CHAPTER 3

Simple Harmonic Motion

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Dynamics of SHM, Phase Difference
(b) Differential Equation for SHM, Condition for Motion to be SHM
(c) Energy in SHM
(d) Spring Mass Systems
(e) Rotational SHM
(f) Simple Pendulum
(g) Compound Pendulum
(h) SHM in other Physical Systems
(i) Composition of SHM
(j) Damped Oscillations
(k) Forced Oscillations & Resonance

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

In this chapter, we shall be studying a special type of periodic motion called Simple Harmonic Motion (SHM). This is a repeating motion of an object in which the object continues to observe to and fro motion about a mean position at fixed time interval (under ideal situations). However, if the time interval is not fixed, then the motion may be called as Oscillatory. The back and forth movements of such an object are called oscillations. We will focus our attention on a special case of periodic motion called simple harmonic motion. It is observed that all periodic motions can be modelled as combinations of simple harmonic motions and hence SHM forms a basic building block for more complicated periodic motion.

PERIODIC MOTION

When a body or a moving particle repeats its motion along a definite path after regular intervals of time, its motion is said to be **Periodic Motion** and interval of time is called **time period** or harmonic motion period T and its reciprocal is called the frequency ν i.e., $\nu = \frac{1}{T}$. The path of periodic motion may be linear, circular, elliptical or any other curve. For example, revolution of earth about the sun, rotation of earth about its own axis.

Mathematically, if any function of time $f(t)$ can be expressed as $f(t+T) = f(t)$, then that function can be regarded as periodic with period T.

OSCILLATORY MOTION

An oscillatory motion need not be periodic and need not have fixed extreme positions. For example, motion of pendulum of a wall clock (because the battery of the wall clock wears out with time).

The oscillatory motions in which energy is conserved can also be called as periodic. Oscillations in which energy is consumed due to some resistive forces and hence total mechanical energy decreases are called as **Damped oscillations.**

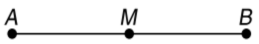
The force/torque (directed towards equilibrium point) acting in oscillatory motion is called restoring force/torque.

SIMPLE HARMONIC MOTION (SHM)

SHM is a special type of oscillatory motion in which the restoring force is proportional to the displacement from the mean position (for small displacement from mean position). It is the simplest (easy to analyse) form of oscillatory motion. The amplitude of oscillations of the particle is very small. At any instant the displacement of a particle in oscillatory motion can be expressed in terms of sinusoidal functions (sine and cosine functions). These functions are called **harmonic functions**. That is why an oscillatory motion is also called a **harmonic motion**.

A simple harmonic motion can be expressed in terms of one single sine or cosine function or a linear combination of sine and cosine functions.

If a particle is moving to and fro about an equilibrium point M (called as the mean position), along a straight line as shown in Figure, then we can call its motion to be SHM, where A and B are extreme positions and $AM = MB =$ Amplitude (a)



However, when a body or a particle is free to rotate about a given axis executing angular oscillations, then this type of SHM can be regarded as angular SHM.

Problem Solving Technique(s)

(a) All Simple Harmonic Motions (SHMs) are periodic but all periodic motions may or may not be an SHM.

(b) All SHMs are oscillatory motions but all oscillatory motions may or may not be SHM.

EQUILIBRIUM POSITION OR MEAN POSITION

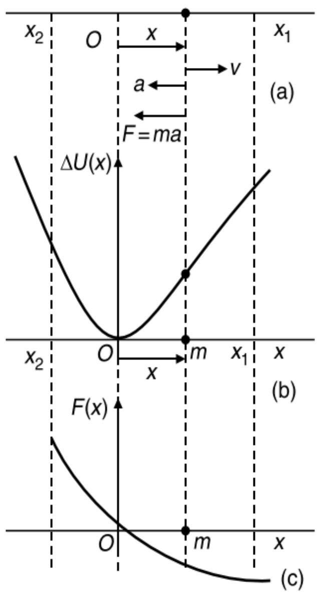
In mechanical oscillations a body oscillates about its mean position which is also its **equilibrium position**. At the equilibrium position no net force (or torque) acts on the oscillating body. The displacement (linear or angular) of an oscillating particle is its distance (linear or angular) from the equilibrium position at any instant.

When a body oscillates along a straight line within two fixed limits, its displacement x changes periodically in both magnitude and direction, its velocity v and acceleration $a = \ddot{x}$ also varies periodically in magnitude and direction. Since $F = ma = m\ddot{x}$, therefore, the force acting on the body also varies in magnitude and direction with time.

In terms of energy, we can say that a particle executing harmonic motion moves back and forth about a point at which the potential energy is minimum (equilibrium position). The force acting on the body at any position is given by

$$F = -\frac{dU}{dx}$$

When a body is displaced from its equilibrium it is acted upon by a restoring force (or torque) which always acts to accelerate the body in the direction of its equilibrium position as shown in Figure.



(a) A body of mass m oscillates harmonically between points x_1 and x_2 about the equilibrium position O.

(b) The potential energy of the body as a function of position. The force acting on the body is

$$F = -\frac{dU}{dx}$$

(c) The force acting on the body as a function of position x. Note that the force is always directed toward the equilibrium position.

DYNAMICS OF SIMPLE HARMONIC MOTION

The restoring force for small displacement x is given by $F = -kx$, where k is called the **force constant of the system**. Force constant is defined as the restoring force per unit displacement. Its SI unit is Nm^{-1}. If m is mass of the body and a is its acceleration, then for simple harmonic motion

$$F = ma = -k\ddot{x}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

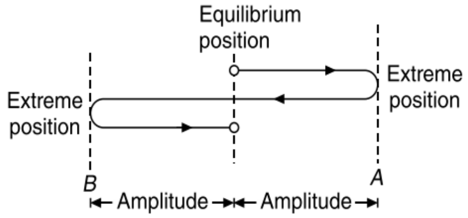
$$\Rightarrow \quad \ddot{x} + \omega^2 x = 0, \text{ where } \omega = \sqrt{k/m}$$

This is the differential equation of SHM.

The general solution to this equation is

$$x = a\sin(\omega t + \phi)$$

where a is the amplitude, $\omega t+\phi$ is the phase (also called as instantaneous phase), ϕ is phase constant or initial phase angle (in radian) also called as epoch.



Consider a particle to execute SHM and assume that the path of the particle is on a straight line, then x is the displacement of particle from the mean position at that instant. The displacement of a particle executing SHM at time t is given by

$$x=a\sin(\omega t+\phi)$$

Also, note that the displacement of a particle executing SHM at time t can be given by

$$y=a\sin(\omega t+\phi)$$

Amplitude, a or A, is the maximum value of displacement of the particle from its equilibrium position i.e., mean position. So, amplitude of SHM is half the separation between the extreme points of SHM. It depends upon the energy of the system.

Angular frequency, ω of SHM is $\omega=\frac{2\pi}{T}=2\pi f$ and its SI unit is rads^{-1}.

Frequency, f or ν is the number of oscillations completed in unit time interval, so we have $f=\nu=\frac{1}{T}=\frac{\omega}{2\pi}$. Its unit is sec^{-1} or Hz.

Time period, T is the smallest time interval after which oscillatory motion gets repeated, so time period T given by

$$T=\frac{1}{\nu}=\frac{1}{f}=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{k}}$$

Mathematically at $t=T$, the displacement x of SHM must satisfy $x(t+T)=x(t)$. For example,

$$\sin\left[\omega\left(t+\frac{2\pi}{\omega}\right)\right]=\sin(\omega t) \text{ and}$$

$$\cos\left[\omega\left(t+\frac{2\pi}{\omega}\right)\right]=\cos(\omega t)$$

ILLUSTRATION 1

Find the period of the function, $y=\sin(\omega t)+\sin(2\omega t)+\sin(3\omega t)$.

SOLUTION

The given function can be written as

$$y=y_1+y_2+y_3$$

where, $y_1=\sin(\omega t)$, $T_1=\frac{2\pi}{\omega}=T$

$$y_2=\sin(2\omega t),\ T_2=\frac{2\pi}{2\omega}=\frac{\pi}{\omega}=\frac{T}{2} \text{ and}$$

$$y_3=\sin(3\omega t),\ T_3=\frac{2\pi}{3\omega}=\frac{T}{3}$$

We see that, $T_1=2T_2$ and $T_1=3T_3$

So, time period of the given function is $T_1=T=\frac{2\pi}{\omega}$, because in time $T=\frac{2\pi}{\omega}$, first function completes one oscillation, the second function two oscillations and the third function completes three oscillations.

ILLUSTRATION 2

Calculate the amplitude and initial phase of a particle in SHM, whose motion equation is given as

$$x=a\sin\omega t+b\cos\omega t$$

SOLUTION

Since $x=a\sin(\omega t)+b\cos(\omega t)$ ...(1)

In this given equation, we can write

$$a=A\cos\phi \quad ...(2)$$

and $b=A\sin\phi$ ...(3)

So, the given equation (1) transforms to

$$x=A\sin(\omega t+\phi) \quad ...(4)$$

Equation (4) is a general equation of SHM having amplitude A and ϕ is the initial phase of the oscillating particle at $t=0$.

Squaring and adding equations (2) and (3), we get amplitude as

$$A=\sqrt{a^2+b^2}$$

Dividing (3) by (2), we get initial phase as

$$\tan\phi=\frac{b}{a}$$

$$\Rightarrow \quad \phi=\tan^{-1}\left(\frac{b}{a}\right)$$

PHASE AND PHASE DIFFERENCE

Phase is the physical quantity which represents the state of motion of particle (e.g., its position and direction of motion at any instant).

The argument $(\omega t+\phi)$ of sinusoidal function is called instantaneous phase of the motion. It gives the position and direction of motion at any instant.

The constant ϕ in equation of SHM is called phase constant or initial phase or epoch. It depends on initial position and direction of velocity.

If $\phi=0$, then the body is initially at the mean position or starts from the mean position, then we have $x=a\sin\omega t$, because at $t=0$, $x=0$.

Similarly, if $\phi = \frac{\pi}{2}$, then the body is initially at the positive extreme position or starts from the positive extreme, then we have $x = a\cos\omega t$, because at $t = 0$, $x = a$.

Also, note that there is a phase difference of $\frac{\pi}{2}$ between the sine and the cosine forms.

If two particles perform SHM and their equations are

$$y_1 = a\sin(\omega t + \phi_1) \text{ and } y_2 = a\sin(\omega t + \phi_2)$$

then, phase difference is given by

$$\Delta\phi = (\omega t + \phi_2) - (\omega t + \phi_1) = \phi_2 - \phi_1$$

The phase changes with time as

$$\Delta\phi = \omega\Delta t = 2\pi f\Delta t = \frac{2\pi}{T}\Delta t$$

We may also define the period T as that time in which the phase changes by 2π.

CASE-I: Same phase or In Phase

Two vibrating particles are said to be in same phase, if the phase difference between them is an even multiple of π or time interval is an even multiple of $\frac{T}{2}$ because 1 time period is equivalent to 2π rad.

CASE-II: Opposite phase or Out of Phase

When the two vibrating particles cross their respective mean positions at the same time moving in opposite directions, then the phase difference between the two vibrating particles is $180°$.

Opposite phase means the phase difference between the particle is an odd multiple of π (i.e., $\pi, 3\pi, 5\pi, 7\pi \ldots$) or the time interval is an odd multiple of $\frac{T}{2}$.

ILLUSTRATION 3

The position of a particle moving along x-axis is given by $x = 0.8\sin(12t + 0.3)$ m, where t is in second. Calculate the amplitude and the period of the motion. Also, find the phase, position, velocity and acceleration at $t = 0.6$ s.

SOLUTION

On comparing the given equation with the standard equation of SHM i.e., $x = A\sin(\omega t + \phi)$, we see that the amplitude is $A = 0.8$ m and the angular frequency $\omega = 12$ rads^{-1}.

$$\Rightarrow \quad T = \frac{2\pi}{\omega} = 0.524 \text{ s.}$$

Velocity and acceleration at any time are given by

$$v = \frac{dx}{dt} = 0.96\cos(12t + 0.3) \text{ ms}^{-1}$$

$$a = \frac{dv}{dt} = -11.5\sin(12t + 0.3) \text{ ms}^{-2}$$

At $t = 0.6$ s, the phase of motion is

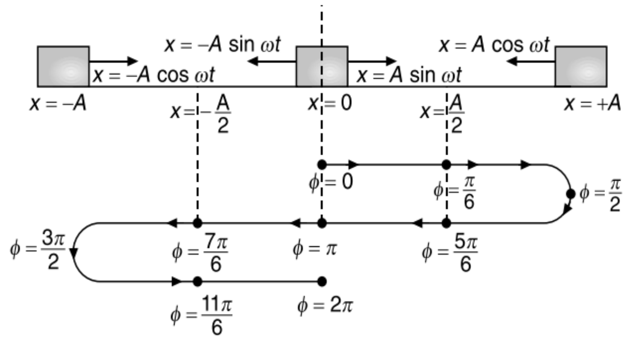
$$\omega t + \phi = (12)(0.6) + 0.3 = 7.5 \text{ rad.}$$

When this is used in the above expressions, we get

$$x = 0.75 \text{ m}, v = 0.333 \text{ ms}^{-1} \text{ and } a = -10.8 \text{ ms}^{-2}$$

EQUATION OF SHM AND PHASE

Consider the equation, $x = A\sin(\omega t + \phi)$, where ϕ is initial phase.



For example, phase difference between two particles at $\frac{A}{2}$ from mean position moving opposite to each other is $\Delta\phi = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$.

Also, we note that

(a) a particle executing SHM takes $\frac{T}{6}$ s to move from extreme position to half way between extreme and mean position.

(b) a particle executing SHM takes $\frac{T}{12}$ s to move from mean position to midway between mean and extreme position.

(c) when two particles executing SHM with time periods T_1 and $T_2 (< T_1)$ start at the same time, then the particles will be in phase after n oscillations of T_2 and $(n-1)$ oscillations of T_1, if

$$nT_2 = (n-1)T_1$$

ILLUSTRATION 4

Two particles move parallel to the x-axis about the origin with the same frequency and amplitude a. At a certain instant they are found at distances $\frac{a}{3}$ from the origin on opposite sides but their velocity is found to be in the same direction. Calculate the phase different between the two particles.

SOLUTION

Let $x_1 = a\sin\omega t$ and $x_2 = a\sin(\omega t + \phi)$ be two SHMs.

$$\Rightarrow \quad \frac{a}{3} = a\sin\omega t \text{ and } -\frac{a}{3} = a\sin(\omega t + \phi)$$

$$\Rightarrow \quad \sin\omega t = \frac{1}{3} \text{ and } \sin(\omega t + \phi) = -\frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{3}\cos\phi + \sqrt{1 - \frac{1}{9}}\sin\phi = -\frac{1}{3}$$

$$\Rightarrow \quad 9\cos^2\phi + 2\cos\phi - 7 = 0$$

$$\Rightarrow \quad \cos\phi = \frac{-2 \pm \sqrt{4+4\times7\times9}}{18} = -1, \ \frac{7}{9}$$

$$\Rightarrow \quad \delta = 180^\circ \text{ or } \delta = \cos^{-1}\left(\frac{7}{9}\right)$$

Now $v_1 = a\omega\cos\omega t$, $v_2 = a\omega\cos(\omega + \phi)$. When we substitute $\phi = 180^\circ$ we find that v_1 and v_2 are of opposite signs. Hence $\phi = 180^\circ$ is not acceptable.

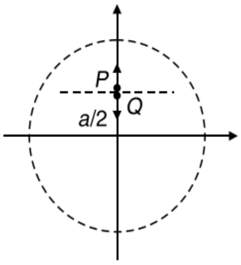
$$\Rightarrow \quad \phi = \cos^{-1}\left(\frac{7}{9}\right)$$

ILLUSTRATION 5

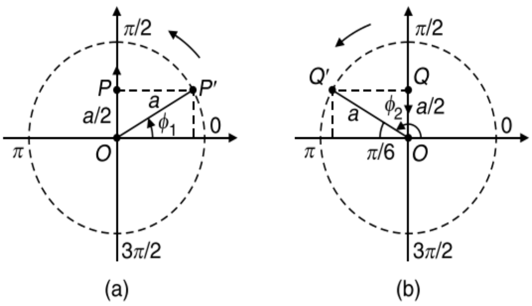
Two particles execute SHM with same amplitude a and same angular frequency ω on same straight line with same mean position. During oscillation, if they cross each while going in opposite direction when at a distance $a/2$ from mean position, then calculate the phase difference between the two SHMs.

SOLUTION

Let particles P (moving up) and Q (moving down) cross each other at $\frac{a}{2}$ from mean position as shown in Figure.



These two respective particles P and Q in SHM along with their corresponding particles P' and Q' in circular motion are shown separately in the two Figures below.



At this instant, phase of P is

$$\phi_1 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Similarly, for particle Q, phase is

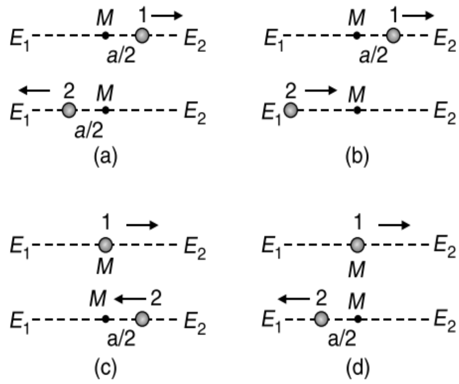
$$\phi_2 = \pi - \sin^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Since both are oscillating at same angular frequency, so their phase difference remains constant and is given by

$$\Delta\phi = \phi_2 - \phi_1 = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$$

ILLUSTRATION 6

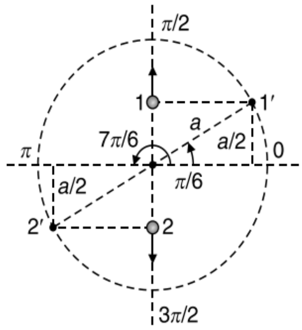
Calculate the phase difference between two particles 1 and 2 executing simple harmonic motion with the same frequency if they are found in the states shown in Figure at four different points of time.



SOLUTION

(a) For particle 1, $\phi_1 = \sin^{-1}\left(\frac{a/2}{a}\right) = \frac{\pi}{6}$

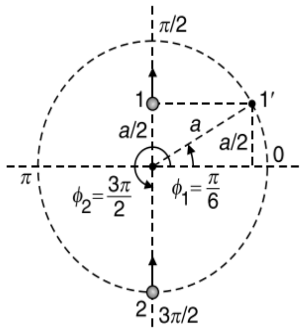
For particle 2, $\phi_2 = \pi + \sin^{-1}\left(\frac{a/2}{a}\right) = \frac{7\pi}{6}$



$$\Rightarrow \quad \Delta\phi_1 = \phi_2 - \phi_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi \text{ radian}$$

(b) For particle 1, $\phi_1 = \sin^{-1}\left(\frac{a/2}{a}\right) = \frac{\pi}{6}$

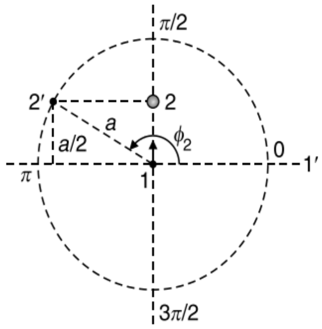
For particle 2, $\phi_2 = \sin^{-1}\left(\frac{-a}{a}\right) = \frac{3\pi}{2}$



$$\Rightarrow \quad \Delta\phi = \phi_2 - \phi_1 = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{4\pi}{3}$$

(c) For particle 1, $\phi_1 = 0$

For particle 2, $\phi_2 = \pi - \sin^{-1}\left(\frac{a/2}{a}\right)$

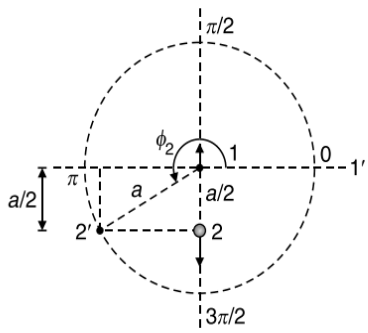


$$\Rightarrow \quad \phi_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\Rightarrow \quad \Delta\phi = \frac{5\pi}{6}$$

(d) For particle 1, $\phi_1 = 0$

For particle 2, $\phi_2 = \pi + \sin^{-1}\left(\frac{a/2}{a}\right) = \frac{7\pi}{6}$



From figure $\phi = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

ILLUSTRATION 7

Two particles are executing SHM of same amplitude and frequency along the same straight line. They pass one another when going in opposite directions, each time their displacement is half of their amplitude. Calculate the phase difference between the particles.

SOLUTION

Let two simple harmonic motions are $x = A\sin(\omega t)$ and $x = A\sin(\omega t + \phi)$

In the first case, we have $\frac{A}{2} = A\sin(\omega t)$

$$\Rightarrow \quad \sin(\omega t) = \frac{1}{2} \text{ OR } \cos(\omega t) = \frac{\sqrt{3}}{2}$$

In the second case, we have

$$\frac{A}{2} = A\sin(\omega t + \phi)$$

$$\Rightarrow \quad \frac{1}{2} = \sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi$$

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{2}\cos\phi + \frac{\sqrt{3}}{2}\sin\phi$$

$$\Rightarrow \quad 1 - \cos\phi = \sqrt{3}\sin\phi$$

$$\Rightarrow \quad (1-\cos\phi)^2 = 3\sin^2\phi$$

$$\Rightarrow \quad (1-\cos\phi)^2 = 3(1-\cos^2\phi)$$

$$\Rightarrow \quad (1-\cos\phi)^2 = 3(1+\cos\phi)(1-\cos\phi)$$

$$\Rightarrow \quad (1-\cos\phi) = 3(1+\cos\phi)$$

$$\Rightarrow \quad 4\cos\phi = -2$$

$$\Rightarrow \quad \cos\phi = -\frac{1}{2}$$

$$\Rightarrow \quad \phi = 120^\circ = \frac{2\pi}{3} \text{ radian}$$

DIFFERENTIAL EQUATION FOR SHM

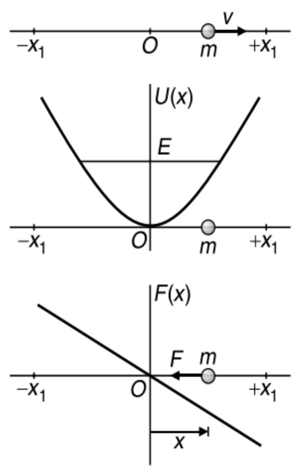
Let us consider a mass m attached to a spring of force constant k oscillating along a straight line. The potential energy of this mass varies with x as

$$U(x) = \frac{1}{2}kx^2 \qquad \ldots(1)$$

The force F acting on the particle is given by

$$F = -\frac{dU}{dx} = -\frac{k}{2}\frac{d}{dx}(x^2) = -kx \qquad \ldots(2)$$

Such an oscillatory motion in which restoring force acting on the particle is directly proportional to the small displacement x from the equilibrium position is called Simple Harmonic Motion. The potential energy function of such a particle is represented by a symmetric curve as shown in the Figure.



Note that the limits of oscillation are equally spaced about the equilibrium position. Applying Newton's Second Law in equation (2), we get

$$F = ma = -kx$$

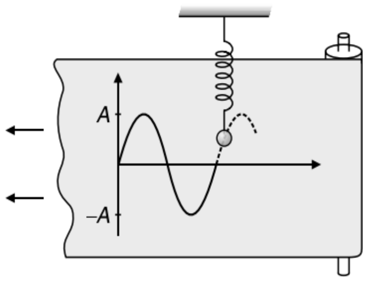
Since acceleration is $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0 \qquad \ldots(3)$$

This equation (3) is called the **characteristic differential equation of SHM**. It gives a relation between a function of the time $x(t)$ and its second derivative $\frac{d^2x}{dt^2}$ i.e., acceleration $(= \ddot{x})$. To find the position of the particle as a function of the time, we must find a function $x(t)$ which satisfies this relation.

It is obvious from a simple experiment in which an oscillating particle traces a sinusoidal curve on a moving strip of paper as shown in Figure.



In general, the equation of a simple harmonic motion may be represented by any of the following functions

$$x = A\sin(\omega t + \phi)$$

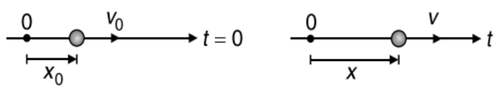
$$x = A\cos(\omega t + \phi)$$

$$x = A\sin(\omega t) + B\cos(\omega t)$$

All the above three equations are the solution to the differential equation $\ddot{x} + \omega^2 x = 0$

EQUATION OF MOTION OF A SIMPLE HARMONIC MOTION

Let us consider a particle of mass m (moving along the x direction) on which a force $F = -kx$ acts, where k is a positive constant and x is the displacement of the particle from the assumed origin as shown in Figure.



The particle then executes SHM with the centre of oscillation i.e., mean position at the origin. We wish to calculate the displacement x and the velocity v of the particle as function of time. Initially, let position of particle be x_0 and its velocity be v_0, so at $t = 0$, we have $x = x_0$ and $v = v_0$. The acceleration a of the particle at any instant is given by

$$a = \frac{F}{m} = -\left(\frac{k}{m}\right)x = -\omega^2 x, \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \quad \frac{dv}{dt} = v\frac{dv}{dx} = -\omega^2 x \qquad \left\{\because \frac{dv}{dt} = v\frac{dv}{dx}\right\}$$

$$\Rightarrow \quad vdv = -\omega^2 xdx$$

Integrating within appropriate limits, we get

$$\int_{v_0}^{v} vdv = \int_{x_0}^{x} -\omega^2 xdx$$

$$\Rightarrow \quad \left(\frac{v^2}{2}\right)\Bigg|_{v_0}^{v} = -\omega^2\left(\frac{x^2}{2}\right)\Bigg|_{x_0}^{x}$$

$$\Rightarrow \quad v^2 - v_0^2 = -\omega^2\left(x^2 - x_0^2\right)$$

$$\Rightarrow \quad v^2 = v_0^2 + \omega^2 x_0^2 - \omega^2 x^2$$

$$\Rightarrow \quad v = \sqrt{\left(v_0^2 + \omega^2 x_0^2\right) - \omega^2 x^2}$$

$$\Rightarrow \quad v = \omega\sqrt{\left(\frac{v_0^2}{\omega^2} + x_0^2\right) - x^2}$$

Since v_0, x_0, ω are constants, so we can write

$$\left(\frac{v_0}{\omega}\right)^2 + x_0^2 = A^2 \qquad \ldots(1)$$

The above equation becomes

$$v = \omega\sqrt{A^2 - x^2} \qquad \ldots(2)$$

$$\Rightarrow \quad \frac{dx}{dt} = \omega\sqrt{A^2 - x^2} \qquad \left\{\because v = \frac{dx}{dt}\right\}$$

$$\Rightarrow \quad \frac{dx}{\sqrt{A^2 - x^2}} = \omega dt \qquad \ldots(3)$$

Since, the displacement is x_0 initially and at time t the displacement is x, so on integrating equation (3), we get

$$\int_{x_0}^{x} \frac{dx}{\sqrt{A^2 - x^2}} = \int_{0}^{t} \omega dt$$

Since $\int \frac{dx}{\sqrt{A^2 - x^2}} = \sin^{-1}\left(\frac{x}{A}\right)$

$$\Rightarrow \quad \left[\sin^{-1}\left(\frac{x}{A}\right)\right]\Bigg|_{x_0}^{x} = \omega t\Big|_{0}^{t}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{A}\right) - \sin^{-1}\left(\frac{x_0}{A}\right) = \omega t \qquad \ldots(4)$$

Writing $\sin^{-1}\left(\frac{x_0}{A}\right) = \delta$, equation (4) becomes

$$\sin^{-1}\left(\frac{x}{A}\right) = \omega t + \delta$$

$$\Rightarrow \quad x = A\sin(\omega t+\delta) \quad \ldots(5)$$

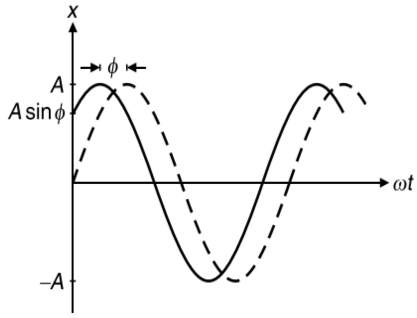
The velocity at time t is

$$v = \frac{dx}{dt} = A\omega\cos(\omega t+\delta) \quad \ldots(6)$$

In equations (5) and (6), δ is the initial phase (can also be represented by ϕ_0) and $\omega t+\delta=\phi$ is called instantaneous phase.

CHARACTERISTICS OF SHM

In equation $x = A\sin(\omega t+\phi)$, the argument $(\omega t+\phi)$ is called the **phase**, where ϕ is called the phase constant. Both the phase and the phase constant are measured in radians. The value of ϕ depends on the position wherefrom we start measuring time.



The characteristic differential equation i.e., $\ddot{x}+\omega^2x=0$ of SHM does not represent one single motion but a group or family of possible motions which have some features in common but differ in other ways. In this case ω is common to all the allowed motions, but A and ϕ may differ among them. The amplitude A and the phase constant ϕ of the oscillation are determined by the initial position and speed of the particle. These two initial conditions will specify A and ϕ exactly. One very important distinctive feature of SHM is the relation between the displacement, velocity and acceleration of oscillatory particle.

$$x = A\sin(\omega t)$$

$$v = \frac{dx}{dt} = \omega A\cos(\omega t) = \omega A\sin\left(\omega t+\frac{\pi}{2}\right)$$

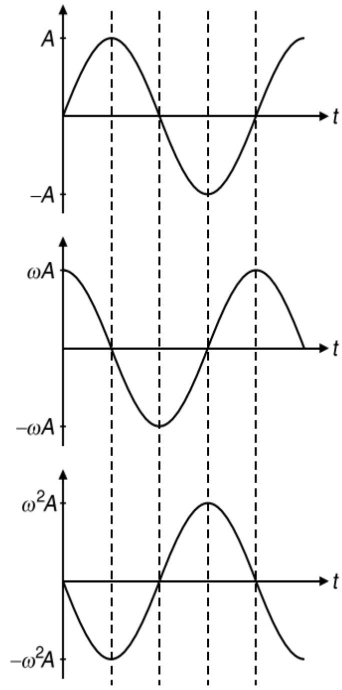
$$a = \frac{d^2x}{dt^2} = -\omega^2A\sin\omega t = \omega^2A\sin(\omega t+\pi)$$

Substituting value of x in above equation, we get

$$\frac{d^2x}{dt^2} = -\omega^2x$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}+\omega^2x=0$$

This is the standard characteristic differential equation of SHM. Notice that the maximum displacement is A, the maximum speed is A and the maximum acceleration is ω^2A. The displacement, velocity and acceleration versus time graphs have been plotted in Figure.



The velocity function is $\frac{\pi}{2}$ ahead of the displacement function and the acceleration function is $\frac{\pi}{2}$ ahead of the velocity function.

CONDITION FOR MOTION TO BE SHM

For SHM is to occur, following three conditions must be satisfied.

(a) There must be a position of **stable equilibrium** also called as **MEAN POSITION.**
At the stable equilibrium potential energy is minimum i.e., $\frac{dU}{dx}=0$ and $\frac{d^2U}{dx^2}>0$

(b) There must be **no dissipation** of energy

(c) The acceleration is proportional to the displacement and opposite in direction, i.e.

$$a = -\omega^2x$$

ILLUSTRATION 8

If a particle moves in a potential energy field $U = U_0 - ax + bx^2$, where a and b are positive constants, obtained an expression for the force acting on it as a function of position. At what point does the force vanish? Is this a point of stable equilibrium? Also Calculate the force constant and frequency of the particle.

SOLUTION

Since, the force and the potential energy are related to each other by the relation

$$F = -\frac{dU}{dx} = a - 2bx$$

$\Rightarrow \quad F = 0$, at $x = \frac{a}{2b}$

$$\Rightarrow \quad \frac{d^2U}{dx^2} = 2b > 0$$

i.e., $x = \frac{a}{2b}$ is a point of minimum potential energy. Hence, the equilibrium is stable. So, $k = 2b$ and

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2b}{m}}$$

VELOCITY OF A PARTICLE IN SHM

Let, $x = A\sin(\omega t)$, then velocity of a particle is

$$v = \frac{dy}{dt} = \frac{d}{dt}[A\sin(\omega t)]$$

$$\Rightarrow \quad v = A\omega\cos(\omega t)$$

$$\Rightarrow \quad v = v_0\cos(\omega t)$$

where, the velocity amplitude is

$$v_0 = A\omega$$

Also, $v = A\omega\cos(\omega t)$

$$\Rightarrow \quad v = A\omega\sqrt{1-\sin^2(\omega t)}$$

$$\Rightarrow \quad v = A\omega\sqrt{1-\frac{x^2}{A^2}} = \omega\sqrt{A^2 - x^2}$$

$$\Rightarrow \quad v^2 = \omega^2A^2 - \omega^2x^2$$

$$\Rightarrow \quad v^2 + \omega^2x^2 = \omega^2A^2$$

$$\Rightarrow \quad \frac{v^2}{\omega^2A^2} + \frac{x^2}{A^2} = 1$$

which is the equation of an ellipse.

Conceptual Note(s)

Velocity vs Displacement curve must be an ellipse

ILLUSTRATION 9

A point moves along the x-axis according to the law $x = a\sin^2\left(\omega t - \frac{\pi}{4}\right)$. Calculate the amplitude and period of oscillations and draw the plot x vs t. Also calculate the velocity projection v_x as a function of the coordinate x and draw the plot v_x vs x.

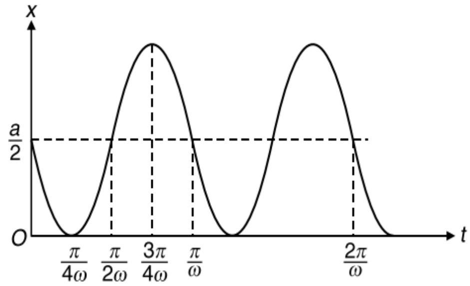
SOLUTION

Given equation is $x = a\sin^2\left(\omega t - \frac{\pi}{4}\right)$

Since $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\Rightarrow \quad x = \frac{a}{2} - \frac{a}{2}\cos\left(2\omega t - \frac{\pi}{2}\right) \quad \ldots(1)$$

From equation (1) the amplitude of SHM is $\frac{a}{2}$, time period is $\frac{\pi}{\omega}$ with mean position at $x = \frac{a}{2}$. Plot of x vs t is shown in Figure.



Differentiating equation (1), we get

$$v = \frac{a}{2}\times 2\omega\sin\left(2\omega t - \frac{\pi}{2}\right) = a\omega\sin\left(2\omega t - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad v = a\omega\sqrt{1-\cos^2\left(2\omega t - \frac{\pi}{2}\right)} = a\omega\sqrt{1-\frac{\left(\frac{a}{2}-x\right)^2}{(a/2)^2}}$$

$$\Rightarrow \quad v_x = 2\omega\sqrt{\left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}-x\right)^2}$$

$$\Rightarrow \quad v_x^2 = 4\omega^2x(a-x)$$

$$\Rightarrow \quad \left(\frac{v_x}{2\omega}\right)^2 + \left(\frac{a}{2}-x\right)^2 = \left(\frac{a}{2}\right)^2$$

$$\Rightarrow \quad \frac{v_x^2}{(a\omega)^2} + \left(\frac{\frac{a}{2}-x}{a/2}\right)^2 = 1 \quad \ldots(2)$$

Equation (2) is an equation of ellipse as shown in Figure.

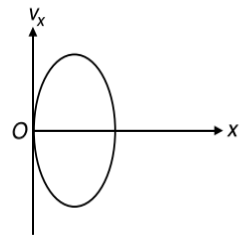


ILLUSTRATION 10

A particle executes SHM in a straight line. The maximum speed of the particle during its motion is v_m. Find the average speed of the particle during its SHM.

SOLUTION

Let $x = A\sin(\omega t)$, so $v = A\omega\cos(\omega t)$

$$\Rightarrow \quad v_{av} = \frac{\int_0^T v\,dt}{\int_0^T dt} = \frac{A\omega}{T}\int_0^T \cos(\omega t)\,dt$$

$$\Rightarrow \quad v_{av} = \frac{\omega A}{T}\left[\left|\int_0^{T/2}\cos(\omega t)\,dt\right| + \left|\int_{T/2}^{T}\cos(\omega t)\,dt\right|\right]$$

$$\Rightarrow \quad v_{av} = \frac{\omega A}{\omega T}(2+2) = \frac{4A}{T} = \frac{4A\omega}{2\pi} = \frac{2A\omega}{\pi}$$

Since, $v_{max} = v_m = A\omega$

$$\Rightarrow \quad v_{av} = \frac{2A\omega}{\pi} = \frac{2v_m}{\pi}$$

POTENTIAL ENERGY OF A PARTICLE IN SHM

The potential energy U at a displacement x is the work done against the restoring force in moving the body from the mean position to this position.

$$PE = U = \int_0^y kx\,dx = \frac{1}{2}kx^2$$

$$\Rightarrow \quad U = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t)$$

The maximum value of potential energy is $U_{max} = \frac{1}{2}m\omega^2A^2$ at the extreme position i.e., at $x = \pm A$ and minimum value of potential energy is $U_{min} = 0$ at the mean position i.e., at $x = 0$.

KINETIC ENERGY OF A PARTICLE IN SHM

If $x = A\sin(\omega t)$, then $v = A\omega\cos(\omega t)$, so kinetic energy K of a particle executing SHM is given by

$$KE = K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t)$$

$$\Rightarrow \quad K = \frac{1}{2}m\omega^2(A^2 - x^2)$$

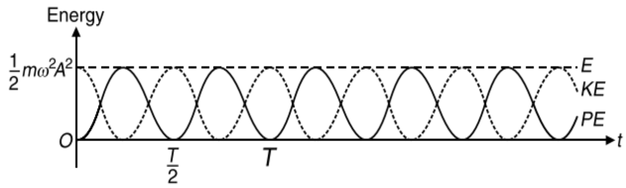
The maximum value of kinetic energy is $K_{max} = \frac{1}{2}m\omega^2A^2$ at the mean position i.e., at $x = 0$ and minimum value of kinetic energy is $K_{min} = 0$ at the extreme position i.e., at $x = \pm A$.

MECHANICAL ENERGY OF A PARTICLE IN SHM

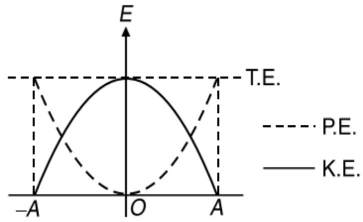
The mechanical energy E of a particle executing SHM is

$$E = K + U = \frac{1}{2}m\omega^2A^2 = \text{constant}$$

So, we observe that total mechanical energy is a constant. The variations of U, K and E with time are shown in Figure.



The variations of U, K and E with displacement are shown in Figure.



At $x = 0$, $U = 0$ and the energy is purely kinetic i.e., $E = K_{max} = \frac{1}{2}m(\omega A)^2$. At extreme points or turning points of the SHM, kinetic energy is zero and the energy is purely potential i.e., $E = U_{max} = \frac{1}{2}m(\omega A)^2$.

Note that the frequency of vibration of the kinetic and potential energy is twice that of the frequency of oscillation.

The instantaneous total mechanical energy of the spring-mass system may be written as

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Differentiating it w.r.t. time, we get

$$\frac{dE}{dt} = \frac{1}{2}m\frac{d}{dt}(v^2) + \frac{1}{2}k\frac{d}{dx}(x^2)$$

$$\Rightarrow \quad 0 = mv\frac{dv}{dt} + kx\frac{dx}{dt}$$

Since, $v = \frac{dx}{dt}$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$, therefore,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is the differential equation of SHM.

ILLUSTRATION 11

In an SHM, at the initial moment of time, the particle's displacement is 4 cm and its velocity is –3 ms^{-1}. The particle's mass is 4 kg and its total energy 50 J. Write down the equation of the SHM and find the distance travelled by the particle in 0.314 s from the start.

SOLUTION

Total energy of the particle is 50 J, so

$$\frac{1}{2}m\omega^2A^2 = 50$$

$\Rightarrow \quad A\omega = \sqrt{\frac{2\times 50}{4}} = 5$

Since $v^2 = \omega^2 A^2 - \omega^2 x^2$

$\Rightarrow \quad (-3)^2 = 25 - \omega^2 x^2$

$\Rightarrow \quad \omega^2 x^2 = 25 - 9$

$\Rightarrow \quad \omega x = \sqrt{16} = 4$

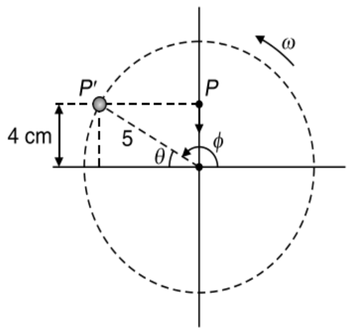
$\Rightarrow \quad \omega(0.04) = 4$

$\Rightarrow \quad \omega = 100 \text{ rads}^{-1}$

Since $A\omega = 5$

$\Rightarrow \quad A = \frac{5}{100} = 0.05 \text{ m} = 5 \text{ cm}$

To calculate the initial phase, we shall map the SHM on a circle as shown in Figure.



Since $\sin\theta = \frac{4}{5}$, i.e. $\theta = 53°$

So, initial phase is $\phi = 180° - 53° = 127°$

Also, $180° = \pi$ radian

$\Rightarrow \quad \phi = 127 \times \frac{\pi}{180}$ radian

$\Rightarrow \quad \phi = 2.2$ radian

Equation of SHM is given by

$$y = 0.05\sin(100t + 2.2) \text{ metre}$$

Time period of SHM is $T = \frac{2\pi}{\omega} = 0.0628$ s

Number of oscillations in 0.5 s are

$$N = \frac{0.314}{0.0628} = 5$$

Hence, total distance travelled in 5 oscillations is

$$l = 5(4A) = 20A = 20(0.05) = 1 \text{ m}$$

ILLUSTRATION 12

A particle of mass m is oscillating in SHM along x-axis about its mean position O with angular frequency ω and amplitude A. At the instant when the particle is passing the position $x = \frac{\sqrt{3}A}{2}$ and going away from O, an impulsive blow is given to it in the direction of motion. The impulse of this blow is of magnitude $J = mA\omega$. Calculate the new amplitude of vibration in terms of A.

SOLUTION

Velocity of particle at $x = \frac{\sqrt{3}A}{2}$ is

$$v = \omega\sqrt{A^2 - x^2} = \frac{A\omega}{2}$$

Applying impulse momentum theorem, we get

$$J = mv_f - mv$$

$\Rightarrow \quad mA\omega = mv_f - m\left(\frac{A\omega}{2}\right)$

$\Rightarrow \quad v_f = \frac{3}{2}A\omega$

If A' be the new amplitude, then we have

$$v_f = \omega\sqrt{A'^2 - x^2}$$

$\Rightarrow \quad \frac{3}{2}A\omega = \omega\sqrt{A'^2 - \frac{3}{4}A^2}$

$\Rightarrow \quad \frac{9}{4}A^2 = \left(A'^2 - \frac{3}{4}A^2\right)$

$\Rightarrow \quad A'^2 = \left(\frac{9}{4} + \frac{3}{4}\right)A^2 = 3A^2$

$\Rightarrow \quad A' = \sqrt{3}A$

ILLUSTRATION 13

The potential energy of a particle oscillating on x-axis is given as

$$U = 20 + (x-2)^2$$

Here U is in joules and x in metres. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not.
(b) Find the mean position.
(c) Find the maximum kinetic energy of the particle.

SOLUTION

(a) $F = -\frac{dU}{dx} = -2(x-2)$

Let us substitute $x - 2 = X$, then we get

$F = -2X$

Since, $F \propto -X$

The motion of the particle is simple harmonic

(b) The mean position of the particle is $X = 0$ or $x - 2 = 0$, which gives $x = 2$ m

(c) Maximum kinetic energy of the particle is,

$K_{max} = E - U_{min} = 36 - 20$

$K_{max} = 16$ J

U_{min} is 20 J at mean position or at $x = 2$ m

Problem Solving Technique(s)

CALCULATING ANGULAR FREQUENCY OF A PARTICLE IN SHM USING TAYLOR'S METHOD:

If mathematical function is given by $y=f(x)$, then by Taylor's theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots \quad \ldots(1)$$

If x is taken as the displacement of a particle from its mean position and the restoring force F acting on the particle depends on this x, then this restoring force $F=f(x)$ acting on particle at a distance x from mean position can be written by using Taylor's expansion series as

$$F=f(x)=f(0)+xf(0)+\frac{x^2}{2!}f''(0)+\ldots \quad \ldots(2)$$

In this case, $f(0)=0$ because at $x=0$ or mean position restoring force is zero and for small displacements of particle higher powers of x can be neglected. Hence restoring force F is given by using equation (2).

$$\Rightarrow \quad F=-xf'(0)$$

$$\Rightarrow \quad m\ddot{x}=-[f'(0)]x$$

$$\Rightarrow \quad \ddot{x}+\left(\frac{f'(0)}{m}\right)x=0$$

Comparing this equation with general differential equation of SHM i.e., $\ddot{x}+\omega^2x=0$, we get

$$\omega=\sqrt{\frac{f'(0)}{m}}$$

$$\Rightarrow \quad T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{f'(0)}}$$

However, if mean position is at x_0 (instead of $x=0$), then

$$\omega=\sqrt{\frac{f'(x_0)}{m}}$$

ILLUSTRATION 14

A particle of mass m is located in a uni-dimensional potential field for which the potential energy of the particle depends on the coordinate x as $U(x)=U_0(1-\cos Cx)$; U_0 and C are constants. Using Taylor's method, calculate the period of small oscillations that the particle performs about the equilibrium position.

SOLUTION

The potential energy of the oscillating particle is given by

$$U=U_0(1-\cos Cx)$$

Force on particle is given by

$$F=\left|\frac{dU}{dx}\right|$$

$$\Rightarrow \quad F=U_0C\sin(Cx)$$

This force is given as a function of displacement of particle x from its mean position, so angular frequency of its SHM from Taylor's method is given by

$$\omega=\sqrt{\frac{f'(0)}{m}} \quad \ldots(1)$$

where, $f'(0)=\left.\frac{dF}{dx}\right|_{x=0}=U_0C^2\cos(Cx)\Big|_{x=0}$

$$\Rightarrow \quad f'(0)=U_0C^2$$

Substituting in equation (1), we get

$$\omega=\sqrt{\frac{U_0C^2}{m}}$$

$$\Rightarrow \quad T=2\pi\sqrt{\frac{m}{U_0C^2}}$$

ILLUSTRATION 15

The potential energy (U) of a particle varies as a function of x-coordinate in a given force field as $U=\frac{a}{x^2}-\frac{b}{x}$, where a and b are positive constants. Calculate the period of small oscillations of the particle about the mean position (in the field).

SOLUTION

Given that $U=\frac{a}{x^2}-\frac{b}{x}$

$$\Rightarrow \quad F=\left|\frac{dU}{dx}\right|=-\frac{2a}{x^3}+\frac{b}{x^2}$$

At mean position say $x=x_0$, $F=0$

$$\Rightarrow \quad -\frac{2a}{x_0^3}+\frac{b}{x_0^2}=0$$

$$\Rightarrow \quad x_0=\frac{2a}{b}$$

According to Taylor's Theorem, we have

$$\omega=\sqrt{\frac{F'(x_0)}{m}}=\sqrt{\frac{F'(2a/b)}{m}}$$

Now, $F=-\frac{2a}{x^3}+\frac{b}{x^2}$

$$\Rightarrow \quad F'=\frac{6a}{x^4}-\frac{2b}{x^3}$$

$$\Rightarrow \quad F'\left(\frac{2a}{b}\right)=\frac{6a}{(2a/b)^4}-\frac{2b}{(2a/b)^3}$$

$$\Rightarrow \quad F'\left(\frac{2a}{b}\right)=\frac{3b^4}{8a^3}-\frac{b^4}{4a^3}=\frac{b^4}{8a^3}$$

$$\Rightarrow \quad \omega=\sqrt{\frac{F'(2a/b)}{m}}=\sqrt{\frac{b^4}{8ma^3}}$$

$$\Rightarrow \quad T=2\pi\sqrt{\frac{8ma^3}{b^4}}$$

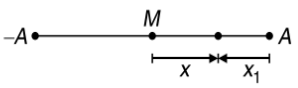
$$\Rightarrow \quad T=\frac{4\pi a}{b^2}\sqrt{2ma}$$

ILLUSTRATION 16

A particle moves with simple harmonic motion in a straight line. In the first second after starting from rest, it travels a distance x_1 and in the next second it travels a distance x_2 in the same direction. Find the amplitude of the motion.

SOLUTION

Starting from rest implies that the particle starts from extreme position as shown in Figure.



So, $x_1 = A - x_1$

$$\Rightarrow \quad x_1 = A - A\cos(\omega t)$$

So, at $t=1$ s, $x_1 = A(1-\cos\omega)$ ...(1)

Similarly, at $t=2$ s, $x_1+x_2 = A(1-\cos 2\omega)$ ...(2)

From equation (1), we get

$$\cos\omega=\frac{A-x_1}{A}$$

From equation (2), we get

$$x_1+x_2=2A\sin^2\omega$$

$$\Rightarrow \quad \sin^2\omega=\frac{x_1+x_2}{2A}$$

Since, $\sin^2\omega+\cos^2\omega=1$

$$\Rightarrow \quad \left(\frac{x_1+x_2}{2A}\right)+\left(\frac{A-x_1}{A}\right)^2=1$$

$$\Rightarrow \quad Ax_1+Ax_2+2A^2+2x_1^2-4Ax_1=2A^2$$

$$\Rightarrow \quad A=\frac{2x_1^2}{3x_1-x_2}$$

ILLUSTRATION 17

A particle executes simple harmonic motion. If the velocities at distances of 4 cm and 5 cm from the equilibrium position are 13 cms^{-1} and 5 cms^{-1} respectively, calculate the period and amplitude.

SOLUTION

Since, $v=\omega\sqrt{A^2-x^2}$

At $x=4$ cm, $v=13$ cms^{-1}

$$\Rightarrow \quad 13=\omega\sqrt{A^2-16} \quad \text{...(1)}$$

At $x=5$ cm, $v=5$ cms^{-1}

$$\Rightarrow \quad 5=\omega\sqrt{A^2-25} \quad \text{...(2)}$$

Divide equation (1) by (2), we get

$$\left(\frac{13}{5}\right)^2=\frac{A^2-16}{A^2-25}$$

$$\Rightarrow \quad 169A^2-4225=5A^2-80$$

$$\Rightarrow \quad 164A^2=4145$$

$$\Rightarrow \quad A^2=25.2$$

$$\Rightarrow \quad A=5.02 \text{ cm}$$

Substituting value of A in equation (1), we get

$$13=\omega\sqrt{A^2-16}$$

$$\Rightarrow \quad 13=\omega\sqrt{25.274-16}$$

$$\Rightarrow \quad \omega=\frac{13}{3.312}=3.925 \text{ rads}^{-1}$$

$$\Rightarrow \quad T=\frac{2\pi}{\omega}=\frac{6.28}{3.925}=1.6 \text{ s}$$

ILLUSTRATION 18

At the moment $t=0$, a particle starts moving along the x axis so that its velocity projection v_x varies as $v_x=35\cos(\pi t)$ cms^{-1}, where t is expressed in seconds. Calculate the distance covered by this particle during 2.80 s from the start.

SOLUTION

Time period of particle is

$$T=\frac{2\pi}{\omega}=2 \text{ s}$$

So, in 2.5 s i.e., $\frac{5T}{4}$, particle covers distance $5A$

Since, maximum speed of the particle is

$$35=A\omega$$

$$\Rightarrow \quad A=\frac{35}{\omega}=\frac{35}{3.14}=11.14 \text{ cm}$$

In remaining 0.3 sec, particle covers a distance s given by

$$s=5A+x_0 \quad \text{...(1)}$$

Since $v^2=\omega^2(A^2-x^2)$ ...(2)

where v at 2.8 s is

$$v=35\cos(3.14\times 2.8)=-28.22 \text{ cms}^{-1}$$

Substituting in (2), we get

$$(28.22)^2 = (3.14)^2\left[(11.14)^2 - x^2\right]$$

$\Rightarrow \quad 80.77 = (11.14)^2 - x^2$

$\Rightarrow \quad x^2 = 43.33$

$\Rightarrow \quad x = 6.59 \text{ cm}$

Hence distance covered is

$$x_0 = A - x = 11.14 - 6.59$$

$\Rightarrow \quad x_0 = 4.55 \text{ cm}$

So, from (1), we get

$$s = 5A + x_0$$

$\Rightarrow \quad s = 5(11.14) + 4.55 = 60.25 \text{ cm}$

ILLUSTRATION 19

In SHM, the distances of a particle from the mean position of its path at three consecutive seconds are observed to be x, y and z. Calculate the period of oscillation.

SOLUTION

Let $x = A\sin(\omega t)$, so we get

$$x = A\sin\omega \quad (\text{at } t = 1 \text{ s})$$

$$y = A\sin 2\omega \quad (\text{at } t = 2 \text{ s})$$

$$z = A\sin 3\omega \quad (\text{at } t = 3 \text{ s})$$

Now, $x + z = A(\sin\omega + \sin 3\omega)$

$\Rightarrow \quad x + z = 2[A\sin(2\omega)]\cos\omega$

$\Rightarrow \quad x + z = 2y\cos\omega$

$\Rightarrow \quad \omega = \cos^{-1}\left(\dfrac{x+z}{2y}\right)$

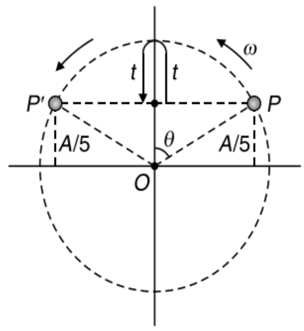
$\Rightarrow \quad T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\cos^{-1}\left(\dfrac{x+z}{2y}\right)}$

ILLUSTRATION 20

A particle executes SHM with amplitude A and angular frequency ω. At an instant when particle is at a distance $A/5$ from mean position and moving away from it. Find the time after which it will come back to this position again and also find the time after which it will pass through mean position.

SOLUTION

The SHM is drawn on circular mapping as shown in Figure.



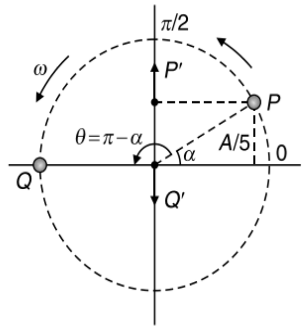
For calculating time taken by particle to return to same position, we see that

$$\theta = \cos^{-1}\left(\frac{A/5}{A}\right) = \cos^{-1}\left(\frac{1}{5}\right)$$

$\Rightarrow \quad t = \dfrac{\theta}{\omega} = \dfrac{\cos^{-1}(1/5)}{\omega}$

$\Rightarrow \quad t_{P\to P'} = 2t = \dfrac{2\cos^{-1}(1/5)}{\omega}$

For the particle to cross the mean position, we have again drawn SHM on circular mapping as shown in Figure.



So, $t = \dfrac{\theta}{\omega} = \dfrac{\pi - \alpha}{\omega} = \dfrac{\pi - \sin^{-1}(1/5)}{\omega}$

ILLUSTRATION 21

A particle performing SHM is in equilibrium at $t = 1$ second and has a speed of 0.25 ms^{-1} at $t = 2$ second. If the period of oscillation is 6 second, calculate the amplitude of oscillation, initial phase and velocity of particle at 6 second.

SOLUTION

Method-I

Let $y = A\sin(\omega t + \phi)$ be the equation of SHM, then at $t = 1$ s, $y = 0$ (because particle at mean or equilibrium)

$\Rightarrow \quad 0 = A\sin[\omega(1) + \phi]$

$\Rightarrow \quad \sin\left(\dfrac{\pi}{3} + \phi\right) = 0$

$\Rightarrow \quad \phi = -\dfrac{\pi}{3}$

Hence initial phase is $\phi = -\dfrac{\pi}{3}$ radian

Now $v = \dfrac{dy}{dt} = A\omega\cos(\omega t + \phi)$

At $t = 2$ s, $v = 0.25\ \text{ms}^{-1}$

$\Rightarrow \quad 0.25 = A\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{2\pi}{3} - \dfrac{\pi}{3}\right)$

$\Rightarrow \quad 0.25 = A\left(\dfrac{\pi}{3}\right)\left(\dfrac{1}{2}\right)$

$\Rightarrow \quad A = \dfrac{3}{2\pi}$ m

At $t = 6$ s, we get

$v = A\omega\cos(6\omega + \phi)$

$\Rightarrow \quad v = \left(\dfrac{3}{2\pi}\right)\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{5\pi}{3}\right)$

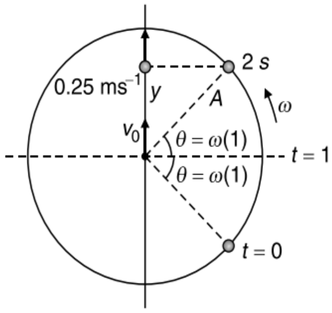
$\Rightarrow \quad v = (0.5)(0.5) = 0.25\ \text{ms}^{-1}$

Method-II

Given that, $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$ rads^{-1}

At $t = 1$ s, $v_0 = A\omega$

Since $\theta = \omega t = \omega(1)$, so from Figure, we get



$$0.25 = A\omega\cos\theta$$

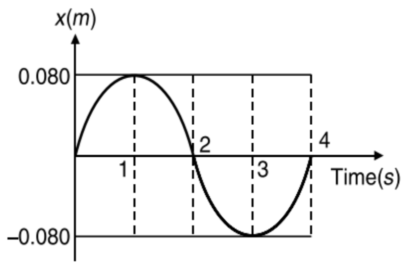
and solve further.

Test Your Concepts-I

Based on SHM Properties

(Solutions on page H.178)

1. An object of mass 0.8 kg is attached to one end of a spring and the system is set into simple harmonic motion. The displacement x of the object as a function of time is shown in the figure.



With the help of given data, calculate the
(a) amplitude A of the motion,
(b) angular frequency ω,
(c) spring constant k,
(d) speed of the object at $t = 1.0$ s and
(e) magnitude of acceleration at $t = 1.0$ s.

2. Prove that $x = Ae^{i\omega t}$ is an equation of SHM, where $i = \sqrt{-1}$.
3. Describe the motion of a particle acted upon by a force
(a) $F = -2(x-2)^3$ (b) $F = -2(x-2)^2$
(c) $F = -2(x-2)$
4. A particle in simple harmonic motion is at rest a distance of 6 cm from its equilibrium position at time $t = 0$. Its period is 2 s. Write expressions for its position x, its velocity v, and its acceleration a as functions of time.
5. When a particle is executing SHM. Calculate the ratio of mean velocity (during motion from one end of the path to the other) and the maximum velocity. Also calculate the ratio of average acceleration (during motion from one extreme to the centre) to the maximum acceleration.
6. (a) The potential energy of a harmonic oscillator of mass 2 kg in its resting position is 5 J, its total energy is 9 J and its amplitude is 1 cm. Calculate its time-period.
(b) A particle of mass 5 g lies in a potential field $V = (8x^2 + 200)$ erg per gram. Calculate its time period.
7. Two particles describe SHM of the same period and same amplitude along the same line about the same equilibrium position O. At a moment when they are at the same displacements their velocities are 1.6 ms^{-1} in opposite directions. At another moment when their displacements are equal in magnitude but on either side of O their velocities are 1.2 ms^{-1} in the same direction. Find the maximum speed of the particles and the phase difference between them.
8. A particle starts its SHM from mean position at $t = 0$. If its time period is T and amplitude A. Calculate the distance travelled by the particle in the time from $t = 0$ to $t = \dfrac{5T}{4}$.

9. A block is executing simple harmonic motion on a frictionless horizontal surface with a amplitude of 0.100 m. At a point 0.060 m away from equilibrium, the speed of the block is $0.360\ \text{ms}^{-1}$.
(a) What is the period?
(b) What is the displacement when the speed is $0.120\ \text{ms}^{-1}$?
(c) A small object whose mass is much less than the mass of the block is placed on the oscillating block. If the small object is just on the verge of slipping at the end point of the path, then find the coefficient of static friction between the small object and the block.

10. Two particles are in SHM with the same amplitude and frequency along the same line and about the same point. If the maximum separation between them is $\sqrt{3}$ times their amplitude, calculate the phase difference between them.

11. A particle of mass m free to move in the x-y plane is subjected to a force whose components are $F_x = -kx$ and $F_y = -ky$ where k is a constant. The particle is released when $t = 0$ at the point $(2, 3)$. Prove that the subsequent motion is simple harmonic along the straight line $2y - 3x = 0$.

12. The position of a particle is given by $x = 4\sin(2t)$, where x is in metres and t is in seconds.
(a) What is the maximum value of x and at what time first after $t = 0$ is this maximum?
(b) Find an expression for the velocity of the particle as a function of time. Also find the initial velocity of the particle.
(c) Find an expression for the acceleration of the particle as a function of time. Also find the initial and the maximum value of the acceleration of the particle.

13. Two particles executing SHM with same angular frequency and amplitudes A and $2A$ on same straight line with same mean position cross each other in opposite direction at a distance $A/3$ from mean position. Find the phase difference in the two SHMs.

14. A 0.2 kg object hangs from an ideal spring of negligible mass. When the object is pulled down 0.1 m below its equilibrium position and released, it vibrates with a period of 1.80 s.
(a) Find its speed as it passes through the equilibrium position.
(b) Find its acceleration when it is 0.050 m above the equilibrium position.
(c) Find the time required by the particle (when moving upwards) to move from a point 0.05 m below its equilibrium position to a point 0.05 m above it.
(d) If the motion of the object is stopped at the mean position and the object is removed from the spring, then find the length by which the spring shortens.

15. A linear harmonic oscillator has a total mechanical energy of 200 J Potential energy of it at mean position is 50 J. Find,
(a) the maximum kinetic energy
(b) the minimum potential energy
(c) the potential energy at extreme positions.

16. A point particle of mass 0.1 kg is executing SHM of amplitude of 0.1 m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} J. Obtain the equation of motion of this particle if this initial phase of oscillation is $45°$.

17. A particle of mass m is located in a unidimensional potential field where the potential energy of the particle depends on the coordinate x as $U(x)(1-\cos bx)$; U_0 and b are constants. Find the period of small oscillations that the particle performs about the equilibrium position.

18. A tray is moved horizontally back and forth in simple harmonic motion at a frequency of $f = 2.00$ Hz. On this tray is an empty cup. Obtained the coefficient of static friction between the tray and the cup, given that the cup begins slipping when the amplitude of the motion is 5.00×10^{-2} m.

19. A 1×10^{-2} kg block is resting on a horizontal frictionless surface and is attached to a horizontal spring whose spring constant is $124\ \text{Nm}^{-1}$. The block is given an initial speed of $8\ \text{ms}^{-1}$ parallel to the spring axis, while the spring is initially unstrained. What is the amplitude of the resulting simple harmonic motion?

20. The period of an oscillating particle of amplitude A is 8 s. At $t = 0$ it is in its equilibrium position.
(a) Calculate the distance it travels in the first 4 s. Compare with the distance it travels in the next 4 s?
(b) The distance travelled in the first 2 s and the next 2 s?
(c) The first second and the next second?

21. When the displacement of a body oscillating on a spring is half its amplitude, what fraction of its total energy is its kinetic energy? At what displacement are its kinetic and potential energies equal?

22. A plank with a body of mass m placed on it starts moving straight up according to the law $y = a(1-\cos\omega t)$, where y is the displacement from the initial position, $\omega = 11\ \text{rads}^{-1}$. Find:
(a) the time dependence of the force that the body exerts on the plank.
(b) the minimum amplitude of oscillation of the plank at which the body starts falling behind the plank.

23. A particle has displacement x given by $x = 3\cos(5\pi t + \pi)$, where x is in metres and t in seconds. Find the
(a) frequency f and the period T of the motion.
(b) greatest distance the particle travels from equilibrium.
(c) position of the particle at time $t = 0$ and at time $t = \frac{1}{2}$ s.

UNDERSTANDING SHM FOR PHYSICAL SYSTEMS

Whenever we have to prove that a particular physical system follows SHM, then we proceed through the following series of steps.

STEP-1 Locate the mean position of the system.

STEP-2 Give the system a small displacement {linear or angular} from the mean position.

STEP-3 For that extremely small displacement find the restoring force (or restoring torque).

STEP-4 This restoring force (or torque) must be directed towards the mean position.

STEP-5 This restoring force (or torque) must be directly proportional to the displacement (or angular displacement) from the mean position.

On positive confirmation of all the five steps we can say that the system follows simple harmonic motion and then time period (T) is given by STEP-6.

STEP-6 $T = 2\pi\sqrt{\frac{x}{|\ddot{x}|}}$ OR $T = 2\pi\sqrt{\frac{\theta}{|\ddot{\theta}|}}$

MASS-SPRING SYSTEM

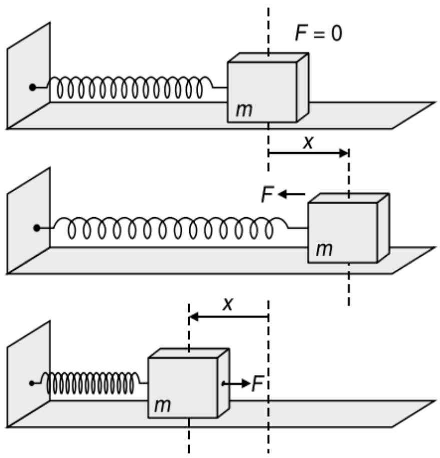
When a spring is compressed or stretched by a small amount, a restoring force is produced in it, which is proportional to the displacement x. So, we have

$$F = -kx$$

The constant k is called the **spring constant or the force constant of the spring**.

Horizontal Oscillations

Consider a light spring of constant k, one end of which is fixed rigidly to a wall and the other end is attached to a body of mass m, which is free to move on a frictionless horizontal surface. The equilibrium position of the system is shown in Figure.



When the body is pulled to the right by a deforming force (not shown in Figure), the restoring force exerted by the spring on the body is directed to the left. When the body is pushed to the left, the restoring force is directed to the right.

$$\Rightarrow \quad F = -kx$$

$$\Rightarrow \quad m\ddot{x} = -kx$$

$$\Rightarrow \quad \ddot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{x}{|\ddot{x}|}} = 2\pi\sqrt{\frac{m}{k}}$$

When the body is released it executes SHM with time period $T = 2\pi\sqrt{\frac{m}{k}}$.

Also, we know that the instantaneous total mechanical energy of the spring-mass system may be written as

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Differentiating it w.r.t. time, we get

$$\frac{dE}{dt} = \frac{1}{2}m\frac{d}{dt}\left(v^2\right) + \frac{1}{2}k\frac{d}{dx}\left(x^2\right)$$

$$\Rightarrow \quad 0 = mv\frac{dv}{dt} + kx\frac{dx}{dt}$$

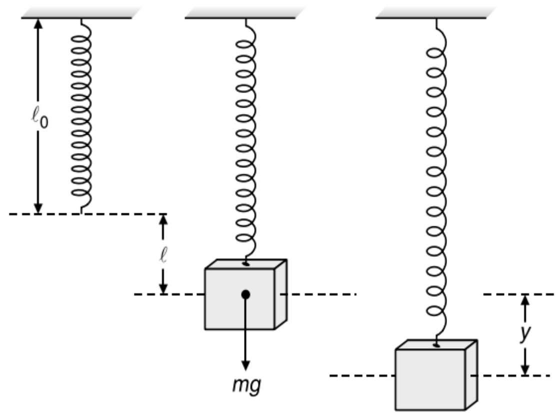
Since, $v = \frac{dx}{dt}$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$, therefore,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is the differential equation of SHM.

Vertical Oscillations

Consider a light spring suspended vertically from a fixed support, having a mass m connected to its lower end as shown in Figure.



In this case the equilibrium position of the spring is that position in which the spring is stretched by a length l such that the restoring force balances the weight mg. Hence,

$$kl = mg$$

$$\Rightarrow \quad k=\frac{mg}{l}$$

When the body is pulled further from this position through a distance y, it executes SHM. If F be the restoring force, then

$$F=k(l+y)-Mg$$

$$\Rightarrow \quad F=kl+ky-Mg$$

$$\Rightarrow \quad F=Mg+ky-Mg$$

$$\Rightarrow \quad F=-ky$$

Negative sign indicates the restoring nature of force.

$$\Rightarrow \quad m\ddot{y}=-ky$$

$$\Rightarrow \quad \ddot{y}+\frac{k}{m}y=0$$

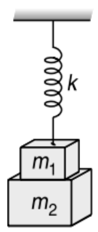
$$\Rightarrow \quad T=2\pi\sqrt{\frac{y}{|\ddot{y}|}}=2\pi\sqrt{\frac{m}{k}}$$

Conceptual Note(s)

It should be noted that the time period in vertical oscillations is same as that in horizontal oscillations. It does not depend on g.

ILLUSTRATION 22

Two masses m_1 and m_2 are suspended together by a massless spring of spring constant k. When the masses are in equilibrium m_1 is removed without disturbing the system. Find the angular frequency and amplitude of oscillation of m_2.



SOLUTION

As m_1 is removed, the mass m_2 will oscillate and so

$$T=2\pi\sqrt{\frac{m_2}{k}}, \text{ i.e., } \omega=\frac{2\pi}{T}=\sqrt{\frac{k}{m_2}}$$

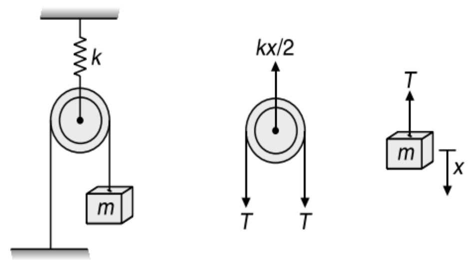
Furthermore, the stretch produced by $m_1 g$ will set an amplitude,
i.e., $m_1 g=kA$ i.e., $A=\left(\frac{m_1 g}{k}\right)$

ILLUSTRATION 23

For the arrangement shown in the figure, find the period of oscillation.

SOLUTION

Obviously, when the block is displaced down by x, the spring will stretch by $\frac{x}{2}$.



From the free body diagram of the pulley,

$$\frac{kx}{2}=2T$$

$$\Rightarrow \quad T=\frac{kx}{4}$$

The net restoring force on the block is T.
Using the Second Law of motion, we get

$$m\frac{d^2x}{dt^2}=-T=-\frac{kx}{4}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}+\left(\frac{k}{4m}\right)x=0$$

Thus, the period of SHM is given by

$$T=2\pi\sqrt{\frac{4m}{k}}$$

Note that we have not taken gravity into account as it does not affect the time period.

ILLUSTRATION 24

A horizontal spring block system of (force constant k) and mass M executes SHM with amplitude A. When the block is passing through its equilibrium position an object of mass m is put on it and the two moves together. Find the new amplitude and frequency of vibration.

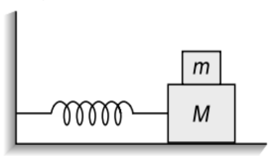
SOLUTION

Since initially mass M and finally $(m+M)$ is oscillating, so we have

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{M}} \text{ and } f'=\frac{1}{2\pi}\sqrt{\frac{k}{(m+M)}}$$

$$\Rightarrow \quad \frac{f'}{f}=\sqrt{\frac{M}{(m+M)}}$$

$$\Rightarrow \quad f'=f\sqrt{\frac{M}{(m+M)}} \qquad \ldots(1)$$



Now by Conservation of Linear Momentum

$$Mv = (m+M)V'$$

At equilibrium, $v = A\omega = 2\pi Af$ $\{\because \omega = 2\pi f\}$

$$M(2\pi fA) = (m+M)2\pi f'A'$$

$$\Rightarrow \frac{A'}{A} = \left(\frac{M}{m+M}\right)\left(\frac{f}{f'}\right)$$

$$\Rightarrow A' = A\sqrt{\frac{M}{(m+M)}}$$ {from equation (1)}

ILLUSTRATION 25

A 2 kgmass is attached to a spring of force constant 600 Nm^{-1} and rests on a smooth horizontal surface. A second mass of 1 kg slides along the surface toward the first at 6 ms^{-1}.

(a) Find the amplitude of oscillation if the masses make a perfectly inelastic collision and remain together on the spring. What is the period of oscillation?

(b) Find the amplitude and period of oscillation if the collision is perfectly elastic.

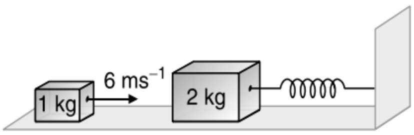
(c) For each case, write down the position x as a function of time t for the mass attached to the spring, assuming that the collision occurs at time $t = 0$. What is the impulse given to the 2 kg mass in each case?

SOLUTION

(a) From Conservation of Linear Momentum,

$$1 \times 6 = (1+2)v$$

$$\Rightarrow v = 2 \text{ ms}^{-1}$$



By Law of Conservation of Mechanical Energy, we have

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}kA^2$$

$$\Rightarrow A = \left(\sqrt{\frac{m}{k}}\right)v = \left(\sqrt{\frac{3}{600}}\right)\times 2$$

$$\Rightarrow A = 0.141 \text{ m} = 14.1 \text{ cm}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3}{600}} = 0.44 \text{ s}$$

(b) For perfectly elastic collision, we have

$$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2 + \left(\frac{2m_1}{m_1 + m_2}\right)v_1$$

$$v_2' = 0 + \left(\frac{2\times 1}{1+2}\right)\times 6 = 4 \text{ ms}^{-1}$$

$$\Rightarrow A = \left(\sqrt{\frac{m}{k}}\right)v_2' = 4\sqrt{\frac{2}{600}}$$

$$\Rightarrow A = 0.23 \text{ m} = 23 \text{ cm}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{600}} = 0.36 \text{ s}$$

(c) In the first case, $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.44} = 14.28 \text{ rads}^{-1}$

and amplitude $A = 14.1$ cm

$$\Rightarrow x = A\sin\omega t = (14.1 \text{ cm})\sin(14.28t)$$

In the second case, $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.36} = 17.45 \text{ rads}^{-1}$

and amplitude $A = 23$ cm

$$\Rightarrow x = 23\sin(17.45t) \text{ cm}$$

Impulse, $J_1 = \Delta P = 2\times 2 = 4$ Ns in the first case

and $J_2 = 4\times 2 = 8$ Ns in the second case.

ILLUSTRATION 26

A 2 kg block is attached to a spring for which $k = 200 \text{ Nm}^{-1}$. It is held at an extension of 5 cm and then released at $t = 0$. Find

(a) the displacement as a function of time

(b) the velocity when $x = +\frac{A}{2}$

(c) the acceleration when $x = +\frac{A}{2}$

SOLUTION

(a) We need to find a, ω, and ϕ in equation. The amplitude is the maximum extension; that is, $A = 0.05$ m

We know the angular frequency of the spring-mass system is given by $\omega = \sqrt{\frac{k}{m}} = 10 \text{ rads}^{-1}$

To find ϕ we note that at $t = 0$ we are given $x = +A$ and $v = 0$.

Thus, from the equation of displacement and velocity, we get

$$x = A\sin(\omega t + \phi)$$

$$\Rightarrow A\sin(0+\phi)$$

$$v = \omega A(\omega t + \phi)$$

$$\Rightarrow 0 = 10A\cos(0+\phi)$$

Since $\sin\phi = 1$ and $\cos\phi = 0$, it follows that $\phi = \frac{\pi}{2}$ rad.

Thus, $x = 0.05\sin\left(10t + \frac{\pi}{2}\right)$ m ...(1)

(b) In order to find the velocity, we have to find the time t, when $x = \frac{A}{2}$. Equation (1) yields $\frac{1}{2} = \sin\left(10t + \frac{\pi}{2}\right)$, from which we infer that

$$\left(10t+\frac{\pi}{2}\right)=\frac{\pi}{6} \quad \text{OR} \quad \left(10t+\frac{\pi}{2}\right)=\frac{5\pi}{6}$$

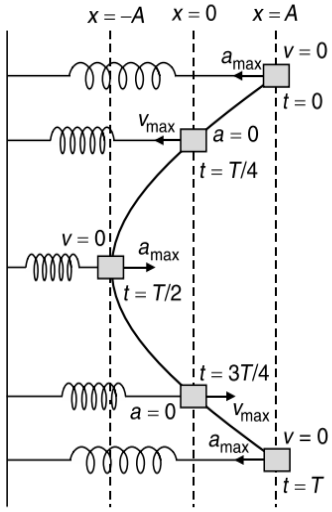
The velocity is given by

$$v=\frac{dx}{dt}=0.5\cos\left(10t+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad v=0.5\cos\left(\frac{\pi}{6}\right)=+0.43 \text{ ms}^{-1} \quad \text{OR}$$

$$v=0.5\cos\left(\frac{5\pi}{6}\right)=-0.43 \text{ ms}^{-1}$$

At a given position, there are two velocities of equal magnitude but opposite directions.



(c) The acceleration at $x=\frac{A}{2}$ may be found from the equation,

$$a=-\frac{k}{m}x=-\omega^2 x=-\left(10 \text{ rads}^{-1}\right)^2\left(\frac{0.05}{2}\text{ m}\right)$$

$$a=-2.5 \text{ ms}^{-2}$$

SHM OF FREE BODIES IN ABSENCE OF EXTERNAL FORCES

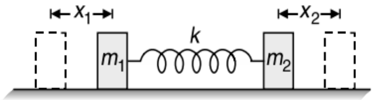
We know for oscillations of a body restoring force must be there due to which the body oscillates about its mean position. In some special cases, it is possible that a system oscillates due to only internal forces and internal forces of system provide the required centripetal force for oscillations. We take an illustrative example to explain such situation.

ILLUSTRATION 27

Two masses m_1 and m_2 are connected by a spring of force constant k and are placed on a frictionless horizontal surface. Show that if the masses are displaced slightly in opposite directions and released, the system will execute simple harmonic motion. Calculate the frequency of oscillation.

SOLUTION

Let masses m_1 and m_2 be displaced by x_1 and x_2 respectively from their equilibrium position in opposite direction so that the total extension in the spring will be $x=x_1+x_2$. Due to this stretch a restoring force kx will act on each mass and so equation of mass m_1 will be



$$m_1\frac{d^2x_1}{dt^2}=-kx \text{, i.e., } \frac{d^2x_1}{dt^2}=-\frac{k}{m_1}x \qquad \ldots(1)$$

while that for m_2 will be

$$m_2\frac{d^2x_2}{dt^2}=-kx \text{, i.e., } \frac{d^2x_2}{dt^2}=-\frac{k}{m_2}x \qquad \ldots(2)$$

But as $x=x_1+x_2$, i.e., $\frac{d^2x}{dt^2}=\frac{d^2x_1}{dt^2}+\frac{d^2x_2}{dt^2}$ ...(3)

So, substituting equation (1) and (2) in (3)

$$\frac{d^2x}{dt^2}=-\left(\frac{1}{m_1}+\frac{1}{m_2}\right)kx$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}=-\frac{k}{m}x \text{ where } \frac{1}{m}=\frac{1}{m_1}+\frac{1}{m_2}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}=-\omega^2 x \text{ where } \omega^2=\frac{k}{m}$$

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}=\frac{1}{2\pi}\sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}$$

ILLUSTRATION 28

A block with a mass of 2 kg hangs without vibrating at the end of a spring of spring constant 500 Nm^{-1}, which is attached to the ceiling of an elevator. The elevator is moving upwards with an acceleration $\frac{g}{3}$. At time $t=0$, the acceleration suddenly ceases.

(a) What is the angular frequency of oscillation of the block after the acceleration ceases?

(b) By what amount is the spring stretched during the time when the elevator is accelerating?

(c) What is the amplitude of oscillation and initial phase angle observed by a rider in the elevator?

Take the upward direction to be positive. take $g=10 \text{ ms}^{-2}$.

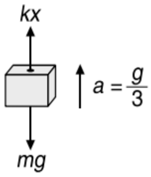
SOLUTION

(a) Angular frequency $\omega=\sqrt{\frac{k}{m}}$

$$\Rightarrow \quad \omega=\sqrt{\frac{500}{2}}$$

$$\Rightarrow \quad \omega=15.81 \text{ rads}^{-1}$$

(b) Equation of motion of the block (white elevator is accelerating) is,

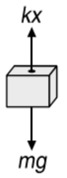


$$kx - mg = ma = m\frac{g}{3}$$

$$\Rightarrow \quad x = \frac{4mg}{3k} = \frac{(4)(2)(10)}{(3)(500)} = 0.053 \text{ m}$$

$$\Rightarrow \quad x = 5.3 \text{ cm}$$

(c) (i) In equilibrium when the elevator has zero acceleration, the equation of motion is,



$$kx_0 = mg$$

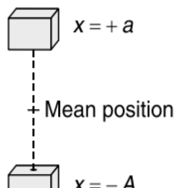
$$\Rightarrow \quad x_0 = \frac{mg}{k} = \frac{(2)(10)}{500} = 0.04 \text{ m}$$

$$\Rightarrow \quad x_0 = 4 \text{ cm}$$

So, amplitude $A = x - x_0 = 5.3 - 4.0$

$$\Rightarrow \quad A = 1.3 \text{ cm}$$

(ii) At time $t = 0$, block is at $x = -A$. Therefore, substituting $x = -A$ and $t = 0$ in equation, we get



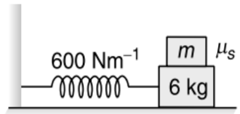
$$x = A\sin(\omega t + \phi)$$

So, the initial phase is

$$\phi = \frac{3\pi}{2}$$

ILLUSTRATION 29

With the assumption of no slipping, determine the mass m of the block which must be placed on the top of a 6 kg cart in order that the system period is 0.75 s. What is the minimum coefficient of static friction μ_s for which the block will not slip relative to the cart if the cart is displaced 50 mm from the equilibrium position and released. Take $g = 9.8 \text{ ms}^{-2}$.



SOLUTION

Since, $T = 2\pi\sqrt{\frac{m+6}{600}}$ $\qquad \left\{\because T = 2\pi\sqrt{\frac{m}{k}}\right\}$

$$\Rightarrow \quad 0.75 = 2\pi\sqrt{\frac{m+6}{600}}$$

$$\Rightarrow \quad m = \frac{(0.75)^2 \times 600}{(2\pi)^2} - 6 = 2.55 \text{ kg}$$

Maximum acceleration of SHM of amplitude A is

$$a_{\max} = \omega^2 A$$

So, maximum force on mass m is $m\omega^2 A$ which is being provided by the force of friction between the mass and cart. Therefore,

$$\mu_s mg \geq m\omega^2 A$$

$$\Rightarrow \quad \mu_s \geq \frac{\omega^2 A}{g}$$

$$\Rightarrow \quad \mu_s \geq \left(\frac{2\pi}{T}\right)^2 \frac{A}{g}$$

$$\Rightarrow \quad \mu_s \geq \left(\frac{2\pi}{0.75}\right)^2\left(\frac{0.05}{9.8}\right) \qquad \{\because A = 50 \text{ mm}\}$$

$$\Rightarrow \quad \mu_s \geq 0.358$$

Thus, the minimum value of μ_s should be 0.358

COUPLED SPRING SYSTEM

Springs in Series

Suppose two springs of force constants k_1 and k_2 are connected in series to a mass m. Let m be displaced to the right through a distance y. If the extensions of the two springs are y_1 and y_2 respectively, then

$$y = y_1 + y_2$$

If F is the restoring force, then

$$F = -k_1 y_1 = -k_2 y_2$$

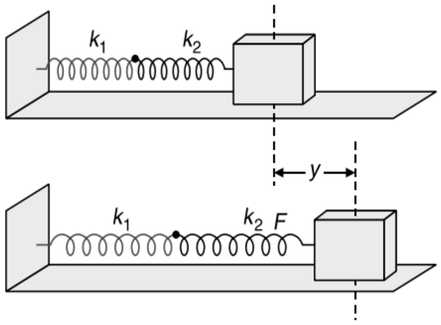
$$\Rightarrow \quad y_1 = -\frac{F}{k_1} \text{ and } y_2 = -\frac{F}{k_2}$$

$$\Rightarrow \quad y = -F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = -F\left(\frac{k_1 + k_2}{k_1 k_2}\right)$$

$$\Rightarrow \quad F = -\left(\frac{k_1 k_2}{k_1 + k_2}\right)y$$

This shows that the effective force constant of the two springs is given by

$$k_{\text{series}} = k_S = \frac{k_1 k_2}{k_1 + k_2} \text{ or } \frac{1}{k_{\text{series}}} = \frac{1}{k_1} + \frac{1}{k_2}$$



such that k_1, $2k_S$, k_2 are in **Harmonic Progression**
The time period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k_S}} = 2\pi\sqrt{m\left(\frac{k_1 + k_2}{k_1 k_2}\right)}$$

If $k_1 = k_2 = k$(say), then $k_S = \frac{k}{2}$

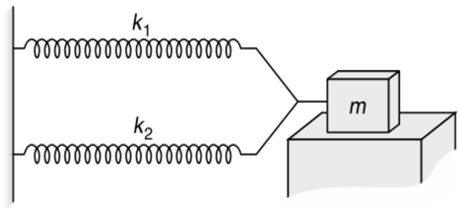
Springs in Parallel

In this case, if m is displaced to the right through a distance y, the extension produced in each spring is the same. If F_1 and F_2 are the restoring forces produced in k_1 and k_2, respectively, then

$$F_1 = -k_1 y \text{ and } F_2 = -k_2 y$$

The total restoring force F is

$$F = F_1 + F_2 = -(k_1 + k_2)y$$



This shows that the effective force constant is

$$k_{\text{parallel}} = k_p = k_1 + k_2$$

The time period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k_P}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

If $k_1 = k_2 = k$(say), then $k_P = 2k$.

Conceptual Note(s)

(a) If n springs are connected in series, then

$$\frac{1}{k_S} = \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_n}$$

(b) If n springs are connected in parallel, then

$$k_P = k_1 + k_2 + \dots + k_n$$

(c) If a spring of spring constant k, natural length ℓ, is divided into parts, then

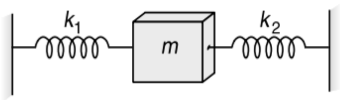
$$k\ell = \text{constant}$$

$$\Rightarrow k_1\ell_1 = k_2\ell_2 = \dots = k_n\ell_n = k\ell$$

where $\ell_1 + \ell_2 + \dots + \ell_n = \ell$

Mass Connected Between Two Springs

If the body is displaced to one side, one of the springs gets extended and the other gets compressed. The restoring forces due to both, say F_1 and F_2, are in the same direction. The total restoring force F is $F_1 + F_2$. Now, if y is the displacement of the body, then



$$F_1 = -k_1 y \text{ and } F_2 = -k_2 y$$

$$\Rightarrow \quad F = -(k_1 + k_2)y$$

Thus, showing that the effective force constant is

$$k_{\text{eff}} = k_1 + k_2$$

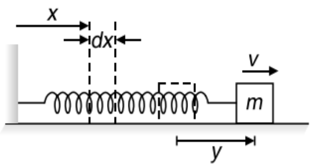
ILLUSTRATION 30

A mass m is connected to a spring of mass m_s and oscillates in SHM on a smooth horizontal surface. The force constant of the spring is k. Find the time period of oscillation.

SOLUTION

Let l be the length of the spring. Let V be the speed of mass m in its displaced position y. Since the spring is not massless, so it will also have some kinetic energy. To find this kinetic energy consider a segment of spring of length dx at a distance x from the fixed end. As the velocity of different segments will be different in an oscillating spring, we assume that velocity of the segment is directly proportional to its distance from the fixed end, so for this segment, we have

$$dm = \frac{m_s}{l}dx \text{ and } v = \left(\frac{x}{l}\right)V$$



So, the kinetic energy of this segment is given by

$$dK_s = \frac{1}{2}(dm)(v^2) = \frac{1}{2}\left(\frac{m_s}{l}dx\right)\left(\frac{x}{l}V\right)^2$$

$$\Rightarrow \quad K_s = \int_0^l dK_s$$

Integrating we get,

$$K_s = \frac{1}{6}m_s V^2$$

The mechanical energy of the system in displaced position of the block will be,

$$E = \begin{pmatrix}\text{Kinetic}\\ \text{Energy}\\ \text{of Mass}\end{pmatrix} + \begin{pmatrix}\text{Kinetic}\\ \text{Energy}\\ \text{of Spring}\end{pmatrix} + \begin{pmatrix}\text{Elastic}\\ \text{Potential}\\ \text{Energy}\\ \text{of Spring}\end{pmatrix}$$

$$\Rightarrow \quad E = \frac{1}{2}mV^2 + \frac{1}{6}m_s V^2 + \frac{1}{2}ky^2$$

Since, E =constant

$$\Rightarrow \quad \frac{dE}{dt} = 0$$

$$\Rightarrow \quad mV\left(\frac{dV}{dt}\right) + \frac{1}{3}m_s V\frac{dV}{dt} + ky\frac{dy}{dt} = 0$$

Substituting $\frac{dV}{dt} = a$ and $\frac{dy}{dt} = V$, we get

$$\left(m + \frac{m_y}{3}\right)a = -ky$$

$$\Rightarrow \quad a \propto -y$$

Therefore, motion is simple harmonic in nature, with time period

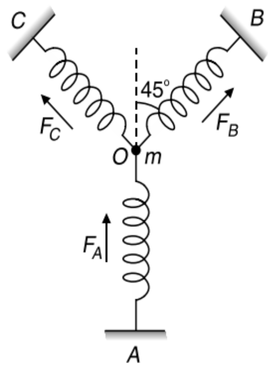
$$T = 2\pi\sqrt{\left|\frac{y}{a}\right|} = 2\pi\sqrt{\frac{m + \frac{m_s}{3}}{k}}$$

ILLUSTRATION 31

A particle of mass m is attached to three identical springs A, B and C each of force constant k as shown in figure. If the particle of mass k is pushed slightly against the spring A and released, find the time period of oscillations.

SOLUTION

When the particle of mass m at O is pushed by y in the direction of A, spring A will be compressed by y while B and C will be stretched by $y' = y\cos 45°$, so the total restoring force on the mass m along OA is given by



$$F = (F_A + F_B\cos 45° + F_C\cos 45°)$$

$$\Rightarrow \quad F = -(ky + 2(ky')\cos 45)$$

$$\Rightarrow \quad F = -(ky + 2k(y\cos 45)\cos 45)$$

$$\Rightarrow \quad F = -k'y, \text{ where } k' = 2k$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{2k}}$$

ILLUSTRATION 32

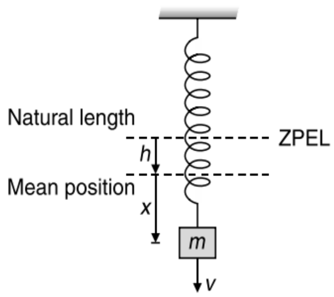
Calculate the angular frequency of vertical oscillations of a block of mass m attached to a spring of spring constant k. (Assume spring to be light).

SOLUTION

Let the extension in the spring at equilibrium be h. Then,

$$mg = kh \qquad \ldots(1)$$

At any instant, let further extension in the spring be x and v be the velocity of block as shown in Figure.



Total mechanical energy of block at this instant is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}k(x+h)^2 - mgx$$

Since total mechanical energy (E) is constant, so

$$\frac{dE}{dt} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2+\frac{1}{2}k(h+x)^2-mgx\right)=0$$

$$\Rightarrow \quad \frac{m}{2}\left(2v\frac{dv}{dt}\right)+\frac{k}{2}\left(2(h+x)\frac{dx}{dt}\right)-mg\left(\frac{dx}{dt}\right)=0$$

$$\Rightarrow \quad mv\frac{d^2x}{dt^2}+k(h+x)v-mgv=0$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2}+kh+kx-mg=0 \qquad \ldots(2)$$

Substituting equation (1) in (2), we get

$$m\frac{d^2x}{dt^2}+kx=0$$

$$\Rightarrow \quad \frac{d^2x}{dt^2}+\frac{k}{m}x=0$$

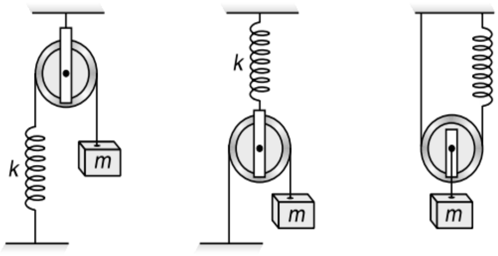
Comparing with standard equation of SHM i.e., $\ddot{x}+\omega^2x=0$, we get

$$\omega=\sqrt{\frac{k}{m}}$$

Since block is given an initial velocity v_0 at mean position, so we have $v_0 = A\omega$, where A is amplitude of oscillations given by $A=\frac{v_0}{\omega}=v_0\sqrt{\frac{m}{k}}$.

ILLUSTRATION 33

Figure shows a system consisting of a massless pulley, a spring of force constant k and a block of mass m. If the block is slightly displaced vertically down from its equilibrium position and released, find the period of its vertical oscillation in cases (a), (b) and (c).

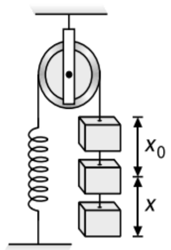


(a) (b) (c)

SOLUTION

(a) In equilibrium $kx_0 = mg$ $\ldots(1)$

When further depressed by an amount x, net restoring force (upwards) is,



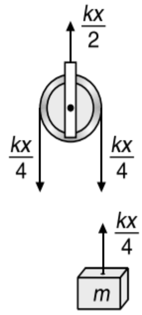
$$F=-[k(x+x_0)-mg]$$

$$\Rightarrow \quad F=-kx \qquad \{\because kx_0=mg\}$$

$$\Rightarrow \quad a=-\frac{k}{m}x$$

$$\Rightarrow \quad T=2\pi\sqrt{\left|\frac{x}{a}\right|}=2\pi\sqrt{\frac{m}{k}}$$

(b) In this case if the mass m moves down a distance x from its equilibrium position, then pulley will move down by $\frac{x}{2}$. So, the extra force in spring will be $k\left(\frac{x}{2}\right)$.



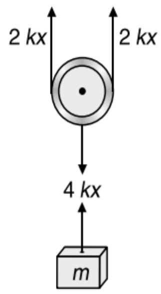
Now, as the pulley is massless, this force $\frac{kx}{2}$ is equal to extra $2T$ or $T=\frac{kx}{4}$. This is also the restoring force of the mass. Hence,

$$F=-\frac{kx}{4}$$

$$\Rightarrow \quad a=-\frac{k}{4m}x$$

$$\Rightarrow \quad T=2\pi\sqrt{\left|\frac{x}{a}\right|}=2\pi\sqrt{\frac{4m}{k}}$$

(c) In this situation if the mass m moves down a distance x from its equilibrium position, the pulley will also move by x and so the spring will stretch by $2x$.



Therefore, the spring force will be $2kx$. The restoring force on the block will be $4kx$. Hence,

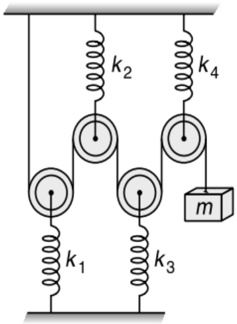
$$F=-4kx$$

$$\Rightarrow \quad a=-\left(\frac{4k}{m}\right)x$$

$$\Rightarrow \quad T=2\pi\sqrt{\left|\frac{x}{a}\right|}=2\pi\sqrt{\frac{m}{4k}}$$

ILLUSTRATION 34

In the arrangement shown in figure, pulleys are small light and springs are ideal. k_1, k_2, k_3 and k_4 are force constants of the springs. Calculate period of small vertical oscillations of block of mass m.



SOLUTION

When the mass m is displaced from its mean position by a distance x, let F be the restoring (extra tension) force produced in the string. By this extra tension further elongation in the springs are $\frac{2F}{k_1}, \frac{2F}{k_2}, \frac{2F}{k_3}$ and $\frac{2F}{k_4}$ respectively. Then,

$$x = 2\left(\frac{2F}{k_1}\right) + 2\left(\frac{2F}{k_2}\right) + 2\left(\frac{2F}{k_3}\right) + 2\left(\frac{2F}{k_4}\right)$$

$$\Rightarrow \quad F\left(\frac{4}{k_1} + \frac{4}{k_2} + \frac{4}{k_3} + \frac{4}{k_4}\right) = -x$$

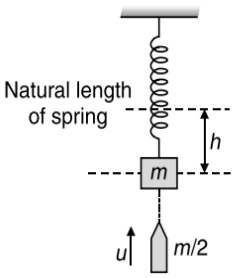
Here negative sign shows the restoring nature of force

$$\Rightarrow \quad a = -\frac{x}{m\left(\frac{4}{k_1} + \frac{4}{k_2} + \frac{4}{k_3} + \frac{4}{k_4}\right)}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} = 4\pi\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4}\right)}$$

ILLUSTRATION 35

Consider spring block pendulum system hanging in equilibrium. A bullet of mass $m/2$ moving at a speed u hits the block of mass m from downward direction and gets embedded in it as shown in Figure.



Calculate the amplitude of oscillation of the block. Also find the time taken by the block to reach its upper extreme position after being hit by bullet.

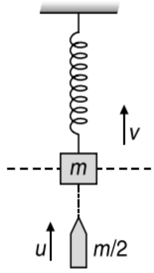
SOLUTION

When the block is in equilibrium, then let the spring have an extension h, such that

$$mg = kh \qquad \ldots(1)$$

When the bullet of mass $m/2$ gets embedded in the block, then the new mass of block becomes $3m/2$. The new mean position of the block will be at a depth h_1 from the old mean position, such that

$$\frac{3}{2}mg = k(h + h_1) \qquad \ldots(2)$$



From equations (1) and (2), we get

$$h_1 = \frac{mg}{2k}$$

Just after impact, due to inelastic collision, the velocity of block becomes v, then by law of conservation of momentum, we get

$$\left(\frac{m}{2}\right)u = \left(\frac{3m}{2}\right)v$$

$$\Rightarrow \quad v = \frac{u}{3}$$

Since the block now executes SHM and at $t = 0$, the block is at a distance $h_1 = \frac{mg}{2k}$ above its mean position having a velocity $\frac{u}{3}$. If amplitude of oscillation is A, then we have

$$\frac{u}{3} = \omega\sqrt{A^2 - \left(\frac{mg}{2k}\right)^2} \qquad \left\{\because\ v = \omega\sqrt{A^2 - x^2}\right\}$$

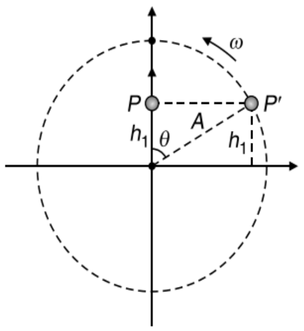
Also, for this new spring block system, we have

$$\omega = \sqrt{\frac{k}{3m/2}} = \sqrt{\frac{2k}{3m}}$$

$$\Rightarrow \quad \frac{u^2}{9} = \frac{2k}{3m}\left(A^2 - \frac{m^2g^2}{4k^2}\right)$$

$$\Rightarrow \quad A = \sqrt{\frac{mu^2}{6k} + \left(\frac{mg}{2k}\right)^2} \qquad \ldots(3)$$

The time taken by particle to reach the topmost point can be obtained by drawing the circular motion representation of SHM as shown in Figure.



This figure shows the position of block P and its corresponding circular motion particle P' at $t = 0$. Block P will reach its upper extreme position when particle P' will traverse the angle θ and reach the topmost point. As P' moves at constant angular velocity ω, it will take a time given by

$$t = \frac{\phi}{\omega} \text{ where, } \phi = \cos^{-1}\left(\frac{h_1}{A}\right) \text{ and } \omega = \sqrt{\frac{2k}{3m}}$$

$$\Rightarrow \quad t = \sqrt{\frac{3m}{2k}}\cos^{-1}\left(\frac{mg}{2kA}\right)$$

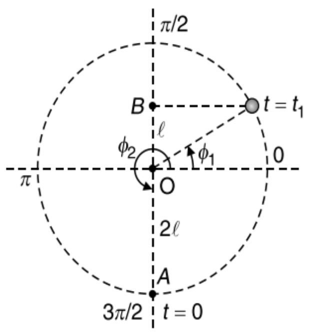
ILLUSTRATION 36

A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is such that in equilibrium the string length is double its natural length. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\frac{l}{g}}\left(\frac{4\pi}{3}+2\sqrt{3}\right)$, where l is the natural length of the string.

SOLUTION

Let k be the equivalent force constant of string, then at equilibrium we have $Mg = k(2l - l)$.

Further extension by $2l$, so that total length becomes $4l$, will make the particle shoot upwards (on being released) upto natural length of spring. Time taken by particle is obtained by drawing the phase diagram as shown in Figure.



Here $\Delta\phi = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{4\pi}{3}$

Time for particle to go from A to B is

$$t_1 = \frac{\Delta\phi}{\omega}, \text{ where } \omega = \sqrt{\frac{g}{l}}$$

$$\Rightarrow \quad t_1 = \frac{4\pi}{3}\sqrt{\frac{l}{g}}$$

At position B, speed of particle is

$$v = \omega\sqrt{(2l)^2 - (l)^2}$$

$$\Rightarrow \quad v = \sqrt{3}\omega l = \sqrt{3gl}$$

After point B, the string will slack and particle will be in free fall motion, so time taken by it to go up and come back to point B is given as

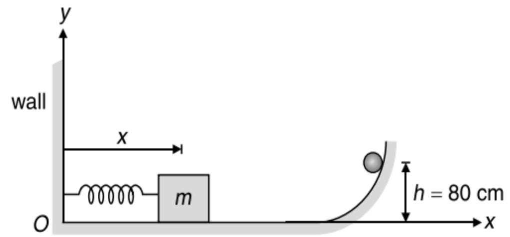
$$t_2 = \frac{2\sqrt{3gl}}{g} = 2\sqrt{3}\sqrt{\frac{l}{g}}$$

Thus, total time after which particle will come back to point A is

$$T = t_1 + t_2 = \sqrt{\frac{l}{g}}\left(\frac{4\pi}{3}+2\sqrt{3}\right)$$

ILLUSTRATION 37

A block of mass $m = 1$ kg is attached to a free end of a spring whose other end is fixed with a wall performing simple harmonic motion as shown in Figure.



The position of the block from O at any instant is $x = 2 + \frac{1}{\sqrt{2}}\sin(2t)$ where x in meter and t is in second.

A shell of same mass is released from smooth the circular path at a height $h = 80$ cm. The shell collides elastically with the block performing SHM and finally reaches up to height 5 cm along circular path. Neglecting friction, find where the collision takes place.

SOLUTION

Initial speed of shell before collision is

$$v_1 = \sqrt{2gh_1} = \sqrt{2\times 10\times 0.8} = 4 \text{ ms}^{-1}$$

Find speed of shell after collision

$$v_2 = \sqrt{2gh_2} = \sqrt{2\times 10\times 0.05} = 1 \text{ ms}^{-1}$$

Since, masses of block and shell are equal and they collide elastically, so v_2 is the block velocity just before collision which is given as

$$v = \frac{dx}{dt} = \sqrt{2}\cos(2t) = 1$$

$$\Rightarrow \quad \cos(2t) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 2t = \frac{\pi}{4} \text{ or } \frac{7\pi}{4}$$

$$\Rightarrow \quad t = \frac{\pi}{8} \text{ or } \frac{7\pi}{8}$$

Position of block at $t = \frac{\pi}{8}$ is

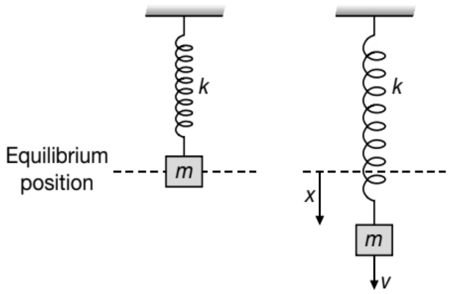
$$x_1 = 2 + \frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}\right) = 2 + \frac{1}{2} = 2.5 \text{ m}$$

Position of block at $t = \frac{7\pi}{8}$ is

$$x_2 = 2 + \frac{1}{\sqrt{2}}\sin\left(\frac{7\pi}{8}\right) = 2 - \frac{1}{2} = 1.5 \text{ m}$$

ILLUSTRATION 38

A spring block system is hanging in equilibrium. The block of system is pulled down by a distance x and imparted a velocity v in downward direction as shown in Figure. Calculate the time it will take to reach its mean position.



(a) (b)

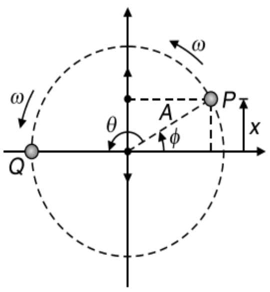
SOLUTION

When the block is pulled down by x (from the mean position) and is given a velocity v downwards, then it executes SHM of amplitude A (say) and $\omega = \sqrt{k/m}$. The velocity of block at distance x from the mean position is

$$v^2 = \omega^2(A^2 - x^2) = \frac{k}{m}(A^2 - x^2)$$

$$\Rightarrow \quad A = \sqrt{\frac{mv^2}{k} + x^2}$$

Taking downward direction as positive, the mapping of this SHM on circular motion is shown in Figure. (The concept behind this mapping is that initially the particle is at distance x from the mean position).



Time taken by the particle to go from start to mean position is

$$t = \frac{\theta}{\omega} = \frac{\pi - \phi}{\omega}$$

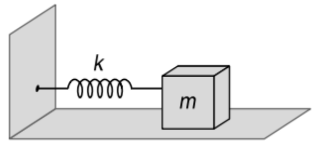
where $\phi = \sin^{-1}\left(\frac{x}{A}\right)$

$$\Rightarrow \quad t = \frac{\pi - \sin^{-1}(x/A)}{\omega} = \sqrt{\frac{m}{k}}\left[\pi - \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \quad t = \sqrt{\frac{m}{k}}\left[\pi - \sin^{-1}\left(\frac{\sqrt{k}x}{\sqrt{mv^2 + kx^2}}\right)\right]$$

ILLUSTRATION 39

A block of mass m lying on a smooth horizontal surface is attached to one end of a spring of force constant k.



The block is now pulled towards right by a distance x_0 and released. When the block passes through a point at a displacement $x_0/2$ from mean position, another block of same mass is gently placed on it which sticks to it due to friction. Calculate the new amplitude of oscillation, the time taken by it to reach its mean position and extreme position on the left side.

SOLUTION

Velocity of block at $\frac{x_0}{2}$ from mean position is

$$v = \omega\sqrt{A^2 - x^2}, \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \quad v = \omega\sqrt{x_0^2 - \frac{x_0^2}{4}} = \frac{\sqrt{3}}{2}x_0\omega \qquad \{\because A = x_0\}$$

$$\Rightarrow \quad v = \frac{\sqrt{3}}{2}x_0\sqrt{\frac{k}{m}} \qquad \text{...(1)}$$

Now, when another block of same mass is placed on this block, then the new angular frequency of system is

$$\omega' = \sqrt{\frac{k}{2m}}$$

If A' be the new amplitude of oscillations of combined system of blocks, then we have

$$v' = \omega'\sqrt{A'^2 - \left(\frac{x_0}{2}\right)^2} \qquad \text{...(2)}$$

where, by law of conservation of linear momentum we get

$$(m + m)v' = mv$$

$$\Rightarrow \quad v' = \frac{v}{2} \qquad \text{...(3)}$$

Substituting (3) in equation (2), we get

$$\left(\frac{v}{2}\right)^2=\left(\frac{k}{2m}\right)\left(A'^2-\frac{x_0^2}{4}\right)$$

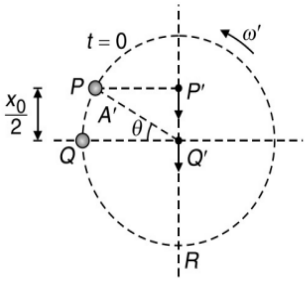
Using equation (1), we get

$$\frac{3}{16}x_0^2\left(\frac{k}{m}\right)=\frac{k}{2m}\left(A'^2-\frac{x_0^2}{4}\right)$$

$$\Rightarrow\quad \frac{3}{8}x_0^2+\frac{x_0^2}{4}=A'^2$$

$$\Rightarrow\quad A'=\left(\frac{\sqrt{5}}{2\sqrt{2}}\right)x_0$$

Initially the combined blocks are at a distance $\frac{x_0}{2}$ from the mean position moving towards the mean position. So, mapping this situation on a circle shown in Figure, we get



$$\theta=\sin^{-1}\left(\frac{x_0/2}{A'}\right)=\sin^{-1}\left(\sqrt{\frac{2}{5}}\right)$$

$$t_{P\to Q}=\frac{\theta}{\omega'}=\frac{\sin^{-1}\left(\sqrt{2/5}\right)}{\omega}$$

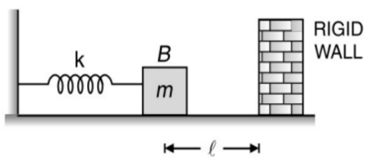
$$\Rightarrow\quad t_{P\to Q}=\sqrt{\frac{2m}{k}}\sin^{-1}\left(\sqrt{\frac{2}{5}}\right)$$

Similarly, time taken by P to reach the other extreme is

$$t_{P\to R}=\frac{\pi/2+\theta}{\omega'}=\sqrt{\frac{2m}{k}}\left[\frac{\pi}{2}+\sin^{-1}\sqrt{\frac{2}{5}}\right]$$

ILLUSTRATION 40

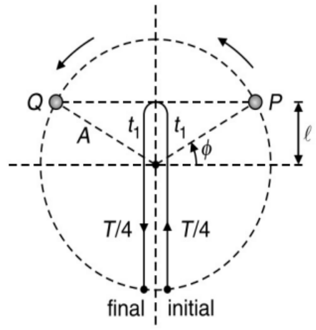
A block B of mass m resting on a smooth horizontal surface, attached to a spring of force constant k which is rigidly fixed on the wall on left side is shown in Figure.



At a distance l to the right of block there is a rigid wall. If block is pushed toward left so that spring is compressed by a distance $5l/3$ and released, it will start its oscillations. If collision of block with the wall is considered to be perfectly elastic. Find the time period of oscillations of the block. (Assume size of block to be small compared to l).

SOLUTION

Since the block is released from rest at a distance $5l/3$ from its mean position, so amplitude of oscillation is $A=\frac{5l}{3}$ and $\omega=\sqrt{\frac{k}{m}}$. However, on the other side the distance available is $l\,(<5l/3)$ due to presence of rigid wall. So, block will move a distance l (away from mean) collide elastically with wall to return to mean position. This situation is mapped on the circle shown in Figure.



Time taken to go from initial to final is

$$t=\frac{T}{4}+t_1+t_1+\frac{T}{4}=2\left(\frac{T}{4}+t_1\right)$$

where $t_1=\frac{\phi}{\omega}=\frac{\sin^{-1}(l/A)}{\omega}=\frac{\sin^{-1}(3/5)}{\omega}$

$$\Rightarrow\quad t_1=\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{3}{5}\right)$$

Hence, $t=2\left(\frac{T}{4}+t_1\right)$

$$\Rightarrow\quad t=2\left[\frac{1}{4}\left(2\pi\sqrt{\frac{m}{k}}\right)+\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{3}{5}\right)\right]$$

$$\Rightarrow\quad t=\sqrt{\frac{m}{k}}\left[\pi+2\sin^{-1}\left(\frac{3}{5}\right)\right]\quad\ldots(1)$$

Please note that, we can also calculate the time period of oscillation by subtracting time to go from P to Q from T.

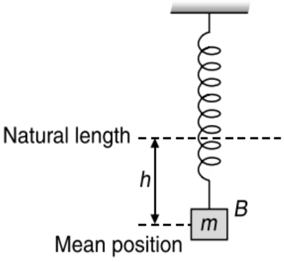
$$\Rightarrow\quad t=T-t_{P\to Q},\text{ where } t_{P\to Q}=\frac{2\theta}{\omega}=\frac{2\cos^{-1}(3/5)}{\omega}$$

$$\Rightarrow\quad t=\sqrt{\frac{m}{k}}\left[2\pi-2\cos^{-1}\left(\frac{3}{5}\right)\right]\quad\ldots(2)$$

(Both these equations (1) and (2) will give same numerical value).

ILLUSTRATION 41

A spring block system is hanging in equilibrium. If a velocity v_0 is imparted to the block of mass m in downward direction, calculate the amplitude of SHM of block and the time after which it will reach a point at half of the amplitude.



SOLUTION

Initially in equilibrium, we have $mg = kh$

The velocity v_0 imparted to the block at the mean position simply implies that this velocity is the maximum velocity, hence $v_0 = A\omega$

where A is amplitude of SHM and $\omega = \sqrt{\frac{k}{m}}$

$$\Rightarrow \quad A = \frac{v_0}{\omega} = v_0\sqrt{\frac{m}{k}}$$

Since the particle starts from the mean position, so we have

$$x = A\sin(\omega t)$$

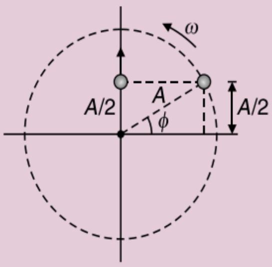
At $x = \frac{A}{2}$, we have $\frac{A}{2} = A\sin(\omega t)$

$$\Rightarrow \quad \sin(\omega t) = \frac{1}{2}$$

$$\Rightarrow \quad \omega t = \frac{\pi}{6}$$

$$\Rightarrow \quad t = \frac{\pi}{6\omega} = \frac{\pi}{6}\sqrt{\frac{m}{k}}$$

We can also calculate this result by mapping the SHM on a circle as shown in Figure.

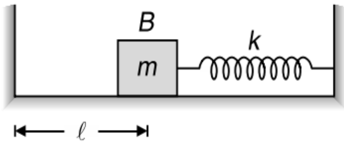


$$\phi = \sin^{-1}\left(\frac{A/2}{A}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad t = \frac{\phi}{\omega} = \frac{\pi}{6\omega}$$

ILLUSTRATION 42

A block B of mass m is resting on a horizontal smooth floor at a distance l from a rigid wall. Block is pushed to the right by a distance $3l/2$ and then released. When the block passes from its mean position, another block of mass M is placed gently on it such that both stick to each other due to friction. Calculate the value of M/m so that the combined block just collides with the left wall. (Ignore size of block compared to l).



SOLUTION

Amplitude is $A = \frac{3l}{2}$ but available space on left is $l\left(< \frac{3l}{2}\right)$.

When block B executes SHM, then $\omega = \sqrt{\frac{k}{m}}$. Applying conservation of linear momentum (at the mean position), we get

$$m(A\omega) = (M+m)A'\omega'$$

Since the combined block just collides with the left wall so, $A' = l$

$$\Rightarrow \quad m\left(\frac{3l}{2}\right)\sqrt{\frac{k}{m}} = (M+m)l\sqrt{\frac{k}{M+m}}$$

$$\Rightarrow \quad \frac{3}{2}\sqrt{m} = \sqrt{M+m}$$

$$\Rightarrow \quad 9m = 4(M+m)$$

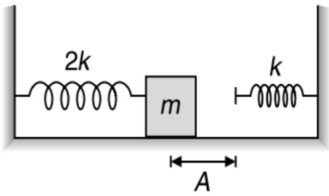
$$\Rightarrow \quad 9m = 4M + 4m$$

$$\Rightarrow \quad 4M = 5m$$

$$\Rightarrow \quad \frac{M}{m} = \frac{5}{4}$$

ILLUSTRATION 43

Initially the springs are unstretched as shown in Figure.



The left spring is now compressed by $2A$ by moving the mass m (which is always attached to it) and then released calculate the time to touch right spring, maximum compression in right spring and equilibrium position of mass m.

SOLUTION

When left spring is compressed by $2A$, then amplitude of oscillations will also be $2A$. On being released, the block will hit the right spring (in time t) when its displacement from the mean position is A i.e., $x = A$.

Since, $x = 2A\sin\left(\omega t + \frac{\pi}{2}\right) = 2A\cos(\omega t)$

$\Rightarrow \quad A = 2A\cos(\omega t)$

$\Rightarrow \quad \cos(\omega t) = \frac{1}{2}$

$\Rightarrow \quad \omega t = \frac{\pi}{3}$

$\Rightarrow \quad t = \frac{\pi}{3\omega} = \frac{\pi}{3}\sqrt{\frac{m}{2k}}$

Now when the mass m strikes the right spring, then it moves under combined influence of both the springs. Let the mass m be in equilibrium when the right spring is compressed by say x_0. Then

$$2k(A - x_0) = kx_0$$

$\Rightarrow \quad x_0 = \frac{2A}{3}$

If x_m be the maximum compression in the right spring, then by Work-Energy Theorem, we have

$$\frac{1}{2}(2k)(2A)^2 = \frac{1}{2}(2k)(A - x_m)^2 + \frac{1}{2}kx_m^2$$

$\Rightarrow \quad 8kA^2 = 2kA^2 + 2kx_m^2 - 4kAx_m + kx_m^2$

$\Rightarrow \quad 3x_m^2 - 4Ax_m - 6A^2 = 0$

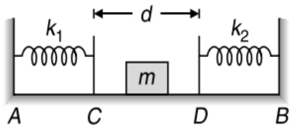
$\Rightarrow \quad x_m = \frac{4A \pm \sqrt{16A^2 - 4(3)(-6A^2)}}{2(3)}$

$\Rightarrow \quad x_m = \frac{4A + \sqrt{88A^2}}{6} = \left(\frac{4 + 2\sqrt{22}}{6}\right)A$

$\Rightarrow \quad x_m = \left(\frac{2 + \sqrt{22}}{3}\right)A$

ILLUSTRATION 44

Two light springs of force constant k_1, k_2 and a block of mass m are in one line AB on a smooth horizontal table such that one end of each spring is fixed on rigid supports and the other end is free as shown in Figure.



The distance between the free ends of the springs is d. If the block moves along AB with a velocity v in between springs, calculate the period of oscillation of the block.

SOLUTION

During motion, the block oscillates as a spring block system while in contact with both left and right spring for half the oscillation. So, the time it is in contact with springs is the sum of half time period of each spring, hence we have

$$\Delta t_1 = \frac{T_1}{2} + \frac{T_2}{2} = \pi\sqrt{\frac{m}{k_1}} + \pi\sqrt{\frac{m}{k_2}}$$

Between points C and D it moves uniformly, so

$$\Delta t_2 = \frac{2d}{v}$$

Total period is $T = \Delta t_1 + \Delta t_2$

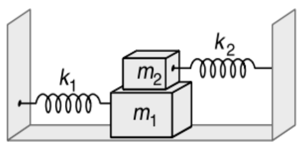
$\Rightarrow \quad T = \pi\left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}}\right) + \frac{2d}{v}$

Test Your Concepts-II

Based on Spring Mass Systems

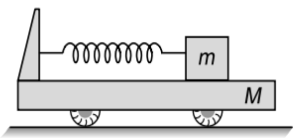
(Solutions on page H.181)

1. A block with mass M attached to a horizontal spring with force constant k is moving with simple harmonic motion having amplitude A_1. At the instant when the block passes through its equilibrium position a lump of putty with mass m is dropped vertically on the block from a very small height and sticks to it.
   (a) Find the new amplitude and period.
   (b) Repeat part (a) for the case in which the putty is dropped on the block when it is at one end of its path.
2. In the shown arrangement, both the springs are in their natural lengths. The coefficient of friction between m_2 and m_1 is μ. There is no friction between m_1 and the surface. If the blocks are displaced slightly, they together perform simple harmonic motion. Obtain



   (a) Frequency of such oscillations.
   (b) The condition if the frictional force on block m_2 is to act in the direction of its displacement from mean position.
   (c) If the condition obtained in (b) is met, what can be maximum amplitude of their oscillations?
3. A mass M attached to a spring oscillates with a period of 2 s. If the mass is increased by 2 kg, the period increases by one second. Calculate the initial mass M.

4. A mass m is attached to a cart of mass M by a spring of force constant k as shown in Figure.



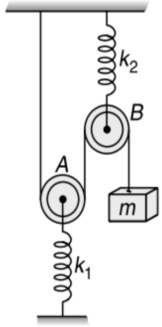
If the friction between m and surface of cart as well as between cart and floor is neglected, calculate the time period of small oscillation of cart-block system.

5. A 0.2 kg object hangs from an ideal spring of negligible mass. When the object is pulled down 0.1 m below its equilibrium position and released, it vibrates with a period of 1.80 s.
   (a) Find its speed as it passes through the equilibrium position.
   (b) Find its acceleration when it is 0.050 m above the equilibrium position.
   (c) Find the time required by the particle (when moving upwards) to move from a point 0.05 m below its equilibrium position to a point 0.05 m above it.
   (d) If the motion of the object is stopped at the mean position and the object is removed from the spring, then find the length by which the spring shortens.

6. A 1×10^{-2} kg block is resting on a horizontal frictionless surface and is attached to a horizontal spring whose spring constant is 124 Nm^{-1}. The block is given an initial speed of 8 ms^{-1} parallel to the spring axis, while the spring is initially unstrained. What is the amplitude of the resulting simple harmonic motion?

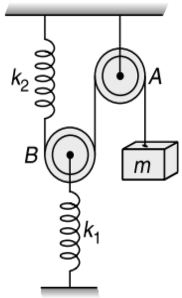
7. A mass m_1 sliding on a frictionless horizontal surface is attached to a spring of force constant k. It oscillates with amplitude A. When the spring is at its greatest extension and the mass is instantaneously at rest, a second mass m_2 is placed on top of m_1.
   (a) What is the smallest value of coefficient of static friction μ_s between the two masses so that m_2 does not slip over m_1.
   (b) Explain how the total energy E, the amplitude A, the angular frequency ω, and the period T are changed by placing m_2 on m_1 in this way, assuming no slipping. Check your results with the expression for the total energy, $E=\frac{1}{2}m\omega^2A^2$.

8. A block of mass m is attached to one end of a light inextensible string passing over a smooth light pulley B and under another smooth light pulley A as shown in the figure. The other end of a string is fixed to a ceiling. A and B are held by springs of spring constants k_1 and k_2. Find angular frequency of small oscillation of the system.

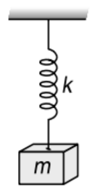


9. An object of mass m is supported by a vertical spring with force constant 1800 Nm^{-1}. When pulled down 2.5 cm and released, it oscillates at 5.5 Hz. Find the
   (a) mass m.
   (b) equilibrium stretching of the spring.
   (c) expressions for $x(t)$, $v(t)$, and $a(t)$.

10. A block of mass m is tied to one end of a string which passes over a smooth fixed pulley A and under a light smooth movable pulley B. The other end of the string is attached to the lower and of a spring of spring constant k_2. Find the period of small oscillations of mass m about its equilibrium positions.



11. A spring mass system is hanging from the ceiling of an elevator in equilibrium. The elevator suddenly starts accelerating upwards with acceleration a, find



   (a) the frequency
   (b) the amplitude of the resulting SHM

12. A spring with force constant $k=150\ \text{Nm}^{-1}$ is suspended from a rigid support. A 1 kg mass is attached at the bottom end and released from rest when the spring is unstretched.
   (a) How far down does the mass move before it starts up again?
   (b) How far below the starting point is the equilibrium position for the 1 kg mass?

(c) What is the period of the oscillation?

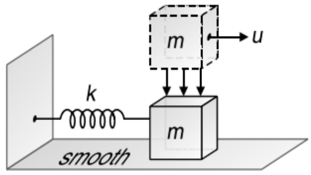
(d) Relative to this equilibrium position, what is the total energy of the mass-spring system?

(e) What is the speed of the mass when it first reaches the equilibrium position? How long after starting does the mass have this speed?

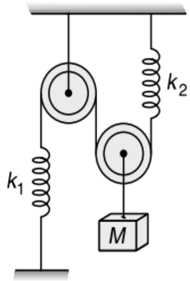
(f) If instead of being attached to the spring the mass had been dropped, when would it have reached the (previous) equilibrium level and what would its speed be? $\left(g = 9.8 \text{ ms}^{-2}\right)$.

13. A spring stretches by 0.018 m when a 2.8 kg object is suspended from its end. How much mass should be attached to this spring so that its frequency of vibration is $f = 3.0$ Hz?

14. A block of mass m resting on a smooth horizontal ground is attached to one end of a spring of force constant k in natural length. Another block of same mass moving with a horizontal velocity u towards right is gently placed on the block such that it sticks to it due to friction. Calculate the time it will take to reach its extreme position. Also find the amplitude of oscillations of the combined mass $2m$.



15. Find the time period of oscillation of M in the arrangement shown in the figure. The pulleys are smooth and massless.



ROTATIONAL SYSTEMS OR ANGULAR SHM

In rotational systems executing SHM, we first give a small angular displacement θ to the system about the mean position and then calculate the restoring torque τ of the system about the mean position as a function of θ. If τ is proportional to this small θ and is directed towards the mean position, then we can say the motion is SHM. Mathematically

$$\tau = -C\theta$$

where, C is a constant of proportionality. If I be the moment of inertia of the system about the specified axis of rotation, then $\tau = I\dfrac{d^2\theta}{dt^2}$.

$$\Rightarrow \quad I\frac{d^2\theta}{dt^2} = -C\theta$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \left(\frac{C}{I}\right)\theta = 0$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \text{ where } \omega = \sqrt{\frac{C}{I}}$$

Also, we can calculate the total mechanical energy E of the system at some intermediate instant and since for SHM

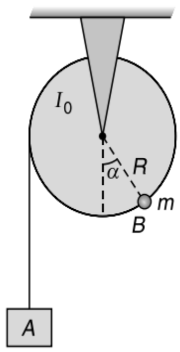
$$E = \text{constant}$$

$$\Rightarrow \quad \frac{dE}{dt} = 0$$

Following illustrations demonstrate the above written methods to calculate the time period of SHM executed by rotational systems.

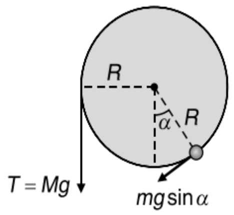
ILLUSTRATION 45

A pulley block system in which a block A is hanging on one side of pulley and on other side a small bead B of mass m is welded on pulley as shown in Figure. The moment of inertia of pulley is I_0 and the system is in equilibrium when bead is at an angle α from the vertical. If the system is slightly disturbed from its equilibrium position, find the time period of its oscillations.



SOLUTION

Let M be the mass of hanging block. In equilibrium, torque due to tension is balanced by torque due to m as shown in Figure. So, we have

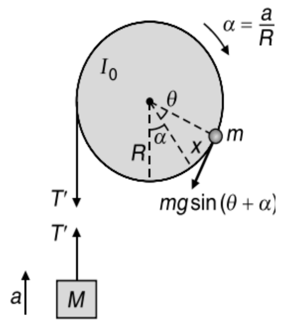


$$TR = (mg\sin\alpha)R$$

$$\Rightarrow \quad Mg = mg\sin\alpha$$

$$\Rightarrow \quad M = m\sin\alpha \qquad \ldots(1)$$

Let the bead be given a small angular displacement θ in the counter clockwise sense. On being released, the pulley rotates in clockwise sense and block moves upwards with an acceleration a as shown in Figure.



Applying Newton's Second Law for translational motion, we get

$$T' - Mg = Ma \qquad \ldots(2)$$

Applying Newton's Second Law for rotational motion, we get

$$mg\sin(\theta+\alpha)R - T'R = I_0\alpha$$

$$\Rightarrow \quad mg\sin(\theta+\alpha) - T' = \frac{I_0 a}{R^2} \qquad \ldots(3)$$

where, I_0 is moment of inertia of bead and pulley about the axis of rotation i.e., $I = mR^2 + I_0$

Substituting (2) in (3), we get

$$\left(M + \frac{I}{R^2}\right)a = mg\sin(\theta+\alpha) - Mg \qquad \ldots(4)$$

Since θ is small, so $\sin\theta \approx \theta$ and $\cos\theta \approx 1$

$$\sin(\theta+\alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\Rightarrow \quad \sin(\theta+\alpha) = \theta\cos\alpha + \sin\alpha \qquad \ldots(5)$$

Substituting value of (5) in (4), we get

$$\left(M + \frac{I_0}{R^2}\right)a = (mg\cos\alpha)\theta + mg\sin\alpha - Mg$$

$$\Rightarrow \quad \left(M + \frac{I_0}{R^2}\right)a = (mg\cos\alpha)\theta \qquad \{\because M = m\sin\alpha\}$$

$$\Rightarrow \quad a = -\frac{mg\cos\alpha}{\left(M + \frac{I}{R^2}\right)}\frac{x}{R}$$

Negative sign indicates that acceleration (and hence restoring force is directed towards mean position).

$$\Rightarrow \quad \ddot{x} = -\left[\frac{mg\cos\alpha}{R\left(M + m + \frac{I_0}{R^2}\right)}\right]x$$

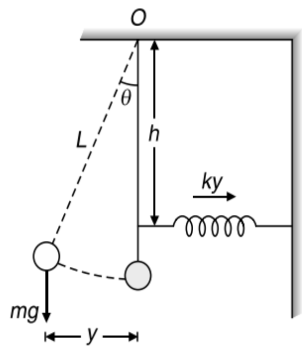
$$\Rightarrow \quad T = 2\pi\sqrt{\frac{x}{|\ddot{x}|}} = 2\pi\sqrt{\frac{R\left(m + m\sin\alpha + \frac{I_0}{R^2}\right)}{mg\cos\alpha}}$$

ILLUSTRATION 46

A simple pendulum of length L and mass m has as spring of force constant k connected to it at a distance h below its point of suspension. Find the frequency of vibrations of the system for small values of amplitude.

SOLUTION

Let the pendulum be given a small angular displacement θ about the mean position as shown in Figure.



Then, the spring will stretch by $y = h\tan\theta$.

So there restoring torque about O will be due to both force of gravity and elastic force of the spring.

$$\Rightarrow \quad \tau = -(mg(L\sin\theta) + k(h\tan\theta)h)$$

Now for small θ, $\tan\theta \approx \sin\theta \approx \theta$

$$\Rightarrow \quad \tau = -(mgL + kh^2)\theta \qquad \ldots(1)$$

Since restoring torque is linear, so motion is angular SHM.

Also, $\tau = I\alpha = mL^2\dfrac{d^2\theta}{dt^2}$, where $I = mL^2$

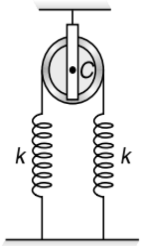
$$\Rightarrow \quad mL^2\frac{d^2\theta}{dt^2} + (mgL + kh^2)\theta = 0 \qquad \{\text{from (1)}\}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \text{ where } \omega^2 = \left(\frac{mgL + kh^2}{mL^2}\right)$$

$$\Rightarrow \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgL + kh^2}{mL^2}}$$

ILLUSTRATION 47

A 7 kg disk is free to rotate about a horizontal axis passing through its centre C. Determine the period of oscillation of the disk if the springs have sufficient tension in them to prevent the string from slipping on the disk as it oscillates. The radius of the disk is 10 cm and spring constant of both the springs is 600 Nm^{-1}.



SOLUTION

Let the angular displacement of the disk be θ. Angular velocity of disk at this instant is ω and compression and elongation in the springs is,

$$x = R\theta$$

Mechanical energy of the system in this position is,

$$E = \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 + \frac{1}{2}kx^2$$

$$\Rightarrow \quad E = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 + kx^2$$

$$\Rightarrow \quad E = \frac{1}{4}mR^2\omega^2 + kR^2\theta^2$$

Since, E is constant, so $\frac{dE}{dt} = 0$

$$\Rightarrow \quad \frac{1}{2}mR^2\omega\left(\frac{d\omega}{dt}\right) + 2kR^2\theta\left(\frac{d\theta}{dt}\right) = 0$$

Substituting $\frac{d\theta}{dt} = \omega$ and $\frac{d\omega}{dt} = \alpha = \ddot{\theta}$, we get

$$m\ddot{\theta} + 4k\theta = 0 \qquad \ldots(1)$$

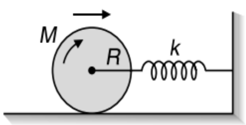
Since, α or $\ddot{\theta}$ is proportional to $-\theta$, the motion is simple harmonic, the time period of which is,

$$T = 2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|} = 2\pi\sqrt{\frac{\theta}{|\ddot{\theta}|}} = 2\pi\sqrt{\frac{m}{4k}} \qquad \{\text{from (1)}\}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{7}{4 \times 600}} = 0.34 \text{ s}$$

ILLUSTRATION 48

A solid cylinder of mass M and radius R is attached to a spring of stiffness k as shown in the figure. The cylinder can roll without slipping on a rough horizontal surface. Show that the centre of mass of the cylinder executes SHM and determine its time period.



SOLUTION

Consider the situation when the spring is extended by x. The total energy of the cylinder plus spring system is

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = \text{constant}$$

where, $I = \frac{1}{2}MR^2$ and $\omega = \frac{v}{R}$

$$\Rightarrow \quad \frac{3}{4}Mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Since, E is constant, so $\frac{dE}{dt} = 0$

$$\Rightarrow \quad \frac{3}{4}M(2v)\frac{dv}{dt} + \frac{1}{2}k(2x)\frac{dx}{dt} = 0$$

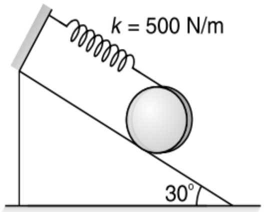
Noting that $v = \frac{dx}{dt}$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{2k}{3M}x = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{3M}{2k}}$$

ILLUSTRATION 49

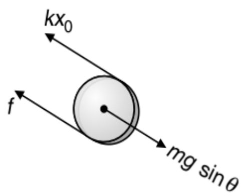
A 14 kg uniform cylinder can roll without sliding on a 30° incline. A belt is attached to the rim of the cylinder and a spring holds the cylinder at rest in the position shown in Figure.



If the cylinder is moved 50 mm down the incline and released, calculate the of small oscillations and the maximum acceleration of centre of cylinder.

SOLUTION

Let x_0 be the extension in the spring in equilibrium position and f the force of friction in the direction shown in figure.



Then for the equilibrium of cylinder

$$kx_0 + f = mg\sin\theta \qquad \ldots(1)$$

and $$(kx_0)R = fR \qquad \ldots(2)$$

From these two equations, we get

$$2kx_0 = mg\sin\theta \qquad \ldots(3)$$

Now, suppose the centre of the cylinder is now pulled down by a distance x, this will further stretch the spring by $2x$. Let v be the linear speed of the centre of the cylinder at that instant, then the angular speed of the cylinder at this instant will be $\omega = \frac{v}{R}$. Total mechanical energy of the system at this instant will be,

$$E = \frac{1}{2}k(x_0 + 2x)^2 - mg(x\sin\theta) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$\Rightarrow \quad E = \frac{1}{2}k(x_0+2x)^2 - mgx\sin\theta + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v^2}{R^2}\right)$$

$$\Rightarrow \quad E = \frac{1}{2}k(x_0+2x)^2 - mgx\sin\theta + \frac{3}{4}mv^2$$

Since, E is constant, so $\frac{dE}{dt} = 0$

$$\Rightarrow \quad 2k(x_0+2x)\frac{dx}{dt} - mg\sin\theta\frac{dx}{dt} + \frac{3}{2}mv\frac{dv}{dt} = 0$$

Since, $\frac{dx}{dt} = v$, $\frac{dv}{dt} = a$ and $2kx_0 = mg\sin\theta$

$$\Rightarrow \quad \frac{3}{2}ma = -4kx$$

Since, $a \propto -x$, motion is simple harmonic

$$\Rightarrow \quad T = 2\pi\sqrt{\left|\frac{x}{a}\right|} = 2\pi\sqrt{\frac{3m}{8k}} = 2\pi\sqrt{\frac{3\times 14}{8\times 5000}} = 0.203 \text{ s}$$

Maximum acceleration of centre of cylinder is

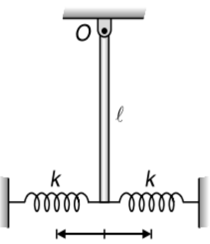
$$a_{max} = \omega^2 A$$

$$\Rightarrow \quad a_{max} = \left(\frac{2\pi}{T}\right)^2 A, \text{ where } A = 50 \text{ mm} = 0.05 \text{ m}$$

$$\Rightarrow \quad a_{max} = \left(\frac{2\pi}{0.203}\right)^2(0.05) = 47.9 \text{ ms}^{-2}$$

ILLUSTRATION 50

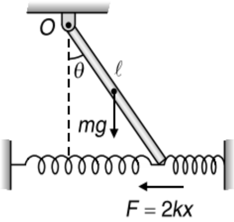
Calculate the frequency of small oscillations of a thin uniform vertical rod of mass m and length l hinged at the point O as shown in Figure.



The combined stiffness of each of the spring is equal to k. The mass of the spring is negligible.

SOLUTION

When the rod is displaced by a small angle θ as shown in Figure, then both the springs are deformed by a distance $x = l\theta$.



We observe that the left spring is stretched and the right spring is compressed, so that both the springs will exert a torque on rod in same sense as the restoring torque. The net restoring torque on rod is given by

$$\tau = -\left[(2kx)(l\sin\theta) + (mg)\left(\frac{l}{2}\sin\theta\right)\right]$$

Since θ is small, so $\sin\theta \approx \theta = \frac{x}{l}$

$$\Rightarrow \quad \tau = -\left(2kxl\theta + \frac{mgl\theta}{2}\right)$$

$$\Rightarrow \quad \tau = -\left(2kl^2\theta + \frac{mgl\theta}{2}\right) \qquad \{\because x = l\theta\}$$

$$\Rightarrow \quad \tau = -\left(2kl^2 + \frac{mgl}{2}\right)\theta$$

Negative sign indicates restoring nature of force. If rod has an angular acceleration α, then, restoring torque is

$$\tau = I_0\alpha$$

$$\Rightarrow \quad \tau = \left(\frac{ml}{3}\right)^2\alpha$$

where, $I_0 = \frac{ml^2}{3}$ is the moment of inertia of rod about an axis passing through O

$$\Rightarrow \quad \frac{ml^2}{3}\alpha = -\left(2kl^2 + \frac{mgl}{2}\right)\theta$$

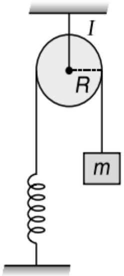
$$\Rightarrow \quad \ddot{\theta} = -\left(\frac{2kl^2 + (mgl/2)}{ml^2/3}\right)\theta \qquad \left\{\because \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}\right\}$$

$$\Rightarrow \quad \ddot{\theta} = -\left(\frac{12kl + 3mg}{2ml}\right)\theta$$

$$\Rightarrow \quad f = \frac{1}{2\pi}\sqrt{\left|\frac{\ddot{\theta}}{\theta}\right|} = \frac{1}{2\pi}\sqrt{\frac{12kl + 3mg}{2ml}}$$

ILLUSTRATION 51

A pulley block system is initially in equilibrium. If the block is displaced down slightly from its equilibrium position and released, calculate the time period of oscillation of the system if the pulley has radius R and moment of inertia I. Assume there is sufficient friction present between pulley and string so that string will not slip over pulley surface.

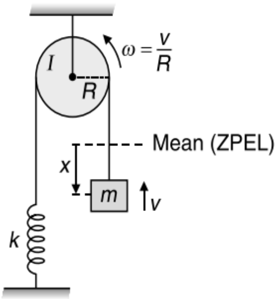


SOLUTION

Initially, in equilibrium let extension in the spring be h. Then, we have

$$mg = kh \quad \ldots(1)$$

When the block is released, then at any instant let it be at a distance x below the mean position and moving with a velocity v towards it as shown in Figure.



Total mechanical energy of block at this instant is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}k(x+h)^2 - mgx \quad \ldots(2)$$

For no slipping of string on pulley surface, we have

$$v = R\omega \quad \ldots(3)$$

So, equation (2) becomes

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v^2}{R^2}\right) + \frac{1}{2}k(x+h)^2 - mgx$$

$$\Rightarrow \quad E = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2 + \frac{1}{2}k(x+h)^2 - mgx$$

Since E is constant, so we have $\frac{dE}{dt} = 0$

$$\Rightarrow \quad \left(m + \frac{I}{R^2}\right)\frac{vdv}{dt} + kv(x+h) - mgv = 0$$

$$\Rightarrow \quad \left(m + \frac{I}{R^2}\right)\frac{d^2x}{dt^2} + kx + kh - mg = 0 \quad \ldots(4)$$

Substituting equation (1) in (4), we get

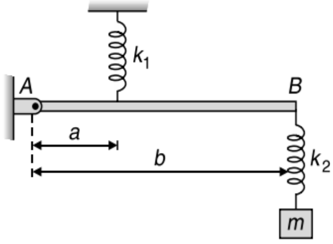
$$\left(m + \frac{I}{R^2}\right)\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \quad \ddot{x} + \left[\frac{k}{\left(m + \frac{I}{R^2}\right)}\right]x = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{x}{|\ddot{x}|}} = 2\pi\sqrt{\frac{m + \frac{I}{R^2}}{k}}$$

ILLUSTRATION 52

Calculate the time period of oscillations of the system shown in Figure. The bar is rigid and light. The springs are also light. Initially in equilibrium bar is horizontal.



SOLUTION

Initially in equilibrium, let h_1 and h_2 be extensions in springs, then

$$(k_1h_1)a = (k_2h_2)b \quad \ldots(1)$$

Also, $k_2h_2 = mg$ ...(2)

When mass m is displaced slightly by x (downwards), then let further extensions in the springs be x_1 and x_2. Given that the bar is light, so we have

$$k_1(x_1 + h_1)a \approx k_2(x_2 + h_2)b$$

$$\Rightarrow \quad k_1x_1a = k_2x_2b \quad \ldots(3)$$

Also, when left spring extends by x_1, then the end B moves down by $x_1\left(\frac{b}{a}\right)$, due to which total extension x is given by

$$x = x_2 + x_1\left(\frac{b}{a}\right) \quad \ldots(4)$$

$$\Rightarrow \quad x = x_2 + \left(\frac{k_2x_2b}{k_1a}\right)\frac{b}{a} = x_2\left(1 + \frac{k_2b^2}{k_1a^2}\right)$$

$$\Rightarrow \quad x_2 = \frac{x}{1 + \left(k_2b^2/k_1a^2\right)}$$

Restoring force (F) acting on mass m is

$$F = -(k_2x_2 - mg)$$

Negative sign indicates that F is directed towards the mean position.

$$\Rightarrow \quad F = -\left[\frac{k_2}{1 + \left(k_2b^2/k_1a^2\right)}x - mg\right]$$

$$\Rightarrow \quad \ddot{x} = -\left[\frac{k_1k_2a^2}{m\left(k_1a^2+k_2b^2\right)}\right]x+g$$

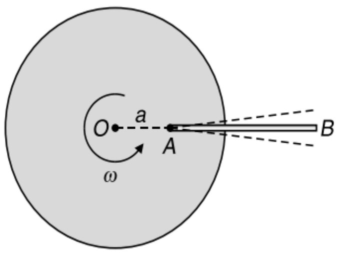
$$\Rightarrow \quad \ddot{x} = -\omega^2x+g$$

$$\Rightarrow \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{k_1k_2a^2}{m\left(k_1a^2+k_2b^2\right)}}$$

$$\Rightarrow \quad T = \frac{2\pi}{a}\sqrt{\frac{m\left(k_1a^2+k_2b^2\right)}{k_1k_2}}$$

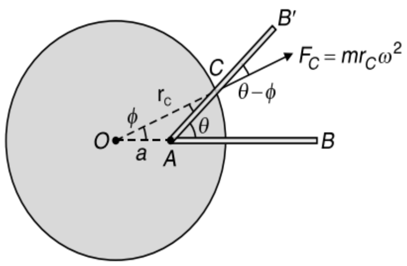
ILLUSTRATION 53

A smooth horizontal disc rotates about the vertical axis O with a constant angular velocity ω. A thin uniform rod AB of length l performs small oscillations about the vertical axis A fitted to the disc at a distance a from the axis of the disc as shown in Figure. Calculate the angular frequency of these oscillations.



SOLUTION

In plane of disc, let the rod be tilted slightly by an angle θ as shown in Figure.



The restoring torque (τ) on the rod is

$$\tau = -\left[\left(mr_C\omega^2\right)\sin(\theta-\phi)\right]\frac{l}{2}$$

Since, $a\phi = \frac{l}{2}(\theta-\phi)$

$$\Rightarrow \quad \phi = \frac{l\theta/2}{a+(l/2)} \qquad \ldots(1)$$

Also, $r_C \approx a+\frac{l}{2}$ $\qquad \{\because \theta \text{ is small}\}$

$$\Rightarrow \quad \tau = -\frac{mr_C\omega^2 l}{2}(\theta-\phi) \qquad \ldots(2)$$

Substituting equation (2) in equation (1), we get

$$\tau = -\frac{mr_C\omega^2 l}{2}\left(1-\frac{l/2}{a+(l/2)}\right)\theta$$

$$\Rightarrow \quad \tau = -\left(\frac{ml\omega^2a}{2}\right)\theta$$

$$\Rightarrow \quad \ddot{\theta}+\left(\frac{ml\omega^2a}{2I}\right)\theta = 0 \qquad \{\because \tau = I\ddot{\theta}\}$$

where, I is the moment of inertia of the rod about A i.e., $I = \frac{ml^2}{3}$

$$\Rightarrow \quad \ddot{\theta}+\left[\frac{ml\omega^2a}{2\left(ml^2/3\right)}\right]\theta = 0$$

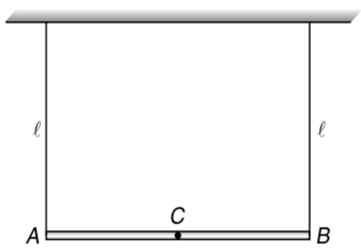
$$\Rightarrow \quad \ddot{\theta}+\left(\frac{3\omega^2a}{2l}\right)\theta = 0$$

If ω_0 is the angular frequency of oscillations, then

$$\omega_0 = \sqrt{\frac{|\ddot{\theta}|}{\theta}} = \sqrt{\frac{3\omega^2a}{2l}}$$

ILLUSTRATION 54

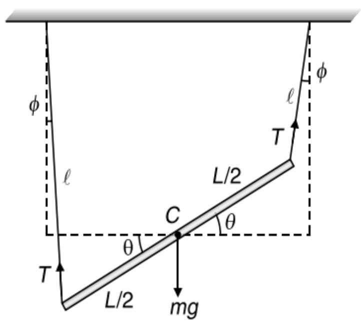
A uniform rod AB of mass m is suspended by two identical strings of length l as shown in Figure.



The rod is turned through a small angle in horizontal plane about and the vertical axis is passing through its centre C. In this process the strings are deviated by a small angle θ_0 from vertical. The rod is then released so that it starts performing angular SHM. Calculate the angular frequency of oscillations and the rod's oscillation energy.

SOLUTION

When the rod is twisted through an angle θ, then let the string deviate through an angle ϕ as shown in Figure.



Assuming length of rod to be L, we have

$$\frac{L}{2}\theta = l\phi \qquad \ldots(1)$$

Since the rod is in vertical equilibrium, so

$$2T\cos\phi = mg$$

where, T is tension in each string holding the rod.
For small ϕ, $\cos\phi \approx 1$, so $2T = mg$

$$\Rightarrow \quad T = \frac{mg}{2} \qquad \ldots(2)$$

The component of tension $T\sin\phi$ acting on each end of rod produces a restoring torque given by

$$\tau = -(2T\sin\phi)\left(\frac{L}{2}\right)$$

Negative sign shows that restoring torque is directed towards mean position. For small ϕ, $\sin\phi \approx \phi$

$$\Rightarrow \quad \tau = I\ddot{\theta} = -TL\phi$$

Using equations (1) and (2), we get

$$\tau = -\left(\frac{mgL}{2}\right)\left(\frac{L}{2l}\right)\theta = -\left(\frac{mgL^2}{4l}\right)\theta$$

$$\Rightarrow \quad I\ddot{\theta} + \frac{mgL^2}{4l}\theta = 0, \text{ where } I = \frac{mL^2}{12}$$

$$\Rightarrow \quad \ddot{\theta} + \left(\frac{mgL^2}{4lI}\right)\theta = 0$$

$$\Rightarrow \quad \omega = \sqrt{\frac{mgL^2}{4lI}} = \sqrt{\frac{mgL^2}{4l\left(mL^2/12\right)}} = \sqrt{\frac{3g}{l}}$$

Since it is given that initially strings are deflected by an angle θ_0, so if β is the angular amplitude of oscillations, then

$$\frac{L}{2}\beta = l\theta_0 \qquad \text{\{from equation (1)\}}$$

$$\Rightarrow \quad \beta = \left(\frac{2l}{L}\right)\theta_0$$

The total energy of oscillation of rod is $E = \frac{1}{2}I\omega^2\beta^2$

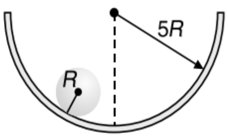
$$\Rightarrow \quad E = \frac{1}{2}\left(\frac{mL^2}{12}\right)\left(\frac{3g}{l}\right)\left(\frac{4l^2}{L^2}\theta_0^2\right) = \frac{1}{2}mgl\theta_0^2$$

Test Your Concepts-III

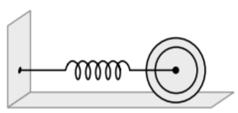
Based on Rotational SHM

(Solutions on page H.183)

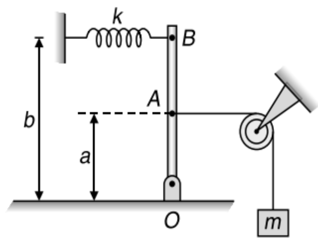
1. A solid sphere of radius R rolls without slipping in a cylindrical trough of radius $5R$. Find the time period of small oscillations.



2. The disk has a weight of 100 N and rolls without slipping on the horizontal surface as it oscillates about its equilibrium position. If the disk is displaced, by rolling it counter clockwise 0.4 rad, determine the equation which describes its oscillatory motion when it is released.

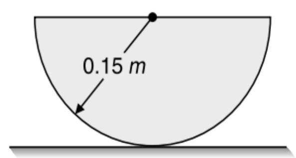


3. A massless rigid rod is hinged at O. A string carrying a mass m at one end is attached to point A on the rod. At another point B of the rod, a horizontal spring of force constant k is attached as shown in Figure.

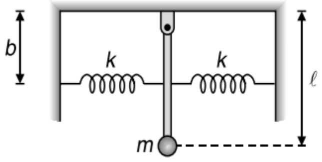


Calculate the period of small vertical oscillations of mass m around its equilibrium position. Consider that the rod is initially vertical and is in equilibrium at that instant.

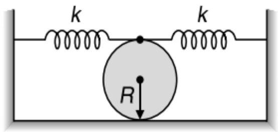
4. Determine the natural period of vibration of the 50 N semi-circular disk. Given that the centre of mass of a semi-circular disk lies at a distance of $\frac{4r}{3\pi}$ from the centre.



5. Calculate the time period of small oscillations of the spring loaded pendulum. The equilibrium position is vertical as shown in Figure. The mass of the rod is negligible and treat the mass as a particle.

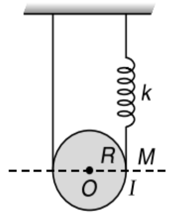


6. A solid uniform cylinder of mass M performs small oscillations in horizontal plane if slightly displaced from its mean position shown in Figure.

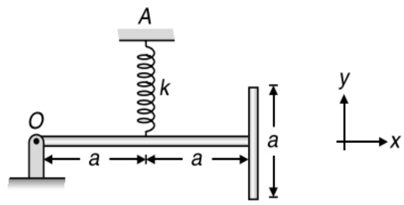


Initially springs are in natural lengths and cylinder does not slip on ground during oscillations due to friction between ground and cylinder. If force constant of each spring is k, then calculate the time period of small oscillations of cylinder.

7. The pulley shown in Figure has a moment of inertia I about its axis and mass m. Calculate the time period of vertical oscillation of its centre of mass. The spring is light, has spring constant k and the string does not slip over the pulley.



8. A T bar of uniform cross section and mass M is supported in a vertical plane by a hinge O and a spring of force constant k at A. Calculate the period for small amplitude rotational oscillations in the xy plane.

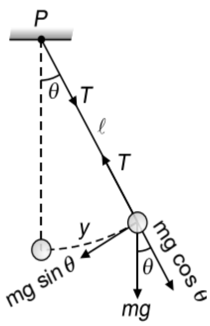


SIMPLE PENDULUM

A simple pendulum consists of a point mass suspended from a fixed point by a light inextensible string. In equilibrium, the mass lies vertically below the point of suspension P. If it is displaced to one side and then released, it oscillates about the equilibrium position.

Suppose at any instant the pendulum makes an angle θ with the vertical. The forces acting on the mass m are its weight mg and tension T in the string. The weight mg may be resolved into two components, the radial component $mg\cos\theta$ and the tangential component $mg\sin\theta$ is the restoring force. Hence

$$F = -mg\sin\theta$$



Note that the restoring force is not proportional to θ but to $\sin\theta$. The motion is, therefore, not simple harmonic. However, if θ is small, then $\sin\theta \approx \theta$

$$\Rightarrow \quad F = -mg\theta$$

The displacement along the arc is $y = \theta l$ and for small θ this is nearly straight-line motion. Hence

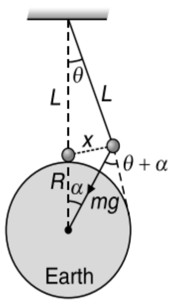
$$\Rightarrow \quad F = m\ddot{y} = -\left(\frac{mg}{l}\right)y$$

$$\Rightarrow \quad \ddot{y} + \frac{g}{l}y = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{y}{|\ddot{y}|}} = 2\pi\sqrt{\frac{l}{g}}$$

SHM OF A PENDULUM OF LARGE LENGTH

In the derivation of a simple pendulum, we had assumed the length of the pendulum to be much less than the radius (R) of earth so that the force mg is always directed vertically downwards. However, if the length (L) of the pendulum is large, then the force mg will be directed towards the centre of the earth as shown in Figure.



When bob is displaced by amount x, restoring force towards mean position will be $mg\sin(\theta+\alpha)$. From free body diagram (FBD), we have

$$F = ma = m\ddot{x} = -mg\sin(\theta+\alpha)$$

For small displacement, θ is small, hence α is also small, i.e., $(\theta+\alpha)$ is small, so $\sin(\theta+\alpha) \approx \theta+\alpha$

$$\Rightarrow \quad ma = m\ddot{x} = -mg(\theta+\alpha)$$

$$\Rightarrow \quad \ddot{x} + g\left(\frac{1}{L}+\frac{1}{R}\right)x = 0$$

$$\Rightarrow \quad \omega = \sqrt{\frac{|\ddot{x}|}{x}} = \sqrt{g\left(\frac{1}{L}+\frac{1}{R}\right)}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{L}+\frac{1}{R}\right)}}$$

CASE-I:

When $L \ll R$, when length of pendulum is small compared to radius of the earth, then

$$g\left(\frac{1}{L}+\frac{1}{R}\right) \approx \frac{g}{L}$$

$$\Rightarrow \quad T \approx 2\pi\sqrt{\frac{L}{g}}$$

CASE-II:

When $L \gg R$, as in the case of infinite pendulum, then

$$g\left(\frac{1}{L}+\frac{1}{R}\right) \approx \frac{g}{R}$$

$$\Rightarrow \quad T \approx 2\pi\sqrt{\frac{R}{g}}$$

ILLUSTRATION 55

A ball is suspended by a thread of length L at the point O on the wall PQ which is inclined to the vertical by an angle α. The thread with the ball is now displaced through a small angle β away from the vertical and also from the wall. If the ball is released, find the period of oscillation of the pendulum when (a) $\beta < \alpha$ (b) $\beta > \alpha$.

Assume the collision on the wall to be perfectly elastic.

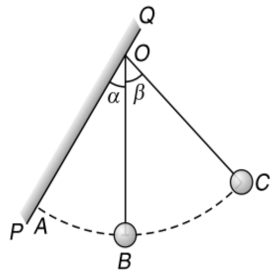
SOLUTION

The motion of simple pendulum is angular SHM; so, its equation of motion will be

$$\theta = \theta_0 \sin\omega t \text{ with } \omega = \sqrt{\frac{g}{L}}$$

(a) When $\beta < \alpha$, i.e., when angular amplitude β is lesser than α, the pendulum will oscillate with its natural frequency, so that

$$T_1 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \qquad \ldots(1)$$



(b) When $\beta > \alpha$, time taken by pendulum to move from B to C and back to B,

$$t_1 = \frac{T}{2} = \frac{1}{2}\left(2\pi\sqrt{\frac{L}{g}}\right) = \pi\sqrt{\frac{L}{g}} \qquad \ldots(2)$$

Now as in case of simple harmonic motion, $\theta = \theta_0 \sin\omega t$ So, time taken by the pendulum to move from equilibrium position B to A,

i.e., for $\theta = \alpha$ when $\theta_0 = \beta$, will be given by $\alpha = \beta\sin\omega t$,

i.e., $t = \frac{1}{\omega}\sin^{-1}\left(\frac{\alpha}{\beta}\right)$

So, time taken by pendulum to move from B to A and back to B,

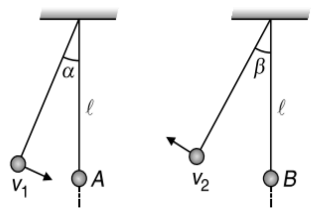
$$t_2 = 2t = 2\sqrt{\frac{L}{g}}\sin^{-1}\left(\frac{\alpha}{\beta}\right) \qquad \left\{\text{since } \omega = \sqrt{\frac{g}{L}}\right\}$$

So, time period of motion,

$$T_2 = t_1 + t_2 = \sqrt{\frac{L}{g}}\left(\pi + 2\sin^{-1}\left(\frac{\alpha}{\beta}\right)\right)$$

ILLUSTRATION 56

Figure shows two identical simple pendulums of length l. One is tilted at an angle α and imparted an initial velocity v_1 toward mean position and at the same time other is projected away from mean position with a velocity v_2 at an initial angular displacement β. Calculate the phase difference in oscillations of these two pendulums.



SOLUTION

Given that first pendulum bob is given a velocity v_1 at a displacement $l\alpha$ from mean position.

Since, $v^2 = \omega^2(A^2 - x^2)$

$$\Rightarrow \quad v_1 = \omega\sqrt{A_1^2 - (l\alpha)^2}$$

where, A_1 is the amplitude of SHM of this bob

For simple pendulum, $\omega = \sqrt{\frac{g}{l}}$

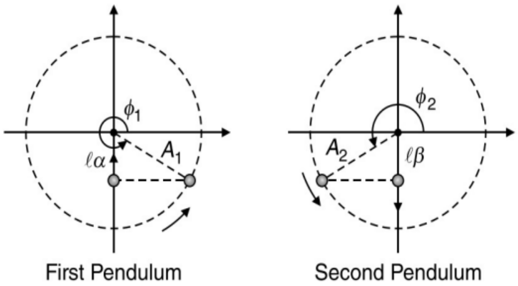
$$\Rightarrow \quad v_1^2 = \frac{g}{l}\left(A_1^2 - l^2\alpha^2\right)$$

$$\Rightarrow \quad A_1 = \sqrt{l^2\alpha^2 + \frac{v_1^2 l}{g}} \qquad \ldots(1)$$

Similarly, if A_2 is the amplitude of SHM of second pendulum, then we have

$$v_2 = \omega\sqrt{A_2^2 - (l\beta)^2}$$

$$\Rightarrow \quad A_2 = \sqrt{l^2\beta^2 + \frac{v_2^2 l}{g}} \qquad \ldots(2)$$



$$\phi_1 = 2\pi - \sin^{-1}\left(\frac{l\alpha}{A_1}\right)$$

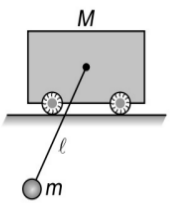
$$\phi_2 = \pi + \sin^{-1}\left(\frac{l\beta}{A_2}\right)$$

$$\Rightarrow \quad \Delta\phi = \pi + \sin^{-1}\left(\frac{l\beta}{A_2}\right) - 2\pi + \sin^{-1}\left(\frac{l\alpha}{A_1}\right)$$

$$\Rightarrow \quad \Delta\phi = \sin^{-1}\left(\frac{l\alpha}{A_1}\right) + \sin^{-1}\left(\frac{l\beta}{A_2}\right) - \pi$$

ILLUSTRATION 57

A trolley of mass M that can slide on frictionless rails has a pendulum bob of mass m connected to it through a massless inextensible string of length l. The pendulum can swing in the vertical plane. Calculate the time period of oscillation of the pendulum if it is left free after giving a small displacement.



SOLUTION

Since there is no external force in the horizontal direction, the centre of mass (CM) will not move along the horizontal direction.

Since the CM is at a distance $l' = \dfrac{Ml}{M+m}$ from m.

The pendulum can be considered to be oscillating from the point of suspension i.e., CM and having string of length l'. So, its time period is

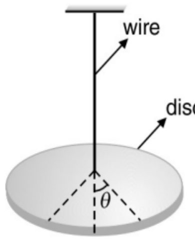
$$T = 2\pi\sqrt{\frac{l'}{g}} = 2\pi\sqrt{\frac{Ml}{(M+m)g}}$$

TORSIONAL PENDULUM

On rotating a body from its position of equilibrium, a restoring torque proportional to angle of rotation comes into play, the body executes angular (or rotational) SHM If τ is torque when the angle of rotation is θ, then

$$\tau \propto \theta$$

$\Rightarrow \quad \tau = -C\theta,$ where C is called **torsional constant**.



If I is the moment of inertia of the body about the specified axis of rotation and α is the angular acceleration, then

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -C\theta$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

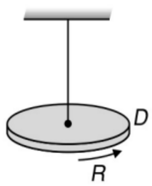
This is the differential equation of angular SHM the time period is clearly

$$T = 2\pi\sqrt{\frac{\theta}{|\ddot{\theta}|}} = 2\pi\sqrt{\frac{I}{C}}$$

A typical torsional pendulum is a disc suspended by a wire attached to the centre of mass of the disc. When the disc is rotated, the wire gets twisted and a restoring torque is produced in it. The disc, therefore, executes angular oscillations on being released.

ILLUSTRATION 58

A torsional pendulum consists of a uniform disc D of mass M and radius R attached to a thin rod of torsional constant C as shown in Figure.



Calculate the amplitude and the energy of small torsional oscillations of the disc, if initially the disc was imparted angular speed ω_0.

SOLUTION

For a torsional pendulum, we have $\omega = \sqrt{\dfrac{C}{I}}$, where $I = \dfrac{1}{2}MR^2$. So, $\omega = \sqrt{\dfrac{2C}{MR^2}}$

Since the disc is imparted an angular speed ω_0 (at mean position), so if β is the angular amplitude, then

$$\omega_0 = \beta\omega = \beta\sqrt{\frac{2C}{MR^2}}$$

$$\Rightarrow \quad \beta = \sqrt{\frac{MR^2}{2C}}\omega_0$$

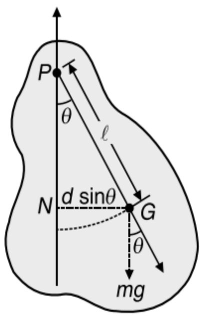
For angular SHM, assuming $U_{min} = 0$, the total energy of oscillation is given by

$$E = \frac{1}{2}I\omega^2\beta^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{2C}{MR^2}\right)\left(\frac{MR^2}{2C}\omega_0^2\right)$$

$$\Rightarrow \quad E = \frac{1}{4}MR^2\omega_0^2$$

PHYSICAL PENDULUM OR COMPOUND PENDULUM

Any rigid body mounted so that it is capable of swinging in a vertical plane about some axis passing through it is called **a physical pendulum**. It is an example of angular SHM



Consider a body of irregular shape pivoted about a horizontal frictionless axis passing through P and displaced from equilibrium position by angle θ. The equilibrium position is that in which the centre of gravity G lies vertically below P. If M is mass of body and I, the moment of inertia of the body about an axis passing through pivot P, then

$$\tau = -(Mg)(l\sin\theta)$$

$$\Rightarrow \quad I\alpha = -(Mgl)\theta \qquad \{\theta \text{ small, so } \sin\theta \approx \theta\}$$

$$\Rightarrow \quad I\ddot{\theta} + (Mgl)\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + \left(\frac{Mgl}{I}\right)\theta = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{\theta}{|\ddot{\theta}|}} = 2\pi\sqrt{\frac{I}{Mgl}}$$

Further by Parallel Axis Theorem,

$$I = I_G + Ml^2$$

$$\Rightarrow \quad I = MK^2 + Ml^2 \qquad \{\because I_G = MK^2\}$$

where K is the radius of gyration of body.

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{MK^2 + Ml^2}{Mgl}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{1}{g}\left(l + \frac{K^2}{l}\right)} = 2\pi\sqrt{\frac{l_{\text{effective}}}{g}}$$

$l_{\text{eff}} = l + \dfrac{K^2}{l}$ is also called **the effective length of the compound pendulum**. When $l = 0$, then $T \to \infty$.

The period of oscillation T of the physical pendulum is minimum when T^2 is minimum i.e., $\dfrac{d}{dl}(T^2) = 0$

$$\Rightarrow \quad \frac{4\pi^2}{g}\frac{d}{dl}\left(l + \frac{K^2}{l}\right) = 0$$

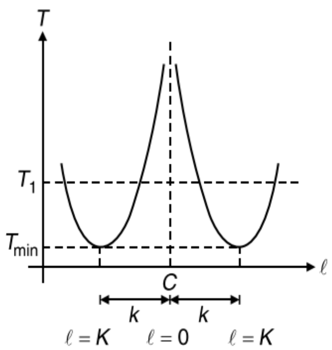
$$\Rightarrow \quad 1 - \frac{K^2}{l^2} = 0$$

$$\Rightarrow \quad l = \pm K$$

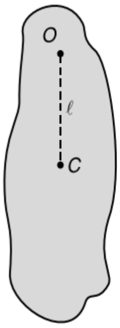
The period of oscillation T of the physical pendulum is minimum when the distance of the point of suspension from the centre of mass of the pendulum is equal to the radius of gyration K of the body.

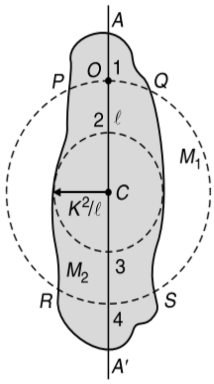
$$T_{\min} = 2\pi\sqrt{\frac{1}{g}\left(K + \frac{K^2}{K}\right)} = 2\pi\sqrt{\frac{2K}{g}}$$

Experimentally, the plot of T vs l is shown in Figure.



We observe that in a compound pendulum when a straight line AA' is drawn passing through the centre of gravity of the body as shown in Figure, then there exist four points 1, 2, 3, 4 on this line about which if a body is suspended, then the time period of small oscillation of body remains same.





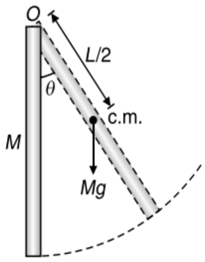
This can also be seen from the plot of T vs l discussed earlier.

ILLUSTRATION 59

A rod of mass M and length L is pivoted about its end O as shown in the figure. Find the period of SHM.

SOLUTION

Restoring torque is $\tau_0 = -Mg\frac{L}{2}\sin\theta$



Using Newton's Second Law and the small angle approximation, we get

$$I_0\frac{d^2\theta}{dt^2} + Mg\frac{L}{2}\theta = 0$$

where, $I_0 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \left(\frac{3g}{2L}\right)\theta = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{2L}{3g}}$$

ILLUSTRATION 60

A uniform rod of mass m and length L performs small oscillations about a horizontal axis passing through its upper end. Find the mean kinetic energy of the rod during its oscillation period if at $t = 0$ it is deflected from vertical by an angle θ_0 and imparted an angular velocity ω_0.

SOLUTION

This rod oscillates like a physical pendulum whose time period is given by $T = 2\pi\sqrt{\frac{I}{Mgd}}$, where I is moment of inertia of rod about axis passing through point of suspension, d is separation between centre of mass of rod and the point of suspension.

In the given case

$$I = \frac{ML^2}{2} \text{ and } d = \frac{L}{2}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{2L}{3g}} \text{ i.e., } \omega = \sqrt{\frac{3g}{2L}}$$

Initially angular displacement of rod is θ_0 and it is imparted an angular velocity ω_0. If angular amplitude be β, then we have

$$\omega_0 = \omega\sqrt{\beta^2 - \theta_0^2} = \sqrt{\frac{3g}{2L}}\sqrt{\beta^2 - \theta_0^2}$$

$$\Rightarrow \quad \beta = \sqrt{\frac{2L\omega_0^2}{3g} + \theta_0^2}$$

Mean kinetic energy of rod during oscillation is

$$\langle K \rangle = \frac{1}{4}I\omega^2\beta^2$$

$$\Rightarrow \quad \langle K \rangle = \frac{1}{4}\left(\frac{ML^2}{3}\right)\left(\frac{3g}{2L}\right)\left(\frac{2L\omega_0^2}{3g} + \theta_0^2\right)$$

$$\Rightarrow \quad \langle K \rangle = \frac{MgL}{8}\left(\frac{2L\omega_0^2}{3g} + \theta_0^2\right)$$

ILLUSTRATION 61

A uniform rod of mass m and length l performs small oscillations about the horizontal axis passing through its upper end. Find the mean kinetic energy of the rod averaged over one oscillation period if at the initial moment it was deflected from the vertical by an angle θ_0 and then imparted an angular velocity Ω.

SOLUTION

Angular frequency ω of a physical pendulum is

$$\omega = \sqrt{\frac{mgd}{I}}$$

where $d = \frac{l}{2}, I = \frac{ml^2}{3}$

$$\Rightarrow \quad \omega = \sqrt{\frac{3g}{2l}}$$

Let β be angular amplitude of rod, then

$$\Omega = \omega\sqrt{\beta^2 - \theta_0^2}$$

$$\Rightarrow \quad \beta^2 = \frac{\Omega^2}{\omega^2} + \theta_0^2$$

Mean kinetic energy of rod is

$$\langle K \rangle = \frac{1}{4}I\omega^2\beta^2 = \frac{1}{4}\left(\frac{ml^2}{3}\right)\left(\frac{3g}{2l}\right)\left(\frac{\Omega^2(2l)}{3g} + \theta_0^2\right)$$

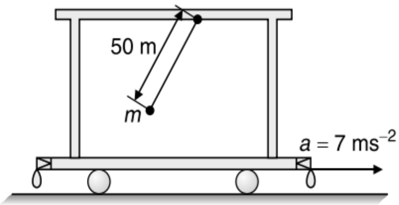
$$\Rightarrow \quad \langle K \rangle = \frac{1}{8}mgl\theta_0^2 + \frac{1}{12}ml^2\Omega^2$$

Test Your Concepts-IV

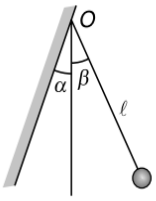
Based on Pendulum Systems

(Solutions on page H.185)

1. A pendulum is secured on a cart rolling without friction down an inclined plane of inclination α. The period of the pendulum on an immobile cart is T_0. How will the period of the pendulum change when the cart rolls down the slope?
2. Pendulum A is a physical pendulum made from a thin, rigid and uniform rod whose length is d. One end of this rod is attached to the ceiling by a frictionless hinge, so the rod is free to swing back and forth. Pendulum B is a simple pendulum whose length is also d. Obtained the ratio $\frac{T_A}{T_B}$ of their periods for small-angle oscillations.
3. Determine the period of small oscillations of a mathematical pendulum, that is ball suspended by a thread $\ell = 20$ cm in length, if it is located in a liquid whose density is $\eta = 3.0$ times less than that of the ball. The resistance of the liquid is to be neglected.
4. A simple pendulum 50 cm long is suspended from the roof of a cart accelerating in the horizontal direction with $a = 7\ \text{ms}^{-2}$ (figure). Find the period of small oscillations of the pendulum about its equilibrium angle.

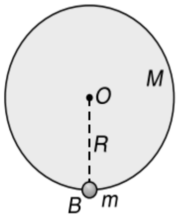


5. A ball is suspended by a thread of length l at the point O on the wall, forming a small angle α with the vertical as shown in Figure.



Then the thread with the ball was deviated through a small angle $\beta(\beta > \alpha)$ and set free. Assuming the collision of the ball against the wall to be perfectly elastic, find the oscillation period of such a pendulum.

6. A simple pendulum of length ℓ is suspended from the ceiling of a cart which is sliding without friction on an inclined plane of inclination θ. What will be the time period of the pendulum?
7. Two pendula begin to swing simultaneously. During the first fifteen oscillations of the first pendulum the other pendulum makes only ten swings. Determine the ratio between the lengths of these pendula.
8. A uniform disc of mass M and radius R is hanging vertically with the help of an axle passing through its centre. A small amount (mass m) of mud is stuck at the bottom end B near rim of the disc. If the disc is now given small angular displacement, find the period of its oscillations. What is the equivalent length of simple pendulum?



9. The angular displacement of simple pendulum is given by

$$\theta = 0.1\pi \sin\left(2\pi t + \frac{\pi}{6}\right) \text{rad}$$

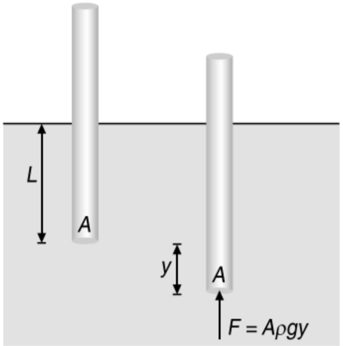
The mass of the bob is 0.4 kg. Calculate

(a) the length of the simple pendulum; and
(b) the velocity of the bob at $t = 0.25$ s

10. A ring radius r is suspended from a point on its circumference. Determine its angular frequency of small oscillations.

OSCILLATIONS OF A FLOATING POLE

Consider a pole of cross-sectional area A and mass M floating in a liquid of density ρ. If the length immersed in the liquid in the equilibrium position is L, then using Archimedes' Principle and Laws of Floatation, the weight of the body is balanced by the upthrust acting on the body, so we have



$$Mg = AL\rho g$$

$$\Rightarrow \quad M = AL\rho$$

If the pole is now pushed down slightly by a distance y, it experiences an additional upthrust equal to $yA\rho g$. This provides the restoring force, tending to bring the pole back to the equilibrium position. So,

$$F = -yA\rho g = -(A\rho g)y$$

$$\Rightarrow \quad M\ddot{y} = -(A\rho g)y$$

$$\Rightarrow \quad \ddot{y} + \left(\frac{A\rho g}{M}\right)y = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{y}{|\ddot{y}|}} = 2\pi\sqrt{\frac{M}{A\rho g}} = 2\pi\sqrt{\frac{L}{g}}$$

where, L is the equilibrium length of the pole immersed in liquid.

ILLUSTRATION 62

A cylinder of mass M, radius r and height h is suspended by a spring whose upper end is fixed. The cylinder is submerged in water such that in equilibrium, the cylinder sinks to half its height. At a certain moment, the cylinder was submerged to $\frac{2}{3}$ of the height and then with no initial velocity started to move vertically. If the spring constant is k and density of water is ρ, then calculate the period of oscillations.

SOLUTION

In equilibrium, the cylinder (of area A) is submerged to a depth $\frac{h}{2}$ and if spring elongation is x_0, then we have

$$kx_0 + \left(A\frac{h}{2}\right)\rho g = Mg \qquad \ldots(1)$$

If cylinder is displaced down further by x and released, then its equation of motion is

$$-\left[k(x_0+x)+\left(\frac{h}{2}+x\right)A\rho g - Mg\right] = Ma$$

$$\Rightarrow \quad a = \ddot{x} = -\left(\frac{k+A\rho g}{M}\right)x$$

The negative sign indicates the restoring nature of force, so we get

$$\omega = \sqrt{\frac{|\ddot{x}|}{x}}$$

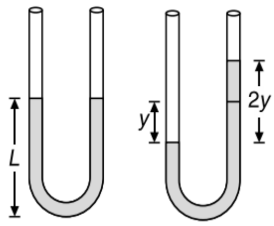
Since $A = \pi r^2$, so we get

$$\omega = \sqrt{\frac{k+(\pi r^2)\rho g}{M}}$$

$$\Rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{k+\pi r^2\rho g}}$$

LIQUID OSCILLATING IN A U-TUBE

Consider a U-tube of cross-sectional area A containing a liquid. Let L be the length of the liquid in each limb of tube. In equilibrium the level of liquid in the two limbs is same. Let the liquid be depressed by a distance y in limb 1. Then it rises through the same distance in limb 2.



Therefore, the liquid level in limb 2 becomes higher by $2y$ than that in limb 1. The restoring force, which tends to bring the liquid back to the original level in limb 2, is given by

$$F = -2yA\rho g = -(2A\rho g)y$$

$$\Rightarrow \quad M\ddot{y} = -(2A\rho g)y$$

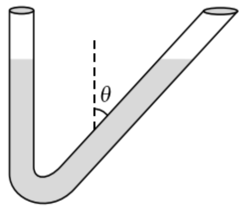
$$\Rightarrow \quad (2L)A\rho\ddot{y} = -(2A\rho g)y$$

$$\Rightarrow \quad \ddot{y} + \frac{g}{L}y = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{y}{|\ddot{y}|}} = 2\pi\sqrt{\frac{L}{g}}$$

ILLUSTRATION 63

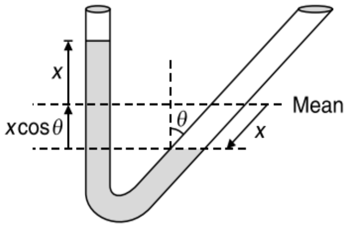
Calculate the period of small oscillations of mercury of mass m poured into a bent tube whose right arm forms an angle θ with the vertical as shown in Figure.



The cross-sectional area of the tube is A and viscosity of mercury is to be neglected.

SOLUTION

Let the mercury be displaced in tube by a distance x as shown in Figure.



The excess pressure due to level difference is

$$\Delta P = (x + x\cos\theta)\rho g$$

Thus, restoring force acting on mercury is

$$F = ma = -A(\Delta P)$$

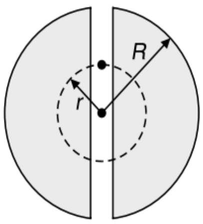
Negative sign tells that the restoring force is directed towards the mean position.

$$\Rightarrow \quad a = \ddot{x} = -\left[\frac{\rho g A}{m}(1+\cos\theta)\right]x$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{x}{|\ddot{x}|}} = 2\pi\sqrt{\frac{m}{A\rho g(1+\cos\theta)}}$$

ILLUSTRATION 64

Suppose a tunnel could be dug through the earth from one side to the other along a diameter, as shown in figure. Show that the motion of the particle is simple harmonic and determine its time period.



SOLUTION

The gravitational force on the particle at a distance r from the centre of the earth arises entirely from that portion of matter of the earth in shells internal to the position of the particle. The external shells exert no force on the particle. The value of gravity at a distance r is given by

$$g' = g\left(\frac{r}{R}\right)$$

where $g = \frac{GM}{R^2}$ is gravity at the surface of earth.

The net force on the particle is

$$F = -mg' = -mg\left(\frac{r}{R}\right)$$

Using Newton's Law,

$$F = m\frac{d^2r}{dt^2} = -\left(\frac{mg}{R}\right)r$$

$$\Rightarrow \quad \frac{d^2r}{dt^2} + \left(\frac{g}{R}\right)r = 0$$

This is the differential equation of SHM and period of oscillation is

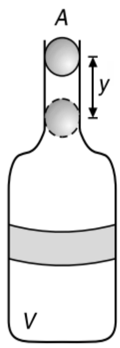
$$T = 2\pi\sqrt{\frac{R}{g}}$$

Since $R = 6.4\times10^6$ m and $g = 9.81\ \text{ms}^{-2}$

$$\Rightarrow \quad T = 84.2\ \text{min}$$

BALL OSCILLATING IN THE NECK OF AN AIR CHAMBER

Consider an air chamber of volume V, having a neck of cross-sectional area A and a ball of mass m fitted smoothly in the neck. If the ball is pressed down slightly by y, the volume of air decreases by Ay.



If B is the Bulk's modulus of air the excess pressure dP produced is given by

$$dP = -B\left(\frac{yA}{V}\right) \qquad \left\{\because B = -\frac{dP}{dV/V}\right\}$$

A restoring force F having magnitude AdP, comes into play and is directed upwards. Thus

$$F = -\left(\frac{BA^2}{V}\right)y$$

$$\Rightarrow \quad m\ddot{y} = -\left(\frac{BA^2}{V}\right)y$$

$$\Rightarrow \quad \ddot{y} + \left(\frac{BA^2}{mV}\right)y = 0$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{y}{|\ddot{y}|}} = 2\pi\sqrt{\frac{mV}{BA^2}}$$

Conceptual Note(s)

(a) If the pressure volume relation is isothermal, then

$$B_{iso} = P_{atm} = P$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{mV}{PA^2}}$$

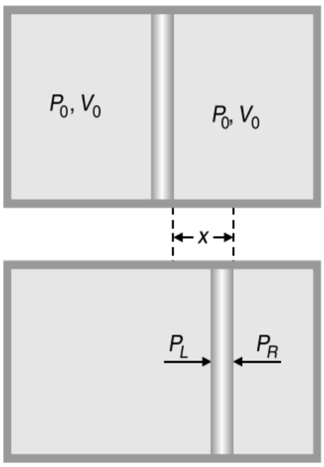
(b) If the pressure volume relation is adiabatic, then

$$B_{ad} = \gamma P_{atm} = \gamma P$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{mV}{\gamma PA^2}}$$

ILLUSTRATION 65

A closed and isolated cylinder contains ideal gas. An adiabatic separator of mass m, cross-sectional area A divides the cylinder into two equal parts, each with volume V_0 and pressure P_0 in equilibrium. Assuming that the separator can move without friction, find the oscillation frequency when the separator is slightly displaced.

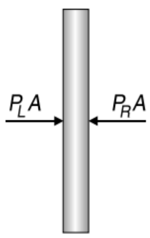


SOLUTION

Let the piston be displaced by a small distance x towards right.

Volume of the right part is $V_R = V_0 - Ax$

Volume of the left part is $V_L = V_0 + Ax$



As the cylinder is isolated from surroundings and separator is adiabatic (non-conducting), the process is adiabatic.

For Right Part

$$P_0V_0^\gamma = P_RV_R^\gamma$$

$$\Rightarrow \quad P_0V_0^\gamma = P(V_0 - Ax)^\gamma$$

$$\Rightarrow \quad P_R = \frac{P_0V_0^\gamma}{V_0^\gamma\left(1-\frac{Ax}{V_0}\right)^\gamma}$$

$$\Rightarrow \quad P_R = P_0\left(1-\frac{Ax}{V_0}\right)^{-\gamma}$$

$$\Rightarrow \quad P_R = P_0\left(1+\frac{\gamma Ax}{V_0}\right) \qquad \left\{\because \frac{Ax}{V_0} \text{ is small quantity}\right\}$$

For Left Part

$$P_0V_0^\gamma = P_LV_L^\gamma = P_L(V_0 + Ax)^\gamma$$

$$\Rightarrow \quad P_L = \frac{P_0V_0^\gamma}{V_0^\gamma\left(1+\frac{Ax}{V_0}\right)^\gamma}$$

$$\Rightarrow \quad P_L \approx P_0\left(1-\frac{\gamma Ax}{V_0}\right)$$

Net restoring force acting on the piston is

$$F = -(P_RA - P_LA)$$

$$\Rightarrow \quad F = -\left[P_0\left(1+\frac{\gamma Ax}{V_0}\right)A - P_0\left(1+\frac{\gamma Ax}{V_0}\right)A\right]$$

$$\Rightarrow \quad F = m\ddot{x} = -\left(\frac{2P_0\gamma A^2}{V_0}\right)x$$

$$\Rightarrow \quad \ddot{x} + \left(\frac{2P_0\gamma A^2}{V_0m}\right)x = 0$$

Comparing with $\ddot{x} + \omega^2x = 0$, we get

$$\omega = \sqrt{\frac{2P_0\gamma A^2}{V_0m}} = \frac{2\pi}{T} = 2\pi f$$

$$\Rightarrow \quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{2P_0\gamma A^2}{V_0m}}$$

ILLUSTRATION 66

A simple pendulum consists of a small sphere of mass m carrying a charge $+q$ suspended by a thread of length l. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. Calculate the period of small oscillations of the pendulum, if the electrostatic force acting on the sphere is less than the gravitational force.

SOLUTION

A simple pendulum with bob of mass m having charge $+q$ with electric field E in the region directed vertically upwards is shown in Figure (a).

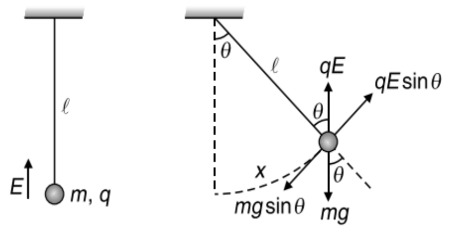


Figure (a) Figure (b)

When bob is in equilibrium, tension in string is

$$T = mg - qE$$

When the bob is given a small displacement x from its mean position as shown in Figure (b), then the restoring force on it is given by

$$F = -(mg\sin\theta - qE\sin\theta)$$

where, θ is the angular displacement of bob from mean position.
For small oscillations, $\sin\theta \approx \theta \approx x/l$

$$\Rightarrow \quad F = -(mg - qE)\theta$$

$$\Rightarrow \quad F = (mg - qE)\frac{x}{l}$$

If a is the acceleration of bob, then

$$a = \frac{F}{m} = -\left(\frac{mg - qE}{ml}\right)x$$

$$\Rightarrow \quad \omega = \sqrt{\frac{|\ddot{x}|}{x}} = \sqrt{\frac{mg - qE}{ml}}$$

$$\Rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{mg - qE}} = 2\pi\sqrt{\frac{l}{(mg - qE)/m}}$$

This result can be written as $T = 2\pi\sqrt{\dfrac{l}{g_{\text{eff}}}}$

where, $g_{\text{eff}} = g - \dfrac{qE}{m}$ is the effective value of acceleration due to gravity under the influence of electric field.

Similarly, if in this problem electric field E is reversed in direction i.e., acts in downward direction, then net effective force on bob in downward direction is increased and effective gravity can be written as

$$g_{\text{eff}} = g + \frac{qE}{m}$$

So, in this case, the time period of a pendulum is written as

$$T = 2\pi\sqrt{\frac{l}{g_{\text{eff}}}} = 2\pi\sqrt{\frac{l}{g + \frac{qE}{m}}}$$

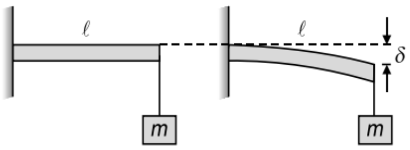
ILLUSTRATION 67

A light wooden rod fixed at one end is kept horizontal. A load of mass 0.4 kg tied to the free end of the rod causes that end to be depressed by $\delta = 2.5$ cm. If this load is disturbed slightly (in vertical direction) from mean position, then calculate the period of oscillations. Take $g = 10\ \text{ms}^{-2}$.

SOLUTION

Due to elasticity in rod it sags by weight of load. If its shear modulus of elasticity is η, then for equilibrium of system we have

$$\eta = \frac{F/A}{\delta/l}$$



where, A is area of cross-section of rod.

$$\frac{Mg}{A} = \frac{\eta\delta}{l} \qquad \ldots(1)$$

When the load is depressed by x, then a restoring force F is developed in rod such that

$$F - Mg = Ma \qquad \ldots(2)$$

where, $\eta = \dfrac{F/A}{(\delta + x)/l}$

$$\Rightarrow \quad F = \frac{\eta A}{l}(\delta + x) \qquad \ldots(3)$$

Substituting (3) in (2), we get

$$\frac{\eta A}{l}(\delta + x) - Mg = Ma$$

$$\Rightarrow \quad a = \ddot{x} = \left(\frac{\eta A}{Ml}\right)x$$

Since this acceleration a (i.e., $\ddot{x}$) is directed towards the mean position, so we have

$$\ddot{x} = -\left(\frac{\eta A}{Ml}\right)x = -\left(\frac{g}{\delta}\right)x \qquad \{\because \text{ of equation (1)}\}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{x}{|\ddot{x}|}} = 2\pi\sqrt{\frac{\delta}{g}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{2.5}{10}} = 3.14\ \text{s}$$

COMPOSITION OF TWO SHM OF THE SAME PERIOD ALONG THE SAME LINE

Let the two SHM's be

$$y_1 = A_1\sin\omega t \text{ and } y_2 = A_2\sin(\omega t + \phi)$$

The resultant displacement

$$y = y_1 + y_2 = A_1\sin\omega t + A_2\sin(\omega t + \phi)$$

$$\Rightarrow \quad y = A_1\sin\omega t + A_2\sin\omega t\cos\phi + A_2\cos\omega t\sin\phi$$

$\Rightarrow \quad y = (A_1 + A_2\cos\phi)\sin\omega t + A_2\sin\phi\cos\omega t \qquad \ldots(1)$

Let $A_1 + A_2\cos\phi = R\cos\theta$ and $A_2\sin\phi = R\sin\theta$.

Substituting in (1), we get

$$y = R[\cos\theta\sin(\omega t) + \sin\theta\cos(\omega t)]$$

$$\Rightarrow \quad y = R\sin(\omega t + \theta)$$

Thus, the resultant motion is also simple harmonic along the same line and has the same time period. Its amplitude R is

$$R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

and it is phase θ ahead of the first motion, where

$$\tan\theta = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$

Problem Solving Technique(s)

If $y_1 = a\sin(\omega t)$ and $y_2 = b\cos(\omega t)$ are two SHM then by the superimposition of these two SHM we get

$$y = y_1 + y_2$$

$$y = a\sin(\omega t) + b\cos(\omega t)$$

$$\Rightarrow \quad y = A\sin(\omega t + \phi)$$

This is also the equation of SHM, having

Amplitude $A = \sqrt{a^2 + b^2}$ and Phase $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

COMPOSITION OF TWO SHM OF SAME PERIOD AT RIGHT ANGLES TO EACH OTHER

Let the two motions at right angles be

$$x = A\sin(\omega t) \qquad \ldots(1)$$

$$y = B\sin(\omega t + \phi) \qquad \ldots(2)$$

along the x and y-axis respectively

Equation (1) gives

$$\sin(\omega t) = \frac{x}{A}$$

$$\Rightarrow \quad \cos(\omega t) = \sqrt{1 - \frac{x^2}{A^2}}$$

Equation (2) gives

$$\frac{y}{B} = \sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi$$

$$\Rightarrow \quad \frac{y}{B} = \left(\frac{x}{A}\right)\cos\phi + \left[\sqrt{1 - \frac{x^2}{A^2}}\right]\sin\phi$$

$$\Rightarrow \quad \frac{y}{B} - \frac{x}{A}\cos\phi = \sqrt{1 - \frac{x^2}{A^2}}\sin\phi$$

Squaring, we get

$$\frac{y^2}{B^2} + \frac{x^2}{A^2}\cos^2\phi - \frac{2xy}{AB}\cos\phi = \left(1 - \frac{x^2}{A^2}\right)\sin^2\phi$$

$$\Rightarrow \quad \frac{y^2}{B^2} + \frac{x^2}{A^2}(\cos^2\phi + \sin^2\phi) - \frac{2xy}{AB}\cos\phi = \sin^2\phi$$

$$\Rightarrow \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\phi = \sin^2\phi$$

This is the equation of an ellipse.

CASE-I: For $\phi = 0$

The equation becomes $\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB} = 0$

$$\Rightarrow \quad \frac{x}{A} - \frac{y}{B} = 0$$

$$\Rightarrow \quad y = \frac{B}{A}x$$

This is the equation of a straight line. Thus, the resultant motion is a SHM along a straight line, passing through the origin, inclined at an angle $\tan^{-1}\left(\frac{B}{A}\right)$ to the x-axis.

CASE-II: For $\phi = \pi$

The equation becomes

$$\Rightarrow \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{2xy}{AB} = 0$$

$$\Rightarrow \quad \frac{x}{A} + \frac{y}{B} = 0$$

$$\Rightarrow \quad y = -\left(\frac{B}{A}\right)x$$

The resultant SHM is along a straight line inclined at $\tan^{-1}\left(\frac{-B}{A}\right)$ to the x axis.

CASE-III: For $\phi = \pi/2$

The equation becomes $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, which is an ellipse

If $A = B$, the equation is $x^2 + y^2 = A^2$, which is a circle.

Problem Solving Technique(s)

A uniform circular motion may thus be regarded as a combination of two similar simple harmonic motions at right angles to each other and differing in phase by $\frac{\pi}{2}$.

CASE-IV: For $\phi = \pi/4$

The equation becomes

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{\sqrt{2}xy}{AB} = \frac{1}{2}$$

which is the equation of an oblique ellipse.

Problem Solving Technique(s)

If two oscillations of different frequencies at right angles are combined, then the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. The resulting curves are called **Lissajous Figures (not in syllabus).**

ILLUSTRATION 68

Two linear simple harmonic motions of equal amplitudes a and frequencies ω and 2ω are impressed on a particle along x and y axis respectively. If the initial phase difference between them is $\frac{\pi}{2}$, calculate the resultant trajectory equation of the particle.

SOLUTION

Let $x = a\cos(\omega t)$, then according to problem, we have

$$y = a\cos\left(2\omega t + \frac{\pi}{2}\right)$$

$$\Rightarrow \quad y = -a\sin(2\omega t)$$

$$\Rightarrow \quad y = -2a\sin(\omega t)\cos(\omega t)$$

$$\Rightarrow \quad y = -2a\left(\sqrt{1-\frac{x^2}{a^2}}\right)\left(\frac{x}{a}\right)$$

$$\Rightarrow \quad a^2y^2 = 4x^2(a^2 - x^2)$$

ILLUSTRATION 69

Three simple harmonic motions in the same direction having the same amplitude a and same period are superposed. If each differs in phase from the next by 45°, then find the resultant equation, resultant amplitude, phase of the resultant motion relative to the first, ratio of the energy of the resultant motion to that possessed by any single motion.

SOLUTION

Let these three simple harmonic motions be represented by

$$y_1 = a\sin\left(\omega t - \frac{\pi}{4}\right)$$

$$y_2 = a\sin\omega t \text{ and}$$

$$y_3 = a\sin\left(\omega t + \frac{\pi}{4}\right)$$

On superimposing, resultant SHM will be

$$y = a\left[\sin\left(\omega t - \frac{\pi}{4}\right) + \sin\omega t + \sin\left(\omega t + \frac{\pi}{4}\right)\right]$$

$$\Rightarrow \quad y = a\left(2\sin\omega t\cos\frac{\pi}{4} + \sin\omega t\right)$$

$$\Rightarrow \quad y = a(1+\sqrt{2})\sin\omega t$$

So, resultant amplitude is $A = (1+\sqrt{2})a$

Since, energy in SHM is proportional to square of the amplitude, so we have

$$\frac{E_{\text{Resultant}}}{E_{\text{Single}}} = \left(\frac{A}{a}\right)^2 = (\sqrt{2}+1)^2 = 3+2\sqrt{2}$$

DAMPED HARMONIC OSCILLATION

A body set into oscillation continues to oscillate for ever with the same amplitude if no dissipative forces are present. Such oscillations are called **free oscillations**. In free oscillations, as there is no dissipation of energy, so the total mechanical energy remains conserved.

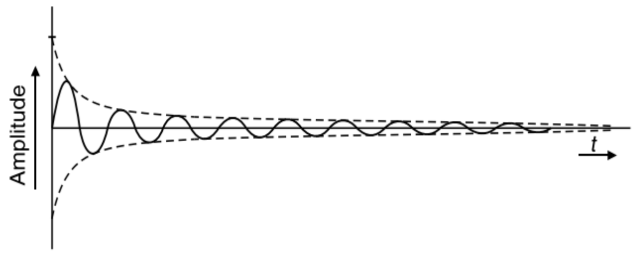
If frictional forces are present, the amplitude of oscillation gradually decreases because energy is dissipated. Such oscillations are **called damped oscillations.** In most cases the damping force is directly proportional to the speed, e.g., viscous drag due to air. The equation of motion of a damped harmonic oscillator can, therefore, be written as

$$F = -ky - bv, \text{ where } b \text{ is damping constant}$$

$$\Rightarrow \quad m\frac{d^2y}{dt} + b\frac{dy}{dt} + ky = 0$$

The solution of this equation for small b is

$$y = Ae^{\frac{-bt}{2m}}\sin(\omega' t + \phi_0), \text{ where } \omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$



We obtain the following results

(a) The amplitude of oscillations decreases exponentially with time. Exponential fall means that if amplitude becomes $\frac{1}{n}$ of initial value in t time, then in next t time, it will become $\frac{1}{n^2}$ of its initial value and so on.

(b) The angular frequency of oscillation is

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

So, linear frequency is given by

$$f = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The frequency is slightly smaller than the frequency of free oscillations, i.e., damping slows down the motion.

(c) If $b \ll \sqrt{km}$, then $\omega' \approx \omega = \sqrt{\frac{k}{m}}$

(d) If we assume $b = 0$, then all the equations of the damped oscillator will reduce to the corresponding equations of an undamped oscillator.

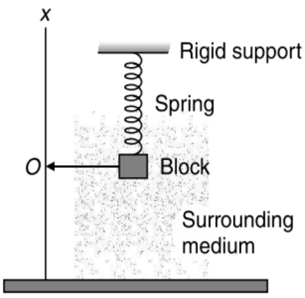
The mechanical energy of the undamped oscillator is $E = \frac{1}{2}kA^2$. However, for a damped oscillator, the amplitude is not constant and varies with time as $A' = Ae^{-\left(\frac{b}{2m}\right)t}$. So, we can say that the mechanical energy will have a value given by

$$E' = \frac{1}{2}k(A')^2 = \frac{1}{2}kA^2\left(e^{-\frac{bt}{2m}}\right)^2 = \frac{1}{2}kA^2e^{-\frac{bt}{m}}$$

This expression makes us conclude that the total energy of the system decreases exponentially with time.

ILLUSTRATION 70

Consider the damped oscillator shown in Figure.



If the mass m of the block is 400 g, $k = 45\ \text{Nm}^{-1}$ and the damping constant b is 80 gs^{-1}. Calculate the period of oscillation, time taken for its amplitude of vibrations to drop to half of its initial value and the time taken for its mechanical energy to drop to half its initial value.

SOLUTION

Since, $km = 45 \times 0.4 = 18\ \text{kgNm}^{-1} = 18\ \text{kg}^2\text{s}^{-2}$

$$\Rightarrow \quad \sqrt{km} = 4.243\ \text{kgs}^{-1}$$

Also, we are given that $b = 0.08\ \text{kgs}^{-1}$

Since $b \ll \sqrt{km}$, so $\omega' \approx \omega = \sqrt{\frac{k}{m}}$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.4}{45}} = 0.6\ \text{s}$$

For $A' = \frac{A}{2}$, we have

$$Ae^{-\frac{bt}{2m}} = \frac{A}{2}, \text{ where } \frac{b}{2m} = \frac{0.08}{2(0.4)} = 0.1$$

$$\Rightarrow \quad t = \frac{\log(2)}{b/2m} = \frac{0.693}{0.1} = 6.93\ \text{s}$$

For $E' = \frac{E}{2}$, we have

$$\frac{1}{2}kA^2e^{-\frac{bt}{m}} = \frac{1}{2}\left(\frac{1}{2}kA^2\right)$$

$$\Rightarrow \quad \frac{1}{2} = e^{-\frac{bt}{m}}, \text{ where } \frac{b}{m} = \frac{0.08}{0.4} = 0.2$$

$$\Rightarrow \quad t = \frac{\log(2)}{b/m} = \frac{0.693}{0.2} = 3.46\ \text{s}$$

This time is just the half of the time in which amplitude decays to half its initial value.

FORCED OSCILLATIONS AND RESONANCE

Suppose a system is made to oscillate by subjecting it to an external periodic force. This is done by linking it in some way with another oscillating system, which is generally called **the driver**. When a periodic force is applied on an oscillating body, then the body begins to vibrate with the frequency of driving force and the oscillating body is called **driven harmonic oscillation**. The resulting oscillations are called **forced oscillations**. The general equation of a damped oscillator, oscillating under influence of an external harmonic force $F_0 \sin(\omega'' t)$ is written as

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0 \sin(\omega'' t)$$

The displacement of body from mean position is

$$y = A'' \sin(\omega'' t - \phi)$$

where A'' is amplitude, given by

$$A'' = \frac{F_0/m}{\sqrt{(\omega^2 - \omega''^2)^2 + \frac{\omega''^2 b^2}{m^2}}}$$

$$\phi = \tan^{-1}\left[\frac{(\omega'' b/m)}{\omega^2 - \omega''^2}\right]$$

and $\omega = \sqrt{\frac{k}{m}}$ is natural frequency of the oscillator.

Problem Solving Technique(s)

(a) The system oscillates with frequency of driver ω'' rather than with its natural frequency ω.

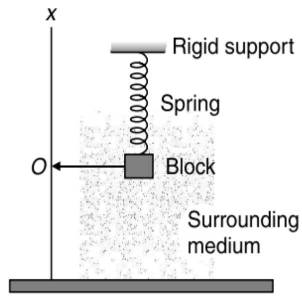
(b) The amplitude of oscillation is small if ω'' is very different from ω. As $\omega'' \to \omega$, the amplitude goes on increasing. When $\omega'' = \omega$, the amplitude is maximum. This situation is called **Resonance**. If in addition, the damping force is absent, then the amplitude tends to become infinite and the system will break down. However, if some damping is present, the amplitude becomes large but remains finite.

Test Your Concepts-V

Based on SHM in Other Physical Systems, Composition of SHM, Damped Oscillations, Forced Oscillations & Resonance

(Solutions on page H.186)

1. A simple pendulum consists of a small sphere of mass m suspended by a thread of length ℓ. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field of strength E directed vertically upwards. With what period will pendulum oscillate if the electrostatic force acting on the sphere is less than the gravitational force?
2. A particle is constrained to move on a smooth circular wire frame of radius r which rotates uniformly about a diameter which is vertical. If in the position of relative rest, the radius drawn to the particle makes an angle α with the vertical, find the period of small oscillations about this position.
3. Calculate the period of small oscillations in a horizontal plane performed by a ball of mass 40 g fixed at the middle of a horizontally stretched string 1 m in length. The tension of the string is assumed to be constant and equal to 10 N.
4. A point mass m is suspended at the end of a massless wire of length l and cross-section A. If Y is the young's modulus of elasticity for the wire, obtain the frequency of oscillation for the simple harmonic motion along the vertical line.
5. Consider the earth as a uniform sphere of mass M and radius R. Imagine a straight smooth tunnel made through the earth which connects any two points on its surface. Show that the motion of a particle of mass m along this tunnel under the action of gravitation would be simple harmonic. Hence, determine the time that a particle would take to go from one end to the other through the tunnel.
6. An ideal gas whose adiabatic exponents is γ, is enclosed in a vertical cylindrical container and supports a freely moving piston of mass M. The piston and the cylinder have equal cross-sectional area A. Atmospheric pressure is P_0 and when the piston is in equilibrium, the volume of the gas is V_0. The piston is now displaced slightly from the equilibrium position. Assuming that the system is completely isolated from its surroundings, show that the piston executes simple harmonic motion and calculate the period of small oscillations.
7. Find the displacement equation of the simple harmonic motion obtained by combining the motions $x_1 = 2\sin\omega t$, $x_2 = 4\sin\left(\omega t + \frac{\pi}{6}\right)$ and $x_3 = 6\sin\left(\omega t + \frac{\pi}{3}\right)$.
8. In damped oscillations, the amplitude of oscillations is reduced to half of its initial value of 5 cm at the end of 25 oscillations. What will be its amplitude when the oscillator completes 50 oscillations?
9. For the damped oscillator shown in figure, $k = 200\ \text{Nm}^{-1}$ and damping constant $b = 40\ \text{gs}^{-1}$.



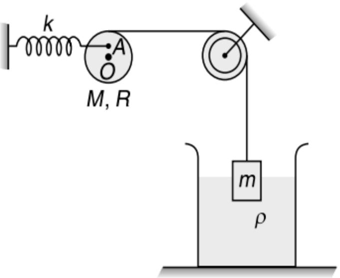
If period of oscillation in the absence of damping is 0.5 s, then calculate mass of block if $b \ll \sqrt{km}$.

10. The amplitude of a damped oscillator becomes half in one minute. The amplitude after 3 minutes will be $1/x$ times of the original. Determine the value of x.

SOLVED PROBLEMS

PROBLEM 1

Figure shows a solid uniform cylinder of radius R and mass M, which is free to rotate about a fixed horizontal axis O and passes through centre of the cylinder. One end of an ideal spring of force constant k is fixed and the other end is hinged to the cylinder at A. Distance OA is equal to $\frac{R}{2}$. An inextensible thread is wrapped round the cylinder and passes over a smooth, small pulley. A block of equal mass M and having cross sectional area A is suspended from free end of the thread. The block is partially immersed in a non-viscous liquid of density ρ.



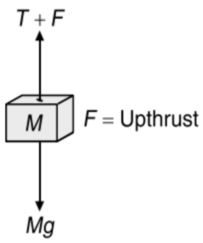
If in equilibrium, spring is horizontal and line OA is vertical, calculate frequency of small oscillations of the system.

SOLUTION

In equilibrium, we have

$$T+F=Mg \quad \ldots(1)$$

When the block is further depressed by x, weight Mg remains unchanged, upthrust F increases by ρAxg and let ΔT be the increase in tension.



If a is the acceleration of block then,

$$\Delta T+\rho Axg=Ma \quad \ldots(2)$$

Restoring torque on the cylinder is given by

$$\tau=\left(\frac{kx}{2}\frac{R}{2}-\Delta TR\right)=\left(\frac{kxR}{4}-(Ma-\rho Axg)R\right)$$

$$\Rightarrow \quad \frac{1}{2}MR^2\alpha=\left[\frac{kR^2\theta}{4}-(MR\alpha-\rho AgR\theta)R\right]$$

$$\Rightarrow \quad \frac{3}{2}MR^2\alpha=\left(\frac{kR^2}{4}+\rho AgR^2\right)\theta$$

$$\Rightarrow \quad \alpha=\frac{-\left(\frac{k}{4}+\rho Ag\right)}{\frac{3}{2}M}\theta$$

Here negative sign has been used for restoring nature of torque. So, we get

$$f=\frac{1}{2\pi}\sqrt{\left|\frac{\alpha}{\theta}\right|}$$

$$\Rightarrow \quad f=\frac{1}{2\pi}\sqrt{\frac{k+4\rho Ag}{6M}}$$

PROBLEM 2

A pendulum clock is mounted in an elevator which starts going up with a constant acceleration $a(<g)$. At a height h the acceleration of the car reverses, its magnitude remaining the same. How soon after the start of the motion will the clock show the right time again?

SOLUTION

Time of ascent $t_1=\sqrt{\frac{2h}{a}}$ $\quad \left\{\because\ h=\frac{1}{2}at^2\right\}$

$$\Rightarrow \quad T\propto\frac{1}{\sqrt{g}}$$

$$\Rightarrow \quad \frac{T'}{T}=\sqrt{\frac{g}{g+a}}$$

$$\Rightarrow \quad T'=T\sqrt{\frac{g}{g+a}}$$

$$\Rightarrow \quad \Delta T=(T-T')=T\left(1-\sqrt{\frac{g}{g+a}}\right)$$

Time gained in time t_1 is

$$\Delta t_1=\left(\frac{\Delta T}{T'}\right)t_1$$

$$\Rightarrow \quad \Delta t_1=\sqrt{\frac{2h}{a}}\left(\sqrt{\frac{g+a}{g}}-1\right) \quad \ldots(1)$$

If t_2 be the time of descent, then in this case, $\frac{T'}{T}=\sqrt{\frac{g}{g-a}}$

$$\Rightarrow \quad T'=T\sqrt{\frac{g}{g-a}}$$

$$\Rightarrow \quad \Delta T=T'-T=T\left(\sqrt{\frac{g}{g-a}}-1\right)$$

Time lost in time t_2 is

$$\Delta t_2 = \left(\frac{\Delta T}{T'}\right)t_2$$

$$\Rightarrow \quad \Delta t_2 = t_2\left(1-\sqrt{\frac{g-a}{g}}\right) \qquad \ldots(2)$$

The clock will show the right time again if,

$$\Delta t_1 = \Delta t_2$$

$$\Rightarrow \quad \sqrt{\frac{2h}{a}}\left(\sqrt{\frac{g+a}{g}}-1\right) = t_2\left(1-\sqrt{\frac{g-a}{g}}\right)$$

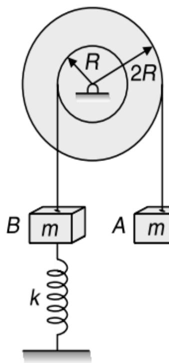
$$\Rightarrow \quad t_2 = \sqrt{\frac{2h}{a}}\left(\frac{\sqrt{g+a}-\sqrt{g}}{\sqrt{g}-\sqrt{g-a}}\right)$$

So, total time $t = t_1 + t_2$

$$\Rightarrow \quad t = \sqrt{\frac{2h}{a}}\left[\frac{\sqrt{g+a}-\sqrt{g-a}}{\sqrt{g}-\sqrt{g-a}}\right]$$

PROBLEM 3

In the figure shown pulley in massless. Initially the blocks are held at a height such that spring is in relaxed position. The block A is released. Find the



(a) amplitude and velocity amplitude of A.

(b) frequency of the oscillation of block B. There is no slipping anywhere.

SOLUTION

(a) (i) Applying Law of Conservation of Mechanical Energy, we get

$$\begin{pmatrix}\text{Decrease in}\\ \text{Gravitational}\\ \text{P.E. of } A\end{pmatrix} = \begin{pmatrix}\text{Increase in}\\ \text{Gravitational}\\ \text{P.E. of } B\end{pmatrix} + \begin{pmatrix}\text{Increase in}\\ \text{Elastic P.E.}\\ \text{of Spring}\end{pmatrix}$$

$$\Rightarrow \quad mg(2x_0) = mgx_0 + \frac{1}{2}kx_0^2 \qquad \ldots(1)$$

where, x_0 = displacement amplitude of A

Solving equation (1), we get

$$x_0 = \frac{2mg}{k}$$

(ii) Again, Applying Law of Conservation of Mechanical Energy at the mean position, we get

$$mgx_0 = mg\frac{x_0}{2} + \frac{1}{2}k\left(\frac{x_0}{2}\right)^2 + \frac{1}{2}mv_m^2 + \frac{1}{2}m\left(\frac{v_m}{2}\right)^2$$

where, v_m = velocity amplitude of A

$$\Rightarrow \quad v_m = 2g\sqrt{\frac{m}{5k}}$$

(b) Maximum velocity of $B = \omega$ (maximum displacement of B)

$$\Rightarrow \quad \frac{v_m}{2} = \omega\left(\frac{x_0}{2}\right)$$

$$\Rightarrow \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{5m}}$$

Conceptual Note(s)

A and B both oscillate simple harmonically with same angular frequency ω but different displacement and velocity amplitudes. Frequency or time period can also be obtained by energy method as under.

$$E = \frac{1}{2}m_Av_A^2 + \frac{1}{2}m_Bv_B^2 - m_Agx_A + m_Bgx_B + \frac{1}{2}k\left(\frac{x_0}{2}+x_B\right)^2$$

$$\Rightarrow \quad E = \frac{1}{2}mv^2 + \frac{1}{2}m\left(\frac{v}{2}\right)^2 - mgx + mg\frac{x}{2} + \frac{1}{2}k\left(\frac{x_0}{2}+\frac{x}{2}\right)^2$$

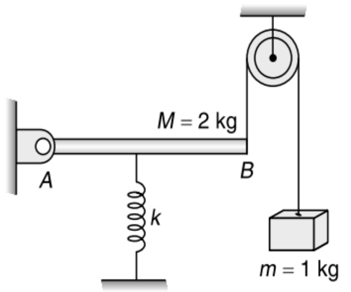
where $x_0 = \frac{2mg}{k}$ and x is the displacement (downwards) of A from its mean position x_0. Since E = constant

$$\Rightarrow \quad \frac{dE}{dt} = 0$$

So, we can calculate the desired frequency.

PROBLEM 4

In the arrangement shown in figure, AB is a uniform rod of length $l = 90$ cm and mass $M = 2$ kg. The rod is free to rotate about a horizontal axis passing through end A. A thread passes over a light, smooth and small pulley. One end of the thread is attached with end B of the rod and the other end carries a block of mass $m = 1$ kg. To keep the system in equilibrium one end of an ideal spring of force constant $K = 7500\ \text{Nm}^{-1}$ is attached with mid-point of the rod and the other end is fixed such that in equilibrium, the spring is vertical and the rod is horizontal. If in equilibrium, part of the thread between end B and pulley is vertical, calculate frequency of small oscillations of the system.



Also, calculate the maximum possible angular amplitude of the rod so that block remains oscillating with the rod without blocking of thread. $\left(g = 10 \text{ ms}^{-2}\right)$.

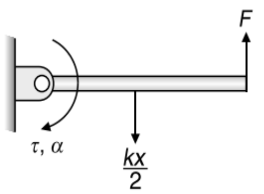
SOLUTION

In equilibrium, the spring is relaxed. When mass m is displaced (downwards) by x from its equilibrium position, then let F be the extra tension in the string. The spring will stretch by $\frac{x}{2}$.

Net restoring torque is given by

$$\tau = \left(\frac{kx}{2}\right)\frac{l}{2} - Fl \qquad \ldots(1)$$

$$\Rightarrow \quad \left(\frac{Ml^2}{3} + ml^2\right)\alpha = -\frac{kl^2\theta}{4} \qquad \{\because\ x = l\theta\}$$



Here negative sign has been introduced due to restoring nature of torque. Since, α is proportional to $-\theta$, motion is simple harmonic in nature. So,

$$f = \frac{1}{2\pi}\sqrt{\left|\frac{\alpha}{\theta}\right|} = \frac{1}{2\pi}\sqrt{\frac{k}{4\left(\frac{M}{3}+m\right)}} = \frac{1}{2\pi}\sqrt{\frac{7500}{4\left(\frac{2}{3}+1\right)}}$$

$$\Rightarrow \quad f = \frac{15\sqrt{5}}{2\pi} \text{ Hz}$$

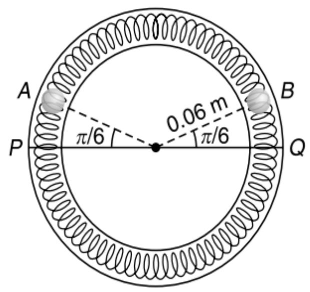
Further, $\omega^2 l\theta_{max} = g$

$$\Rightarrow \quad \theta_{max} = \frac{g}{\omega^2 l} = \frac{10}{\left(15\sqrt{5}\right)^2(0.9)}$$

$$\Rightarrow \quad \theta_{max} = 9.88 \times 10^{-3} \text{ rad}$$

PROBLEM 5

Two identical balls A and B, each of mass 0.1 kg, are attached to two identical massless springs. The spring-mass system is constrained to move inside a rigid smooth pipe bent in the form of a circle as shown in figure. The pipe is fixed in a horizontal plane. The centres of the ball can move in a circle of radius 0.06 metre. Each spring has a natural length of 0.06π metre and spring constant 0.1 Nm^{-1}. Initially, both the balls are displaced by an angle $\theta = \frac{\pi}{6}$ radian with respect to the diameter PQ of the circle (as shown in figure) and released from rest.



(a) Calculate the frequency of oscillation of ball A.

(b) Find the speed of ball A when A and B are at the two ends of the diameter PQ.

(c) What is the total energy of the system?

SOLUTION

(a) Given: Mass of each ball A and B, $m = 0.1$ kg

Radius of circle, $R = 0.06$ m

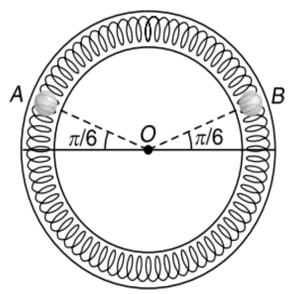
Natural length of spring,

$$l_0 = 0.06\pi = \pi R \qquad \{\text{Half circle}\}$$

and spring constant, $k = 0.1 \text{ Nm}^{-1}$

In the stretched position elongation in each spring $(x = R\theta)$

Spring in lower side is stretched by $2x$ and on upper side compressed by $2x$.



Therefore, each spring will exert a force $2kx$ on each block.

Hence, a restoring force, $F = 4kx$ will act on A in the direction shown in figure.

Restoring torque of this force about origin,

$$\tau = -F \cdot R = -(4kx)R = -(4kR\theta)R$$

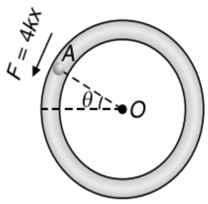
$$\Rightarrow \quad \tau = -4kR^2 \cdot \theta \qquad \ldots(1)$$

Since, $\tau \propto -\theta$, each ball executes angular SHM about origin O.

Equation (1) can be rewritten as

$$I\alpha = -4kR^2\theta \text{ or } \left(mR^2\right)\alpha = -4kR^2\theta$$

$$\Rightarrow \quad \alpha = -\left(\frac{4k}{m}\right)\theta$$



So, frequency of oscillation is given by

$$f=\frac{1}{2\pi}\sqrt{\left|\frac{\text{acceleration}}{\text{displacement}}\right|}$$

$$\Rightarrow\ f=\frac{1}{2\pi}\sqrt{\left|\frac{\alpha}{\theta}\right|}=\frac{1}{2\alpha}\sqrt{\left|\frac{4k}{m}\right|}$$

Substituting the values, we get

$$f=\frac{1}{2\pi}\sqrt{\frac{4\times 0.1}{0.1}}=\frac{1}{\pi}\text{ Hz}$$

(b) In stretched position, potential energy of the system is

$$\text{P.E.}=2\left(\frac{1}{2}k\right)(2x)^2=4kx^2$$

and in mean position, both the balls have only kinetic energy. Hence

$$\text{K.E.}=2\left(\frac{1}{2}mv^2\right)=mv^2$$

By Law of Conservation of Energy, we get

$$(\text{Loss in KE})=(\text{Gain in PE})$$

$$\Rightarrow\ 4kx^2=mv^2$$

$$\Rightarrow\ v=2x\sqrt{\frac{k}{m}}=2R\theta\sqrt{\frac{k}{m}}$$

Substituting the values

$$v=2(0.06)\left(\frac{\pi}{6}\right)\sqrt{\frac{0.1}{0.1}}$$

$$\Rightarrow\ v=0.0628\text{ ms}^{-1}$$

(c) Total energy of the system,

$E=$P.E. in stretched position

$E=$K.E. in mean position

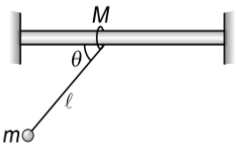
$$\Rightarrow\ E=mv^2=(0.1)(0.0628)^2\text{ J}$$

$$\Rightarrow\ E=3.9\times 10^{-4}\text{ J}$$

PROBLEM 6

A bead of mass M can slide on a smooth straight horizontal wire and a particle of mass m is attached to the body by a light string of length l. The particle is held in contact with the wire with the string taut and is then left fall. When the string is inclined to the wire at an angle θ, find the

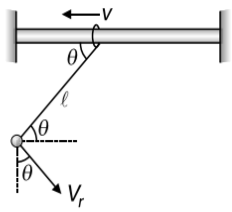
(a) distance x slipped along the wire by the bead and
(b) angular velocity ω of the string.



SOLUTION

(a) Displacement of m relative to M is

$$x_r=l(1-\cos\theta)$$



Let x be the displacement of M (towards left). Absolute displacement of m will be x_r-x (towards right). For the centre of mass to remain stationary, we have

$$m(x_r-x)=Mx$$

$$\Rightarrow\ x=\left(\frac{m}{M+m}\right)l(1-\cos\theta)$$

(b) By Law of Conservation of Mechanical Energy, we get

$$mgl\sin\theta=\frac{1}{2}Mv^2+\frac{1}{2}m\left[(v_r\sin\theta-v)^2+v_r^2\cos^2\theta\right]\quad\ldots(1)$$

Applying Law of Conservation of Linear Momentum, we get

$$m(v_r\sin\theta-v)=Mv$$

$$\Rightarrow\ v=\frac{mv_r\sin\theta}{M+m}\quad\ldots(2)$$

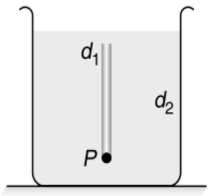
From equations (1) and (2), we get

$$v_r=\sqrt{\frac{2gl\sin\theta(M+m)}{M+m\cos^2\theta}}$$

$$\Rightarrow\ \omega=\frac{v_r}{l}=\sqrt{\frac{2g(M+m)\sin\theta}{(M+m\cos^2\theta)l}}$$

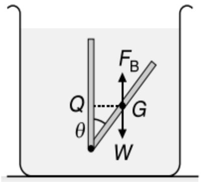
PROBLEM 7

A thin rod of length L and uniform cross-section is pivoted at its lowest point P inside a stationary homogeneous and non-viscous liquid (as shown in figure). The rod is free to rotate in a vertical plane about a horizontal axis passing through P. The density d_1 of the material of the rod is smaller than the density d_2 of the liquid. The rod is displaced by small angle θ from its equilibrium position and then released. Show that the motion of the rod is simple harmonic and determine its angular frequency in terms of the given parameters.



SOLUTION

Let S be the area of cross-section of the rod. In the displaced position, as shown in figure,



The weight (W) and upthrust (F_B) both pass through its centre of gravity G.

Here, $W=(\text{Volume})\times(\text{Density of rod})\times g$

$$\Rightarrow \quad W=(SL)(d_1)g$$

The buoyant force F_B is given by

$$F_B=(\text{Volume})\times(\text{Density of liquid})\times g$$

$$\Rightarrow \quad F_B=(SL)(d_2)g$$

Given that $d_1<d_2$. Therefore, $W<F_B$

Therefore, net force acting at G is given by

$$F=F_B-W=(SLg)(d_2-d_1)\text{upwards}$$

Restoring torque of this force about point P is

$$\tau=F\times\tau_{\perp}=(SLg)(d_2-d_1)(QG)$$

$$\Rightarrow \quad \tau=-(SLg)(d_2-d_1)\left(\frac{L}{2}\sin\theta\right)$$

Here, negative sign shows the restoring nature of torque. Since θ is small, so $\sin\theta\approx\theta$

$$\Rightarrow \quad \tau=-\left(\frac{SL^2g(d_2-d_1)}{2}\right)\theta \qquad \ldots(1)$$

From equation (1), we get

$$\tau\propto-\theta$$

Hence, motion of the rod is simple harmonic.

Further, we know that $\tau=I\dfrac{d^2\theta}{dt^2}=I\alpha$

Rewriting equation (1) as

$$\Rightarrow \quad I\frac{d^2\theta}{dt^2}=-\left(\frac{SL^2g(d_2-d_1)}{2}\right)\theta \qquad \ldots(2)$$

where, I =moment of inertia of rod about an axis passing through P given by

$$I=\frac{ML^2}{3}=\frac{(SLd_1)L^2}{3}$$

Substituting this value of I in equation (2), we get

$$\frac{d^2\theta}{dt^2}=-\left(\frac{3}{2}\frac{g(d_2-d_1)}{d_1L}\right)\theta$$

Comparing this equation with standard differential equation of SHM, i.e.,

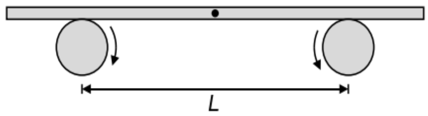
$$\frac{d^2\theta}{dt^2}=-\omega^2\theta$$

The angular frequency of oscillation is given by

$$\omega=\sqrt{\frac{3g(d_2-d_1)}{2d_1L}}$$

PROBLEM 8

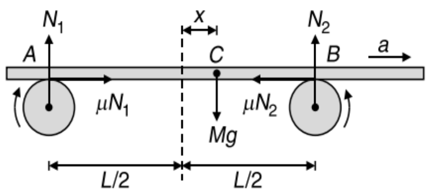
A uniform plate of mass M stays horizontally and symmetrically on two wheels rotating in opposite directions as shown in Figure.



The separation between the wheels is L. The friction coefficient between each wheel and the plate is μ. Find the time period of oscillation of the plate if it is slightly displaced along its length and released.

SOLUTION

In equilibrium the normal reactions due to two wheels are equal. Hence, the frictional forces are also equal and balance each other. When the plate is displaced by x towards right, the normal N_2 due to right wheel increases and N_1 due to left wheel decreases. The resultant friction $\mu(N_2-N_1)$ is towards left resulting in oscillatory motion. In displaced position, $N_1+N_2=Mg$



Balancing moments about A, we get

$$mg\left(\frac{L}{2}+x\right)=N_2L$$

$$\Rightarrow \quad N_2=Mg\left(\frac{1}{2}+\frac{x}{L}\right)$$

Balancing moments about B, we get

$$N_1=Mg\left(\frac{1}{2}-\frac{x}{L}\right)$$

So, restoring force F is given by

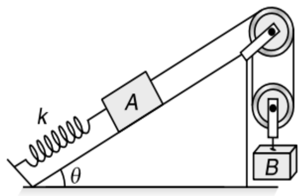
$$F = M\ddot{x} = -\mu(N_1 - N_2) = -\frac{2\mu Mg}{L}x$$

$$\Rightarrow \quad \omega^2 = \frac{2\mu g}{L}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{L}{2\mu g}}$$

PROBLEM 9

Calculate the angular frequency of the system shown in figure. Friction is absent everywhere and the threads, spring and pulleys are massless. Given that $m_A = m_B = m$.



SOLUTION

Let x_0 be the extension in the spring in equilibrium. Then for equilibrium of block A and B, we have

$$T = kx_0 + mg\sin\theta \qquad \ldots(1)$$

and $$2T = mg \qquad \ldots(2)$$

where, T is the tension in the string.
Now, suppose A is further displaced by a distance x from its mean position and v be its speed at this moment. Then B lowers by $\frac{x}{2}$ and speed of B at this instant will be $\frac{v}{2}$. Total energy of the system in this position is given by

$$E = \frac{1}{2}k(x+x_0)^2 + \frac{1}{2}m_Av^2 + \frac{1}{2}m_B\left(\frac{v}{2}\right)^2 + m_Agh_A - m_Bgh_B$$

$$\Rightarrow \quad E = \frac{1}{2}k(x+x_0)^2 + \frac{1}{2}mv^2 + \frac{1}{8}mv^2 + mgx\sin\theta - mg\frac{x}{2}$$

$$\Rightarrow \quad E = \frac{1}{2}k(x+x_0)^2 + \frac{5}{8}mv^2 + mgx\sin\theta - mg\frac{x}{2}$$

Since, E is constant, so

$$\frac{dE}{dt} = 0$$

$$\Rightarrow \quad 0 = k(x+x_0)\frac{dx}{dt} + \frac{5}{4}mv\left(\frac{dv}{dt}\right) + mg(\sin\theta)\left(\frac{dx}{dt}\right) - \frac{mg}{2}\left(\frac{dx}{dt}\right)$$

Substituting, $\frac{dx}{dt} = v = \dot{x}$, $\frac{dv}{dt} = a$ and from equations (1) and (2), using $kx_0 + mg\sin\theta = \frac{mg}{2}$, we get

$$\frac{5}{4}ma = -kx$$

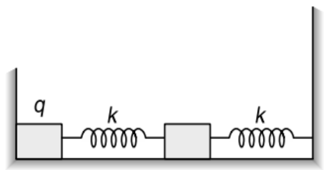
Since, $a \propto -x$, so, motion is simple harmonic, time period of which is

$$T = 2\pi\sqrt{\left|\frac{x}{a}\right|} = 2\pi\sqrt{\frac{5m}{4k}}$$

$$\Rightarrow \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{4k}{5m}}$$

PROBLEM 10

One end of each of two identical springs, natural length 9 cm and force constant $k = 45\ \text{Nm}^{-1}$ is attached with a small particle of mass $m = 30$ g. Other end of right spring is fixed with a wall and other end of left spring is attached with a fixed block having a positive charge $q = 1\ \mu\text{C}$ as shown in figure. The particle rests over a smooth horizontal plane and springs are non-deformed.



Calculate deformation of springs when positive charge $q = 1\ \mu\text{C}$ is given to the particle and equilibrium is attained. Also calculate the frequency of small oscillations of the particle.

SOLUTION

In equilibrium

$$(\text{Net Repulsive Force}) = (\text{Total Spring Force})$$

$$\Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(0.09+x)^2} = 2kx$$

$$\Rightarrow \quad x = 0.01\ \text{m} = 1\ \text{cm}$$

At a distance r, the electrostatic repulsion between the particles is

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2}$$

$$\Rightarrow \quad dF_e = -\frac{2q^2}{4\pi\varepsilon_0 r^3}dr$$

In equilibrium $r = 9 + 1 = 10\ \text{cm} = 0.1\ \text{m}$
When the particle is displaced by a distance y from equilibrium position, we have

$$\text{Net restoring force} = -\left[2ky + \frac{2q^2}{(4\pi\varepsilon_0)r^3}y\right]$$

$$\Rightarrow \quad ma = -\left[2k + \frac{2q^2}{(4\pi\varepsilon_0)r^3}\right]y$$

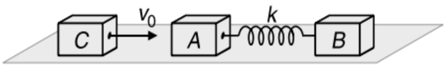
$$\Rightarrow \quad f=\frac{1}{2\pi}\sqrt{\left|\frac{a}{y}\right|}$$

Substituting the values, we get

$$f=\frac{30}{\pi}\text{ Hz}$$

PROBLEM 11

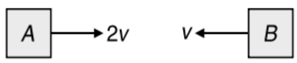
Two blocks A and B of masses $m_1 = 3$ kg and $m_2 = 6$ kg respectively are connected with each other by a spring of force constant $k = 200\text{ Nm}^{-1}$ as shown in figure. Blocks are pulled away from each other by $x_0 = 3$ cm and then released. When spring is in its natural length and blocks are moving towards each other, another block of mass $m = 3$ kg moving with velocity $v_0 = 0.4\text{ ms}^{-1}$ (towards right) collides with A and gets stuck to it. Neglecting friction, calculate



(a) velocities v_1 and v_2 of the blocks A and B respectively just before collision and their angular frequency.
(b) velocity of centre of mass of the system, after collision,
(c) amplitude of oscillations of combined body,
(d) loss of energy during collision.

SOLUTION

(a) $\frac{1}{2}kx_0^2=\frac{1}{2}\mu v_r^2$



where, μ = reduced mass $=\frac{6\times 3}{6+3}=2$ kg

$$\Rightarrow \quad v_{\text{relative}}=\sqrt{\frac{k}{\mu}}x_0=\left(\sqrt{\frac{200}{2}}\right)(3\times 10^{-2})$$

$$\Rightarrow \quad v_{\text{relative}}=0.3\text{ ms}^{-1}=2v+v$$

$$\Rightarrow \quad v=0.1\text{ ms}^{-1}$$

$$\Rightarrow \quad v_1=0.2\text{ ms}^{-1}\text{ and } v_2=0.1\text{ ms}^{-1}$$

Angular frequency $\omega=\sqrt{\frac{k}{\mu}}=\sqrt{\frac{200}{2}}=10\text{ rads}^{-1}$

(b) $v_{\text{cm}}=\frac{m_1v_1+m_2v_2+m_3v_3}{m_1+m_2+m_3}$

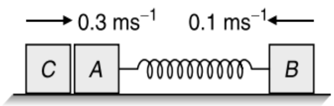
$$\Rightarrow \quad v_{\text{cm}}=\frac{(3)(0.2)-(6)(0.1)+3(0.4)}{3+6+3}$$

$$\Rightarrow \quad v_{\text{cm}}=0.1\text{ ms}^{-1}\text{ (towards right)}$$

(c) After collision velocity of block A and C is

$$v_0=\frac{(3\times 0.2)+(3)(0.4)}{3+3}=0.3\text{ ms}^{-1}$$

and velocity of block B is $v_2 = 0.1\text{ ms}^{-1}$



The spring will compress till velocity of all the blocks become equal to the centre of mass. Applying Law of Conservation of Mechanical Energy, we get

$$\frac{1}{2}(3+3)(0.3)^2+\frac{1}{2}(6)(0.1)^2=\frac{1}{2}(3+3+6)(0.1)^2+\frac{1}{2}kA^2$$

$$\Rightarrow \quad A=0.048\text{ m}$$

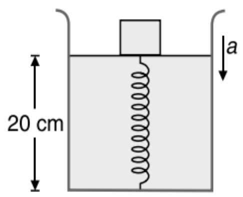
$$\Rightarrow \quad A=4.8\text{ cm}$$

(d) $\Delta E=\frac{1}{2}(3)(0.4)^2+\frac{1}{2}(3)(0.2)^2-\frac{1}{2}(3+3)(0.3)^2$

$$\Rightarrow \quad \Delta E=0.24+0.06-0.27=0.03\text{ J}$$

PROBLEM 12

A rectangular tank having base $15\text{ cm}\times 20\text{ cm}$ is filled with water (density $\rho = 1000\text{ kgm}^{-3}$) upto 20 cm height. One end of an ideal spring of natural length 20 cm and force constant $k = 280\text{ Nm}^{-1}$ is fixed to the bottom of a tank so that spring, remains vertical. This system is in an elevator moving downwards with acceleration $a = 2\text{ ms}^{-2}$. Cubical block of side $l = 10$ cm and mass $m = 2$ kg is gently placed over in figure.



(a) Calculate compression of the spring in equilibrium position.
(b) If block is attached to spring and slightly pushed down from equilibrium position and released, calculate frequency of its vertical oscillations. $(g = 10\text{ ms}^{-2})$.

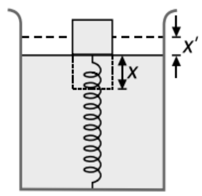
SOLUTION

(a) Let in equilibrium compression of spring is x. Water of volume l^2x is displaced from its original position and level of liquid in the tank rises by x', then,

$$l^2x=(A-l^2)x'$$

Substituting the values, we get

$$x'=0.5x$$



Equation of motion for the block is,

$$mg - \text{upthrust} - \text{spring force} = ma$$

$$\Rightarrow \quad mg - (x + x')l^2\rho g - kx = ma$$

Substituting the values, we get

$$x = 0.04 \text{ m}$$

$$\Rightarrow \quad x = 4 \text{ cm}$$

(b) If the block is slightly pushed downwards by the amount y, both upthrust and spring force increase.

$$\begin{pmatrix}\text{Net}\\\text{Restoring}\\\text{Force}\end{pmatrix} = \begin{pmatrix}\text{Increase}\\\text{in}\\\text{Upthrust}\end{pmatrix} + \begin{pmatrix}\text{Increase}\\\text{in Spring}\\\text{Force}\end{pmatrix}$$

$$\Rightarrow \quad F = -\left[(y + 0.5y)l^2\rho g + ky\right]$$

Substituting the values, we get

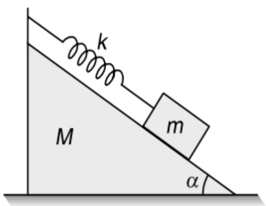
$$F = -400y$$

$$\Rightarrow \quad a = -\frac{400}{2}y = -200y$$

$$\Rightarrow \quad f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{y}\right|} = \frac{5\sqrt{2}}{\pi} \text{ Hz}$$

PROBLEM 13

Consider a block of mass m to be placed on a wedge of mass M, having wedge angle α. Assuming all the contacts to the frictionless, find the angular frequency of wedge and block system.



SOLUTION

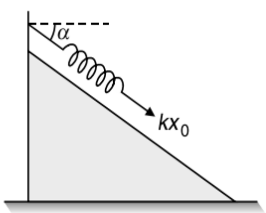
In equilibrium, we have

$$kx_0 \cos\alpha = N\sin\alpha \quad \text{...(1)}$$

$$N = mg\cos\alpha \quad \text{...(2)}$$

From these two equations, we get

$$mg\sin\alpha = kx_0 \quad \text{...(3)}$$



When further displaced by x (downwards). Let v_r be the velocity of m w.r.t. M and v be the velocity of M at this instant. Applying Law of Conservation of Linear Momentum in horizontal direction, we get

$$Mv = m(v_r\cos\alpha - v)$$

$$\Rightarrow \quad v = \frac{mv_r\cos\alpha}{M+m}$$

Further total energy E of system is given by

$$E = \frac{1}{2}k(x + x_0)^2 - mgx\sin\alpha + \frac{1}{2}Mv^2 + \frac{1}{2}m\left[(v_r\cos\alpha - v)^2 + (v_r\sin\alpha)^2\right]$$

$$\Rightarrow \quad E = \frac{1}{2}k(x + x_0)^2 - mgx\sin\alpha + \frac{1}{2}\frac{Mm^2}{(M+m)^2}v_r^2\cos^2\alpha + \frac{1}{2}m\left[v_r^2 + v^2 - 2vv_r\cos\alpha\right]$$

$$\Rightarrow \quad E = \frac{1}{2}k(x + x_0)^2 - mgx\sin\alpha + \frac{1}{2}\frac{m^2}{(M+m)}v_r^2\cos^2\alpha + \frac{1}{2}mv_r^2 - \frac{m^2v_r^2\cos^2\alpha}{(M+m)}$$

$$\Rightarrow \quad E = \frac{1}{2}k(x + x_0)^2 - mgx\sin\alpha + \frac{1}{2}m\left[1 - \frac{m\cos^2\alpha}{M+m}\right]v_r^2$$

$$\Rightarrow \quad E = \frac{1}{2}k(x + x_0)^2 - mgx\sin\alpha + \frac{1}{2}m\left[\frac{M + m\sin^2\alpha}{M+m}\right]v_r^2$$

Since, E =constant

$$\Rightarrow \quad \frac{dE}{dt} = 0$$

$$\Rightarrow \quad 0 = k(x + x_0)\frac{dx}{dt} + mv_r\left(\frac{dv_r}{dt}\right)\left(\frac{M + m\sin^2\alpha}{M+m}\right) - mg\sin\alpha\left(\frac{dx}{dt}\right)$$

Substituting, $mg\sin\alpha = kx_0$, $\frac{dx}{dt} = v_r$ and $\frac{dv_r}{dt} = a_r$, we get

$$m\left(\frac{M + m\sin^2\alpha}{M+m}\right)a_r = -kx$$

$$\Rightarrow \quad a_r = \frac{-k(M+m)}{m(M + m\sin^2\alpha)}x$$

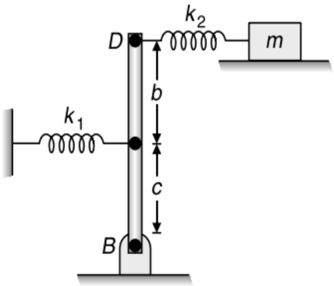
Comparing with $a_r = -\omega^2 x$, we get

$$\omega = \sqrt{\frac{k(M+m)}{m(M + m\sin^2\alpha)}}$$

Please note that, here x is the displacement of block relative to the wedge.

PROBLEM 14

Find the angular frequency of motion of block m for small motion of rod BD when we neglect the inertial effects of the rod BD. Spring constants are k_1 and k_2. Neglect friction forces.



SOLUTION

Let extension in springs be x_1 and x_2. Then $x_1 = c\theta$ but $x_2 \neq (b+c)\theta$, because the block will also move towards the right.

Suppose the block moves towards right by x, then

$$x = x_2 + (b+c)\theta \qquad \ldots(1)$$

Since, $\Sigma\tau_B = 0$

$$\Rightarrow \quad k_1 x_1 c = k_2 x_2 (b+c)$$

$$\Rightarrow \quad k_1 (c\theta) c = k_2 [x - (b+c)\theta](b+c)$$

$$\Rightarrow \quad \left[k_1 c^2 + k_2 (b+c)^2\right]\theta = k_2 x (b+c)$$

$$\Rightarrow \quad \theta = \frac{k_2 (b+c) x}{k_1 c^2 + k_2 (b+c)^2} \qquad \ldots(2)$$

If F is the restoring force on the block, then

$$F = k_2 x_2$$

$$\Rightarrow \quad x_2 = \frac{F}{k_2} \qquad \ldots(3)$$

From equation (1), (2) and (3), we get

$$x = \frac{F}{k_2} + (b+c)\theta$$

$$\Rightarrow \quad x = \frac{F}{k_2} + \frac{(b+c)k_2 x_2 (b+c)}{k_1 c^2}$$

$$\Rightarrow \quad x = \frac{F}{k_2} = \frac{(b+c)F(b+c)}{k_1 c^2}$$

$$F = \frac{k_1 k_2 c^2}{k_1 c^2 + k_2 (b+c)^2} x$$

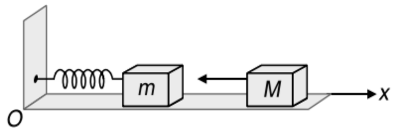
So, the effective spring constant is given by

$$k_{\text{eff}} = \frac{k_1 k_2 c^2}{k_1 c^2 + k_2 (b+c)^2}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{k_1 k_2 c^2}{m\left[k_1 c^2 + k_2 (b+c)^2\right]}}$$

PROBLEM 15

One end of an ideal spring is fixed to a wall at origin O and axis of spring is parallel to x-axis. A block of mass $m = 1$ kg is attached to free end of the spring and it is performing S.H.M. Equation of position of the block in coordinate system shown in figure is $x = 10 + 3\sin(10 \cdot t)$. Here t is in second and x in cm. Another block of mass $M = 3$ kg, moving towards the origin with velocity $30\ \text{cmsec}^{-1}$ collides with the block performing S.H.M. at $t = 0$ and gets stuck to it. Calculate the



(a) new amplitude of oscillations,
(b) new equation for position of the combined body,
(c) loss of energy during collision. Neglect friction.

SOLUTION

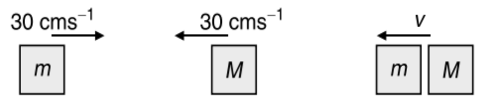
(a) Since, $\omega^2 = \dfrac{k}{m}$ $\qquad \left\{\because \ \omega = \sqrt{\dfrac{k}{m}}\right\}$

$$\Rightarrow \quad k = m\omega^2 = (1)(10)^2 = 100\ \text{Nm}^{-1}$$

At $t = 0$, block of mass m is at mean position $(x = 10\ \text{cm})$ and moving towards positive x-direction with velocity $A\omega$ or $30\ \text{cms}^{-1}$

By Law of Conservation of Linear Momentum, we get

$$(M+m)v = M(30) - m(30)$$



Substituting the values, we have

$$v = 15\ \text{cms}^{-1}$$

$$\Rightarrow \quad v = 0.15\ \text{ms}^{-1}$$

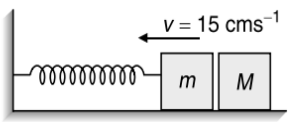
By Law of Conservation of Mechanical Energy, we get

$$\frac{1}{2}(M+m)v^2 = \frac{1}{2}kA^2$$

$$\Rightarrow \quad A = \left(\sqrt{\frac{M+m}{k}}\right)v = \left(\frac{4}{100}\right)^{\frac{1}{2}}(0.15)$$

$$\Rightarrow \quad A = 0.03\ \text{m}$$

$$\Rightarrow \quad A = 3\ \text{cm}$$



(b) $\omega' = \sqrt{\dfrac{k}{M+m}} = \sqrt{\dfrac{100}{4}} = 5\ \text{radsec}^{-1}$

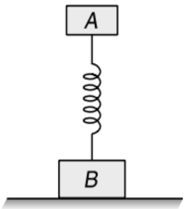
$$\Rightarrow \quad x' = 10 - 3\sin 5t$$

(c) $\Delta E = \frac{1}{2}(1)(0.3)^2 + \frac{1}{2}(3)(0.3)^2 - \frac{1}{2}(4)(0.15)^2$

$\Rightarrow \quad \Delta E = 0.135 \text{ J}$

PROBLEM 16

A body A of mass $m_1 = 1.00$ kg and a body B of mass $m_2 = 4.10$ kg are interconnected by a spring as shown in figure. The body A performs free vertical harmonic oscillations with amplitude $a = 1.6$ cm and frequency $\omega = 25 \text{ rads}^{-1}$. Neglecting the mass of the spring, find the maximum and minimum values of force that this system exerts on the bearing surface.



SOLUTION

Maximum Force

$$k = \omega^2 m_1 = (25)^2(1) = 625 \text{ Nm}^{-1}$$

Acceleration of centre of mass in extreme position,

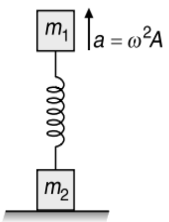
$$a_{CM} = \frac{m_1 a}{m_1 + m_2}(\text{upwards}) = \frac{m_1 \omega^2 A}{m_1 + m_2}$$

Now, $N_{max} - (m_1 + m_2)g = \frac{m_1 \omega^2 A}{(m_1 + m_2)} \times (m_1 + m_2)$

$\Rightarrow \quad N_{max} = (m_1 + m_2)g + m_1 \omega^2 A$

$\Rightarrow \quad N_{max} = (1 + 4.10)9.8 + (1)(25)^2(1.6 \times 10^{-2})$

$\Rightarrow \quad N_{max} = 59.98$ newton



Minimum Force

$a_{cm} = \frac{m_1 a}{m_1 + m_2}$ {downwards}

Since, $(m_1 + m_2)g - N_{min} = (m_1 + m_2)a_{cm}$

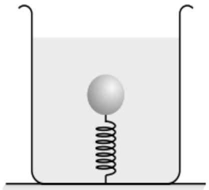
$\Rightarrow \quad N_{min} = (m_1 + m_2)g - m_1 \omega^2 A$

$\Rightarrow \quad N_{min} = (1 + 4.10)9.8 - (1)(25)^2(1.6 \times 10^{-2})$

$\Rightarrow \quad N_{min} = 39.98$ newton

PROBLEM 17

As a submerged body moves through a fluid, the particles of the fluid, flow around the body and thus acquire kinetic energy. In the case of a sphere moving in an ideal fluid, the total kinetic energy acquired by the fluid is $\frac{1}{4}\rho V v^2$ where ρ is the mass density of fluid, V the volume of sphere and v is the velocity of the sphere. Consider a 0.5 kg hollow spherical shell of radius 8 cm which is held submerged in a tank of water by a spring of force constant 500 Nm^{-1}.

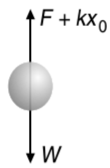


(a) Neglecting fluid friction, determine the period of vibration of the shell when it is displaced vertically and then released.

(b) Solve part (a) assuming that the tank is accelerated upward at the constant rate of 8 ms^{-2}. Density of water is 10^3 kgm^{-3}.

SOLUTION

(a) Let F be the upthrust and W the weight of the sphere. In equilibrium let x_0 be the compression of the spring, then



$$F = kx_0 = W$$

$$\Rightarrow \quad kx_0 = W - F \quad \ldots(1)$$

If the sphere is further compressed by x, then total energy of the system is given by

$$E = -(W - F)x + \frac{1}{2}k(x + x_0)^2 + \frac{1}{2}mv^2 + \frac{1}{4}\rho V v^2$$

Since, friction is absent, total energy remains constant, hence

$$\frac{dE}{dt} = 0$$

$$\Rightarrow \quad 0 = -(W - F)\cdot\frac{dx}{dt} + k(x + x_0)\frac{dx}{dt} + mv\left(\frac{dv}{dt}\right) + \frac{1}{2}\rho V v\left(\frac{dv}{dt}\right) \quad \ldots(2)$$

From equations (1) and (2), with substitutions $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = \ddot{x}$, we get

$$\ddot{x}+\frac{k}{\left(m+\frac{\rho V}{2}\right)}x=0$$

$\Rightarrow \ddot{x} \propto -x$

Oscillations are simple harmonic, time period of which is given by

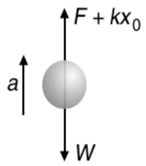
$$T=2\pi\sqrt{\left|\frac{x}{\ddot{x}}\right|}=2\pi\sqrt{\frac{m+\frac{1}{2}\rho V}{k}}$$

$$\Rightarrow T=2\pi\sqrt{\frac{0.5+\frac{1}{2}\times 10^3\times\frac{4}{3}\times\pi\times(0.08)^3}{500}}$$

$\Rightarrow T = 0.352$ s

(b) When it is accelerated upwards with an acceleration a, then

$$F'=\frac{F(g+a)}{g}$$



Now, $F'+kx_0-W=\left(\frac{W}{g}\right)a$

$$\Rightarrow kx_0=\frac{W}{g}\cdot a+W-F\left(1+\frac{a}{g}\right)$$

$$\Rightarrow kx_0=(W-F)+\frac{a}{g}(W-F)$$

$$\Rightarrow kx_0=(W-F)\left(1+\frac{a}{g}\right) \quad \ldots(3)$$

When displaced downwards, total energy is given by

$$E=-(W-F)\frac{(g+a)}{g}x+\frac{1}{2}k(x+x_0)^2+\frac{1}{2}mv^2+\frac{1}{4}\rho Vv^2$$

Substituting $\frac{dE}{dt}=0$

$$\Rightarrow 0=-(W-F)\left(1+\frac{a}{g}\right)\frac{dx}{dt}+k(x+x_0)\frac{dx}{dt}+ mv\left(\frac{dv}{dt}\right)+\frac{1}{2}\rho vV\left(\frac{dv}{dt}\right) \quad \ldots(4)$$

From equations (4) and (3) we get the same result as was obtained in part (a), i.e.,

$T = 0.352$ s

PROBLEM 18

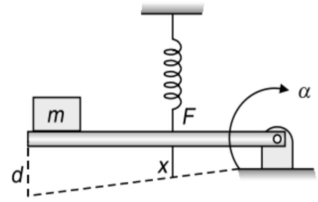
A rod AB of mass M is attached as shown in figure to a spring of constant k. A small block of mass m is placed on the rod at its free end A. In equilibrium rod is horizontal, i.e., spring is stretched.

(a) If end A is moved down through a small distance d and released, determine the period of vibration.

(b) Determine the largest allowable value of d if the block m is to remain at all times in contact with the rod.

SOLUTION

(a) Since, $\frac{d}{l}=\frac{x}{b}$



$$\Rightarrow x=\left(\frac{b}{l}\right)d$$

Unbalanced force (extra force) is given by

$$F=kx=k\left(\frac{b}{l}\right)d$$

Restoring torque, $\tau=F\cdot b=\left(\frac{kb^2}{l}\right)d$

$$\Rightarrow \alpha\left(\frac{Ml^2}{3}+ml^2\right)=-(kb^2)\theta \text{ as } d=l\theta$$

$$\Rightarrow T=2\pi\sqrt{\left|\frac{\theta}{\alpha}\right|}=2\pi\sqrt{\frac{\frac{Ml^2}{3}+ml^2}{kb^2}}$$

(b) Maximum acceleration of block $\le g$

$$a_{max}=\omega^2 d_{max}=\frac{kb^2}{\frac{Ml^2}{3}+ml^2}d_{max}$$

For m to remain in contact with the rod at all the times, we have

$$a_{max}\le g$$

$$\Rightarrow \frac{kb^2 d_{max}}{\left(\frac{Ml^2}{3}+ml^2\right)}\le g$$

$$\Rightarrow d_{max}\le\frac{g\left(\frac{Ml^2}{3}+ml^2\right)}{kb^2}$$

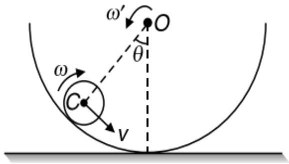
PROBLEM 19

Consider a solid homogeneous cylinder of radius r rolling inside a fixed hollow cylinder of radius R. Find the frequency of small oscillations of the inner cylinder about the stable equilibrium position.

SOLUTION

For pure rolling to take place, we have

$$v = \omega r \qquad \ldots(1)$$



Let ω' be the angular velocity of C w.r.t. O, then

$$\omega' = \frac{v}{R-r} = \omega\left(\frac{r}{R-r}\right)$$

$$\Rightarrow \quad \frac{d\omega'}{dt} = \left(\frac{r}{R-r}\right)\frac{d\omega}{dt}$$

$$\Rightarrow \quad \alpha' = \left(\frac{r}{R-r}\right)\alpha \qquad \ldots(2)$$

Since, $\alpha = \dfrac{a}{r}$ $\qquad \ldots(3)$

and $a = \dfrac{g\sin\theta}{1+\dfrac{I}{mr^2}}$ were $I = \dfrac{1}{2}mr^2$ $\qquad \ldots(4)$

So, from equations (2), (3) and (4), we get

$$\alpha' = \frac{2g\sin\theta}{3(R-r)}$$

For small oscillations $\sin\theta \approx \theta$ and being restoring in nature, so

$$\alpha' = -\frac{2g\theta}{3(R-r)}$$

$$\Rightarrow \quad f = \frac{1}{2\pi}\sqrt{\left|\frac{\alpha'}{\theta}\right|} = \frac{1}{2\pi}\sqrt{\frac{2g}{3(R-r)}}$$