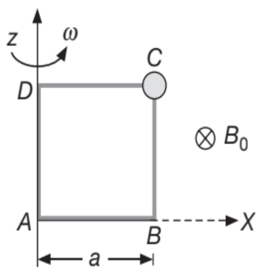
$$\omega = \sqrt{\frac{\mu_0 I^2}{\pi\lambda d^2}}$$

$$\Rightarrow \quad 2\pi f = \sqrt{\frac{\mu_0 I^2}{\pi\lambda d^2}}$$

$$\Rightarrow \quad f = \frac{I}{2\pi d}\sqrt{\frac{\mu_0}{\pi\lambda}}$$

PROBLEM 30

An insulated square frame $ABCD$ of side a is able to rotate about one of its side taken as positive z-axis. A magnetic field B is present in the region given by $\vec{B} = B_0\hat{j}$. A small bead of mass m and charge q movable alongside CB is initially near C, when frame lies in x-z plane. Now, frame is rotated by constant angular velocity ω about z-axis. Whole system lies in gravity free space. If after time t, the bead reaches point B, find B_0 in terms of t.



SOLUTION

Bead moves with speed $v = a\omega$ in x-y plane

Velocity of bead, when frame has rotated through angle $\theta = \omega t$

$$\vec{v} = a\omega\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) + v_z\hat{k}$$

Force on bead

$$F = q\vec{v}\times\vec{B} = \left[a\omega\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right) + v_z\hat{k}\right]\times B_0\hat{j}$$

Force in z-direction $= a\omega q B_0\left(-\hat{k}\right)\sin\theta$

Acceleration of bead $\vec{a}_z = -\frac{a\omega q B_0\sin\theta}{m}\left(-\hat{k}\right)$

$$\Rightarrow \quad \frac{dv_z}{dt} = -\frac{a\omega q B_0}{m}\sin\omega t$$

$$\Rightarrow \quad v_z = -\frac{a\omega q B_0}{m}\int_0^t \sin\omega t\,dt = -\frac{aB_0}{m}(1-\cos\omega t)$$

$$\Rightarrow \quad \frac{dz}{dt} = -\frac{aqB_0}{m}(1-\cos\omega t)$$

$$\Rightarrow \quad dz = -\frac{aqB_0}{m}\int_0^t(1-\cos\omega t)$$

$$\Rightarrow \quad z = -\frac{aqB_0}{m}\left[t - \frac{\sin\omega t}{\omega}\right]_0^1 = -\frac{aqB_0}{m}\left[t - \frac{\sin\omega t}{\omega}\right]$$

When bead reaches B, then we have $z = -a$

$$\Rightarrow \quad -a = -\frac{aqB_0}{m}\left(t - \frac{\sin\omega t}{\omega}\right)$$

$$\Rightarrow \quad B_0 = \frac{m\omega}{q(\omega t - \sin\omega t)}$$

PROBLEM 31

The force on a magnetic dipole $\vec{M}$ aligned with a non-uniform magnetic field in the x direction is given by $F_x = |\vec{M}|\frac{dB}{dx}$. Consider two flat loops of wire each have radius R and carry current I.

(a) The loops are arranged coaxially and separated by a variable distance x, large compared to R. Show that the magnetic force between them varies as $\frac{1}{x^4}$.

(b) Evaluate the magnitude of this force if $I = 10$ A, $R = 0.5$ cm and $x = 5$ cm.

SOLUTION

(a) The magnetic field produced by one loop at the center of the second loop for $x \gg R$ is given by

$$B = \frac{\mu_0 I R^2}{2x^3} = \frac{\mu_0 I(\pi R^2)}{2\pi x^3} = \frac{\mu_0 M}{2\pi x^3}$$

where the magnetic moment of either loop is $M = I(\pi R^2)$. Therefore,

$$|F_x| = M\frac{dB}{dx} = \mu\left(\frac{\mu_0 M}{2\pi}\right)\left(\frac{3}{x^4}\right) = \frac{3\mu_0(\pi R^2 I)^2}{2\pi x^4}$$

$$\Rightarrow |F_x| = \frac{3\pi}{2}\left(\frac{\mu_0 I^2 R^4}{x^4}\right)$$

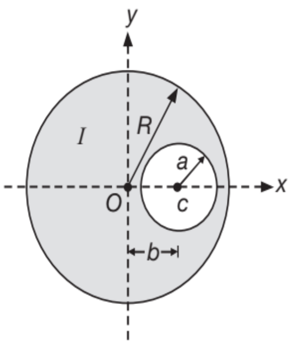
(b) $|F_x| = \dfrac{3\pi}{2}\left(\dfrac{\mu_0 I^2 R^4}{x^4}\right)$

$$\Rightarrow |F_x| = \frac{3\pi}{2}\frac{(4\pi\times10^{-7})(10)^2(5\times10^{-3})^4}{(5\times10^{-2})^4}$$

$$\Rightarrow |F_x| \cong 6\times10^{-8}\text{ N}$$

PROBLEM 32

A very long straight conductor has a circular cross-section of radius R and carries a current I. Inside the conductor there is a cylindrical cavity of radius a whose axis is parallel to the axis of the conductor and a distance b from its centre. Let the z-axis be the axis of the conductor, and let the axis of the cavity be at $x = b$. Find the magnetic field



(a) on the x-axis at $x = 2R$

(b) on the y-axis at $y = 2R$

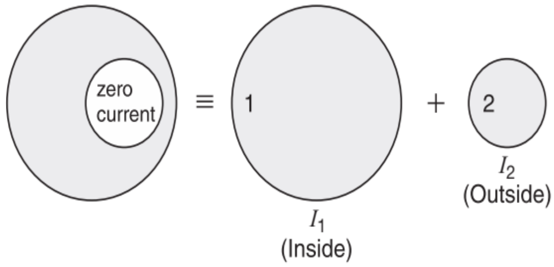
(c) at a point P inside the cylindrical cavity. Also show that the field at P is constant both in magnitude and direction.

SOLUTION

(a) Let us take help from the concept of current density. Since current density (current per unit area) in the wire is,

$$J = \frac{I}{\pi R^2 - \pi a^2} = \frac{I}{\pi(R^2 - a^2)}$$

The given arrangement is obtained by superimposing a uniform inward cylindrical current distribution of radius R and an inward cylindrical current distribution of radius a, so that net current in the cavity of radius a is zero.



Current void (zero current space) obtained from superposition of 1 and 2

$$I_1 = J(\pi R^2) \text{ and } I_2 = J(\pi a^2)$$

Magnetic field at $x = 2R$ due to current I_1 is,

$$B_1 = \frac{\mu_0}{2\pi}\frac{I_1}{2R} \quad \{\text{along negative } y\text{-direction}\}$$

Magnetic field at $x = 2R$ due to current I_2 is,

$$B_2 = \frac{\mu_0}{2\pi}\frac{I_2}{(2R-b)} \quad \{\text{along positive } y\text{-direction}\}$$

Therefore, net magnetic field at $x = 2R$ is,

$$B = B_2 - B_1 \quad \{\text{along positive } y\text{-direction}\}$$

Substituting, the values, we get

$$B = \frac{\mu_0 J}{2\pi}\left[\frac{\pi a^2}{2R-b} - \frac{\pi R^2}{2R}\right] = \frac{\mu_0 J}{2}\left(\frac{a^2}{2R-b} - \frac{R}{2}\right)$$

$$B = \frac{\mu_0 I}{2\pi(R^2 - a^2)}\left(\frac{a^2}{2R-b} - \frac{R}{2}\right)$$

(b) At $y = 2R$, magnetic field due to current I_1 is,

$$B_1 = \frac{\mu_0}{2\pi}\frac{I_1}{2R} \quad \{\text{along positive } x\text{-direction}\}$$

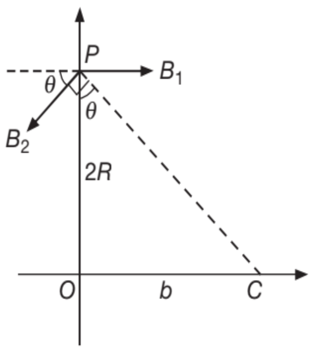
and due to current I_2, magnetic field is B_2 in a direction perpendicular to CP. Here

$$B_2 = \frac{\mu_0}{2\pi}\frac{I_2}{\sqrt{(2R)^2 + b^2}} = \frac{\mu_0}{2\pi}\frac{I_2}{\sqrt{4R^2 + b^2}}$$

Now this has two components one along negative x-axis and another along negative y-axis. Thus,

$$B_{2x} = -B_2\cos\theta = -\frac{\mu_0}{2\pi}\frac{(I_2)(2R)}{(4R^2 + b^2)}$$

and $B_{2y} = -B_2\sin\theta = -\dfrac{\mu_0}{2\pi}\dfrac{(I_2)(b)}{(4R^2 + b^2)}$



Therefore, net magnetic field along x-axis is,

$$B_x = B_1 + B_{2x} = \frac{\mu_0}{2\pi}\frac{I_1}{2R} - \frac{\mu_0}{2\pi}\frac{(I_2)(2R)}{(4R^2 + b^2)}$$

Substituting the value of I_1 and I_2, we have

$$B_x = \frac{\mu_0 J}{2}\left[\frac{R}{2} - \frac{2Ra^2}{4R^2 + b^2}\right]$$

and $B_y = -\dfrac{\mu_0 J}{2}\left[\dfrac{a^2 b}{4R^2 + b^2}\right]$

$\Rightarrow \vec{B} = B_x\hat{i} + B_y\hat{j}$

$$\Rightarrow \vec{B} = \frac{\mu_0 J}{2}\left[\left(\frac{R}{2} - \frac{2Ra^2}{4R^2 + b^2}\right)\hat{i} - \left(\frac{a^2 b}{4R^2 + b^2}\right)\hat{j}\right],$$

where $J = \dfrac{I}{\pi(R^2 - a^2)}$

(c) Let the point P inside the cavity be at a distance r_1 from O and r_2 from C.

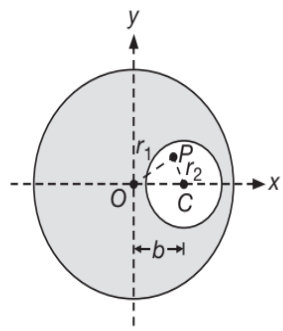
At point P magnetic field due to I_1 is B_1 (perpendicular to OP) and is $\vec{B}_2$ due to I_2 (perpendicular to CP) in the directions shown. Although $\vec{B}_1$ and $\vec{B}_2$ are actually at P, but for getting a clear picture and better understanding they are drawn at O and C respectively. Let B_x be the x component of resultant of $\vec{B}_1$ and $\vec{B}_2$ and B_y its y-component. Then,

$$B_x = B_1\sin\alpha - B_2\sin\beta$$

$$B_x = \left(\frac{\mu_0}{2\pi}\frac{I_1}{R^2}r_1\right)\sin\alpha - \left(\frac{\mu_0}{2\pi}\frac{I_2}{a^2}\cdot r_2\right)\sin\beta$$

$$B_x = \left(\frac{\mu_0}{2\pi}\frac{J\pi R^2}{R^2}\cdot r_1\sin\alpha\right) - \left(\frac{\mu_0}{2\pi}\frac{J\cdot\pi a^2}{a^2}r_2\sin\beta\right)$$

$$B_x = \frac{\mu_0 J}{2}(r_1\sin\alpha - r_2\sin\beta) \qquad \ldots(1)$$

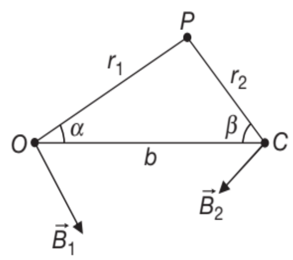


In ΔOPC, we have, from LAMI's Theorem

$$\frac{r_1}{\sin\beta} = \frac{r_2}{\sin\alpha}$$

$\Rightarrow r_1\sin\alpha - r_2\sin\beta = 0$

$\Rightarrow B_x = 0$



Now, $B_y = -(B_1\cos\alpha + B_2\cos\beta)$

$$B_y = -\frac{\mu_0 J}{2}(r_1\cos\alpha + r_2\cos\beta) \qquad \ldots(2)$$

From ΔOPC, we can see that

$$r_1\cos\alpha + r_2\cos\beta = b$$

$$\Rightarrow B_y = -\frac{\mu_0 Jb}{2} = -\frac{\mu_0 Ib}{2\pi(R^2 - a^2)} = \text{constant}$$

Thus, we can see that net magnetic field at point P is along negative y direction and is constant in magnitude.

CHECK POINT

At point C magnetic field due to I_2 is zero (i.e. $B_2 = 0$) while that due to I_1 is $\frac{\mu_0}{2\pi}\frac{I_1}{R^2}b$ in negative y-direction.

Substituting, $I_1 = J(\pi R^2)$ we get,

$$B = B_2 = \frac{\mu_0 Ib}{2\pi(R^2 - a^2)} \text{\{along negative } y\text{-direction\}}$$

This agrees with the result derived above.

PROBLEM 33

In a certain region of space there exists a uniform and constant electric field of magnitude E along the positive y-axis of a co-ordinate system. A charged particle of mass m and charge $-q(q>0)$ is projected with speed $2v_0$ at an angle of $60°$ with the positive x-axis in x-y plane from the origin. When the x-coordinate of the particle becomes $\frac{\sqrt{3}mv_0^2}{qE}$, a uniform and constant magnetic field of strength B is also switched on along the positive y-axis. Find the co-ordinate of the particle as a function of time t after its projection.

SOLUTION

According to the problem, we have

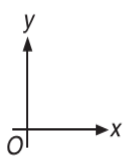
$$\vec{E} = E\hat{j}$$

$$\vec{v} = 2v_0\cos 60°\hat{i} + 2v_0\sin 60°\hat{j} = v_0\hat{i} + v_0\sqrt{3}\hat{j}$$

$$\vec{a} = -\frac{qE}{m}\hat{j}$$

$$\vec{R}(t) = v_0 t\hat{i} + \left(v_0\sqrt{3}t - \frac{1}{2}\frac{qE}{m}t^2\right)\hat{j}$$

Given, $x = v_0 t = \frac{\sqrt{3}mv_0^2}{qE}$



$$\Rightarrow \quad t = \frac{\sqrt{3}mv_0}{qE}$$

For $t \le \frac{\sqrt{3}mv_0}{qE}$

$$x(t) = v_0 t \text{ and } v_y(t) = v_0\sqrt{3} - \frac{qE}{m}t$$

$$\Rightarrow \quad v_y\left(t = \frac{\sqrt{3}mv_0}{2E}\right) = v_0\sqrt{3} - \left(\frac{qE}{m}\right)\frac{\sqrt{3}mv_0}{qE} = 0$$

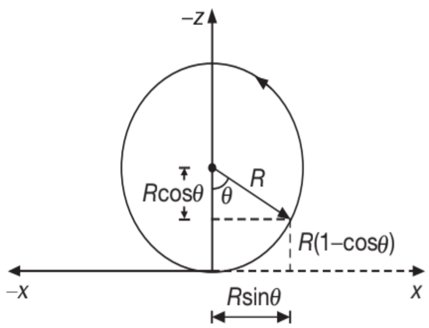
For $t \ge \frac{\sqrt{3}mv_0}{qE}$, magnetic field is also present

The particle will start moving on helical path. The cross-section of the helix will be in x-z plane

$$R = \frac{mv_0}{qB}$$

$$\omega = \frac{qB}{m}$$

$$\theta = \omega t = \frac{qBt}{m}$$



For $t \ge \frac{\sqrt{3}mv_0}{qE}$, we get

$$x(t) = \frac{\sqrt{3}mv_0^2}{qE} + R\sin\left(\frac{qB}{m}t\right),$$

$$y(t) = \frac{3mv_0^2}{2qE} - \frac{1}{2}\left(\frac{qE}{m}\right)t^2 \text{ and}$$

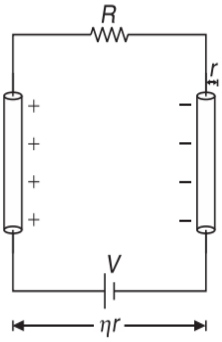
$$z(t) = -R\left[1 - \cos\left(\frac{qB}{m}t\right)\right]$$

PROBLEM 34

Two long parallel wires of negligible resistance are connected at one end to a resistance R and at the other end to a constant voltage source of voltage V. The distance between the axes of the wires is η times greater than the cross-sectional radius of each wire. At what value of resistance R, does the resultant force of interaction between the wires will become zero?

SOLUTION

The situation described in the question is shown in figure.



The two wires form a wire capacitor where capacitance is given by

$$C = \frac{\pi\varepsilon_0 l}{\ln\eta}$$

Charge (q) on the capacitor in steady state is

$$q = CV = \frac{\pi\varepsilon_0 lV}{\ln\eta} \quad \ldots(1)$$

Also, we observe that due to voltage source, a constant current flow in opposite directions through the wires which repel each other by a magnetic force. In this magnetic force gets nullified by the force of attraction between the two wires due to opposite charges on them, then the resultant force of interaction between the wires will become zero.
The force of attraction is given as

$$F_e = qE = q\left(\frac{q}{2\pi\varepsilon_0 l(\eta r)}\right) \quad \ldots(2)$$

Substituting the value of q from equation (1) in equation (2), we get

$$F_e = \frac{\left(\frac{\pi\varepsilon_0 lV}{\ln\eta}\right)^2}{2\pi\varepsilon_0 l(\eta r)} \quad \ldots(3)$$

Since the current flows is in the opposite direction in the two wires, so the net magnetic force of repulsion between these wires is given by

$$F_m = \frac{\mu_0 I^2 l}{2\pi(\eta r)} = \frac{\mu_0\left(\frac{V}{R}\right)^2 l}{2\pi\eta r} \quad \ldots(4)$$

Since the net force between the two wires is zero, so we have

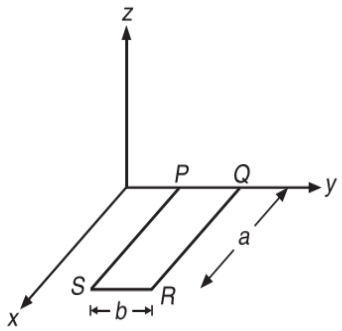
$$F_e = F_m$$

$$\Rightarrow \frac{\mu_0\left(\frac{V}{R}\right)^2 l}{2\pi\eta r} = \frac{\left(\frac{\pi\varepsilon_0 lV}{\ln\eta}\right)^2}{2\pi\varepsilon_0 l(\eta r)}$$

$$\Rightarrow R = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\ln\eta}{\pi}$$

PROBLEM 35

A rectangular loop $PQRS$ made from a uniform wire has length a, width b and mass m. It is free to rotate about the arm PQ, which remains hinged along a horizontal line taken as the y-axis (shown in figure). Take the vertically upward direction as the z-axis. A uniform magnetic field $\vec{B} = (3\hat{i} + 4\hat{k})B_0$ exists in the region. The loop is held in the x-y plane and a current I is passed through it. The loop is now released and is found to stay in the horizontal position in equilibrium.



(a) What is the direction of the current I in PQ?

(b) Find the magnetic force on the arm RS.

(c) Find the expression for I in terms of B_0, a, b and m.

SOLUTION

(a) Let the direction of current in wire PQ is from P to Q and its magnitude be I.

The magnetic moment of the given loop is $\vec{M} = -Iab\hat{k}$

Torque on the loop due to magnetic forces is

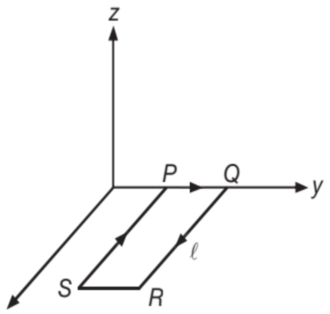
$$\vec{\tau}_1 = \vec{M}\times\vec{B} = (-Iab\hat{k})\times(3\hat{i} + 4\hat{k})B_o\hat{i}$$

$$\Rightarrow \vec{\tau}_1 = -3IabB_o\hat{j}$$

Torque of weight of the loop about axis PQ is

$$\vec{\tau}_2 = \vec{r}\times\vec{F} = \left(\frac{a}{2}\hat{i}\right)\times(-mg\hat{k}) = \frac{mga}{2}\hat{j}$$

We see that when the current in the wire PQ is from P to Q, $\vec{\tau}_1$ and $\vec{\tau}_2$ are in opposite direction, so they can cancel each other and the loop may remain in equilibrium. So, the direction of current I in wire PQ is from P to Q.



(b) Further for equilibrium of the loop:

$$|\vec{\tau}_1| = |\vec{\tau}_2|$$

$$\Rightarrow \quad 3IabB_o = \frac{mga}{2}$$

$$\Rightarrow \quad I = \frac{mg}{6bB_o}$$

(c) Magnetic force on wire RS is

$$\vec{F} = I(\vec{l} \times \vec{B}) = I\left[(-b\hat{j}) \times \left\{(3\hat{i} + 4\hat{k})B_o\right\}\right]$$

$$\Rightarrow \quad \vec{F} = IbB_o(3\hat{k} - 4\hat{i})$$

PROBLEM 36

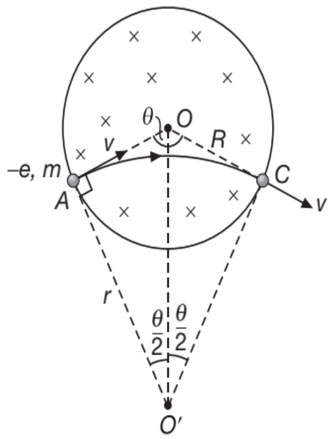
A direct current flowing through the winding of a long cylindrical solenoid of radius R produces a uniform magnetic induction B in it. An electron travelling at velocity v enters into the solenoid along the radial direction between its turns at right angles to the solenoid axis. After a certain time, the electron deflected by magnetic field leaves the solenoid. Calculate the time which the electron spends inside the solenoid.

SOLUTION

The magnetic field due to the solenoid is directed along its axis. The magnetic force on the electron at any instant will be in a plane perpendicular to solenoid axis and the trajectory of the electron in the solenoid will be an arc of a circle. The radius r of the circular arc is given by

$$r = \frac{mv}{qB} = \frac{mv}{eB} \qquad \ldots(1)$$

The trajectory followed by the electron is shown in figure.



If electron leaves the solenoid at point C after suffering a deviation θ, then from figure we have

$$\angle AO'O = \frac{1}{2}\angle AO'C = \frac{\theta}{2}$$

$$\Rightarrow \quad \tan\left(\frac{\theta}{2}\right) = \frac{R}{r} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$\tan\left(\frac{\theta}{2}\right) = \frac{eBR}{mv}$$

$$\Rightarrow \quad \theta = 2\tan^{-1}\left(\frac{eBR}{mv}\right)$$

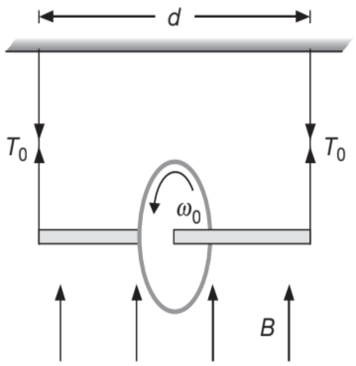
Since the magnitude of velocity remains constant over the entire trajectory, so the transit time of electron inside the solenoid is given by

$$t = \frac{r\theta}{v} = \left(\frac{m}{eB}\right)\theta$$

$$\Rightarrow \quad t = \frac{2m}{eB}\tan^{-1}\left(\frac{eBR}{mv}\right)$$

PROBLEM 37

A ring of radius R having uniformly distributed charge Q is mounted on a rod suspended by two identical strings. The tension in strings in equilibrium is T_0. Now a vertical magnetic field is switched on the ring is rotated at constant angular velocity w. Find the maximum ω with which the ring can be rotated if the strings can withstand a maximum tension of $\frac{3T_0}{2}$.



SOLUTION

In equilibrium, $2T_o = mg$

$$\Rightarrow \quad T_o = \frac{mg}{2} \qquad \ldots(1)$$

Magnetic moment, $M = IA = \left(\frac{\omega}{2\pi}Q\right)\left(\pi R^2\right)$

$$\Rightarrow \quad \tau = MB\sin 90^o = \frac{\omega BQR^2}{2}$$

Let T_1 and T_2 be the tensions in the two strings when magnetic field is switched on $(T_1 > T_2)$.

For translational equilibrium of ring in vertical direction.

$$T_1 + T_2 = mg \qquad \ldots(2)$$

For rotational equilibrium,

$$(T_1 - T_2)\frac{D}{2} = \tau = \frac{\omega BQR^2}{2}$$

$$\Rightarrow \quad T_1 - T_2 = \frac{\omega BQR^2}{2} \qquad \ldots(3)$$

Solving equations (2) and (3), we have

$$T_1 = \frac{mg}{2} + \frac{\omega BQR^2}{2D}$$

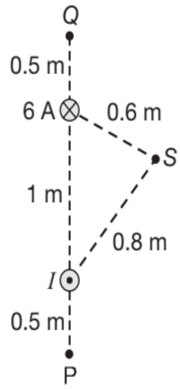
As $T_1 > T_2$ and maximum values of T_1 can be $\frac{3T_o}{2}$, so we get

$$\frac{3T_o}{2} = T_o + \frac{\omega_{max} BQR^2}{2D} \qquad \left(\frac{mg}{2} = T_o\right)$$

$$\Rightarrow \quad \omega_{max} = \frac{DT_o}{BQR^2}$$

PROBLEM 38

Two long, straight, parallel wires are 1 m apart. The wire on the left carries a current I_1 of 6 A into the plane of the paper.



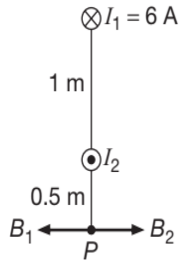
(a) What must the magnitude and direction of the current I_2 be for the net field at point P to be zero?

(b) What are the magnitude and direction of the net field at Q?

(c) What is the magnitude of the net field at S?

SOLUTION

(a) The directions of the fields at point P due to the two wires are shown in figure $\vec{B}_1$ and $\vec{B}_2$ must be equal and opposite for the resultant field at P to be zero. $\vec{B}_2$ is to the right so I_2 is out of the page.



$$B_1 = \frac{\mu_0 I_1}{2\pi r_1} = \frac{\mu_0}{2\pi}\left(\frac{6\text{ A}}{1.5\text{ m}}\right) \text{ and}$$

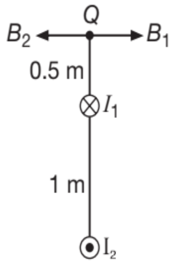
$$B_2 = \frac{\mu_0 I_2}{2\pi r_2} = \frac{\mu_0}{2\pi}\left(\frac{I_2}{0.5\text{ m}}\right)$$

$$B_1 = B_2$$

$$\Rightarrow \quad \frac{\mu_0}{2\pi}\left(\frac{6\ A}{1.5\text{ m}}\right) = \frac{\mu_0}{2\pi}\left(\frac{I_2}{0.5\text{ m}}\right)$$

$$\Rightarrow \quad I_2 = \left(\frac{0.5\text{ m}}{1.5\text{ m}}\right)(6\text{ A}) = 2\text{ A}$$

(b) The directions of the fields at point Q are shown in figure



$$B_1 = \frac{\mu_0 I_1}{2\pi r_1}$$

$$\Rightarrow B_1 = \left(2\times10^{-7}\ \text{T}\cdot\text{mA}^{-1}\right)\left(\frac{6\ \text{A}}{0.5\ \text{m}}\right)$$

$$\Rightarrow B_1 = 2.4\times10^{-6}\ \text{T} \text{ and } B_2 = \frac{\mu_0 I_2}{2\pi r_2}$$

$$\Rightarrow B_2 = \left(2\times10^{-7}\ \text{T}\cdot\text{mA}^{-1}\right)\left(\frac{2\ \text{A}}{1.5\ \text{m}}\right)$$

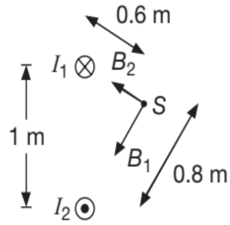
$$\Rightarrow B_2 = 2.67\times10^{-7}\ \text{T}$$

$\vec{B}_1$ and $\vec{B}_2$ are in opposite directions and $B_1 > B_2$. Hence

$$B = B_1 - B_2 = 2.4\times10^{-6}\ \text{T} - 2.67\times10^{-7}\ \text{T}$$

$B = 2.13\times10^{-6}\ \text{T}$, and $\vec{B}$ is to the right

(c) The directions of the fields at point S are shown in figure



$$B_1 = \frac{\mu_0 I_1}{2\pi r_1}$$

$$\Rightarrow B_1 = \left(2\times10^{-7}\ \text{T}\cdot\text{mA}^{-1}\right)\left(\frac{6\ \text{A}}{0.6\ \text{m}}\right) = 2\times10^{-6}\ \text{T}$$

and $B_2 = \dfrac{\mu_0 I_2}{2\pi r_2}$

$$\Rightarrow B_2 = \left(2\times10^{-7}\ \text{TmA}^{-1}\right)\left(\frac{2\ \text{A}}{0.8\ \text{m}}\right) = 5\times10^{-7}\ \text{T}$$

$\vec{B}_1$ and $\vec{B}_2$ are right angles to each other, so the magnitude of their resultant is given by

$$B = \sqrt{B_1^2 + B_2^2} = \sqrt{\left(2\times10^{-6}\ \text{T}\right)^2 + \left(5\times10^{-7}\ \text{T}\right)^2}$$

$$B = 2.06\times10^{-6}\ \text{Ta}$$