CHAPTER

2 Capacitance and Applications

Learning Objectives

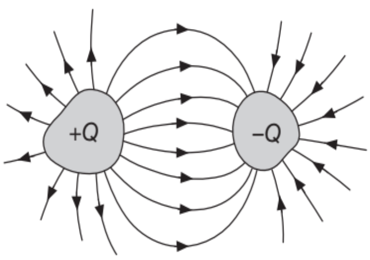
After reading this chapter, you will be able to understand concepts and problems based on:

(a) Capacitors
(b) Capacitance
(c) Energy Concepts
(d) Effect of Dielectric Insertion
(e) Capacitor Circuits
(f) Spherical Capacitor and Cylindrical Capacitor

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

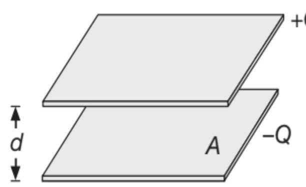
CAPACITORS: INTRODUCTION

A capacitor is a device which stores electric charge. Capacitors vary in shape and size, but the basic configuration involves two conductors carrying equal but opposite charges (shown in figure). Capacitors have many important applications in electronics. Some examples include storing electric potential energy, delaying voltage changes when coupled with resistors, filtering out unwanted frequency signals, forming resonant circuits and making frequency dependent and independent voltage dividers when combined with resistors.



In the uncharged state, the charge on either one of the conductors in the capacitor is zero. During the charging process, a charge Q is moved from one conductor to the other one, giving one conductor a charge $+Q$, and the other one a charge $-Q$ as a result of which a potential difference ΔV is created, with the positively charged conductor at a higher potential than the negatively charged conductor. Note that whether charged or uncharged, the net charge on the capacitor as a whole is zero.

The simplest example of a capacitor consists of two conducting plates of area A, which are parallel to each other, and separated by a distance $d(\ll A)$, as shown in figure (called as a Parallel Plate Capacitor).



Experimentally it has been verified that the amount of charge Q stored in a capacitor is linearly proportional to ΔV, the electric potential difference between the plates. Thus, we may write

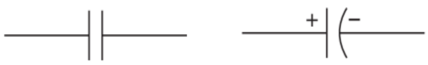
$$Q = C|\Delta V|$$

$$\Rightarrow \quad C = \frac{Q}{|\Delta V|}$$

where C is a positive proportionality constant called capacitance. Physically, capacitance is a measure of the capacity of storing electric charge for a given potential difference ΔV. The SI unit of capacitance is the farad (F).

$$1\ \mathrm{F} = 1\ \text{farad} = 1\ \text{coulomb volt}^{-1} = 1\ \mathrm{CV}^{-1}$$

Figure (a) shows the symbol which is used to represent capacitors in circuits. For a polarized fixed capacitor which has a definite polarity. Figure (b) is rarely used.



(a) (b)

CALCULATION OF CAPACITANCE

As discussed, the method for the calculation of capacitance involves integration of the electric field between two conductors or the plates which are just equipotential surfaces to obtain the potential difference ΔV. Thus,

$$\Delta V = -\int_b^a \vec{E}\cdot d\vec{r}$$

Since, by definition, we have $C = \dfrac{Q}{|\Delta V|}$

$$\Rightarrow \quad C = \frac{Q}{|\Delta V|} = \frac{Q}{\left|-\int_b^a \vec{E}\cdot d\vec{r}\right|}$$

DEFINITION OF C

$$C = \frac{Q}{\Delta V} \qquad \ldots(1)$$

If $\Delta V = 1$ volt, then

$$C = Q \quad \text{(numerically)}$$

i.e., capacitance of a conductor is numerically equal to the charge that raises its potential by 1 volt.

Practically farad is a very big unit for capacitance and generally smaller units such as μF (microfarad), nF (nanofarad), pF (picofarad) are used.

$$1\ \mu\mathrm{F} = 10^{-6}\ \mathrm{F}$$

$$1\ \mathrm{nF} = 10^{-9}\ \mathrm{F}$$

$$1\ \mathrm{pF} = 10^{-12}\ \mathrm{F} = 1\ \mu\mu\mathrm{F}$$

CGS unit of capacitance is statfarad (SF) and

$$1\ \text{farad} = 9\times 10^{11}\ \text{statfarad}$$

Dimensional formula for capacitance is,

$$[C] = M^{-1}L^{-2}T^{4}A^{2}$$

ENERGY STORED IN A CHARGED CAPACITOR

Whenever any conductor is charged, the charge produces an electric field outside it. Now if, further similar charge is to be given to the conductor some work has to be done by the external agency against the electrostatic forces of repulsion. This work done is stored as the electrostatic potential energy which is stored in the electrostatic field of the conductor.

Let us consider that charge is given in a stepwise manner to the conductor. If V be the potential of the conductor at the instant when it has a charge q, then

$$V = \frac{q}{C}$$

If dq is the additional infinitesimal charge that is to be given to the conductor and dW is the corresponding work done, then

$$dW = V\,dq$$

$$\Rightarrow \quad dW = \frac{q}{C}\,dq$$

Total work done to charge the conductor to Q is

$$W = \int dW = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{C}\left.\frac{q^2}{2}\right|_0^Q$$

$$\Rightarrow \quad W = \frac{Q^2}{2C}$$

If V_0 is the potential that develops on the conductor finally on account of charge Q then

$$V_0 = \frac{Q}{C}$$

$$\Rightarrow \quad W = \frac{1}{2}CV_0^2 = \frac{Q^2}{2C} = \frac{1}{2}QV_0$$

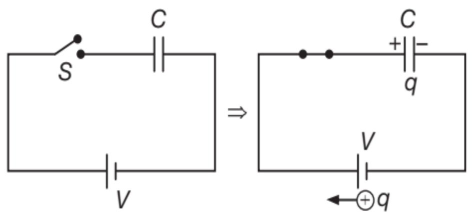
ILLUSTRATION 1

Prove that while charging a capacitor half of the energy supplied by the battery dissipates in the form of heat.

SOLUTION

When the switch S is closed, the charge stored in the capacitor is

$$q = CV$$



Charge transferred from the battery is also q. So, energy supplied by the battery $= qV = (CV)(V) = CV^2$.

However the capacitor stores only $\frac{1}{2}CV^2$ which is exactly half this energy. So, the remaining half or 50% or $\frac{1}{2}CV^2$ must have been dissipated as heat.

CAPACITANCE OF AN ISOLATED SPHERE

Let a conducting sphere of radius R acquire a potential V when a charge Q is given to it. The potential acquired by the sphere is

$$V = \frac{Q}{4\pi\varepsilon_0 R}$$

$$\Rightarrow \quad C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$

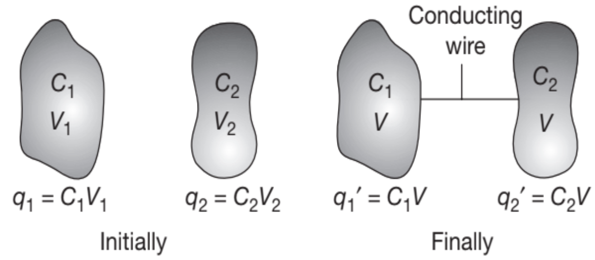
In CGS system $\frac{1}{4\pi\varepsilon_0} = 1$

So, $C = R$ (numerically).

Hence, in cgs system the capacitance of an isolated sphere equals its radius.

CHARGE SHARING BETWEEN TWO CHARGED CONDUCTORS

Consider two charged conductors having capacitance C_1 and C_2 at potential V_1 and V_2 respectively. Let the two be connected by a conducting wire such that the charge now starts flowing from the conductor at higher potential to the conductor at lower potential till both the conductors acquire the same potential.



Total initial charge $= q_1 + q_2 = C_1V_1 + C_2V_2$

If V is the common potential after charge sharing takes place then

Total final charge $= q_1' + q_2' = C_1V + C_2V$

By Law of Conservation of Charge

Total Initial Charge = Total Final Charge

$$\Rightarrow \quad C_1V_1 + C_2V_2 = (C_1 + C_2)V$$

$$\Rightarrow \quad V = \frac{C_1V_1 + C_2V_2}{C_1 + C_2} \qquad \ldots(1)$$

Further after sharing the new potential is the same. So,

$$\frac{q_1'}{q_2'} = \frac{C_1V}{C_2V} = \frac{C_1}{C_2}$$

i.e., charges are shared in the ratio of the capacitances after common potential (V) is acquired.

Further, there is always a loss in energy during the sharing process as some energy gets converted to heat.

Loss $= -\Delta U =$ Total Initial Energy − Total Final Energy

$$\Rightarrow \quad \text{Loss} = -\Delta U = \left(\frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2\right) - \left(\frac{1}{2}C_1V^2 + \frac{1}{2}C_2V^2\right)$$

$$\Rightarrow \quad \text{Loss} = -\Delta U = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2 - \frac{1}{2}(C_1 + C_2)V^2$$

From (1) put value of V above we get

$$\text{Loss} = -\Delta U = \frac{1}{2}\left(\frac{C_1C_2}{C_1 + C_2}\right)(V_1 - V_2)^2$$

ILLUSTRATION 2

A capacitor of capacitance $\mathbf{C_0}$ is charged to a potential $\mathbf{V_0}$ and then isolated. A small capacitor C is then charged from C_0, discharged and then charged again.

If the process is being repeated n times, the potential of the large capacitor has now fallen to V. Calculate C.

SOLUTION

Let V_1 be the potential of C_0 after first charging, then

$$(C+C_0)V_1 = C_0V_0$$

$$\Rightarrow \quad V_1 = \frac{C_0V_0}{C+C_0}$$

Let V_2 be the potential of C_0 after second charging then

$$C_0V_1 = (C+C_0)V_2$$

$$\Rightarrow \quad V_2 = \left(\frac{C_0}{C+C_0}\right)\left(\frac{C_0}{C+C_0}\right)V_0 = \left(\frac{C_0}{C+C_0}\right)^2 V_0 \text{ and}$$

so on

$$\Rightarrow \quad V_n = V = \left(\frac{C_0}{C+C_0}\right)^n V_0$$

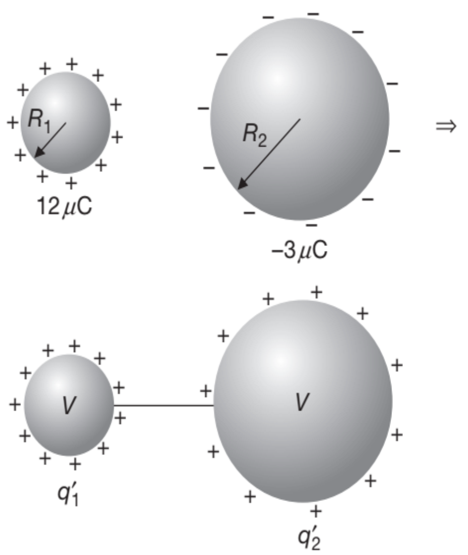
$$\Rightarrow \quad C = \left[\left(\frac{V_0}{V}\right)^{\frac{1}{n}} - 1\right]C_0$$

ILLUSTRATION 3

Two isolated spherical conductors have radii 6 cm and 12 cm respectively. They have charges of 12 μC and –3 μC. Find the charges after they are connected by a conducting wire. Also find the common potential after redistribution.

SOLUTION

Net charge $= (12-3)\ \mu\text{C} = 9\ \mu\text{C}$



Since the charge is distributed in the ratio of their capacitance (or radii in case of spherical conductors), i.e.,

$$\frac{q_1'}{q_2'} = \frac{R_1}{R_2} = \frac{6}{12} = \frac{1}{2}$$

$$\Rightarrow \quad q_1' = \left(\frac{1}{1+2}\right)(9) = 3\ \mu\text{C}$$

$$\text{and} \quad q_2' = \left(\frac{2}{1+2}\right)(9) = 6\ \mu\text{C}$$

The common potential is given by

$$V = \frac{q_1+q_2}{C_1+C_2} = \frac{(9\times10^{-6})}{4\pi\varepsilon_0(R_1+R_2)}$$

$$\Rightarrow \quad V = \frac{(9\times10^{-6})(9\times10^{9})}{(18\times10^{-2})}$$

$$\Rightarrow \quad V = 4.5\times10^5\ \text{V}$$

ILLUSTRATION 4

A capacitor of 20 μF charged to 500 V, is connected in parallel to another capacitor of 10 μF charged to 200 V, find the common potential.

SOLUTION

The common potential,

$$V_{\text{common}} = \frac{C_1V_1 + C_2V_2}{C_1+C_2}$$

Here $C_1 = 20\ \mu\text{F} = 20\times10^{-6}\ \text{F},\ V_1 = 500\ \text{V}$

$$C_2 = 10\ \mu\text{F} = 10\times10^{-6}\ \text{F},\ V_2 = 200\ \text{V}$$

$$\Rightarrow \quad V_{\text{common}} = \frac{20\times10^{-6}\times500 + 10\times10^{6}\times200}{20\times10^{-6}+10\times10^{-6}}$$

$$\Rightarrow \quad V_{\text{common}} = 400\ \text{V}$$

Remark(s)

CAPACITOR OR CONDENSER

An arrangement which has capability of collecting (and storing) charge and whose capacitance can be varied is called **a capacitor (or condenser)**.

In a capacitor two charged conductors are placed near each other so that the potential of each conductor is determined not only by its own charge but also by the magnitude and sign of the charge on the neighbouring conductor. The capacitance of a capacitor depends

(a) directly on the size of the conductors of the capacitor.
(b) directly on the dielectric constant K of the medium between the conductors.
(c) inversely on the distance of separation between the conductors.

The capacitance of a capacitor also depends on the presence of conductors present in neighbourhood and is independent of the material of metallic conductor.

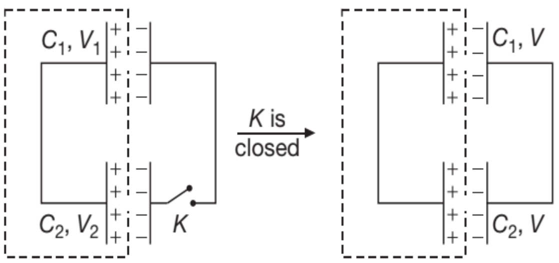
SHARING OF CHARGES BETWEEN CAPACITORS AND COMMON POTENTIAL

Just like conductors, when two isolated capacitors (either both of them are charged or one of them is charged) are connected to each other, then redistribution of charge takes place due to potential difference between them. The charge flows from higher potential to lower potential till both of them acquire the same potential called common potential.

Consider two capacitors, C_1 and C_2 charged to potentials, V_1 and V_2. Let the capacitors be connected such that

CASE-1:

Their similar plates are connected to each other (i.e. positive plate of one capacitor is connected to positive plate of other capacitor and so on).



Consider the part of circuit inside dotted loop. This part is isolated from other part. Therefore, total charge of this part remains constant. Hence

$$C_1V_1 + C_2V_2 = C_1V + C_2V$$

$$\Rightarrow \quad V = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$$

During this redistribution of charge, there is a loss of energy in the form of heat and electromagnetic radiations, given by

Loss of energy is $-\Delta U = U_i - U_f$.

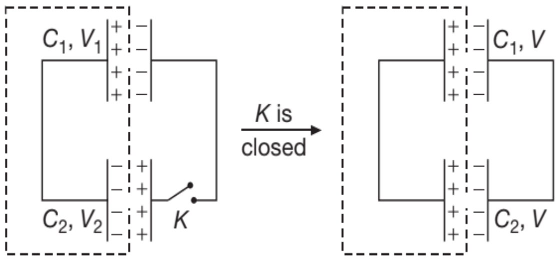
$$\Rightarrow \quad -\Delta U = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2 - \frac{1}{2}(C_1 + C_2)V^2$$

Substituting $V = \dfrac{C_1V_1 + C_2V_2}{C_1 + C_2}$ in above equation, we get

$$\text{Loss} = -\Delta U = \frac{1}{2}\left(\frac{C_1C_2}{C_1 + C_2}\right)(V_1 - V_2)^2$$

CASE-2:

Their dissimilar plates are connected together (i.e. positive plate of one capacitor is connected to the negative plate of other capacitor and so on).



Again applying the Law of Conservation of Charge, we get

$$C_1V_1 - C_2V_2 = C_1V + C_2V$$

$$\Rightarrow \quad V = \frac{C_1V_1 - C_2V_2}{C_1 + C_2}$$

So, loss in energy is given by

$$\text{Loss} = -\Delta U = \frac{1}{2}\frac{C_1C_2}{C_1 + C_2}(V_1 + V_2)^2$$

ILLUSTRATION 5

A parallel plate capacitor of capacitance C has been charged, so that the potential difference between its plates is V. Now, the plates of this capacitor are connected to another uncharged capacitor of capacitance 2C. Find the common potential acquired by the system and loss of energy.

SOLUTION

In this problem, we have

$$C_1 = C,\ C_2 = 2C,\ V_1 = V \text{ and } V_2 = 0$$

Common potential is given by

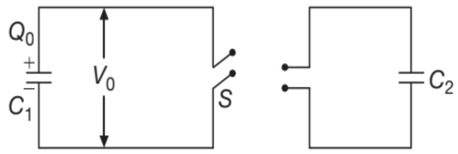
$$V_{\text{Common}} = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$$

$$V_{\text{Common}} = \frac{CV + 0}{3C} = \frac{V}{3}$$

and $\text{Loss} = -\Delta U = \frac{1}{2}\frac{C \times 2C}{3C} \times (V-0)^2 = \frac{1}{3}CV^2$

ILLUSTRATION 6

In the circuit shown, $C_1 = 8\ \mu\text{F}$, $C_2 = 4\ \mu\text{F}$ and $V_0 = 120$ V. When the switch S is closed, the charged capacitor C_1 gets connected to an uncharged capacitor C_2. Calculate the



(a) charge on each capacitor after switching S.
(b) total energy after switching S.

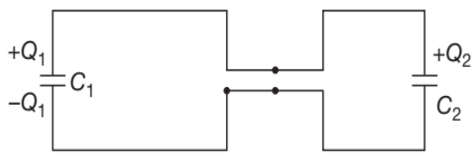
SOLUTION

(a) Before switching on S, $Q_0 = C_1V_0 = 960\ \mu\text{C}$ and

Energy stored $U = \frac{1}{2}Q_0V_0$

$$U = \frac{1}{2}(960 \times 10^{-6}\ \text{C})(120\ \text{V})$$

$$U = 0.058\ \text{J}$$



(b) After we close switch S, we have

$$Q_0 = Q_1 + Q_2$$

In the steady state, we have

$$Q_1 = C_1V \quad \ldots(1)$$

$$Q_2 = C_2V \quad \ldots(2)$$

Adding equation (1) and (2), we get

$$(C_1 + C_2)V = Q_1 + Q_2 = Q_0$$

$$\Rightarrow \quad V = \frac{Q_0}{C_1 + C_2} = \frac{960\ \mu\text{C}}{12\ \mu\text{F}} = 80\ \text{V}$$

$$\Rightarrow \quad Q_1 = 640\ \mu\text{C},\ Q_2 = 320\ \mu\text{C}$$

The total energy stored is given by

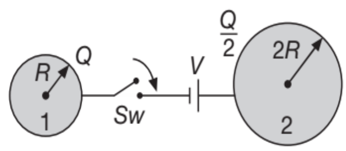
$$U = \frac{1}{2}Q_1V + \frac{1}{2}Q_2V = \frac{1}{2}Q_0V$$

$$\Rightarrow \quad U = \frac{1}{2}(960 \times 10^{-6})(80\ \text{V}) = 0.038\ \text{J}$$

The energy after closing the switch is less than the original energy of 0.058 J. This difference has been converted to energy of some other form (e.g. heat energy or energy radiated as electromagnetic waves).

ILLUSTRATION 7

Two spheres 1 and 2 of radii R and $2R$ having charges Q and $\frac{Q}{2}$, respectively are connected with a cell of potential difference V as shown in Figure. When the switch Sw is closed, calculate the final charge on each sphere.

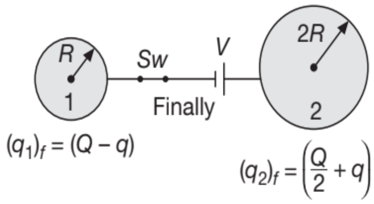


SOLUTION

When the switch is closed, the potential difference between the spheres should be V. Let charge q flow from the sphere of radius R to sphere of radius $2R$. Then

$$\frac{(q_2)_{\text{final}}}{C_2} - \frac{(q_1)_{\text{final}}}{C_1} = V$$

$$C_2 = 4\pi\varepsilon_0(2R) \text{ and } C_1 = 4\pi\varepsilon_0 R$$



$$\Rightarrow \quad \frac{\left(\frac{Q}{2}+q\right)}{4\pi\varepsilon_0(2R)} - \frac{(Q-q)}{4\pi\varepsilon_0(R)} = V$$

$$\Rightarrow \quad \frac{Q}{2} + q - 2Q + 2q = 4\pi\varepsilon_0(2R)V$$

$$\Rightarrow \quad 3q = 8\pi\varepsilon_0 RV + \frac{3Q}{2}$$

$$\Rightarrow \quad q = \frac{8\pi\varepsilon_0 RV}{3} + \frac{Q}{2}$$

So the final charges on each of the spheres are

$$(q_1)_f = Q - q = \frac{Q}{2} - \frac{8\pi\varepsilon_0 RV}{3}$$

and $(q_2)_f = \frac{Q}{2} + q = Q + \frac{8\pi\varepsilon_0 RV}{3}$

Test Your Concepts-I

Based on General Capacitance

(Solutions on page H.126)

1. A capacitor of $20\,\mu$F charged to 500 V, is connected in parallel to another capacitor of $10\,\mu$F charged to 200 V, find the common potential.

2. Calculate the heat generated when a condenser of $100\,\mu$F capacity and charged to 200 V is discharged through a 2 Ω resistance.

3. The capacitance of a variable radio capacitor can be changed from 50 pF to 950 pF by turning the dial from 0° to 180°. With the dial set at 180°, the capacitor is connected to a 400 V battery. After charging, the capacitor is disconnected from the battery and the dial is turned at 0°.

(a) What is the potential difference across the capacitor when the dial reads 0°?

(b) How much work is required to turn the dial, if friction is neglected?

4. A battery of 10 V is connected to a capacitor of capacitance 0.1 F. The battery is now removed and this capacitor is connected to a second uncharged capacitor. If the charge distributes equally on these two capacitors, find the total energy stored in the two capacitors. Further compare this energy with the initial energy stored in the capacitors.

5. Consider two conducting spheres with radii R_1 and R_2 separated by a distance much greater than radius of either. A total charge Q is shared between the spheres, subject to the condition that the electric potential energy of the system has the smallest possible value. The spheres being very far apart, you can assume that the charge of each is uniformly distributed over the surface of each.

(a) Determine the values of charges on the spheres in terms of Q, R_1 and R_2.

(b) Find the potential difference between the surfaces of the spheres.

6. A capacitor of capacitance C_0 is charged to a potential V_0 and then isolated. A small capacitor C is then charged from C_0, discharged and charged again, the process being repeated 10 times. The potential of the capacitor C_0 has now fallen to V. Calculate C.

PRINCIPLE OF A PARALLEL PLATE CAPACITOR

Consider a conducting plate A which is given a charge Q (as shown in Figure 1(a)) such that its potential rises to V. Then

$$C = \frac{Q}{V}$$

Let us place another identical conducting plate B parallel to it such that charge is induced on plate B (as shown in Figure 1(b)).

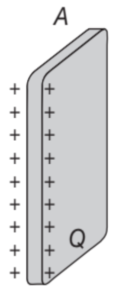


Figure 1(a)

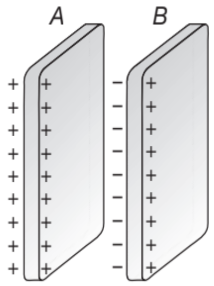


Figure 1(b)

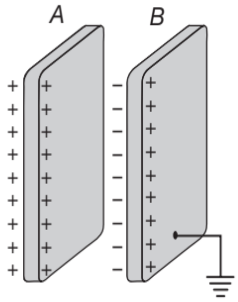


Figure 1(c)

If V_- is the potential at A due to induced negative charge on B and V_+ is the potential at A due to induced positive charge on B, then

$$C' = \frac{Q}{V'} = \frac{Q}{V + V_+ - V_-}$$

Since $V' < V$ (as the induced negative charge lies closer to the plate A in comparison to induced positive charge).

$$\Rightarrow \quad C' > C$$

Further, if B is earthed from the outer side (as shown in Figure 1(c)) then $V'' = V - V_-$ as the entire positive charge flows to the earth. So

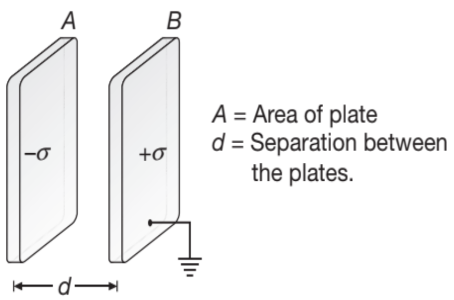
$$C'' = \frac{Q}{V''} = \frac{Q}{V - V_-}$$

$$\Rightarrow \quad C'' \gg C$$

So, if an identical earthed conductor is placed in the viscinity of a charged conductor then the capacitance of the charged conductor increases appreciably. This is the principle of a parallel plate capacitor.

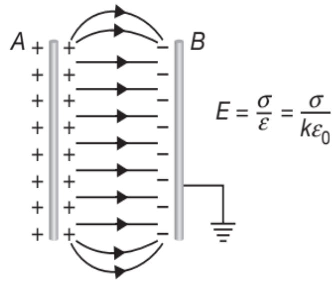
PARALLEL PLATE CAPACITOR

It consists of two metallic plates A and B each of area A at separation d. Plate A is positively charged and plate B is earthed. If K is the dielectric constant of the material medium and E is the field that exists between the two plates, then



$$E = \frac{\sigma}{\varepsilon} = \frac{\sigma}{K\varepsilon_0}$$

$$\Rightarrow \quad \frac{V}{d} = \frac{q}{K\varepsilon_0 A} \qquad \left\{\because E = \frac{V}{d} \text{ and } \sigma = \frac{q}{A}\right\}$$



$$\Rightarrow \quad C = \frac{q}{V} = \frac{K\varepsilon_0 A}{d}$$

If medium between the plates is air or vacuum, then

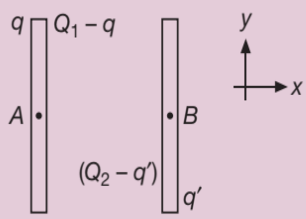
$$K = 1$$

$$\Rightarrow \quad C_0 = \frac{\varepsilon_0 A}{d}$$

Remark(s)

Suppose charges Q_1 and Q_2 are given to the two plates. The resulting charge distribution on the various surfaces is shown in the figure and is derived using the concept that no field exists in the thickness of the conductor.

$$E_A = 0$$



$$\Rightarrow \quad \frac{1}{2\varepsilon_0 A}\left[q - (Q_1 - q) - (Q_2 - q') - q'\right] = 0$$

$$\Rightarrow \quad q - Q_1 + q - Q_2 + q' - q' = 0$$

$$\Rightarrow \quad 2q - Q_1 - Q_2 = 0$$

$$\Rightarrow \quad q = \frac{Q_1 + Q_2}{2}$$

Similarly $E_B = 0$

$$\Rightarrow \frac{1}{2\varepsilon_0 A}\left[q+(Q_1-q)+(Q_2-q')-q'\right]=0$$

$$\Rightarrow Q_1+Q_2-2q'=0$$

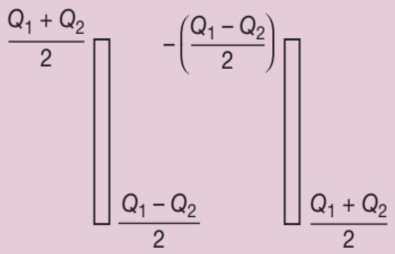
$$\Rightarrow q'=\frac{Q_1+Q_2}{2}$$

$$\Rightarrow Q_1-q=\frac{2Q_1-Q_1-Q_2}{2}=\frac{Q_1-Q_2}{2}$$

and $Q_2-q'=Q_2-\left(\frac{Q_1+Q_2}{2}\right)$

$$\Rightarrow Q_2-q'=\frac{2Q_2-Q_1-Q_2}{2}=\frac{Q_2-Q_1}{2}=-\left(\frac{Q_1-Q_2}{2}\right)$$

So, the distribution of charges is as shown below



The charges on the inner surfaces are equal and opposite and the field between the plates is,

$$E=\frac{\sigma}{\varepsilon_0}=\frac{(q_1-q_2)}{2A\varepsilon_0}$$

Thus, the potential difference between the plates is,

$$V=Ed=\frac{(q_1-q_2)d}{2A\varepsilon_0}$$

In order to find $C=\frac{q}{V}$ we must now use the value $\frac{(q_1-q_2)}{2}$ for q because this is the charge responsible for creating the field between the two plates. The outer charges $\frac{(q_1+q_2)}{2}$ do not come in the picture.

Thus, we get $C=\frac{\varepsilon_0 A}{d}$ as before.

ELECTROSTATIC FORCE BETWEEN THE PLATES OF A PARALLEL PLATE CAPACITOR

The plates of the capacitor each carry equal and opposite charges, hence they must attract each other with a force, say F.

At any instant let the plate separation be x, then

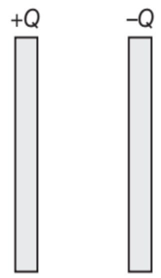
$$C=\frac{\varepsilon_0 A}{x}$$

Also $U=\frac{Q^2}{2C}$

$$\Rightarrow \quad U=\left(\frac{Q^2}{2\varepsilon_0 A}\right)x$$

Let the plates be moved towards each other through dx, such that the new separation between the plates is $(x-dx)$. If U_f is the final potential energy, then

$$U_f=\frac{Q^2}{2C'}=\frac{Q^2}{2\varepsilon_0 A}(x-dx)$$



If dU is the change in potential energy, then

$$dU=U_f-U_i$$

$$\Rightarrow \quad dU=\frac{Q^2}{2\varepsilon_0 A}(x-dx)-\frac{Q^2}{2\varepsilon_0 A}x$$

$$\Rightarrow \quad dU=-\frac{Q^2}{2\varepsilon_0 A}dx$$

Further since

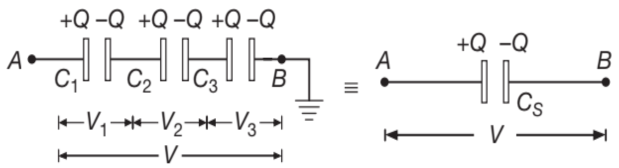
$$F=-\frac{dU}{dx}$$

$$\Rightarrow \quad F=\frac{Q^2}{2\varepsilon_0 A}=\left(\frac{\sigma^2}{2\varepsilon_0}\right)A=\left(\frac{1}{2}\varepsilon_0 E^2\right)A$$

$$\left\{\because Q=\sigma A,\ E=\frac{\sigma}{\varepsilon_0}\right\}$$

From here too, we observe that electrostatic pressure $P_e = \frac{1}{2}\varepsilon_0 E^2 = \frac{\sigma^2}{2\varepsilon_0}$

CAPACITORS IN SERIES



In this arrangement of capacitors the charge has no alternative path(s) to flow.

(a) The charges on each capacitor are equal

i.e., $Q = C_1V_1 = C_2V_2 = C_3V_3$ ...(1)

(b) The total potential difference across AB is shared by the capacitors in the inverse ratio of the capacitances.

$$V = V_1 + V_2 + V_3 \quad ...(2)$$

If C_S is the net capacitance of the series combination, then

$$\frac{Q}{C_S} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \Rightarrow \frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

Further $V_1 = \frac{Q}{C_1}$ and $V = \frac{Q}{C_S}$

$$\Rightarrow \frac{V_1}{V} = \frac{\frac{1}{C_1}}{\frac{1}{C_S}}$$

$$\Rightarrow V_1 = \left(\frac{\frac{1}{C_1}}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}}\right)V = \left(\frac{C_S}{C_1}\right)V$$

Similarly, $$V_2 = \left(\frac{\frac{1}{C_2}}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}}\right)V = \left(\frac{C_S}{C_2}\right)V$$

and $$V_3 = \left(\frac{\frac{1}{C_3}}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}}\right)V = \left(\frac{C_S}{C_3}\right)V$$

(c) C_S has a value smaller than the least capacitance of the circuit, provided all capacitors are connected in series.

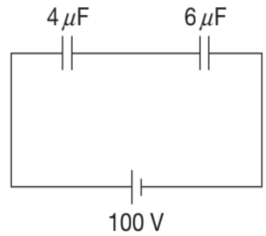
ILLUSTRATION 8

In the circuit shown in figure, find

(a) the equivalent capacitance

(b) the charge stored in each capacitor and

(c) the potential difference across each capacitor.

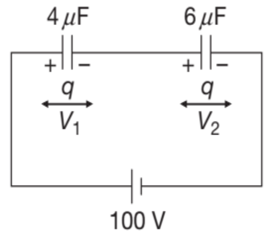


SOLUTION

(a) The equivalent capacitance

$$C = \frac{C_1C_2}{C_1+C_2}$$

$$\Rightarrow C = \frac{(4)(6)}{4+6} = 2.4\ \mu\text{F}$$



(b) The charge q, stored in each capacitor is,

$$q = CV = (2.4 \times 10^{-6})(100)\ \text{C}$$

$$\Rightarrow q = 240\ \mu\text{C}$$

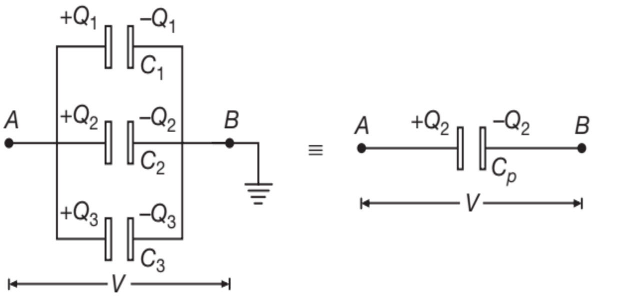
(c) In series combination, $V \propto \frac{1}{C}$ $\{\because q = \text{constant}\}$

$$\Rightarrow \frac{V_1}{V_2} = \frac{C_2}{C_1}$$

$$\Rightarrow V_1 = \left(\frac{C_2}{C_1+C_2}\right)V = \left(\frac{6}{4+6}\right)(100) = 60\ \text{V}$$

So, $V_2 = V - V_1 = 100 - 60 = 40$ V

CAPACITORS IN PARALLEL



In such an arrangement of capacitors the charge has an alternative path(s) to flow.

(a) The potential difference across each capacitor is same and equals the total potential applied.

i.e., $V = V_1 = V_2 = V_3$ ...(1)

$$\Rightarrow \quad V = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_3}{C_3} \quad \text{...(2)}$$

(b) The total charge Q is shared by each capacitor in the direct ratio of the capacitances.

$$Q = Q_1 + Q_2 + Q_3 \quad \text{...(3)}$$

If C_p is the net capacitance for the parallel combination of capacitors then

$$C_pV = C_1V + C_2V + C_3V$$

$$\Rightarrow \quad C_p = C_1 + C_2 + C_3$$

Further $Q_1 = C_1V$, $Q_2 = C_2V$, $Q_3 = C_3V$, $Q = C_pV$

$$\Rightarrow \quad \frac{Q_1}{Q} = \frac{C_1}{C_p}$$

$$\Rightarrow \quad Q_1 = \left(\frac{C_1}{C_p}\right)Q$$

Similarly $Q_2 = \left(\frac{C_2}{C_p}\right)Q$

$$Q_3 = \left(\frac{C_3}{C_p}\right)Q$$

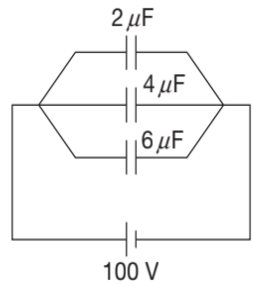
(c) C_p has a value greater than the greatest capacitance of the circuit.

ILLUSTRATION 9

In the circuit shown in figure, find

(a) the equivalent capacitance and

(b) the charge stored in each capacitor.



SOLUTION

(a) The capacitors are in parallel. Hence the equivalent capacitance is,

$$C = C_1 + C_2 + C_3$$

$$\Rightarrow \quad C = (2 + 4 + 6) = 12\ \mu\text{F}$$

(b) Total charge drawn from the battery

$$q = CV = 12 \times 100\ \mu\text{C} = 1200\ \mu\text{C}$$

This charge will be distributed in the ratio of their capacitances. Hence,

$$q_1 : q_2 : q_3 = C_1 : C_2 : C_3 = 2 : 4 : 6$$

$$\Rightarrow \quad q_1 = \left(\frac{2}{2+4+6}\right) \times 1200 = 200\ \mu\text{C}$$

$$\Rightarrow \quad q_2 = \left(\frac{4}{2+4+6}\right) \times 1200 = 400\ \mu\text{C}$$

and $\quad q_3 = \left(\frac{6}{2+4+6}\right) \times 1200 = 600\ \mu\text{C}$

Problem Solving Technique(s)

(a) If C_1, C_2, C_3,... are capacitors connected in series and if total potential across all is V, then potential across each capacitor is

$$V_1 = \left(\frac{\frac{1}{C_1}}{\frac{1}{C_S}}\right)V;\ V_2 = \left(\frac{\frac{1}{C_2}}{\frac{1}{C_S}}\right)V;\ V_3 = \left(\frac{\frac{1}{C_3}}{\frac{1}{C_S}}\right)V$$

and so on, where $\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_n}$

(b) If C_1, C_2, C_3,... are capacitors connected in parallel and if Q is total charge on the combination, then charge on each capacitor is

$$Q_1 = \left(\frac{C_1}{C_P}\right)Q;\ Q_2 = \left(\frac{C_2}{C_P}\right)Q;\ Q_3 = \left(\frac{C_3}{C_P}\right)Q$$

and so on, where $C_P = C_1 + C_2 + C_3 + ... + C_n$

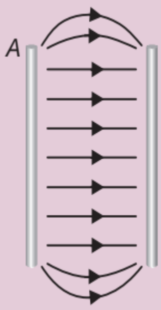
(c) The current through a capacitor is zero as long as the voltage across it does not change with time (or as long as capacitor is in the steady state i.e., charge on the capacitor has build up to attain a constant maximum value).

(d) It is not possible to change the voltage across a capacitor by a finite amount in zero time as this process requires an infinite amount of current to flow through the capacitor.

(e) A capacitor can store a finite amount of energy in it even if the current flowing through it is zero when voltage across the capacitor is constant.

(f) A true mathematical (theoretical) model of a capacitor should never dissipate energy but can only store it whereas a physical (practical) model never does so.

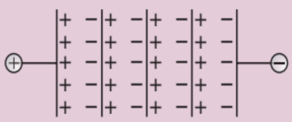
(g) A parallel plate capacitor must have plates of large area (A) in comparison to the distance of separation (d) between the plates to avoid **Fringing Effect**. Fringing Effect is the bending of field lines at the corners of a capacitors with large d (see figure).



Fringing effect for a capacitor with large *d*.

(h) n plates arranged as shown in figure constitute $(n-1)$ capacitors in parallel, each of value $\left(\frac{\varepsilon_0 A}{d}\right)$, so that

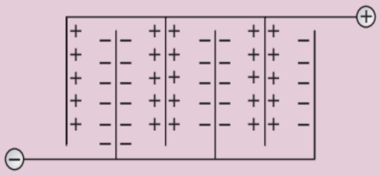
$$C_P = (n-1)\frac{\varepsilon_0 A}{d}$$



In this situation except two extreme plates each plate is common to two adjacent capacitors.

(i) n plates arranged as shown in figure constitute $(n-1)$ capacitors in series each of value $\left(\frac{\varepsilon_0 A}{d}\right)$, so that

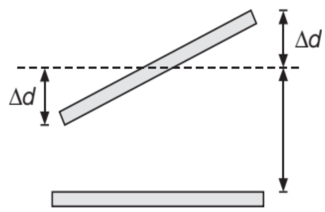
$$C_S = \frac{\varepsilon_0 A}{d(n-1)}$$



In this situation except two extreme plates each plate is common to two adjacent capacitors.

ILLUSTRATION 10

An air-dielectric capacitor is formed by two non-parallel plates, each of area A. An edge view of the arrangement is shown in figure. Note that the top plate is tilted relative to the bottom plate so that on one edge the plate separation is $d + \Delta d$, while on the other edge it is $d - \Delta d$. Assuming that $\Delta d \ll d$ and that d is small compared with the length of the plate, show that $C = \frac{\varepsilon_0 A}{d}\left[1 + \frac{1}{3}\left(\frac{\Delta d}{d}\right)^2\right]$.



SOLUTION

Consider an infinitesimal element of thickness dx at a distance x from O.

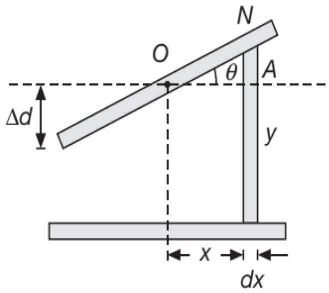
Then the capacitor is thought of being made from such infinitesimal elements having their capacitances in parallel. If dC be the capacitance of such an element, then

$$dC = \frac{\varepsilon_0 (dA)}{y}$$

Since, $\Delta d \ll d$, so

$$AN \cong x\theta$$

$$\Rightarrow \quad dC = \frac{\varepsilon_0 (b\,dx)}{d + x\theta}$$



where b = breadth of the plate and l = length of the plate

So, total capacitance C is given by

C = sum of capacitances of infinitesimal elements

$$\Rightarrow \quad C = \int dC = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{\varepsilon_0 b\,dx}{d + x\theta}$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 b}{\theta} \log_e (d + x\theta) \Big|_{-\frac{l}{2}}^{\frac{l}{2}}$$

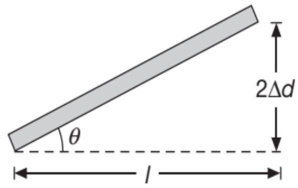
$$\Rightarrow \quad C = \frac{\varepsilon_0 b}{\theta}\left[\log_e\left(d + \frac{l\theta}{2}\right) - \log_e\left(d - \frac{l\theta}{2}\right)\right]$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 b}{\theta}\left[\log\left(1 + \frac{l\theta}{2d}\right) - \log\left(1 - \frac{l\theta}{2d}\right)\right]$$

Using $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$..........

and $\log_e(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\right)$

$$\Rightarrow \quad C = \frac{\varepsilon_0 b}{\theta}\left\{\left[\frac{l\theta}{2d} - \frac{1}{2}\left(\frac{l\theta}{2d}\right)^2 + \frac{1}{3}\left(\frac{l\theta}{2d}\right)^3 + ...\right] + \left[\frac{l\theta}{2d} + \frac{1}{2}\left(\frac{l\theta}{2d}\right)^2 + \frac{1}{3}\left(\frac{l\theta}{2d}\right)^3 + ...\right]\right\}$$



$$\Rightarrow \quad C = \frac{\varepsilon_0 b}{\theta}\left[2\left(\frac{l\theta}{2d}\right) + 2\frac{1}{3}\left(\frac{l\theta}{2d}\right)^3\right]$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 b}{\theta}\left[\frac{l\theta}{d} + \frac{1}{12}\frac{l^3\theta^3}{d^3}\right]$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 lb}{d}\left[1 + \frac{1}{12}\frac{l^2}{d^2}\theta^2\right]$$

But $\frac{\varepsilon_0 lb}{d} = \frac{\varepsilon_0 A}{d}$

and $\theta \simeq \frac{2\Delta d}{l}$

$$\Rightarrow \quad C = \frac{\varepsilon_0 A}{d}\left[1 + \frac{1}{12}\frac{l^2}{d^2}\left(\frac{4(\Delta d)^2}{l^2}\right)\right]$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 A}{d}\left[1 + \frac{1}{3}\left(\frac{\Delta d}{d}\right)^2\right]$$

ILLUSTRATION 11

A parallel plate square capacitor has the space between the plates filled with a medium whose dielectric constant increases uniformly with distance x from one of its plate as $K = K_1 + \alpha x$. If d is distance between the plates and K_1 and K_2 are dielectric constants of the medium at the two square plates, find the capacity of the capacitor.

SOLUTION

Since $K = K_1 + \alpha x$

and at $x = d$, $K = K_2$

$$\Rightarrow \quad \alpha = \frac{K_2 - K_1}{d}$$

The capacitance of the elementary capacitor is

$$dC = \frac{A\varepsilon_0 (K_1 + \alpha x)}{dx}$$

Since these elementary capacitors are joined in series, so

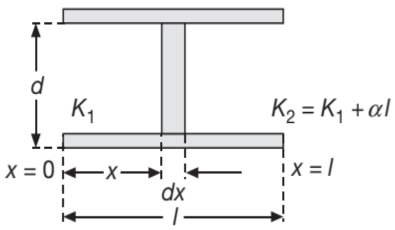
$$\frac{1}{C} = \int \frac{1}{dC} = \frac{1}{A\varepsilon_0}\int_0^d \frac{dx}{(K_1 + \alpha x)} = \frac{1}{A\varepsilon_0}\left(\frac{\ln(K_1 + \alpha x)}{\alpha}\right)\Bigg|_0^d$$

$$\Rightarrow \quad \frac{1}{C} = \frac{1}{A\varepsilon_0 \alpha}\ln\left(\frac{K_1 + \alpha d}{K_1}\right) = \frac{d}{A\varepsilon_0 (K_2 - K_1)}\ln\left(\frac{K_2}{K_1}\right)$$

$$\Rightarrow \quad C = \frac{A\varepsilon_0 (K_2 - K_1)}{d\ln\left(\frac{K_2}{K_1}\right)}$$

ILLUSTRATION 12

A parallel plate square capacitor has the space between the plates filled with a dielectric whose dielectric constant increases linearly with distance x from one edge to the other as $K = K_1 + \alpha x$, where K_1 and K_2 are dielectric constants of the medium at the two edges of the square plates, find the capacity of the capacitor.



SOLUTION

Since $K = K_1 + \alpha x$ and at $x = l$, we have

$$K = K_2$$

$$\Rightarrow \quad \alpha = \frac{K_2 - K_1}{l}$$

The capacitance of the elementary capacitor is

$$dC = \frac{\varepsilon_0 (l dx)(K_1 + \alpha x)}{d}$$

Since, these elementary capacitors are joined in parallel, so

$$C = \int dC = \frac{\varepsilon_0 l}{d}\int_0^l (K_l + \alpha x)dx = \frac{\varepsilon_0 l^2}{d}\left(K_1 + \frac{\alpha l}{2}\right)$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 l^2}{d}\left[K_1 + \frac{1}{2}\left(\frac{K_2 - K_1}{l}\right)l\right]$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 A}{d}\left(\frac{K_1 + K_2}{2}\right)$$

ELECTROSTATIC ENERGY DENSITY (u_E)

For a parallel plate capacitor, we have

$$C = \frac{\varepsilon_0 A}{d} \text{ and } V = Ed, \quad \text{where } E = \frac{\sigma}{\varepsilon_0}$$

Since, $U = \frac{1}{2}CV^2$

$$\Rightarrow \quad U = \frac{1}{2}\frac{\varepsilon_0 A}{d}E^2 d^2$$

$$\Rightarrow \quad U = \left(\frac{1}{2}\varepsilon_0 E^2\right)(Ad)$$

$$\Rightarrow \quad U = \frac{1}{2}\varepsilon_0 E^2 \tau$$

where $\tau (= Ad)$ is volume of the capacitor

$$\Rightarrow \quad \frac{U}{\tau} = u_E = \frac{\text{Electrostatic Energy}}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2 = \frac{\sigma^2}{2\varepsilon_0}$$

This energy is stored in the capacitor in the form of electrostatic field. Since,

$$\frac{\text{Electrostatic Energy}}{\text{Volume}} = \text{Electrostatic Pressure}$$

$$\Rightarrow \quad \text{Electrostatic Pressure} = \frac{1}{2}\varepsilon_0 E^2 = \frac{\sigma^2}{2\varepsilon_0} = u_E$$

Also we note that if we take $d\tau$ as the volume element, then

$$u_E = \int u_E d\tau$$

I, have taken a new symbol (τ) for the volume, because we may confuse V for potential.

Remark(s)

The potential energy U of a charged conductor or a capacitor is stored in the electric field. The energy per unit volume is called the energy density (u_E). Energy density (u_E) in a dielectric medium is given by,

$$u_E = \frac{1}{2}\varepsilon_0 K E^2$$

This relation shows that the energy stored per unit volume depends on E^2. If E is the electric field in a space of volume $d\tau$, then the total stored energy in an electrostatic field is given by,

$$U = \frac{1}{2}\varepsilon_0 K \int E^2\, d\tau$$

and if E is uniform throughout the volume (electric field between the plates of a capacitor is uniform), then the total stored energy can be given by,

$$U = \frac{1}{2}K\varepsilon_0 E^2 \tau$$

ENERGY FOR SERIES AND PARALLEL COMBINATIONS

Series Combination

For a series combination of capacitors Q = constant and

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$$

$$\Rightarrow \quad \frac{Q^2}{2C_S} = \frac{Q^2}{2C_1} + \frac{Q^2}{2C_2} + \frac{Q^2}{2C_3} + \ldots$$

$$\Rightarrow \quad U_S = U_1 + U_2 + U_3 + \ldots$$

i.e., if U_1 is energy stored in capacitor C_1, if U_2 is energy stored in capacitor C_2 and so on, and U_S is total energy for series combination of C_1, C_2, C_3,… then,

$$\begin{pmatrix}\text{Total Energy}\\ \text{for the series}\\ \text{combination}\end{pmatrix} = \begin{pmatrix}\text{Sum of individual}\\ \text{energies stored in}\\ \text{each capacitor}\end{pmatrix}$$

Parallel Combination

For a parallel combination of capacitors V =constant and

$$C_P = C_1 + C_2 + C_3 + \ldots$$

$$\Rightarrow \quad \frac{1}{2}C_P V^2 = \frac{1}{2}C_1 V^2 + \frac{1}{2}C_2 V^2 + \frac{1}{2}C_3 V^2 + \ldots$$

$$\Rightarrow \quad U_P = U_1 + U_2 + U_3 + \ldots$$

i.e., if U_1 is energy stored in capacitor C_1, if U_2 is energy stored in capacitor C_2 and so on, and U_P is total energy for parallel combination of C_1, C_2, C_3,… then too

$$\begin{pmatrix}\text{Total Energy}\\ \text{for the parallel}\\ \text{combination}\end{pmatrix} = \begin{pmatrix}\text{Sum of individual}\\ \text{energy stored in}\\ \text{each capacitor}\end{pmatrix}$$

Problem Solving Technique(s)

So, to conclude, if capacitors are connected in series or in parallel the total energy across the combination (any one) is equal to the sum of the individual energies stored in each capacitor.

CALCULATING THE NET CAPACITANCE OF CIRCUITS

To calculate the net capacitance of a circuit/network, the steps given below should be followed.

STEP-1: Identify the two points across which the equivalent capacitance is to be calculated.

STEP-2: Imagine a battery to be connected between these points.

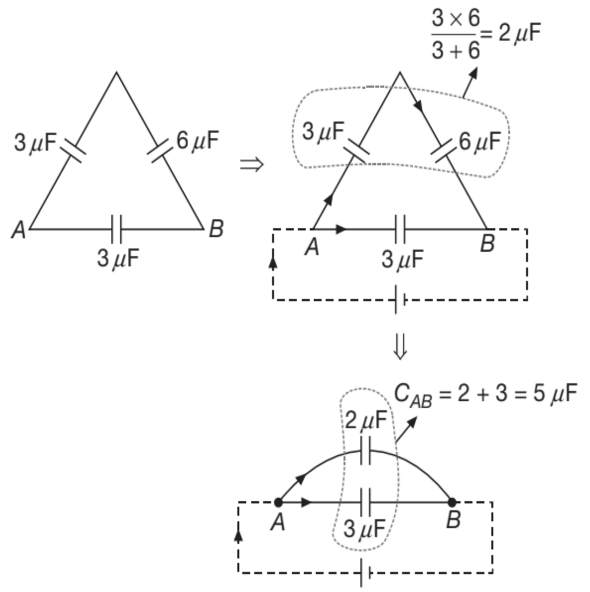
STEP-3: Start solving the circuit from the reference point which is farthest from the points between which the equivalent capacitance has to be calculated. This point is likely to be not a node.

SIMPLE CIRCUITS

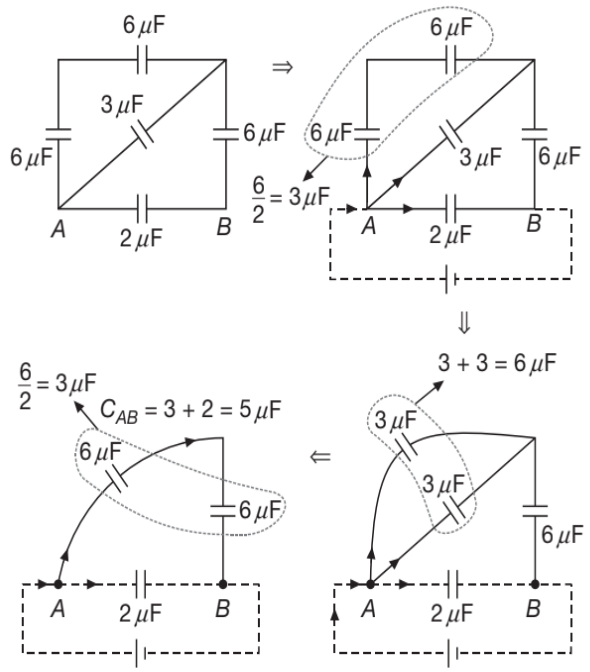
Analyse the circuit carefully to conclude which pair of capacitors are in series and which are in parallel (This all should be done keeping in mind the points across which net capacitance has to be calculated).

Find their net capacitance and again draw an equivalent diagram to apply the above specified technique repeatedly so as to get the total capacitance between the points A and B as Illustrated below.

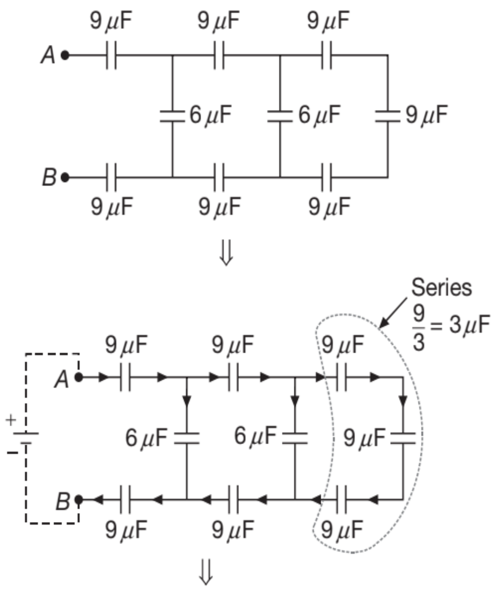
(i)



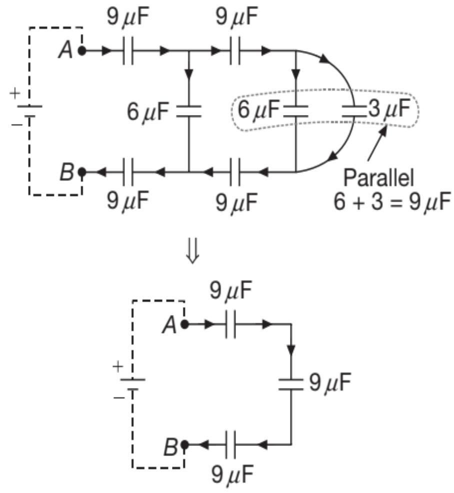
(ii)



(iii)



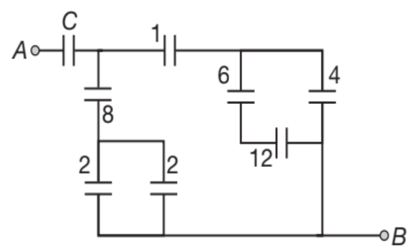
(Continued)



By similar process $C_{AB} = 3\ \mu\text{F}$

ILLUSTRATION 13

Find C if the equivalent capacitance between the points A and B in the circuit shown is $1\ \mu\text{F}$. All the capacitances are in μF.



SOLUTION

After a brief simple analysis, we conclude the following information

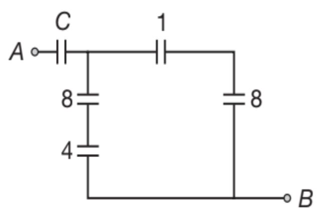
(a) 2 and 2 are in parallel to give $4\ \mu\text{F}$

(b) 6 and 12 are in series to give $\dfrac{6 \times 12}{6 + 12} = 4\ \mu\text{F}$

(c) This $4\ \mu\text{F}$ (net capacitance of 6 and 12) is in parallel combination to another 4, so as to give us $8\ \mu\text{F}$

So, using the above information, we redraw the above circuit

Further, we conclude



(d) 1 and 8 are in series, so we get $\frac{1\times 8}{1+8}=\frac{8}{9}\ \mu\text{F}$

(e) 8 and 4 are in series, so we get $\frac{8\times 4}{8+4}=\frac{32}{12}\ \mu\text{F}$

(f) $\frac{8}{9}\ \mu\text{F}$ and $\frac{32}{12}\ \mu\text{F}$ are in a parallel combination to give net capacitance $\frac{8}{9}+\frac{32}{12}=\frac{32}{9}\ \mu\text{F}$

Finally $\frac{32}{9}\ \mu\text{F}$ and C are in series between A and B, so as to give $1\ \mu\text{F}$. So, the final equivalent circuit is shown in figure.

$$\Rightarrow \quad \frac{\frac{32}{9}C}{\frac{32}{9}+C}=1$$

$$\Rightarrow \quad 32C=32+9C$$

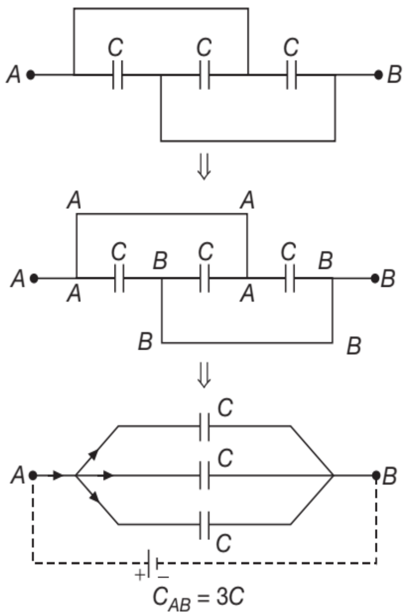
$$\Rightarrow \quad 23C=32$$

$$\Rightarrow \quad C=\frac{32}{23}\ \mu\text{F}$$

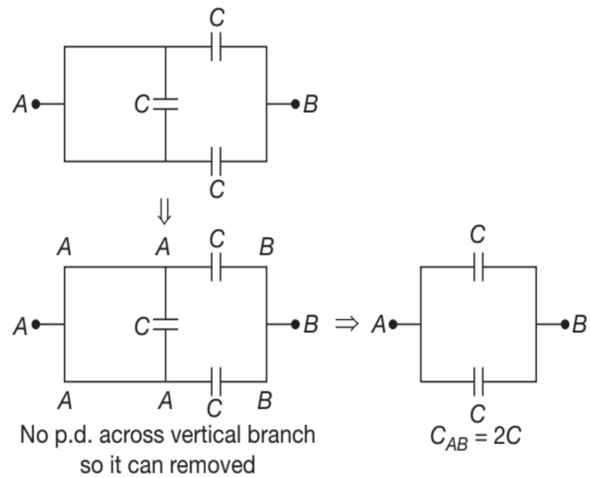
CIRCUITS WITH EXTRA WIRES

If there is no capacitor or resistor in any branch of a circuit, then every point of this branch will be at same potential. Suppose equivalent capacitance is to be determined in following cases.

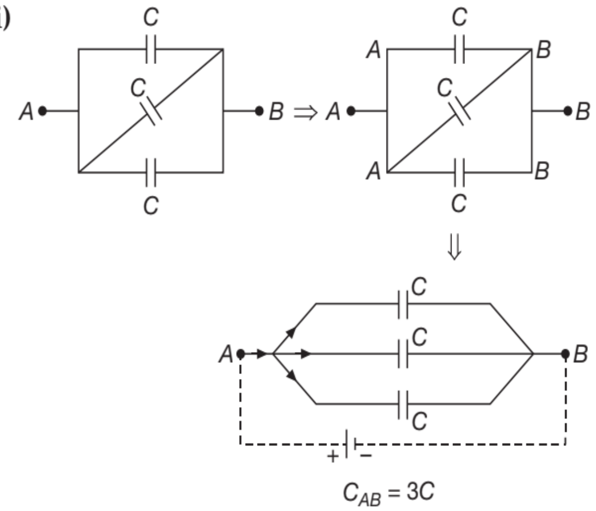
(i)



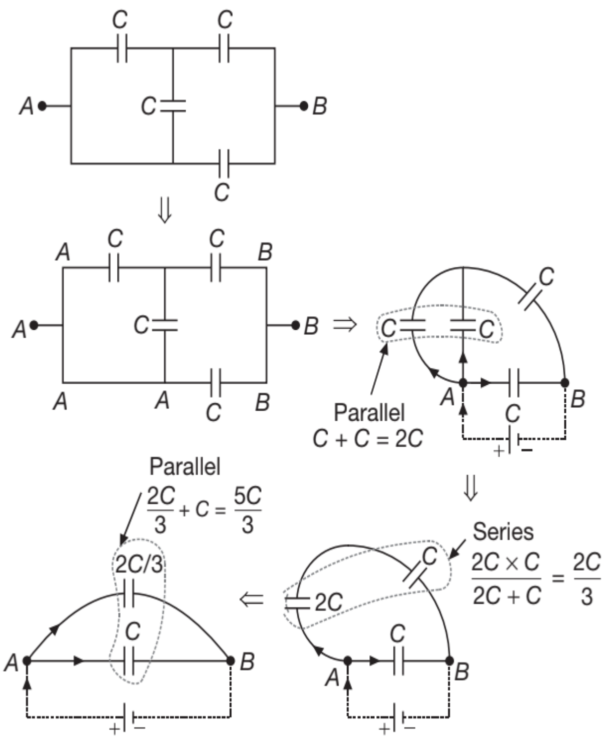
(ii)



(iii)

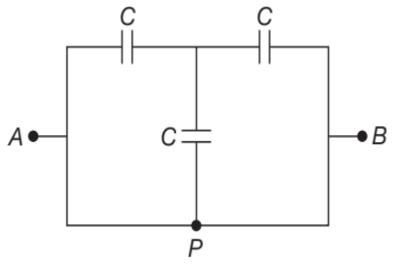


(iv)



Hence equivalent capacitance between A and B is $\frac{5C}{3}$.

(v) Since there is no capacitor in the path APB, the points A, P and B are electrically same i.e., the input and output points are directly connected (short circuited).



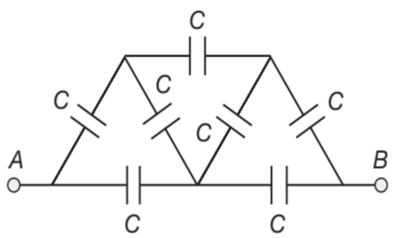
Thus, entire charge will prefer to flow along path APB. It means that the capacitors connected in the circuit will not receive any charge for storing. Thus equivalent capacitance of this circuit is zero.

CONCEPT OF LINE OF SYMMETRY

Line of symmetry (L.O.S.) is an imagination of our mind to divide a highly symmetric circuit into two equal halves such that the points of the circuit through which LOS passes are at equal potential.

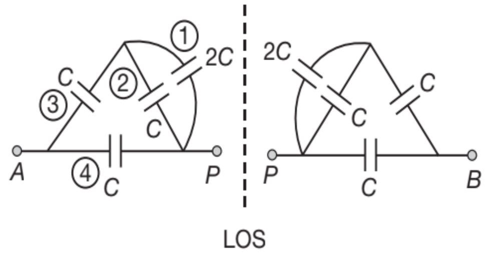
ILLUSTRATION 14

Find the net capacitance of the circuit shown between the points A and B.



SOLUTION

This circuit is highly symmetric and so we can consider the line of symmetry to pass through the circuit to divide it into two equal (identical) halves. If line of symmetry passes through a branch possessing a capacitor, then on each side of Line of Symmetry (LOS) the capacitance will become $2C$ ($2C$ and $2C$ in series will gives C), as shown.



Now, the concept of Line of Symmetry makes our job easy to calculate capacitance across AP. (1) and (2) are in parallel further in series with (3), whose resultant capacitance is in parallel with (4).

Resultant of (1) and (2) is $3C$

Resultant of $3C$ and (3) is $\frac{3C}{4}$.

Resultant of $\frac{3C}{4}$ and (4) is $\frac{7C}{4}$.

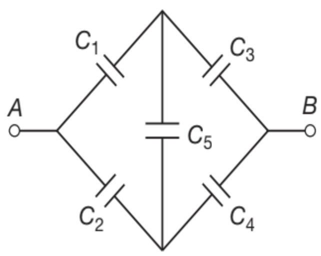
So total capacitance across AB is

$$C_{AB} = \frac{C_{AP}}{2}$$

$$\Rightarrow \quad C_{AB} = \frac{7C}{8}$$

BALANCED WHEATSTONE BRIDGE

If in a circuit, five capacitors are arranged as shown in following figure, the circuit is called Wheatstone Bridge type circuit.



It is called a Balanced Wheatstone Bridge (BWSB), when

$$\frac{C_1}{C_2} = \frac{C_3}{C_4}$$

In this situation we observe that no charge flows through C_5 and hence C_5 can be removed from the circuit to get the equivalent capacitance between A and B.

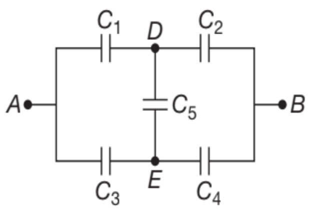
Conceptual Note(s)

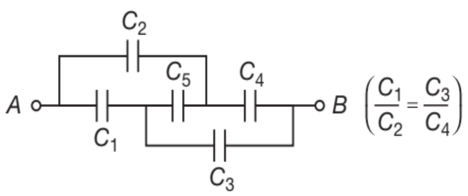
If $\frac{C_1}{C_2}=\frac{C_3}{C_4}$, then no charge exists in the branch containing C_5 (also called fifth branch) and hence it can be omitted (just neglect it as if it was never present).

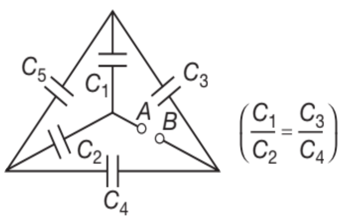
As a consequence of this we can say that C_1 and C_3 are in series to give $\frac{C_1C_3}{C_1+C_3}$, C_2 and C_4 are in series to give $\frac{C_2C_4}{C_2+C_4}$ and both in parallel. So, net capacitance

$$C_{net}=\frac{C_1C_3}{C_1+C_3}+\frac{C_2C_4}{C_2+C_4}$$

Other shapes of Balanced Wheatstone Bridge are indicated below.

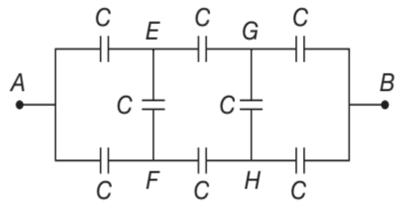






EXTENDED WHEATSTONE BRIDGE

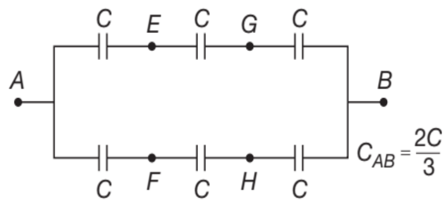
The given figure consists of two Wheatstone Bridges connected together. One bridge is connected between points $AEGHFA$ and the other is connected between points $EGBHFE$.



This circuit is known as Extended Wheatstone Bridge and it has two branches EF and GH to the left and right of which symmetry in the ratio of capacitances can be seen.

$$\frac{C}{C}=\frac{C}{C}=\frac{C}{C}=1$$

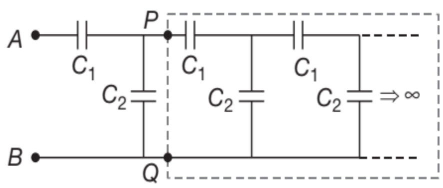
It can be seen that ratio of capacitances in branches AE and EG is same as that between the capacitances of the branches AF and FH. Thus, in the bridge $AEGHFA$; the branch EF can be removed. Similarly in the bridge $EGBHFE$ branch GH can be removed.

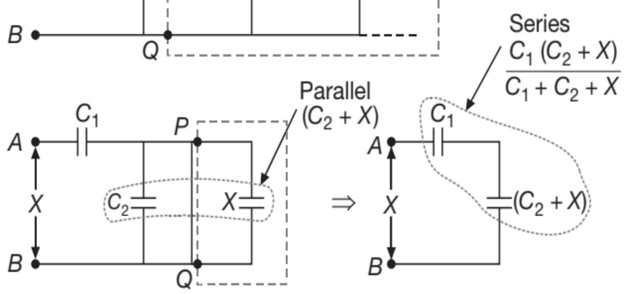


INFINITE CHAIN OF CAPACITORS

In the following infinite circuits, the equivalent capacitance between A and B is to be calculated.

Suppose the effective capacitance between A and B is X. Since the network is infinite, so even if we remove one repetitive unit of capacitors from the chain, then too the remaining network would still have infinite pair of capacitors, i.e., effective capacitance between P and Q will still be X





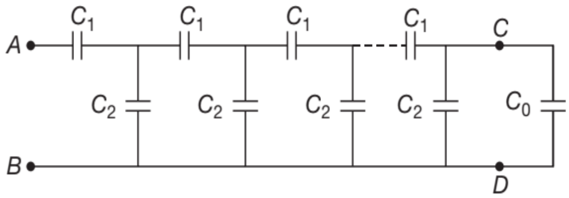
Hence equivalent capacitance between A and B

$$C_{AB} = \frac{C_1(C_2+X)}{C_1+C_2+X} = X$$

$$\Rightarrow \quad C_{AB} = \frac{C_2}{2}\left[\sqrt{\left(1+4\frac{C_1}{C_2}\right)}-1\right]$$

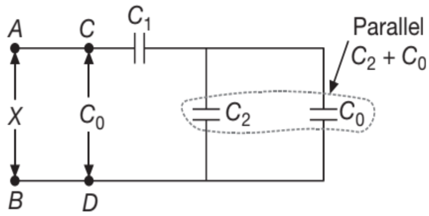
ILLUSTRATION 15

For what value of C_0 in the circuit shown below will the net effective capacitance between A and B be independent of the number of sections in the chain?



SOLUTION

Suppose there are n sections between A and B and the network is terminated by C_0 with equivalent capacitance X.



Now if we add one more sections to the network between D and C (as shown in figure), the equivalent capacitance of the network X will be independent of number of sections if the capacitance between D and C still remains C_0 i.e.,

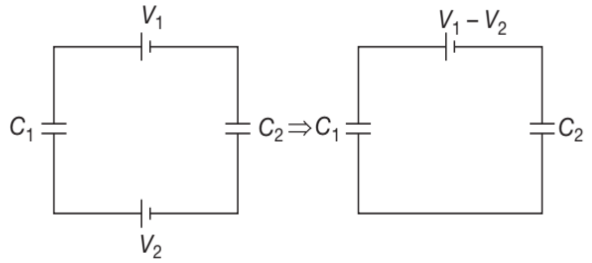
$$C_0 = \frac{C_1 \times (C_2+C_0)}{C_1+C_2+C_0}$$

$$\Rightarrow \quad C_0^2 + C_2C_0 - C_1C_2 = 0$$

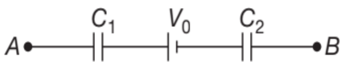
On simplification $C_0 = \frac{C_2}{2}\left[\sqrt{\left(1+4\frac{C_1}{C_2}\right)}-1\right]$

NETWORK WITH MORE THAN ONE CELL

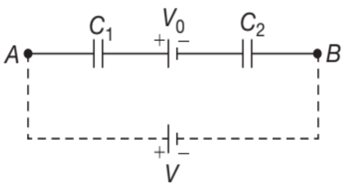
(a) Potential difference across C_1 is $\left(\frac{C_2}{C_1+C_2}\right)(V_1 - V_2)$ and potential difference across C_2 is $\left(\frac{C_1}{C_1+C_2}\right)(V_1 - V_2)$



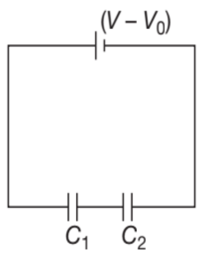
(b) Consider the arrangement shown.



This arrangement is connected across a battery of voltage $V(>V_0)$ as shown.



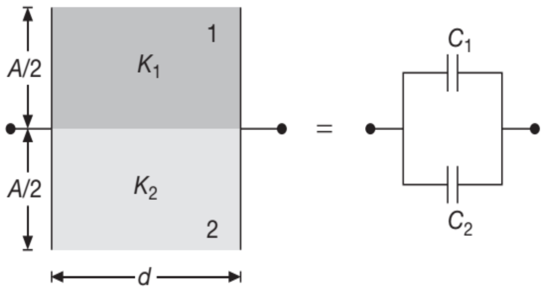
Then, potential difference between the ends of this arrangement is $(V - V_0)$.



CASE OF COMPOUND DIELECTRICS

If several dielectric medium filled between the plates of a parallel plate capacitor in different ways as shown.

(a) In the arrangement shown, these two capacitors are in parallel and



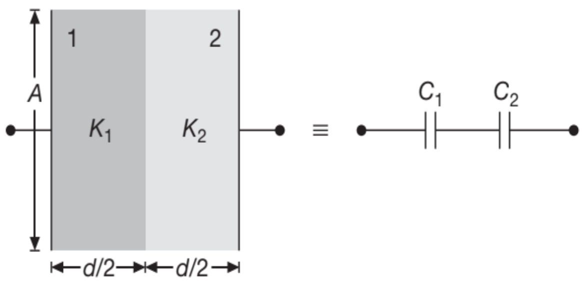
$$C_1 = \frac{K_1 \varepsilon_0 A}{2d} \text{ and } C_2 = \frac{K_2 \varepsilon_0 A}{2d}$$

Since, $C_{eq} = C_1 + C_2$

$$\Rightarrow \quad C_{eq} = \left(\frac{K_1 + K_2}{2}\right) \cdot \frac{\varepsilon_0 A}{d}$$

$$\Rightarrow \quad K_{eq} = \frac{K_1 + K_2}{2}$$

(b) The system can be assumed to be made up of two capacitors C_1 and C_2 which may be said to connected in series



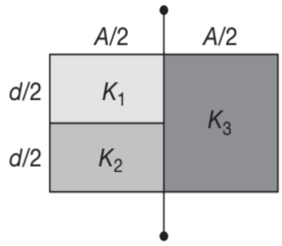
$$C_1 = \frac{K_1 \varepsilon_0 A}{d/2}, \; C_2 = \frac{K_2 \varepsilon_0 A}{d/2}$$

Since, $\dfrac{1}{C_{eq}} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$

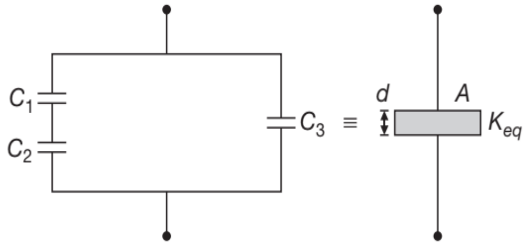
$$\Rightarrow \quad C_{eq} = \left(\frac{2K_1 K_2}{K_1 + K_2}\right) \frac{\varepsilon_0 A}{d}$$

$$\Rightarrow \quad K_{eq} = \frac{2K_1 K_2}{K_1 + K_2}$$

(c)



The equivalent capacitor circuit diagram for this arrangement is shown in figure.



where $C_1 = K_1 \dfrac{\varepsilon_0 (A/2)}{(d/2)} = K_1 \left(\dfrac{\varepsilon_0 A}{d}\right)$

$$C_2 = K_2 \frac{\varepsilon_0 (A/2)}{d/2} = K_2 \left(\frac{\varepsilon_0 A}{d}\right)$$

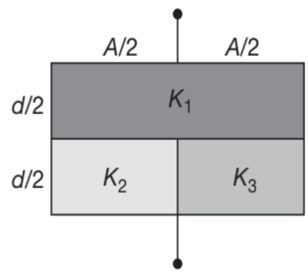
$$C_3 = K_3 \frac{\varepsilon_0 (A/2)}{d} = K_3 \left(\frac{\varepsilon_0 A}{2d}\right)$$

So, $C_{eq} = \dfrac{C_1 C_2}{C_1 + C_2} + C_3 = \dfrac{\varepsilon_0 A}{d}\left(\dfrac{K_1 K_2}{K_1 + K_2} + \dfrac{K_3}{2}\right)$

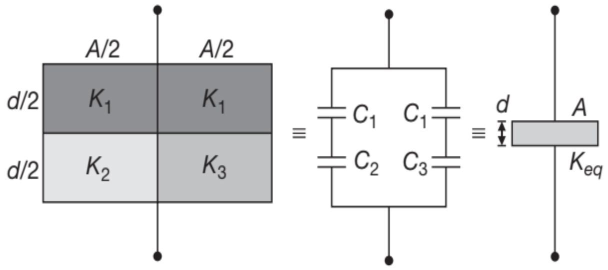
$$C_{eq} = K_{eq}\left(\frac{\varepsilon_0 A}{d}\right)$$

$$\Rightarrow \quad K_{eq} = \frac{K_1 K_2}{K_1 + K_2} + \frac{K_3}{2}$$

(d) To calculate equivalent capacitance in this case, we have



to proceed as follows by redrawing the arrangement



where $C_1 = K_1 \dfrac{\varepsilon_0 (A/2)}{(d/2)} = K_1 \left(\dfrac{\varepsilon_0 A}{d}\right)$

$$C_2 = K_2 \frac{\varepsilon_0 (A/2)}{(d/2)} = K_2 \left(\frac{\varepsilon_0 A}{d}\right)$$

$$C_3 = K_3 \frac{\varepsilon_0 (A/2)}{(d/2)} = K_3 \left(\frac{\varepsilon_0 A}{d}\right)$$

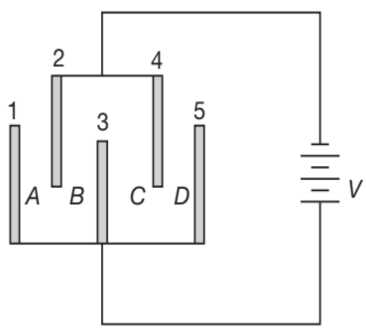
So, $C_{eq} = \dfrac{C_1 C_2}{C_1 + C_2} + \dfrac{C_1 C_3}{C_1 + C_3}$

$$\Rightarrow \quad C_{eq} = \frac{\varepsilon_0 A}{d}\left[\frac{K_1 K_2}{K_1 + K_2} + \frac{K_1 K_3}{K_1 + K_3}\right] = K_{eq}\left(\frac{\varepsilon_0 A}{d}\right)$$

$$\Rightarrow \quad K_{eq} = \frac{K_1 K_2}{K_1 + K_2} + \frac{K_1 K_3}{K_1 + K_3}$$

ILLUSTRATION 16

Five identical capacitor plates each of area A are arranged in such a way that adjacent plates are at a distance d apart. The plates are connected to source of emf V. Calculate the magnitude and nature of charge on plate 1 and 4 respectively.



SOLUTION

These five plates constitute four identical capacitors in parallel each of capacitance C given by

$$C = \left(\frac{\varepsilon_0 A}{d}\right)$$

Now plate-1 connected to positive terminal of battery and part of one capacitor

$$q_1 = +\left(\frac{\varepsilon_0 A V}{d}\right)$$

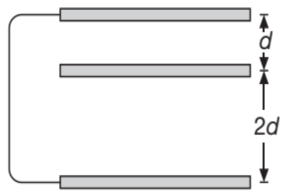
Since plate 4 is connected to negative terminal and is common between two capacitors in parallel, so

$$q_4 = q_C + q_D = -2q_1 = \left(-2\left[\frac{\varepsilon_0 A V}{d}\right]\right)$$

$$q_4 = -\left(\frac{2\varepsilon_0 A V}{d}\right)$$

ILLUSTRATION 17

Two large parallel metal plates are oriented horizontally and separated by a distance $3d$. A grounded conducting wire joins them, and initially each plate carries no charge. Now a third identical plate carrying charge Q is inserted between the two plates, parallel to them and located a distance d from the upper plate, as in Figure.



(a) What induced charge appears on each of the two original plates?

(b) What potential difference appears between the middle plate and each of the other plates? Each plate has area A.

SOLUTION

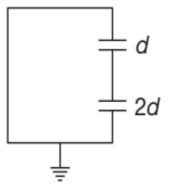
Imagine the centre plate is split along its midplane and pulled apart. We have two capacitors in parallel, supporting the same ΔV and carrying total charge Q.

The upper has a capacitance $C_1 = \dfrac{\varepsilon_0 A}{d}$ and the lower has a capacitance $C_2 = \dfrac{\varepsilon_0 A}{2d}$.

Charge flows from ground onto each of the outside plates of capacitors, so that

$$Q_1 + Q_2 = Q \qquad \text{...(1)}$$

and $\Delta V_1 = \Delta V_2 = \Delta V$



$$\Rightarrow \quad \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$\Rightarrow \quad \frac{Q_1 d}{\varepsilon_0 A} = \frac{Q_2 2d}{\varepsilon_0 A}$$

$$\Rightarrow \quad Q_1 = 2Q_2$$

From (1), we get

$$2Q_2 + Q_2 = Q$$

$$\Rightarrow \quad Q_2 = \frac{Q}{3}$$

(a) $Q_2 = \frac{Q}{3}$

So, on the lower plate the charge is $-\frac{Q}{3}$

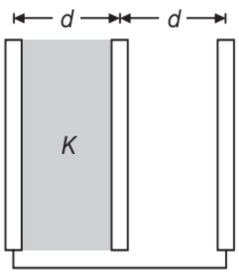
$$Q_1 = \frac{2Q}{3}$$

So, on the upper plate the charge is $-\frac{2Q}{3}$

(b) $\Delta V = \frac{Q_1}{C_1} = \frac{2Qd}{3\varepsilon_0 A}$

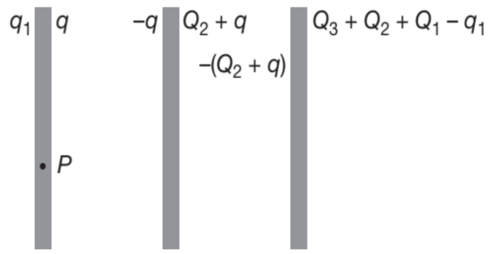
ILLUSTRATION 18

Three large conducting plates are placed a distance d apart in air. The space between the first two plates is completely filled with a dielectric slab of dielectric constant $K = 2$ as shown in Figure. The plates are given charges $Q_1 = 7Q$, $Q_2 = 3Q$ and $Q_3 = 2Q$ respectively. The outer two plates are now connected with a conducting wire. Find the charges on all the six surfaces.



SOLUTION

Charges on the surfaces facing each other will be equal and opposite (from Gauss's Law). Let the distribution be as shown in the following diagram and the charges on the six surfaces are as shown.



As electric field inside the conductor (let at point P) is zero.

$$q_1 = Q_3 + Q_2 + Q_1 - q_1$$

$$\Rightarrow \quad q_1 = \frac{Q_3 + Q_2 + Q_1}{2}$$

Since, the outer plates are connected hence, their potential will be same.

$$\Rightarrow \quad \frac{Q_2 + q}{\varepsilon_0 A} d + \frac{q}{K\varepsilon_0 A} d = 0$$

$$\Rightarrow \quad Q_2 + q + \frac{q}{K} = 0$$

$$\Rightarrow \quad KQ_2 + (K+1)q = 0$$

$$\Rightarrow \quad q = \frac{-KQ_2}{K+1} = -2Q \qquad \{\because Q_2 = 3Q\}$$

So, the final charge configuration on the six surfaces will be as shown here.

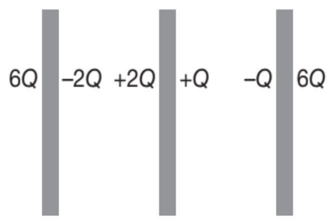
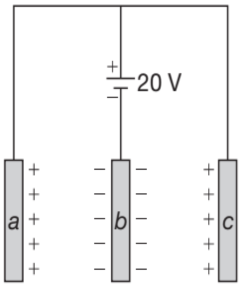


ILLUSTRATION 19

Each of the three plates shown in Figure has $200\ \text{cm}^2$ area on one side, and the gap between the adjacent plates is $0.2\ \text{mm}$. The emf of the battery is $20\ \text{V}$. Calculate the distribution of charge on various surfaces of the plates and the equivalent capacitance of the system between the terminal points?



SOLUTION

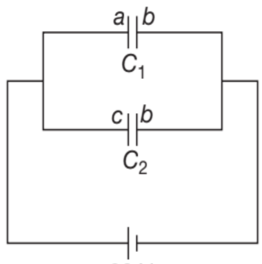
As the potentials of a and c are equal, the capacitors C_{ab} and C_{bc} are in parallel. Therefore, equivalent capacitance is

$$C = C_{ab} + C_{bc} = \frac{2\varepsilon_0 A}{d}$$

where, $C_{ab} = C_{bc} = \frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 \times 2\times 10^{-2}}{0.2\times 10^{-3}} = 100\varepsilon_0$

$$\Rightarrow \quad C = \frac{2\times \varepsilon_0 \times 2\times 10^{-2}}{0.2\times 10^{-3}} = 200\varepsilon_0$$

The charge on plate b is negative on both faces. Thus, the charge on faces a and c is



$$q_a = q_c = \frac{\varepsilon_0 A}{d} V$$

$$\Rightarrow \quad q_a = q_c = 100\varepsilon_0 \times 20$$

$$\Rightarrow \quad q_a = q_c = 2000\varepsilon_0$$

$$\Rightarrow \quad Q = 2\times 2000\varepsilon_0$$

$$\Rightarrow \quad Q = 4000\varepsilon_0$$

Test Your Concepts-II

Based on Series and Parallel Combination of Capacitors

(Solutions on page H.127)

1. Two identical parallel-plate capacitors, each with capacitance C, are charged to potential difference ΔV and connected in parallel. Then the plate separation in one of the capacitors is doubled.

(a) Find the total energy of the system of two capacitors before the plate separation is doubled.

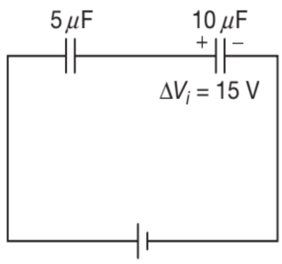
(b) Find the potential difference across each capacitor after the plate separation is doubled.

(c) Find the total energy of the system after the plate separation is doubled.

(d) Reconcile the difference in the answers to parts (a) and (c) with the Law of Conservation of Energy.

2. Two identical parallel plate capacitors are first connected in series and then in parallel. In each case, the plates of one capacitor are brought closer by a distance Δd and the plates of the other capacitor are moved away by same distance Δd. How does the total capacitance of the system change in both cases by the above displacement of plates?

3. A 10 μF capacitor is charged to 15 V. It is next connected in series with an uncharged 5 μF capacitor. The series combination is finally connected across a 50 V battery, as shown in Figure. Find the new potential differences across the 5 μF and 10 μF capacitors.



4. Each plate of a parallel plate capacitor has an area A. What amount of work has to be performed to slowly increase the distance between the plates from x_1 to x_2, if

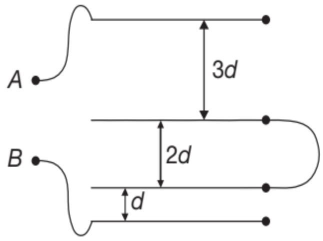
(a) the charge of the capacitor, q is kept constant in the process.

(b) the voltage across the capacitor, V, is kept constant in the process.

5. Two capacitors A and B are connected in series across a 100 V supply and it is observed that the potential difference across them are 60 V and 40 V. A capacitor of 2 μF capacitance is now connected in parallel with A and the potential difference across B rises to 90 V. Determine the capacitance of A and B.

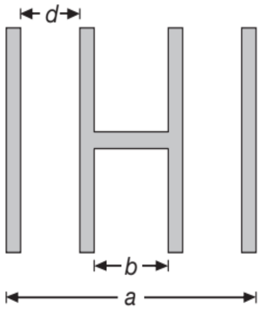
6. Two capacitors are in parallel and the energy of the combination is 0.1 J, when the differences of potential between terminals is 2 V. With the same two condensers now connected in series, the energy is 1.6×10^{-2} J for the same difference of potential across the series combination. Calculate the capacitances of the capacitors.

7. If the area of each plate is A and the successive separation are $3d$, $2d$ and d. Then find the equivalent capacitance across A and B.



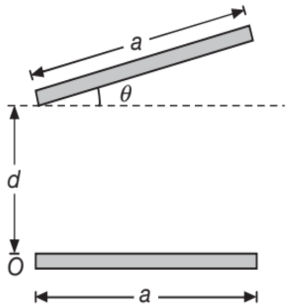
8. Two capacitors A and B each having a dielectric of dielectric constant 2 are connected in series. When they are connected across a 230 V D.C. supply it is found that the potential difference across A is 130 V and that across B is 100 V. If the dielectric in the smaller capacitor is replaced by another dielectric of dielectric constant 5, what will be the new values of the potential difference across each?

9. Two capacitors are joined in series shown in figure. The central rigid H-shaped part is movable. Find the equivalent capacitance of the combination and hence show that it is independent of position of the central H-shaped part. The area of each plate is A.

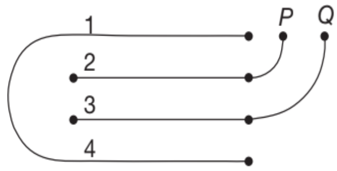


10. A capacitor has square plates, each of side a making an angle θ with each other as shown in figure. Show that for small θ, the capacitance C is given by

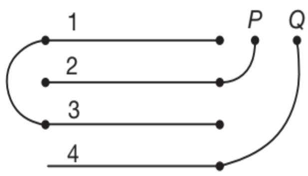
$$C = \frac{\varepsilon_0 a^2}{d}\left(1 - \frac{a\theta}{2d}\right)$$



11. Find the effective capacitance of the following arrangement of four identical metallic plates between P and Q. The area of each plate is A and separation between successive plates is d.



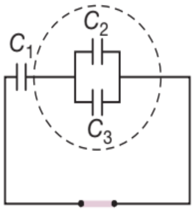
ARRANGEMENT 1



ARRANGEMENT 2

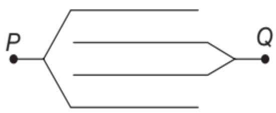
12. A radio capacitor of variable capacitance is made of n plates each of area A and separated from each other by a distance d. The alternate plates are connected together. One group of alternate plates is fixed while the other is movable. Find the maximum capacitance of the capacitor.

13. (a) Find the equivalent capacitance of the combination shown in figure, where $C_1 = 6\ \mu F$, $C_2 = 4\ \mu F$ and $C_3 = 8\ \mu F$.

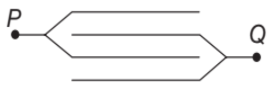


(b) If a battery is connected between a and b, determine the charge on each capacitor and the potential difference across each, if $V_{ab} = 12$ V.

14. Find the capacitance of the following combinations between P and Q. Area of the each plate is A and separation of successive plates is d.

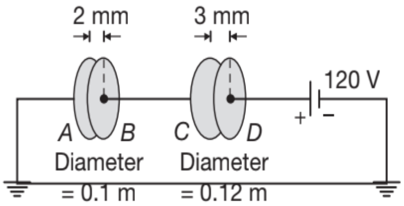


ARRANGEMENT 1



ARRANGEMENT 2

15. The circular plates A and B of a parallel plate air capacitor have a diameter of 0.1 m and are 2 mm apart. The plates C and D, of a similar capacitor have a diameter 0.12 m and are 3 mm apart. Plate A is earthed and the plates B and D are connected together. Plate C is connected to the positive pole of a 120 V battery whose negative terminal is earthed. Calculate



(a) the combined capacitance of the arrangement and

(b) the energy stored in it.

Take $\frac{1}{\varepsilon_0} = 36\pi \times 10^9\ \text{Nm}^2\text{C}^{-2}$

16. Using the concept of energy density, find the total energy stored in a

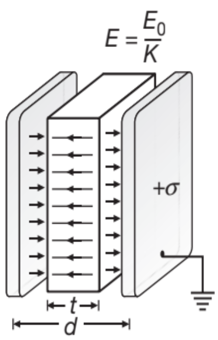
(a) parallel plate capacitor

(b) charged spherical conductor

DIELECTRIC SLAB INSERTED IN A PARALLEL PLATE CAPACITOR

When the space between the parallel plate capacitor is partly filled with a dielectric of thickness $t(<d)$

If no slab is introduced between the plates of the capacitor, then a field E_0 given by $E_0 = \frac{\sigma}{\varepsilon_0}$, exists in a space d.



On inserting the slab of thickness t, a field $E = \frac{E_0}{K}$ exists inside the slab of thickness t and a field E_0 exists in remaining space $(d-t)$. If V is total potential then

$$V = E_0(d-t) + Et$$

$$\Rightarrow \quad V = E_0\left[d - t + \left(\frac{E}{E_0}\right)t\right]$$

But $\frac{E_0}{E} = K$ = Dielectric Constant

$$\Rightarrow \quad V = \frac{\sigma}{\varepsilon_0}\left[d - t + \frac{t}{K}\right]$$

$$\Rightarrow \quad V = \frac{q}{A\varepsilon_0}\left[d - t + \frac{t}{K}\right]$$

$$\Rightarrow \quad C = \frac{q}{V} = \frac{\varepsilon_0 A}{d - t\left(1 - \frac{1}{K}\right)}$$

So, on introducing a dielectric slab of thickness t and dielectric constant K the capacitance increases by the same amount as the effective air spacing between the plates is made $t\left(1 - \frac{1}{K}\right)$

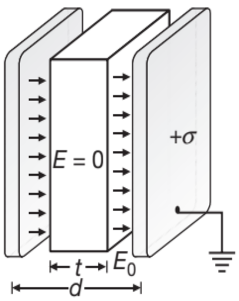
CONDUCTING SLAB INSERTED IN A PARALLEL PLATE CAPACITOR

When the space between the parallel plate capacitor is partly filled by a conducting slab of thickness $t(< d)$.

If no conducting slab is introduced between the plates, then a field $E_0 = \frac{\sigma}{\varepsilon_0}$ exists in a space d. If C_0 be the capacitance (without the introduction of conducting slab), then

$$C_0 = \frac{\varepsilon_0 A}{d}$$

On inserting the slab, field inside it is zero and so a field $E_0 = \frac{\sigma}{\varepsilon_0}$ now exists in a space $(d - t)$



$$\Rightarrow \quad V = E_0(d - t)$$

$$\Rightarrow \quad V = \frac{\sigma}{\varepsilon_0}(d - t)$$

$$\Rightarrow \quad V = \frac{q}{A\varepsilon_0}(d - t)$$

$$\Rightarrow \quad C = \frac{q}{V} = \frac{\varepsilon_0 A}{d - t}$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 A}{d\left(1 - \frac{t}{d}\right)}$$

$$\Rightarrow \quad C = \frac{C_0}{\left(1 - \frac{t}{d}\right)}$$

Since $d - t < d$

$$\Rightarrow \quad C > C_0$$

i.e., Capacitance increases on insertion of conducting slab between the plates of capacitor.

If $t = d$, then $C \to \infty$ i.e., if a conducting slab occupies the complete space between the plates of the capacitor, then $C \to \infty$.

CHARGE INDUCED ON A DIELECTRIC AND GAUSS'S LAW FOR DIELECTRICS

Consider a parallel plate capacitor which has a surface charge density σ and $-\sigma$ on each plate. If E_0 is the electric field between the plates of air capacitor, then

$$E_0 = \frac{\sigma}{\varepsilon_0}$$

Let a dielectric of dielectric constant K be now introduced between the plates of the capacitor. If σ_p is induced surface charge density (see figure) due to the already existing field, then electric field due to induced charge is

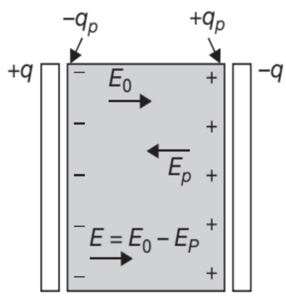
$$E_p = \frac{\sigma_p}{\varepsilon_0}$$

So, resultant dielectric field within the plates is

$$E = E_0 - E_p$$

$$\Rightarrow \quad E = \frac{1}{\varepsilon_0}(\sigma - \sigma_p) \qquad \ldots(1)$$

$$\text{Also } E = \frac{\sigma}{K\varepsilon_0} \qquad \ldots(2)$$



Compare (1) and (2), we get

$$\frac{1}{\varepsilon_0}(\sigma-\sigma_p)=\frac{\sigma}{K\varepsilon_0}$$

$$\Rightarrow \quad \sigma_p=\sigma\left(1-\frac{1}{K}\right)$$

$$\Rightarrow \quad \frac{q_p}{A}=\frac{q}{A}\left(1-\frac{1}{K}\right)$$

$$\Rightarrow \quad q_p=q\left(1-\frac{1}{K}\right)$$

Since according to Gauss's Law, for dielectric

$$\oint \vec{E}\cdot d\vec{A}=\frac{1}{\varepsilon_0}(\Sigma q_{\text{enc}})=\frac{1}{\varepsilon_0}(q-q_p)$$

$$\Rightarrow \quad \oint \vec{E}\cdot d\vec{A}=\frac{1}{\varepsilon_0}\left[q-q\left(1-\frac{1}{K}\right)\right]=\frac{q}{K\varepsilon_0}$$

$$\Rightarrow \quad \oint \vec{E}\cdot d\vec{A}=\frac{q}{K\varepsilon_0}=\frac{q}{\varepsilon} \qquad \{\because\ \varepsilon=\varepsilon_r\varepsilon_0=K\varepsilon_0\}$$

$$\Rightarrow \quad \oint K\varepsilon_0\vec{E}\cdot d\vec{A}=q$$

$$\Rightarrow \quad \oint \vec{D}\cdot d\vec{A}=q$$

where $\vec{D}=K\varepsilon_0\vec{E}$ is called the **Electric Displacement Vector**.

Problem Solving Technique(s)

Since $q_p=q\left(1-\frac{1}{K}\right)$

For a conductor $K\to\infty$. Hence,

$$q_p=q,\ \sigma_p=\sigma \text{ and } E=0$$

Hence, we may conclude the above discussion as under

(a) $E_{\text{vacuum}}=E_0=\frac{\sigma}{\varepsilon_0}=\frac{q}{A\varepsilon_0}$

(b) $E_{\text{dielectric}}=\frac{E_0}{K}$ $\quad\{K=\text{dielectric constant}\}$

(c) $E_{\text{conductor}}=0$ $\quad\{\text{as } K\to\infty\}$

Regarding dielectrics, it is worth noting that:

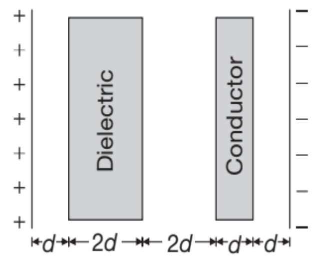
(a) These are non-conductors upto a certain value of field depending on its nature. If the field exceeds this limiting value called **dielectric strength,** dielectric loses its insulating property and begins to conduct.

(b) These have either permanent dipole moment (polar-dielectrics, e.g., water) or acquire induced dipole moment **(non-polar dielectrics)** when placed in an electric field.

(c) The dielectric constant of polar dielectric depends on its temperature and due to thermal agitation with rise in temperature decreases.

ILLUSTRATION 20

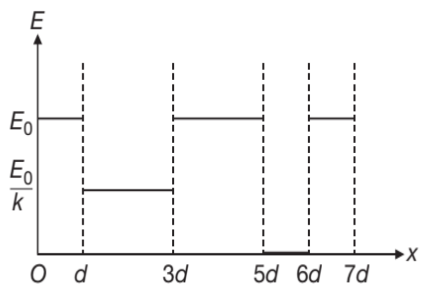
Two parallel conducting plates of area A charge $+q$ and $-q$ are as shown. A dielectric slab of dielectric constant K, thickness $2d$ and a conducting plate of thickness d is inserted between them. Taking $x=0$ at positive plate and $x=7d$ at negative plate, plot E vs x and V vs x graphs. Here E is the electric field and V is the potential and consider the potential at the positive plate to be V_0.

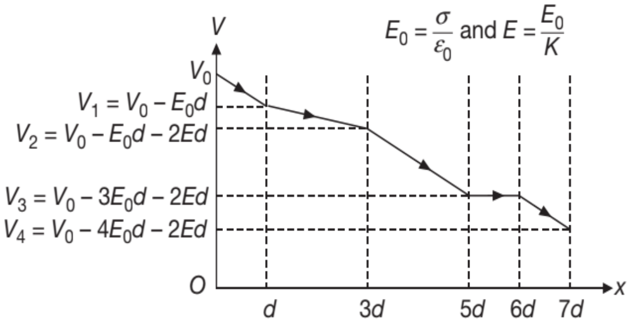


SOLUTION

The electric field in air/vacuum is $E_0=\frac{\sigma}{\varepsilon_0}=\frac{q}{A\varepsilon_0}$.

In dielectric, field is $E=\frac{E_0}{K}$ and in conductor field is zero. Hence, the E-x graph is as shown in figure.





Using $V = Ed$ (in uniform field) the V-x graph is shown (slope of this graph gives electric field in that region).

ILLUSTRATION 21

A parallel plate capacitor with air as dielectric has a plate area of 200 cm^2 and plate separation of 4 mm. Calculate the percentage change in the capacitance if a layer of varnish $(k = 3)$ of thickness 0.1 mm is given on the inside of both plates.

SOLUTION

$$C = \frac{\varepsilon_0 A}{d - t + \frac{t}{k}} = \frac{C_0}{1 - \frac{t}{d} + \frac{t}{kd}} \qquad \left\{\because C_0 = \frac{\varepsilon_0 A}{d}\right\}$$

where $t = 2(0.1 \text{ mm}) = 0.2$ mm, $d = 4$ mm and $K = 3$

Putting values, we get $C_0 = 44.25$ pF

Now, $C = \dfrac{44.25}{1 - \frac{0.2}{4} + \frac{0.2}{(3)(4)}}$

$$\Rightarrow \quad C = \frac{44.25}{1 - \frac{1}{20} + \frac{1}{60}} \text{ pF}$$

$$\Rightarrow \quad C = \frac{44.25 \times 60}{60 - 3 + 1} = \frac{2655}{58}$$

$$\Rightarrow \quad C = 45.77\%$$

So, %age change is

$$\frac{\Delta C}{C_0} \times 100 = \left(\frac{45.77 - 44.25}{44.25}\right) \times 100 = 3.4\%$$

So, percentage increase in the value of capacitance is 3.4%

EFFECT OF INSERTION OF DIELECTRIC IN A PARALLEL PLATE CAPACITOR

The effect of insertion of a dielectric on other physical quantities such as charge, potential difference, electric field and energy associated with a capacitor depends on the fact that whether the charged capacitor is isolated (i.e., charge is kept constant) or is attached to the battery (i.e. potential is kept constant).

If q_0, C_0, V_0, E_0 and U_0 represent the charge, capacitance, potential difference, electric field and energy associated with charged air capacitor respectively. On introduction of a dielectric slab of dielectric constant K between the plates let the respective quantities becomes q, C, V, E and U.

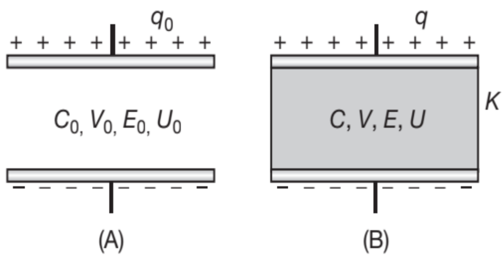
CASE-1: When charge is kept constant OR Capacitor is Isolated OR Battery is Disconnected

(a) Charge remains unchanged, i.e., $q = q_0$, as in an isolated system charge is conserved.

(b) Capacitance increases and becomes $C = KC_0$, because due to the presence of a dielectric, the capacitance becomes K times.

(c) Potential difference between the plates decreases and becomes $V = \left(\frac{V_0}{K}\right)$

$$\left(\because V = \frac{q}{C} = \frac{q_0}{KC_0}, \text{ as } q = q_0 \text{ and } C = KC_0\right)$$



(d) Field between the plates decreases and becomes

$$E = \left(\frac{E_0}{K}\right)$$

$$\Rightarrow \quad E = \frac{V}{d} = \frac{V_0}{Kd} = \frac{E_0}{K} \qquad \left\{\because V = \frac{V_0}{K} \text{ and } E_0 = \frac{V_0}{d}\right\}$$

(e) Energy stored in the capacitor decreases and becomes $U = \left(\frac{U_0}{K}\right)$

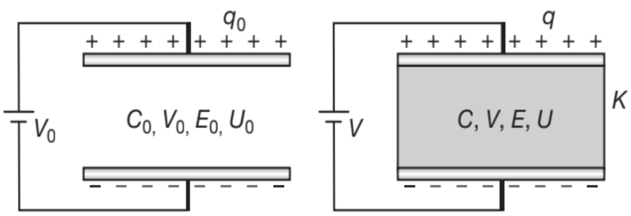
Since $U = \dfrac{q^2}{2C}$

$$\Rightarrow \quad U = \frac{q_0^2}{2KC_0} = \frac{U_0}{K} \qquad \{\because q = q_0 \text{ and } C = KC_0\}$$

CASE-2: When potential is kept constant OR battery remains attached

(a) Potential difference remains constant, i.e., $V = V_0$, because battery is a source of constant potential difference.

(b) Capacitance increases and becomes $C = KC_0$, because by presence of a dielectric capacitance becomes K times.



(A) (B)

(c) Charge on capacitor increases, i.e., $q = Kq_0$, because

$$q = CV$$

$$\Rightarrow \quad q = (KC_0)V = Kq_0 \qquad \{\because q_0 = C_0V\}$$

(d) Electric field remains unchanged, so $E = E_0$

$$E = \frac{V}{d} = \frac{V_0}{d} = E_0 \quad \left\{\because V = V_0 \text{ and } \frac{V_0}{d} = E_0\right\}$$

(e) Energy stored in the capacitor increases and becomes $U = KU_0$

$$\text{Since, } U = \frac{1}{2}CV^2 = \frac{1}{2}(KC_0)(V_0)^2$$

$$\Rightarrow \quad U = \frac{1}{2}KU_0 \qquad \left\{\because C = KC_0 \text{ and } U_0 = \frac{1}{2}C_0V_0^2\right\}$$

Problem Solving Technique(s)

While solving problems of this type always keep in mind that, whenever a battery is disconnected then,

q = constant

and if battery remains attached, then V = constant

ILLUSTRATION 22

An air capacitor of capacity $C = 10\ \mu\text{F}$ is connected to a constant voltage battery of $12\ \text{V}$. Now the space between the plates is filled with a liquid of dielectric constant $k = 5$. Calculate the charge that flows from battery to the capacitor.

SOLUTION

Initially charge on the capacitor $Q_i = 10 \times 12 = 120\ \mu\text{C}$

When dielectric medium is filled, so capacitance becomes k times, i.e., new capacitance is

$$C' = kC = 5 \times 10 = 50\ \mu\text{C}$$

Final charge on the capacitor $Q_f = 50 \times 12 = 600\ \mu\text{C}$

Hence additional charge supplied by the battery is

$$\Delta Q = Q_f - Q_i = 480\ \mu\text{C}$$

ILLUSTRATION 23

The capacitance of a variable radio capacitor can be changed from $50\ \text{pF}$ to $950\ \text{pF}$ by turning the dial from $0°$ to $180°$. With the dial set at $180°$, the capacitor is connected to a $400\ \text{V}$ battery. After charging, the capacitor is disconnected from the battery and the dial is turned at $0°$.

(a) What is the potential difference across the capacitor when the dial reads $0°$?

(b) How much work is required to turn the dial, if friction is neglected?

SOLUTION

When the dial is at $0°$, the capacitance of the capacitor is given by

$$C_1 = 50\ \text{pF} = 50 \times 10^{-12}\ \text{F}$$

When dial is at $180°$, the capacitance is given by

$$C_2 = 950\ \text{pF} = 950 \times 10^{-12}\ \text{F}$$

The potential difference across capacitor C_2, is given by

$$V_2 = 400\ \text{V}$$

Charge on capacitor C_2 is

$$q = C_2V_2 = 950 \times 10^{-12} \times 400$$

$$\Rightarrow \quad q = 380 \times 10^{-9}\ \text{C}$$

(a) When battery is disconnected the charge remains the same and so, $q=$ constant. Let V_1 be the potential difference across capacitor when dial reads 0°. Then

$$q = C_1V_1$$

$$\Rightarrow \quad 380\times10^{-9} = 50\times10^{-12}\times V_1$$

$$\Rightarrow \quad V_1 = \frac{380\times10^{-9}}{50\times10^{-12}} = 7600\text{ V}$$

(b) Work required to turn the dial from 180° to 0° is
$W =$ Gain in energy of capacitor

$$\Rightarrow \quad W = \frac{q^2}{2C_1} - \frac{q^2}{2C_2} = \frac{q^2}{2}\left(\frac{1}{C_1}-\frac{1}{C_2}\right)$$

$$\Rightarrow \quad W = \frac{q^2(C_2-C_1)}{2C_1C_2}$$

$$\Rightarrow \quad W = \frac{(380\times10^{-9})^2(950-50)\times10^{-12}}{2\times50\times10^{-12}\times950\times10^{-12}}$$

$$\Rightarrow \quad W = 1.368\times10^{-3}\text{ J}$$

ILLUSTRATION 24

Two parallel capacitors each of capacitance C are connected in series to a battery of emf V. Now, one of the capacitors is filled completely with a dielectric of dielectric constant K.

(a) Calculate the ratio of the electric field strength in the capacitor to the electric field when the dielectric is introduced. Does the field increase or decrease after insertion of dielectric?

(b) Find the amount of charge that flows through the battery.

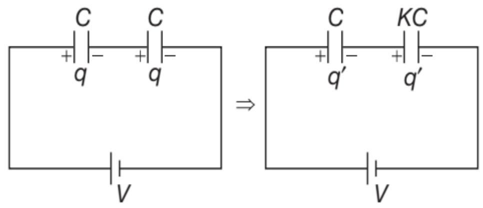
SOLUTION

(a) Net capacitance without inserting the slab is, $\frac{C}{2}$

$$\Rightarrow \quad q = \left(\frac{C}{2}\right)V$$

Initial electric field E is given by

$$E = \frac{\text{Potential Difference}}{\text{separation between the plates}} = \frac{\left(\frac{V}{2}\right)}{d} = \frac{V}{2d}$$



Net capacitance after inserting the slab is

$$C' = \left(\frac{K}{K+1}\right)C$$

$$\Rightarrow \quad q' = CV\left(\frac{K}{K+1}\right)$$

Final electric field E' is given by

$$E' = \frac{\text{Potential Difference}}{\text{Separation between the plates}}$$

$$\Rightarrow \quad E' = \frac{V\left(\frac{1}{K+1}\right)}{d}\frac{V}{(K+1)d}$$

So, electric field decreases by a factor

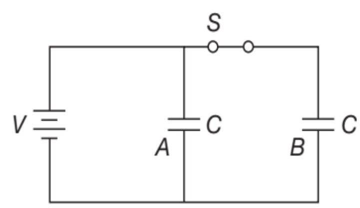
$$\frac{E}{E'} = \frac{\left(\frac{V}{2d}\right)}{\left(\frac{V}{(K+1)d}\right)} = \frac{K+1}{2}$$

(b) Charge that flows through the battery is $q'-q$

$$\text{Charge Flowing} = \frac{CV(K-1)}{2(K+1)}$$

ILLUSTRATION 25

Figure shows two identical parallel plate capacitors connected to a battery with the switch S closed. The switch is now opened and the free space between the plates of the capacitors is filled with a dielectric of dielectric constant (or relative permittivity) 3. Find the ratio of the total electrostatic energy stored in both capacitors before and after the introduction of the dielectric.



SOLUTION

With the switch S closed, the potential difference across capacitors A and B is same. So,

$$V=\frac{Q_A}{C}=\frac{Q_B}{C}$$

The initial charges on the capacitors are given by

$$Q_A=Q_B=CV$$

When dielectric is introduced, the new capacitance of either capacitor.

$$C'=KC=3C$$

Now, when the switch S is opened, let the potential difference across capacitor A be V volt and that across the capacitor B be V' volt.

When dielectric is introduced with the switch open (i.e., battery disconnected) then, the charge on capacitor B remains unchanged, so

$$Q_B=CV=C'V'$$

$$\Rightarrow \quad V'=\frac{C}{C'}V=\frac{V}{3}\text{ volt}$$

Initial energy of both capacitors

$$U_i=\frac{1}{2}CV^2+\frac{1}{2}CV^2=CV^2$$

Final energy of both capacitors

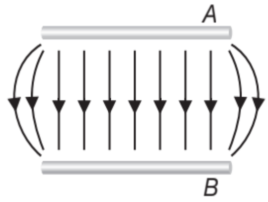
$$U_f=\frac{1}{2}C'V^2+\frac{1}{2}C'V'^2$$

$$\Rightarrow \quad U_f=\frac{1}{2}(3C)V^2+\frac{1}{2}(3C)\left(\frac{V}{3}\right)^2=\frac{5}{3}CV^2$$

$$\Rightarrow \quad \frac{U_i}{U_f}=\frac{CV^2}{\frac{5}{3}CV^2}=\frac{3}{5}$$

FORCE ON A DIELECTRIC SLAB BEING INSERTED BETWEEN CAPACITOR PLATES

When a dielectric slab is being inserted between the plates of an initially charged capacitor, then due to the electric field between the plates of the capacitor, bound charges of opposite nature are induced inside the dielectric. Also it is observed that while inserting the dielectric slab, fringing of electric field lines at the edges of the plates takes place. (*Fringing Effect is the bending of field lines at the corners of a capacitor as shown*).



Fringing effect for a capacitor with large d.

A component of the electric field along the surface of the dielectric pulls the dieletric slab inside the capacitor. Thus force on the dielectric slab is given by

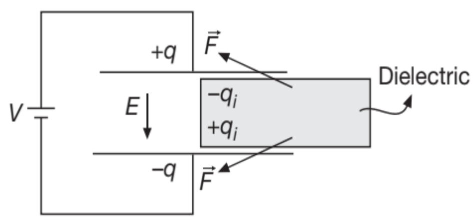
$$F=\left|\frac{dU}{dx}\right|$$

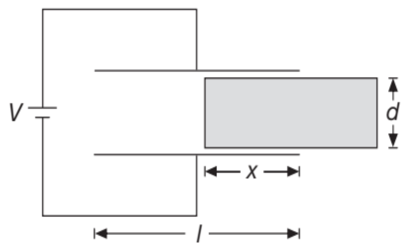
Let us discuss the two cases under which the dielectric slab can be introduced between the plates of the capacitor.

CASE-1: FOR CAPACITOR CONNECTED TO BATTERY (i.e. V = Constant)

Consider a dielectric slab inserted partially between the plates of a capacitor. Due to positive charge on the upper plate, some negative charge is induced on the top surface of the dielectric. Similarly, some positive charge is induced at the bottom surface of the slab. The negative induced charge is attracted towards positive charge on upper plate and the positive induced charge is attracted towards negative charge on lower plate as shown. As a result, the net force on the slab acts inwards and the slab is attracted into the capacitor.

To calculate this force, let us consider a parallel plate capacitor having plates of length l, width b and separation d. Let the capacitor be connected to a battery of voltage V. Let us find the force on a slab of thickness d and dielectric constant K when a portion of length $x(<l)$ lies inside the plates.





The capacitance of the capacitor so formed is calculated by assuming the capacitor in this position to be made up of two capacitors in parallel, one without a dielectric (of length $(l-x)$, separation d and width b) and the other with a dielectric (of length x, separation d and width b).

The capacitance of the portion of capacitor without dielectric is,

$$C_1 = \frac{\varepsilon_0 b(l-x)}{d}$$

and that of capacitor with dielectric is

$$C_2 = \frac{K\varepsilon_0 bx}{d}$$

So, the net capacitance of the capacitor in this position is,

$$C = C_1 + C_2$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 b}{d}[l + x(K-1)] \qquad \ldots(1)$$

The electric field inside the capacitor attracts the dielectric slab with a force F (say). For the dielectric slab to slowly enter the capacitor, an external force $F_{ext} = F$ must be applied opposite to F. Due to this external force, the dielectric slab doesnot gain any kinetic energy while entering the capacitor plates.

Let the slab move further inside the capacitor by an infinitesimal distance dx. Due to this the capacitance of the capacitor also increases from C to $C + dC$. Since the capacitor is connected to the battery, so potential difference V remains constant and hence the battery has to supply a charge (dQ) to the capacitor given by

$$dQ = V(dC)$$

During this process work done by the battery is

$$dW_{\text{battery}} = VdQ = (dC)V^2$$

Also work done by the external force F during this displacement dx is

$$dW_{\text{external}} = Fdx\cos(180°) = -Fdx$$

Total work done on the capacitor is

$$dW = dW_{\text{battery}} + dW_{\text{external}}$$

$$\Rightarrow \quad dW = (dC)V^2 - Fdx$$

This work done dW is equal to the increase in the energy $dU = \frac{1}{2}(dC)V^2$ stored in the capacitor (*assuming that no heat losses are produced while inserting the dielectric inside the capacitor*). So, we have

$$\frac{1}{2}(dC)V^2 = (dC)V^2 - Fdx$$

$$\Rightarrow \quad Fdx = \frac{1}{2}(dC)V^2$$

$$\Rightarrow \quad F = \frac{1}{2}V^2\left(\frac{dC}{dx}\right)$$

From (1), we get $dC = \frac{\varepsilon_0 b}{d}(K-1)dx$

$$\Rightarrow \quad \frac{dC}{dx} = \frac{\varepsilon_0 b}{d}(K-1)$$

$$\Rightarrow \quad F = \frac{\varepsilon_0 bV^2(K-1)}{2d}$$

Thus, the electric field attracts the dielectric into the capacitor with the force F given by

$$F = \frac{\varepsilon_0 bV^2(K-1)}{2d}$$

Also, we note that this force is constant and is independent of the distance x.

CASE-2: FOR AN ISOLATED CAPACITOR (i.e., q = Constant)

When the battery is removed after charging the capacitor then q remains constant. In that case,

$$U = \frac{q^2}{2C}$$

$$\Rightarrow \quad dU = d\left[\frac{q^2 d}{2\varepsilon_0 b\{l + x(K-1)\}}\right]$$

$$\Rightarrow \quad dU = -\frac{q^2 d(K-1)}{2\varepsilon_0 b}\left(\frac{1}{l + x(K-1)}\right)^2 dx$$

Here, since work done by battery is zero and so only the electrostatic force does the work.

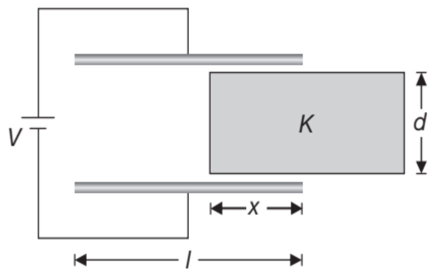
Hence, $F = -\frac{dU}{dx}$

$$\Rightarrow \quad F = \frac{q^2 d(K-1)}{2\varepsilon_0 b}\left(\frac{1}{l + x(K-1)}\right)^2$$

This force also acts inwards, but in this case it is not constant and is a function of x.

ILLUSTRATION 26

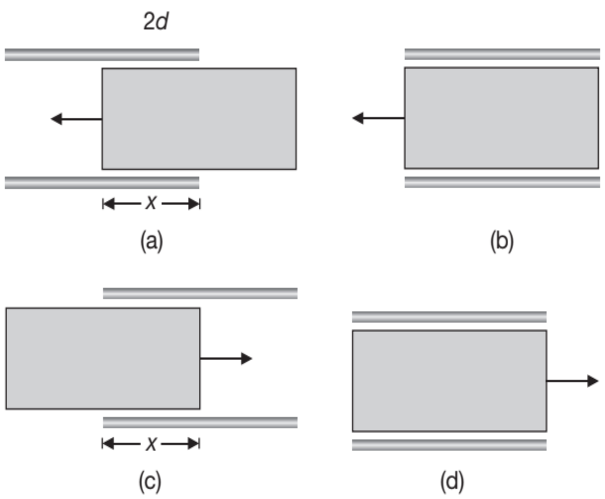
In the situation shown in figure the length of the plates is l, area of the plates is A and separation between then is d. A dielectric slab of dielectric constant K, mass m and thickness d is released from rest. Prove that the slab will execute periodic motion and find its time period. Assume friction to be absent between inner portion of the plates and the outer surface of the slab. Also see that initially the slab has its length x inserted in the plates of the capacitor.



SOLUTION

As derived and seen earlier that the electric forces pull the slab inwards with a constant force, given by

$$F = \frac{\varepsilon_0 b V^2 (K-1)}{2d}$$



where, b = width of plate $= \frac{A}{l}$

So, acceleration of slab, $a = \frac{F}{m}$

$$\Rightarrow \quad a = \frac{\varepsilon_0 \left(\frac{A}{l}\right) V^2 (K-1)}{2md}$$

$$\Rightarrow \quad a = \frac{\varepsilon_0 A V^2 (K-1)}{2mld}$$

At the instant when the slab is fully inside the plates, we get the state of equilibrium for the slab. The slab will execute periodic motion in the phases as shown in figure.

So, time taken to reach from position (a) to position (b) equals t (say).

Then, $t = \frac{T}{4}$

Since, $s = \frac{1}{2} a t^2$

$$\Rightarrow \quad t = \sqrt{\frac{2s}{a}} \qquad \{\because \ s = l - x\}$$

$$\Rightarrow \quad t = \sqrt{\frac{2(l-x)}{\frac{\varepsilon_0 A V^2 (K-1)}{2mld}}} = \sqrt{\frac{4(l-x)mld}{\varepsilon_0 A V^2 (K-1)}}$$

The desired time period is

$$T = 4t = 8\sqrt{\frac{(l-x)mld}{\varepsilon_0 A V^2 (K-1)}}$$

DIELECTRIC BREAKDOWN

When a dielectric material is subjected to a strong electric field then there comes a breakdown field strength at which it loses its insulating or dielectric ability and the dielectric becomes a conductor. This occurs when the electric field is so strong that electrons are ripped loose from their molecules. This maximum electric field magnitude that a material can withstand without the occurrence of breakdown is called its **Dielectric Strength**. The dielectric strength of dry air is about $3 \times 10^6\ \text{Vm}^{-1}$.

ELECTRIC ENERGY DENSITY OF DRY AIR

The breakdown field strength at which dry air loses its insulating ability and allows a discharge to pass through is $E_b = 3\times10^6\ \text{Vm}^{-1}$. At this field strength, the electric energy density is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\left(8.85\times10^{-12}\right)\left(3\times10^6\right)^2 = 40\ \text{Jm}^{-3}$$

ILLUSTRATION 27

Four identical plane capacitors with an air dielectric are connected in series. The intensity of the field at which the air is punctured is $E_a = 3\times10^4\ \text{Vcm}^{-1}$. The distance between the plates $d = 0.7$ cm.

(a) What maximum voltage can be fed to this battery of capacitors?

(b) What will this maximum voltage be if one of the capacitors is replaced by a similar one in which glass is used as a dielectric?

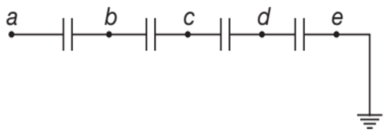
The permitivity of glass $K = 7$ and the puncturing field intensity of glass $E_g = 9\times10^4\ \text{Vcm}^{-1}$.

SOLUTION

(a) Potential across each capacitor $= V_0 = Ed$

$$\Rightarrow\quad V_0 = \left(3\times10^4\times10^2\ \text{Vm}^{-1}\right)\left(0.7\times10^{-2}\ \text{m}\right)$$

$$\Rightarrow\quad V_0 = 2.1\times10^4\ \text{V}$$



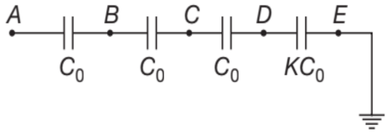
Now $\Delta V = \Delta V_{ab} + \Delta V_{bc} + \Delta V_{cd} + \Delta V_{de}$

$$\Rightarrow\quad \Delta V = 2.1\times10^4 + 2.1\times10^4 + 2.1\times10^4 + 2.1\times10^4$$

$$\Rightarrow\quad (\Delta V)_{\text{Max}} = 4\left(2.1\times10^4\right) = 8.4\times10^4\ \text{V}$$

(b) For glass, breakdown voltage is given by

$$V_g = 6.3\times10^4\ \text{V}$$



If the glass capacitor is connected between the points D and E, then

$$q = C_0\left(\Delta V_{AB}\right) = C_0\left(\Delta V_B C\right) = C_0\left(\Delta V_{CD}\right) = 7C_0\left(\Delta V_{DE}\right)$$

$$\Rightarrow\quad \Delta V_{AB} = \Delta V_{BC} = \Delta V_{CD} = K\Delta V_{DE}$$

$$\Rightarrow\quad \Delta V_{AB} = \Delta V_{BC} = \Delta V_{CD} = 7\Delta V_{DE}$$

OPTION 1

Take $\Delta V_{AB} = 2.1\times10^4$ V and find ΔV_{DE}

From the above equality, we get

$$\Delta V_{DE} = \frac{\Delta V_{AB}}{7} = 0.3\times10^4\ \text{V}$$

OPTION 2

Take $\Delta V_{DE} = 6.3\times10^4$ V (The breakdown voltage of glass), then

$$\Delta V_{AB} = \Delta V_{BC} = \Delta V_{CD} = 7\left(6.3\times10^4\ \text{V}\right)$$

$$\Rightarrow\quad \Delta V_{AB} = \Delta V_{BC} = \Delta V_{CD}$$

$$\Delta V_{AB} = 4.41\times10^5\ \text{V} > 2.1\times10^4\ \text{V}$$

So, as per OPTION 2, all the first three capacitors are going to breakdown, so, we have to accept

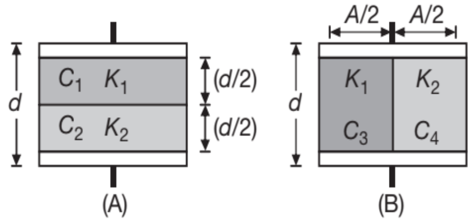
$$\Delta V_{DE} = 0.3\times10^4\ \text{V (for glass capacitor)}$$

Test Your Concepts-III

Based on Dielectrics and Breakdown

(Solutions on page H.132)

1. A capacitor is filled with two dielectrics of same dimensions but of dielectric constant 2 and 3 respectively. Find the ratio of capacities in the two arrangements.



2. A parallel-plate capacitor of plate separation d is charged to a potential difference ΔV_0. A dielectric slab of thickness d and dielectric constant κ is introduced between the plates while the battery remains connected to the plates.

(a) Show that the ratio of energy stored after the dielectric is introduced to the energy stored in the empty capacitor is $\dfrac{U}{U_0} = \kappa$. Give a physical explanation for this increase in stored energy.

(b) What happens to the charge on the capacitor?

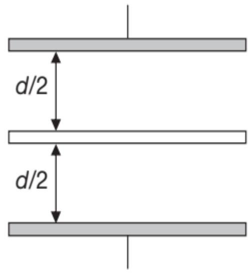
3. Two parallel capacitors each of capacitance C are connected in series to a battery of emf V. Now, one of the capacitors is filled completely with a dielectric of dielectric constant K.

(a) Calculate the ratio of the electric field strength in the capacitor to the electric field when the dielectric is introduced. Does the field increase or decrease after insertion of dielectric?

(b) Find the amount of charge that flows through the battery.

4. A parallel plate capacitor with air as dielectric has a plate area of 200 cm^3 and plate separation of 4 mm. Calculate the percentage change in the capacitance if a layer of varnish $(k = 3)$ of thickness 0.1 mm is given on the inside of both plates.

5. A thin conducting plate is inserted in halfway between the plates of a parallel plate capacitor.

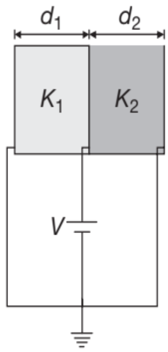


(a) Compute the capacitance before the plate was inserted.

(b) What is the capacitance after the plate was inserted?

(c) What does the value of capacitance becomes when the thin plate and the upper plate are shortened (i.e., connected by a conducting wire)? Area of each plate is A.

6. A capacitor is composed of three parallel conducting plates. All three plates are of the same area A. The first pair of plates are kept a distance d_1 apart, and the space between them is filled with a medium of a dielectric K_1. The corresponding data for the second pair are d_2 and K_2, respectively.



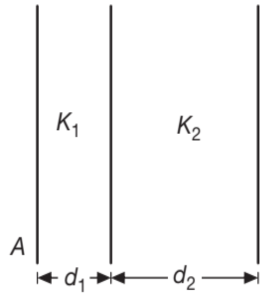
The middle plate is connected to the positive terminal of a constant voltage source V, and the external plates are connected the other terminal of V.

(a) Find the capacitance of the system.

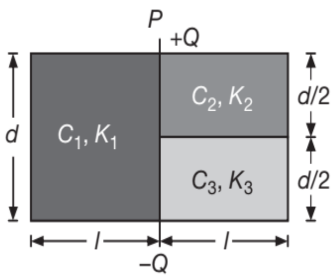
(b) What is the surface charge density on the middle plate?

(c) Compute the energy density in the medium K_1.

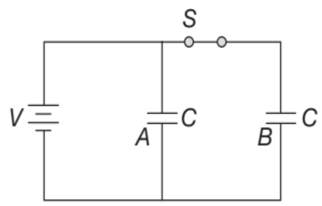
7. Find the capacitance of a system of three parallel plates, each of area A, separated by d_1 and d_2. The space between them is filled with dielectrics of relative dielectric constants K_1 and K_2.



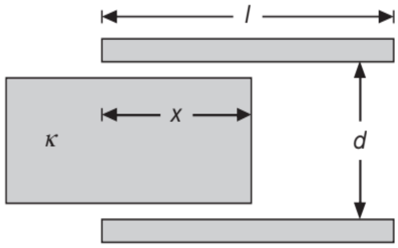
8. A parallel plate capacitor is constructed using three different dielectric materials as shown in figure. The parallel plates across which a potential difference is applied, are of area $A = 1\,\text{cm}^2$ and are separated by a distance $d = 2$ mm. If $K_1 = 4$, $K_2 = 6$ and $K_3 = 2$, find capacitance across P and Q. (Take $\varepsilon_0 = 8.8 \times 10^{-12}$ $\text{C}^2\text{N}^{-1}\text{m}^{-2}$)



9. Figure shows two identical parallel plate capacitors connected to a battery with the switch S closed. The switch is now opened and the free space between the plates of the capacitors is filled with a dielectric of dielectric constant (or relative permittivity) 3. Find the ratio of the total electrostatic energy stored in both capacitors before and after the introduction of the dielectric.



10. A capacitor is constructed from two square plates of sides ℓ and separation d. A material of dielectric constant κ is inserted a distance x into the capacitor, as shown in Figure. Assume that d is much smaller than x.



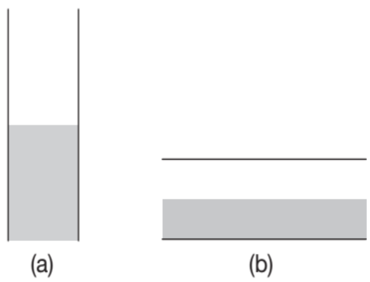
(a) Find the equivalent capacitance of the device.

(b) Calculate the energy stored in the capacitor, letting ΔV represent the potential difference.

(c) Find the direction and magnitude of the force exerted on the dielectric, assuming a constant potential difference ΔV. Ignore friction.

(d) Obtain a numerical value for the force assuming that $\ell = 5$ cm, $\Delta V = 2000$ V, $d = 2$ mm and the dielectric is glass $(\kappa = 4.5)$.

11. A vertical parallel-plate capacitor is half filled with a dielectric for which the dielectric constant is 2. When this capacitor is positioned horizontally, what fraction of it should be filled with the same dielectric in order for the two capacitors to have equal capacitance?



KIRCHHOFF'S LAWS FOR CAPACITOR CIRCUITS

Kirchhoff's Junction Law (KJL)

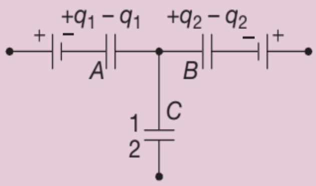
According to KJL, charge can never accumulate at a junction i.e., at the junction

$$\sum q_{\text{junction}} = 0$$

KJL is based on Law of Conservation of Charge.

Problem Solving Technique(s)

This law is helpful in determining the nature of charge on an unknown capacitor plate.



Charge on capacitor C can be determined by using this rule. As no charge must accumulate at the junction O, so if x is charge on plate 1 of C, then

$$-q_1 + q_2 + x = 0$$

$$\Rightarrow \quad x = q_1 - q_2$$

i.e., plate 1 has a charge $(q_1 - q_2)$ and plate 2 has a charge $-(q_1 - q_2)$.

Kirchhoff's Loop Law (KLL)

In a closed loop/circuit (a closed loop is the one which starts and ends at the same point), the algebraic sum of potential differences across each element of a closed loop/circuit is zero.

$$\Rightarrow \quad \sum V = 0$$

Conventions Followed to Apply Loop Law

(a) In a loop, across a battery, if we travel from negative terminal of battery to the positive terminal then there is a potential rise and a +ve sign is applied with voltage of the battery.

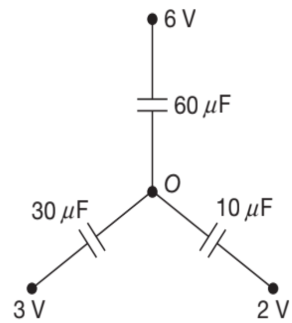
(b) In a loop, across a battery, if we travel from positive terminal of the battery to the negative terminal then there is a potential fall and a –ve sign is applied with voltage of the battery.

(c) In a loop, across a capacitor, if we go from negative plate to the positive plate of the capacitor then there is a potential rise and a +ve sign is to be taken with potential difference across the capacitor i.e., $\Delta V = +\dfrac{q}{C}$.

(d) In a loop, across a capacitor, if we go from positive plate to the negative plate of the capacitor then there is a potential fall and a –ve sign is to be taken with the potential difference across the capacitor i.e., $\Delta V = -\dfrac{q}{C}$.

ILLUSTRATION 28

Three uncharged capacitors having capacitances $60\ \mu\text{F}$, $30\ \mu\text{F}$ and $10\ \mu\text{F}$ are connected to each other as shown. Calculate the potential at the point O.



SOLUTION

According to Kirchhoff's Junction Law (KJL), charge cannot accumulate at the junction (O), so

$$\Sigma q_0 = 0, \text{ where } q = C\Delta V$$

Let V_0 be the potential of junction O, then

$$60(6 - V_0) + 10(2 - V_0) + 30(3 - V_0) = 0$$

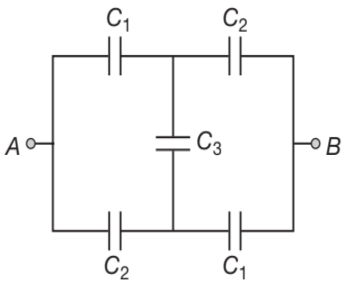
$$\Rightarrow \quad 360 + 20 + 90 = (60 + 10 + 30)V_0$$

$$\Rightarrow \quad V_0 = \frac{470}{100}$$

$$\Rightarrow \quad V_0 = 4.7\ \text{V}$$

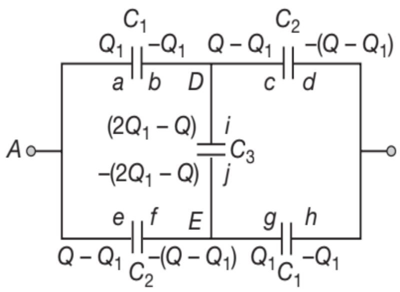
ILLUSTRATION 29

Find the equivalent capacitance between the point A and B in figure.



SOLUTION

Let us connect a battery between the points A and B. The charge distribution is shown in figure. Suppose the positive terminal of the battery supplies a charge $+Q$ and the negative terminal a charge $-Q$. The charge Q is divided between plates a and e.



Let a charge Q_1 goes to the plate a and the rest $Q-Q_1$ goes to the plate e. The charge $-Q$ supplied by the negative terminal is divided between plates d and h. Using the symmetry of the figure, charge $-Q_1$ goes to the plate h (as it has a capacitance C_1) and $-(Q-Q_1)$ to the plate d (as it has a capacitance C_2). This is because if we look into the circuit from A or from B, the circuit looks identical. The division of charge at A and at B should, therefore, be similar. The charges on the other plates may be written easily. The charge on the plate i is $2Q_1-Q$ which ensures that the total charge on plates b, c and i remains zero as these three plates form an isolated system.

We have $V_A-V_B=(V_A-V_D)+(V_D-V_B)$

$$\Rightarrow \quad V_A-V_B=\frac{Q_1}{C_1}+\frac{Q-Q_1}{C_2} \qquad \ldots(1)$$

Also, $V_A-V_B=(V_A-V_D)+(V_D-V_E)+(V_E-V_B)$

$$\Rightarrow \quad V_A-V_B=\frac{Q_1}{C_1}+\frac{2Q_1-Q}{C_3}+\frac{Q_1}{C_1} \qquad \ldots(2)$$

We have to eliminate Q_1 from these equations to get the equivalent capacitance $\frac{Q}{(V_A-V_B)}$.

The first equation may be written as

$$V_A-V_B=Q_1\left(\frac{1}{C_1}-\frac{1}{C_2}\right)+\frac{Q}{C_2}$$

$$\Rightarrow \quad \frac{C_1C_2}{C_2-C_1}(V_A-V_B)=Q_1+\frac{C_1}{C_2-C_1}Q \qquad \ldots(3)$$

The second equation may be written as

$$V_A-V_B=2Q_1\left(\frac{1}{C_1}+\frac{1}{C_3}\right)-\frac{Q}{C_3}$$

$$\Rightarrow \quad \frac{C_1C_2}{2(C_1+C_3)}(V_A-V_B)=Q_1-\frac{C_1}{2(C_1+C_3)}Q \qquad \ldots(4)$$

Subtracting (4) from (3)

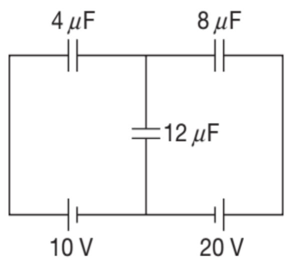
$$(V_A-V_B)\left[\frac{C_1C_2}{C_2-C_1}-\frac{C_1C_3}{2(C_1+C_3)}\right]=\left[\frac{C_1}{C_2-C_1}+\frac{C_1}{2(C_1+C_3)}\right]Q$$

$$\Rightarrow \quad (V_A-V_B)[2C_1C_2(C_1+C_3)-C_1C_3(C_2-C_1)]=C_1[2(C_1+C_3)+(C_2-C_1)]Q$$

$$\Rightarrow \quad C=\frac{Q}{V_A-V_B}=\frac{2C_1C_2+C_2C_3+C_3C_1}{C_1+C_2+2C_3}$$

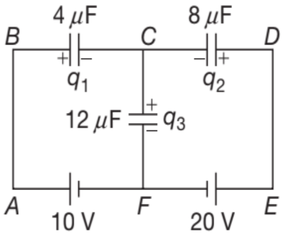
ILLUSTRATION 30

Find the charges on the three capacitors shown in figure.



SOLUTION

Let the charges in three capacitors be as shown in figure



Charge supplied by 10 V battery is q_1 and that from 20 V battery is q_2. Then,

$$q_1 + q_2 = q_3 \qquad \ldots(1)$$

This relation can also be obtained in a different manner. The charges on the three plates which are in contact add to zero. Because these plates taken together form an isolated system which can't receive charges from the batteries. Thus,

$$q_3 - q_1 - q_2 = 0$$

$$\Rightarrow \quad q_3 = q_1 + q_2$$

Applying Second Law in Loops *BCFAB* and *CDEFC*, we have

$$-\frac{q_1}{4} - \frac{q_3}{12} + 10 = 0$$

$$\Rightarrow \quad q_3 + 3q_1 = 120 \qquad \ldots(2)$$

and $\quad \dfrac{q_2}{8} - 20 + \dfrac{q_3}{12} = 0$

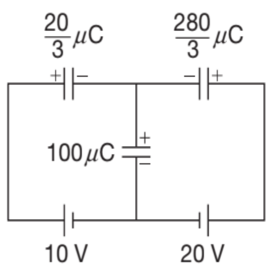
$$\Rightarrow \quad 3q_2 + 2q_3 = 480 \qquad \ldots(3)$$

Solving the above three equations, we have

$$q_1 = \frac{20}{3}\ \mu\text{C},$$

$$q_2 = \frac{280}{3}\ \mu\text{C} \text{ and } q_3 = 100\ \mu\text{C}$$

Thus, charges on different capacitors in μC, are shown in figure.



FLOW OF CHARGE

Consider a circuit having a switch, say S. Whenever the switch S in the circuit is opened or closed (or transferred from one mode to the other), then a charge flow takes place through certain points/branches of the circuit. To solve such like problems, we may follow the following steps.

STEP-1: Find the charges on different capacitors for the initial and final positions/modes of the switch.

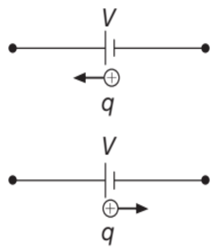
STEP-2: The charge flowing through a point/branch is calculated by the difference between the charges in the initial and the final positions/modes of the switch.

$$q_{\substack{\text{flowing through}\\ \text{a point/branch}}} = \left|\sum q_i - \sum q_f\right|$$

ENERGY SUPPLIED/CONSUMED BY THE BATTERY

We must take a note that whenever a charge $+q$ leaves the positive terminal of the battery of emf V, then work is done by the battery or the battery supplies an energy given by

$$E_{\text{supplied}} = W_{\text{by the battery}} = qV$$



However, if a charge $+q$ enters the positive terminal of the battery then work is done on the battery or the battery consumes an energy given by

$$E_{\text{consumed}} = W_{\text{on the battery}} = qV$$

While solving numerical problems we must keep in mind that the energy supplied by the battery will be taken as positive and the energy consumed by the battery will be taken as negative.

GENERATION OF HEAT

Again, let us consider a circuit having a switch S (as discussed earlier). Now, whenever the switch S in the circuit is opened or closed (or transferred from open mode to the close mode), then some amount of heat will be generated in the circuit. If W_{supplied} be the energy supplied by the battery/batteries, W_{consumed} be the energy consumed by the battery/batteries, ΣU_i be the initial energy stored in all the capacitors and ΣU_f be the final energy stored in all the capacitors, then heat generated is given by

$$\Delta H = \underbrace{\left(W_{\text{supplied}} - W_{\text{consumed}}\right)}_{\text{by the battery/batteries}} + \underbrace{\left(\sum U_i - \sum U_f\right)}_{\text{for the capacitors}}$$

because, we have

$$W_{\text{battery}} = \Delta U_C + \Delta H$$

ILLUSTRATION 31

A capacitor of capacitance C which is initially charged upto a potential difference V is connected with a battery of emf $\frac{V}{2}$ such that the positive terminal of battery is connected with positive plate of capacitor. After a long time, calculate the

(a) charge flow through the battery
(b) work done by battery
(c) heat dissipated in the circuit during the process of charging

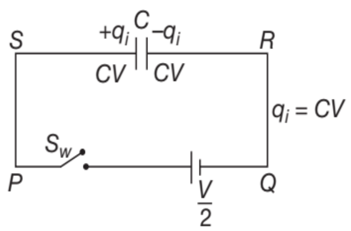
SOLUTION

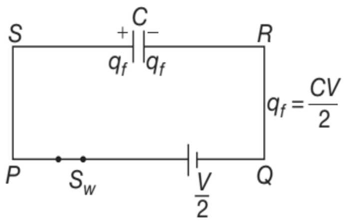
(a) Initial charge on the capacitor is $q_i = CV$

Final charge on the capacitor is $q_f = \frac{CV}{2}$

Charge flown $= \Delta q = q_f - q_i = -\frac{CV}{2}$

i.e. charge must have flown into the battery.





(b) Since a charge $\frac{CV}{2}$ flows into the battery i.e., a charge $\frac{CV}{2}$ enters the positive terminal of the battery so energy is consumed by the battery and is given by $W_{\text{battery}} = (\Delta q)V_{\text{battery}} = -\frac{CV^2}{4}$.

(c) Finally, change in energy of capacitor is

$$\Delta U_c = U_{\text{final}} - U_{\text{initial}}$$

$$\Rightarrow \quad \Delta U_c = \frac{1}{2}C\left(\frac{V}{2}\right)^2 - \frac{1}{2}CV^2$$

$$\Rightarrow \quad \Delta U_c = \frac{1}{8}CV^2 - \frac{1}{2}CV^2 = -\frac{3}{8}CV^2$$

Since $W_{\text{battery}} = \Delta U_c + \Delta H$

$$\Rightarrow \quad \Delta H = W_{\text{battery}} - \Delta U_c$$

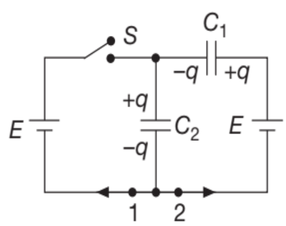
$$\Rightarrow \quad \Delta H = -\frac{CV^2}{4} - \left(-\frac{3}{8}CV^2\right)$$

$$\Rightarrow \quad \Delta H = \frac{3}{8}CV^2 - \frac{CV^2}{4}$$

$$\Rightarrow \quad \Delta H = \frac{CV^2}{8}$$

ILLUSTRATION 32

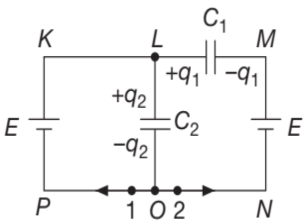
What charges will flow after shorting of the switch S in the circuit illustrated in figure through sections 1 and 2 in the directions indicated by the arrows?



SOLUTION

The charge distribution, when the switch is closed, is shown here.
In figure, the two capacitors are connected in series and hence their equivalent capacitance is given by

$$C_S = \frac{C_1 C_2}{(C_1 + C_2)}$$



If q is the charge on either of the capacitors, then

$$q=\left(\frac{C_1C_2}{C_1+C_2}\right)E$$

For the closed loop $LKPOL$, applying Kirchhoff's Loop Law, we get

$$E-\frac{q_2}{C_2}=0 \qquad \{\because\ -\Delta V=0\}$$

$$\Rightarrow \quad q_2=C_2E \qquad \ldots(1)$$

For closed loop $MLONM$, applying Kirchhoff's Loop Law, we get

$$-\frac{q_1}{C_1}+\frac{q_2}{C_2}-E=0 \qquad \ldots(2)$$

Substituting the value of q_2 from equation (1) in equation (2), we get

$$-\frac{q_1}{C_1}+\frac{C_2E}{C_2}-E=0$$

$$\Rightarrow \quad q_1=0$$

So, charge flowing through 1 is given by

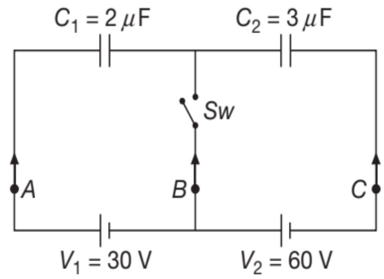
$$\Delta q_1=C_2E$$

and the charge that flows through section 2 is given by

$$\Delta q_2=-q_1-q=-\left(\frac{C_1C_2}{C_1+C_2}\right)E$$

ILLUSTRATION 33

In the circuit shown, calculate the charge that flows through A, B and C (in the directions shown), when the switch Sw is closed. Also calculate the loss in energy.

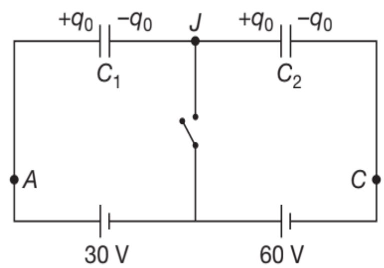


SOLUTION

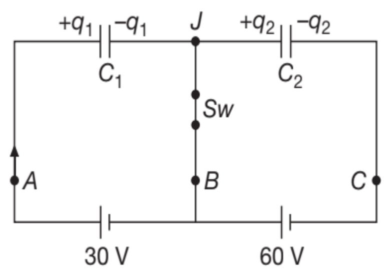
When the switch Sw is open, then both capacitors $2\ \mu$F and $3\ \mu$F are in series. So, their equivalent capacitance is $C_{eq}=\frac{(2)(3)}{2+3}=\frac{6}{5}\ \mu$F

Since both in series, so charge on both will be the same, say q_0. So,

$$q_0=\left(C_{eq}\right)V_{total}=\left(\frac{6}{5}\right)(30+60)=108\ \mu\text{C}$$



When the switch Sw is closed then let q_1 and q_2 be the charges on capacitors C_1 and C_2 (as shown).



Now, $q_1=C_1V_1=60\ \mu$C and

$$q_2=C_2V_2=180\ \mu\text{C}$$

Now initial charge on left plate of C_1 is $q_0=108\ \mu$C and final charge on left plate of C_1 is $q_1=60\ \mu$C, which makes us conclude that a charge of $-48\ \mu$C is being sent (via A) to left plate of capacitor C_1. Hence charge flowing from A in the direction shown is

$$\Delta q_1 = (q_1)_{final} - (q_1)_{initial}$$

$$\Rightarrow \quad \Delta q_1 = 60 - 108 = -48\ \mu C$$

Similarly, final charge on right plate of C_2 is $-q_2 = -180\ \mu C$ whereas initial charge on right plate of C_2 was $-q_0 = -108\ \mu C$. This simply means that a charge $-72\ \mu C$ flows (via C) to right plate of capacitor C_2. Hence charge flowing through A in the direction shown is

$$\Delta q_2 = (q_2)_{final} - (q_2)_{initial}$$

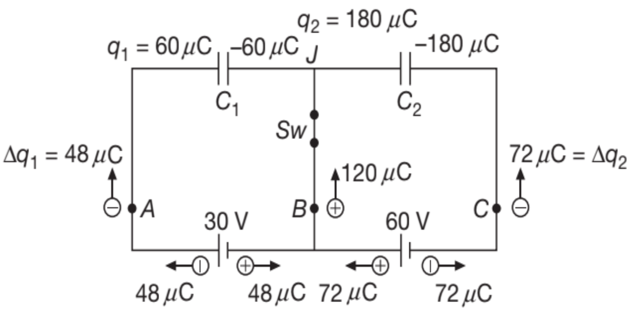
$$\Rightarrow \quad \Delta q_2 = -180 - (-108) = -72\ \mu C$$

Initial charge at the junction J is $+q_0 - q_0 = \text{zero}$

However, final charge at the junction J is $(q_2 - q_1)$

$$\Rightarrow \quad q_2 - q_1 = 180 - 60 = 120\ \mu C$$

So, finally charges flowing in the circuit are shown in figure.



Now, we observe that energy is supplied by $V_1 = 60$ V battery and consumed by $V_2 = 30$ V battery. So,

$$\text{Energy supplied} = |\Delta q_2| V_2 = (72 \times 10^{-6})(60)$$

$$\Rightarrow \quad \text{Energy supplied} = 4.32 \text{ mJ}$$

$$\text{Energy consumed} = |\Delta q_1| V_1 = (48 \times 10^{-6})(30)$$

$$\Rightarrow \quad \text{Energy consumed} = 1.44 \text{ mJ}$$

Initial energy stored by capacitors is

$$\Sigma U_{initial} = \frac{1}{2} C_{eq} V_{total}^2 = \frac{1}{2}\left(\frac{6}{5} \times 10^{-6}\right)(90)^2$$

$$\Rightarrow \quad \Sigma U_{initial} = 4.86 \text{ mJ}$$

Final energy stored by the capacitors is

$$\Sigma U_{final} = \frac{1}{2}(2 \times 10^{-6})(30)^2 + \frac{1}{2}(3 \times 10^{-6})(60)^2$$

$$\Rightarrow \quad \Sigma U_{final} = 6.3 \text{ mJ}$$

So, loss in energy is given by

$$\text{Loss} = (E_{supplied} - E_{consumed}) + (\Sigma U_i - \Sigma U_f)$$

$$\Rightarrow \quad \text{Loss} = (4.32 - 1.44) + (4.86 - 6.3)$$

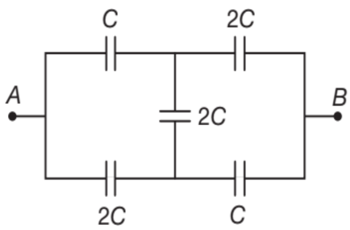
$$\Rightarrow \quad \text{Loss} = 1.44 \text{ mJ}$$

Test Your Concepts-IV

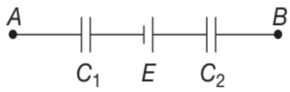
Based on Capacitor Circuits, Kirchhoff's Laws, Charge Flown and Generation of Heat

(Solutions on page H.135)

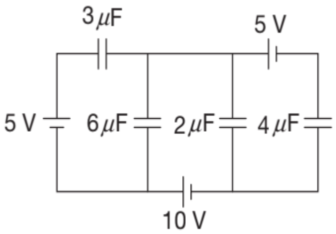
1. Find the equivalent capacitance between A and B.



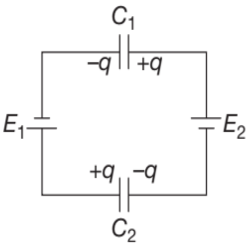
2. A circuit has section AB shown in figure. The emf of the source equals $E = 10$ V, the capacitor capacitances are equal to $C_1 = 1\ \mu F$ and $C_2 = 2\ \mu F$, and the potential difference $V_A - V_B = 5$ V. Find the voltage across each capacitor.



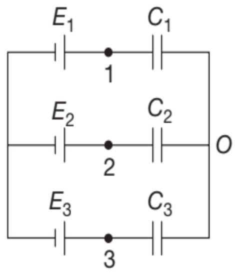
3. In the circuit shown, find the charges on 6 μF and 4 μF capacitors.



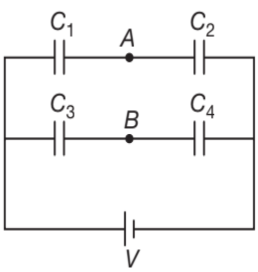
4. Find the charges on capacitor C_1 and C_2 in the circuit shown in figure.



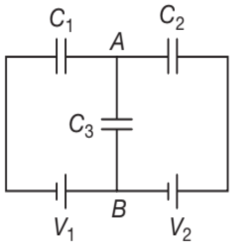
5. Determine the potential at point 1 of the circuit shown in figure, assuming the potential at the point O to be equal to zero. Using the symmetry of the formula obtained, write the expressions for the potential at points 2 and 3.



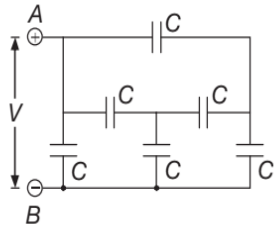
6. Calculate the potential difference between points A and B in the circuit shown in Figure. Under what condition is it equal to zero?



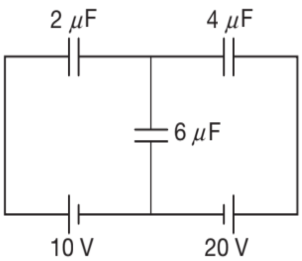
7. Calculate the potential difference between points A and B in the circuit shown in figure.



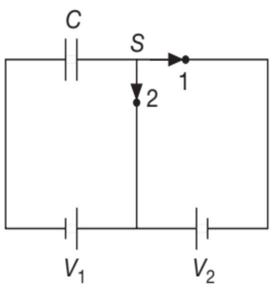
8. Determine the capacitance C_{AB} of the battery of identical capacitors shown in Figure.



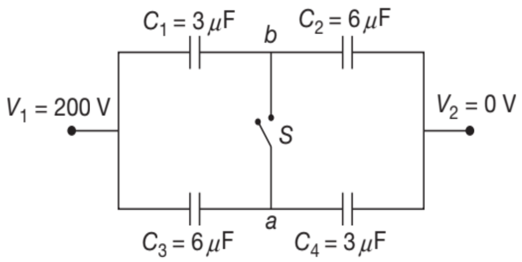
9. Find the charges on the three capacitors shown in figure.



10. What amount of heat will be generated in the circuit shown in Figure after the switch S is shifted from position 1 to position 2?



11. The arrangement shows a capacitor circuit with a switch S.

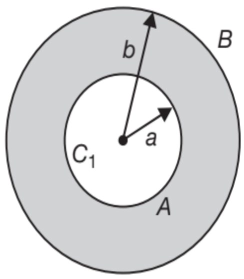


(i) Calculate the potential difference between a and b when the switch S is open and when S is closed.

(ii) Also calculate the charge that flows through the switch when it is closed.

SPHERICAL CAPACITOR

It consists of two concentric spherical conductors of radii a and b $(a < b)$ respectively. The space between the two conductors is filled with dielectric material of dielectric constant K.



The inner conductor is given a charge q. Depending upon which conductor (inner or outer) is earthed, we get the net capacitance as discussed in the following cases.
Let C_1 be the capacitance in between the two conductors and C_2 be capacitance outside both.

To Find C_1

Imagine the outer surface of B to be earthed. Then $-q$ is the charge induced on the inner surface of B.

If V is the potential difference between the two surfaces, then

$$V = \frac{q}{4\pi\varepsilon_0 Ka} + \frac{-q}{4\pi\varepsilon_0 Kb}$$

$$\Rightarrow \quad V = \frac{q}{4\pi\varepsilon_0 K}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow \quad C_1 = \frac{q}{V} = 4\pi\varepsilon_0 K\left(\frac{ab}{b-a}\right) \qquad \ldots(1)$$

To Find C_2

Imagine A to be made open circuited (i.e. made non conducting), then

$$C_2 = 4\pi\varepsilon_0 Kb \qquad \ldots(2)$$

CASE-1: When battery is connected to B and A is earthed. Then C_1 and C_2 are in parallel

$$\Rightarrow \quad C = C_1 + C_2$$

$$\Rightarrow \quad C = 4\pi\varepsilon_0 K\left(\frac{ab}{b-a}\right) + 4\pi\varepsilon_0 Kb$$

$$\Rightarrow \quad C = 4\pi\varepsilon_0 K\left(\frac{b^2}{b-a}\right)$$

CASE-2: When battery is connected to A, then C_1 and C_2 are in series.

$$\Rightarrow \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\Rightarrow \quad \frac{1}{C} = \frac{b-a}{ab}\frac{1}{4\pi\varepsilon_0 K} + \frac{1}{4\pi\varepsilon_0 Kb}$$

$$\Rightarrow \quad \frac{1}{C} = \frac{1}{4\pi\varepsilon_0 Kb}\left(\frac{b-a}{a} + 1\right)$$

$$\Rightarrow \quad \frac{1}{C} = \frac{1}{4\pi\varepsilon_0 Kb}\left(\frac{b}{a}\right)$$

$$\Rightarrow \quad C = 4\pi\varepsilon_0 Ka$$

CASE-3: When battery connected to A and B is earthed. Then C_2 can be omitted as it will not receive any charge.

So, $\quad C = C_1$

$$\Rightarrow \quad C = 4\pi\varepsilon_0 K\left(\frac{ab}{b-a}\right)$$

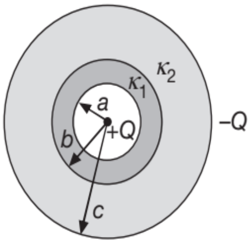
CASE-4: When battery connected to B and A is open circuited (or made non conducted) then C_1 can be omitted (as it is open circuited). So,

$$C = C_2$$

$$\Rightarrow \quad C = 4\pi\varepsilon_0 Kb$$

ILLUSTRATION 34

Consider a conducting spherical shell with an inner radius a and outer radius c. Let the space between two surfaces be filled with two different dielectric materials so that the dielectric constant is κ_1 between a and b and κ_2 between b and c, as shown in figure. Determine the capacitance of this system.



SOLUTION

The system can be treated as two capacitors connected in series, since the total potential difference across the capacitors is the sum of potential differences across individual capacitors. The equivalent capacitance for a spherical capacitor of inner radius r_1 and outer radius r_2 filled with dielectric with dielectric constant κ is given by

$$C = 4\pi\varepsilon_0\kappa\left(\frac{r_1r_2}{r_2-r_1}\right)$$

Thus, the equivalent capacitance of this system is

$$\frac{1}{C} = \frac{1}{\dfrac{4\pi\varepsilon_0\kappa_1 ab}{(b-a)}} + \frac{1}{\dfrac{4\pi\varepsilon_0\kappa_2 bc}{(c-b)}} = \frac{\kappa_2 c(b-a)+\kappa_1 a(c-b)}{4\pi\varepsilon_0\kappa_1\kappa_2 abc}$$

$$\Rightarrow \quad C = \frac{4\pi\varepsilon_0\kappa_1\kappa_2 abc}{\kappa_2 c(b-a)+\kappa_1 a(c-b)}$$

Check Point

Let us check the limiting value of C when $\kappa_1, \kappa_2 \to 1$. In this case, the above expression reduces to

$$C = \frac{4\pi\varepsilon_0 abc}{c(b-a)+a(c-b)} = \frac{4\pi\varepsilon_0 abc}{b(c-a)} = \frac{4\pi\varepsilon_0 ac}{(c-a)}$$

which agrees with equation for a spherical capacitor of inner radius a and outer radius c.

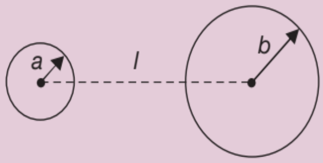
Remark(s)

(a) Capacitance of a two sphere capacitor of radii a and b when spheres are placed at large distance is

$$C = 4\pi\varepsilon_0\left(\frac{ab}{a+b}\right), \text{ for } \ell \gg a,\ \ell \gg b$$

(b) For a spherical capacitor of inner radius a and outer radius b, we have

$$C = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$



If both a and b are made extremely large but $b-a=d$ is kept fixed, then

$$ab \simeq a^2 \left(\text{or } b^2\right)$$

$$\Rightarrow \quad C = \varepsilon_0\left(\frac{4\pi a^2}{d}\right) = \frac{\varepsilon_0 A}{d},$$

where A is area of sphere $\left(\simeq 4\pi a^2 \simeq 4\pi b^2\right)$

The above result is the capacitance of a parallel plate capacitor of plate area A and plate separation d.

ILLUSTRATION 35

In case of two conducting spherical shells having radii a and $b(>a)$ calculate the capacity of the system if

(a) shells are concentric and inner is given a charge while outer earthed

(b) shells are concentric and the outer is given a charge while the inner is earthed

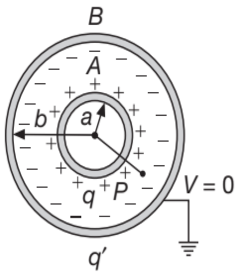
(c) shells carry equal and opposite charges and are separated by a distance d. Can you try to solve this part, using the concept of self energy and interaction energy?

SOLUTION

(a) Since the outer sphere is earthed hence, its potential will be zero, i.e., $V_B = 0$. If q' is the charge induced on shell B, then

$$\frac{(q+q')}{4\pi\varepsilon_0 b} = 0$$

$$\Rightarrow \quad q' = -q$$



So electric field at a point P between the shells is

$$E = E_A + E_B = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \{\text{as } E_B = E_{in} = 0\}$$

$$\Rightarrow \quad \frac{-dV}{dr} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \left\{\text{as } E = -\frac{dV}{dr}\right\}$$

$$\Rightarrow \quad -\int_V^0 dV = \frac{q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2}$$

$$\Rightarrow \quad V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Since, $C = \frac{q}{V}$

$$\Rightarrow \quad C = \frac{4\pi\varepsilon_0 ab}{b-a} \qquad \ldots(1)$$

From this it is clear that

(i) As $b \to \infty$ equation (1) reduces to $C \to 4\pi\varepsilon_0 a$. So, we can say that, a spherical conductor is a spherical capacitor with its other plate of infinite radius.

(ii) As a and b both become very large such that $b-a$ has a finite and small value, say d, then $4\pi ab \simeq 4\pi a^2 \simeq 4\pi b^2 = A$ and so equation (1) reduces to $C = \left(\frac{\varepsilon_0 A}{d}\right)$ and hence we observe that a spherical capacitor behaves as a parallel plate capacitor when its spherical surfaces have large radii and are close to each other:

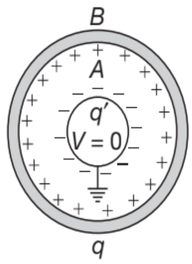
(b) In this situation $V_A = 0$ so if q' is the charge induced on shell A, then

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{a} + \frac{q}{b}\right) = 0$$

$$\Rightarrow \quad q' = -\frac{a}{b}q$$

Since no field will exist at P due to B as P lies inside the conductor, so the field at a point P, between the shells is

$$E = E_A + E_B = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2}$$



$$\Rightarrow \quad -\frac{dV}{dr} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{a}{b}q\right)\frac{1}{r^2}$$

$$\left\{\because \; E = -\frac{dV}{dr} \text{ and } q' = -\frac{a}{b}q\right\}$$

$$\Rightarrow \quad -\int_0^V dV = -\frac{q}{4\pi\varepsilon_0}\frac{a}{b}\int_a^b \frac{dr}{r^2}$$

$$\Rightarrow \quad V = \frac{q}{4\pi\varepsilon_0}\frac{(b-a)}{b^2}$$

So, $C_1 = \frac{q}{V} = \frac{4\pi\varepsilon_0 b^2}{(b-a)}$

$$\Rightarrow \quad C_1 = \frac{b}{a}C(>C) \qquad \{\text{as } b > a\} \qquad \ldots(2)$$

Also, we note that in this situation $q_B \neq -q_A$ and hence the system is not a capacitor. However, we observe that

$$C_1 = \frac{4\pi\varepsilon_0 b^2}{(b-a)} = \frac{4\pi\varepsilon_0 ab}{(b-a)} + 4\pi\varepsilon_0 b = C_{AB} + C_B$$

This system is equivalent to a spherical capacitor (of inner radius a and outer radius b and a spherical conductor of radius b connected in parallel. This is because the charge q given to the outer shell distributes in such a way that $\left(\frac{a}{b}\right)q$ remains on its inner side while the remaining $\left[q - \left(\frac{a}{b}\right)q\right]$ lies on its outer side. Hence,

$$\left[\frac{q_{\text{in}}}{q_{\text{out}}}\right]_B = \frac{\left(\frac{a}{b}\right)q}{q - \left(\frac{a}{b}\right)q} = \frac{a}{(b-a)}$$

and the system becomes equivalent to two capacitors in parallel having a common potential V.

(c) To find the capacitance in this situation, let us consider a point P at a distance r from centre of A on the line joining the centres of two spheres. If E be the net field at this point P, then

$$E = \frac{q}{4\pi\varepsilon_0 r^2} + \frac{q}{4\pi\varepsilon_0 (d-r)^2}$$

$$\Rightarrow \quad -\frac{dV}{dr} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r^2} + \frac{1}{(d-r)^2}\right] \quad \left\{\text{as } E = -\frac{dV}{dr}\right\}$$

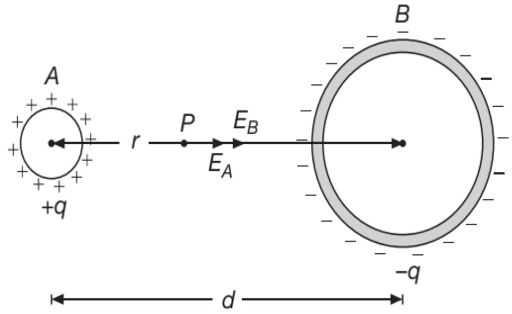
$$\Rightarrow \quad -\int_{V_A}^{V_B} dV = \frac{q}{4\pi\varepsilon_0}\int_a^{(d-b)}\left[\frac{1}{r^2} + \frac{1}{(d-r)^2}\right]dr$$

$$\Rightarrow \quad V_A - V_B = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r} + \frac{1}{(d-r)}\right]\Bigg|_a^{(d-b)}$$

$$\Rightarrow \quad V_A - V_B = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a} + \frac{1}{b} - \frac{1}{(d-a)} - \frac{1}{(d-b)}\right]$$

For $d \gg a$ and $d \gg b$, we get

$$\Rightarrow \quad V_A - V_B = V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a} + \frac{1}{b} - \frac{2}{d}\right]$$



Since $C = \frac{q}{|\Delta V|}$

$$\Rightarrow \quad C_2 = \frac{q}{V} = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{d}\right)} \qquad \ldots(3)$$

Using Concept of Self Energy and Interaction Energy

The self energy of a spherical conductor is

$$U_S = \frac{1}{2}qV = \frac{1}{2}\left[\frac{q^2}{4\pi\varepsilon_0 R}\right] \qquad \left\{\text{as } V = \frac{q}{4\pi\varepsilon_0 R}\right\}$$

while interaction energy of two point charges separated by a distance r

$$U_I = qV = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

So total potential energy of the system is

$$U = \begin{pmatrix}\text{Self}\\\text{Energy}\\\text{of } A\end{pmatrix} + \begin{pmatrix}\text{Self}\\\text{Energy}\\\text{of } B\end{pmatrix} + \begin{pmatrix}\text{Interaction}\\\text{Energy}\\\text{of } AB\end{pmatrix}$$

$$\Rightarrow \quad U = U_A + U_B + U_{AB}$$

$$\Rightarrow \quad U = \frac{q^2}{8\pi\varepsilon_0 a} + \frac{(-q)^2}{8\pi\varepsilon_0 b} + \frac{q(-q)}{4\pi\varepsilon_0 d}$$

$$\Rightarrow \quad U = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{d}\right)$$

Since, we have $U = \left(\frac{q^2}{2C}\right)$

$$\Rightarrow \quad C_2 = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{d}\right)}$$

This result is identical to the result obtained earlier, using the method of electric field.

Now, if $d \to \infty$ then

$$C_2' = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$\Rightarrow \quad C_2' < C_2$$

Also, $\frac{1}{C_2'} = \frac{1}{4\pi\varepsilon_0 a} + \frac{1}{4\pi\varepsilon_0 b} = \frac{1}{C_A} + \frac{1}{C_B}$

and hence the given system becomes equivalent to two capacitors $C_A\left(=4\pi\varepsilon_0 a\right)$ and $C_B\left(=4\pi\varepsilon_0 b\right)$ in series.

Also, we observe that in this ILLUSTRATION, (a) and (c) represent capacitors (as here the two conductors have equal and opposite charges) while arrangement given in (b) is not a capacitor (because q_B is not equal to $-q_A$).

ILLUSTRATION 36

An isolated conductor initially free from charge is charged by repeated contacts with a plate which after each contact is replenished to a charge Q. If q is the charge on the conductor after first operation prove that the maximum charge which can be given to the conductor in this way is $\frac{Qq}{Q-q}$.

SOLUTION

Let C_1, be the capacitance of the plate and C_2 that of the conductor. After first contact charge on conductor is q. Therefore, charge on plate will remain $Q-q$. As the charge redistributes in the ratio of capacitances, so

$$\frac{Q-q}{q} = \frac{C_1}{C_2} \qquad \ldots(1)$$

Let q_{max} be the maximum charge which can be given to the conductor. Then flow of charge from the plate to the conductor will stop when, both get equal potential. So,

$$V_{conductor} = V_{plate}$$

$$\Rightarrow \quad \frac{q_{max}}{C_2} = \frac{Q}{C_1}$$

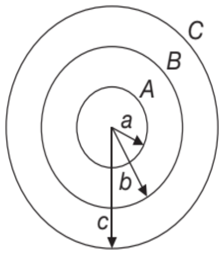
$$\Rightarrow \quad q_{max} = \left(\frac{C_2}{C_1}\right)Q$$

Substituting $\frac{C_2}{C_1}$ from equation (1), we get

$$q_{max} = \frac{Qq}{Q-q}$$

ILLUSTRATION 37

Three concentric conducting shells A, B and C of radii a, b and c are as shown in figure. Find the capacitance of the assembly between A and C. Now if, the space between A and B is filled with a dielectric material of dielectric constant K, then find the new capacitance between A and C.



SOLUTION

Consider a positive charge q on A and a negative charge $-q$ on C. Find the potential difference V between A and C. The desired capacitance will be

$$C = \frac{q}{V}$$

While calculating V, notice that net charge on B is zero. Moreover, we can apply the generator principle while calculating V.

$$V = V_A - V_C = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{c}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{c-a}{ac}\right)$$

So, the desired capacitance is,

$$C = \frac{q}{V} = 4\pi\varepsilon_0\left(\frac{ac}{c-a}\right)$$

Now when the dielectric is filled between A and B, the electric field will change in this region. Therefore the potential difference and hence the capacitance of the system will change. So, first find the electric field $E(r)$ in the region $a \le r \le c$. Then find the potential difference (V) between A and C and finally the capacitance of the system will be,

$$C=\frac{q}{V}$$

where, $q=$ charge on A

$$E(r)=\begin{cases}\dfrac{q}{4\pi\varepsilon_0 Kr^2} & \text{for} \quad a\le r\le b\\[2ex] \dfrac{q}{4\pi\varepsilon_0 r^2} & \text{for} \quad b\le r\le c\end{cases}$$

Since $\Delta V=\int dV=-\int \vec{E}\cdot d\vec{r}$

So, the potential difference between A and C is

$$V=V_A-V_C=-\int_a^b \frac{q}{4\pi\varepsilon_0 Kr^2}dr-\int_b^c \frac{q}{4\pi\varepsilon_0 r^2}dr$$

$$\Rightarrow \quad V=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{K}\left(\frac{1}{a}-\frac{1}{b}\right)+\left(\frac{1}{b}-\frac{1}{c}\right)\right]=$$

$$\frac{q}{4\pi\varepsilon_0}\left[\frac{(b-a)}{Kab}+\frac{(c-b)}{bc}\right]$$

$$\Rightarrow \quad V=\frac{q}{4\pi\varepsilon_0 Kabc}\left[c(b-a)+Ka(c-b)\right]$$

The desired capacitance is,

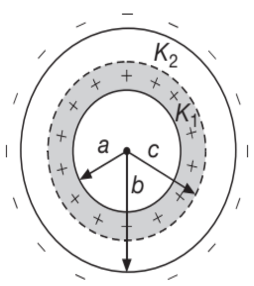
$$C=\frac{q}{V}=\frac{4\pi\varepsilon_0 Kabc}{Ka(c-b)+c(b-a)}$$

ILLUSTRATION 38

A capacitor is formed of two concentric spherical conducting shells of radii a and b with $b>a$. If the medium between the spherical shells has a dielectric constant K_1, from a to c and K_2 from c to b, find the capacitance of the spherical capacitor.

SOLUTION

Let the charge on inner sphere be $+Q$ and that on the outer sphere be $-Q$.



$$V=-\int E dr=\frac{-Q}{4\pi\varepsilon_0}\left[\int_r^a \frac{dr}{K_1 r^2}+\int_b^r \frac{dr}{K_2 r^2}\right]$$

$$\Rightarrow \quad V=\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{K_1}\left(\frac{1}{a}-\frac{1}{r}\right)+\frac{1}{K_2}\left(\frac{1}{r}-\frac{1}{b}\right)\right]$$

$$\Rightarrow \quad V=\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(\frac{1}{K_2}-\frac{1}{K_1}\right)+\left(\frac{1}{K_1 a}-\frac{1}{K_2 b}\right)\right]$$

$$\Rightarrow \quad C=\frac{Q}{V}=4\pi\varepsilon_0\left[\frac{1}{r}\left(\frac{1}{K_2}-\frac{1}{K_1}\right)+\left(\frac{1}{K_1 a}-\frac{1}{K_2 b}\right)\right]^{-1}$$

CYLINDRICAL CAPACITOR

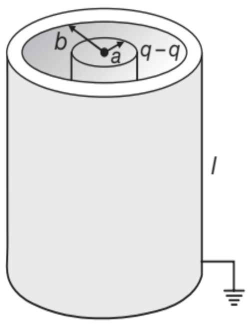
It consists of two coaxial metallic cylinders of inner radius a and outer radius b. The outer surface of the cylinder of radius b (outer one) is earthed. The space between the two cylinders is filled with material of dielectric constant K. Let inner cylinder be given a charge per unit length of $\lambda\left(=\frac{q}{l}\right)$. A charge $-q$ is induced on length l at inner surface of outer cylinder

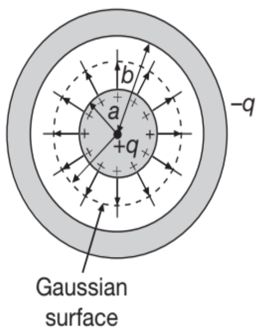
$$E=\frac{\lambda}{2\pi\varepsilon_0 r} \text{ for } a<r<b$$

$$\Rightarrow \quad -\frac{dV}{dr}=\frac{\lambda}{2\pi\varepsilon_0 Kr}$$

$$\Rightarrow \quad \int_{\text{inner surface}}^{\text{outer surface}} dV=-\frac{\lambda}{2\pi\varepsilon_0 K}\int_{r=a}^{r=b}\frac{dr}{r}$$

$$\Rightarrow \quad V_{\substack{\text{outer}\\ \text{surface}}}-V_{\substack{\text{inner}\\ \text{surface}}}=-\frac{\lambda}{2\pi\varepsilon_0 K}\log_e\left(\frac{b}{a}\right)$$





Since, inner surface is at higher potential and outer at lower potential, so

$$V_{\substack{\text{inner}\\ \text{surface}}} - V_{\substack{\text{outer}\\ \text{surface}}} = \frac{\lambda}{2\pi\varepsilon_0 K}\log_e\left(\frac{b}{a}\right)$$

$$\Rightarrow \quad V_{\substack{\text{inner}\\ \text{surface}}} - V_{\substack{\text{outer}\\ \text{surface}}} = \frac{q}{2\pi\varepsilon_0 lK}\log_e\left(\frac{b}{a}\right)$$

$$\Rightarrow \quad C = \frac{q}{V_{\substack{\text{inner}\\ \text{surface}}} - V_{\substack{\text{outer}\\ \text{surface}}}} = \frac{2\pi\varepsilon_0 lK}{\log_e\left(\frac{b}{a}\right)}$$

$$\Rightarrow \quad C = \frac{2\pi\varepsilon_0 lK}{\log_e\left(\frac{b}{a}\right)}$$

ENERGY STORED IN A SPHERICAL SHELL

The electric field associated with a spherical shell of radius a is given by

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r > a \\ \vec{0} & r < a \end{cases}$$

The corresponding energy density is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

outside the sphere, and zero inside it. Since the electric field is non-vanishing outside the spherical shell, we must integrate over the entire region of space from $r = a$ to $r \to \infty$. In spherical coordinates the infinitesimal volume element $d\tau = 4\pi r^2 dr$, we have

$$U = \int u_E\, d\tau = \int_a^\infty \left(\frac{Q^2}{32\pi^2\varepsilon_0 r^4}\right)4\pi r^2 dr = \frac{Q^2}{8\pi\varepsilon_0}\int_a^\infty \frac{dr}{r^2}$$

$$\Rightarrow \quad U = \frac{Q^2}{8\pi\varepsilon_0 a} = \frac{1}{2}QV$$

where $V = \dfrac{Q}{4\pi\varepsilon_0 a}$ is the electric potential on the surface of the shell, with $V(\infty) = 0$.

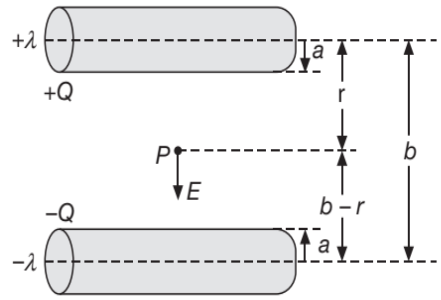
We can readily verify that the energy of the system is equal to the work done in charging the sphere. To show this, suppose at some instant the sphere has charge q and is at a potential $V = \dfrac{q}{4\pi\varepsilon_0 a}$. The work required to add an additional charge dq to the system is $dW = Vdq$. Thus, the total work is

$$W = \int dW = \int V dq = \int_0^Q dq\left(\frac{q}{4\pi\varepsilon_0 a}\right) = \frac{Q^2}{8\pi\varepsilon_0 a}$$

ILLUSTRATION 39

Two long straight wires with equal cross-sectional radii a are located parallel to each other in air. The distance between their axes equals b. Find the mutual capacitance of the wires per unit length under the condition $b \gg a$.

SOLUTION



Let the charges on the two wires of length l be $+Q$ and $-Q$. The charge per unit length is $+\lambda$ and $-\lambda$ where $\lambda = \dfrac{Q}{l}$. The electric field at point P due to both wires is from positively charged wire towards the negatively charged wire and is given by

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} + \frac{\lambda}{2\pi\varepsilon_0 (b-r)}$$

The potential difference between the two wires is

$$V = -\int E dr = -\frac{\lambda}{2\pi\varepsilon_0}\int_{b-a}^{a}\left(\frac{1}{r} + \frac{1}{b-r}\right)dr$$

$$\Rightarrow \quad V = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln r - \ln(b-r)\right)\Big|_a^{b-a} = \frac{Q}{\pi\varepsilon_0 l}\ln\left(\frac{b-a}{a}\right)$$

$$\Rightarrow \quad C = \frac{Q}{V} = \frac{\pi\varepsilon_0 l}{\ln\left(\frac{b-a}{a}\right)}$$

Therefore, capacitance per unit length is

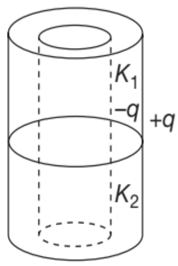
$$\frac{C}{l} = \frac{\pi\varepsilon_0}{\ln\left(\frac{b-a}{a}\right)}$$

Test Your Concepts-V

Based on Spherical and Cylindrical Capacitors

(Solutions on page H.139)

1. Two conducting spheres A and B each of radius $R = 9$ mm are separated by a very large distance $d = 9$ m. Calculate the capacitance of the system.
2. A spherical capacitor has the inner sphere of radius 2 cm and the outer one of 4 cm. If the inner sphere is earthed and the outer one is charged with a charge of 2 μC and isolated, calculate,
   (a) the potential to which the outer sphere is raised.
   (b) the charge retained on the outer surface of the outer sphere. Give an account for the remaining charge.
3. N drops of mercury of equal radii and possessing equal charges combine to form a big drop. Compare the charge, capacitance, potential of bigger drop and energy stored by bigger with the corresponding quantities of individual drop.
4. Find the capacitance of a spherical capacitor made of two concentric spherical conductors of inner radius a and outer radius b, when the outer conductor is charged and the inner conductor is earthed.
5. Two conducting spheres are placed concentrically. The inner sphere is earthed and the outer given a charge 30 nC. The radii of sphere are 8 cm and 10 cm. What is the charge on the inner sphere? Will there be any charge on the inner surface of the outer sphere? Explain.
6. A cylindrical capacitor of inner and outer radii a and b respectively is filled with two dielectrics of constants K_1 and K_2. Each dielectric occupies half the length of the cylinder. Find the capacity of the system between the inner and outer cylinders.



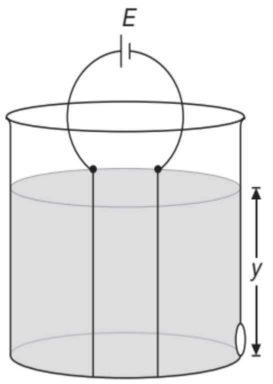
SOLVED PROBLEMS

PROBLEM 1

A parallel plate capacitor is placed in a cylindrical tank filled with a liquid of dielectric constant K. The area of cross-section of the tank is A and height of the liquid is equal to the length of the square plate of plate area l^2. The separation between the plates is d. A small orifice of area a is opened at the bottom of the tank at $t = 0$. If the capacitor in the process remains connected with a battery of emf E and assuming the level of liquid in the capacitor remains same as outside, find the current in the circuit as a function of time

SOLUTION

According to Equation of Continuity, Rate of Change of volume is av, where, v is the velocity of efflux given by



$$v = \sqrt{2gy}$$

$$\Rightarrow \quad A\left(-\frac{dy}{dt}\right) = av = a\sqrt{2gy}$$

$$\Rightarrow \quad -y^{-\frac{1}{2}}dy = \left(\frac{a\sqrt{2g}}{A}\right)dt$$

$$\Rightarrow \quad -\int_l^y y^{-\frac{1}{2}}dy = \left(\frac{a\sqrt{2g}}{A}\right)\int_0^t dt$$

$$\Rightarrow \quad 2\left(\sqrt{l} - \sqrt{y}\right) = \frac{a\sqrt{2g}}{A}t$$

$$\Rightarrow \quad \sqrt{y} = \sqrt{l} - \frac{a\sqrt{2g}}{2A}t$$

$$\Rightarrow \quad y = \left(\sqrt{l} - \frac{a\sqrt{2g}}{2A}t\right)^2 \quad \ldots(1)$$

Capacitance of the capacitor at time t is,

$$C = \frac{\varepsilon_0 l(l-y)}{d} + \frac{K\varepsilon_0 yl}{d}$$

{$\because$ the plate is a square plate of side l}

Since, $q = EC$ and by definition of current,

$$I = \frac{dq}{dt}$$

Since here, q is decreasing with the passage of time, so

$$I = -\frac{dq}{dt}$$

$$\Rightarrow \quad I = \left(-\frac{dq}{dt}\right) = -E\left(\frac{dC}{dt}\right) = -E\left(\frac{K\varepsilon_0 l}{d} - \frac{\varepsilon_0 l}{d}\right)\frac{dy}{dt} \quad \ldots(2)$$

From equation (1),

$$\frac{dy}{dt} = -\frac{a\sqrt{2g}}{A}\left(\sqrt{l} - \frac{a\sqrt{2g}}{2A}t\right)$$

Substituting in equation (2), we get

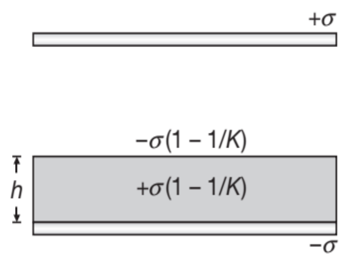
$$I = \frac{\varepsilon_0 El}{d}\left(\frac{a}{A}\sqrt{2gl} - \frac{a^2 gt}{A^2}\right)(K-1)$$

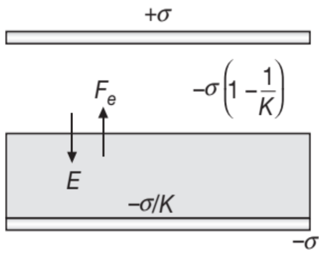
PROBLEM 2

A parallel plate capacitor is located horizontally so that one of its plate is just submerged into liquid while the other is over the surface. The permitivity of the liquid is equal to K and its density is ρ. To what height will the level of liquid in the capacitor rise after its plates get a charge of surface density σ.

SOLUTION

Let us consider the equilibrium of liquid inside the parallel plate capacitor. For that





Weight of the liquid (acting downwards) = electrostatic force on the liquid (acting upwards)

$$\Rightarrow \quad Ah\rho g=\sigma\left(1-\frac{1}{K}\right)AE \qquad \ldots(1)$$

where, $E=\frac{\sigma}{2\varepsilon_0}+\frac{\sigma}{2K\varepsilon_0}$ ...(2)

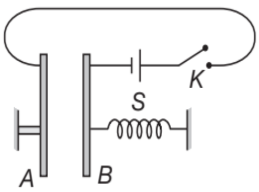
Substituting equation (2) in equation (1), we get

$$h=\frac{\sigma\left(1-\frac{1}{K}\right)\left(\frac{\sigma}{2\varepsilon_0}+\frac{\sigma}{2K\varepsilon_0}\right)}{\rho g}$$

$$\Rightarrow \quad h=\frac{\sigma^2\left(K^2-1\right)}{2K^2\varepsilon_0\rho g}$$

PROBLEM 3

One plate A of a parallel plate capacitor AB is fixed, while the other is attached to the wall by a spring and can move parallel to the plate A. The two plates are joined to a cell through a key K which is initially open. When the key is closed permanently, the plate separation d decreases by $\eta=10\%$. What will be the percentage fractional decrease between the plates if the key is closed for such a short interval that the separation between A and plate B does not change noticeably?



SOLUTION

When key is closed permanently, voltage between A and B remains constant. Now charge on capacitor changes because the capacitance changes as plate B moves towards plate A due to force of attraction between them.

The force of attraction between two capacitor plates is $F=\frac{1}{2}\varepsilon_0E^2A$

where A = Area of each plate and E = Electric field between plates.

Since plates move towards each other such that separation decreases by η. Hence, new separation d' becomes $d'=d-\eta d$.

If E' be the new electric field between plates, then

$$\Rightarrow \quad E'=\frac{V}{d-\eta d}$$

In equilibrium, force of attraction between plates is balanced by outward elastic force produced in the spring.

$\Rightarrow$ Force of attraction = Elastic force

$$\left(\frac{1}{2}\varepsilon_0E'^2\right)A=K(d-d')$$

$$\Rightarrow \quad \frac{1}{2}\varepsilon_0\frac{V^2}{d^2(1-\eta)^2}A=K\eta d \qquad \ldots(1)$$

where ηd is the displacement from original position and K is the force constant of spring.

When key is closed for very short time, the capacitor collects some charge during that short interval at constant voltage and then B will move towards A, with constant voltage. If Q be the charge collected, then

$$Q=CV=\left(\frac{\varepsilon_0A}{d}\right)V$$

Let new separation in equilibrium position be $(d-x_2)$

Force of attraction $=\frac{Q^2}{2\varepsilon_0A}=\frac{1}{2}\varepsilon_0\frac{V^2A}{d^2}$

where $E''=\frac{V}{d-x_2}\cong\frac{V}{d}$ $\qquad\{\because x_2\ll d\}$

In equilibrium $\frac{1}{2}\varepsilon_0\frac{V^2A}{d^2}=Kx_2$ ...(2)

Dividing (2) by (1)

$$\frac{Kx_2}{K\eta d}=\frac{\left(\frac{\frac{1}{2}\varepsilon_0V^2A}{d^2}\right)}{\left(\frac{\frac{1}{2}\varepsilon_0V^2A}{d^2(1-\eta)^2}\right)}$$

$$\Rightarrow \quad \frac{x_2}{\eta d} = (1-\eta)^2$$

$$\Rightarrow \quad \frac{x_2}{d} = \eta(1-\eta)^2$$

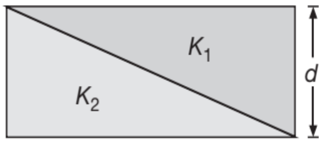
So, fractional change is

$$\frac{x_2}{d} = \eta(1-\eta)^2 = 0.1(1-0.1)^2 = 0.081$$

$\Rightarrow$ Percentage Fractional Change $= 8.1\%$

PROBLEM 4

The capacitance of a parallel plate capacitor with plate area A and separation d is C. The space between the plates is filled with two wedges of dielectric constants K_1 and K_2 respectively. Find capacitance of the resultant capacitor.

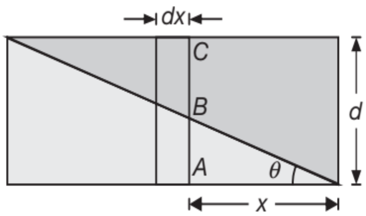


SOLUTION

Let the length of capacitor is l and breadth be b and d is distance between two plates and area of capacitor is A.

So $A = lb$

In this problem, first we will find the capacitance of small area considering a strip of width dx then integrate over. $x = 0$ to $x = l$ to get total capacitance of capacitor where l is length of each capacitor plate.



Hence consider a strip at distance x from right of width dx.

Now AB has K_1 and BC has K_2 dielectric constant

$$AB = x\tan\theta$$

$$BC = d - x\tan\theta$$

capacitance of strip

$$dC = \frac{\varepsilon_0 A_S}{\frac{AB}{K_1} + \frac{BC}{K_2}} \qquad \{\because A_S = \text{area of strip} = bdx\}$$

$$\Rightarrow \quad dC = \frac{\varepsilon_0 bdx}{\frac{x\tan\theta}{K_1} + \frac{d - x\tan\theta}{K_2}}$$

$$\Rightarrow \quad C = \int_0^C dC = \int_0^l \frac{\varepsilon_0 bdx}{\left(\frac{x\tan\theta}{K_1} + \frac{d - x\tan\theta}{K_2}\right)}$$

$$\Rightarrow \quad C = \varepsilon_0 bK_1K_2 \int_0^l \frac{dx}{K_1 d + (K_2 - K_1)x\tan\theta}$$

$$\Rightarrow C = \frac{\varepsilon_0 bK_1K_2}{(K_2 - K_1)\tan\theta}\left[\log_e\left(K_1 d + (K_2 - K_1)x\tan\theta\right)\right]_0^l$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 bK_1K_2}{(K_2 - K_1)\tan\theta}\log_e\left[\frac{K_1 d + (K_2 - K_1)l\tan\theta}{K_1 d}\right]$$

But $\tan\theta = \frac{d}{l}$ and $A = lb$

$$\Rightarrow \quad C = \frac{\varepsilon_0 bK_1K_2}{(K_2 - K_1)\frac{d}{l}}\log_e\left[\frac{K_1 d + (K_2 - K_1)d}{K_1 d}\right]$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 AK_1K_2}{(K_2 - K_1)d}\log_e\left(\frac{K_2}{K_1}\right)$$

PROBLEM 5

Consider a parallel plate capacitor whose plate separation is maintained by a dielectric of dielectric constant K and thickness d when the potential across the capacitor is zero. The dielectric strength of dielectric is E_0 and it is compressible having Young's modulus Y. The capacitance of the capacitor as the limit $V \to 0$ is C_0.

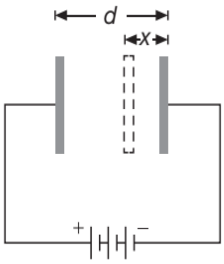
(a) Derive an expression for the capacitance as a function of voltage across the capacitor.

(b) Calculate the maximum voltage that can be applied to the capacitor?

Assume that dielectric constant of capacitor does not change under compression.

SOLUTION

(a) Let the voltage applied and compression in the dielectric be V and x respectively. Let us now consider an infinitesimal compression dx, during which the charge supplied by battery is dq.



Variable voltage source

From Conservation of Energy, we have

$$\begin{pmatrix}\text{Work}\\\text{done}\\\text{by}\\\text{battery}\end{pmatrix}=\begin{pmatrix}\text{Electrostatic}\\\text{energy}\\\text{in}\\\text{dielectric}\end{pmatrix}+\begin{pmatrix}\text{Elastic}\\\text{potential}\\\text{energy}\\\text{in dielectric}\end{pmatrix}$$

$$\Rightarrow \quad Vdq=\frac{1}{2}Vdq+\frac{YA}{d}xdx$$

$$\Rightarrow \quad \frac{1}{2}Vdq=\frac{YA}{d}xdx \qquad \ldots(1)$$

Since, $q=KCV$

$$\Rightarrow \quad dq=KV(dC) \qquad \ldots(2)$$

Substitute equation (2) in (1), we get

$$\frac{1}{2}KV^2dC=\frac{YA}{d}xdx \qquad \ldots(3)$$

Capacitance of the capacitor is given by

$$C=\frac{K\varepsilon_0A}{(d-x)} \qquad \ldots(4)$$

$$\Rightarrow \quad \frac{dC}{dx}=\frac{K\varepsilon_0A}{(d-x)^2} \qquad \ldots(5)$$

Now substituting equation (4) in (3), we get

$$\frac{1}{2}KV^2\frac{K\varepsilon_0A}{(d-x)^2}=\frac{YA}{d}x$$

$$\Rightarrow \quad V^2=\frac{2Y(d-x)^2}{K^2\varepsilon_0d} \qquad \ldots(6)$$

For a small compression x, we have, $x\ll d$,

$$\Rightarrow \quad d-x\approx d$$

$$\Rightarrow \quad V^2\approx\frac{2Yd^2}{K^2\varepsilon_0d}x$$

$$\Rightarrow \quad x=\left(\frac{K^2\varepsilon_0}{2Yd}\right)V^2 \qquad \ldots(7)$$

Substitute equation (7) in (4), we get

$$C=\frac{K\varepsilon_0A}{d-\dfrac{V^2K^2\varepsilon_0}{2Yd}}=\frac{K\varepsilon_0d}{d\left(1-\dfrac{V^2K^2\varepsilon_0}{2Yd^2}\right)}$$

For $x\ll d$, we have

$$C=\frac{K\varepsilon_0A}{d}\left(1-\frac{V^2K^2\varepsilon_0}{2Yd^2}\right)^{-1}\approx\frac{K\varepsilon_0A}{d}\left(1+\frac{V^2K^2\varepsilon_0}{2Yd^2}\right)$$

Since at $V=0$, we have $C\to C_0=\dfrac{K\varepsilon_0A}{d}$

$$\Rightarrow \quad C=C_0\left(1+\frac{V^2K^2\varepsilon_0}{2Yd^2}\right)=C_0\left(1+\frac{x}{d}\right)$$

(b) Let x_0 be the compression when the dielectric breakdown occurs. The potential difference at this position is given by

$$V_0=E_0(d-x_0)$$

Substitute this in equation (6), we get

$$E_0^2=\frac{2Y}{K^2\varepsilon_0d}x_0$$

$$\Rightarrow \quad x_0=\frac{E_0^2K^2\varepsilon_0d}{2Y}$$

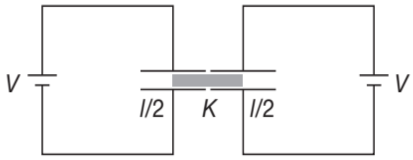
Substitute x_0 in expression for V_0, we get

$$V_0=E_0\left(d-\frac{E_0^2K^2\varepsilon_0d}{2Y}\right)$$

$$\Rightarrow \quad V_0=E_0d\left(1-\frac{E_0^2K^2\varepsilon_0}{2Y}\right)$$

PROBLEM 6

In figure, the capacitors have plate area $A(=l\times b)$ and separation 'd'. Now, if the slab of mass m is displaced slightly, calculate the time period of the oscillation. Is the motion simple harmonic?



SOLUTION

Let the slab be displaced to the left by a distance x, then the work done W is given by

W = change in stored energy + work done by battery

Let C_0 be the initial capacitance of capacitor, then

$$C_0 = \frac{\varepsilon_0 lb}{2d} + \frac{\varepsilon_0 kbl}{2d}$$

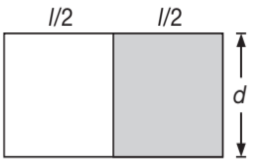
Let C_1 and C_2 be the final capacitances, then

$$C_1 = \frac{\varepsilon_0 b}{d}\left(\frac{l}{2} - x\right) + \frac{\varepsilon_0 bk}{d}\left(\frac{l}{2} + x\right) \text{ and}$$

$$C_2 = \frac{\varepsilon_0 b}{d}\left(\frac{l}{2} + x\right) + \frac{\varepsilon_0 bk}{d}\left(\frac{l}{2} - x\right)$$

Since $F = \dfrac{2\varepsilon_0 b(K-1)V^2}{2d}$ {As done earlier}

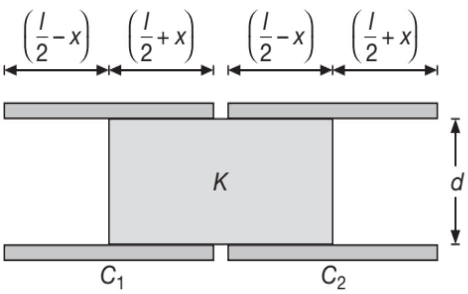
$$\Rightarrow \quad a = \frac{F}{m} = \frac{\varepsilon_0 b}{md}(k-1)V^2$$



$C_0 = C_{air} + C_{dielectric}$

Hence, motion is oscillatory but not SHM because acceleration is constant. For constant acceleration motion,

$$x = \frac{1}{2}at^2$$

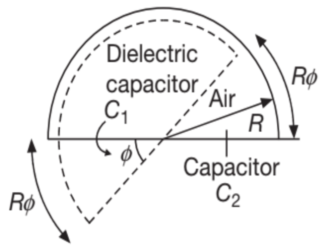


$$\Rightarrow \quad t = \sqrt{\frac{2x}{a}}$$

$$\text{Time period } T = 4t = \sqrt{\frac{32mxd}{\varepsilon_0 b(k-1)V^2}}$$

PROBLEM 7

A capacitor consists of two stationary plates shaped as a semi-circle of radius R and a movable plate made of dielectric with permittivity K and capable of rotating about an axis O between the stationary plates. The thickness of the movable plate is equal to d which is practically the separation between the stationary plates. A potential difference V is applied to the capacitor. Find the magnitude of the moment of forces relative to the axis O acting on the movable plate in the position shown in figure.



SOLUTION

Let C_0 be the initial capacitance of the condenser.

$$C_0 = \frac{K\varepsilon_0\left(\frac{\pi R^2}{2}\right)}{d} = \frac{K\varepsilon_0 \pi R^2}{2d} \quad \ldots(1)$$

When the dielectric is turned through an angle ϕ it divides the capacitor into an air capacitor and a dielectric capacitor. Let C_1 be the new capacity of dielectric capacitor. Then

$$C_1 = \frac{K\varepsilon_0\left(\frac{\pi R^2}{2} - \frac{R^2\phi}{2}\right)}{d} \quad \ldots(2)$$

If C_2 be the capacity of air, then

$$C_2 = \frac{\varepsilon_0\left(\frac{R^2\phi}{2}\right)}{d} = \frac{\varepsilon_0 R^2\phi}{2d} \quad \ldots(3)$$

The arrangement is equivalent to two capacitors (of capacitance C_1 and C_2) connected in parallel because they have common plates. So,

$$C = C_1 + C_2$$

$$\Rightarrow \quad C = \frac{K\varepsilon_0\left(\pi R^2\right)}{2d} - \frac{K\varepsilon_0 R^2\phi}{2d} + \frac{\varepsilon_0 R^2\phi}{2d}$$

$$\Rightarrow \quad C = \frac{\varepsilon_0 R^2}{2d}\left[K\pi + (1-K)\phi\right] \qquad \ldots(4)$$

Energy of capacitor $= \frac{1}{2}CV^2$

$$\Rightarrow \quad U = \frac{1}{4}\frac{\varepsilon_0 R^2}{d}\left[K\pi + (1-K)\phi\right]V^2$$

For a conservative force,

$$\tau = -\frac{\partial U}{\partial \phi}$$

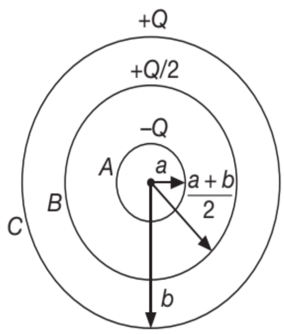
$$\Rightarrow \quad \tau = -\frac{\partial}{\partial \phi}\left[\frac{1}{4}\frac{\varepsilon_0 R^2}{d}\left\{K\pi + (1-K)\phi\right\}\right]V^2$$

$$\Rightarrow \quad \tau = -\frac{\varepsilon_0 R^2}{4d}(K-1)V^2$$

The negative sign shows that the torque acts opposite to angular displacement.

PROBLEM 8

Figure shows three conducting spherical shells A, B and C with charges $-Q$, $+\frac{Q}{2}$ and $+Q$ respectively. Calculate the capacitance of the system between points A and C.



SOLUTION

Potential at A, V_A is given by

$$(V_A) = \begin{pmatrix}\text{Potential} \\ \text{at } A \text{ due} \\ \text{to charge} \\ \text{on } A\end{pmatrix} + \begin{pmatrix}\text{Potential} \\ \text{at } A \text{ due} \\ \text{to charge} \\ \text{on } B\end{pmatrix} + \begin{pmatrix}\text{Potential} \\ \text{at } A \text{ due} \\ \text{to charge} \\ \text{on } C\end{pmatrix}$$

$$\Rightarrow \quad V_A = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{a} + \frac{\frac{Q}{2}}{4\pi\varepsilon_0\left(\frac{a+b}{2}\right)} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{b}$$

$$\Rightarrow \quad V_A = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a} + \frac{1}{(a+b)}\right)$$

Potential at C, V_C is given by

$$(V_C) = \begin{pmatrix}\text{Potential} \\ \text{at } C \text{ due} \\ \text{to charge} \\ \text{on } A\end{pmatrix} + \begin{pmatrix}\text{Potential} \\ \text{at } C \text{ due} \\ \text{to charge} \\ \text{on } B\end{pmatrix} + \begin{pmatrix}\text{Potential} \\ \text{at } C \text{ due} \\ \text{to charge} \\ \text{on } C\end{pmatrix}$$

$$\Rightarrow \quad V_C = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{b} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{2b} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{b} = \frac{Q}{8\pi\varepsilon_0 b}$$

Potential difference, $\Delta V = V_A - V_C$

$$\Rightarrow \quad \Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a} + \frac{1}{(a+b)}\right) - \frac{Q}{8\pi\varepsilon_0 b}$$

$$\Rightarrow \quad \Delta V = \frac{Q}{8\pi\varepsilon_0}\left(\frac{a^2 - 2b^2 + ab}{ab(a+b)}\right)$$

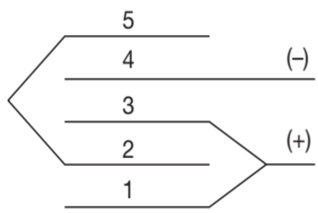
Capacitance of the arrangement is

$$C_{AC} = \frac{Q}{\Delta V} = \frac{Q}{V_A - V_C}$$

$$\Rightarrow \quad C_{AC} = \frac{Q}{\left\{\frac{Q}{8\pi\varepsilon_0}\left(\frac{a^2 - 2b^2 + ab}{ab(a+b)}\right)\right\}} = \frac{8\pi\varepsilon_0 ab(a+b)}{\left(a^2 - 2b^2 + ab\right)}$$

PROBLEM 9

Five identical conducting plates 1, 2, 3, 4 and 5 are fixed parallel to and equidistant from each other as shown in Figure. Plates 2 and 5 are connected by a conductor while 1 and 3 are joined by another conductor. The junction of 1 and 3 and the plate 4 are connected to a source of constant emf V_0. If d is the distance between any two successive plates and A is the area of either face of each plate. Find

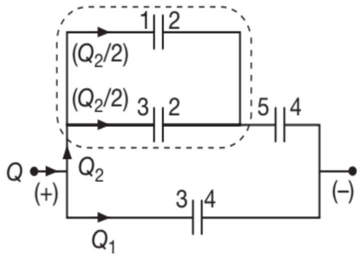


(a) the effective capacitance of the system between the terminals of the source.

(b) the charges on plates 3 and 5.

SOLUTION

(a) The equivalent circuit is shown here. The system consists of four capacitors, C_{12}, C_{32}, C_{34} and C_{54}. The capacitance of each capacitor is $C_0 = \dfrac{\varepsilon_0 A}{d}$.



The capacitors C_{12} and C_{32} are in parallel, and hence their net capacitance is $C_0 + C_0 = 2C_0$. The capacitor C_{54} is in series with the parallel combination of C_{12} and C_{32}. Hence the resultant capacitance will be $\dfrac{C_0 \times 2C_0}{C_0 + 2C_0}$

Since, C_{34} is again in parallel with the combination of C_{12}, C_{32}, C_{54}. Hence the effective capacitance C is

$$C = C_0 + \frac{C_0 \times 2C_0}{C_0 + 2C_0} = \frac{5}{3}C_0 = \frac{5}{3}\left(\frac{\varepsilon_0 A}{d}\right)$$

(b) Charge on the plate 5 = Charge on the upper half of parallel combination

$$\Rightarrow \quad Q_5 = V_0\left(\frac{2}{3}C_0\right) = \frac{2}{3}\left(\frac{\varepsilon_0 A V_0}{d}\right)$$

Charge on plate 3 on the surface facing 4 is

$$Q_{34} = V_0 C_0 = \frac{\varepsilon_0 A V_0}{d}$$

Charge on plate 3 on the surface facing 2 = (potential difference across 3 – 2 combination) C_0

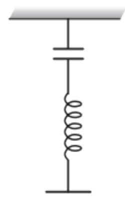
$$Q_{32} = V_0 \frac{C_0}{C_0 + 2C_0} C_0 = \varepsilon_0 \frac{AV_0}{3d}$$

$$\Rightarrow \quad Q_{32} = \frac{\varepsilon_0 A V_0}{d} + \varepsilon_0 \frac{AV_0}{3d}$$

$$\Rightarrow \quad Q_{32} = \frac{\varepsilon_0 A V_0}{d}\left(1 + \frac{1}{3}\right) = \frac{4}{3}\left(\frac{\varepsilon_0 A V_0}{d}\right)$$

PROBLEM 10

A parallel plate capacitor with air as a dielectric is arranged horizontally, such that its one plate is fixed and the other plate is connected with a perpendicular spring. The area of each plate is A. In the steady position, the separation between the plates is d_0. When the capacitor is connected with an electric source with the voltage V, a new equilibrium appears, such that the new separation between the plates becomes d_1. Assuming mass of the upper plate to be m, find the



(a) spring constant K.

(b) maximum voltage for a given K in which an equilibrium is possible.

(c) angular frequency of the oscillating system around the equilibrium value d_1 (amplitude of the oscillation << d_1).

SOLUTION

(a) Let the initial extension in the spring, in equilibrium situation be x_0. Then, for equilibrium of lower plate 2, we must have kx_0 (acting upwards) balancing mg (acting downwards)

$$Kx_0 = mg$$

When the voltage source is connected, then the plate separation changes from d_0 to d_1, as a result of which the extension of the spring becomes $(d_0 - d_1)$, so that net extension in the spring becomes $x_0 + (d_0 - d_1)$

When equilibrium is attained again, we have

$$K\left[x_0 + (d_0 - d_1)\right] = mg + \frac{1}{2}\varepsilon_0 E^2 A$$

Since, $Kx_0 = mg$

$$\Rightarrow \quad K(d_0 - d_1) = \frac{1}{2}\varepsilon_0\left(\frac{V}{d_1}\right)^2 A \qquad \ldots(1)$$

$$\Rightarrow \quad K = \frac{\varepsilon_0 AV^2}{2d_1^2(d_0 - d_1)}$$

(b) From equation (1), we have

$$V^2 = \frac{2Kd_1^2(d_0 - d_1)}{\varepsilon_0 A} = \frac{2K}{\varepsilon_0 A}\left(d_0 d_1^2 - d_1^3\right) \ldots(2)$$

Differentiating w.r.t. d_1, we get

$$2V\left(\frac{dV}{dd_1}\right) = \frac{2K}{A\varepsilon_0}\left(2d_1 d_0 - 3d_1^2\right)$$

For V to be maximum, we have

$$\left(\frac{dV}{dd_1}\right) = 0$$

$$\Rightarrow \quad 2d_1 d_0 - 3d_1^2 = 0$$

$$\Rightarrow \quad d_1 = \frac{2}{3}d_0$$

Substituting $d_1 = \frac{2d_0}{3}$ in (2), we get

$$V_{max}^2 = \frac{2K\left(\frac{2}{3}d_0\right)^2\left(d_0 - \frac{2}{3}d_0\right)}{\varepsilon_0 A}$$

$$\Rightarrow \quad V_{max} = \frac{2d_0}{3}\sqrt{\frac{2Kd_0}{3A\varepsilon_0}} \qquad \ldots(3)$$

(c) Let a small displacement x be given to the upper plate in the downward direction from the equilibrium position. Then the net force on the plate is

$$F = -K\left[x_0 + (d_0 - d) + x\right] + mg + \frac{1}{2}\varepsilon_0 A\left[\frac{V^2}{(d_1 - x)^2}\right]$$

$$\left\{\because\ E = \frac{V}{d_1 - x}\right\}$$

$$\Rightarrow \quad F = -K(d_0 - d) - Kx + \frac{1}{2}\frac{\varepsilon_0 AV^2}{d_1^2}\left(1 - \frac{x}{d_1}\right)^{-2}$$

$$\Rightarrow \quad F = -K(d_0 - d) - Kx + \frac{1}{2}\frac{\varepsilon_0 AV^2}{d_1^2}\left(1 + \frac{2x}{d_1}\right)$$

$$\left\{\because\ \text{For } x \ll d_1,\ \left(\frac{1-x}{d_1}\right)^{-2} \simeq 1 + \frac{2x}{d_1}\right\}$$

$$\Rightarrow \quad F = -K(d_0 - d) - Kx + K(d_0 - d_1)\left(1 + \frac{2x}{d_1}\right)$$

$$\Rightarrow \quad F = -Kx\left(\frac{3d_1 - 2d_0}{d_1}\right)$$

$$\Rightarrow \quad m\ddot{x} = -Kx\left(\frac{3d_1 - 2d_0}{d_1}\right)$$

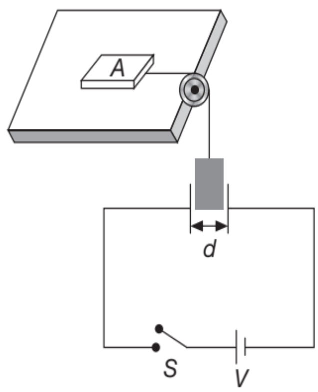
$$\Rightarrow \quad \ddot{x} + \frac{K}{m}\left(\frac{3d_1 - 2d_0}{d_1}\right)x = 0$$

Compare with the standard equation of SHM, $\ddot{x} + \omega^2 x = 0$, we get

$$\omega = \sqrt{\frac{K}{m}\left(\frac{3d_1 - 2d_0}{d_1}\right)}$$

PROBLEM 11

A block A of mass m kept on a rough horizontal surface is connected to a dielectric slab of mass $\frac{m}{6}$ and dielectric constant k by means of a light and inextensible string passing over a fixed pulley as shown in figure. The dielectric can completely fill the space between the parallel plate capacitor of plate area $l \times l$ and separation between the plates d kept in vertical position. Initially switch S is open and length of the dielectric inside the capacitor is b. The coefficient of friction between the block A and the surface is $\mu = \frac{1}{4}$. Ignore any other friction.



(a) Find the minimum value of the emf V of the battery so that after closing the switch the block A will move.

(b) If $V = 2V_{\min}$ find the speed of the block A when the dielectric completely fills the space between the plates of the capacitor.

SOLUTION

(a) The forces acting on the dielectric are electrostatic attractive force of field of capacitor and its weight. The block will slip when

$$F_E + mg \geq \mu Mg$$

$$\Rightarrow \quad F_E \geq \frac{M}{4}g - \frac{M}{6}g$$

Since, $F_E = \frac{\varepsilon_0 l}{2d}(K-1)V^2$

$$\Rightarrow \quad \frac{1}{2}\frac{\varepsilon_0 l}{d}(K-1)V^2 \geq \frac{Mg}{12}$$

$$\Rightarrow \quad V_{\min} = \sqrt{\frac{Mg}{12} \times \frac{2d}{\varepsilon_0 l(K-1)}} = \sqrt{\frac{Mgd}{6\varepsilon_0 l(K-1)}}$$

(b) Now $V = 2V_{\min}$. In this case the block will accelerate. So, writing equations of motion for the dielectric and the block, we get

For dielectric

$$F_E' + mg - T = ma \quad \ldots(1)$$

For block

$$T - \mu Mg = Ma \quad \ldots(2)$$

Equation (1) and (2) give

$$a = \frac{F_E' + (m - \mu M)g}{m + M}$$

Since, $F_E = \frac{\varepsilon_0 l}{2d}(K-1)V^2$

So, at $V = 2V_{\text{MIN}} = 2\sqrt{\frac{Mgd}{6\varepsilon_0 (K-1)l}}$, we get

$$F_E = \frac{\varepsilon_0 l}{2d}(K-1) \times 4 \times \frac{Mgd}{6\varepsilon_0 l(K-1)} = \frac{Mg}{3}$$

$$\left\{ \because V = 2V_{\text{MIN}} = 2\sqrt{\frac{Mgd}{6\varepsilon_0 l(K-1)}} \right\}$$

$$\text{Thus } a = \frac{\frac{Mg}{3} + \left(\frac{M}{6} - \frac{M}{4}\right)g}{\frac{M}{6} + M} = \frac{\frac{Mg}{3} - \frac{Mg}{12}}{\frac{7M}{6}} = \frac{3g}{14}$$

$$\Rightarrow \quad a = \frac{3g}{14}$$

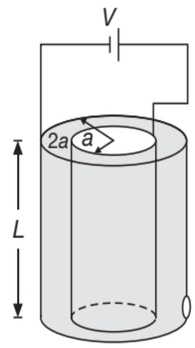
Using equation of motion, $v^2 = 2as$, we get

$$v^2 = 2\left(\frac{3g}{14}\right)(l - b)$$

$$\Rightarrow \quad v = \sqrt{\frac{3}{7}g(l-b)}$$

PROBLEM 12

A cylindrical capacitor of inner radius a, outer radius $2a$ and length L is kept with its axis vertical. Lower cross-section of the capacitor is sealed with very thin dielectric material and the curved space is filled completely with the oil of dielectric constant K. The plates are connected to a battery of emf V. Suddenly an orifice of cross-sectional area A is pierced at the bottom of the cylinder. Find the expression for current in the connecting wires (of negligible resistance) as a function of time.



SOLUTION

Let the liquid column inside the cylinder occupies a height x after time t. So, the air column gets a height $(L - x)$. The arrangement shown is now a parallel combination of two capacitors of capacitances C_1 and C_2 where

$$C_1 = \frac{2\pi\varepsilon_0 xK}{\log_e(2)} \quad \ldots(1)$$

$$C_2 = \frac{2\pi\varepsilon_0 (L-x)}{\log_e(2)} \quad \ldots(2)$$

Hence, equivalent capacitance C is given by

$$C = C_1 + C_2 = \frac{2\pi\varepsilon_0}{\log_e(2)}[L + x(K-1)] \qquad \ldots(3)$$

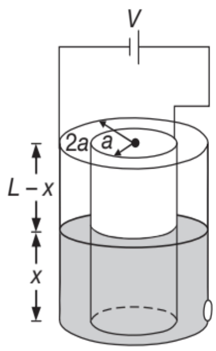
The cross-sectional area of cylinder in which the liquid is present is

$$A' = \pi(4a^2 - a^2) = 3\pi a^2$$

Using equation of continuity, we get

$$A\sqrt{2gx} = -A'\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \quad dt = -3\frac{dx}{\sqrt{2gx}} \qquad \{\because A' = 3\pi a^2 \text{ and } A = \pi a^2\}$$



Integrating, we get

$$\int_0^t dt = -\frac{3}{\sqrt{2g}}\int_L^x \frac{dx}{\sqrt{x}}$$

$$\Rightarrow \quad t = -\frac{3}{\sqrt{2g}}\left[2(\sqrt{x})\Big|_L^x\right] = 3\sqrt{\frac{2}{g}}(\sqrt{L} - \sqrt{x})$$

$$\Rightarrow \quad t = 3\sqrt{\frac{2}{g}}(\sqrt{L} - \sqrt{x})$$

$$\Rightarrow \quad x = \left(\sqrt{L} - \frac{t}{3}\sqrt{\frac{g}{2}}\right)^2$$

$$\Rightarrow \quad x = L - \frac{2}{3}\sqrt{\frac{gL}{2}}t + \frac{g}{18}t^2 \qquad \ldots(4)$$

This variation in x will change the value of C and hence q changes with t so as to produce a current, say I. To calculate C as a function of t, substitute x from equation (4) in equation (3), we get

$$C = \frac{2\pi\varepsilon_0}{\log_e(2)}\left[L + \left(L - \left(\frac{2}{3}\right)\sqrt{\frac{gL}{2}}t + \frac{g}{18}t^2\right)(K-1)\right]$$

$$\Rightarrow \quad C = \frac{2\pi\varepsilon_0}{\log_e(2)}\left[KL - \frac{1}{3}\sqrt{2gL}(K-1)t + \frac{g}{18}(K-1)t^2\right]$$

Now, the charge on capacitor is $Q = CV$

So, current in the connecting wires is given by

$$I = \frac{dQ}{dt} = V\left(\frac{dC}{dt}\right) \qquad \{\text{Since, } V \text{ is a constant}\}$$

where $\dfrac{dC}{dt} = \dfrac{2\pi\varepsilon_0}{\log_e(2)}(K-1)\left(\dfrac{gt}{18} - \dfrac{\sqrt{2gL}}{3}\right)V$

$$\Rightarrow \quad I = \frac{2\pi\varepsilon_0(K-1)V}{\log_e(2)}\left(\frac{gt}{18} - \frac{\sqrt{2gL}}{3}\right)$$

PROBLEM 13

A charge Q is imparted to two identical plane capacitors connected in parallel. At the moment of time $t = 0$ the distance between the plates of the first capacitor begins to increase uniformly according to the law $d_1 = d_0 + vt$, and the distance between the plates of the second capacitor to decrease uniformly according to the law $d_2 = d_0 - vt$.

(a) Neglecting the resistance of the connecting wires, find the current in the circuit when the plates of the capacitors move.

(b) Find the work performed by an electrostatic field when the distance between the plates of the first capacitor increases and that between the plates of the second capacitor simultaneously decreases by a.

SOLUTION

(a) Let us denote the charges on the first and second capacitors at time t by q_1 and q_2. So, we must have

$$q_1 + q_2 = Q \text{ and} \qquad \ldots(1)$$

$$\frac{q_1}{C_1} = \frac{q_2}{C_2} \qquad \ldots(2)$$

Since, at time t

$$C_1 = \frac{\varepsilon_0 A}{(d_0 + vt)} \text{ and } C_2 = \frac{\varepsilon_0 A}{(d_0 - vt)}$$

$$\Rightarrow \quad \frac{q_1}{q_2} = \frac{d_0 - vt}{d_0 + vt} \qquad \ldots(3)$$

$$\Rightarrow \quad q_1 = Q\left(\frac{d_0 - vt}{2d_0}\right) \text{ and } q_2 = Q\left(\frac{d_0 + vt}{2d_0}\right)$$

The reduction in the charge on the first capacitor is equal to the increase of the charge on the second capacitor. The current is

$$I = -\frac{\Delta q_1}{\Delta t} = \frac{\Delta q_2}{\Delta t} = \frac{Qv}{2d_0}$$

The current will flow from the positively charged plate of the first capacitor to the positively charged plate of the second capacitor.

(b) Since the resistance of the connecting wires is zero, the quantity of heat liberated is also zero. Hence the change in the electrostatic energies of the two capacitors will be equal to the work performed by the electrostatic field.
At time t, the energies of the first and the second capacitors will be

$$U_1 = \frac{q_1^2}{2C_1} = \frac{1}{8\varepsilon_0}\frac{Q^2}{Ad_0^2}(d_0 - vt)^2(d_0 + vt)$$

$$U_1 = \frac{Q^2}{8\varepsilon_0 Ad_0^2}(d_0 - a)^2(d_0 + a)$$

and $$U_2 = \frac{q_2^2}{2C_2} = \frac{1}{8\varepsilon_0}\frac{Q^2}{Ad_0^2}(d_0 + vt)^2(d_0 - vt)$$

$$U_2 = \frac{Q^2}{8\varepsilon_0 Ad_0^2}(d_0 + a)^2(d_0 - a)$$

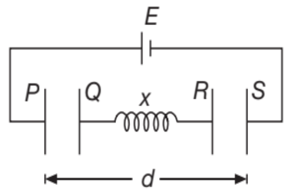
The total energy is given by

$$U = U_1 + U_2 = \frac{1}{4\varepsilon_0}\frac{Q^2}{Ad_0}(d_0^2 - a^2)$$

Therefore, the energy will drop by $\Delta U = \frac{Q^2}{4\varepsilon_0 Ad_0}a^2$ during the time t. This change will be equal to the work W performed by the electrostatic field.

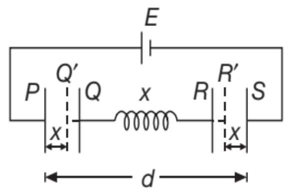
PROBLEM 14

Two parallel plate capacitors with area A are connected through a conducting spring of natural length l in series as shown. Plates P and S have fixed positions at separation d. Now the plates are connected by a battery of emf E as shown. If the extension in the spring in equilibrium is equal to the separation between the plates, find the spring constant k.



SOLUTION

Let charge on capacitors be q and separation between plates P and Q and R and S be x at any time. Distance between plates P and Q and R and S is same because the force acting on them is same.



Capacitance of capacitor PQ is given by

$$C_1 = \frac{\varepsilon_0 A}{x}$$

Capacitance of capacitor RS is given by

$$C_2 = \frac{\varepsilon_0 A}{x}$$

From Kirchhoff's Voltage Law (KVL), we have

$$\frac{q}{C_1} + \frac{q}{C_2} = E$$

$$\Rightarrow \quad q = \frac{\varepsilon_0 AE}{2x}$$

At this moment extension in spring, $y = d - 2x - l$ and force on plate Q towards P is given by

$$F_1 = \frac{q^2}{2A\varepsilon_0} = \frac{\varepsilon_0^2 A^2 E^2}{8Ax^2\varepsilon_0} = \frac{A\varepsilon_0 E^2}{8x^2} \quad \ldots(1)$$

Spring force on plate Q due to extension in spring is given by

$$F_2 = ky$$

At equilibrium,
Separation between the plates = Extension in the spring
Thus $x = y = d - 2x - l$

$$\Rightarrow \quad x = \frac{d-l}{3} \qquad \text{...(2)}$$

$$\Rightarrow \quad F_1 = F_2 = kx = ky \qquad \text{...(3)}$$

From equations (1) and (3), we get

$$\frac{A\varepsilon_0 E^2}{8x^2} = ky = kx$$

$$\Rightarrow \quad x = \left(\frac{A\varepsilon_0 E^2}{8k}\right)^{\frac{1}{3}} \qquad \text{...(4)}$$

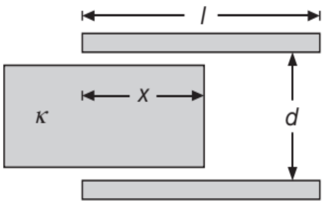
From equations (2) and (4), we get

$$\left(\frac{d-l}{3}\right)^3 = \frac{A\varepsilon_0 E^2}{8k}$$

$$\Rightarrow \quad k = \frac{27\varepsilon_0 AE^2}{8(d-l)^3}$$

PROBLEM 15

A capacitor is constructed from two square plates of sides l and separation $d(<<l)$, as suggested in figure. The plates carry charges $+Q_0$ and $-Q_0$. A block of metal has a width l, a length l and a thickness slightly less than d. It is inserted a distance x into the capacitor. The charges on the plates are not disturbed as the block slides in. In a static situation, a metal prevents an electric field from penetrating inside it.



(a) Calculate the stored energy as a function of x.
(b) Find the direction and magnitude of the force that acts on the metallic block.
(c) The area of the advancing front face of the block is essentially equal to ld. Considering the force on the block as acting on this face, find the stress (force per area) on it.
(d) For comparison, express the energy density in the electric field between the capacitor plates in terms of Q_0, l, d and ε_0.

SOLUTION

Since, the metal can be thought of as a dielectric having $\kappa \to \infty$, so the portion of the capacitor nearly filled by metal has capacitance $\frac{\kappa\varepsilon_0(lx)}{d} \to \infty$ and stored energy $\frac{Q^2}{2C} \to 0$

The unfilled portion has capacitance $\frac{\varepsilon_0 l(l-x)}{d}$

The charge on this portion is $Q = \frac{(l-x)Q_0}{l}$

(a) The stored energy is

$$U = \frac{Q^2}{2C} = \frac{\left[(l-x)\frac{Q_0}{l}\right]^2}{2\varepsilon_0 \frac{l(l-x)}{d}} = \frac{Q_0^2(l-x)d}{2\varepsilon_0 l^3}$$

(b) $F = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{Q_0^2(l-x)d}{2\varepsilon_0 l^3}\right) = +\frac{Q_0^2 d}{2\varepsilon_0 l^3}$

$$\Rightarrow \quad F = \frac{Q_0^2 d}{2\varepsilon_0 l^3} \text{ to the right (into the capacitor)}$$

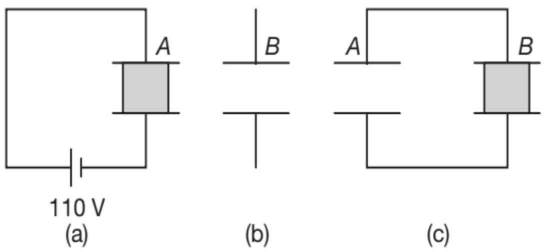
(c) Stress $= \frac{\text{Force}}{\text{Area}} = \frac{F}{ld} = \frac{Q_0^2}{2\varepsilon_0 l^4}$

(d) Energy density

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{1}{2}\varepsilon_0\left(\frac{Q_0}{\varepsilon_0 l^2}\right)^2 = \frac{Q_0^2}{2\varepsilon_0 l^4}$$

PROBLEM 16

Two parallel plate capacitors A and B have the same separation $d = 8.85\times10^{-4}$ m between the plates. The plate areas of A and B are 0.04 m^2 and 0.02 m^2 respectively. A slab of dielectric constant (relative permittivity) $K = 9$ has dimensions such that it can exactly fill the space between the plates of capacitor B.



(a) The dielectric slab is placed inside A as shown in figure (a). A is then charged to a potential difference of 110 V. Calculate the capacitance of A and the energy stored in it.

(b) The battery is disconnected and then the dielectric slab is removed from A. Find the work done by the external agency in removing the slab from A.

(c) The same dielectric slab is now placed inside B, filling it completely. The two capacitors A and B are then connected as shown in figure (c). Calculate the energy stored in the system.

SOLUTION

(a) Capacitor A is a combination of two capacitors C_K and C_O in parallel. Hence,

$$C_A = C_K + C_O = \frac{K\varepsilon_0 A}{d} + \frac{\varepsilon_0 A}{d} = (K+1)\frac{\varepsilon_0 A}{d}$$

Here, $A = 0.02\ \text{m}^2$. Substituting the values, we have

$$C_A = (9+1)\frac{8.85\times10^{-12}(0.02)}{(8.85\times10^{-4})}\ \text{F}$$

$$\Rightarrow \quad C_A = 2\times10^{-9}\ \text{F}$$

Energy stored in capacitor A, when connected with a 110 V battery is

$$U_A = \frac{1}{2}C_A V^2 = \frac{1}{2}(2\times10^{-9})(110)^2$$

$$\Rightarrow \quad U_A = 1.21\times10^{-5}\ \text{J}$$

(b) Charge stored in the capacitor

$$q_A = C_A V = (2\times10^{-9})(110)$$

$$\Rightarrow \quad q_A = 2.2\times10^{-7}\ \text{C}$$

Now, this charge remains constant even after battery is disconnected. But when the slab is removed, capacitance of A will get reduced. Let it be C'_A

$$C'_A = \frac{\varepsilon_0(2A)}{d} = \frac{(8.85\times10^{-12})(0.04)}{8.85\times10^{-4}}\ \text{F}$$

$$\Rightarrow \quad C'_A = 0.4\times10^{-9}\ \text{F}$$

Energy stored in this case would be

$$U'_A = \frac{1}{2}\frac{(q_A)^2}{C'_A} = \frac{1}{2}\frac{(2.2\times10^{-7})^2}{(0.4\times10^{-9})}\ \text{J}$$

$$\Rightarrow \quad U'_A = 6.05\times10^{-5}\ \text{J} > U_A$$

Therefore, work done to remove the slab would be

$$W = U'_A - U_A = (6.05-1.21)\times10^{-5}\ \text{J}$$

$$\Rightarrow \quad W = 4.84\times10^{-5}\ \text{J}$$

(c) Capacity of B when filled with dielectric is given by

$$C_B = \frac{K\varepsilon_0 A}{d} = \frac{(9)(8.85\times10^{-12})(0.02)}{(8.85\times10^{-4})}\ \text{F}$$

$$\Rightarrow \quad C_B = 1.8\times10^{-9}\ \text{F}$$

These two capacitors are in parallel. Therefore, net capacitance of the system is

$$C = C'_A + C_B = (0.4+1.8)\times10^{-9}\ \text{F}$$

$$\Rightarrow \quad C = 2.2\times10^{-9}\ \text{F}$$

Charge stored in the system is $q = q_A = 2.2\times10^{-7}\ \text{C}$

Therefore, energy stored, $U = \frac{1}{2}\frac{q^2}{C}$

$$\Rightarrow \quad U = \frac{1}{2}\frac{(2.2\times10^{-7})^2}{(2.2\times10^{-9})}$$

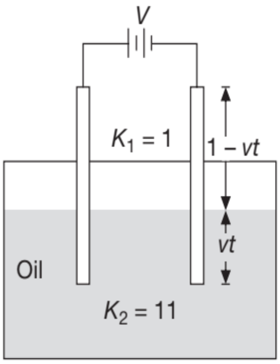
$$\Rightarrow \quad U = 1.1\times10^{-5}\ \text{J}$$

PROBLEM 17

Two square metal plates of side 1 m are kept 0.01 m apart like a parallel plate capacitor in air in such a way that one of their edges is perpendicular to an oil surface in a tank filled with an insulating oil. The plates are connected to a battery of emf 500 V. The plates are then lowered vertically into the oil at a speed of $0.001\ \text{ms}^{-1}$. Calculate the current drawn from the battery during the process. (Dielectric constant of oil $= 11$, $\varepsilon_0 = 8.85\times10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-1}$).

SOLUTION

Let a be the side of the square plate.



As shown in figure, C_1 and C_2 are in parallel. Therefore, total capacity of capacitors in the position shown is

$$C = C_1 + C_2$$

$$\Rightarrow \quad C = \frac{\varepsilon_o a(a-x)}{d} + \frac{K\varepsilon_o ax}{d}$$

$$\Rightarrow \quad q = CV = \frac{\varepsilon_o aV}{d}(a - x + Kx)$$

As plates are lowered in the oil, C increases or charge stored will increase.

Therefore, $I = \frac{dq}{dt} = \frac{\varepsilon_o aV}{d}(K-1)\frac{dx}{dt}$

Substituting the values

$$\varepsilon_o = 8.85 \times 10^{-12} \text{ C}^2\text{Nm}^{-2}$$

$a = 1$ m, $V = 500$ V, $d = 0.01$ m, $K = 1$ and

$$\frac{dx}{dt} = \text{speed of plate} = 0.001 \text{ ms}^{-1}$$

We get current

$$I = \frac{(8.85 \times 10^{-12})(500)1(11-1)(0.001) \text{ A}}{(0.01)}$$

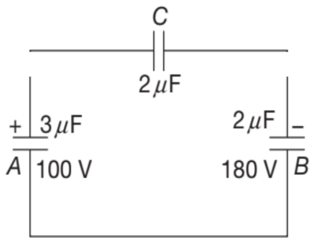
$$\Rightarrow \quad I = 4.43 \times 10^{-9} \text{ A}$$

PROBLEM 18

Two capacitors A and B with capacities $3\ \mu$F and $2\ \mu$F are charged to a potential difference of 100 V and 180 V respectively. The plates of the capacitors are connected as shown in the figure with one wire of each capacitor free. The upper plate of A is positive and that of B is negative. An uncharged $2\ \mu$F capacitor C with lead wires falls on the free ends to complete the circuit. Calculate:

(a) the final charge on the three capacitors and

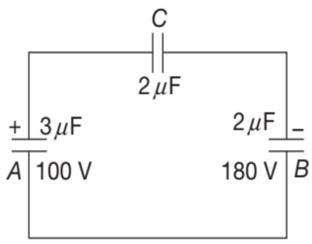
(b) the amount of electrostatic energy stored in the system before and after completion of the circuit.



SOLUTION

(a) Charge on capacitor A, before joining with an uncharged capacitor

$$q_A = CV = (100)(3)\ \mu\text{C} = 300\ \mu\text{C}$$



Similarly, charge on capacitor B

$$q_B = (180)(2)\ \mu\text{C} = 360\ \mu\text{C}$$

Let q_1, q_2 and q_3 be the charges on the three capacitors after joining them as shown in figure alongside (q_1, q_2 and q_3 are in microcoulombs).

From conservation of charge

Net charge on plates 2 and 3 before joining = net charge after joining

$$\Rightarrow \quad 300 = q_1 + q_2 \qquad \ldots(1)$$

Similarly, net charge on plates 4 and 5 before joining = net charge after joining.

$$-360 = -q_2 - q_3$$

$$\Rightarrow \quad 360 = q_2 + q_3 \qquad \ldots(2)$$

Applying Kirchhoff's Second Law in closed loop ABCDA

$$\frac{q_1}{3} - \frac{q_2}{2} + \frac{q_3}{2} = 0$$

$$\Rightarrow \quad 2q_1 - 3q_2 + 3q_3 = 0 \qquad \ldots(3)$$

Solving equations (1), (2) and (3), we get

$$q_1 = 90\ \mu\text{C},\ q_2 = 210\ \mu\text{C and } q_3 = 150\ \mu\text{C}$$

Therefore, final charges on the three capacitors are as shown.

(b) (i) Electrostatic energy stored before, completing the circuit

$$U_i = \frac{1}{2}\left(3\times10^{-6}\right)(100)^2 + \frac{1}{2}\left(2\times10^{-6}\right)(180)^2$$

$$\left\{\because U = \frac{1}{2}CV^2\right\}$$

$$\Rightarrow \quad U_i = 4.74\times10^{-2}\ \text{J}$$

$$\Rightarrow \quad U_i = 47.4\ \text{mJ}$$

(ii) Electrostatic energy stored after, completing the circuit

$$U_f = \frac{1}{2}\frac{\left(90\times10^{-6}\right)^2}{\left(3\times10^{-6}\right)} + \frac{1}{2}\frac{\left(210\times10^{-6}\right)^2}{\left(2\times10^{-6}\right)} + \frac{1}{2}\frac{\left(150\times10^{-6}\right)^2}{\left(2\times10^{-6}\right)}$$

$$\Rightarrow \quad U_f = 1.8\times10^{-2}\ \text{J}$$

$$\Rightarrow \quad U_f = 18\ \text{mJ}$$