iCON

CHAPTER 4

Mechanical Waves

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Characteristics of Wave Motion
(b) Equation of Harmonic Wave
(c) Characteristic Wave Equation
(d) Particle Velocity, Wave Slope, Particle Acceleration
(e) Transverse Wave in a String and Properties
(f) Sound Wave and Properties
(g) Superposition of Waves
(h) Concept of Interference
(i) Stationary Waves, Organ Pipes
(j) Beats
(k) Doppler Effect of Sound and Light

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

A wave is a disturbance from an equilibrium condition that propagates from one region of space to another without the transfer of matter. We can also say that wave is a disturbance which transfers from one part of the medium to the other without actual transfer of matter as a whole.

Mechanical Waves

Such waves require a material medium for propagation and are governed by Newton's Laws. These are the most common to be observed.

EXAMPLE
sound, waves on the surface of water, waves in a stretched string, compressional waves in a spring, seismic waves etc.

Electromagnetic Waves

Such waves do not require a material medium for propagation as they can travel through vacuum as well as through certain media. These waves are less familiar but are used constantly. All electromagnetic waves travel through vacuum at a same speed of 3×10^{8} ms^{-1}.

EXAMPLE: Light, X-rays, heat, radio waves etc.

Matter Waves

These waves are associated with electrons, photons, other fundamental particles as well as atoms and molecules. Since these particles form fundamental matter, so these waves are called **Matter Waves.**

CHARACTERISTICS OF WAVE MOTION

(a) In wave motion, there is transfer of energy and momentum from one place to another without any bulk motion of the medium. The particles of the medium simply vibrate about their mean positions and the disturbance propagates due to elastic and inertial properties of the medium.

(b) All the particles of the medium have the same kind of motion. However, there is a systematic phase change from one particle to another. The movement of a particle begins a little later than its predecessor.

(c) The wave velocity depends only on the nature of the medium, i.e., its elastic and inertial properties. It does not depend on the nature of the source, i.e., on the shape and size of the disturbance to be transmitted.

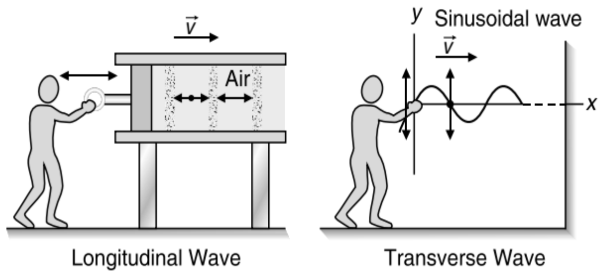
(d) On the other hand, the velocity of the particles (also called **particle velocity**) of the medium changes as they move from the mean position to the extreme position.

TYPES OF WAVE MOTION

Based on oscillations of particles and the direction of propagation, we are discussing two types of waves.

(a) Longitudinal Waves
(b) Transverse Waves

Longitudinal Waves are waves in which the particles of the medium oscillate along the direction of propagation. They travel in the form of compressions and rarefactions. A sound wave is set up in an air-filled pipe by moving a piston back and forth. Because the oscillations of an element of the air (represented by the dot) are parallel to the direction in which the wave travels, the wave is a longitudinal wave.



Transverse Waves are waves in which particles of the medium oscillate at right angles to the direction of propagation. They travel in the form of crests and troughs. A sinusoidal wave is sent along the string. A typical string element moves up and down continuously as the wave passes. This too is a transverse wave.

WAVE MOTION PARAMETERS

Wave Length (λ)

It is the spatial period of the wave at a given instant, i.e., it is the distance between two consecutive particles which vibrate in the same phase. Thus, in longitudinal waves, the distance between two successive compressions or rarefactions is equal to λ and in transverse waves, the distance between two successive crests or troughs is equal to λ.

Wave Number ($1/\lambda$)

The reciprocal of the wavelength is called Wave number and wave number is the measure of the number of waves present in a unit distance in the direction of the propagation of the wave. Mathematically, wave number $\bar{\nu} = 1/\lambda$

Time Period (T)

It is time in which the particle of a medium completes one vibration (to and fro) about its mean position.

Frequency (ν or f)

It is the number of vibrations performed by a source in unit time, so the number of crests (or troughs or compressions, or rarefactions) that pass a fixed point in unit time is called the **frequency** $(\nu \text{ or } f)$. It is the reciprocal of the time period T and has SI unit hertz (Hz). If v is the speed of the wave, then $v = \nu\lambda = f\lambda$

Amplitude (A)

Amplitude is maximum displacement suffered by particles of a medium about their mean position.

Phase (ϕ)

Phase of a wave is position of a point in time on a wave form. It is expressed in radian or sometimes degree. Please keep in mind that $180° = \pi$ radian

Path Difference (Δx)

It is the difference in the path travelled by two waves. It is generally measured in terms of the wavelength of the associated waves. It is measured in metre.

Phase Difference ($\Delta\phi$)

It is difference in the phase travelled by two waves. It is generally measured in terms of path difference of the associated waves. It is measured in radian.

ILLUSTRATION 1

Calculate the velocity of sound in a gas, in which the difference in frequencies of two waves of wavelength 1 m and 1.01 m is 4 Hz.

SOLUTION

Let frequencies of two waves be f_1 and f_2. Then

$$f_1 - f_2 = 4.$$

Since, $v = f\lambda$, so we have

$$\frac{v}{\lambda_1} - \frac{v}{\lambda_2} = v\left(\frac{1}{1.0} - \frac{1}{1.01}\right) = 4$$

$$\Rightarrow \quad v = \frac{4 \times 1.01}{0.01} = 404 \text{ ms}^{-1}$$

RELATION BETWEEN PATH DIFFERENCE (Δx) AND PHASE DIFFERENCE ($\Delta\phi$)

In wave motion, the wave travels a distance λ in one period and there is a phase difference of 2π after one period. Also, we observe that a

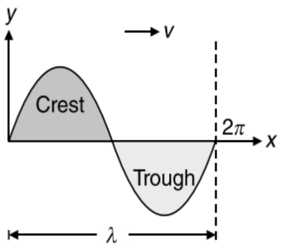
path difference of λ corresponds to phase of 2π

path difference of 1 corresponds to phase of $\frac{2\pi}{\lambda}$

path difference of Δx corresponds to phase of

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x \textbf{ OR } \Delta x = \left(\frac{\lambda}{2\pi}\right)\Delta\phi$$

where Δx is in meter and $\Delta\phi$ is in radian.



The term $\frac{2\pi}{\lambda} = k$ is called the Propagation constant.

$$\Rightarrow \quad \Delta\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x = k\Delta x$$

ILLUSTRATION 2

A wave of frequency 500 Hz has a wave velocity of $350\ \text{ms}^{-1}$. Calculate the distance between two points which are $60°$ out of phase. Also calculate the phase difference between two points 10^{-3} s apart.

SOLUTION

Since, $\lambda = \frac{v}{f} = \frac{350}{500} = 0.7$ m

So, $\Delta\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x$

$$\Rightarrow \quad \Delta x = \frac{\lambda(\Delta\phi)}{2\pi} = \frac{(0.7)(\pi/3)}{(2\pi)} = 0.116 \text{ m}$$

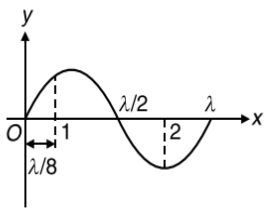
Also, $\Delta\phi = \omega\Delta t$

$$\Rightarrow \quad \Delta\phi = \left(\frac{2\pi}{T}\right)\Delta t = (2\pi f)\Delta t$$

$$\Rightarrow \quad \Delta\phi = (2\pi)(500)(10^{-3}) = \pi$$

ILLUSTRATION 3

Calculate the phase difference between the particle 1 and 2 located as shown in Figure.



SOLUTION

Path difference between the particles is

$$\Delta x = \left(\frac{\lambda}{2} - \frac{\lambda}{8}\right) + \frac{\lambda}{4} = \frac{5\lambda}{4}$$

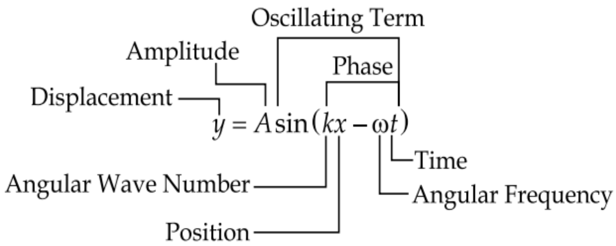
$$\Rightarrow \quad \Delta\phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{\lambda}\left(\frac{5\lambda}{8}\right) = \frac{10\pi}{8} = \frac{2\pi}{5}$$

EQUATION OF A HARMONIC WAVE

When a wave travels through a medium, the particles of the medium oscillate about their mean positions. If oscillations are simple harmonic, the wave is called a **Harmonic Wave**. Such waves are generated by sources that execute S.H.M. Consider a one-dimensional wave travelling towards the positive direction of x-axis. The displacement y of a particle at path difference x at time t is given by

$$y = A\sin(\omega t - kx)$$

where, $\omega = 2\pi/T$ and $k = 2\pi/\lambda$ are respectively called as the Angular Frequency and **Angular Wave Number (also called as Propagation Constant)**.



Two more alternative forms of this equation are

$$y = A\sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right]$$

$$y = A\sin\left[\frac{2\pi}{\lambda}(vt - x)\right]$$

If a wave is travelling towards the negative x-axis then x is replaced by $-x$, i.e.,

$$y = A\sin(\omega t + kx)$$

$$y = A\sin\left[2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)\right]$$

$$y = A\sin\left[\frac{2\pi}{\lambda}(vt + x)\right]$$

ILLUSTRATION 4

The equation of a progressive wave is $y = 1.5\sin(328t - 1.27x)$, where y, x are in cm and t is in second. Calculate the amplitude, frequency, time period and wavelength of the wave.

SOLUTION

Comparing with standard equation of a progressive wave we get, amplitude $A = 1.5$ cm, angular frequency $\omega = 2\pi/T = 328\ \text{rads}^{-1}$ and propagation constant $k = 2\pi/\lambda = 1.27\ \text{radcm}^{-1}$.

$$\Rightarrow \quad T = \frac{2\pi}{328} = \frac{2\times3.14}{328} = 0.019 \text{ s}$$

So, frequency $f=\frac{1}{T}=\frac{328}{2\pi}=\frac{328}{6.28}=52.23$ Hz

$\Rightarrow \quad \lambda=\frac{2\pi}{1.27}=\frac{2\times 3.14}{1.27}=4.94$ cm

CHARACTERISTIC WAVE EQUATION

The equation of a plane progressive wave is

$$y=A\sin(\omega t-kx)$$

The particle velocity v_P is given by

$$v_P=\frac{\partial y}{\partial t}=A\omega\cos(\omega t-kx) \quad \ldots(1)$$

Slope of displacement curve or strain is given by

$$\frac{\partial y}{\partial x}=-Ak\cos(\omega t-kx) \quad \ldots(2)$$

Differentiating (1) w.r.t. t,

$$\frac{\partial^2 y}{\partial t^2}=A\omega^2\sin(\omega t-kx) \quad \ldots(3)$$

Differentiating (2), w.r.t. x,

$$\frac{\partial^2 y}{\partial x^2}=-Ak^2\sin(\omega t-kx) \quad \ldots(4)$$

Dividing (3) by (4), we have

$$\frac{\partial^2 y/\partial t^2}{\partial^2 y/\partial x^2}=\frac{\omega^2}{k^2}=v^2$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \ldots(5)$$

This is the differential equation of wave motion, travelling with speed v.
The general solution of this equation is of the form

$$y(x,\ t)=f(ax\pm bt) \quad \ldots(6)$$

Thus, any *finite function* of x and t which satisfies equation (5) or which can be written as equation (6) represents a wave. The only condition is that it should be finite everywhere and at all times. Further, if these conditions are satisfied, then speed of wave (v) is given by,

$$v=\frac{|\text{Coefficient of } t|}{|\text{Coefficient of } x|}=\frac{b}{a}$$

The plus $(+)$ sign between ax and bt implies that the wave is travelling along positive x-direction and minus $(-)$ sign shows that it is travelling along negative x-direction.

Problem Solving Technique(s)

Analytically, any finite function of space and time, i.e., any $y(x,t)=y$ which satisfies $\frac{\partial^2 y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ must represent the progressive wave equation.

Functions such as $y=A\sin(\omega t)$ or $y=A\sin(kx)$ do not satisfy above equation, hence do not represent waves.
On the other hand, functions such as

$A\sin(kx-\omega t)$, $A\sin(kx)\sin(\omega t)$,

$A\sin(kx-\omega t)+B\cos(kx+\omega t)$, $\sqrt{ax+bt}$,

$(ax-bt)^2$, $Ae^{-B(x-vt)^2}$ Or $A\cos^2(kx-\omega t)$

all satisfy the wave equation, and hence these are wave functions. Also note that, for a function to be wave function, the three quantities x, t and v must appear in the combinations $(x-vt)$ or $(x+vt)$. Thus, $(x-vt)^2$ is acceptable but $(x^2-v^2t^2)$ is not. Negative sign between t and x implies that the wave is travelling along positive x-axis and vice-versa.

ILLUSTRATION 5

Which of the following four functions (out of 1, 2, 3 and 4) represent travelling waves?

1. $y=(x+5t)^3$ **2.** $y=\tan(2x+3t)$

3. $y=e^{-(4t+2x)^2}$ **4.** $y=\frac{1}{x+3t}$

SOLUTION

All these functions are of type $y=f(ax+bt)$.
However, function 1 is infinite for large values of x and t, function 2 is infinite when $2x+3t$ is close to $\frac{\pi}{2}$ and function 4 is infinite for very small negative value of $(4t+2x)$.
Only function 3 i.e., $y=e^{-(4t+2x)}$ is finite for all x and t, hence only function 3 is an appropriate representation of a plane progressive wave.

ILLUSTRATION 6

Does the function $y=y_0e^{-(x-vt)^2/k}$ represents a travelling wave? Also check mathematically.

SOLUTION

Yes, the function is of the form $f(x-vt)$ and is defined for every value of x and t. Since,

$$y=y_0e^{-(x-vt)^2/k}$$

$$\Rightarrow \quad \frac{\partial y}{\partial t}=\frac{2y_0v}{k}(x-vt)e^{-(x-vt)^2/k}$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2}=\frac{2y_0v^2}{k}e^{-(x-vt)^2/k}\left[-1+\frac{2}{k}(x-vt)^2\right] \quad \ldots(1)$$

Also, $y=y_0e^{-(x-vt)^2/k}$

$$\Rightarrow \quad \frac{\partial y}{\partial x} = -\frac{2y_0}{k}(x - vt)e^{-(x-vt)^2/k}$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{2y_0}{k}e^{-(x-vt)^2/k}\left[-1 + \frac{2}{k}(x - vt)^2\right] \quad \ldots(2)$$

Comparing (1) and (2), we get $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

Hence, the given function also satisfies the differential form of wave equation.

PARTICLE VELOCITY, WAVE SLOPE AND PARTICLE ACCELERATION IN A SINUSOIDAL WAVE

In a plane progressive harmonic wave the particles of the medium oscillate simple harmonically about their mean position. Therefore, all formulae that we have read in SHM also apply to particles here. For example, maximum particle velocity is $\pm A\omega$ at mean position and it is zero at extreme positions etc. Similarly, maximum particle acceleration is $\pm A\omega^2$ at extreme positions and zero at mean position. *However, wave velocity is different from the particle velocity. This depends on certain characteristics of the medium. Unlike the particle velocity which oscillates simple harmonically (between $+A\omega$ and $-A\omega$) the wave velocity is constant for given characteristics of the medium. The particle velocity is* $v_P = \frac{\partial y}{\partial t}$ *and wave slope (also called as strain) is* $\frac{\partial y}{\partial x}$.

Consider a sinusoidal wave

$$y = y(x, t) = A\sin(kx - \omega t) \quad \ldots(1)$$

$$\text{Then, } v_P = \frac{\partial y}{\partial t} = \frac{\partial y(x, t)}{\partial t} = -A\omega\cos(kx - \omega t) \quad \ldots(2)$$

$$\text{and } \frac{\partial y}{\partial x} = \frac{\partial y(x, t)}{\partial x} = Ak\cos(kx - \omega t) \quad \ldots(3)$$

From (2) and (3), we get

$$\frac{\partial y}{\partial t} = -\left(\frac{\omega}{k}\right)\frac{\partial y}{\partial x} = -v\left(\frac{\partial y}{\partial x}\right) \quad \left\{\because \frac{\omega}{k} = v\right\}$$

$$\Rightarrow \quad v_P = -v(\text{Wave Slope}) \quad \ldots(4)$$

i.e., particle velocity at a given position and time is equal to negative of the product of wave velocity with slope of the wave at that point at that instant.

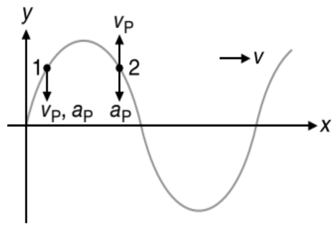
Particle acceleration is given by

$$a_P = \frac{\partial v}{\partial t} = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t) = -\omega^2 y(x, t)$$

So, the acceleration of the particle equals $-\omega^2$ times its displacement, similar to the result we obtained for SHM.

$$\Rightarrow \quad a_P = -\omega^2(\text{displacement}) \quad \ldots(5)$$

Figure shows the velocity (v_P) and acceleration (a_P) given by Equations (4) and (5) for two points 1 and 2 on a string when a sinusoidal wave is travelling in it along the positive x-direction.



At 1, slope of the curve is positive. Hence from equation (4) particle velocity (v_P) is negative or downwards. Similarly, displacement of the particle is positive, so from equation (5) acceleration will be negative or downwards.

At 2, slope is negative while displacement is positive. Hence v_P will be positive (upwards) and a_P is negative (downwards). Also note that the direction of v_P will change if the wave travels along negative x-direction.

ILLUSTRATION 7

If $y(x, t) = 0.05\sin\left(\frac{\pi}{2}(10x - 40t) - \frac{\pi}{4}\right)$ m, then find the wavelength, frequency and wave velocity. Also calculate the particle velocity and acceleration at $x = 0.5$ m and $t = 0.05$ s

SOLUTION

The equation may be rewritten as,

$$y(x, t) = 0.05\sin\left(5\pi x - 20\pi t - \frac{\pi}{4}\right) \text{ m}$$

Comparing this with equation of plane progressive harmonic wave, $y(x, t) = A\sin(kx - \omega t + \phi)$ we get

Wave number, $k = \frac{2\pi}{\lambda} = 5\pi$ radm^{-1} i.e., $\lambda = 0.4$ m

Angular frequency, $\omega = 2\pi f = 20\pi$ rads^{-1}

$$\Rightarrow \quad f = 10 \text{ Hz}$$

Wave velocity, $v = f\lambda = \frac{\omega}{k} = 4$ ms^{-1} in $+x$ direction

The particle velocity and acceleration are given by

$$v_P = \frac{\partial y}{\partial t} = -(20\pi)(0.05)\cos\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad v_P = \frac{\partial y}{\partial t} = 2.22 \text{ ms}^{-1}$$

$$a_P = \frac{\partial^2 y}{\partial t^2} = -(20\pi)^2(0.05)\sin\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad a_P = 140 \text{ ms}^{-2}$$

ILLUSTRATION 8

A sinusoidal wave travelling in the positive direction on a stretched string has amplitude 2 cm, wavelength 1 m and wave velocity 5 ms^{-1}. At $x = 0$ and $t = 0$, it is given that $y = 0$ and $\frac{\partial y}{\partial t} < 0$. Find the wave function $y(x,\ t)$.

SOLUTION

We start with a general form for a wave, moving along $+x$ axis, i.e.,

$$y(x,\ t) = A\sin(\omega t - kx + \phi)$$

The amplitude given is $A = 2\text{ cm} = 0.02\text{ m}$

The wavelength is $\lambda = 1\text{ m}$, so $k = \frac{2\pi}{\lambda} = 2\pi\text{ radm}^{-1}$

Angular frequency $\omega = vk = 10\pi\text{ rads}^{-1}$ $\{\because\ v = \omega/k\}$

$\Rightarrow\quad y(x, t) = (0.02)\sin[2\pi(5t - x) + \phi]$

For $x = 0$ and $t = 0$, we have $y = 0$ and $\partial y/\partial t < 0$

So, $\quad 0.02\sin\phi = 0$ $\{\because\ y = 0\}$

and $\quad -0.2\pi\cos\phi < 0$ $\{\because \partial y/\partial t < 0\}$

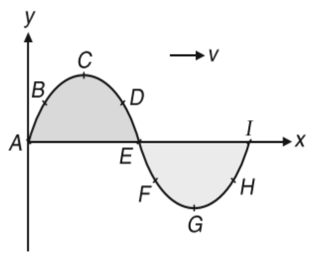
From these conditions, we may conclude that

$$\phi = 2n\pi \text{ where } n = 0, 2, 4, 6, \ldots..$$

$\Rightarrow\quad y(x,\ t) = 0.02\sin(10\pi t - 2\pi x)$

ILLUSTRATION 9

A transverse wave is travelling along a string from left to right. Figure represents shape of string (snap shot) at a given instant. At this instant, which points have an upward velocity, have downward velocity, have zero velocity and have maximum magnitude of velocity?



SOLUTION

For a travelling wave, the particle velocity v_P and the wave velocity v are related by

$$v_P = -v(\text{Wave Slope})$$

That is, the particle velocity is proportional to the negative of the slope of y-x curve.

Points having upward velocity, means v_P must be positive. It implies the slope must be negative, which is at points D, E and F.

Points having downward velocity, means v_P must be negative. It implies the slope must be positive, which is at points A, B, H and I.

Points having zero velocity, means the slope must be zero, which is at points C and G.

Points having maximum velocity, means the magnitude of slope must be maximum, which is points A, E and I.

The shape of the string after a small time Δt is shown by dotted line. The distances moved by different points in this time are indicated by arrows. Thus, the length of an arrow is proportional to magnitude of the velocity of that point. This diagram confirms the above conclusions.

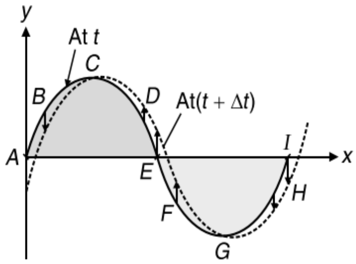


ILLUSTRATION 10

For a wave given by $y = a\sin(\omega t - kx)$, four points A at $x = 0$, B at $x = \pi/4k$, C at $x = \pi/2k$ and D at $x = 3\pi/4k$ are taken. For a particle at each of these points at $t = 0$, describe whether the particle is moving or not and in what direction and describe whether the particle is speeding up, slowing down or instantaneously not accelerating.

SOLUTION

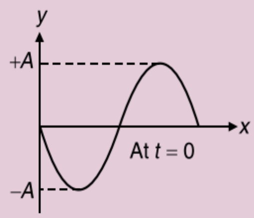
Since $y = a\sin(\omega t - kx)$, so particle velocity (v_P) and particle acceleration (a_P) are given by

$$v_P(x,\ t) = \frac{\partial y}{\partial t} = a\omega\cos(\omega t - kx)$$

$$a_P(x,\ t) = \frac{\partial^2 y}{\partial t^2} = -a\omega^2\sin(\omega t - kx)$$

For point A, at $t = 0$ and $x = 0$, we get $v_P = +a\omega$ and $a_P = 0$ i.e., particle is moving up but its acceleration is zero.

Direction of velocity can be obtained by an alternate method. At $t = 0$, $y = A\sin(-kx) = -A\sin kx$ i.e., y-x graph is as shown in Figure.



At $x = 0$, slope is negative. Therefore, particle velocity is positive $(v_P = -v \times \text{slope})$ as the wave is travelling along positive x-direction.

For point B, at $t = 0$ and $x = \pi/4k$, we get

$$v_P = a\omega\cos(-\pi/4) = +a\omega/\sqrt{2} \qquad \{\because kx = \pi/4\}$$

$$a_P = -a\omega^2\sin(-\pi/4) = +a\omega^2/\sqrt{2}$$

Velocity of particle is positive, i.e., the particle is moving upwards (along positive y-direction). Since v_P and a_P are in the same direction (both are positive), so the particle is speeding up.

For point C, at $t = 0$ and $x = \pi/2k$, we get

$$v_P = a\omega\cos(-\pi/2) = 0 \qquad \{\because kx = \pi/2\}$$

$$a_P = -a\omega^2\sin(-\pi/2) = a\omega^2$$

i.e., particle is stationary or at its extreme position $(y = -a)$. So, it is speeding up at this instant.

For point D, at $t = 0$ and $x = 3\pi/4k$, we get

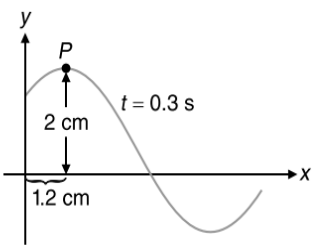
$$v_P = a\omega\cos(-3\pi/4) = -a\omega/\sqrt{2} \qquad \{\because kx = 3\pi/4\}$$

$$a_P = -a\omega^2\sin(-3\pi/4) = +a\omega^2/\sqrt{2}$$

Velocity of particle is negative i.e., the particle is moving downwards. Since v_P and a_P are in opposite directions, so, the particle is slowing down.

ILLUSTRATION 11

Figure shows a snapshot of a sinusoidal travelling wave taken at $t = 0.3$ s. The wavelength is 7.5 cm and the amplitude is 2 cm. If the crest P was at $x = 0$ at $t = 0$, write the equation of travelling wave.



SOLUTION

Given $\lambda = 7.5$ cm, so $k = \dfrac{2\pi}{\lambda} = 0.84\ \text{cm}^{-1}$

Since wave has travelled 1.2 cm in 0.3 s. So

Wave Speed $v = \dfrac{1.2}{0.3} = 4\ \text{cms}^{-1}$

$$\Rightarrow \quad \omega = vk = 3.36\ \text{rads}^{-1} \qquad \{\because v = \omega/k\}$$

Since the wave is travelling along positive x-direction and crest (maximum displacement) is at $x = 0$ at $t = 0$, we can write the wave equation as,

$$y(x, t) = A\cos(kx - \omega t)$$

$$\Rightarrow \quad y(x, t) = A\cos(\omega t - kx) \qquad \{\because \cos(-\theta) = \cos\theta\}$$

So, the required equation of the wave is

$$y(x, t) = 2\cos(0.84x - 3.36t)\ \text{cm}$$

GROUP VELOCITY

Speed of a single wave travelling in a medium is

$$v = \nu\lambda = \frac{\omega}{k}$$

and is called the **wave velocity** or **phase velocity**.

But if a number of waves having slightly different wavelengths travel in same medium, then medium is said to be dispersive and energy is not transmitted in medium with wave velocity or phase velocity, but it travels with **group velocity** v_g given by

$$v_g = \frac{d\omega}{dk} = v - \lambda\frac{dv}{d\lambda}$$

ILLUSTRATION 12

Waves passing through a certain medium have a dispersion relation, $\omega(k) = \sqrt{\omega_0^2 + \alpha^2 k^2}$. Here α and ω_0 are constants. Find phase velocity and group velocity in this medium.

SOLUTION

The phase velocity is,

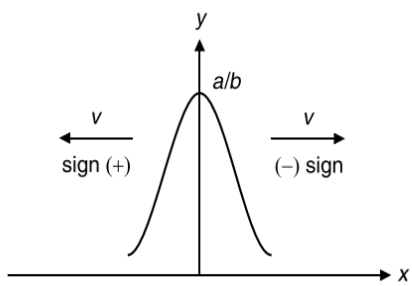
$$v = \frac{\omega}{k} = \frac{1}{k}\sqrt{\omega_0^2 + \alpha^2 k^2}$$

The group velocity is given by,

$$v_g = \frac{d\omega}{dk} = \frac{\alpha^2 k}{\sqrt{\omega_0^2 + \alpha^2 k^2}} = \left(\frac{k}{\omega}\right)\alpha^2 = \frac{\alpha^2}{v}$$

A SYMMETRICAL WAVE PULSE

A wave pulse is a disturbance localised only in a small part of space at a given instant and its shape does not change during propagation. Pulse will be symmetric if at $t = 0$, $y(x) = y(-x)$



A symmetrical wave pulse shown has an equation

$$y=\frac{a}{b+(x\mp vt)^2}$$

where, a and b are constants. Speed of pulse is

$$v=\frac{|\text{Coefficient of } t|}{|\text{Coefficient of } x|}$$

Please note that, here the coefficient of x must be 1 and then the amplitude of the wave pulse is a/b. Negative sign is for pulse propagating along positive x-axis, positive sign for a pulse propagating along negative x-axis and v is the pulse speed.

ILLUSTRATION 13

A wave pulse on a horizontal string is represented by the function $y(x,\ t)=\frac{5}{1+(x-2t)^2}$ (cgs units). Plot this function at $t=0$, 2.5 s and 5 s.

SOLUTION

At the given times, the function representing the wave pulse is given by

$$y(x,\ 0)=\frac{5}{1+x^2}$$

$$\Rightarrow\quad y(x,\ 2.5\text{ s})=\frac{5}{1+(x-5)^2}$$

$$\Rightarrow\quad y(x,\ 5\text{ s})=\frac{5}{1+(x-10)^2}$$

The maximum of $y(x,\ 0)$ is 5 cm which it is located at $x=0$ and the pulse is also centred at $x=0$. At $t=2.5$ s and 5 s, the centre of the pulse has moved to $x=5$ cm and 10 cm, respectively. So, in each 2.5 s time interval, the pulse moves 5 cm in the positive x-direction. Its velocity is therefore +2 cms^{-1} a value that is also evident from the given function shown in Figure.

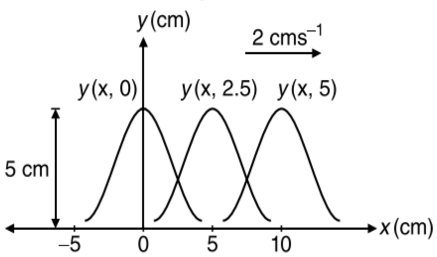


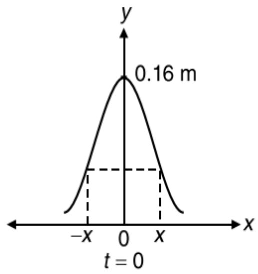
ILLUSTRATION 14

If $y(x,\ t)=\frac{0.8}{[(4x+5t)^2+5]}$ represents a moving pulse where x and y are in metre and t in second. Then select the correct alternative(s).

(A) Pulse is moving in positive x-direction.
(B) In 2 s it will travel a distance of 2.5 m.
(C) Its maximum displacement is 0.16 m.
(D) It is a symmetric pulse.

SOLUTION

Shape of pulse at $t=0$, $x=0$ is shown in figure.



$$y(0,\ 0)=\frac{0.8}{5}=0.16\text{ m}$$

From the figure it is clear that $y_{max}=0.16$ m

From the given equation, we see that at $t=0$

$$y(x)=\frac{0.8}{16x^2+5}\text{ and }y(-x)=\frac{0.8}{16x^2+5}$$

Since $y(x)=y(-x)$, so pulse is symmetric.

Speed of pulse is

$$v=\frac{|\text{Coefficient of } t|}{|\text{Coefficient of } x|}=\frac{5}{4}=1.25\text{ ms}^{-1}$$

So, it will travel a distance of 2.5 m in 2 s

Also, note that at $t=1$ s, $x=-1.25$ m and at $t=2$ s, $x=-2.5$ m, so we can say that the wave is travelling along negative x direction. **So, OPTIONS (B), (C) and (D) are correct.** Also, we can compare this with

$$y=\frac{a}{b+(x\pm vt)^2},\text{ where } a=\frac{0.8}{16}\text{ and } b=\frac{5}{16}$$

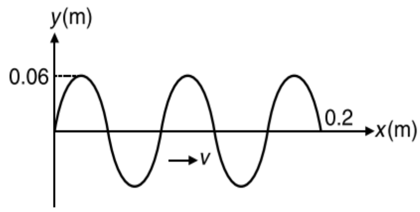
to get the desired results.

Test Your Concepts-I

Based on Wave Equation & Properties

(Solutions on page H.233)

1. For the wave shown in figure, write the wave equation if its position is shown at $t = 0$. Given, speed of wave is $v = 300\ \text{ms}^{-1}$.



2. A transverse harmonic wave of amplitude 0.02 m is generated at $x = 0$ at one end of a long horizontal string by a tuning fork of frequency 500 Hz. At a given instant of time, the displacement of the particle at $x = 0$ is zero, the displacement of a particle at $x = 0.1$ m is $-0.01\sqrt{3}$ m and that of a particle at $x = 0.8$ m is $0.01\sqrt{3}$. Find the velocity of the wave.

3. A pulse is propagating on a long stretched string along its length taken as positive x-axis. Shape of the string at $t = 0$ is given by

$$y = \begin{cases} \sqrt{a^2 - x^2}, & \text{when } |x| \geq a \\ 0, & \text{when } |x| \leq 0 \end{cases}$$

Study the propagation of this pulse, if it is travelling in positive x-direction with speed v.

4. Verify that the equation, $y = a\sin(\omega t - kx)$ satisfies the wave equation $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$. Find speed of wave and the direction in which it is travelling.

5. A progressive wave of frequency 500 Hz is travelling with a velocity of $360\ \text{ms}^{-1}$. How far apart are two points $60°$ out of phase?

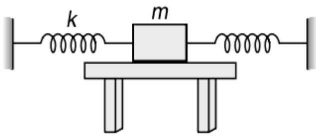
6. Out of the following functions, which one(s) represent(s) a wave?

(a) $\frac{1}{x + vt}$ (b) $\ln(x + vt)$

(c) $(x - vt)^2$ (d) $e^{-(x-vt)^2}$

7. A wave pulse moving to the right along the x-axis is represented by the wave function $y(x, t) = \frac{2}{(x - 3t)^2 + 1}$, where x and y are measured in cm and t is measured in seconds. Plot the wave function at $t = 0$, $t = 1$ s and $t = 2$ s.

8. One end of each of two identical springs, each of force-constant $0.5\ \text{Nm}^{-1}$, are attached on the opposite sides of a wooden block of mass 0.01 kg. The other ends of the springs are connected to separate rigid supports such that the springs are unstretched and are collinear in a horizontal plane. as shown in Figure.



To the wooden piece is fixed a pointer which touches a vertically moving plane paper. The wooden piece, kept on a smooth horizontal table is now displaced by 0.02 m along the line of springs and released. If the speed of paper is $0.1\ \text{ms}^{-1}$, find the equation of path traced by pointer on paper and the distance between two consecutive maxima on this path.

9. Equation of a transverse wave in a stretched string is given by $y = 2\sin\left[2\pi\left(\frac{x}{30} - \frac{t}{0.01}\right)\right]$, where y, x are in cm, t is in second. Calculate the amplitude, frequency, wavelength and wave velocity.

10. A travelling wave pulse is given by, $y = \frac{10}{5 + (x + 2t)^2}$, where x and y are in metre and t in second. In which direction and with what velocity is the pulse propagating? What is the amplitude of pulse?

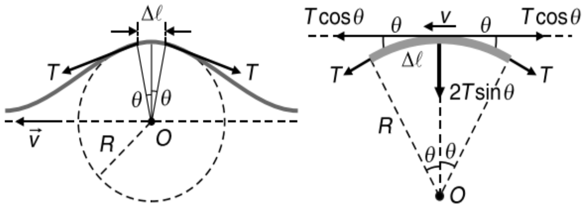
11. The wave function of a pulse is given by $y = \frac{3}{2 + (x - 4t)^2}$, where y is in metres and t is seconds. Determine the wave velocity of the pulse and indicate the direction of propagation of the wave.

12. Two pulses travelling on a same string are described by functions $y_1 = \frac{5}{(3x - 4t)^2 + 2}$ and $y_2 = \frac{-5}{(3x + 4t - 6)^2 + 2}$. In which direction does each pulse travel? At what time and point do the two waves cancel?

SPEED OF A TRANSVERSE WAVE IN A STRING

Let us consider a single symmetrical pulse travelling along a string with a speed v from left to right. If the amplitude of the pulse is small compared to the length of the string, the tension T will be approximately constant along the string.

To obtain the speed v of a wave on a stretched string, it is convenient to select a reference frame in which the pulse remains stationary i.e., we run along with the pulse keeping it constantly in our view. In this reference frame, it will appear as if the string is moving past us from right to left with speed v as shown in Figure.



Consider a small string element of length Δl, mass Δm within the pulse, such that it forms an arc of a circle or radius R and subtends an angle 2θ at the centre of that circle.

At the instant shown, this element is moving with speed v in a circular path, so it has a centripetal acceleration v^2/R. Forces acting on the segment are the tension T at each end. Horizontal components of tension T are equal and opposite, so they cancel. Vertical components of tension T point radially inward towards centre of the circular arc and provide the centripetal acceleration such that

$$2T\sin\theta = \frac{(\Delta m)v^2}{R} \qquad \ldots(1)$$

For small θ, $\sin\theta \approx \theta$ and if μ is the mass per unit length of the string, then mass of the element of length $\Delta l = 2R\Delta\theta$ is

$$\Delta m = \mu\Delta l = 2\mu R\theta$$

So, from equation (1), we get

$$2T\theta = (2\mu R\theta)\left(\frac{v^2}{R}\right)$$

$$\Rightarrow \quad v = \sqrt{\frac{T}{\mu}}$$

Conceptual Note(s)

If D is the diameter of the string, ℓ is its length and ρ is its density, then

$$\Rightarrow \quad m = \frac{\pi(D/2)^2\ell\rho}{\ell} = \frac{\pi D^2\rho}{4}$$

$$\Rightarrow \quad v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\pi D^2\rho/4}} = \frac{2}{D}\sqrt{\frac{T}{\pi\rho}} \qquad \ldots(1)$$

Since μ(the mass per unit length of string) is

$$\mu = \frac{m}{\ell} = \frac{A\ell\rho}{\ell} = A\rho$$

where A is area of cross-section of string, then from equation (1), we get

$$v = \sqrt{\frac{T}{\rho A}} = \sqrt{\frac{\text{Stress}}{\rho}} \qquad \left\{\because \frac{T}{A} = \text{Stress}\right\}$$

ILLUSTRATION 15

A harmonic wave with a wavelength of 20 cm, an amplitude of 3 cm and a velocity of 2 ms^{-1} travels on a string to the left along the x-axis. The mass per unit length of the string is 0.25 gm^{-1}.

(a) What are the frequency and period of the wave motion?
(b) How much is the tension in the string?
(c) What is the position function for the wave?

SOLUTION

(a) The frequency, $f = \frac{v}{\lambda} = \frac{2}{0.2} = 10 \text{ Hz}$

The period $T = \frac{1}{f} = \frac{1}{10} = 0.1 \text{ s}$

(b) We know that $v = \sqrt{\frac{T}{\mu}}$

$$\Rightarrow \quad T = \mu v^2 = (0.25\times10^{-3})\times(2)^2 = 10^{-3} \text{ N}$$

(c) In order to write the wave equation, we need to know and ω

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.2} = 10\pi \text{ radm}^{-1}$$

and $\omega = 2\pi f = 2\pi(10) = 20\pi \text{ rads}^{-1}$

Thus, the wave equation is $y = A\sin(kx + \omega t)$

$$\Rightarrow \quad y = 0.03\sin[(10\pi)x + (20\pi)t]$$

where x and y are in metre and t is in second.

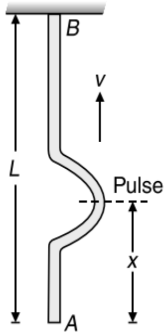
ILLUSTRATION 16

A uniform rope of mass 0.1 kg and length 2.45 m hangs from a ceiling. Calculate the speed of transverse wave in the rope at a point 0.5 m distant from the lower end. Also calculate the time taken by a transverse wave to travel the full length of the rope. If a particle is dropped from the ceiling at the instant wave starts from the free end of rope, then calculate the distance from the free end where the wave crosses the particle. Take $g = 9.8 \text{ ms}^{-2}$.

SOLUTION

If M is the mass of the string of length L, the mass of length x of the string will be

$$M_1 = \mu x = \frac{M}{L}x$$



The tension in the string at a distance x from lower end A is $T = M_1 g = \left(\frac{M}{L}x\right)g$

$$\Rightarrow \quad v = \sqrt{\frac{T}{m}} = \sqrt{\frac{Mxg}{L(M/L)}} = \sqrt{xg} \qquad \ldots(1)$$

For $x = 0.5$ m, we get $v = \sqrt{0.5 \times 9.8} = 2.21 \text{ ms}^{-1}$

The tension and hence the velocity of the wave is different at different points of the string. So, if at point x the wave travels distance dx in time dt,

$$v = \frac{dx}{dt} = \sqrt{gx} \qquad \{\text{from (1)}\}$$

$$\Rightarrow \quad t = \int dt = \int_0^L \frac{dx}{\sqrt{gx}} = \frac{1}{\sqrt{g}}\int_0^L x^{-1/2}dx$$

$$\Rightarrow \quad t = 2\sqrt{\frac{L}{g}} = 2\sqrt{\frac{2.45}{9.8}} = 1 \text{ s}$$

Let y be the distance from the free end where the particle crosses the pulse. Then

for the particle, $t = \sqrt{\frac{2(L-y)}{g}}$ $\qquad \left\{\because L - y = \frac{1}{2}gt^2\right\}$

for the pulse, $t = \sqrt{\frac{2y}{g}}$

At the instant of crossing t is same, so we get

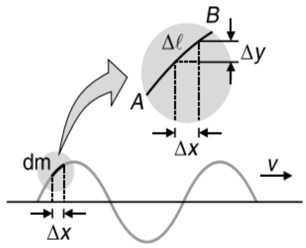
$$\sqrt{\frac{2(L-y)}{g}} = 2\sqrt{\frac{y}{g}}$$

$$\Rightarrow \quad 2L - 2y = 4y$$

$$\Rightarrow \quad y = \frac{2L}{6} = \frac{L}{3} = \frac{2.45}{3} \approx 0.82 \text{ m}$$

ENERGY DENSITY OF A TRANSVERSE WAVE

Consider a transverse wave to propagate in a string. When this wave moves along the string, it carries energy along the direction of travel. To calculate the energy density associated with the wave, let us consider a small segment of the string of lengthΔx as shown in Figure.



While oscillating, this segment has potential energy due to its stretching and kinetic energy due to its motion. The potential energy of a segment AB is the work done by the tension T in stretching the string, which is $T(\Delta l - \Delta x)$, where Δl is the length of the stretched segment and Δx is its original length. From the figure we see that

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Rightarrow \quad \Delta l = \Delta x\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]^{\frac{1}{2}} \approx \Delta x\left[1 + \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

$$\Rightarrow \quad \Delta l - \Delta x \approx \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2 \Delta x$$

So, potential energy of this segment is

$$U = \frac{1}{2}T(\Delta l - \Delta x)^2 = \frac{1}{2}T\Delta x\left(\frac{\Delta y}{\Delta x}\right)^2$$

For very small increments i.e., when $\Delta x \to 0$, then $\frac{\Delta y}{\Delta x} \to \frac{\partial y}{\partial x}$

(since y is a simultaneous function of x and t, so this partial derivative indicates that we are interested in the variation of y with position at a fixed time). So, potential energy is

$$dU = \frac{1}{2}Tdx\left(\frac{\partial y}{\partial x}\right)^2$$

Potential energy per unit length is

$$\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \qquad \ldots(1)$$

The kinetic energy dK of the segment is

$$dK = \frac{1}{2}dm\left(\frac{\partial y}{\partial t}\right)^2$$

So, kinetic energy per unit length is

$$\frac{dK}{dx} = \frac{1}{2}\left(\frac{dm}{dx}\right)\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

Since, $\frac{\partial y}{\partial t} = -v\left(\frac{\partial y}{\partial x}\right)$

$$\Rightarrow \quad \frac{dK}{dx}=\frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2, \text{ where } v=\sqrt{\frac{T}{\mu}}$$

$$\Rightarrow \quad \frac{dK}{dx}=\frac{1}{2}\mu\left(\frac{T}{\mu}\right)\left(\frac{\partial y}{\partial x}\right)^2=\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \qquad \ldots(2)$$

The first interesting thing we observe is that the potential energy per unit length is equal to the kinetic energy per unit length of a transverse wave carried by the string.

So, total energy per unit length of the segment is

$$\frac{dE}{dx}=\frac{dU}{dx}+\frac{dK}{dx}=2\left[\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\right]=T\left(\frac{\partial y}{\partial x}\right)^2$$

$$\Rightarrow \quad \frac{dE}{dx}=T\left(\frac{\partial y}{\partial x}\right)^2=\mu v^2\left(\frac{\partial y}{\partial x}\right)^2 \qquad \left\{\because v=\sqrt{\frac{T}{\mu}}\right\}$$

Assuming $y=A\sin(\omega t-kx)$, we get

$$\frac{\partial y}{\partial x}=-Ak\cos(\omega t-kx)$$

$$\Rightarrow \quad \frac{dE}{dx}=\mu v^2\left(\frac{\partial y}{\partial x}\right)^2=\mu v^2A^2k^2\cos^2(\omega t-kx)$$

$$\Rightarrow \quad \frac{dE}{dx}=\mu\omega^2A^2\cos^2(\omega t-kx) \qquad \left\{\because v=\frac{\omega}{k}\right\}$$

If S be the cross-sectional area of the string, then energy density associated with this segment is

$$\frac{dE}{Sdx}=\frac{\mu\omega^2A^2}{S}\cos^2(\omega t-kx)$$

Since, $\mu=\dfrac{dm}{dx}=\dfrac{\rho(Sdx)}{dx}=S\rho$

$$\frac{dE}{Sdx}=\rho\omega^2A^2\cos^2(\omega t-kx) \qquad \{\because \mu=S\rho\}$$

However, a wave can be very long, so the energy associated with each wavelength of the wave is

$$E=\int_0^\lambda dE=\mu\omega^2A^2\int_0^\lambda\cos^2(\omega t-kx)dx$$

If we take a snapshot of the wave at time $t=0$, then energy of a given element is

$$E=\int_0^\lambda dE=\mu\omega^2A^2\int_0^\lambda\cos^2(kx)dx$$

$$\Rightarrow \quad E=\int_0^\lambda dE=\frac{\mu\omega^2A^2}{2}\left[\int_0^\lambda dx+\int_0^\lambda\cos(2kx)dx\right]$$

$$\Rightarrow \quad E=\int_0^\lambda dE=\frac{\mu\omega^2A^2}{2}\left[\lambda+\frac{1}{2k}\sin(2kx)\Big|_0^\lambda\right]$$

$$\Rightarrow \quad E=\int_0^\lambda dE=\frac{\mu\omega^2A^2}{2}\left[\lambda+\frac{\sin(2k\lambda)}{2k}\right]$$

Since, $\sin(2k\lambda)=\sin(4\pi)=0 \qquad \{\because k=2\pi/\lambda\}$

$$\Rightarrow \quad E=\int_0^\lambda dE=\frac{1}{2}\mu\omega^2A^2\lambda$$

We can also say that the potential energy associated with each wavelength of the wave is equal to the kinetic energy and is given by

$$U=K=\frac{1}{2}\mu\omega^2A^2\int_0^\lambda\cos^2(\omega t-kx)dx=\frac{1}{4}\mu\omega^2A^2\lambda$$

POWER AND INTENSITY OF WAVE TRAVELLING THROUGH A STRETCHED STRING

If a wave is travelling in a stretched string of mass per unit length μ, then the power of the wave is given by

$$P_{\text{ins}}=\frac{dE}{dt}=\left(\frac{dE}{dx}\right)\left(\frac{dx}{dt}\right)=v\left(\frac{dE}{dx}\right)$$

Since, $\dfrac{dE}{dx}=\mu\omega^2A^2\cos^2(\omega t-kx)$

$$\Rightarrow \quad P_{\text{ins}}=\mu\omega^2A^2v\cos^2(\omega t-kx)$$

However, average power is given by

$$P_{\text{av}}=\langle P\rangle=\mu\omega^2A^2v\langle\cos^2(\omega t-kx)\rangle$$

Since, $\langle\cos^2(\omega t-kx)\rangle=\langle\sin^2(\omega t-kx)\rangle=1/2$

$$\Rightarrow \quad P_{\text{av}}=\frac{1}{2}\mu\omega^2A^2v$$

Note that $P\propto A^2$ and $P\propto\omega^2$

These results hold for all types of waves.

Intensity (I) is defined as the energy transmitted per second per unit area normal to the direction of propagation of the wave or the power transmitted across a unit area normal to the direction of propagation. Mathematically

$$I=\frac{1}{S}\left(\frac{dE}{dt}\right)=\frac{P}{S}=\left(\frac{dE}{Sdx}\right)\left(\frac{dx}{dt}\right)=uv$$

where, $u=\dfrac{dE}{Sdx}=\dfrac{dE}{dV}$ is energy density of the wave.

Since, we have discussed earlier, that energy density (u) of the wave is proportional to $A^2\omega^2$, so

$$I\propto A^2\omega^2$$

When a wave front advances from a distance r_1 from a point source to a distance r_2, its surface area increases from $4\pi r_1^2$ to $4\pi r_2^2$. If there is no dissipation of energy, then $P=4\pi r_1^2I_1=4\pi r_2^2I_2$, *where I_1 and I_2 being the intensities at distances r_1 and r_2, respectively.*

$$\Rightarrow \quad \frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

ABOUT AMPLITUDE

For a plane progressive wave, the amplitude remains constant i.e., it does not vary with distance.

For a spherical wave (i.e., wave starting from a point source), the amplitude varies inversely with distance from position of source i.e.,

$$A \propto \frac{1}{r} \qquad \left\{\because I \propto A^2 \propto \frac{1}{r^2}\right\}$$

For a cylindrical wave (i.e., wave starting from a linear source), the amplitude varies inversely as the square root of distance from the axis of source i.e.,

$$A \propto \frac{1}{\sqrt{r}} \qquad \left\{\because I \propto A^2 \propto \frac{1}{r}\right\}$$

ILLUSTRATION 17

A stretched string is forced to transmit transverse waves by means of an oscillator coupled to one end. The string has a diameter of 4 mm. The amplitude of the oscillation is 10^{-4} m and the frequency is 10 Hz. Tension in the string is 100 N and mass density of wire is 4.2×10^3 kgm^{-3}. Calculate the equation of the waves along the string, energy per unit volume of the wave, average energy flow per unit time across any section of the string and power required to drive the oscillator.

SOLUTION

Speed of transverse wave in a string of density ρ, area of cross section S is given by

$$v = \sqrt{\frac{T}{\rho S}} \qquad \{\because \mu = \rho S\}$$

$$\Rightarrow \quad v = \sqrt{\frac{100}{(4.2\times10^3)(\pi/4)(4\times10^{-3})^2}}$$

$$\Rightarrow \quad v = 43.53 \text{ ms}^{-1}$$

Since, $\omega = 2\pi f = 20\pi \text{ rads}^{-1} = 62.83 \text{ rads}^{-1}$

$$\Rightarrow \quad k = \frac{\omega}{v} = 1.44 \text{ m}^{-1}$$

Equation of wave along the string is given by

$$y(x,\ t) = A\sin(\omega t - kx)$$

$$\Rightarrow \quad y(x,\ t) = 10^{-4}\sin(62.83t - 1.44x)$$

Energy per unit volume of the string is

$$u = \frac{1}{2}\rho\omega^2A^2 = \left(\frac{1}{2}\right)(4.2\times10^3)(62.83)^2(10^{-4})^2$$

$$\Rightarrow \quad u = 8.29\times10^{-2} \text{ Jm}^{-3}$$

Average energy flow per unit time is power, so

$$P = \left(\frac{1}{2}\rho\omega^2A^2\right)(sv) = (u)(sv)$$

$$\Rightarrow \quad P = (8.29\times10^{-2})\left(\frac{\pi}{4}\right)(4\times10^{-3})^2(43.53)$$

$$\Rightarrow \quad P = 4.53\times10^{-5} \text{ W}$$

So, the power required to drive the oscillator is obviously 4.53×10^{-5} W

REFLECTION OF STRING WAVE

If incident wave is $y = a\sin(\omega t - kx)$, then the equation of reflected wave takes the form

$$y = a'\sin(\omega t + kx)$$

where a' is new amplitude of reflected wave.

When string wave is reflected from a denser medium or rigid boundary, the wave suffers an additional phase change of π or a path change of $\lambda/2$. The equation of reflected wave in this case takes the form

$$y = a'\sin(\omega t + kx + \pi) = -a'\sin(\omega t + kx)$$

A mechanical wave is reflected and refracted at a boundary separating two media according to the usual laws of reflection and refraction.

Conceptual Note(s)

So, in reflection the following rules apply

(a) A wave coming from a medium where the wave velocity is larger, i.e., from a rarer to a denser medium, suffers a phase change of π or a path change of $\lambda/2$ (called **STOKE'S LAW**) on reflection.

(b) A wave coming from a denser to a rarer medium is reflected without any phase change.

(c) These rules apply in reflection of light also.

When a pulse travelling along a string reaches the end, it is reflected. If the end is fixed as shown in Figure 1, the pulse returns inverted.

This is because as the leading edge reaches the wall, the string pulls up the wall. According to Newton's Third Law, the wall will exert an equal and opposite force on the string at all instants. This force is therefore, directed first down and then up. It produces a pulse that is inverted but otherwise identical to the original.

The motion of free end can be studied by tying a ring at the end of string so that the ring can slide smoothly on the rod. The ring and rod maintain the tension but exert no transverse force.

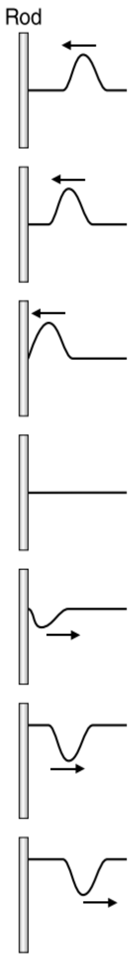


Figure 1

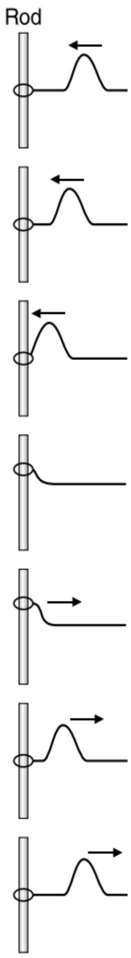


Figure 2

When a wave arrives at this free end, the ring slides along the rod. The ring reaches a maximum displacement. At this position the ring and the string come momentarily to rest as in the fourth drawing from the top in Figure 2. But the string is stretched in this position, giving increased tension, so the free end of the string is pulled back down and again a reflected pulse is produced, but now the direction of the displacement is the same as for the initial pulse.

WAVE PROPERTIES AFTER REFLECTION / REFRACTION (TRANSMISSION)

Any type of wave is associated with the following physical quantities:

(i) speed of wave (v)

(ii) frequency (f), time period (T) and angular frequency (ω)

(iii) wavelength (λ) and wave number (k)

(iv) amplitude (A) and intensity I and

(v) phase (ϕ)

Now, let us see what happens to these physical quantities when they are either reflected or transmitted.

S. No.	Wave Property	Reflection	Refraction (Transmission)
1.	v	does not change	changes
2.	f, T, w	do not change	do not change
3.	λ, k	do not change	change
4.	A, I	change	change
5.	ϕ	$\Delta\phi = 0$, from a rarer medium $\Delta\phi = \pi$, from a denser medium	does not change

PARTIAL REFLECTION AND TRANSMISSION

When a pulse encounters the boundary between a light string and a heavy string then partial reflection and partial transmission of the pulse takes place. Since the tensions are the same, the relative magnitudes of the wave velocities are determined by the mass densities. In Figure 1, the pulse approaches from the light string. The heavy string behaves somewhat like a wall but it can move, and so part of the original pulse is transmitted to the heavy string.

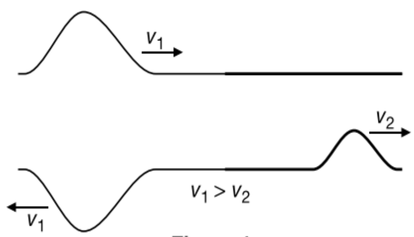


Figure 1

In Figure 2, the pulse approaches from a heavy string. The light string offers little resistance and now approximates a free end. Consequently, the reflected pulse is not inverted.

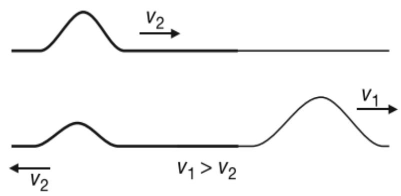


Figure 2

BOUNDARY CONDITIONS

As observed, when a travelling wave encounters the boundary of another medium, it is partially reflected and partially transmitted.

In this section let us investigate the division of the incident wave into reflected and transmitted components, as specified by the boundary conditions. For arriving at the results, we shall be considering two strings of linear mass density μ_1 and μ_2 joined at $x = L$ as shown in Figure 1. The rules obtained with waves on a string can also be generalized to the reflection and transmission of other types of waves.

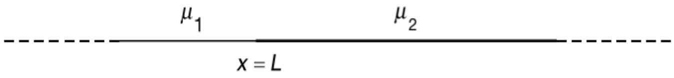


Figure 4.1

Our boundary is the junction between two strings of linear mass densities μ_1 and μ_2 as shown in figure. Both strings are under the same tension T.

Let the incident pulse at the boundary is represented by $y_i(x - v_1 t)$.

So, the reflected pulse is represented by $y_r(x + v_1 t)$ and the transmitted pulse by $y_t(x - v_2 t)$.

Both the incident and the reflected pulses travel on the left string at a speed $v_1 = \sqrt{\dfrac{T}{\mu_1}}$ while the transmitted pulse propagates to the right string at a speed $v_2 = \sqrt{\dfrac{T}{\mu_2}}$.

According to the Superposition Principle, the net vertical displacement of the left string is given by $y_i(x - v_1 t) + y_r(x + v_1 t)$.

Now, there are two independent conditions, the pulses must satisfy at the boundary.

CONDITION-1

The net vertical displacement on both sides of the boundary must be the same at all times as shown in Figure 2.

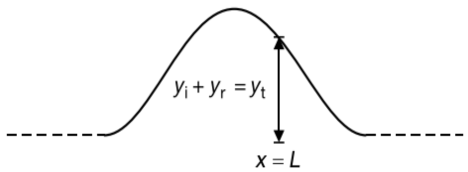


Figure 2

This is expressed mathematically as,

$$y_i(L - v_1 t) + y_r(L + v_1 t) = y_t(L - v_2 t) \qquad \ldots(1)$$

where L, the position of the boundary has been substituted for x because $x = L$ at the boundary.

CONDITION-2

This condition is a consequence of the fact that the two strings must exert equal and opposite forces on each other at the junction. As Figure 3 illustrates $\vec{F}_1 = -\vec{F}_2$ only holds if the slopes of the two strings are the same at the junction. We therefore, have, $\left\{\dfrac{\partial}{\partial x}\left[y_i(x - v_1 t) + y_r(x + v_1 t)\right]\right\}_{x=L} =$

$$\left\{\frac{\partial}{\partial x}\left[y_t(x - v_2 t)\right]\right\}_{x=L} \qquad (2)$$

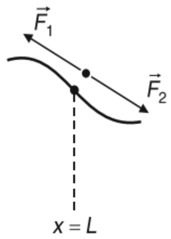


Figure 3

Solving Equations (1) and (2), we get

$$y_t(L - v_2 t) = \left(\frac{2v_2}{v_1 + v_2}\right) y_i(L - v_1 t)$$

and $$y_r(L + v_1 t) = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) y_i(L - v_1 t)$$

The above result can also be explained as under. Suppose the equations of incident wave, reflected wave and transmitted wave are,

$$y_i = A_i \sin(kx - \omega t)$$

$$y_r = A_r \sin(kx + \omega t)$$

and $$y_t = A_t \sin(kx - \omega t)$$

where, $A_r = \left(\dfrac{v_2 - v_1}{v_1 + v_2}\right) A_i$ and $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right) A_i$

The above relations can also be derived from Energy Conservation Principle as shown below

By Law of Conservation of Energy, we have

$$\begin{pmatrix}\text{Average}\\\text{Incident}\\\text{Power}\end{pmatrix} = \begin{pmatrix}\text{Average}\\\text{Reflected}\\\text{Power}\end{pmatrix} + \begin{pmatrix}\text{Average}\\\text{Transmitted}\\\text{Power}\end{pmatrix}$$

$$\Rightarrow \quad P_i = P_r + P_t$$

$$\Rightarrow \quad \frac{1}{2}\mu_1 \omega^2 A_i^2 v_1 = \frac{1}{2}\mu_1 \omega^2 A_r^2 v_1 + \frac{1}{2}\mu_2 \omega^2 A_t^2 v_2$$

Since $v = \sqrt{\dfrac{T}{\mu}}$

$$\Rightarrow \quad \left(\frac{T}{v_1^2}\right)\omega^2 A_i^2 v_1 = \left(\frac{T}{v_1^2}\right)\omega^2 A_r^2 v_1 + \left(\frac{T}{v_2^2}\right)\omega^2 A_t^2 v_2$$

$$\Rightarrow \quad \frac{A_i^2}{v_1} = \frac{A_r^2}{v_1} + \frac{A_t^2}{v_2} \qquad \ldots(1)$$

Further $A_i + A_r = A_t \qquad \ldots(2)$

Solving these two equations for A_r and A_t, we get

$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_i \text{ and } A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$$

CASE 1:

When $v_1 < v_2$, i.e., medium 1 is denser and medium 2 is rarer, then both A_r and A_t are positive and also $A_t > A_i$ as shown in Figure 1.

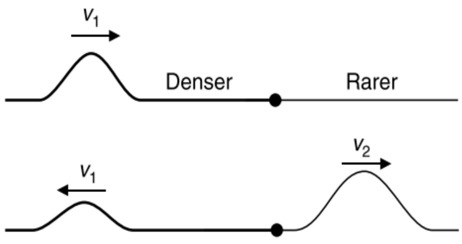


Figure 1 : For $v_1 < v_2$

CASE 2:

When $v_1 > v_2$ i.e., medium 1 is rarer and 2 is denser, then A_r = negative and $|A_r|$ and $|A_t|$ both are individually less than $|A_i|$. The negative value of A_r indicates that the reflected wave suffers a phase change of π as shown in Figure 2.

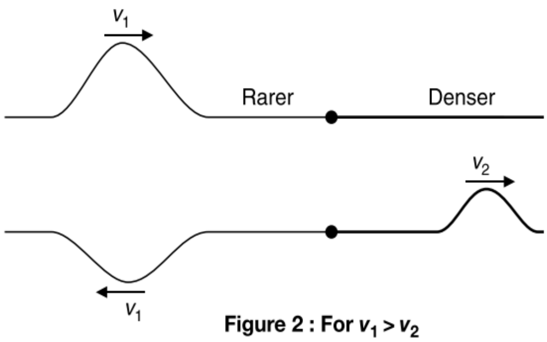


Figure 2 : For $v_1 > v_2$

CASE 3:

Free end of a string: In this case $\mu_2 \to 0$, so $v_2 \to \infty$ and so we get

$$A_t = 0 \text{ and } A_r = A_i$$

CASE 4:

Fixed end of a string: Since the fixed end is equivalent to a string of infinite linear mass density, $v_2 = \sqrt{\dfrac{T}{\mu_2}} \to 0$ and so we get

$$A_t = 0 \text{ and } A_r = -A_i$$

ILLUSTRATION 18

Two wires of different densities are soldered together end to end and are then stretched under tension T. The wave speed in the first wire is twice that in the second wire.

(a) If the amplitude of incident wave is A, what are amplitudes of reflected and transmitted waves?

(b) Assuming no energy loss in the wire, find the fraction of the incident power that is reflected at the junction and fraction of the same that is transmitted.

SOLUTION

(a) Since, $v_1 = 2v_2$ and $v = \sqrt{\dfrac{T}{\mu}}$

$$\Rightarrow \mu_1 = \frac{\mu_2}{4}$$

Since, $A_r = \left(\dfrac{v_2 - v_1}{v_1 + v_2}\right)A = \left(\dfrac{v_2 - 2v_2}{2v_2 + v_2}\right)A = -\dfrac{A}{3}$

and $A_t = \left(\dfrac{2v_2}{v_1 + v_2}\right)A = \left(\dfrac{2v_2}{2v_2 + v_2}\right)A = \dfrac{2}{3}A$

Here negative sign with A_r implies that the reflected wave suffers a phase change of π.

(b) Fraction of incident power reflected is

$$f_r = \frac{P_r}{P_i} = \frac{\frac{1}{2}\mu_1\omega^2 A_r^2 v_1}{\frac{1}{2}\mu_1\omega^2 A_i^2 v_1} = \left(\frac{A_r}{A_i}\right)^2 = \frac{1}{9}$$

Fraction of incident power transmitted is

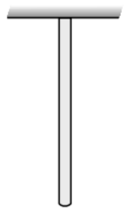
$$f_t = 1 - f_r = 1 - \frac{1}{9} = \frac{8}{9}$$

Test Your Concepts-II

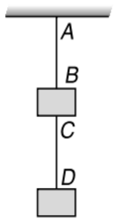
Based on Transverse Wave in a String & Properties

(Solutions on page H.234)

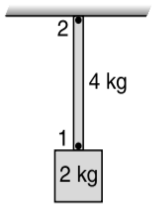
1. A wire of variable mass per unit length $\mu = \mu_0 x$, is hanging from the ceiling as shown in figure. The length of wire is ℓ_0. A small transverse disturbance is produced at its lower end. Find the time after which the disturbance will reach to the other ends.



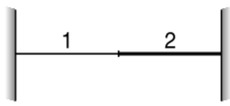
2. A wave pulse starts propagating in the positive x-direction along a non-uniform wire of length l with a mass per unit length given by $\mu = \mu_0 + \alpha x$ and under a tension 100 N. Find the time taken by the pulse to travel from the lighter end $(x = 0)$ to the heavier end where μ_0 and α are constant.
3. Two blocks each having a mass of 3.2 kg are connected by a wire CD and the system is suspended from the ceiling by another wire AB as shown in Figure. The linear mass density of the wire AB is 10 gm^{-1} and that of CD is 8 gm^{-1}. Calculate the speed of a transverse wave pulse produced in AB and in CD.



4. A transverse wave travelling in a string is given by the equation $y = A\sin(kx - \omega t)$. Calculate the potential energy per unit volume possessed by this wave.
5. A uniform rope of length 10 cm and mass 4 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope, what is the wavelength of the pulse when it reaches the top of the rope?



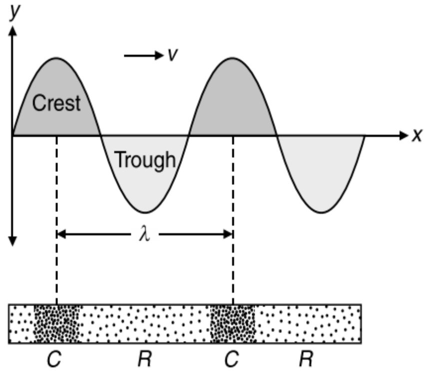
6. The speed of a transverse wave travelling in a wire having a length 50 cm and mass 5 g is 80 ms^{-1}. The area of cross-section of the wire is 1 mm^2 and its Young's modulus is 16×10^{11} Nm^{-2}. Calculate the extension of the wire over its natural length.
7. One end of 12 m long rubber tube with a total mass of 900 g is fastened to a fixed support. A cord attached to the other and passes over a pulley and supports an object with a mass of 5 kg. The tube is struck a transverse blow at one end. Find the time required for the pulse to reach the other end. Take $g = 9.8$ ms^{-2}.
8. Energy is transmitted at a rate P_1 by a wave of frequency v_1, on a string with tension T_1. Assuming the amplitude to remain the same, calculate the new transmission rate in terms of P_1 if tension is increased to $4T_1$ and the wave frequency is decreased to $\frac{v_1}{4}$.
9. A steel wire of length 64 cm weighs 5 g. If it is stretched by a force of 8 N, calculate the speed of a transverse wave passing in it.
10. Two wires of different densities but same area of cross section are soldered together at one end and are stretched to a tension T. The velocity of a transverse wave in one wire is double of that in the second wire. Calculate the ratio of density of the first wire to that of the second wire.
11. A long wire PQR is made by joining two wires PQ and QR of equal radii. PQ has a length 4.8 m and mass 0.06 kg. QR has a length 2.56 m and mass 0.2 kg. The wire PQR is under a tension of 80 N. A sinusoidal wave pulse of amplitude 3.5 cm is sent along the wire PQ from the end P. No power is dissipated during the transmission of the wave pulse. Calculate
    (a) The time taken by the wave pulse to reach the other end R and
    (b) The amplitude of the reflected and transmitted wave pulse after the incident wave pulse crosses the joint Q.
12. Two strings 1 and 2 are taut between two fixed supports (as shown in figure) such that the tension in both strings is same. Mass per unit length of 2 is more than that of 1. Explain which string is denser for a transverse travelling wave.



13. A string of length 40 cm and weighing 10 g is attached to a spring at one end and to a fixed wall at the other end. The spring has a spring constant of 160 Nm^{-1} and is stretched by 1 cm. If a wave pulse is produced on the string near the wall, calculate the time taken by the wave pulse to reach the spring.

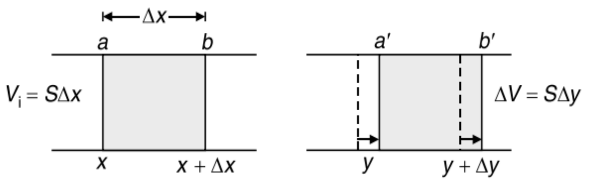
RELATION BETWEEN DISPLACEMENT WAVE AND PRESSURE WAVE FOR A LONGITUDINAL WAVE

A transverse wave propagates by means of crests and troughs and there is no change in pressure in a transverse wave.



A longitudinal wave propagates by means of compressions and rarefactions and there is always a pressure variation along longitudinal wave.

Consider a harmonic displacement wave moving through air contained in a long tube of cross sectional area S as shown in Figure.



The volume of gas that has a thickness Δx in the horizontal direction is $V_i = S\Delta x$. The change in volume ΔV is $S\Delta y$, where Δy is the difference between the value of y at $x+\Delta x$ and the value of y at x. From the definition of Bulk Modulus, the pressure variation in the gas is given by

$$\Delta P = -B\left(\frac{\Delta V}{V_i}\right) = -B\left(\frac{S\Delta y}{S\Delta x}\right) = -B\left(\frac{\Delta y}{\Delta x}\right)$$

When Δx approaches zero, the ratio $\frac{\Delta y}{\Delta x}$ becomes $\frac{\partial y}{\partial x}$ (This partial derivative indicates that we are interested in the variation of y with position at a fixed time). So, we get

$$\Delta P = -B\frac{\partial y}{\partial x} \quad \text{...(1)}$$

This is the equation which relates the displacement equation with the pressure equation.

If, $y = A\sin(\omega t - kx)$ ...(2)

then, $\frac{\partial y}{\partial x} = kA\cos(\omega t - kx)$ ...(3)

and from Equations (1) and (3), we get

$$\Delta P = BAk\cos(\omega t - kx) = (\Delta P)_m \cos(\omega t - kx) \quad \text{...(4)}$$

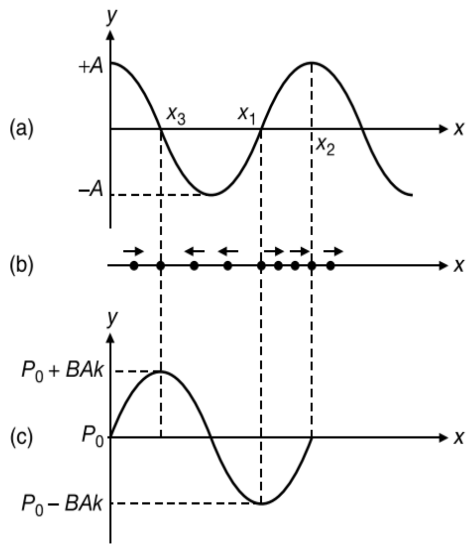
where, $(\Delta P)_m = BAk$ is the amplitude of pressure variation.

The pressure being maximum at compressions and minimum at rarefactions. In the loudest sound the change in pressure is 28 Nm^{-2} and amplitude is nearly 10^{-5} m, while in the faintest sound the change in pressure is 0.002 Nm^{-2} and amplitude is nearly 10^{-11} m.

Thus, a sound wave may be considered either as a displacement wave or as a pressure wave. We note that the pressure wave is 90° out of phase with the displacement wave.

From Equations (2) and (4), we observe that the pressure equation is 90° out of phase with displacement equation. When the displacement is zero, the pressure variation is either maximum or minimum and vice-versa.

Figure (a) shows displacement of air molecules from equilibrium in a harmonic sound wave versus position at some instants. Points x_1 and x_3 are points of zero displacement. Now from, Figure (b), we observe that, just to the left of x_1, the displacement is negative, indicating that the gas molecules are displaced to left, away from point x_1 at this instant.



Just to the right of x_1, the displacement is positive, indicating that the molecules are displaced to the right, which is again away from point x_1. So, at point x_1 the pressure of the gas is minimum. So, if P_0 be the atmospheric pressure (normal pressure), then pressure at x_1 will be,

$$P(x_1) = P_0 - (\Delta P)_m = P_0 - BAk$$

At point x_3, the pressure (and hence the density also) is maximum because the molecules on both sides of that point are displaced toward point x_3. Hence,

$$P(x_3) = P_0 + (\Delta P)_m = P_0 + BAk$$

At point x_2 the pressure (and hence the density) does not change because the gas molecules on both sides of that point have equal displacements in the same direction and hence

$$P(x_2) = P_0$$

From Figure (a) and (c), we observe that the pressure change and displacement are 90° out of phase.

ILLUSTRATION 19

The pressure variation equation of a sound wave in air is given by

$$\Delta P = (0.02\ \text{Nm}^{-2})[\sin(500\ \text{s}^{-1})t - (3\ \text{m}^{-1})x]$$

Calculate the frequency, wavelength and the speed of sound wave in air. If the equilibrium pressure of air is $1.01 \times 10^5\ \text{Nm}^{-2}$, then calculate the maximum and minimum pressure at a point as the wave passes through that point.

SOLUTION

Since $\Delta P = 0.02\sin(500t - 3x)$ ...(1)

$\Rightarrow \quad \Delta P = (\Delta P)_{\text{max}} \sin(\omega t - kx)$ ...(2)

Comparing equations (1) and (2), we get $k = 3\ \text{m}^{-1}$, $\Delta P_{\text{max}} = 0.02\ \text{Nm}^{-2}$ and $\omega = 500\ \text{rads}^{-1}$

So, $f = \dfrac{\omega}{2\pi} = \dfrac{500}{2\pi} = \dfrac{250}{\pi}$ Hz

Since $k = \dfrac{2\pi}{\lambda}$, so $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{3}$ m

$\Rightarrow \quad v = f\lambda = \dfrac{250}{\pi} \times \dfrac{2\pi}{3} = \dfrac{500}{3}\ \text{ms}^{-1}$

If P_0 be the atmospheric pressure, then

$$P_{\text{max}} = P_0 + (\Delta P)_{\text{max}} = (1.01 \times 10^5 + 0.02)\ \text{Nm}^{-2}$$

$\Rightarrow \quad P_{\text{max}} = 101000.02\ \text{Nm}^{-2}$

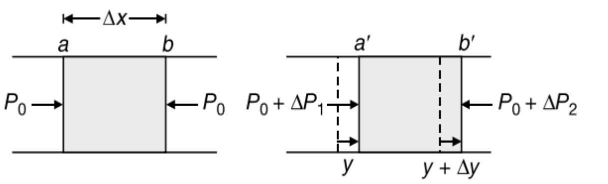
Also, $P_{\text{min}} = P_0 - (\Delta P)_{\text{max}} = (1.01 \times 10^5 - 0.02)\ \text{Nm}^{-2}$

$\Rightarrow \quad P_{\text{min}} = 100999.98\ \text{Nm}^{-2}$

SPEED OF A LONGITUDINAL WAVES

Let us calculate the speed at which a longitudinal pulse propagates through a fluid (gas/liquid). To derive the expression for the wave equation, we shall be applying Newton's Second Law to the motion of an element of the fluid.

Consider a fluid element ab of thickness Δx confined to a tube of cross-sectional area S as shown in figure.



Assume that the equilibrium pressure of the fluid is P_0. Due to the disturbance, the section a of the element moves a distance y from its mean position and section b moves a distance $y + \Delta y$ to a new position b'. The pressure on the left side of the element becomes $P_0 + \Delta P_1$ and on the right side it becomes $P_0 + \Delta P_2$. If ρ is the equilibrium density, the mass of the element is $\rho S \Delta x$, because when the element moves its mass does not change, even though its volume and density do change.

The net force acting on the element is,

$$F = (\Delta P_1 - \Delta P_2)S$$

The net acceleration of the element is $a = \partial^2 y / \partial t^2$

Applying Newton's Second Law to the motion of the element, we get

$$F = (\Delta P_1 - \Delta P_2)S = \rho S \Delta x \frac{\partial^2 y}{\partial t^2} \quad \text{...(1)}$$

Dividing both sides by Δx and observing that in the limit as $\Delta x \to 0$, we get

$$\frac{\Delta P_1 - \Delta P_2}{\Delta x} \to \frac{\partial P}{\partial x}$$

So, equation (1) becomes,

$$-\frac{\partial P}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2} \quad \text{...(2)}$$

The excess pressure ΔP may be written as

$$\Delta P = -B\frac{\partial y}{\partial x} \quad \text{\{as discussed already\}}$$

When this is used in Equation (2), we get the wave equation as

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{B}\left(\frac{\partial^2 y}{\partial t^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = \frac{B}{\rho}\frac{\partial^2 y}{\partial x^2}$$

Comparing this equation with the wave equation $\dfrac{\partial^2 y}{\partial t^2} = v^2 \dfrac{\partial^2 y}{\partial x^2}$, we get the speed of longitudinal wave in a fluid (liquid or gas) to be

$$v = \sqrt{\frac{B}{\rho}}$$

When a longitudinal wave propagates in a solid rod or bar, the rod expands sideways slightly when it is compressed longitudinally and the speed of a longitudinal wave in a rod is given by

$$v = \sqrt{\frac{Y}{\rho}}$$

The speed of longitudinal waves in a medium of elasticity E and density ρ is given by

$$v = \sqrt{\frac{E}{\rho}}$$

For Liquids and Gases, E is the Bulk modulus of elasticity (B). For Solids, E is the Young's modulus (Y). So, we have

$$v_{\text{solid}} = \sqrt{\frac{Y}{\rho}} \text{ and } v_{\text{liquid/gas}} = \sqrt{\frac{B}{\rho}}$$

If the solid is extended, the speed will depend on the Bulk modulus B and the Shear modulus η and is given by

$$v = \sqrt{\frac{B + \frac{4\eta}{3}}{\rho}}$$

Conceptual Note(s)

(a) The density of a solid is much larger than that of a gas but the elasticity is larger by a greater factor. Hence longitudinal waves in a solid travel much faster than that in a gas.

(b) In a liquid the speed lies in between the two i.e.

$$v_{\text{solid}} > v_{\text{liquid}} > v_{\text{gas}}$$

(c) The speed at which a longitudinal pulse propagates through a fluid (gas/liquid) of Bulk's modulus B having density ρ is given by

$$v_{\text{liquid/gas}} = \sqrt{\frac{B}{\rho}}$$

(d) When a longitudinal wave propagates in a solid rod or bar, the rod expands sideways slightly when it is compressed longitudinally and the speed of a longitudinal wave in a rod is given by

$$v_{\text{solid}} = \sqrt{\frac{Y}{\rho}}$$

(e) The speed of longitudinal waves in a medium of elasticity E and density ρ is given by

$$v = \sqrt{\frac{E}{\rho}}$$

For Liquids and Gases, E is the Bulk modulus of elasticity (B) and for Solids, E is the Young's modulus (Y).

(f) If the solid is extended, the speed will depend on the Bulk modulus B and the Shear modulus η and is given by

$$v = \sqrt{\frac{B + \frac{4\eta}{3}}{\rho}}$$

ILLUSTRATION 20

The volumetric strain of water at a pressure of $10^5\ \text{Nm}^{-2}$ is 5×10^{-5}. Calculate the speed of sound in water. Density of water is $10^3\ \text{kgm}^{-3}$.

SOLUTION

Bulk modulus of water is

$$B = \frac{\text{Normal stress}}{\text{Volume strain}} = \frac{10^5}{5 \times 10^{-5}} = 2 \times 10^9\ \text{Nm}^{-2}$$

Since, density of water is $\rho = 10^3\ \text{kgm}^{-3}$

So, speed of sound in water is $v = \sqrt{\frac{B}{\rho}}$

$$\Rightarrow \quad v = \sqrt{\frac{2 \times 10^9}{10^3}} = 1.41 \times 10^3\ \text{ms}^{-1}$$

ILLUSTRATION 21

The Bulk modulus and modulus of rigidity for aluminium are $7.5 \times 10^{10}\ \text{Nm}^{-2}$ and $2.1 \times 10^{10}\ \text{Nm}^{-2}$ respectively. Determine the velocity of the waves in the medium. Density of aluminium is $2.7 \times 10^3\ \text{kgm}^{-3}$.

SOLUTION

Given, Bulk modulus, $B = 7.5 \times 10^{10}\ \text{Nm}^{-2}$

Modulus of rigidity, $\eta = 2.1 \times 10^{10}\ \text{Nm}^{-2}$

Density, $\rho = 2.7 \times 10^3\ \text{kgm}^{-3}$

So, velocity of longitudinal waves in aluminium is

$$v = \sqrt{\frac{B + \frac{4}{3}\eta}{\rho}} = \sqrt{\frac{7.5 \times 10^{10} + \frac{4}{3} \times 2.1 \times 10^{10}}{2.7 \times 10^3}}$$

$$\Rightarrow \quad v = 6.18 \times 10^3\ \text{ms}^{-1}$$

RELATION BETWEEN PRESSURE WAVE AND DENSITY WAVE

Let us now find the relation between pressure wave and density wave from the definition of Bulk Modulus (B), we have

$$B = -\frac{dP}{dV/V} \quad \ldots(1)$$

Since, volume $\text{Volume} = \frac{\text{Mass}}{\text{Density}} = \frac{M}{\rho}$

$$\Rightarrow \quad dV = -\left(\frac{m}{\rho^2}\right)d\rho = -\left(\frac{V}{\rho}\right)d\rho$$

$$\Rightarrow \quad \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Substituting in equation (1), we get

$$d\rho = \frac{\rho(dP)}{B} = \frac{dP}{v^2} \qquad \left\{\because v = \sqrt{\frac{B}{\rho}}\right\}$$

This can be re-written as

$$\Delta\rho = \left(\frac{\rho}{B}\right)\Delta P = \frac{\Delta P}{v^2}$$

So, this relation relates the pressure equation with the density equation. For example, if we have

$$\Delta P = (\Delta P)_m \sin(kx - \omega t)$$

then, $\Delta\rho = (\Delta\rho)_m \sin(kx - \omega t)$

where, $(\Delta\rho)_m = \frac{\rho}{B}(\Delta P)_m = \frac{(\Delta P)_m}{v^2}$

So, the density equation is in phase with the pressure equation and is 90° out of phase with the displacement equation.

ILLUSTRATION 22

The pressure variation in a sound wave is given by

$$\Delta P = 8\cos\left(4.00x - 3000t + \frac{\pi}{4}\right)$$

Calculate the displacement amplitude. The density of the medium is 10^3 kgm^{-3}.

SOLUTION

Since, $\Delta P_0 = BAK = \rho v^2 Ak \qquad \{\because B = \rho v^2\}$

$$\Rightarrow \quad \Delta P_0 = \frac{\rho\omega^2}{k^2}Ak = \frac{\rho\omega^2 A}{k} \qquad \left\{\because v^2 = \frac{\omega^2}{k^2}\right\}$$

$$\Rightarrow \quad A = \frac{(\Delta P_0)k}{\rho\omega^2} = \frac{8\times 4.00}{10^3\times(3000)^2} = 3.55\times 10^{-9}\ \text{m}$$

ILLUSTRATION 23

Find the displacement amplitude of a sound wave having a frequency of 100 Hz and a pressure amplitude of 10 Pa. If another sound wave of frequency 300 Hz has a displacement amplitude of 10^{-7} m, find the pressure amplitude of this wave. Speed of sound in air is 340 ms^{-1} and density of air is 1.29 kgm^{-3}.

SOLUTION

Since, $(\Delta P)_m = BAk$, where $k = \frac{\omega}{v} = \frac{2\pi f}{v}$

Since, $B = \rho v^2 \qquad \{\because v = \sqrt{B/\rho}\}$

$$\Rightarrow \quad (\Delta P)_m = (\rho v^2)(A)\left(\frac{2\pi f}{v}\right) = \frac{(\Delta P)_m}{2\pi v\rho f} \qquad \ldots(1)$$

$$\Rightarrow \quad A = \frac{10}{2\times 3.14\times 100\times 1.29\times 340} = 3.63\times 10^{-5}\ \text{m}$$

From above equation (1), we get

$$(\Delta P)_m = 2\pi f\rho v A$$

$$\Rightarrow \quad (\Delta P)_m = 2\times 3.14\times 300\times 1.29\times 340\times 10^{-7}$$

$$\Rightarrow \quad (\Delta P)_m = 8.26\times 10^{-2}\ \text{Nm}^{-2}$$

ABOUT NATURE OF WAVES

For propagation of transverse mechanical waves, the medium must be rigid. Therefore, within solids the waves are transverse. A gas (air) has no rigidity, therefore the waves in a gas are always longitudinal. The waves on the surface of water are a combination of longitudinal and transverse waves, such types of waves are usually called **ripples**. In these waves the particles of medium vibrate up and down and back and forth simultaneously, describing ellipse in a vertical plane. The waves in a stretched string are always transverse. The wave on the surface of solids (e.g. rails) are longitudinal.

POWER AND INTENSITY OF WAVE TRAVELLING THROUGH A MEDIUM

When a particle executes harmonic oscillations, there exists oscillation energy which keeps on changing between kinetic and potential forms. In a progressive waves, this energy is transferred through the medium with a velocity v. The amplitude of the particle velocity is given as

$$v_0 = A\omega$$

If ρ is the density of the medium, the energy per unit volume is

$$U = \frac{1}{2}\rho v_0^2 = \frac{1}{2}\rho A^2\omega^2$$

If S is the area of cross-section of the medium, the energy associated with a volume $S\Delta x$, will be

$$\Delta E = U\,\Delta V = \frac{1}{2}\rho A^2\omega^2 S\,\Delta x$$

So, power (rate of transmission of energy) becomes

$$P = \frac{\Delta E}{\Delta t} = \frac{1}{2}\rho v\omega^2 A^2 S \qquad \left\{\text{as } \frac{\Delta x}{\Delta t} = v\right\}$$

The intensity is defined as average energy transmitted per unit normal area per second i.e., power per unit area. Hence, intensity

$$I = \frac{\Delta E}{S\Delta t} = \frac{P}{S} = \frac{1}{2}\rho v\omega^2 A^2$$

Further, as in case of sound wave, the displacement amplitude is related to the pressure amplitude through the relation $p_0 = \rho v A\omega$, so

$$I = \frac{1}{2}\rho v\omega^2\left(\frac{p_0}{\rho v\omega}\right)^2 = \frac{p_0^2}{2\rho v}$$

So, we observe that $I \propto A^2$or $I \propto p_0^2$ and also $I \propto \omega^2$

The SI unit of intensity is Wm^{-2}.

ILLUSTRATION 24

An electric siren of 100 W sends out sound waves in air at a frequency of 1 kHz. Assuming that the sound energy travels equally in all directions in space, calculate. The intensity of sound at a point 100 m away from the source. If the velocity of sound v is 350 ms^{-1} and density of air $\rho = 1.29\ \text{kgm}^{-3}$, determine the displacement amplitude of the air particles at that point.

SOLUTION

The intensity of sound at the point 100 m away from the source is

$$I = \frac{P}{\text{Area}} = \frac{P}{4\pi r^2} = \frac{100}{4\times 3.141\times(100)^2}$$

$$\Rightarrow \quad I = 7.96\times 10^{-4}\ \text{Wm}^{-2}$$

Now, we know that

$$I = \frac{1}{2}\rho v\omega^2 A^2$$

where, $I = 7.96\times 10^{-4}\ \text{Wm}^{-2}$, $\rho = 1.29\ \text{kgm}^{-3}$,

$v = 350\ \text{ms}^{-1}$ and $\omega = 2\pi\times 1\times 10^3\ \text{rads}^{-1}$

$$\Rightarrow \quad A^2 = \frac{2I}{\rho v\omega^2} = \frac{2\times 7.96\times 10^{-4}}{1.29\times 350\times(2\pi\times 10^3)^2}$$

$$\Rightarrow \quad A^2 = 8.94\times 10^{-14}$$

$$\Rightarrow \quad A = 2.99\times 10^{-7}\ \text{m}$$

INTENSITY OF SOUND WAVE

Travelling sound waves, like all other travelling waves, transfer energy from one region of space to another. As discussed earlier, we define the intensity of a wave (denoted by I) to be the time average rate at which energy is transported by the wave, per unit area across a surface perpendicular to the direction of propagation. Since, we have already derived that

$$I = \frac{1}{2}\rho A^2\omega^2 v \quad \ldots(1)$$

For a sound wave, we have

$$(\Delta P)_m = BAk = BA\left(\frac{\omega}{v}\right)$$

$$\Rightarrow \quad \omega = \frac{(\Delta P)_m v}{BA}$$

Substituting this value in Equation (1), we get

$$I = \frac{1}{2}\rho A^2\frac{(\Delta P)_m^2 v^2}{B^2A^2}v \quad \{\because \rho v^2 = B\}$$

$$\Rightarrow \quad I = \frac{v(\Delta P)_m^2}{2B} \quad \ldots(2)$$

So, intensity of a sound wave can be calculated by using either of the Equations (1) or (2)

ILLUSTRATION 25

A point sound source is situated in a medium having Bulk modulus $1.6\times 10^5\ \text{Nm}^{-2}$. An observer standing at a distance 10 m from the source, writes down the equation for the wave as $y = A\sin(15\pi x - 6000\pi t)$; where y and x are in metre and t is in second. The maximum pressure, amplitude tolerable to the observer's ear is 24π Pa, then find the

(a) density of the medium
(b) displacement amplitude A of the waves received by the observer
(c) maximum power of the sound source.

SOLUTION

Since $k = 15\pi\ \text{m}^{-1}$ and $\omega = 6000\pi\ \text{sec}^{-1}$

$$\Rightarrow \quad v = \omega/k = 400\ \text{ms}^{-1}$$

(a) Since, $v = \sqrt{B/\rho}$

$$\Rightarrow \rho = \frac{B}{v^2} = \frac{1.6\times 10^5}{(400)^2} = 1\ \text{kgm}^{-3}$$

(b) Pressure amplitude is $(\Delta P)_{\text{max}} = BAk$

$$\Rightarrow A = \frac{(\Delta P)_{\text{max}}}{Bk} = \frac{24\pi}{1.6\times 10^5\times 15\pi} = 10^{-5}\ \text{m}$$

$$\Rightarrow A = 10\ \mu\text{m}$$

(c) Intensity at distance r from a point source is given by

$$I = \frac{W}{4\pi r^2}. \text{ Also, } I = \frac{(\Delta P)_{\text{max}}^2}{2\rho v}$$

$$\Rightarrow \frac{W}{4\pi(10)^2} = \frac{(24\pi)^2}{2\times 1\times 400}$$

$$\Rightarrow W = 8.93\times 10^3\ \text{watt}$$

ILLUSTRATION 26

A point like sound source emits sound of frequency 2000 Hz, uniformly in all directions. Its intensity at a distance of 6 m is 0.960 mW m^{-2}.

(a) Find the intensity at 30 m.
(b) At 6 m from the source, find the displacement and pressure amplitudes. (Atmospheric pressure $= 1\times 10^5$ Pa and density of air $= 1.3\ \text{kgm}^{-3}$)

SOLUTION

(a) $I \propto \dfrac{1}{r^2}$

So, $0.960 \propto \dfrac{1}{6^2}$ …(1)

and $I \propto \frac{1}{30^2}$ ...(2)

Dividing (2) and (1), we get

$$\frac{I}{0.960} = \frac{6^2}{30^2}$$

$$\Rightarrow I = 0.960\left(\frac{1}{5}\right)^2 = 0.038 \text{ mW m}^{-2}$$

(b) Now, $I = \frac{1}{2}\rho\omega^2 vA^2$

$$0.960\times10^{-3} = \frac{1}{2}\times1.3\times(2\pi\times2000)^2\times332\times A^2$$

$$\Rightarrow A \approx 171 \text{ nm}$$

Also, $I = \frac{v(\Delta P_0)^2}{2B}$

$$\Rightarrow \Delta P_0 = \sqrt{\frac{2BI}{v}} = \sqrt{\frac{2\times1.41\times10^5\times0.960\times10^{-3}}{332}}$$

where, $B = \gamma P$ and $\gamma_{air} \approx 1.41$

$$\Rightarrow \Delta P_0 = 0.89 \text{ Pa}$$

SUPERSONIC AND SHOCK WAVES

When an object moves with a velocity greater than that of sound, it is termed as **Supersonic**. When such a supersonic body or plane travels in air, it produces energetic disturbance which moves in backward direction and diverges in the form of a cone. Such disturbances are called the **Shock Waves**.

The speed of supersonic is measured in **Mach number**. One Mach number is the speed of sound.

$$\text{Mach number} = \frac{\text{Speed of Supersonic}}{\text{Speed of Sound}}$$

INTENSITY LEVEL AND LOUDNESS OF SOUND

The term loudness describes the human perception of sound. A sound wave of higher intensity is perceived as louder than a wave of lower intensity. However, the relation is not linear. The sensation of loudness is roughly proportional to logarithm of intensity. The intensity level is defined by an arbitrary scale that corresponds roughly to sensation of loudness. Human ear responds to sound intensities over a wide range (from 10^{-12} Wm^{-2} to 1 Wm^{-2}). Therefore, instead of specifying intensity of sound in Wm^{-2}, we use a logarithmic scale of intensity called sound level, *SL* or *L* which is expressed in decibel (dB) and is given by

$$L(\text{in dB}) = L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where $I_0 = 10^{-12}$ Wm^{-2} is the threshold of hearing of human ear (the weakest sound audible to human ear). Unit of loudness level is decibel (dB) and

1 bel = 10 decibel

The decibel is a dimensionless unit.

A sound of intensity I_0 has

$$L = 10\log_{10}\left(\frac{I_0}{I_0}\right) = 0 \text{ dB}$$

The upper range of human hearing also called threshold of pain is 1 Wm^{-2} and it corresponds to

$$L = 10\log_{10}\left(\frac{1}{10^{-12}}\right) = 120 \text{ dB}$$

A sound 10 times more intense than I_0 has intensity level 10 dB, a sound 100 times more intense than I_0 has intensity level 20 dB and so on. *So, as the intensity of sound increases (or decreases) by a factor of 10, the intensity level increases (or decreases) by* 10 dB. The normal ear can distinguish between intensities that differ by about 1 dB.

The normal conversation is about 60 dB, city traffic noise is about 70 – 90 dB and a jet aircraft produces as much as 150 dB which can damage the ear.

NOISE LEVELS DUE TO DIFFERENT SOURCES

Source	Noise level in decibel	Intensity (Wm^{-2})
Threshold of hearing	0	10^{-12}
Rustle of leaves	10	10^{-11}
Average whispering	30	10^{-10}
Quiet radio in home	40	10^{-8}
Quiet automobile	50	10^{-7}
Ordinary conversation	60	3.2×10^{-6}
Busy Street traffic	70–90	10^{-5}
Noise factory	90	10^{-3}
Motorcycle rider and orchestra	100	10^{-2}
Threshold of pain	120	1
Machine Gun/Jack Hammer	130	10
Nearby Jet Airplane	150	10^{3}

If two intensity levels are I_1 and I_2, then number of decibels L_1 and L_2 for intensities I_1 and I_2 are

$$L_1 = 10\log_{10}\left(\frac{I_1}{I_0}\right) \text{ and } L_2 = 10\log_{10}\left(\frac{I_2}{I_0}\right)$$

$$\Rightarrow \quad L_2 - L_1 = 10\left[\log_{10}\left(\frac{I_2}{I_0}\right) - \log_{10}\left(\frac{I_1}{I_0}\right)\right]$$

$$\Rightarrow \quad L_2 - L_1 = 10\log_{10}\left(\frac{I_2}{I_1}\right)$$

So, if two sounds have intensity ratio 2, then they differ in sound level by

$$\Delta L = 10\log_{10}(2) = 10 \times 0.301 = 3 \text{ dB}$$

ILLUSTRATION 27

The loudness of a source of sound at a given location increases by 10 dB. By how many times does its intensity increase?

SOLUTION

Let the initial loudness be L_1 decibel, so we have

$$L_1 = 10\log_{10}\left(\frac{I}{I_0}\right)$$

The new loudness is now $L_1 + 10$. So, let us assume that the new intensity be nI i.e., becomes n times.

$$\Rightarrow \quad L_1 + 10 = 10\log_{10}\left(\frac{nI}{I_0}\right)$$

$$\Rightarrow \quad L_1 + 10 = 10\log_{10}\left(\frac{I}{I_0}\right) + 10\log_{10} n$$

$$\Rightarrow \quad L_1 + 10 = L_1 + 10\log n$$

$$\Rightarrow \quad \log n = 1$$

$$\Rightarrow \quad n = 10$$

Hence the loudness increases by 10 decibel when the intensity increases to 10 times the initial value. Similarly, the loudness decreases by 10 decibels when intensity drops to one tenth of the initial value.

ILLUSTRATION 28

What is the maximum possible sound level in dB of sound waves in air? Given that density of air is 1.3 kgm^{-3}, $v = 332 \text{ ms}^{-1}$ and atmospheric pressure $P = 1.01 \times 10^5 \text{ Nm}^{-2}$.

SOLUTION

For maximum possible sound intensity, pressure amplitude of wave will be equal to atmospheric pressure. So, we have $p_0 = P = 1.01 \times 10^5 \text{ Nm}^{-2}$

Since $I = \dfrac{p_0^2}{2\rho v}$, where $\rho = 1.3 \text{ kgm}^{-3}$, $v = 232 \text{ ms}^{-1}$

$$\Rightarrow \quad I = \frac{p_0^2}{2\rho v} = \frac{(1.01\times10^5)^2}{2\times1.3\times332} = 1.18\times10^7 \text{ Wm}^{-2}$$

Since, $L(\text{in dB}) = 10\log_{10}\left(\dfrac{I}{I_0}\right)$

$$\Rightarrow \quad L(\text{in dB}) \approx 10\log_{10}\left(\frac{10^7}{10^{-12}}\right) = 190 \text{ dB}$$

ILLUSTRATION 29

A fighter jet (subsonic) flies over head at an altitude of 100 m. The intensity of sound is observed to be 150 dB. At what altitude should this plane fly so that the intensity drops to the level of the threshold of pain (i.e., 120 dB)? Ignore the finite time required for the sound to reach the ground. Assume the jet to be a point source of sound.

SOLUTION

Since $L_2 - L_1 = 10\log_{10}\left(\dfrac{I_2}{I_1}\right)$

$$\Rightarrow \quad L_2 - L_1 = 120 - 150 = 10\log_{10}\left(\frac{I_2}{I_1}\right)$$

$$\Rightarrow \quad \log_{10}\left(\frac{I_2}{I_1}\right) = -3$$

$$\Rightarrow \quad \frac{I_2}{I_1} = \frac{r_1^2}{r_2^2} = 10^{-3} \qquad \left\{\because I \propto \frac{1}{r^2}\right\}$$

$$\Rightarrow \quad \frac{(100)^2}{r_2^2} = 10^{-3}$$

$$\Rightarrow \quad r_2^2 = 10^7$$

$$\Rightarrow \quad r_2 = 1000\sqrt{10} \text{ m} = 3162 \text{ m}$$

NEWTON'S FORMULA FOR SPEED OF SOUND IN A GAS

Newton assumed that the propagation of sound in a gas takes place under isothermal conditions, i.e., a compressed layer of air gradually loses heat to the surroundings and a rarefied layer gains heat from the surroundings, so that the temperature of air remains constant. Since for isothermal process

$$PV = \text{constant}$$

$$\Rightarrow \quad d(PV) = 0$$

$$\Rightarrow \quad PdV + VdP = 0$$

$$\Rightarrow \quad -\frac{dP}{dV/V} = P = B_{iso}$$

Since, $v = \sqrt{\dfrac{B}{\rho}}$ $\qquad \left\{\because B = -\dfrac{dP}{dV/V}\right\}$

$$\Rightarrow \quad v = \sqrt{\frac{B_{iso}}{\rho}} = \sqrt{\frac{P_{atm}}{\rho_{air}}} \qquad \text{\{Newton's Formula\}}$$

At STP, $P_{atm} = 1.013 \times 10^5\ \text{Nm}^{-2}$, $\rho_{air} = 1.29\ \text{kgm}^{-3}$

$$\Rightarrow \quad v = \sqrt{\frac{1.013 \times 10^5}{1.29}} \approx 280\ \text{ms}^{-1}$$

This value is about 16% less than the actual value, which is about 332 ms^{-1}. This large discrepancy shows that Newton made some error while calculating the speed of sound.

ILLUSTRATION 30

The speed of sound in air at *NTP* is 332 ms^{-1}. Calculate the percentage error in speed of sound as calculated from Newton's formula. Given that the density of air at 1 atm pressure is 1.293 kgm^{-3}.

SOLUTION

From Newton's formula, $v = \sqrt{\frac{P}{\rho}}$

where, $P = 1.01 \times 10^5$ Pa

$$\Rightarrow \quad v = \sqrt{\frac{1.01 \times 10^5}{1.293}} = 280\ \text{ms}^{-1}$$

Difference in velocity $= 332 - 280 = 52\ \text{ms}^{-1}$

So, percentage error $= \frac{52}{332} \times 100 = 15.7\%$

LAPLACE'S CORRECTION FOR SPEED OF SOUND IN A GAS

Laplace pointed out that Newton's assumption of "**isothermal**" propagation is not correct. *The reason is that the compressions and rarefactions follow each other so rapidly that there is no time for the compressed layer (which is at a higher temperature) and the rarefied layer (which is at a lower temperature) to equalise their temperatures with the surroundings.*
As a result, the sound propagates under adiabatic and not under isothermal conditions. Since for an adiabatic process $PV^\gamma = \text{constant}$

$$\Rightarrow \quad d\left(PV^\gamma\right) = 0$$

$$\Rightarrow \quad P\gamma V^{\gamma-1} dV + V^\gamma dP = 0$$

$$\Rightarrow \quad B_{adi} = -\frac{dP}{dV/V} = \gamma P_{atm}, \text{ where } \gamma = \frac{C_P}{C_V}$$

$$\Rightarrow \quad v = \sqrt{\frac{\gamma P}{\rho}}$$

For air $\gamma = 1.4$, hence $v = 280\sqrt{1.4} \approx 330\ \text{ms}^{-1}$, which agrees with the experimentally calculated value.

EFFECT OF EXTERNAL FACTORS ON SPEED OF SOUND

The speed of sound in a gas is given by

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

Effect of Change in Pressure

There is no effect of change of pressure (at constant temperature). This is because density also changes proportionately such that $P/\rho = \text{constant}$.

Effect of Change in Density

Since $v \propto \frac{1}{\sqrt{\rho}}$, so if ρ_1 and ρ_2 are the densities of two gases for which γ is the same, then the speeds of sound in them, v_1 and v_2 respectively, are related as

$$\frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}}$$

Effect of Change in Temperature

Since $v = \sqrt{\frac{\gamma RT}{M}}$ i.e., $v \propto \sqrt{T}$, so $\frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}}$

If v_t is velocity of sound at t °C and v_0 is velocity of sound at 0 °C, then we have

$$v_t = v_0\sqrt{\frac{273+t}{273}} = v_0\left(1+\frac{t}{273}\right)^{\frac{1}{2}}$$

For $\frac{t}{273} \ll 1$, we have $\left(1+\frac{t}{273}\right)^{\frac{1}{2}} \approx 1+\frac{1}{2}\left(\frac{t}{273}\right)$

$$\Rightarrow \quad v_t \approx v_0\left(1+\frac{t}{546}\right)$$

Taking $v_0 = 332\ \text{ms}^{-1}$ we get $v_t \approx (332+0.61t)\ \text{ms}^{-1}$

Thus for $\frac{t}{273} \ll 1$, the velocity increases by about 0.61 ms^{-1} for every degree rise of temperature.

Effect of Atomicity of Gas

Since $v = \sqrt{\frac{\gamma RT}{M}}$ i.e., $v \propto \sqrt{\frac{\gamma}{M}}$

So, $\frac{v_{He}}{v_{O_2}} = \sqrt{\left(\frac{\gamma_{He}}{\gamma_{O_2}}\right)\left(\frac{M_{O_2}}{M_{He}}\right)} = \sqrt{\left(\frac{5/3}{7/5}\right)\left(\frac{32}{4}\right)} = \sqrt{\frac{200}{21}}$

Effect of Change in Humidity

Density of water vapour is less than that of dry air (water vapours have a tendency to rise up). At S.T.P. it is 0.8 kgm^{-3} whereas the density of dry air at S.T.P. is 1.29 kgm^{-3}. Hence, the speed of sound in air increases with increase of humidity.

Effect of Wind Velocity

If $\vec{v}_s$ be the velocity of sound in still air and wind blows at a speed $\vec{v}_w$ then velocity of sound $\vec{v}'$ in the medium is $\vec{v}' = \vec{v}_s + \vec{v}_w$

(a) If wind blows along the sound, then effective speed of sound becomes $v' = v_s + v_w$

(b) If wind blows opposite to sound, then effective speed of sound becomes $v' = v_s - v_w$

(c) If wind blows along the sound making an angle θ with it, then the effective speed of sound becomes $v' = v_s + v_w \cos\theta$

(d) If wind blows opposite to the sound making an angle θ with it, then the effective speed of sound becomes $v' = v_s - v_w \cos\theta$

Also keep in mind that the component of velocity of wind perpendicular to the velocity of sound does not affect the velocity of sound.

ILLUSTRATION 31

The velocity of sound in hydrogen at STP is 1400 ms^{-1}. Find the velocity of sound in a mixture with 3 parts by volume of oxygen and 2 parts by volume of hydrogen at 819 °C?

SOLUTION

$$v_{H_2} = \sqrt{\frac{\gamma RT}{M_{H_2}}} = \sqrt{\frac{\gamma R(273)}{M_{H_2}}} = 1400 \text{ ms}^{-1} \qquad \ldots(1)$$

Let the number of moles of hydrogen in the mixture be $2n$. The number of moles of oxygen will then be $3n$. The mean molar mass of a mixture is given by

$$M_{mix} = \frac{n_1M_1 + n_2M_2}{n_1 + n_2} = \frac{2nM_{H_2} + 3nM_{O_2}}{2n + 3n}$$

$$\Rightarrow \quad M_{mix} = \frac{2M_{H_2} + 3(16M_{H_2})}{5} \qquad \{\because M_{O_2} = 16M_{H_2}\}$$

$$\Rightarrow \quad M_{mix} = \frac{50M_{H_2}}{5} = 10M_{H_2}$$

Since both these gases are diatomic, γ of the mixture remains unchanged i.e. $\gamma_{mix} = \gamma = 7/5$

At $T = 819\ °C = 1092\ K = 4(273\ K)$, speed of sound in mixture is given by

$$v_{mix} = \sqrt{\frac{\gamma_{mix}RT}{M_{mix}}} = \sqrt{\frac{\gamma R(1092)}{10M_{H_2}}} = \frac{2}{\sqrt{10}}\sqrt{\frac{\gamma R(273)}{M_{H_2}}}$$

From equation (1), we get

$$v_{mix} = \frac{2}{\sqrt{10}}(1400) = \frac{2800}{\sqrt{10}} = 280\sqrt{10}\ \text{ms}^{-1}$$

ILLUSTRATION 32

Consider two points A and B such that the distance between them is d and temperature between them varies linearly from T_1 to T_2. What will be the time taken by sound waves to travel this distance if velocity of sound propagation in air varies as $v = \beta\sqrt{T}$.

SOLUTION

Since, $T(x) = T_1 + \left(\frac{T_2 - T_1}{d}\right)x$

Further $v = \beta\sqrt{T}$

$$\Rightarrow \quad v = \frac{dx}{dt} = \beta\sqrt{T_1 + \left(\frac{T_2 - T_1}{d}\right)x}$$

$$\Rightarrow \quad \int_0^t dt = \frac{1}{\beta}\int_0^d \frac{dx}{\sqrt{T_1 + \left(\frac{T_2 - T_1}{d}\right)x}}$$

$$\Rightarrow \quad t = \frac{2d}{\beta\left(\sqrt{T_1} + \sqrt{T_2}\right)}$$

ILLUSTRATION 33

At what temperature does the velocity of sound in air increases by 2% in comparison with velocity at 0 °C?

SOLUTION

If v_0 be the speed of sound at 0 °C and v_t be the speed at t °C, then

$$\frac{v_t}{v_0} = \sqrt{\frac{T}{T_0}} = \sqrt{\frac{t + 273}{273}}$$

According to problem, we have

$$v_t = v_0 + \frac{2}{100}v_0 = v_0(1 + 0.02) = (1.02)v_0$$

$$\Rightarrow \quad \frac{(1.02)v_0}{v_0} = \sqrt{\frac{t + 273}{273}}$$

$$\Rightarrow \quad t + 273 = (1.02)^2 \times 273 = 1.0404 \times 273$$

$$\Rightarrow \quad t = 284 - 273 = 11\ °C$$

Test Your Concepts-III

Based on Sound Waves & Properties

(Solutions on page H.235)

1. A point A is located at a distance $r = 1.5$ m from a point source of sound of frequency 600 Hz. The power of the source is 0.8 watt. Speed of sound in air is 340 ms^{-1} and density of air is 1.29 kgm^{-3}. Find at the point A,
   (a) the pressure oscillation amplitude $(\Delta P)_m$
   (b) the displacement oscillation amplitude A.
2. Calculate the ratio of speed of sound in hydrogen gas to the rms speed of hydrogen molecules.
3. A dog while barking delivers about 1 mW of power. Assuming this power to be uniformly distributed over a hemispherical area, calculate the sound level at a distance of 5 m. Also calculate the sound level if instead of one dog, five dogs start barking at the same time each delivering 1 mW of power.
4. Determine the speed of sound waves in water, and find the wavelength of a wave having a frequency of 242 Hz. Take $B_{water} = 2 \times 10^9$ Pa.
5. A sound wave propagating in air has a frequency of 5000 Hz. Calculate the percentage change in wavelength when the wave front, initially in a region where $T = 27$ °C, enters a region where the temperature decreases to 10 °C.
6. A window whose area is 2 m^2 opens on a street where the street noise results at the window an intensity level of 60 dB. How much acoustic power enters the window through sound waves? Now, if a sound absorber is fitted at the window, how much energy from the street will it collect in a day?
7. At what temperature will the speed of sound in hydrogen be the same as in oxygen at 100 °C? Molar masses of oxygen and hydrogen are in the ratio 16 : 1.
8. For aluminium, the bulk modulus of elasticity is 7.5×10^{10} Nm^{-2} and density is 2.70×10^3 kgm^{-3}. Calculate the velocity of longitudinal waves in aluminium.
9. Find the maximum increase in pressure when a sound wave produces an energy flow of 10^{-3} $Wattm^{-2}$. Velocity of sound = 340 ms^{-1}, density of air = 1.293 kgm^{-3}.
10. A plane longitudinal wave having angular frequency $\omega = 500$ sec^{-1} is travelling in positive x-direction in a medium of density $\rho = 1$ kgm^{-3} and bulk modulus 4×10^4 Nm^{-2}. The loudness at a point in the medium is observed to be 20 dB. Assuming at $x = 0$ initial phase of the medium particles to be zero, find the
    (a) maximum pressure change in the medium
    (b) equation of the wave.
11. A gas is a mixture of two parts by volume of hydrogen and one part by volume of nitrogen. If the velocity of sound in hydrogen at 0 °C is 1300 ms^{-1}, find the velocity of sound in the gaseous mixture at 27 °C.
12. For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about 6×10^{-5} Pa. Calculate the corresponding intensity in Wm^{-2}. Take speed of sound in air as 344 ms^{-1} and density of air 1.2 kgm^{-3}.
13. The faintest sound that the human ear can detect at frequency 1 kHz corresponds to an intensity of about 10^{-12} Wm^{-2}. Determine the pressure amplitude and the maximum displacement associated with this sound assuming the density of the air = 1.3 kgm^{-3} and velocity of sound in air = 332 ms^{-1}.

REFLECTION OF SOUND WAVES

Reflection of sound waves from a rigid boundary (e.g. closed end of an organ pipe) is analogous to reflection of a string wave from rigid boundary in which reflection is accompanied by an inversion i.e. an abrupt phase change of π. This is consistent with the requirement of displacement amplitude to remain zero at the rigid end, because a medium particle at the rigid end cannot vibrate. Since the excess pressure and displacement corresponding to a same sound wave vary by $\pi/2$ in term of phase, so a displacement minima at the rigid end will be a point of pressure maxima. *This implies that the reflected pressure wave from the rigid boundary will have same phase as the incident wave, i.e., a compression pulse is reflected as a compression pulse and a rarefaction pulse is reflected as a rarefaction pulse.*

On the other hand, reflection of sound wave from a low pressure region (like open end of an organ pipe) is analogous to reflection of string wave from a free end. This point corresponds to a displacement maxima, so that incident and reflected displacement wave at this point must be in phase. This would imply that this point would be a minima for pressure wave (i.e. pressure at this point remains at its average value), and hence the reflected pressure wave would be out of phase by π with respect to the incident wave. i.e. a compression pulse is reflected as a rarefaction pulse and vice-versa.

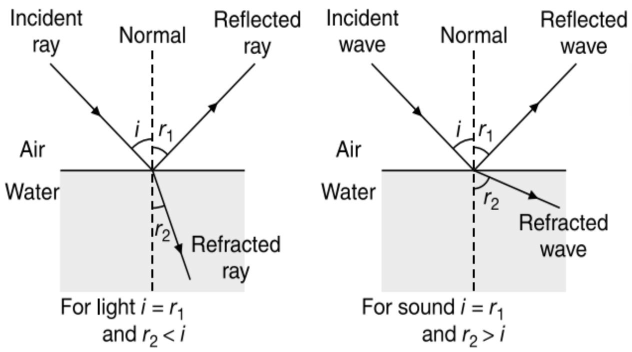
REFRACTION

A medium is said to be denser (relative to the other) if the speed of wave in this medium is less than the speed of the wave in the other medium. Rather we can say speed of a wave in a denser medium is less than its speed in the rarer medium. Thus, it is the speed of wave which decides whether the medium is denser or rarer for that particular wave and

$$v_{\text{denser}} < v_{\text{rarer}}$$

If a medium is denser for one type of wave then at the same time the same medium can be rarer for the other type of wave. For example, water is denser for electromagnetic (light) waves compared to air, because the speed of electromagnetic waves is less in water than in air. At the same time, for sound wave water is a rarer medium because speed of sound wave in water is more. The laws of refraction (or transmission from one medium to the other) and reflection remain the same i.e.,

The ray bends towards the normal if it travels from a rarer medium to a denser medium and vice-versa.



When sound wave passes from one homogeneous medium to another homogeneous medium, it deviates from its path. This phenomenon is called **refraction**. No phase change exists during refraction. If i and r are the angles of incidence and refraction, then **Snell's Law** states

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2} = \text{constant}$$

where v_1 and v_2 are the velocities of sound in first and second medium respectively.

In reflection and refraction the frequency remains unchanged.

ECHO

The most common experience of sound reflection is the echo heard in large halls and in the neighbourhood of hills. An **echo** is simply the repetition of speaker's own voice caused by reflection at a distant surface e.g. a cliff, a row of buildings or any other extended surface.

If t is the time interval between production of sound from source and its echo at the site of source, then the distance between source and reflector(s) is given by

$$2s = vt$$

$$\Rightarrow \quad s = \frac{vt}{2}$$

where v is the velocity of sound.

As the persistence of hearing for human ear is 0.1 sec, therefore in order that an echo of short sound (e.g. shot or clapping) may be heard distinctly, the echo must come 0.1 sec later than the direct sound. In the case of articulate sound, it has been found that one can hear or pronounce distinctly not more than 5 syllables per second. Therefore, for monosyllabic sound the minimum time interval between sound and its echo is $\frac{1}{5}$ s, for disyllabic and trisyllabic sounds it is $\frac{2}{5}$ s and $\frac{3}{5}$ s respectively and so on.

If we hear the echo of monosyllabic sound that means sound travels from source to reflector and back from reflector to source in a collective time of $\frac{1}{5}$ s i.e. time to go from source to reflector is $\frac{1}{10}$ s.

Multiple echoes are produced when there are several reflecting surfaces. The stethoscope speaking tubes, whispering gallery and sounding boards are based upon the reflection of sound.

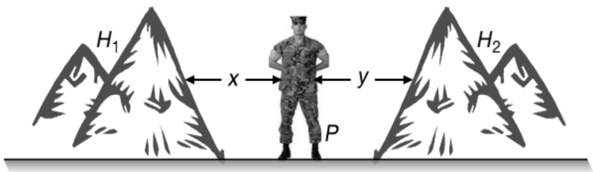
ILLUSTRATION 34

A person, standing between two parallel hills, fires a gun. He hears the first echo after 1.5 s and the second after 2.5 s. If the speed of sound is 332 ms^{-1}, calculate the distance between the hills. When will he hear the third echo?

SOLUTION

Let the person P be at a distance x from hill H_1 and at a distance y from H_2 as shown in figure. The time interval between the original sound and echoes from H_1 and H_2 will be respectively.

$$t_1 = \frac{2x}{v} \text{ and } t_2 = \frac{2y}{v}$$



Therefore, the distance between the hills is

$$x + y = \frac{v}{2}(t_1 + t_2) = \frac{332}{2}(1.5 + 2.5) = 664 \text{ m}$$

Now, as I echo will be from H_1 after time t_1, while II echo from H_2 after time t_2, III echo will be produced due to reflection of sound of I echo from H_2 or of II echo from H_1. Thus, the time after which III echo will be heard is

$$t_3 = t_1 + t_2 = 1.5 + 2.5 = 4 \text{ s}$$

The sound from both H_1 and H_2 will reach simultaneously. Note that it is an example of multiple or successive echoes which are not harmonic, as $t_1 = 1.5$ s, $t_2 = 2.5$ s and $t_3 = 4$ s.

ILLUSTRATION 35

The echo of a gunshot is heard 5 s after it is fired. Calculate the distance of the surface which reflects the sound. The velocity of sound is 340 ms^{-1}.

SOLUTION

Distance of the surface which reflects the sound is given by

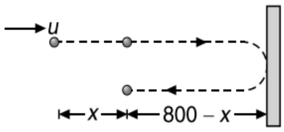
$$s = \frac{v \times t}{2} = \frac{340 \times 5}{2} = 170 \times 5 = 850 \text{ m}$$

ILLUSTRATION 36

An engine approaches a hill with a constant speed. When it is at a distance of 0.8 km it blows a whistle, whose echo is heard by the driver after 4 s. If speed of engine in air is 330 ms^{-1}. Calculate the speed of engine.

SOLUTION

Lets draw the situation according to the problem.



Distance travelled by sound when it again meets the person is

$$s = 800 + (800 - x)$$

$$\Rightarrow \quad s = 1600 - ut_0 = 1800 - u \times 4$$

$$\Rightarrow \quad \frac{1600 - 4u}{v} = 4$$

$$\Rightarrow \quad 1600 - 4u = 4v$$

$$\Rightarrow \quad 1600 - 4u = 4 \times 330$$

$$\Rightarrow \quad 4u = 1600 - 1320$$

$$\Rightarrow \quad 4u = 280$$

$$\Rightarrow \quad u = 70 \text{ ms}^{-1}$$

DIFFRACTION

When a wave is deflected by the corners of an obstacle (or is deviated from its straight line path), the phenomenon is called **diffraction**. For diffraction the wavelength of wave must be of the same order as the dimension of diffracting obstacle. For example sound has wavelength of the order of 1 m, so it is diffracted by doors and windows while light has wavelength of 4000 Å to 7800 Å, so it is diffracted by thin edge of blade and diffraction grating having nearly 15000 rulings per inch.

SUPERPOSITION OF WAVES: INTRODUCTION

When two or more wave trains travel in a region simultaneously they superpose each other, resulting in a variation of intensity in space and/or with time. This phenomenon is called **phenomenon of superposition of waves**.

SUPERPOSITION PRINCIPLE

If two or more waves arrive at a point simultaneously then the net displacement at that point is the algebraic sum of the displacement due to individual waves, i.e.

$$y = y_1 + y_2 + \dots\dots\dots\dots + y_n$$

where $y_1, y_2 \dots\dots\dots\dots, y_n$ are the displacements due to individual waves and y is the resultant displacement.

Suppose that two waves travel simultaneously along the same stretched string. Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that the string would experience if each wave travelled alone. The displacement of the string when the waves overlap is then the algebraic sum of the individual wave equations.

$$y'(x, t) = y_1(x, t) + y_2(x, t)$$

This summation of displacements along the string means that superimposing simply means that the waves algebraically add to produce a resultant wave (or net wave).

This is another example of the principle of superposition, which says that when several effects occur simultaneously, their net effect is the sum of the individual effects.

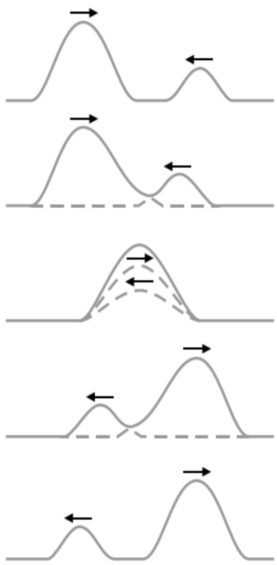


Figure shows a sequence of snapshots of two pulses travelling in opposite directions on the same stretched string. When the pulses overlap, the resultant pulse is their sum and we see the resultant wave, not the individual waves.

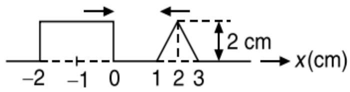
Also, note that the overlapping/superposition of the waves do not in any way alter the travel of each other and continue to move further individually after superposition as they were doing so before superposition.

Let us discuss three important cases.

	Interference	Beats	Stationary Waves
a)	Two waves of same amplitude or nearly same amplitude,	Two waves of same amplitude,	Two waves of same amplitude,
b)	having constant phase difference or no phase difference (such waves are called **coherent waves**),	having no phase difference,	having no phase difference or constant phase difference,
c)	moving in the same direction,	moving in same direction,	moving in opposite direction,
d)	having same velocity and frequency, on superposition will give rise to interference in which,	with same velocity, having slightly different frequency (Difference in frequency must not be greater than 10 Hz for human ear to observe), on superposition will give rise to beats in which,	with same velocity, having same frequency (or λ), on superposition will give rise to stationary waves in which,
e)	maxima and minima of intensity are obtained called constructive and destructive interference respectively. The position of Maxima and Minima remain fixed on screen and hence the phenomenon is called **Sustained interference**.	maxima and minima of intensity varying periodically with time are obtained.	maxima and minima of intensity varying periodically with path difference (x). The maxima of intensity are called Antinodes and minima of intensity are called Nodes.

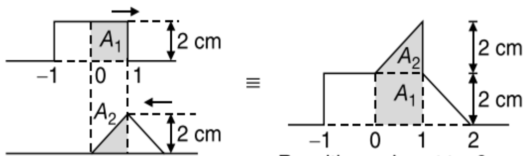
ILLUSTRATION 37

In the arrangement shown in Figure a rectangular pulse and triangular pulse approaching each other. The pulse speed is $0.5\ \text{cms}^{-1}$. Sketch the resultant pulse at $t = 2$ s.



SOLUTION

In 2 s each pulse will travel a distance of 1 cm. The two pulse overlap between 0 and 1 cm. So, A_1 and A_2 can be added as shown in Figure.



Resulting pulse at t = 2 s

ILLUSTRATION 38

Sources separated by 20 m vibrate according to the equations $y_1 = 0.06\sin\pi t$ and $y_2 = 0.02\sin\pi t$. They send out waves along a rod with speed $3\ \text{ms}^{-1}$. Find the equation of motion of a particle 12 m from first source and 8 m from the second if y_1, y_2 are in metre.

SOLUTION

Since, $k = \dfrac{\omega}{v} = \dfrac{\pi}{3}$

$$\Rightarrow \quad y_1 = 0.06(\pi t - kx) = 0.06\sin\left(\pi t - \frac{\pi}{3}\times 12\right)$$

$$\Rightarrow \quad y_1 = 0.06\sin(\pi t - 4\pi)$$

Similarly, $y_2 = 0.02\sin(\pi t - kx') = 0.02\sin\left(\pi t - \dfrac{\pi}{3}\times 8\right)$

$$\Rightarrow \quad y_2 = 0.02\sin\left(\pi t - \frac{8\pi}{3}\right)$$

By Principle of Superposition $y = y_1 + y_2$

$$\Rightarrow \quad y = 0.06\sin(\pi t)\cos(4\pi) - 0.06\cos(\pi t)\sin(4\pi) + 0.02\sin(\pi t)\cos\left(\frac{8\pi}{3}\right) - 0.02\cos(\pi t)\sin\left(\frac{8\pi}{3}\right)$$

$$\Rightarrow \quad y = 0.05\sin(\pi t) - 0.0173\cos(\pi t)$$

INTERFERENCE

When two coherent waves of the same frequency moving in the same direction superpose, they give rise to variation of intensity in space – positions of maximum and minimum intensities appear alternately in the whole region of superposition. The positions of maxima and minima are fixed and this phenomenon is called **Sustained Interference**. The use of the term "**interference**" is generally restricted to this case only. Let the two waves be

$$y_1 = A_1\sin(\omega t - kx) \text{ and } y_2 = A_2\sin(\omega t - kx + \phi)$$

On superposition $y = y_1 + y_2$

$$\Rightarrow \quad y = A_1 \sin(\omega t - kx) + A_2 \sin(\omega t - kx + \phi)$$

$$\Rightarrow \quad y = A_1 \sin(\omega t - kx) + A_2 [\sin(\omega t - kx)\cos\phi + \cos(\omega t - kx)\sin\phi]$$

$$\Rightarrow \quad y = (A_1 + A_2 \cos\phi)\sin(\omega t - kx) + (A_2 \sin\phi)\cos(\omega t - kx)$$

Let $A_1 + A_2 \cos\phi = R\cos\theta$ and $A_2 \sin\phi = R\sin\theta$

$$\Rightarrow \quad y = R[\cos\theta \sin(\omega t - kx) + \sin\theta \cos(\omega t - kx)]$$

$$\Rightarrow \quad y = R\sin(\omega t - kx + \theta)$$

where, the resultant amplitude is

$$R = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos\phi}$$

and the phase angle $\theta = \tan^{-1}\left(\dfrac{A_2 \sin\phi}{A_1 + A_2 \cos\phi}\right)$

the resultant wave is also a harmonic wave of the same frequency.

CASE-I: CONSTRUCTIVE INTERFERENCE

Resultant Amplitude (R) is maximum
when $\cos\phi = +1$,

$$\Rightarrow \quad \phi = 2n\pi \text{ where } n = 0, 1, 2, \ldots\ldots$$

$$\Rightarrow \quad x = (2n)\frac{\lambda}{2}, \text{ where } n = 0, 1, 2, 3, \ldots\ldots$$

and $R_{max} = A_1 + A_2$

*The interference is said to be **constructive** when phase difference is an even multiple of π or path difference is an even multiple of $\lambda/2$.*

CASE-II: DESTRUCTIVE INTERFERENCE

The Resultant Amplitude is Minimum,
when, $\cos\phi = -1$,

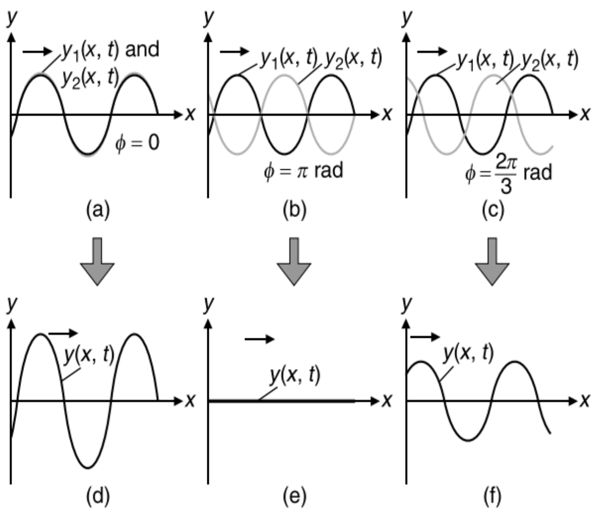
$$\Rightarrow \quad \phi = (2n+1)\pi \text{ where } n = 0,\ 1,\ 2\ldots\ldots..$$

$$\Rightarrow \quad x = (2n+1)\frac{\lambda}{2}, \text{ where } n = 0, 1, 2, 3, \ldots..$$

and $R_{min} = |A_1 - A_2|$

The interference is said to be destructive when phase difference is an odd multiple of π or path difference is an odd multiple of $\lambda/2$.

Two identical sinusoidal waves, $y_1(x, t)$ and $y_2(x, t)$, travel along a string in the positive direction of an x-axis. They interfere to give a resultant wave $y(x, t)$. The resultant wave is what is actually seen on the string. The phase difference (ϕ) between the two interfering waves is (a) 0 rad or 0°, (b) π rad or 180° and (c) $\dfrac{2\pi}{3}$ rad or 120°. The corresponding resultant waves are shown in (d), (e) and (f).



INTENSITY FACTOR

Since the intensity of a wave is proportional to the square of the amplitude, we have

$$I_R \propto R^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos\phi$$

This gives

$$I_{max} \propto (A_1 + A_2)^2 \text{ and } I_{min} \propto (A_1 - A_2)^2$$

$$\Rightarrow \quad \frac{I_{max}}{I_{min}} = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2 = \left(\frac{r+1}{r-1}\right)^2$$

where $r = \dfrac{A_1}{A_2}$ is called the **amplitude ratio**. If I_1 and I_2 are intensities of two interfering waves, then

$$I_R = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$$

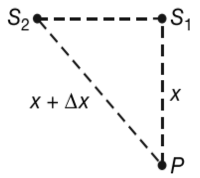
If $I_1 = I_2$, then $I_R = 4I\cos^2\left(\dfrac{\phi}{2}\right)$

In this case $I_{max} = 4I$ and $I_{min} = 0$

COHERENT SOURCES

Two sources which are either in phase or have a constant phase difference between them are called coherent sources. For the two sources to be coherent their frequencies must be same. But the reverse is not always true, i.e., the two different sources having the same frequency are not always coherent.

If the phase difference of the sources changes erratically with time, even if they have the same frequency the sources are said to be incoherent. So, light sources composed of a large number of same kind of atoms, which emit light of the same frequency are not coherent, because there are many atoms involved in each source and they do not oscillate in phase and hence are incoherent. Thus, two different light sources cannot be coherent.



Consider two coherent point sources of sound S_1 and S_2 which oscillate in phase with the same angular frequency ω. A point P is situated at a distance x from S_1 and $x + \Delta x$ from S_2, so that the path difference between the two waves reaching P from S_1 and S_2 is Δx. The displacement equations of two waves arriving at P are

$$y_1 = A_1 \sin(\omega t - kx)$$

and $y_2 = A_2 \sin[(\omega t - kx) + \phi]$

where, $\phi = \left(\dfrac{2\pi}{\lambda}\right)\Delta x$ is phase difference between two waves reaching P.

If the sources are incoherent, the phase difference between the sources keep on changing. At any point P, sometimes constructive and sometimes destructive interference takes place. If the intensity due to each source is I, the resultant intensity rapidly and randomly changes between $4I$ and zero, so that the average observable intensity is $2I$. If intensities due to individual sources is I_1 and I_2 the resultant intensity is

$$I = I_1 + I_2 \text{ (for incoherent sources)}$$

No interference effect is therefore observed. For observable interference, the sources must be coherent.

A good way to obtain a pair of coherent sources is to obtain two sound waves from the same source by dividing the original wave along two different paths and then combining them. The two waves then differ in phase only because of different paths travelled.

ILLUSTRATION 39

Two waves having intensity ratio of $100:1$, interfere with each other. Find the ratio of intensities at the maxima and minima

SOLUTION

The amplitude ratio,

$$\gamma = \frac{A_1}{A_2} = \frac{\sqrt{I_1}}{\sqrt{I_2}} = \sqrt{\frac{100}{1}} = 10$$

$$\Rightarrow \quad \frac{I_{max}}{I_{min}} = \frac{(\gamma+1)^2}{(\gamma-1)^2} = \frac{(10+1)^2}{(10-1)^2} = \frac{121}{81} = 1.494$$

ILLUSTRATION 40

Two coherent sound sources, each having a frequency of 400 Hz, are at distances $x_1 = 0.2$ m and $x_2 = 0.48$ m from a point. Calculate the intensity of the resultant wave at this point, if each wave has intensity of 60 Wm^{-2} and the velocity of the waves in the medium is 448 ms^{-1}.

SOLUTION

Path difference,

$$\Delta x = x_2 - x_1 = 0.48 - 0.2 = 0.28 \text{ m}$$

The corresponding phase difference is given as

$$\phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{(v/f)}\Delta x = \frac{2\pi f}{v}\Delta x$$

$$\Rightarrow \quad \phi = \frac{2\pi \times 400}{448} \times 0.28 = \frac{\pi}{2} \text{ rad}$$

Therefore, the intensity of the resultant wave is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi = I_0 + I_0 + 2I_0\cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad I = 2I_0 = 2 \times 60 = 120 \text{ Wm}^{-2}$$

ILLUSTRATION 41

Find the resultant amplitude and phase of a point at which N sinusoidal waves interfere. All the waves have same amplitude A and their phases increase in arithmetic progression of common difference ϕ.

SOLUTION

The diagram for their sum is shown in figure for $N = 6$. The resultant amplitude is A_R. The apex angle of every isosceles triangle is ϕ. So, the angle subtended by the resultant is $N\phi$. Since the heads of the vectors are all at the same distance r from the apex of the polygon, so

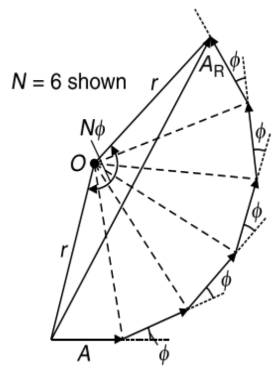


Figure 1

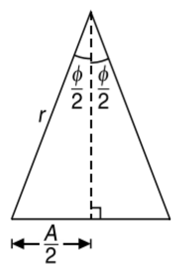


Figure 2

From Figure 1, we get

$$\frac{A_R}{2} = r\sin\left(\frac{N\phi}{2}\right) \quad \text{...(1)}$$

and from Figure 2, we get

$$\frac{A}{2} = r\sin\left(\frac{\phi}{2}\right) \quad \text{...(2)}$$

Dividing Equation (1) by (2), we get

$$\Rightarrow \quad A_R = A\frac{\sin(N\phi/2)}{\sin(\phi/2)}$$

Conceptual Note(s)

Consider $N = 2$ and $\phi = 90°$, then

$$A_R = \sqrt{2}A$$

Similarly, the above result can also be checked for other special cases.

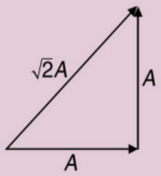
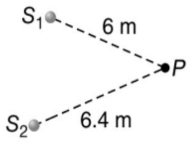


ILLUSTRATION 42

Two sources are placed from a person P as shown in figure. The speed of sound in air is 320 ms^{-1}. If sound signal is continuously varied from 500 Hz to 2500 Hz, for which frequency listener will hear minimum sound intensity.



SOLUTION

Path difference between waves arriving at point P is

$$\Delta x = S_2P - S_1P = 6.4 - 6 = 0.4 \text{ m}$$

For minimum sound intensity, we have

$$\Delta x = (2n+1)\frac{\lambda}{2} = \frac{(2n+1)}{2}\frac{v}{f}$$

$$\Rightarrow \quad f = \frac{(2n+1)v}{2\Delta x} = \frac{(2n+1)320}{2(0.4)} = 400(2n+1) \text{ Hz}$$

For $n = 0, 1, 2, \ldots\ldots$ we have

$f = 400$ Hz (for $n = 0$)

$f = 1200$ Hz (for $n = 1$)

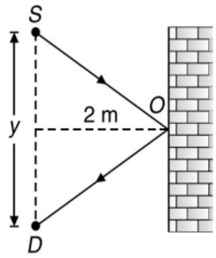
$f = 2000$ Hz (for $n = 2$)

$f = 2800$ Hz (for $n = 3$)

The frequencies are 1200 Hz and 2000 Hz

ILLUSTRATION 43

A source of sound emitting waves at 360 Hz is placed in front of a vertical wall, at a distance 2 m from it. A detector is also placed in front of the wall at the same distance from it. Find the minimum distance between the source and the detector for which the detector detects a maximum of sound. Take speed of sound in air = 360 ms^{-1}.



SOLUTION

Let the detector be at a distance of y metre from the source. Now the detector receives waves from two different paths.

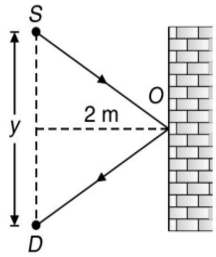
(i) Path SD

(ii) Path SOD

$$SD = y \text{ and } SOD = 2\sqrt{4 + \frac{y^2}{4}} = \sqrt{y^2 + 16}$$

So, path difference is given by

$$\Delta x = \sqrt{y^2 + 16} - y$$



For constructive interference at D i.e., for minimum value of y, we have

$$\Delta x = \lambda$$

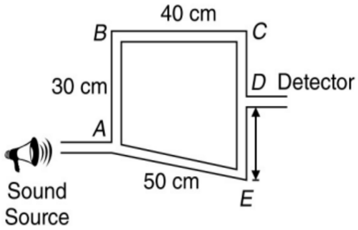
$$\Rightarrow \quad \sqrt{y^2 + 16} - y = \lambda = \frac{v}{f} = \frac{360}{360} = 1 \text{ m}$$

Solving this, we get

$$y = 7.5 \text{ m}$$

ILLUSTRATION 44

In the shown Figure $ABCDE$ is a tube which is open at A and D. A source of sound A is placed in front of A. If frequency of the source can be varied from 2000 Hz to 4000 Hz. Find frequencies at which a detector placed in front of D receives a maximum of intensity. (Given speed of sound = 340 ms^{-1})



SOLUTION

The path difference between the sounds reaching the detector from two parts of the tube is

$$\Delta x = ABCD - AED$$

$$\Rightarrow \quad \Delta x = [(30+40+30)-(30+50)] \text{ cm}$$

$$\Rightarrow \quad \Delta x = 20 \text{ cm} = 0.2 \text{ m}$$

If sound has frequency f and wavelength λ, then

$$\lambda = \frac{340}{f} \text{ m}$$

For maximum intensity at detector, we have

$$\Delta x = n\lambda, \text{ where } n = 1, 2, 3, \ldots\ldots$$

$$\Rightarrow \quad 0.2 \text{ m} = n\left(\frac{340}{f}\right)$$

$$\Rightarrow \quad f = \frac{n \times 3400}{2} \text{ Hz}$$

$$\Rightarrow \quad f = n \times 1700 \text{ Hz}$$

Now, for $n = 1$, $f = 1700$ Hz

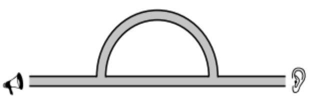
For $n = 2$, $f = 3400$ Hz

For $n = 3$, $f = 5100$ Hz and so, on

Since, frequency of the source is varied between 2000 Hz to 4000 Hz, so the frequency at which maxima is received by the detector is $f = 3400$ Hz.

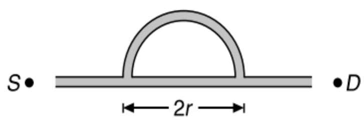
ILLUSTRATION 45

Figure shows a tube structure in which a sound signal is sent from one end and received at the other end. The semi-circular part has a radius of 20 cm. The frequency of the sound source can be varied electronically between 1000 and 4000 Hz. Find the frequencies at which maxima of intensity are detected. The speed of sound in air = 340 ms^{-1}.



SOLUTION

According to the question, the path difference between the waves arriving at D is given by



$$\Delta x = \pi r - 2r = (\pi - 2)r$$

For maximum intensity, we have path difference

$$\Delta x = (2n)\frac{\lambda}{2} = n\lambda$$

$$\Rightarrow \quad (\pi - 2)r = n\lambda = \frac{nv}{f}$$

$$\Rightarrow \quad f = \frac{nv}{(\pi-2)r} = \frac{n \times 340}{(3.14-2)\times 0.2} = 1491n \text{ Hz}$$

For $n = 1$, $f = 1491$ Hz

For $n = 2$, $f = 2982$ Hz

For $n = 3$, $f = 4473$ Hz

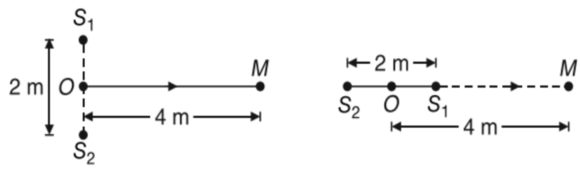
The frequencies in the given range are 1491 Hz and 2982 Hz

ILLUSTRATION 46

Two speakers connected to the same source of fixed frequency are placed 2 m apart in a box. A sensitive microphone placed at a distance of 4 m from their mid-point along the perpendicular bisector shows maximum response. The box is slowly rotated till the speakers are in line with the microphone. The distance between the mid-point of the speakers and microphone remains unchanged. Exactly 5 maximum responses are observed in the microphone in doing this. Calculate the wavelength of the sound wave.

SOLUTION

As shown in Figure (a), initially $S_1M = S_2M$, $\Delta x = 0$. Hence, there is principal maxima at M.



(a) (b)

On rotation of speakers about O, when S_1 and S_2 are in line with microphone M as shown in Figure (b), 5 maximum response are observed. Therefore, we should have

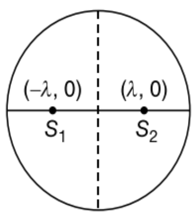
$$S_2M - S_1M = 5\lambda$$

$$\Rightarrow \quad S_2S_1 = 5\lambda$$

$$\Rightarrow \quad \lambda = \frac{S_2S_1}{5} = \frac{2}{5} = 0.4 \text{ m}$$

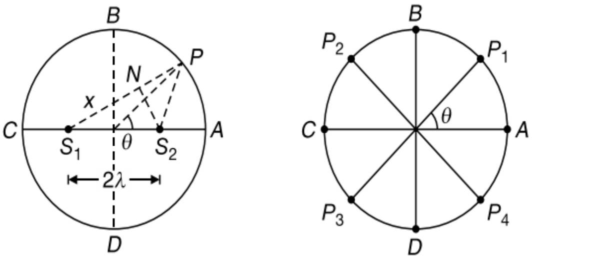
ILLUSTRATION 47

Two identical coherent sources S_1 and S_2 which emit sounds of wavelength λ in same phase are placed in opposite side of the centre of a circle of large radius (Considered to be origin) at $(-\lambda, 0)$ and $(\lambda, 0)$ as shown. Calculate angular position on the circle at which constructive interference occurs.



SOLUTION

From the shown Figure path difference between two sound waves coming from S_1 and S_2 at P.



$$\Delta x = S_1S_2\cos\theta = 2\lambda\cos\theta \qquad \{\because S_1S_2 = 2\lambda\}$$

For maxima i.e., constructive interference, we have

$$\Delta x = n\lambda$$

$$\Rightarrow \quad 2\lambda\cos\theta = n\lambda$$

$$\Rightarrow \quad \cos\theta = \frac{n}{2}$$

Since, $-1 \le \cos\theta \le 1$

$$\Rightarrow \quad -1 \le \frac{n}{2} \le 1$$

$$\Rightarrow \quad -2 \le n \le 2$$

n	–2	–1	0	1	2
$\cos\theta = n/2$	–1	–1/2	0	1/2	1
θ	180°	120° 240°	90° 270°	60° 300°	0°
Points	C	P_2, P_3	B, D	P_1, P_4	A

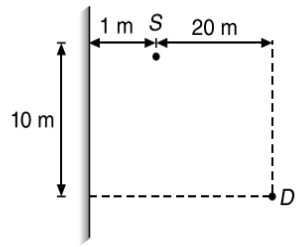
So, a total of 8 maxima are obtained on the circle.

Test Your Concepts-IV

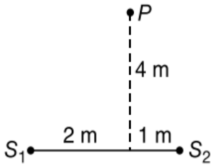
Based on Interference

(Solutions on page H.237)

1. Two harmonic waves are represented in *SI* units by, $y_1(x,t) = 0.2\sin(x-3t)$ and
$$y_2(x,t) = 0.2\sin(x-3t+\phi)$$
Write the expression for the sum $y = y_1 + y_2$ for $\phi = \frac{\pi}{2}$ rad. If the phase difference ϕ between waves is unknown and amplitude of their sum is 0.32 m, calculate ϕ.
2. Two waves of equal frequencies have their amplitudes in the ratio of 3:5. They are superimposed on each other. Calculate the ratio of minimum and maximum intensities of the resultant wave.
3. Three component sinusoidal waves progressing in the same direction along the same path have the same period but their amplitudes are A, $\frac{A}{2}$ and $\frac{A}{3}$. The phases of the variation at any position x on their path at time $t = 0$ and $0, -\frac{\pi}{2}$ and $-\pi$ respectively. Find the amplitude and phase of the resultant wave.
4. Sound waves from a tuning fork A reach a point P by two separate paths ABP and ACP. When ACP is greater than ABP by 11.5 cm, there is silence at P. When the difference is 23 cm the sound becomes loudest at P and when 34.5 cm there is silence again and so on. Calculate the minimum frequency of the fork if the velocity of sound is taken to be 331.2 ms^{-1}.
5. A wave is represented by $y_1 = 10\cos(5x+25t)$ where x is measured in meters and t in seconds. A second wave for which $y_2 = 20\cos\left(5x+25t+\frac{\pi}{3}\right)$ interferes with the first wave. Calculate the amplitude and phase of the resultant wave.
6. Two loudspeakers S_1 and S_2 vibrating in the same phase, each emit sounds of frequency 220 Hz uniformly in all directions. S_1 has an acoustic output of 1.2×10^{-3} watt and S_2 has 1.8×10^{-3} watt. Consider a point P such that $S_1P = 0.75$ m and $S_2P = 3$ m. How are the phases arriving at P related? What is the intensity at P when both S_1 and S_2 are on? Speed of sound in air is 330 ms^{-1}.
7. A sound source capable of producing sound of variable frequencies is located at a distance of 1 m from a reflecting wall as shown in Figure. A detector D is lying at a point shown in Figure. The various distances are also indicated. If the speed of sound in air is 340 ms^{-1}, find the frequencies of sound within the audible range, which will have maximum intensity at detector?

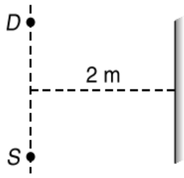


8. Two sound sources S_1 and S_2 emit pure sinusoidal waves in phase. If the speed of sound is 350 ms^{-1}, then for what frequencies,
(a) constructive interference occurs at P?
(b) destructive interference occurs at P?



9. A sound wave of wavelength 40 cm enters the tube shown in Figure from the source. What must be the smallest radius r of the semi-circular part of the tube, so that minima would be received by the detector?

10. A source emitting sound of frequency 180 Hz is placed in front of an obstacle at a distance 2 m from it. A detector is also placed in front of obstacle at the same distance from it.

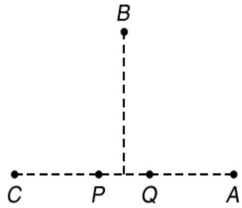


(a) Find the minimum distance between the source and detector for which the detector detects a maximum sound.

(b) How much further to the right must the obstacle be moved if the two waves are to be out of phase by 180°.

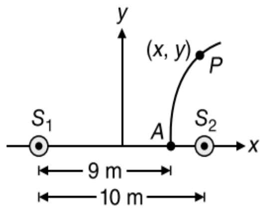
Speed of sound in air = $360\ \text{ms}^{-1}$.

11. Sources P and Q are two equally intense coherent sources emitting radiation of wavelength 20 m as shown in Figure.



The separation between P and Q is 5 m and the phase of P is ahead of that of Q by 90°. A, B and C are three distinct points of observation, each equidistant from the midpoint of PQ. Calculate the ratio of intensities of radiation arriving at A, B and C.

12. Two identical speakers 10 m apart are driven by the same oscillator with a frequency of $f = 21.5$ Hz Figure.



Explain why a receiver at point A records a minimum in sound intensity from the two speakers. If the receiver is moved in the plane of the speakers, what path should it take so that the intensity remains at a minimum? That is, determine the relationship between x and y (the coordinates of the receiver) that causes the receiver to record a minimum in sound intensity. Take the speed of sound to be $343\ \text{ms}^{-1}$.

STATIONARY WAVES

When two coherent waves of equal frequencies and equal amplitudes travelling through a region in opposite directions superpose, the resultant effect is a wave, which does not travel either way with time. These waves are called **stationary waves** or **standing waves** & their resultant amplitude varies periodically with distance. The name stationary for such type of wave is justified because there is no flow of energy along the stationary wave.
In practice, a stationary wave is formed when a wave train reflected at a boundary superimposes with the incident wave. The incident and the reflected waves then interfere to produce a stationary wave.
Stationary waves can be of two types

TRANSVERSE WAVES

Transverse stationary waves are formed in strings stretched between two points.

LONGITUDINAL WAVES

Longitudinal stationary waves are formed in air columns, e.g., in an organ pipe (open of closed).

STATIONARY WAVES PRODUCED ON REFLECTION FROM THE FREE END (RARER MEDIUM)

Let the incident wave be $y_i = A\sin(\omega t - kx)$

Then reflected wave is given by $y_r = A\sin(\omega t + kx)$

By Principle of Superposition, we have

$$y = y_i + y_r = A[\sin(\omega t - kx) + \sin(\omega t + kx)]$$

$$\Rightarrow \quad y = 2A\cos(kx)\sin(\omega t) = R\sin(\omega t)$$

This equation represents a stationary wave. We note that all the particles execute S.H.M. with a frequency equal to the frequency of the interfering waves. However,

unlike a progressive wave, the resultant amplitude is R given by

$$R = 2A\cos(kx)$$

$$\Rightarrow \quad I_R = R^2 = 4A^2\cos^2(kx)$$

Resultant intensity (I_R) is not the same for all the particles but varies with the location x of the particle. I_R **has maximum value** $4A^2$ when

$$\cos^2(kx) = 1$$

$$\Rightarrow \quad \cos(kx) = \pm 1$$

$$\Rightarrow \quad kx = n\pi, \text{ where } n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad x = \frac{n\lambda}{2}, \text{ where } n = 0, 1, 2, 3, \ldots \qquad \left\{\because k = \frac{2\pi}{\lambda}\right\}$$

The points where the amplitude is maximum are called **Antinodes.** Antinodes are obtained at x values given by

$$x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \frac{4\lambda}{2}, \ldots.$$

Distance between two successive antinodes is $\lambda/2$.
I_R **has the minimum value** i.e., zero when

$$\cos(kx) = 0$$

$$\Rightarrow \quad kx = (2n+1)\frac{\pi}{2}, \text{ where } n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad x = (2n+1)\frac{\lambda}{4}, \text{ where } n = 0, 1, 2, 3, \ldots$$

The points where the amplitude is zero are called **Nodes**. Nodes are obtained at x values given by

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \frac{7\lambda}{4}, \ldots.$$

Conceptual Note(s)

Here we also take a note that an antinode is always formed at the free end (i.e. at $x = 0$).

The formation of the reflected pulse is similar to the overlap of two pulses travelling in opposite directions. The net displacement at any point is given by the Principle of Superposition.

Figure shows two pulses with the same shape, travelling in opposite directions but not inverted relative to each other. Note that at one instant, the displacement of the free end is double the pulse height. Also note that an antinode is always formed at the free end.

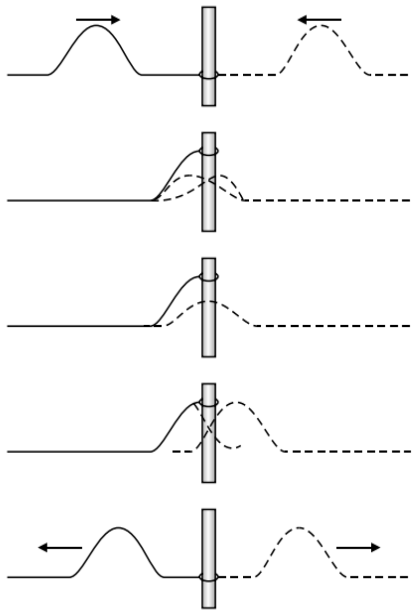


ILLUSTRATION 48

The standing wave $y = 2A\sin kx\cos\omega t$ in an elastic medium is the result of superposition of two travelling waves y_1 and y_2. If $y_1 = A\sin(\omega t - kx)$, determine the wave y_2.

SOLUTION

Since, $y = 2A\sin kx\cos\omega t$

$$\Rightarrow \quad y = A\sin(kx + \omega t) + A\sin(kx - \omega t)$$

Also, $y_1 = A\sin(\omega t - kx) = -A\sin(kx - \omega t)$
By the principle of superposition,

$$y = y_1 + y_2$$

$$\Rightarrow \quad y_2 = y - y_1 = [A\sin(kx + \omega t) + A\sin(kx - \omega t)] - [-A\sin(kx - \omega t)]$$

$$\Rightarrow \quad y_2 = A\sin(kx + \omega t) + 2A\sin(kx - \omega t)$$

$$\Rightarrow \quad y_2 = A\sin(\omega t + kx) - 2A\sin(\omega t - kx)$$

ILLUSTRATION 49

The vibration of a string of length 60 cm fixed at both ends are represented by the equation

$$y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi)$$

where x and y are in cm and t in second.

(a) What is the maximum displacement at $x = 5$ cm?
(b) Where are the nodes located along the string?
(c) What is the velocity of the particle at $x = 7.5$ cm and $t = 0.25$ s?
(d) Write down the equations of component waves whose superposition gives the above wave

SOLUTION

(a) For the point at $x = 5$ cm,

$$y = 4\sin\left(\frac{5\pi}{15}\right)\cos 96\pi t$$

Therefore, the maximum displacement is

$$A_s = 4\sin\left(\frac{5\pi}{15}\right) = 4\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}\text{ cm}$$

(b) At nodes, the amplitude of the stationary wave is zero. That is,

$$4\sin\left(\frac{\pi x}{15}\right) = 0$$

$$\Rightarrow \frac{\pi x}{15} = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\Rightarrow x = 0, 15, 30, 45, 60 \text{ cm}$$

The length of the string being 60 cm, its both ends are nodes.

(c) The velocity of a particle of the string is given as

$$v_p = \frac{dy}{dt} = -(96\pi)4\sin\left(\frac{\pi x}{15}\right)\sin(96\pi t)$$

Thus, the velocity of the particle at $x = 7.5$ cm and $t = 0.25$ s is given as

$$v_p = -(96\pi)4\sin\left(\frac{7.5\pi}{15}\right)\sin(96\pi \times 0.25)$$

$$\Rightarrow v_p = -384\sin\left(\frac{\pi}{2}\right)\sin(24\pi) = -384 \times 1 \times 0 = 0$$

(d) $y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$

$$\Rightarrow y = 2\left[\sin\left(\frac{\pi x}{15} + 96\pi t\right) + \sin\left(\frac{\pi x}{15} - 96\pi t\right)\right]$$

$$\Rightarrow y = 2\sin\left(\frac{\pi x}{15} + 96\pi t\right) + 2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$$

So, $y_1 = 2\sin\left(\frac{\pi x}{15} + 96\pi t\right)$ and

$$y_2 = 2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$$

STATIONARY WAVES PRODUCED ON REFLECTION FROM FIXED END (DENSER MEDIUM)

Let the incident wave be $y_i = A\sin(\omega t - kx)$

Then the reflected wave is given by Stroke's Law as

$$y_r = -A\sin(\omega t + kx)$$

On superposition $y = y_i + y_r$

$$\Rightarrow y = A[\sin(\omega t - kx) - \sin(\omega t + kx)]$$

$$\Rightarrow y = -2A\sin(kx)\cos(\omega t) = R\cos(\omega t)$$

where $R = -2A\sin(kx)$

$$\Rightarrow I_R = R^2 = 4A^2\sin^2(kx)$$

I_R **has the maximum value** $4A^2$ when

$$\sin^2(kx) = 1$$

$$\Rightarrow \sin(kx) = \pm 1$$

$$\Rightarrow kx = (2n+1)\frac{\pi}{2}, \text{ where } n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow x = (2n+1)\frac{\lambda}{4}, \text{ where } n = 0, 1, 2, 3, \ldots \left\{\because k = \frac{2\pi}{\lambda}\right\}$$

The points where the amplitude is maximum are called **Antinodes.** Antinodes are obtained at x values given by

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \frac{7\lambda}{4}, \ldots$$

Distance between two successive antinodes is $\lambda/2$.

I_R **has the minimum value i.e.**, zero when

$$\sin(kx) = 0$$

$$\Rightarrow kx = n\pi, \text{ where } n = 0, 1, 2, \ldots$$

$$\Rightarrow x = n\frac{\lambda}{2} \qquad \left\{\because k = \frac{2\pi}{\lambda}\right\}$$

The points where the amplitude is zero are called **Nodes**. Nodes are obtained at x values given by

$$x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \frac{4\lambda}{2}, \ldots$$

Conceptual Note(s)

(a) Here we also take a note that a node is always formed at the fixed end (i.e. at $x = 0$).

(b) The distance between two successive nodes is $\frac{\lambda}{2}$. The distance between a node and the nearest consecutive antinode is $\frac{\lambda}{4}$. Nodes and Antinodes are formed alternately.

(c) In a longitudinal stationary wave, there is maximum **variation** of pressure, and hence density, at the **nodes**. There is **no variation** of pressure and density at the **antinodes**. Therefore, **Antinodes** may be called **Pressure Nodes** and **Nodes** may be called **Pressure Antinodes.**

The formation of the reflected pulse is similar to the overlap of two pulses travelling in opposite directions. The net displacement at any point is given by the Principle of Superposition.

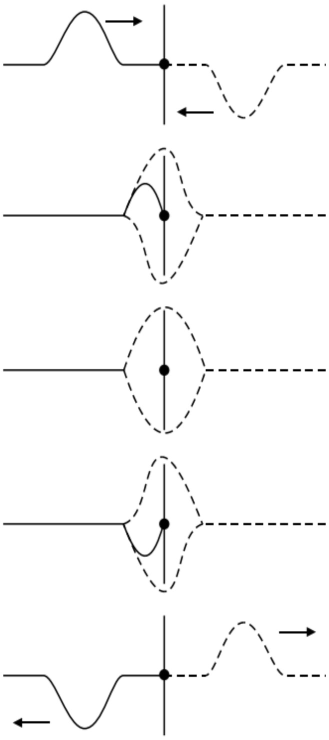


Figure shows two pulses with the same shape, one inverted with respect to the other, travelling in opposite directions. Because these two pulses have the same shape the net displacement of the point where the string is attached to the wall is zero at all times and hence a node is always obtained at the fixed end.

ILLUSTRATION 50

A standing wave is formed by two harmonic waves, $y_1 = A\sin(kx - \omega t)$ and $y_2 = A\sin(kx + \omega t)$ travelling on a string of density ρ, area of cross-section S, in opposite directions. Find the total mechanical energy between two adjacent nodes on the string.

SOLUTION

The distance between two adjacent nodes is $\frac{\lambda}{2}$ or $\frac{\pi}{k}$
Volume of string between two nodes is

$$V = \begin{pmatrix}\text{area of}\\ \text{cross-section}\end{pmatrix}\begin{pmatrix}\text{distance between}\\ \text{two nodes}\end{pmatrix}$$

$$\Rightarrow \quad V = (S)\left(\frac{\pi}{k}\right)$$

Energy density (energy per unit volume) of a travelling wave is given by

$$u = \frac{1}{2}\rho A^2\omega^2$$

A standing wave is formed by two identical waves travelling in opposite directions. So, energy stored between two nodes in a standing wave is twice the energy stored in distance π/k of a travelling wave.

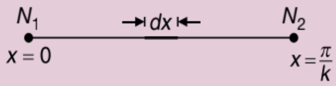
$$\Rightarrow \quad E = 2(\text{Energy Density})(\text{Volume})$$

$$\Rightarrow \quad E = 2\left(\frac{1}{2}\rho A^2\omega^2\right)\left(\frac{\pi S}{k}\right) = \frac{\rho A^2\omega^2\pi S}{k}$$

Alternate Method:
The equation of the standing wave

$$y = y_1 + y_2 = 2A\sin(kx)\cos(\omega t) = A(x)\cos(\omega t)$$

where, $A(x) = 2A\sin(kx)$



i.e., first node is at $x = 0$ and the next node is at $x = \frac{\pi}{k}$. Let us take an infinitesimal element of length dx at distance x from N_1. Mass of this element is $dm(=\rho S dx)$ and this element can be treated as point mass oscillating simple harmonically with angular frequency ω and amplitude $2A\sin(kx)$.
Hence, energy of this element is

$$dE = \frac{1}{2}(dm)(2A\sin kx)^2(\omega^2)$$

$$\Rightarrow \quad dE = \frac{1}{2}(\rho S dx)(2A\sin kx)^2\omega^2$$

Integrating this from $x = 0$ to $x = \frac{\pi}{k}$, we get

$$E = \frac{\rho A^2\omega^2\pi S}{k}$$

ILLUSTRATION 51

Consider a wave propagating in the negative x-direction whose frequency is 100 Hz. At $t = 5$ s, displacement associated with the wave is given by $y = 0.5\cos(0.1x)$ where x and y are measured in centimetre and t in second. Obtain the displacement (as a function of x) at $t = 10$ s. What is the wavelength and velocity associated with the wave?

SOLUTION

A wave travelling in negative x-direction can be represented as,

$$y(x, t) = A\cos(\omega t + kx + \phi)$$

At $t = 5$ s, $y(x, t = 5) = A\cos(5\omega + kx + \phi)$

Comparing this with the given equation, we get

$$A = 0.5 \text{ cm}, k = 0.1 \text{ cm}^{-1} \text{ and}$$

$$5\omega + \phi = 0 \qquad \ldots(1)$$

Since, $\lambda = \frac{2\pi}{k}$ i.e., $\lambda = \frac{2\pi}{0.1} = 20\pi$ cm

Also, $\omega = 2\pi f = 200\pi$ rads^{-1}

$$\Rightarrow \quad v = \frac{\omega}{k} = \frac{200\pi}{0.1} = 2000\pi \text{ cms}^{-1}$$

From Equation (1), we get $\phi = -5\omega$

At $t = 10$ s, we have

$$y = y(x, t = 10) = 0.5\cos(0.1x + 10\omega - 5\omega)$$

$$\Rightarrow \quad y = 0.5\cos(0.1x + 5\omega)$$

Substituting $\omega = 200\pi$, we get

$$y = y(x, t = 10) = 0.5\cos(0.1x + 1000\pi)$$

$$\Rightarrow \quad y = 0.5\cos(0.1x)$$

COMPARISON OF PROGRESSIVE AND STATIONARY WAVE

The following table compares a progressive wave with a stationary wave.

Sl. No.	Progressive	Stationary
1.	The wave advances with a constant speed.	The wave does not advance but remains confined in a particular region.
2.	The amplitude is the same for all the particles in the path of the wave.	The amplitude varies according to position, being zero at the nodes and maximum at the antinodes.
3.	All particles within one wavelength have different phases.	Phase of all particles between two adjacent nodes is the same. Particles in adjacent segments of length $\frac{\lambda}{2}$ have opposite phases.
4.	Energy is transmitted in the direction of propagation of the wave.	Energy is associated with the wave, but there is no transfer of energy across any section of the medium.

Conceptual Note(s)

(a) At nodes displacement of particles is always zero, so they are permanently at rest. But strain $\frac{\partial y}{\partial x}$ at nodes is maximum, so pressure and hence energy is maximum at nodes.

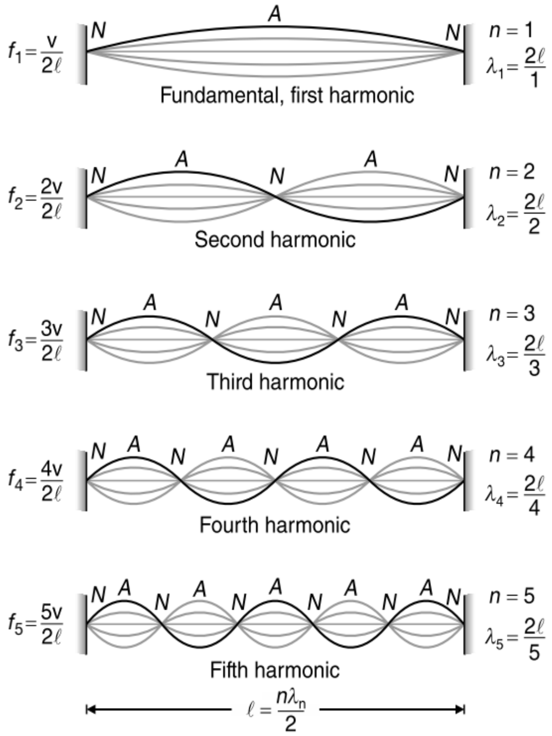
(b) At antinodes the displacement is maximum and strain $\frac{\partial y}{\partial x}$ is zero, so pressure and hence energy is minimum at antinodes.

(c) All particles between two consecutive nodes vibrate in same phase while the particles on opposite side of a node vibrate in opposite phase.

STATIONARY WAVES IN A STRING FIXED AT BOTH ENDS AND MODES OF VIBRATION OF A STRETCHED STRING

Consider a string of length l, stretched under a tension T between two fixed points. If the string is plucked and then released, a transverse wave travels along the string and is reflected at the ends.

A stationary wave is thus set up in the string. Since the end points are fixed, they are nodes. A string can vibrate in several modes. The first five modes and their frequencies are Shown in Figure.



Fundamental Mode or First Harmonic

String plucked at $\frac{l}{2}$ to get 1 loop

$$l = 2\left(\frac{\lambda_1}{4}\right) = \frac{v}{2f_1} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_1 = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

Second Harmonic or First Overtone

String plucked at $\frac{l}{4}$ to get 2 loops

$$l = 4\left(\frac{\lambda_2}{4}\right) = \frac{v}{f_2} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_2 = \frac{v}{l} = 2\left(\frac{v}{2l}\right) = 2\left(\frac{1}{2l}\sqrt{\frac{T}{\mu}}\right) = 2f_1$$

Third Harmonic or Second Overtone

String plucked at $\frac{l}{6}$ to get 3 loops

$$l = 6\left(\frac{\lambda_3}{4}\right) = \frac{3v}{2f_3} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_3 = \frac{3v}{2l} = 3\left(\frac{v}{2l}\right) = 3\left(\frac{1}{2l}\sqrt{\frac{T}{\mu}}\right) = 3f_1$$

In general, if the string of length l is plucked at length $\frac{l}{2n}$, then it vibrates in n segments (loops) such that $l = \frac{n\lambda_n}{2}$ and hence we get the nth harmonic whose frequency is given by

$$f_n = n\left(\frac{1}{2l}\sqrt{\frac{T}{m}}\right) = nf_1.$$

If H represents Harmonic, O represents overtone and f_n represents the frequency of the n^{th} harmonic, then they are related to each other as shown in table.

H	1	2	3	4	5	...	n	$n+1$
f_n	f_1	$2f_1$	$3f_1$	$4f_1$	$5f_1$	⋯	nf_1	$(n+1)f_1$
O	–	1	2	3	4	...	$n-1$	n

Conceptual Note(s)

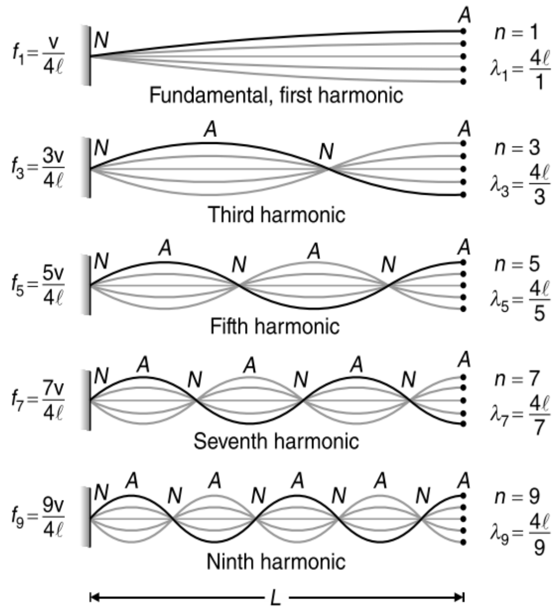
(a) Overtones are the frequencies which are counted leaving the first harmonic i.e. first overtone will be assigned to the frequency that happens the be the next immediate present frequency after first harmonic.
e.g., if third harmonic is present immediately after first harmonic then third harmonic is designated as first overtone.

(b) Whenever we are asked to compare the overtones we compare their respective harmonics.

(c) Harmonics may be either odd or even or both odd and even, but overtones are always both odd and even.

STATIONARY WAVES IN A STRING FIXED AT ONE END AND MODES OF VIBRATION OF A STRETCHED STRING

Similarly, stationary waves can be produced in string fixed at one end. *The first five modes and their frequencies are shown in Figure.*



In each mode of vibration shown, there is an odd number of quarter wave lengths in the length l such that

$$l = n\left(\frac{\lambda_n}{4}\right), n = 1, 3, 5, ...$$

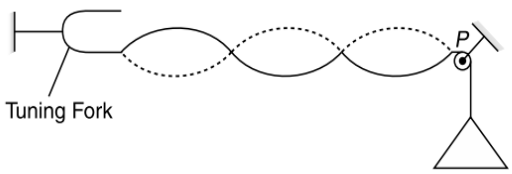
$$\Rightarrow \quad \lambda_n = \frac{v}{f_n} = \frac{4l}{n}$$

$$\Rightarrow \quad f_n = n\left(\frac{v}{4l}\right) = nf_1$$

where, $f_1 = \frac{v}{4l}$ is the fundamental frequency. The natural frequencies of this system are in the ratio $1:3:5:7:9:...$ which means that only odd harmonics are present and even harmonics are absent.

MELDE'S EXPERIMENT

One end of a horizontal string is attached to a prong of an electrically maintained tuning fork. The string passes over a pulley P and to the other end a desired weight can be attached. When the fork vibrates, a stationary wave is formed on the string. If the tension and/or the length of the string are suitably adjusted, the string can be made to vibrate in one of its modes with large amplitude. When this happens, string is in resonance with tuning fork and their frequencies are equal.



If n_1 and n_2 are the number of loops corresponding to tensions T_1 and T_2, then

$$n_1^2 T_1 = n_2^2 T_2$$

i.e. $n^2 T = \text{constant}$

Thus, if the number of loops is to be increased the tension should be decreased.

SONOMETER

It consists of a thin wire mounted on a large hollow wooden box. One end of the wire is fixed at A and the other end E carries a load. The wire passes over two sliding bridges B and C and a pulley D. Any disturbance created on the wire gets reflected by the bridges and hence a transverse stationary wave is formed between B and C. The length BC and tension are adjusted to produce resonance with a source of sound, e.g., tuning fork.

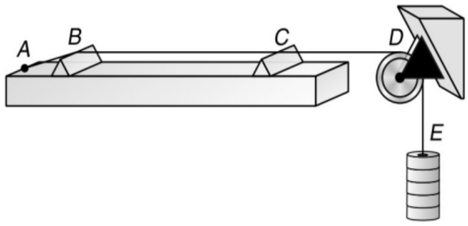


ILLUSTRATION 52

The fundamental frequency of a sonometer wire increases by 6 Hz if its tension is increased by 44% keeping the length constant. Find the change in the fundamental frequency of the sonometer wire, when the length of the wire is increased by 20% keeping the original tension in the wire.

SOLUTION

The fundamental frequency is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

Keeping the length of given wire constant, we have

$$\frac{f'}{f} = \left(\frac{T'}{T}\right)^{1/2}$$

Here, $f' = f + 6$ and $T' = T + 0.44T = 1.44T$

$$\Rightarrow \quad \frac{(f+6)}{f} = \sqrt{\frac{1.44T}{T}}$$

$$\Rightarrow \quad f = 30 \text{ Hz}$$

Now, keeping the original tension same the length of given wire is changed to

$$L'' = L + 0.020L = 1.20L$$

$$\Rightarrow \quad \frac{f''}{f} = \frac{L}{L''} = \frac{1}{1.20}$$

$$\Rightarrow \quad f'' = \frac{30}{1.2} = 25 \text{ Hz}$$

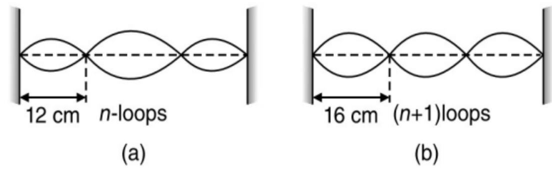
$$\Rightarrow \quad \Delta f'' - f = 25 - 30 = -5 \text{ Hz}$$

Thus, the fundamental frequency will decrease by 5 Hz.

ILLUSTRATION 53

A string fixed at both ends has consecutive standing wave modes for which the distances between adjacent nodes are 12 cm and 10 cm respectively. What the minimum possible length of the string?

SOLUTION



Let length of string be l then

From Figure (a) $12n = l$ ...(1)

From Figure (b) $10(n+1) = l$ ...(2)

From equation (1) and (2), we get

$$12n = 10(n+1)$$

$$\Rightarrow \quad 2n = 10$$

$$\Rightarrow \quad n = 5$$

From equation (1), we get $12 \times 5 = l$

$$\Rightarrow \quad l = 60 \text{ cm}$$

Therefore, the minimum possible length of the string can be 60 cm

ILLUSTRATION 54

The total length of a sonometer wire between fixed ends is 110 cm. Thus, bridges are placed to divide the length of the wire in the ratio 6 : 3 : 2. What is the minimum common frequency with which three parts can vibrate? Also calculate the ratio of number of loops formed in three parts. The tension in the wire is 400 N and the mass per unit length is 0.01 kgm^{-1}.

SOLUTION

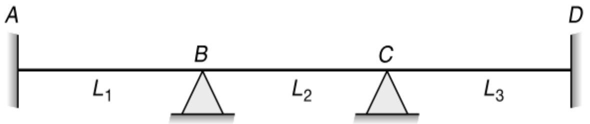
The length of each segment is given by

$$L_1 = \left(\frac{6}{6+3+2}\right)(110 \text{ cm}) = 0.6 \text{ m}$$

$$L_2 = \left(\frac{3}{6+3+2}\right)(110 \text{ cm}) = 0.3 \text{ m}$$

$$L_3 = \left(\frac{2}{6+3+2}\right)(110 \text{ cm}) = 0.2 \text{ m}$$

Let f_1, f_2 and f_3 be the fundamental frequencies of the segments AB, BC and CD respectively.



Since $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\mu}}$, so $fL = \dfrac{1}{2}\sqrt{\dfrac{T}{\mu}}$ = constant

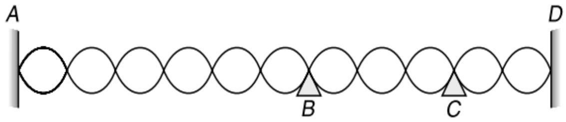
Also, $f_1L_1 = f_2L_2 = f_3L_3$ i.e., $0.6f_1 = 0.3f_2 = 0.2f_3$

Obviously, the sixth harmonic of AB, third harmonic of BC and second harmonic of CD coincide. Hence the lowest common frequency is given by

$$6f_1 = 3f_2 = 2f_3 = f_c$$

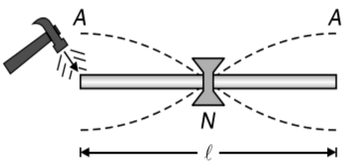
$$\Rightarrow \quad f_c = 6f_1 = 6\times\frac{1}{2L_1}\sqrt{\frac{T}{\mu}} = \frac{6}{2\times 0.6}\sqrt{\frac{400}{0.01}} = 1000 \text{ Hz}$$

Since nth *harmonic contains* n *loops, so AB vibrates in 6 loops, BC in 3 loops and CD in 2 loops*. Ratio of number of loop in three segments is $6:3:2$.



VIBRATIONS OF A CLAMPED ROD

Let us discuss the oscillations of a rod clamped at its middle point on its length as shown in Figure.



Let the rod be gently hit at its one end due to which it begins to oscillate. In natural oscillations the rod vibrates at its lowest frequency and maximum wavelength, which we call fundamental mode of oscillations. With maximum wavelength, when transverse stationary waves setup in the rod, the free ends vibrate as antinodes and the clamped end as a node. If λ be wavelength of wave, then $l = \lambda_1/2$.

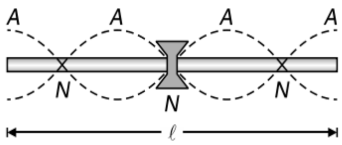
$$\Rightarrow \quad \lambda_1 = 2l \qquad \ldots(1)$$

The frequency of fundamental oscillations of a rod clamped at midpoint is

$$f_1 = \frac{v}{\lambda_1} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \qquad \ldots(2)$$

where, Y is the Young's modulus and ρ is the density of the material of rod.

The next higher frequency at which rod vibrates is shown in Figure.



In this case if λ_3 be the wavelength of the waves in rod, then we have $l = 3\lambda_3/2$ and hence

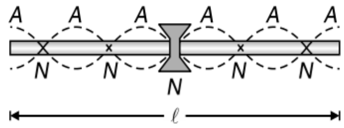
$$\lambda_3 = \frac{2l}{3} \qquad \ldots(3)$$

So, in this case, the oscillation frequency of rod is

$$f_3 = \frac{v}{\lambda_3} = 3\left(\frac{1}{2l}\sqrt{\frac{Y}{\rho}}\right) = 3f_1 \qquad \ldots(4)$$

This is called third harmonic or first overtone frequency of the clamped rod.

Similarly, the next higher frequency of oscillation is again shown in Figure



In this case if λ_5 be the wavelength of the waves in rod, we have $l = 5\lambda_5/2$ and hence

$$\lambda_5 = \frac{2l}{5} \qquad \ldots(5)$$

So, in this case, the oscillation frequency of rod is

$$f_5 = \frac{v}{\lambda_5} = 5\left(\frac{1}{2l}\sqrt{\frac{Y}{\rho}}\right) = 5f_1 \qquad \ldots(6)$$

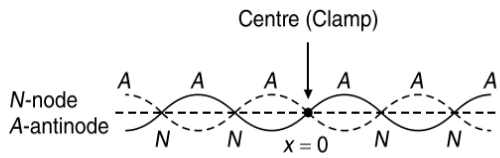
This is called fifth harmonic or second overtone frequency of the clamped rod.

ILLUSTRATION 55

A metallic rod of length 1 m is rigidly clamped at its mid-point. Longitudinal stationary waves are set up in the rod in such a way that there are two nodes on either side of the mid-point. The amplitude of an antinode is 2×10^{-6} m. Write the equation of motion at a point 2 cm from the mid-point and equations of the constituent waves in the rod. Take Young's modulus and density of the material of the rod to be 2×10^{11} Nm^{-2} and 8000 kgm^{-3})

SOLUTION

The wave pattern is shown in the Figure below.



From the diagram, we see that $1 \text{ m} = \dfrac{5\lambda}{2}$

$$\Rightarrow \quad \lambda = \frac{2}{5} = 0.4 \text{ m.}$$

The amplitude at the antinode is $2A = 2\times10^{-6}$

$$\Rightarrow \quad A = 1\times10^{-6}$$

The wave velocity v is given by

$$v=\sqrt{\frac{Y}{\rho}}=\sqrt{\frac{2\times10^{11}}{8000}}=5000\ \text{ms}^{-1}$$

Since $v=f\lambda$

$\Rightarrow \quad f(0.4)=5000$

$\Rightarrow \quad f=12500$ Hz

Also, $k=\dfrac{2\pi}{\lambda}=\dfrac{2\pi}{0.4}=5\pi$ and

$$\omega=2\pi f=2\pi(12,500)=25000\pi$$

The equation of the stationary wave is therefore

$$y=2A\sin kx\cos\omega t$$

$\Rightarrow \quad y=2\times10^{-6}\sin(5\pi x)\cos(25000\pi t)$

At $x=0.02$ m, we have

$$y=2\times10^{-6}\sin(5\pi\times2\times10^{-2})\cos(25000\pi t)$$

$\Rightarrow \quad y=2\times10^{-6}\sin\left(\dfrac{\pi}{10}\right)\cos 25000\pi t$

The equations of the two waves are,

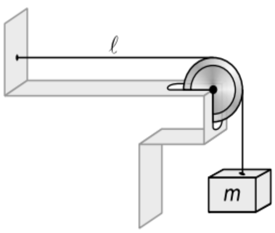
$$y_1=A\sin(kx-\omega t)\text{ and }y_2=A\sin(\omega t+kx)$$

So, $\quad y_1=10^{-6}\sin(5\pi x-25000\pi t)$ and

$$y_2=10^{-6}\sin(25000\pi t+5\pi x)$$

ILLUSTRATION 56

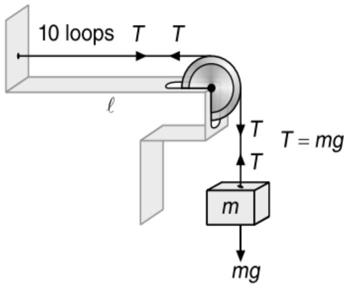
Figure shows a string stretched by a block going over a pulley. The string vibrates in its tenth harmonic in unison with a particular tuning fork. When a beaker containing water is brought under the block so that the block is completely dipped into the beaker, the string vibrates in its eleventh harmonic. Find the density of the material of the block.



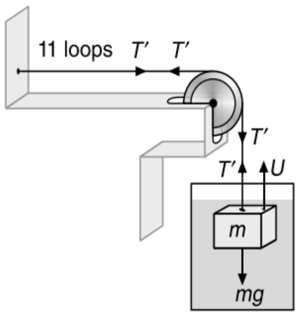
SOLUTION

According to the problem, we have $f=10f_1$

$\Rightarrow \quad f=\dfrac{10}{2l}\sqrt{\dfrac{T}{\mu}}=\dfrac{10}{2l}\sqrt{\dfrac{mg}{\mu}}$ ...(1)



When block is dipped in the beaker, then FBD of situation is shown



$$T'+U=mg$$

$\Rightarrow \quad T'=mg-U=mg-\rho_\omega\left(\dfrac{m}{\rho_b}\right)g=mg\left(1-\dfrac{\rho_w}{\rho_b}\right)$

where ρ_b is density of material of block and ρ_w is density of water.

$\Rightarrow \quad f=\dfrac{11}{2l}\sqrt{\dfrac{T'}{\mu}}=\dfrac{11}{2l}\sqrt{\dfrac{mg}{\mu}\left(1-\dfrac{\rho_w}{\rho_b}\right)}$ ...(2)

Equating (1) and (2), we get

$$\frac{10}{2l}\sqrt{\frac{mg}{\mu}}=\frac{11}{2l}\sqrt{\frac{mg}{\mu}\left(1-\frac{\rho_w}{\rho_b}\right)}$$

$\Rightarrow \quad 10=11\sqrt{1-\dfrac{\rho_w}{\rho_b}}$

$\Rightarrow \quad \dfrac{\rho_w}{\rho_b}=1-\dfrac{100}{121}=\dfrac{21}{121}$

$\Rightarrow \quad \rho_b=\dfrac{121}{21}\rho_w=\dfrac{121}{21}\ \text{gcc}^{-1}=5.8\ \text{gcc}^{-1}$

LONGITUDINAL STATIONARY WAVES IN AIR COLUMNS/ORGAN PIPES

An air filled pipe is called an organ pipe. There are two types of organ pipes

1. An open pipe which has both the ends open
2. A closed pipe which has one end closed and the other end open

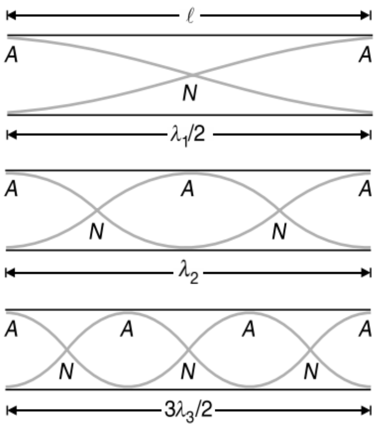
If a tuning fork is placed near one end, which is open, a longitudinal wave travels in the air column. This is reflected back from the other end. The incident and the reflected waves superpose to produce a stationary wave. *It is important to note that a node is always formed at the closed end and an antinode at the open end.*

STATIONARY SOUND WAVES IN AN OPEN ORGAN PIPE

The concept of stationary waves can also be applied to sound waves in an air column (an air-filled pipe) called an organ pipe. Standing waves are the result of interference between longitudinal sound waves traveling in opposite directions.

Many other aspects of standing sound wave patterns are similar to those of string waves, like closed end of the pipe gets a displacement node (or pressure antinode) and the open end of the pipe gets a displacement antinode (or a pressure node).

The first three modes of vibration of an open pipe (open at both ends) are shown in Figure.



First Harmonic (Fundamental Mode)

$$l = 2\left(\frac{\lambda_1}{4}\right) = \frac{v}{2f_1} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_1 = \frac{v}{2l}$$

where v is the velocity of longitudinal wave.

Second Harmonic (First Overtone)

$$l = 4\left(\frac{\lambda_2}{4}\right) = \frac{v}{f_2} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_1 = \frac{v}{l} = 2\left(\frac{v}{2l}\right) = 2f_1$$

Third Harmonic (Second overtone)

$$l = 6\left(\frac{\lambda_3}{4}\right) = \frac{3v}{2f_3} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_3 = \frac{3v}{2l} = 3\left(\frac{v}{2l}\right) = 3f_1$$

Thus, in an open pipe all the harmonics (both odd and even) are present. For the nth harmonic

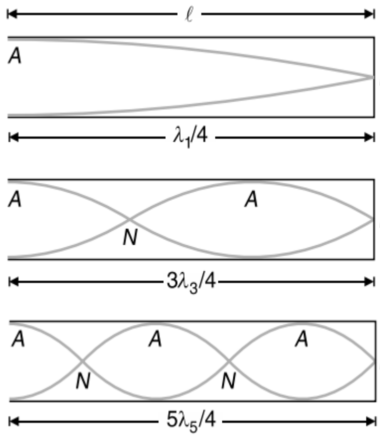
$$f_n = n\left(\frac{v}{2l}\right) = nf_1$$

If H represents Harmonic, O represents overtone and f_n represents the frequency of the n^{th} harmonic, then they are related to each other as shown in table.

H	1	2	3	4	5	...	n	n + 1
f_n	f_1	$2f_1$	$3f_1$	$4f_1$	$5f_1$	$\cdots$	nf_1	$(n+1)f_1$
O	–	1	2	3	4	...	$n-1$	n

CLOSED PIPE

The first three modes of vibration of a closed pipe (closed at one end and open at the other) are shown in Figure.



First Harmonic (Fundamental Mode)

$$l = 1\left(\frac{\lambda_1}{4}\right) = \frac{v}{4f_1} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_1 = \frac{v}{4l}$$

where v is the velocity of longitudinal wave .

Third Harmonic (First Overtone)

$$l = 3\left(\frac{\lambda_2}{4}\right) = \frac{3v}{4f_3} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_3 = \frac{3v}{4l} = 3\left(\frac{v}{4l}\right) = 3f_1$$

Please note here, that this frequency (the next immediate present frequency after fundamental harmonic) is thrice the fundamental and hence it has to be called as Third Harmonic or First Overtone.

Fifth Harmonic (Second overtone)

$$l = 5\left(\frac{\lambda_3}{4}\right) = \frac{5v}{4f_5} \qquad \{\because v = f\lambda\}$$

$$\Rightarrow \quad f_5 = \frac{5v}{4l} = 5\left(\frac{v}{4l}\right) = 5f_1$$

Again, please note here, that this frequency is five times the fundamental and hence it has to be called as Fifth Harmonic or Second Overtone.

So, in a closed organ pipe only odd harmonics are present (even harmonics are absent).

If H represents Harmonic, O represents overtone and f_n represents the frequency of the n^{th} harmonic, then they are related to each other as shown in table.

H	1	3	5	7	9	...	n	$n+1$
f_n	f_1	$3f_1$	$5f_1$	$7f_1$	$9f_1$	$\cdots$	nf_1	$(2n+1)f_1$
O	–	1	2	3	4	...	$\frac{n-1}{2}$	n

Here the nth overtone is $(2n+1)$th harmonic, having frequency

$$f_{n\text{th overtone}} = (2n+1)\frac{v}{4l}$$

As an example, ratio of the seventh overtone to the second overtone in a closed organ pipe should be

$$\frac{f_{7\text{th overtone}}}{f_{2\text{nd overtone}}} = \frac{2(7)+1}{2(2)+1} = \frac{15}{5} = 3$$

END CORRECTION IN ORGAN PIPES

In the above discussion we have assumed that the antinode is formed exactly at the open end of a pipe. *However, the antinode is formed actually a little distance outside the open end. This is because the air outside is not a rigid boundary and therefore gets slightly compressed due to which it reflects back the incident wave.*

The correct effective length of a closed pipe is thus $l+e$, and that of an open pipe is $l+2e$, where e is called the end correction having approximate value $e = 0.3D$, where D is the internal diameter of the pipe. Hence, for a broad pipe the end correction is more compared to that of a narrow pipe.

Whenever we calculate the harmonic frequencies of oscillations of air column in organ pipe, then we must take into account the end corrections due to which the fundamental frequency of a closed organ pipe (having only one open end) of length l is

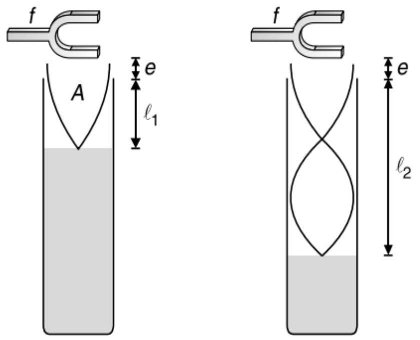
$$f_{\text{closed}} = \frac{v}{4(l+e)}$$

and fundamental frequency of an open organ pipe (having both ends open) of length l is

$$f_{\text{open}} = \frac{v}{2(l+2e)}$$

RESONANCE TUBE

It is used to determine velocity of sound in air by resonating the tuning fork of known frequency say f with the organ pipe. It also helps to calculate the end correction to be applied.



If l_1 and l_2 are lengths of first and second resonances also called as first resonance and second resonance lengths respectively, then we have

$$l_1 + e = \frac{\lambda}{4} \qquad \ldots(1)$$

$$l_2 + e = \frac{3\lambda}{4} \qquad \ldots(2)$$

Subtracting (1) from (2),

$$l_2 - l_1 = \frac{\lambda}{2} = \frac{v}{2f}$$

where v is the speed of sound in air at room temperature. So, we get

$$v = 2f(l_2 - l_1)$$

Dividing (2) by (1), we get $\frac{l_2+e}{l_1+e} = 3$ ...(3)

$$\Rightarrow \quad e = \frac{l_2 - 3l_1}{2}$$

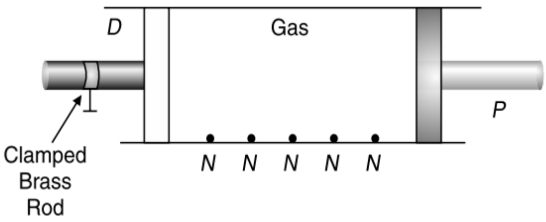
From equation (3), we get

$$l_2 = 3l_1 + 2e$$

So, second resonance is obtained at length slightly more than thrice the length of first resonance.

KUNDT'S TUBE

Kundt's tube can be used to compare the speeds of sound in different gases. A horizontal glass tube fitted with a smoothly moving piston P at one end and a loosely fit disc D at the other is filled with a gas. A horizontal brass rod, clamped at the middle is attached to D. A thin layer of lycopodium powder is spread inside the tube.



The outer half of the brass rod is rubbed along its length in one direction only by means of a wet or a rinsed cloth. This sets the rod and hence the air inside the tube into longitudinal vibrations. The piston is moved until resonance occurs. The lycopodium powder gets collected at the nodes in small heaps. The distance between two heaps gives $\frac{\lambda}{2}$.

Let λ_A and λ_B be the wavelengths corresponding to two gases. If v_A and v_B be the velocities of sound in them respectively, then, since frequency is the same in both the cases, we have

$$\frac{v_A}{v_B}=\frac{\lambda_A}{\lambda_B}$$

ILLUSTRATION 57

The air column in a pipe closed at one end is made to vibrate in its second overtone by a tuning fork of frequency 440 Hz. The speed of sound in air is 330 ms^{-1}. End correction may be neglected. Let p_0 denote the mean pressure at any point in the pipe, and Δp_0 the maximum amplitude of pressure variation.

(a) Find the length L of the air column?
(b) What is the amplitude of pressure variation at the middle of the column?
(c) What are the maximum and minimum pressure at the open end of the pipe?
(d) What are the maximum and minimum pressure at the closed-end of the pipe?

SOLUTION

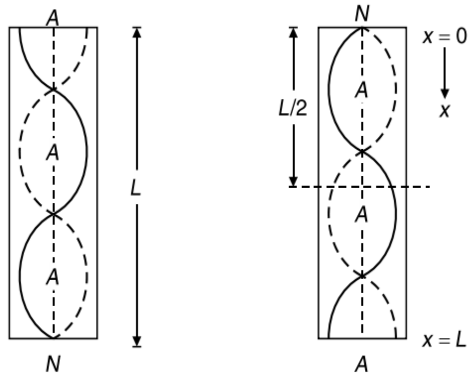
(a) In case of a closed organ pipe, the fundamental frequency is $\left(\frac{v}{4L}\right)$ and only odd harmonics are present.

Therefore, the second overtone will mean fifth harmonic i.e., $f=\frac{5v}{4L}$

$$\Rightarrow\ L=\frac{5v}{4f}=\frac{5\times 330}{4\times 440}=\frac{15}{16}\text{ m}$$

(b) At the position of displacement antinode there is pressure node and vice-versa. The variation of the pressure amplitude of standing pressure waves along the length of the column $x=0$ at its open end will be

$$p=\Delta p_0\sin kx=\Delta p_0\sin\left(\frac{2\pi}{\lambda}x\right)\qquad\left\{\text{as}\quad k=\frac{2\pi}{\lambda}\right\}$$



Displacement wave pattern   Pressure wave pattern

Now, for second overtone $L=\left(\frac{5}{4}\right)\lambda$, so at the middle, $x=\frac{L}{2}=\frac{5}{8}\lambda$

$$\Rightarrow\ p=\Delta p_0\sin\frac{2\pi}{\lambda}.\left(\frac{5}{8}\lambda\right)=\Delta p_0\sin\left(\frac{5}{4}\pi\right)$$

$$\Rightarrow\ |p|=\Delta p_0\times\frac{1}{\sqrt{2}}=\frac{\Delta p_0}{\sqrt{2}}$$

At the open end, $x=0$, $p=0$.

$$\Rightarrow\ p_{\max}=p_{\min}=p_0\pm 0=p_0$$

(b) The closed end is an antinode for the pressure wave. Therefore,

$$p_{\max}=p_0+\Delta p_0 \text{ and } p_{\min}=p_0-\Delta p_0$$

ILLUSTRATION 58

A tuning fork having frequency of 340 Hz is vibrated just above a cylindrical tube. The height of the tube is 120 cm. Water is slowly poured in. What is the minimum height of water required for resonance? $\left(v=340\ \text{ms}^{-1}\right)$

SOLUTION

As the tuning fork is in resonance with air column in the pipe closed at one end.

$$f=n\frac{v}{4L}\text{ with } n=1,3,5,\ldots$$

Therefore, the length of air column in the pipe,

$$L=\frac{nv}{4f}=n\frac{340\times 100}{4\times 340}=25n\text{ cm with } n=1,3,5,\ldots$$

$$\Rightarrow\quad L=25\text{ cm},75\text{ cm},125\text{ cm},\ldots.$$

Now, as the tube is 120 cm, the length of air column must be lesser than 120 cm. It can be only 25 cm or 75 cm. Further if h is the height of water filled in the tube.

$$L + h = 120 \text{ cm}$$

$\Rightarrow \quad h = 120 - L$

So h will be minimum when L is maximum, i.e., when $L = 75$ cm

$\Rightarrow \quad h_{min} = 120 - 75 = 45$ cm

ILLUSTRATION 59

A resonance air column resonates with a tuning fork of frequency 512 Hz at column lengths 16 cm and 49.2 cm. Find the end correction, and the velocity of sound wave in air.

SOLUTION

The resonance air column has one closed end, which is adjustable. Thus, if e is the end correction we have

$$L_1 + e = \frac{\lambda}{4} \text{ and } L_2 + e = \frac{3\lambda}{4}$$

$\Rightarrow \quad \lambda = 4(L_1 + e)$ and $\lambda = \frac{4}{3}(L_2 + e)$

$\Rightarrow \quad 3(L_1 + e) = (L_2 + e)$

$\Rightarrow \quad e = \frac{L_2 - 3L_1}{2} = \frac{0.492 - 3 \times 0.16}{2} = 0.006 \text{ m}$

Now, $v = f\lambda = f \times 4(L_1 + e)$

$$v = 512 \times 4 \times (0.16 + 0.006) = 340 \text{ ms}^{-1}$$

ILLUSTRATION 60

The water level in a vertical glass tube 1 m long can be adjusted to any position in the tube. A tuning fork vibrating at 660 Hz is held just over the open top end of the tube. At what positions of the water level will there be resonance. Assume the speed of sound to be 330 ms^{-1}.

SOLUTION

Resonance corresponds to a pressure antinode at closed end and pressure node at open end. Further, the distance between a pressure node and a pressure antinode is $\frac{\lambda}{4}$. For resonance, the length of air column l, must be

$$l = n\left(\frac{\lambda}{4}\right) = n\left(\frac{v}{4f}\right)$$

where, $n = 1, 3, 5, \ldots$

$\Rightarrow \quad l_1 = (1)\left(\frac{330}{4 \times 660}\right) = 0.125 \text{ m}$

$\Rightarrow \quad l_2 = 3l_1 = 0.375 \text{ m}$

$\Rightarrow \quad l_3 = 5l_1 = 0.625 \text{ m}$

$\Rightarrow \quad l_4 = 7l_1 = 0.875 \text{ m}$

$\Rightarrow \quad l_5 = 9l_1 = 1.125 \text{ m}$

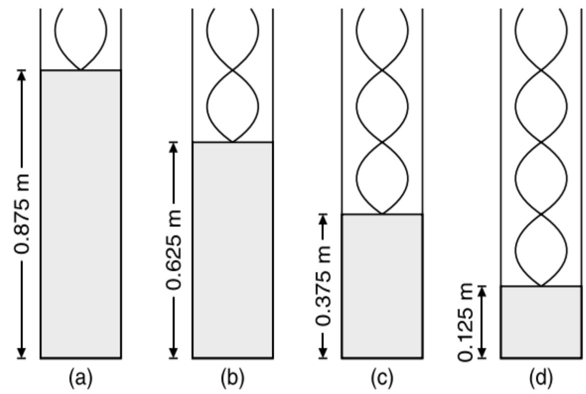
Since, $l_5 > 1$ m (the length of tube), the length of air columns can have the values from l_1 to l_4 only. Therefore, level of water at resonance will be

$$(1 - 0.125) \text{ m} = 0.875 \text{ m}$$

$$(1 - 0.375) \text{ m} = 0.625 \text{ m}$$

$$(1 - 0.625) \text{ m} = 0.375 \text{ m}$$

and $(1 - 0.875) \text{ m} = 0.125 \text{ m}$



In all the four cases shown in figure, the resonance frequency is 660 Hz but first one is the fundamental tone or first harmonic. Second is first overtone or third harmonic and so on.

ILLUSTRATION 61

The first overtone of an open organ pipe together with the first overtone of a closed organ pipe produces beats with a beat frequency 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz. Find the lengths of the pipes. Speed of sound in air, $v = 330$ ms^{-1}.

SOLUTION

Let the lengths of the closed and open organ pipes be l_1 and l_2. The fundamental frequency of the closed organ pipe is given by

$$f_1 = \frac{v}{4l_1}$$

$\Rightarrow \quad 110 = \frac{330}{4l_1}$

$\Rightarrow \quad l_1 = \frac{330}{4 \times 110} = 0.75 \text{ m}$

The frequency of the first overtone (third harmonic) of this pipe is thrice that of the fundamental mode. Hence it equals 330 Hz. This produces 2.2 beats per second with the first overtone (second harmonic) of an open organ pipe. Hence the frequency of second harmonic, f_2 of the open organ pipe must be

$$f_2 = (330 \pm 2.2) \text{ Hz}$$

$\Rightarrow f_2 = 332.2$ Hz	$f_2 = 327.8$ Hz
$\Rightarrow 2\left(\frac{v}{2l_2}\right) = 332.2$	$\frac{2v}{2l_2} = 327.8$
$\Rightarrow \frac{330}{l_2} = 332.2$	$\frac{330}{l_2} = 327.8$
$\Rightarrow l_2 = 0.99$ m	$l_2 = 1.0067$ m

ILLUSTRATION 62

An air column in a pipe closed at one end is made to vibrate in its second overtone by a tuning fork of frequency 440 Hz. The speed of sound in air is 330 ms^{-1}. The end correction may be neglected. Let P_0 denote the mean pressure at any point in the pipe and ΔP_0 the maximum amplitude of pressure variation.

(a) Find the length of the air column.
(b) What is the amplitude of pressure variation at the mid-point of the pipe?
(c) What are the maximum and minimum pressures at the open end of the pipe?
(d) What are the maximum and minimum pressure at the closed end of the pipe?

SOLUTION

(a) For the second overtone (fifth harmonic), the frequency is given by

$$f = \frac{5v}{4l}$$

$$\Rightarrow 440 = \frac{5\times 330}{4l}$$

$$\Rightarrow l = \frac{5\times 330}{4\times 440} = \frac{15}{16}\text{ m}$$

(b) Since, $\Delta P = \Delta P_0 \cos(kx)$

$$\Rightarrow \frac{15}{16} = \frac{5\lambda}{4}$$

$$\Rightarrow \lambda = \frac{3}{4}\text{ m}$$

$$\Rightarrow k = \frac{2\pi}{\lambda} = \frac{2\pi}{3/4} = \frac{8\pi}{3}$$

At mid-point, $x = \frac{5\lambda}{8} = \frac{5}{8}\times\frac{3}{4} = \frac{15}{32}$

$$\Delta P = \pm\Delta P_0 \cos(kx)$$

$$\Rightarrow \Delta P = \pm\Delta P_0 \cos\left(\frac{8\pi}{3}\times\frac{15}{32}\right)$$

$$\Rightarrow \Delta P = \pm\Delta P_0 \cos\left(\frac{5\pi}{4}\right)$$

$$\Rightarrow \Delta P = \pm\frac{\Delta P_0}{\sqrt{2}}$$

(c) The open end is a displacement antinode and therefore a pressure node. Therefore, the amplitude of pressure variation at the open end is zero. Hence,

$$P_{max} = P_{min} = P_0$$

(d) The closed end is a displacement node and therefore a pressure antinode, so

$$P_{max} = P_0 + \Delta P_0 \text{ and } P_{min} = P_0 - \Delta P_0$$

BEATS

If two coherent sources of slightly different frequencies send waves simultaneously in same region, then at each point there is a variation of intensity with time. In the case of sound waves, alternate loud and low sounds are heard. These variations in loudness are called **beats**. The time from each loud sound to the next loud sound is called **one beat period** and the number of such repetitions per second is called **beat frequency**.

Let the two waves be

$$y_1 = A\sin(2\pi f_1 t) \text{ and } y_2 = A\sin(2\pi f_2 t)$$

According to Principle of Superposition $y = y_1 + y_2$

$$\Rightarrow y = A\left[\sin(2\pi f_1 t) + \sin(2\pi f_2 t)\right]$$

$$\Rightarrow y = \left[2A\cos\left\{2\pi\left(\frac{f_1 - f_2}{2}\right)t\right\}\right]\sin\left\{2\pi\left(\frac{f_1 + f_2}{2}\right)t\right\}$$

$$\Rightarrow y = R\sin(2\pi f t)$$

where, $R = 2A\cos\left\{2\pi\left(\frac{f_1 - f_2}{2}\right)t\right\}$ and $f = \frac{f_1 + f_2}{2}$

Intensity $\propto (\text{Amplitude})^2$

$$\Rightarrow I_R \propto R^2$$

$$\Rightarrow I_R = 4a^2\cos^2\left[2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]$$

CONDITION FOR MAXIMA

I_R is MAXIMUM, when

$$\cos^2\left[2\pi\left(\frac{f_1 - f_2}{2}\right)t\right] = 1$$

$$\Rightarrow \cos\left[2\pi\left(\frac{f_1 - f_2}{2}\right)t\right] = \pm 1$$

$$\Rightarrow 2\pi\left(\frac{f_1 - f_2}{2}\right)t = n\pi;\ n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad t=\frac{n}{|f_1-f_2|};\ n=0,1,2,3,\ldots$$

So, we observe that MAXIMA are obtained at time values given by

$$t=0,\ \frac{1}{|f_1-f_2|},\ \frac{2}{|f_1-f_2|},\ \frac{3}{|f_1-f_2|},\ldots$$

CONDITION FOR MINIMA

I_R is MINIMUM, when

$$\cos^2\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]=0$$

$$\Rightarrow \quad \cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]=0$$

$$\Rightarrow \quad 2\pi\left(\frac{f_1-f_2}{2}\right)t=(2n+1)\frac{\pi}{2};\ n=0,1,2,3,\ldots$$

$$\Rightarrow \quad t=\frac{(2n+1)}{2|f_1-f_2|};\ n=0,1,2,3,\ldots$$

So, we observe that MINIMA are obtained at time values given by

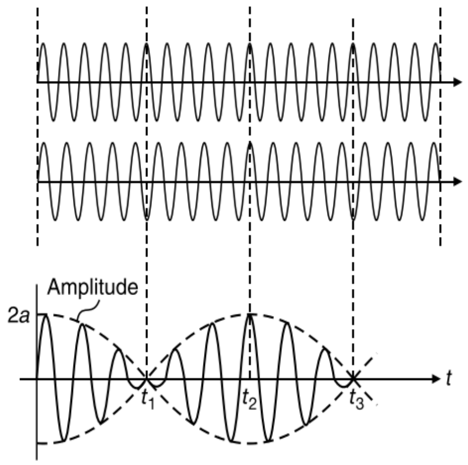
$$t=\frac{1/2}{|f_1-f_2|},\ \frac{3/2}{|f_1-f_2|},\ \frac{5/2}{|f_1-f_2|},\ldots$$

Beat time (t_b) is the time interval between two consecutive maxima or minima. So,

$$t_b=\frac{1}{|f_1-f_2|}$$

Further, beat frequency (f_b) is the reciprocal of beat time t_b. Hence,

$$f_b=\frac{1}{t_b}=|f_1-f_2|$$



Conceptual Note(s)

We note that

(a) The resultant wave is also a harmonic wave with a frequency $f=\frac{1}{2}(f_1+f_2)$

(b) The amplitude R is not constant but varies harmonically with a frequency $\frac{1}{2}|f_1-f_2|$. The number of beats per second is twice this frequency because the ear perceives the magnitude of displacement and not its sign. Thus, the beat frequency f_b is $f_b=|f_1-f_2|$

(c) If $|f_1-f_2|$ is small, the beats can be heard. However, if $|f_1-f_2|$ is large (more than 7), the beats produced are too rapid to be heard distinctly.

(d) Beats are used for tuning musical instruments and also to determine an unknown frequency.

TO FIND UNKNOWN FREQUENCY

Suppose we have two tuning forks A and B. The frequency of A is n Hz and that of B is **unknown**. When both are sounded together, they give x beats per second. Then possible frequency of B is $(n+x)$ or $(n-x)$ Hz.

If on loading B, the number of beats decreases, then frequency of B before loading was $(n+x)$ and if the number of beats increases, then frequency of B before loading will be $(n-x)$.

ILLUSTRATION 63

Two turning forks A and B produce 4 beats per second when sounded simultaneously. The frequency of A is known to be 256 Hz. When B is loaded with a little wax 4 beats per second are again produced. Find the frequency of B before and after loading.

SOLUTION

Since the beat frequency is 4 per second, the possible frequencies of fork B before loading are

$(256+4)$ Hz and $(256-4)$ Hz

or 260 Hz and 252 Hz

After loading B with wax, again 4 beats/second are heard. Therefore, after loading, the possible frequencies are

$(256+4)$ Hz and $(256-4)$ Hz

or 260 Hz and 252 Hz

But, by loading a fork its frequency can only decrease. Hence, before loading fork B, its frequency can only be 260 Hz, so that after loading it decreases to 252 Hz.

ILLUSTRATION 64

Two tuning forks A and B when set vibrating, give 4 beats/s. If a prong of the fork A is filed, the beats are reduced to 2 s^{-1}. Determine the frequency of A, if that of B is 250 Hz.

SOLUTION

There are four beats between A and B therefore the possible frequencies of A are 246 Hz or 254 Hz (that is 250 ± 4) Hz. Now when the prong of A is filed, its frequency becomes greater than the original frequency.
Assuming that the original frequency of A is 254, then on filing its frequency will become greater than 254.
The beats between A and B will be more than 4. But it is given that the beats are reduced to 2, therefore, 254 is not possible.
Therefore, the required frequency must be 246 Hz.
(This is true, because on filing the frequency may increase to 248, giving 2 beats with Q of frequency 250 Hz).

ILLUSTRATION 65

Wavelength of two notes in air is $\left(\frac{90}{175}\right)$ m and $\left(\frac{90}{173}\right)$ m respectively. Each of these notes produce 4 beats per second with a third note of a fixed frequency. Calculate the velocity of sound in air.

SOLUTION

Given: $\lambda_1 = \frac{90}{175}$ m and $\lambda_2 = \frac{90}{173}$

If f_1 and f_2 are the corresponding frequencies and v is the velocity of sound in air, we have

$$v = f_1\lambda_1 \quad \text{and} \quad v = f_2\lambda_2$$

$$\Rightarrow \quad f_1 = \frac{v}{\lambda_1} \quad \text{and} \quad f_2 = \frac{v}{\lambda_2}$$

Since $\lambda_1 < \lambda_2$, we must have $f_1 < f_2$. If f is the frequency of the third note, then

$$f_1 - f = 4 \text{ and } \quad f - f_2 = 4$$

$$\Rightarrow \quad f_1 - f_2 = 8$$

$$\Rightarrow \quad \frac{v}{\lambda_1} - \frac{v}{\lambda_2} = 8$$

$$\Rightarrow \quad v\left(\frac{175}{90} - \frac{173}{90}\right) = 8$$

$$\Rightarrow \quad v = 360 \text{ ms}^{-1}$$

ILLUSTRATION 66

A column of air and a tuning fork produces 4 beats/s. When sounding together, the tuning fork gives the lower note. The temperature of air is 15 °C. When the temperature falls to 10 °C, the two produce 3 beats/s. Find the frequency of the tuning fork.

SOLUTION

The frequency of the air column is given by $f = \frac{v}{\lambda}$, where v is the velocity of sound in air and λ is the wavelength. Since

$$v = \sqrt{\frac{\gamma RT}{M}}, \text{ so } f \text{ is dependent of temperature}$$

$$\Rightarrow \quad f \propto \sqrt{T}$$

Let the frequency of the tuning fork be f.
Then, the frequency of the air column at 15 °C is $f_1 = f + 4$ and the frequency of the air column at 10 °C is $f_2 = f + 3$
Since the frequency decreases with decrease temperature, so we have

$$\frac{f+4}{f+3} = \sqrt{\frac{288}{283}}$$

$$\Rightarrow \quad \frac{f+4}{f+3} = 1.00879$$

$$\Rightarrow \quad f + 4 = 1.00879f + 3.02638$$

$$\Rightarrow \quad 8.79 \times 10^{-3} f = 0.97362$$

$$\Rightarrow \quad f = 110.76 \text{ Hz}$$

ILLUSTRATION 67

Two wires are fixed on a sonometer. Their tensions are in the ratio 8 : 1, their lengths are in the ratio 36 : 35, the diameters are in the ratio 4 : 1 and densities are in the ratio 1 : 2. Find the frequencies of the beats produced if the note of the higher pitch has a frequency of 360 Hz.

SOLUTION

Given: $\frac{T_1}{T_2} = \frac{8}{1}, \frac{L_1}{L_2} = \frac{36}{35}, \frac{D_1}{D_2} = \frac{4}{1}, \frac{\rho_1}{\rho_2} = \frac{1}{2}$

If μ_1 and μ_2 are the linear densities, we have

$$\mu_1 = \pi\left(D_1^2/4\right)\rho_1 \text{ and } \mu_2 = \pi\left(D_2^2/4\right)\rho_2$$

Therefore, the ratio of linear densities is

$$\frac{\mu_1}{\mu_2} = \left(\frac{D_1}{D_2}\right)^2 \frac{\rho_1}{\rho_2} = \left(\frac{4}{1}\right)^2 \frac{1}{2} = \frac{8}{1}$$

The fundamental frequencies of the two wires are

$$f_1 = \frac{1}{2L_1}\sqrt{\frac{T_1}{\mu_1}} \text{ and } f_2 = \frac{1}{2L_2}\sqrt{\frac{T_2}{\mu_2}}$$

$$\Rightarrow \quad \frac{f_1}{f_2}=\frac{L_2}{L_1}\sqrt{\frac{T_1}{T_2}\times\frac{\mu_2}{\mu_1}}=\frac{35}{36}\sqrt{\frac{8}{1}\times\frac{1}{8}}=\frac{35}{36}$$

Since, $f_2 > f_1$, we have $f_2 = 360$ Hz

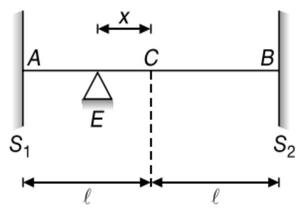
$$\Rightarrow \quad f_1=\left(\frac{35}{36}\right)\times 360 = 350 \text{ Hz}$$

Therefore, the beat frequency is

$$f_b = f_1 - f_2 = 350 \sim 360 = 10 \text{ Hz}$$

ILLUSTRATION 68

A string AB of length $2l$ is kept taut between two rigid supports S_1 and S_2. The tension in the string is T and its mass per unit length is μ. A knife edge E is placed at a distance $x(\ll l)$ from the midpoint C of AB. The two segments of the string are now made to vibrate in their fundamental modes. What beat frequency will be heard by the observer?



SOLUTION

Frequency of the section AE is $f_{AE}=\dfrac{1}{2(l-x)}\sqrt{\dfrac{T}{\mu}}$

Frequency of section EB is $f_{EB}=\dfrac{1}{2(l+x)}\sqrt{\dfrac{T}{\mu}}$

So, beat frequency is $f_b = f_{AE} - f_{EB}$

$$\Rightarrow \quad f_b=\frac{1}{2(l-x)}\sqrt{\frac{T}{\mu}}-\frac{1}{2(l+x)}\sqrt{\frac{T}{\mu}}$$

$$\Rightarrow \quad f_b=\frac{1}{2}\sqrt{\frac{T}{\mu}}\left[\frac{1}{l-x}-\frac{1}{l+x}\right]$$

$$\Rightarrow \quad f_b=\frac{x}{l^2-x^2}\sqrt{\frac{T}{\mu}}$$

$$\Rightarrow \quad f_b \approx \frac{x}{l^2}\sqrt{\frac{T}{\mu}} \qquad \{\because x \ll l, \text{ so } x^2 \text{ is negligible}\}$$

PITCH (FREQUENCY)

The term pitch refers to the attribute of a sound sensation that distinguishes a "**high**" note from a "**low**" note. In physical description, it is identified with the frequency – the higher the frequency, the higher is the pitch. The pitch is the characteristic which distinguishes between a shrill (or sharp) sound and a grave (or flat) sound. A sound of **high pitch** is said to be **shrill** and of **low pitch** a **grave sound**. The pitch does not depend on intensity and loudness but depends on the frequency. The pitch of a sound changes due to **Doppler's effect**.

QUALITY (TIMBRE)

A musical instrument vibrates with many frequencies at the same time – a lowest frequency, called the **fundamental**, and its multiples, called **overtones**. The quality (or timbre) of any musical sound is determined by the number of overtones and their relative intensities. The sounds of different instruments are said to differ in quality. The quality of sound enables us to distinguish between two sounds having same loudness and pitch. The quality of sound depends on the presence of overtones. Due to quality of sound one can recognise the voice of his friend without seeing him.

OCTAVE AND INTERVAL

Higher (upper) octave of frequency f is $2f$

Lower octave of frequency f is $\dfrac{f}{2}$

Interval is the ratio of frequencies of successive notes.

MUSICAL SOUND AND MUSICAL SCALE

A sound produced by periodic vibrations is called a **musical sound**. A musical sound can be decomposed into harmonic components having simple frequency ratio like $1:2$, $2:3$, $3:5$ etc.

A musical scale consists of a series of notes whose fundamental frequencies have specified ratios. There are eight notes which fix an octave – the frequency ratio of the first and the eighth is $1:2$. The note of the lowest frequency is called the Keynote. There are two types of musical scales.

Major diatonic Scale: The frequency ratio of adjacent notes are either $\dfrac{9}{8}$ or $\dfrac{10}{9}$ or $\dfrac{16}{15}$ (simple ratio).

Whole tones: Interval between two tones whose frequencies bear the ratio $\dfrac{9}{8}$ or $\dfrac{10}{9}$.

Half tone: The interval between two notes of frequency ratio $\dfrac{16}{15}$ is called a **half tone**.

Symbol	Indian Name	Western Name	Frequency in the base 256 hz	Frequency Ratio of Intervals
C	*Sa*	*DO*	256	9/8 (Whole)
D	*Re*	*RE*	288	10/9 (Whole)
E	*Ga*	*MI*	320	16/15 (Half)
F	*Ma*	*FA*	341.3	9/8 (Whole)
G	*Pa*	*SOL*	384	10/9 (Whole)
A	*Dha*	*LA*	426.7	9/8 (Whole)
B	*Ni*	*SI*	480	16/15 (Half)
C_1	*Sa*	*DO*	512	

Equally Tempered Scale: It consists of 13 keys and 12 intervals. The intervals are equal and each being $2^{1/2}$.

Key	C	Cs	D	Ds	E	F	Fs	G	Gs	A	As	B	C′
	2^0	$2^{1/12}$	$2^{2/12}$	$2^{3/12}$	$2^{4/12}$	$2^{5/12}$	$2^{6/12}$	$2^{7/12}$	$2^{8/12}$	$2^{9/12}$	$2^{10/12}$	$2^{11/12}$	

i.e., that is the notes form a geometric progression.

AUDIBLE, INFRASONIC AND ULTRASONIC WAVES

Longitudinal mechanical waves possess large range of frequencies. The frequency range 20 Hz to 20,000 Hz (or 20 kHz) causes the sensation of hearing in human beings and is, therefore, called the **Audible Range**. Waves of frequencies below 20 Hz are called **Infrasonic Waves** and those above 20,000 Hz are called **Ultrasonic Waves**.
Audible waves are generated by vibrating strings, air columns, plates and membranes.
Infrasonic waves are usually generated by large sources. e.g., earthquake waves.

Ultrasonic waves are produced by piezoelectric effect, magnetostriction method and Galton's whistle. Some animals can hear ultrasound. Ultrasonics have various applications, e.g., determining depth of mines and sea, medical diagnosis and therapy, echo sounding, finding flaws in materials, stimulated plant growth destruction of living cells suspended in liquids kill bacteria and smaller animals like rats, frogs, fishes, removal of grease and dirt etc.

Test Your Concepts-V

Based on Stationary Waves & Beats

(Solutions on page H.239)

1. The displacement of a standing wave on a string is given by
$$y(x, t) = 0.4\sin(0.5x)\cos(30t)$$
where, x and y are in centimetre. Calculate the frequency, amplitude and wave speed of the component waves. Also calculate particle velocity at $x = 2.4$ cm, $t = 0.8$ s.
2. A string fixed at both ends has consecutive standing wave modes for which the distances between adjacent nodes are 18 cm and 16 cm respectively.
(a) What is the minimum possible length of the string in cm?
(b) If the tension is 10 N and the linear mass density is 4 gm^{-1}, what is the fundamental frequency, in Hz, to the nearest two digit integer.
3. The vibrations of a string fixed at both ends are described by the equation
$$y = (5\text{ mm})\sin\left[(1.57\text{ cm}^{-1})x\right]\sin\left[(314\text{ s}^{-1})t\right]$$
(a) What is the maximum displacement of the particle at $x = 5.66$ cm?
(b) What are the wavelengths and the wave speeds of the two transverse waves that combine to give the above vibration?
(c) What is the velocity of the particle at $x = 5.66$ cm at time $t = 2$ s?
(d) If the length of the string is 10 cm, locate the nodes and the antinodes. How many loops are formed in the vibration.
4. Consider a steel wire of length 1 m, mass 0.1 kg and cross-sectional area 10^{-6} m^2 in which longitudinal vibrations are setup.
(a) What is the fundamental frequency?
(b) The same wire is rigidly fixed at both ends and the temperature is lowered by 20 °C. If now transverse waves are set up by plucking it at the mid-point, calculate the fundamental frequency and compare this with the frequency in the first case.

Given: Coefficient of linear expansion of steel is $1.21\times10^{-5}\ (^{\circ}C)^{-1}$ and Young's modulus of steel is $2\times10^{11}\ Nm^{-2}$.

5. A sonometer wire has a length of 114 cm between two fixed ends. Where should two bridges be placed so as to divide the wire into three segments whose fundamental frequencies are in the ratio 1:3:4?

6. A wire having a linear density of $0.05\ gcm^{-1}$ is stretched between two rigid supports with a tension of 450 N. It is observed that the wire resonates at a frequency of 420 Hz. The next higher frequency at which the same wire resonates is 490 Hz. Find the length of the wire.

7. In a stationary wave pattern that forms as a result of reflection of waves from an obstacle the ratio of the amplitude at an antinode and a node is $\beta=1.5$. What percentage of the energy passes across the obstacle?

8. A metallic rod of length 1 m is rigidly clamped at one of its end. The other end of the rod is free. Longitudinal stationary waves are set up in the rod in such a way that there are total six antinodes observed along the rod. If antinode amplitude is 4×10^{-6} m, Young's modulus is $6.4\times10^{10}\ Nm^{-2}$ and density of the rod is $5\times10^{3}\ kgm^{-3}$, calculate the
   (a) wavelength of the constituent waves.
   (b) frequency of the constituent waves.
   (c) equation of the motion at the mid-point of the rod.

9. A string is stretched with tension T between two rigid supports. A wave $y=A\sin(kx-\omega t)$ is produced in the string at left end. Stationary waves are formed due to reflection from the rigid supports. Calculate the
   (a) instantaneous kinetic energy contained in the string between two nodes.
   (b) maximum kinetic energy.
   (c) average kinetic energy over a cycle.

10. Calculate the velocity of sound in a gas in which two wavelength 204 cm and 208 cm produce 20 beats in 6 seconds.

11. An open organ pipe has a fundamental frequency of 300 Hz. The first overtone of a closed organ pipe has the same frequency as the first overtone of the open pipe. How long is each pipe? Given: Speed of sound in air is $v=330\ ms^{-1}$.

12. A tube 1 m long is closed at one end. A stretched wire is placed near the open end. The wire is 0.3 m long and has a mass of 0.01 kg. It is held fixed at both ends and vibrates in its fundamental mode. It sets the air column in the tube into vibration at its fundamental frequency by resonance. Calculate the frequency of oscillation of the air column and tension in the wire if speed of sound in air is $v=330\ ms^{-1}$.

13. The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency of 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz. Find the lengths of the pipes. Speed of sound in air is $v=330\ ms^{-1}$

14. A 60 cm long flute can be considered to be a pipe open at both ends.
   (a) What is the fundamental frequency when all the holes are covered?
   (b) How far from the mouthpiece should a hole be uncovered for the fundamental frequency to be 330 Hz?
   Take speed of sound in air as $340\ ms^{-1}$.

15. A tuning fork produces 3 beats per second when sounded together with a fork of frequency 364 Hz. When the first fork is loaded with a little wax then the number of beats becomes two per second. What is the frequency of the first fork?

16. A string 25 cm long and having a mass of 2.5 g is under tension. A pipe closed at one end is 40 cm long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decrease the beat frequency. If the speed of sound in air is $320\ ms^{-1}$, find the tension in the string.

17. Write the equation for the fundamental standing sound waves of antinode amplitude A, in a tube that is open at both ends. If the tube is 80 cm long and speed of the wave is $330\ ms^{-1}$.

DOPPLER EFFECT

You must have noticed the variation in the sound of a vehicle's horn as the vehicle moves past you. The frequency (or pitch) of the sound heard by you as the vehicle approaches you is higher than the frequency heard by you as it moves away from you. This experience is one example of the **Doppler effect.**

So, when a wave source and a listener are moving relative to each other, the received frequency is not the same as the frequency of the source. This effect is called Doppler effect and was first described by Austrian scientist Christian Andreas Doppler.

Doppler effect is also valid for electromagnetic waves. Doppler effect is observed when

(a) listener is moving but source is stationary
(b) source is moving but listener is stationary
(c) both the source and listener are moving

Case a) above involves the relative motion of the observer with respect to the medium while case b) does not. This fact makes the two cases different. *The basic effect of motion* *of the source is a change in wavelength whereas the basic effect of motion of the listener/observer/detector is a change in the number of wave crests received per second i.e., the frequency or pitch. Also, note that in the following discussion, all motions are relative to the medium.*

Conceptual Note(s)

(a) Doppler effect is symmetric in electromagnetic waves as they do not require any medium to propagate but asymmetric in case of sound waves. Therefore, the observed frequency of sound when the observer is at rest and source is moving is different from that when the observer is moving and source is at rest. This can be easily understood from the first two cases discussed below.

(b) When the listener (detector) moves relative to air and the source is stationary relative to air, then the motion changes the frequency at which the listener intercepts wavefronts and thus changes the frequency of the sound wave heard by the listener.

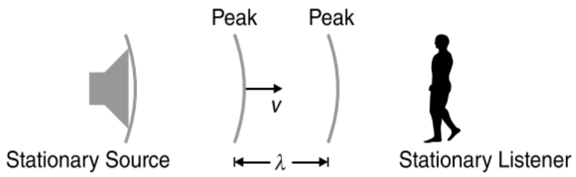
(c) When the source moves relative to the air and the listener (detector) is stationary relative to the air, then motion changes the wavelength of the sound wave and thus changes the frequency of the sound wave heard by the listener (since we know that frequency is inversely proportional to wavelength).

(d) The apparent frequency is more than the actual frequency if the source is moving towards the observer or the observer is moving towards the source. On the other hand, the apparent frequency is less than the actual frequency if the source is moving away from observer or observer is moving away from source.

(e) Doppler's effect of sound is asymmetric i.e., a source moving towards a stationary listener is not equivalent to a listener moving with same velocity towards a stationary source.

LISTENER AND SOURCE BOTH STATIONARY

For the Non-Doppler case, where the sound source and listener/receiver/detector/observer are both at rest with respect to the air (the medium in which the sound propagates) as shown in Figure.



If source produces waves of wavelength λ, frequency f that travel at wave speed v, then

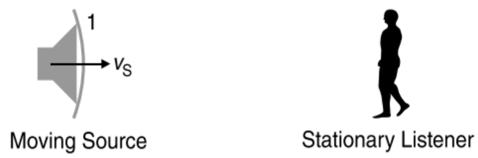
$$v = f\lambda$$

By stationary, we mean "stationary with respect to air/medium through which the sound wave propagates". Thus, when we say "a stationary observer", then no wind must be blowing.

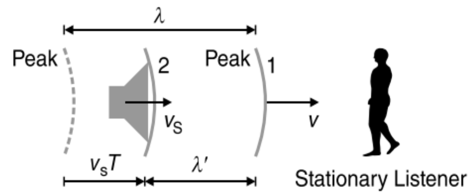
SOURCE APPROACHING A STATIONARY LISTENER

Let us find the equation for the shift in frequency, when the wave source is moving at speed v_S towards a stationary listener/receiver/detector/observer.

Let wave peak 1 be emitted by the source at a certain instant of time. Exactly one wave period T later, peak 1 has moved a distance $vT = \lambda$ (one wavelength) and then peak 2 is emitted as shown in Figure.



(a) Wave peak 1 is emitted



(b) Wave peak 2 is emitted exactly one wave period T later

Since, peak 2 is not emitted at the same position in space as peak 1 was emitted, because in time T the wave source also has moved distance v_ST (assume that $v_S \ll v$). So, with each successive wave peak, the source is constantly advancing such that each wavelength gets diminished by the distance v_ST, due to which the listener perceives waves of wavelength λ' where

$$\lambda' = \lambda - v_ST \quad ...(1)$$

Can we think about the other wave characteristics, such as wave speed and frequency?

Once the wave source releases a wave into the air, then the wave speed is determined by the properties of the medium (density, elastic response, temperature, etc.) and not on the manner the wave was produced. The wave forgets its history, i.e., whether its source was moving or not, hence the wave speed is still v.

Although the source is producing waves of frequency f it cannot be this frequency that the listener perceives, but a frequency f' which satisfies the equation $v = f'\lambda'$

Since $\lambda' = \lambda - v_ST$, so we get

$$v = f'(\lambda - v_ST) \quad ...(2)$$

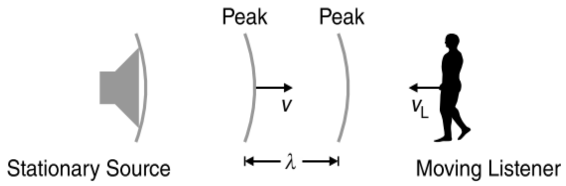
This frequency f' determines the pitch of sound heard by the listener. This frequency is also called the apparent frequency. Substituting $T = 1/f$ and $\lambda = v/f$ in equation (2), we get

$$v = f'\left(\frac{v}{f} - \frac{v_S}{f}\right)$$

$$\Rightarrow \quad f' = f\left(\frac{v}{v - v_S}\right)$$

LISTENER APPROACHING A STATIONARY SOURCE

Suppose the listener is approaching a stationary source with a speed v_L as shown in Figure.



The listener would measure the wavelength of the wave as λ just as in case both source and listener were at rest. However, since the listener is in motion relative to the air in which the waves have a speed v, so the listener perceives a higher speed of the waves relative to himself given by

$$v' = v_{\text{rel}} = v + v_L$$

Since $v' = f'\lambda$, so we get

$$f'\lambda = v + v_L \qquad \ldots(1)$$

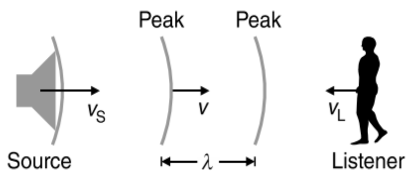
Also, $\lambda = \dfrac{v}{f}$, so equation (1) becomes

$$f'\left(\frac{v}{f}\right) = v + v_L$$

$$\Rightarrow \quad f' = f\left(\frac{v + v_L}{v}\right)$$

SOURCE AND LISTENER APPROACHING

When source and listener are approaching each other at shown in Figure, then by just combining the arguments given in the previous cases, we can write



$$f'(\lambda - v_S T) = v + v_L \qquad \ldots(1)$$

where, $T = \dfrac{1}{f'}$, $\lambda = \dfrac{v}{f}$

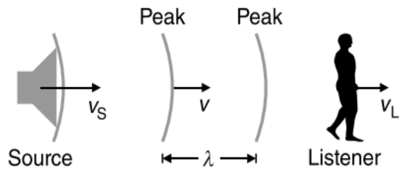
Substituting in equation (1), we get

$$f'\left(\frac{v}{f} - \frac{v_S}{f}\right) = v + v_L$$

$$\Rightarrow \quad f' = f\left(\frac{v + v_L}{v - v_S}\right)$$

SOURCE APPROACHING LISTENER AND BOTH MOVING IN SAME DIRECTION

When source is approaching listener and both are moving in the same direction as shown in Figure, then we can write



$$f'(\lambda - v_S T) = v - v_L \qquad \ldots(1)$$

where, $T = \dfrac{1}{f'}$, $\lambda = \dfrac{v}{f}$

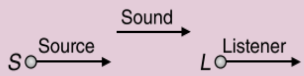
Substituting in equation (1), we get

$$f'\left(\frac{v}{f} - \frac{v_S}{f}\right) = v - v_L$$

$$\Rightarrow \quad f' = f\left(\frac{v - v_L}{v - v_S}\right)$$

Problem Solving Technique(s)

To learn about the general formula and apply it to various situations we proceed by executing the following set of instructions.



(a) Sound must travel from source (S) to listener (L).

(b) All velocities along the direction of sound are taken as positive.

(c) All velocities opposite to the direction of sound are taken as negative.

(d) The apparent frequency is taken as

$$f' = f\left(\frac{v - v_L}{v - v_S}\right)$$

(e) The above formula has to be suitably modified while applying to various situations.

(f) Above formula is valid only for velocities less than the velocity of sound.

The various possible cases and corresponding apparent frequencies f' are listed in the given table. Here, f is the actual frequency emitted by the source, f' is the apparent frequency heard by the listener, v_S is the speed of the source, v_L is the speed of the listener and v is the speed of sound.

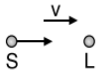
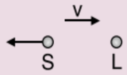
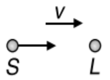
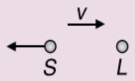
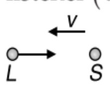
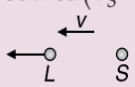
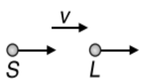
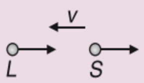
Sl. No.	Situation and Diagram	Apparent frequency
1.	Source moving towards a stationary listener 	$f' = \left(\frac{v}{v - v_S}\right) f \ (f' > f)$
2.	Source moving away from a stationary listener 	$f' = \left(\frac{v}{v + v_S}\right) f \ (f' < f)$
3.	Listener moving towards a stationary source 	$f' = \left(\frac{v + v_L}{v}\right) f \ (f' > f)$
4.	Listener moving away from a stationary source 	$f' = \left(\frac{v - v_L}{v}\right) f \ (f' < f)$
5.	Source moving towards a receding listener $(v_L > v_S)$ 	$f' = \left(\frac{v - v_L}{v - v_S}\right) f \ (f' < f)$
6.	Listener moving towards a receding source $(v_S > v_L)$ 	$f' = \left(\frac{v + v_L}{v + v_S}\right) f \ (f' < f)$
7.	Source and listener moving towards each other 	$f' = \left(\frac{v + v_L}{v - v_S}\right) f \ (f' > f)$
8.	Source and listener moving away from each other 	$f' = \left(\frac{v - v_L}{v + v_s}\right) f \ (f' < f)$

ILLUSTRATION 69

A van is moving with a speed of 3 ms^{-1} towards a large wall. A person standing on the line of motion of the van observes the van to be moving away from him. The horn of the van is now blown. The frequency of the sound produced by the horn is 600 Hz. What is

(a) The frequency of sound heard by the person for the sound produced directly by the horn?

(b) The frequency of sound reflected by the wall?

(c) The beat frequency heard by the driver. The speed of sound in air is 330 ms^{-1}?

SOLUTION

(a) Since the van moves away from the person, the source is receding from the observer. Hence, the frequency v_a heard is given by,

$$f_a = f_0\left(\frac{v}{v + v_S}\right)$$

$$\Rightarrow f_a = 600\left(\frac{330}{330 + 3}\right) = 594.6 \text{ Hz}$$

$$\Rightarrow f_a \approx 595 \text{ Hz}$$

(b) The frequency of sound reflected by the wall is the same as the frequency of sound received by the wall. Now, the source is approaching the wall. Hence the frequency f_w of sound reflected by the wall is given by

$$f_w = f_0\left(\frac{v}{v - v_S}\right)$$

$$\Rightarrow f_w = 600\left(\frac{330}{330 - 3}\right)$$

$$\Rightarrow f_w = 605.5 \text{ Hz} = 606 \text{ Hz}$$

(c) The beat frequency heard is the difference between the two frequencies.

So, beat frequency is $f_b = 606 - 595$

$$\Rightarrow f_b = 11 \text{ Hz}$$

Incidentally this beat frequency is not perceptible to the human ear because of the phenomenon of persistence of hearing. This persistence of hearing is about $\frac{1}{10}$th of a second. Therefore, the maximum beat frequency we can hear is about 10 Hz.

ILLUSTRATION 70

Two sources A and B are producing notes of frequency 680 Hz. A detector moves from A to B with a constant velocity 2 ms^{-1}. Find number of beats heard by the detector per second, if velocity of sound v is 340 ms^{-1}.

SOLUTION

The frequency of source A received by the detector

$$f_1 = \left(\frac{v - v_0}{v}\right) f \qquad \ldots(1)$$

The frequency of source B received by the observer

$$f_2 = \left(\frac{v + v_0}{v}\right) f \qquad \ldots(2)$$

Now number of beats heard by the detector per second i.e., beat frequency is $f_b = |f_1 - f_2|$

$$\Rightarrow \quad f_b = \left[\left(\frac{v+v_0}{v}\right) - \left(\frac{v-v_0}{v}\right)\right] = \frac{2v_0 f}{v}$$

$$\Rightarrow \quad f_b = \frac{2 \times 2 \times 680}{340} = 8 \text{ Hz}$$

Problem Solving Technique(s)

(a) Change in frequency measured by a stationary observer when a moving source crosses him

$$\Delta f = f_{\text{approach}} - f_{\text{recede}} = \left(\frac{2vv_s}{v^2 - v_s^2}\right) f$$

If $v_S \ll v$, then $v^2 - v_S^2 \approx v^2$

$$\Rightarrow \quad \Delta f \approx \left(\frac{2v_S}{v}\right) f$$

(b) Change in frequency measured by an observer as he passes by a stationary source

$$\Delta f = f_{\text{approach}} - f_{\text{recede}} = \left(\frac{2v_L}{v}\right) f$$

ILLUSTRATION 71

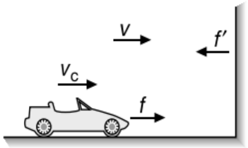
A car while blowing a horn of frequency f is approaching a wall with velocity v_C. If velocity of sound in air is v, then find number of beats heard by the driver per second.

SOLUTION

Frequency of the whistle heard by the driver directly from the source is

$$f_1 = f \qquad \ldots(1)$$

Frequency received by the wall is $f' = \left(\frac{v}{v - v_c}\right) f$



Since same frequency is reflected by the wall (as it is stationary), so let the driver receive the reflected frequency as $f_2 = f''$, then

$$f_2 = f'' = \left(\frac{v + v_C}{v}\right) f' = \left(\frac{v + v_C}{v}\right)\left(\frac{v}{v - v_C}\right) f$$

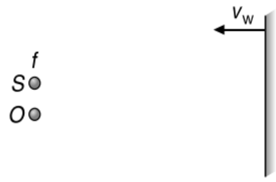
$$\Rightarrow \quad f_2 = \left(\frac{v + v_C}{v - v_C}\right) f \qquad \ldots(2)$$

Hence, number of beats heard by the driver per second i.e., beat frequency is $f_b = |f_1 - f_2|$

$$\Rightarrow \quad f_b = \left| f - \left(\frac{v + v_c}{v - v_c}\right) f \right| = \frac{2v_c f}{v - v_c}$$

ILLUSTRATION 72

A source S of frequency f and an observer are at rest and a wall is moving towards them with velocity v_w as shown.



If velocity of sound is v, calculate the number of beats heard by the observer per second.

SOLUTION

Frequency received by the observer directly from the source is

$$f_1 = f \qquad \ldots(1)$$

Frequency received by the wall is

$$f' = \left(\frac{v + v_w}{v}\right) f$$

Now as the wall is moving towards the observer, frequency heard by the observer after sound gets reflected from the wall is

$$f_2 = \left(\frac{v}{v - v_w}\right) f' = \left(\frac{v + v_w}{v - v_w}\right) f \qquad \ldots(2)$$

Number of beats heard by the observer per second i.e., beat frequency is $f_b = |f_1 - f_2|$

$$\Rightarrow \quad f_b = \left|1 - \frac{v + v_w}{v - v_w}\right| f = \left|\frac{v - v_w - v - v_w}{v - v_w}\right| f$$

$$\Rightarrow \quad f_b = \frac{2v_w f}{v - v_w}$$

ILLUSTRATION 73

A source of sound of frequency 256 Hz is moving rapidly towards a wall with a velocity of 5 ms^{-1}. Calculate the number of beats heard per second if sound travels at a speed of 330 ms^{-1}?

SOLUTION

CASE-I: When observer is between wall and the source

In this case, the observer is stationary and the source in motion towards observer. The apparent frequency for stationary observer is given by

$$f' = f\left(\frac{v}{v - v_s}\right)$$

where, $f = 256$ Hz, $v = 330\ \text{ms}^{-1}$, $v_s = 5\ \text{ms}^{-1}$

$$\Rightarrow \quad f' = 256\left(\frac{330}{330-5}\right) = \frac{256 \times 330}{325}$$

$$\Rightarrow \quad f' = 259.93\ \text{Hz}$$

So, the apparent frequency received by the observer from source is 259.93 Hz.

Also, the observer receives sound reflected from the wall. Due to reflection, there is no change in the frequency of the sound and hence the frequency of reflected sound is also 259.93 Hz.

Hence, beat frequency is

$$f_b = |\Delta f| = 259.93 - 259.93 = 0\ \text{Hz}$$

CASE-II: When source is between wall and observer

For reflected sound, source is moving towards wall, so

$$f' = n\left(\frac{v}{v - v_s}\right) = \left(\frac{330}{330-5}\right)256 = 259.9\ \text{Hz}$$

For direct sound, source is moving away from the observer, so

$$f'' = f\left(\frac{v}{v + v_s}\right) = \left(\frac{330}{330+5}\right)256 = 252.2\ \text{Hz}$$

Hence, beat frequency is $f_b = |\Delta f| = f' - f''$

$$\Rightarrow \quad f_b = 259.9 - 252.2 = 7.7 \approx 8\ \text{Hz}$$

ILLUSTRATION 74

A source of sonic oscillations of frequency $f_0 = 1700$ Hz and a receiver are located at the same point. At the moment $t = 0$, the source starts receding from the receiver with constant acceleration $a = 10\ \text{ms}^{-2}$. Assuming the velocity of sound to be equal to $v = 340\ \text{ms}^{-1}$, find the oscillation frequency registered by the stationary receiver $t = 10$ seconds after the start of motion.

SOLUTION

$$\text{Since, } f = \left(\frac{v}{v + v_s}\right) f_0 \qquad \ldots(1)$$

Let the sound emitted at time t_0 reaches the observer at time t. The time taken for the sound waves to reach the observer is $(t - t_0)$. So, the distance travelled by sound is $l = v(t - t_0)$

This is also equal to the distance of the source from the observer at time t_0. So,

$$v(t - t_0) = \frac{1}{2}at_0^2$$

$$\Rightarrow \quad at_0^2 + 2vt_0 - 2vt = 0$$

$$\Rightarrow \quad t_0 = \frac{-2v \pm \sqrt{4v^2 + 8avt}}{2a}$$

$$\Rightarrow \quad at_0 = v_s = \sqrt{v^2 + 2avt} - v$$

$$\Rightarrow \quad v + v_s = \sqrt{v^2 + 2avt}$$

$$\Rightarrow \quad f = \left(\frac{v}{\sqrt{v^2 + 2avt}}\right) f_0 \qquad \text{\{because of (1)\}}$$

$$\Rightarrow \quad f = \frac{f_0}{\sqrt{1 + \frac{2at}{v}}}$$

Substituting $f_0 = 1700$ Hz, $a = 10\ \text{ms}^{-2}$, $t = 10$ s and $v = 340\ \text{ms}^{-1}$, we get

$$f = 1349\ \text{Hz}$$

ILLUSTRATION 75

Two trains A and B simultaneously start moving along parallel tracks from a station along same direction. A starts with constant acceleration $2\ \text{ms}^{-2}$ from rest while B with the same acceleration but with initial velocity of $4\ \text{ms}^{-1}$. After 20 s passenger of A hears whistle of B. If frequency of whistle is 1194 Hz and velocity of sound in air is $322\ \text{ms}^{-1}$, calculate frequency observed by the passenger.

SOLUTION

Let t be the time when wave is emitted by B which is being received by A in 20 s. At 20 s train A will be at a distance

$$s_1 = \frac{1}{2} \times 2 \times (20)^2 = 400\ \text{m}$$

from the starting point.

At time t, train B will be at a distance

$$s_2 = 40t + \frac{1}{2} \times 2 \times t^2 = 40t + t^2$$

Now the distance $s_2 - s_1$ is travelled by the wave in time $(20 - t)$ sec with speed $322\ \text{ms}^{-1}$

$$\Rightarrow \quad 322(20 - t) = s_2 - s_1 = (40t + t^2) - 400$$

$$\Rightarrow \quad t^2 + 362t - 6840 = 0$$

$$\Rightarrow \quad t = \frac{-362 \pm \sqrt{(362)^2 + 4(6840)}}{2} = 18\ \text{s}$$

Velocity of source at $t = 18$ s is given by

$$v_s = u + at = (40) + 18(2) = 76\ \text{ms}^{-1}$$

{away from listener/observer}

and velocity of listener at $t = 20$ s is given by

$$v_L = at = (2)(20) = 40 \text{ ms}^{-1} \quad \{\text{towards the source}\}$$

So, apparent frequency $f_1 = \left(\dfrac{v + v_L}{v + v_s}\right) f$

$$\Rightarrow \quad f_1 = \left(\frac{322 + 40}{322 + 76}\right) \times 1194$$

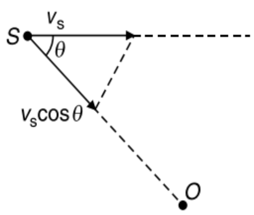
$$\Rightarrow \quad f_1 = 1086 \text{ Hz}$$

Conceptual Note(s)

In such type of problems, we first find the time t when the emitted wave was being received by the listener, in this case 20 sec. Now substitute v_L in the formula $f' = \left(\dfrac{v \pm v_L}{v \mp v_s}\right) f$ at time when wave is received and v_s when it is emitted (with proper signs). For example, in this equation the wave which is emitted at 18 s is being received by the listener at 20 s. So, we have put v_L at 20 s while v_s at 18 s.

SOURCE NOT MOVING TOWARDS OBSERVER

The previous cases were derived assuming that the motion is along the line joining the source and the observer. If the motion is along some other direction, the components of velocities along the line joining source and observer considered.



If at any instant the line joining the moving source and stationary observer makes an angle θ with the direction of motion of source, v_s is replaced by $v_s \cos\theta$.

$$f' = f\left(\frac{v}{v - v_s \cos\theta}\right)$$

Problem Solving Technique(s)

Note that **no Doppler's effect** will be observed in following cases:

(a) If the source and observer both move in the same direction with same speed, i.e.,

$$v_s = v_0 = u \text{ (say)}$$

Then, $f' = f\left(\dfrac{v + v_w - u}{v + v_w - u}\right) = f$

(b) If one is at the centre of a circle while the other is moving on it with uniform speed. In this situation, the component of u along the line of sight, i.e., radius, will be $u\cos 90° = 0$.

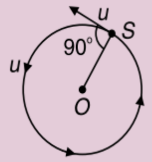
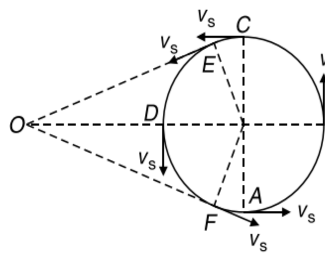


ILLUSTRATION 76

An observer is standing at O, far away from the circle A, B, C, D as shown and a source of frequency f is moving on the circle with constant speed v_S. Find maximum and minimum frequency ward by the observer.



SOLUTION

Observer will receive maximum frequency when the source is at E as in this case the source will be moving exactly towards the observer. Therefore

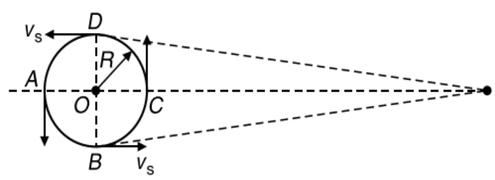
$$f_{\max} = \left(\frac{v}{v - v_S}\right) f$$

Observer will receive minimum frequency when the source is at F as in this case the source is moving exactly away from the observer. Therefore, minimum frequency received by the observer

$$f_{\min} = \left(\frac{v}{v + v_S}\right) f$$

ILLUSTRATION 77

A whistle of frequency 540 Hz rotates in a circle of radius 2 m at an angular speed of 15 rads^{-1}. What is the lowest and highest frequency heard by a listener, a long distance away at rest with respect to the centre of the circle? Can the apparent frequency be ever equal to actual? Given that the velocity of sound in air is 330 ms^{-1}.



SOLUTION

The linear speed of the source,

$$v_s = R\omega = 2\times 15 = 30 \text{ ms}^{-1}$$

Here, observer is at rest and the source is moving. Therefore, the apparent frequency is

$$f' = f\left(\frac{v}{v - v_s}\right)$$

The frequency f' will be lowest when the source is moving away from the observer, i.e., when the whistle is at point D i.e., $v_s = -30 \text{ ms}^{-1}$

$$f'_{(\text{lowest})} = f\left(\frac{v}{v - v_s}\right) = 540\left(\frac{330}{330+30}\right) = 495 \text{ Hz}$$

The frequency f' will be highest when the source is moving towards the observer, i.e., when the whistle is at point B i.e., $v_s = +30 \text{ ms}^{-1}$

$$f'_{(\text{highest})} = f\left(\frac{v}{v - v_s}\right) = 540\left(\frac{330}{330-30}\right) = 594 \text{ Hz}$$

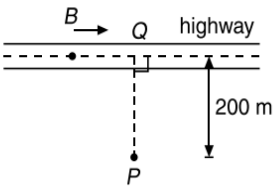
When the whistle is A or C, the velocity of source along the line of sight is $v_s \cos 90° = 0$, and hence

$$f' = f\left(\frac{v}{v-0}\right) = f$$

Hence, at point A and C, the apparent frequency will be equal to the actual frequency.

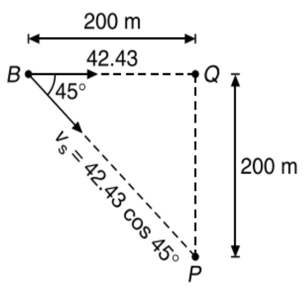
ILLUSTRATION 78

A person P is standing at a perpendicular distance of 200 m from point Q on a highway. A bus B is moving with a speed of 42.43 ms^{-1} towards point Q on the highway. The driver of the bus blows a horn of frequency 1300 Hz. What is the frequency of sound received by the person P, when the bus is distant 200 m from Q. Speed of sound in air is 330 ms^{-1}.



SOLUTION

The velocity component of the source along the line joining the source and the observer is



$$v_S = 42.43 \cos 45°,$$

$$\Rightarrow \quad v_S = \frac{42.43}{\sqrt{2}} \approx 30 \text{ ms}^{-1}$$

Since, $f_a = f_0\left(\frac{v}{v - v_S}\right)$

$$\Rightarrow \quad f_a = 1300\left(\frac{330}{330-30}\right) = 1430 \text{ Hz}$$

EFFECT OF MOTION OF MEDIUM

If v_m is the speed of the medium, then velocity of sound v is replaced by $v \pm v_m$, where "+" (positive) sign is taken if the medium moves in the direction of propagation of the wave, and "−" (negative) sign is taken if the medium moves opposite to the direction of propagation of the wave. There is no effect of motion of the medium alone.

WIND EFFECT

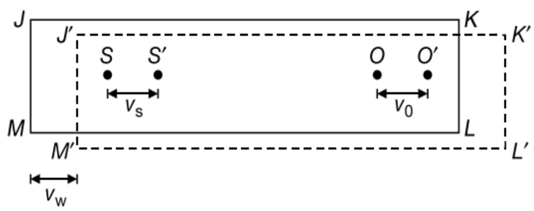
In case wind is blowing with speed v_w, the above formulae are modified by replacing v by $v + v_w$, if the wind blows in the same direction as v and $v - v_w$, if the wind blows in opposite direction to v.

Observer, Source and Medium in Motion

This is the most general case. All the formulae derived above, for both approaching and receiving source from observer, can be combined into a **single formula**,

$$f' = f\left(\frac{v + v_w - v_0}{v + v_w - v_s}\right)$$

While using the above formula, we should place the source S, to the left of the observer O, as shown in figure. Any velocity directed rightward is taken +ve and leftward −ve.



Conceptual Note(s)

If the wind blows with velocity w having direction making an angle θ with the direction of travel of sound towards observer, then velocity of sound v should be replaced by $v + w\cos\theta$.

If wind blows in the direction of sound, $\theta = 0$, so $\cos\theta = 1$ but if wind blows opposite to direction of sound, then $\theta = \pi$, so $\cos\theta = -1$.

ILLUSTRATION 79

A man standing near a railway track hears the whistle of an engine, which is approaching him with a velocity of 36 kmh^{-1}. What is the actual frequency of the whistle, if the apparent pitch of the whistle as heard by the man is 800 Hz? Velocity of sound is 350 ms^{-1}.

SOLUTION

Using the formula for most general case,

$$f' = f\left(\frac{v+v_w-v_0}{v+v_w-v_s}\right)$$

Keeping source to the left of observer, we get

$v = 350\ \text{ms}^{-1}$, $v_s = 36\ \text{kmh}^{-1} = 10\ \text{ms}^{-1}$, $v_0 = 0$, $v_w = 0$ and $f' = 800$ Hz

$$\Rightarrow \quad 800 = f\left(\frac{350}{350-10}\right)$$

$$\Rightarrow \quad f = \frac{800\times(350-10)}{350} = 777.14\ \text{Hz}$$

ILLUSTRATION 80

A source and observer are approaching one another with a relative velocity of 30 ms^{-1}. The actual frequency of the sound emitted by the source is 1000 Hz. If the velocity of sound is 350 ms^{-1}, find the apparent pitch as heard by the observer in each of the following cases:

(a) Only the source moving, and the observer is stationary.
(b) Only the observer is moving and the source is stationary.
(c) The source is moving towards the observer with a velocity of 80 ms^{-1} and the observer is moving away from the source with a velocity of 50 ms^{-1}.

SOLUTION

Using the most general formula, and keeping the source on the left of the observer

$$f' = f\left(\frac{v+v_w-v_0}{v+v_w-v_s}\right)$$

(a) Here, $f = 1000$ Hz, $v_s = 30\ \text{ms}^{-1}$, $v = 350\ \text{ms}^{-1}$, $v_0 = 0$, $v_w = 0$

$$\Rightarrow \quad f' = 1000\left(\frac{350-0}{350-30}\right) = \frac{1000\times 350}{320}$$

$$\Rightarrow \quad f' = 1093.75\ \text{Hz}$$

(b) Here, $f = 1000$ Hz, $v_s = 0$, $v_0 = -30\ \text{ms}^{-1}$, $v = 350\ \text{ms}^{-1}$, $v_w = 0$

$$\Rightarrow \quad f' = 1000\left(\frac{350-(-30)}{350-0}\right) = \frac{1000\times 380}{350}$$

$$\Rightarrow \quad f' = 1085.7\ \text{Hz}$$

(c) Here, $f = 1000$ Hz, $v_s = 80\ \text{ms}^{-1}$, $v_0 = 50\ \text{ms}^{-1}$, $v = 350\ \text{ms}^{-1}$, $v_w = 0$

$$\Rightarrow \quad f' = 1000\left(\frac{350-50}{350-80}\right) = \frac{1000\times 300}{270}$$

$$\Rightarrow \quad f' = 1111.11\ \text{Hz}$$

Please note that, although the relative velocity between the source and observer is same in all three cases, still the apparent frequencies are different.

ILLUSTRATION 81

Two trains travelling at 108 kmh^{-1} are approaching one another. The first train gives a whistle of frequency 850 Hz. The wind is blowing at 36 kmh^{-1} in a direction from the first train towards the second. If the velocity of sound in still air is 340 ms^{-1}, find the apparent pitch of the whistle by a person in the second train

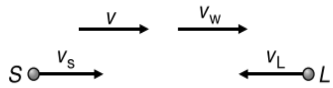
(a) before the trains meet, and
(b) after the trains have passed one another.

SOLUTION

We shall use the most general equation,

$$f' = f\left(\frac{v+v_w-v_L}{v+v_w-v_S}\right)$$

(a) Before the trains meet:



Here, $f = 850$ Hz, $v = 340\ \text{ms}^{-1}$,

$v_w = 36\ \text{kmh}^{-1} = 10\ \text{ms}^{-1}$,

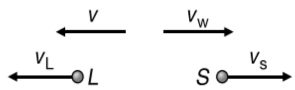
$v_S = 108\ \text{kmh}^{-1} = 30\ \text{ms}^{-1}$,

$v_L = -108\ \text{kmh}^{-1} = -30\ \text{ms}^{-1}$

So, $$f' = 850\left(\frac{340+10-(-30)}{340+10-30}\right) = \frac{850\times 380}{320}$$

$$\Rightarrow \quad f' = 1009.375\ \text{Hz}$$

(b) After the trains have passed one another:



Here, $f = 850$ Hz, $v = 340\ \text{ms}^{-1}$,

$v_w = -36\ \text{kmh}^{-1} = -10\ \text{ms}^{-1}$

$v_S = -108\ \text{kmh}^{-1} = -30\ \text{ms}^{-1}$

$v_L = 108\ \text{kmh}^{-1} = 30\ \text{ms}^{-1}$

$$\Rightarrow \quad f' = 850\left(\frac{340-10-30}{340-10-(-30)}\right) = \frac{850\times 300}{360}$$

$$\Rightarrow \quad f' = 708.33\ \text{Hz}$$

ILLUSTRATION 82

Two vehicles A and B are moving towards each other with same speed u. They blow horns of the same frequency f. Wind is blowing at speed w in the direction of motion of A. The driver of vehicle A hears the sound of horn blown by vehicle B and the sound of horn of his own vehicle after reflection from the vehicle B. Find the frequency and wavelength of both sounds received by A. Velocity of sound is v.

SOLUTION

From Doppler's Effect, the frequency of sound coming directly from B is given by

$$f_1 = \left[\frac{(v-w)+u}{(v-w)-u}\right]f$$

The wavelength is given by

$$\lambda_1 = \frac{\text{Velocity of sound w.r.t. observer}}{\text{Frequency of sound heard by observer}}$$

$$\Rightarrow \quad \lambda_1 = \frac{(v-w)+u}{f_1} = \frac{(v-w)+u}{\left[\frac{(v-w)+u}{(v-w)-u}\right]f} = \frac{(v-w)-u}{f}$$

The frequency of sound incident on B is given by

$$f' = \left[\frac{(v+w)+u}{(v+w)-u}\right]f$$

Now, B will act as a source of this frequency moving with velocity u towards the observer. The frequency of the sound reflected by B as heard by driver A is

$$f_2 = \left[\frac{(v-w)+u}{(v-w)-u}\right]f'$$

$$\Rightarrow \quad f_2 = \left[\frac{(v+w)+u}{(v+w)-u}\right]\left[\frac{(v-w)+u}{(v-w)-u}\right]f$$

$$\Rightarrow \quad \lambda_2 = \frac{(v-w)+u}{f_2} = \frac{(v-w)+u}{\left[\frac{(v+w)+u}{(v+w)-u}\right]\left[\frac{(v-w)+u}{(v-w)-u}\right]f}$$

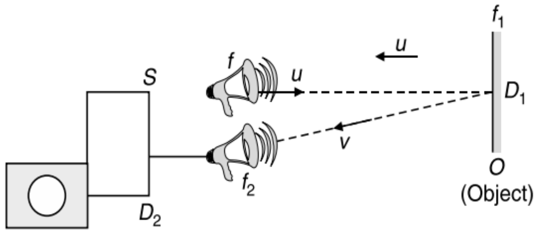
$$\Rightarrow \quad \lambda_2 = \frac{(v-w-u)(v+w-u)}{(v+w+u)f}$$

PRINCIPLE OF SONAR (OR RADAR)

The sound waves sent by the source S are reflected by the object O. The detector D_2 receives two notes one directly from the source S and other from the reflector O (which also acts as a virtual source). If these two frequencies are slightly different, their superposition produces beats. The beat frequency can be used to determine the speed by the object.

If the source S is at rest and a reflector is moving towards the source with speed u, the reflector (acting as detector D_1) will 'hear' the frequency

$$f_1 = f\left(\frac{v+u}{v}\right)$$



Now, this frequency is reflected by the object, which acts as a source moving with a speed u towards the detector D_2. Therefore, the frequency received back by D_2 at the site of the source is

$$f_2 = f_1\left(\frac{v}{v-u}\right) = f\left(\frac{v+u}{v}\right)\left(\frac{v}{v-u}\right) = f\left(\frac{v+u}{v-u}\right)$$

If $u \ll v$ using binomial theorem, we get

$$f_2 = f\left(1+\frac{u}{v}\right)\left(1-\frac{u}{v}\right)^{-1} = f\left(1+\frac{u}{v}\right)\left(1-(-1)\frac{u}{v}\right)$$

$$f_2 = f\left(1+\frac{u}{v}\right)^2 = f\left(1+\frac{2u}{v}\right)$$

So, beat frequency, $f_b = f_2 - f_1 = f\left(\frac{u}{v}\right)$

Thus, the speed of the moving object is given as

$$u = \frac{v}{2}\left(\frac{f_b}{f}\right)$$

ILLUSTRATION 83

A radar gun is a device which produces ultrasonic sound waves of a frequency f_0. These waves are then reflected by a moving object and received back by the device. The frequency of the waves thus received is f. Estimate the speed x of the moving object assuming it to be moving straight towards the radar gun. The speed of sound in air is $v(\gg x)$.

SOLUTION

Frequency, f' of sound waves received by the moving object is given by

$$f' = f_0\left(\frac{v+x}{v}\right)$$

The moving object reflects the sound waves of frequency f'. Since it is moving, so it acts as a moving source and hence frequency of sound waves f received from it by receiving unit of radar gun is given by

$$f = f'\left(\frac{v}{v-x}\right) = f_0\left(\frac{v+x}{v}\right)\left(\frac{v}{v-x}\right) = f_0\left(\frac{v+x}{v-x}\right)$$

$$\Rightarrow \quad f = f_0\left[\frac{1+(x/v)}{1-(x/v)}\right] = f_0\left(1+\frac{x}{v}\right)\left(1-\frac{x}{v}\right)^{-1}$$

Since $x/v \ll 1$, so from Binomial approximation, we get

$$\Rightarrow \quad f \approx f_0\left(1+\frac{x}{v}\right)\left(1+\frac{x}{v}\right) \approx f_0\left(1+\frac{2x}{v}\right)$$

$$\Rightarrow \quad \frac{f}{f_0} = 1+\frac{2x}{v}$$

$$\Rightarrow \quad \frac{f-f_0}{f} = \frac{2x}{v}$$

$$\Rightarrow \quad x = \frac{(f-f_0)v}{2}$$

DOPPLER'S EFFECT OF LIGHT

Till now we have studied that the Doppler effect observed in sound is asymmetric, i.e., the frequency change depends on whether the source is moving or the listener is moving, irrespective of the fact that they have same relative velocity. Doppler effect occurs in light also. However, the Doppler effect observed in light is symmetric, i.e., it depends only on the relative velocity of the source and the observer, irrespective of which of the two is moving.

This difference occurs because sound requires a material medium and v, v_S and v_L are measured relative to the medium. Light, on the other hand, does not require a medium for propagation and the speed of light is the same for any observer whether observer and/or the source is moving.

The formula for apparent frequency f' observed for a source emitting light of frequency f and moving with non-relativistic velocity $v_r\,(\ll c)$ with respect to the observer is

$$f' \approx f\left(1 \pm \frac{v_r}{c}\right)$$

Hence, we conclude that when a source approaches observer or vice versa, then frequency increases and wavelength decreases such that

$\frac{\Delta f}{f} = \pm\frac{v_r}{c}$	$\frac{\Delta\lambda}{\lambda} = \mp\frac{v_r}{c}$
$(+)$ Approach	$(-)$ Approach
$(-)$ Recede	$(+)$ Recede

RED SHIFT AND BLUE SHIFT

Since, from Doppler's Effect for Light we have

$$\frac{\Delta f}{f} = \pm\frac{v}{c} \text{ or } \frac{\Delta\lambda}{\lambda} = \mp\frac{v}{c}$$

Consider a source of light approaching an observer (or vice-versa), then the frequency must increase and hence wavelength should decrease. Since blue colour has minimum wavelength, so decrease in wavelength implies that a shift occurs towards the blue, hence this shift is termed as **Blue Shift**.

Further if a source of light recedes from an observer (or vice versa), then frequency must decrease and hence wavelength should increase. Since red colour has maximum wavelength, so increase in wavelength implies that the shift occurs towards the red colour and hence this shift is termed as **Red Shift**.

Our universe always exhibits red shift and hence it must be expanding in all the directions.

ILLUSTRATION 84

Assuming the earth to be moving towards a stationary star at a speed of 30 kms^{-1}, calculate the apparent wavelength of light emitted from the star if the real wavelength of light emitted by the star is 5875 Å.

SOLUTION

Here earth (observer) is approaching the star, so we gave

$$\Delta f = +\left(\frac{v}{c}\right)f \text{ or } \Delta\lambda = -\left(\frac{v}{c}\right)\lambda$$

where, v is the relative velocity of the source and c is the velocity of light.

So, the observer will notice an increase in frequency or decrease in wavelength.

$$\Rightarrow \quad |\Delta\lambda| = \left(\frac{30\times10^3}{3\times10^8}\right)5875\text{ Å} = 5875\times10^{-4}\text{ Å}$$

Hence, altered wavelength is

$$\lambda' = \lambda - \Delta\lambda$$

$$\Rightarrow \quad \lambda' = 5875\text{ Å} - (5875\times10^{-4}\text{ Å})$$

$$\Rightarrow \quad \lambda' = 0.9999\times5875\text{ Å}$$

$$\Rightarrow \quad \lambda' = 5874.4125\text{ Å}$$

ILLUSTRATION 85

How fast would you have to go through a red light to have it appear green if the respective wavelengths of red and green light are 620 nm and 540 nm? Is it possible to achieve this speed on earth?

SOLUTION

Apparent wavelength is

$$\lambda_{app} = \lambda' = \lambda\left(1-\frac{v}{c}\right)$$

$$\Rightarrow \quad 540 = 620\left(1-\frac{v}{c}\right)$$

$$\Rightarrow \quad 1-\frac{v}{c} = \frac{27}{31}$$

$$\Rightarrow \quad \frac{v}{c} = \frac{4}{31}$$

$$\Rightarrow \quad v = \frac{4}{31}(3\times10^8) = 3.87\times10^7\text{ ms}^{-1}$$

Since this speed in much larger than escape speed on earth, so this speed is not attainable.

ACOUSTICS OF BUILDINGS

Designing a building has to be done keeping in mind certain acoustical facts.

(a) There should not be **penetration** of sound between rooms, especially in hotels and studios of radio stations. For this the walls must be covered with sound absorbing materials and doors must have heavy curtains. Just the opposite is the requirement of a **'whispering gallery'**, in which the walls are made completely non-absorbing so that even whispers can be heard at diametrically opposite ends of a large gallery around a dome.

(b) **Echo:** In large auditoriums, echo creates lots of problems. If the reflected sound of any syllable comes after the next syllable directly reaching a listener, then there is confusion due to overlapping. Since the average interval between two syllables spoken by a person is about 0.2 s, those walls and ceilings should be made non-reflecting from where the sound comes back in more than 0.2 s.

(c) **Reverberation:** A sound once produced persists for some time due to repeated reflections in the hall. This phenomenon is called **Reverberation**. If the hall has many doors and windows and absorbing materials like curtains and furniture, the sound dies out quickly. Sound is also absorbed by the audience in the hall. The time taken by the sound intensity to fall by a factor of 10^{-6} is called the **Reverberation Time** of the hall. The reverberation time should neither be too large nor too small. For conference and lecture halls the reverberation time should be about 1 s. For a musical concert it should be a little larger.

Sabines reverberation formula for time is

$$T = \frac{CV}{AS}$$

where C = constant, V = Volume of hall,

A = Absorption coefficient,

S = Total surface area of enclosed space.

The reverberation time for a hall of 10^4 m^3 volume is between 1 s to 1.5 s.

(d) Lecture halls must possess **even distribution of sound.** For such halls, dome, curved surfaces and projections should be designed with great care. One method is to make the wall in front of the audience parabolic with the speaker at its focus. The reflected sound from this wall will then go equally everywhere. The other walls and ceilings should not be curved. Otherwise, they can produce improper focussing, increased intensity at some points and no sound at some other points (**Dead Centres**). Destructive interference can also take place between direct and reflected sound at some places.

Test Your Concepts-VI

Based on Doppler's Effect

(Solutions on page H.242)

1. Two tuning forks with natural frequencies 340 Hz each move relative to a stationary observer. One fork moves away from the observer while the other moves towards him at the same speed. The observer hears beats of frequency 3 Hz. Find the speed of the tuning fork. Given that the velocity of sound in air is 340 ms^{-1}.
2. A source of sound of frequency 1000 Hz moves to the right with a speed of 32 ms^{-1} relative to the ground. To its right is a reflecting surface moving to the left with a speed of 64 ms^{-1} relative to the ground. Given that the speed of sound in air is 332 ms^{-1}. Find the
   (a) wavelength of the sound emitted in air by the source.
   (b) number of waves arriving per second at the reflecting surface.
   (c) speed of the reflected waves.
   (d) wavelength of the reflected waves.
3. A whistle of frequency 540 Hz rotates in a circle of radius 2 m at a linear speed of 30 ms^{-1}. What is the lowest and highest frequency heard by an observer a long distance away at rest with respect to the centre of circle. Take speed of sound in air to be 330 ms^{-1}. Can the apparent frequency be ever equal to actual?
4. A siren emitting a sound of frequency 1000 Hz moves away from you toward a cliff at a speed of 10 ms^{-1}. Find the
   (a) frequency of the sound, coming directly from the siren heard by you.
   (b) frequency of the sound, reflected off the cliff heard by you.
   (c) beat frequency heard by you.
   Take the speed of sound in air to be 330 ms^{-1}.
5. A sonic source starts vibrating with a frequency of 1000 Hz and at the same time starts moving towards a receiver with an acceleration of 10 ms^{-2}. If the velocity of sound in air is 350 ms^{-1}
   (a) find the wavelength of sound emitted exactly after source starts accelerating

(b) find the average frequency measured by the receiver over 1 sec after start.

6. A radar wave has a frequency of 7.8×10^9 Hz. The reflected wave from an airplane shows a frequency difference of 2.6×10^3 Hz on the higher side. Calculate velocity of the airplane in the line of sight.

7. Two cars A and B depart simultaneously from the same position and in same direction on a straight road. A starts with initial velocity $2\ \text{ms}^{-1}$ and acceleration $2\ \text{ms}^{-2}$ while B start with initial velocity $2\ \text{ms}^{-1}$. The driver of car A hears a sound of frequency 352 Hz emitted by car B after 10 sec, after the start, find the actual frequency of the sound as emitted by B.
Given: Velocity of the sound is $v = 330\ \text{ms}^{-1}$.

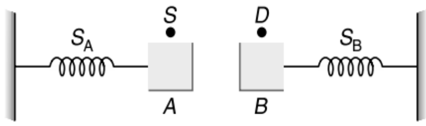
8. A stationary source emits sound of frequency $f_0 = 1000$ Hz. If a wind blows at the speed of $0.1v$, where v is the speed of sound, then find
(a) the percentage change in the wavelength towards the wind side of the source.
(b) the percentage change in the frequency for a stationary observer on the wind-side of the source.
(c) what happens when there is no wind, but the observer moves at speed $0.1v$ towards the source.

9. A sonometer wire under a tension of 64 N vibrating in its fundamental mode is in resonance with a vibrating tuning fork. The vibrating portion of the sonometer wire has a length of 10 cm and a mass of 1 g. The vibrating tuning fork is now moved away from the vibrating wire with a constant speed and an observer standing near the sonometer hears one beat per second. Calculate the speed with which the tuning fork is moved. The speed of sound in air is $300\ \text{ms}^{-1}$.

10. A source of sonic oscillations of frequency $f_0 = 1000$ Hz moves at right angles to the wall with a velocity $u = 0.17\ \text{ms}^{-1}$. Two stationary receivers R_1 and R_2 are located on a straight line, coinciding with the trajectory of the source, in the following succession : R_1 source R_2 wall. Which receiver registers the beats and what is the beat frequency? The velocity of sound is equal to $v = 340\ \text{ms}^{-1}$.

11. The wavelength of light coming from a distant galaxy is found to be 0.5% more than that coming from a source on earth. Calculate the velocity of the galaxy.

12. A source S emitting sound of 300 Hz is fixed on block A which is attached to the free end of a spring S_A as shown in the figure. The detector D fixed on block B attached to the free end of spring S_B detects this sound. The blocks A and B are simultaneously displaced towards each other through a distance of 1 m and then left to vibrate. Find the maximum and minimum frequencies of sound detected by D if the vibrational frequency of each block is 2 Hz. Take speed of sound $= 340\ \text{ms}^{-1}$.



13. A railway engine is moving from east to west with a velocity of $72\ \text{kmh}^{-1}$. At the same time you are moving from west to east towards the engine in a car with a velocity of $54\ \text{kmh}^{-1}$. The wind is blowing from west to east with a velocity of $36\ \text{kmh}^{-1}$. If the velocity of sound in still air is $330\ \text{ms}^{-1}$ and the actual frequency of the whistle of the engine is 1200 Hz, what will be the apparent frequency heard by you?

14. A fighter plane moving in a vertical loop with constant speed of radius R. The centre of the loop is at a height h directly overhead of an observer standing on the ground. The observer receives maximum frequency of the sound produced by the plane when it is nearest to him. Find the speed of the plane if velocity of sound in air is v.

SOLVED PROBLEMS

PROBLEM 1

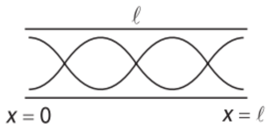
A 3 m long organ pipe open at both ends is driven to third harmonic standing wave. If the maximum amplitude of pressure oscillations is 1 percent of mean atmospheric pressure $(P_0 = 10^5 \text{ Nm}^{-2})$. Find the maximum amplitude of particle displacement and density oscillations. Speed of sound $v = 332 \text{ ms}^{-1}$ and density of air $\rho = 1.03 \text{ kgm}^{-3}$.

SOLUTION

Given length of pipe is $l = 3$ m, so for third harmonic

$$3\left(\frac{\lambda}{2}\right) = l$$

$$\Rightarrow \quad \lambda = \frac{2l}{3} = \frac{2\times 3}{3} = 2 \text{ m}$$



The angular frequency is

$$\omega = 2\pi f = \frac{2\pi v}{\lambda} = \frac{(2\pi)(332)}{2} = 332\pi \text{ rads}^{-1}$$

The particle displacement $y(x,\ t)$ can be written as

$$y(x,\ t) = A\cos(kx)\sin(\omega t)$$

where, $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{2l/3} = \dfrac{3\pi}{l}$

and $\quad \omega = kv = \dfrac{3\pi v}{l} \qquad \left\{\because v = \dfrac{\omega}{k}\right\}$

$$\Rightarrow \quad y(x,\ t) = A\cos\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi vt}{l}\right)$$

The longitudinal oscillations of an air column can be viewed as oscillations of particle displacement or pressure wave or density wave. Pressure variation is related to particle displacement as

$$P(x) = -B\frac{\partial y}{\partial x} \qquad \{B = \text{Bulk modulus}\}$$

$$\Rightarrow \quad P(x) = \left(\frac{3BA\pi}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi vt}{l}\right)$$

The amplitude of pressure variation is

$$\Delta P_{\max} = \frac{3BA\pi}{l} = \frac{3\rho v^2 A\pi}{l} \qquad \{\because B = \rho v^2\}$$

$$\Rightarrow \quad A = \frac{\Delta P_{\max} l}{3\rho v^2 \pi}$$

Given that, $\Delta P_{\max} = P_0/100 = 10^3 \text{ Nm}^{-2}$

$$\Rightarrow \quad A = \frac{(10^3)(3)}{(3)(1.03)(332)^2(\pi)} = 0.0028 \text{ m} = 0.28 \text{ cm}$$

From definition of Bulk modulus (B), we have

$$B = -\frac{dP}{dV/V} \qquad \ldots(1)$$

Since, volume $V = \dfrac{m}{\rho}$

$$\Rightarrow \quad dV = -\frac{m}{\rho^2}d\rho = -\frac{Vd\rho}{\rho}$$

$$\Rightarrow \quad \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Substituting in Equation (1), we get

$$d\rho = \frac{\rho(dP)}{B}$$

So, maximum amplitude of density oscillation is

$$\Delta\rho_{\max} = \frac{\rho}{B}\Delta P_{\max} = \frac{\Delta P_{\max}}{v^2} \qquad \left\{\because \frac{\rho}{B} = \frac{1}{v^2}\right\}$$

$$\Rightarrow \quad \Delta\rho_{\max} = \frac{10^3}{(332)^2} = 9\times 10^{-3} \text{ kgm}^{-3}$$

PROBLEM 2

A string 1 carries a harmonic wave. At a junction with string 2 it is partly reflected and partly transmitted. The linear mass density of the second string is four times that of the first string and that the boundary between the two strings is at $x = 0$. If the expression for the incident wave is, $y_i = A_i\cos(k_1 x - \omega_1 t)$

(a) What are the expressions for the transmitted and the reflected waves in terms of A_i, k_1 and ω_1?

(b) Show that the average power carried by the incident wave is equal to the sum of the average power carried by the transmitted and reflected waves.

SOLUTION

(a) Since, $v = \sqrt{\dfrac{T}{\mu}}$, $T_2 = T_1$ and $\mu_2 = 4\mu_1$

$$\Rightarrow \quad v_2 = \frac{v_1}{2} \qquad \ldots(1)$$

Also, the frequency does not change, so

$$\omega_1 = \omega_2 \qquad \ldots(2)$$

Since, $k = \dfrac{\omega}{v}$, so the wave number of harmonic waves in two strings are related as

$$k_2 = \frac{\omega_2}{v_2} = \frac{\omega_1}{v_1/2} = 2\left(\frac{\omega_1}{v_1}\right) = 2k_1 \qquad \ldots(3)$$

The amplitudes are given by

$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right)A_i = \left[\frac{2(v_1/2)}{v_1 + (v_1/2)}\right] = \frac{2}{3}A_i \qquad \ldots(4)$$

and $A_r = \left(\dfrac{v_2 - v_1}{v_1 + v_2}\right)A_i = \left[\dfrac{(v_1/2) - v_1}{v_1 + (v_1/2)}\right]A_i$

$$\Rightarrow \quad A_r = -\frac{A_i}{3} \qquad \ldots(5)$$

So, from Equations (2), (3) and (4), the transmitted wave can be written as,

$$y_t = \frac{2}{3}A_i\cos(2k_1x - \omega_1 t)$$

Similarly, the reflected wave is expressed as,

$$y_r = -\frac{A_i}{3}\cos(k_1x + \omega_1 t)$$

$$\Rightarrow \quad y_r = \frac{A_i}{3}\cos(k_1x + \omega_1 t + \pi)$$

The reflected ray suffers and additional phase change of π because the wave is reflected at the fixed end.

(b) The average power of a harmonic wave on a string of cross-sectional area S is given by

$$P = \frac{1}{2}\rho A^2\omega^2 S v = \frac{1}{2}A^2\omega^2\mu v \qquad \{\because \rho S = \mu\}$$

So, $P_i = \dfrac{1}{2}\omega_1^2 A_i^2 \mu_1 v_1 \qquad \ldots(6)$

$$P_t = \frac{1}{2}\omega_1^2\left(\frac{2}{3}A_i\right)^2(4\mu_1)\left(\frac{v_1}{2}\right) = \frac{4}{9}\omega_1^2 A_i^2\mu_1 v_1 \qquad \ldots(7)$$

and $P_r = \dfrac{1}{2}\omega_2^2\left(-\dfrac{A_i}{3}\right)^2(\mu_1)(v_1) = \dfrac{1}{18}\omega_1^2 A_i^2\mu_1 v_1 \qquad \ldots(8)$

From Equations (6), (7) and (8), we observe that

$$P_i = P_t + P_r$$

PROBLEM 3

On a stormy day Ram asked Shyam to go to his friend's house which is towards north from his house. He asked him to go directly towards North. After sometime Ram observed Shyam going towards 37° West of North as wind was blowing at 36 kmhr^{-1} towards west. After 15 min Ram blew a whistle so as to give signal to Shyam to come back. Calculate the distance between Ram's house and Shyam when he heard the signal. If natural frequency of the whistle was 400 Hz, calculate the frequency as heard by Shyam. Assume that speed of sound in air is 330 ms^{-1} and Shyam is travelling at a speed of 16.67 ms^{-1}.

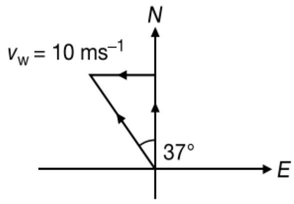
SOLUTION

Let Shyam hear the whistle after a time t, then total distance travelled by Shyam is

$$x = 16.67(15 \times 60) + 16.67t = 15000 + 16.67t$$

Distance travelled by sound in time t is

$$y = [330 + 10\cos(53°)]t = 336t$$



Since, $x = y$

$$\Rightarrow \quad 15000 + 16.67t = 336t$$

$$\Rightarrow \quad t = \frac{45000}{958} = 47\ \text{s}$$

So, distance is

$$x = y = 15000 + (16.67)(47) = 15783.3\ \text{m}$$

$$\Rightarrow \quad x = y = 15.78\ \text{km}$$

So, $f' = \left(\dfrac{330 + 10(3/5) - 16.67}{330 + 10(3/5)}\right)(400) = 380.15\ \text{Hz}$

PROBLEM 4

A standing wave is produced in a steel wire of mass 100 g tied to two fixed supports. The length of the string is 2 m and strain in it is 0.4%. The string vibrates in four loops. Assume one end of the string to be at $x = 0$, all particle to be at rest at $t = 0$ and maximum amplitude to be 3 mm, find

(a) wavelength and frequency of the wave
(b) equation of the standing wave
(c) equation of the travelling waves whose superposition is the given standing wave. Also find the velocity of these travelling waves.
(d) maximum kinetic energy of the wire.

(Given $\pi^2 = 10$, density of steel $= 4 \times 10^3\ \text{kgm}^{-3}$, Young's modulus of steel $= 1.6 \times 10^{11}\ \text{Nm}^{-2}$)

SOLUTION

(a) Since the string vibrates in four loops, so

$$4\left(\frac{\lambda}{2}\right) = 2$$

$$\Rightarrow \quad \lambda = 1\ \text{m}$$

Since, $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{T}{\rho S}}$

Also, $Y = \dfrac{T/S}{\Delta l/l}$

$$\Rightarrow \quad \frac{T}{S} = Y\left(\frac{\Delta l}{l}\right)$$

Substituting the values, we get

$$v = 400\ \text{ms}^{-1}$$

$$\Rightarrow \quad f = \frac{v}{\lambda} = \frac{400}{1} = 400\ \text{Hz}$$

(b) $y = A\sin(kx)\cos(\omega t)$

Substituting $A = 3$ mm, $k = \dfrac{2\pi}{\lambda}$ and $\omega = 2\pi f$

$\Rightarrow\ y = (3\text{ mm})\sin(2\pi x)\cos(800\pi t)$

(c) $y_1 = 1.5\sin(2\pi x - 800\pi t)$ and

$y_2 = 1.5\sin(2\pi x + 800\pi t)$

with $v = 400\text{ ms}^{-1}$

(d) Velocity of particle at

$$x = \frac{\partial y}{\partial t} = -\omega A\sin(kx)\sin(\omega t)$$

Kinetic energy of a small element of length dx at that position is

$$dK = \frac{1}{2}(\mu dx)v^2 = \frac{1}{2}\frac{m}{l}v^2 dx$$

So, total K.E. is $K = \dfrac{1}{2}\dfrac{m}{l}\omega^2 A^2\displaystyle\int_0^l \sin^2 kx dx$

$\Rightarrow\ K = \dfrac{1}{4}m\omega^2 A^2$

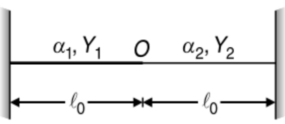
$\Rightarrow\ K = 1.44$ J

PROBLEM 5

Two wires 1 and 2 of the same cross-sectional area $A = 10\text{ mm}^2$ and the same length but made of different materials are welded together and their ends are rigidly clamped between two walls, as shown in the figure. The respective Young's Moduli and the linear coefficients of thermal expansion are

$$Y_1 = 10^9\text{ Nm}^{-2},\ Y_2 = 2\times 10^9\text{ Nm}^{-2}$$

$$\alpha_1 = 6\times 10^{-4}\ ^\circ\text{C},\ \alpha_2 = 3\times 10^{-4}\ ^\circ\text{C}$$



(a) If the temperature of the system is reduced by 20 °C, then
  (i) find the tension in each wire
  (ii) find the displacement of the joint O

(b) Find the first overtone frequency of the system if joint is a node and the mass per unit length of the wires are

$\mu_1 = 0.3\text{ kgm}^{-1}$ and $\mu_2 = 0.075\text{ kgm}^{-1}$; $l_0 = 1$ m

SOLUTION

(a) Since, we know that

$$\text{Stress} = \frac{T}{S} = Y\alpha\Delta\theta$$

$\Rightarrow\ T = YS\alpha\Delta\theta$

where S is area of cross-section of the wire

$\Rightarrow\ T_1 = (10^9)(10\times 10^{-6})(6\times 10^{-4})(20)$

$\Rightarrow\ T_1 = 120$ N

Similarly, using the same formula, we get

$T_2 = 120$ N

So, displacement of the joint O is zero.

(b) Let n_1 loops be formed in wire 1 and n_2 loops in wire 2. Then $f_1 = f_2$, so we get

$$n_1\left(\frac{v_1}{2l_1}\right) = n_2\left(\frac{v_2}{2l_2}\right)$$

$$\Rightarrow\ \frac{n_1}{\sqrt{\mu_1}} = \frac{n_2}{\sqrt{\mu_2}} \qquad \left\{\because v \propto \frac{1}{\sqrt{\mu}}\right\}$$

$$\Rightarrow\ \frac{n_1}{n_2} = \sqrt{\frac{\mu_1}{\mu_2}} = \sqrt{\frac{0.3}{0.075}} = \frac{2}{1}$$

This ratio implies that in fundamental mode, 2 loops are formed on wire 1 and 1 loop is formed on wire 2. In first overtone frequency 4 loops are formed on wire 1 and 2 loops on wire 2.
Thus, the first overtone frequency is

$$f = 4\left(\frac{v_1}{2l_0}\right) = 4\left(\frac{1}{2l_0}\sqrt{\frac{T}{\mu_1}}\right) = 4\left(\frac{1}{2}\sqrt{\frac{120}{0.3}}\right)$$

$\Rightarrow\ f = 40$ Hz

PROBLEM 6

A boat is travelling in a river with a speed 10 ms^{-1} along the stream flowing with a speed 2 ms^{-1}. From this boat, a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.

(a) What will be the frequency detected by a receiver kept inside the river downstream?

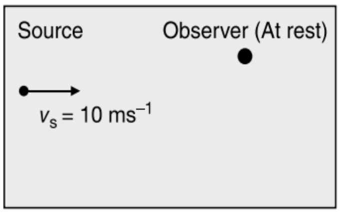
(b) The transmitter and the receiver are now pulled up into air. The air is blowing with a speed 5 ms^{-1} in the direction opposite the river stream. Determine the frequency of the sound detected by the receiver.

Given : Temperature of the air and water = 20 °C, Density of river water $= 10^3\text{ kgm}^{-3}$, Bulk modulus of the water is 2.088×10^9 Pa, Gas constant $R = 8.31\text{ Jmol}^{-1}\text{K}^{-1}$, Mean molecular mass of air $= 28.8\times 10^{-3}\text{ kgmol}^{-1}$, $\dfrac{C_P}{C_V}$ for air is 1.4

SOLUTION

Velocity of sound in water is $v_w = \sqrt{\dfrac{B}{\rho}}$

$$\Rightarrow\ v_w = \sqrt{\frac{2.088\times 10^9}{10^3}} = 1445\text{ ms}^{-1}$$



Frequency of sound in water will be

$$f_0 = \frac{v_w}{\lambda_w} = \frac{1445}{14.45 \times 10^{-3}}\ \text{Hz}$$

$$\Rightarrow \quad f_0 = 10^5\ \text{Hz}$$

(a) Frequency of sound detected by receiver (observer) at rest is given by

$$f_1 = f_0\left(\frac{v_w + v_r}{v_w + v_r - v_s}\right)$$

$$\Rightarrow \quad f_1 = (10^5)\left(\frac{1445+2}{1445+2-10}\right)\ \text{Hz}$$

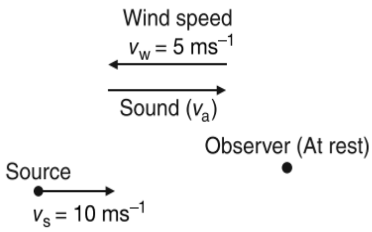
$$\Rightarrow \quad f_1 = 1.0069 \times 10^5\ \text{Hz}$$

(b) Velocity of sound in air is given by

$$v_a = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{(1.4)(8.31)(20+273)}{28.8\times 10^{-3}}}$$

$$\Rightarrow \quad v_a = 344\ \text{ms}^{-1}$$

Since, frequency does not depend on the medium. Therefore, frequency in air is also $f_0 = 10^5$ Hz.



Frequency of sound detected by receiver (observer) in air is given by

$$f_2 = f_0\left(\frac{v_a - v_w}{v_a - v_w - v_s}\right) = 10^5\left(\frac{344-5}{344-5-10}\right)\ \text{Hz}$$

$$\Rightarrow \quad f_2 = 1.0304 \times 10^5\ \text{Hz}$$

PROBLEM 7

The following equations represent transverse waves:

$$z_1 = A\cos(kx - \omega t),\ z_2 = A\cos(kx + \omega t) \text{ and}$$

$$z_3 = A\cos(ky - \omega t).$$

Identify the combination of the wave which will produce

(a) a standing wave

(b) a wave travelling in the direction making an angle of 45° with the positive x and positive y-axis.

In each case find the position at which the resultant intensity is always zero.

SOLUTION

(a) A standing wave is produced due to superposition of two waves of equal amplitude and frequency travelling in opposite direction with same speed. Here, the amplitude, the frequency and the velocity of all waves are equal; but z_1 is travelling along positive x-axis, z_2 along negative x-axis and z_3 along positive y-axis. So, superposition of z_1 and z_2 will produce stationary waves,

$$z = z_1 + z_2 = A\cos(\omega t - kx) + A\cos(\omega t + kx)$$

or $\quad z = 2A\cos kx \cos\omega t$

Now, as the amplitude of standing wave is $(2A\cos kx)$, the velocity is

$$I = A_s^2 = 4A^2\cos^2 kx$$

which will zero, if

$$\cos^2 kx = 0$$

$$\Rightarrow \quad kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\Rightarrow \quad x = \frac{\pi}{2k}, \frac{3\pi}{2k}, \frac{5\pi}{2k}, \ldots$$

$$\Rightarrow \quad x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots \qquad \left\{\because k = \frac{2\pi}{\lambda}\right\}$$

(b) The waves z_1 and z_3 are travelling along positive x-axis and positive y-axis respectively, with the same speed,

$$v = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda$$

Therefore, the superposition of z_1 and z_3 will produce a wave travelling in a direction making an angle of 45° with positive x and y-axis. The resulting wave is given as

$$z = z_1 + z_3 = A\cos(\omega t - kx) + A\cos(\omega t - ky)$$

$$\Rightarrow \quad z = 2A\cos\left(\frac{k|x-y|}{2}\right)\cos\left[\omega t - k\left(\frac{x+y}{2}\right)\right]$$

The amplitude of this wave is

$$A_s = 2A\cos\left(\frac{k|x-y|}{2}\right)$$

$$\Rightarrow \quad I \propto \cos^2\left(\frac{k|x-y|}{2}\right)$$

Therefore, the intensity will be zero for

$$\left(\frac{k|x-y|}{2}\right) = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\Rightarrow \ |x-y| = \frac{\pi}{k}, \frac{3\pi}{k}, \frac{5\pi}{k}, \ldots$$

$$\Rightarrow \ |x-y| = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots \qquad \left\{\because k = \frac{2\pi}{\lambda}\right\}$$

PROBLEM 8

An aluminium wire of cross-sectional area 10^{-6} m^2 is joined to a steel wire of the same cross-sectional area. This compound wire is stretched on a sonometer pulled by a weight of 10 kg. The total length of the compound wire between the bridges is 1.5 m of which the aluminium wire is 0.6 m and the rest is steel wire. Transverse vibrations are set-up in the wire by using an external source of variable frequency. Find the lowest frequency of excitation for which the standing waves are formed such that the joint in the wire is a node. What is the total number of nodes at this frequency? Density of aluminium is 2.6×10^3 kgm^{-3} and that of steel is 1.04×10^4 kgm^{-3}. Take $g = 10$ ms^{-2}.

SOLUTION

Let n_a loops are formed in aluminium wire and n_s in steel. Then $f_a = f_s$

$$\Rightarrow \quad n_a\left(\frac{v_a}{2l_a}\right) = n_s\left(\frac{v_s}{2l_s}\right)$$

$$\Rightarrow \quad \frac{n_a}{n_s} = \left(\frac{v_s}{v_a}\right)\left(\frac{l_a}{l_s}\right)$$

Since, $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho S}} \propto \frac{1}{\sqrt{\rho}}$

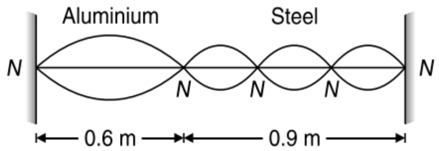
$$\Rightarrow \quad \frac{v_s}{v_a} = \sqrt{\frac{\rho_a}{\rho_s}}$$

$$\Rightarrow \quad \frac{n_a}{n_s} = \sqrt{\frac{\rho_a}{\rho_s}}\frac{l_a}{l_s}$$

Substituting the values, we get

$$\frac{n_a}{n_s} = \sqrt{\frac{2.6\times10^3}{1.04\times10^4}}\left(\frac{0.6}{0.9}\right) = \frac{1}{3}$$

i.e., at lowest frequency, one loop is formed in aluminium wire and three loops are formed in steel wire as shown in figure.



So, $f_{\text{min}} = n_a\left(\frac{v_a}{2l_a}\right) = n_a\left(\frac{1}{2l_a}\sqrt{\frac{T}{\rho_a S}}\right)$

$$\Rightarrow \quad f_{\text{min}} = (1)\left(\frac{1}{2\times0.6}\sqrt{\frac{100}{2.6\times10^3\times10^{-6}}}\right) = 163.4 \text{ Hz}$$

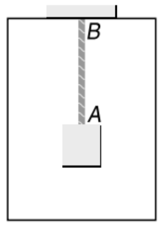
Total number of nodes are five as shown in figure.

Conceptual Note(s)

In such type of problems nature of junction will be known to us. Then we have to equate frequencies on the two sides. By equating the frequencies, we find $\frac{n_1}{n_2}$. Suppose this comes out to be 0.6. Write it, $\frac{n_1}{n_2} = \frac{3}{5}$. At lowest oscillation frequency 3 loops are formed on side 1 and 5 loops on side 2. At next higher frequency 4 loops will be formed on side 1 and 10 on side 2 and so on.

PROBLEM 9

A block of mass M is attached with a string of mass m and length l as shown in figure. The whole system is placed on a planet whose mass and radius is three times the mass and radius of earth. Find the



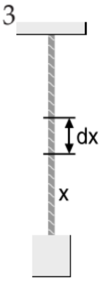
(a) time taken by a wave pulse to travel from one end A to B of the string.

(b) ratio of maximum and minimum velocity of wave pulse.

Assume the acceleration due to gravity on the earth to be g.

SOLUTION

(a) Since, $g' = \frac{3Gm}{(3R)^2} = \frac{g}{3}$



Consider a small element dx at a distance x from one end, then

$$v = \sqrt{\frac{T}{\mu}}$$

where, $\mu = \frac{m}{l}$ and $T = \left(M + \frac{m}{l}x\right)g'$

$$\Rightarrow \quad v = \frac{dx}{dt} = \sqrt{\frac{\left(M+\frac{m}{l}x\right)g}{3\left(\frac{m}{l}\right)}}$$

$$\Rightarrow \int_0^l \frac{dx}{\sqrt{Ml+mx}} = \sqrt{\frac{g}{3m}}\int_0^t dt$$

$$\Rightarrow t = 2\left(\frac{\sqrt{M+m}-\sqrt{M}}{\sqrt{m}}\right)\sqrt{\frac{3l}{g}}$$

(b) $v_{\min} = \sqrt{\frac{Mlg}{3m}}$ and $v_{\max} = \sqrt{\frac{(M+m)lg}{3m}}$

$$\Rightarrow \frac{v_{\max}}{v_{\min}} = \sqrt{1+\frac{m}{M}}$$

PROBLEM 10

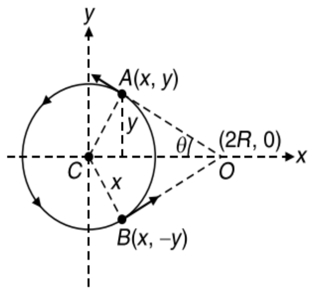
A source is moving along a circle $x^2+y^2=R^2$ with constant speed $v_s = \frac{330\pi}{6\sqrt{3}}$ ms^{-1} in clockwise direction while an observer is stationary at point $(2R,\ 0)$ with respect to the center of circle. Frequency emitted by the source is f.

(a) Find the coordinates of source when observer records the maximum and minimum frequency.

(b) Find the value of maximum and minimum frequency. Take speed of sound $v = 330$ ms^{-1}.

SOLUTION

(a) When the source is at B, the observed frequency is maximum and when it is at A, the observed frequency is minimum.



Let the co-ordinates of point A be $(x,\ y)$, then the equation of the circle is given by

$$x^2+y^2=R^2 \qquad \ldots(1)$$

$$d\left(x^2+y^2\right)=0$$

$$\Rightarrow 2xdx+2ydy=0$$

$$\Rightarrow \left(-\frac{dy}{dx}\right) = \frac{x}{y} = \tan\theta = \frac{y}{2R-x}$$

$$\Rightarrow 2Rx - x^2 = y^2$$

$$\Rightarrow 2Rx = x^2+y^2 = R^2$$

$$\Rightarrow x = \frac{R}{2}$$

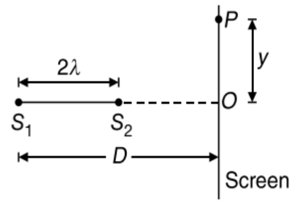
Substituting $x=\frac{R}{2}$ in (1), we get $y = \frac{\sqrt{3}}{2}R$

$$\Rightarrow A = \left(\frac{R}{2},\ \frac{\sqrt{3}}{2}R\right) \text{ and } B = \left(\frac{R}{2},\ -\frac{\sqrt{3}}{2}R\right)$$

(b) $f_{\max} = f_B = \left(\frac{v}{v-v_s}\right)f$, $f_{\min} = f_A = \left(\frac{v}{v+v_s}\right)f$

PROBLEM 11

Two coherent narrow slits emitting sound of wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ. The sound is detected by moving a detector on the screen at a distance $D(\gg\lambda)$ from the slit S_1 as shown in figure. Find the distance y such that the intensity at P is equal to intensity at O.



SOLUTION

METHOD-I:

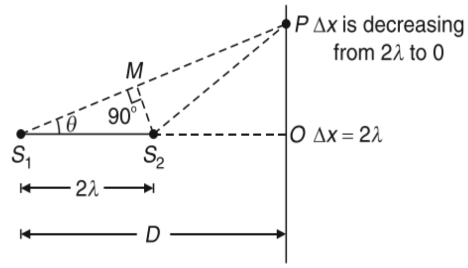
At point O on the screen the path difference between the sound waves reaching from S_1 and S_2 is 2λ (an even multiple of $\frac{\lambda}{2}$), so constructive interference is obtained at O. At a very large distance from point O on the screen the path difference is zero.

So, we can conclude that as we move away from point O on the screen path difference decreases from 2λ to zero. At O constructive interference is obtained (where $\Delta x = 2\lambda$). So, next constructive interference will be obtained where $\Delta x = \lambda$. Hence,

$$S_1P - S_2P = \lambda$$

$$\Rightarrow \sqrt{D^2+y^2} - \sqrt{y^2+(D-2\lambda)^2} = \lambda$$

$$\Rightarrow \sqrt{D^2+y^2} - \lambda = \sqrt{y^2+(D-2\lambda)^2}$$



Squaring both sides, we get

$$D^2+y^2+\lambda^2-2\lambda\sqrt{D^2+y^2} = y^2+D^2+4\lambda^2-4\lambda D$$

$$\Rightarrow 2\sqrt{D^2+y^2} = 4D-3\lambda$$

Since, $D \gg \lambda$, so we get

$$4D - 3\lambda \approx 4D$$

$$\Rightarrow \quad 2\sqrt{D^2 + y^2} = 4D$$

$$\Rightarrow \quad \sqrt{D^2 + y^2} = 2D$$

Again, squaring both sides, we get

$$D^2 + y^2 = 4D^2$$

$$\Rightarrow \quad y = \sqrt{3}D$$

METHOD-II:

Let $\Delta x = \lambda$ at angle θ as shown, so the path difference between the waves is

$$\Delta x = S_1M = 2\lambda\cos\theta$$

$$\Rightarrow \quad 2\lambda\cos\theta = \lambda \qquad \{\because \Delta x = \lambda\}$$

$$\Rightarrow \quad \theta = 60°$$

Now, $PO = S_1O\cot(30°)$

$$\Rightarrow \quad y = \sqrt{3}D$$

PROBLEM 12

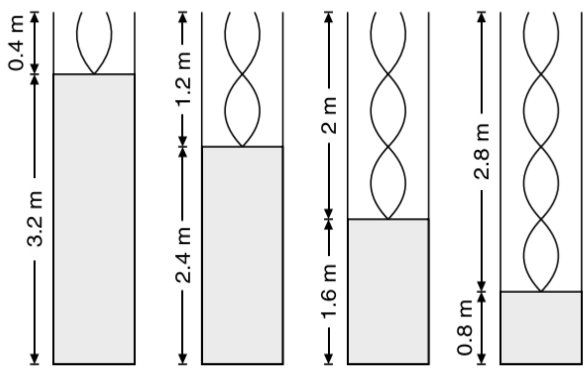
A 3.6 m long pipe resonates with a source of frequency 212.5 Hz when water level is at certain heights in the pipe. Find the heights of water level (from the bottom of the pipe) at which resonances occur. Neglect end correction. Now the pipe is filled to a height $H(= 3.6 \text{ m})$. A small hole is drilled very close to its bottom and water is allowed to leak. Obtain an expression for the rate of fall of water level in the pipe as a function of H. If the radii of the pipe and the hole are 2×10^{-2} m and 1×10^{-3} m respectively, calculate the time interval between the occurrence of first two resonances. Given that the speed of sound in air is 340 ms^{-1} and $g = 10 \text{ ms}^{-2}$.

SOLUTION

Speed of sound, $v = 340 \text{ ms}^{-1}$. Let l_0 be the length of air column corresponding to the fundamental frequency, then

$$\frac{v}{4l_0} = 212.5$$

$$\Rightarrow \quad l_0 = \frac{v}{4(212.5)} = \frac{340}{4(212.5)} = 0.4 \text{ m}$$



In closed pipe only odd harmonics are obtained. Now let l_1, l_2, l_3, l_4 etc., be the lengths corresponding to the 3rd harmonic, 5th harmonic, 7th harmonic, etc. Then

$$3\left(\frac{v}{4l_1}\right) = 212.5$$

$$\Rightarrow \quad l_1 = 1.2 \text{ m}$$

$$5\left(\frac{v}{4l_2}\right) = 212.5$$

$$\Rightarrow \quad l_2 = 2 \text{ m}$$

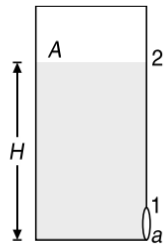
$$7\left(\frac{v}{4l_3}\right) = 212.5$$

$$\Rightarrow \quad l_3 = 2.8 \text{ m and } 9\left(\frac{v}{4l_4}\right) = 212.5$$

$$\Rightarrow \quad l_4 = 3.6 \text{ m}$$

So, the heights of corresponding water levels are $(3.6 - 0.4)$ m, $(3.6 - 1.2)$ m, $(3.6 - 2)$ m and $(3.6 - 2.8)$ m

Heights of corresponding water levels are 3.2 m, 2.4 m, 1.6 m and 0.8 m.



Let A and a be the area of cross-sections of the pipe and hole respectively. Then

$$A = \pi(2\times10^{-2})^2 = 1.26\times10^{-3} \text{ m}^2$$

and $a = \pi(10^{-3})^2 = 3.14\times10^{-6} \text{ m}^2$

Since, velocity of efflux, $v = \sqrt{2gH}$

Applying Equation of Continuity at (1) and (2), we get

$$a\sqrt{2gH} = A\left(\frac{-dH}{dt}\right)$$

So, rate of fall of water level in the pipe is given by

$$\left(\frac{-dH}{dt}\right) = \frac{a}{A}\sqrt{2gH}$$

Substituting the values, we get

$$-\frac{dH}{dt} = \frac{3.14\times10^{-6}}{1.26\times10^{-3}}\sqrt{2\times10\times H}$$

$$\Rightarrow \quad -\frac{dH}{dt} = (1.11\times10^{-2})\sqrt{H}$$

$$\Rightarrow \quad \frac{dH}{\sqrt{H}} = -(1.11\times10^{-2})dt$$

Between first two resonances, the water level falls from 3.2 m to 2.4 m.

$$\Rightarrow \quad \int_{3.2}^{2.4}\frac{dH}{\sqrt{H}} = -(1.11\times10^{-2})\int_0^t dt$$

$\Rightarrow \quad 2\left(\sqrt{2.4}-\sqrt{3.2}\right)=-\left(1.11\times10^{-2}\right)t$

$\Rightarrow \quad t \approx 43$ second

PROBLEM 13

A train approaching a hill at a speed of 40 kmh^{-1} sounds a whistle of frequency 580 Hz when it is at a distance of 1 km from a hill. A wind with a speed of 40 kmh^{-1} is blowing in the direction of motion of the train. Find the

(a) frequency of the whistle as heard by an observer on the hill and

(b) distance from the hill at which the echo from the hill is heard by the driver and its frequency.

Given: Velocity of sound in air $v = 1200\ \text{kmh}^{-1}$.

SOLUTION

(a) Since, $f' = \left(\dfrac{v+w}{v+w-v_s}\right)f$

$$\Rightarrow f' = \left(\frac{1200+40}{1200+40-40}\right)(580) \approx 599\ \text{Hz}$$

(Also note that all velocities are in kmph, so do not bother to convert them to ms^{-1})

(b) For echo to be heard by the driver, the source is to be considered at the hill having frequency 599 Hz and the listener approaching with a speed of 40 kmhr^{-1} and the wind blowing at a speed of 40 kmhr^{-1} opposite to the direction of the waves. The frequency f'' is given by

$$f'' = \left(\frac{1200-40+40}{1200-40}\right)(599) \approx 620\ \text{Hz}$$

Time taken by the wave to reach the hill,

$$t_1 = \frac{1}{1200+40} = \frac{1}{1240}\ \text{hr}$$

During this time the train moves towards the hill by a distance of

$$x = 40 \times \frac{1}{1240}\ \text{km} = \frac{1}{31}\ \text{km}$$

So, the distance between train and hill now is

$$\left(1-\frac{1}{31}\right) = \frac{30}{31}\ \text{km}$$

Let the echo be heard after t hr after this instant, then

$$40t + (1200-40)t = \frac{30}{31}$$

$$\Rightarrow t = \frac{1}{1240}\ \text{hr}$$

The distance travelled by the train in this time is

$$x' = 40 \times \frac{1}{1240} = \frac{1}{31}\ \text{km}$$

Therefore, the echo is heard by the driver when the train has moved a total distance

$$s = x + x' = \frac{1}{31} + \frac{1}{31} = \frac{2}{31}\ \text{km}$$

So, distance from the hill is

$$l = 1 - s = 1 - \frac{2}{31} = 0.935\ \text{km}$$

PROBLEM 14

Two sound sources driven in phase by the same amplifier are 2 m apart on the y-axis. At a point a very large distance from the y-axis, constructive interference is first heard at an angle $\theta_1 = 0.14$ rad with the x-axis and is next heard at $\theta_2 = 0.283$ rad. What is the

(a) wavelength of the sound waves from the sources?

(b) frequency of the sources?

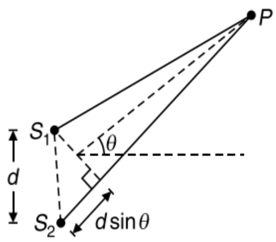
(c) smallest angle for which the sound waves cancel?

Take speed of sound in air as 340 ms^{-1}.

SOLUTION

(a) At a point for away from the sources the path difference is

$$\Delta x = d\sin\theta$$



First constructive interference is obtained where

$$\Delta x = \lambda$$

$$\Rightarrow d\sin\theta_1 = \lambda \qquad \text{...(1)}$$

Next constructive interference is obtained when, we get

$$\Delta x = 2\lambda$$

$$\Rightarrow d\sin\theta_2 = 2\lambda \qquad \text{...(2)}$$

Subtracting Equation (1) from (2), we get

$$\lambda = d\left(\sin\theta_2 - \sin\theta_1\right)$$

Substituting $d = 2\ \text{m}$,

$$\theta_1 = 0.14\ \text{rad} = 8^\circ$$

and $\theta_2 = 0.283\ \text{rad} = 16.21^\circ$

we get, $\lambda = 0.279\ \text{m}$

(b) Frequency of the sources

$$f = \frac{v}{\lambda} = \frac{340}{0.279} = 1219\ \text{Hz}$$

(c) The smallest angle where the sound waves cancel each other i.e., interference destructively is where, we have

$$\Delta x = d\sin\theta = \frac{\lambda}{2}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{\lambda}{2d}\right) = \sin^{-1}\left(\frac{0.279}{2\times 2}\right)$$

$$\Rightarrow \quad \theta = 4^\circ = 0.07 \text{ rad}$$

PROBLEM 15

A thin string is held at one end and oscillates vertically so that, $y(x=0, t) = 8\sin 4t(\text{cm})$. Neglect the gravitational force. The string's linear mass density is 0.2 kgm^{-1} and its tension is 1 N. The string passes through a bath filled with 1 kg water. Due to friction heat is transferred to the bath. The heat transfer efficiency is 50%. Calculate how much time passes before the temperature of the bath rises one degree kelvin.

SOLUTION

Comparing the given equation with equation of a travelling wave,

$y = A\sin(kx \pm \omega t)$ at $x = 0$ we get

$A = 8 \text{ cm} = 8\times 10^{-2} \text{ m}$

$\omega = 4 \text{ rads}^{-1}$

Speed of travelling wave,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{1}{0.2}} = 2.236 \text{ ms}^{-1}$$

Since, $\mu = \frac{m}{l} = \rho S = 0.2 \text{ kgm}^{-1}$

where S is area of cross-section of string

The average power over a period is,

$$P = \frac{1}{2}(\rho S)\omega^2 A^2 v$$

Substituting the values, we get

$$P = \frac{1}{2}(0.2)(4)^2(8\times 10^{-2})^2(2.236)$$

$$\Rightarrow \quad P = 2.29\times 10^{-2} \text{ Js}^{-1}$$

The power transferred to the bath is,

$$P' = 0.5P = 1.145\times 10^{-2} \text{ Js}^{-1}$$

To raise the temperature of 1 kg water by 1 degree kelvin, let the time taken be t, then

$$P't = mc\Delta T$$

where, c =specific heat of water $= 4.2\times 10^3 \text{ Jkg}^{-1}\text{K}^{-1}$

$$\Rightarrow \quad t = \frac{mc\Delta T}{P'} = \frac{(1)(4.2\times 10^{-3})(1)}{1.145\times 10^{-2}}$$

$$\Rightarrow \quad t = 3.6\times 10^5 \text{ s} \approx 4.2 \text{ day}$$

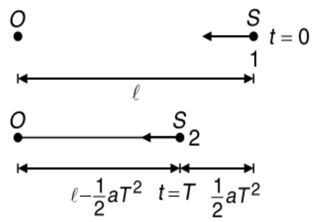
PROBLEM 16

A source emitting a sound of frequency f is placed at a large distance from an observer. The source starts moving towards the observer with a uniform acceleration a. Find the frequency heard by the observer corresponding to the wave emitted just after the source starts. The speed of sound in the medium is v.

SOLUTION

Suppose at $t = 0$, distance between source and observer is l. First wave pulse (say p_1) is emitted at this instant. This pulse will reach the observer after a time

$$t_1 = \frac{l}{v} \quad \ldots(1)$$



Source will emit the next pulse (say p_2) after a time $T\left(=\frac{1}{f}\right)$. During this time the source will move a distance $\frac{1}{2}aT^2$ towards the observer. This pulse p_2 will reach the observer in a time

$$t_2 = T + \frac{\left(l - \frac{1}{2}aT^2\right)}{v} \quad \ldots(2)$$

The changed time period as observed by the observer is

$$T' = t_2 - t_1 = T + \frac{l}{v} - \frac{1}{2}\frac{aT^2}{v} - \frac{l}{v}$$

Substituting $T' = \frac{1}{f'}$ and $T = \frac{1}{f}$ in the above equation, we get

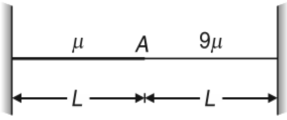
$$f' = \frac{2vf^2}{2vf - a}$$

Conceptual Note(s)

We have written $t_2 = T + \frac{\left(\ell - \frac{1}{2}aT^2\right)}{v}$, because the pulse p_2 is emitted after time T and then it takes a time $\frac{\left(\ell - \frac{1}{2}aT^2\right)}{v}$ to reach the observer. So, it reaches the observer in a time $T + \frac{\left(\ell - \frac{1}{2}aT^2\right)}{v}$.

PROBLEM 17

A light string is tied at one end to a fixed support and to a heavy string of equal length L at the other end as shown in figure. Mass per unit length of the strings are μ and 9μ and the tension is T. Find the possible values of frequencies such that point A is a node/antinode.



SOLUTION

When A is a Node

Let n_1 and n_2 be the number of complete loops formed on left and right side of point A, then

$$f_1 = f_2$$

$$\Rightarrow \quad n_1\left(\frac{v_1}{2L}\right) = n_2\left(\frac{v_2}{2L}\right), \text{ where } v \propto \frac{1}{\sqrt{\mu}}$$

$$\Rightarrow \quad \frac{n_1}{n_2} = \left(\frac{v_2}{v_1}\right) = \sqrt{\frac{\mu_1}{\mu_2}} = \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots, \text{ etc.}$$

So, the possible frequencies are,

$$\frac{v_1}{2L}, 2\left(\frac{v_1}{2L}\right), \frac{3v_1}{2L}, \ldots, \text{ etc.} \qquad \left\{\because v_1 = \sqrt{\frac{T}{\mu}}\right\}$$

$$\text{OR} \quad \frac{1}{2L}\sqrt{\frac{T}{\mu}}, \frac{1}{L}\sqrt{\frac{T}{\mu}}, \frac{3}{2L}\sqrt{\frac{T}{2\mu}}, \ldots, \text{ etc.}$$

When A is an Antinode

Let n_1 and n_2 be the number of complete loops on left and right side of point A, then

$$n_1\left(\frac{\lambda_1}{2}\right) + \frac{\lambda_1}{4} = L$$

$$\Rightarrow \quad f_1 = \frac{v_1}{L}\left(\frac{n_1}{2} + \frac{1}{4}\right) \qquad \{\because v = f\lambda\}$$

$$\text{and} \quad n_2\left(\frac{\lambda_2}{2}\right) + \frac{\lambda_2}{4} = L$$

$$\Rightarrow \quad f_2 = \frac{v_2}{L}\left(\frac{n_2}{2} + \frac{1}{4}\right)$$

Substituting, $f_1 = f_2$ we get

$$\frac{2n_1 + 1}{2n_2 + 1} = \frac{1}{3}$$

This condition is met for

$n_1 = 1, n_2 = 4$

$n_1 = 2, n_2 = 7$

$n_1 = 3, n_2 = 10$ etc.

… … …

Therefore, the possible frequencies are

$$\frac{v_1}{L}\left(\frac{1}{2} + \frac{1}{4}\right), \frac{v_1}{L}\left(\frac{2}{2} + \frac{1}{4}\right), \frac{v_1}{L}\left(\frac{3}{2} + \frac{1}{4}\right), \ldots, \text{ etc.}$$

$$\text{OR} \quad \frac{3}{4L}\sqrt{\frac{T}{\mu}}, \frac{5}{4L}\sqrt{\frac{T}{\mu}}, \frac{7}{4L}\sqrt{\frac{T}{\mu}}, \ldots \text{ etc.}$$

PROBLEM 18

A string of linear mass density 5×10^{-3} kgm^{-1} is stretched under a tension of 65 N between two rigid supports 60 cm apart.

(a) If the string is vibrating in its second overtone so that the amplitude at one of its antinodes is 0.25 cm, what are the maximum transverse speed and acceleration of the string at antinodes?

(b) What are these quantities at a distance 5 cm from an node?

SOLUTION

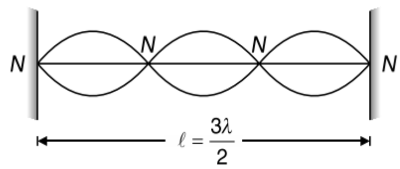
(a) In second overtone $l = \dfrac{3\lambda}{2}$

$$\Rightarrow \quad \lambda = \frac{2l}{3} = \frac{2 \times 60}{3} = 40 \text{ cm} = 0.4 \text{ m}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = 5\pi \text{ m}^{-1}$$

$$\text{Since, } v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{65}{5 \times 10^{-3}}} = 114 \text{ ms}^{-1}$$

$$\Rightarrow \quad \omega = kv = 570\pi \text{ rads}^{-1}$$



Maximum transverse speed at antinode is $v_{max} = A_0\omega$, where A_0 is amplitude of antinode

$$A_0 = 0.25 \text{ cm} = 2.5 \times 10^{-3} \text{ m}$$

So, maximum speed is

$$v_{max} = (2.5 \times 10^{-3})(570\pi) \text{ ms}^{-1} = 4.48 \text{ ms}^{-1}$$

Maximum acceleration is $a_{max} = \omega^2 A_0$

$$\Rightarrow \quad a_{max} = (570\pi)^2 (2.5 \times 10^{-3}) \text{ ms}^{-2}$$

$$\Rightarrow \quad a_{max} = 8 \times 10^3 \text{ ms}^{-2}$$

(b) At a distance x from the node, the amplitude is

$$A = A_0 \sin(kx) = (2.5 \times 10^{-3})\sin(5\pi x) \text{ metre}$$

where x is in metre.

At $x = 5$ cm $= 5 \times 10^{-2}$ m, we have

$$A = (2.5 \times 10^{-3})\sin(5\pi \times 5 \times 10^{-2})$$

$\Rightarrow \; A = 1.8 \times 10^{-3}$ m

So, maximum speed is $v_{max} = A\omega$

$\Rightarrow \; v_{max} = (1.8 \times 10^{-3})(570\pi)$ ms^{-1} = 3.22 ms^{-1}

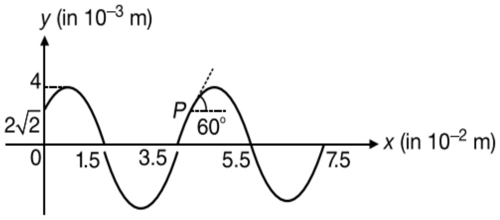
Also, maximum acceleration is $a_{max} = \omega^2 A$

$\Rightarrow \; a_{max} = (570\pi)^2 (1.8 \times 10^{-3})$ ms^{-2}

$\Rightarrow \; a_{max} = 5.8 \times 10^3$ ms^{-2}

PROBLEM 19

The figure shows a snapshot of a vibrating string at $t = 0$. The particle P is observed moving up with velocity $20\sqrt{3}$ cms^{-1}. The tangent at P makes an angle 60° with x-axis. Calculate the



(a) direction in which the wave is moving
(b) equation of the wave
(c) total energy carried by the wave per cycle of the string, assuming that μ, the mass per unit length of the string = 50 gm^{-1}.

SOLUTION

(a) Since, $v_P = -v\left(\dfrac{dy}{dx}\right)$. Also v_p and slope at P both are positive, v must be negative. Hence, wave is moving along negative x-axis.

(b) Since, $y = A\sin(\omega t - kx + \phi)$ ...(1)

and $k = \dfrac{2\pi}{\lambda} = \dfrac{\pi}{2}$ cm^{-1}, $A = 4 \times 10^{-3}$ m = 0.4 cm

At $t = 0$, $x = 0$, $y = 2\sqrt{2} \times 10^{-3}$ m $= 0.2\sqrt{2}$ cm

Substituting in equation (1), we get

$$\phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Also, at $t = 0$, $x = 1.5$ cm, $y = 0$

Substituting in equation (1), we get

$$\phi = \frac{3\pi}{4}$$

Further, $20\sqrt{3} = -v\tan(60°)$

$\Rightarrow \; v = -20$ cms^{-1}

$\Rightarrow \; f = \dfrac{v}{\lambda} = 5$ Hz

$\Rightarrow \; \omega = 2\pi f = 10\pi$

$\Rightarrow \; y = 0.4\sin\left(10\pi t - \dfrac{\pi}{2}x + \dfrac{3\pi}{4}\right)$

(c) Since, $P = 2\pi^2 A^2 f^2 \mu v$

So, energy carried per cycle is

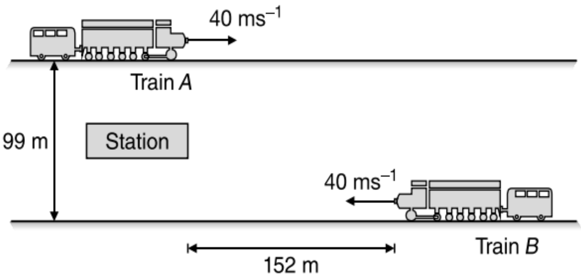
$$E = PT = \frac{P}{f} = 2\pi^2 A^2 f \mu v$$

Substituting the values, we get

$$E = 1.6 \times 10^{-5} \text{ J}$$

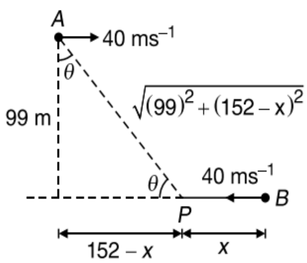
PROBLEM 20

Train A crosses a station with a speed of 40 ms^{-1} and whistles a short pulse of natural frequency 596 Hz. Another train B is approaching towards the same station with the same speed along a parallel track. Two tracks are 99 m apart. When train A whistles, the train B is 152 m away from the station as shown in Figure. If velocity of sound in air is 330 ms^{-1}, calculate the frequency of the pulse heard by driver of train B.



SOLUTION

From Figure if driver of train B receives the sound at point P as shown in Figure.



So, time taken by sound to go from A to P equals the time taken by the train to go from B to P i.e.,

$$\frac{\sqrt{(99)^2 + (152 - x)^2}}{330} = \frac{x}{40}$$

$\Rightarrow \; (4)^2\left[(99)^2 + (152)^2 + x^2 - 304x\right] = (33)^2 x^2$

$\Rightarrow \; 1073x^2 + 4864x - 526480 = 0$

$\Rightarrow \; x = \dfrac{-4864 + \sqrt{(4864)^2 + 4(1073)(526480)}}{2(1073)}$

$\Rightarrow \; x = \dfrac{-4864 + 47784}{2(1073)} = 20$ m

$$\Rightarrow \quad \cos\theta = \frac{132}{\sqrt{(99)^2+(132)^2}} = \frac{132}{165} = \frac{4}{5} = 0.8$$

Apparent frequency is given by

$$f_{app} = 596\left(\frac{330+40\cos\theta}{330-40\cos\theta}\right)$$

$$\Rightarrow \quad f_{app} = 596\left(\frac{330+32}{330-32}\right) = 724 \text{ Hz}$$

PROBLEM 21

A wire of length 2 m is fixed at both ends and is vibrating in its fundamental mode. The tension in the wire is 40 N and the mass of the wire is 0.1 kg. At the midpoint of the wire the amplitude is 2 cm.

(a) Find the maximum kinetic energy of the wire.
(b) At the instant the transverse displacement is given by $(0.02 \text{ m})\sin\left(\frac{\pi x}{2}\right)$, what is the kinetic energy of the wire.
(c) At what position on the wire does the kinetic energy per unit length have its largest value?
(d) Where does the potential energy per unit length have its maximum value?

SOLUTION

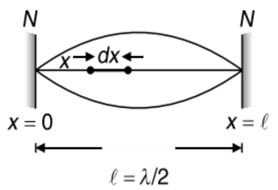
(a) In fundamental mode, $l = \frac{\lambda}{2}$

$$\Rightarrow \quad \lambda = 2l = 4 \text{ m}$$

$$\Rightarrow \quad k = \frac{2\pi}{\lambda} = \frac{\pi}{2} \text{ m}^{-1}$$

Since, $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{40}{0.1/2}} = 28.3 \text{ ms}^{-1}$

$$\Rightarrow \quad \omega = vk = 44.5 \text{ rads}^{-1}$$



At the instant the transverse displacement is given by $R = (0.02 \text{ m})\sin\left(\frac{\pi x}{2}\right)$, we have

$$R = A\sin(kx) = 2\times10^{-2}\sin\left(\frac{\pi x}{2}\right) \text{ m}$$

Maximum kinetic energy of an element dx is

$$dK = \frac{1}{2}(dm)(R^2)(\omega^2), \text{ where } dm = \mu dx$$

So, maximum kinetic energy of the wire is

$$K_{max} = \int dK = \int_0^1 \frac{1}{2}(\mu dx)R^2\omega^2$$

$$\Rightarrow \quad K_{max} = \frac{1}{2}A^2\omega^2\mu\int_0^1 \sin^2\left(\frac{\pi x}{2}\right)dx$$

$$\Rightarrow \quad K_{max} = \frac{1}{4}A^2\omega^2\mu\int_0^1 [1-\cos(\pi x)]dx$$

$$\Rightarrow \quad K_{max} = 9.9\times10^{-3}\int_0^2 [1-\cos(\pi x)]dx$$

$$\Rightarrow \quad K_{max} = 9.9\times10^{-3}\left[x - \frac{1}{\pi}\sin(\pi x)\right]\Bigg|_0^2$$

$$\Rightarrow \quad K_{max} \approx 0.02 \text{ J}$$

(b) At the given instant, we have

$$y = 0.02\sin\left(\frac{\pi x}{2}\right)$$

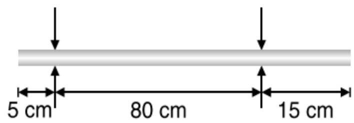
We observe that at, $y = 0$ at $x = 0$ and $y = 0.02$ m at $x = 1$ m i.e., at the centre (or at the position of antinode) $y = 0.02$ m (the amplitude at that position). This implies that all the particles are at their extreme positions. Hence, the kinetic energy of the string is zero.

(c) Kinetic energy per unit length have its largest value at antinode or at $x = 1$ m.

(d) Potential energy per unit length have also its largest value at antinode or at $x = 1$ m

PROBLEM 22

A metal rod of length 1 m is clamped at two points as shown in the figure. Distance of the clamp from the two ends are 5 cm and 15 cm respectively. Find the minimum and next higher frequency of natural longitudinal oscillation of the rod. Given that Young's modulus of elasticity and density of aluminium are $Y = 1.6\times10^{11}$ Nm^{-2} and $\rho = 2500$ kgm^{-3} respectively.



SOLUTION

Speed of longitudinal waves in the rod

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{1.6\times10^{11}}{2500}} = 8000 \text{ ms}^{-1}$$

At the clamped position, nodes will be formed. Between the clamps integral number of loops will be formed. Hence,

$$n_1\left(\frac{\lambda}{2}\right) = 80$$

$$\Rightarrow \quad n_1\lambda = 160 \qquad \ldots(1)$$

Between P and R, P is a fixed end and R is the free end. It means the number of loops between P and R will be an odd multiple of $\frac{\lambda}{4}$. So,

$$(2n_2-1)\frac{\lambda}{4}=5$$

$$\Rightarrow \quad (2n_2-1)\lambda=20 \qquad \ldots(2)$$

Also, between Q and S, we have

$$(2n_3-1)\frac{\lambda}{4}=15$$

$$\Rightarrow \quad (2n_3-1)\lambda=60 \qquad \ldots(3)$$

From Equations (1) and (2), we get

$$\frac{n_1}{2n_2-1}=\frac{160}{20}=8 \qquad \ldots(4)$$

and from Equations (1) and (3), we get

$$\frac{n_1}{2n_3-1}=\frac{160}{60}=\frac{8}{3} \qquad \ldots(5)$$

For minimum frequency n_1, n_2 and n_3 should be least from Equations (4) and (5)

So, we get, $n_1=8$, $n_2=1$, $n_3=2$

$$\Rightarrow \quad \lambda=\frac{20}{2n_2-1}=20 \text{ cm} \qquad \text{\{from Equation (2)\}}$$

$$\Rightarrow \quad \lambda=0.2 \text{ m}$$

$$\Rightarrow \quad f_{\min}=\frac{v}{\lambda}=\frac{8000}{0.2}=40 \text{ kHz}$$

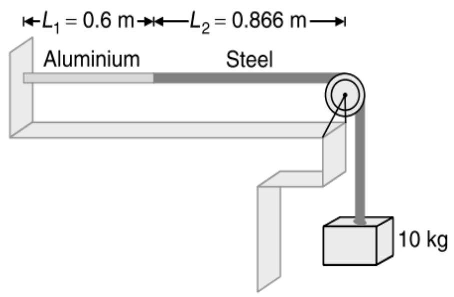
Next higher frequency corresponds to

$$n_1=24,\ n_2=2 \text{ and } n_3=5$$

$$f=120 \text{ kHz}$$

PROBLEM 23

An aluminium wire of length $L=60$ cm and of cross-sectional area 0.01 cm^2 is connected to a steel wire of the same cross-sectional area. The compound wire is loaded with a block of mass 10 kg, as shown in the figure, so that the distance L_2, from the joint to the supporting pulley is 86.6 cm Transverse waves are set up in the composite wire by using an external source of variable frequency. Density of aluminium is 2.6 gcm^{-3} and density of steel is 7.8 gcm^{-3}.



(a) Find the lowest frequency of excitation for which standing waves are observed such that the joint in the wire is a node.

(b) What is total number of nodes observed at this frequency excluding the two at the ends of the wire?

SOLUTION

The distance between two consecutive nodes is equal to $\lambda/2$. The whole wire vibrates with the frequency of the tuning fork. The velocity of the wave in each part of the wire is $\sqrt{T/\mu}$. Since the mass per unit length in both the wires are different, so velocity of the wave in two parts of the wire is also different. Hence the wavelength is different in both parts of the wire (because frequency cannot be different)

Area of cross-section, $A=0.01\text{ cm}^2=10^{-6}\text{ m}^2$

For aluminium wire, $\rho_1=2.6\text{ gcm}^{-3}=2.6\times10^3\text{ kgm}^{-3}$

So, mass per unit length is

$$\mu_1=\rho_1 A=(2.6\times10^3)(10^{-6})=2.6\times10^{-3}\text{ kgm}^{-1}$$

For steel wire, $\rho_2=7.8\text{ gcm}^{-3}=7.8\times10^3\text{ kgm}^{-3}$

So, mass per unit length is

$$\mu_2=\rho_2 A=7.8\times10^3\times10^{-6}=7.8\times10^{-3}\text{ kgm}^{-1}$$

Let the aluminium wire vibrate in n_1 harmonic, and the steel wire in n_2 harmonic. Then

For Aluminium Wire

$$n_1\left(\frac{\lambda_1}{2}\right)=L_1 \text{ i.e., } n_1\lambda_1=2L_1$$

$$\Rightarrow \quad f_1=\frac{v_1}{\lambda_1}=\frac{n_1}{2L_1}\sqrt{\frac{T_1}{\mu_1}}$$

For Steel Wire

$$n_2\left(\frac{\lambda_2}{2}\right)=L_2 \text{ i.e., } n_2\lambda_2=2L_2$$

$$\Rightarrow \quad f_2=\frac{v_2}{\lambda_2}=\frac{n_2}{2L_2}\sqrt{\frac{T_2}{\mu_2}}$$

Since tension $T_1=T_2$ and whole wire vibrates with same frequency, so we have

$$f_1=f_2$$

$$\Rightarrow \frac{n_1}{2L_1}\sqrt{\frac{T}{\mu_1}} = \frac{n_2}{2L_2}\sqrt{\frac{T}{\mu_2}}$$

$$\Rightarrow \frac{n_1}{2\times 0.6}\sqrt{\frac{T}{2.6\times 10^{-3}}} = \frac{n_2}{2\times 86.6}\sqrt{\frac{T}{7.8\times 10^{-3}}}$$

$$\Rightarrow \frac{n_1}{n_2} = \frac{2}{5}$$

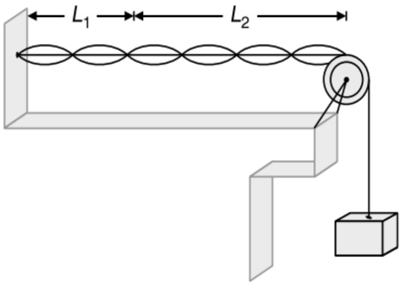
Therefore, the lowest frequency of excitation causes,

$$n_1 = 2 \quad \text{and} \quad n_2 = 5$$

(a) Now, the lowest frequency of excitation is

$$f = \frac{n_1}{2L_1}\sqrt{\frac{T}{\mu_1}} = \frac{2}{2\times 0.6}\sqrt{\frac{10\times 9.81}{2.6\times 10^{-3}}} = 323.74 \text{ Hz}$$

(b) The vibrating looks as shown in the figure



Therefore, total number of nodes observed is 6.

PROBLEM 24

The air column in a pipe closed at one end is made to vibrate in its second overtone by tuning fork of frequency 440 Hz. The speed of sound in air is 330 ms^{-1}. End corrections may be neglected. Let P_0 denote the mean pressure at any point in the pipe and ΔP_0 the maximum amplitude of pressure variation.

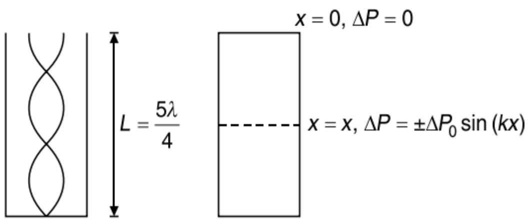
(a) Find the length L of the air column.
(b) What is the amplitude of pressure variation at the middle of the column?
(c) What are the maximum and minimum pressures at the open end of the pipe?
(d) What are the maximum and minimum pressures at the closed end of the pipe?

SOLUTION

(a) Frequency of second overtone of the closed pipe

$$f_5 = 5\left(\frac{v}{4L}\right) = 440 \text{ Hz} \qquad \{\text{Given}\}$$

$$\Rightarrow L = \frac{5v}{4\times 440} \text{ m}$$



Substituting v = speed of sound in air = 330 ms^{-1}

$$\Rightarrow L = \frac{5\times 330}{4\times 440} = \frac{15}{16} \text{ m}$$

$$\Rightarrow \lambda = \frac{4L}{5} = \frac{4\left(\frac{15}{16}\right)}{5} = \frac{3}{4} \text{ m}$$

(b) Since, open end is a displacement antinode, therefore, it would be a pressure node. So, at $x = 0$; $\Delta P = 0$
Hence, pressure amplitude at $x = x$ can be written as

$$\Delta P = \pm\Delta P \sin(kx)$$

where $k = \frac{2\pi}{\lambda} = \frac{2\pi}{3/4} = \frac{8\pi}{3} \text{ m}^{-1}$

Therefore, pressure amplitude at

$$x = \frac{L}{2} = \frac{\frac{15}{16}}{2} \text{ m} = \frac{15}{32} \text{ m will be given by}$$

$$\Delta P = \pm\Delta P_0 \sin\left(\frac{8\pi}{3}\right)\left(\frac{15}{32}\right) = \pm\Delta P_0 \sin\left(\frac{5\pi}{4}\right)$$

$$\Rightarrow \Delta P = \pm\frac{\Delta P_0}{\sqrt{2}}$$

(c) Open end is a pressure node, i.e., $\Delta P = 0$

Hence, $P_{max} = P_{min} = \text{Mean pressure}(P_0)$

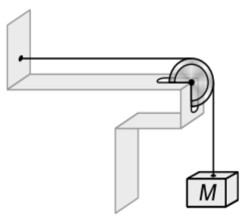
(d) Closed end is a displacement node or pressure anti-node, so we have

$$P_{max} = P_0 + \Delta P_0$$

$$P_{min} = P_0 - \Delta P_0$$

PROBLEM 25

A string of length 1 m fixed at one end and on the other end a block of mass $M = 4$ kg is suspended. The string is set into vibrations and represented by equation $y = 6\sin\left(\frac{\pi x}{10}\right)\cdot\cos 100\pi t$ where x and y are in cm and t in seconds.



(a) Find the number of loops formed in the string.
(b) Find the maximum displacement of a point at $x = \frac{5}{3}$ cm.
(c) Calculate maximum kinetic energy of the string.
(d) Write down the equations of the component waves whose superposition gives the wave.

SOLUTION

(a) Comparing the given equation with

$y = A\sin(kx)\cos(\omega t)$, we get

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{10}\text{ cm}^{-1} = 10\pi(\text{m}^{-1})$$

$$\lambda = 20 \text{ cm} = 0.2 \text{ m}$$

$$\Rightarrow \frac{\lambda}{2} = 10 \text{ cm}$$

Number of loops is $p = \dfrac{l}{\frac{\lambda}{2}} = 10$

(b) $A(x) = 6\sin\left(\dfrac{\pi x}{10}\right)$

Maximum displacement at $x = \dfrac{5}{3}$ cm is

$$A(x) = 6\sin\left[\left(\frac{\pi}{10}\right)\left(\frac{5}{3}\right)\right] = 3 \text{ cm}$$

(c) Considering an elemental length dx of string at a distance x from the left end. Maximum kinetic energy of this element is given by

$$dK = \frac{1}{2}(\mu dx)\left(\frac{A}{2}\right)(\omega^2)$$

$$\Rightarrow dK = \frac{1}{2}(\mu dx)(A\sin kx)^2\omega^2$$

$$\Rightarrow dK = \frac{1}{2}(\mu A^2\sin^2 kx)\left(\frac{T}{\mu}k^2\right)dx \qquad \left\{\because \omega^2 = \frac{T}{\mu}k^2\right\}$$

$$\Rightarrow dK = \frac{1}{4}TA^2k^2(1-\cos 2kx)dx$$

So, total kinetic energy of the string is

$$K = \int_0^l dK$$

$$\Rightarrow K = \frac{1}{4}TA^2k^2\int_0^l(1-\cos(2kx))dx$$

$$\Rightarrow K = \frac{1}{4}TA^2k^2\left(x - \frac{\sin 2kx}{2}\right)\Bigg|_0^l$$

$$\Rightarrow K = \frac{TA^2k^2l}{4}$$

Substituting the values, we get

$$K = \frac{(4\times10)(6\times10^{-2})^2(10\pi)^2(1)}{4} = 142.12 \text{ J}$$

(d) The equations of component waves are given by

$$y_1 = 3\sin\left(\frac{\pi x}{10} - 100\pi t\right)$$

and $y_2 = 3\sin\left(\dfrac{\pi x}{10} + 100\pi t\right)$

PROBLEM 26

The linear mass density of a non-uniform wire under constant tension decreases gradually along the wire so that an incident wave is transmitted without reflection. The wire is uniform for $-\infty < x \le 0$. In this region, a transverse wave has the form $y(x, t) = 0.003\cos(25x - 50t)$ where y and x are in metres and t is in seconds. From $x = 0$ to $x = 20$ m the linear mass density decreases gradually from μ_1 to $\mu_1/4$. For $20 < x \le \infty$, the linear mass density is $\mu = \mu_1/4$. Calculate

(a) the wave velocity for large value of x.
(b) amplitude of the wave for large value of x.
(c) $y(x, t)$ for $x > 20$ m.

SOLUTION

(a) Speed of wave for $x = 0$ is, $\dfrac{\omega}{k} = \dfrac{50}{25} = 2 \text{ ms}^{-1}$

Since, $v = \sqrt{\dfrac{T}{\mu}}$ i.e., $v \propto \dfrac{1}{\sqrt{\mu}}$

For $x > 20$ m, μ becomes $\dfrac{1}{4}$th its value at $x = 0$. Hence, speed of wave will become two times.
So, for $x > 20$ m, speed of wave will become 4 ms^{-1}.

(b) Equating the average power at $x = 0$ and for $x > 20$ m, we get

$$\frac{1}{2}\mu_1\omega^2A_1^2v_1 = \frac{1}{2}\left(\frac{\mu_1}{4}\right)\omega^2A_2^2v_2$$

$$\Rightarrow A_2 = 2A_1\sqrt{\frac{v_1}{v_2}} = 2\times0.003\sqrt{\frac{2}{4}}$$

$$\Rightarrow A_2 = 0.0042 \text{ m}$$

(c) For $x > 20$ m, ω will remain same while k will become half so that speed becomes two times, because $v = \omega/k$. So, for $x > 20$ m, we have $A = 0.0042$ m,

$k = \dfrac{25}{2} = 12.5 \text{ m}^{-1}$, $\omega = 50 \text{ rads}^{-1}$.

$$\Rightarrow y(x, t) = 0.0042\cos(12.5x - 50t)$$