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CHAPTER 1

Electrostatics

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Coulomb's Law, Electric Field and Dipole
(b) Gauss Law
(c) Electrostatic Potential, Potential Energy and Conductors

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

COULOMB'S LAW, ELECTRIC FIELD, DIPOLE AND APPLICATIONS

INTRODUCTION

Life without electricity and its effects would have been hard to imagine. The electromagnetic force between charged particles is one of the fundamental forces of nature. Here we shall begin by describing some of the basic properties of electrostatic forces and then smoothly moving on to the basic electrical phenomenon and applications along with a rigorous mathematical treatment through ILLUSTRATIVE EXAMPLES.

The branch, known as electromagnetism is concerned with the nature of electric and magnetic phenomenon and the connection between them. I shall be presenting this together with some applications of interest to you from IIT-JEE view point exclusively.

When a glass rod is rubbed with silk, it acquires power to attract light bodies such as small pieces of paper. The bodies which acquire this power are said to be charged. If these charges do not move they are called **static charges** and the branch of Physics which deals with static charges is called **electrostatics**.

ELECTRIC CHARGE

Charge is a scalar quantity which is categorised into two types.

(a) Positive charge (anciently called **Vitreous**)
(b) Negative charge (anciently called **Resinous**)

A body having no charge, is said to be neutral in nature i.e., on a neutral body the sum of positive charges is equal to the sum of negative charges.

The positive charge means deficiency of electrons, whereas the negative charge on a body implies excess of electrons.
The S.I. unit of charge is coulomb (C).

Units of Charge

(a) S.I. unit of charge is coulomb (C). One coulomb of charge is that charge which when placed at rest in vacuum at a distance of one metre from an equal and similar stationary charge repels it and is repelled by it with a force of 9×10^9 N.

(b) In the Cgs system there are two units of charge statcoulomb (SC) or esu, abcoulomb or emu

$$1 \text{ coulomb} = 3\times10^{9} \text{ esu} = \frac{1}{10} \text{ emu.}$$

(c) $1 \text{ coulomb} = 3\times10^{9} \text{ SC} = \frac{1}{10} \text{ abcoulomb}$

(d) Practical units of charge are ampere hour with $1 \text{ Ahr} = 3600 \text{ C}$ and faraday (F) with $1 \text{ F} = 96500 \text{ C}$

(e) Smallest unit of charge is statcoulomb (SC) and largest is faraday (F).

CONDUCTORS AND INSULATORS

Objects can broadly be classified as

(a) **Conductors:** Objects which allow charges to flow through them are called **conductors**. This category generally comprises metals with few exceptions from non-metals too (like graphite).

(b) **Insulators:** Objects which do not allow charges to flow through them are called **insulators**.

METHODS OF CHARGING

(a) **By Friction:** In this method, when two bodies are rubbed together some electrons are transferred from one body to other. As a result, one body becomes positively charged while the other becomes negatively charged. Electricity so obtained by rubbing or friction is known as frictional electricity.

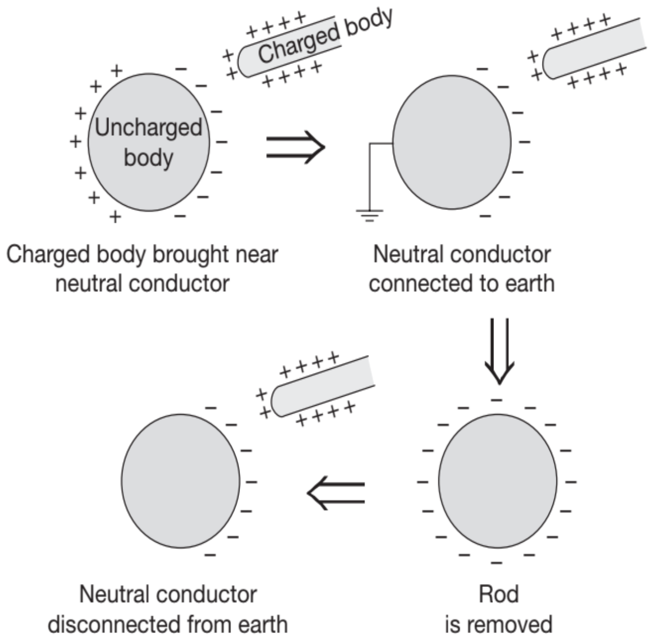
EXAMPLE:

When glass rod is rubbed with silk, the rod becomes positively charged while the silk becomes negatively charged. Clouds also become charged by friction

Remark(s)

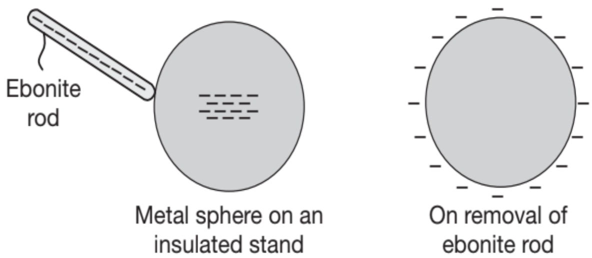
When two bodies have to be charged through friction, one of them has to be an insulator.

(b) **By Induction:** Whenever a charged body (conductor or insulator) is brought near a neutral conductor, the charged body will attract opposite charge and repel similar charge present in neutral conductor. This process is called **charging by induction**. Charging of body by Induction is shown here. Assume the neutral conductor to be placed on an insulated stand, not shown.



This process of induction is not applicable when the sphere were made of an insulated material such as plastic.

(c) **By Conduction (charging by contact):** When a charged body is in direct contact with an uncharged body, charge flows from former to latter till both are at same potential. This flow of charges is due to mutual repulsions between same kind of charges on charged body. When a negatively charged ebonite rod is rubbed on a metal object (say sphere) placed on an insulated stand (not shown), some of the electrons that are already in excess on the rod are transferred to the sphere. As soon as the electrons reach the metal sphere they start repelling each other and get spread over the surface of the sphere.



Remark(s)

(a) Nature of induced charge is always opposite to inducing charge.

(b) Induced charge can be lesser than or equal to inducing charge

$$q_{\text{induced}} = -q_{\text{inducing}}\left(1-\frac{1}{K}\right)$$, where K is the dielectric constant.

(c) If a charged body is brought near uncharged gold leaf electroscope or similar charged body brought near charged electroscope, leaves diverge. If oppositely charged body brought near charged electroscope, leaves converge.

PROPERTIES OF ELECTRIC CHARGE

(a) Like charges repel each other and unlike charges attract each other.

(b) Charge Conservation

The algebraic sum of all the charges in an isolated system is a constant. In crude language we can say that charge can neither be created nor be destroyed, however it can simply be transferred from one body to the other.

(c) Relativistic Invariance

Charge on a body is relativistically invariant. i.e., charge on the body at rest equals the charge on the body at relativistic speeds (speeds comparable to the speed of light). However charge density is not relativistically invariant. Mathematically,

$$(q)_{\text{at rest}} = (q)_{\text{in motion}}$$

or $$(q)_{\text{at rest}} = (q)_{\text{at relativistic speeds}} = (q)_{\text{at non relativistic speeds}}$$

Physically, we just want to say that the charge on a body is simply independent of the velocity of the body. However, this may not be the case with charge density.

(d) Charge Quantisation

Charge on a body q must always exist as an integral multiple of some fundamental unit of charge (called electronic charge) e, where $e = 1.6\times10^{-19}$ C.

Mathematically, $q = \pm ne$, $n = 1, 2, 3, ...$

From here we conclude that a neutral body can have +1 C of charge when it falls deficient of 6.25×10^{18} electrons.

Remark(s)

Quarks are particles which have fractional charges (e.g. $\frac{2e}{3}, -\frac{e}{3}$ etc) but they never exist independently i.e., they always exist in groups so as to make total charge on a body as an integral multiple of e. So existence of quarks does not violate charge quantisation.

(e) A charged body can attract light uncharged body. (Due to charging by induction)

(f) Charges are always added algebraically.

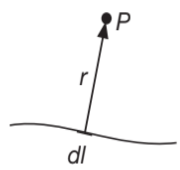
EXAMPLE:

If a neutral body is first given a charge of +5 C and subsequently a charge of –7 C, then it will finally have a charge of +5 C – 7 C = –2 C.

CONCEPT OF CHARGE DISTRIBUTION(S)

Sometimes we observe that the charge, instead of being pointlike, is distributed over the entire body. In such a case we shall be using the concept of charge distributions as discussed.

CASE-1: Charge distributed on thin ropes, threads, or thin rods.



Line charge, λ

In this kind of situation(s), we shall be using the concept of Linear Charge Density λ, defined as charge per unit length. Mathematically, we have

$$\lambda = \frac{dq}{dl} = \frac{dq}{dx} \quad \text{i.e.} \quad dq = \lambda dx$$

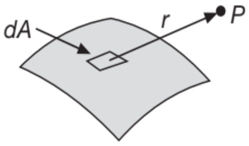
If the charge is distributed uniformly over the thin rod of length L, then we have $Q = \lambda L$.

However, if the charge is distributed non-uniformly over the length of the rod, then we have

$$Q = \int \lambda dx$$

This expression has to be integrated within suitable limits for getting the total charge on the rod.

CASE-2: Charge distributed on thin sheets, or thin surfaces



Surface charge, σ

In this kind of situation(s), we shall be using the concept of Surface Charge Density σ, defined as charge per unit area. Mathematically, we have

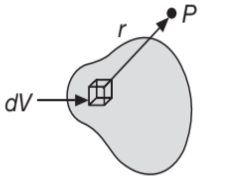
$$\sigma = \frac{dq}{dA} \quad \text{i.e.} \quad dq = \sigma dA$$

If the charge is distributed uniformly over the entire surface of area A, then we have $Q = \sigma A$.
For example, for a uniformly charged shell of radius R, the charge on the shell is $Q = \sigma A = \sigma(4\pi R^2)$.
However, if the charge is distributed non-uniformly over the area of the surface, then we have

$$Q = \int \sigma dA$$

This expression has to be integrated within suitable limits for getting the total charge on the surface.

CASE-3: Charge distributed on thin sheets, or thin surfaces



Volume charge, ρ

In this kind of situation(s), we shall be using the concept of Surface Charge Density σ, defined as charge per unit area. Mathematically, we have

$$\sigma = \frac{dq}{dA} \quad \text{i.e.} \quad dq = \sigma dA$$

If the charge is distributed uniformly over the entire surface of area A, then we have $Q = \sigma A$.
For example, for a uniformly charged shell of radius R, the charge on the shell is $Q = \sigma A = \sigma(4\pi R^2)$.
However, if the charge is distributed non-uniformly over the area of the surface, then we have

$$Q = \int \sigma dA$$

This expression has to be integrated within suitable limits for getting the total charge on the surface.

ILLUSTRATION 1

A spherical region of space has a distribution of charge such that the volume charge density varies with the radial distance from the centre r as $\rho = \rho_0 r^2$, $0 \le r \le R$ where ρ_0 is a positive constant. Calculate the total charge Q on sphere of radius R.

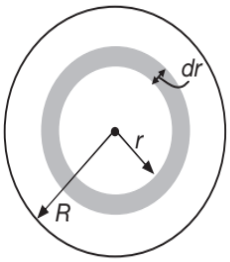
SOLUTION

Consider an infinitesimal concentric spherical element of thickness dr, radius r. If dq is the charge on this element, then

$$dq = \rho dV \qquad \{\text{where } dV = 4\pi r^2 dr\}$$

$$\Rightarrow \quad dq = (\rho_0 r^2)(4\pi r^2 dr)$$

$$\Rightarrow \quad dq = 4\pi\rho_0 r^4 dr$$



$$\Rightarrow \quad Q = \int_0^Q dq = 4\pi\rho_0 \int_0^R r^4 dr$$

$$\Rightarrow \quad Q = 4\pi\rho_0 \left[\frac{r^5}{5} \bigg|_0^R \right]$$

$$\Rightarrow \quad Q = \frac{4}{5}\pi\rho_0 R^5$$

COULOMB'S LAW

The magnitude of the force (F) of attraction or repulsion between two point charges q_1 and q_2 placed in vacuum at separation r is

(a) directly proportional to the product of the magnitude of the two charges.

$$F \propto q_1 q_2 \qquad \text{...(1)}$$

(b) inversely proportional to the square of the distance of separation between them.

$$F \propto \frac{1}{r^2} \text{ (called Inverse Square Law)} \qquad \text{...(2)}$$

Combining (1) and (2), we get

$$F \propto \frac{q_1 q_2}{r^2}$$

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$$\Rightarrow \quad F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} \qquad ...(3)$$

where $\frac{1}{4\pi\varepsilon_0} = 9\times10^9\ \text{Nm}^2\text{C}^{-2}$

and $\varepsilon_0 = 8.85\times10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}$ is the absolute permittivity of free space or vacuum.
If the charges are placed in a medium, then

$$F_{\text{med}} = \frac{1}{4\pi\varepsilon}\frac{q_1q_2}{r^2} \qquad ...(4)$$

where ε (read as Epsilon) is the permittivity of the medium given by,

$$\varepsilon = \varepsilon_0\varepsilon_r$$

where ε_r = Relative permittivity of the medium, called the **dielectric constant** (K). So

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = K$$

$$\Rightarrow \quad F_{\text{med}} = \frac{1}{4\pi\varepsilon_0 K}\frac{q_1q_2}{r^2}$$

$$\Rightarrow \quad F_{\text{med}} = \frac{F_{\text{vacuum/air}}}{K}$$

IMPORTANT POINTS REGARDING COULOMB'S LAW

(a) It is applicable only for point charges.

(b) The constant of proportionality K in SI units in vacuum is expressed as $\frac{1}{4\pi\varepsilon_0}$ and in any other medium expressed as $\frac{1}{4\pi\varepsilon}$. If charges are dipped in a medium then electrostatic force on one charge is $\frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1q_2}{r^2}$. ε_0 and ε are called permittivity of vacuum and absolute permittivity of the medium respectively. The ratio $\frac{\varepsilon}{\varepsilon_0} = \varepsilon_r$ is called relative permittivity of the medium, which is a dimensionless quantity.

(c) The value of relative permittivity ε_r is constant for medium and can have values between 1 to ∞. For vacuum, by definition it is equal to 1. For air it is nearly equal to 1 and may be taken to be equal to 1 for calculations. For metals the value of ε_r is ∞ and for water is 81. The material in which more charge can induce ε_r will be higher.

(d) The value of $\frac{1}{4\pi\varepsilon_0} = 9\times10^9\ \text{Nm}^2\text{C}^{-2}$.

$$\Rightarrow \quad \varepsilon_0 = 8.855\times10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}$$

(e) Dimensional formula of ε is $M^{-1}L^{-3}T^4A^2$

(f) The force acting on one point charge due to the other point charge is always along the line joining these two charges. It is equal in magnitude and opposite in direction on two charges, irrespective of the medium, in which they lie.

(g) The force is conservative in nature i.e., work done by electrostatic force in moving a point charge along a close loop of any shape is zero.

(h) Since the force is a central force, in the absence of any other external force, angular momentum of one particle w.r.t. the other particle (in two particle system) is conserved.

(i) In vector form formula can be given as below.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1q_2}{|\vec{r}|^3}\vec{r} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1q_2}{|\vec{r}|^2}\hat{r}$$

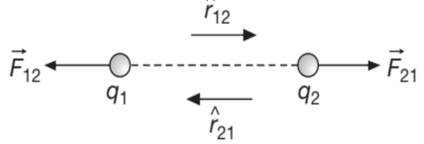
(q_1 & q_2 are to be substituted with sign.)

Here $\vec{r}$ is position vector of the test charge (on which force is to be calculated) with respect to the source charge (due to which force is to be calculated).

COULOMB'S LAW IN VECTOR FORM

Consider two point charges q_1 and q_2 separated by a distance r. If $q_1q_2 > 0$ i.e., if both q_1 and q_2 are positive or both q_1 and q_2 are negative then the charges repel each other.

$\vec{F}_{12}$ is the force exerted on charge q_1 by charge q_2 and $q_1q_2 > 0$



$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}\hat{r}_{21}$$

If $\vec{F}_{21}$ is the force exerted on q_2 due to q_1

then $\vec{F}_{21} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1q_2}{r^2}\hat{r}_{12}$

Both $\hat{r}_{21}$ and $\hat{r}_{12}$ have same magnitude i.e., unity and are oppositely directed

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}(-\hat{r}_{21}) = -\vec{F}_{12}$$

$$\vec{F}_{21} = -\vec{F}_{12}$$

So the forces exerted by charges on each other are equal in magnitude opposite in direction

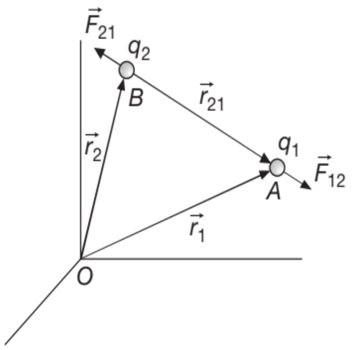
COULOMB'S LAW IN POSITION VECTOR FORM

Consider two point charges q_1 and q_2 located at points A and B respectively. According to Coulomb's Law the force $\vec{F}_{12}$ exerted on charge q_1 by q_2 is given by

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{21}^2}\hat{r}_{21}$$

Since, $\hat{r}_{21} = \dfrac{\vec{r}_{21}}{|\vec{r}_{21}|}$

$$\Rightarrow \quad \vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{21}^2}\frac{\vec{r}_{21}}{r_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{21}^3}\vec{r}_{21}$$



Applying Triangle Law of vectors to vector triangle OBA

$$\vec{r}_2 + \vec{r}_{21} = \vec{r}_1$$

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2$$

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)$$

Similarly $\vec{F}_{21} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1q_2}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1)$

If the charges are attracting each other i.e., are of opposite nature, then

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2) = -\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1)$$

and similarly for $\vec{F}_{12}$

So, if charge q_1, is lying at position vector $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and a charge q_2 is lying at position vector $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then

(a) For like charges, we have

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2) \text{ and}$$

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1)$$

(b) For unlike charges, we have

$$\vec{F}_{12} = -\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2) \text{ and}$$

$$\vec{F}_{21} = -\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1)$$

Problem Solving Technique(s)

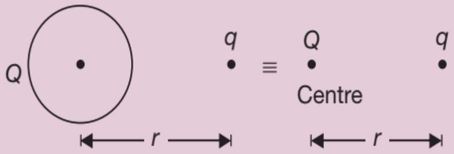
(a) The electrostatic force (F) is central (it is directed along the line joining the centres of two charges) and spherically symmetric (it is a function only of r).

(b) F is a force obeying the inverse Square Law.

(c) While applying Coulomb's Law, *I* must remind you to remember the following two points.

Firstly, the charges are assumed to be at rest. (As we shall see later that charges in motion also produce and experience magnetic forces).

Secondly, the charges are assumed to be point particles. However, when the charge is distributed **uniformly over a spherical surface**, then force on a point charge outside the surface may be computed from Coulomb's Law by treating the charge on the sphere as if it were concentrated at its centre and so



Furthermore, if the dimensions of two charged bodies are small in comparison to their separation, then too Coulomb's Law (as applied for point charges) will provide an approximate value for the force between them. In all other cases, integral calculus finds its applications by suitably applying the concepts of charge distributions (linear charge distributions, surface charge distributions or volume charge distributions, discussed later).

(d) While calculating the force between the two charges from Coulomb's Law, never take into account, the sign of the two charges. The sign just indicates the nature of the force. However it is advisable to take the sign when writing Coulomb's Law in vector form.

(e) When we are to calculate the force on charge 1 due to charge 2, then we assume charge 2 to be fixed and vice versa, unless and otherwise stated.

$$\left|\vec{F}_{12}\right| = m_1 a_1$$

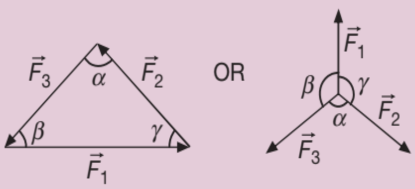
$$\left|\vec{F}_{21}\right| = m_2 a_2$$

i.e., $m_1 a_1 = m_2 a_2$ (by Newton's Third Law)

where m_1 and m_2 are the masses of particles having charges q_1 and q_2 respectively.

(f) When two identical bodies having charges q_1 and q_2 respectively are brought in contact and separated, then the charge on each body is $\dfrac{q_1+q_2}{2}$.

(g) Some problems can also be solved by using the concept of Lami's Theorem, according to which "if a body is in equilibrium under the influence of three concurrent forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ i.e., $\vec{F}_1+\vec{F}_2+\vec{F}_3=\vec{0}$, then



$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

(h) For two charges q_1 and q_2 having position vectors $\vec{r}_1$ and $\vec{r}_2$ the electrostatic force $\vec{F}_{12}$ is given by the expression

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\left|\vec{r}_1-\vec{r}_2\right|^3}\left(\vec{r}_1-\vec{r}_2\right)$$

and $$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\left|\vec{r}_2-\vec{r}_1\right|^3}\left(\vec{r}_2-\vec{r}_1\right)$$

with q_1 and q_2 substituted in the above expressions along with the nature of the charge signified by the signs on them.

(i) If we are to find the electrostatic force on a charge q_0 at a point P, due to a charge distribution (may be uniform or non uniform), then we calculate the force on q_0 due to an infinitesimal element of the distribution and then integrate it within appropriate limits.

ILLUSTRATION 2

Two point charges Q_1 and Q_2 are 3 m apart and their combined charge is $20\ \mu$C.

(a) If one repels other with force of 0.075 N. Calculate the two charges.

(b) If one attracts the other with force of 0.525 N. Find the magnitude of the charges.

SOLUTION

(a) Since combined charge of Q_1 and Q_2 is

$$Q_1 + Q_2 = 20\ \mu\text{C} \quad \ldots(1)$$

The force is repulsive, so according to Coulomb's Law

$$0.075 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2} = 9\times10^9\left[\frac{Q_1 Q_2}{(3)^2}\right]$$

$$\Rightarrow\ Q_1 Q_2 = 75\times10^{-12}\,\text{C}^2 = 75\ \mu\text{C}^2$$

Substituting value of Q_2 from (1)

$$Q_1\left(20-Q_1\right) = 75$$

$$\Rightarrow\ Q_1^2 - 20Q_1 + 75 = 0$$

$$\Rightarrow\ Q_1 = 15, 5$$

So the Q's are $5\ \mu$C and $15\ \mu$C

(b) Force is attractive, so one charge is negative so force equation is

$$-0.525 = 9\times10^{9}\frac{Q_1Q_2}{9}$$

$$\Rightarrow \quad Q_1Q_2 = -525\ \mu C^2$$

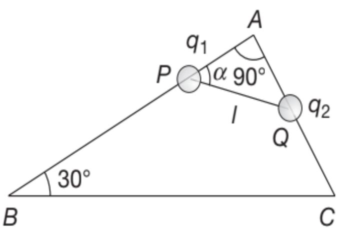
$$\Rightarrow \quad Q_1(20-Q_1) = -525$$

$$\Rightarrow \quad Q_1^2 - 20Q_1 - 525 = 0$$

From here Q values are $35\ \mu C$ and $-15\ \mu C$ or $-35\ \mu C$ and $15\ \mu C$

ILLUSTRATION 3

A rigid insulated wire frame in the form of a right-angled triangle ABC is set in a vertical plane as shown in the figure. Two beads of equal masses m each and carrying the charges q_1 and q_2 are connected by a cord of length l and can slide without friction on the wires. Considering the case when the beads are stationary determine



(a) the angle α

(b) the tension in the cord and

(c) the normal reaction on the beads. If the cord is now cut, what are the values of the charges for which the beads continue to remain stationary?

SOLUTION

(a) The forces acting on bead at P having charge q_1 are

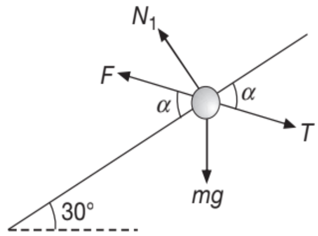
(i) Weight mg acting vertically downward

(ii) Tension T in the string along the length PQ

(iii) The electric force $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{l^2}$ between the beads along the length PQ

(iv) Normal reaction N_1 of wire on bead.

For P



For bead P to be in equilibrium,

$$mg\cos60^\circ = (T-F)\cos\alpha \qquad \ldots(1)$$

and $N_1 = mg\cos30^\circ + (T-F)\sin\alpha \qquad \ldots(2)$

For bead Q to be in equilibrium,

$$mg\sin60^\circ = (T-F)\sin\alpha \qquad \ldots(3)$$

and $N_2 = mg\cos60^\circ + (T-F)\cos\alpha \qquad \ldots(4)$

Dividing (3) by (1)

$$\tan60^\circ = \tan\alpha$$

Hence, $\alpha = 60^\circ$

(b) Hence, From (3),

$$T - F = mg$$

$$\Rightarrow \quad T = mg + F = mg + \frac{q_1q_2}{4\pi\varepsilon_0 l^2}$$

From (2) and (4), we have

$$N_1 = mg\cos30^\circ + (T-F)\sin60^\circ$$

$$\Rightarrow \quad N_1 = mg\cos30^\circ + mg\sin60^\circ = \sqrt{3}\ mg$$

and $N_2 = mg\sin30^\circ + mg\cos60^\circ = mg$

(c) If string is cut, Tension $T = 0$

Hence we have, $\frac{-q_1q_2}{4\pi\varepsilon_0 l^2} = mg$

$$q_1q_2 = -4\pi\varepsilon_0 mgl^2$$

Hence if the beads are to remain in equilibrium after the string is cut, q_1 and q_2 should have opposite charges satisfying the above condition.

ILLUSTRATION 4

A particle of mass m carrying charge q_1 is revolving around a fixed charge $-q_2$ in a circular path of radius r. Calculate the period of revolution and its speed also.

SOLUTION

$$\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} = mr\omega^2 = \frac{4\pi^2 mr}{T^2}$$

$$\Rightarrow \quad T^2 = \frac{(4\pi\varepsilon_0)r^2(4\pi^2 mr)}{q_1q_2}$$

$$\Rightarrow \quad T = 4\pi r\sqrt{\frac{\pi\varepsilon_0 mr}{q_1q_2}}$$

Since, $\frac{q_1q_2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r}$

$$\Rightarrow \quad v = \sqrt{\frac{q_1q_2}{4\pi\varepsilon_0 mr}}$$

ILLUSTRATION 5

What would be the interaction force between two copper spheres, each of mass 1 g, separated by the distance 1 m, if the total electronic charge in them differed from the total charge of the nuclei by one percent?

SOLUTION

Total number of atoms in the sphere of mass $1\text{ g} = \frac{1}{63.54}\times 6.023\times 10^{23}$

So the total nuclear charge

$$Q = \frac{6.023\times 10^{23}}{63.54}\times 1.6\times 10^{-19}\times 29$$

Now the charge on the sphere (q) = Total nuclear charge − Total electronic charge

$$\Rightarrow \quad q = \frac{6.023\times 10^{23}}{63.54}\times 1.6\times 10^{-19}\times\frac{29\times 1}{100} = 4.298\times 10^2\text{ C}$$

Hence force of interaction between these two spheres,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{(4.398\times 10^2)^2}{1^2}\text{N}$$

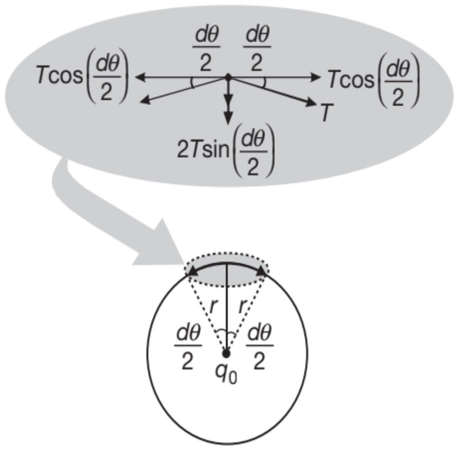
$$F = 9\times 10^9\times 19.348\text{ N} = 1.74\times 10^{15}\text{ N}$$

ILLUSTRATION 6

A thin wire ring of radius r has an electric charge q. What will be the increment of the force stretching the wire if a point charge q_0 is placed at the ring's centre?

SOLUTION

Let us consider an element of arc length dl having a charge dq. Then $dq = \left(\frac{q}{2\pi r}\right)dl$.



If dF is the force of repulsion between the element and the charge q_0 at the centre, then

$$dF = \frac{1}{4\pi\varepsilon_0}\frac{q_0 dq}{r^2} \qquad \text{(radially outwards)}$$

For equilibrium to be there

$$dF = 2T\sin\left(\frac{d\theta}{2}\right)$$

$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\frac{q_0 dq}{r^2} \simeq 2T\left(\frac{d\theta}{2}\right)$$

$$\left\{\because \text{ for small angle, } \sin\left(\frac{d\theta}{2}\right) \simeq \frac{d\theta}{2}\right\}$$

$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\frac{q_0\left(\frac{q}{2\pi r}\right)dl}{r^2} = T\left(\frac{dl}{r}\right)$$

$$\Rightarrow \quad T = \frac{qq_0}{8\pi^2\varepsilon_0 r^2}$$

ILLUSTRATION 7

Two small equally charged spheres, each of mass m, are suspended from the same point by light silk threads of length l. The separation between the spheres is $x \ll l$. Calculate the rate $\frac{dq}{dt}$ with which the charge leaks off each sphere if their velocity of approach varies as $v = \frac{\alpha}{\sqrt{x}}$, where α is a positive constant.

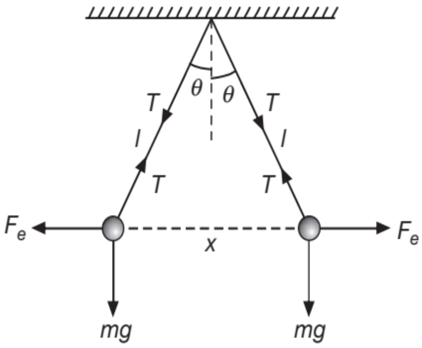
SOLUTION

In this problem, let us first calculate the value of x in terms of other known parameters. Let us consider the particles to be in equilibrium, then

$$T\cos\theta = mg \quad \ldots(1)$$

$$T\sin\theta = \frac{q^2}{4\pi\varepsilon_0 x^2} \quad \ldots(2)$$

$$\Rightarrow \quad \tan\theta = \frac{q^2}{4\pi\varepsilon_0 mgx^2}$$



Since $x \ll l$

$\therefore$ $\tan\theta$ is very small and hence $\tan\theta \simeq \theta = \dfrac{x}{2l}$

$$\Rightarrow \quad \frac{x}{2l} = \frac{q^2}{4\pi\varepsilon_0 mg\,x^2}$$

$$\Rightarrow \quad x^3 = \frac{q^2 l}{2\pi\varepsilon_0 mg} \quad \ldots(3)$$

$$\Rightarrow \quad x = \left(\frac{q^2 l}{2\pi\varepsilon_0 mg}\right)^{\frac{1}{3}}$$

Take derivative of (3) w.r.t. time, we get

$$3x^2\frac{dx}{dt} = \left(\frac{l}{2\pi\varepsilon_0 mg}\right)2q\left(\frac{dq}{dt}\right)$$

But according to the problem, $v = \dfrac{dx}{dt} = \dfrac{\alpha}{\sqrt{x}}$

$$\Rightarrow \quad 3x^2\frac{\alpha}{\sqrt{x}} = \left(\frac{x^3}{q^2}\right)2q\left(\frac{dq}{dt}\right)$$

$$\Rightarrow \quad \frac{dq}{dt} = \frac{3\alpha}{2}\left(\frac{q}{x^{3/2}}\right)$$

But $\dfrac{x^3}{q^2} = \dfrac{l}{2\pi\varepsilon_0 mg}$

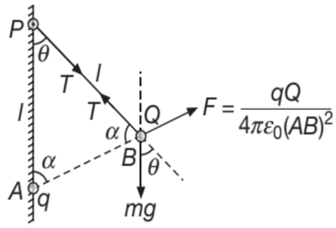
$$\Rightarrow \quad \frac{dq}{dt} = \frac{3\alpha}{2}\sqrt{\frac{2\pi\varepsilon_0 mg}{l}}$$

ILLUSTRATION 8

A particle A having a charge q is fixed on a vertical (insulated) wall. A second particle B of mass m, charge Q is suspended by a silk thread of length l from a point P on the wall that is at a distance l above the A. Calculate the angle made by the thread with the vertical, when B stays in equilibrium.

SOLUTION

Before starting the problem, let us draw a diagram that gives a visualisation of the problem.



Let the angle of suspension be θ. If $\angle PAB = \alpha$, then $\angle PBA = \alpha$ (because angles opposite to equal sides are equal). Also, in ΔPAB

$$2\alpha + \theta = \pi$$

$$\Rightarrow \quad \alpha = \frac{\pi}{2} - \frac{\theta}{2} \quad \ldots(1)$$

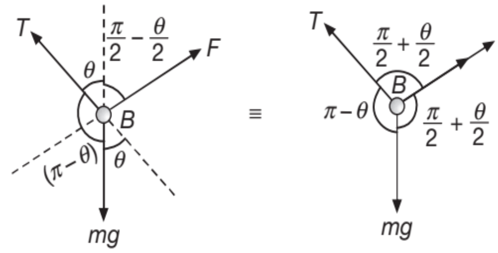
Further from the diagram, we observe three forces to be acting on B,

(a) Weight mg (acting vertically downwards)

(b) Tension T (acting along QP)

(c) Coulombic force $F = \dfrac{Qq}{4\pi\varepsilon_0 (AB)^2}$ along AB

Free body diagram of B is shown here



Now, we can make the use of Lami's Theorem and get

$$\frac{mg}{\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)}=\frac{F}{\sin(\pi-\theta)}$$

$$\Rightarrow \quad \frac{mg}{\cos\left(\frac{\theta}{2}\right)}=\frac{F}{\sin\theta} \qquad \ldots(2)$$

where $F=\frac{Qq}{4\pi\varepsilon_0(AB)^2}$

In triangle PAB

$$\frac{l}{\sin\alpha}=\frac{AB}{\sin\theta}$$

$$\Rightarrow \quad \frac{l}{\cos\left(\frac{\theta}{2}\right)}=\frac{AB}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} \qquad \left\{\because \alpha=\frac{\pi}{2}-\frac{\theta}{2}\right\}$$

$$\Rightarrow \quad AB=2l\sin\left(\frac{\theta}{2}\right) \qquad \ldots(3)$$

$$\Rightarrow \quad \frac{mg}{\cos\left(\frac{\theta}{2}\right)}=\frac{Qq}{4\pi\varepsilon_0\left(4l^2\sin^2\left(\frac{\theta}{2}\right)\right)\sin\theta}$$

$$\Rightarrow \quad mg=\frac{Qq}{32\pi\varepsilon_0 l^2\sin^3\left(\frac{\theta}{2}\right)}$$

$$\left\{\because \sin\theta=2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right\}$$

$$\Rightarrow \quad \sin^3\left(\frac{\theta}{2}\right)=\frac{Qq}{32\pi\varepsilon_0 mgl^2}$$

$$\Rightarrow \quad \sin\left(\frac{\theta}{2}\right)=\left(\frac{Qq}{32\pi\varepsilon_0 mgl^2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \theta=2\sin^{-1}\left[\left(\frac{Qq}{32\pi\varepsilon_0 mgl^2}\right)^{\frac{1}{3}}\right]$$

ILLUSTRATION 9

Three particles, each of mass 1 g and carrying a charge q are suspended from a common point by three insulated massless strings, each 100 cm long. If the particles are in equilibrium and are located at the corners of an equilateral triangle of side length 3 cm, calculate the charge q on each particle. (Take $g=10\ \text{ms}^{-2}$).

SOLUTION

After drawing the diagram and applying Coulomb's Law, we get

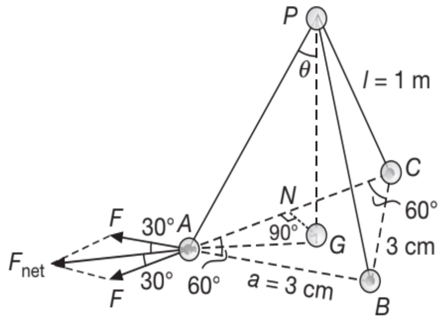
$$F=F_{AB}=F_{BC}=F_{AC}=\frac{q^2}{4\pi\varepsilon_0 a^2}$$

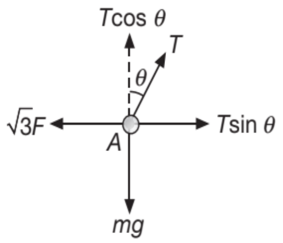
Also,

$$F_{\text{net}}=\sqrt{F^2+F^2+2F^2\cos 60}$$

$$\Rightarrow \quad F_{\text{net}}=\sqrt{3}F=\frac{\sqrt{3}q^2}{4\pi\varepsilon_0 a^2}$$

Let us now draw the free body diagram for A. The following forces are acting on A (see figure).





(a) Tension T in the thread AP acting upwards (as shown)

(b) Force $\sqrt{3}F$, where $F=\frac{q^2}{4\pi\varepsilon_0 a^2}$

(c) Weight mg (vertically down)

For A to be in equilibrium, we get

$$T\sin\theta=\sqrt{3}F$$

$$T\cos\theta=mg$$

$$\Rightarrow \quad \tan\theta=\frac{\sqrt{3}F}{mg} \qquad \ldots(1)$$

But since, we observe from the data provided in the problem, that $AP \gg AB$. So θ must be very small and hence

$$\tan\theta \simeq \theta=\frac{AG}{AP}$$

So, now we need to find AG (where G denotes the centroid of the equilateral triangle ABC).

In ΔAGN,

$$\cos 30 = \frac{AN}{AG}$$

$$\Rightarrow \quad AG = \frac{\left(\frac{a}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{a}{\sqrt{3}} \qquad \left\{\because AN = \frac{a}{2}\right\}$$

So, from equation (1), we get

$$\frac{\frac{a}{\sqrt{3}}}{l} = \frac{\sqrt{3}\frac{q^2}{4\pi\varepsilon_0 a^2}}{mg}$$

$$\Rightarrow \quad q^2 = \frac{4\pi\varepsilon_0 mga^3}{3l}$$

$$\Rightarrow \quad q^2 = \frac{1}{9\times10^9}\times\frac{\left[\frac{1}{1000}\times10\times\left(\frac{3}{100}\right)^3\right]}{3(1)}$$

$$\left\{\begin{aligned}\because m &= \frac{1}{1000}\text{ kg}\\ a &= \frac{3}{100}\text{ m}\end{aligned}\right\}$$

$$\Rightarrow \quad q^2 = 10^{-17} = 10\times10^{-18}$$

$$\Rightarrow \quad q = 3.17\times10^{-9}\text{ C}$$

Test Your Concepts-I

Based on Coulomb's Law

(Solutions on page H.3)

1. Two similar point charges q_1 and q_2 are placed at a distance r apart in air. A dielectric slab of thickness $t(\ll r)$ having dielectric constant K is placed between the charges. Calculate the Coulomb force of repulsion between the charges. Now assume that a slab of thickness half the separation between the charges is placed between the charges and the Coulomb's repulsive force is reduced in the ratio $9:4$. Calculate K for such a slab.
2. A copper atom consists of copper nucleus that has 29 electrons surrounding it. The atomic weight of copper is 63.5 gmol^{-1}. Now take two pieces of copper each weighing 10 g and transfer an electron from one piece to other for every 1000 atoms in that piece. Calculate the Coulomb force between the two pieces after the transfer of electrons, if the pieces are 1 cm apart. Given $N_A = 6.0\times10^{23}\text{ mol}^{-1}$, $e = 1.6\times10^{-19}$ C.
3. A wire of length L is placed along x-axis with one end at the origin. The linear charge density of the wire varies with distance x from the origin as $\lambda = \lambda_0\left(\frac{x^2}{L}\right)$, where λ_0 is a positive constant. Determine the total charge Q on the rod.
4. Two identical balls each having a density ρ are suspended from a common point by two insulating strings of equal length. Both the balls have equal mass and charge. In equilibrium each string makes an angle θ with vertical. Now, both the balls are immersed in a liquid. As a result of immersion in the liquid the angle θ does not change. The density of the liquid is σ. Find the dielectric constant of the liquid.
5. Calculate the ratio of the electrostatic force to the gravitation force between two electrons, between two protons. At what value of the specific charge (ratio of charge to mass) of a particle would these forces become equal (in their absolute values) in the case of interaction of identical particles?
6. A positive point charge $50\ \mu$C is located in the plane xy at the point with radius vector $\vec{r}_0 = 2\hat{i} + 3\hat{j}$, where $\hat{i}$ and $\hat{j}$ are the unit vectors of the x and y axes. Find the electrostatic force $\vec{F}$ and its magnitude on a charge of $2\ \mu$C placed at the point with radius vector $\vec{r} = 8\hat{i} - 5\hat{j}$. Here $\vec{r}_0$ and $\vec{r}$ are expressed in metre.
7. A charge $q = 1\ \mu$C is placed at a point $P(1,\ 2,\ -4)$m. Find the electric force on a $-17\ \mu$C charge at point $Q(4,\ 6,\ -16)$m to the nearest approximation.
8. A ring of radius 0.1 m is made out of a thin metallic wire of area of cross-section 10^{-6} m^2. The ring has

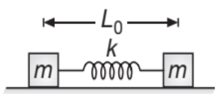
a uniform charge of π coulomb. Find the change in the radius of the ring when a charge of 10^{-8} coulomb is placed at the centre of the ring. Young's modulus of the metal is 2×10^{11} Nm^{-2}.

9. Three identical small balls, each of mass 0.1 g, are suspended at one point on silk threads having a length of $\ell = 20$ cm. What charge should be imparted to each ball so that every thread makes an angle of $\alpha = 30°$ with the vertical?

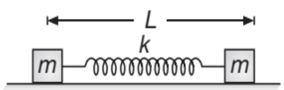
10. A small point mass m has a charge q, which is constrained to move inside a narrow frictionless cylinder. At the base of the cylinder is a point mass of charge Q having the same sign as q. Show that if the mass m is displaced by a small amount from its equilibrium position and released, it will exhibit simple harmonic motion with angular frequency $\omega = \sqrt{\frac{2g}{\ell}}$ where ℓ is the equilibrium position of charge q.

11. A small bead of mass m, charge $-q$ is constrained to move along a frictionless wire. A positive charge Q lies at a distance L from the wire. Initially the bead constrained to move along the wire is just above $+Q$. Now if the bead is displaced a distance x, where $x \ll L$, and released, it will exhibit simple harmonic motion. Obtain an expression for the time period of simple harmonic motion of the bead.

12. Two identical metallic blocks resting on a frictionless horizontal surface are connected by a light metallic spring having a spring constant k and an unstretched length L_0. A total charge Q is slowly placed on the system, causing the spring to stretch to an equilibrium length L, as shown. Determine the value of Q, assuming that all the charge resides on the blocks and assuming the blocks as point charges.



INITIALLY



FINALLY

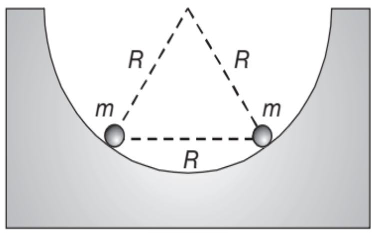
13. Two identical oppositely charged metallic spheres placed 0.5 m apart, attract each other with a force of 0.108 N. When connected to each other by a copper wire for a short while, they begin to repel each other with a force of 0.036 N. Find the initial charge on each one of them.

14. A charge of 50 μC has been distributed between two spheres, such that they repel each other with a force of 1 N with their centres 2 m apart. Find the distribution.

15. In an H atom, an electron revolves round a proton in a circular orbit of 0.53 Å. Calculate the radial acceleration, the angular velocity and the period of revolution of the electron.

16. Show that the gravitational force is negligible in comparison to the electrostatic force in an Hydrogen atom in which an electron is revolving around the proton in a circle of radius 0.53 Å.

17. Two identical beads each have a mass m and charge q. When placed in a hemispherical bowl of radius R with frictionless, non-conducting walls, the beads move, and at equilibrium they are a distance R apart (shown in figure). Determine the charge on each bead.

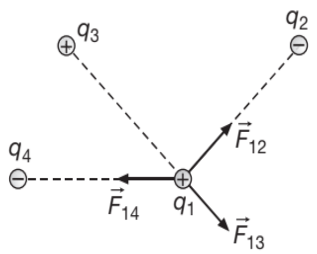


18. Two particles each of charge 10^{-7} C and mass 5 g, stay in limiting equilibrium on a horizontal surface. The particles have a separation of 10 cm between them. Assume the coefficient of friction between each particle and the table to be μ. Calculate μ.

19. A charge Q is divided into two parts. Calculate the ratio of the charges on the divided parts such that force between the divided parts is the maximum.

PRINCIPLE OF SUPERPOSITION

This principle is used to calculate the force on a charge due to an assembly of charges placed near it. According to this principle, "the net force on a charge due to the others in the assembly is calculated by taking the vector sum of all the forces acting on that charge, one at a time". Figure below shows an arrangement of four interacting charged particles.



So, net force on q_1, according to the principle is

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}$$

In general, the net force on q_1 due to assembly of N charges is

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \dots\dots\dots + \vec{F}_{1N}$$

Here, we must note one thing that force between any pair of charges is simply not affected by the presence of remaining others.

Also, this principle will also be used to find the electric field due to an assembly of charges at a point.

Problem Solving Technique(s)

Suppose we wish to calculate the net force on q_1 due to q_2, q_3 and q_4 (in the arrangement already shown), then we follow the following series of steps.

STEP-1: Find out whether the force due to a given charge on q_1 is attractive or repulsive. Draw the force vector with its tail at q_1, either towards q_1 (in case of repulsion) or away from q_1 (in case of attraction).

STEP-2: Find out the magnitude of each force, using Coulomb's Law. Do not take into account the sign of the charges (the signs account for the direction of force on q_1, already that has been taken into account in Step 1). For this step to be completed, you will also require the separation between q_1 and others i.e., r_{12}, r_{13}, r_{14} etc., then

$$F_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2},\ F_{13} = \frac{q_1 q_3}{4\pi\varepsilon_0 r_{13}^2},\ F_{14} = \frac{q_1 q_4}{4\pi\varepsilon_0 r_{14}^2}$$

STEP-3: Now select a convenient coordinate axis, so as to calculate the components of these forces (say x and y components). Then in 2D space (say xy plane),

$$F_{1x} = (F_{12})_x + (F_{13})_x + (F_{14})_x$$

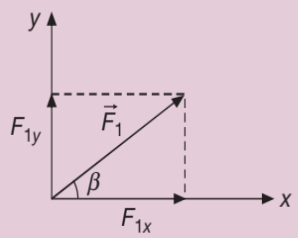
and $$F_{1y} = (F_{12})_y + (F_{13})_y + (F_{14})_y$$

Here, please note that all vectors must be expressed in terms of unit vectors $(\hat{i}, \hat{j} \text{ or } \hat{k})$ notations, unless otherwise indicated.

STEP-4: As a final step, if we wish to calculate the magnitude of the net force on q_1, then

$$F_1 = \sqrt{(F_{1x})^2 + (F_{1y})^2}$$

where $$\vec{F}_1 = (F_{1x})\hat{i} + (F_{1y})\hat{j}$$



This vector can be diagramatically expressed as

So, if we say that, let $\vec{F}_1$ makes an angle β with x-axis, then the direction is given by

$$\tan\beta = \frac{F_{1y}}{F_{1x}}$$

STEP-5: If need be, then we can extend the same to 3D space, where we modify the equations as

$$F_{1x} = (F_{12})_x + (F_{13})_x + (F_{14})_x$$

$$F_{1y} = (F_{12})_y + (F_{13})_y + (F_{14})_y$$

$$F_{1z} = (F_{12})_z + (F_{13})_z + (F_{14})_z$$

and then

$$F_1 = \sqrt{F_{1x}^2 + F_{1y}^2 + F_{1z}^2}$$

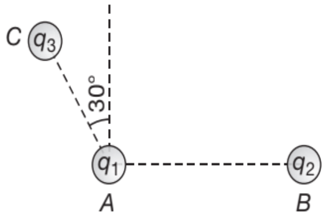
where

$$\vec{F}_1 = (F_{1x})\hat{i} + (F_{1y})\hat{j} + (F_{1z})\hat{k}$$

To have a basic understanding of all the steps, see the following Illustrative Examples.

ILLUSTRATION 10

Figure below shows three charges q_1, q_2 and q_3 placed at A, B and C respectively. If $q_1 = -1\ \mu C$, $q_2 = +3\ \mu C$, $q_3 = -2\ \mu C$, AB = 15 cm and $AC = 10$ cm. Calculate the net force that acts on q_1 placed at A.

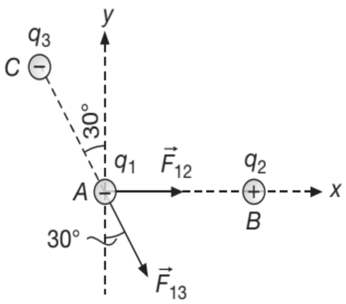


SOLUTION

Here we shall use the concept of Principle of Superposition along with the problem solving strategy we have learnt already.
Before we start, let us consider the origin to be at A and the reconstruction of the diagram is shown.
So, we have drawn/shown all the forces that are acting on the charge q_1.

$$F_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2} = \frac{(9\times10^9)(10^{-6})(3\times10^{-6})}{(15\times10^{-2})^2} = 1.2\ \text{N}$$

$\Rightarrow$ $F_{12} = 1.2$ N (attractive in nature)



Similarly

$$F_{13} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{13}^2}$$

$$\Rightarrow \quad F_{13} = \frac{(9\times10^9)(10^{-6})(2\times10^{-6})}{(10\times10^{-2})^2} = 1.8\ \text{N}$$

$\Rightarrow$ $F_{13} = 1.8$ N (repulsive in nature)

Since both the forces have different directions, so we shall apply principle of superposition in component form along x and y-axis.

$$\Rightarrow \quad F_{1x} = (F_{12})_x + (F_{13})_x = F_{12} + F_{13}\sin(30°)$$

and $$F_{1y} = (F_{12})_y + (F_{13})_y = 0 - F_{13}\cos(30°)$$

$$\Rightarrow \quad F_{1x} = 1.2 + 1.8\sin(30°)$$

$$\Rightarrow \quad F_{1x} = 2.1\ \text{N}$$

and $$F_{1y} = -1.8\frac{\sqrt{3}}{2} = -1.6\ \text{N}$$

(Negative sign shows that F_{1y} acts in the downward direction)

$$\Rightarrow \quad \vec{F}_1 = (2.1\hat{i} - 1.6\hat{j})\ \text{N}$$

and $$|\vec{F}_1| = \sqrt{(2.1)^2 + (1.6)^2} = \sqrt{4.41 + 2.56} = 2.64\ \text{N}$$

ILLUSTRATION 11

Four identical free charges each of value q are located at the corners of a square of side a. What must be the charge Q that has to be placed at the centre of the system so that the system stays in equilibrium?

SOLUTION

Let us here consider the equilibrium of any charge q at 1 (say).
For equilibrium of charge at 1, we have

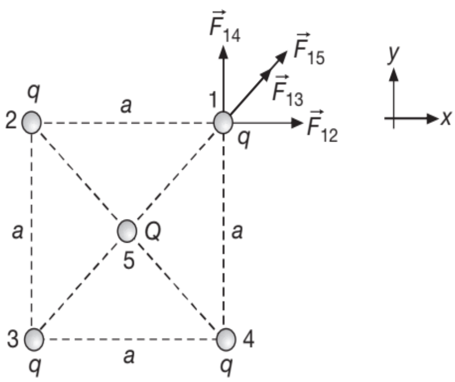
$$\sum \vec{F}_1 = \vec{0}$$

$$\Rightarrow \quad \sum F_{1x} = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad \sum F_{1y} = 0 \qquad \ldots(2)$$

Out of these two equations any one can be selected as both of these will give identical results. So, lets take

$$\sum F_{1x} = 0$$



Now, $F_{1x} = \sum F_{1x} = (F_{12})_x + (F_{13})_x + (F_{14})_x + (F_{15})_x$

$$\Rightarrow \quad F_{1x} = F_{12} + F_{13}\cos(45^\circ) + 0 + F_{15}\cos(45^\circ)$$

$$\Rightarrow \quad F_{1x} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{a^2} + \frac{q^2}{(\sqrt{2}a)^2}\frac{1}{\sqrt{2}} + \frac{Qq}{\left(\frac{a}{\sqrt{2}}\right)^2}\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad F_{1x} = \frac{1}{4\pi\varepsilon_0 a^2}\left(q^2 + \frac{q^2}{2\sqrt{2}} + \sqrt{2}Qq\right)$$

So, for $F_{1x} = 0$, we get

$$q\left(1 + \frac{1}{2\sqrt{2}}\right) + \sqrt{2}Q = 0$$

$$\Rightarrow \quad Q = -(1 + 2\sqrt{2})\frac{q}{4}$$

We shall get the same result even if we would have taken $F_{1y} = 0$ $\left(\text{or } \sum F_{1y} = 0\right)$

EQUILIBRIUM OF THREE CHARGES

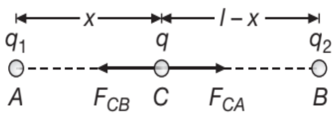
Equilibrium of q when q_1 and q_2 are fixed at a separation l from each other.

CASE-1: q_1 and q_2 similar in nature

Here, the point C, where the net force due to q_1 and q_2 on q will be zero must lie between the charges q_1 and q_2 on the line joining them, towards the charge of smaller magnitude.

If $AC = r_1$, $CB = r_2$ and $AB = l$ (given)

then $|\vec{F}_{CA}| = |\vec{F}_{CB}|$



$$\Rightarrow \quad \frac{q_1 q}{4\pi\varepsilon_0 r_1^2} = \frac{q_2 q}{4\pi\varepsilon_0 r_2^2}$$

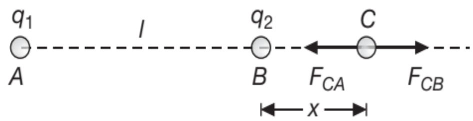
$$\Rightarrow \quad \frac{q_1}{q_2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{x}{l-x}\right)^2$$

CASE-2: q_1 and q_2 opposite in nature and $|q_2| < |q_1|$

Here the point C, where the net force due to q_1 and q_2 on q will be zero must lie on the line joining the two charges q_1 and q_2 but outside them towards the charge of smaller magnitude i.e. q_2.

If again $AC = r_1$, $BC = r_2$ and $AB = l$ (given), then

$$|\vec{F}_{CA}| = |\vec{F}_{CB}|$$



$$\Rightarrow \quad \frac{q_1 q}{4\pi\varepsilon_0 r_1^2} = \frac{q_1 q}{4\pi\varepsilon_0 r_2^2}$$

$$\Rightarrow \quad \frac{q_1}{q_2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{l+x}{x}\right)^2$$

Conceptual Note(s)

Electrostatic Equilibrium

The point where the resultant force on a charged particle becomes zero is called equilibrium position.

(a) Stable Equilibrium: A charge is initially in equilibrium position and is displaced by a small distance. If the charge tries to return back to the same equilibrium position then this equilibrium is called position of stable equilibrium.

(b) Unstable Equilibrium: If charge is displaced by a small distance from its equilibrium position and the charge has no tendency to return to the same equilibrium position. Instead it goes away from the equilibrium position.

(c) Neutral Equilibrium: If charge is displaced by a small distance and it is still in equilibrium condition then it is called neutral equilibrium.

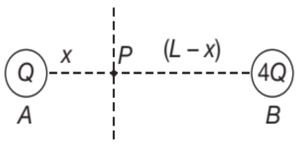
However if, q_1 and q_2 are not fixed, then equilibrium of q will exactly be the same as done before, but if we have to keep q_1 and q_2 also in equilibrium, then we have to make the net force on q_1 (at A) due to q_2 (at B) and q (at C) to be zero and then repeat the same for q_2 also to get the desired results. See ***Illustration(s)*** *below to learn how to perform in these kind of problems.*

ILLUSTRATION 12

Two free charges $+Q$ and $+4Q$ are placed at a separation L. Find the magnitude, sign and the location of the third charge that makes the system to stay in equilibrium.

SOLUTION

Since both the charges have similar nature, so irrespective of a nature of third charge, the location of the third charge must be somewhere between $+Q$ and $+4Q$ on the line joining them. Let the net force be zero at P at a distance x from $+Q$.



$\Rightarrow \quad |\vec{F}_{PA}| = |\vec{F}_{PB}|$

$\Rightarrow \quad \dfrac{Qq}{4\pi\varepsilon_0 x^2} = \dfrac{(4Q)q}{4\pi\varepsilon_0 (L-x)^2}$

$\Rightarrow \quad \dfrac{L-x}{x} = \pm 2$

$\dfrac{L-x}{x} = 2$	$\dfrac{L-x}{x} = -2$
$\Rightarrow \quad L = 3x$	$\Rightarrow \quad L - x = -2x$
$\Rightarrow \quad x = \dfrac{L}{3}$	$\Rightarrow \quad x = -L$ But this coordinate lies outside A and B so, this value of x is REJECTED.

$\Rightarrow \quad q$ must be placed at $\dfrac{L}{3}$ from Q or $\dfrac{2L}{3}$ from $4Q$.

To find the charge q, such that the system is in equilibrium, we proceed further by taking either

$$\sum F_A = 0 \quad \text{OR} \quad \sum F_B = 0$$

So let us take

$$\sum F_A = 0$$

$\Rightarrow \quad F_{AP} + F_{AB} = 0$

$\Rightarrow \quad \dfrac{Qq}{4\pi\varepsilon_0 \left(\dfrac{L}{3}\right)^2} + \dfrac{(4Q)Q}{4\pi\varepsilon_0 L^2} = 0$

$\Rightarrow \quad 9q + 4Q = 0$

$\Rightarrow \quad q = -\dfrac{4Q}{9}$

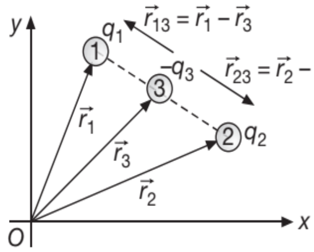
- Please note that we could also have taken $\sum F_B = 0$, to get the same result.
- Also, note that even if $q = +\dfrac{4Q}{9}$, then still it will be in equilibrium, but then the system would not be in equilibrium.

ILLUSTRATION 13

Two positive charges q_1 and q_2 are located at points 1 and 2 with radius vectors $\vec{r}_1$ and $\vec{r}_2$. Find a negative charge q_3 and the radius vector $\vec{r}_3$ of a point 3 at which it has to be placed for the force acting on each of the three charges to be equal to zero.

SOLUTION

Before we start the problem, we must keep in mind that the radius vector of any point P is actually the position vector (PV) of the point P which is the vector drawn from the origin to the point P.
So, let the origin be at O (see figure)



For equilibrium of $-q_3$, we have

$$\vec{F}_{31} + \vec{F}_{32} = \vec{0}$$

$\Rightarrow \quad \dfrac{q_1 q_3}{4\pi\varepsilon_0 |\vec{r}_1 - \vec{r}_3|^3}(\vec{r}_1 - \vec{r}_3) + \dfrac{q_2 q_3}{4\pi\varepsilon_0 |\vec{r}_2 - \vec{r}_3|^3}(\vec{r}_2 - \vec{r}_3) = \vec{0}$ ...(1)

Also, from figure, we see that

$$\hat{r}_{13} = -\hat{r}_{23}$$

$\Rightarrow \quad \dfrac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|} = -\dfrac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2 - \vec{r}_3|}$ ...(2)

Put (2) in (1), we get

$$\frac{q_1}{|\vec{r}_1 - \vec{r}_3|^2} = \frac{q_2}{|\vec{r}_2 - \vec{r}_3|^3}$$

$$\Rightarrow \quad \pm\sqrt{q_1}(\vec{r}_2-\vec{r}_3)=\sqrt{q_2}(\vec{r}_1-\vec{r}_3)$$

Taking positive sign, we get

$$\sqrt{q_1}\vec{r}_2-\sqrt{q_2}\vec{r}_1=\left(\sqrt{q_1}-\sqrt{q_2}\right)\vec{r}_3$$

$$\Rightarrow \quad \vec{r}_3=\frac{\sqrt{q_1}\vec{r}_2-\sqrt{q_2}\,\vec{r}_1}{\sqrt{q_1}-\sqrt{q_2}}$$

This must be REJECTED as it will lie outside q_1 and q_2.

Taking negative sign, we get

$$\sqrt{q_1}\vec{r}_2+\sqrt{q_2}\vec{r}_1=\left(\sqrt{q_1}+\sqrt{q_2}\right)\vec{r}_3$$

$$\Rightarrow \quad \vec{r}_3=\frac{\sqrt{q_1}\vec{r}_2+\sqrt{q_2}\vec{r}_1}{\sqrt{q_1}+\sqrt{q_2}}$$

Clearly, this position vector or radius vector will lie on the line joining q_1 and q_2, between them.

$$\text{So, } \vec{r}_3=\frac{\sqrt{q_1}\vec{r}_2+\sqrt{q_2}\vec{r}_1}{\sqrt{q_1}+\sqrt{q_2}}$$

For q_1 to be in equilibrium, let charge placed on 3 be q_3, then

$$\vec{F}_{12}+\vec{F}_{13}=\vec{0}$$

$$\Rightarrow \quad \frac{q_1q_2}{4\pi\varepsilon_0|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)+\frac{q_1q_3}{4\pi\varepsilon_0|\vec{r}_1-\vec{r}_3|^3}(\vec{r}_1-\vec{r}_3)=\vec{0}$$

While writing the above expression, we have not taken any nature of q_3. The answer will have its sign attached to q_3. So, as of now q_3 is assumed to be positive and then lets see what happens. Also, as per the assumption made

$$\hat{r}_{12}=\hat{r}_{13}$$

$$\Rightarrow \quad \frac{q_2}{|\vec{r}_1-\vec{r}_2|^2}=-\frac{q_3}{|\vec{r}_1-\vec{r}_3|^2} \qquad \left\{\because \frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|}=\frac{\vec{r}_1-\vec{r}_3}{|\vec{r}_1-\vec{r}_3|}\right\}$$

$$\Rightarrow \quad q_3=-q_2\left|\frac{\vec{r}_1-\vec{r}_3}{\vec{r}_1-\vec{r}_2}\right|^2 \qquad \ldots(3)$$

$$\text{Now, } \vec{r}_1-\vec{r}_3=\vec{r}_1-\left(\frac{\sqrt{q_1}\vec{r}_2+\sqrt{q_2}\vec{r}_1}{\sqrt{q_1}+\sqrt{q_2}}\right)$$

$$\Rightarrow \quad \vec{r}_1-\vec{r}_3=\frac{\sqrt{q_1}\vec{r}_1+\sqrt{q_2}\vec{r}_1-\sqrt{q_1}\vec{r}_2-\sqrt{q_2}\vec{r}_1}{\sqrt{q_1}+\sqrt{q_2}}$$

$$\Rightarrow \quad \vec{r}_1-\vec{r}_3=\frac{\sqrt{q_1}(\vec{r}_1-\vec{r}_2)}{\sqrt{q_1}+\sqrt{q_2}}$$

$$\Rightarrow \quad \left|\frac{\vec{r}_1-\vec{r}_3}{\vec{r}_1-\vec{r}_2}\right|^2=\frac{q_1}{\left(\sqrt{q_1}+\sqrt{q_2}\right)^2}$$

Using (3), we get

$$\Rightarrow \quad q_3=-\frac{q_1q_2}{\left(\sqrt{q_1}+\sqrt{q_2}\right)^2}$$

ILLUSTRATION 14

Two identical positive point charges, each having a charge Q are fixed at a separation $2a$. A point charge q lies midway between the fixed charges. Show that for small

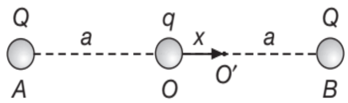
(a) displacement (compared to a) along the line joining the fixed charges, the charge q executes SHM, if it is positive in nature.

(b) lateral displacement, the charge q executes SHM, if it is negative in nature.

Compare the periods of oscillations in the above two cases.

SOLUTION

(a) When q is displaced along the line joining the fixed charges. Let the displacement of q be $x(\ll a)$.



The restoring force F acting on q is

$$F=-\frac{Qq}{4\pi\varepsilon_0}\left[\frac{1}{(a-x)^2}-\frac{1}{(a+x)^2}\right]$$

(Negative sign in the above expression indicates that F is directed towards the mean position)

$$F=-\frac{Qq}{4\pi\varepsilon_0}\frac{4ax}{(a^2-x^2)^2}$$

Since $x\ll a$

$$\therefore \quad F=-\frac{Qq}{4\pi\varepsilon_0}\frac{4x}{a^3}$$

$$\left\{\because \text{for } x\ll a,\ (a^2-x^2)^2\cong a^4\right\}$$

$$\Rightarrow \quad m\ddot{x}+\left(\frac{Qq}{\pi\varepsilon_0 a^3}\right)x=0$$

$$\Rightarrow \quad \ddot{x}+\left(\frac{Qq}{\pi m\varepsilon_0 a^3}\right)x=0$$

Compare with the standard equation of SHM, that is

$$\ddot{x}+\omega_1^2 x=0$$

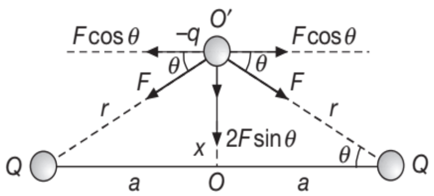
$$\Rightarrow \quad \omega_1=\sqrt{\frac{Qq}{\pi\varepsilon_0 m a^3}}$$

$$\Rightarrow \quad T_1=2\pi\sqrt{\frac{\pi\varepsilon_0 m a^3}{Qq}} \qquad \ldots(1)$$

(b) When q is replaced by $-q$ and is given a lateral displacement $OO'=x(\ll a)$. Then the forces acting on $-q$ are shown in figure.

where $F=\dfrac{Qq}{4\pi\varepsilon_0 r^2}$ and $r=\sqrt{a^2+x^2}$

The charge $-q$ is attracted towards Q with a force F (given above) on resolution, the component $F\cos\theta$ cancels, so that the net force $2F\sin\theta$ restores $-q$ back to mean position O.



$$\Rightarrow \quad F_{net}=-2F\sin\theta$$

$$\Rightarrow \quad m\ddot{x}=-2\frac{Qq}{4\pi\varepsilon_0\left(a^2+x^2\right)}\sin\theta$$

But $\sin\theta=\dfrac{x}{\sqrt{a^2+x^2}}$

$$\Rightarrow \quad F_{net}=-2\frac{Qqx}{4\pi\varepsilon_0\left(a^2+x^2\right)^{3/2}}$$

Since $x\ll a$

$$\Rightarrow \quad \left(a^2+x^2\right)^{3/2}\cong a^3$$

$$\Rightarrow \quad F_{net}=-2\left(\frac{Qq}{4\pi\varepsilon_0 a^3}\right)x$$

$$\Rightarrow \quad m\ddot{x}+\left(\frac{Qq}{2\pi\varepsilon_0 a^3}\right)x=0$$

$$\Rightarrow \quad \ddot{x}+\left(\frac{Qq}{2\pi\varepsilon_0 m a^3}\right)x=0$$

Again compare with standard equation of SHM, we get

$$\ddot{x}+\omega_2^2 x=0$$

$$\Rightarrow \quad \omega_2=\sqrt{\frac{Qq}{2\pi\varepsilon_0 m a^3}}$$

$$\Rightarrow \quad T_2=2\pi\sqrt{\frac{2\pi\varepsilon_0 m a^3}{Qq}} \qquad \ldots(2)$$

Finally, from (1) and (2), we get

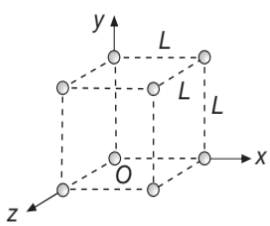
$$\frac{T_1}{T_2}=\frac{1}{\sqrt{2}}$$

Test Your Concepts-II

Based on Principle of Superposition

(Solutions on page H.7)

1. Eight charges each of magnitude Q are located at the corners of a cube of side L (see figure). One corner is at the origin and the edges lie along the rectangular axes. Calculate the net electrostatic force on the charge at $\vec{r}=L\hat{i}+L\hat{j}+L\hat{k}$.



2. Four equal positive charges, each of charge Q are arranged at the corners of a square of side ℓ. A unit positive charge of mass m is placed at point P at height h above the centre of the square. Calculate Q, so that the unit charge is in equilibrium.
3. Two equal positive point charges Q are separated by a distance $2a$. A point test charge q_0 is located in a plane normal to the line joining the two charges and midway between them. Find the radius r of a circle of symmetry in this plane for which the force on the test charge has a maximum value. Find the direction of the force on this charge if it is assumed to be positive. Also find F_{max}.
4. Three identical spheres each having a charge $2q$ and radius R are kept such that each touches the other two. Find the magnitude of the electric force on any sphere due to the other two.
5. Five point charges each of value $+q$ are placed on five vertices of a regular hexagon of side a. Calculate the magnitude of the force on a point charge $-q$ placed at the centre of the hexagon.
6. (a) Twelve equal charges, q, are situated at the corners of a regular 12 sided polygon (for instance, one on each numeral of a clock face). What is the net force on a test charge Q at the center?
   (b) Suppose one of the 12 q's is removed (say the one at "6 o'clock"). What is the force on Q? Explain your reasoning carefully.
   (Assume the distance between q and Q pair(s) to be a)
7. Four identical charges, each having a charge $+q$ are fixed at the corners of a square of side L. A fifth point charge $-Q$ lies a distance x along the line perpendicular to the plane of the square. Find the force exerted by the charges on $-Q$. Show that for $z \ll a$ the motion of $-Q$ is simple harmonic. Also calculate the time period of oscillations of $-Q$ if it has a mass m?
   (Neglect gravitational force.)
8. Consider a regular polygon with 24 sides. The distance from the center to each vertex is a. Twenty three identical charges each having a charge q are placed at the vertices of the polygon. A single charge Q is placed at the centre of the polygon experiences a force F. What is the magnitude and direction of the force experienced by the charge Q?
9. Three charges, each of magnitude 100 μC are located in vacuum at the corners A, B and C of an equilateral triangle of side length 3 m. If charges placed at A and C are positive and the one placed at B is negative, then find the magnitude and direction of force F experienced by the charge placed at C.

ELECTROSTATIC FIELD $(\vec{E})$

The region of space around a source charge (q) in which it can exert a force on a test charge (q_0).

Mathematically, electric field is the force experience per unit test charge q_0 placed in the electrostatic influence of source charge q.

$$\vec{E} = \frac{\vec{F}}{q_0}$$

Electric field strength is a vector quantity directed away from a positive charge and towards the negative charge. SI unit of electric field is newton/coulomb (NC^{-1}) or volt/metre(Vm^{-1}).

The dimensional formula for E is $MLT^{-3}A^{-1}$ i.e., $[E] = MLT^{-3}A^{-1}$

Since $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r}$

$$\Rightarrow \quad \vec{E} = \frac{\vec{F}}{q_0} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

is the electric field due to a source point charge q at a distance r from it.

Problem Solving Technique(s)

(a) For calculating the electric field due to a charge at a point *P* at distance *r* we proceed as follows.

STEP-1: Place a charge of +1 C at point P.

STEP-2: Calculate the force between q and +1 C.

STEP-3: This force is the value of E due to q at the point P.

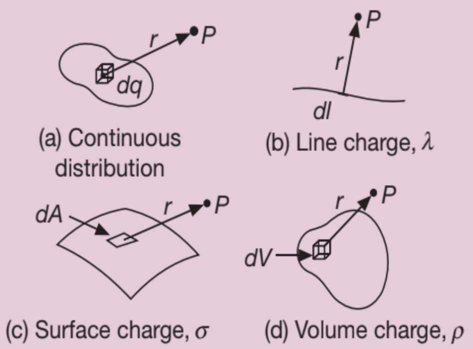
STEP-4: The direction of force experienced by a charge of +1 C is the direction of E due to q at the point P.

(b) If we are to find the electrostatic field due to assembly of charges $q_1, q_2, ..., q_n$ at a point P, then we apply Superposition Principle and then

$$\vec{E}_p = \vec{E}_1 + \vec{E}_2 + ... + \vec{E}_n$$

where we find $\vec{E}_1, \vec{E}_2, ... \vec{E}_n$ at point P by the technique mentioned above.

(c) If we are to find the electric field at a point P due to a uniform charge distribution, then we calculate the field due to an infinitesimal element of the distribution at the point P and then integrate it within appropriate limits.



$$E = \int dE = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}$$

For charge distributed on a wire with linear charge density λ,

$$E = \frac{1}{4\pi\varepsilon_0}\int_\ell \frac{\lambda d\ell}{r^2}$$

For charge distributed on a surface with surface charge density σ,

$$E = \frac{1}{4\pi\varepsilon_0}\int_A \frac{\sigma dA}{r^2}$$

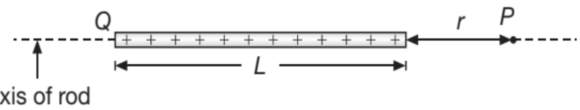
For charge distributed on a volume with volume charge density ρ,

$$E = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho dV}{r^2}$$

If the charge distributions are uniform, then charge densities can be taken out of the integral to get desired results.

ILLUSTRATION 15

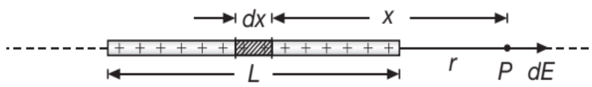
Calculate the electric field due to a uniformly charged rod of length L at a point P at distance r from one end of the rod. Assume total charge on the rod to be Q.



SOLUTION

For this we consider an infinitesimal element of length dx on rod as shown in figure. Charge on the infinitesimal elemental length dx is

$$dq = \frac{Q}{L}dx$$



This dq (an infinitesimal element) can be regarded as a point charge, hence the electric field dE due to this element at point P is dE given by

$$dE = \frac{dq}{4\pi\varepsilon_0 x^2}$$

$$\Rightarrow \quad dE = \frac{\left(\frac{Q}{L}\right)dx}{4\pi\varepsilon_0 x^2}$$

The net electric field strength at point P can be given by integrating this expression over the whole length of rod as

$$E_P = \int dE = \int_r^{r+L} \frac{Qdx}{4\pi\varepsilon_0 Lx^2}$$

$$\Rightarrow \quad E_P = \frac{Q}{4\pi\varepsilon_0 L}\int_r^{r+L} x^{-2}dx$$

$$\Rightarrow \quad E_P = \frac{Q}{4\pi\varepsilon_0 L}\left[-\frac{1}{x}\right]_r^{r+L}$$

$$\Rightarrow \quad E_P = \frac{Q}{4\pi\varepsilon_0 L}\left[\frac{1}{r}-\frac{1}{r+L}\right]$$

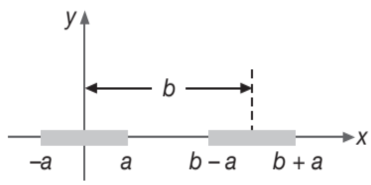
$$\Rightarrow \quad E_P = \frac{Q}{4\pi\varepsilon_0 r(r+L)}$$

SPECIAL CASE: This special case can be taken as a check point too. For $x \gg L$ i.e., for distances far away from the rod, the rod with charge Q must be have as a point charge. In this expression, when we put $r \gg L$, then $E \cong \dfrac{Q}{4\pi\varepsilon_0 r^2}$ (= field due to a point charge Q at a distance r from it).

ILLUSTRATION 16

Two identical thin rods of length $2a$ carry equal charges $+Q$ that is distributed uniformly along their lengths. The rods lie along the x-axis with their centers separated by a distance $b > 2a$. Show that the magnitude of the force exerted by the left rod on the right one is given by

$$F = \frac{Q^2}{16\pi\varepsilon_0 a^2}\log_e\left(\frac{b^2}{b^2-4a^2}\right)$$



SOLUTION

This problem is a very special one where we shall learn to calculate the force between two extended bodies using the concept of electric field. For this let us divide the problem in two parts.

PART I

Here we shall imagine the rod to the right to be absent and then let us calculate the field due to the rod on the left at a hypothetical point P (say) at a distance d from one end of the rod. Let us consider an element of length dx at a distance x from the point P. If dE be the electric field due to the rod at P, then

$$dE = \frac{dq}{4\pi\varepsilon_0 x^2} = \left(\frac{\lambda}{4\pi\varepsilon_0}\right)x^{-2}dx$$

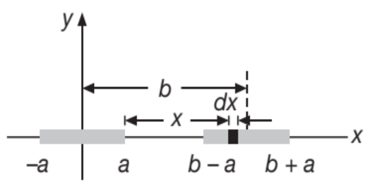
$$\Rightarrow \quad E = \frac{\lambda}{4\pi\varepsilon_0}\left(\int_d^{d+2a} x^{-2}dx\right) = \frac{\lambda(2a)}{4\pi\varepsilon_0(d+2a)}$$

$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_0 d(d+2a)}$$

PART II

Now after we have calculated the field due to one rod, we actually observe that the second rod will be lying in the field of the first rod. For finding the force on this rod let us again consider an infinitesimal element of length dx at a distance x from the nearest end of the left rod. If dF be the force due to this rod then

$$dF = Edq = \frac{Q(\lambda dx)}{4\pi\varepsilon_0 x(x+2a)}$$



Since $\lambda = \dfrac{Q}{2a}$

$$\Rightarrow \quad dF = \left(\frac{Q}{4\pi\varepsilon_0}\right)\left(\frac{Q}{2a}\right)\left[\frac{dx}{x(x+2a)}\right]$$

$$\Rightarrow \quad F = \int dF = \frac{Q^2}{8\pi\varepsilon_0 a}\int_{b-2a}^{b}\frac{dx}{x(x+2a)}$$

$$\Rightarrow \quad F = \frac{Q^2}{8\pi\varepsilon_0 a}\left[-\frac{1}{2a}\log_e\left(\frac{2a+x}{x}\right)\right]\Bigg|_{b-2a}^{b}$$

$$\Rightarrow \quad F = \frac{Q^2}{16\pi\varepsilon_0 a^2}\left[-\log_e\left(\frac{2a+b}{b}\right)+\log_e\left(\frac{b}{b-2a}\right)\right]$$

$$\Rightarrow \quad F = \frac{Q^2}{16\pi\varepsilon_0 a^2}\log_e\left[\frac{b^2}{(b-2a)(b+2a)}\right]$$

$$\Rightarrow \quad F = \frac{Q^2}{16\pi\varepsilon_0 a^2}\log_e\left(\frac{b^2}{b^2-4a^2}\right)$$

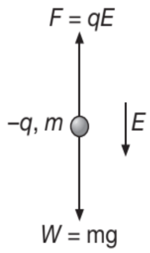
ILLUSTRATION 17

Calculate the electric field intensity which would be just sufficient to balance the weight of a particle of charge $-10\ \mu\text{C}$ and mass $10\ \text{mg}$. (Take $g = 10\ \text{ms}^{-2}$)

SOLUTION

Force on a charge q in an electric field $\vec{E}$ is

$$\vec{F} = q\vec{E}$$



According to given problem, we have

$$|\vec{F}| = |\vec{W}|$$

$$\Rightarrow \quad |q|E = mg$$

$$\Rightarrow \quad E = \frac{mg}{|q|} = 10\ \text{NC}^{-1}, \text{ in downward direction.}$$

ELECTROSTATIC LINES OF FORCE: PROPERTIES

A line of force is an imaginary path straight or curved such that the tangent to it at any point gives the direction of electrostatic field at that point. A field line is an imaginary line along which a unit positive charge would move when set free. The lines of force are drawn such that the number of lines per unit area of cross-section, (area held normally to the field lines) is proportional to magnitude of $\vec{E}$.

Pattern of Field Lines

The pattern of electric field lines can be obtained by considering the following points.

(a) Symmetry: For every point above the line joining the two charges there is an equivalent point below it. Therefore, the pattern must be symmetrical about the line joining the two charges.

(b) Near Field: Very close to a charge, the field due to that charge predominates. Therefore, the lines are radial and spherically symmetric.

(c) Far Field: Far from the system of charges, the pattern should look like that of a single point charge of value $Q = \Sigma Q_i$. Thus, the lines should be radially outward, unless $Q = 0$.

(d) Null Point: This is a point at which $\vec{E} = \vec{0}$, and no field lines should pass through it.

Properties of Field Lines

(a) Field lines always come (emanate) out of positive charge and enter the negative charge.

(b) The number of lines per unit area through a surface held normally to the line is observed to be proportional to the magnitude of the electric field existing in a given region.

(c) Field lines never cross each other; otherwise the field would be pointing in two different direction at the same point.

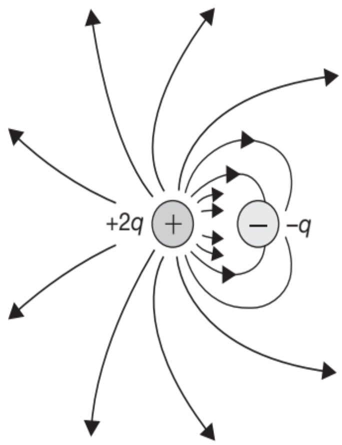
(d) Field lines never form closed loops.

(e) Field lines are always directed from higher potential to lower potential.

(f) Field lines never exist inside a conductor.

(g) If N_1 is the number of field lines coming out of a charge q_1 and N_2 is the number of field lines entering q_2, then

$$\frac{|q_1|}{|q_2|} = \frac{N_1}{N_2}$$



The electric field lines for a point charge $+2q$ and a second point charge $-q$. Note that two lines leave the charge $+2q$ for every one that terminates on $-q$.

(h) If N_1 is the number of field lines coming out of a charge q_1 and N_2 is the number of field lines coming out of charge q_2, then

$$\frac{q_1}{q_2} = \frac{N_1}{N_2}$$

This relation also exists if field lines are entering both the charges

(i) Tangent to field line at a point gives the direction of field at that point.

(j) Field lines exhibit longitudinal (length wise) contraction, thus indicating that unlike charges attract each other. (See Figure 1)

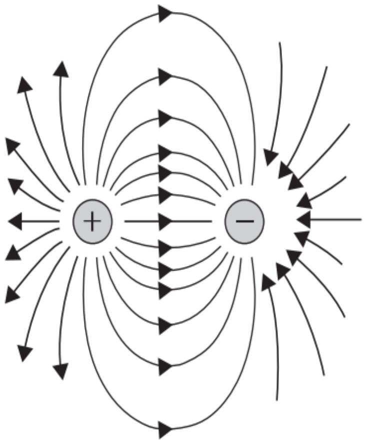


Figure 1 The electric field lines for two equal and opposite point charges an electric dipole. Note that the number of lines that leave the positive charge equals the number that terminate at the negative charge.

(k) Field lines exhibit lateral (sideways) expansion, thus indicating that like charges repel each other. (See Figure 2)

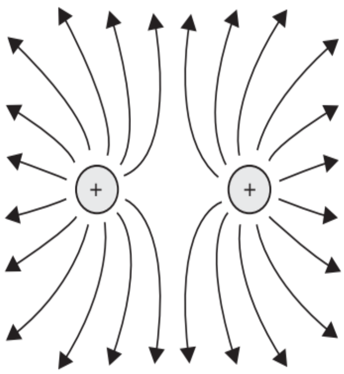


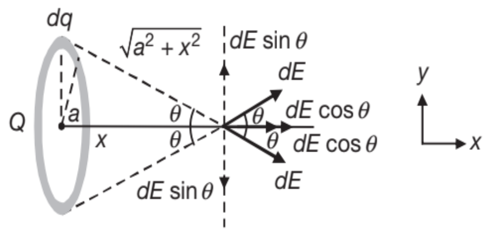
Figure 2 The electric field lines for two equal positive point charge

(l) Field lines always enter or leave a surface at right angles.

ELECTRIC FIELD AT THE AXIS OF A CIRCULAR UNIFORMLY CHARGED RING

Consider a ring of uniform charge density λ. If a is the radius of the ring and Q is the total charge on the ring, then

$$\lambda = \frac{Q}{2\pi a} \qquad \text{...(1)}$$



Consider two elements of length dl placed symmetrically on the two diametrically opposite ends. If dE is the field due to each such element, then $dE\sin\theta$ components will cancel out such that the net field is just due to $dE\cos\theta$ components summed up over the entire ring. So,

$$E_{\text{net}} = E_x = \int dE\cos\theta = \int \frac{dq}{4\pi\varepsilon_0\left(\sqrt{a^2+x^2}\right)^2}\cos\theta$$

$$\Rightarrow \quad E_x = \frac{\lambda\cos\theta}{4\pi\varepsilon_0\left(a^2+x^2\right)}\int_0^{2\pi a} dl$$

$$\Rightarrow \quad E_x = \frac{Qx}{4\pi\varepsilon_0\left(a^2+x^2\right)^{3/2}} \text{(along } +x \text{ axis)}$$

$$\left\{\because \cos\theta = \frac{x}{\sqrt{a^2+x^2}}\right\}$$

Remark(s)

(a) For the point P to lie at far off distance from the centre of the ring i.e., for $x >> a$ we have $\left(x^2+a^2\right)^{3/2} \cong x^3$.

$$\Rightarrow \quad E_x \simeq \frac{Q}{4\pi\varepsilon_0 x^2}$$

This result under the specified condition just matches with the field due to a point charge at a distance x from it.

(b) For this field to be a MAXIMUM

$$\frac{dE}{dx} = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[x\left(x^2+a^2\right)^{-3/2}\right] = 0$$

$$\Rightarrow \quad \left(x^2+a^2\right)^{-3/2} + x\left(-\frac{3}{2}\right)\left(x^2+a^2\right)^{-5/2}(2x) = 0$$

$$\Rightarrow \quad 1 = \frac{3x^2}{x^2 + a^2}$$

$$\Rightarrow \quad x^2 + a^2 = 3x^2$$

$$\Rightarrow \quad x = \pm \frac{a}{\sqrt{2}}$$

i.e., the field will attain a maximum value at $x = \pm \frac{a}{\sqrt{2}}$ along the axis of the ring and the maximum value equals E_{max} given by

$$E_{max} = \frac{Q\frac{a}{\sqrt{2}}}{4\pi\varepsilon_0\left(a^2 + \frac{a^2}{2}\right)^{3/2}}$$

$$\Rightarrow \quad E_{max} = \frac{2Q}{4\pi\varepsilon_0 3\sqrt{3}a^2}$$

$$\Rightarrow \quad E_{max} = \frac{Q}{6\sqrt{3}\pi\varepsilon_0 a^2}$$

Problem Solving Technique(s)

For **symmetrical charge distributions,** if we are asked to calculate the electric field at a point which lies symmetrically with respect to the charge distribution, then we proceed as follows :

STEP-1: Consider an infinitesimal element having a charge *dq* (say) that lies at a distance *r* from the point *P*. If *dE* is the electric field due to this element then

$$dE = \frac{dq}{4\pi\varepsilon_0 r^2}$$

STEP-2: Now consider another identical mirror infinitesimal element having same charge and located at the same distance from *P*. Then field due to this element will also be *dE*.

STEP-3: On resolving *dE* due to both the elements along suitable axis we observe that one set of components cancels and the net electric field will then be calculated by taking the integral of the component of the electric field due to the contribution of the single element.

$$E_{net} = \int (\text{contribution due to a single element})$$

However, in the case of **unsymmetrical charge distributions**, if we have to calculate the electric field at any point P, then we proceed as follows :

STEP-1: Consider an infinitesimal element having a charge *dq* (say) that lies at a distance *r* from the point *P*. If *dE* is the electric field due to this element then

$$dE = \frac{dq}{4\pi\varepsilon_0 r^2}$$

STEP-2: Resolve dE along the selected axes (say *x* and *y*) so as to get the components dE_x and dE_y.

STEP-3: Integrate dE_x and dE_y separately to get E_x and E_y.

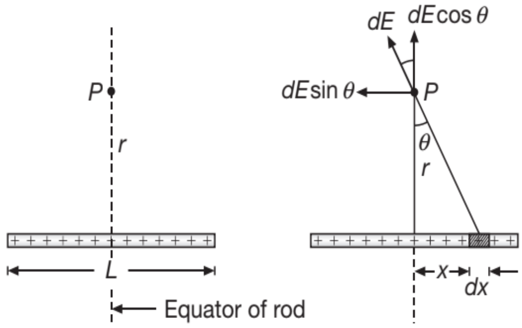
STEP-4: Then $\vec{E} = E_x\hat{i} + E_y\hat{j}$ such that $|\vec{E}| = E = \sqrt{E_x^2 + E_y^2}$ and if $\vec{E}$ makes an angle β with *x*-axis, then $\tan\beta = \frac{E_y}{E_x}$.

ILLUSTRATION 18

Calculate the electric field due to a uniformly charged rod of length L at a point P that lies at a distance r on its perpendicular bisector. Assume charge on the rod to be Q. Discuss the result when $r \gg L$ and $L \gg r$. In the case when $L \gg r$, also draw a plot of $\frac{r}{L}$ (along x-axis) vs $\frac{E}{E_0}$ (on y-axis) where $E_0 = \frac{Q}{4\pi\varepsilon_0 L^2}$.

SOLUTION

For this we consider an element of length dx at a distance x from centre of rod as shown in Figure.



The charge on this element is

$$dq=\frac{Q}{L}dx$$

If the strength of electric field at point P due to this infinitesimal charge dq is dE, then

$$dE=\frac{1}{4\pi\varepsilon_0}\frac{dq}{\left(r^2+x^2\right)}$$

Let us again take a symmetrical element such that it is just the mirror image of this element. Then we can see that the component $dE\sin\theta$ will get cancelled and net electric field at point P will be due to integration of $dE\cos\theta$ only.

Thus net electric field strength at point P can be given as

$$E=E_P=\int dE\cos\theta$$

$$\Rightarrow \quad E=\frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{+L/2}\frac{Q\,dx}{L\left(r^2+x^2\right)}\times\frac{r}{\sqrt{r^2+x^2}}$$

$$\Rightarrow \quad E=\frac{1}{4\pi\varepsilon_0}\frac{Qr}{L}\int_{-L/2}^{+L/2}\frac{dx}{\left(r^2+x^2\right)^{3/2}}$$

Let the integral be $I=\int\frac{dx}{\left(r^2+x^2\right)^{3/2}}$

To evaluate this let us substitute

$$x=r\tan\theta$$

$$\Rightarrow \quad dx=r\sec^2\theta\,d\theta$$

$$\Rightarrow \quad I=\int\frac{r\sec^2\theta\,d\theta}{r^3\sec^3\theta}$$

$$\Rightarrow \quad I=\int\cos\theta\,d\theta$$

$$\Rightarrow \quad I=\sin\theta$$

Since $x=r\tan\theta$

$$\Rightarrow \quad \sin\theta=\frac{x}{\sqrt{x^2+r^2}}$$

So, $E=E_P=\frac{1}{4\pi\varepsilon_0}\frac{Q}{Lr}\left(\frac{x}{x^2+r^2}\right)_{-\frac{L}{2}}^{+\frac{L}{2}}$

$$\Rightarrow \quad E=E_P=\frac{1}{4\pi\varepsilon_0}\frac{Q}{Lr}\left(\frac{\frac{L}{2}}{\sqrt{\frac{L^2}{4}+r^2}}+\frac{\frac{L}{2}}{\sqrt{\frac{L^2}{4}+r^2}}\right)$$

$$\Rightarrow \quad E=\frac{2Q}{4\pi\varepsilon_0 r\sqrt{L^2+4r^2}}$$

In the limit where $r\gg L$ the above expression reduces to the "point-charge" limit and we get

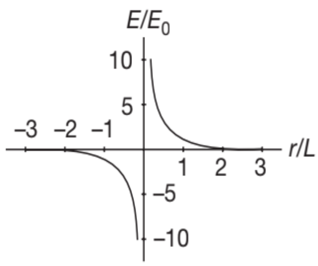
$$E\simeq\frac{Q}{4\pi\varepsilon_0 r^2}$$

On the other hand, when $L\gg r$, we have

$$E\simeq\frac{Q}{2\pi\varepsilon_0 rL}=\frac{\lambda}{2\pi\varepsilon_0 r}$$

In this infinite length limit, the system has cylindrical symmetry. In this case, an alternative approach based on Gauss's Law will also be derived at a later stage.

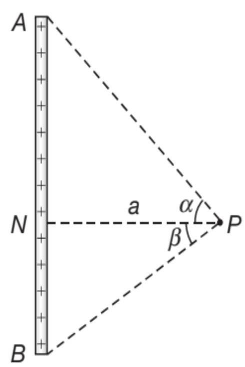
The characteristic behaviour of $\frac{E}{E_0}$ (with $E_0=\frac{Q}{4\pi\varepsilon_0 l^2}$) as a function of $\frac{r}{L}$ is shown in figure.



Electric field of a non-conducting rod as a function of r/L

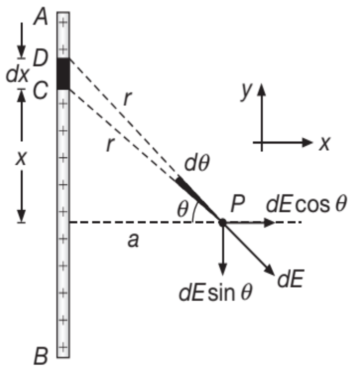
ILLUSTRATION 19

Calculate the electric field due to a uniformly charged rod AB having charge density λ at a point P at perpendicular distance a from the rod, as shown in figure.



SOLUTION

Here it is very important to note the unsymmetrical placement of the point P and hence we must calculate the fields E_x and E_y separately. For this, let us consider an infinitesimal element of length dx at a distance x as shown.



The net electric field at point P due to this infinitesimal element is dE. This dE is resolved into components.

(a) $dE_x = dE\cos\theta$

(b) $dE_y = dE\sin\theta$

The net field E can be calculated by finding E_x and E_y from the above expressions. So

$$E_x = \int dE\cos\theta, \text{ where } dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{r^2}$$

$$\Rightarrow \quad E_x = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{dx\cos\theta}{r^2} = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{dx\cos\theta}{(a^2+x^2)}$$

Also, we observe that

$$\tan\theta = \frac{x}{a}$$

$$\Rightarrow \quad x = a\tan\theta$$

$$\Rightarrow \quad dx = a\sec^2\theta d\theta$$

$$\Rightarrow \quad E_x = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{a\sec^2\theta\cos\theta d\theta}{a^2\sec^2\theta}$$

$$\Rightarrow \quad E_x = \frac{\lambda}{4\pi\varepsilon_0 a}\int_{-\beta}^{\alpha}\cos\theta d\theta$$

$$\Rightarrow \quad E_x = \frac{\lambda}{4\pi\varepsilon_0 a}(\sin\beta+\sin\alpha)$$

Similarly, let us calculate the value of E_y, given by

$$E_y = \int dE_y = \int dE\sin\theta$$

$$\Rightarrow \quad E_y = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{dx\sin\theta}{(a^2+x^2)}$$

Since $x = a\tan\theta$

$$\Rightarrow \quad dx = a\sec^2\theta d\theta$$

$$\Rightarrow \quad E_y = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{a\sec^2\theta\sin\theta d\theta}{a^2\sec^2\theta}$$

$$\Rightarrow \quad E_y = \frac{\lambda}{4\pi\varepsilon_0 a}\int_{-\beta}^{\alpha}\sin\theta d\theta$$

$$\Rightarrow \quad E_y = \frac{\lambda}{4\pi\varepsilon_0 a}\left(-\cos\theta|_{-\beta}^{\alpha}\right)$$

$$\Rightarrow \quad E_y = \frac{\lambda}{4\pi\varepsilon_0 a}(\cos\beta-\cos\alpha)$$

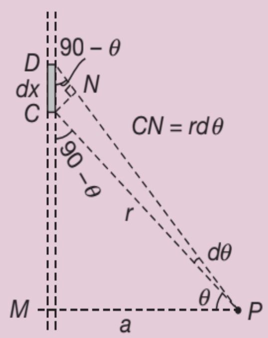
So, the electric field due to a rod of length having uniform charge density λ at a point P, that subtends an angle α at one end and β at the other is given by

$$E_x = \frac{\lambda}{4\pi\varepsilon_0 a}(\sin\alpha+\sin\beta)$$

and $$E_y = \frac{\lambda}{4\pi\varepsilon_0 a}(\cos\beta-\cos\alpha)$$

Misconception Removal

While attempting this problem, we may think that $dx = rd\theta$, but that would be a wrong step. From the magnified diagram of the element shown $PC = r$ and $PN = r$. So its CN that equals $rd\theta$ and not CD. Since CD is extremely small, so $\angle MCP = \angle CDP = 90 - \theta$. Now, we observe the triangle NCD, then



$$\sin(90-\theta)=\frac{CN}{CD}=\frac{r\,d\theta}{dx}$$

$$\Rightarrow \quad dx=\frac{r\,d\theta}{\cos\theta}$$

$$\Rightarrow \quad dx=\frac{a}{\cos^2\theta}d\theta$$

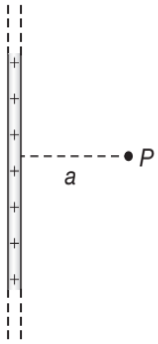
$$\Rightarrow \quad dx=a\sec^2\theta\, d\theta$$

Otherwise we would have got $dx=a\sec\theta d\theta$ which definitely would not fetch us correct result for E.

So, we can also extend the results obtained to more look alike situations as the special cases of the above discussion.

Situation 1:

For an infinite rod having uniform charge density λ, electric field at point P is calculated by taking $\alpha \longrightarrow \frac{\pi}{2}$ and $\beta \longrightarrow \frac{\pi}{2}$.



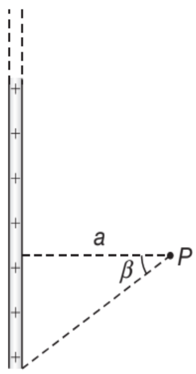
$$\Rightarrow \quad E_x=\frac{\lambda}{2\pi\varepsilon_0 a} \text{ and } E_y=0$$

Situation 2:

For a semi-infinite rod having uniform charge density λ, electric field at point P is calculated by taking

$$\alpha \longrightarrow \frac{\pi}{2} \text{ and } \beta=\beta$$

$$\Rightarrow \quad E_x=\frac{\lambda}{4\pi\varepsilon_0 a}(1+\sin\beta) \text{ and } E_y=\frac{\lambda}{4\pi\varepsilon_0}\cos\beta$$

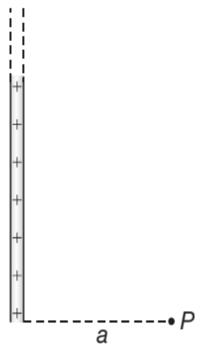


Situation 3:

For a semi-infinite rod having uniform density λ, the electric field at point P (which lies on the line joining P to wire normally and passing through the known end of wire) is calculated by taking.

$$\alpha \longrightarrow \frac{\pi}{2} \text{ and } \beta \longrightarrow 0$$

$$\Rightarrow \quad E_x=\frac{\lambda}{4\pi\varepsilon_0 a} \text{ and } E_y=\frac{\lambda}{4\pi\varepsilon_0 a}$$



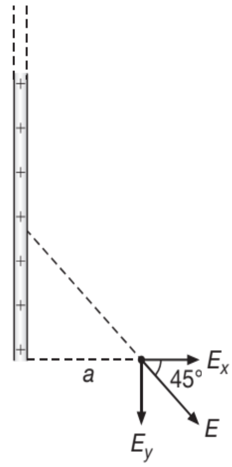


ILLUSTRATION 20

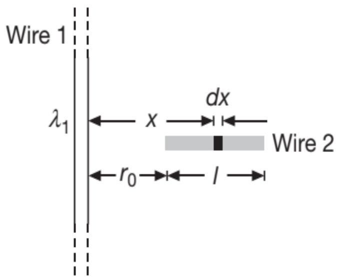
A segment of a charged wire of length l, charge density λ_2, and an infinitely long charged wire, charge density λ_1, lie in a plane at right angles to each other. The separation between the wires is r_0. Determine the force of interaction between the wires.

SOLUTION

Electric field near a long wire is given by

$$E=\frac{\lambda}{2\pi\varepsilon_0 r}$$

The second wire lies in the non uniform field of first wire so that each element of second wire experiences a different magnitude of field due to the first wire.



Let us consider an infinitesimal element of length dx having a charge $dq=\lambda_2 dx$, at a distance x from the long wire. The infinitesimal force acting on this element dF is given by

$$dF = Edq = \left(\frac{\lambda_1}{2\pi\varepsilon_0 x}\right)\lambda_2 dx$$

The force acting on each element depends on x, the separation between wire 1 and wire 2.
Integrating the expression for dF within the limits $x = r_0$ to $x = r_0 + l$, we obtain

$$F = \int_{r_0}^{r_0+l} \frac{\lambda_1\lambda_2}{2\pi\varepsilon_0}\frac{dx}{x} = \frac{\lambda_1\lambda_2}{2\pi\varepsilon_0}\log_e\left(1+\frac{l}{r_0}\right)$$

ILLUSTRATION 21

A system consists of a thin charged wire ring of radius R and a very long uniformly charged thread oriented along the axis of the ring, with one of its ends coinciding with the centre of the ring. The total charge of the ring is equal to Q. The charge of the thread (per unit length) is equal to λ. Find the interaction force between the ring and the thread.

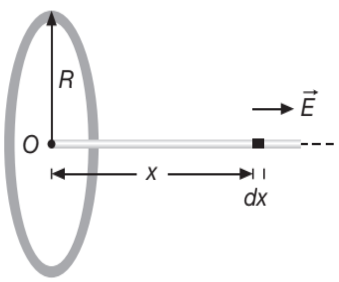
SOLUTION

The electric field strength due to ring at a point on its axis (say x-axis) at distance x from the centre of the ring is given by

$$E(x) = \frac{Qx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

And from symmetry $\vec{E}$ at every point on the axis is directed along the x-axis

Let us consider an element dx on thread which carries the charge $dq = \lambda dx$. The electric force experienced by the element in the field of ring.



$$dF = (\lambda\, dx)E(x) = \frac{\lambda Qx\, dx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

So, $F = \int_0^\infty \frac{\lambda Qx\, dx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$

On integrating we get, $F = \frac{\lambda Q}{4\pi\varepsilon_0 R}$

ILLUSTRATION 22

Calculate the electric field at the centre of a uniformly charged wire in the form of an arc of radius r. The wire subtends an angle ϕ at the centre.

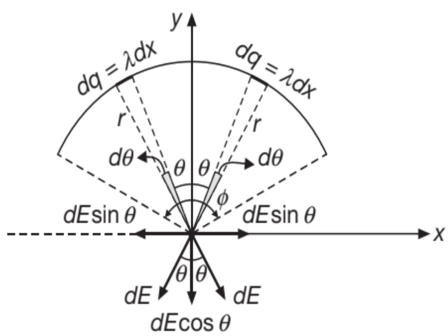
SOLUTION

Consider an element $dq(=\lambda\, dx)$ of the wire that makes an angle θ with y-axis and subtends an angle $d\theta$ at the centre. Then

$$dq = \lambda(r\, d\theta)$$

Since, $dE = \frac{dq}{4\pi\varepsilon_0 r^2} = \frac{\lambda(r\, d\theta)}{4\pi\varepsilon_0 r}$

$$\Rightarrow \quad dE = \frac{\lambda\, d\theta}{4\pi\varepsilon_0 r}$$



Again take another element which is the mirror image of the element already taken. On resolution, we observe that the x components cancel, while the net field is equal to integral of contribution due to a single element. Hence

$$E = E_y = \int dE\cos\theta$$

$$\Rightarrow \quad E = \int dE\cos\theta$$

$$\Rightarrow \quad E = \frac{\lambda}{4\pi\varepsilon_0 r}\int_{-\frac{\phi}{2}}^{+\frac{\phi}{2}} \cos\theta d\theta$$

$$\Rightarrow \quad E = \frac{\lambda}{4\pi\varepsilon_0 r}\sin\theta\Big|_{-\frac{\phi}{2}}^{\frac{\phi}{2}}$$

$$\Rightarrow \quad E = \frac{\lambda}{4\pi\varepsilon_0 r}\left[\sin\left(\frac{\phi}{2}\right) - \sin\left(-\frac{\phi}{2}\right)\right]$$

$$\Rightarrow \quad E = \frac{\lambda}{4\pi\varepsilon_0 r}2\sin\left(\frac{\phi}{2}\right)$$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}\sin\left(\frac{\phi}{2}\right)$$

Problem Solving Technique(s)

(a) For semi-circle, $\phi = \pi$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

(b) For Quarter-circle $\phi = \frac{\pi}{2}$

$$\Rightarrow \quad E = \frac{\lambda}{2\sqrt{2}\pi\varepsilon_0 r}$$

(c) For circle, $\phi = 2\pi$

$$\Rightarrow \quad E = 0$$

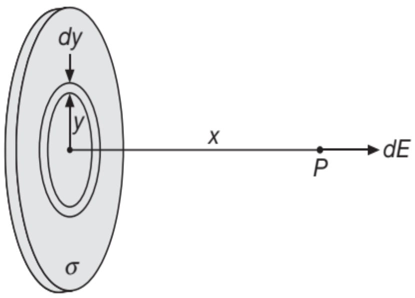
ILLUSTRATION 23

Calculate the electric field at point P that lies on the axis of a uniformly charged disc of radius R having charge density σ at a distance x from the centre.

SOLUTION

To find electric field at point P due to this disc let us consider an elemental ring of radius y and width dy. If dq be the charge on this infinitesimal element of area $dA = 2\pi y dy$, then

$$dq = \sigma dA = \sigma(2\pi y dy)$$



Now we know that electric field strength due to a ring of radius R, charge Q, at a distance x from its centre on its axis can be given as

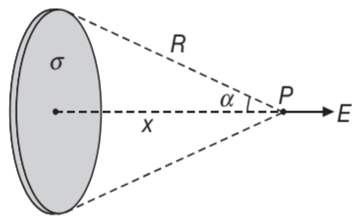
$$E_{\text{axis}} = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{(x^2+R^2)^{3/2}} \qquad \text{(Derived earlier)}$$

So, due to the infinitesimal elemental ring the electric field strength dE at point P is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{(dq)x}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \quad dE = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi y\, dy\, x}{(x^2+y^2)^{3/2}}$$

Please note that, here once point P is taken, then value of x remains fixed. So,



$$E = E_{\text{axis}} = \int_0^R dE$$

$$\Rightarrow \quad E = \int dE = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{\sigma 2\pi xy\, dy}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \quad E = \frac{\sigma\pi x}{4\pi\varepsilon_0}\int_0^R \frac{2y\, dy}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \quad E = \frac{\sigma x}{4\pi\varepsilon_0}\left[-\frac{2}{\sqrt{x^2+y^2}}\Big|_{y=0}^{y=R}\right]$$

$$\Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right]$$

Also, we observe here that, if we assume the complete disc to subtend an apex angle α at P, then, the field can also be expressed as

$$E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{R^2+x^2}}\right] = \frac{\sigma}{2\varepsilon_0}(1-\cos\alpha)$$

$$\left(\text{Because } \cos\alpha = \frac{x}{\sqrt{R^2+x^2}}\right)$$

So, finally, we get

$$E=\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{\sqrt{x^2+R^2}}\right]=\frac{\sigma}{2\varepsilon_0}(1-\cos\alpha)$$

Remark(s)

(a) Please note that here we have used

$$dE=\frac{(dq)x}{4\pi\varepsilon_0\left(a^2+x^2\right)^{3/2}} \qquad ...(1)$$

But actually the result obtained from this expression is

$$E=\frac{\sigma x}{2\varepsilon_0}\left[\frac{1}{x}-\frac{1}{\sqrt{R^2+x^2}}\right] \qquad ...(2)$$

where it would not be possible to calculate E at $x=0$. So, to be precise with the boundaries here, we just calculate $E(\text{at } x=0)$ and $E(\text{as } x\to 0)$ and both will yield different results.

$E(\text{at } x=0)=0$ {from expression (2)}

and $E(\text{as } x\to 0)=\frac{\sigma}{2\varepsilon_0}$ (i.e., very close to the disc)

{from expression (1)}

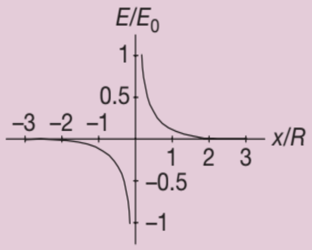
So, we can say that E has two different values i.e., at the centre and just outside the disc and hence E shows discontinuity at $x=0$.

(b) So, $$E_x=\begin{cases}\frac{\sigma}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{R^2+x^2}}\right), & x>0\\ \frac{\sigma}{2\varepsilon_0}\left(-1-\frac{x}{\sqrt{R^2+x^2}}\right), & x<0\end{cases}$$

(c) On the other hand we may also consider the limit where $R\gg x$. Physically this means that the plane is very large, or the field point P is extremely close to the surface of the plane. The electric field in this limit becomes, in unit vector notation,

$$\vec{E}=\begin{cases}\frac{\sigma}{2\varepsilon_0}\hat{i}, & x>0\\ -\frac{\sigma}{2\varepsilon_0}\hat{i}, & x<0\end{cases}$$

The plot of the electric field in this limit is shown in figure

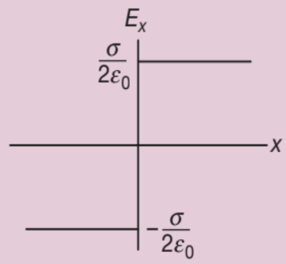


Notice the discontinuity in electric field as we cross the plane. The discontinuity is given by

$$\Delta E_x=E_{x+}-E_{x-}=\frac{\sigma}{2\varepsilon_0}-\left(-\frac{\sigma}{2\varepsilon_0}\right)=\frac{\sigma}{\varepsilon_0}$$

As we shall see during the study of Gauss's Law that if a given surface has a charge density σ, then the normal component of the electric field across that surface always exhibits a discontinuity with $\Delta E_n=\frac{\sigma}{\varepsilon_0}$.

(d) The electric field $\frac{E_x}{E_0}$, where $E_0=\frac{\sigma}{2\varepsilon_0}$ as a function of $\frac{x}{R}$ is shown in figure



Electric field of an infinitely large non-conducting plane

(e) To show that the "point-charge" limit is retained for $x\gg R$, we make use of the expansion series, in which

$$1-\frac{x}{\sqrt{x^2+R^2}}=1-\left(1+\frac{R^2}{x^2}\right)^{-1/2}=$$

$$1-\left(1-\frac{1}{2}\frac{R^2}{x^2}+....\right)\approx\frac{R^2}{2x^2}$$

This gives

$$E_x=\frac{\sigma}{2\varepsilon_0}\frac{R^2}{2x^2}=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\pi R^2}{x^2}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}$$

which is indeed the expected "point-charge" result.

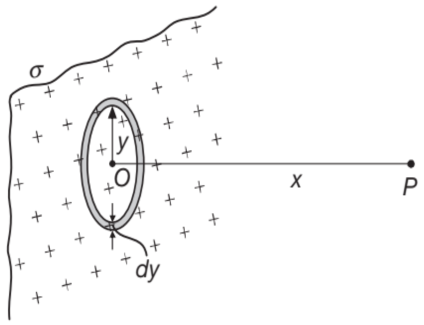
ILLUSTRATION 24

Calculate the electric field strength due to a uniformly charged infinite sheet having charge density σ at a point P at perpendicular distance x from the sheet.

SOLUTION

Let us consider an infinitesimal elemental ring of radius y having width dy with centre at O as shown. If dq be the charge on this ring, then

$$dq = \sigma(2\pi y\,dy)$$



The electric field strength due to this elemental ring at point P at distance x is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dqx}{\left(x^2+y^2\right)^{3/2}}$$

$$\Rightarrow \quad E = \frac{1}{4\pi\varepsilon_0}\int_0^{\infty}\frac{\sigma(2\pi y\,dy)x}{\left(x^2+y^2\right)^{3/2}}$$

$$\Rightarrow \quad E = \frac{\pi\sigma x}{4\pi\varepsilon_0}\int_0^{\infty}\frac{2y\,dy}{\left(x^2+y^2\right)^{3/2}}$$

$$\Rightarrow \quad E = \frac{\sigma x}{4\varepsilon_0}\int_0^{\infty}\left(x^2+y^2\right)^{-3/2}2y\,dy$$

Since, $\int\left[f(x)\right]^n f'(x)dx = \frac{\left[f(x)\right]^{n+1}}{n+1}$, $n \neq -1$

$$\Rightarrow \quad E = \frac{\sigma x}{4\varepsilon_0}\left(\left.\frac{\left(x^2+y^2\right)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right|_{y=0}^{y\to\infty}\right)$$

$$\Rightarrow \quad E = -\frac{2\sigma x}{4\varepsilon_0}\left(\left.\frac{1}{\sqrt{x^2+y^2}}\right|_{y=0}^{y\to\infty}\right)$$

$$\Rightarrow \quad E = -\frac{\sigma x}{2\varepsilon_0}\left(0-\frac{1}{x}\right)$$

$$\Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0} \qquad \{\text{independent of value of } x\}$$

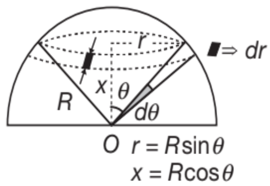
ILLUSTRATION 25

Calculate the electric field due to a thin uniformly charged hemispherical shell (of radius R having uniform charge density σ) at its centre.

SOLUTION

Let us consider an infinitesimal elemental ring on its surface having angular width $d\theta$ at an angle θ from its axis as shown. The surface area dA of this ring is

$$dA = (2\pi R\sin\theta)(Rd\theta)$$



Charge on this elemental ring is

$$dq = \sigma dA = \sigma\left(2\pi R^2\sin\theta d\theta\right)$$

Now due to this ring, electric field strength at centre C is

$$dE = \frac{1}{4\pi\varepsilon_0}\left[\frac{dq(R\cos\theta)}{\left(R^2\sin^2\theta+R^2\cos^2\theta\right)^{3/2}}\right]$$

$$\left\{\because\ dE = \frac{(dq)x}{4\pi\varepsilon_0\left(a^2+x^2\right)^{3/2}}\right\}$$

$$\Rightarrow \quad dE = \frac{1}{4\pi\varepsilon_0}\left[\frac{\sigma\left(2\pi R^2\sin\theta d\theta\right)R\cos\theta}{R^3}\right]$$

$$\Rightarrow \quad dE = \frac{\pi\sigma}{4\pi\varepsilon_0}\sin(2\theta)d\theta$$

Net electric field at center is obtained by integrating the above expression for dE between the limits zero to $\frac{\pi}{2}$. So,

$$E = \int dE = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi/2}\sin 2\theta\, d\theta$$

$$\Rightarrow \quad E=\frac{\sigma}{4\varepsilon_0}\left(-\frac{\cos 2\theta}{2}\Big|_0^{\pi/2}\right)$$

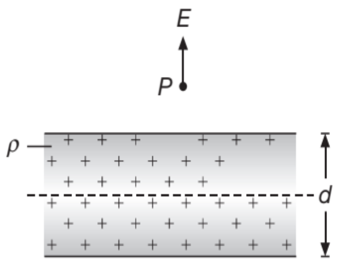
$$\Rightarrow \quad E=\frac{\sigma}{4\varepsilon_0}\left(\frac{1}{2}+\frac{1}{2}\right)$$

$$\Rightarrow \quad E=\frac{\sigma}{4\varepsilon_0}$$

ELECTRIC FIELD DUE TO A LARGE THICK CHARGED SHEET

CASE-1: OUTSIDE THE SHEET

Consider a large sheet of thickness d having uniform charge density ρ. On both sides of sheet electric field strength is directed, away from the sheet.



The electric field strength at a point P in front of the sheet is

$$E=\frac{\sigma}{2\varepsilon_0} \quad \ldots(1)$$

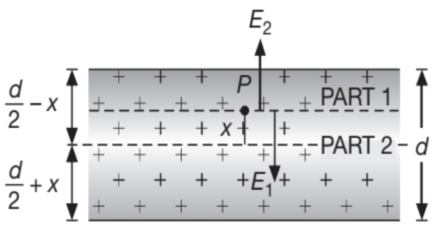
where σ is the charge per unit surface area of the sheet. Further

$$\sigma=\rho d$$

$$\Rightarrow \quad E=\frac{\rho d}{2\varepsilon_0} \quad \ldots(2)$$

CASE-2: INSIDE THE SHEET

To find electric field strength at an interior point P of the sheet at a distance x from its centre as shown, we divide the sheet in two sheet parts. One of thickness $\left(\frac{d}{2}-x\right)$ and other of thickness $\left(\frac{d}{2}+x\right)$ as shown.



Due to the thinner sheet **(PART 1)** electric field at point P is in downward direction, say it is E_1 which is given by using equation (2).

$$E_1=\frac{\rho\left(\frac{d}{2}-x\right)}{2\varepsilon_0}$$

Similarly due to thicker sheet **(PART 2)** electric field at P is in upward direction, say it is E_2 which is given again by using equation (2).

$$E_2=\frac{\rho\left(\frac{d}{2}+x\right)}{2\varepsilon_0}$$

Net electric field at point P will be $E_P\,(=E)$ given by

$$E_P=E=E_2-E_1$$

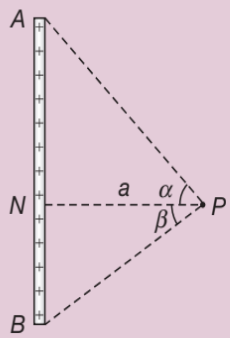
$$\Rightarrow \quad E=E_P=\frac{\rho\left(\frac{d}{2}+x\right)}{2\varepsilon_0}-\frac{\rho\left(\frac{d}{2}-x\right)}{2\varepsilon_0}$$

$$\Rightarrow \quad E=\frac{\rho x}{\varepsilon_0}$$

Conceptual Note(s)

Electrostatic Field in Some Cases

(a) The electric field due to a uniformly charged rod *AB* having charge density λ at a point *P* at perpendicular distance *a* from the rod, as shown in figure is



$$E_x=\frac{\lambda}{4\pi\varepsilon_0 a}(\sin\alpha+\sin\beta)$$

and $$E_y=\frac{\lambda}{4\pi\varepsilon_0 a}(\cos\beta-\cos\alpha)$$

So, we can also extend the result obtained to more look alike situations.

(b) For an infinite rod having uniform charge density λ, electric field at point P is calculated by taking $\alpha \longrightarrow \frac{\pi}{2}$ and $\beta \longrightarrow \frac{\pi}{2}$.

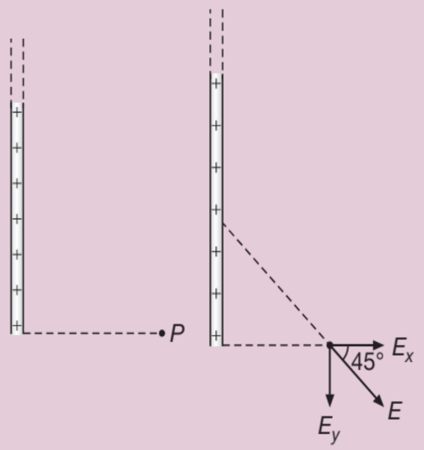
$$\Rightarrow \quad E_x = \frac{\lambda}{2\pi\varepsilon_0 a} \text{ and } E_y = 0$$

(c) For a semi-infinite rod having uniform charge density λ, electric field at point P is calculated by taking $\alpha \longrightarrow \frac{\pi}{2}$ and $\beta = \beta$

$$\Rightarrow \quad E_x = \frac{\lambda}{4\pi\varepsilon_0 a}(1+\sin\beta) \text{ and } E_y = \frac{\lambda}{4\pi\varepsilon_0}\cos\beta$$

(d) For a semi-infinite rod having uniform density λ, the electric field at point P (which lies on the line joining P to wire normally and passing through the known end of wire) is calculated by taking $\alpha \longrightarrow \frac{\pi}{2}$ and $\beta \longrightarrow 0$

$$\Rightarrow \quad E_x = \frac{\lambda}{4\pi\varepsilon_0 a} \text{ and } E_y = \frac{\lambda}{4\pi\varepsilon_0 a}$$



(e) The electric field at the centre of a uniformly charged wire in the form of a circular arc of radius r subtending an angle ϕ at the centre is $E = \frac{\lambda}{2\pi\varepsilon_0 r}\sin\left(\frac{\phi}{2}\right)$, along the angle bisector.

(i) For semi-circle, $\phi = \pi$

So, $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$

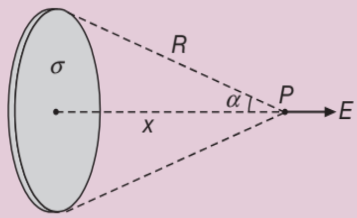
(ii) For Quarter-circle, $\phi = \frac{\pi}{2}$

$$\Rightarrow \quad E = \frac{\lambda}{2\sqrt{2}\pi\varepsilon_0 r}$$

(iii) For circle, $\phi = 2\pi$

$$\Rightarrow \quad E = 0$$

(f) The electric field at point P that lies on the axis of a uniformly charged disc of radius R having charge density σ at a distance x from the centre is



$$E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{R^2+x^2}}\right] = \frac{\sigma}{2\varepsilon_0}(1-\cos\alpha)$$

If Q is the total charge on the disc, then

$$E = \frac{2Q}{4\pi\varepsilon_0 R^2}\left[1 - \frac{x}{\sqrt{x^2+R^2}}\right]$$

(g) The electric field strength due to a uniformly charged infinite sheet having charge density σ at a point P at perpendicular distance x from the sheet is $E = \frac{\sigma}{2\varepsilon_0}$ (independent of value of x).

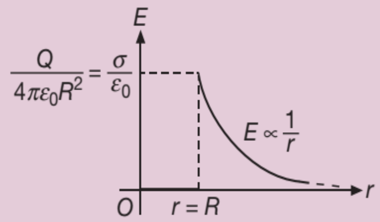
(h) The electric field due to a thin uniformly charged shell (of radius R having uniform charge density σ) at its centre is $E = \frac{\sigma}{4\varepsilon_0}$.

(i) Due to a charged spherical conductor (spherical shell) of charge Q, radius R at a distance r from its centre.

$$E = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2} & \text{for } r \geq R \\ 0 & \text{for } r < R \end{cases}$$

If the conducting sphere has uniform surface charge density σ, then

$$E = \begin{cases} \dfrac{\sigma}{\varepsilon_0}\dfrac{R^2}{r^2} & \text{for } r \geq R \\ 0 & \text{for } r < R \end{cases}$$

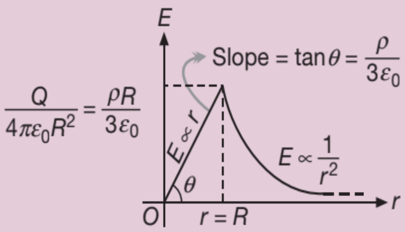


(j) Due to a uniformly charged non-conducting sphere of charge Q, radius R at a distance r from the centre of the sphere.

$$E = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2} & \text{for } r \geq R \\ \dfrac{Qr}{4\pi\varepsilon_0 R^3} & \text{for } r < R \end{cases}$$

If the sphere has uniform volume charge density ρ, then

$$E = \begin{cases} \dfrac{\rho R^3}{3\varepsilon_0 r^2} & \text{for } r \geq R \\ \dfrac{\rho r}{3\varepsilon_0} & \text{for } r < R \end{cases}$$



(k) Due to an infinite thread having uniform charge distribution λ at distance r from it.

$$E = \frac{2\lambda}{4\pi\varepsilon_0 r}$$

(l) Due to an infinite thin conducting plane sheet of charge with surface charge density σ.

$$E = \frac{\sigma}{\varepsilon_0}$$

(m) Due to an infinite non conducting plane sheet of charge with surface charge density σ.

$$E = \frac{\sigma}{2\varepsilon_0}$$

Suppose Q charge is given to

Conducting plate | Non-conducting plate

Electric field for both the cases

$$E = \frac{Q}{2A\varepsilon_0}$$

$E = \dfrac{\sigma_{\text{conducting}}}{\varepsilon_0}$

Where $\sigma_{\text{conducting}} = \dfrac{Q}{2A}$

Because Q is distributed in area 2A.

$E = \dfrac{\sigma_{\text{non-conducting}}}{2\varepsilon_0}$

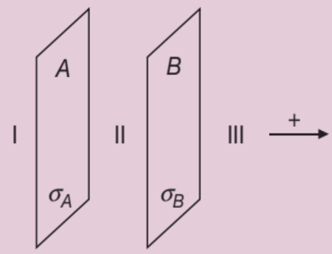
Where $\sigma_{\text{non-conducting}} = \dfrac{Q}{A}$

Because Q is distributed in area A.

(n) Near a conductor of any shape,

$$E = \frac{\sigma}{\varepsilon_0}$$

(o) Due to two parallel plane thin sheets with surface charge density σ_A and σ_B in the regions specified in figure



$$E_{I} = -\frac{1}{2\varepsilon_0}(\sigma_A + \sigma_B)$$

$$E_{II} = \frac{1}{2\varepsilon_0}(\sigma_A - \sigma_B)$$

$$E_{III} = \frac{1}{2\varepsilon_0}(\sigma_A - \sigma_B)$$

If $\sigma_A = +\sigma$ and $\sigma_B = -\sigma$, then

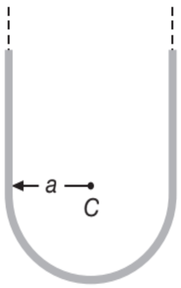
$$E_{I} = E_{III} = 0 \text{ and } E_{II} = \frac{\sigma}{\varepsilon_0}$$

(p) At the axis of a uniformly charged ring of radius R at a distance x from the centre of the ring.

$$E=\frac{Qx}{4\pi\varepsilon_0\left(R^2+x^2\right)^{3/2}}$$

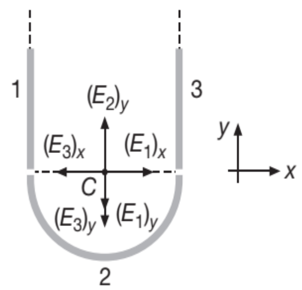
ILLUSTRATION 26

In the given arrangement find the electric field at C in the figure. Here the U-shaped wire uniformly charged with linear charge density λ.



SOLUTION

Here we shall divide the arrangement into three parts as shown and then apply the principle of superposition.



Here we have used the results obtained previously

$$(E_1)_x=\frac{\lambda}{4\pi\varepsilon_0 a} \quad \text{(along } +x \text{ axis)}$$

$$(E_1)_y=\frac{\lambda}{4\pi\varepsilon_0 a} \quad \text{(along } -y \text{ axis)}$$

Similarly

$$(E_3)_x=\frac{\lambda}{4\pi\varepsilon_0 a} \quad \text{(along } -x \text{ axis)}$$

$$(E_3)_y=\frac{\lambda}{4\pi\varepsilon_0 a} \quad \text{(along } -y \text{ axis)}$$

and $(E_2)_x=0$

$$(E_2)_y=\frac{\lambda}{2\pi\varepsilon_0 a} \quad \text{(along } +y \text{ axis)}$$

So, $\vec{E}_C=\vec{E}_1+\vec{E}_2+\vec{E}_3$

$\Rightarrow \quad \vec{E}_C=\vec{0}$

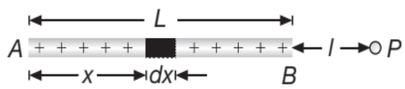
ILLUSTRATION 27

Calculate the electric field due to a rod of length L at a point P at a distance l from its end, when the charge density of the rod varies with distance x from the other end of the rod as $\lambda=\lambda_0 x$ where λ_0 is a positive constant.

SOLUTION

The rod AB shown here is of length L having linear charge density that depends on distance x from end A of rod as

$$\lambda=\lambda_0 x$$



and we have to calculate electric field strength at point P. For this let us consider an infinitesimal element of length dx at a distance x from the end A as shown. Here charge on this element is

$$dq=\lambda dx=\lambda_0 x\,dx$$

Now electric field strength E at point P due to infinitesimal charge dq is dE (say), given by

$$dE=\frac{1}{4\pi\varepsilon_0}\frac{dq}{(L+l-x)^2}$$

Net electric field at P due to complete rod (from A to B) is calculated by integrating the above expression within limits taken from zero to L. Thus we have

$$E=\int dE=\frac{1}{4\pi\varepsilon_0}\int_0^L\frac{\lambda_0 x\,dx}{(L+l-x)^2}=\frac{\lambda_0}{4\pi\varepsilon_0}\int_0^L\frac{x\,dx}{(L+l-x)^2}$$

$$\Rightarrow \quad E=\frac{\lambda_0}{4\pi\varepsilon_0}\int_0^L\frac{x\,dx}{(L+l-x)^2}$$

To calculate the integral, let us denote the integral by

$$I = \int \frac{x\,dx}{(L+l-x)^2}$$

Substituting $L+l-x = y$

$$\Rightarrow \quad -dx = dy$$

$$\Rightarrow \quad I = \int \frac{-[(L+l)-y]dy}{y^2}$$

$$\Rightarrow \quad I = -(L+l)\int y^{-2}\,dy + \int \frac{y}{y^2}dy$$

$$\Rightarrow \quad I = -(L+l)\frac{y^{-2+1}}{-2+1} + \log_e y$$

$$\Rightarrow \quad I = \frac{(L+l)}{y} + \log_e y$$

$$\Rightarrow \quad I = \frac{L+l}{L+l-x} + \log_e(L+l-x)$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\pi\varepsilon_0}\int_0^L \frac{x\,dx}{(L+l-x)^2}$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\pi\varepsilon_0}\left[\left(\frac{L+l}{L+l-x}\right)\Bigg|_0^L + \log_e(L+l-x)\Bigg|_0^L\right]$$

$$\Rightarrow \quad E = \frac{\lambda_0(L+l)}{4\pi\varepsilon_0}\left[\frac{1}{L+l-L} - \frac{1}{L+l}\right] + \frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{L+l-L}{L+l}\right)$$

$$\Rightarrow \quad E = \frac{\lambda_0(L+l)}{4\pi\varepsilon_0}\left[\frac{1}{l} - \frac{1}{L+l}\right] + \frac{\lambda_0}{4\pi\varepsilon_0}\log_e\left(\frac{l}{L+l}\right)$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\pi\varepsilon_0}\frac{(L+l)L}{l(L+l)} + \frac{\lambda_0}{4\pi\varepsilon_0}\log_e\left(\frac{l}{L+l}\right)$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\pi\varepsilon_0}\left[\frac{L}{l} + \log_e\left(\frac{l}{L+l}\right)\right]$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\pi\varepsilon_0}\left[\frac{L}{l} - \log_e\left(1+\frac{L}{l}\right)\right]$$

ILLUSTRATION 28

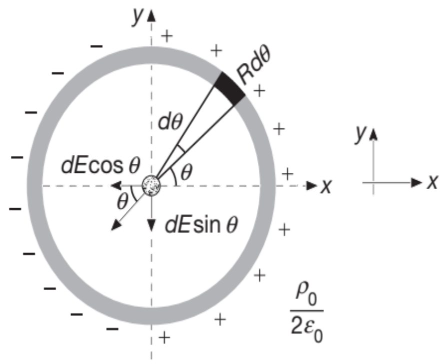
A thin non conducting ring of radius R has linear charge density λ varying as $\lambda = \lambda_0 \cos\phi$, where λ_0 is a positive constant and ϕ is the azimuthal angle. Calculate the magnitude of the electric field strength at

(a) the centre of the ring

(b) on the axis of the ring as a function of the distance x from its centre. Investigate the obtained function at $x \gg R$.

SOLUTION

(a) Here by observing the function $\lambda = \lambda_0 \cos\theta$, we can state that the first and fourth quadrant are positively charged and second and third quadrant are negatively charged.



To find electric field strength at the centre of the ring, we consider an infinitesimal element on ring having polar width $d\theta$ and making an angle θ with x-axis. The charge on this infinitesimal element is

$$dq = \lambda\,dx$$

$$\Rightarrow \quad dq = \lambda(R d\theta)$$

$$\Rightarrow \quad dq = (\lambda_0 \cos\theta)(R d\theta) \qquad \ldots(1)$$

The electric field strength at the centre of the ring due to this element is

$$dE = \frac{1}{4\pi\varepsilon_0}\left(\frac{dq}{R^2}\right) \qquad \ldots(2)$$

To find net electric field at centre of ring due to this semicircular distribution, we integrate the components of this electric field taken for the semi-circumference of ring. Here components $dE\sin\theta$ will cancel each other and the component $dE\cos\theta$ will be contributing to the net field. Thus net electric field strength at centre due to semicircular distribution will be given as

$$E = E_{\substack{\text{due to semi-circular}\\ \text{positive charge}\\ \text{distribution}}} + E_{\substack{\text{due to semi-circular}\\ \text{negative charge}\\ \text{distribution}}}$$

$$\Rightarrow \quad E = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dE\cos\theta$$

$$\Rightarrow \quad E = \frac{2\lambda_0}{4\pi\varepsilon_0 R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$\Rightarrow \quad E = \frac{\lambda_0}{2\pi\varepsilon_0 R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\pi\varepsilon_0 R}\left(\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin(2\theta)}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right)$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\pi\varepsilon_0 R}(\pi + 0)$$

$$\Rightarrow \quad E = \frac{\lambda_0}{4\varepsilon_0 R}$$

(b) Take an infinitesimal element dl at an azimuthal angle ϕ. This element subtends an angle $d\phi$ at the centre O. The charge on this element

$$dq = \lambda\, dl = \lambda(R d\phi)$$

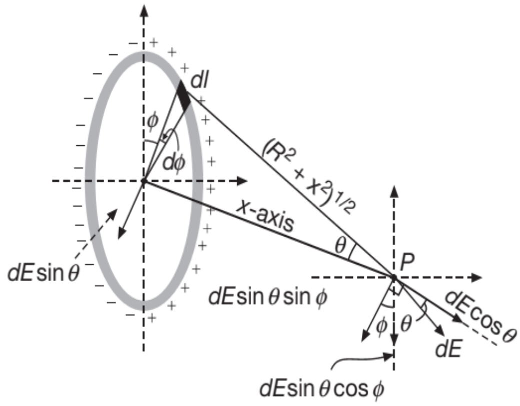
$$\Rightarrow \quad dq = (\lambda_0 \cos\phi)(R d\phi)$$

$$\{\because\ \lambda = \lambda_0\cos\phi \text{ and } dl = R d\phi\}$$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(R^2+x^2)}$$

Electric field due to positive half ring at P on the axis is

$$E_1 = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} dE\sin\theta\cos\phi$$



$$\Rightarrow \quad E_1 = \frac{1}{4\pi\varepsilon_0}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{dq}{(R^2+x^2)} \times \frac{R}{(R^2+x^2)^{1/2}}\cos\phi$$

$$\Rightarrow \quad E_1 = \frac{1}{4\pi\varepsilon_0}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\lambda_0 R\cos\phi\, d\phi \times R\cos\phi}{(R^2+x^2)^{3/2}}$$

$$\Rightarrow \quad E_1 = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 R^2}{(R^2+x^2)^{3/2}}\left(2\int_0^{\frac{\pi}{2}}\left[\frac{1+\cos 2\phi}{2}\right]d\phi\right)$$

$$\Rightarrow \quad E_1 = \frac{2\lambda_0 R^2}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad E_1 = \frac{\lambda_0 R^2}{8\varepsilon_0 (R^2+x^2)^{3/2}}$$

Similarly, for negative part, we have

$$E_2 = \frac{\lambda_0 R^2}{8\varepsilon_0 (R^2+x^2)^{3/2}}$$

So, net electric field at point P is given by

$$E_{net} = E_1 + E_2 = \frac{\lambda_0 R^2}{4\varepsilon_0 (R^2+x^2)^{3/2}}$$

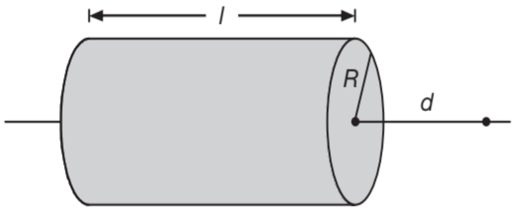
The direction will be perpendicular to the axis of ring

For $x \gg R$,

$$E_{net} = \frac{\lambda_0 R^2}{4\varepsilon_0 (x^3)}$$

ILLUSTRATION 29

(a) Consider a uniformly charged thin walled right circular cylindrical shell having total charge Q, radius R, and length l. Determine the electric field at a point a distance d from the right side of the cylinder as shown in figure.

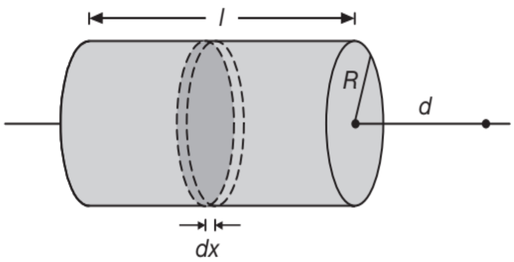


(b) Now consider a solid cylinder with the same dimensions and carrying the same charge, uniformly distributed through its volume. Find the field it creates at the same point.

SOLUTION

(a) We define $x = 0$ at the point where we are to find the field. One ring, with thickness dx, has charge $\frac{Qdx}{l}$ and produces, at the chosen point, a field

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{x}{(x^2+R^2)^{\frac{3}{2}}}\frac{Qdx}{l}\hat{i}$$



The total field is

$$E = \int dE = \frac{Q}{4\pi\varepsilon_0 l}\int_d^{d+\ell}\frac{xdx}{(x^2+R^2)^{\frac{3}{2}}}$$

$$\Rightarrow \quad E = \frac{Q}{8\pi\varepsilon_0 l}\int_d^{d+\ell}(x^2+R^2)^{-\frac{3}{2}}2xdx$$

Since $\int (x^2+R^2)^{-\frac{3}{2}}2xdx = -\frac{2}{\sqrt{x^2+R^2}}$

$$\Rightarrow \quad E = -\frac{Q}{4\pi\varepsilon_0 l}\left(\frac{1}{\sqrt{x^2+R^2}}\right)\Bigg|_d^{d+\ell}$$

$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_0 l}\left(\frac{1}{\sqrt{d^2+R^2}} - \frac{1}{\sqrt{(d+l)^2+R^2}}\right)$$

(b) Think of the cylinder as a stack of disks, each with thickness dx, charge $\frac{Qdx}{l}$, and charge per area $\sigma = \frac{Qdx}{\pi R^2 l}$. One disk produces a field

$$d\vec{E} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)\hat{i}$$

So, the field due to the entire cylinder is given by

$$E = \int dE = \frac{Q}{2\pi\varepsilon_0 R^2 l}\int_d^{d+\ell}\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)dx$$

$$\Rightarrow \quad E = \frac{Q}{2\pi\varepsilon_0 R^2 l}\left(\int_d^{d+l}dx - \frac{1}{2}\int_d^{d+l}(x^2+R^2)^{-\frac{1}{2}}2xdx\right)$$

$$\Rightarrow \quad E = \frac{Q}{2\pi\varepsilon_0 R^2 l}\left(x - \sqrt{x^2+R^2}\right)\Bigg|_d^{d+l}$$

$$\Rightarrow \quad E = \frac{Q}{2\pi\varepsilon_0 R^2 l}\left(l + \sqrt{d^2+R^2} - \sqrt{(d+l)^2+R^2}\right)$$

Test Your Concepts-III

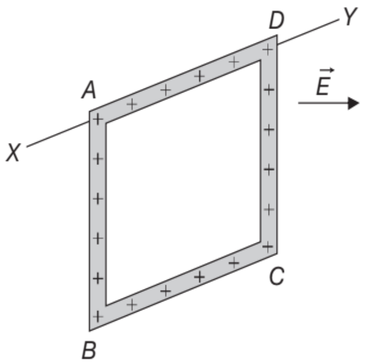
Based on Electric Field

(Solutions on page H.10)

1. Four particles each having a charge q, are placed on the four vertices of a regular pentagon. The distance of each corner from the centre is a. Find the electric field at the centre of the pentagon.

2. A circular wire loop of radius R carries a total charge Q distributed uniformly over its length. A small length $x(\ll R)$ of the wire is cut off. Find the electric field at the centre due to the remaining wire.

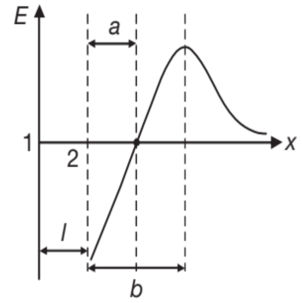
3. An electric field of 10^5 NC^{-1} points due North at a certain spot. Find the magnitude and direction of the force that acts on a charge of +2 μC and −5 μC at this spot?
4. Three identical positive charges q are arranged at the vertices of an equilateral triangle of side ℓ. Calculate the intensity of the field at the vertex of a regular tetrahedron of which the triangle is the base.
5. A square frame $ABCD$ is made of four thin rods, each of length L and mass m and charged with a charge q. The frame is hanging from one of its sides as shown in figure along a horizontal axis XY. At $t=0$ a horizontal electric field is switched on in horizontal direction, perpendicular to the plane of the frame. Find the minimum value of E so that the square frame rotates upto horizontal level.



6. A particle of mass m and charge $-Q$ is constrained to move along the axis of a ring of radius R. The ring carries a uniform charge density $+\lambda$ along its length. Initially the particle is in the plane of the ring where the force on it is zero. Show that the period of oscillation of the particle when it is displaced slightly from its equilibrium position is given by

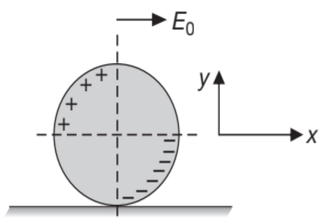
$$T = 2\pi\sqrt{\frac{2\varepsilon_0 mR^2}{\lambda Q}}$$

7. Two point charges are positioned at points 1 and 2. The field intensity to the right of the charge Q_2 on the line that passes through the two charges varies according to a law that is represented schematically in the figure. The field intensity is assumed to be positive if its direction coincides with the positive direction on the x-axis. The distance between the charges is ℓ.

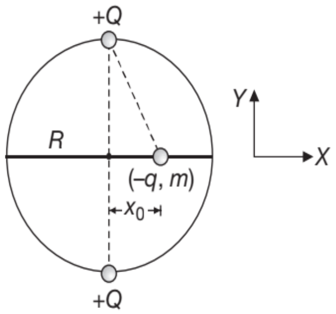


(a) Find the sign of each charge.
(b) Find the ratio of the absolute values of the charges $\left|\frac{Q_1}{Q_2}\right|$
(c) Find the value of b where the field intensity is maximum.

8. A non-conducting ring of mass m and radius R is charged as shown. The charge density, i.e., charge per unit length is λ. It is then placed on a rough non-conducting horizontal plane. At time $t=0$, a uniform electric field $\vec{E} = E_0\hat{i}$ is switched on and the ring starts rolling without sliding. Determine the frictional force (magnitude and direction) acting on the ring when it starts moving.



9. A thin insulating wire is stretched along the diameter of an insulated circular hoop of radius R. A small bead of mass m and charge $-q$ is threaded onto the wire. Two small identical charges are tied to the hoop at points opposite to each other, so that the line connecting them is perpendicular to the thread (shown in figure). The bead is released at a point which is a distance x_0 from the centre of the hoop.



(a) Calculate the resultant force acting on the charged bead.

(b) Describe (qualitatively) the motion of the bead after it is released.

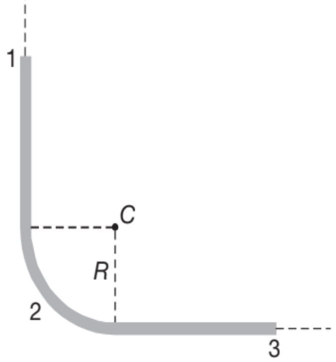
(c) Write the exact equation that governs the motion of the bead along the thread.

(d) After the assumption that $\frac{x_0}{R} \ll 1$ is used to obtain an approximate equation of motion, find the displacement and velocity of the bead as functions of time.

(e) When will the velocity of the bead vanish for the first time?

10. A non-conducting ring of mass m and radius R is lying at rest in the vertical XY plane on a smooth non-conducting horizontal XZ plane. Charge $+q$ and $-q$ are distributed uniformly on the ring, on the two sides of the vertical diameter of the ring. A constant and uniform electric field $\vec{E}$ is set up along the x direction. The ring is given a small angular rotation about an axis perpendicular to its plane and released. Find the period of oscillation of the ring.

11. In the given arrangement find electric field at C. Complete wire is uniformly charged at linear charge density λ.

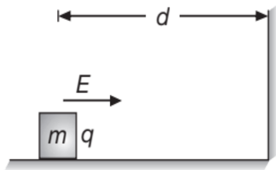


12. A pitch ball covered with tin foil having a mass m hangs by a fine silk thread of length ℓ in a horizontal electric field E. When the ball is given an electric charge q, it stands out a distance d from the vertical line. Show that the electric field is given by

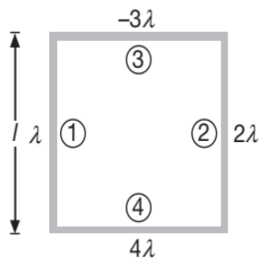
$$E = \frac{mgd}{q\sqrt{\ell^2 - d^2}}$$

13. A bob of mass m carrying a positive charge q is suspended from a light inextensible string of length ℓ inside a parallel plate condenser with its plates making an angle β with the horizontal. The upper plate of the condenser is negatively charged and the intensity of electric field inside the condenser is E. Find the period of vibration of the pendulum and the angle between the thread in equilibrium position and the vertical.

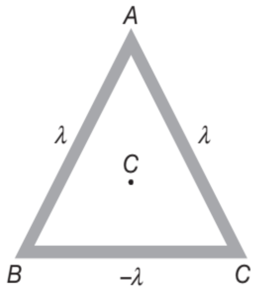
14. A block of mass m containing a net positive charge q is placed on a smooth horizontal table which terminates in a vertical wall as shown in figure. The distance of the block from the wall is d. A horizontal electric field E towards right is switched on. Assuming collisions (if any) to be only of elastic nature, find the time period of resulting oscillatory motion. Is it a simple harmonic motion.



15. In the given arrangement of a charged square frame find electric field at centre. The linear charged density is as shown in figure.



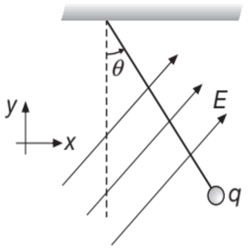
16. In the figure shown, three charged rods each of length L, having charge densities λ, $-\lambda$ and λ form a triangle. Calculate the electric field due to the arrangement at its centre.



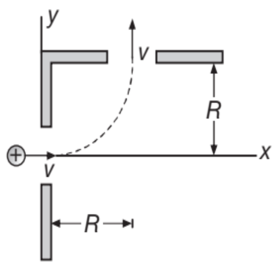
17. A charged cork ball of mass m is suspended on a light string in the presence of a uniform electric field as shown in figure. When an electric field given by $E=\left(A\hat{i}+B\hat{j}\right)\text{NC}^{-1}$ is switched on (where A and B are positive numbers) the ball attains equilibrium making an angle θ with the vertical. Find

(a) the charge on the ball and

(b) the tension in the string.



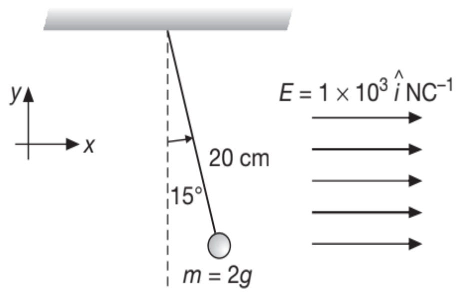
18. A physicist studying the properties of ions in the upper atmosphere wishes to construct an apparatus having the following characteristics. In the setup used, an electric field is applied such that a beam of ions, each having charge q, mass m, and initial velocity $v\hat{i}$, is turned through an angle of $90°$ such that each ion undergoes displacement $R\hat{i}+R\hat{j}$. The ions enter a chamber as shown in figure, and leave through the exit port with the same speed they had when they entered the chamber. The electric field acting on the ions is to have constant magnitude.



(a) Suppose the electric field is produced by two concentric cylindrical electrodes (not shown in the diagram) and hence is radial. Calculate the magnitude of the electric field.

(b) If the field is produced by two flat plates and is uniform in direction, what value should the field have in this case?

19. A small, 2 g plastic ball is suspended by a 20 cm long string in a uniform electric field as shown in Figure. If the ball is in equilibrium when the string makes a $15°$ angle with the vertical, what is the net charge on the ball?



20. Consider n equal positive point charges each of magnitude $\dfrac{Q}{n}$ placed symmetrically around a circle of radius R. Calculate the magnitude of the electric field at a point a distance x on the line passing through the centre of the circle and perpendicular to the plane of the circle.

21. A line of charge starts at $x=+x_0$ and extends to positive infinity. The linear charge density is $\lambda=\dfrac{\lambda_0 x_0}{x}$. Determine the electric field at the origin.

22. A rod lies along x-axis with one end at origin and other at $x\to\infty$. The rod has a uniform charge density of $\lambda\ \text{Cm}^{-1}$. Find the electric field at the point $x=-a$ on x-axis. Find E_x and E_y for a point on the y-axis where $y=b$.

MOTION OF A CHARGED PARTICLE IN AN ELECTRIC FIELD

CASE-1:

In uniform electric field when a positively charge particle is **released from rest**, it starts moving in the direction of electric field with uniform acceleration a given by

$$a = \frac{qE}{m}$$

Then $v = at = \left(\frac{qE}{m}\right)t$

$$y = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{qE}{m}\right)t^2$$

and kinetic energy,

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{q^2E^2}{m}\right)t = yqE$$

CASE-2:

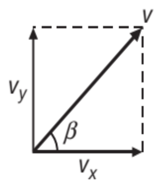
In uniform electric field when a positively charge particle is released with **non-zero initial velocity** u **perpendicular** to the electric field, then
Velocity at any instant

Also, $\vec{v} = \vec{u} + \vec{a}t$

$$\Rightarrow \quad \vec{v} = u\hat{i} + \left(\frac{qEt}{m}\right)\hat{j} \quad \left\{\because \vec{E} = E\hat{j}(\text{say}), \text{ then } \vec{u} = u\hat{i}\right\}$$

Comparing with

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$



we have

$$v_x = u$$

$$v_y = \frac{qEt}{m}$$

Since, $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$

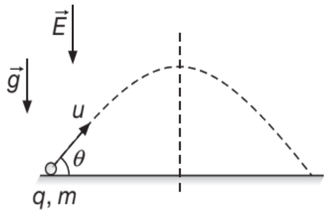
$$\Rightarrow \quad v = \sqrt{u^2 + \frac{q^2E^2t^2}{m^2}}$$

If β is the angle made by v with x-axis, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{qEt}{mu}$$

CASE-3:

If the particle is thrown in a uniform electric field in the direction different from the direction of electric field, it will follow a parabolic trajectory like projectile motion. Consider the following case.



A particle of mass m and charge q is thrown from ground at an angle θ with initial speed u. A uniform electric field $\vec{E}$ exists in downward direction as shown. Here during motion of particle we can consider the effective acceleration to be

$$a = g + \frac{qE}{m}$$

Here in above example we can use all the concepts of projectile motion by replacing g by a. If electric field in space is in upward direction, effective acceleration due to gravity will be

$$a = g - \frac{qE}{m}$$

ILLUSTRATION 30

A particle of mass 9×10^{-31} kg and a negative charge of 1.6×10^{-19} C is projected horizontally with a velocity of 10^5 ms^{-1} into a region between two infinite horizontal parallel plates of metal. The distance between the plates is 0.3 cm and the particle enters 0.1 cm below the top plate. The top and bottom plates are connected respectively to the positive and negative terminals of a 30 V battery. Calculate the component of the velocity of the particle just before it hits one of the plates.

SOLUTION

$$E = \frac{V}{d}$$

$$\Rightarrow \quad E = \frac{30}{3 \times 10^{-3}} = 10^4 \text{ NC}^{-1} \qquad \{\because d = 3 \times 10^{-3} \text{ m}\}$$

Force on the particle of negative charge moving between the plates is

$$F = eE = 1.6\times10^{-19}\times10^{4} = 1.6\times10^{-15}\ \text{N}$$

The direction of force will be towards the positive plate i.e., upwards

Acceleration of the particle is

$$a=\frac{eE}{m}$$

$$\Rightarrow \quad a=\frac{(1.6\times10^{-15})}{(9\times10^{-31})}$$

$$\Rightarrow \quad a = 1.77\times10^{15}\ \text{ms}^{-2}$$

The electric intensity E is acting in the vertical direction so the horizontal velocity v of the particle remains same. If y is the displacement of the particle, in upward direction, then

$$y=\frac{1}{2}at^2$$

where $y = 0.1\ \text{cm} = 10^{-3}\ \text{m}$, $a = 1.77\times10^{15}\ \text{ms}^{-2}$

$$\Rightarrow \quad 10^{-3}=\frac{1}{2}\times(1.77\times10^{15})(t^2)$$

$$\Rightarrow \quad t = 1.063\times10^{-9}\ \text{s}$$

Component of velocity in the direction of field is given by

$$v_y = at$$

$$\Rightarrow \quad v_y=(1.77\times10^{15})(1.063\times10^{-9})$$

$$\Rightarrow \quad v_y = 1.881\times10^{6}\ \text{ms}^{-1} \text{ and } v_x = 10^{5}\ \text{ms}^{-1}$$

ILLUSTRATION 31

A projectile of mass m having charge q is thrown with initial velocity u at an angle α with the horizontal. An electric field of strength E exists, making an angle β with the downward vertical away from the point of projection. Find the time of flight and range of projectile.

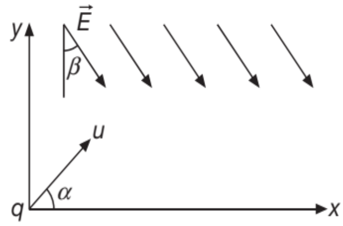
SOLUTION

Initial velocity is given by (in components)

$$u_y = u\sin\alpha,\ u_x = u\cos\alpha$$

and acceleration $a_y=\left(\dfrac{qE}{m}\cos\beta+g\right)$

and $a_x=\dfrac{qE\sin\beta}{m}$



Time of flight $T=\dfrac{2v_y}{a_y}=\dfrac{2u\sin\alpha}{\left(\dfrac{qE}{m}\cos\beta+g\right)}$

In x direction range of projectile is given by

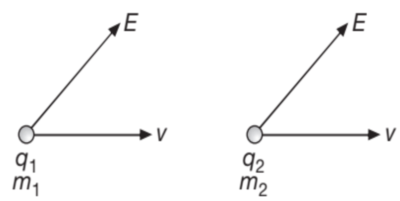
$$x = u_x t+\frac{1}{2}a_x t^2$$

$$\Rightarrow \quad R = u_x T+\frac{1}{2}a_x T^2$$

$$\Rightarrow \quad R = u\cos\alpha\left(\frac{2u\sin\alpha}{\dfrac{qE}{m}\cos\beta+g}\right)+\frac{1}{2}\left(\frac{qE\sin\beta}{m}\right)\left(\frac{2u\sin\alpha}{\dfrac{qE}{m}\cos\beta+g}\right)^2$$

ILLUSTRATION 32

Two balls of charge q_1 and q_2 initially have a velocities of equal magnitude and same direction. After a uniform electric field has been applied for a certain time interval, the direction of first ball changes by 60° and the velocity magnitude is reduced by half. The direction of velocity of the second ball changes thereby 90°.



Initial condition

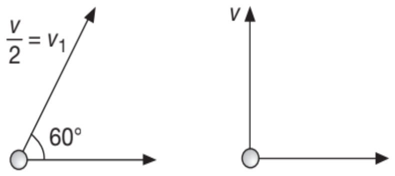
(a) Determine the magnitude of the charge to mass ratio of the second ball if it is equal to α_1 for the first ball.

(b) In what ratio will the velocity of the second ball change?
Ignore the electrostatic interaction between the balls

SOLUTION

(a) Let the electric field acting on each ball be given by

$E = E_x\hat{i} + E_y\hat{j}$



Final condition

From Impulse Momentum equation, we have

Impulse = Change in momentum

Let the final velocities of the balls be v_1 and v_2.

Noting that $v_1 = \frac{v}{2}$, we have

$$q_1\left(E_x\hat{i} + E_y\hat{j}\right)\Delta t = m_1\left(\frac{v}{2}\cos 60°\hat{i} + \frac{v}{2}\sin 60°\hat{j}\right) - m_1 v\hat{i} \quad \ldots(1)$$

$$q_2\left(E_x\hat{i} + E_y\hat{j}\right)\Delta t = m_2\left(v_2\cos 90°\hat{i} + v_2\sin 90°\hat{j}\right) - m_2 v\hat{i} \quad \ldots(2)$$

On comparing the x and y-components on both sides of equation (1), we get

$$\frac{q_1}{m_1}E_x\Delta t = -\frac{3}{4}v \text{ and} \quad \ldots(3)$$

$$\frac{q_1}{m_1}E_y\Delta t = \frac{\sqrt{3}}{4}v \quad \ldots(4)$$

Similarly, from equation (2), we get

$$\frac{q_2}{m_2}E_x\Delta t = -v \text{ and} \quad \ldots(5)$$

$$\frac{q_2}{m_2}E_y\Delta t = v_2 \quad \ldots(6)$$

From equations (3) and (5), by dividing the equations expressing x-components, we get

$$\frac{\left(\frac{q_1}{m_1}\right)}{\left(\frac{q_2}{m_2}\right)} = \frac{3}{4} \quad \ldots(7)$$

$$\Rightarrow \frac{q_2}{m_2} = \frac{4}{3}\frac{q_1}{m_1} = \frac{4}{3}\alpha_1$$

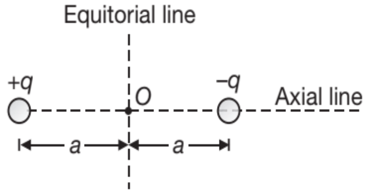
(b) Also $\dfrac{\frac{q_1}{m_1}}{\frac{q_2}{m_2}} = \dfrac{\sqrt{3}v}{4v_2}$

$$\Rightarrow \frac{\sqrt{3}v}{4v_2} = \frac{3}{4}$$

$$\Rightarrow v_2 = \frac{v}{\sqrt{3}}$$

ELECTRIC DIPOLE AND DIPOLE MOMENT

Two equal and opposite charges separated by a distance together constitute a dipole.



Dipole moment $\left(\vec{p}\right)$ is defined as the simple product of magnitude of either charge and the distance of separation between the two charges.

$\vec{p} = q\left(2\vec{a}\right)$

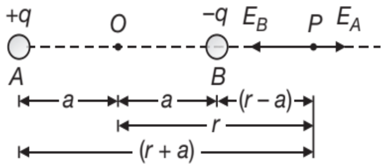
Dipole moment $\vec{p}$ always points from $-q$ to $+q$. Its SI unit is coulombmetre(Cm).

ELECTRIC FIELD DUE TO A DIPOLE AT A POINT LYING ON THE AXIAL LINE (END ON POSITION)

The electric field due to a dipole at point P at distance r from the centre of the dipole is

$$E_{\text{axial}} = E_B - E_A$$

$$\Rightarrow E_{\text{axial}} = \frac{q}{4\pi\varepsilon_0\left(r-a\right)^2} - \frac{q}{4\pi\varepsilon_0\left(r+a\right)^2}$$



$$\Rightarrow \quad E_{\text{axial}} = \frac{1}{4\pi\varepsilon_0}\frac{4raq}{\left(r^2-a^2\right)^2}$$

Since, $p = q(2a)$. So,

$$E_{\text{axial}} = \frac{1}{4\pi\varepsilon_0}\frac{2pr}{\left(r^2-a^2\right)^2}$$

For $\quad r >> a$

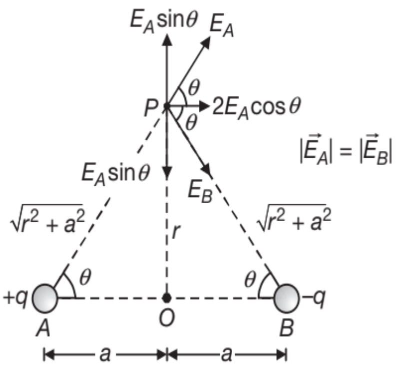
$$E_{\text{axial}} \simeq \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}$$

ELECTRIC FIELD DUE TO A DIPOLE AT A POINT LYING ON THE EQUITORIAL LINE (BROAD SIDE ON POSITION)

The electric field due to the dipole at the point P at distance r from O is

$$E_{\text{equitorial}} = 2E_A\cos\theta \text{ where } E_A = \frac{q}{4\pi\varepsilon_0\left(r^2+a^2\right)}$$

(Because the components $E_A\sin\theta$ and $E_A\sin\theta$ cancel out)



$$\Rightarrow \quad E_{\text{equitorial}} = 2\frac{q}{4\pi\varepsilon_0\left(r^2+a^2\right)}\frac{a}{\sqrt{r^2+a^2}}$$

$$\left\{\because \cos\theta = \frac{a}{\sqrt{a^2+r^2}}\right\}$$

$$\Rightarrow \quad E_{\text{equitorial}} = \frac{1}{4\pi\varepsilon_0}\frac{p}{\left(r^2+a^2\right)^{3/2}} \quad \left\{\begin{matrix}\text{from positive to}\\ \text{negative charge}\end{matrix}\right\}$$

Vectorially $\vec{E}_{\text{equitorial}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\left(-\vec{p}\right)}{\left(r^2+a^2\right)^{3/2}}$

For $\quad r >> a$

$$E_{\text{equitorial}} \simeq \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}$$

Remark(s)

So we conclude that at points lying far away from the centre of the dipole $\{\text{for } r >> a\}$

$$E_{\text{axial}} \simeq 2E_{\text{equitorial}}$$

ILLUSTRATION 33

The electric field due to a short dipole at a distance r, on the axial line, from its mid-point is the same as that of electric field at a distance r', on the equatorial line, from its mid-point. Determine the ratio $\dfrac{r}{r'}$.

SOLUTION

$$\frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r'^3}$$

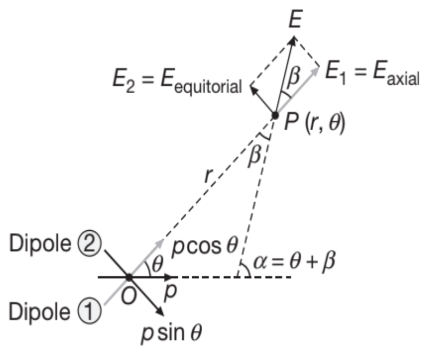
$$\Rightarrow \quad \frac{2}{r^3} = \frac{1}{r'^3}$$

$$\Rightarrow \quad \frac{r^3}{r'^3} = 2$$

$$\Rightarrow \quad \frac{r}{r'} = 2^{\frac{1}{3}}$$

ELECTRIC FIELD DUE TO A DIPOLE AT ANY POINT $P(r, \theta)$

Consider a point $P(r, \theta)$ at a large distance r from the centre O of the dipole at an angle θ with $\vec{p}$ (as shown).



On resolving $\vec{p}$ in two components $p_1 = p\cos\theta$ and $p_2 = p\sin\theta$, we observe that the point P lies at the axial line of Dipole ① (having dipole moment $p_1 = p\cos\theta$) and on the equatorial line of Dipole ② (having dipole moment $p_2 = p\sin\theta$). So,

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{2p_1}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3} \text{ and}$$

$$E_2 = \frac{1}{4\pi\varepsilon_0}\frac{p_2}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}$$

$$\Rightarrow \quad E = \sqrt{E_1^2 + E_2^2}$$

$$\Rightarrow \quad E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{1+3\cos^2\theta}$$

If β is the angle made by the net field E with E_1 then,

$$\tan\beta = \frac{E_2}{E_1} = \frac{1}{2}\tan\theta$$

$$\Rightarrow \quad \tan\beta = \frac{1}{2}(\tan\theta)$$

So, we observe that the net field E makes an angle $\alpha = \theta + \beta$ with the dipole moment $\vec{p}$.

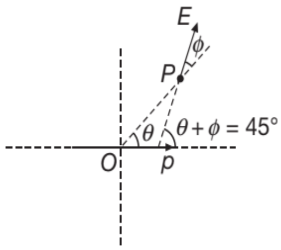
ILLUSTRATION 34

Find the locus of all the points where the resultant $\vec{E}$ field will always have a bearing of $45°$ with the axis of a short dipole.

SOLUTION

$$\theta + \phi = \frac{\pi}{4} \text{ (Given)} \quad \ldots(1)$$

Since, $\tan\phi = \frac{1}{2}\tan\theta$ (from the theory of dipole) …(2)



So, from (1) and (2), we get

$$\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1}{2}\tan\theta$$

$$\Rightarrow \quad \tan^2\theta + 3\tan\theta - 2 = 0$$

Solving this Quadratic in $\tan\theta$, we get

$$\tan\theta = \frac{-3 \pm \sqrt{17}}{2} = 0.56 \text{ or } -3.56$$

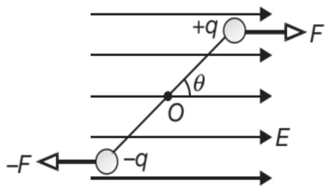
$$\Rightarrow \quad \theta = 29°18'$$

TORQUE ON A DIPOLE PLACED IN A UNIFORM ELECTRIC FIELD

Suppose an electric dipole is placed in a uniform external electric field $\vec{E}$ where the dipole moment makes an angle θ with the field. The forces on the two charges are equal and opposite as shown, each having a magnitude

$$F = qE$$

Thus, we see that the net force on the dipole is zero. However, the two forces produce a net torque on the dipole, and the dipole tends to rotate such that its axis gets aligned with the field.



The torque due to the force on the positive charge about an axis through O is given by $Fa\sin\theta$ where, $a\sin\theta$ is the moment arm of F about O. This force tends to produce a clockwise rotation. Likewise, the torque on the negative charge about O is also $Fa\sin\theta$, and so the net torque τ about O is given by

$$\tau = 2Fa\sin\theta$$

Since $F = qE$ and $p = 2aq$

$$\Rightarrow \quad \tau = 2aqE\sin\theta$$

$$\Rightarrow \quad \tau = pE\sin\theta$$

It is convenient to express the torque in vector form as the cross product of the vectors $\vec{p}$ and $\vec{E}$, so vectorially,

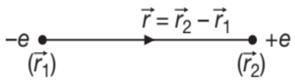
$$\vec{\tau} = \vec{p} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_x & p_y & p_z \\ E_x & E_y & E_z \end{vmatrix}$$

ILLUSTRATION 35

A charge e is placed at (1, 2 , 1) and another charge $-e$ is placed at (0, 1, 0) such that they form an electric dipole. There exists a uniform electric field $E=\left(2\hat{i}+3\hat{j}\right)$. Calculate the dipole moment vector and torque experienced by the dipole.

SOLUTION

(i) $\vec{p}=q\vec{r}$



$\Rightarrow \quad \vec{p}=e\left(\vec{r}_2-\vec{r}_1\right)$

Now, $\vec{r}_1=\hat{j}$

$\vec{r}_2=\hat{i}+2\hat{j}+\hat{k}$

$\Rightarrow \quad \vec{r}_2-\vec{r}_1=\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow \quad \vec{p}=e\left(\hat{i}+\hat{j}+\hat{k}\right)$

and $\quad |\vec{p}|=\sqrt{3}e$

(ii) $\vec{\tau}=\vec{p}\times\vec{E}$

$\Rightarrow \quad \vec{\tau}=e\left(\hat{i}+\hat{j}+\hat{k}\right)\times\left(2\hat{i}+3\hat{j}\right)$

$$\vec{\tau}=e\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2 & 3 & 0\end{vmatrix}=e\left(-3\hat{i}+2\hat{j}+\hat{k}\right)\text{Nm}$$

POTENTIAL ENERGY OF A DIPOLE PLACED IN A UNIFORM ELECTRIC FIELD

Work must be done by an external agent to rotate the dipole through a given angle in the field. This work done is then stored as potential energy in the system, that is, the dipole and the external field. The work dW required to rotate the dipole through an angle $d\theta$ is given by

$$dW=\tau\,d\theta$$

Since,

$$\tau=pE\sin\theta$$

This work is transformed into potential energy U. We find this for a rotation from θ_0 to θ. So,

$$U=\int_{\theta_0}^{\theta}\tau\,d\theta=\int_{\theta_0}^{\theta}pE\sin\theta d\theta=pE\int_{\theta_0}^{\theta}\sin\theta d\theta$$

$$\Rightarrow \quad U=pE\left(-\cos\theta\right)\Big|_{\theta_0}^{\theta}=pE\left(\cos\theta_0-\cos\theta\right)$$

The term involving $\cos\theta_0$ is a constant that depends on the initial orientation of the dipole. It is convenient to choose $\theta_0=90^\circ$, so that $\cos\theta_0=\cos 90^\circ=0$. In this case, we can express U as

$$U=-pE\cos\theta$$

This is equivalent to the dot product of the vectors $\vec{p}$ and $\vec{E}$. So,

$$U=-\vec{p}\cdot\vec{E}=-\left(p_xE_x+p_yE_y+p_zE_z\right)$$

ILLUSTRATION 36

When an electric dipole is placed in a uniform electric field making angle θ with electric field, it experiences a torque τ. Calculate the minimum work done in changing the orientation to 2θ.

SOLUTION

$$\tau=pE\sin\theta$$

$$\Rightarrow \quad pE=\frac{\tau}{\sin\theta}$$

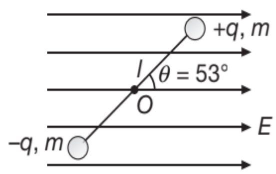
$$W=\Delta U=-pE\cos\theta_2+pE\cos\theta_1$$

$$\Rightarrow \quad W=pE\left(\cos\theta-\cos 2\theta\right) \qquad \left\{\because \theta_1=\theta,\ \theta_2=2\theta\right\}$$

$$\Rightarrow \quad W=\frac{\tau}{\sin\theta}\left(\cos\theta-\cos 2\theta\right)$$

ILLUSTRATION 37

In the arrangement shown, if the dipole is released from $\theta=53^\circ$, then calculate



(i) Its angular acceleration just after releasing.

(ii) Its angular velocity when it passes through stable equilibrium.

(iii) The work required to rotate it by 180°

SOLUTION

(i) $\tau_{\text{net}}=pE\sin\left(53^\circ\right)=I\alpha$

$$\Rightarrow \quad \alpha = \frac{(ql)E\left(\frac{4}{3}\right)}{\frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2}$$

$$\Rightarrow \quad \alpha = \frac{\frac{4}{3}qlE}{\frac{7}{12}ml^2} = \frac{16}{7}\left(\frac{qE}{ml}\right)$$

(ii) By Law of Conservation of Energy, we get

$$(K+U)_{\text{initial}} = (K+U)_{\text{final}}$$

$$\Rightarrow 0 + (-pE(\cos 53°)) = \frac{1}{2}I\omega^2 + (-pE\cos 0°)$$

Where $I = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = \frac{7}{12}ml^2$

$$\Rightarrow \omega = \sqrt{\frac{4pE}{5I}} = \sqrt{\frac{4qlE}{5\left(\frac{7}{12}ml^2\right)}} = \sqrt{\frac{48qE}{35ml}}$$

(iii) $W = U_f - U_i$

$$\Rightarrow \quad W = (-pE\cos(180° + 53°)) - (-pE\cos 53°)$$

$$W = (ql)E\left(\frac{4}{3}\right) + (ql)E\left(\frac{4}{3}\right)$$

$$\Rightarrow \quad W = \left(\frac{8}{3}\right)qlE$$

ILLUSTRATION 38

A light massless rod of length l lies in x-y plane with its centre at origin and it makes an angle θ with x-axis. A particle of mass m and charge $-q$ is attached at its left end and another particle of mass m and charge q at the other end. Write the dipole moment vector. Now an electric field of constant magnitude E and directed along x-axis is switched on. Write an expression for the torque on the rod at the initial positon. What is the angular speed of the rod at the instant when it becomes parallel to x-axis ?

SOLUTION

The dipole moment has a magnitude

$$p = q(2a) = ql$$

And it is directed along a unit vector $\cos\theta\hat{i} + \sin\theta\hat{j}$

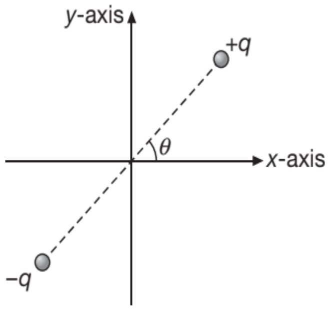
$$\Rightarrow \quad \vec{p} = ql[\cos\theta\hat{i} + \sin\theta\hat{j}]$$

The electric field is given as

$$\vec{E} = E\hat{i}$$

Since $\vec{\tau} = \vec{p} \times \vec{E}$

$$\Rightarrow \quad \vec{\tau} = ql(\cos\theta\hat{i} + \sin\theta\hat{j}) \times E\hat{i} = -(qlE\sin\theta)\hat{k}$$



To calculate angular speed, applying Law of Conservation of Mechanical Energy, i.e.

$$U_i + K_i = U_f + K_f$$

where, $U_i = -\vec{p}\cdot\vec{E} = -pE\cos\theta$

$$U_f = -\vec{p}\cdot\vec{E} = -pE$$

{$\because$ rod becomes parallel to x-axis}

$K_i = 0$ and

$$K_f = \frac{1}{2}I\omega^2 = \frac{1}{2}\left[m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2\right]\omega^2$$

$$\Rightarrow \quad -pE\cos\theta = -pE + \frac{ml^2}{4}\omega^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{4qE(1-\cos\theta)}{ml}}$$

SMALL OSCILLATIONS OF A DIPOLE PLACED IN A UNIFORM ELECTRIC FIELD

Let the dipole possess a moment of inertia I about O. Since

$$\tau = -pE\sin\theta$$

where negative sign indicates that this is the restoring torque which tries to restore dipole to its mean position.

For small θ, $\sin\theta \cong \theta$

$$\Rightarrow \quad \tau = -pE\theta$$

Further $\tau = I\alpha = I\ddot{\theta}$

$$\Rightarrow \quad I\ddot{\theta} = -pE\theta$$

$$\Rightarrow \quad \ddot{\theta} + \left(\frac{pE}{I}\right)\theta = 0$$

This equation shows that the dipole will execute oscillations when given a small angular displacement from the mean position. The time period of oscillations is T given by

$$T = 2\pi\sqrt{\frac{\theta}{|\ddot{\theta}|}} = 2\pi\sqrt{\frac{I}{pE}} = 2\pi\sqrt{\frac{I_{CM}}{pE}}$$

where I_{CM} is the moment of inertia of the dipole about the centre of mass of the dipole.

FORCE ON AN ELECTRIC DIPOLE IN NON-UNIFORM ELECTRIC FIELD

In a non-uniform electric field, if a dipole is placed at a point where electric field is $\vec{E}$, then the interaction energy of dipole at this point is given by

$$U = -\vec{p}\cdot\vec{E}$$

The force on dipole due to electric field is

$$\vec{F} = -\vec{\nabla}U$$

For unidirectional variation in electric field, we get

$$|\vec{F}| = -\frac{d}{dx}(\vec{p}\cdot\vec{E})$$

If the dipole is placed in the direction of electric field, then

$$F = -p\frac{dE}{dx}$$

ILLUSTRATION 39

An electric dipole is placed at distance x from an infinitely long rod of linear charge density λ.

(a) What is net force acting on dipole?

(b) What is work done in rotating dipole through $180°$?

(c) If dipole is slightly rotated about its equilibrium position. Find the time period of oscillation.

SOLUTION

(a) We know that electric field at a distance x from the rod

$$E = \frac{\lambda}{2\pi\varepsilon_0 x} \qquad \{\text{linear charge distribution}\}$$

directed along the axis of the dipole. The direction can be determined by the fact that electric field is always perpendicular to equipotential surface.

Net force on dipole due to electric field of rod

$$F = q\frac{dE}{dx}$$

As we know potential difference $= dV = -Edr$

Change in electrostatic Potential Energy

$$= dU = \frac{dV}{q} = -Edr$$

$$dU = -qEdr$$

$$F = -\frac{dU}{dr} = \frac{pdE}{dr} \qquad \{\because U = -pE \text{ as } p \parallel E\}$$

As dipole is lying along x-axis

$$F = p\frac{dE}{dx} = 2aq\frac{d}{dx}\left[\frac{\lambda}{2\pi\varepsilon_0 x}\right]$$

$$F = \frac{\lambda aq}{\pi\varepsilon_0 x^2}$$

(b) Workdone = Difference in P.E. at two positions

$$\Rightarrow W = -pE\cos 180° - (-pE\cos 0)$$

$$\Rightarrow W = 2(2aq)\left(\frac{\lambda}{2\pi\varepsilon_0 x}\right) = \frac{2\lambda aq}{\pi\varepsilon_0 x}$$

(c) When the dipole is slightly disturbed from its equilibrium position by an angle θ.

Restoring torque $= -pE\sin\theta$

$$\{\sin\theta \simeq \theta \text{ for small displacement}\}$$

$$\Rightarrow \tau = -pE\theta = -2aq \times \frac{\lambda\theta}{2\pi\varepsilon_0 x} = \frac{-\lambda a\theta q}{\pi\varepsilon_0 x}$$

Now torque is also given as

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

where I is the moment of inertia and α is angular acceleration

$$I\frac{d^2\theta}{dt^2}=-\frac{\lambda a\theta q}{\pi\varepsilon_0 x} \qquad \{\text{for dipole } I=2ma^2\}$$

$$\Rightarrow\ 2ma^2\frac{d^2\theta}{dt^2}=\frac{\lambda a\theta q}{\pi\varepsilon_0 x}$$

$$\Rightarrow\ \frac{d^2\theta}{dt^2}+\frac{\lambda q\theta}{2\pi\varepsilon_0 \max}=0$$

Comparing it with standard equation of S.H.M.

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0,\text{ where } \omega=\sqrt{\frac{\lambda q}{2\pi\varepsilon_0 \max}}$$

$$\Rightarrow\ T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{2\pi\varepsilon_0 \max}{\lambda q}}$$

FORCE ON A DIPOLE IN THE SURROUNDING OF A LONG CHARGED WIRE

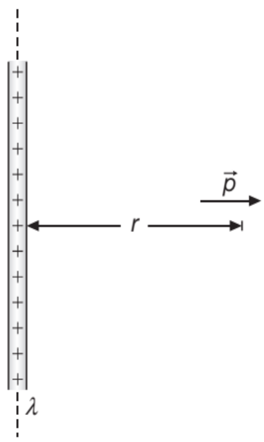
In the situation shown in figure, the electric field strength due to the wire, at the position of dipole is

$$E=\frac{2\lambda}{4\pi\varepsilon_0 r}=\frac{\lambda}{2\pi\varepsilon_0 r}$$

Thus force on dipole is

$$F=-p\left(\frac{dE}{dr}\right)=-p\left[-\frac{\lambda}{2\pi\varepsilon_0 r^2}\right]$$

$$\Rightarrow\ F=\frac{\lambda p}{2\pi\varepsilon_0 r^2}$$



Here negative charge of dipole is placed close to wire and hence net force on the dipole due to wire will be attractive.

ELECTRIC FORCE BETWEEN TWO DIPOLES

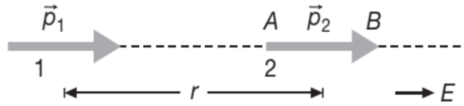
Electrostatic force between two dipoles of dipole moments p_1 and p_2 lying at a separation $r(r\gg a)$ is

$$F=\begin{cases}\dfrac{1}{4\pi\varepsilon_0}\dfrac{6p_1p_2}{r^4}, & \text{when dipoles are placed coaxially to each other}\\[2ex] \dfrac{1}{4\pi\varepsilon_0}\dfrac{3p_1p_2}{r^4}, & \text{when dipoles are placed perpendicular to each other}\end{cases}$$

The above results have been derived here to give you a view of the fundamentals related to dipoles. Please keep in mind that the above results are obtained on the basis of assumption that the centres of the dipoles lie far apart from each other i.e., $r\gg 2a$ for each one of them.

CASE-1:

When the dipoles are placed coaxially



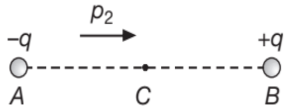
Consider two dipoles with dipole moments $\vec{p}_1$ and $\vec{p}_2$ (as shown) with their centres at distance $r(\gg 2a)$.

The electric field of the dipole 1 on the left side exerts a net force on the dipole 2 placed to its right. The electric field due to 1 at the centre of 2 is

$$E=\frac{1}{4\pi\varepsilon_0}\frac{2p_1}{r^3} \qquad \{\text{from } A \text{ to } B\}$$

$$\Rightarrow\ dE=\frac{2p_1}{4\pi\varepsilon_0}d\left(r^{-3}\right)$$

$$\Rightarrow\ dE=-\frac{6p_1}{4\pi\varepsilon_0 r^4}dr \qquad \ldots(1)$$



C is midpoint of dipole 2 with AB = dr

Let the $-q$ charge of dipole be placed at A and the $+q$ charge be placed at B. So, the electric field at A (where $-q$ lies) is

$$E_A=E+|dE|$$

{obviously field due to dipole 1 is stronger at A in comparison to B}

and $E_B = E - |dE|$

As a result of this, force on $-q$ at A is

$$F_1 = qE_A \qquad \text{\{towards left\}}$$

and that on $+q$ at B is

$$F_2 = qE_B \qquad \text{\{towards right\}}$$

$\Rightarrow$ Net force on the dipole is

$$F = F_1 - F_2 \qquad \text{\{towards left\}}$$

$$\Rightarrow F = qE_A - qE_B$$

$$\Rightarrow F = q\left[\left(E+|dE|\right)-\left(E-|dE|\right)\right]$$

$$\Rightarrow F = 2q|dE|$$

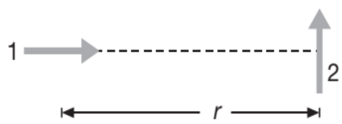
$$\Rightarrow F = 2q\left(\frac{6p_1}{4\pi\varepsilon_0 r^4}\right)dr$$

$$\Rightarrow F = \frac{6p_1(2qdr)}{4\pi\varepsilon_0 r^4}$$

$$\Rightarrow F = \frac{6p_1p_2}{4\pi\varepsilon_0 r^4} \qquad \{\because 2q(dr) = p_2\}$$

CASE-2:

When the dipoles are placed perpendicular to each other.



Proceeding exactly the same way as in case-1, we get

$$F = \frac{3p_1p_2}{4\pi\varepsilon_0 r^4}$$

All assumptions here are exactly the same as before, other than this that the field due to dipole 1 at the centre of 2 will be

$$E = \frac{p_1}{4\pi\varepsilon_0 r^3}$$

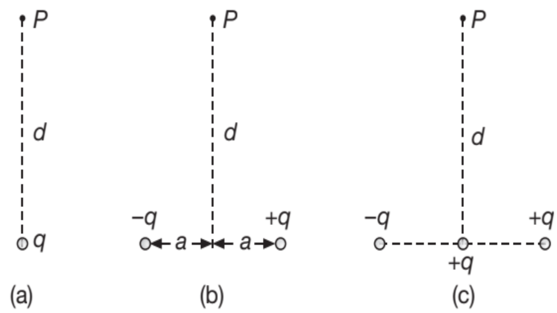
and hence we get

$$F = \frac{3p_1p_2}{4\pi\varepsilon_0 r^4}$$

as the force between the dipoles placed perpendicular to each other.

ILLUSTRATION 40

Find the magnitude of the electric field at the point P in the configuration shown in figure for $d \gg a$. Take $2qa = p$.



SOLUTION

(a) The electric field due to a point charge at P is given by

$$E_P = \frac{q}{4\pi\varepsilon_0 d^2}$$

(b) As $d \gg a$ a combination of $-q$ and $+q$ can be treated as a dipole. Hence, the electric field at P is

$$E_P = \frac{q(2a)}{4\pi\varepsilon_0 d^3} \qquad \text{(towards left)}$$

Direction will be horizontal towards left

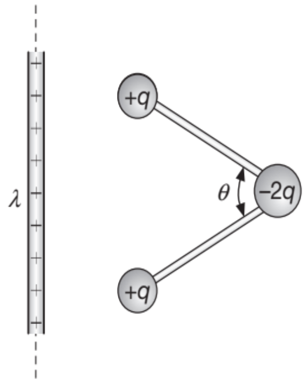
(c) This case is the super position of above two cases

Thus we have $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q}{d^2}\hat{j} + \left(-\dfrac{p}{d^3}\right)\hat{i}\right]$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0 d^2}\left(q\hat{i} - \frac{p}{d}\hat{j}\right)$$

ILLUSTRATION 41

A water molecule is placed at a distance l from the line carrying linear charge density λ. Find the maximum force exerted on the water molecule. The shape of water molecule and the partial charges on H and O atoms are shown in figure. Assume l to be much larger than the dimensions of the molecule.



SOLUTION

The figure can be resolved as combination of 2 dipoles

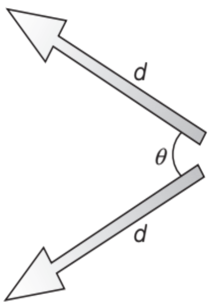
Dipole moments of each $p = qd$

Here total dipole moment of system is

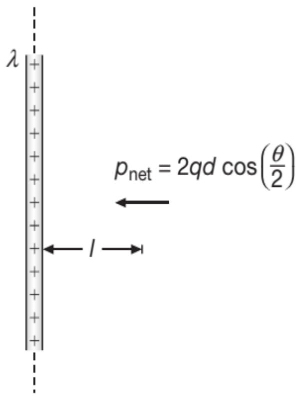
$$p_{\text{net}} = 2qd\cos\left(\frac{\theta}{2}\right)$$

Now

$$|\vec{F}| = \vec{p}_{\text{net}}\left(\frac{d\vec{E}}{dx}\right)$$



For maximum force, the angle between $\vec{p}_{\text{net}}$ and $\dfrac{d\vec{E}}{dx}$ is $0°$



$$\Rightarrow \quad F_{\max} = 2qd\cos\left(\frac{\theta}{2}\right)\left[\frac{d}{dx}\left(\frac{\lambda}{2\pi\varepsilon_0 x}\right)\right]$$

$$\Rightarrow \quad F_{\max} = 2qd\cos\left(\frac{\theta}{2}\right)\times\frac{\lambda}{2\pi\varepsilon_0}\left(-\frac{1}{x^2}\right)$$

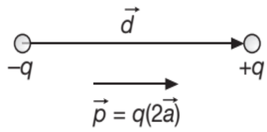
$$\Rightarrow \quad F_{\max} = \frac{\lambda qd\cos\left(\frac{\theta}{2}\right)}{\pi\varepsilon_0 x^2}$$

$$\Rightarrow \quad |\vec{F}_{\max}| = \frac{\lambda qd\cos\left(\frac{\theta}{2}\right)}{\pi\varepsilon_0 x^2}$$

$$F_{\max} = \frac{\lambda qd\cos\left(\frac{\theta}{2}\right)}{l^2}$$

CONCEPT OF DISTRIBUTED DIPOLE

Till now we have discussed about an electric dipole that has two equal and opposite charges separated by a small distance $2\vec{a}$ as shown. Its dipole moment is defined, as $\vec{p} = q(2\vec{a})$ from $-q$ to $+q$.



Sometimes the two charges of dipole are not concentrated at its ends. The system, instead of having two charges has more than two charges.

In such a case we select a convenient origin. Then we locate the coordinates of the charges. Consider a system having charges q_1, q_2 and q_3 at points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) then

$$p_x = q_1x_1 + q_2x_2 + q_3x_3,$$

$$p_y = q_1y_1 + q_2y_2 + q_3y_3$$

and $\quad p_z = q_1z_1 + q_2z_2 + q_3z_3$

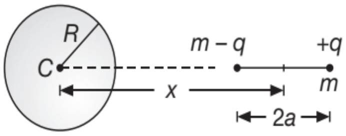
and the net dipole moment is then given by

$$\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$$

where $|\vec{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$

ILLUSTRATION 42

In figure shown, an electric dipole lies at a distance x from the centre of the axis of a charged ring of radius R with charge Q uniformly distributed over it.



(a) Find the net force acting on the dipole.

(b) What is the work done in rotating the dipole through $180°$?

(c) The dipole is slightly rotated about its equilibrium position. Find the time period of oscillation. Assume that the dipole is linearly restrained.

SOLUTION

(a) Field at a distance x on the axis of a ring is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{\left(R^2+x^2\right)^{3/2}}$$

The net force on the dipole is

$$F = p\frac{dE}{dx} = 2aq\frac{1}{4\pi\varepsilon_0}Q\frac{d}{dx}\left[\frac{x}{\left(R^2+x^2\right)^{3/2}}\right]$$

$$F = \frac{aQq}{2\pi\varepsilon_0}\frac{R^2-2x^2}{\left(R^2+x^2\right)^{5/2}}$$

(b) Work done in rotating a dipole is equal to change in its potential energy.

$$W = \Delta U$$

$$\Rightarrow \quad U_i = -\vec{p}\cdot\vec{E} = -pE\cos(0°) = -pE$$

and $$U_f = -\vec{p}\cdot\vec{E} = -pE\cos(180°) = +pE$$

$$\Rightarrow \quad W = 2pE$$

$$\Rightarrow \quad W = \frac{2(2a)}{4\pi\varepsilon_0}\frac{Qx}{\left(R^2+x^2\right)^{3/2}} = \frac{aQqx}{\pi\varepsilon_0\left(R^2+x^2\right)^{3/2}}$$

(c) Restoring torque $\tau = -pE\sin\theta = -pE\theta$ (for small θ)

$$\tau = I\frac{d^2\theta}{dt^2} = 2ma^2\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{pE}{2ma^2}\theta$$

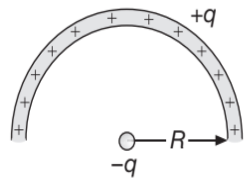
$$\Rightarrow \quad T = 2\pi\sqrt{\frac{I}{pE}} = 2\pi\sqrt{\frac{2ma^2}{pE}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{2ma^2}{2aq}\frac{\left(R^2+x^2\right)^{3/2}4\pi\varepsilon_0}{Qx}}$$

$$\Rightarrow \quad T = \sqrt{\frac{16\pi^2\varepsilon ma}{qQx}\left(R^2+x^2\right)^{3/2}}$$

ILLUSTRATION 43

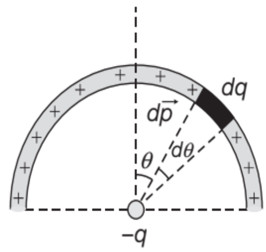
On a half ring a charge $+q$ is uniformly distributed and another equal negative charge $-q$ is placed at the centre of ring.



In this system negative charge is a point charge and positive charge is distributed on the ring. This system is called, distributed dipole. Calculate the dipole moment of this system.

SOLUTION

For this we consider a polar element of angular width $d\theta$ at an angle θ from the vertical axis as shown.



Charge on this element is

$$dq = \frac{q}{\pi}d\theta$$

Now in the point charge $-q$, we consider an element charge $-dq$ which forms a dipole with the polar element of charge $+dq$. This element dipole moment can be given as

$$dp = dqR = \frac{q}{\pi}Rd\theta$$

Total dipole moment of system can be given by integrating all such elemental dipole moments as

$$p_{\text{total}} = \int dp\cos\theta$$

{The components $dp\sin\theta$ cancel out}

$$\Rightarrow \quad p_{\text{total}} = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{q}{\pi} R\cos\theta d\theta$$

$$\Rightarrow \quad p_{\text{total}} = \frac{qR}{\pi}[\sin\theta]\Big|_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}$$

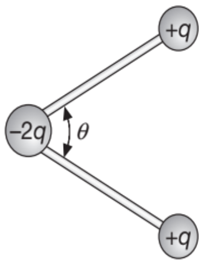
$$\Rightarrow \quad p_{\text{total}} = \frac{2qR}{\pi}$$

Test Your Concepts-IV

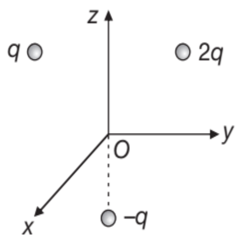
Based on Dipoles and Dipole Moment

(Solutions on page H.15)

1. Three charges $-2q$, q and q are arranged in the configuration shown. Calculate resultant dipole moment of the system.



2. Three point charges $2q$, q and $-q$ are located respectively at $(0, a, a)$, $(0, -a, a)$ and $(0, 0, -a)$ as shown. Find the dipole moment of this distribution.

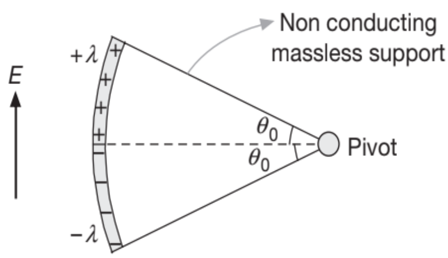


3. A small freely oriented dipole with dipole moment p lies at the centre of a uniformly charged hemisphere, charge density σ and radius R. Determine the potential energy of the dipole and the period of small oscillations about an axis perpendicular to the axis of the hemisphere. The moment of inertia of the dipole about the rotation axis is I.

4. Figure shows an assembly of two charged rods each of mass m, length L connected by non-conducting massless rods of length d. The system is free to rotate about the axis shown. The system is given an anticlockwise small angular displacement. Show that the motion is simple harmonic and determine its time period.

What is the work done required to turn the system through an angle (a) $\frac{\pi}{2}$ (b) π ?

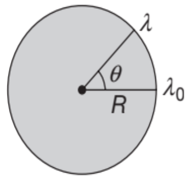
5. Figure shows a charged rod of mass m bent in the form of an arc of a circle of radius R. The charge distribution on the rod is shown in the figure. The assembly is kept in a uniform electric field.



(a) Show that for small angular displacement the system will perform SHM. Determine its time period.
(b) Considering the size of the system to be very small, determine the work done required to turn it through 180°. Neglect any displacement of centre of charge.

6. A dipole with an electric moment p is located at a distance r from a long thread charged uniformly with a linear density λ. Find the force $\vec{F}$ acting on the dipole if the vector $\vec{p}$ is oriented
(a) along the thread
(b) along the radius vector $\vec{r}$
(c) at right angles to the thread and the radius vector $\vec{r}$.

7. A thin non-conducting ring of radius R has a linear charge density $\lambda = \lambda_0 \cos\theta$, where λ_0 is the value of λ at $\theta = 0$. Find the net electric dipole moment for this charge distribution.



8. A positive point charge q is fixed at origin. A small dipole with a dipole moment $\vec{p}$ is placed along the x-axis far away from the origin with $\vec{p}$ pointing along positive x-axis. Find
(a) the kinetic energy of the dipole when it reaches a distance d from the origin, and
(b) the force experienced by the charge q at this moment.

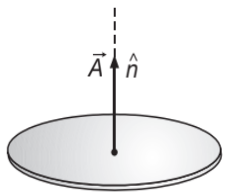
GAUSS'S LAW AND APPLICATIONS

AREA AS A VECTOR

The physical quantity, area is a pseudo vector and we treat area of a surface as a vector quantity whose direction is along the outward normal to the surface.

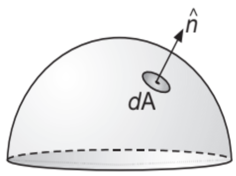
The area vector $\vec{A}$ of a surface which has surface area A can be written as

$$\vec{A} = A\hat{n}$$



Where $\hat{n}$ is the unit vector in the direction along outward normal to the surface.

When a surface is three dimensional we consider a small infinitesimal area element dA on this surface and direction of this elemental area vector is along the local outward normal to the surface at the point where elemental area is chosen as shown.



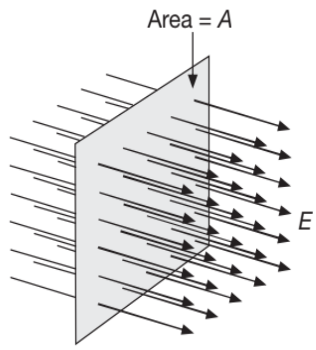
So, $\vec{A} = (dA)\hat{n}$

Here $\hat{n}$ is the unit vector in the direction along the outward normal at elemental area dA.

ELECTRIC FLUX

Consider an electric field that is uniform in both magnitude and direction, as shown in figure. The field lines penetrate a rectangular surface of area A, whose plane is oriented perpendicular to the field. Since the number of lines per unit area (in other words, the line density) is proportional to the magnitude of the electric field. Therefore, the total number of lines penetrating the surface is proportional to the product EA. This product of the magnitude of the electric field E and surface area A perpendicular to the field is called the electric flux ϕ_E (Greek phi)

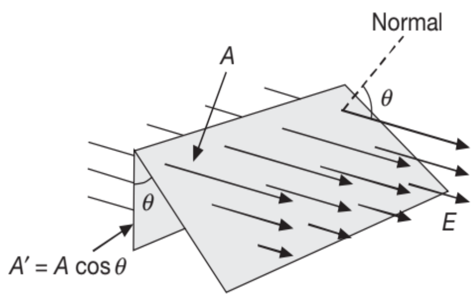
$$\phi_E = EA$$



Field lines representing a uniform electric field penetrating a plane of area A perpendicular to the field. The electric flux ϕ_E through this area is equal to EA.

From the SI units of E and A, we see that ϕ_E has SI units Nm^2C^{-1}. Electric flux is proportional to the number of electric field lines penetrating some surface.

If the surface under consideration is not perpendicular to the field, the flux through it must be less than EA. We can understand this by considering figure, where the normal to the surface of area A is at an angle θ to the uniform electric field. Note that the number of lines that cross this area A is equal to the number that cross the area A', which is a projection of area A onto a plane oriented perpendicular to the field. From figure we see that the two areas are related by $A' = A\cos\theta$.

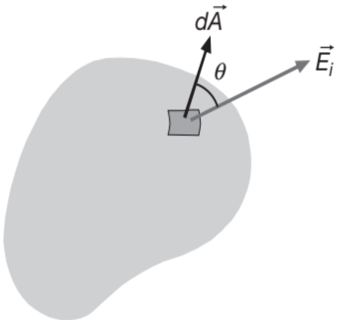


Field lines representing a uniform electric field penetrating an area A that is at an angle θ to the field. Because the number of lines that go through A' is the same as the number that go through A, the flux through A' is equal to the flux through A and is given $\Phi_E = EA\cos\theta$.

Because the flux through A equals the flux through A', we conclude that the flux through A is

$$\phi_E = EA' = EA\cos\theta \qquad \dots(1)$$

From this result, we see that the flux through a surface of fixed area A has a maximum value EA when the surface is perpendicular to the field (when the normal to the surface is parallel to the field, that is, $\theta = 0°$ in figure); the flux is zero when the surface is parallel to the field (when the normal to the surface is perpendicular to the field, that is $\theta = 90°$).



A small element of surface area $d\vec{A}$. The electric field makes an angle θ with the vector $d\vec{A}$, defined as being normal to the surface element, and the flux through the element is equal to $EdA\cos\theta$.

$$\phi_E = \int_{\text{surface}} \vec{E}\cdot d\vec{A} \qquad \dots(2)$$

Equation (2) is a surface integral, which means it must be evaluated over the surface in question. In general, the value of Φ_E depends both on the field pattern and on the surface. Using the symbol $\oint$ to represent an integral over a closed surface, we can write the net flux ϕ_E through a closed surface as

$$\phi_E = \oint \vec{E}\cdot d\vec{A} = \oint E_n dA \qquad \dots(3)$$

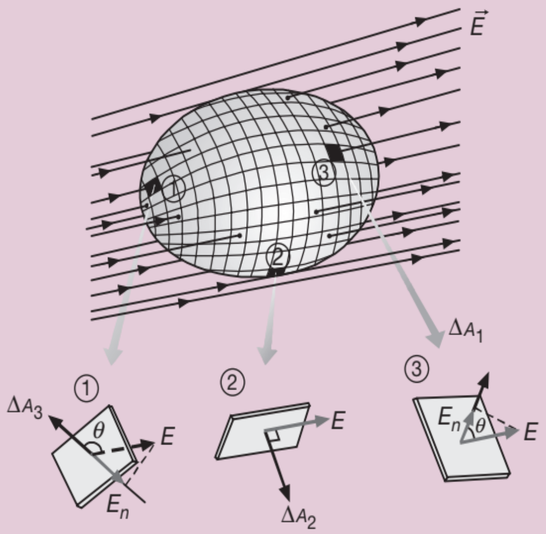
where E_n represents the component of the electric field normal to the surface. If the field is normal to the surface at each point and constant in magnitude, the calculation is straightforward.

Remark(s)

(a) We are often interested in evaluating the flux through a closed surface, which is defined as one that divides space into an inside and an outside region, so that one cannot move from one region to the other without crossing the surface. The surface of a sphere, for example, is a closed surface. Consider the closed surface in figure. The vectors $\Delta\vec{A}_i$ point in different directions for the various surface elements, but at each point they are normal to the surface and, by convention, always point outward.

For element ①, where the field lines are crossing the surface from outside to inside, $180° > \theta > 90°$ and the flux is negative because $\cos\theta$ is negative.

For element ②, the field lines graze the surface (perpendicular to the vector $\Delta\vec{A}_2$); thus, $\theta = 90°$ and the flux is zero.



A closed surface in an electric field. The area vectors ΔA_i are, by convention, normal to the surface and point outward. The flux through an area element can be negative (element ①), zero (element ②), or positive (element ③).

For element ③, the field lines are crossing the surface from the inside to the outside and $\theta < 90°$; hence, the flux $\Delta\phi_E = \vec{E}\cdot\Delta\vec{A}_1$ through this element is positive.

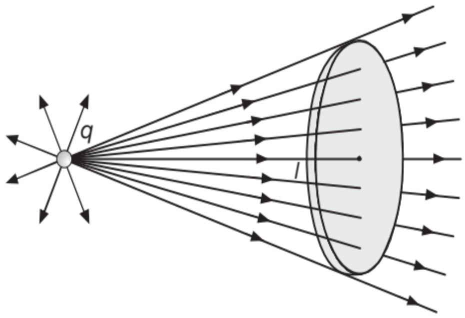
(b) The net flux through the surface is proportional to the net number of lines leaving the surface, where the net number means the number leaving the surface minus the number entering the surface. If more lines are leaving than entering, the net flux is positive. If more lines are entering than leaving, the net flux is negative.

ILLUSTRATION 44

Calculate the electric flux through the surface of the disc of radius R due to a point charge q placed at a point P on its axis at a distance l from its centre.

SOLUTION

Since a point charge q has electric field originating in radially outward direction.

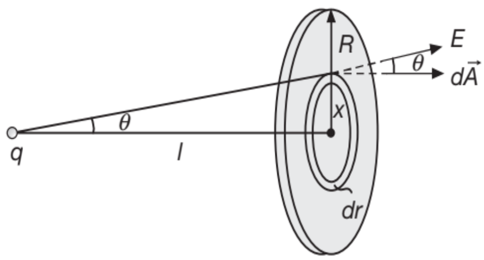


To calculate this flux, we consider an infinitesimal ring on disc surface of radius x and width dx as shown. Area of this strip is

$$dA = (2\pi x)dx$$

The electric field due to q at this elemental ring is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{(x^2+l^2)}$$



If $d\phi$ is the flux due to this elemental ring, then

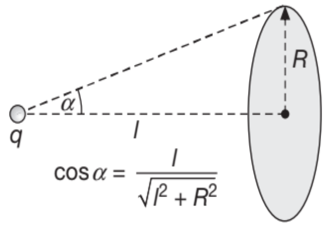
$$d\phi = \vec{E}\cdot d\vec{A} = EdA\cos\theta$$

$$\Rightarrow \quad d\phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{(l^2+x^2)}(2\pi x\,dx)\left(\frac{l}{\sqrt{l^2+x^2}}\right)$$

$$\Rightarrow \quad d\phi = \frac{ql}{2\varepsilon_0}\frac{x\,dx}{(l^2+x^2)^{3/2}}$$

$$\Rightarrow \quad \phi = \int d\phi = \frac{ql}{2\varepsilon_0}\int_0^R \frac{x\,dx}{(l^2+x^2)^{3/2}}$$

$$\Rightarrow \quad \phi = \frac{ql}{2\varepsilon_0}\left[\frac{1}{2}\int_0^R (l^2+x^2)^{-\frac{3}{2}}\,2x\,dx\right]$$



$$\text{Since,} \quad \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}, \; n \neq -1$$

$$\text{So,} \quad \phi = \frac{ql}{2\varepsilon_0}\left[\frac{1}{2}\frac{(l^2+x^2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\Bigg|_0^R\right]$$

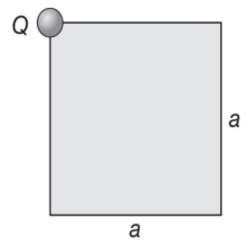
$$\Rightarrow \quad \phi = -\frac{ql}{2\varepsilon_0}\left[\frac{1}{\sqrt{l^2+x^2}}\Bigg|_0^R\right]$$

$$\Rightarrow \quad \phi = \frac{ql}{2\varepsilon_0}\left[\frac{1}{l} - \frac{1}{\sqrt{R^2+l^2}}\right]$$

$$\Rightarrow \quad \phi = \frac{q}{2\varepsilon_0}\left(1 - \frac{l}{\sqrt{R^2+l^2}}\right) = \frac{q}{2\varepsilon_0}(1-\cos\alpha)$$

ILLUSTRATION 45

A point charge Q is placed at the corner of a square of side a, then find the flux through the square.



SOLUTION

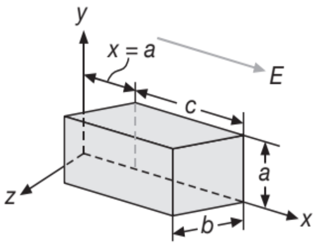
The electric field due to Q at any point of the square will be along the plane of square and the electric field line are perpendicular to square, so $\phi = 0$.
In other words we can say that no line is crossing the square so flux $= 0$.

ILLUSTRATION 46

A closed surface with dimensions $a=b=0.4$ m and $c=0.6$ m is located as in figure. The left edge of the closed surface is located at position $x=a$. The electric field throughout the region is non-uniform and given by $E=(3+2x^2)\hat{i}$ NC^{-1}, where x is in meter.

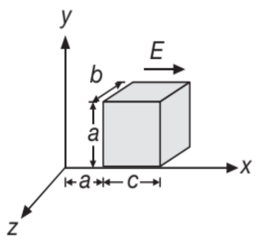
(a) Calculate the net electric flux leaving the closed surface.

(b) What net charge is enclosed by the surface?



SOLUTION

The electric field throughout the region is directed along x, therefore, E will be perpendicular to dA over the four faces of the surface which are perpendicular to the yz plane, and E will be parallel to dA over the two faces which are parallel to the yz plane. Therefore



$$\phi_E = -\left(E_x\big|_{x=a}\right)A + \left(E_x\big|_{x=a+c}\right)A$$

$$\Rightarrow \quad \phi_E = -(3+2a^2)ab + (3+2(a+c)^2)ab$$

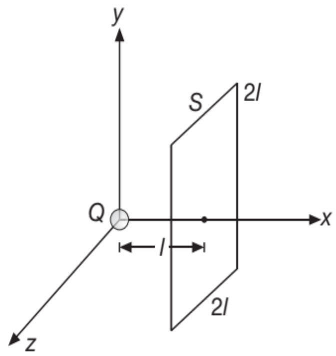
$$\Rightarrow \quad \phi_E = 2abc(2a+c)$$

Substituting the given values for a, b and c, we find $\phi_E = 0.269$ Nm^{2}C^{-1}

$$Q = \varepsilon_0\phi_E = 2.38\times10^{-12}\text{ C} = 2.38\text{ pC}$$

ILLUSTRATION 47

(a) Compute the electric flux through a square surface of edges $2l$ due to a charge $+Q$ located at a perpendicular distance l from the center of the square, as shown in figure.

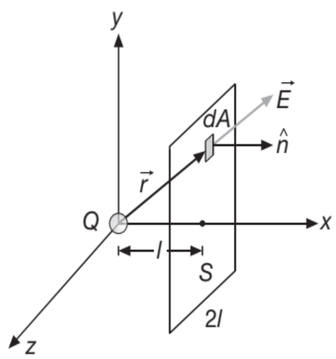


(b) Using the result obtained in (a), if the charge $+Q$ is now at the centre of a cube of side $2l$, what is the total flux emerging from all the six faces of the closed surface?

SOLUTION

(a) The electric field due to the charge $+Q$ is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right)$$



where $r=(x^2+y^2+z^2)^{1/2}$ in Cartesian coordinates. On the surface S, $x=l$ and the area element is $d\vec{A} = dA\hat{i} = (dydz)\hat{i}$. So,

$$\vec{E}\cdot d\vec{A} = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right)\cdot(dydz)\hat{i}$$

$$\Rightarrow \quad \vec{E}\cdot d\vec{A} = \frac{Ql}{4\pi\varepsilon_0 r^3}dydz$$

Thus, the electric flux through S placed at $x=l$ is

$$\phi_E = \oint_S \vec{E}\cdot d\vec{A} = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l}dz\int_{-l}^{l}\frac{dy}{(l^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow \phi_E = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l} dz \frac{y}{(l^2+z^2)(l^2+y^2+z^2)^{1/2}}\Bigg|_{-l}^{l}$$

$$\Rightarrow \phi_E = \frac{Ql}{2\pi\varepsilon_0}\int_{-l}^{l} \frac{l\,dz}{(l^2+z^2)(2l^2+z^2)^{1/2}}$$

$$\Rightarrow \phi_E = \frac{Q}{2\pi\varepsilon_0}\tan^{-1}\left(\frac{z}{\sqrt{z^2+2l^2}}\right)\Bigg|_{-l}^{l}$$

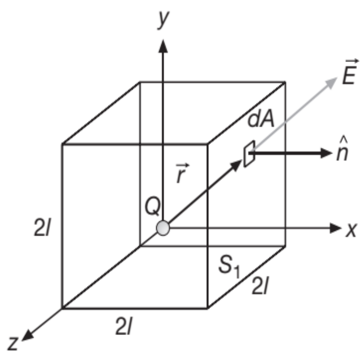
$$\Rightarrow \phi_E = \frac{Q}{2\pi\varepsilon_0}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$\Rightarrow \phi_E = \frac{Q}{2\pi\varepsilon_0}\left(\frac{\pi}{6}+\frac{\pi}{6}\right)=\frac{Q}{6\varepsilon_0}$$

In evaluating the above integrals, the following results have been used.

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$$

$$\int \frac{dx}{(x^2+a^2)(x^2+b^2)^{1/2}} = \frac{1}{a(b^2-a^2)^{1/2}}\tan^{-1}\sqrt{\frac{b^2-a^2}{a^2(x^2+b^2)}},\ b^2>a^2$$



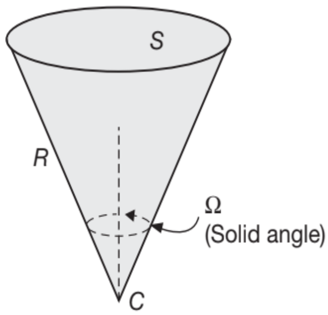
(b) From symmetry arguments, the flux through each face must be the same. Thus, the total flux through the cube is just six times that through one face. So,

$$\Phi_E = 6\left(\frac{Q}{6\varepsilon_0}\right)=\frac{Q}{\varepsilon_0}$$

The result shows that the electric flux Φ_E passing through a closed surface is proportional to the charge enclosed. In addition, the result further strengthens the notion that Φ_E is independent of the shape of the closed surface.

CONCEPT OF SOLID ANGLE

The three dimensional angle enclosed by the lateral surface of a cone at its vertex (as shown in figure) is the **Solid Angle**.

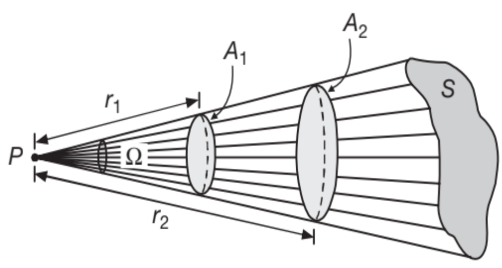


Solid angle is also defined as the three dimensional angle subtended by a spherical section at its centre of curvature. As in the figure shown point C is the centre of curvature of a spherical section S of radius R which subtends a solid angle Ω (Greek symbol omega) at point C.

SOLID ANGLE SUBTENDED DUE TO A RANDOM SURFACE AT A GIVEN POINT

Consider a random surface S as shown in figure. To find the solid angle subtended by this surface at a point P, let us join all the points of the periphery or boundary of the surface S to the point P by straight lines. This gives a cone with vertex at P.

Now by taking centre at P, we construct several spherical sections on this cone of different radii as shown. Let the area of spherical section which is of radius r_1 be A_1 and the area of section of radius r_2 be A_2.



Mathematically we observe that the ratio of area of any sphere enclosed by the cone to the square of radius of that sphere is a constant and this constant is called solid angle Ω.
In the figure shown the solid angle Ω subtended at P is given by

$$\Omega = \frac{A_1}{r_1^2} = \frac{A_2}{r_2^2}$$

Solid angle is a dimensionless physical quantity and its S.I. unit is steradian.

One steradian is the solid angle subtended at the centre of sphere by the surface of the sphere having area equal to square of the radius of sphere. Also,

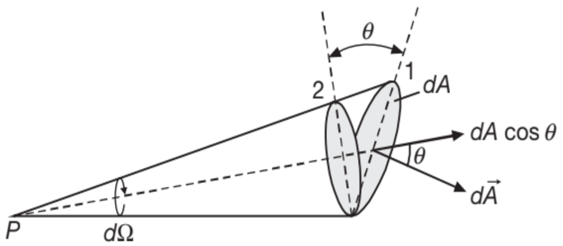
$$\Omega_{max} = \frac{\text{Area of the sphere}}{(\text{Radius of sphere})^2} = \frac{4\pi r^2}{r^2}$$

$$\Rightarrow \quad \Omega_{max} = 4\pi \text{ steradian}$$

SOLID ANGLE SUBTENDED BY A SURFACE NOT NORMAL TO AXIS OF CONE OR NOT LYING ON SURFACE OF SPHERE

Consider a small surface AB of area $d\vec{A}$ as shown. Let PQ be the axis of cone formed by this surface. Here PQ is not normal to the surface 1 (in light grey shade). The solid angle Ω subtended at point P is given by

$$d\Omega = \frac{dA\cos\theta}{r^2} \qquad \ldots(1)$$



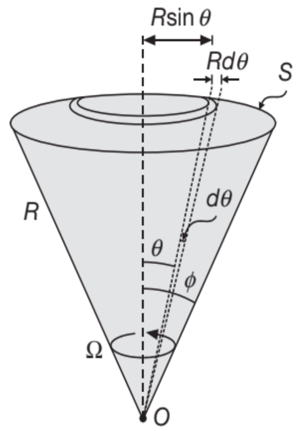
where θ is the angle between surface area vector $d\vec{A}$ and the axis of cone PQ as shown in the Figure.
Since, the total solid angle that any closed surface can subtend is 4π steradian.
So,

$$\oint \frac{dA\cos\theta}{r^2} = 4\pi$$

RELATION BETWEEN HALF ANGLE OF CONE AND SOLID ANGLE AT VERTEX

Consider a spherical section S of radius R, which subtends a half angle ϕ (in radian) at the centre of curvature. To find the area of this section, we consider an infinitesimal circular strip on this section having angular width $d\theta$ and radius $R\sin\theta$. The surface area dA of this strip is

$$dA = (2\pi R\sin\theta)(Rd\theta)$$



The total area of spherical section is calculated by integrating the area of this elemental strip within limits from 0 to ϕ.

Total area of spherical section is

$$A = \int dA = \int_0^{\phi} 2\pi R^2 \sin\theta\, d\theta$$

$$\Rightarrow \quad A = 2\pi R^2 [-\cos\theta]_0^{\phi}$$

$$\Rightarrow \quad A = 2\pi R^2 (1-\cos\phi) \qquad \ldots(1)$$

If this section subtends a solid angle Ω at its centre O is Ω then, by definition

$$\Omega = \frac{A}{R^2} \qquad \ldots(2)$$

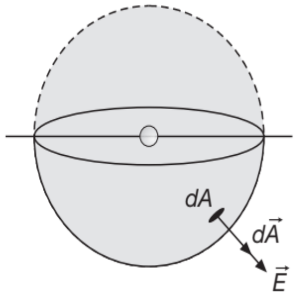
From (1) and (2), we get

$$\Omega = 2\pi(1-\cos\phi) \qquad \ldots(3)$$

This equation gives the relation in half angle of a cone ϕ and the solid angle enclosed by the lateral surface of cone at its vertex.

ELECTRIC FLUX PRODUCED BY A POINT CHARGE

The figure shows a point charge placed at the centre of a spherical surface of radius R. For convenience and picture clarity only lower half of sphere is drawn in the figure.



Since the charge q lies inside the sphere, so the total flux that originates comes out from the spherical surface. To find the total flux, let us consider an elemental area dA on surface. The electric field on the points on surface of sphere is

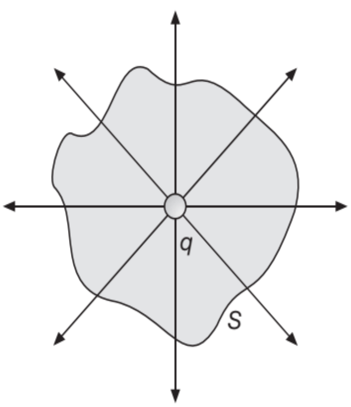
$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$

The electric flux coming out from the surface dA is

$$d\phi = \vec{E}\cdot d\vec{A} = EdA\cos 0$$

{because $\vec{E}$ and $d\vec{A}$ are parallel}

$$\Rightarrow \quad d\phi = \frac{q}{4\pi\varepsilon_0 R^2}dA$$



So, total flux coming out from the spherical surface is

$$\phi = \int d\phi = \int \frac{q}{4\pi\varepsilon_0 R^2}dA$$

At every point on spherical surface the magnitude of electric field remains same and hence we have

$$\phi = \frac{q}{4\pi\varepsilon_0 R^2}\oint dA$$

$$\Rightarrow \quad \phi = \frac{q}{4\pi\varepsilon_0 R^2}\left(4\pi R^2\right)$$

$$\Rightarrow \quad \phi = \frac{q}{\varepsilon_0}$$

Thus total flux, originating from charge q is $\frac{q}{\varepsilon_0}$. Similarly a charge $-q$ absorbs $\frac{q}{\varepsilon_0}$ electric lines (flux) into it. Figure shows a charge q enclosed in a closed surface S of random shape. Here we can say that the total electric flux emerging out from the surface S is the flux that originates from charge q and hence flux emerging from surface is

$$\phi_S = \frac{q}{\varepsilon_0}$$

Please note that, the above result is independent of the shape of surface it only depends on the amount of charge enclosed by the surface. Also field lines leaving a surface will give positive flux and the field lines entering a surface give negative flux.

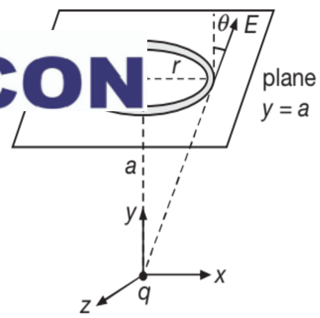
ILLUSTRATION 48

A point charge q is fixed at the origin. Calculate the electric flux through the infinite plane $y = a$.

SOLUTION

Consider a ring of radius r on the plane $y = a$. The thickness of ring is dr. Electric field on the ring due to charge q is,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(r^2 + a^2\right)}$$



Area of the ring $dA = (2\pi r)\,dr$

$$\cos\theta = \frac{a}{\sqrt{r^2+a^2}}$$

Flux passing through the infinitesimal ring is

$$d\phi_e = EdA\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2+a^2}(2\pi r\,dr)\frac{a}{\sqrt{r^2+a^2}}$$

$$\Rightarrow \quad d\phi_e = \frac{q}{2\varepsilon_0}\frac{ar\,dr}{(r^2+a^2)^{3/2}}$$

Total flux passing through the plane is given by

$$\phi_e = \int_0^\infty d\phi_e = \frac{qa}{2\varepsilon_0}\int_0^\infty \frac{r\,dr}{(r^2+a^2)^{3/2}}$$

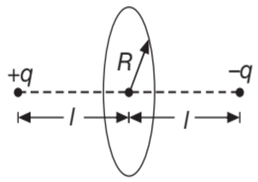
$$\Rightarrow \quad \phi_e = \frac{q}{2\varepsilon_0} \qquad \left\{\because \int_0^\infty \frac{r\,dr}{(r^2+a^2)^{3/2}} = \frac{1}{a}\right\}$$

Test Your Concepts-V

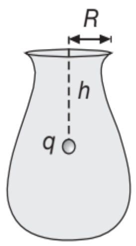
Based on Flux

(Solutions on page H.18)

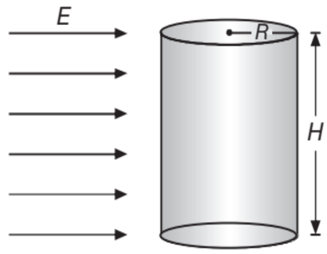
1. Two point charges q and $-q$ separated by a distance 2ℓ. Find the flux of electric field strength vector across the circle of radius R placed with its centre coinciding with the midpoint of line joining the two charges in the perpendicular plane.



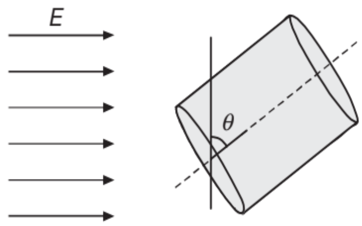
2. A point charge q is kept at the centre of the cylinder of length L and radius R. Calculate the flux through the curved surface of the cylinder.
3. A very long uniformly charged thread oriented along the axis of a circle of radius R rests on its centre with one of the ends. The charge of the thread per unit length is equal to λ. Find the flux of the vector E across the circle area.
4. Consider the situation shown in figure. A point charge q is placed at a depth $h = \sqrt{3}R$ exactly below the centre of mouth of a vessel whose open end is circular having a radius R. Calculate the electric flux through the lateral surface of this vessel.



5. A cylinder of height H and radius R is placed in a uniform electric field as shown in figure. Find flux crossing the cylinder.



6. A cylinder of height H and radius R is placed in a uniform electric field (as shown) such that the axis of the cylinder makes an angle θ with the vertical. Calculate the flux through the cylinder.



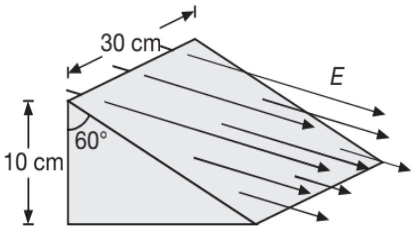
7. A non-uniform electric field is given by the expression $E = ay\hat{i} + bz\hat{j} + cx\hat{k}$, where a, b, and c are constants. Determine the electric flux through a rectangular surface in the xy plane, extending from $x = 0$ to $x = \ell$.
8. An electric field having a magnitude of $3.5\ \text{kNC}^{-1}$ is applied along the x-axis. Calculate the electric flux through a rectangular plane 0.35 m wide and 0.7 m long, assuming that the plane

(a) is parallel to the *yz* plane.
(b) is parallel to the *xy* plane.
(c) contains the *y*-axis, and its normal makes an angle of 60° with the *x*-axis.

9. On a day, above the Earth's surface, when a thunderstorm is brewing a vertical electric field of magnitude 2×10^4 NC^{-1} exists. A car having a rectangular size of 6 m by 3 m is travelling along the road sloping downward at 60°. Determine the electric flux through the bottom of the car.

10. A loop of radius 40 cm is rotated in a uniform electric field until the position of maximum electric flux is found. The flux in this position is measured to be 5.04×10^5 Nm^2 C^{-1}. Calculate the magnitude of the electric field.

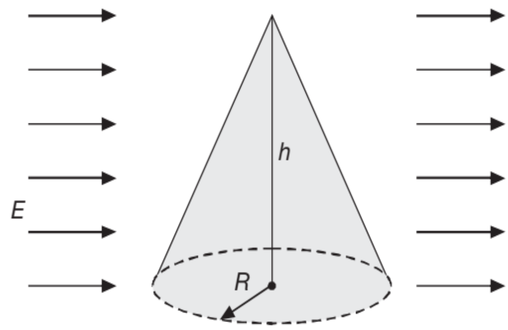
11. Consider a closed triangular box resting within a horizontal electric field of magnitude $E=7.80\times10^4$ NC^{-1} as shown.



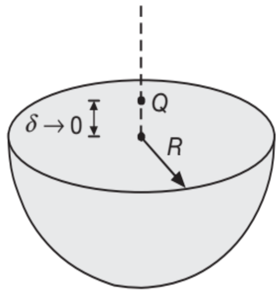
Calculate the electric flux through the
(a) vertical rectangular surface.
(b) slanted surface.
(c) entire surface of the box.

12. A pyramid with horizontal square base, 6 m on each side, and a height of 4 m is placed in a vertical electric field of 52 NC^{-1}. Calculate the total electric flux through the pyramid's four slanted surfaces.

13. A cone with base radius *R* and height *h* is located on a horizontal table. A horizontal uniform field *E* penetrates the cone, as shown in Figure. Determine the electric flux that enters the left-hand side of the cone.



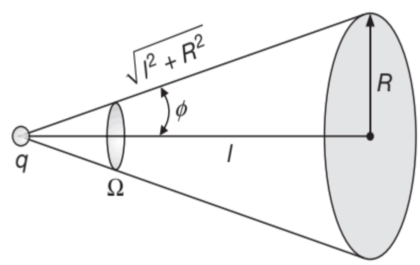
14. A point charge *Q* is located just above the center of the flat face of a hemisphere of radius *R* as shown in figure . What is the electric flux through



(a) the curved surface and
(b) the flat face?

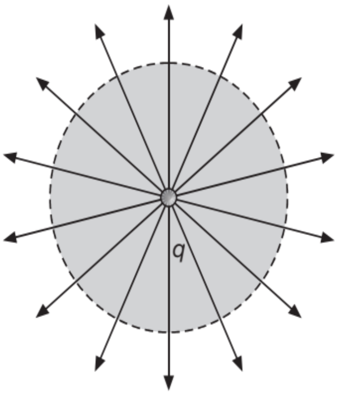
ELECTRIC FLUX CALCULATION DUE TO A POINT CHARGE USING SOLID ANGLE

Consider a point charge q, placed at a distance l from the centre of a circular disc of radius R. Let us find the electric flux passing through the surface of disc due to the charge q.



Since a total flux $\frac{q}{\varepsilon_0}$ originates from a point charge q in all directions

We can say that from a point charge q, $\frac{q}{\varepsilon_0}$ flux originates in a solid angle 4π.



The solid angle enclosed by cone subtended by the disc at the point charge is given by

$$\Omega = 2\pi(1-\cos\phi)$$

$$\Rightarrow \quad \Omega = 2\pi\left(1-\frac{l}{\sqrt{l^2+R^2}}\right) \quad \left\{\because \cos\phi = \frac{l}{\sqrt{l^2+R^2}}\right\}$$

The flux of q associated with the surface of the disc

$$\phi = \left(\frac{\text{Total flux associated with closed surface due to the charge}}{4\pi}\right)\left(\begin{matrix}\text{Solid angle}\\ \text{subtended}\end{matrix}\right)$$

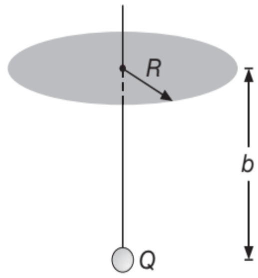
$$\phi_{\text{disc}} = \frac{\left(\frac{q}{\varepsilon_0}\right)}{4\pi}\times\Omega = \left(\frac{q}{4\pi\varepsilon_0}\right)\Omega$$

$$\Rightarrow \quad \phi_{\text{disc}} = \frac{q}{4\pi\varepsilon_0}\times 2\pi\left(1-\frac{l}{\sqrt{l^2+R^2}}\right)$$

$$\Rightarrow \quad \phi_{\text{disc}} = \frac{q}{2\varepsilon_0}\left(1-\frac{l}{\sqrt{l^2+R^2}}\right) = \frac{q}{2\varepsilon_0}(1-\cos\phi)$$

ILLUSTRATION 49

A point charge Q is located on the axis of a disk of radius R at a distance b from the plane of the disk. Show that if one fourth of the electric flux from the charge passes through the disk, then $R = \sqrt{3}b$.



SOLUTION

Since $\phi_{E,\text{ disc}} = \frac{Q}{2\varepsilon_0}\left[1-\frac{b}{\sqrt{R^2+b^2}}\right]$

According to the problem, we have

$$\phi_{E,\text{ disc}} = \frac{1}{4}\left(\frac{Q}{\varepsilon_0}\right)$$

$$\Rightarrow \quad \frac{Q}{4\varepsilon_0} = \frac{Q}{2\varepsilon_0}\left(1-\frac{b}{\sqrt{R^2+b^2}}\right)$$

$$\Rightarrow \quad \frac{1}{2} = 1-\frac{b}{\sqrt{R^2+b^2}}$$

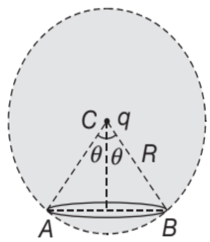
$$\Rightarrow \quad \frac{b^2}{R^2+b^2} = \frac{1}{4}$$

$$\Rightarrow \quad 4b^2 = R^2+b^2$$

$$\Rightarrow \quad R = \sqrt{3}\,b$$

ILLUSTRATION 50

A point charge q is placed on the top of a cone of semi vertex angle θ. Show that the electric flux associated with the base of the cone due to the charge is $\frac{q(1-\cos\theta)}{2\varepsilon_0}$.



SOLUTION

Consider a Gaussian surface, a sphere with its centre at the top and radius the slant length of the cone. The flux through the whole sphere is $\frac{q}{\varepsilon_0}$. Therefore, the flux through the base of the cone can be calculated by using the following formula,

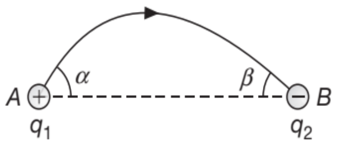
$$\phi = \frac{q}{4\pi\varepsilon_0}\Omega$$

Since, $\Omega = 2\pi(1-\cos\theta)$

$$\Rightarrow \quad \phi = \frac{q}{2\varepsilon_0}(1-\cos\theta)$$

ILLUSTRATION 51

Two charges $+q_1$ and $-q_2$ are placed at A and B respectively. A line of force emanates from q_1 at angle α with the line AB. At what angle will it terminate at $-q_2$?

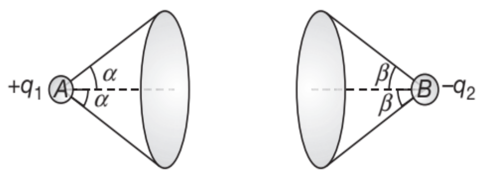


SOLUTION

A line can leave $+q_1$ in a cone of apex angle α and then enter $-q_2$ in a cone of apex angle β.

So, flux due to the charge $+q_1$ is $\phi_1 = \frac{q_1}{2\varepsilon_0}(1-\cos\alpha)$

and that due to the charge $-q_2$ is $\phi_2 = \frac{q_2}{2\varepsilon_0}(1-\cos\beta)$.



Since, we know that only one line is leaving q_1 to enter $-q_2$. So, we can say

$$\frac{\phi_1}{\phi_2} = \frac{N_1}{N_2} = 1$$

$$\Rightarrow \quad \frac{q_1}{2\varepsilon_0}(1-\cos\alpha) = \frac{q_2}{2\varepsilon_0}(1-\cos\beta)$$

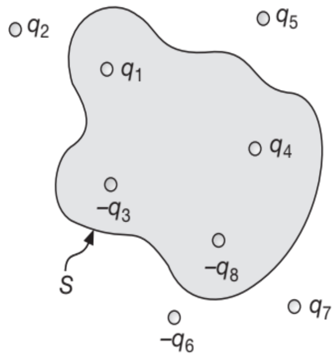
$$\Rightarrow \quad q_1\left[2\sin^2\left(\frac{\alpha}{2}\right)\right] = q_2\left[2\sin^2\left(\frac{\beta}{2}\right)\right]$$

$$\Rightarrow \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{q_1}{q_2}}\sin\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow \quad \beta = 2\sin^{-1}\left[\sqrt{\frac{q_1}{q_2}}\sin\left(\frac{\alpha}{2}\right)\right]$$

GAUSS'S LAW

This law gives the mathematical relation between the electric flux associated with a closed surface (number of field lines entering or leaving a closed surface) and the charge enclosed by the surface.



Gauss's Law states that the total flux due to a closed surface is equal to the $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface.

Mathematically Gauss's Law is written as

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{\Sigma q_{\text{enc}}}{\varepsilon_0}$$

Here the sign $\oint$ represents the integration over a closed surface S which encloses a total charge Σq_{enc}.

Let us consider a surface S shown which encloses four charges q_1, $-q_3$, q_4 and $-q_8$. For the surface S, if we find surface integral of electric field $\oint_S \vec{E}\cdot d\vec{A}$, it gives the total electric flux coming out from the surface, which can be given as

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0}(q_1 + q_4 - q_3 - q_8) \qquad \{\text{Gauss's Law}\}$$

Remark(s)

(a) Here electric field $\vec{E}$ is the net electric field at the points on the surface of S. Remember that the electric field we use to find the flux must be the net electric field of the system due to all the charges inside and outside the Gaussian surface, but the total flux associated with the surface is due to the charges that are enclosed inside the closed surface i.e. the Gaussian surface.

(b) Using Gauss's Law we can find electric field strength due to some symmetrical distribution of charges.

(c) For application of Gauss's Law, we choose a closed imaginary surface over which we apply Gauss's Law, called Gaussian surface.

(d) Gauss's Law can be used to calculate electric field strength, for this we first choose a proper Gaussian surface on which the electric field strength is to be calculated.

(e) Sometimes, when a random Gaussian surface is chosen then the integral $\oint \vec{E} \cdot d\vec{A}$ involves complex calculations. To make these calculations easier, we choose a Gaussian surface keeping following points in mind.

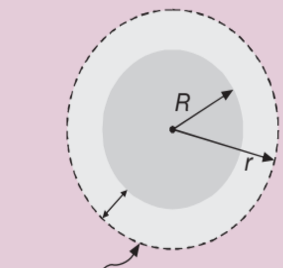
(i) The Gaussian surface should be chosen in such a way that at every point of surface the magnitude of electric field is either uniform or zero.

(ii) The surface should be chosen in such a way that at every point of surface electric field strength is either parallel or perpendicular to the surface.

Keeping the above two points e) (i) and e) (ii) in mind, let us conclude what type of Gaussian surface should be selected in different situations

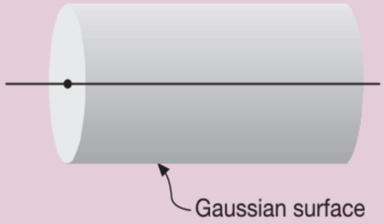
A. SPHERICAL SYMMETRY

Make the Gaussian surface a concentric sphere.



B. CYLINDRICAL SYMMETRY

Make the Gaussian surface a coaxial cylinder.



C. PLANE SYMMETRY

Use a Gaussian Pillbox which passes through the surface.

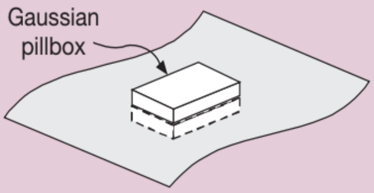
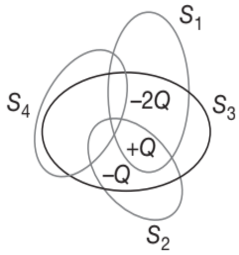


ILLUSTRATION 52

Four closed surfaces, S_1, S_2, S_3 and S_4, together with the charges $-2Q$, Q, and $-Q$ are sketched in figure. Find the electric flux through each surface.



SOLUTION

$$\phi_E = \frac{q_{\text{enc}}}{\varepsilon_0}$$

Through S_1

$$\phi_E = \frac{-2Q+Q}{\varepsilon_0} = -\frac{Q}{\varepsilon_0}$$

Through S_2

$$\phi_E = \frac{+Q-Q}{\varepsilon_0} = 0$$

Through S_3

$$\phi_E = \frac{-2Q+Q-Q}{\varepsilon_0} = -\frac{2Q}{\varepsilon_0}$$

Through S_4

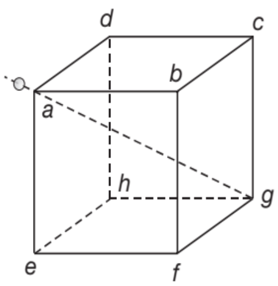
$$\phi_E = 0$$

Conceptual Note(s)

(a) Electric field intensity at the Gaussian surface is due to all the charges present inside as well as outside the Gaussian surface.
(b) Flux through Gaussian surface is independent of its shape.
(c) Flux through Gaussian surface depends only on total charge present inside Gaussian surface.
(d) Flux through Gaussian surface is independent of position of charges inside Gaussian surface.
(e) Flux due to field lines entering a closed surface is taken as negative and flux due to field lines leaving a surface is taken as positive. This is because $\hat{n}$ is taken positive in outward direction.
(f) In a Gaussian surface $\phi = 0$ does not imply $E = 0$ at every point of the surface but $E = 0$ at every point implies $\phi = 0$.

ILLUSTRATION 53

The line ag in figure is a diagonal of a cube. A point charge q is located at the vertex a of the cube. Determine the electric flux through each of the sides of the cube which meet at the point a.



SOLUTION

No charge is inside the cube. The net flux through the cube is zero. Positive flux comes out through the three faces meeting at g. These three faces together fill solid angle equal to one-eighth of a sphere as seen from q, and together pass flux $\frac{1}{8}\left(\frac{q}{\varepsilon_0}\right)$.

Each face containing a intercepts equal flux going into the cube so,

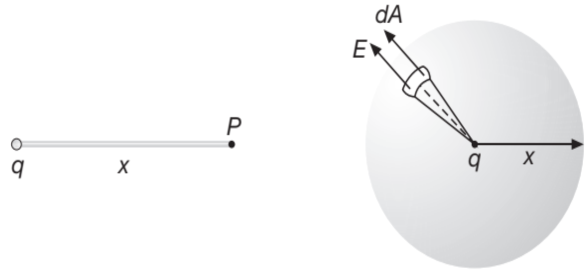
$$\Phi_{E,\text{ net}} = 0$$

$$\Rightarrow \quad 3\Phi_{E,\text{ abcd}} + \frac{q}{8\varepsilon_0} = 0$$

$$\Rightarrow \quad \Phi_{E,\text{ abcd}} = -\frac{q}{24\,\varepsilon_0}$$

ELECTRIC FIELD DUE TO A POINT CHARGE *q* AT A POINT P AT DISTANCE *x*

To find electric field strength at P, we first consider a Gaussian surface so that point P lies on its surface.



Here, the Gaussian surface taken has to be spherical such that at every point on this surface electric field due to the charge q is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2} \qquad \text{\{directed radially outwards\}}$$

According to Gauss's Law

$$\oint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \quad \oint E\,dA\cos(0^\circ) = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \quad E\int dA = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \quad E(4\pi x^2) = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}$$

Also, we observe that for the Gaussian surface selected the electric field vector is parallel to $d\vec{A}$ and also the magnitude of $\vec{E}$ is uniform at every point, thus the integral $\oint \vec{E}\cdot d\vec{A}$ can be easily evaluated.

Problem Solving Technique(s)

Gauss's Law provides a convenient tool for evaluating electric field. However, its application is limited only to systems that possess certain symmetry, namely, systems with cylindrical, planar and spherical symmetry. In the table below, we give some examples of systems in which Gauss's Law is applicable for determining electric field, with the corresponding Gaussian surfaces:

System	Symmetry	Gaussian Surface
Sphere, Spherical shell	Spherical	Concentric Sphere
Infinite rod	Cylindrical	Coaxial Cylinder
Infinite plane	Planar	Gaussian "Pillbox"

The following steps may be useful when applying Gauss's Law.

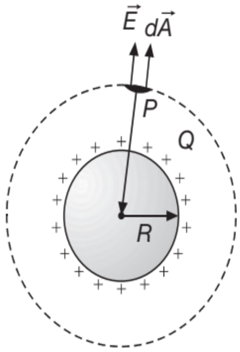
(a) Identify the symmetry associated with the charge distribution.
(b) Determine the direction of the electric field, and a "Gaussian surface" on which the magnitude of the electric field is constant over portions of the surface.
(c) Divide the space into different regions associated with the charge distribution. For each region, calculate q_{enc}, the charge enclosed by the Gaussian surface.
(d) Calculate the electric flux ϕ_E (or ϕ) through the Gaussian surface for each region.
(e) Now $\phi_E = \dfrac{q_{enc}}{\varepsilon_0}$ (due to Gauss's Law) and deduce the magnitude of the electric field.

Let us apply this technique to the situations discussed hereafter.

ELECTRIC FIELD STRENGTH OF A CHARGED CONDUCTING SPHERE OR CONDUCTING SHELL

CASE-1: OUTSIDE $(x > R)$

To find electric field at an outer point at a distance $x(>R)$ from the centre of sphere, we consider a spherical Gaussian surface of radius x.



Due to symmetry the electric field strength at every point of this surface is E. Using Gauss's Law we have

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \quad \oint E\,dA\cos 0 = \frac{1}{\varepsilon_0}(q_{enc})$$

Here we have

$$E\oint dA = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad E(4\pi x^2) = \frac{Q}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_0 x^2}$$

CASE-2: AT THE SURFACE

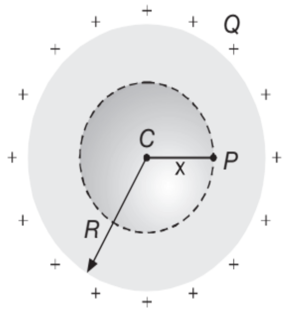
Similarly for points on the surface, we can consider a spherical Gaussian surface of radius R which gives electric field strength on the sphere surface as

$$E = \frac{Q}{4\pi\varepsilon_0 R^2}$$

In both these cases, it seems as if the net field is as if the charge Q is concentrated at the centre of the sphere.

CASE-3: INSIDE THE SPHERE $(x < R)$

To find electric field strength at an interior point of the sphere, we consider an inner spherical Gaussian surface of radius $x(x < R)$.



Here if we apply Gauss's Law for this surface, we have

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

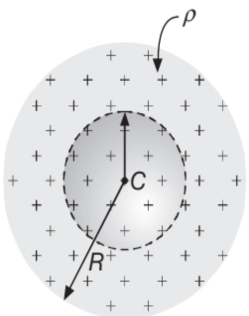
$$\Rightarrow \quad \vec{E} = \vec{0}$$

{as all the charge resides on surface so $q_{enc} = 0$}

ELECTRIC FIELD DUE TO A NON-CONDUCTING UNIFORMLY CHARGED SPHERE

CASE-1 and 2: For points outside and at the surface of sphere, the electric field strength is calculated by using Gauss's Law similar to the previous case of conducting sphere. Here too we get the net field at these points (outside and at the surface) as if the net charge is concentrated at the centre.

CASE-3: For interior points of sphere, we consider a spherical Gaussian surface of radius $x(<R)$ as shown.



If we apply Gauss's Law for this surface, then we get

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

Here enclosed charge q_{enc} will be calculated as follows

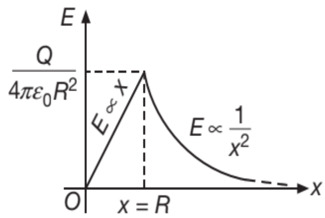
$$q_{enc} = \rho\left(\frac{4}{3}\pi x^3\right)$$

Thus $E(4\pi x^2) = \dfrac{\rho\left(\frac{4}{3}\pi x^3\right)}{\varepsilon_0}$

$$\Rightarrow \quad E = \frac{\rho x}{3\varepsilon_0} \qquad \{\text{for } x < R\}$$

One important thing that we observe regarding the electric field due to conducting sphere is that it exhibits its discontinuity at $x = R$ where as the field due to the non conducting uniformly charged sphere is continuous in nature and the field due to a uniformly charged sphere is given by

$$E = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 x^2} & x \geq R \text{ (outside and at the surface)} \\ \left(\dfrac{\rho}{3\varepsilon_0}\right)x & x < R \text{ (inside)} \\ \text{zero} & x = 0 \text{ (at the centre)} \end{cases}$$



So, to conclude, we observe that for a charged conducting sphere or a charged conducting shell, the net field is given by

$$E = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 x^2} & x \geq R \text{ (outside and at the surface)} \\ \text{zero} & x < R \text{ (inside)} \end{cases}$$

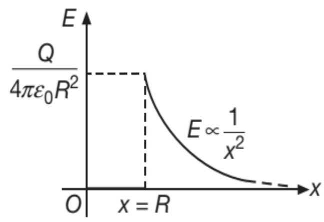


ILLUSTRATION 54

A ball of radius R carries a positive charge whose volume density depends on a separation r from the ball's centre as $\rho=\rho_0\left(1-\frac{r}{R}\right)$, where ρ_0 is a constant. Assuming the permittivities of the ball and the environment is equal to unity, find

(a) the magnitude of the electric field strength as a function of the distance r both inside and outside the ball.

(b) the maximum intensity $E_{\max}$ and the corresponding distance r_m.

SOLUTION

(a) Let the ball be divided into a number of infinitesimal spherical shells. Consider one shell at a distance r and of thickness dr. Volume (dV) of this shell is given by

$$dV=4\pi r^2 dr$$

Also $dq=\rho dV$

Given that $\rho=\rho_0\left(1-\frac{r}{R}\right)$

$$\Rightarrow \quad dq=\rho_0\left(1-\frac{r}{R}\right)\times 4\pi r^2 dr$$

So, total charge enclosed by the sphere of radius r is given by

$$\int_0^q dq=4\pi\rho_0\int_0^r\left(1-\frac{r}{R}\right)r^2 dr$$

$$\Rightarrow \quad q=4\rho_0\int_0^r\left(1-\frac{r^3}{R}\right)dr$$

$$\Rightarrow \quad q=4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^4}{4R}\right]_0^r=4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^4}{4R}\right]$$

By Gauss's Theorem, we have

$$E\left(4\pi r^2\right)=\frac{q_{\text{enc}}}{\varepsilon_0}$$

For $r<R$ i.e., inside

$$q_{\text{enc}}=4\pi\rho_0\left(\frac{r^3}{3}-\frac{r^4}{4R}\right)$$

$$\Rightarrow \quad E=\frac{4\pi\rho_0}{4\pi\varepsilon_0 r^2}\left(\frac{r^3}{3}-\frac{r^4}{4R}\right)$$

$$\Rightarrow \quad E=\frac{4\pi\rho_0}{4\pi\varepsilon_0}\left(\frac{r}{3}-\frac{r^2}{4R}\right)$$

$$\Rightarrow \quad E=\frac{\rho_0 r}{3\varepsilon_0}\left(1-\frac{3r}{4R}\right) \qquad \ldots(1)$$

When $r>R$ then

$$q=\int_0^R\rho_0\left(r-\frac{r}{R}\right)4\pi r^2 dr=\frac{4\pi\rho_0 R^3}{12}$$

$$\Rightarrow \quad E=\frac{\rho_0 R^3}{12\varepsilon_0 r^2}$$

(b) For electric field to be maximum we have

$$\frac{dE}{dr}=0$$

Since $E=\frac{\rho_0}{3\varepsilon_0}\left(r-\frac{3r^2}{4R}\right)$

$$\Rightarrow \quad \frac{dE}{dr}=\frac{\rho_0 R}{9\varepsilon_0}\left(1-\frac{6r}{4R}\right)=0$$

$$\Rightarrow \quad \left(1-\frac{6r}{4R}\right)=0$$

$$\Rightarrow \quad r=r_m=\frac{2R}{3}$$

Thus maximum electric field will be obtained by putting

$r=r_m=\frac{2R}{3}$ in equation (1)

$$\Rightarrow \quad E_{\max}=\frac{\rho_0}{3\varepsilon_0}\left(\frac{2R}{3}-\frac{3}{4R}\times\frac{4R^2}{9}\right)=\frac{\rho_0 R}{9\varepsilon_0}$$

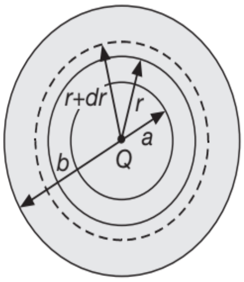
$$\Rightarrow \quad E_{\max}=\frac{\rho_0 R}{9\varepsilon_0}$$

ILLUSTRATION 55

A system consists of a ball of radius R carrying a spherically symmetric charge Q and the surrounding space filled with a charge of volume density $\rho=\frac{\alpha}{r}$, where α is a constant, r the distance from the centre of the ball. Find the ball's charge at which the magnitude of the electric field strength vector is independent of r outside the ball. How high is this strength? The permittivities of the ball are assumed to be unity.

SOLUTION

Let us draw a Gaussian surface of radius $r\,(a<r<b)$ concentric with the spheres of radii a and b respectively.



Let the charge enclosed between the radii r and a be q. Then

$$q=\int \rho(r)dV$$

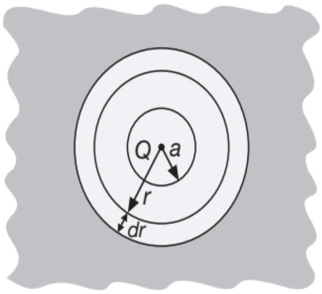
$$\Rightarrow \quad q=\int \rho(r)4\pi r^2 dr$$

$$\Rightarrow \quad q=4\pi\int_a^r \frac{\alpha}{r}r^2 dr$$

$$\Rightarrow \quad q=\frac{4\pi\alpha}{2}\left(r^2-a^2\right)$$

$$\Rightarrow \quad q=2\pi\alpha\left(r^2-a^2\right)$$

Also, we see that as per the instructions given in problem, the ball already possesses a charge Q. So, by Gauss's Theorem, total charge enclosed inside Gaussian surface is $(q+Q)$.



and $\oint_S \vec{E}\cdot d\vec{A}=\frac{1}{\varepsilon_0}(Q+q)$

$$\Rightarrow \quad E\left(4\pi r^2\right)=\frac{1}{\varepsilon_0}\left[Q+2\pi\alpha\left(r^2-a^2\right)\right]$$

$$\Rightarrow \quad E=\frac{\alpha}{2\varepsilon_0}+\frac{Q-2\pi\alpha a^2}{4\pi\varepsilon_0 r^2}$$

For E to be of constant magnitude, we have

$$Q-2\pi\alpha a^2=0$$

$$\Rightarrow \quad \alpha=\frac{Q}{2\pi a^2}$$

and $E=\frac{\alpha}{2\varepsilon_0}$

ELECTRIC FIELD DUE TO AN INFINITELY LONG THIN CHARGED WIRE

Let us find electric field strength due to a long charged wire having linear charge density λ at a point P situated at a distance r from the wire.

For this application of Gauss's Law, we consider a coaxial cylindrical Gaussian surface of length l and radius x as shown. Applying Gauss's Law on this surface, we get

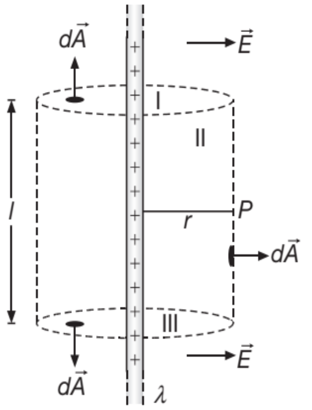
$$\oint \vec{E}\cdot d\vec{A}=\frac{q_{\text{enc}}}{\varepsilon_0} \qquad \ldots(1)$$

Now, we see the closed Gaussian surface to be made of three parts I, II and III. Two flat circular faces (I and III) called Lids and one cylindrical lateral surface I called curved surface. Let us split the closed surface integration in three parts, then

$$\oint \vec{E}\cdot d\vec{A}=\int_I \vec{E}\cdot d\vec{A}+\int_{II} \vec{E}\cdot d\vec{A}+\int_{III} \vec{E}\cdot d\vec{A}$$

Here we observe that for part I and III, electric field strength vector is perpendicular to the area vector as shown in figure hence no flux will be associated with these parts. Thus we have

Flux due to Lids $=\int_I \vec{E}\cdot d\vec{A}=\int_{III} \vec{E}\cdot d\vec{A}=0$



But according to Gauss's Law given in equation (1), we have

$$\int_{II} \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

{Since the enclosed charged is $q_{enc} = \lambda l$}

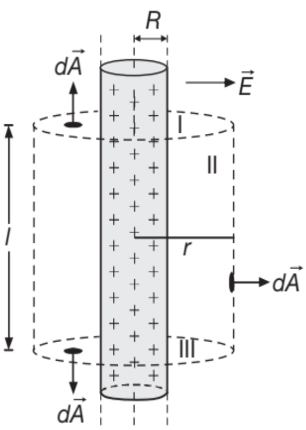
Since for lateral surface or curved surface, we observe that $\vec{E}$ is parallel to $d\vec{A}$ and is constant on the entire lateral surface, so

$$\phi = \oint \vec{E}\cdot d\vec{A} = E\int_{II} dA = \frac{\lambda l}{\varepsilon_0}$$

$$\Rightarrow \quad E(2\pi r l) = \frac{\lambda l}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

ELECTRIC FIELD DUE TO A LONG UNIFORMLY CHARGED CONDUCTING CYLINDER

Figure shows a long cylinder of radius R which is uniformly charged on its surface having surface charge density σ.



At all the interior points of a metal/conducting body electric field strength is zero. For finding electric field strength at outer points at a distance r from the axis of the cylinder, we consider a cylindrical Gaussian surface of radius r and length l as shown in figure. Now on applying Gauss's Law we get

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

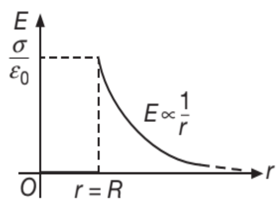
Here enclosed charge in the cylindrical Gaussian surface is

$$q_{enc} = \sigma(2\pi R l)$$

Here also similar to previous case the electric flux through the circular faces i.e., lids is zero, hence according to Gauss's Law, we have

$$\oint \vec{E}\cdot d\vec{A} = \int_{II} \vec{E}\cdot d\vec{A} = \frac{\sigma(2\pi R l)}{\varepsilon_0}$$

$$\Rightarrow \quad E\int_{II} dA = \frac{\sigma(2\pi R l)}{\varepsilon_0}$$



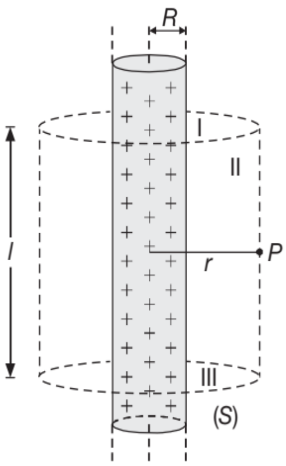
$$\Rightarrow \quad E(2\pi r l) = \frac{\sigma(2\pi R l)}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{\sigma R}{\varepsilon_0 r} \qquad \{\text{for } r > R\}$$

$$\text{So, } E = \begin{cases} \dfrac{\sigma R}{\varepsilon_0 r} & \text{for } r \ge R \text{ (outside and at the surface)} \\ 0 & \text{for } r < R \text{ (inside)} \end{cases}$$

ELECTRIC FIELD DUE TO A LONG UNIFORMLY CHARGED NON-CONDUCTING CYLINDER

Let us take a long cylinder of radius R, charged uniformly having volume charge density ρ. To find the electric field strength at a distance r from the axis of the cylinder we again consider a cylindrical Gaussian surface (S) shown in figure.



Applying Gauss's Law on this surface, we get

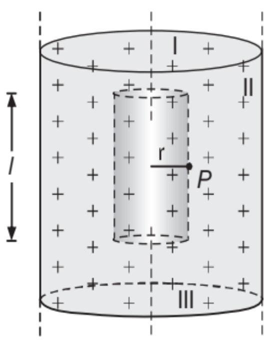
$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \quad \oint_{II} \vec{E} \cdot d\vec{A} = \frac{\rho(\pi R^2 l)}{\varepsilon_0} \qquad \{\text{Since } q_{enc} = \rho(\pi R^2 l)\}$$

(Because, as in earlier cases flux contribution due to circular faces/lids is still zero)

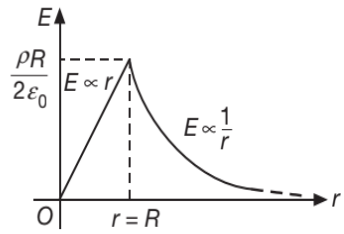
$$\Rightarrow \quad E\int dA = \frac{\rho \pi R^2 l}{\varepsilon_0}$$

$$\Rightarrow \quad E(2\pi r l) = \frac{\rho \pi R^2 l}{\varepsilon_0} \Rightarrow E = \frac{\rho R^2}{2\varepsilon_0 r}$$



To find electric field inside the cylinder at a distance r from the axis, we consider a small cylindrical Gaussian surface of radius r and length l. Applying Gauss's Law for this surface, we get

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$



$$\Rightarrow \quad E\int_{II} dA = \frac{\rho \pi r^2 l}{\varepsilon_0}$$

$$\Rightarrow \quad E(2\pi r l) = \frac{\rho \pi r^2 l}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{\rho r}{2\varepsilon_0}$$

Again we observe that here E is continuous, where as it exhibited discontinuity in case of the conducting cylinder.

ILLUSTRATION 56

An infinitely long insulating cylinder of radius R has a volume charge density that varies with the radius as

$$\rho = \rho_0\left(a - \frac{r}{b}\right)$$

where ρ_0, a and b are positive constants and r is the distance from the axis of the cylinder. Use Gauss's Law to determine the magnitude of the electric field at radial distances

(a) $r < R$ **(b)** $r > R$

SOLUTION

In this case the charge density is not uniform, and Gauss's Law is written as $\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0}\int \rho dV$. We use a Gaussian surface which is a cylinder of radius r, length l and is coaxial with the charge distribution.

(a) When $r < R$, this becomes $E(2\pi r l) = \frac{\rho_0}{\varepsilon_0}\int_0^r \left(a - \frac{r}{b}\right) dV$.

The element of volume is a cylindrical shell of radius r, length l and thickness dr so that $dV = 2\pi r l dr$.

$$E(2\pi rl) = \left(\frac{2\pi r^2 l\rho_0}{\varepsilon_0}\right)\left(\frac{a}{2} - \frac{r}{3b}\right)$$

So, inside the cylinder, we get

$$E = \frac{\rho_0 r}{2\varepsilon_0}\left(a - \frac{2r}{3b}\right)$$

(b) When $r > R$, Gauss's Law becomes

$$E(2\pi rl) = \frac{\rho_0}{\varepsilon_0}\int_0^R \left(a - \frac{r}{b}\right)(2\pi rl dr)$$

So, outside the cylinder, we get

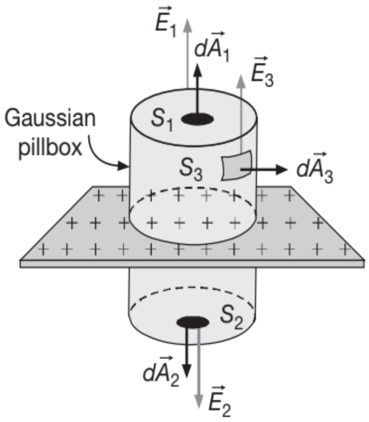
$$E = \frac{\rho_0 R^2}{2\varepsilon_0 r}\left(a - \frac{2R}{3b}\right)$$

ELECTRIC FIELD DUE TO AN INFINITE NON-CONDUCTING THIN UNIFORMLY CHARGED SHEET

To find the electric field strength at a point P at distance r infront of the charged sheet we consider a cylindrical Gaussian surface (also called as Gaussian Pill box) as shown in figure of face area A. Applying Gauss's Law for this surface, we have

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \quad \int_I \vec{E} \cdot d\vec{A} + \int_{II} \vec{E} \cdot d\vec{A} + \int_{III} \vec{E} \cdot d\vec{A} = \frac{\sigma A}{\varepsilon_0} \quad \{\because q_{enc} = \sigma A\}$$



In this case $\int_{II} \vec{E} \cdot d\vec{A} = 0$, as the lateral surface of cylinder is parallel to the direction of electric field strength. So, $\vec{E}$ becomes perpendicular to $\vec{A}$ for curved surface and hence no flux is associated with the curved surface.

$$\phi_{total} = \int_I EdA + \int_{II} EdA = \frac{\sigma A}{2\varepsilon_0}$$

$$\Rightarrow \quad 2EA = \frac{\sigma A}{\varepsilon_0}$$

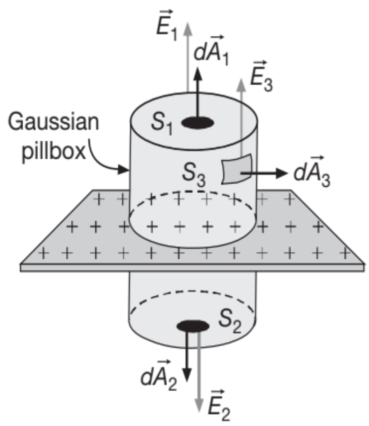
{As electric field is uniform on both sides}

$$\Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0}$$

{This field is independent of the distance of point P from the sheet}

ELECTRIC FIELD DUE TO AN INFINITE CHARGED CONDUCTING SHEET

Figure shows a large charged conducting sheet, charged on both the surfaces with surface charge density σ. There is no charge within the volume of the sheet and also the electric field inside the metal sheet is zero. To find electric field strength at a point P in front of the sheet we consider a cylindrical Gaussian surface (or Gaussian Pill Box) as shown.



Applying Gauss's Law to this surface, we get

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \quad \int_I \vec{E} \cdot d\vec{A} + \int_{II} \vec{E} \cdot d\vec{A} + \int_{III} \vec{E} \cdot d\vec{A} = \frac{2\sigma A}{\varepsilon_0}$$

{Here $q_{enc} = 2\sigma A$}

Since the field inside the sheet is zero, so the Gaussian Pill box will enclose a charge σA at the left and σA at the right too. So that the total charge enclosed is

$$q_{enc} = 2\sigma A$$

$$\Rightarrow \quad \phi_{total} = \int_{I} \vec{E} \cdot d\vec{A} + \int_{II} \vec{E} \cdot d\vec{A} + \int_{III} \vec{E} \cdot d\vec{A} = \frac{2\sigma A}{\varepsilon_0}$$

$$\Rightarrow \quad 2EA = \frac{2\sigma A}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{\sigma}{\varepsilon_0}$$

{Again here E is independent of the distance of point P from the sheet}

ILLUSTRATION 57

A sphere of radius R is charged with a non-uniform charge density which varies with the distance x from the centre as

$$\rho = \frac{\rho_0}{x}$$

where ρ_0 is a positive constant. Find the electric field strength at a point P situated at a distance r from centre of sphere

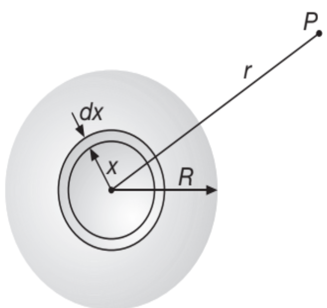
(a) outside it
(b) inside it
(c) also plot the variation of E with r.

SOLUTION

(a) $E_P = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

{where Q is the total charge of sphere}

For outer points we see that the entire charge of sphere to be concentrated at its centre. We can also prove this again by taking a spherical Gaussian surface as done in earlier cases. Let us now calculate Q. This is done by integrating the charge of an elemental shell of radius x and thickness dx as shown in figure.



The charge dq in this infinitesimal shell is

$$dq = \rho dV$$

$$\Rightarrow \quad dq = \rho\left(4\pi x^2\, dx\right)$$

$$\Rightarrow \quad dq = \frac{\rho_0}{x}\left(4\pi x^2\, dx\right)$$

$$\Rightarrow \quad dq = 4\pi\rho_0 x\, dx$$

$$\Rightarrow \quad Q = \int dq = \int_0^R 4\pi\rho_0 x\, dx$$

$$\Rightarrow \quad Q = 4\pi\rho_0 \left[\frac{x^2}{2}\right]_0^R$$

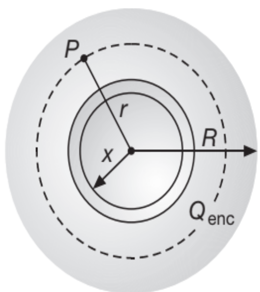
$$\Rightarrow \quad Q = 2\pi\rho_0 R^2$$

Thus electric field strength at point outside the sphere at distance r from the centre is

$$E_p = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\left(2\pi\rho_0 R^2\right)}{r^2} = \frac{\rho_0 R^2}{2\varepsilon_0 r^2}$$

$$\Rightarrow \quad E_{outside} = \frac{\rho_0 R^2}{2\varepsilon_0 r^2}$$

(b) To find electric field strength at an interior point at distance r from the centre of sphere, we first find the charge enclosed within the inner sphere of radius r such that point P is on the surface. Thus enclosed charge is



$$Q_{enc} = \int_0^r \frac{\rho_0}{R}\left(4\pi x^2 dx\right)$$

$$\Rightarrow \quad Q = 2\pi\rho_0 r^2$$

Here electric field strength at point will be due to the inner charged core having charge Q_{enc} as if it were again concentrated at the centre. So

$$E_{\text{inside}} = E_P = \frac{1}{4\pi\varepsilon_0}\frac{Q_{\text{enc}}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\left(2\pi\rho_0 r^2\right)}{r^2}$$

$$\Rightarrow \quad E_P = \frac{\rho_0}{2\varepsilon_0}$$

{Independent of distance r from the centre}

(c) Since

$$E = \begin{cases} \dfrac{\rho_0 R^2}{2\varepsilon_0 r^2} & \text{for } r \ge R \\ & \text{(outside and at the surface)} \\ \dfrac{\rho_0}{2\varepsilon_0} & \text{for } r < R \\ & \text{(inside the sphere)} \end{cases}$$

Also, we observe that

$$E_{\text{surface}} = \frac{\rho_0}{2\varepsilon_0} \qquad \{\text{by putting } r = R\}$$

So, the field has a continuity as shown

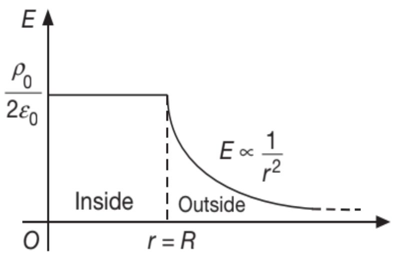


ILLUSTRATION 58

A region between yz plane and another plane parallel to yz plane at $x = d$ has charge density ρ which varies with distance x from origin O as

$$\rho = \rho_0 x^2$$

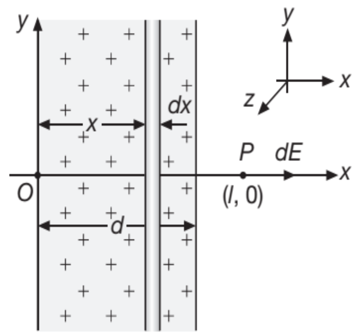
where ρ_0 is a positive constant. Calculate the electric field strength due to this slab at point $P(l, O)$.

SOLUTION

Let us consider an elemental sheet of thickness dx at a distance x from the yz plane. Since we know that due to a sheet of thickness d the electric field is $E = \dfrac{\rho d}{2\varepsilon_0}$. Electric field strength at P due to this sheet is

$$dE = \frac{\rho dx}{2\varepsilon_0}$$

$$\Rightarrow \quad dE = \frac{\rho_0 x^2}{2\varepsilon_0}dx \qquad \left\{\because \rho = \rho_0 x^2\right\}$$



Net electric field at P is given by

$$E_P = \int dE = \int_0^d \frac{\rho_0 x^2}{2\varepsilon_0}dx$$

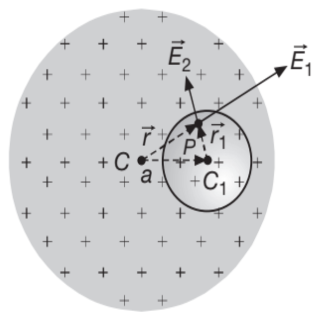
$$\Rightarrow \quad E_P = \frac{\rho_0 d^3}{6\varepsilon_0}$$

ILLUSTRATION 59

Inside a ball charged uniformly with volume density ρ, there is a spherical cavity. The centre of the cavity is displaced with respect to the centre of the ball by $\vec{a}$. Calculate the field strength inside the cavity, assuming the permittivity equal to unity.

SOLUTION

Consider a point P in the cavity at a position vector $\vec{r}$ from the centre of sphere and at a position vector $\vec{r}_1$ from the centre of cavity as shown.



If $\vec{E}$ be the electric field strength at P due to the complete charge of the sphere (inside cavity also) then we know electric field strength inside a uniformly charged sphere is given as

$$\vec{E} = \frac{\rho\vec{r}}{3\varepsilon_0}$$

Similarly if we assume that the charge is only there in the region of cavity and hence this will also be a uniformly charged small sphere. If $\vec{E}_1$ be the electric field only due to the cavity charge then,

$$\vec{E}_1 = \frac{\rho\vec{r}_1}{3\varepsilon_0}$$

Now the electric field due to the charged sphere in the cavity at point P can be given as

$$\vec{E}_{net} = \vec{E} - \vec{E}_1$$

{As now charge of cavity is removed}

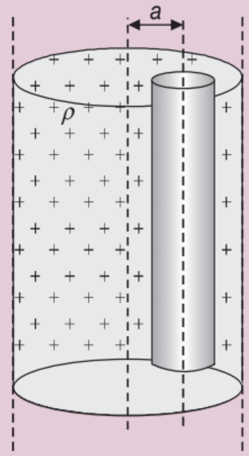
$$\Rightarrow \quad \vec{E}_{net} = \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{r}_1) \qquad \{\because \vec{a} + \vec{r}_1 = \vec{r}\}$$

$$\Rightarrow \quad \vec{E}_{net} = \frac{\rho\vec{a}}{3\varepsilon_0} \qquad \{\text{As } \vec{r} - \vec{r}_1 = \vec{a}\}$$

Problem Solving Technique(s)

"Cavity can be considered to be negative charge embedded in positive charge so that the common region having size of cavity has no charge on it".

This shows that the net electric field inside the cavity is uniform and in the direction of $\vec{a}$ i.e., along the line joining the centre of spheres and cavity.



Similarly we can find the electric field strength inside a cylindrical cavity of a long uniformly charged cylinder. If cavity axis is displaced from axis of cylinder by a displacement vector $\vec{a}$, then by the analysis we have already done for a sphere, we can say that the electric field strength inside the cavity is also uniform and hence will be given as

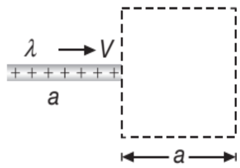
$$\vec{E} = \frac{\rho\vec{a}}{2\varepsilon_0}$$

Test Your Concepts-VI

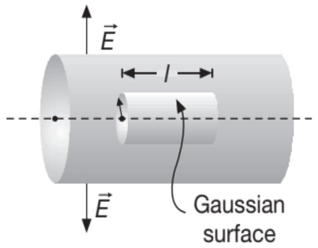
Based on Gauss's Law

(Solutions on page H.20)

1. An electric dipole is placed at the centre of a sphere. Find the electric flux passing through the sphere.
2. A point charge q is placed at the centre of a cube of edge a. Calculate the flux linked
   (a) with all the faces of the cube.
   (b) with each face of the cube.
   If charge is not at the centre, then what will be the answers for parts (a) and (b).
3. The intensity of an electric field depends only on the coordinates x and y as $\vec{E} = \frac{\alpha(x\hat{i} + y\hat{j})}{x^2 + y^2}$, where, α is a constant and $\hat{i}$ and $\hat{j}$ are the unit vectors of the x and y axes. Calculate the charge within a sphere of radius R having its centre at the origin.
4. Figure shows an imaginary cube of side a. A uniformly charged rod of length a moves towards right at a constant speed v. At $t = 0$, the right end of the rod just touches the left face of the cube. Plot a graph between electric flux passing through the cube versus time.



5. A long cylinder (shown in figure) carries a charge density that is proportional to the distance x from the axis as $\rho = kx$, for some constant k. Find the electric field inside this cylinder.

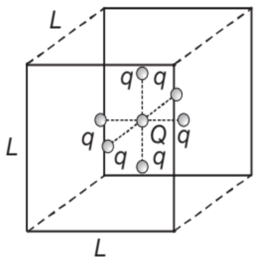


6. **(a)** A point charge q is located a distance d from an infinite plane. Determine the electric flux through the plane due to the point charge.

(b) If a point charge q is located a very small distance from the centre of a very large square on the line perpendicular to the square and going through its centre. Determine the approximate electric flux through the square due to the point charge.

(c) Explain why the answers to parts (a) and (b) are identical.

7. A positive point charge Q is located at the centre of a cube of edge L. In addition, six other identical negative point charges q are positioned symmetrically around Q as shown in figure. Determine the electric flux through one face of the cube. Also find the condition when no flux is associated with the cube.



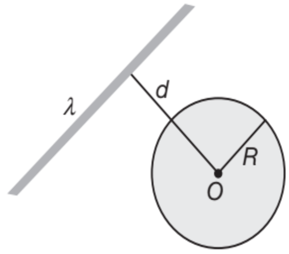
8. An insulating solid sphere of radius a has a uniform volume charge density and carries a total positive charge Q. A spherical Gaussian surface of radius r, which shares a common center with the insulating sphere, is inflated starting from $r = 0$.

(a) Find an expression for the electric flux passing through the surface of the Gaussian sphere as a function of r for $r < a$.

(b) Find an expression for the electric flux for $r > a$.

(c) Plot the flux versus r.

9. An infinitely long line charge having a uniform charge per unit length λ lies a distance d from point O as shown in figure. Determine the total electric flux through the surface of a sphere of radius R having centre at O resulting from this line charge. Consider both cases, where $R < d$ and $R > d$.

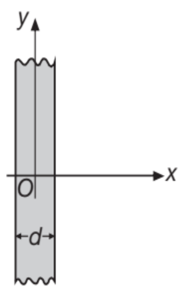


10. A solid insulating sphere of radius R has a non-uniform charge density that varies with r according to the expression $\rho = \rho_0 r^2$, where ρ_0 is a constant and $r < R$ is measured from the centre of the sphere. Show that the magnitude of the electric field

(a) outside $(r > R)$ the sphere is $E = \dfrac{\rho_0 R^5}{5\varepsilon_0 r^2}$.

(b) inside $(r < R)$ the sphere is $E = \dfrac{\rho_0 r^3}{5\varepsilon_0}$.

11. A slab of insulating material has a non-uniform positive charge density $\rho = \rho_0 x^2$, where x is measured from the centre of the slab as shown in Figure, and ρ_0 is a constant. The slab is infinite in the y and z directions. Derive expressions for the electric field in the



(a) exterior regions of the slab $\left(\text{i.e., } x > \dfrac{d}{2} \text{ and } x < -\dfrac{d}{2}\right)$

(b) interior region of the slab $\left(\text{i.e., } -\dfrac{d}{2} < x < \dfrac{d}{2}\right)$.

12. A spherically symmetric charge distribution has a charge density given by $\rho = \frac{\rho_0}{r}$, where ρ_0 is constant. Find the electric field as a function of r.

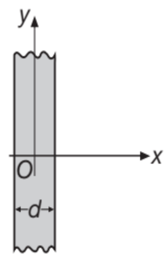
13. (a) Using the mathematical similarity between Coulomb's Law and Newton's Law of Universal Gravitation, show that Gauss's Law for gravitation expressed as

$$\oint \vec{g} \cdot d\vec{A} = -4\pi G \sum m_{enc}$$

where $\sum m_{enc}$ is the net mass enclosed by the Gaussian surface and $\vec{g} = \frac{\vec{F}_g}{m}$ represents the gravitational field at any point on the Gaussian surface.

(b) Determine the gravitational field at a distance r from the centre of the Earth where $r < R_E$, assuming that the Earth's mass density is uniform.

14. A slab of insulating material (infinite in two of its three dimensions) has a uniform positive charge density ρ. An edge view of the slab is shown in figure.



(a) Show that the magnitude of the electric field a distance x from its centre and inside the slab is $E = \frac{\rho x}{\varepsilon_0}$.

(b) Suppose an electron of charge $-e$ and mass m can move freely within the slab. It is released from rest at a distance x from the centre. Show that the electron exhibits simple harmonic motion with a period

$$T = 2\pi\sqrt{\frac{m\varepsilon_0}{\rho_e}}$$

15. Inside an infinitely long circular cylinder charged uniformly with volume density ρ there is a circular cylindrical cavity. The distance between the axes of the cylinder and the cavity is equal to $\vec{a}$. Find the electric field strength $\vec{E}$ inside the cavity. The permittivity is assumed to be equal to unity.

ELECTROSTATIC POTENTIAL, POTENTIAL ENERGY, CONDUCTORS AND APPLICATIONS

ELECTROSTATIC POTENTIAL ENERGY AND POTENTIAL

Electrostatic Potential Energy

Potential energy of a system of particles is defined as the work done in assembling the system of charges in a given configuration against the interaction forces of particles. Potential energy of a system of particles is a concept associated with conservative fields only. Since, electric field is also conservative in nature so, we associate the concept of potential energy with it.

When all particles of a system are separated far apart by infinite distance there exists no interaction between them. This state, we take as reference of Zero Potential Energy (ZPE).

Electrostatic potential energy is categorised in two ways.

(a) Electrostatic interaction energy of charged particles of a system.

(b) Electrostatic self energy of a charged object.

Electrostatic Interaction Energy

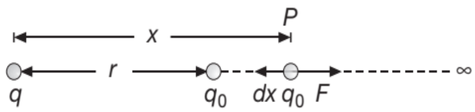
Electrostatic interaction energy of a system of charged particles is defined as the external work required to assemble the particles from infinity to a given configuration (as that required in system).

When charged particles are lying at infinity, their potential energy is taken to be zero, because no interaction exists between them. When these charges are brought close to a given configuration, external work is required if the force between these particles is repulsive and hence energy is supplied to the system so that final potential energy of system will be positive. If the force between the particles is attractive, work will be done by the system and final potential energy of system will be negative.

In giving these arguments please keep this thing in mind that work done by a conservative force equals the decrease in potential energy of the system.

ELECTROSTATIC INTERACTION ENERGY OF A SYSTEM OF TWO CHARGED PARTICLES

Figure shows two positive charges q and q_0 separated by a distance r. The electrostatic interaction energy of this system can be given as work done in bringing test charge q_0 from infinity to the given point P at a distance r from the source charge q. Let, at any instant the test charge q_0 be at a distance x from the source charge q. Let q_0 be displaced through dx towards q. The electrostatic force of repulsion F acts on q_0 away from q.



If dW be the work done, then

$$dW = \vec{F} \cdot d\vec{x}$$

$$\Rightarrow \quad dW = \frac{qq_0}{4\pi\varepsilon_0 x^2} dx \cos(180°)$$

$$\Rightarrow \quad dW = -\frac{qq_0}{4\pi\varepsilon_0} x^{-2} dx$$

$$\Rightarrow \quad W = -\frac{qq_0}{4\pi\varepsilon_0} \int_0^r x^{-2} dx$$

$$\Rightarrow \quad W = -\frac{qq_0}{4\pi\varepsilon_0} \left(\frac{x^{-2+1}}{-2+1} \bigg|_\infty^r \right)$$

$$\Rightarrow \quad W = \frac{qq_0}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{\infty} \right)$$

$$\Rightarrow \quad W_{\infty \to P} = \frac{qq_0}{4\pi\varepsilon_0 r} = W$$

So, work done by the external force in bringing the test charge q_0 from ∞ to the point P under the influence of source charge q is

$$W = W_{\infty \to P} = \frac{qq_0}{4\pi\varepsilon_0 r}$$

This work done is stored in the form of electrostatic interaction energy (U) of the system. So,

$$U = \frac{qq_0}{4\pi\varepsilon_0 r}$$

Please note that if the two charges are of opposite nature, then

$$U = -\frac{qq_0}{4\pi\varepsilon_0 r}$$

ILLUSTRATION 60

Two charges, $+q$ and $-q$, each of mass m, are revolving in a circle of radius R, under mutual electrostatic force. Find (a) speed of each charge (b) kinetic energy of system (c) potential energy of the system (d) total energy of the system

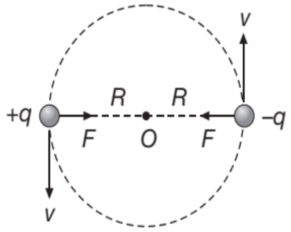
SOLUTION

In this situation, the centripetal force is provided by the electrostatic force of attraction.

(a) By Newton's second law we get,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2R)^2} = \frac{mv^2}{R}$$

$$\Rightarrow \quad v = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(4R)m}}$$



(b) Kinectic energy of the system is $K = \frac{1}{2}mv^2 + \frac{1}{2}mv^2$

$$\Rightarrow \quad K = \frac{q^2}{4\pi\varepsilon_0(4R)} = \frac{q^2}{16\pi\varepsilon_0 R}$$

(c) Potential energy of the system is $U = \frac{1}{4\pi\varepsilon_0}\frac{q(-q)}{2R}$

$$\Rightarrow \quad U = \frac{-1}{4\pi\varepsilon_0}\frac{q^2}{(2R)} = -\frac{q^2}{8\pi\varepsilon_0 R}$$

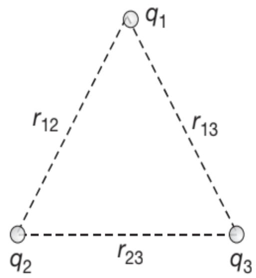
(d) Total energy of the system is $E = K.E. + P.E.$

$$\Rightarrow \quad E = \frac{-q^2}{4\pi\varepsilon_0(4R)} = -\frac{q^2}{16\pi\varepsilon_0 R}$$

So, we observe that $T.E. = -(K.E.) = \frac{1}{2}(P.E.)$

ELECTROSTATIC INTERACTION ENERGY FOR A SYSTEM OF PARTICLES

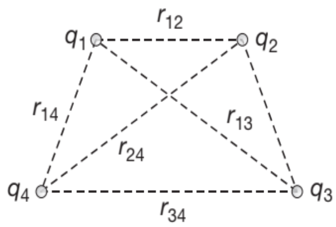
When more than two charged particles are there in a system, the interaction energy can be given by sum of interaction energy of all the pairs of particles with no pair repeated.



For example if a system of three particles having charges q_1, q_2 and q_3 is given as shown in figure. The total interaction energy of this system can be given as

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{r_{12}} + \frac{q_2q_3}{r_{23}} + \frac{q_1q_3}{r_{13}}\right] = U_{12} + U_{13} + U_{23}$$

Similarly, for an assembly of four charges q_1, q_2, q_3 and q_4 (having total number of six interaction = ${}^4C_2$), we have



$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{r_{12}} + \frac{q_2q_3}{r_{23}} + \frac{q_3q_4}{r_{34}} + \frac{q_4q_1}{r_{14}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_4}{r_{24}}\right]$$

$$\Rightarrow \quad U = U_{12} + U_{23} + U_{34} + U_{41} + U_{13} + U_{24}$$

From above we see that the total potential energy is simply the sum of contributions due to distinct pairs. Generalising to N charges, we get

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ (j>i)}}^{N}\frac{q_iq_j}{r_{ij}}$$

where $j > i$ assures that the pairs are not repeated. Other way round, we can count every pair twice and divide the result by $\frac{1}{2}$. So,

$$U = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_iq_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N}q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{q_i}{r_{ij}}\right)$$

$$\Rightarrow \quad U = \frac{1}{2}\sum_{i=1}^{N}q_iV(r_i)$$

where $V(r_i)$ is the potential due to all other charges at the location of charge q_i (i.e., r_i)

ILLUSTRATION 61

Calculate the work required to be done to make an arrangement of three particles each having a charge $+q$ such that the particles lie at the vertices of an equilateral triangle of side a. What work will be done by electric field when the particles are shifted away so that the side of triangle becomes $2a$?

SOLUTION

The work required to make an arrangement of charges is equal to potential energy of the system

$$\Rightarrow \quad W = \frac{q^2}{4\pi\varepsilon_0 a}\times 3 = \frac{3q^2}{4\pi\varepsilon_0 a}$$

Work done by electric field equals the decrease in potential energy. So, we have

$$W_{E\text{-Field}} = -\Delta U = U_i - U_f$$

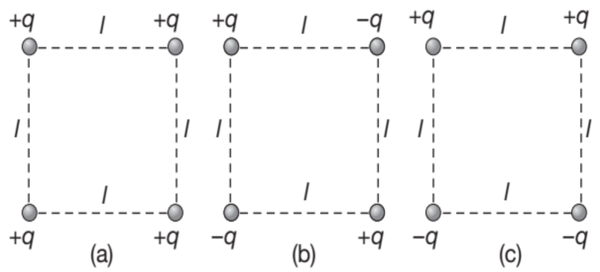
Since $U_i = \dfrac{q^2}{4\pi\varepsilon_0 (a)}\times 3$ and

$$U_f = \frac{q^2}{4\pi\varepsilon_0 (2a)}\times 3$$

$$\Rightarrow \quad W_{E\text{-Field}} = U_i - U_f = \frac{3q^2}{4\pi\varepsilon_0 a} - \frac{3q^2}{8\pi\varepsilon_0 a} = \frac{3q^2}{8\pi\varepsilon_0 a}$$

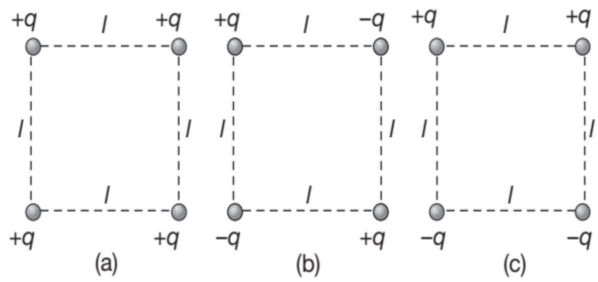
ILLUSTRATION 62

Determine the interaction energy of the point charges located at the corners of a square of side l in the figures shown.



SOLUTION

Interaction energy of any two point charges q_1 and q_2 at separation r is given by $\dfrac{q_1 q_2}{4\pi\varepsilon_0 r}$



Let interaction energy of the system in Figure (a), (b) and (c) be U_a, U_b and U_c respectively, then

$$U_a = 4\frac{q^2}{4\pi\varepsilon_0 l} + 2\frac{q^2}{4\pi\varepsilon_0 (\sqrt{2}l)} = \frac{q^2}{4\pi\varepsilon_0 l}(4+\sqrt{2})$$

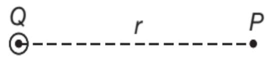
$$U_b = 4\frac{-q^2}{4\pi\varepsilon_0 l} + 2\frac{q^2}{4\pi\varepsilon_0 (\sqrt{2}l)} = \frac{q^2}{4\pi\varepsilon_0 l}(-4+\sqrt{2})$$

and $$U_c = 2\frac{q^2}{4\pi\varepsilon_0 l} - \frac{2q^2}{4\pi\varepsilon_0 l} - \frac{2q^2}{4\pi\varepsilon_0 (\sqrt{2}l)} = -\frac{\sqrt{2}q^2}{4\pi\varepsilon_0 l}$$

ELECTROSTATIC POTENTIAL (V)

The electrostatic potential at a point P due to a source charge is the work done per unit test charge in bringing the test charge (q_0) from infinity to the point P under the electrostatic influence of source charge Q. So,

$$V_P = \text{Potential at point } P = \frac{W_{\infty\to P}}{q_0}$$



Since $W_{\infty\to P} = \dfrac{Qq_0}{4\pi\varepsilon_0 r}$

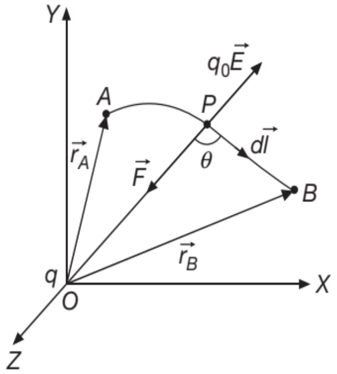
$$\Rightarrow \quad V_P = \frac{W_{\infty\to P}}{q_0} = \frac{Q}{4\pi\varepsilon_0 r}$$

Electrostatic potential is a scalar physical quantity with SI unit joule per coulomb (JC^{-1}) and $1\ \text{JC}^{-1} = 1\ \text{volt} = 1\ \text{V}$.

ELECTROSTATIC POTENTIAL DIFFERENCE (ΔV)

Consider a charge $+q$ to be located at the origin O. Let us arbitrarily choose two points A and B. Let $\vec{E}$ be the electric field strength at point P on this curve. The test charge q_0 will experience an electric force $q_0\vec{E}$ directed radially away from charge $+q$. In order to prevent the test charge from accelerating in the direction of $q_0\vec{E}$, the external agent is required to apply a force $\vec{F}$ in the opposite direction. So,

$$\vec{F} = -q_0\vec{E}$$



If dW is the infinitesimally small amount of work done by external agent in giving small displacement $d\vec{l}$, then

$$dW = \vec{F}\cdot d\vec{l}$$

$$\Rightarrow \quad W_{AB} = \int_A^B \vec{F}\cdot d\vec{l}$$

$$\Rightarrow \quad W_{AB} = \int_A^B -q_0\vec{E}\cdot d\vec{l} = -q_0\int_A^B \vec{E}\cdot d\vec{l}$$

$$\Rightarrow \quad \frac{W_{AB}}{q_0} = -\int_A^B \vec{E}\cdot d\vec{l}$$

But $V_B - V_A = \dfrac{W_{AB}}{q_0}$

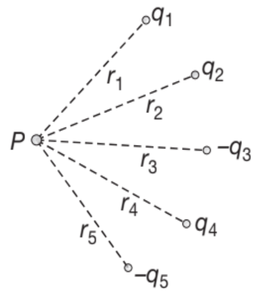
$$\Rightarrow \quad -\int_A^B \vec{E}\cdot d\vec{l} = V_B - V_A$$

$$\Rightarrow \quad V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} = \frac{W_{A\to B}}{q_0}$$

So, electrostatic potential difference between two points is work done by external agent in moving unit positive charge from one point to another.

POTENTIAL DUE TO AN ASSEMBLY OF CHARGES

Before moving further with this discussion, let us know this and keep in mind that electrostatic potential is a scalar quantity and is positive due to positive charges and negative due to negative charges. Let us now calculate the electrostatic potential due to an assembly of charges at point P shown.



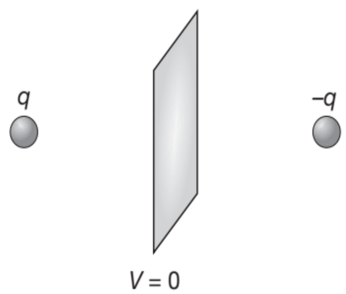
The electrostatic potential V at point P due to the assembly of charges shown is the algebraic sum of the potential due to each of the charge in the assembly. So,

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} - \frac{q_3}{r_3} + \frac{q_4}{r_4} - \frac{q_5}{r_5}\right)$$

POINTS WITH ZERO POTENTIAL DUE TO TWO POINT CHARGES

CASE-1: If both charges are like, then there is no point at a finite distance where net potential zero.

CASE-2: If the charges are equal and unlike then the net potential is zero at all points on the equatorial plane as shown below.



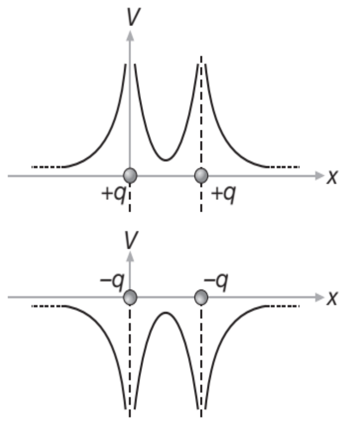
CASE-3: If the charges are unequal and unlike then all such points where resultant potential is zero lie on a closed surface.

However, on the line joining the two charges, only two such points exist, one between the charges and the other outside the charges. Both the above points lie nearer to charge of smaller magnitude.

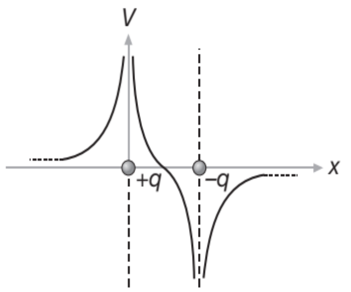
GRAPHICAL REPRESENTATION OF POTENTIAL OF A SYSTEM OF TWO POINT CHARGES

As we move on the line joining two charges from one charge towards the other, the variation of potential V with distance x is shown below.

CASE-1: For like charges of equal magnitude.



CASE-2: For unlike charges of equal magnitude.



CASE-3: For unlike charges of unequal magnitude.

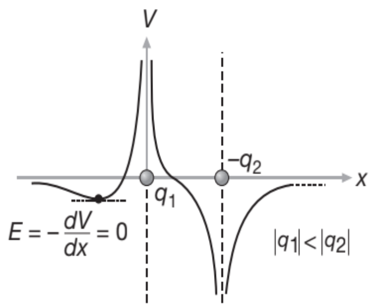


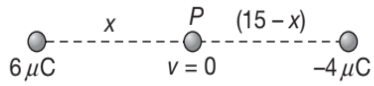
ILLUSTRATION 63

Two charges $6\,\mu\text{C}$ and $-4\,\mu\text{C}$ are located 15 cm apart. At what point on the line joining the two charges is the electric potential zero? Take the potential at infinity to be zero.

SOLUTION

CASE-1: Between the charges, let $V = 0$ at distance x from 6×10^{-6} C charge. So, we have

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{6\times10^{-6}}{x} - \frac{4\times10^{-6}}{15-x}\right) = 0$$



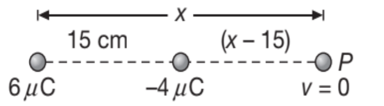
$$\Rightarrow \quad \frac{3}{x} = \frac{2}{15-x}$$

$$\Rightarrow \quad x = 9 \text{ cm}$$

CASE-2: Outside the charges, let $V = 0$ at distance x from 6×10^{-6} C charge. So, we have

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{6\times10^{-6}}{x} - \frac{4\times10^{-6}}{x-15}\right) = 0$$

$$\Rightarrow \quad x = 45 \text{ cm}$$



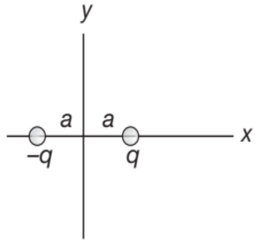
So, electric potential is zero at 9 cm and 45 cm away from the positive charge on the side of the negative charge. Note that the formula for potential used in the calculation required choosing potential to be zero at infinity.

Conceptual Note(s)

The electric field for such an arrangement of charges is zero only at one point where as the potential is zero for two different points (other than ∞).

ILLUSTRATION 64

Consider a system of two charges shown in figure. The arrangement is also called an Electric Dipole. Find the electric potential at an arbitrary point on the x-axis and make a plot of $\frac{V(x)}{V_0}$ Vs $\frac{x}{a}$ where $V_0 = \frac{q}{4\pi\varepsilon_0 a}$.



Electric dipole

SOLUTION

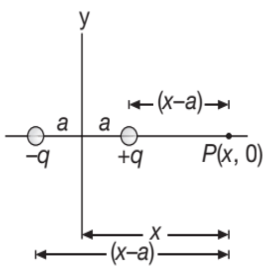
The electric potential can be found by the superposition principle. At a point $P(x,\ 0)$ on the x-axis, we have

$$V(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x-a|} + \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|x+a|}$$

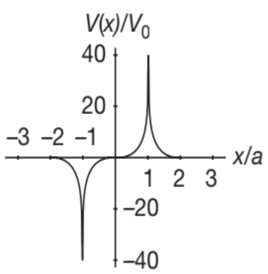
$$\Rightarrow \quad V(x) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|}\right]$$

The above expression is rewritten as

$$\frac{V(x)}{V_0} = \frac{1}{\left|\frac{x}{a}-1\right|} - \frac{1}{\left|\frac{x}{a}+1\right|}$$



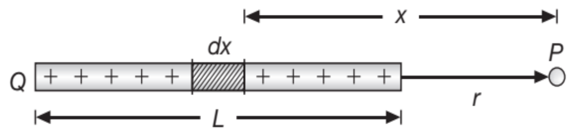
where $V_0 = \frac{q}{4\pi\varepsilon_0 a}$. The plot of the dimensionless electric potential as a function of $\frac{x}{a}$ is shown in figure.



As can be seen from the graph, $V(x)$ diverges at $\frac{x}{a} = \pm 1$, where the charges are located.

ELECTROSTATIC POTENTIAL AT THE AXIS OF A UNIFORMLY CHARGE ROD

Consider a non-conducting charged rod of length L, having a uniform charge Q. Let us find the electrostatic potential at a point P due to the rod at a distance r from one end of the rod.



For this we consider an infinitesimal element of length dx at a distance x from the point P. Charge on this element is

$$dq = \frac{Q}{L}dx$$

The potential dV due to this element at point P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x}$$

$$\Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{Lx}dx$$

Net electric potential at point P is obtained by integrating this expression

$$\Rightarrow \quad V = \int dV = \frac{1}{4\pi\varepsilon_0}\int_r^{r+L}\frac{Q}{Lx}dx$$

$$\Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\log_e x\Big|_r^{r+L}$$

$$\Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\log_e\left(\frac{r+L}{r}\right)$$

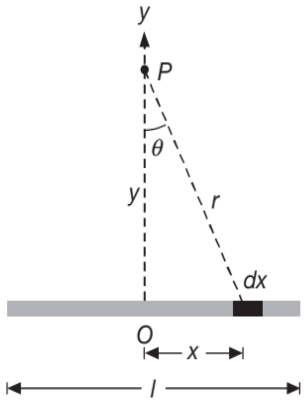
If charge density of the rod is λ, then $\lambda = \frac{Q}{L}$. So,

$$V = \frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{r+L}{r}\right)$$

ELECTROSTATIC POTENTIAL AT POINT P LYING ON PERPENDICULAR BISECTOR OF ROD

Consider a non-conducting rod of length l having a uniform charge density λ. The electric potential at P, a perpendicular distance y above the midpoint of the rod is calculated by considering a differential element of length dx which carries a charge $dq(=\lambda\,dx)$ as shown in figure. The source element is located at $(x,\ 0)$, while the observation point P is located on the y-axis at $(0,\ y)$. The distance of P from element is $r=\sqrt{x^2+y^2}$. Its contribution to the potential dV is given by

$$dV=\frac{1}{4\pi\varepsilon_0}\frac{dq}{r}=\frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx}{\left(x^2+y^2\right)^{1/2}}$$



A non-conducting rod of length l and uniform charge density λ.

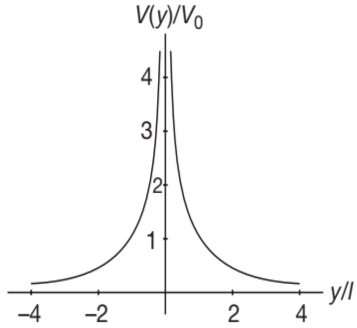
Taking V to be zero at infinity, the total potential due to the entire rod is

$$V=\frac{\lambda}{4\pi\varepsilon_0}\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{dx}{\sqrt{x^2+y^2}}=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left[x+\sqrt{x^2+y^2}\right]\Bigg|_{-\frac{l}{2}}^{\frac{l}{2}}$$

Using $\displaystyle\int\frac{dx}{\sqrt{x^2+y^2}}=\log_e\left(x+\sqrt{x^2+y^2}\right)$

$$\Rightarrow\quad V=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{\left(\frac{l}{2}\right)+\sqrt{\left(\frac{l}{2}\right)^2+y^2}}{-\left(\frac{l}{2}\right)+\sqrt{\left(\frac{l}{2}\right)^2+y^2}}\right]$$

A plot of $\dfrac{V(y)}{V_0}$, where $V_0=\dfrac{\lambda}{4\pi\varepsilon_0}$, as a function of $\dfrac{y}{l}$ is shown in figure.



In the limit $l\gg y$, the potential becomes

$$V=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{\frac{l}{2}+\frac{l}{2}\sqrt{1+\left(\frac{2y}{l}\right)^2}}{-\left(\frac{l}{2}\right)+\frac{l}{2}\sqrt{1+\left(\frac{2y}{l}\right)^2}}\right]$$

$$\Rightarrow\quad V=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{1+\sqrt{1+\left(\frac{2y}{l}\right)^2}}{-1+\sqrt{1+\left(\frac{2y}{l}\right)^2}}\right]$$

Since $\sqrt{1+\left(\dfrac{2y}{l}\right)^2}\simeq 1+\dfrac{2y^2}{l^2}$

$$\Rightarrow\quad V=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{2+\frac{2y^2}{l^2}}{\frac{2y^2}{l^2}}\right]$$

Since, $\dfrac{y^2}{l^2}\ll 1$

$$\Rightarrow\quad 2+\frac{2y^2}{l^2}\simeq 2$$

$$\Rightarrow\quad V\approx\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{2}{\frac{2y^2}{l^2}}\right)=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{l^2}{y^2}\right)$$

$$\Rightarrow\quad V=\frac{\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{l}{y}\right)$$

The corresponding electric field can be obtained by using

$$E_y = -\frac{\partial V}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0 y}\frac{\frac{l}{2}}{\sqrt{\left(\frac{l}{2}\right)^2 + y^2}}$$

which is in complete agreement with the result.

ILLUSTRATION 65

On a thin rod of length $l = 1$ m, lying along the x-axis with one end at the origin $x = 0$, there is uniformly distributed charge per unit length $\lambda = Kx$, where $K = 10^{-9}$ Cm^{-2}. Find the work done in displacing a charge $q = 1000\ \mu\text{C}$ from a point $A\left(0,\ \sqrt{0.44}\right)$ m to $B(0,\ l)$ m.

SOLUTION

Let us calculate the potential due to the rod at A and then at B.

Potential at $A\left(0,\ \sqrt{0.44}\right)$ m

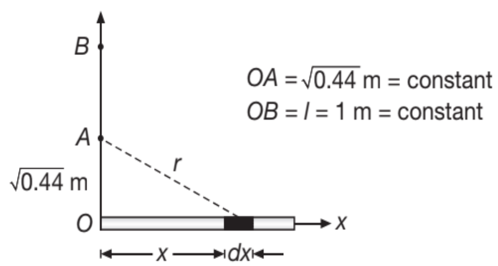
Potential at A due to rod is calculated by taking an infinitesimal element of length dx at a distance x from $O(0,\ 0)$ on the rod. Then

$$dV = \frac{dq}{4\pi\varepsilon_0 r} = \frac{\lambda\, dx}{4\pi\varepsilon_0\sqrt{OA^2 + x^2}} \qquad \ldots(1)$$

$$\Rightarrow \quad dV = \frac{kx\, dx}{4\pi\varepsilon_0\sqrt{0.44 + x^2}}$$

$$\Rightarrow \quad \int_{V_0}^{V_A} dV = \frac{k}{4\pi\varepsilon_0}\left[\sqrt{0.44 + x^2}\Big|_0^{l=1\text{ m}}\right]$$

$$\Rightarrow \quad V_A - V_0 = \frac{k}{4\pi\varepsilon_0}\left[\sqrt{1.44} - \sqrt{0.44}\right] \qquad \ldots(2)$$



Potential at $B(0,\ 1)$ m

$$dV = \frac{dq}{4\pi\varepsilon_0 r'} = \frac{kx\, dx}{4\pi\varepsilon_0\sqrt{OB^2 + x^2}} = \frac{kx\, dx}{4\pi\varepsilon_0\sqrt{1 + x^2}}$$

$$\Rightarrow \quad V_B - V_0 = \frac{k}{4\pi\varepsilon_0}\left[\sqrt{1 + x^2}\Big|_0^1\right]$$

$$\Rightarrow \quad V_B - V_0 = \frac{k}{4\pi\varepsilon_0}\left(\sqrt{2} - 1\right) \qquad \ldots(3)$$

Now, by definition, $W_{A\to B} = q\left(V_B - V_A\right)$

$$\Rightarrow \quad W_{A\to B} = W = \frac{qk}{4\pi\varepsilon_0}\left[\sqrt{2} - 1 - \sqrt{1.44} + \sqrt{0.44}\right]$$

(Please observe, that V_0 will cancel in the process)

$$\Rightarrow \quad W_{A\to B} = \frac{qk}{4\pi\varepsilon_0}[1.414 - 1 - 1.2 + 0.66]$$

$$\Rightarrow \quad W_{A\to B} = 9\times10^9 \times 1000\times10^{-6} \times 10^{-9}[-0.126]$$

$$\Rightarrow \quad W_{A\to B} = -1.1\times10^{-3}\ \text{J}$$

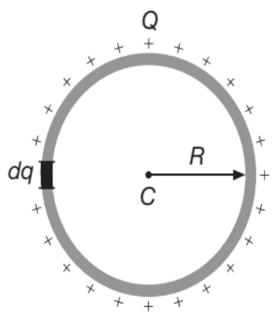
ELECTROSTATIC POTENTIAL DUE TO A CHARGED RING AT ITS CENTRE

Let us first find potential dV at centre C due to an infinitesimal charge dq on ring which is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R}$$

Total potential at C is $V = \int dV$

$$\Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{R} = \frac{Q}{4\pi\varepsilon_0 R}$$



Since all the infinitesimal dq's of the ring are situated at same distance R from the ring centre C so, we can directly say that the total electric potential at centre of ring is $V_C = \dfrac{Q}{4\pi\varepsilon_0 R}$.

Here we must see that even if charge Q is non-uniformly distributed on ring, the electric potential C will remain same (we shall discuss this with an example below) because in that case too

$$V=\frac{1}{4\pi\varepsilon_0 R}\int dq=\frac{\int dq}{4\pi\varepsilon_0 R}$$

$$\Rightarrow \quad V_{\text{centre}}=\frac{Q_{\text{total on ring}}}{4\pi\varepsilon_0 R}$$

ILLUSTRATION 66

Consider a charged ring of radius R. Let the charge on this ring be varying according to the relation $\lambda=\lambda_0\cos^2\phi$, where λ_0 is a positive constant and ϕ is the azimuthal angle. Find the electrostatic potential due to this ring at the centre.

SOLUTION

Let us first calculate the total charge possessed by the ring. Since,

$$dq=\int \lambda\, dx$$

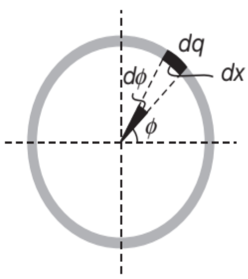
$$\Rightarrow \quad dq=\int \lambda R d\phi$$

$$\Rightarrow \quad dq=\lambda_0 R\int_0^{2\pi}\cos^2\phi\, d\phi$$

$$\Rightarrow \quad q=\int dq=\frac{\lambda_0 R}{2}\int_0^{2\pi}\left[1+\cos(2\phi)\right]d\phi$$

$$\Rightarrow \quad q=\frac{\lambda_0 R}{2}\left[\int_0^{2\pi}d\phi+\int_0^{2\pi}\cos 2\phi\, d\phi\right]$$

$$\Rightarrow \quad q=\frac{\lambda_0 R}{2}\left[2\pi+\frac{\sin(2\phi)}{2}\Bigg|_0^{2\pi}\right]$$



$$\Rightarrow \quad q=\lambda_0 R\pi+0$$

$$\Rightarrow \quad q=\lambda_0 \pi R$$

Since, $dV=\dfrac{dq}{4\pi\varepsilon_0 R}$

$$\Rightarrow \quad V=\frac{1}{4\pi\varepsilon_0 R}\int dq$$

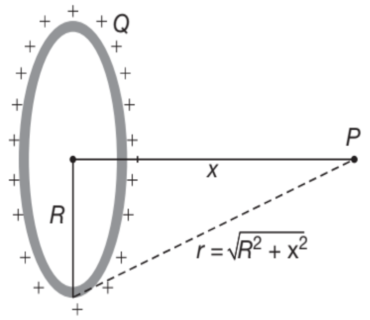
$$\Rightarrow \quad V=\frac{\lambda_0\pi R}{4\pi\varepsilon_0 R}$$

$$\Rightarrow \quad V=\frac{\lambda_0}{4\varepsilon_0}$$

ELECTROSTATIC POTENTIAL AT A POINT *P* ON THE AXIS OF THE RING AT DISTANCE *X* FROM ITS CENTRE

If we wish to find the electrostatic potential at a point P lying on the axis of ring, we can directly calculate the result because here too, all points of ring are at same distance $\sqrt{R^2+x^2}$ from the point P. So, potential at P is

$$V_P=\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+x^2}}$$



VARIATION OF ELECTROSTATIC POTENTIAL ON THE AXIS OF A CHARGED RING

Since $\quad V_{\text{centre}}=\dfrac{Q}{4\pi\varepsilon_0 R}$

and $\quad V_{\text{axis}}=\dfrac{Q}{4\pi\varepsilon_0\sqrt{R^2+x^2}}$

From the above two expressions, we conclude that electrostatic potential is maximum at the centre and goes on decreasing when we move away from the centre on its axis.

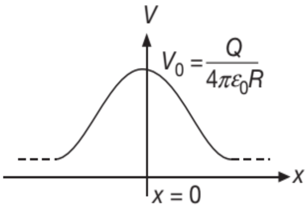


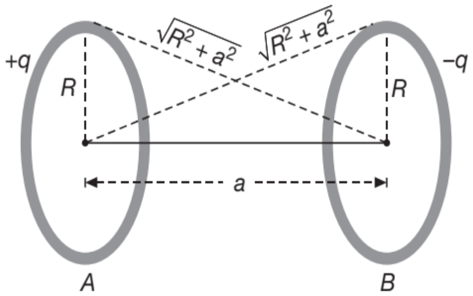
ILLUSTRATION 67

There are two thin wire rings, each of radius R, whose axes coincide. The charges of the rings are q and $-q$. Find the potential difference between the centres of the rings separated by a distance a. Also calculate the work done to move a charge q_0 from centre of first ring to the centre of other ring.

SOLUTION

Net potential at centre of ring A is

$$V_A = \begin{pmatrix}\text{Potential at } A\\ \text{due to itself}\end{pmatrix} + \begin{pmatrix}\text{Potential at } A\\ \text{due to } B\end{pmatrix}$$



$$\Rightarrow \quad V_A = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R} + \frac{(-q)}{\sqrt{R^2+a^2}}\right)$$

$$\Rightarrow \quad V_A = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{\sqrt{R^2+a^2}}\right]$$

Similarly net potential at centre of ring B is

$$V_B = \begin{pmatrix}\text{Potential at } B\\ \text{due to itself}\end{pmatrix} + \begin{pmatrix}\text{Potential at } B\\ \text{due to } A\end{pmatrix}$$

$$\Rightarrow \quad V_B = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{R} + \frac{1}{\sqrt{R^2+a^2}}\right]$$

Thus potential difference,

$$\Delta V = V_B - V_A$$

$$\Rightarrow \quad \Delta V = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{R} + \frac{1}{\sqrt{R^2+a^2}} - \frac{1}{R} + \frac{1}{\sqrt{R^2+a^2}}\right]$$

$$\Rightarrow \quad \Delta V = \frac{2q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{R}\right]$$

$$\Rightarrow \quad \Delta V = \frac{q}{2\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{R}\right]$$

Since, $W_{A\to B} = q_0(V_B - V_A)$

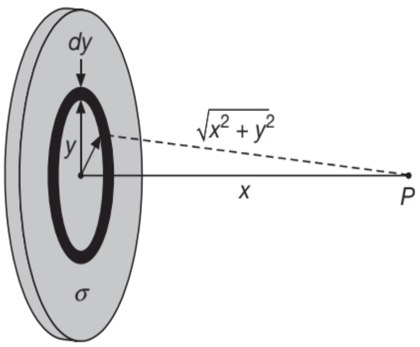
$$\Rightarrow \quad W_{A\to B} = \frac{qq_0}{2\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{R}\right]$$

ELECTROSTATIC POTENTIAL DUE TO A UNIFORMLY CHARGED DISC AT A POINT P ON ITS AXIS

Consider a uniformly charged disc of radius R with surface charge density σ. We wish to find electric potential at point P lying on its axis at distance x from the centre. For this we consider an infinitesimal elemental ring of radius y and thickness dy concentric with the disc. Then charge on this infinitesimal element is

$$dq = \sigma\,(\text{Area of element})$$

$$\Rightarrow \quad dq = \sigma(2\pi y\,dy)$$



The electric potential at point P due to this infinitesimal element is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+y^2}}$$

(Please note here that once the point P is taken, then x becomes a fixed value)

$$\Rightarrow \quad dV = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi y\,dy)}{\sqrt{x^2+y^2}}$$

Net electric potential at point P due to entire disc is calculated by integrating this expression.

$$\Rightarrow \quad V = \int dV = \int_0^R \frac{\sigma}{2\varepsilon_0} \frac{y\,dy}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow \quad V = \frac{\sigma}{2\varepsilon_0} \int_0^R \frac{y\,dy}{\sqrt{x^2 + y^2}}$$

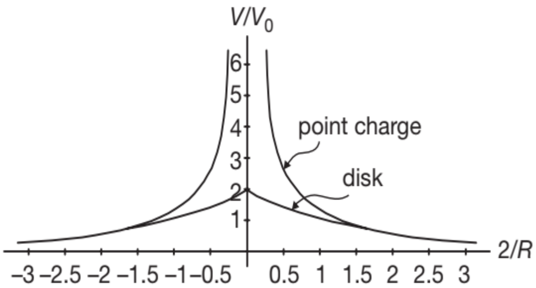
$$\Rightarrow \quad V = \frac{\sigma}{4\varepsilon_0} \int_0^R \left(x^2 + y^2\right)^{-\frac{1}{2}} 2y\,dy$$

Using $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1$

$$\Rightarrow \quad V = \frac{\sigma}{4\varepsilon_0} \left. \frac{\left(x^2 + y^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right|_0^R$$

$$\Rightarrow \quad V = \frac{\sigma}{2\varepsilon_0} \left[\sqrt{x^2 + y^2} \Big|_0^R \right]$$

$$\Rightarrow \quad V_P = \frac{\sigma}{2\varepsilon_0} \left[\sqrt{R^2 + x^2} - x \right]$$



Comparison of the electric potentials of a non-conducting disk and a poin charge. The electric potential is measured in terms of $V_0 = Q/4\pi\varepsilon_0 R$

Also, at the centre of the disc, we have $x = 0$, so

$$V_{\text{centre}} = V_C = \frac{\sigma R}{2\varepsilon_0}$$

In the limit $x \gg R$, we have

$$\sqrt{R^2 + x^2} = x\left(1 + \frac{R^2}{x^2}\right)^{\frac{1}{2}} \simeq x\left(1 + \frac{R^2}{2x^2} + \ldots\right)$$

$$\Rightarrow \quad V = \frac{\sigma}{2\varepsilon_0}\left[x + \frac{R^2}{2x} - x\right] = \frac{\sigma R^2}{4\varepsilon_0 x} = \frac{\sigma\left(\pi R^2\right)}{4\pi\varepsilon_0 x}$$

$$\Rightarrow \quad V = \frac{Q}{4\pi\varepsilon_0 x} = V_{\substack{\text{due to a point charge} \\ \text{at distance } x \text{ from it}}}$$

ILLUSTRATION 68

A non-conducting disc of radius a and uniform positive surface charge density σ is placed on the ground with its axis vertical. A particle of mass m and positive charge q is dropped, along the axis of the disc from a height H with zero initial velocity. The particle has $\frac{q}{m} = \frac{4\varepsilon_0 g}{\sigma}$.

(a) Find the value of H if the particle just reaches the disc.

(b) Sketch the potential energy of the particle as a function of its height and find its equilibrium position.

SOLUTION

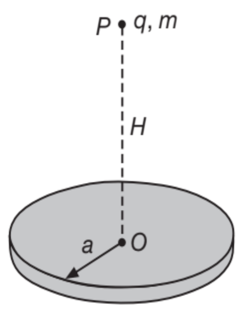
Since, $V_p = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{a^2 + H^2} - H\right]$

Potential at centre, (O) will be

$$V_O = \frac{\sigma a}{2\varepsilon_0} \qquad \{H = 0\}$$

(a) Particle is released from P and it just reaches point O. Therefore, from conservation of mechanical energy decrease in gravitational potential energy = increase in electrostatic potential energy

$(\Delta \text{KE} = 0 \text{ because } K_i = K_f = 0)$



$$\Rightarrow \quad mgH = q\left[V_O - V_P\right]$$

$$\Rightarrow \quad gH = \left(\frac{q}{m}\right)\left(\frac{\sigma}{2\varepsilon_0}\right)\left[a - \sqrt{a^2+H^2} + H\right] \qquad \ldots(1)$$

Since, $\frac{q}{m} = \frac{4\varepsilon_0 g}{\sigma}$

$$\Rightarrow \quad \frac{q\sigma}{2\varepsilon_0 m} = 2g$$

Substituting in equation (1), we get

$$gH = 2g\left[a + H - \sqrt{a^2+H^2}\right]$$

$$\Rightarrow \quad \frac{H}{2} = (a+H) - \sqrt{a^2+H^2}$$

$$\Rightarrow \quad \sqrt{a^2+H^2} = a + \frac{H}{2} \text{ or } a^2+H^2 = a^2 + \frac{H^2}{4} + aH$$

$$\Rightarrow \quad \frac{3}{4}H^2 = aH$$

$$\Rightarrow \quad H = \frac{4}{3}a \text{ and } H = 0$$

$$\Rightarrow \quad H = \left(\frac{4}{3}\right)a$$

(b) If U be the potential energy, U_e be the electrostatic energy and U_g be the gravitational potential energy of the particle at height h, then

$$U = U_e + U_g$$

where $U_e = qV = \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{a^2+H^2} - H\right]$

and $U_g = mgH$

$$\Rightarrow \quad U = \frac{\sigma q}{2\varepsilon_0}\left[\sqrt{a^2+H^2} - H\right] + mgH \qquad \ldots(2)$$

At equilibrium position

$$F = \frac{-dU}{dH} = 0$$

Differentiating equation (2) w.r.t. H, we get

$$mg + \frac{\sigma q}{2\varepsilon_0}\left[\left(\frac{1}{2}\right)(2H)\frac{1}{\sqrt{a^2+H^2}} - 1\right] = 0$$

$$\left\{\because \frac{\sigma q}{2\varepsilon_0} = 2mg\right\}$$

$$\Rightarrow \quad mg + 2mg\left[\frac{H}{\sqrt{a^2+H^2}} - 1\right] = 0$$

$$\Rightarrow \quad 1 + \frac{2H}{\sqrt{a^2+H^2}} - 2 = 0$$

$$\Rightarrow \quad \frac{2H}{\sqrt{a^2+H^2}} = 1$$

$$\Rightarrow \quad \frac{H^2}{a^2+H^2} = \frac{1}{4}$$

$$\Rightarrow \quad 3H^2 = a^2$$

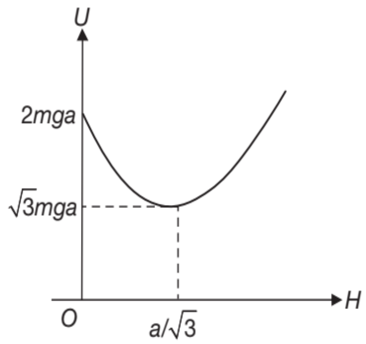
$$\Rightarrow \quad H = \frac{a}{\sqrt{3}}$$

From equation (2), we can see that,

$U = 2mga$ at $H = 0$ and

$U = U_{\min} = \sqrt{3}mga$ at $H = \frac{a}{\sqrt{3}}$

Therefore, U-H graph will be as shown.

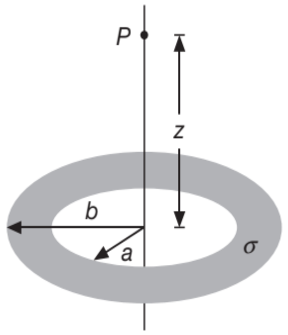


Also we note that at $H = \frac{a}{\sqrt{3}}$, U is minimum.

Therefore, $H = \frac{a}{\sqrt{3}}$ is the stable equilibrium position.

ELECTRIC POTENTIAL OF AN ANNULUS

Consider an annulus of uniform charge density σ, as shown in figure. The electric potential at a point P along the symmetric axis is calculated by taking an infinitesimal elemental ring of radius y and thickness dy concentric with the annulus. If dq be the charge on this infinitesimal element, then



An annulus of uniform charge density

$$dq = \sigma dA = \sigma(2\pi y\, dy)$$

Its contribution to the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi y\, dy)}{\sqrt{z^2+y^2}}$$

Integrating over the entire annulus and using

$$\int \frac{y\, dy}{\sqrt{y^2+x^2}} = \sqrt{y^2+x^2}$$

We obtain

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_a^b \frac{\sigma(2\pi y\, dy)}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \quad V = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_a^b \frac{y\, dy}{\sqrt{x^2+y^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right]$$

$$\Rightarrow \quad V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right)$$

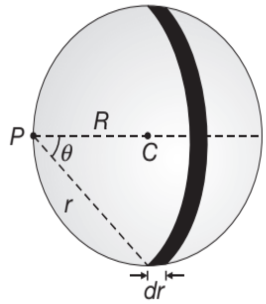
In the limit $a \to 0$ and $b \to R$, the potential becomes

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-|z|\right]$$

which coincides with the result of a non-conducting disk of radius R

ELECTROSTATIC POTENTIAL ON THE EDGE OF A UNIFORMLY CHARGED DISC

Consider a uniformly charged disc of radius R, having surface charge density σ. Let us take a point P on the edge of the disc.



For this, assume the disc to be made up of a large number of concentric rings with P as the centre. Let us consider one such infinitesimal ring of radius r, thickness dr and charge dq. Then potential due to this infinitesimal ring is

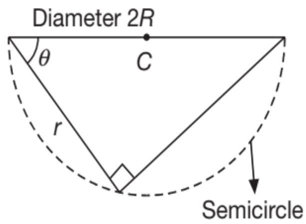
$$dV = \frac{dq}{4\pi\varepsilon_0 r}$$

where $dq = \sigma(\text{Area of infinitesimal element})$

$$\Rightarrow \quad dq = \sigma(2r\theta)dr \qquad \{\because \text{ length of element } = 2r\theta\}$$

$$\Rightarrow \quad dV = \frac{2\sigma r\theta\, dr}{4\pi\varepsilon_0 r} = \frac{\sigma}{2\pi\varepsilon_0}\theta\, dr$$

Also, we observe that (see figure)



$$r = 2R\cos\theta$$

$$\Rightarrow \quad dr = -2R\sin\theta\, d\theta$$

$$\Rightarrow \quad dV = -\frac{\sigma}{2\pi\varepsilon_0}2R\theta\sin\theta\, d\theta$$

$$\Rightarrow \quad V = \int_{\frac{\pi}{2}}^{0} dV = -\frac{\sigma R}{\pi\varepsilon_0}\int_{\frac{\pi}{2}}^{0}\theta\sin\theta\, d\theta$$

$$\text{Since } -\int_b^a f(x)dx = \int_a^b f(x)dx$$

$$\Rightarrow \quad V = \frac{\sigma R}{\pi\varepsilon_0}\int_0^{\frac{\pi}{2}}\theta\sin\theta\, d\theta$$

Using ILATE property to calculate the integral, we get

$$V = \frac{\sigma R}{\pi \varepsilon_0}\left[\theta \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta - \int_0^{\frac{\pi}{2}} (-\cos\theta)\, d\theta\right]$$

$$\Rightarrow \quad V = \frac{\sigma R}{\pi \varepsilon_0}\left[-\theta\cos\theta\Big|_0^{\frac{\pi}{2}} + \sin\theta\Big|_0^{\frac{\pi}{2}}\right]$$

$$\Rightarrow \quad V = \frac{\sigma R}{\pi \varepsilon_0}\left[0 + \sin\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$\Rightarrow \quad V = \frac{\sigma R}{\pi \varepsilon_0}$$

Also, we know that at the centre of the disc

$$V_C = \frac{\sigma R}{2\varepsilon_0}$$

Hence the potential at the centre is more than that at the edge

ILLUSTRATION 69

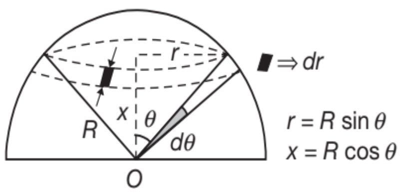
Find the electric field potential at the centre of a hemisphere of radius R having uniform surface charge density σ.

SOLUTION

Consider an infinitesimal ring element as shown. The charge on this element is given by

$$dq = (2\pi R \sin\theta)(R\, d\theta)\sigma$$

{Because radius of ring is $r = R\sin\theta$}



The potential due to this element at the centre O of the hemisphere

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R} = \frac{1}{4\pi\varepsilon_0} \times (2\pi R\sigma \sin\theta)\, d\theta$$

$$\Rightarrow \quad dV = \frac{\sigma R}{2\varepsilon_0}\sin\theta\, d\theta$$

Potential due to the hemisphere is

$$V = \frac{\sigma R}{2\varepsilon_0}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = \frac{\sigma R}{2\varepsilon_0}\left[-\cos\theta\Big|_0^{\frac{\pi}{2}}\right]$$

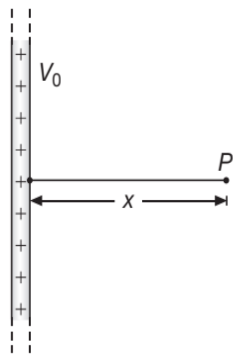
$$\Rightarrow \quad V = \frac{\sigma R}{2\varepsilon_0}$$

ILLUSTRATION 70

The electrostatic potential at surface of thin non-conducting sheet with charge density σ is V_0. Find the potential at a distance x from infinite sheet.

SOLUTION

Potential at point P at a distance x is



$$V_P = V_0 - \int_0^x E\, dx$$

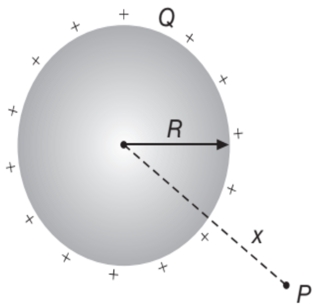
$$\Rightarrow \quad V_P = V_0 - \int_0^x \frac{\sigma}{2\varepsilon_0} dx$$

$$\Rightarrow \quad V_P = V_0 - \frac{\sigma}{2\varepsilon_0}x$$

ELECTROSTATIC POTENTIAL DUE TO A CHARGED CONDUCTING SPHERE/ CHARGED SHELL

For points located outside the charged sphere we can assume the entire charge to be concentrated at its centre. Thus electrostatic potential at a distance r from the centre of sphere, outside it will be

$$V_{\text{outside}} = V = \frac{Q}{4\pi\varepsilon_0 r}$$



At the points on surface of sphere, the potential is given by

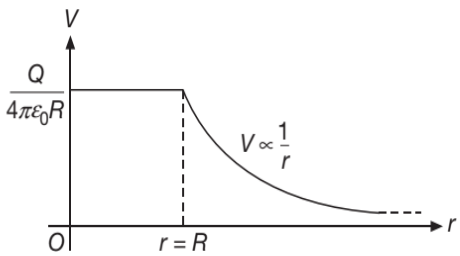
$$V_{\text{surface}} = V_S = \frac{Q}{4\pi\varepsilon_0 R}$$

At the interior points of sphere the electric field is zero at all points inside, so, we can say that the inner portion of the conducting sphere / charged shell is an equipotential region and hence at every interior point potential is same as that on its surface. Thus we have

$$V_{\text{inside}} = V_{\text{surface}} = \frac{Q}{4\pi\varepsilon_0 R}$$

The variation of V with distance r from the centre is shown in figure

Also, the same results are valid for a uniformly charged hollow sphere



Problem Solving Technique(s)

To find the electric potential due to a conducting sphere (or shell) we should keep in mind the following two points:

(a) Electric potential on the surface and at any point inside the sphere is

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R}$$

(R = radius of sphere)

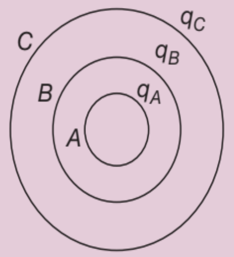
(b) Electric potential at any point outside the sphere is

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r}$$

$\{r = \text{distance of the point from the centre}\}$

For example, in the figure shown, potential at A is

$$V_A = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_A} + \frac{q_B}{r_B} + \frac{q_C}{r_C}\right]$$



Similarly, potential at B is

$$V_B = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_B} + \frac{q_B}{r_B} + \frac{q_C}{r_C}\right]$$

and potential at C is ,

$$V_C = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_A}{r_C} + \frac{q_B}{r_C} + \frac{q_C}{r_C}\right]$$

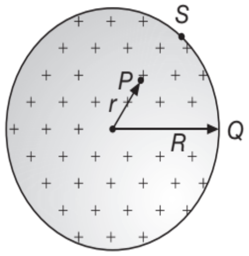
ELECTROSTATIC POTENTIAL DUE TO A NON-CONDUCTING UNIFORMLY CHARGED SPHERE

Here too, for outer and surface points the potential remains same as that of a conducting sphere (as if the entire charge is concentrated at the centre). So,

$$V_{\text{outside}} = V_o = \frac{Q}{4\pi\varepsilon_0 r} \qquad \{\text{for } r > R\}$$

$$V_{\text{surface}} = V_S = \frac{Q}{4\pi\varepsilon_0 R} \qquad \{\text{for } r = R\}$$

For an interior point, unlike to a conducting sphere, potential will not remain uniform as electric field exists inside this region. Inside a uniformly charged sphere electric field is in radially outward direction, so as we move away from centre, in the direction of electric field the potential decreases.



Consider a point P at a distance r from the centre of sphere. The potential difference between points P and S is

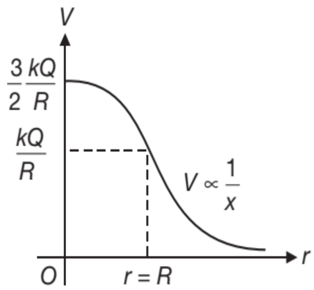
$$V_S - V_P = -\int_r^R \frac{Qrdr}{4\pi\varepsilon_0 R^3}$$

Since $V_S = \frac{Q}{4\pi\varepsilon_0 R}$

$$\Rightarrow \quad V_P - \frac{Q}{4\pi\varepsilon_0 R} = \frac{Q}{8\pi\varepsilon_0 R^3}\left(R^2 - r^2\right)$$

$$\Rightarrow \quad V_P = \frac{Q}{8\pi\varepsilon_0 R^3}\left(R^2 - r^2\right) + \frac{Q}{4\pi\varepsilon_0 R}$$

$$\Rightarrow \quad V_P = \frac{Q}{8\pi\varepsilon_0 R^3}\left(3R^2 - r^2\right)$$



Also we observe that, potential at centre of sphere $(r = 0)$ is

$$V_C = \frac{3Q}{8\pi\varepsilon_0 R} = \frac{3}{2}V_S$$

$$\Rightarrow \quad V_{\text{centre}} = \frac{3}{2}V_{\text{surface}}$$

Thus at centre, potential is maximum and is equal to $\frac{3}{2}$ times that on the surface

If we assume the charge density of the sphere to be ρ, then we get

$$V = \begin{cases} \dfrac{\rho R^3}{3\varepsilon_0 r} & r \geq R \ \text{(outside and at surface)} \\[2ex] \dfrac{\rho}{6\varepsilon_0}\left(3R^2 - r^2\right) & r < R \ \text{(inside)} \end{cases}$$

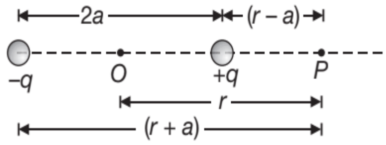
POTENTIAL DUE TO A DIPOLE

On Axial Line

Consider a point P lying on axial line of the dipole at a distance r from its centre. Then potential at this point is

$$V_P = -\frac{q}{4\pi\varepsilon_0 (r + a)} + \frac{q}{4\pi\varepsilon_0 (r - a)}$$

$$\Rightarrow \quad V_P = \frac{q}{4\pi\varepsilon_0}\frac{2a}{\left(r^2 - a^2\right)}$$



$$\Rightarrow \quad V = \frac{p}{4\pi\varepsilon_0\left(r^2 - a^2\right)}$$

For $a << r$ we have

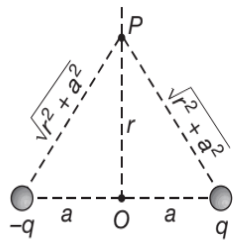
$$V \simeq \frac{p}{4\pi\varepsilon_0 r^2}$$

On Equitorial Line

Consider a point P lying on the equatorial line of the dipole at a distance r from its centre. Then potential at this point is

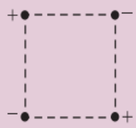
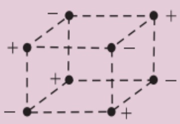
$$V_P = \frac{q}{4\pi\varepsilon_0\sqrt{r^2 + a^2}} + \frac{-q}{4\pi\varepsilon_0\sqrt{r^2 + a^2}}$$

$$\Rightarrow \quad V_P = 0$$



Remark(s)

As a matter of fact we do have four basic kinds of categories for charge distributions shown below.

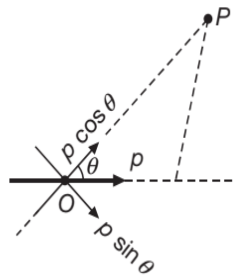
NAME	REPRESENTATION	VARIATION OF V AT FAR OFF DISTANCE
Monopole	+ •	$V - \frac{1}{r}$
Dipole		$V - \frac{1}{r^2}$
Quadrupole		$V - \frac{1}{r^3}$
Octopole		$V - \frac{1}{r^4}$

POTENTIAL AT POINT $P(r, \theta)$ DUE TO A SMALL DIPOLE

Consider a short dipole of dipole moment p. Let us calculate the electric potential due to this small dipole at the point P. For doing this we resolve the dipole moment p in two components

(i) $p\cos\theta$ along the line joining centre of dipole (O) to point P.

(ii) $p\sin\theta$ perpendicular to the line joining centre of dipole (O) to point P (as shown).



The point P lies on the axial line of dipole having dipole moment $p\cos\theta$ and equatorial line of dipole having dipole moment $p\sin\theta$.

$$V_{\text{axial}} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} \text{ and } V_{\text{equitorial}} = 0$$

$$\Rightarrow \quad V = V_{\text{axial}} + V_{\text{equitorial}} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

BINDING ENERGY OF A DIPOLE

The energy required to be supplied to the dipole so that the charges forming it become separated from each other's electrostatic influence.

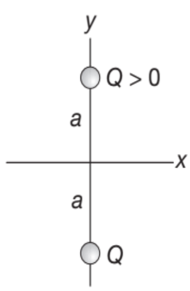
$$U = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2a}$$

Test Your Concepts-VII

Based on Electrostatic Potential and Energy

(Solutions on page H.24)

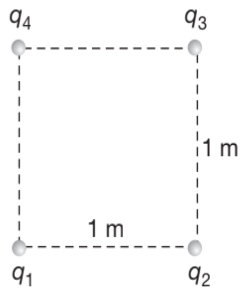
1. Two point charges of equal magnitude are located along the y-axis equal distances above and below the x-axis, as shown in figure.



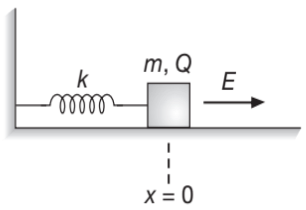
(a) Plot a graph of the potential at points along the x-axis over the interval $-3a < x < 3a$. You should plot the potential in units of $\frac{Q}{4\pi\varepsilon_0 a}$.

(b) Let the charge located at $-a$ be negative and plot the potential along the y-axis over the interval $-4a < y < 4a$.

2. Four charges $q_1 = 1\ \mu C$, $q_2 = 2\ \mu C$, $q_3 = -3\ \mu C$ and $q_4 = 4\ \mu C$ are kept on the vertices of a square of side 1 m. Find the electric potential energy of this system of charges.

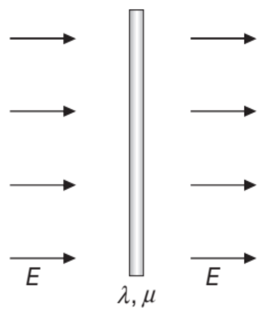


3. How much work is done (by a battery, generator, or some other source of potential difference) in moving Avogadro's number of electrons from an initial point where the electric potential is 9 V to a point where the potential is –5 V? (The potential in each case is measured relative to a common reference point).
4. Three point charges $q_1 = 1\,\mu C$, $q_2 = -2\,\mu C$ and $q_3 = 3\,\mu C$ are placed at $(1\,m, 0, 0)$, $(0, 2\,m, 0)$ and $(0, 0, 3\,m)$ respectively. Find the electrostatic potential due to the charges at origin.
5. A block having mass m and charge $+Q$ is connected to a spring having constant k. The block lies on a frictionless horizontal surface, and the system is immersed in a uniform electric field of magnitude E, directed as shown in figure. If the block is released from rest when the spring is unstretched (at $x = 0$)



(a) By what maximum amount does the spring expand?
(b) What is the equilibrium position of the block?
(c) Show that the block's motion is simple harmonic, and determine its period.
(d) By what maximum amount does the spring expand, if the coefficient of kinetic friction between block and surface is μ_k.

6. A charge $Q = 10\,\mu C$ is distributed uniformly over the circumference of a ring of radius 3 m placed on x-y plane with its centre at origin. Find the electric potential at a point $P(0, 0, 4\,m)$.
7. Consider a planet where free-fall acceleration is the same as that on Earth but there is also a strong downward electric field that is uniform close to the planet's surface. A 2 kg ball having a charge of $+5\,\mu C$ is thrown upward at a speed of $20.1\,ms^{-1}$, and it hits the ground after an interval of 4.1 s. What is the potential difference between the starting point and the top point of the trajectory?
8. Two equal point charges are fixed at $x = -a$ and $x = +a$ on the x-axis. Another point charge Q is placed at the origin. Find the change in electrical energy of Q (approximately) when it is displaced by a small distance along the x-axis.
9. An insulating rod having linear charge density $\lambda = 40\,\mu Cm^{-1}$ and linear mass density $\mu = 0.1\,kgm^{-1}$ is released from rest in a uniform electric field $E = 100\,Vm^{-1}$ directed perpendicular to the rod. Determine the speed of the rod after it has travelled a distance of 2 m.



10. Two point charges are located on the x-axis, $q_1 = -1\,\mu C$ at $x = 0$ and $q_2 = +1\,\mu C$ at $x = 1\,m$.
    (a) Find the work that must be done by an external force to bring a third point charge $q_3 = +1\,\mu C$ from infinity to $x = 2$ m.
    (b) Find the total potential energy of the system of three charges.
11. Consider two concentric spherical metal shells of radii a and b, where $b > a$. The outer shell has charge Q but the inner shell is grounded. Find the charge on the inner shell.
12. At a certain distance from a point charge, the magnitude of the electric field is $500\,Vm^{-1}$ and the electric potential is –3 kV.
    (a) What is the distance to the charge?
    (b) What is the magnitude of the charge?

13. Find out the points on the line joining two charges $+q$ and $-3q$ (kept at a distance of 1 m) where electric potential is zero.

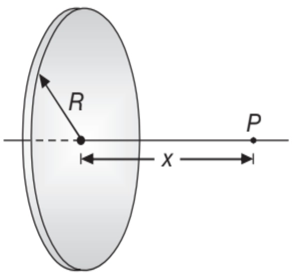
14. A light unstressed spring has length d. Two identical particles, each with charge q, are connected to the opposite ends of the spring. The particles are held stationary a distance d apart and then released at the same time. The system then oscillates on a horizontal frictionless table. The spring has a bit of internal kinetic friction, so the oscillation is damped. The particles eventually stop vibrating when the distance between them is $3d$. Find the increase in internal energy that appears in the spring during the oscillations. Assume that the system of the spring and two charges is isolated.

15. Consider a ring of radius R with the total charge Q spread uniformly over its perimeter. What is the potential difference between the point at the center of the ring and a point on its axis a distance $2R$ from the center?

16. A uniform electric field E_0 is directed along positive y-direction. Find the change in electric potential energy of a positive test charge q_0 when it is displaced in this field from $y_i = a$ to $y_f = 2a$ along the y-axis.

17. A wire having a uniform linear charge density λ is bent into the shape shown in figure. Find the electric potential at point O.

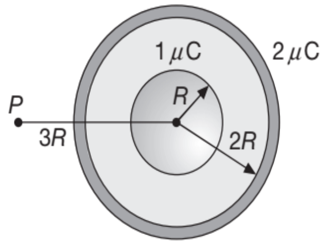
18. A disk of radius R has a non-uniform surface charge density $\sigma = \sigma_0 r$, where σ_0 is a positive constant and r is measured from the center of the disk. Find the potential at P.



19. An infinite number of charges each equal to q are placed along the x axis at $x = 1$, $x = 4$, $x = 8$, ... and so on. Find the potential and electric field at the point $x = 0$ due to this set of charges.

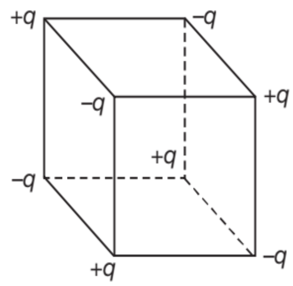
What will be potential and electric field if in the above set up the consecutive charge have opposite sign?

20. Two concentric spheres of radii R and $2R$ are charged. The inner sphere has a charge of $1\,\mu$C and the outer sphere has a charge of $2\,\mu$C of the same sign. The potential is 9000 V at a distance $3R$ from the common center. What is the value of R?



21. A point charge Q is held fixed at origin. A second point charge $-q$ having mass m is placed on the x-axis, at $(2\ell, 0)$ from the origin. The second point charge is released from rest. What is its speed when it is at $\left(\frac{\ell}{4}, 0\right)$ from the origin?

22. Eight point charges are placed at the corners of a cube of edge a as shown in figure. Find the work done in disassembling this system of charges.

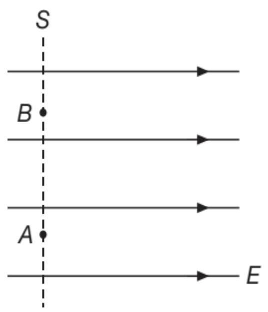


23. A conducting bubble of radius a, thickness $t\,(t \ll a)$ has potential V. Now the bubble collapses into a droplet. Find the potential of the droplet.

24. Two point charges $2\,\mu$C and $-4\,\mu$C are kept 6 m apart. At what points on the line joining the charges, the electric potential will be zero.

EQUIPOTENTIAL SURFACES: INTRODUCTION

Consider a situation in which a charge is displaced from a point A to B on a surface S which is perpendicular to the direction of electric field. Then work done in displacing the charge will be zero, because electric force is normal to the direction of displacement.

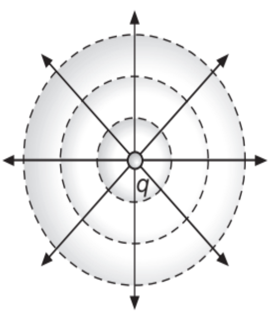


Since no work is done in moving the charge from A to B, hence we can say that A and B are at same potential or we can say that all the points lying on surface S are at same potential. So, we call the surface S to be an Equipotential Surface.

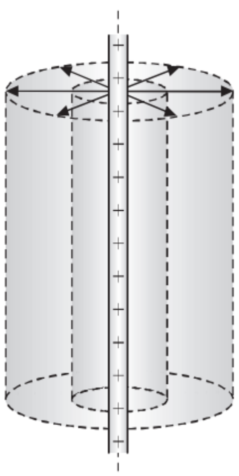
Following figures show equipotential surfaces in the surrounding of point charge, a long charged wire and at far off distance from both.

Mathematically equipotential surface is the locus of all points which have the same potential due to a charge distribution.

Any surface in an electric field at every point of which the direction of electric field is normal to the surface can be regarded as equipotential surface.

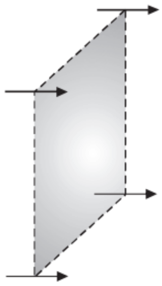


Spherical equipotential surfaces for point charge



Cylindrical equipotential surfaces for line charge

(Continued)



Planar equipotential surface

EQUIPOTENTIAL SURFACE: PROPERTIES

Since, we have already read that an equipotential surface is the locus of all points which have the same electrostatic potential due to a charge distribution. Equipotential surfaces possess the following properties.

(a) The work done in moving a charge along an equipotential surface is always zero.

(b) An electric field line must cut the equipotential surface at right angles.

(c) The tangential component of the electric field along the equipotential surface is zero, otherwise non-zero work would be done to move a charge from one point on the surface to another point on it.

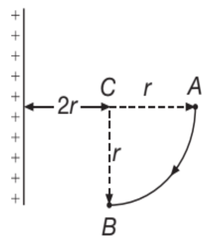
(d) No two equipotential surfaces cross each other.

(e) An equipotential surface for a point charge must be a family of concentric spheres, for a line charge must be a family of coaxial cylinders.

(f) The equipotential surface for a point charge or a line charge at far off distance is a family of planes perpendicular to the field lines.

ILLUSTRATION 71

A charge q_0 is transported from point A to B along the arc AB with centre at C as shown in figure near a long charged wire with linear density λ lying in the same plane. Find the work done in doing so.



SOLUTION

Since, electric field is conservative in nature, so work done does not depend upon the path followed between the two points.

$$\Rightarrow \quad W_{A\to B} = W_{A\to C} + W_{C\to B}$$

But $W_{C\to B} = 0$ (because B and C are at same potentials or B and C lie on the same equipotential line)

$$\Rightarrow \quad W_{A\to B} = W_{A\to C}$$

and $W_{A\to C} = -q_0 \int_{3r}^{2r} \vec{E}\cdot d\vec{r}$

$$\Rightarrow \quad W_{A\to C} = \frac{q_0}{2\pi\varepsilon_0}\int_{2r}^{3r} \frac{\lambda}{r}dr \qquad \left\{\because E = \frac{\lambda}{2\pi\varepsilon_0 r}\right\}$$

$$\Rightarrow \quad W_{A\to C} = \frac{q_0\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{3}{2}\right)$$

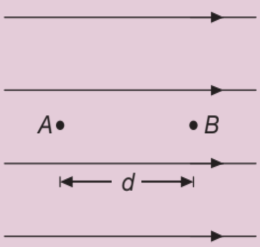
Remark(s)

(a) A region in which at every point electric field is zero, can be regarded as an equipotential region.

(b) As we have already discussed whenever charge is given to a metal body, it is distributed on its outer surface in such a way that net electric field at every interior point of body is zero. Thus if inside a metal body, a charge is displaced, no work is done in the process because electric field at every point is zero. Hence we can say that the whole metal body is equipotential at the inside.

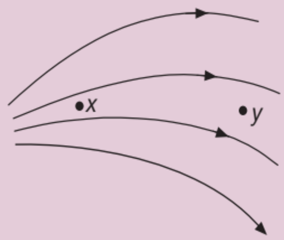
(c) For uniform electric field as shown in figure, we have

$$V_A - V_B = Ed \qquad \{\text{As } E = \text{constant}\}$$



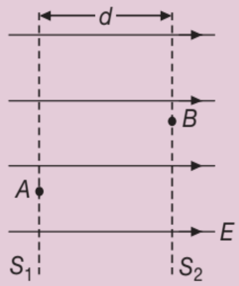
(d) For a non-uniform electric field like shown in figure, we have (between two points),

$$V_x - V_y = \int_x^y E dx$$



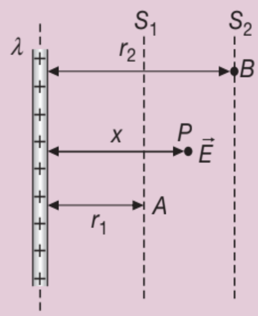
(e) Figure shows two equipotential surfaces in a uniform electric field E. If we have to calculate the potential difference between two points A and B as shown in figure, we simply find the potential difference between the two equipotential surfaces on which the points lie. So,

$$V_A - V_B = Ed$$



(f) Similarly, the figure shows a line charge with linear charge density λ. To find potential difference between points A and B which lie on equipotential surfaces S_1 and S_2, we find the potential difference between these surfaces and for that we consider a point P at a distance x from wire as shown. The electric field at point P is

$$E = \frac{\lambda}{2\pi\varepsilon_0 x}$$



Now the potential difference between surface S_1 and S_2 is given by

$$V_B - V_A = -\int_{r_1}^{r_2} E\,dx$$

$$\Rightarrow V_A - V_B = \int_{r_1}^{r_2} \frac{\lambda}{2\pi\varepsilon_0 x}dx$$

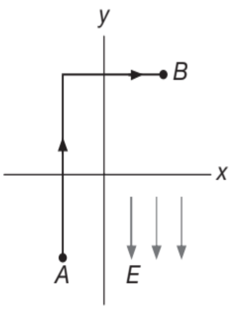
$$\Rightarrow V_A - V_B = \frac{\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r_2}{r_1}\right)$$

Test Your Concepts-VIII

Based on Equipotential Surfaces

(Solutions on page H.30)

1. A uniform electric field of magnitude $325\ \text{Vm}^{-1}$ is directed in the negative y direction in figure. The coordinates of point A are $(-20, -30)$ cm, and those of point B are $(40, 50)$ cm. Calculate the potential difference $V_B - V_A$, along the path shown.

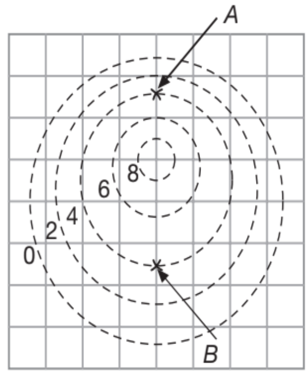


2. Figure shows several equipotential lines each labelled by its potential in volt. The distance between the lines of the square grid represents 2 cm.

(a) Where the magnitude of the field is larger, at A or at B? Explain.

(b) What is the electric field at B?

(c) Represent what the field looks like by drawing at least eight field lines in all directions of the page.



3. A point charge q is located at $x = -R$, and a point charge $-2q$ is located at the origin. Prove that the equipotential surface that has zero potential is a sphere centered at $\left(-\frac{4R}{3}, 0, 0\right)$ and having a radius $r = \frac{2R}{3}$.

4. Three infinitely long linear charges of charge density λ, λ and -2λ are placed in space. A point in space is specified by its perpendicular distance r_1, r_2 and r_3 respectively from the linear charges. Prove that for the points which are equipotential, we have

$$\frac{r_1 r_2}{r_3^2} = \text{constant}$$

RELATION BETWEEN ELECTROSTATIC FIELD AND POTENTIAL

Since $dV = -\vec{E}\cdot d\vec{l}$

In cartesian coordinates, we have

$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$ and $d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$, so we get

$$dV = -\left(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$\Rightarrow \quad dV = -\left(E_x dx + E_y dy + E_z dz\right)$$

$$\Rightarrow \quad E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y} \text{ and } E_z = -\frac{\partial V}{\partial z}$$

Now, by introducing a new differential quantity called the "del (gradient) operator $(\vec{\nabla})$", we get electrostatic field as the negative gradient of potential. So,

$$\vec{E} = -\vec{\nabla}V \qquad \ldots(1)$$

where $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$\Rightarrow \quad \vec{E} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

where

$\frac{\partial V}{\partial x} =$ (Partial Derivative of V w.r.t. x OR Derivative of V w.r.t. x keeping y and z constant)

$\frac{\partial V}{\partial y} =$ (Partial Derivative of V w.r.t. y OR Derivative of V w.r.t. y keeping x and z constant)

$\frac{\partial V}{\partial z} =$ (Partial Derivative of V w.r.t. z OR Derivative of V w.r.t. z keeping x and y constant)

Just to make you understand the concept discussed here, I have a small problem discussed here for your comfort.

PROBLEM: Find the electrostatic field $\vec{E}$ for the potential $V = kxy$.

SOLUTION: Since

$$\vec{E} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right),$$

where $\frac{\partial V}{\partial x} = ky\left(\frac{\partial x}{\partial x}\right) = ky$; $\frac{\partial V}{\partial y} = kx\left(\frac{\partial y}{\partial y}\right) = kx$

and $\frac{\partial V}{\partial z} = 0$

$$\Rightarrow \quad \vec{E} = -k\left(y\hat{i} + x\hat{j}\right)$$

Notice that $\vec{\nabla}$ operates on a scalar quantity (electric potential) and results in a vector quantity (electric field). Mathematically, we can think of $\vec{E}$ as the negative of the gradient of the electric potential V. Physically, the negative sign implies that if V increases as a positive charge moves along some direction, say x, with $\frac{\partial V}{\partial x} > 0$, then there is a non vanishing component of $\vec{E}$ in the opposite direction $(-E_x \neq 0)$, because E always goes from higher V to lower V.

If the charge distribution possesses spherical symmetry, then the resulting electric field is a function of the radial distance r, i.e., $\vec{E} = E_r\hat{r}$. In this case, $dV = -E_r dr$. If $V(r)$ is known, then $\vec{E}$ may be obtained as

$$\vec{E} = E_r\hat{r} = -\left(\frac{dV}{dr}\right)\hat{r}$$

For example, the electric potential due to a point charge q is $V(r) = \frac{q}{4\pi\varepsilon_0 r}$. Using the above formula, the electric field is simply

$$\vec{E} = \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)\hat{r}$$

ILLUSTRATION 72

Suppose that the electric potential in some region of space is given by

$$V(x,\ y,\ z) = V_0 \exp(-az)\cos ax,$$

where a is a positive constant. Find the electric field everywhere.

SOLUTION

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$E_x = -\frac{\partial V}{\partial x} = -V_0 \exp(-az)\frac{\partial}{\partial x}[\cos(ax)]$$

$$\Rightarrow \quad E_x = \frac{\partial V}{\partial x} = +aV_0 \sin(ax)\exp(-az)$$

$$E_y = \frac{\partial V}{\partial y} = 0$$

{∵ the given expression has no dependence only}

$$E_z = -\frac{\partial V}{\partial z} = -V_0 \cos(ax)\frac{\partial}{\partial z}[\exp(-az)]$$

$$\Rightarrow \quad E_z = -\frac{\partial V}{\partial z} = aV_0 \cos(ax)\exp(-az)$$

$$\Rightarrow \quad E = E_x\hat{i} + E_z\hat{k} \qquad \{\because E_y = 0\}$$

$$\Rightarrow \quad \vec{E} = aV_0 \exp(-az)[\sin(ax)\hat{i} + \cos(ax)\hat{k}]$$

ILLUSTRATION 73

Find the potential function $V(x, y)$ of an electrostatic field $\vec{E} = 2axy\,\hat{i} + a(x^2 - y^2)\hat{j}$ where a is a constant.

SOLUTION

Let V_0 be the potential at origin.

Since, $dV = -\vec{E}\cdot d\vec{\ell}$

$$\Rightarrow \int_{(0,0)}^{(x,y)} dV = -\int_{(0,0)}^{(x,y)} \left[2axy\,\hat{i} + a(x^2 - y^2)\hat{j}\right]\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$\Rightarrow V(x,y) - V(0,0) = -\int_{(0,0)}^{(x,y)} \left(2axy\,dx + ax^2\,dy\right) - ay^2 dy$$

$$\Rightarrow \quad V(x,y) - V_0 = -\int_{(0,0)}^{(x,y)} \left\{d(ax^2y) - ay^2dy\right\}$$

$$\Rightarrow \quad V(x,y) - V_0 = \left[\left(-ax^2y + \frac{ay^3}{3}\right)\Bigg|_{(0,0)}^{(x,y)}\right]$$

$$\Rightarrow \quad V(x,y) = V_0 - ax^2y + \frac{ay^3}{3}$$

Remark(s)

(a) For an attractive system U is always **NEGATIVE**.
(b) For a repulsive system U is always **POSITIVE**.
(c) For a stable system U must be **MINIMUM**

i.e., $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} > 0$

Since $F = -\frac{dU}{dx}$

$$\Rightarrow \quad F = -\frac{dU}{dx} = 0 \qquad \text{(FOR A STABLE SYSTEM)}$$

(d) If we are given $V(x, y, z)$, then

$$\vec{E} = -\vec{\nabla}V = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}\right)$$

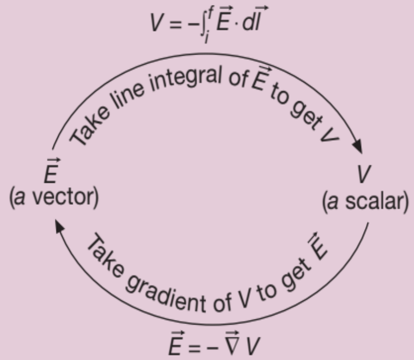
(e) If we are given $\vec{E}(x, y, z) = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$, then

$$V = -\int \vec{E}\cdot d\vec{\ell}$$

where, $d\vec{\ell} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\Rightarrow \quad V = -\left[\int E_x dx + \int E_y dy + \int E_z dx\right]$$

The integrals are to be calculated within specified limits.



PROBLEM: An electrostatic field is given by $\vec{E} = -k(y\hat{i} + x\hat{j})$. Find the electrostatic potential generating such a field.

SOLUTION: Since $V = -\int \vec{E}\cdot d\vec{r}$

Let $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$$\Rightarrow \quad V = \int k(y\hat{i} + x\hat{j})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$\Rightarrow \quad V = k\int y\,dx + x\,dy$$

$$\Rightarrow \quad V = k\int d(xy) \qquad \{\because d(xy) = x\,dy + y\,dx\}$$

$$\Rightarrow \quad V = k(xy) + \text{constant}$$

(f) Electrostatic potential at a point due to a positive charge is positive and due to a negative charge is negative.

(g) Electrostatic potential due an assembly of charges $q_1, q_2, ..., q_n$ at a point P at distance $r_1, r_2, ..., r_n$ respectively from the charges is given by

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

(h) If two points A and B are at potentials V_A and V_B, then work done in taking a test charge q_0 from A to B is $W_{A\to B} = q_0(V_B - V_A)$

(i) A positive charge always moves from higher potential to lower potential whereas a negative charge always moves from lower potential to higher potential.

(j) Commonly used unit of electrostatic energy is electronvolt (eV) and $1\text{ eV} = 1.6\times10^{-19}$ J.

(k) If a charge q of mass m accelerates through a potential V, then velocity of q is calculated by using the Work-Energy Theorem, according to which work done equals the change in kinetic energy i.e.,

$$qV = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = \sqrt{\frac{2qV}{m}}$$

(l) Consider two points A and B situated in a uniform electric field at a distance d such that the line joining A and B is parallel to the field. If V be the potential difference between them, then

$$V = Ed\,(\text{in magnitude})$$

Test Your Concepts-IX

Based on Relation Between Electrostatic Field and Potential

(Solutions on page H.31)

1. Over a certain region of space, the electric potential is $V = 5x - 3x^2y + 2yz^2$. Find the expressions for the x, y and z components of the electric field over this region. Also calculate the magnitude of the field at the point P that has coordinates $(1, 0, -2)$ m?
2. When an uncharged conducting sphere of radius a is placed at the origin of an xyz cartesian coordinate system that lies in an initially uniform electric field $E = E_0\hat{k}$, the resulting electric potential is $V(x, y, z) = V_0$ for points inside the sphere and

$$V(x, y, z) = V_0 - E_oz + E_0a^3\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

for points outside the sphere, where V_0 is the (constant) electric potential on the conductor. Use this equation to determine the x, y and z components of the resulting electric field.

3. In a region of space, the electric potential is represented by $V = 2x + 3y - z$. Obtain an expression for the electric field strength.
4. In a region of space the electric field is given by $\vec{E} = (x\hat{i} - 2y\hat{j} + z\hat{k})\text{ Vm}^{-1}$. Calculate the potential difference V_{AB} between A(2, 1, 0) m and B(0, 2, 4) m.
5. Find potential difference V_{AB} between $A(0, 0, 0)$ m and $B(1, 1, 1)$ m in an electric field given by
   (a) $\vec{E} = y\hat{i} + x\hat{j}$
   (b) $\vec{E} = 3x^2y\hat{i} + x^3\hat{j}$.
   What can you say about the nature of the fields? Explain.
6. Determine the electric field strength vector if the potential of this field depends on x, y coordinates as
   (a) $\varphi = a(x^2 - y^2)$
   (b) $\varphi = axy$

where a is a positive constant. Draw the approximate shape of these fields using lines of force (in the x, y plane).

7. The potential of a certain electrostatic field has the form $V = a(x^2 + y^2) + bz^2$, where a and b are constants. Find the magnitude and direction of the electric field strength vector. What shape will the equipotential surface have for $a > 0$, $b > 0$?

8. Determine the potential $V(x, y, z)$ of an electrostatic field $E = ay\hat{i} + (ax + bz)\hat{j} + by\hat{k}$, where a and b are constants, $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors of the axes x, y, z.

MOTION OF CHARGED PARTICLES AND CONSERVATION LAWS

For a variety of problems, we have to understand the Conservation Laws that can be applied to the electrostatic system of particles.

CASE-1: Law of Conservation of Linear Momentum

For an electrostatic system of charges, when no external force acts on the system, then we observe that no net internal forces will be acting on the system because electrostatic forces always form an action reaction pair. So, by a suitable selection of a system or subsystem we can apply Law of Conservation of Linear Momentum.

Total Initial Momentum = Total Final Momentum

CASE-2: Law of Conservation of Energy

Since electrostatic forces are conservative in nature, so, in the absence of an external force and a dissipative force (such as forces of friction) we can easily use this law as

$$\begin{pmatrix}\text{Total Initial Energy} \\ \text{of the system}\end{pmatrix} = \begin{pmatrix}\text{Total Final Energy} \\ \text{of the system}\end{pmatrix}$$

Here we need to read the problem carefully for the energies possessed by the system (as selected) initially and finally.

Another point which should be kept in mind is that electric field is a conservative field. Work done by electric field on a moving charge only depends on the positions of the charge. If a charge is freely moving in electric field, workdone by electric field on charge is equal to the change in kinetic energy of the charge i.e.,

$$W = \Delta K$$

In the presence of an external force and the dissipative non conservative force, the law will be suitably modified as below i.e.,

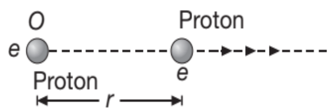
$$W_{\text{ext}} + W_{nc} = \Delta U + \Delta K$$

CASE-3: Law of Conservation of Angular Momentum

Since electrostatic forces are central in nature, so torque due to electrostatic force will be zero. So, we can easily use Law of Conservation of Angular Momentum too.

ILLUSTRATION 74

A proton is fixed at origin. Another proton is released from rest, from a point at a distance r from origin. Taking charge of proton as e and mass as m, find the speed of the proton (a) at a distance $2r$ from origin (b) at a large distance from origin.



SOLUTION

(a) The proton moves away under electrostatic repulsion. So, by Conservation of Energy, we have

$$U_i + K_i = U_f + K_f$$

$$\Rightarrow \frac{e^2}{4\pi\varepsilon_0 r} + 0 = \frac{e^2}{4\pi\varepsilon_0 (2r)} + \frac{1}{2}mv^2$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{e^2}{4\pi\varepsilon_0 (2r)}$$

$$\Rightarrow v = \sqrt{\frac{e^2}{4\pi\varepsilon_0 rm}}$$

(b) Similarly, we have

$$U_i + K_i = U_\infty + K_\infty$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{e^2}{4\pi\varepsilon_0 r}$$

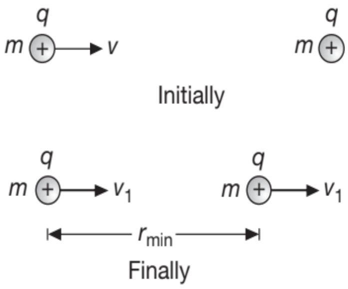
$$\Rightarrow v = \sqrt{\frac{2e^2}{4\pi\varepsilon_0 rm}}$$

ILLUSTRATION 75

A charge q of mass m initially lying at infinity is projected head on with speed v toward another stationary particle of same mass and charge, initially at rest. Find the distance of closest approach of the two particles.

SOLUTION

As the moving charge comes closer to the charge initially at rest, due to electric repulsion it also starts moving and the first one starts retarding.



Since no external force is acting on system, total momentum remains conserved. The two charges get close to each other and their separation will be minimum when each will move with equal common speed say v_1. When their speeds are equal say v_1, then by Law of Conservation of Linear Momentum, we have

$$mv = mv_1 + mv_1$$

$$\Rightarrow \quad v_1 = \frac{v}{2}$$

To find the minimum separation between the two particles, using Law of Conservation of Energy,

Total Initial Energy = Total Final Energy

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_1^2 + \frac{q^2}{4\pi\varepsilon_0 r_{min}}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = m\left(\frac{v}{2}\right)^2 + \frac{q^2}{4\pi\varepsilon_0 r_{min}} \qquad \left\{\because v_1 = \frac{v}{2}\right\}$$

$$\Rightarrow \quad \frac{mv^2}{4} = \frac{q^2}{4\pi\varepsilon_0 r_{min}}$$

$$\Rightarrow \quad r_{min} = \frac{q^2}{\pi\varepsilon_0 mv^2}$$

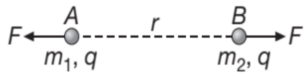
ILLUSTRATION 76

Two particles, each of charge q, and masses m_1 and m_2, are released from rest, from points A and B, r distance apart. Calculate the final relative velocity of separation of the particles at large separation from each other. (Neglect gravitational interaction of the particles)

SOLUTION

Both particles move under mutual electrostatic repulsion. Let v_1 and v_2 are final speed (when the separation becomes large). So, $v_{rel} = v_1 + v_2$

Since no external force is acting on particles, so by Law of Conservation of Linear Momentum, we have



$$-m_1v_1 + m_2v_2 = 0$$

$$\Rightarrow \quad m_1v_1 = m_2v_2 = p(\text{say})$$

By Law of Conservation of Mechanical Energy we have

$$U_i + K_i = U_f + K_f$$

$$\Rightarrow \quad \frac{q^2}{4\pi\varepsilon_0 r} + 0 + 0 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + 0$$

$$\Rightarrow \quad \frac{p^2}{2m_1} + \frac{p^2}{2m_2} = \frac{q^2}{4\pi\varepsilon_0 r}$$

$$\Rightarrow \quad p^2 = \left(\frac{2m_1m_2}{m_1+m_2}\right)\frac{q^2}{4\pi\varepsilon_0 r}$$

$$\Rightarrow \quad p = \sqrt{\left(\frac{2m_1m_2}{m_1+m_2}\right)\frac{q^2}{4\pi\varepsilon_0 r}}$$

$$\Rightarrow \quad v_1 = \frac{p}{m_1} = \frac{1}{m_1}\sqrt{\left(\frac{2m_1m_2}{m_1+m_2}\right)\frac{q^2}{4\pi\varepsilon_0 r}}$$

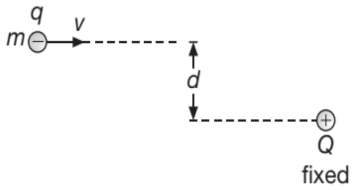
and $v_2 = \dfrac{p}{m_2} = \dfrac{1}{m_2}\sqrt{\left(\dfrac{2m_1m_2}{m_1+m_2}\right)\dfrac{q^2}{4\pi\varepsilon_0 r}}$

$$\Rightarrow \quad v_r = v_1 + v_2$$

$$\Rightarrow \quad v_r = \sqrt{\frac{2q^2}{4\pi\varepsilon_0 r}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$$

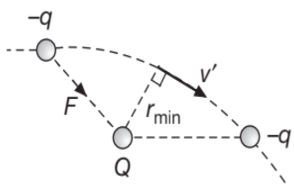
ILLUSTRATION 77

Figure shows a charge $+Q$ fixed at a position in space. From a large distance another charged particle of charge $-q$ and mass m is thrown with a speed v towards $+Q$ with an impact parameter d (it is the perpendicular distance between the velocities of the two particles) as shown. Find the distance of closest approach of the two particles.



SOLUTION

Here we can see that when $-q$ moves toward $+Q$, an attractive force acts on $-q$ toward $+Q$. Here as the line of action of force passes through the fix charge, no torque act on $-q$ relative to the fix point charge $+Q$, thus here we can say that with respect to $+Q$, the angular momentum of $-q$ must remain constant. The charge $-q$ will be closest to $+Q$ when it is moving perpendicularly to the line joining the two charges as shown.



If the closest separation in the two charges is r_{min} then, from conservation of angular momentum we have

$$mvd = mv' r_{min} \quad \ldots(1)$$

Further, by Law of Conservation of Energy, we have

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 - \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_{min}}$$

From (1), we get

$$v' = \frac{vd}{r_{min}}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{1}{2}mv^2\left(\frac{d^2}{r_{min}^2}\right) - \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_{min}} \quad \ldots(2)$$

$$\Rightarrow \quad r_{min}^2 + \left(\frac{2Qq}{4\pi\varepsilon_0 mv^2}\right)r_{min} - d^2 = 0$$

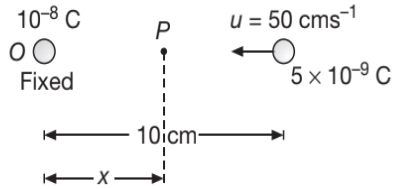
$$\Rightarrow \quad r_{min} = \frac{Qq}{2\pi\varepsilon_0 mv^2}\left[\sqrt{1+\frac{16\pi^2\varepsilon_0^2 m^2 v^4 d^2}{Q^2q^2}} - 1\right]$$

ILLUSTRATION 78

A particle of mass 40 mg having a charge of 5×10^{-9} C is moving directly towards a fixed positive point charge of magnitude 10^{-8} C. When it is at a distance of 10 cm from the fixed positive point charge it has a velocity of 50 cms^{-1}. At what distance from the fixed point charge will the particle come momentarily to rest? Is the acceleration constant during the motion?

SOLUTION

Let it come to momentary rest at point P at distance x from O. Then, by Law of Conservation of Energy.



$$(U+K)_{initial} = (U+K)_{final}$$

$$\Rightarrow \quad \frac{Qq}{4\pi\varepsilon_0 r_0} + \frac{1}{2}mu^2 = \frac{Qq}{4\pi\varepsilon_0 x} + \frac{1}{2}m(0)^2$$

where $Q = 10^{-8}$ C

$$q = 5\times10^{-9}\ \text{C}$$

$$r_0 = 10\ \text{cm} = \frac{10}{100}\ \text{m}$$

$$m = 40\ \text{mg} = \frac{40}{1000}\ \text{g} = 40\times10^{-6}\ \text{kg}$$

$$u = 50\ \text{cms}^{-1} = 0.5\ \text{ms}^{-1}$$

$$\Rightarrow \quad \frac{9\times10^9\times5\times10^{-17}}{0.1} + \frac{1}{2}(40\times10^{-6})(0.5)^2 = \frac{9\times10^9\times5\times10^{-17}}{x}$$

$$\Rightarrow \quad 45\times10^{-7} + 50\times10^{-7} = \frac{45\times10^{-8}}{x}$$

$$\Rightarrow \quad x = \frac{45\times10^{-8}}{95\times10^{-8}} = 0.473 \text{ m}$$

$$\Rightarrow \quad x = 4.73 \text{ cm}$$

The force of repulsion increases as the particle approaches the fixed charge. So the acceleration is not constant during the motion.

ILLUSTRATION 79

Four point charges $+8\,\mu\text{C}$, $-1\,\mu\text{C}$, $-1\,\mu\text{C}$ and $+8\,\mu\text{C}$ are fixed at the points $-\sqrt{\frac{27}{2}}$ m, $-\sqrt{\frac{3}{2}}$ m, $+\sqrt{\frac{3}{2}}$ m and $+\sqrt{\frac{27}{2}}$ m respectively on the Y-axis. A particle of mass 6×10^{-4} kg and charge $+0.1\,\mu\text{C}$ moves along the $-x$ direction. Its speed at $x\to\infty$ is v_0. Find the least value of v_0 for which the particle will cross the origin. Find also the kinetic energy of the particle at the origin. Assume that space is gravity free.

SOLUTION

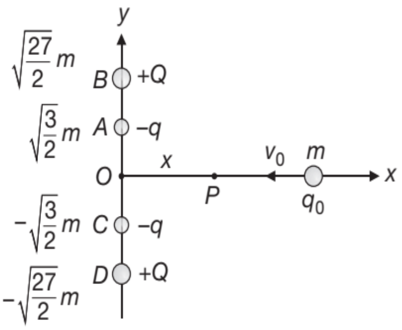
In the figure,

$$q = 1\,\mu\text{C} = 10^{-6}\text{ C},$$

$$q_0 = +0.1\,\mu\text{C} = 10^{-7}\text{ C},$$

$$m = 6\times10^{-4}\text{ kg}$$

and $Q = 8\,\mu\text{C} = 8\times10^{-6}$ C



Let P be any point at a distance x from origin O. Then

$$AP = CP = \sqrt{\frac{3}{2}+x^2}$$

$$BP = DP = \sqrt{\frac{27}{2}+x^2}$$

Electric potential at point P will be

$$V = \frac{2Q}{4\pi\varepsilon_0(BP)} - \frac{2q}{4\pi\varepsilon_0(AP)}$$

$$\Rightarrow \quad V = 2\times9\times10^9\left[\frac{8\times10^{-6}}{\sqrt{\frac{27}{2}+x^2}} - \frac{10^{-6}}{\sqrt{\frac{3}{2}+x^2}}\right]$$

$$\Rightarrow \quad V = 1.8\times10^4\left[\frac{8}{\sqrt{\frac{27}{2}+x^2}} - \frac{1}{\sqrt{\frac{3}{2}+x^2}}\right] \qquad \ldots(1)$$

Since, electric field at P is given by $E = -\frac{dV}{dx}$, so

$$E = -1.8\times10^4\left[(8)\left(-\frac{1}{2}\right)\left(\frac{27}{2}+x^2\right)^{-\frac{3}{2}} - (1)\left(-\frac{1}{2}\right)\left(\frac{3}{2}+x^2\right)^{-\frac{3}{2}}\right](2x)$$

So, $E = 0$ on x-axis either at $x = 0$ or when

$$\frac{8}{\left(\frac{27}{2}+x^2\right)^{3/2}} = \frac{1}{\left(\frac{3}{2}+x^2\right)^{3/2}}$$

$$\Rightarrow \quad \frac{(4)^{3/2}}{\left(\frac{27}{2}+x^2\right)^{3/2}} = \frac{1}{\left(\frac{3}{2}+x^2\right)^{3/2}}$$

$$\Rightarrow \quad \left(\frac{27}{2}+x^2\right) = 4\left(\frac{3}{2}+x^2\right)$$

This equation gives $x = \pm\sqrt{\frac{5}{2}}$ m

The least value of kinetic energy of the particle at infinity should be enough to take the particle upto $x = +\sqrt{\frac{5}{2}}$ m because at $x = +\sqrt{\frac{5}{2}}$ m, $E = 0$, so electrostatic force on charge q_0 is zero or $F_e = 0$.

For at $x > \sqrt{\frac{5}{2}}$ m, E is repulsive (towards positive x-axis)

and for $x < \sqrt{\frac{5}{2}}$ m, E is attractive (towards negative x-axis)

Now, from equation (1), potential at $x=\sqrt{\frac{5}{2}}$ m

$$V=1.8\times10^4\left[\frac{8}{\sqrt{\frac{27}{2}+\frac{5}{2}}}-\frac{1}{\sqrt{\frac{3}{2}+\frac{5}{2}}}\right]$$

$\Rightarrow \quad V=2.7\times10^4$ V

Applying energy conservation at $x\to\infty$ and $x=\sqrt{\frac{5}{2}}$ m

$$\frac{1}{2}mv_0^2=q_0V \qquad \ldots(2)$$

$$\Rightarrow \quad v_0=\sqrt{\frac{2q_0V}{m}}$$

Substituting the values $v_0=\sqrt{\frac{2\times10^{-7}\times2.7\times10^4}{6\times10^{-4}}}$

$$v_0=3\text{ ms}^{-1}$$

$\therefore$ Minimum value of v_0 is 3 ms^{-1}

From equation (1), potential at origin $(x=0)$ is

$$V_0=1.8\times10^4\left[\frac{8}{\sqrt{\frac{27}{2}}}-\frac{1}{\sqrt{\frac{3}{2}}}\right]=2.4\times10^4\text{ V}$$

Let T be the kinetic energy of the particle at origin. Applying energy conservation at $x=0$ and at $x=\infty$

$$T+q_0V_0=\frac{1}{2}mv_0^2$$

But $\frac{1}{2}mv_0^2=q_0V$ {from equation (2)}

$\Rightarrow \quad T=q_0(V-V_0)$

$\Rightarrow \quad T=(10^{-7})(2.7\times10^4-2.4\times10^4)$

$\Rightarrow \quad T=3\times10^{-4}$ J

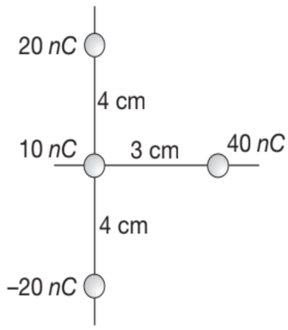
Please note here that, $E=0$ or F on q_0 is zero at $x=0$ and $x=\pm\sqrt{\frac{5}{2}}$ m. Of these $x=0$ is stable equilibrium position and $x=\pm\sqrt{\frac{5}{2}}$ m is unstable equilibrium position.

Test Your Concepts-X

Based on Motion of Charged Particles in Electric Field

(Solutions on page H.32)

1. Consider an electron to be released from rest in a uniform electric field whose magnitude is 6 kVm^{-1}. Find,
   (a) through what potential difference will it have passed after moving 1 cm?
   (b) how fast will the electron be moving after it has travelled 1 cm?
2. **(a)** Calculate the speed of a proton that is accelerated from rest through a potential difference of 120 V.
   (b) Calculate the speed of an electron that is accelerated through the same potential difference.
3. Two particles, having charges 20 nC and –20 nC, are placed at the points with coordinates (0, 4) cm and (0, –4) cm, as shown. A particle with charge 10 nC is located at the origin.



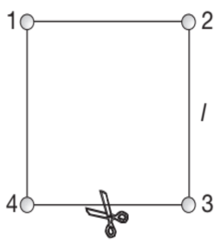
   (a) Find the electric potential energy of the configuration of the three fixed charges.
   (b) A fourth particle, with a mass of 2×10^{-13} kg and a charge of 40 nC, is released from rest at the point (3, 0) cm. Find its speed after it has moved freely to a very large distance away.

4. In 1911 Ernest Rutherford and his assistants Geiger and Marsden conducted an experiment in which they scattered alpha particles from thin sheets of gold. An alpha particle, having charge $+2e$ and mass 6.67×10^{-27} kg, is a product of certain radioactive decays. The results of the experiment led Rutherford to the idea that most of the mass of an atom is in a very small nucleus, with electrons in orbit around it his planetary model of the atom. Assume an alpha particle, initially very far from a gold nucleus, is fired with a velocity of 2×10^{7} ms^{-1} directly toward the nucleus (charge $+79e$). How close does the alpha particle get to the nucleus before turning around? Assume the gold nucleus remains stationary.

5. Four identical particles each have charge q and mass m. They are released from rest at the vertices of a square of side L. How fast is each charge moving at the instant when their distance from the center of the square doubles?

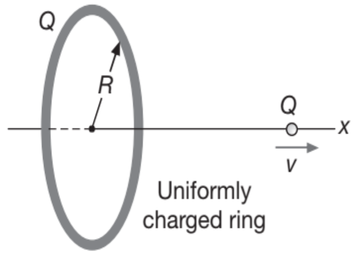
6. An electron is released from rest on the axis of a uniform positively charged ring, 0.1 m from the ring's center. If the linear charge density of the ring is $0.1\,\mu$Cm^{-1} and the radius of the ring is 0.2 m, how fast will the electron be moving when it reaches the center of the ring?

7. Four balls, each with mass m, are connected by four non-conducting strings to form a square with side ℓ, as shown in Figure. The assembly is placed on a horizontal nonconducting frictionless surface. Balls 3 and 4 each have charge q, and balls 1 and 2 are uncharged. Find the maximum speed of balls 3 and 4 after the string connecting them is cut.

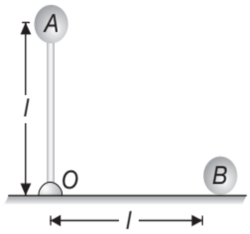


8. The x-axis is the symmetry axis of a stationary uniformly charged ring of radius R and charge Q. A point charge Q of mass M is located initially at the center of the ring. When it is displaced slightly, the point charge accelerates along the x-axis to infinity. Show that the ultimate speed of the point charge is

$$v=\sqrt{\frac{Q^2}{2\pi\varepsilon_0 MR}}$$



9. Figure shows a small ball A of mass m and charge q connected at one end of light rod of length ℓ. Another identical ball B is placed on smooth ground at a distance ℓ from the point O, where the rod is clamped. A slight jerk on ball A makes the rod start rotating clockwise. At the instant when A strikes the ground, ball B is moving with a velocity u towards right and is at a position 2ℓ away from the point O. Calculate the kinetic energy of the ball A when it strikes the floor.

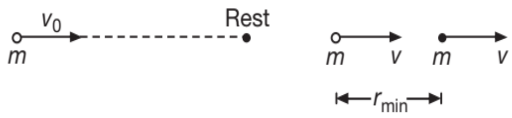


10. A charged particle of charge $+q$, mass m is thrown towards another fixed charge $+Q$ from a very large distance away with speed v_0 with an impact parameter ℓ. Find the distance of closest approach of the two charges.

11. The potential difference between two large parallel plates is varied as $V=at$, where a is a constant and t is time. An electron starts from rest at $t=0$ from the plate which is at negative potential. If the distance between the plates is ℓ, mass of electron is m and charge on electron is $-e$ then find the velocity of the electron when it reaches the other plate.

12. A point charge $-q$ revolves around a fixed charge $+Q$ in elliptical orbit. The minimum and maximum distance of q from Q are r_1 and r_2, respectively. The mass of revolving particle is m. $Q > q$ and assume no gravitational effects. Find the velocity of q at positions when it is at a distance r_1 and r_2 from Q.

13. A proton of mass m approaches a free proton from a very large distance with a velocity v_0 along the straight line joining their centres. Calculate the distance of closest approach between the two protons.



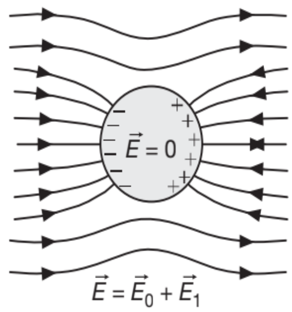
CONDUCTORS AND THEIR PROPERTIES

Insulators such as glass or paper are materials in which electrons are attached to some particular atoms and cannot move freely. On the other hand conductors (such as metals) have electrons capable of moving around freely. If a resultant electric field exists in the conductor then free charges will experience a force which will set a current flow. When no current flows, the resultant force and the electric field must be zero. Thus, under electrostatic conditions the value of $\vec{E}$ at all points within a conductor is zero. This idea, together with the Gauss's Law can be used to prove several interesting facts regarding a conductor.

The Electric Field is Zero Inside a Conductor

If we place a solid spherical conductor in a constant external field $\vec{E}_0$, the positive and negative charges will move toward the polar regions of the sphere (the regions on the left and right of the sphere in figure shown), thereby inducing an electric field $\vec{E}_i$. Inside the conductor, $\vec{E}_i$ points in the direction opposite to $\vec{E}_0$. Since charges are mobile, they will continue to move until $\vec{E}_i$ completely cancels $\vec{E}_0$ inside the conductor. At electrostatic equilibrium, $\vec{E}$ must vanish inside a conductor. Outside the conductor, the electric field $\vec{E}_i$ due to the induced charge distribution corresponds to a dipole field, and the total electric field is simply the resultant of both, so $\vec{E} = \vec{E}_0 + \vec{E}_i$. The field lines are depicted in figure.

Thus if a metal body is placed in an electric field, charge induction on the body surface starts and the continuous flow of electrons takes place inside the metal body (conductor) till the net electric field inside the body becomes zero.



Placing a conductor in a uniform electric field $\vec{E}$

Hence $E - E_i = 0$

Thus we can say that when all charges in a metal body are at rest, net electric field inside it is zero or the electric field inside the body due to the charges on its surface always balances the external electric field in the body. This is the condition of ELECTROSTATIC EQUILIBRIUM.

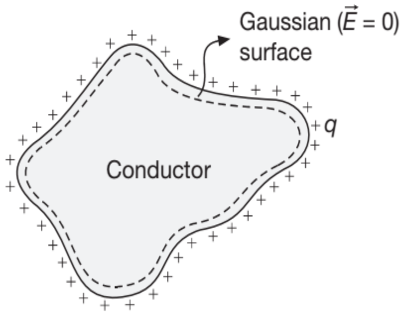
Net Excess Charge on a Conductor Resides on Its Outer Surface

Consider a charged conductor carrying a charge q and no currents are flowing in it i.e., in electrostatic equilibrium. Now consider a Gaussian surface inside the conductor everywhere on which $\vec{E} = 0$ (because it lies inside the conductor). Since, according to Gauss's Law,

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

Since $\vec{E} = \vec{0}$, so $q_{\text{enc}} = 0$ (because it lies inside the conductor)

Thus, the sum of all charges inside the Gaussian surface is zero.



This Gaussian surface has been taken just inside the surface of the conductor. So, any charge on the conductor must reside on the surface of the conductor. In other words,

"Under electrostatic conditions (condition of electrostatic equilibrium) the excess charge on a conductor resides on its outer surface".

The Tangential Component of $\vec{E}$ is Zero on the Surface of a Conductor

We have already seen that for an isolated conductor, the electric field is zero in its interior. Any excess charge placed on the conductor must then distribute itself on the surface, as implied by Gauss's Law.

Consider the line integral $\oint \vec{E}\cdot d\vec{l}$ around a closed path $abcda$ shown in figure.

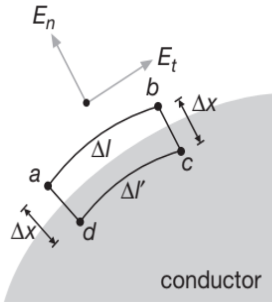
Since the electric field $\vec{E}$ is conservative, the line integral around the closed path $abcda$ must be zero.

$$\Rightarrow \oint_{abcda} \vec{E}\cdot d\vec{l} = E_t(\Delta l) + E_n(\Delta x)\cos(180) + 0(\Delta l) + E_n(\Delta x)\cos 0 = 0$$

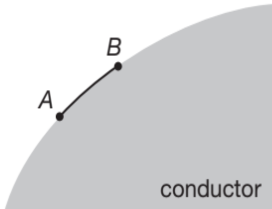
$$\Rightarrow \oint_{abcda} \vec{E}\cdot d\vec{l} = E_t(\Delta l) = 0$$

where E_t and E_n are the tangential and the normal components of the electric field, respectively. We have oriented the infinitesimal segment ab so that it is parallel to E_t. Also here we take $ad = bc$. If we take $ad \neq bc$ then too since we have taken an infinitesimal loop $abcda$ so in the limit where both Δx and $\Delta x' \to 0$, we will have $E_t \Delta l = 0$. However, since the length element Δl is finite, we conclude that the tangential component of the electric field on the surface of a conductor vanishes.

$$E_t = 0 \text{ (on the surface of a conductor)}$$



This implies that the surface of a conductor in electrostatic equilibrium is an equipotential surface. Consider two points A and B on the surface of a conductor as shown,



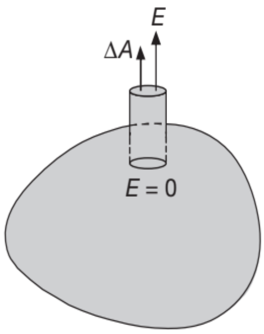
Since the tangential component $E_t = 0$, the potential difference is

$$V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} = 0$$

because $\vec{E}$ is perpendicular to $d\vec{l}$. Thus, points A and B are at the same potential with $V_A = V_B$.

Electric Field at any Point Close to the Charged Conductor is σ/ε_0

Consider a charged conductor of irregular shape. In general, surface charge density will vary from place to place. At a small surface ΔA, let us assume it to be a constant σ. Now construct a Gaussian surface in the form of a cylinder also called as Gaussian Pill Box of cross-section ΔA. One plane face of the cylinder is inside the conductor and other outside the conductor close to it. The surface inside the conductor does not contribute to the flux as $\vec{E}$ is zero everywhere inside the conductor. The curved surface outside the conductor also does not contribute to flux as $\vec{E}$ is always normal to the charged conductor and hence parallel to the curved surface. Thus the only contribution to the flux is through the plane face outside the conductor.



So, from Gauss's Law,

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

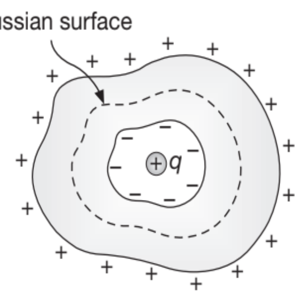
$$\Rightarrow \quad E\Delta A = \frac{(\sigma)(\Delta A)}{\varepsilon_0}$$

$$\Rightarrow \quad E = E_n = \frac{\sigma}{\varepsilon_0}$$

(This result holds good for a conductor of any arbitrary shape)

Conductor with Charge Inside a Cavity

Consider a hollow conductor shown in figure. Suppose the net charge carried by the conductor is $+Q$. In addition, there is a charge q inside the cavity. What is the charge on the outer surface of the conductor?



Conductor with a cavity

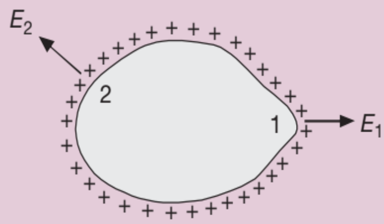
Since the electric field inside a conductor must be zero, the net charge enclosed by the Gaussian surface shown in figure must be zero. This implies that a charge $-q$ must have been induced on the cavity surface. Since the conductor itself has a charge $+Q$, the amount of charge on the outer surface of the conductor must be $Q+q$. (Think about the concept of charge conservation to get this as an answer).

The Potential of a Charged Conductor Throughout Its Volume is Same

In any region in which $\vec{E}=0$ at all points, such as the region vary far from all charges or the interior of a charged conductor, the line integral of $\vec{E}$ is zero along any path. It means that any two points in the conductor are at the same potential or have zero potential difference or the interior of a charged conductor is an equipotential region.

Remark(s)

(a) Electric field changes discontinuously at the surface of a conductor. Just inside the conductor it is zero and just outside the conductor it is $\frac{\sigma}{\varepsilon_0}$. In fact, the field gradually decreases from $\frac{\sigma}{\varepsilon_0}$ to zero in a small thickness of about 4 to 5 atomic layers at the surface.



(b) For a nonuniform conductor the surface charge density (σ) varies inversely as the radius of curvature (ρ) of that part of the conductor, i.e.,

$$\sigma \propto \frac{1}{\text{radius of curvature } (\rho)}$$

For case in the figure, $\rho_1 < \rho_2$

$$\Rightarrow \quad \sigma_1 > \sigma_2$$

$$\Rightarrow \quad E_1 > E_2 \qquad \left\{\because E = \frac{\sigma}{\varepsilon_0}\right\}$$

ILLUSTRATION 80

Consider two concentric conducting spheres. The outer sphere is hollow and initially has a charge $-7Q$ on it. The inner sphere is solid and has a charge $+2Q$ on it.

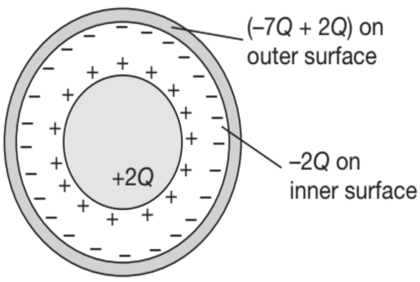
(a) Calculate the charge on the outer and inner surface of the outer sphere.

(b) If a wire is connected between the inner and outer spheres, after electrostatic equilibrium is established, how much total charge is on the outside sphere? How much charge is on the outer and inner surface of outer sphere? Does the electric field at the surface of the inside sphere change when the wire is connected?

(c) Now, if we return to the original condition as specified in (a) and connect the outer sphere to the ground with a wire and then disconnect it. How much total charge will be on the outer sphere? How much charge will be on the inner and outer surface of the outer sphere?

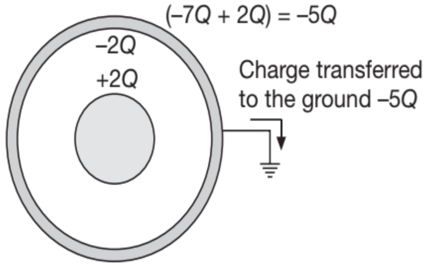
SOLUTION

(a) The charges on the inner sphere induce equal magnitude of charge, but opposite in sign, on the inner surface of outer sphere. Sum of all the induced charges is always zero. Therefore, an equal amount of charge must come on the outer surface. Thus outer and inner surface of outer sphere have charges $-5Q$ and $-2Q$ respectively.



(a)

(b) In electrostatic equilibrium, charge does not reside inside a conductor. So, when outer and inner spheres are connected by a wire, the entire charge is transferred to the outer surface from inner sphere.



(b)

Total charge on outer surface of outer sphere is $-5Q$ and the total charge on inner surface is zero.

The electric field at the surface of the inner sphere goes to zero after connection. Consider a Gaussian surface just on the surface of inner sphere. So, according to Gauss's Law

$$\phi_E = E\left(4\pi r^2\right) = \frac{Q_{\text{enclosed}}}{\varepsilon_0} = 0$$

$$\left(\text{as } Q_{\text{enclosed}} = 0\right)$$

$$\Rightarrow \quad E = 0$$

(c) When the outer sphere is grounded the charge on the surface is transferred to ground, thus charge is reduced to zero. The final charge distribution is shown in figure (b).

ILLUSTRATION 81

A long, straight wire is surrounded by a hollow metal cylinder whose axis coincides with that of the wire. The wire has a charge per unit length of λ, and the cylinder has a net charge per unit length of 2λ. From this information, use Gauss's Law to find

(a) the charge per unit length on the inner and outer surfaces of the cylinder and

(b) the electric field outside the cylinder, a distance r from the axis.

SOLUTION

(a) Inside surface: consider a cylindrical surface within the metal. Since E inside the conducting shell is zero, the total charge inside the gaussian surface must be zero, so the inside $\dfrac{\text{charge}}{\text{length}} = -\lambda$

$$0 = \lambda l + q_{in}$$

So $\frac{q_{in}}{l} = -\lambda$

$\Rightarrow \quad 2\lambda l = q_{in} + q_{out}$

Outside surface: The total charge on the metal cylinder is $q_{out} = 2\lambda l + \lambda l$ so the outside $\frac{\text{charge}}{\text{length}}$ is 3λ

(b) $E = \frac{3\lambda}{2\pi \epsilon_0 r}$ radially outward $\left\{\because E = \frac{\lambda_{enc}}{2\pi\varepsilon_0 r}\right\}$

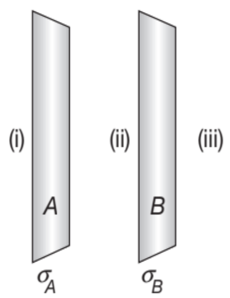
Test Your Concepts-XI

Based on Conductors

(Solutions on page H.35)

1. Two identical conducting spheres each having a radius of 0.5 cm are connected by a light 2 m long conducting wire. A charge of 60 μC is placed on one of the conductors. Assume that the surface distribution of charge on each sphere is uniform. Determine the tension in the wire.

2. Two infinite parallel planes carry equal but opposite uniform charge densities σ_A and σ_B (shown in figure). Find the field in each of the three regions:



(i) to the left of both
(ii) between them
(iii) to the right of both.
Also find the field in the three regions for $\sigma_A = +\sigma$ and $\sigma_B = -\sigma$.

3. In a particular region of air at an altitude of 500 m above the ground the electric field is found to be 120 NC^{-1} directed downwards. At 600 m above the ground the electric field is 100 NC^{-1} downwards. Calculate the average volume charge density in the layer of air between these two elevations. Is it positive or negative?

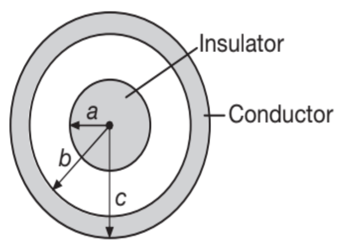
4. The electric field on the surface of an irregularly shaped conductor varies from 56 kNC^{-1} to 112 kNC^{-1}. Calculate the local surface charge density at the point on the surface where the radius of curvature of the surface is
(a) greatest and
(b) smallest

5. A conducting spherical shell of inner radius a and outer radius b carries a net charge Q. A point charge q is placed at the center of this shell. Determine the surface charge density on
(a) the inner surface of the shell and
(b) the outer surface of the shell.

6. A hollow conducting sphere is surrounded by a larger concentric spherical conducting shell. The inner sphere has charge $-Q$, and the outer shell has net charge $+3Q$. The charges are in electrostatic equilibrium. Using Gauss's Law, find the charges and the electric fields everywhere.

7. A positive point charge is at a distance $\frac{R}{2}$ from the center of an uncharged thin conducting spherical shell of radius R. Sketch the electric field lines set up by this arrangement both inside and outside the shell.

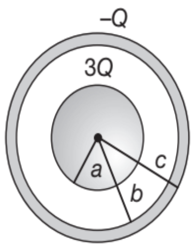
8. A solid, insulating sphere of radius a has a uniform charge density ρ and a total charge Q. Concentric with this sphere is an uncharged, conducting hollow sphere whose inner and outer radii are b and c, as shown in figure.



(a) Find the magnitude of the electric field in the regions $r < a$, $a < r < b$, $b < r < c$, and $r > c$.

(b) Determine the induced charge per unit area on the inner and outer surfaces of the hollow sphere.

9. A solid insulating sphere of radius a carries a net positive charge $3Q$, uniformly distributed throughout its volume. Concentric with this sphere is a conducting spherical shell with inner radius b and outer radius c, and having a net charge $-Q$, as shown in Figure



(a) Construct a spherical gaussian surface of radius $r > c$ and find the net charge enclosed by this surface.

(b) Find the magnitude and the direction of electric field at $r > c$.

(c) Find the electric field in the region with radius r where $b < r < c$.

(d) Construct a spherical gaussian surface of radius r, where $b < r < c$, and find the net charge enclosed by this surface.

(e) Construct a spherical gaussian surface of radius r, where $a < r < b$, and find the net charge enclosed by this surface.

(f) Find the electric field in the region $a < r < b$.

(g) Construct a spherical gaussian surface of radius $r < a$, and find an expression for the net charge enclosed by this surface, as a function of r.

(h) Find the electric field in the region $r < a$.

(i) Determine the charge on the inner surface of the conducting shell.

(j) Determine the charge on the outer surface of the conducting shell.

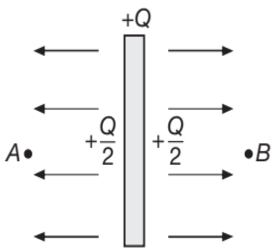
(k) Make a plot of the magnitude of the electric field versus r.

CHARGE DISTRIBUTION ON A SYSTEM OF PARALLEL PLATES

CASE-1:

Consider a large conducting plate that has been given a charge $+Q$. Due to symmetry, this charge is distributed equally on both sides of its surfaces as shown. If surface area of each face is A, charge density on the surface is given by

$$\sigma = \frac{Q}{2A}$$

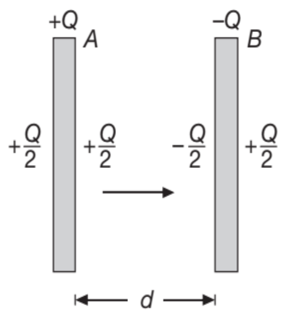


The electric field, on both sides of the plate, at points A and B is given by

$$E_A = E_B = \frac{\sigma}{\varepsilon_0} = \frac{Q}{2A\varepsilon_0}$$

CASE-2:

Now, if another uncharged plate is placed in front of this charged sheet as shown in figure. On the sides facing each other, plate of this plate an equal and opposite charge $-\frac{Q}{2}$ is induced and on the back surface $+\frac{Q}{2}$ is induced. The electric field, between the plates is $\frac{\sigma}{\varepsilon_0}$, where σ is the charge density of the surface. In figure the direction of electric field in different region is indicated by arrows.



Now we find the potential difference between the two plates A and B as

$$V_A - V_B = \frac{\sigma}{\varepsilon_0} d$$

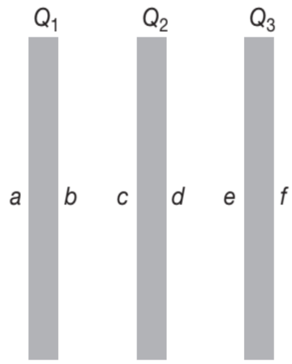
{As electric field is uniform between the plates}

$$\Rightarrow \quad V_A - V_B = \frac{Q}{2A\varepsilon_0} d \qquad \left\{\because \sigma = \frac{Q}{2A}\right\}$$

CASE-3:

Consider three parallel metallic plates each of area A which are kept as shown and charges Q_1, Q_2 and Q_3 are given to them. Let us find the resulting charge distribution on the six surfaces. Neglecting edge effects as usual, we observe that the plate separations do not affect the distribution of charge here. Let the plates 1, 2 and 3 have charges Q_1, Q_2 **and** Q_3. If we assume the charge on side a to be Q_a on the side c to be Q_c and that on the side e to be Q_e, then we have the charges on the left out surfaces given by

$$Q_b = Q_1 - Q_a,\ Q_d = Q_2 - Q_c \text{ and } Q_f = Q_3 - Q_e.$$



Electric field at point P is zero

$$E_P = 0$$

At P, charge Q_a will give an electric field towards right. All other charges Q_b, Q_c …, etc., will give the electric field towards left. So,

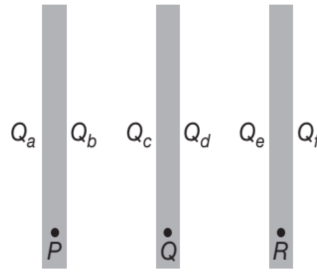
$$\frac{[Q_a - (Q_1 - Q_a) - Q_c - (Q_2 - Q_c) - Q_e - (Q_3 - Q_e)]}{2\varepsilon_0 A} = 0$$

$$\Rightarrow \quad 2Q_a - Q_1 - Q_2 - Q_3 = 0$$

$$\Rightarrow \quad Q_a = \frac{Q_1 + Q_2 + Q_3}{2}$$

Similarly the condition, $E_R = 0$ will give the result,

$$Q_f = \frac{Q_1 + Q_2 + Q_3}{2}$$



From here we may conclude that, half of the sum of all charges appears on each of the two outermost surfaces of the system of plates.

Further we have a condition,

$$E_Q = 0$$

$$\frac{[Q_a + (Q_1 - Q_a) + Q_c - (Q_2 - Q_c) - Q_e - (Q_3 - Q_e)]}{2\varepsilon_0 A} = 0$$

$$\Rightarrow \quad Q_1 + 2Q_c - Q_2 - Q_3 = 0$$

$$\Rightarrow \quad Q_c = \frac{Q_2 + Q_3 - Q_1}{2}$$

and $Q_b = Q_1 - Q_a = \dfrac{Q_1 - Q_2 - Q_3}{2} = -Q_c$

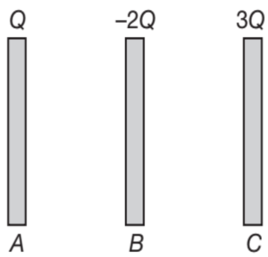
Similarly, we can show that, $Q_d = -Q_e$.

From here we can find another important result that the pairs of opposite surfaces like b, c and d, e carry equal and opposite charges and the outer surfaces carry equal charges.

The properties can often be applied to similar problems, provided additional conditions or constraints are not applied. These conditions (or constraints) can be in the form of some plates being connected together by conducting wires or some plates being held at fixed potentials through the use of batteries.

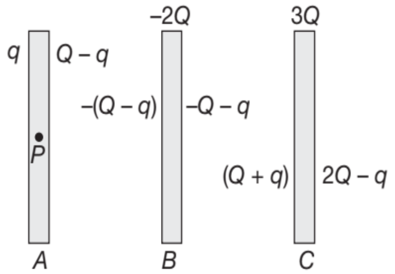
ILLUSTRATION 82

Consider a system of parallel plates A, B and C. The plates are given charges Q, $-2Q$ and $3Q$ respectively. Find the distribution of charge on the faces of the three plates.



SOLUTION

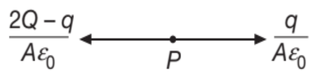
Let us assume that after final distribution of charge on the left face of plate A, charge is q then on right side of it remaining charge will be $(Q-q)$ as shown. Always remember that "In a system of large parallel conducting plates, the facing surfaces of plates always carry equal and opposite charges".



So, using this concept, we distribute the charges on the plates as shown in figure. Let us consider a point P inside the plate A. The net electric field at point P must be zero as it is inside a conducting body. This fact helps us to determine the charge q and hence the complete charge distribution.

Also, we observe that the electric field at point P is only due to the left face of A and right face of C, because all other charges produces electric field at point P in equal and opposite direction

Thus we have electric field at point P as



Since the point P lies inside a conductor, so net electric field at point P has to be zero.

$$\Rightarrow \quad \frac{2Q-q}{A\varepsilon_0}-\frac{q}{A\varepsilon_0}=0$$

$$\Rightarrow \quad q=Q$$

So, using this concept, we calculate the final distribution of charges on the system of parallel plates shown in figure.

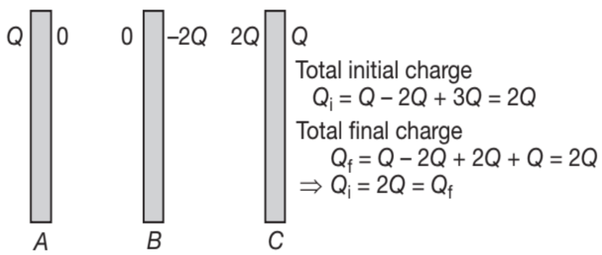
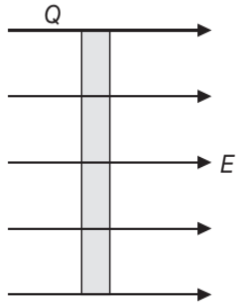


ILLUSTRATION 83

An isolated conducting sheet of area A and carrying a charge Q is placed in a uniform electric field E, such that electric field is perpendicular to sheet and covers the entire sheet. Find the charges appearing on the two surfaces of the sheet.

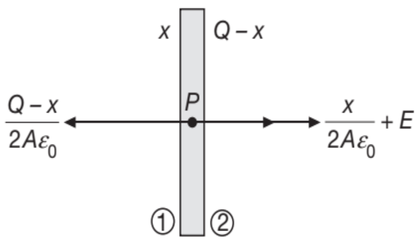


SOLUTION

Let charge on left side of plate be x, then $(Q-x)$ is charge on right side of plate.

Since inside the plate, at point P, we have

$$E_P=0$$



$$\Rightarrow \quad \frac{x}{2A\varepsilon_0}+E=\frac{Q-x}{2A\varepsilon_0}$$

$$\Rightarrow \quad \frac{x}{A\varepsilon_0}=\frac{Q}{2A\varepsilon_0}-E$$

$$\Rightarrow \quad x=\frac{Q}{2}-\varepsilon_0 AE \text{ and } (Q-x)=\frac{Q}{2}+\varepsilon_0 AE$$

So charge on one side is $\left(\frac{Q}{2}-EA\varepsilon_0\right)$ and other side $\left(\frac{Q}{2}+EA\varepsilon_0\right)$

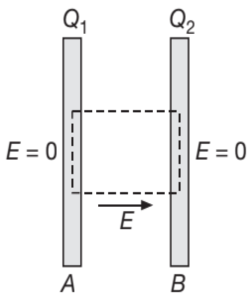
If $Q=0$ i.e. the sheet is neutral, then a charge $-\varepsilon_0 AE$ and $+\varepsilon_0 AE$ appear on the two surfaces of conducting sheet.

ILLUSTRATION 84

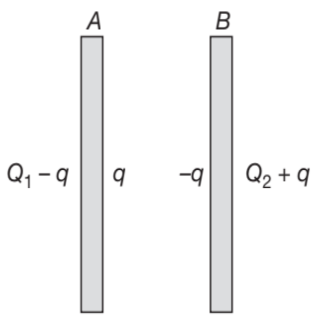
Two conducting plates A and B are placed parallel to each other. A is given a charge Q_1 and B a charge Q_2. Prove that the charges on the inner facing surfaces are of equal magnitude and opposite sign. Also calculate the charges on the inner and outer surfaces of the plates.

SOLUTION

Consider a Gaussian surface as shown in figure. Two faces of this closed surface lie completely inside the conductor where the electric field is zero. The flux through these faces is, therefore, zero. The other parts of the closed surface which are outside the conductor are parallel to the electric field and hence the flux on these parts is also zero. The total flux of the electric field through the closed surface is, therefore zero. From Gauss's law, the total charge inside this closed surface should be zero. The charge on the inner surface of A should be equal and opposite to that on the inner surface of B.



The distribution of charges is shown in figure. To find the value of q, consider the field at a point P inside the plate A. Suppose, the surface area of the plate (one side) is A. Since we have $E = \dfrac{q}{2\varepsilon_0 A}$, So at P



field due to charge $Q_1 - q$ is $\dfrac{Q_1 - q}{2A\varepsilon_0}$ (downwards)

field due to charge $+q$ is $\dfrac{q}{2A\varepsilon_0}$ (upwards)

field due to charge $-q$ is $\dfrac{q}{2A\varepsilon_0}$ (downwards)

and field due to charge $Q_2 + q$ is $\dfrac{Q_2 + q}{2A\varepsilon_0}$ (upwards)

The net electric field at P due to all the four charged surfaces is (in the downward direction)

$$E_p = \frac{Q_1 - q}{2A\varepsilon_0} - \frac{q}{2A\varepsilon_0} + \frac{q}{2A\varepsilon_0} - \frac{Q_2 + q}{2A\varepsilon_0}$$

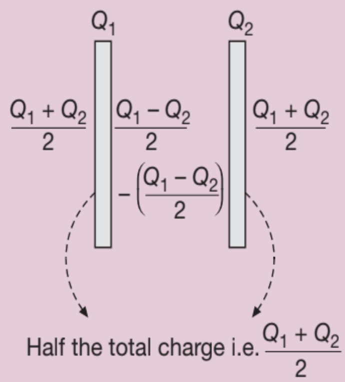
As the point P lies inside the conductor so, this field should be zero. Hence,

$$Q_1 - q - q + q - Q_2 - q = 0$$

$$\Rightarrow \quad q = \frac{Q_1 - Q_2}{2}$$

Problem Solving Technique(s)

So, when charged conducting plates are placed parallel to each other, then the outermost surfaces get equal charge (that is exactly half the total charge on the plates) and the plate surfaces facing each other have equal and opposite charge.



For Example:

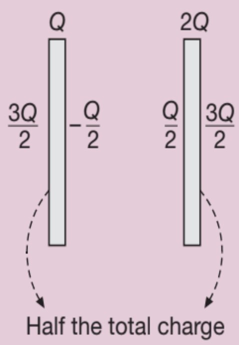
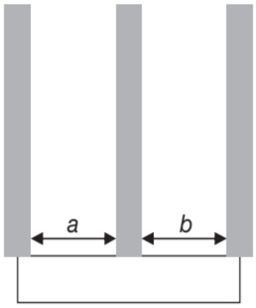


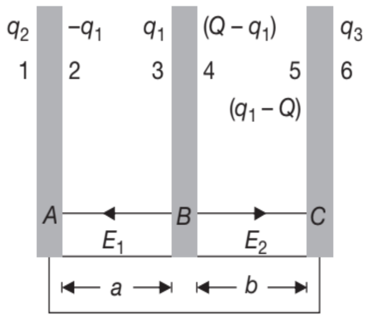
ILLUSTRATION 85

Three identical metallic plates are kept parallel to one another at a separation of a and b. The outer plates are connected by a thin conducting wire and a charge Q is placed on the central plate. Find the final charges on all the six faces of the plates shown.



SOLUTION

Let the charge distribution in all the six faces be as shown in figure. While distributing the charge on different faces, we have used the fact that two opposite faces have equal and opposite charges on them.



Net charge on plates A and C is zero. Hence,

$$q_2 - q_1 + q_3 + (q_1 - Q) = 0$$

$$\Rightarrow \quad q_2 + q_3 = Q \qquad \ldots(1)$$

Further A and C are at same potentials, if E_1 be the field between the plates A and B and that between B and C is E_2, then,

$$V_B - V_A = V_B - V_C$$

$$\Rightarrow \quad E_1 a = E_2 b$$

$$\Rightarrow \quad \left(\frac{q_1}{A\varepsilon_0}\right)a = \left(\frac{Q-q_1}{A\varepsilon_0}\right)b \qquad \{A = \text{Area of plates}\}$$

$$\Rightarrow \quad q_1 a = (Q - q_1)b$$

$$\Rightarrow \quad q_1 = \frac{Qb}{a+b} \qquad \ldots(2)$$

Electric field inside any conducting plate (say inside C) is zero. Therefore,

$$\frac{q_2}{2A\varepsilon_0} - \frac{q_1}{2A\varepsilon_0} + \frac{q_1}{2A\varepsilon_0} + \frac{Q-q_1}{2A\varepsilon_0} + \frac{q_1-Q}{2A\varepsilon_0} - \frac{q_3}{2A\varepsilon_0} = 0$$

$$\Rightarrow \quad q_2 - q_3 = 0 \qquad \ldots(3)$$

Solving equations, (1), (2) and (3), we get

$$q_1 = \frac{Qb}{a+b}, \; q_2 = q_3 = \frac{Q}{2}$$

Hence, charge on different faces are as follows :

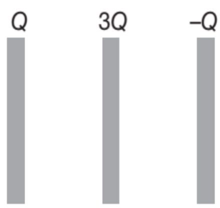
Face	Charge	Face	Charge
1	$q_2 = \frac{Q}{2}$	4	$(Q-q_1) = \frac{Qa}{a+b}$
2	$-q_1 = -\frac{Qb}{a+b}$	5	$(q_1-Q) = -\frac{Qa}{a+b}$
3	$q_1 = \frac{Qb}{a+b}$	6	$q_3 = \frac{Q}{2}$

Test Your Concepts-XII

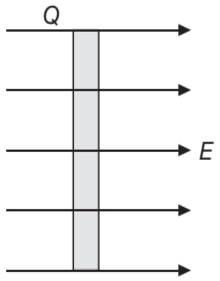
Based on Charge Distribution

(Solutions on page H.37)

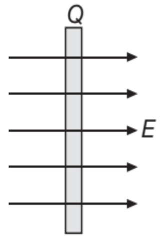
1. Figure shows three metallic plates with charges $-Q$, $+3Q$ and Q respectively. Determine the final charges on all the surfaces.



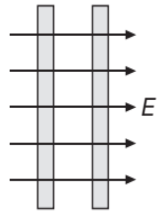
2. Prove that if an isolated large conducting sheet (isolated means no charges are near the sheet) is given a charge, then the charge distributes equally on its two surfaces.
3. An isolated conducting sheet of area A and carrying a charge Q is placed in a uniform electric field E, such that electric field is perpendicular to sheet and covers the entire sheet. Find the charges appearing on the two surfaces of the sheet. Also calculate the charge appearing on each face of the sheet if the sheet is neutrals.



4. An isolated conducting sheet of area A and carrying a charge Q is placed in a uniform electric field E, such that electric field is perpendicular to sheet and covers all the sheet. Calculate the resultant electric field on the left and right side of the plate.



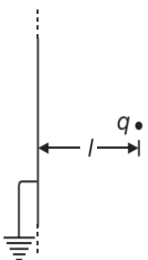
5. Two uncharged parallel conducting sheets each of area A are placed in a uniform electric field E at a finite distance from each other such that electric field is perpendicular to sheets and covers the entire sheets arrangement. Find out charges appearing on its two surfaces.



6. There are two large parallel metallic plates S_1 and S_2 carrying surface charge densities σ_1 and σ_2 respectively $(\sigma_1 > \sigma_2)$ placed at a distance d apart in vacuum. Find the work done by the electric field in moving a point charge q a distance $a(a < d)$ from S_1 towards S_2 along a line making an angle $\frac{\pi}{4}$ with the normal to the plates.

METHOD OF IMAGES

This method can be used for Boundary Value Problems in which we are asked to find electrostatic potential which results from a given charge configuration in the neighbourhood of an array (arrangement) of grounded conductors in the region outside the conductors. **Here, by Grounded Conductors we mean that conductors along the surface of which electrostatic potential V is set to zero.**



Such problems can be solved by reducing to an equivalent simpler problem. We begin by "removing" the conductor(s) and then introducing a new auxiliary system of charges. The **auxiliary** charges are also called **IMAGE** charges. The reason behind this nomenclature is that the auxiliary charges are considered as being practically the mirror images of the real charges (at least when the conductor is spherical or planar) in the sense that their positions and relative magnitudes are related in a manner exactly similar to the relations between the positions and relative linear magnification of the object and its corresponding image in Geometrical optics (also called Gaussian ray optics).

EXAMPLE

If we are asked to find the force between an infinite earthed conductor and a point charge q placed at

perpendicular distance ℓ from the earthed conductor (see figure), than we proceed as follows.
Firstly, the conductor being earthed implies $V = 0$.
So, we redraw the situation in which we replace the conductor and introduce an IMAGE charge $-q$ as shown.
The force between the two charges (object charge q and image charge $-q$) is the electrostatic force between the infinite grounded conductor and q. So,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2\ell)^2}$$

$$\Rightarrow F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\ell^2} \text{ (attractive in nature)}$$

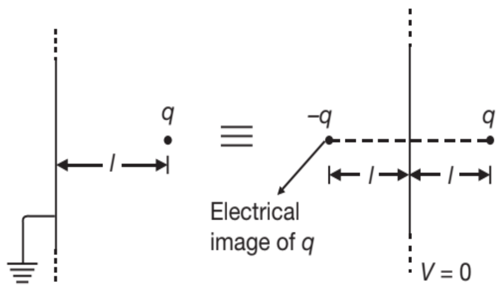


ILLUSTRATION 86

Calculate the work done in taking the charge q to infinity, when, initially it is at a distance l from an infinite conducting plane.

SOLUTION

While taking the charge to infinity let it (at any instant) be at a distance x from the grounded surface. The force with which it is attracted towards grounded surface is given by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2x)^2}$$

$$\Rightarrow \quad F = \frac{1}{4\pi\varepsilon_0(4x^2)}$$

Let it be further moved through dx away from grounded surface (or towards infinity), then

$$dW = Fdx\cos(180^\circ)$$

$$\Rightarrow \quad dW = -\frac{q^2}{4\pi\varepsilon_0(4x^2)}dx$$

$$\Rightarrow \quad W = \int dW = -\frac{q^2}{16\pi\varepsilon_0}\int_l^\infty x^{-2}dx$$

$$\Rightarrow \quad W = -\frac{q^2}{16\pi\varepsilon_0}\left[\frac{x^{-2+1}}{-2+1}\Big|_l^\infty\right]$$

$$\Rightarrow \quad W = \frac{q^2}{16\pi\varepsilon_0}\left(\frac{1}{\infty}-\frac{1}{l}\right)$$

$$\Rightarrow \quad W = -\frac{q^2}{16\pi\varepsilon_0 l}$$

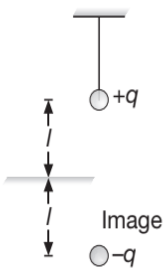
So, this much amount of energy i.e., $\frac{q^2}{16\pi\varepsilon_0 l}$ is to be supplied to the charge (system) to set it free from the electrostatic influence of the grounded conductor.

ILLUSTRATION 87

A small ball is suspended over an infinite horizontal conducting plane by means of an insulating elastic thread of stiffness k. As soon as the ball was charged, it descended by x and its new separation from the plane became equal to l. Find the charge of the ball.

SOLUTION

When the ball is charged, for the equilibrium of ball, electric force on it must counter balance the excess spring force, exerted, on the ball due to the extension in the string.



Thus $F_{el} = F_{spr}$

The force on the charge q is considered due to attraction by the electrical image

$$\Rightarrow \quad \frac{q^2}{4\pi\varepsilon_0(2l)^2} = kx$$

$$\Rightarrow \quad q = 4l\sqrt{\pi\varepsilon_0 kx}$$

ILLUSTRATION 88

Two point charges, q and $-q$, are separated by a distance L, both being located at a distance $\frac{L}{2}$ from the infinite conducting plane. Find

(a) the modulus of the vector of the electric force acting on each charge;

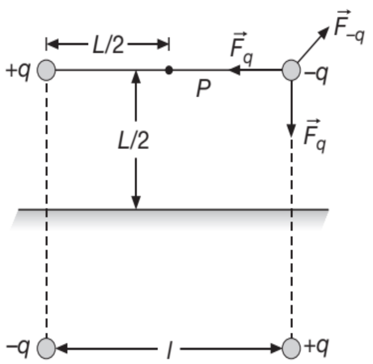
(b) the magnitude of the electric field strength vector at the mid-point between these charges.

SOLUTION

(a) Using the concept of electrical image, it is clear that the magnitude of the force acting on each charge,

$$|\vec{F}| = \sqrt{2}\frac{q^2}{4\pi\varepsilon_0 L^2} - \frac{q^2}{4\pi\varepsilon_0\left(\sqrt{2}L\right)^2}$$

$$\Rightarrow \quad |\vec{F}| = \frac{q^2}{8\pi\varepsilon_0 L^2}\left(2\sqrt{2}-1\right)$$



(b) Also, from the figure, magnitude of electric field strength at P

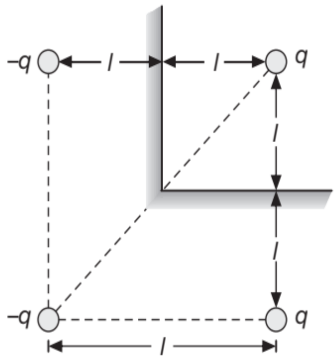
$$E = 2\left(1-\frac{1}{5\sqrt{5}}\right)\frac{q}{\pi\varepsilon_0 L^2}$$

ILLUSTRATION 89

A point charge q is located between two mutually perpendicular conducting half-planes. Its distance from each half-plane is equal to l. Find the modulus of the vector of the force acting on the charge.

SOLUTION

Using the concept of electrical image, it is easily seen that the force on the charge q is,



$$F = \frac{\sqrt{2}q^2}{4\pi\varepsilon_0(2l)^2} + \frac{(-q)^2}{4\pi\varepsilon_0\left(2\sqrt{2}l\right)^2}$$

$$\Rightarrow \quad F = \frac{\left(2\sqrt{2}-1\right)q^2}{32\pi\varepsilon_0 l^2} \text{ (It is attractive)}$$

CHARGE DISTRIBUTION ON METAL OBJECT

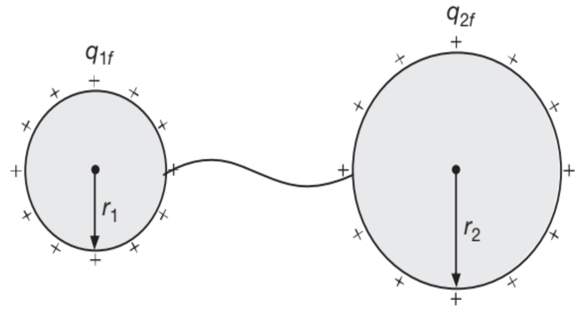
Whenever a charge is given to a metal body then, due to mutual repulsion, it automatically spreads on the outer surface of the body. Let us consider an example for understanding the distribution of charges in detail.

Figure shows two conducting metal spheres of radii r_1 and r_2 at large separation having charges q_1 and q_2 initially.

If the two spheres are connected by a metal wire, charge flow takes place from the sphere at higher potential to the one at lower potential till final potential of the two spheres become equal.

After final distribution the charges on the two spheres be $(q_1)_f$ and $(q_2)_f$ respectively such that

$$\frac{(q_1)_f}{4\pi\varepsilon_0 r_1} = \frac{(q_2)_f}{4\pi\varepsilon_0 r_2} \qquad \ldots(1)$$



If after distribution the final surface charge densities on the two spheres are σ_1 and σ_2, then we have

$$(q_1)_f = \sigma_1(4\pi r_1^2)$$

and $$(q_2)_f = \sigma_2(4\pi r_2^2)$$

Now from equation (1) we get

$$\frac{\sigma_1(4\pi r_1^2)}{4\pi\varepsilon_0 r_1} = \frac{\sigma_2(4\pi r_2^2)}{4\pi\varepsilon_0 r_2}$$

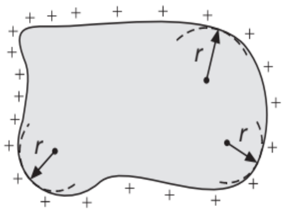
$$\Rightarrow \quad \sigma_1 r_1 = \sigma_2 r_2$$

$$\Rightarrow \quad \frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1}$$

Thus we can say $\sigma \propto \frac{1}{r}$

$$\Rightarrow \quad \sigma r = \text{constant}$$

This shows that for charged metal bodies at same potential, the surface charge density is inversely proportional to the radius of curvature of the body.



Consider a randomly shaped metal body that has been given some charge as shown, then charge is distributed on the outer surface of this body in such a way that at sharp edges of body (where radius of curvature is less), charge density is more and on the broader parts of body (where radius of curvature is large) charge density is less because we already know that at every point of a metal body potential is same.

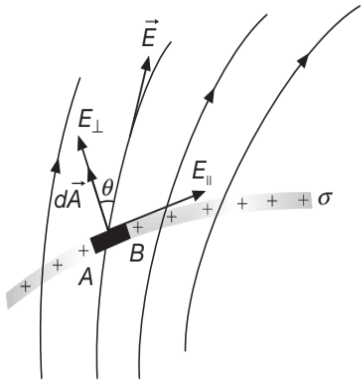
ELECTRIC PRESSURE ON A CHARGED SURFACE DUE TO EXTERNAL ELECTRIC FIELD

Consider a surface uniformly charged with charge density σ placed in electric field E. Let us consider a small portion AB having area dA, on the surface. Charge on AB is given by

$$dq = \sigma\, dA$$

If the external electric field on the surface AB is E in the direction shown, then it has two components, one parallel to the surface, other perpendicular to the surface, given by

$$E_\perp = E\cos\theta \text{ and } E_\parallel = E\sin\theta$$



Due to $E_\parallel$, the force on surface is tangential which only stretches the surface and due to $E_\perp$, the surface experiences an outward force and hence an outward pressure. The outward force on AB due to external electric field is

$$dF = E_\perp dq$$

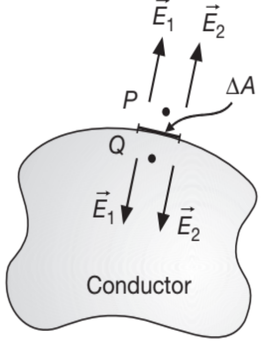
$$\Rightarrow \quad dF = (\sigma E_\perp)dA$$

Thus outward electrostatic pressure P_e is given by

$$P_e = \frac{dF}{dA} = \sigma E_\perp$$

MECHANICAL FORCE ON A CHARGED CONDUCTOR

Similar charges repel each other, hence the charge on any part of surface of the conductor is repelled by the charge on its remaining part. The surface of the conductor thus experiences a mechanical force.

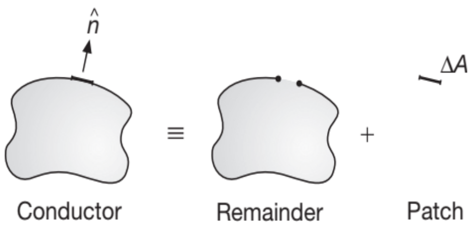


Also, we have seen that at the boundary of a conductor with uniform charge density σ, the tangential component of the electric field is zero, and hence is continuous. But the normal component of the electric field exhibits discontinuity with $E_n = \frac{\sigma}{\varepsilon_0}$ just outside and zero inside the conductor. Let us now consider a small patch of charge on the surface of conductor as shown.

A question comes to our mind "what is the force experienced by this patch"? Since by Newton's Third Law, the patch cannot exert the force on itself and hence the force on the patch must come only from the remaining part of the conductor **(called as remainder)**. Assume the patch to be a flat surface of area A on the surface of the conductor.

The electric field $(\vec{E})$ at any point P near the conductor surface is the sum of the electric field due to a small part of the surface (called as Patch) of area say A immediately in the neighbourhood of the point under consideration and due to the remaining part of the surface (called as Remainder). Let $\vec{E}_{\text{Patch}}$ and $\vec{E}_{\text{Remainder}}$ be the field intensities due to these parts respectively.
Then, total electric field,

$$\vec{E} = \vec{E}_{\text{Patch}} + \vec{E}_{\text{Remainder}}$$



Now, since $\vec{E}$ has a magnitude $\frac{\sigma}{\varepsilon_0}$ at any point P just outside the conductor and is zero at point Q just inside the conductor. Thus,

at P, $\vec{E}_{\text{Patch}} + \vec{E}_{\text{Remainder}} = \frac{\sigma}{\varepsilon_0}$ and

at Q, $E_{\text{Patch}} - \vec{E}_{\text{Remainder}} = \vec{0}$

$$\Rightarrow \quad \vec{E}_{\text{Patch}} = \vec{E}_{\text{Remainder}}$$

$$\Rightarrow \quad \left|\vec{E}_{\text{Patch}}\right| = \left|\vec{E}_{\text{Remainder}}\right| = \frac{\sigma}{2\varepsilon_0}$$

Hence, the force experienced by small surface of area A due to the charge on the rest of the surface is

$$\Rightarrow \quad F = qE_{\text{Remainder}} = (\sigma A)(E_{\text{Remainder}}) = \frac{\sigma^2 A}{2\varepsilon_0}$$

Thus, the force acting on the Patch is

$$\vec{F} = q\vec{E}_{\text{Remainder}} = \left(\frac{\sigma^2 A}{2\varepsilon_0}\right)\hat{n}$$

This is precisely the force needed to drive the charges on the surface of a conductor to an equilibrium state where the electric field just outside the conductor has a value $\frac{\sigma}{\varepsilon_0}$ and vanishes inside. The most interesting thing we must note here that irrespective of the sign of σ, the force always tends to pull the patch into the field.
Using the result obtained above, we may define the electrostatic pressure on the patch as

$$\frac{\text{Force}}{\text{Area}} = \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0 E^2 \qquad \left\{\text{as } E = \frac{\sigma}{\varepsilon_0}\right\}$$

$$\Rightarrow \quad \text{Electrostatic Pressure} = \frac{1}{2}\varepsilon_0 E^2 = \frac{\sigma^2}{2\varepsilon_0}$$

Here, we also note that the pressure is being transmitted via the electric field.
Also, this is the value of the energy density u_e due to the field. So,

$$u_e = \frac{1}{2}\varepsilon_0 E^2 = \frac{\sigma^2}{2\varepsilon_0}$$

(Please keep in mind that actually pressure and energy density possess the same dimensional formula i.e., $ML^{-1}T^{-2}$)

ELECTROSTATIC ENERGY DENSITY (u_e)

Electrostatic energy density is defined as the energy stored in unit volume in any electric field. Its mathematical formula is given by

$$\text{Energy density } u_e = \frac{1}{2}\varepsilon E^2$$

where $E =$ electric field intensity at that point and

$\varepsilon = \varepsilon_o \varepsilon_r$ electric permittivity of medium

ILLUSTRATION 90

Calculate the energy stored in an imaginary cubical volume of side a placed in front of an infinitely large nonconducting sheet of uniform charge density σ.

SOLUTION

Since electrostatic energy stored is

$$U=\int \frac{1}{2}\varepsilon_0 E^2 d\tau \text{ where } dr \text{ is small volume}$$

$$\Rightarrow \quad U=\frac{1}{2}\varepsilon_0 E^2 \int d\tau \text{, where } E=\frac{\sigma}{2\varepsilon_0}=\text{constant}$$

$$\Rightarrow \quad U=\frac{1}{2}\varepsilon_0\left(\frac{\sigma^2}{4\varepsilon_0^2}\right)a^3=\frac{\sigma^2 a^3}{8\varepsilon_0}$$

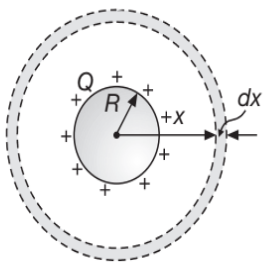
ILLUSTRATION 91

Using the concept of energy density, show that the total energy stored by a shell having charge Q and radius R is $\frac{Q^2}{8\pi\varepsilon_0 R}$.

SOLUTION

Since we have discussed that in space wherever electric field exists, there must be some field energy stored which has energy density, given as

$$u_e=\frac{1}{2}\varepsilon_0 E^2$$



Here we can see that when the sphere was uncharged, there was no electric field in its surroundings. But when the sphere becomes fully charged, electric field will exist in its surrounding from its surface to infinity. Let us calculate the field energy associated with this charged conducting sphere.
We know electric field due to a sphere at outer points varies with distance from centre as

$$E=\frac{Q}{4\pi\varepsilon_0 x^2}$$

To find the total field energy due to this sphere, we consider an elemental spherical shell of radius x and thickness dx as shown. The volume enclosed in this shell is

$$d\tau=\left(4\pi x^2\right)dx$$

Here we have taken symbol for volume as τ (and not V) as V is the symbol for potential.

Thus the field energy stored in the volume of this elemental shell is

$$u_e=\frac{dU}{d\tau}=\left(\frac{1}{2}\varepsilon_0 E^2\right)$$

$$\Rightarrow \quad dU=\frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 x^2}\right)^2\left(4\pi x^2\right)dx$$

$$\Rightarrow \quad dU=\frac{Q^2}{8\pi\varepsilon_0 x^2}dx$$

Thus total field energy associated with the sphere can be calculated by integrating this expression from surface of sphere to infinity (as electric field inside the sphere is zero).
Total field energy in the surrounding of sphere is

$$U=\int dU=\frac{Q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{1}{x^2}dx$$

$$\Rightarrow \quad U=\frac{Q^2}{8\pi\varepsilon_0}\left[-\frac{1}{x}\Big|_R^\infty\right]$$

$$\Rightarrow \quad U_{R\to\infty}=\frac{Q^2}{8\pi\varepsilon_0 R}$$

ILLUSTRATION 92

A non-conducting sphere of radius R has a charge Q distributed uniformly on it. Using the concept of energy density, find the

(a) energy stored outside the sphere.
(b) energy stored inside the sphere and
(c) total energy stored inside and outside the sphere.

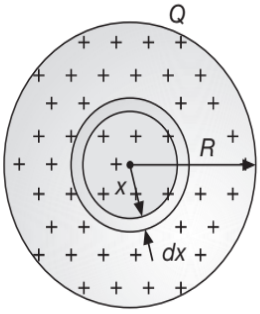
SOLUTION

(a) We know in outside region of a non-conducting uniformly charged sphere, every point is same as that of a conducting sphere of same radius.

Thus field energy in the surrounding of this sphere from surface to infinity is also

$$U_{R\to\infty} = \frac{Q^2}{8\pi\varepsilon_0 R}$$

(b) As in the case of conducting sphere (where $E_{\text{inside}} = 0$), in non-conducting sphere, at interior points $E \neq 0$.
Thus field energy also exists in the interior region. This can be calculated by considering an elemental shell of radius x and thickness dx inside the sphere as shown.



Here field energy in the volume of this elemental shell is

$$dU = \left(\frac{1}{2}\varepsilon_0 E^2\right) d\tau$$

where $d\tau = 4\pi x^2 dx$ and $E = E_{\text{inside}} = \dfrac{Qx}{4\pi\varepsilon_0 R^3}$

$$\Rightarrow \quad dU = \frac{1}{2}\varepsilon_0 \left(\frac{Qx}{4\pi\varepsilon_0 R^3}\right)^2 \left(4\pi x^2\, dx\right)$$

$$\Rightarrow \quad dU = \frac{Q^2}{8\pi\varepsilon_0 R^6} x^4\, dx$$

Total field energy inside the sphere can be given as

$$U = \int dU = \frac{Q^2}{8\pi\varepsilon_0 R^6} \int_0^R x^4 dx$$

$$\Rightarrow \quad U = \frac{Q^2}{8\pi\varepsilon_0 R^6}\left(\frac{R^5}{5}\right)$$

$$\Rightarrow \quad U_{0\to R} = \frac{Q^2}{40\pi\varepsilon_0 R}$$

(c) So, total self-energy of this sphere, also called ***Electrostatic Self Energy*** is

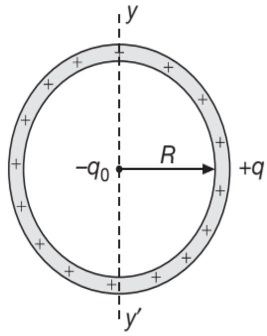
$$U_{\text{self}} = U_{0\to R} + U_{R\to\infty}$$

$$\Rightarrow \quad U_{\text{self}} = \frac{Q^2}{40\pi\varepsilon_0 R} + \frac{Q^2}{8\pi\varepsilon_0 R}$$

$$\Rightarrow \quad U_{\text{self}} = U_{0\to R} + U_{R\to\infty} = \frac{3Q^2}{20\pi\varepsilon_0 R}$$

ILLUSTRATION 93

Consider a thin spherical shell of radius R having charge q distributed uniformly on it. At the centre of shell a negative point charge $-q_0$ is placed. The shell is cut in two identical hemispherical portions along a diametrical section yy' as shown. Due to mutual repulsion the two hemispherical parts tend to move away from each other but due to the attraction of $-q_0$, they may stay in equilibrium. Find the condition of equilibrium of the hemispherical shells. Calculate minimum value of q_0 to attain this equilibrium.



SOLUTION

Let the outward electric pressure at every point of the spherical shell, due to its own charge q be P_1. Then

$$P_1 = \frac{\sigma^2}{2\varepsilon_0}$$

$$\Rightarrow \quad P_1 = \frac{1}{2\varepsilon_0}\left(\frac{q}{4\pi R^2}\right)^2 = \frac{q^2}{32\pi^2\varepsilon_0 R^4} \quad \left\{\because\ \sigma = \frac{q}{4\pi R^2}\right\}$$

Due to charge $-q_0$, the electric field on the surface of shell is

$$E = \frac{q_0}{4\pi\varepsilon_0 R^2} \qquad \text{\{radially inwards\}}$$

This electric field pulls every point of the shell in an inward direction. Let the inward pressure on the surface of shell due to this negative point charge at centre be P_2. Then

$P_2 = \sigma E$ {as done already}

$$\Rightarrow \quad P_2 = \left(\frac{q}{4\pi R^2}\right)\left(\frac{q_0}{4\pi\varepsilon_0 R^2}\right) = \frac{qq_0}{16\pi^2\varepsilon_0 R^4}$$

For equilibrium of hemispherical shell or for the shells not to separate,

$$P_2 \ge P_1$$

$$\Rightarrow \quad \frac{qq_0}{16\pi^2\varepsilon_0 R^4} \ge \frac{q^2}{32\pi^2\varepsilon_0 R^4}$$

$$\Rightarrow \quad q_0 \ge \frac{q}{2}$$

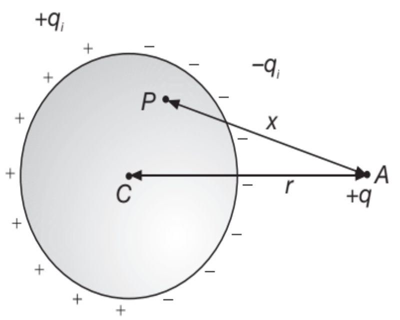
$$\Rightarrow \quad q_{\text{min}} = \frac{q}{2}$$

ELECTROSTATIC FIELD AND POTENTIAL DUE TO INDUCED CHARGES

Electrostatic Field

Consider the situation in which a metal sphere of radius R is placed at a distance r from the point charge $+q$. Consider a point P in the sphere at a distance x from $+q$ where we wish to find the electric field and potential due to the induced charges on the surface of sphere.

The charge $+q$ at A will induce a negative and a positive charge $-q_i$ and $+q_i$ as shown at the nearer and farther end respectively.



Now since P lies inside the metal body,

so $\left(\vec{E}_P\right)_{\text{NET}} = \vec{0}$

i.e., the total field at E due to the point charge $+q$ (at A) and due to the charges induced on the sphere must be zero.

$$\Rightarrow \quad \vec{E}_{\substack{\text{at } P \text{ due to} \\ +q}} + \vec{E}_{\substack{\text{at } P \text{ due to} \\ \text{induced charges}}} = \vec{0}$$

$$\Rightarrow \quad \vec{E}_{\substack{\text{at } P \text{ due to} \\ \text{induced charges}}} = -\vec{E}_{\substack{\text{at } P \text{ due to} \\ +q}}$$

$\Rightarrow \quad \left|E_{\substack{\text{at } P \text{ due to} \\ \text{induced charges}}}\right| = \left|E_{\substack{\text{at } P \text{ due} \\ \text{to } +q}}\right| = \dfrac{q}{4\pi\varepsilon_0 x^2}$ (in magnitude) and $E_{\substack{\text{at } P \text{ due to} \\ \text{induced charges}}}$ will have a direction opposite to the direction of field at P due to $+q$. So, field due to induced charges is $\dfrac{q}{4\pi\varepsilon_0 x^2}$ directed from P to A i.e., towards $+q$.

Electrostatic Potential

Since, inside the sphere at every point $E = 0$ and due to this the entire region is equipotential. Hence the potential at the centre of sphere, will be only due to the charge $+q$, because potential due to induced charges at the centre will be zero (due to symmetrical placement of induced charges about C). So, net potential at the centre of sphere due to q at A is

$$V_C = \frac{q}{4\pi\varepsilon_0 r}$$

Since the entire sphere is equipotential, so potential at point P must be equal to the potential at point C. However, at P the potential due to induced charges will be non-zero because induced charges are not equidistant from point P. Thus net potential at point P will be

$V_P =$ Potential at P due to $+q$ at A + Potential due to induced charges

$$\Rightarrow \quad V_P = \frac{q}{4\pi\varepsilon_0 x} + V_i$$

and since

$$V_P = V_C = \frac{q}{4\pi\varepsilon_0 r}$$

(because the sphere is equipotential inside) Hence

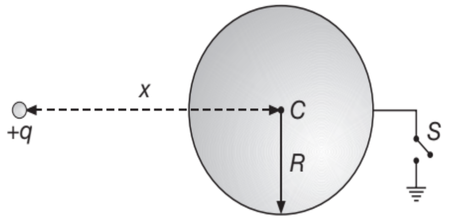
$$V_P = \frac{q}{4\pi\varepsilon_0 r} = \frac{q}{4\pi\varepsilon_0 x} + V_i$$

$$\Rightarrow \quad V_i = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{x}\right)$$

EARTHING OF CHARGED OR UNCHARGED METAL BODIES

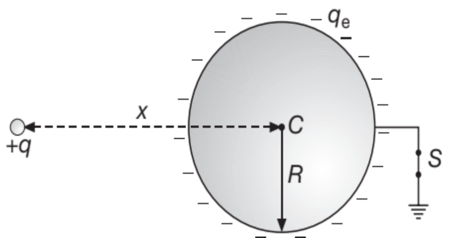
As far as electrical behaviour of the earth is concerned, the earth is assumed to be a very large conducting sphere of radius 6400 km. If some charge Q is given to earth, its potential V_E becomes

$$V_E = \frac{Q}{4\pi\varepsilon_0 R} \qquad \{\text{where } R = \text{Radius of Earth}\}$$



As R is very large V_E becomes a negligible value. Thus for very small bodies whose dimensions are negligible compared to earth we can assume the earth to be at zero potential.

If we connect a small body to earth then charge flow takes place between earth and the body till both acquire the same potential i.e., zero potential. The potential of earth will always remain zero, no matter if charge flows into earth or from earth. This implies that if a body at some positive potential is connected to earth, earth will supply some negative charge to this body so that the final potential of body becomes zero.



Consider a solid uncharged conducting sphere of radius R. A point charge q is placed in front of the sphere at a distance x as shown. Due to q, the potential at sphere is

$$V = \frac{q}{4\pi\varepsilon_0 x}$$

Here we have ignore the charges induced due to q because potential at C due to induced charges on the sphere is zero. If we close the switch S, earth supplies a charge q_E on to the sphere to make its final potential zero. Thus the final potential on sphere can be taken as

$$V = \frac{q}{4\pi\varepsilon_0 x} + \frac{q_E}{4\pi\varepsilon_0 R} = 0$$

$$\Rightarrow \quad q_E = -\frac{qR}{x}$$

Here it is obvious that earth has supplied a negative charge so as to develop a negative potential on sphere to nullify the initial positive potential on it due to q.

Remark(s)

Always remember whenever a metal body is connected to earth, we consider that earth supplies a charge to it (say q_E) to make its final potential zero due to all the charges including the charge on body and the charges in its surrounding.

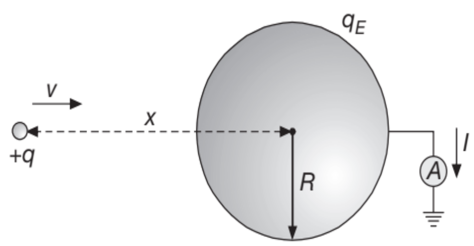
Problem Solving Technique(s)

Earthing a conductor: Potential of earth is often taken to be zero. If a conductor is connected to the earth, the potential of the conductor becomes equal to that of the earth, i.e., zero. If the conductor was at some other potential, charges will flow from it to the earth or from the earth to it to bring its potential to zero.

CURRENT DUE TO MOVEMENT OF CHARGES THROUGH EARTH

Consider a metal ball of radius R connected to earth through an ammeter. Let a charge $+q$ be moving towards the ball at speed v, due to which the potential of ball increases continuously. When the charge is at a distance x from ball, the potential of ball due to this charge is given by

$$V = \frac{q}{4\pi\varepsilon_0 x}$$



Since the metal ball is connected to the earth, so the earth supplies a charge q_E (say) to make the potential of the ball to be zero. This q_E is given by (as done earlier)

$$q_E = -\frac{qR}{x}$$

Since, x is continuously decreasing hence the charge on ball must increase continuously. As a result of this the ammeter shows a current I, which is given by

$$I = \frac{dq_E}{dt} = \frac{d}{dt}\left(-\frac{qR}{x}\right) = \frac{qR}{x^2}\left(\frac{dx}{dt}\right)$$

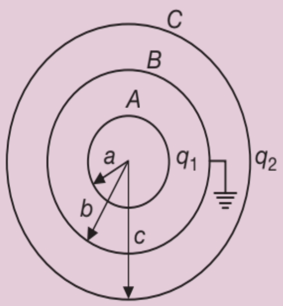
$$\Rightarrow \quad I = \left(\frac{qR}{x^2}\right)v$$

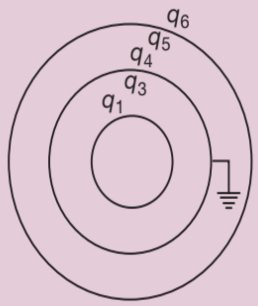
So, in the surrounding of an earthed conducting body whenever one or more charges are in motion, a continuous current flows in the wires connecting body to earth.

Problem Solving Technique(s)

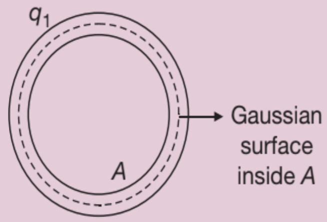
Charges appearing on different surfaces of concentric spherical conducting shells:

Figure shows three concentric thin spherical conducting shells *A*, *B* and *C* of radii *a*, *b* and *c*. The shells *A* and *C* are given charges q_1 and q_2 and the shell *B* is earthed. We are interested in finding the charges on inner and outer surfaces of *A*, *B* and *C*. To solve such type of problems following points should keep in mind:





(a) The entire charge q_1 will come on the outer surface of *A* unless some charge is kept inside *A*.



To understand this, let us consider a Gaussian surface (a sphere) through the material of *A*. As the electric field in a conducting material is zero. The flux through this Gaussian surface is zero. Using Gauss's Law, the total charge enclosed must be zero.

(b) Similarly if we draw a Gaussian surface through the material of *B*, then we have

$$q_3 + q_1 = 0 \Rightarrow q_3 = -q_1$$

and if we draw a Gaussian surface through the material of *C*, then

$$q_5 + q_4 + q_3 + q_1 = 0 \Rightarrow q_5 = -q_4$$

(c) Since q_2 charge was given to shell C, so $q_5 + q_6 = q_2$

(d) Since B is earthed, so potential of B must be zero. Hence

$$V_B = 0$$

$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{b} + \frac{q_3 + q_4}{b} + \frac{q_5 + q_6}{c}\right) = 0$$

Using the above conditions we can find charges on different surfaces, we can summarise the above points as under

(i) Net charge inside a closed Gaussian surface drawn in any **conducting** shell is zero.

(ii) Potential of the conductor which is earthed is zero.

(iii) If two conductors are connected, they are at same potential.

(iv) Charge remains constant on all conductors except those which are earthed.

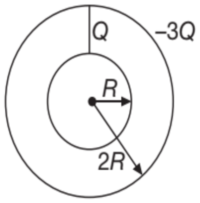
(v) Charge on the inner surface of the innermost shell is zero **provided no charge is kept inside it**. In all other shells charge resides on both the surfaces.

(vi) Equal and opposite charges appear on surfaces facing each other.

Based on this, try to understand the following ILLUSTRATIONS.

ILLUSTRATION 94

The two conducting spherical shells as shown in figure are joined by a conducting wire. After a long time, the wire is cut and the charge stops flowing. Calculate the final charge on each sphere after that.



SOLUTION

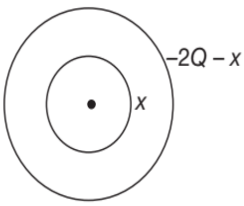
Since the shells had been connected by a conducting wire for a long time, so after the conducting wire is removed then both shells must be at the same potential. If x be the charge that has flown from inner shell to outer shell, then

Potential of inner shell is

$$V_{in} = \frac{Kx}{R} + \frac{K(-2Q-x)}{2R} = \frac{k(x-2Q)}{2R}$$

and potential of outer shell is

$$V_{out} = \frac{Kx}{2R} + \frac{K(-2Q-x)}{2R} = \frac{-KQ}{R}$$



Since $V_{out} = V_{in}$

$$\Rightarrow \quad \frac{-KQ}{R} = \frac{K(x-2Q)}{2R}$$

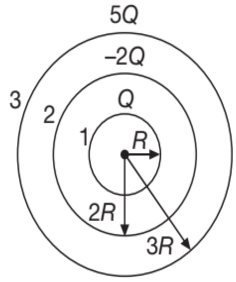
$$\Rightarrow \quad -2Q = x - 2Q$$

$$\Rightarrow \quad x = 0$$

So final charge on inner spherical shell is zero and on outer spherical shell is $-2Q$, i.e. entire charge on inner shell flows to the outer shell.

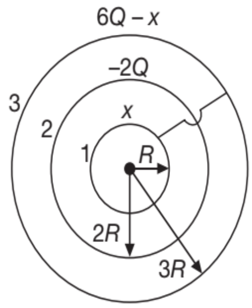
ILLUSTRATION 95

Calculate the final charge on each spherical shell after joining the inner most shell and outer most shell by a conducting wire.



SOLUTION

Let the charge on the innermost sphere be x. Then finally,



Potential of shell 1 = Potential of shell 3

$$\frac{Kx}{R} + \frac{K(-2Q)}{2R} + \frac{K(6Q-x)}{3R} = \frac{Kx}{3R} + \frac{K(-2Q)}{3R} + \frac{K(6Q-x)}{3R}$$

$$\Rightarrow \quad 3x - 3Q + 6Q - x = 4Q$$

$$\Rightarrow \quad 2x = Q$$

$$\Rightarrow \quad x = \frac{Q}{2}$$

Charge on innermost shell $= \dfrac{Q}{2}$

Charge on outermost shell $= \dfrac{5Q}{2}$

Charge on middle shell $= -2Q$

Final charge distribution is as shown in figure.

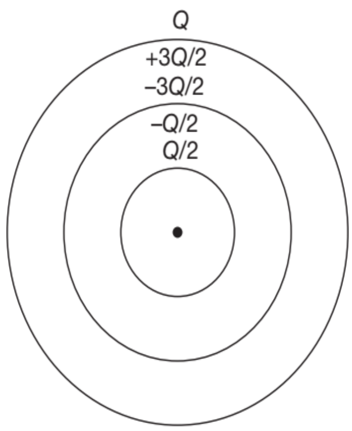
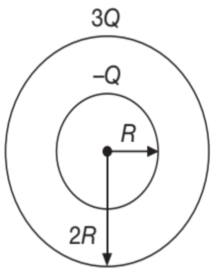


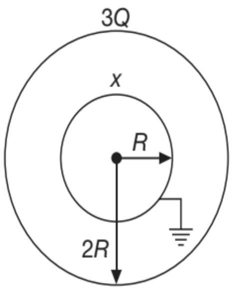
ILLUSTRATION 96

Two conducting hollow spherical shells of radii R and $2R$ carry charges $-Q$ and $3Q$ respectively. How much charge will flow into the earth if inner shell is grounded?



SOLUTION

When inner shell is grounded, then the potential of inner shell becomes zero.



$$\Rightarrow \quad \frac{Kx}{R}+\frac{K(3Q)}{2R}=0$$

$$\Rightarrow \quad x=\frac{-3Q}{2}$$

Since the charge has increased by

$$\Delta q=\frac{-3Q}{2}-(-Q)=\frac{-Q}{2}$$

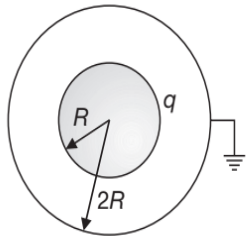
Hence charge flown into the Earth is $\frac{Q}{2}$

ILLUSTRATION 97

A charge q is distributed uniformly on the surface of a sphere of radius R. It is covered by an earthed concentric hollow conducting sphere of radius $2R$. Find the charges on inner and outer surfaces of hollow sphere when

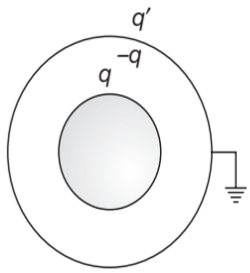
(a) thickness of shell is negligible

(b) the shell has considerable thickness.



SOLUTION

(a) The charge on the inner surface should be $-q$, because if we draw a closed Gaussian surface through the material of the hollow sphere the total charge enclosed by this Gaussian surface should be zero. Let q' be the charge on the outer surface of the hollow sphere.



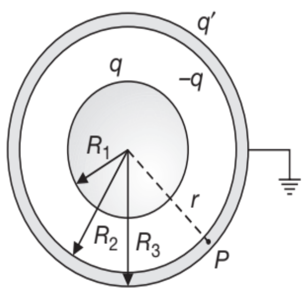
Since, the outer sphere is earthed, its potential should be zero. The potential on it is due to the charges q, $-q$ and q', Hence,

$$V=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{2R}-\frac{q}{2R}+\frac{q'}{2R}\right)=0$$

$$\Rightarrow \quad q'=0$$

Therefore, there will be no charge on the outer surface of the hollow sphere.

(b)



In this case, we can set $V=0$ at any point on the hollow sphere. Let us select a point P a distance r from the centre. Where $R_2<r<R_3$. So,

$$V_P=0$$

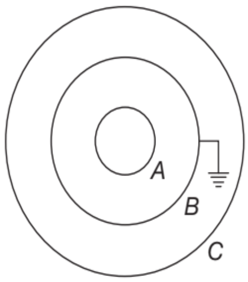
$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}-\frac{q}{r}+\frac{q'}{R_3}\right)=0$$

$$\Rightarrow \quad q'=0$$

i.e., in this case also there will be no charge on the outer surface of the hollow sphere.

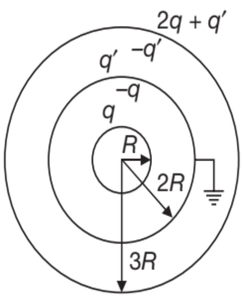
ILLUSTRATION 98

Figure shows three concentric thin spherical shells A, B and C of radii R, $2R$ and $3R$. The shell B is earthed and A and C are given charges q and $2q$ respectively. Find the charges appearing on all the surfaces of A, B and C.



SOLUTION

Since there is no charge inside A. The whole charge q given to the shell A will appear on its outer surface. Charge on its inner surface will be zero. Moreover if a Gaussian surface is drawn on the material of shell B net charge enclosed by it should be zero. Therefore, charge on its inner surface will be $-q$. Now let q' be the charge on its outer surface, then charge on the inner surface of C will be $-q'$ and on its outer surface will be, $2q-(-q')=2q+q'$ as total charge on C is $2q$.



Shell B is earthed. Hence, its potential should be zero.

$$V_B=0$$

$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{2R}-\frac{q}{2R}+\frac{q'}{2R}-\frac{q'}{3R}+\frac{2q+q'}{3R}\right)=0$$

Solving this equation, we get

$$q'=-\frac{4}{3}q$$

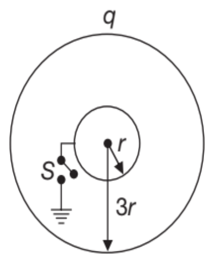
$$\Rightarrow \quad 2q+q'=2q-\frac{4}{3}q=\frac{2}{3}q$$

Therefore, charges on different surfaces in tabular form are given below:

A		B		C	
INNER	OUTER	INNER	OUTER	INNER	OUTER
0	q	$-q$	$-\frac{4}{3}q$	$\frac{4}{3}q$	$\frac{2}{3}q$

ILLUSTRATION 99

Figure shows two conducting thin concentric shells of radii r and $3r$. The outer shell carries charge q. Inner shell is neutral. Find the charge that will flow from inner shell to earth after the switch S is closed.



SOLUTION

Let q' be the charge on inner shell when it is earthed.

$$V_{\text{inner}}=0$$

$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{r}+\frac{q}{3r}\right)=0$$

$$\Rightarrow \quad q'=-\frac{q}{3}$$

i.e., $+\frac{q}{3}$ charge will flow from inner shell to earth.

FIELD ENERGY DENSITY OF ELECTRIC FIELD

An energy is associated with every region where electric field is present. This energy is called the field energy of the electric field. Using the concept of field energy we can calculate the amount of energy stored in the space where electric field exists. To calculate the Field Energy Density (FED) we have two cases to be taken into account.

CASE-1:

Field Energy Density of a charged body, also called **Self Energy**.

"Self energy of a charged body is the total field energy, associated with the electric field due to the body itself in its surrounding".

CASE-2:

Field Energy Density of a system of charged bodies.

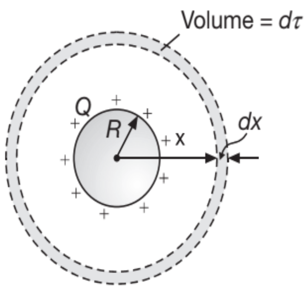
Here, we have to take Field Energy Density as the total of self energies associated with the bodies and the interaction energies of the pairs of charged bodies.

$$\Rightarrow\ U = \left(\sum \begin{matrix}\text{self energy}\\ \text{of all charged}\\ \text{bodies}\end{matrix}\right) + \left(\sum \begin{matrix}\text{interaction}\\ \text{energy of all}\\ \text{pairs of}\\ \text{charged bodies}\end{matrix}\right)$$

ELECTROSTATIC SELF ENERGY OF A CHARGED CONDUCTING SPHERE

We have discussed that in space wherever electric field exist, there must be some field energy stored which has energy density, given as

$$U = \frac{1}{2}\varepsilon_0 E^2$$



Here we can see that when the sphere was uncharged, there was no electric field in its surroundings. But when the sphere becomes fully charged, electric field will exist in its surrounding from its surface to infinity. Let us calculate the field energy associated with this charged conducting sphere.

We know electric field due to a sphere at outer points varies with distance from centre as

$$E = \frac{Q}{4\pi\varepsilon_0 x^2}$$

To find the total field energy due to this sphere, we consider an elemental spherical shell of radius x and thickness dx as shown. The volume enclosed in this shell is

$$d\tau = \left(4\pi x^2\right)dx$$

Thus the field energy stored in the volume of this elemental shell is

$$dU = \left(\frac{1}{2}\varepsilon_0 E^2\right)d\tau$$

$$\Rightarrow\quad dU = \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 x^2}\right)^2\left(4\pi x^2\right)dx$$

$$\Rightarrow\quad dU = \frac{Q^2}{8\pi\varepsilon_0 x^2}dx$$

Thus total field energy associated with the sphere can be calculated by integrating this expression from surface of sphere to infinity (as electric field inside the sphere is zero).

Total field energy in the surrounding of sphere is

$$U = \int dU = \frac{Q^2}{8\pi\varepsilon_0}\int_R^{\infty}\frac{1}{x^2}dx$$

$$\Rightarrow\quad U = \frac{Q^2}{8\pi\varepsilon_0}\left[-\frac{1}{x}\bigg|_R^{\infty}\right]$$

$$\Rightarrow\quad U_{R\to\infty} = \frac{Q^2}{8\pi\varepsilon_0 R} \qquad \ldots(1)$$

Alternatively, whenever a system of charges is assembled, some work is done and this work is stored in the form of electrical potential energy of the system. Let consider an example of charging a conducting sphere of radius R.

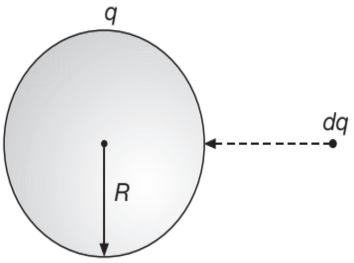
In the process of charging we bring charge to the sphere from infinity in steps of elemental charges dq. The charge on sphere opposes the elemental charge being brought to it. Let us assume that at an instant sphere has a charge q, due to which it has a potential given as

$$V = \frac{q}{4\pi\varepsilon_0 R}$$

If now a charge dq is brought to its surface from infinity, work done in this process can be given as

$$dU = Vdq$$

$$\Rightarrow \quad dU = \left(\frac{q}{4\pi\varepsilon_0 R}\right)dq$$



Total work done in charging the sphere is

$$\Rightarrow \quad U = \int dU = \frac{1}{4\pi\varepsilon_0 R}\int_0^Q qdq$$

$$\Rightarrow \quad U_{R\to\infty} = \frac{Q^2}{8\pi\varepsilon_0 R} \qquad \ldots(2)$$

Equation (1) gives the total work done in charging the sphere of radius R.
Here we can see that again result in equation (2) is same as that in equation (1)

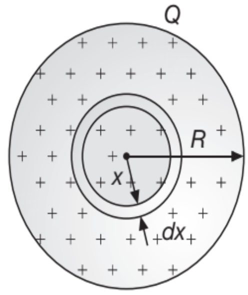
We can conclude by this that whatever work is done in charging a body is stored in its surrounding in the form of its field energy and can be regarded as self energy of that body. Once a body is charged in a given configuration, its self energy is fixed, if the body is now displaced or moved in any manner keeping its shape and charge distribution constant, its self energy does not charge.

ELECTROSTATIC SELF ENERGY OF A UNIFORMLY CHARGED NON-CONDUCTING SPHERE

METHOD 1:

We know in outside region of a non-conducting uniformly charged sphere, every point is same as that of a conducting sphere of same radius. Thus field energy in the surrounding of this sphere from surface to infinity is also

$$U_{R\to\infty} = \frac{Q^2}{8\pi\varepsilon_0 R}$$



As in the case of conducting sphere (where $E_{\text{inside}} = 0$), in non-conducting sphere, at interior points $E \neq 0$. Thus field energy also exists in the interior region. This can be calculated by considering an elemental shell of radius x and thickness dx inside the sphere as shown.
Here field energy in the volume of this elemental shell is

$$dU = \left(\frac{1}{2}\varepsilon_0 E^2\right)dV$$

where $E = E_{\text{inside}} = \dfrac{Qx}{4\pi\varepsilon_0 R^3}$

$$\Rightarrow \quad dU = \frac{1}{2}\varepsilon_0\left(\frac{Qx}{4\pi\varepsilon_0 R^3}\right)^2\left(4\pi x^2\,dx\right)$$

$$\Rightarrow \quad dU = \frac{Q^2}{8\pi\varepsilon_0 R^6}x^4\,dx$$

Total field energy inside the sphere can be given as

$$U = \int dU = \frac{Q^2}{8\pi\varepsilon_0 R^6}\int_0^R x^4 dx$$

$$\Rightarrow \quad U = \frac{Q^2}{8\pi\varepsilon_0 R^6}\left(\frac{R^5}{5}\right)$$

$$\Rightarrow \quad U_{0\to R} = \frac{Q^2}{40\pi\varepsilon_0 R}$$

So, total self energy of this sphere is

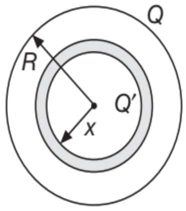
$$U_{\text{self}} = U_{0\to R} + U_{R\to\infty}$$

$$\Rightarrow \quad U_{\text{self}} = \frac{Q^2}{40\pi\varepsilon_0 R} + \frac{Q^2}{8\pi\varepsilon_0 R}$$

$$\Rightarrow \quad U_{\text{self}} = U_{0\to R} + U_{R\to\infty} = \frac{3Q^2}{20\pi\varepsilon_0 R}$$

METHOD 2:

Let us assemble the sphere from infinitesimal concentric shells.



Now to add an infinitesimal spherical element of radius x, thickness dx having charge dq on a charged spherical core of radius x having charge q', the work that has to be done is given by

$$dU = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q'dq}{x}\right)$$

where $Q' = \left(\frac{4}{3}\pi x^3\right)\rho$, $dq = \left(4\pi x^2\,dx\right)\rho$ and $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$

$$\Rightarrow \quad dU = \frac{1}{4\pi\varepsilon_0}\frac{16\pi^2\rho^2}{3}x^4 dx$$

$$\Rightarrow \quad U = \frac{1}{4\pi\varepsilon_0}\frac{16\pi^2\rho^2}{3}\int_0^R x^4 dx$$

$$\Rightarrow \quad U = \frac{1}{4\pi\varepsilon_0}\left(\frac{16\pi^2}{3}\right)\left(\frac{Q^2}{\frac{16\pi^2}{9}R^6}\right)\left(\frac{R^5}{5}\right)$$

$$\Rightarrow \quad U = \frac{3Q^2}{20\pi\varepsilon_0 R}$$

This work done in assembling the system is stored in the form of electrostatic self energy.

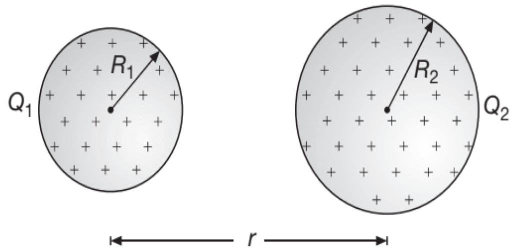
TOTAL ELECTROSTATIC ENERGY OF A SYSTEM OF CHARGED BODIES

Total electrostatic potential energy a system of charges will be given as

U = Total of self energy of all charged bodies + Total of interaction energy of all pairs of charged bodies

$$\Rightarrow \quad U = \begin{pmatrix}\text{Total self}\\ \text{energy of}\\ \text{all charged}\\ \text{bodies}\end{pmatrix} + \begin{pmatrix}\text{Total interaction}\\ \text{energy of all}\\ \text{pairs of charged}\\ \text{bodies}\end{pmatrix}$$

Let us understand this concept by taking this simple example. Figure shows uniformly charged non-conducting sphere of radius R_1 having charge Q_1 and a charged conducting sphere of radius R_2 having charge Q_2 respectively separated by a distance r.



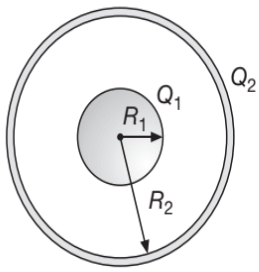
The total electrostatic energy of this system, will be

$$U = U_{\text{self}} + U_{\text{interaction}}$$

$$\Rightarrow \quad U = \frac{3Q_1^2}{20\pi\varepsilon_0 R_1} + \frac{Q_2^2}{8\pi\varepsilon_0 R_2} + \frac{Q_1 Q_2}{4\pi\varepsilon_0 r}$$

ELECTROSTATIC ENERGY OF A SYSTEM OF CONCENTRIC SHELLS

Consider two concentric shells of radii R_1 and R_2 having uniform charges Q_1 and Q_2.



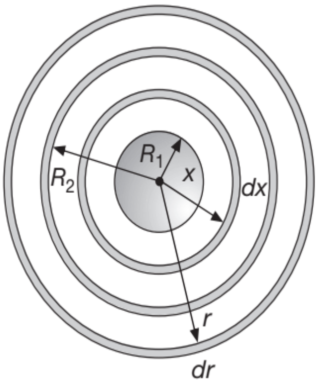
Here the total energy of this system will be

$$U_{\text{total}} = \begin{pmatrix}\text{self energy}\\ \text{of inner}\\ \text{shell}\end{pmatrix} + \begin{pmatrix}\text{self energy}\\ \text{of outer}\\ \text{shell}\end{pmatrix} + \begin{pmatrix}\text{interaction}\\ \text{energy of}\\ \text{the two}\\ \text{shells}\end{pmatrix}$$

$$\Rightarrow \quad U_{\text{total}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1^2}{2R_1} + \frac{Q_2^2}{2R_2} + \frac{Q_1 Q_2}{R_2}\right)$$

The above result can also be obtained by following method.

Since, the total electrostatic energy of a system is stored in the form of field energy of the system hence here we can calculate the total electrostatic energy of the system by integrating the field energy density in the space surrounding the shells where the electric field exists.



Total field energy in the electric field associated with the system shown in figure can be given as

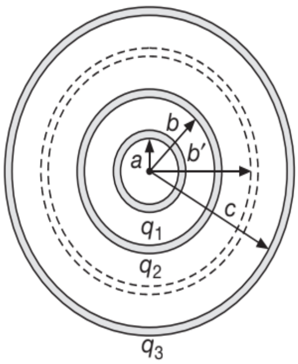
$$U=\int_{R_1}^{R_2}\frac{1}{2}\varepsilon_0\left(\frac{Q_1}{4\pi\varepsilon_0 x^2}\right)^2 4\pi x^2 dx+\int_{R_2}^{\infty}\frac{1}{2}\varepsilon_0\left(\frac{(Q_1+Q_2)}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$U=\frac{Q_1^2}{8\pi\varepsilon_0}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{(Q_1+Q_2)^2}{8\pi\varepsilon_0}\left(\frac{1}{R_2}\right)$$

$$\Rightarrow\quad U=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1^2}{2R_1}+\frac{Q_2^2}{2R_2}+\frac{Q_1Q_2}{R_2}\right)$$

ILLUSTRATION 100

Three shells are shown carrying charge q_1, q_2 and q_3 and of radii a, b and c respectively. If the middle shell expands from radius b to $b'(b'<c)$. Find the work done by electric field in process.



SOLUTION

Since work done by a conservative field equals the decrease in potential energy, so we have

$$\Rightarrow\quad\begin{pmatrix}\text{Workdone}\\ \text{by}\\ \text{field}\end{pmatrix}=\begin{pmatrix}\text{Total initial}\\ \text{energy}\\ \text{of system}\end{pmatrix}-\begin{pmatrix}\text{Total final}\\ \text{energy of}\\ \text{system}\end{pmatrix}$$

Now, total initial energy of system is

$$U_i=SE_1+SE_2+SE_3+IE_{12}+IE_{13}+IE_{23}$$

(SE denotes self energy and IE denotes interaction energy)

$$\Rightarrow\quad U_i=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1^2}{2a}+\frac{q_2^2}{2b}+\frac{q_3^2}{2c}+\frac{q_1q_3}{c}+\frac{q_1q_2}{b}+\frac{q_2q_3}{c}\right)$$

Total final energy of system

$$U_f=SE_1+SE_2+SE_3+IE_{12}+IE_{13}+IE_{23}$$

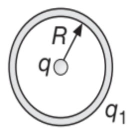
$$\Rightarrow\quad U_f=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1^2}{2a}+\frac{q_2^2}{2b'}+\frac{q_3^2}{2c}+\frac{q_1q_3}{c}+\frac{q_1q_2}{b'}+\frac{q_2q_3}{c}\right)$$

Work done by field $=U_i-U_f$

$$\Rightarrow\quad W=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_2^2}{2b}-\frac{q_2^2}{2b'}+\frac{q_1q_2}{b}-\frac{q_1q_2}{b'}\right)$$

ILLUSTRATION 101

Figure shows a shell of radius R having charge q_1 uniformly distributed over it.

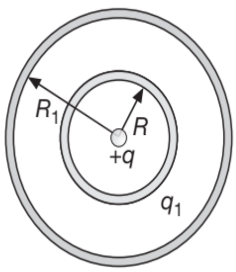


A point charge q is placed at the centre of shell. Find work required to increase radius of shell from R to $R_1(>R)$.

SOLUTION

Work done $=$ Decrease in Potential Energy

$$\Rightarrow\quad \text{Work}=U_i-U_f$$



Now $U_i = (SE)_q + (SE)_{q_1} + (IE)$

$$\Rightarrow \quad U_i = (SE)_q + \frac{q_1^2}{8\pi\varepsilon_0 R} + \frac{q_1 q}{4\pi\varepsilon_0 R}$$

and $U_f = (SE)_q + \dfrac{q_1^2}{8\pi\varepsilon_0 R_1} + \dfrac{q_1 q}{4\pi\varepsilon_0 R_1}$

$\Rightarrow$ Work done $= U_i - U_f$

$$\Rightarrow \quad W = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1^2}{2R} + \frac{q_1 q}{R} - \frac{q_1^2}{2R_1} - \frac{q_1 q}{R_1}\right)$$

ILLUSTRATION 102

Find the electrostatic energy stored in a cylindrical shell of length l, inner radius a and outer radius b, coaxial with a uniformly charged wire with linear charge density λ.

SOLUTION

For this we consider an elemental shell of radius x and width dx. The volume of this shell dV can be given as

$$dV = (2\pi x l)dx$$

The electric field due to the wire at the shell is

$$E = \frac{\lambda}{2\pi\varepsilon_0 x}$$

The electrostatic field energy stored in the volume of this shell is

$$dU = \left(\frac{1}{2}\varepsilon_0 E^2\right)dV$$

$$\Rightarrow \quad dU = \frac{1}{2}\varepsilon_0\left(\frac{\lambda}{2\pi\varepsilon_0 x}\right)^2(2\pi x l)dx$$

The total electrostatic energy stored in the above mentioned volume can be obtained by integrating the above expression within limits from a to b as

$$U = \int dU = \int_a^b \frac{1}{2}\varepsilon_0\left(\frac{\lambda}{2\pi\varepsilon_0 x}\right)^2(2\pi x l)dx$$

$$\Rightarrow \quad U = \frac{\lambda^2 l}{4\pi\varepsilon_0}\int_a^b \frac{1}{x}dx$$

$$\Rightarrow \quad U = \frac{\lambda^2 l}{4\pi\varepsilon_0}\left(\log_e x\Big|_a^b\right)$$

$$\Rightarrow \quad U = \frac{\lambda^2 l}{4\pi\varepsilon_0}\log_e\left(\frac{b}{a}\right)$$

Test Your Concepts-XIII

Based on Concept of Self and Interaction Energy

(Solutions on page H.39)

1. A spherical shell of radius R_1 with uniform charge q is expanded to a radius R_2. Calculate the work performed by the electric forces in this process.
2. A spherical shell of radius R_1 with a uniform charge q has a point charge q_0 at its centre. Find the work performed by the electric forces during the shell expansion from radius R_1 to radius R_2.
3. A spherical shell is uniformly charged with the surface density σ. Using the Energy Conservation Law, find the magnitude of the electric force acting on a unit area of the shell.
4. Calculate the work that must be done to charge a spherical shell of radius R to a total charge Q.

SOLVED PROBLEMS

PROBLEM 1

A rectangular tank of mass m_0 and charge Q over it is placed over a smooth horizontal floor. A horizontal electric field E exist in the region. Rain drops are falling vertically in the tank at the constant rate of n drops per second. Mass of each drop is m. Find the time taken by tank to reach to half the maximum speed.

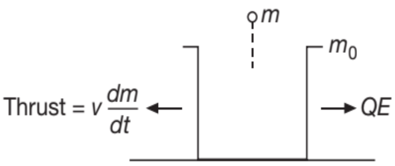
SOLUTION

Applying Newton's Second Law, we get

$$QE - v\frac{dm}{dt} = M\frac{dv}{dt}$$

$$\Rightarrow \quad QE - v(nm) = (m_0 + nmt)\frac{dv}{dt} \qquad \ldots(1)$$

When v becomes $v_{\max}$, then $\frac{dv}{dt} = 0$



$$\Rightarrow \quad QE = v_{\max}(nm)$$

$$\Rightarrow \quad v_{\max} = v_0 = \frac{QE}{nm}$$

From (1), we get

$$\frac{QE - nmv}{m_0 + nmt} = \left(\frac{dv}{dt}\right)$$

$$\Rightarrow \quad \int_0^t \frac{dt}{m_0 + mnt} = \int_0^{\frac{v_0}{2}} \frac{dv}{QE - nmv}$$

$$\Rightarrow \quad \frac{1}{mn}\log\left(\frac{m_0 + nmt}{m_0}\right) = -\frac{1}{nm}\log\left(\frac{QE - \frac{mnv}{2}}{QE}\right)$$

$$\Rightarrow \quad \frac{1 + nmt}{m_0} = \frac{QE}{QE - mn\left(\frac{v}{2}\right)}$$

$$\Rightarrow \quad \frac{m_0 = nmt}{m_0} = \frac{2QE}{2QE - mn\left(\frac{v_0}{2}\right)}$$

$$\Rightarrow \quad \frac{m_0 + nmt}{m_0} = \frac{2QE}{2QE - mn\left(\frac{QE}{mn}\right)} \qquad \left\{\because\ v_0 = \frac{QE}{nm}\right\}$$

$$\Rightarrow \quad \frac{m_0 + nmt}{m_0} = \frac{2QE}{QE}$$

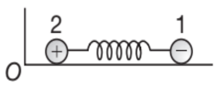
$$\Rightarrow \quad 1 + \frac{nmt}{m_0} = 2$$

$$\Rightarrow \quad \left(\frac{nm}{m_0}\right)t = 1$$

$$\Rightarrow \quad t = \frac{m_0}{nm}$$

PROBLEM 2

Two small identical balls lying on a horizontal plane are connected by a weightless spring. One ball is fixed at the origin and the other is free. The balls are charged identically as a result of which the spring length increases 'n' times. Determine the ratio of new frequency (when balls are charged) to old frequency (when balls were uncharged) when the free ball is displaced slightly from its mean position.



SOLUTION

When the balls are uncharged, $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where k is force constant of the spring and m = mass of the oscillating ball (ball 1). When charged we have in the equilibrium position of ball 1,

$$\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(\eta l)^2} = k(\eta l - l) = kl(\eta - 1)$$

$$\Rightarrow \quad l^3 = \frac{q^2}{4\pi\varepsilon_0\eta^2(\eta - 1)k}$$

When the ball 1 is displaced by a small distance from the equilibrium position to the right, the unbalanced force to the right is given by

Resultant force to the right is

$$F=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(\eta l+x)^2}-k(\eta l+x-l)$$

From Newton's Law, we have

$$m\frac{d^2x}{dt^2}=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2l^2}\left(1+\frac{x}{\eta l}\right)^{-2}-kl(\eta-1)-kx$$

$$\Rightarrow\quad m\frac{d^2x}{dt^2}=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2l^2}\left(1-\frac{2x}{\eta l}\right)-kl(\eta-1)-kx$$

$$\Rightarrow\quad m\frac{d^2x}{dt^2}=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2l^2}-\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\eta^2l^2}\frac{2x}{\eta l}-kl(\eta-1)-kx$$

$$\Rightarrow\quad m\frac{d^2x}{dt^2}=-\left(\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{\eta^3l^3}+k\right)x$$

$$\Rightarrow\quad m\frac{d^2x}{dt^2}=-\left(\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{\eta^3\dfrac{q^2}{4\pi\varepsilon_0\eta^2(\eta-1)k}}+k\right)x$$

$$\Rightarrow\quad m\frac{d^2x}{dt^2}=-\left(\frac{2(\eta-1)}{\eta}k+k\right)x=-\frac{3\eta-2}{\eta}kx$$

$$\Rightarrow\quad \frac{d^2x}{dt^2}=-\frac{3\eta-2}{\eta}\frac{k}{m}x$$

$$\Rightarrow\quad \omega^2=\frac{3\eta-2}{\eta}\frac{k}{m}$$

$$\Rightarrow\quad f=\frac{1}{2\pi}\sqrt{\frac{3\eta-2}{\eta}\frac{k}{m}}$$

$$\Rightarrow\quad \frac{f}{f_0}=\sqrt{\frac{3\eta-2}{\eta}}$$

Thus the frequency is increased $\sqrt{\frac{3\eta-2}{\eta}}$ times

Here $\eta=2$ and so frequency increases $\sqrt{2}$ times.

PROBLEM 3

A point charge q is located at the centre of a thin ring of radius R with uniformly distributed charge $-q$. Find the magnitude of the electric field strength vector at the point lying on the axis of the ring at a distance x from its centre if $x \gg R$.

SOLUTION

Electric field at P due to ring

$$E_1=\frac{1}{4\pi\varepsilon_0}\frac{qx}{(x^2+R^2)^{3/2}}$$

{charged toward centre as it is negatively}

Electric field at P due to $+q$

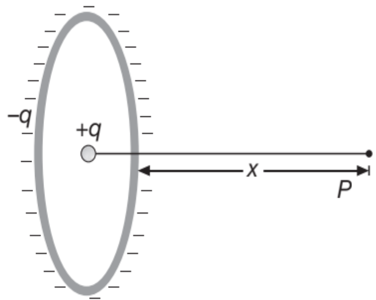
$$E_2=\frac{q}{4\pi\varepsilon_0x^2} \qquad \text{\{away from centre\}}$$

Thus net field at P is

$$\vec{E}_{\text{net}}=\vec{E}_2+\vec{E}_1 \qquad \text{\{vectorially\}}$$

$$\Rightarrow\quad E_{\text{net}}=E_2-E_1$$

$$\Rightarrow\quad E_{\text{net}}=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x^2}-\frac{x}{(x^2+R^2)^{3/2}}\right)$$



For $x \gg R$,

$$E_{\text{net}}=\frac{q}{4\pi\varepsilon_0x^2}\left[1-\left(1+\frac{R^2}{x^2}\right)^{-\frac{3}{2}}\right]$$

$$\Rightarrow\quad E_{\text{net}}=\frac{q}{4\pi\varepsilon_0x^2}\left[1-\left(1-\frac{3}{2}\frac{R^2}{x^2}\right)\right]$$

{using Binomial Approximation}

$$\Rightarrow\quad F_{\text{net}}=\frac{3qR^2}{8\pi\varepsilon_0x^4}$$

PROBLEM 4

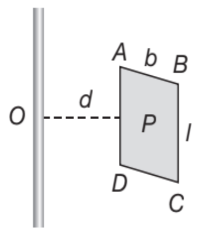
Find the electric flux crossing the wire frame $ABCD$ of length l width b and whose centre is at a distance $OP=d$ from an infinite line of charge with linear charge density λ. Consider that the plane of frame is perpendicular to the line OP.

SOLUTION

$$\phi_e = 2\int_0^{b/2} d\phi_e = 2\int_0^{b/2}\left(\frac{\lambda}{2\pi\varepsilon_0 r}\right)(l\,dx)\cos\theta$$

where, $r = \sqrt{d^2 + x^2}$

and $\cos\theta = \frac{d}{r} = \frac{d}{\sqrt{d^2+x^2}}$



$$\Rightarrow \quad \phi = 2\int_0^{b/2}\left(\frac{\lambda l d}{2\pi\varepsilon_0}\right)\frac{dx}{d^2+x^2}$$

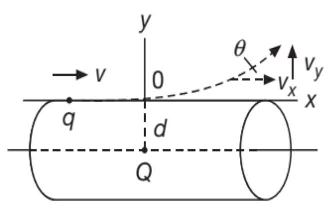
$$\Rightarrow \quad \phi = \frac{\lambda l}{\pi\varepsilon_0}\tan^{-1}\left(\frac{b}{2d}\right)$$

PROBLEM 5

A particle of mass m and charge q moves at high speed along the x axis. It is initially near $x \to -\infty$, and it ends up near $x \to +\infty$. A second charge Q is fixed at the point $x = 0$, $y = -d$. As the moving charge passes the stationary charge, its x component of velocity does not change appreciably, but it acquires a small velocity in the y direction. Determine the angle through which the moving charge is deflected.

SOLUTION

The vertical velocity component of the moving charge increases according to



$$m\frac{dv_y}{dt} = F_y$$

$$\Rightarrow \quad m\frac{dv_y}{dx}\frac{dx}{dt} = qE_y$$

Now $\frac{dx}{dt} = v_x$ has the nearly constant value v

$$\Rightarrow \quad dv_y = \frac{q}{mv}E_y dx$$

$$\Rightarrow \quad v_y = \int_0^{v_y} dv_y = \frac{q}{mv}\int_{-\infty}^{\infty} E_y dx \qquad \ldots(1)$$

The radially outward component of the electric field varies along the x-axis, and is described by using Gauss's Theorem

$$\int_{-\infty}^{\infty} E_y dA = \int_{-\infty}^{\infty} E_y (2\pi d)dx = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} E_y dx = \frac{Q}{2\pi d\varepsilon_0}$$

and hence from equation (1), we get

$$v_y = \frac{qQ}{mv2\pi d\varepsilon_0}$$

The angle of deflection is described by

$$\tan\theta = \frac{v_y}{v} = \frac{qQ}{2\pi\varepsilon_0 dmv^2}$$

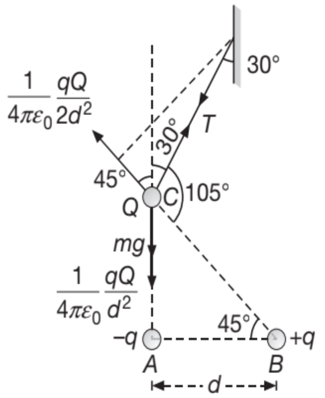
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{qQ}{2\pi\varepsilon_0 dmv^2}\right)$$

PROBLEM 6

Two small balls A and B with charges $-q$ and $+q$ respectively are fixed on a horizontal plane at a distance d from each other. A third ball C with charge $+Q$ is suspended from a string. The string makes an angle of $30°$ with the vertical when the ball C is in equilibrium at a height d vertically above the ball A. When the ball C is in an identical situation above ball B, find the angle which the string now makes with the vertical.

SOLUTION

Since ball C is in equilibrium, the sum of torques of all the forces about the point of suspension must be zero.



$$\frac{KqQ}{2d^2}l\sin 75^\circ=\left(\frac{KqQ}{d^2}+mg\right)l\sin 30^\circ \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{KqQ}{2d^2}\cos 15^\circ-\frac{KqQ}{2d^2}=\frac{mg}{2}$$

$$\Rightarrow \quad Q=\frac{mg}{\frac{Kq}{2d^2}(\cos 15^\circ-1)\times 2}$$

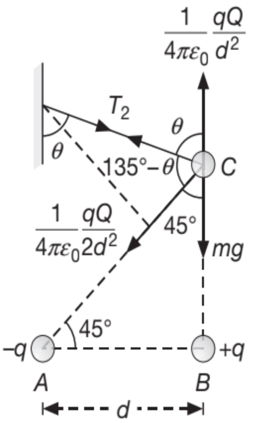
$$\Rightarrow \quad Q=\frac{mgd^2}{Kq(\cos 15^\circ-1)} \qquad \ldots(2)$$

Similarly, in second case,

$$\left(\frac{KqQ}{d^2}-mg\right)l\sin\theta=\frac{KqQ}{2d^2}l\sin(135^\circ-\theta)$$

Since $\sin(135^\circ-\theta)=\cos(45^\circ-\theta)$

$$\Rightarrow \quad \left(\frac{KqQ}{d^2}-mg\right)l\sin\theta=\frac{KqQ}{2d^2}l\cos(45^\circ-\theta) \qquad \ldots(3)$$



Substituting the value of charge Q from equation (2) in (3), we get

$$\left(\frac{mg}{(\cos 15^o-1)}-mg\right)\sin\theta=\frac{mg\cos(45^\circ-\theta)}{2(\cos 15^\circ-1)}$$

$$\Rightarrow \quad \left(\frac{1-(\cos 15^\circ-1)}{(\cos 15^\circ-1)}\right)=\frac{\cos 45^\circ\cos\theta+\sin 45^\circ\sin\theta}{2(\cos 15^\circ-1)\sin\theta}$$

$$\Rightarrow \quad (2-\cos 15^\circ)=\frac{\cos 45^\circ\cot\theta+\sin 45^\circ}{2}$$

$$\Rightarrow \quad (\cot\theta+1)=2\sqrt{2}(2-\cos 15^\circ)$$

$$\Rightarrow \quad \cot\theta=2\sqrt{2}(2-\cos 15^\circ)-1$$

Since $\cos(15^\circ)=\cos(45^\circ-30^\circ)$

$$\Rightarrow \quad \cos(15^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$$

$$\Rightarrow \quad \cos(15^\circ)=\frac{1}{2\sqrt{2}}(\sqrt{3}+1)$$

$$\Rightarrow \quad \cot\theta=2\sqrt{2}\left(2-\frac{1}{2\sqrt{2}}(\sqrt{3}+1)\right)-1$$

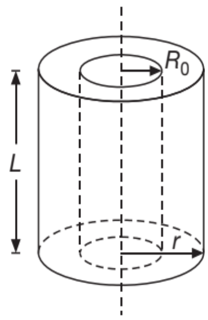
$$\Rightarrow \quad \cot\theta=(4\sqrt{2}-\sqrt{3}-1)-1$$

$$\Rightarrow \quad \cot\theta=4\sqrt{2}-\sqrt{3}-2$$

$$\Rightarrow \quad \theta=\cot^{-1}(4\sqrt{2}-\sqrt{3}-2)$$

PROBLEM 7

A straight infinitely long cylinder of radius R_0 is uniformly charged with charge density σ. The cylinder serves as a source of electrons, with the velocity vector of emitted electrons perpendicular to its surface. What must be the electron velocity to ensure that the electron can move away from the axis of the cylinder to a distance greater than r.



SOLUTION

Let us, first of all determine electric field at a distance r from the cylinder. For that we consider a coaxial cylinder as a Gaussian surface, then according to Gauss's Law, we have

$$\varepsilon_0(2\pi rL)E=(2\pi R_0L)\sigma$$

$$\Rightarrow \quad E = \frac{\sigma R_0}{\varepsilon_0 r} \qquad \ldots(1)$$

Applying Law of Conservation of Energy, we get

$$\frac{1}{2} m_e v_0^2 - eV_0 = -eV \qquad \ldots(2)$$

where V_0 is the potential of cylinder and V is the potential at a distance r from the cylinder's axis.

Employing the relationship $E = -\frac{dV}{dr}$, we have

$$\frac{R_0 \sigma}{\varepsilon_0 r} = -\frac{dV}{dr} \qquad \ldots(3)$$

Integrating, we get

$$V = -\frac{R_0 \sigma}{\varepsilon_o} \log_e r + C \qquad \ldots(4)$$

where C is constant of integration. Also, from equation (4), we get

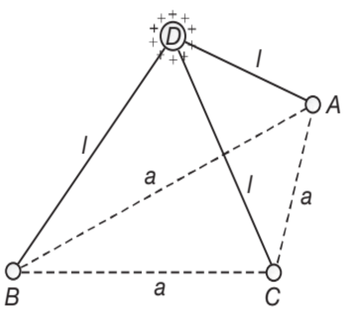
$$V_0 = -\frac{R_0 \sigma}{\varepsilon_0} \log_e (R_0) + C \qquad \ldots(5)$$

Substituting values of V and V_0 in (2), we get

$$v_0 = \sqrt{\frac{2eR_0\sigma}{\varepsilon_0 m_e} \log_e \left(\frac{r}{R_0}\right)}$$

PROBLEM 8

Three small identical neutral metal balls are at the vertices of an equilateral triangle. The balls are connected to a large charged sphere held above the plane of the triangle. The first and the second ball acquire charge q_1 and q_2 respectively. How much is q_3? The connecting wires are very thin.



SOLUTION

When a small ball is connected to a large charged sphere, the potential of the sphere will remain unchanged. Let l be the distance of each small ball from the large sphere D, r = radius of each small ball, a = each side of the equilateral triangle. Let Q be the charge on the sphere and V be its potential. First consider sharing of charge between the sphere and the first ball, i.e., between D and A. Then

$$\begin{pmatrix}\text{Potential}\\ \text{of small}\\ \text{sphere } A\end{pmatrix} = \begin{pmatrix}\text{Potential}\\ \text{due to}\\ \text{its own}\\ \text{charge}\end{pmatrix} + \begin{pmatrix}\text{Potential}\\ \text{due to}\\ \text{charge on the}\\ \text{sphere } D\end{pmatrix}$$

$$\Rightarrow \quad V_A = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{l}$$

But potential of the ball is same as that of the sphere

$$\Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r} + \frac{Q}{l}\right) \qquad \ldots(1)$$

Next consider sharing of charge between $(A+D)$ and B.

The potential of B is $V_B = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_2}{r} + \frac{q_1}{a} + \frac{Q}{l}\right)$

But potential of B is same as that of $(A+D)$, so

$$V_B = V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_2}{r} + \frac{q_1}{a} + \frac{Q}{l}\right) \qquad \ldots(2)$$

Consider sharing of charge between $(A+D+B)$ and C.

$$\Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_3}{r} + \frac{q_1}{a} + \frac{q_2}{a} + \frac{Q}{l}\right) \qquad \ldots(3)$$

From equations (1) and (2), we get

$$\frac{q_1}{r} = \frac{q_2}{r} + \frac{q_1}{a}$$

$$\Rightarrow \quad \frac{r}{a} = \frac{q_1 - q_2}{q_1}$$

From equations (1) and (3), we get

$$\frac{q_1}{r} = \frac{q_3}{r} + \frac{q_1 + q_2}{a}$$

$$\Rightarrow \quad \frac{r}{a} = \frac{q_1 - q_3}{q_1 + q_2}$$

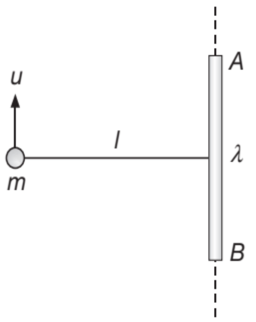
$$\Rightarrow \quad \frac{q_1 - q_2}{q_1} = \frac{q_1 - q_3}{q_1 + q_2}$$

$$\Rightarrow \quad q_1^2 - q_2^2 = q_1^2 - q_1 q_3$$

$$\Rightarrow \quad q_3 = \frac{q_2^2}{q_1}$$

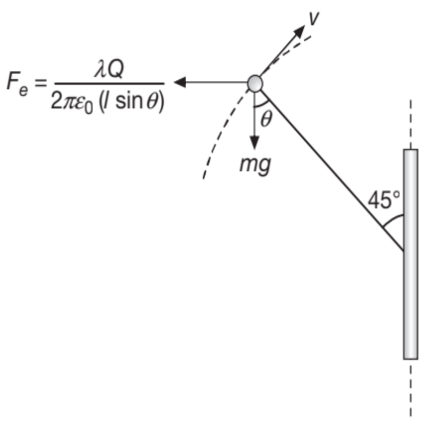
PROBLEM 9

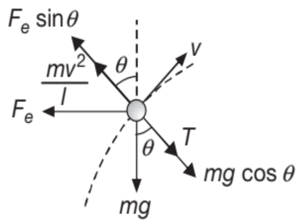
AB is a vertical rigid thin infinite wire carrying a linear charge of density $\lambda = 10\ \mu\text{Cm}^{-1}$. A particle having mass $m = 2$ g and charge is fixed to the wire by means of a light, insulating and inextensible string having length $l = 2\sqrt{2}$ m. Find the vertical velocity u with which it should be projected under gravity from the shown position so that the string slacks when its angle with vertical becomes 45°.



SOLUTION

Let us visualise the situation by drawing a diagram showing all the forces acting on the particle.





The electric field, when the particle is at a perpendicular distance $l\sin\theta$ from the wire, is

$$E = \frac{\lambda}{2\pi\varepsilon_0 (r_\perp)} = \frac{\lambda}{2\pi\varepsilon_0 (l\sin\theta)}$$

So, the electrostatic force F_e on the charge Q is

$$F_e = \frac{\lambda Q}{2\pi\varepsilon_0 (l\sin\theta)}$$

If the string slacks at $\theta = 45°$, then $T = 0$

$$\Rightarrow \quad mg\cos\theta = \frac{mv^2}{l} + \frac{\lambda Q\sin\theta}{2\pi\varepsilon_0 (l\sin\theta)} \qquad \ldots(1)$$

Using Work Energy principle, we have

$$\begin{pmatrix}\text{Work done}\\ \text{by gravity}\end{pmatrix} + \begin{pmatrix}\text{Work done}\\ \text{by electrostatic}\\ \text{forces}\end{pmatrix} = \begin{pmatrix}\text{Change}\\ \text{in } KE\end{pmatrix}$$

$$\Rightarrow \quad -mgl\cos\theta + \frac{1}{2\pi\varepsilon_0}\int_{l}^{l\sin\theta} \frac{Q(\lambda dx)}{x} = \frac{1}{2}m(v^2 - u^2)$$

$$\Rightarrow \quad -mgl\cos\theta + \left(\frac{\lambda Q}{2\pi\varepsilon_0}\right)\log_e(\sin\theta) = \frac{1}{2}m(v^2 - u^2)$$

Using (1), we get

$$-2mgl\cos\theta + \frac{\lambda Q}{\pi\varepsilon_0}\log_e(\sin\theta)$$

$$= mgl\cos\theta - \frac{\lambda Q}{2\pi\varepsilon_0} - mu^2$$

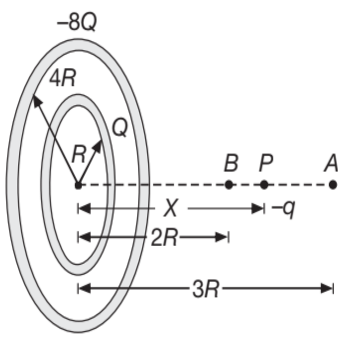
$$\Rightarrow \quad mu^2 = 3ml\cos\theta - 2\frac{1}{4\pi\varepsilon_0}\lambda Q\left[2\log_e(\sin\theta) + 1\right]$$

$$\Rightarrow \quad u = \sqrt{\frac{3gl}{\sqrt{2}} - \frac{\lambda Q}{2\pi\varepsilon_0 m}(-\log_e 2 + 1)}$$

$$\Rightarrow \quad u = 5.7\ \text{ms}^{-1}$$

PROBLEM 10

Two thin concentric rings are placed in a gravity free region in yz- plane, one of radius R carries a charge $+Q$ and the second of radius $4R$ and charge $-8Q$ distributed uniformly over it. Find the minimum velocity with which a point charge of mass m and charge $-q$ should be projected from a point at a distance $3R$ from the centre of the ring on its axis so that it will reach to the centre of the rings.



SOLUTION

Note that at a point distant 3*R* charge $-q$ will have repulsive force but on points close to the centre it will have an attraction force. First we have to find out a point where the electric field is zero because, beyond that, the charge will have attraction force and will be attracted to the centre of the rings. Electric field at a point *P* distant *x* from the centre.

$$E_P = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{(R^2+x^2)^{\frac{3}{2}}} - \frac{1}{4\pi\varepsilon_0}\frac{8Qx}{(16R^2+x^2)^{\frac{3}{2}}} = 0$$

On solving for x, we get

$$x = 2R$$

We have to give the charge sufficient minimum KE so that it could reach this point, because for $x < 2R$ the field will be attractive.

Potential at A is

$$V_A = \frac{1}{4\pi\varepsilon_o}\left[\frac{Q}{\sqrt{9R^2+R^2}} - \frac{8Q}{\sqrt{9R^2+16R^2}}\right]$$

$$\Rightarrow \quad V_A = -\frac{1}{4\pi\varepsilon_0}\left(\frac{16-\sqrt{10}}{10}\right)\frac{Q}{R}$$

Potential at point B is

$$V_B = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{4R^2+R^2}} - \frac{1}{4\pi\varepsilon_0}\frac{8Q}{\sqrt{4R^2+16R^2}} = -\frac{1}{4\pi\varepsilon_0}\frac{3}{\sqrt{5}}\frac{Q}{R}$$

From Conservation of Energy, we have

$$W_{A\to B} = q(V_B - V_A)$$

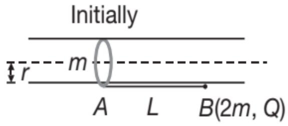
$$\text{Now } \frac{1}{2}mv^2 = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R}\left[\frac{3}{\sqrt{5}} - \frac{16-\sqrt{10}}{10}\right]$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R}\left[\frac{6\sqrt{5}+\sqrt{10}-16}{10}\right]$$

$$\Rightarrow \quad v_{\min} = \sqrt{\frac{Qq}{20\pi\varepsilon_0 mR}(6\sqrt{5}+\sqrt{10}-16)}$$

PROBLEM 11

In the figure shown R is a long smooth fixed non-conducting rod of uniform linear charge density λ and radius of cross-section *r*. *A* is non-conducting uncharged smooth ring of mass m which fits completely on the rod and can move horizontally on the rod. B is a non-conducting charged small particle of mass $2m$ and charge Q and connected to A by an inextensible light string of length l. B is released from rest from the position shown in figure. Find velocity of B and tension in the string when the string becomes vertical. Assume gravitational acceleration g and $\lambda Q > 0$.

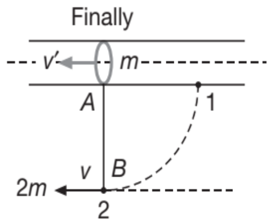


SOLUTION

The particle of charge q, mass $2m$ is released from rest (from 1) and it goes to 2, where it has a horizontal velocity v. Also, $2m$ is attached through string to m (the smooth ring) capable of moving along the rod. So, let the velocity of the ring, when the string becomes vertical be v'. Then by Law of Conservation of Linear Momentum (applied only along the horizontal)

$$mv' = (2m)v$$

$$\Rightarrow \quad v' = 2v$$



Further in going from 1 to 2, we have

$$W_{1\to 2} = q(V_2 - V_1) + \text{ Work done by Gravitational Force}$$

$$\Rightarrow \quad \Delta K = Q\left[\frac{\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r+L}{r}\right)\right] + (2m)gL$$

$$\begin{pmatrix}\text{Change}\\ \text{in } KE\end{pmatrix} = \begin{pmatrix}\text{Work done}\\ \text{by electrostatic}\\ \text{forces}\end{pmatrix} + \begin{pmatrix}\text{Work done}\\ \text{by gravity}\end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{2}mv'^2 + \frac{1}{2}(2m)v^2 = \frac{Q\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r+L}{r}\right) + (2m)gL$$

$$\Rightarrow \quad \frac{1}{2}m(4v^2) + \frac{1}{2}(2m)v^2 = \frac{Q\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r+L}{r}\right) + 2mgL$$

$$\Rightarrow \quad 3mv^2 = \frac{Q\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r+L}{r}\right) + 2mgL$$

$$\Rightarrow \quad v = \sqrt{\frac{2gL}{3} + \frac{Q\lambda}{6\pi\varepsilon_0 m}\log_e\left(\frac{r+L}{r}\right)}$$

Now, tension in the string, when it becomes vertical is

$$T = (2m)g + \frac{m\left(v_{\text{net}}^2\right)}{L} + QE$$

where $v_{\text{net}} = v + v' = 3v$ and $E = \dfrac{\lambda}{2\pi\varepsilon_0(r+L)}$

$$\Rightarrow \quad T = 2mg + \frac{9mv^2}{L} + \frac{\lambda Q}{2\pi\varepsilon_0(r+L)}$$

PROBLEM 12

A uniform electric field of strength $10^6\ \text{Vm}^{-1}$ is directed vertically downwards. A particle of mass 0.01 kg and charge 10^{-6} coulomb is suspended by an inextensible thread of length 1 m. The particle is displaced slightly from its mean position and released. Calculate the time period of its oscillation. What minimum velocity should be given to the particle at rest so that it completes a full circle in vertical plane without the thread getting slack? Calculate the maximum and minimum tension in the thread in this situation.

SOLUTION

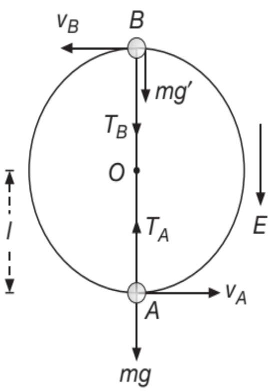
The situation is shown in figure

Acceleration a due to electric field

$$a = \frac{qE}{m} = \frac{10^{-6}\times 10^{6}}{0.01} = 100\ \text{ms}^{-2}$$

Thus effective acceleration would be

$$g' = g + \frac{qE}{m} = 9.8 + 100 = 109.8\ \text{ms}^{-2}$$



Now time period of particle would be given as

$$T = 2\pi\sqrt{\left(\frac{l}{g'}\right)} = 2\pi\sqrt{\left(\frac{1}{109.8}\right)}$$

$$T = 0.6\ \text{s}$$

Now from figure, at point A we have

$$\frac{mv_A^2}{l} = T_A - mg' = T_A - (qE + mg) \quad \ldots(1)$$

and $$\frac{mv_B^2}{l} = T_B + (qE + mg)$$

To complete the circle at point B tension T_B should be zero, thus we have

$$\frac{mv_{B_{\min}}^2}{l} = qE + mg \quad \ldots(2)$$

Using Work Energy Theorem at points A and B, we have

$$\frac{1}{2}mv_A^2 - 2mgl - 2gEl = \frac{1}{2}mv_B^2 + U_B \quad \ldots(3)$$

From equation (2) and (3), we get

$$v_A^2 = \frac{5(qE+mg)l}{m} = 5g'l = 5\times 109.8\times 1$$

$$\left\{\text{where } g' = \frac{qE+mg}{m}\right\}$$

$$\Rightarrow \quad v_A = 23.42\ \text{ms}^{-1}$$

Now, $T_A = (qE + mg) + \dfrac{mv_A^2}{l}$

$$T_A = 6(qE+mg) = 6(1+0.098)$$

$$T_A = 6.588\ \text{N}$$

PROBLEM 13

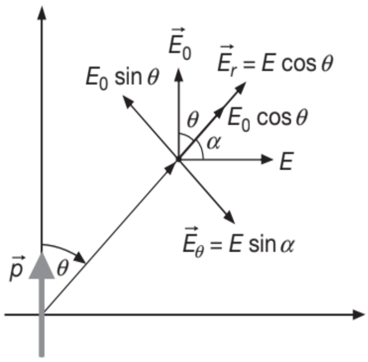
A point electric dipole with a moment p is placed in the external uniform electric field whose strength equals E_0 with p parallel to E_0. In this case one of the equipotential surfaces enclosing the dipole forms a sphere. Find radius of this sphere.

SOLUTION

The electric field due to dipole is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{R^3}\sqrt{1+3\cos^2\theta}$$

and its direction with the radius vector $\vec{r}$ is given by $\tan\alpha = \frac{1}{2}\tan\theta$. Now given that external field $\vec{E}_0$ is parallel to $\vec{p}$. So that $\vec{E}_r$ and $\vec{E}_\theta$ are the radial and transverse component of electric field due to the dipole and $E_0 \sin\theta$ and $E_0 \cos\theta$ are two rectangular components of external field E_0. Since at equipotential surface, there is no tangential components of electric field. So $E_0 \sin\theta$ must be equal and opposite to $E_0 \sin\alpha$



Hence $E_0 \sin\theta = E\sin\alpha$

$$\Rightarrow \quad E_0 = \frac{E\sin\alpha}{\sin\theta}$$

$$\Rightarrow \quad E_0 = \frac{1}{4\pi\varepsilon_0}\left[\frac{p}{R^3}\sqrt{1+3\cos^2\theta}\right]\frac{\sin\theta}{\sqrt{1+3\cos^2\theta}} \times \frac{1}{\sin\theta}$$

$$\left\{\because \ \tan\alpha = \frac{1}{2}\tan\theta\right\}$$

$$\Rightarrow \quad E_0 4\pi\varepsilon_0 R^3 = p$$

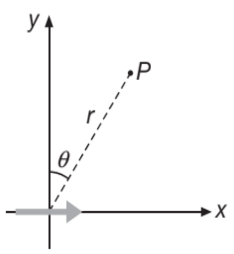
$$\Rightarrow \quad R = \sqrt[3]{\frac{p}{4\pi\varepsilon_0 E_0}}$$

PROBLEM 14

A short electric dipole is situated at the origin of coordinate axis with its axis along x-axis and equator along y-axis. It is found that the magnitudes of the electric field intensity and electric potential due to the dipole are equal at a point distant $r = \sqrt{5}$ m from origin. Find the position vector of this point.

SOLUTION

Consider a point P at distance r and angle θ from equation. Since



$$|E_P| = |V_P|$$

$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{1+3\sin^2\theta} = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^2}$$

$$\Rightarrow \quad \frac{\sqrt{1+3\sin^2\theta}}{\sqrt{5}} = \sin\theta$$

$$\Rightarrow \quad 1+3\sin^2\theta = 5\sin^2\theta$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = 45^\circ$$

Positive vector $\vec{r}$ of point P is

$$\vec{r} = \sqrt{5}\left(\cos 45\hat{i} + \sin 45\hat{j}\right)6 = \sqrt{\frac{5}{2}}\left(\hat{i}+\hat{j}\right)$$

$$\Rightarrow \quad \vec{r} = \sqrt{\frac{5}{2}}\left(\hat{i}+\hat{j}\right)$$

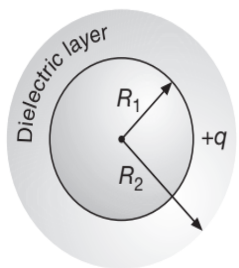
PROBLEM 15

Find the potential of an isolated ball shaped conductor of radius R_1 surrounded by an adjacent concentric layer of dielectric with dielectric constant K and outer radius R_2.

SOLUTION

Let a charge q be assumed to be given to the conductor

Since, $V = -\int E \cdot dr$



$$\Rightarrow \quad V = -\left[\int_{\infty}^{R_2}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr + \int_{R_2}^{R_1}\frac{1}{4\pi\varepsilon_0 K}\frac{q}{r^2}dr\right]$$

$$\Rightarrow \quad V = -\frac{q}{4\pi\varepsilon_0}\left[\int_{\infty}^{R_2}\frac{dr}{r^2} + \int_{R_2}^{R_1}\frac{dr}{Kr^2}\right]$$

$$\Rightarrow \quad V=\frac{q}{4\pi\varepsilon_0}\left[\left(-\frac{1}{r}\right)_{\infty}^{R_2}+\left(-\frac{1}{Kr}\right)_{R_2}^{R_1}\right]$$

$$\Rightarrow \quad V=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R_2}+\frac{1}{KR_1}-\frac{1}{KR_2}\right]$$

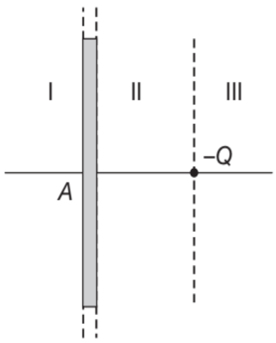
$$\Rightarrow \quad V=\frac{q}{4\pi\varepsilon_0}\left[\frac{KR_1+R_2-R_1}{KR_1R_2}\right]$$

$$\Rightarrow \quad V=\frac{q}{4\pi\varepsilon_0}\left[\frac{R_1(K-1)+R_2}{KR_1R_2}\right]$$

$$\Rightarrow \quad V=\frac{q}{4\pi\varepsilon_0}\left[\left(\frac{K-1}{K}\right)\frac{1}{R_2}+\frac{1}{KR_1}\right]$$

PROBLEM 16

An infinitely long conducting wire of charge density $+\lambda$ and a point charge $-Q$ are at a distance from each other. In which of the three Regions (I, II or III) are there points that lie on the line passing through point charge perpendicular to the conductor and at which the field is zero?



SOLUTION

FOR REGION I

Now we check the region I, take a point to the left of wire at a distance x from it. The resultant field is

$$\vec{E}_R=\frac{\lambda}{2\pi\varepsilon_0 x}(-\hat{i})+\frac{Q}{4\pi\varepsilon_0(a+x)^2}\hat{i}$$

The two fields point in the opposite directions, so resultant field can be zero if,

$$\frac{\lambda}{2\pi\varepsilon_0 x}=\frac{Q}{4\pi\varepsilon_0(a+x)^2}$$

$$\Rightarrow \quad x^2+\left(2a-\frac{Q}{2\lambda}\right)x+a^2=0$$

$$\Rightarrow \quad x=\frac{1}{2}\left(\frac{Q}{2\lambda}-2a\right)\pm\sqrt{\frac{1}{4}\left(\frac{Q}{2\lambda}-2a\right)^2-a^2}$$

If the discriminant of the quadratic equation is real, we have two points where the field is zero. Discriminant is positive for $Q\geq 8a\lambda$.

FOR REGION II

In the region (II) the electric field of wire and point charge point in the same direction, positive x-axis. So no point can exist where the field is zero.

FOR REGION III

Now we take a point to the right of the point charge, at a distance x from it. Resultant field at this point is

$$\vec{E}_R=\frac{\lambda}{2\pi\varepsilon_0(x+a)}\hat{i}+\frac{Q}{4\pi\varepsilon_0 x^2}(-\hat{i})$$

Resultant field is zero if

$$\frac{\lambda}{(a+x)}=\frac{Q}{2x^2}$$

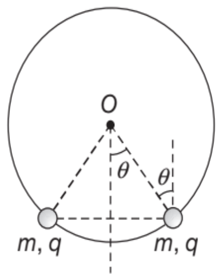
On solving the quadratic in x, we have

$$x=\frac{Q}{4\lambda}\pm\sqrt{\frac{Q^2}{16\lambda^2}+\frac{aQ}{2\lambda}}$$

The negative sign infront of the radical has no meaning because it would mean that the point is to the left of point charge, where field of wire and point charge are added, the magnitudes of the two fields are zero.

PROBLEM 17

Figure shows two identical beads of mass m and charge q. The beads can slide smoothly on a wire frame kept in a vertical frame.



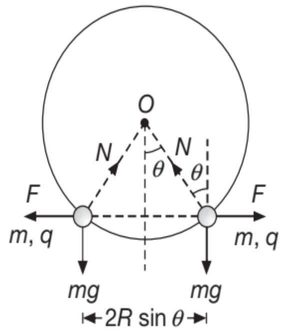
(a) Determine angular position θ w.r.t. vertical diameter.

(b) Now the beads are given a small angular displacement. Show that they perform simple harmonic motion.

SOLUTION

(a) Each bead is in equilibrium under the action of three forces

Coulomb force of repulsion, Weight mg and Normal reaction, N



From conditions of equilibrium, we have

$$N\sin\theta = F = \frac{q^2}{4\pi\varepsilon_0(2R\sin\theta)} \quad \ldots(1)$$

and $N\cos\theta = mg$ ...(2)

From equations (1) and (2), we have

$$\tan\theta = \frac{q^2}{4\pi\varepsilon_0 mg(2R\sin\theta)} \quad \ldots(3)$$

(b) After a small angular displacement, we may consider the Coulombic force F to be approximately constant.

In equilibrium, net torque about centre of circle is zero. So we have

$$-mgR\sin\theta + FR\cos\theta = 0 \quad \ldots(4)$$

After a small angular displacement ϕ of right bead, we have

$$-mgR\sin(\theta+\phi) + FR\cos(\theta+\phi) = I\alpha = mR^2\alpha \quad \ldots(5)$$

$$\Rightarrow \quad -mgR(\sin\theta\cos\phi + \cos\theta\sin\phi) + FR(\cos\theta\cos\phi - \sin\theta\sin\phi) = mR^2\alpha \quad \ldots(6)$$

For small ϕ, $\sin\phi \simeq \phi$ and $\cos\phi \simeq 1$, so equation (6) reduces to

$$-mgR(\sin\theta + \phi\cos\theta) + FR(\cos\theta - \phi\sin\theta) = mR^2\alpha \quad \ldots(7)$$

From equations (4) and (7), we get

$$-(mgR\cos\theta + FR\sin\theta)\phi = mR^2\alpha$$

$$\Rightarrow \quad \alpha = -\left(\frac{mgR\cos\theta + FR\sin\theta}{mR^2}\right)\phi$$

$$\Rightarrow \quad \ddot{\phi} + \left(\frac{mR\cos\theta + FR\sin\theta}{mR^2}\right)\phi = 0$$

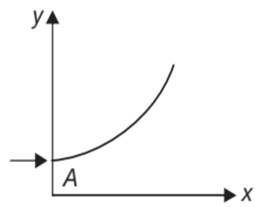
Comparing with standard equation of SHM i.e., $\ddot{\phi} + \omega^2\phi = 0$, we get

$$\omega = \sqrt{\frac{mgR\cos\theta + \dfrac{q^2}{4\pi\varepsilon_0(2R\sin\theta)^2}\sin\theta}{mR^2}}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{8\pi\varepsilon_0 R^3 mg\sin^2\theta + q^2}{16\pi\varepsilon_0 mR^2\sin\theta}}$$

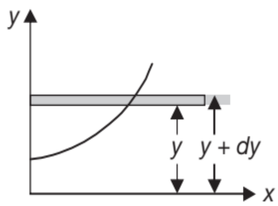
PROBLEM 18

In an insulating medium (di-electric constant $K = 1$) volumetric charge density varies with y-coordinate according to the law $\rho = ay$. A particle of mass m having positive charge q is at point $A(0,\ y_0)$ and projected with velocity $\vec{v} = v_0\hat{i}$ as shown in figure. Neglecting gravity and frictional resistance of the medium and assuming electric field strength to be zero at $y = 0$, calculate the slope of the trajectory of the particle as a function of y.



SOLUTION

As charge density varies with y-coordinate only, electric field must be directed along y-axis. Consider a thin layer of medium, having thickness dy at a distance y from x-axis as shown in figure.



Let strength of electric field at positions y and $(y+dy)$ be E and $(E + dE)$ respectively.

Then for area A of the layer, according to Gauss's Law,

$$(E+dE)A-EA=\frac{\rho A\,dy}{\varepsilon_0}$$

$$\Rightarrow \quad dE=\frac{\rho}{\varepsilon_0}dy=\frac{a}{\varepsilon_0}y\,dy$$

Integrating above equation with in appropriate limits, we get

$$\int_0^E dE=\frac{a}{\varepsilon_0}\int_0^y ydy$$

$$\Rightarrow \quad E=\frac{ay^2}{2\varepsilon_0} \qquad \ldots(1)$$

There is no force on the particle along x-axis, velocity v_0 of the particle along x-axis remains constant

Due to electric field E, the particle accelerates along positive y-direction with acceleration a_y, given by

$$a_y=\frac{qE}{m}=\frac{qay^2}{2m\varepsilon_0}$$

Let y-component of velocity of particle be v, then from Newton's Second Law we get

$$v\frac{dv}{dy}=\frac{qay^2}{2m\varepsilon_0}$$

$$\Rightarrow \quad v\,dv=\frac{qa}{2m\varepsilon_0}y^2\,dy$$

Since at point $A(y=y_0)$, y-component of velocity is $v=0$

$$\Rightarrow \quad \int_0^v v\,dv=\frac{qa}{2m\varepsilon_0}\int_{y_0}^y y^2dy$$

$$\Rightarrow \quad v=\sqrt{\frac{qa}{3m\varepsilon_0}\left(y^3-y_0^3\right)}$$

$$\text{Slope of trajectory }=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{v}{v_0}$$

$$\Rightarrow \quad \text{Slope }=\frac{dy}{dx}=\sqrt{\frac{qa}{3m\varepsilon_0 v_0^2}\left(y^3-y_0^3\right)}$$

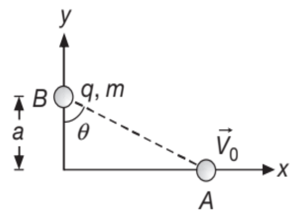
PROBLEM 19

A charged particle having a charge q moves along the x-axis with a constant velocity v_0. Another particle B with charge q and mass m is lying on the y-axis at $y=a$. The particle B is constrained to move along the y-axis, while the particle A moves along the x-axis.

Assuming that the velocity v_0 is very large, find the impulse imparted to B along the y-axis as the particle A moves from $-\infty$ to ∞ assuming that the motion of particle B is negligible.

SOLUTION

First of all, let us determine the angular speed of A relative to B at angular position θ shown in figure.



For particle A

$$\frac{dx}{dt}=\frac{d}{dt}(a\tan\theta)=v_0 \qquad \ldots(1)$$

$$\Rightarrow \quad a\sec^2\theta\frac{d\theta}{dt}=v_0$$

$$\Rightarrow \quad \left(\frac{d\theta}{dt}\right)=\frac{v_0}{a}\cos^2\theta \qquad \ldots(2)$$

Since particle A is moving very fast, we can assume that when it crosses along the x-axis, the y-component of the force on B due to A is

$$F_y=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2\sec^2\theta}\times\cos\theta \qquad \ldots(3)$$

$$\text{Impulse on }B=\int F_y dt=\int\frac{F_y d\theta}{\left(\frac{d\theta}{dt}\right)}$$

$$\Rightarrow \quad B=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\frac{a}{v_0}\int\frac{\cos^3\theta}{\cos^2\theta}d\theta$$

$$\Rightarrow \quad B=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{av_0}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta$$

The impulse delivered to B, is given by

$$\Delta P_y = \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{av_0} = \frac{q^2}{2\pi\varepsilon_0 (av_0)} \quad \ldots(4)$$

PROBLEM 20

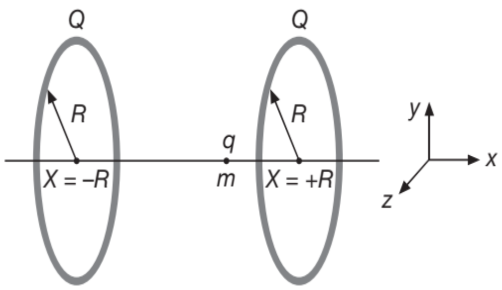
A particle of mass m and charge $+q$ is constrained to move along the x-axis. Two charged rings of radius R, charge Q are placed with their centres at $(-R, 0)$ and $(R, 0)$ as shown.

(a) Obtain an expression for the potential due to the rings as a function of x for $-R < x < R$.

(b) Show that in this region $V(x)$ is minimum at $x = 0$.

(c) Show that for $x \ll R$, the potential is of the form $V(x) = V(0) + \alpha x^2$.

(d) Derive an expression for the time period of oscillation of the mass m when it is displaced slightly from the origin and released.



SOLUTION

Electric potential due to a ring on its axis is given by

$$V(x) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(R^2+x^2)^{\frac{1}{2}}}$$

(a) At any point $(x, 0, 0)$ the potential due to both rings

$$V(x) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+(R-x)^2}} + \frac{1}{\sqrt{R^2+(R+x)^2}}\right]$$

(b) For minimum $V(x)$,

$$\frac{d}{dx}[V(x)] = 0$$

$$\Rightarrow \quad \frac{dV(x)}{dx} = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{2}\frac{2(R-x)(-1)}{[R^2+(R-x)^2]^{\frac{3}{2}}} - \frac{1}{2}\frac{2(R+x)}{[R^2+(R+x)^2]^{\frac{3}{2}}}\right]$$

$$\Rightarrow \quad \frac{dV(x)}{dx} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{R-x}{[R^2+(R-x)^2]^{\frac{3}{2}}} - \frac{R+x}{[R^2+(R+x)^2]^{\frac{3}{2}}}\right]$$

The expression at $x = 0$ is

$$\frac{dV}{dx} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{R}{(2R^2)^{\frac{3}{2}}} - \frac{R}{(2R^2)^{\frac{3}{2}}}\right] = 0$$

Thus $V(x)$ is minimum at $x = 0$

$$V_{\min} = V(0) = \frac{Q}{4\pi\varepsilon_0}\frac{\sqrt{2}}{R}$$

(c) For $x \ll R$,

$$V(x) = \frac{Q}{4\pi\varepsilon_0}\left\{[R^2+(R-x)^2]^{-\frac{1}{2}} + [R^2+(R+x)^2]^{-\frac{1}{2}}\right\}$$

$$\Rightarrow \quad V(x) = \frac{Q}{4\pi\varepsilon_0 R}\left\{\left[1+\left(1-\frac{x}{R}\right)^2\right]^{-\frac{1}{2}} + \left[1+\left(1+\frac{x}{R}\right)^2\right]^{-\frac{1}{2}}\right\}$$

Now we apply the binomial approximation, we have $(1+x)^n \simeq 1+nx$ when $|x| \ll 1$

$$\left(1-\frac{x}{R}\right)^2 \simeq 1-\frac{2x}{R} \text{ and } \left(1+\frac{x}{R}\right)^2 \simeq 1+\frac{2x}{R}$$

$$\Rightarrow \quad V(x) = \frac{Q}{4\pi\varepsilon_0 R}\left[\left(2-\frac{2x}{R}\right)^{-\frac{1}{2}} + \left(2+\frac{2x}{R}\right)^{-\frac{1}{2}}\right]$$

Now we apply the binomial expansion,

$$\left(1-\frac{x}{R}\right)^{-\frac{1}{2}}=1-\frac{x}{2R}+\frac{3}{8}\frac{x^2}{R^2}-\dots$$

$$\left(1+\frac{x}{R}\right)^{-\frac{1}{2}}=1+\frac{x}{2R}+\frac{3}{8}\frac{x^2}{R^2}+\dots$$

where, we ignore the cubic and higher powers of x.

$$V(x)=\frac{Q}{4\pi\varepsilon_0\sqrt{2}R}\left[\left(1-\frac{x}{R}\right)^{-\frac{1}{2}}+\left(1+\frac{x}{R}\right)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \quad V(x)=\frac{Q}{4\pi\varepsilon_0\sqrt{2}R}\left[2+\frac{3}{4}\frac{x^2}{R^2}\right]$$

$$\Rightarrow \quad V(x)=\frac{Q}{4\pi\varepsilon_0}\frac{\sqrt{2}}{R}+\frac{Q}{4\pi\varepsilon_0}\frac{3}{4\sqrt{2}R^3}x^2$$

$$V(x)=V_0(0)+ax^2\text{, where } a=\frac{3Q}{4\pi\varepsilon_0 4\sqrt{2}R^3}$$

(d) The charged particle is displaced from the origin by a distance x (say)

For $x \ll R$, the potential function is

$$V(x)=V(0)+ax^2$$

Since $E_x=-\frac{dV}{dx}$

$$\Rightarrow \quad E_x=-2ax$$

The force experienced by the charge is given by

$$F=qE_x=-Q(2ax)$$

So, acceleration of the charge is

$$a=-\frac{2aQ}{m}x$$

$$\Rightarrow \quad \ddot{x}+\left(\frac{2Qa}{m}\right)x=0$$

Now, we compare a with standard expression of SHM, $a=-\omega^2 x$, or $(\ddot{x}+\omega^2 x=0)$, we get

$$\omega=\sqrt{\frac{2aQ}{m}}=\frac{2\pi}{T}$$

$$\Rightarrow \quad T=2\pi\sqrt{\frac{m}{2aQ}}$$

PROBLEM 21

A thin rod extends along the z-axis from $z=-d$ to $z=d$. The rod carries a positive charge Q uniformly distributed along its length $2d$ with charge density $\lambda=\frac{Q}{2d}$.

(a) Find the electric potential $V(z)$ at a point $z>d$ along the z-axis.

(b) Find the change in potential energy if an electron moves from $z=4d$ to $z=3d$.

(c) Assume that the electron started from rest at the point $z=4d$, then find its velocity at $z=3d$.

SOLUTION

(a) For simplicity, let's assume the potential to be zero at infinity, $V(\infty)=0$.

Consider an infinitesimal charge element $dq=\lambda dz$ located at a distance z along the z-axis. Its contribution to the electric potential at a point $z>d$ is

$$dV=\frac{\lambda}{4\pi\varepsilon_0}\frac{dz}{z}$$

Integrating over the entire length of the rod, we obtain

$$V(z)=\frac{\lambda}{4\pi\varepsilon_0}\int_{z+d}^{z-d}\frac{dz}{z}=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{z+d}{z-d}\right)$$

(b) Using the result derived in (a), the electrical potential at $z=4d$ is

$$V|_{z=4d}=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{4d+d}{4d-d}\right)=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{5}{3}\right)$$

Similarly, the electrical potential at $z=3d$ is

$$V|_{z=3d}=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{3d+d}{3d-d}\right)=\frac{\lambda}{4\pi\varepsilon_0}\log_e 2$$

The electric potential difference between the two points is given by

$$\Delta V=V|_{z=3d}-V|_{z=4d}=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{6}{5}\right)>0$$

Using the fact that the electric potential difference ΔV is equal to the change in potential energy per unit charge, we have

$$\Delta U=q\Delta V=-\frac{e\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{5}{6}\right)$$

where $q=-e$ is the charge of the electron.

(c) If the electron starts out at rest from $z = 4d$, then the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_f^2$$

By conservation of energy

$$\Delta U + \Delta K = 0$$

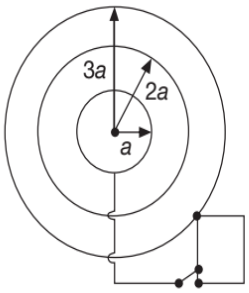
$$\Rightarrow \quad \Delta K = -\Delta U = \frac{e\lambda}{4\pi\varepsilon_0}\log_e\left(\frac{6}{5}\right) > 0$$

Thus, the magnitude of the velocity at $z = 3d$ is

$$v_f = \sqrt{\frac{2e\lambda}{4\pi\varepsilon_0 m}\log_e\left(\frac{6}{5}\right)}$$

PROBLEM 22

Three concentric conducting shells of A, B and C radii a, $2a$ and $3a$ are shown in figure. The charge on the shell A, B and C is Q. When the key K is closed, find the charges on the innermost and outermost shells and ratio of charge densities of the shells.



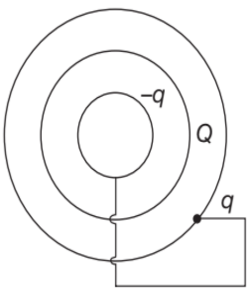
SOLUTION

When the key is closed, then the innermost and outermost shells will acquire the same potential. Let the charge on the outer shell be q and that on the inner shell be $-q$, the total charge on inner and outer shells is zero.

Potential on innermost shell,

V_a = Sum of potentials due to $-q$, Q and q

$$\Rightarrow \quad V_a = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{a} + \frac{Q}{2a} + \frac{q}{3a}\right)$$



Similarly, potential on the outermost shell,

$$V_c = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{3a} + \frac{Q}{3a} + \frac{q}{3a}\right)$$

Since $V_a = V_c$, so we get

$$-\frac{q}{a} + \frac{Q}{2a} + \frac{q}{3a} = -\frac{q}{3a} + \frac{Q}{3a} + \frac{q}{3a} \qquad \ldots(1)$$

Equation (1) now becomes

$$-\frac{q}{a} + \frac{Q}{2a} = -\frac{q}{3a} + \frac{Q}{3a} \qquad \ldots(2)$$

$$\Rightarrow \quad q = \frac{Q}{4}$$

Thus charge on outermost shell is $q = \frac{Q}{4}$

Charge on innermost shell is $-q = -\frac{Q}{4}$, so

$$\sigma_A = \frac{1}{4\pi a^2}\left(-\frac{Q}{4}\right)$$

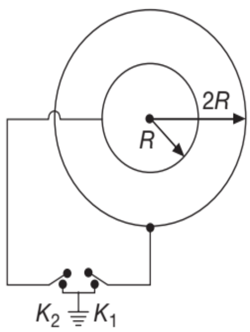
$$\sigma_B = \frac{Q}{4\pi(4a^2)}$$

$$\sigma_C = \frac{+Q}{4\pi(9a^2)}$$

$$\Rightarrow \quad \sigma_A : \sigma_B : \sigma_C = -9 : 9 : 1$$

PROBLEM 23

Two concentric shells of radii R and $2R$ are shown in figure. Initially a charge q is imparted to the inner shell. Now key K_1 is closed and opened and then key K_2 is closed and opened. After the keys K_1 and K_2 are alternately closed n times each, find the potential difference between the shells. Note that finally the key K_2 remains closed.



SOLUTION

When K_1 is closed first time, outer sphere is earthed and the potential on it becomes zero.
Let the charge on it be q_1'

V_1' = Potential due to charge on inner sphere and that due to charge on outer sphere

$$V_1' = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{2R}+\frac{q_1'}{2R}\right]=0$$

$$\Rightarrow \quad q_1' = -q$$

When K_2 is closed first time, the potential V_2' on inner sphere becomes zero as it is earthed. Let the new charge on inner sphere be q_2'

$$0 = \frac{1}{4\pi\varepsilon_0}\frac{q_2'}{R}+\frac{1}{4\pi\varepsilon_0}\frac{(-q)}{(2R)}$$

$$\Rightarrow \quad q_2' = \frac{q}{2}$$

Now when K_1 will be closed second time, charge on outer sphere will be $-q_2'$, i.e., $-\frac{q}{2}$

After one event involving closure and opening of K_1 and K_2, charge is reduced to half its initial value.
Similarly, when K_1 will be closed nth time, charge on outer sphere will be $-\frac{q}{2^{n-1}}$ as each time charge will be reduced to half the previous value.
After closing K_2 nth time, charge on inner shell will be negative of half the charge on outer shell, i.e., $\left(+\frac{q}{2^n}\right)$ and potential on it will be zero.

For potential of outer shell,

$$V_0 = \frac{1}{4\pi\varepsilon_0}\frac{\left(+\frac{q}{2^n}\right)}{2R}+\frac{1}{4\pi\varepsilon_0}\frac{\left(-\frac{q}{2^{n-1}}\right)}{2R}$$

$$\Rightarrow \quad V_0 = \frac{-q(-1+2)}{4\pi\varepsilon_0 2^{n+1}R} = \frac{-q}{4\pi\varepsilon_0 2^{n+1}R}$$

Potential difference $= V_0 - V_i = \frac{-q}{4\pi\varepsilon_0 2^{n+1}R} - 0$

$$\Rightarrow \quad V_0 - V_i = \frac{-q}{4\pi\varepsilon_0 2^{n+1}R}$$

PROBLEM 24

Two isolated metallic solid spheres of radii R and $2R$ are charged such that both of these have same charge density σ. The spheres are located far away from each other and connected by a thin conducting wire. Find the new charge density on the bigger sphere.

SOLUTION

Let q_1 and q_2 be the charges on the two spheres before connecting them.

Then, $q_1 = \sigma(4\pi R^2)$ and

$$q_2 = \sigma(4\pi)(2R)^2 = 16\sigma\pi R^2$$

Therefore, total charge (q) on both the spheres is

$$q = q_1 + q_2 = 20\sigma\pi R^2$$

Now, after connecting, the charge is distributed in the ratio of their capacities, which in turn depends on the ratio of their radii $(C = 4\pi\varepsilon_o R)$.

$$\therefore \quad \frac{q'_1}{q'_2} = \frac{R}{2R} = \frac{1}{2}$$

$$\Rightarrow \quad q'_1 = \frac{q}{3} = \frac{20}{3}\sigma\pi R^2$$

and $$q'_2 = \frac{2q}{3} = \frac{40}{3}\sigma\pi R^2$$

Therefore, surface charge densities on the spheres are:

$$\sigma_1 = \frac{q'_1}{4\pi R^2} = \frac{(20/3)\sigma\pi R^2}{4\pi R^2} = \frac{5}{3}\sigma$$

and $$\sigma_2 = \frac{q'_2}{4\pi(2R)^2} = \frac{(40/3)\sigma\pi R^2}{16\pi R^2} = \frac{5}{6}\sigma$$

Hence, surface charge density on the bigger sphere is σ_2 i.e., $\left(\frac{5}{6}\right)\sigma$.

PROBLEM 25

A conducting sphere S_1 of radius r is attached to an insulating handle. Another conducting sphere S_2 of radius R is mounted on an insulating stand S_2 is initially uncharged.

S_1 is given a charge Q brought into contact with S_2 and removed. S_1 is recharged such that the charge on it is again Q and it is again brought into contact with S_2 and removed. This procedure is repeated n times.

(a) Find the electrostatic energy of S_2 after n such contacts with S_1.

(b) What is the limiting value of this energy as $n \to \infty$?

SOLUTION

Capacities of conducting spheres are in the ratio of their radii. Let C_1 and C_2 be the capacities of S_1 and S_2, then

$$\frac{C_2}{C_1} = \frac{R}{r}$$

(a) Charges are distributed in the ratio of their capacities. Let in the first contact, charge acquired by S_2, is q_1.

Therefore, charge on S_1 will be $Q - q_1$. Say it is q'_1.

$$\Rightarrow \quad \frac{q_1}{q'_1} = \frac{q_1}{Q - q_1} = \frac{C_2}{C_1} = \frac{R}{r}$$

It implies that Q charge is to be distributed in S_2 and S_1 in the ratio of R/r.

$$\Rightarrow \quad q_1 = Q\left(\frac{R}{R+r}\right) \qquad \ldots(1)$$

In the second contact, S_1 again acquires the same charge Q.

Therefore, total charge in S_1 and S_2 will be

$$Q + q_1 = Q\left(1 + \frac{R}{R+r}\right)$$

This charge is again distributed in the same ratio.

Therefore, charge on S_2 in second contact,

$$q_2 = Q\left(1 + \frac{R}{R+r}\right)\left(\frac{R}{R+r}\right)$$

$$\Rightarrow \quad q_2 = Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2\right]$$

Similarly,

$$q_3 = Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2 + \left(\frac{R}{R+r}\right)^3\right]$$

and $$q_n = Q\left[\frac{R}{R+r} + \left(\frac{R}{R+r}\right)^2 + \ldots + \left(\frac{R}{R+r}\right)^n\right]$$

$$\Rightarrow \quad q_n = Q\frac{R}{r}\left[1 - \left(\frac{R}{R+r}\right)^n\right] \qquad \ldots(2)$$

$$\left\{\because\ S_n = \frac{a(1-r^n)}{(1-r)}\right\}$$

Therefore, electrostatic energy of S_2 after n such contacts is

$$U_n = \frac{q_n^2}{2C} = \frac{q_n^2}{2(4\pi\varepsilon_o R)}$$

$$\Rightarrow \quad U_n = \frac{q_n^2}{8\pi\varepsilon_o R}$$

where q_n can be written from equation (2).

(b) $$q_n = \frac{QR}{R+r}\left[1 + \frac{R}{R+r} + \ldots\ldots + \ldots. + \left(\frac{R}{R+r}\right)^{n-1}\right]$$

Since, the sum of infinite GP is $S_\infty = \dfrac{a}{1-r}$

So, when $n \to \infty$, then

$$q_\infty = \frac{QR}{R+r}\left[\frac{1}{1 - \frac{R}{R+r}}\right]$$

$$\Rightarrow \quad q_\infty = \frac{QR}{R+r}\left(\frac{R+r}{R+r-R}\right)$$

$$\Rightarrow \quad q_\infty = \frac{QR}{R+r}\left(\frac{R+r}{r}\right) = Q\frac{R}{r}$$

$$\Rightarrow \quad U_\infty = \frac{q_\infty^2}{8\pi\varepsilon_0 R} = \frac{Q^2R^2/r^2}{8\pi\varepsilon_o R}$$

$$\Rightarrow \quad U_\infty = \frac{Q^2 R}{8\pi\varepsilon_o r^2}$$

PROBLEM 26

Two fixed charges $-2Q$ and Q are located at the points with coordinates $(-3a, 0)$ and $(+3a, 0)$ respectively in the x-y plane.

(a) Show that all points in the x-y plane where the electric potential due to the two charges is zero, lie on a circle. Find its radius and the location of its centre.

(b) Give the expression $V(x)$ at a general point on the x-axis and sketch the function $V(x)$ on the whole x-axis.

(c) If a particle of charge $+q$ starts form rest at the centre of the circle, show by a short quantitative argument that the particle eventually crosses the circle. Find its speed when it does so.

SOLUTION

(a) Let $P(x, y)$ be a general point of x-y plane. Electric potential at point P would be,

$$V = (\text{potential due to } Q) + (\text{potential due to} - 2Q)$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{(3a-x)^2+y^2}} + \frac{-2Q}{\sqrt{(3a+x)^2+y^2}}\right) \quad \ldots(1)$$

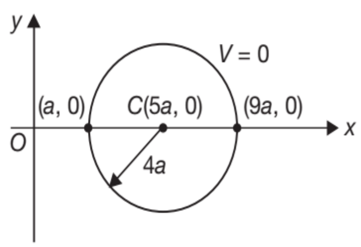
Given $V = 0$

$$\Rightarrow \quad 4\left[(3a-x)^2+y^2\right] = (3a+x)^2+y^2$$

Solving and rearranging, we get

$$(x-5a)^2+y^2 = (4a)^2$$

which is the equation of a circle of radius $4a$ and centre at $(5a, 0)$



(b) On x- axis, potential will be undefined (or say $\pm\infty$) at $x = 3a$ and $x = -3a$, because charge Q and $-2Q$ are placed at these two points. So, between $-3a < x < 3a$ we can find potential by putting $y = 0$ in equation (1). Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{3a-x} - \frac{2}{3a+x}\right), \text{ for } -3a < x < 3a$$

For $-3a < x < 3a$, we observe that

$V = 0$ at $x = a$

$V \to -\infty$ as $x \to -3a$

and $V \to +\infty$ as $x \to 3a$

For $x > 3a$, there is again a point where potential will become zero so for $x > 3a$, we can write

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{x-3a} - \frac{2}{3a+x}\right) \text{ for } x > 3a$$

and observe that, $V = 0$ at $x = 9a$

For $x < -3a$, we can write

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{3a-x} - \frac{2}{3a-x}\right) \text{ for } x < -3a$$

In this region potential will be zero only when $x \to -\infty$

So, we can summarise $V(x)$ as under

(i) At $x = 3a$, $V \to +\infty$

(ii) At $x = -3a$, $V \to -\infty$

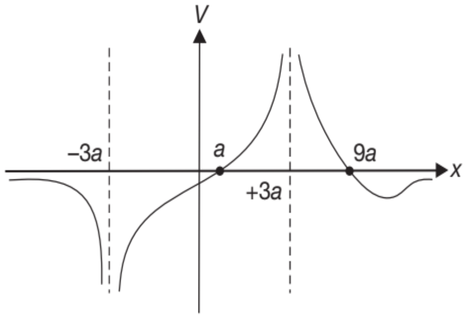
(iii) For $x < -3a$, $V(x) = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{3a-x} - \dfrac{2}{3a+x}\right)$

(iv) For $-3a < x < 3a$, expression of $V(x)$ is same i.e.,

$$V(x) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{3a-x} - \frac{2}{3a+x}\right)$$

(v) For $x > 3a$, $V(x) = \dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{1}{x-3a} - \dfrac{2}{3a+x}\right)$

Potential on x-axis is zero at two places at $x = a$ and $x = 9a$. The V-x graph is shown here.



(c) Potential at centre i.e., at $x = 5a$ will be,

$$V_C = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{2a} - \frac{2}{8a}\right) = \frac{Q}{16\pi\varepsilon_o a} = \text{positive}$$

Potential on the circle will be zero.

Since, potential at centre > potential on circumference on it, the particle will cross the circle because positive charge moves from higher potential to lower potential. Speed of particle, while crossing the circle would be,

$$v = \sqrt{\frac{2q(\Delta V)}{m}} = \sqrt{\frac{Qq}{8\pi\varepsilon_o ma}}$$

Here, ΔV is the potential difference between the centre and circumference of the circle.

PROBLEM 27

A positive charge Q is uniformly distributed throughout the volume of a dielectric sphere of radius R. A point mass having charge $+q$ and mass m is fired towards the centre of the sphere with velocity v from a point at distance $x(x > R)$ from the centre of the sphere. Calculate the minimum velocity v so that it penetrates a distance $\frac{R}{2}$ inside the sphere. Neglect any resistance other that electrostatic interaction. Assume that the charge on small mass remains constant through out the motion.

SOLUTION

By Law of Conservation of Energy

Total Energy at A = Total Energy at B

$$\Rightarrow \quad K_A + U_A = K_B + U_B$$

Energy of particle at A

$$E_A = \frac{1}{2}mv^2 + \frac{Qq}{4\pi\varepsilon_0 x}$$

At any point P inside the sphere at a distance r from its centre, the potential energy is given by

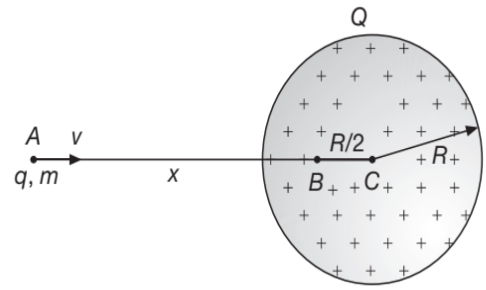
$$U_P = qV_P$$

$$\text{where, } V_{\text{inside}} - V_P = \frac{Q}{8\pi\varepsilon_0 R^3}\left(3R^2 - r^2\right)$$

So, energy of particle at B is

$$E_B = 0 + \frac{Qq}{8\pi\varepsilon_0 R^3}\left(3R^2 - \left(\frac{R}{2}\right)^2\right)$$

$$\Rightarrow \quad E_B = \frac{Q}{8\pi\varepsilon_0 R^3}\left(\frac{11R^2}{4}\right) = \frac{11}{32}\frac{Qq}{\pi\varepsilon_0 R}$$



$$\Rightarrow \quad \frac{1}{2}mv^2 + \frac{Qq}{4\pi\varepsilon_0 x} = \frac{11}{32}\frac{Qq}{\pi\varepsilon_0 R}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{Qq}{4\pi\varepsilon_0}\left[\frac{11}{8R} - \frac{1}{x}\right]$$

$$\Rightarrow \quad v = \sqrt{\frac{Qq}{2\pi\varepsilon_0 m}\left(\frac{11}{8R} - \frac{1}{x}\right)}$$