CHAPTER 3

Rotational Dynamics

Learning Objectives

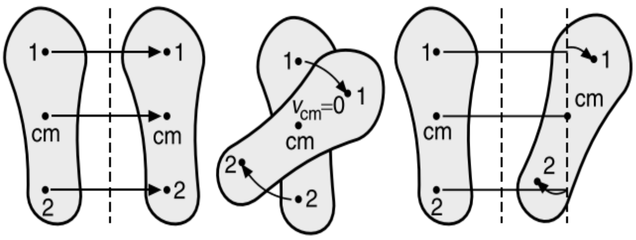
After reading this chapter, you will be able to understand concepts and problems based on:

(a) Moment of Inertia
(b) Parallel and Perpendicular Axis Theorem
(c) Rotational Kinematics
(d) Combined Effect of Rotation and Translation Motion
(e) Instantaneous Axis of Rotation
(f) Pure Rolling
(g) Conservation of Energy
(h) Concept of Torque
(i) General Motion of a Rigid Body
(j) Concept of Toppling
(k) Shifting of Normal Reaction
(l) Work-Energy Principle
(m) Modified Newton's Second Law for Fixed Axis Rotation
(n) Uniform and Accelerated Pure Rolling
(o) Angular Momentum and its Conservation
(p) Rolling with Slipping.

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

In the previous chapters we have studied the translator motion. In this chapter we will study the rotational motion of a rigid body about a fixed axis.

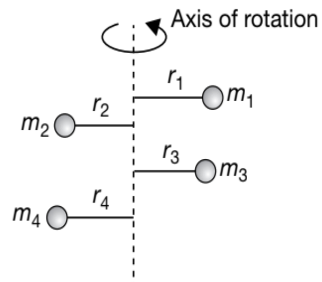


Translation   Rotation   Translation + Rotation

A rigid body is defined as an object that has fixed size and shape. In other words, the relative positions of its constituent particles remain constant. In actual, a rigid body does not exist – it is a useful idealization. By the term **fixed axis**, we mean that the axis must be fixed relative to the body and fixed in direction relative to an inertial position.

MOMENT OF INERTIA (*I*)

Moment of Inertia (MI) of a rigid n-particle system about an axis shown in Figure is the sum of the products of the masses of the particles with the square of their respective distances from the axis of rotation.



A discrete system of particles

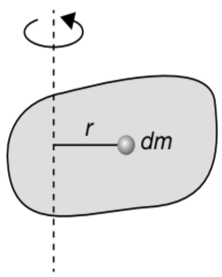
$\Rightarrow \quad I = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2 = \Sigma m_i r_{i\perp}^2$

where m_i is the mass of the i^{th} particle and $r_{i\perp}$ is the perpendicular distance of the i^{th} particle from the Axis of Rotation (AOR). It is a tensor quantity with SI unit kgm^2.

Moment of Inertia depends upon the mass of the body and the manner the mass is distributed in the system with respect to the AOR. So, the moment of inertia depends on the location of the axis, that is, on how the mass of the body is distributed relative to the axis. Thus, a body does not possess a unique moment of inertia and different axes through the body are associated with different moments of inertia.

The role played by MI in rotational dynamics is actually the same as that played by inertia in linear dynamics. Moment of Inertia opposes any change in rotational motion and hence is also called **rotational inertia**.

To find the Moment of Inertia of a continuous mass distribution, we consider an element of mass dm at a perpendicular distance r from the AOR as shown in Figure.



The moment of inertia of this infinitesimal element is dI, given by the moment of inertia of the whole body is given by

$$I = \int r_\perp^2 dm = \int r^2 dm$$

where $r_\perp$ is the perpendicular distance of the element dm from the axis of rotation.

The integration has then to be carried within specified limits.

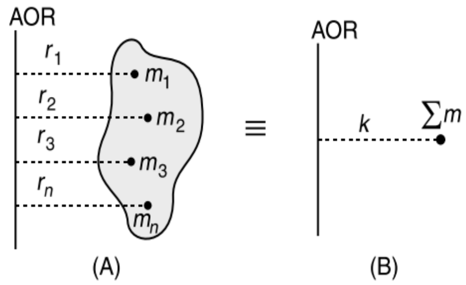
RADIUS OF GYRATION (*k*)

Consider the Figures shown. Figure (A) has the system with masses m_1, m_2, ..., m_n at distances r_1, r_2,..., r_n from AOR. If we take a point mass $M = \Sigma m = m_1 + m_2 + ... + m_n$ and place it at a distance k from AOR, then we must get same value of Moment of Inertia from both the cases. So,

$$\Sigma m_i r_i^2 = Mk^2$$

$$\Rightarrow \quad k = \sqrt{\frac{m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2}{M}}$$

So, radius of gyration for a system is that distance from the AOR, where if the entire mass of the system is placed, then we get same value of MI as obtained from the original mass distribution.



SI unit of k is metre.

Further, for a symmetrical mass distribution we have

$$m_1 = m_2 = ... = m_n = m \text{ (say)}$$

$$M = \Sigma m = nm$$

$$\Rightarrow \quad k = \sqrt{\frac{r_1^2 + r_2^2 + ... + r_n^2}{n}}$$

So, for a symmetrical mass distribution, the radius of gyration equals the Root Mean Squares (RMS) of the distances of the particles from the AOR.

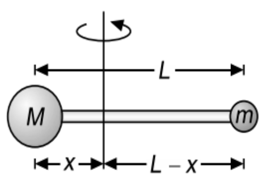
(The operation Root Mean Square is always to be read as Root of Mean of Squares).

Finally, in general for a system of total mass M rotating about an axis, we have

$$k = \sqrt{\frac{I}{M}}$$

ILLUSTRATION 1

Two balls with masses M and m are connected by a rigid rod of length L and negligible mass as in Figure.



For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axis passes through the centre of mass. Show that this moment of inertia is $I=\mu L^2$, where $\mu=\dfrac{mM}{m+M}$ is the reduced mass of system.

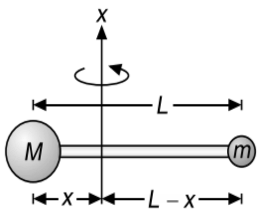
SOLUTION

$$I=Mx^2+m(L-x)^2$$

For I to be minimum, $\dfrac{dI}{dx}=0$

$$\Rightarrow \quad \frac{dI}{dx}=2Mx-2m(L-x)=0$$

$$\Rightarrow \quad x=\frac{mL}{M+m}=(\text{Centre of Mass of System})$$



MI about an axis passing through x is

$$I_{CM}=M\left(\frac{mL}{M+m}\right)^2+m\left(1-\frac{m}{M+m}\right)^2L^2$$

$$\Rightarrow \quad I_{CM}=\left(\frac{Mm}{M+m}\right)L^2=\mu L^2,\ \text{where } \mu=\frac{Mm}{M+m}$$

So, we see that the moment of inertia of the system is minimum about the axis passing through the axis of rotation.

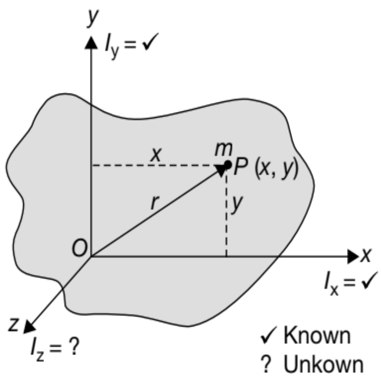
PERPENDICULAR AXIS THEOREM

This theorem is distinct for a 2-D body (such as a lamina or a planar body) and a 3-D body (such as cube, prism etc).

For a Lamina: The theorem states that the moment of inertia of a lamina about an axis perpendicular to the plane of lamina (say z-axis) is equal to the sum of the moments of inertia of lamina about two mutually perpendicular axis lying in its own plane (say x and y-axis) intersecting each other at the point where the two perpendicular axis meet.

If I_x and I_y be the moment of inertia of lamina about two mutually perpendicular axis x and y lying in its plane and I_z be the moment of inertia of lamina about third axis perpendicular to lamina and passing through intersection of x and y, then

$$I_z=I_x+I_y$$



Proof of Perpendicular Axis Theorem

Consider a lamina that lies in x-y plane. So, z axis will be perpendicular to x, y and hence normal to lamina. Now consider a particle of mass m lying at the point $P(x, y)$

Now, MI of particle about x axis is my^2

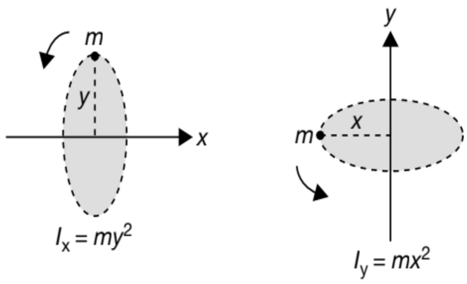
MI of body/lamina about x axis is

$$I_x=\Sigma my^2 \qquad \ldots(1)$$

Similarly, MI of particle about y axis is mx^2

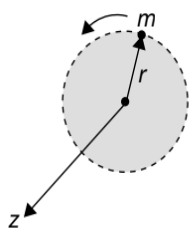
MI of body/lamina about y axis is

$$I_y=\Sigma mx^2 \qquad \ldots(2)$$



Finally, MI of particle about z axis is mr^2

MI of body/lamina about z axis is



$$I_z=\Sigma mr^2 \qquad \ldots(3)$$

Since, $r^2=x^2+y^2$

$$\Rightarrow \quad I_z=\Sigma mx^2+\Sigma my^2$$

$$\Rightarrow \quad I_z=I_y+I_x$$

Conceptual Note(s)

For a 3-D body: The theorem, here states that the sum of the moments of inertia of a three-dimensional body about three mutually perpendicular axes is equal to twice the summation Σmr^2 about the origin, where $r^2 = x^2 + y^2 + z^2$

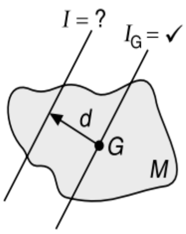
Mathematically,

$$I_x + I_y + I_z = 2\Sigma mr^2$$

$$\Rightarrow \quad I_x + I_y + I_z = 2\Sigma m\left(x^2 + y^2 + z^2\right)$$

PARALLEL AXIS THEOREM

This theorem has the same statement for a planar (laminar) body as well as a three-dimensional body.



It states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of gravity (CG) (or centre of mass (CM)) plus the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.

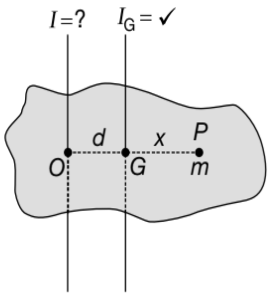
Mathematically, $I = I_G + (\Sigma m)d^2$

$$\Rightarrow \quad I = I_G + Md^2$$

where, M is the total mass of the body, I_G is the MI of body about CG and d is the perpendicular distance between two parallel axes.

Proof of Parallel Axis Theorem

Consider a body of mass M having centre of gravity G. Let I_G be the MI of body about an axis passing through the centre of gravity (G). Let us find the MI of the body about another axis parallel to axis passing through centre of gravity (G) at perpendicular distance d from it as shown in Figure.



For this let us consider a particle of mass m lying at a point P at perpendicular distance x from G. Then

$$I = I_G + Md^2$$

MI of particle about G is mx^2

MI of body about G is $I_G = \Sigma mx^2$ ...(1)

MI of particle about O is $m(x+d)^2$

MI of body about O is $I = \Sigma m(x+d)^2$ ...(2)

$$\Rightarrow \quad I = \Sigma m\left(x^2 + d^2 + 2xd\right)$$

$$\Rightarrow \quad I = \Sigma mx^2 + \Sigma md^2 + \Sigma m(2xd)$$

$$\Rightarrow \quad I = I_G + d^2\Sigma m + 2d\Sigma mx \qquad \text{...(3)}$$

Now, since a body always balances itself when held from centre of gravity (CG), i.e., net torque due to weights of all the particles about CG is zero. Hence

$$\Sigma(mg)x = g\Sigma mx = 0$$

$$\Rightarrow \quad \Sigma mx = 0$$

$$\Rightarrow \quad I = I_G + Md^2 \qquad \text{\{from equation (3)\}}$$

Conceptual Note(s)

The moment of inertia of a body is minimum about an axis that passes through the centre of mass. So $\dfrac{dI}{dx} = 0$ will give the location of the centre of mass of the system.

For example, if the moment of inertia I of a system varies with x as $I = \left(8x^2 - 4x + 5\right)$ kgm^2, then to find the location of the centre of mass of the system, we have

$$\frac{dI}{dx} = 0$$

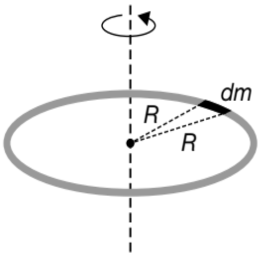
$$\Rightarrow \quad \frac{d}{dx}\left(8x^2 - 4x + 5\right) = 0$$

$$\Rightarrow \quad 16x - 4 = 0$$

$$\Rightarrow \quad x = 4 \text{ m}$$

MOMENT OF INERTIA OF A THIN RING

CASE-1: About an axis passing through centre perpendicular to plane of ring

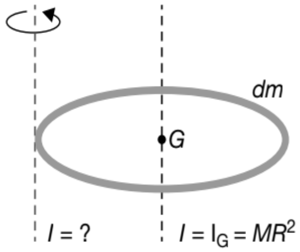


Consider an element of mass dm on perimeter of ring. Then MI of element about said AOR is

$$dI = R^2 dm$$

$$\Rightarrow \quad I = R^2 \int dm = MR^2$$

CASE-2: About an axis tangential to ring and perpendicular to the plane of ring



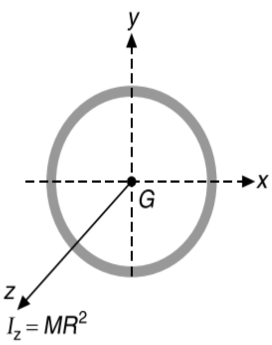
Using Parallel Axis Theorem, we get

$$I = I_G + Md^2$$

$$\Rightarrow \quad I = MR^2 + MR^2 = 2MR^2$$

CASE-3: About the diameter

Consider two diameters mutually perpendicular to each other i.e., along x and y axis (say) as shown in Figure.



Then by Perpendicular Axis Theorem, we have

$$I_z = I_x + I_y \text{ (by perpendicular axis theorem)}$$

Since both diameters are identical, so

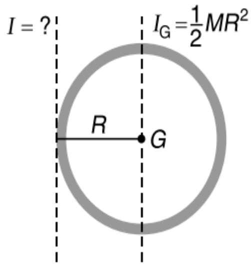
$$I_x = I_y = I_d$$

Also, $I_z = MR^2$

$$\Rightarrow \quad MR^2 = I_d + I_d$$

$$\Rightarrow \quad I_d = \frac{1}{2}MR^2$$

CASE-4: About an axis tangential to ring lying in its plane



Using Parallel Axis Theorem, we get

$$I = I_G + Md^2$$

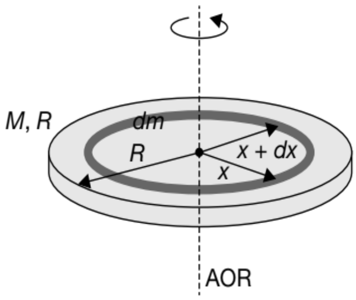
Since $I_G = \frac{1}{2}MR^2$

$$\Rightarrow \quad I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

MOMENT OF INERTIA OF A THIN DISC

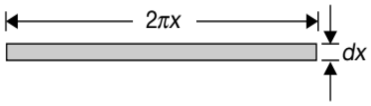
CASE-1: About axis passing through centre perpendicular to its plane

Consider a disc of mass M, radius R having surface mass density $\sigma\left(= \frac{M}{\pi R^2}\right)$. Consider a circular concentric element of inner radius x and outer radius $x + dx$ (i.e., thickness dx) as shown in Figure.



If dm be the mass of element of area dA, then

$$dA = (2\pi x)dx$$



Mass of element is

$$dm = \sigma A = \frac{M}{\pi R^2} dA = \frac{M}{\pi R^2}(2\pi x dx)$$

$$\Rightarrow \quad dm = \frac{2M}{R^2} x dx$$

So, MI of infinitesimal ring element about the specified AOR is

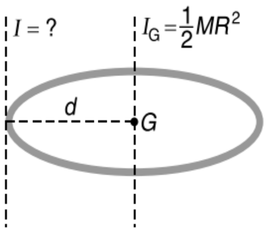
$$dI = (dm)x^2$$

$$\Rightarrow \quad dI = \frac{2M}{R^2} x^3 dx$$

$$\Rightarrow \quad I = \frac{2M}{R^2}\int_0^R x^3 dx = \frac{2M}{R^2}\left(\frac{x^4}{4}\bigg|_0^R\right) = \frac{2M}{R^2}\left(\frac{R^4}{4} - 0\right)$$

$$\Rightarrow \quad I = \frac{1}{2} MR^2$$

CASE-2: About an axis tangential to disc perpendicular to disc

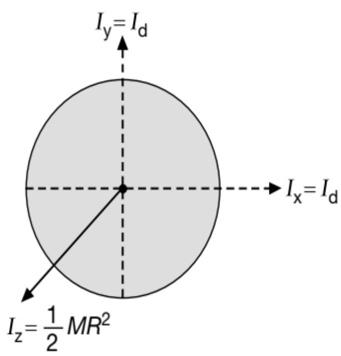


Using Parallel Axis Theorem, we get

$$I = I_G + Md^2$$

$$\Rightarrow \quad I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

CASE-3: About the diameter



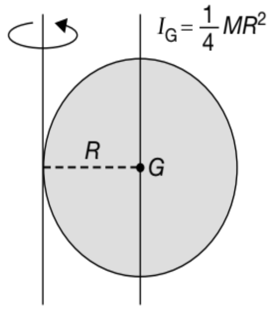
Using Perpendicular Axis Theorem, we get

$$I_x + I_y = I_z$$

$$\Rightarrow \quad I_d + I_d = \frac{1}{2}MR^2$$

$$\Rightarrow \quad I_d = \frac{1}{4}MR^2$$

CASE-4: About an axis tangential to disc lying in its plane



Using Parallel Axis Theorem, we get

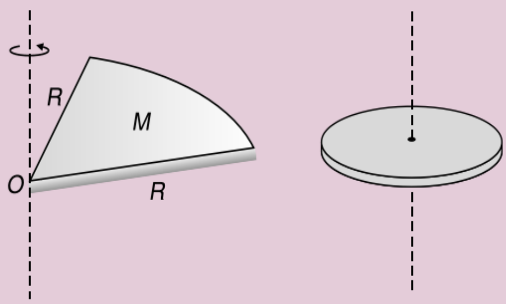
$$I = I_G + Md^2$$

$$\Rightarrow \quad I = \frac{1}{4}MR^2 + MR^2$$

$$\Rightarrow \quad I = \frac{5}{4}MR^2$$

Conceptual Note(s)

Moment of inertia of a part of a rigid body (symmetrically cut from the whole mass) is the same as that of the whole body. e.g., in Figure (a) moment of inertia of the section shown (a part of a circular disc) about an axis perpendicular to its plane and passing through point O is $\frac{1}{2}MR^2$ as the moment of inertia of the complete disc is also $\frac{1}{2}MR^2$.



(a) (b)

This can be shown as in Figure. Suppose the given section is $\frac{1}{n}$th part of the disc, then mass of the disc will be nM.

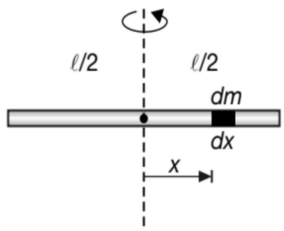
$$I_{disc} = \frac{1}{2}(nM)R^2$$

$$\Rightarrow \quad I_{section} = \frac{1}{n}I_{disc} = \frac{1}{2}MR^2$$

MOMENT OF INERTIA OF A THIN ROD

CASE-1: About axis passing through centre perpendicular to its length

Consider a thin uniform rod of mass M, length l having mass per unit length λ.



Consider an element of length dx, mass dm at perpendicular distance x from AOR. If dm be the mass of this element, then

$$dm = \lambda dx = \frac{M}{l}dx$$

Moment of inertia due to this element about said axis is

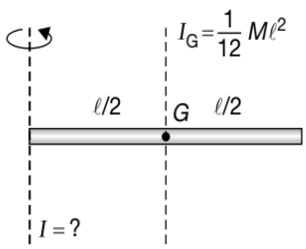
$$dI = x^2 dm = x^2\left(\frac{M}{l}dx\right)$$

$$\Rightarrow \quad I = \frac{M}{l}\int_{-l/2}^{l/2} x^2 dx = \frac{M}{l}\left(\frac{x^3}{3}\Big|_{-l/2}^{l/2}\right)$$

$$\Rightarrow \quad I = \frac{M}{3l}\left[\frac{l^3}{8} - \left(-\frac{l^3}{8}\right)\right] = \frac{M}{3l}\left(\frac{l^3}{4}\right)$$

$$\Rightarrow \quad I = \frac{Ml^2}{12}$$

CASE-2: About an axis passing through end and perpendicular to its length



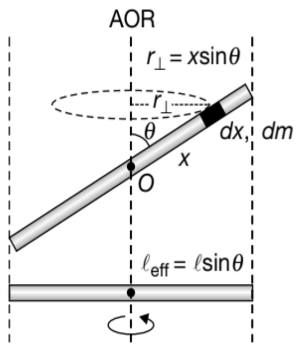
Using Parallel Axis Theorem, we get

$$I = I_G + Md^2$$

$$\Rightarrow \quad I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4}$$

$$\Rightarrow \quad I = \frac{Ml^2}{3}$$

CASE-3: About an axis passing through centre, inclined to the rod at an angle θ.



$$I = \frac{1}{12}M\ell_{eff}^2 = \frac{1}{12}M\ell^2\sin^2\theta$$

Consider an element of length dx at a distance x from centre of rod. If dm be mass of element then

$$dm = \frac{M}{l}dx$$

Now, when the rod rotates about the said AOR, then the element rotates about AOR in a circle of radius $r_\perp = x\sin\theta$. Since

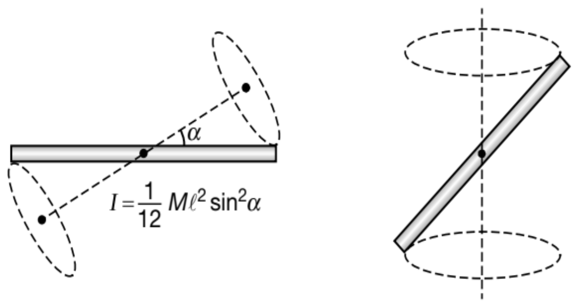
$$dI = (dm)r_\perp^2$$

where, $dm = \frac{M}{l}dx$

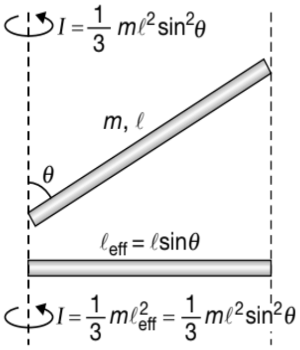
$$\Rightarrow \quad dI = \frac{M}{l}x^2\sin^2\theta dx$$

$$\Rightarrow \quad I = \frac{M}{l}\sin^2\theta\int_{-l/2}^{l/2} x^2 dx = \frac{M}{12}l^2\sin^2\theta$$

$$\Rightarrow \quad I = \frac{1}{12}M(l\sin\theta)^2 = \frac{1}{12}Ml_{eff}^2$$



Similarly, if the axis would have passed through one end of rod as shown, then $I=\frac{1}{3}Ml^2\sin^2\theta$



MOMENT OF INERTIA OF A SHELL ABOUT DIAMETER AND TANGENT

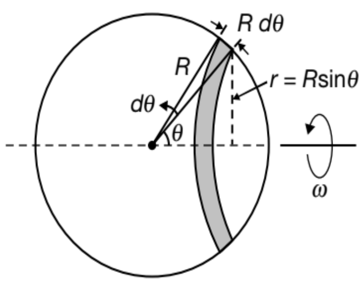
Consider a uniform shell of mass M, radius R. Let us find the moment of inertia of shell about diameter shown. For that, let us consider an infinitesimal ring element of radius $r(=R\sin\theta)$ thickness $Rd\theta$ having mass dm. Then

$$dm=\left(\frac{M}{4\pi R^2}\right)(\text{Area of Element})$$

$$\Rightarrow \quad dm=\left(\frac{M}{4\pi R^2}\right)(2\pi r)(Rd\theta)$$

$$\Rightarrow \quad dm=\left(\frac{M}{4\pi R^2}\right)(2\pi R\sin\theta)(Rd\theta)$$

$$\Rightarrow \quad dm=\frac{M}{2}\sin\theta d\theta$$



The moment of inertia of this ring element about diameter is

$$dI=r^2dm=\left(R^2\sin^2\theta\right)\left(\frac{M}{2}\sin\theta d\theta\right)$$

$$\Rightarrow \quad I=\int dI=\frac{MR^2}{2}\int_0^{\pi}\sin^3\theta d\theta$$

$$\Rightarrow \quad I=\frac{MR^2}{2}\int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta d\theta$$

Since, $d(\cos\theta)=-\sin\theta d\theta$

$$\Rightarrow \quad I=-\frac{MR^2}{2}\int_0^{\pi}\left(1-\cos^2\theta\right)d(\cos\theta)$$

$$\Rightarrow \quad I=-\frac{MR^2}{2}\left[\int_0^{\pi}d(\cos\theta)-\int_0^{\pi}d\left(\frac{\cos^3\theta}{3}\right)\right]$$

$$\Rightarrow \quad I=-\frac{MR^2}{2}\left[\cos\theta\Big|_0^{\pi}-\frac{1}{3}\cos^3\theta\Big|_0^{\pi}\right]$$

$$\Rightarrow \quad I=-\frac{MR^2}{2}\left[-2-\frac{1}{3}(-2)\right]=MR^2\left(1-\frac{1}{3}\right)$$

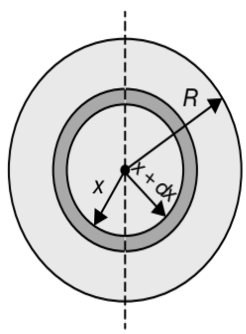
$$\Rightarrow \quad I=\frac{2}{3}MR^2$$

Applying Parallel Axis Theorem, the MI of the shell about the diameter is

$$I=I_{cm}+Md^2=\frac{2}{3}MR^2+MR^2=\frac{5}{3}MR^2$$

MOMENT OF INERTIA OF A SPHERE ABOUT DIAMETER AND TANGENT

Consider a uniform sphere of mass M and radius R. Let us find the moment of inertia of the sphere about diameter as shown in Figure.



For that, let us consider an infinitesimal shell element of inner radius x, outer radius $x+dx$ and mass dm. If ρ be the mass per unit volume of sphere, then

$$\rho=\frac{M}{\frac{4}{3}\pi R^3}=\frac{3M}{4\pi R^3}$$

If dV be the volume of infinitesimal shell element, then

$$dm = \rho dV = \rho\left(4\pi x^2 dx\right)$$

$$\Rightarrow \quad dm = \left(\frac{3M}{4\pi R^3}\right)\left(4\pi x^2 dx\right)\rho = \left(\frac{3M}{R^3}\right)\left(x^2 dx\right)$$

If dI be the moment of inertia of element about said axis, then

$$dI = \frac{2}{3}(dm)x^2 = \frac{2}{3}\left(\frac{3M}{R^3}x^2 dx\right)x^2 = \frac{2M}{R^3}x^4 dx$$

$$\Rightarrow \quad I = \frac{2M}{R^3}\int_0^R x^4 dxI = \frac{2M}{R^3}\left(\frac{R^5}{5}\right)$$

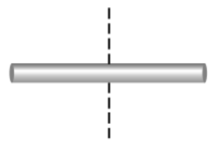
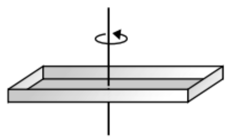
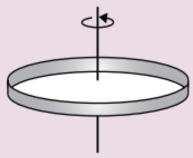
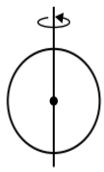
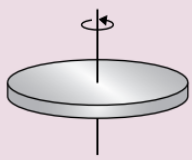
$$\Rightarrow \quad I = \frac{2}{5}MR^2$$

Applying Parallel Axis Theorem, the MI of the sphere about the diameter is

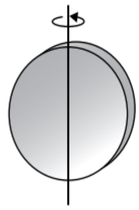
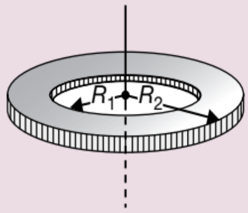
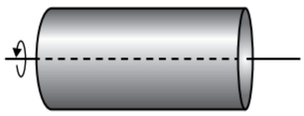
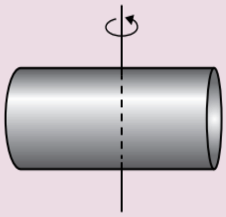
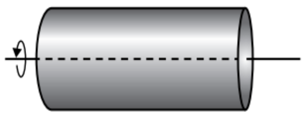
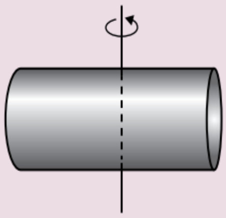
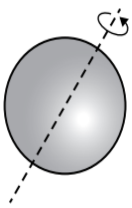
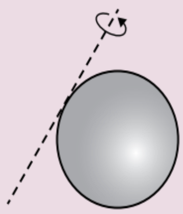
$$I = I_{\text{cm}} + Md^2 = \frac{2}{5}MR^2 + MR^2$$

$$\Rightarrow \quad I = \frac{7}{5}MR^2$$

TABLE : MOMENT OF INERTIA OF DIFFERENT BODIES

Body	Axis of Rotation	Diagram Showing Axis of Rotation	Moment of Inertia
1. Uniform thin bar	a) Through centre of gravity and perpendicular to length.		$\frac{Ml^2}{12}$
	b) Through one end and perpendicular to length.		$\frac{Ml^2}{3}$
2. Rectangular lamina	Passing through its C.G. and perpendicular to its plane of length and breadth.		$\frac{M\left(l^2+b^2\right)}{12}$
3. Ring or Hoop	a) Passing through its centre and perpendicular to its plane.		MR^2
	b) About diameter.		$\frac{MR^2}{2}$
4. Disc	a) Passing through its centre and perpendicular to its plane.		$\frac{MR^2}{2}$

(Continued)

Body	Axis of Rotation	Diagram Showing Axis of Rotation	Moment of Inertia
	b) About diameter.		$\frac{MR^2}{4}$
5. Hollow disc/ Annular disc of radii R_1 and R_2	Passing through its centre and perpendicular to its plane.		$I = \frac{M\left(R_1^2 + R_2^2\right)}{2}$
6. Solid cylinder	a) About its own geometric axis.		$\frac{MR^2}{2}$
	b) Passing through C.G. and perpendicular to its geometric axis.		$M\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$
7. Hollow cylinder	a) About its own geometrical axis.		MR^2
	b) Passing through C.G. and perpendicular to length.		$M\left(\frac{l^2}{12} + \frac{R^2}{2}\right)$
8. Thin spherical shell	a) About diameter.		$\frac{2}{3}MR^2$
	b) About tangent.		$\frac{5}{3}MR^2$

(Continued)

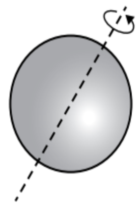
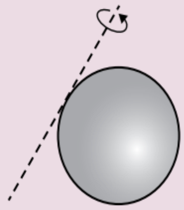
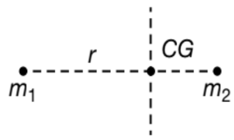
Body	Axis of Rotation	Diagram Showing Axis of Rotation	Moment of Inertia
9. Solid sphere	a) About diameter.		$\frac{2}{5}MR^2$
	b) About tangent.		$\frac{7}{5}MR^2$
10. Diatomic molecule	Passing through centre of gravity and perpendicular to bond length.		$\left(\frac{m_1 m_2}{m_1+m_2}\right)r^2$

ILLUSTRATION 2

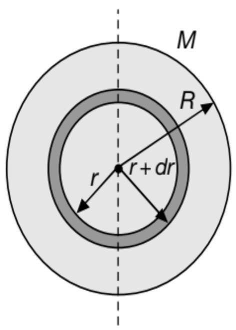
The density of a sphere of mass M, radius R, varies with distance r from its centre as $\rho = \rho_0\left(1+\frac{r}{R}\right)$, where ρ_0 is a positive constant. Calculate the moment of inertia of this sphere about its diameter.

SOLUTION

Consider an infinite shell element of mass dm having inner radius r and outer radius $r+dr$ i.e., thickness dr. If dV be the volume of the element, then $dV = 4\pi r^2 dr$

$$\Rightarrow \quad dm = \rho dV$$

$$\Rightarrow \quad dm = \rho_0\left(1+\frac{r}{R}\right)4\pi r^2 dr \qquad \ldots(1)$$



If dI be the moment of inertia of this element about diameter, then

$$dI = \frac{2}{3}(dm)r^2$$

$$\Rightarrow \quad dI = \frac{2}{3}\rho_0\left(1+\frac{r}{R}\right)4\pi r^4 dr$$

$$\Rightarrow \quad dI = \frac{8\pi\rho_0}{3}r^4 dr + \frac{8\pi\rho_0}{3R}r^5 dr$$

$$\Rightarrow \quad I = \frac{8\pi\rho_0}{3}\left(\frac{R^5}{5}\right) + \frac{8\pi\rho_0}{3R}\left(\frac{R^6}{6}\right)$$

$$\Rightarrow \quad I = \frac{8\pi\rho_0 R^5}{3}\left(\frac{1}{5}+\frac{1}{6}\right)$$

$$\Rightarrow \quad I = \frac{88\pi\rho_0 R^5}{90} \qquad \ldots(2)$$

However, from (1), we get, on integrating

$$M = \int dm = 4\pi\rho_0\left(\int_0^R r^2 dr + \int_0^R \frac{r^3}{R}dr\right)$$

$$\Rightarrow \quad M = 4\pi\rho_0\left(\frac{R^3}{3}+\frac{R^3}{4}\right)$$

$$\Rightarrow \quad M = 4\pi\rho_0 R^3\left(\frac{7}{12}\right) = \frac{7}{3}\pi\rho_0 R^3$$

$$\Rightarrow \quad \pi\rho_0 R^3 = \frac{3M}{7} \qquad \ldots(3)$$

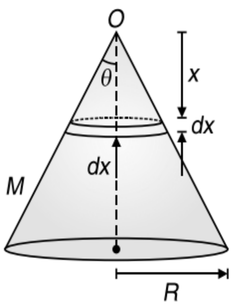
Substituting (3) in (2), we get

$$I = \frac{88}{90}\left(\frac{3M}{7}\right)R^2$$

$$\Rightarrow \quad I = \frac{44}{105}MR^2$$

MOMENT OF INERTIA OF A HOLLOW CONE

To calculate the MI of a hollow cone of mass M, radius R and height H, we consider elemental strips of vertical width dx at a distance x from the vertex of the cone as shown in Figure.



The actual width of this strip is $dx\sec\theta$, where θ is the half angle of the cone. So, area dA of the strip is

$$dA = (2\pi r)(dx\sec\theta)$$

where, r is the radius of the ring. Since,

$$\tan\theta = \frac{r}{x} = \frac{R}{H}$$

$$\Rightarrow \quad r = \left(\frac{R}{H}\right)x$$

If the cone mass has a mass M, then mass dm of the elemental strip ring is given by

$$dm = \sigma dA = \left(\frac{M}{\pi R\sqrt{R^2+H^2}}\right)(2\pi r dx\sec\theta)$$

The moment of inertia of this ring is

$$dI = r^2 dm$$

$$\Rightarrow \quad I = \int_0^H r^2 dm = \sigma\int_0^H \left(\frac{R^2x^2}{H^2}\right)dA$$

$$\Rightarrow \quad I = \sigma\left(\frac{R^2}{H^2}\right)\int_0^H x^2(2\pi r dx\sec\theta)$$

$$\Rightarrow \quad I = \left(\frac{2\pi\sigma R^3}{H^3}\right)\left(\frac{\sqrt{R^2+H^2}}{H}\right)\int_0^H x^3 dx$$

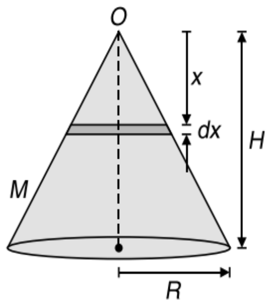
$$\Rightarrow \quad I = \left(\frac{2\pi MR^3}{\pi RH^3\sqrt{R^2+H^2}}\right)\left(\frac{\sqrt{R^2+H^2}}{H}\right)\left(\frac{H^4}{4}\right)$$

$$\Rightarrow \quad I = \frac{2MR^2}{H^4}\int_0^H x^3 dx = \frac{2MR^2}{H^4}\left(\frac{x^4}{4}\right)\Bigg|_0^H$$

$$\Rightarrow \quad I = \frac{1}{2}MR^2$$

MOMENT OF INERTIA OF A SOLID CONE

To calculate the moment of inertia of a solid cone of mass M, radius R and height H, we consider an infinitesimal disc at a distance x from the vertex of the cone as shown in Figure.



The mass dm of this disc is

$$dm = \left(\frac{M}{\pi R^2 H/3}\right)(\pi r^2 dx) = \left(\frac{3Mr^2}{R^2 H}\right)dx$$

where, r is the radius of the infinitesimal disc. Since,

$$\tan\theta = \frac{r}{x} = \frac{R}{H}$$

$$\Rightarrow \quad r = \left(\frac{R}{H}\right)x$$

The moment of inertia of this infinitesimal disc about the specified axis of rotation (AOR) is

$$dI = \frac{1}{2}(dm)r^2$$

Moment of inertia of complete cone is evaluated by integrating the above expression for the total height of the cone. So, we get

$$I = \int dI = \frac{1}{2}\int(dm)r^2 = \frac{1}{2}\int\left(\frac{3M}{R^2H}\right)r^4 dx$$

$$\Rightarrow \quad I = \int_0^H \frac{1}{2}\left(\frac{3M}{R^2H}\right)\left(\frac{Rx}{H}\right)^4 dx = \frac{3MR^2}{2H^4}\int_0^H x^4 dx$$

$$\Rightarrow \quad I = \frac{3MR^2}{2H^4}\left(\frac{x^5}{5}\right)\Bigg|_0^H = \frac{3MR^2}{2H^4}\left(\frac{H^5}{5}\right)$$

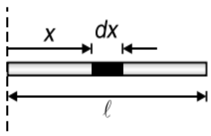
$$\Rightarrow \quad I = \frac{3}{10}MR^2$$

ILLUSTRATION 3

A rod of length l is pivoted about an end. The linear mass density of rod varies with distance x from its one end as $\lambda = (ax^2 + b)$ kgm^{-1}. Calculate the moment of inertia of the rod about axis passing through this end and perpendicular to the rod.

SOLUTION

Since linear mass density of the rod varies with distance, so here we cannot use the expression $\frac{Ml^2}{3}$ because it is only applicable for uniformly distributed mass along the length of rod pivoted at an end.
So, let us consider an infinitesimal element of mass dm, length dx from the AOR at a distance x as shown in Figure.



Since, $dm = \lambda dx$

$$\Rightarrow \quad dm = (ax^2 + b)dx$$

During rotation of rod about the specified AOR, this dm revolves in a circle of radius x, hence its moment of inertia dI is given by

$$dI = x^2 dm$$

$$\Rightarrow \quad dI = (ax^2 + b)x^2 dx$$

The moment of inertia of the whole rod is obtained by integrating the above expression within limits from zero to l. So, we get

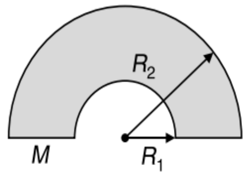
$$I = \int_0^l (ax^2 + b)x^2 dx$$

$$\Rightarrow \quad I = \left(\frac{ax^5}{5} + \frac{bx^3}{3}\right)\Bigg|_0^l$$

$$\Rightarrow \quad I = \frac{al^5}{5} + \frac{bl^3}{3}$$

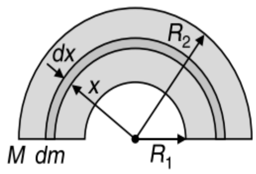
ILLUSTRATION 4

Find the moment of inertia of a semi-circular annular half disc (of mass M, outer radius R_2, inner radius R_1), about an axis passing through its centre and perpendicular to its plane as shown in Figure.



SOLUTION

Consider an infinitesimal half ring of radius x, width dx and mass dm as shown in Figure.



So, $dm = \sigma dA = \left(\frac{M}{\pi R_2^2 - \pi R_1^2}\right)(\pi x dx)$

Moment of inertia of this elemental half ring about the specified is given by

$$dI = x^2 dm$$

$$\Rightarrow \quad I = \int dI = \left(\frac{M\pi}{\pi R_2^2 - \pi R_1^2}\right)\int_{R_1}^{R_2} x^3 dx$$

$$\Rightarrow \quad I = \left(\frac{M}{R_2^2 - R_1^2}\right)\left(\frac{x^4}{4}\right)\Bigg|_{R_1}^{R_2} = \frac{M}{4(R_2^2 - R_1^2)}(R_2^4 - R_1^4)$$

$$\Rightarrow \quad I = \frac{1}{4}M(R_1^2 + R_2^2)$$

ILLUSTRATION 5

Calculate the moment of inertia of a rod whose linear density changes from λ to $\eta\lambda$ from the thinner end to the thicker end. The mass of the rod is equal to M and length L. Consider the axis of rotation perpendicular to the rod and passing through the thinner end and express your answer in terms of M, L and η.

SOLUTION

Consider an element at a distance x from thinner end of width dx, we use its mass dm as

$$dm = \left[\frac{\lambda(\eta-1)}{L}x+\lambda\right]dx$$

Total mass of rod is

$$M = \int dm\left(\frac{\lambda(\eta-1)}{L}x+\lambda\right)dx$$

$$\Rightarrow \quad M = \frac{\lambda(\eta-1)}{L}\left(\frac{L^2}{2}\right)+\lambda L$$

$$\Rightarrow \quad M = \frac{\lambda(\eta-1)L}{2}+\lambda L$$

$$\Rightarrow \quad M = \frac{1}{2}\lambda(\eta+1)L \qquad \ldots(1)$$

Moment of inertia of element is

$$dI = x^2 dm$$

Moment of inertia of rod is

$$I = \int dI = \int_0^L \left[\lambda+\frac{\lambda(\eta-1)}{L}x\right]x^2 dx$$

$$\Rightarrow \quad I = \frac{\lambda L^3}{3}+\frac{\lambda(\eta-1)}{L}\frac{L^4}{4}$$

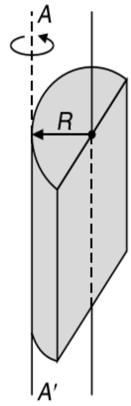
$$\Rightarrow \quad I = \frac{1}{12}\lambda L^3(3\eta+1) \qquad \ldots(2)$$

Substituting equation (1) in (2), we get

$$I = \frac{1}{6}ML^2\left(\frac{3\eta+1}{\eta+1}\right)$$

ILLUSTRATION 6

Find moment of inertia of the half cylinder of mass M shown in Figure, about the axis AA'.



SOLUTION

From Parallel Axis Theorem, we have

$$I = I_{cm} + Md^2$$

The moment of inertia of the half cylinder about the axis through its centre of mass at distance $4R/3\pi$ from the axis passing through the centre of mass is

$$I_{cm} = I - M\left(\frac{4R}{3\pi}\right)^2, \text{ where } I = \frac{1}{2}MR^2$$

$$\Rightarrow \quad I_{cm} = \frac{1}{2}MR^2 - \frac{16}{9\pi^2}MR^2 = MR^2\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)$$

The moment of inertia about axis AA' is again calculated by using the parallel axis theorem.

$$\Rightarrow \quad I_{AA'} = I_{cm} + M\left(R-\frac{4R}{3\pi}\right)^2 = I_{cm} + MR^2\left(1-\frac{4}{3\pi}\right)^2$$

$$\Rightarrow \quad I_{AA'} = MR^2\left[\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)+\left(1+\frac{16}{9\pi^2}-\frac{8}{3\pi}\right)\right]$$

$$\Rightarrow \quad I_{AA'} = \left(\frac{3}{2}-\frac{8}{3\pi}\right)MR^2$$

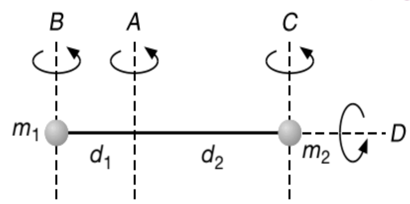
$$\Rightarrow \quad I_{AA'} = \left(\frac{9\pi-16}{6\pi}\right)MR^2$$

Test Your Concepts-I

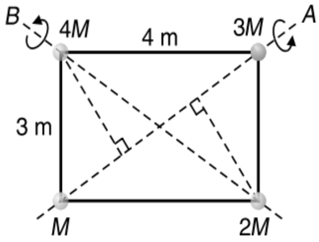
Based on Moment of Inertia and Applications

(Solutions on page H.147)

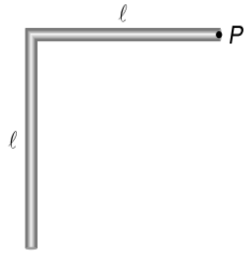
1. Many molecules have a simple diatomic, dumb-bell structure. Let us find the moments of inertia about four axes. We treat the bodies as point particles with mass $m_1 = 3$ kg and $m_2 = 5$ kg. Take $d_1 = 1$ m and $d_2 = 2$ m in Figure.



2. Four-point masses lie at the corners of a rectangle with sides of length 3 m and 4 m, as shown in Figure. Find the moment of inertia about each of the diagonals. Take $M = 1\text{kg}$.



3. Calculate the moment of inertia of the two uniform joint rods having mass m each about point P as shown in Figure using parallel axis theorem.



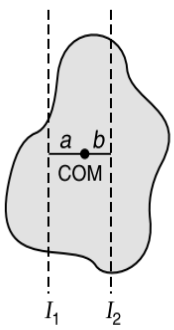
4. A non-uniform rod AB has a mass M and length $2L$. The mass per unit length of the rod varies with the distance x from the end A as $\lambda = \lambda_0 x$ where λ_0 is a positive constant. Find the moment of inertia of this rod about an axis perpendicular to the rod

(a) through A

(b) through the mid-point of AB.

5. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and I_2 the moment of inertia of the ring formed by the same rod about an axis tangent to the ring and perpendicular to the plane of the ring. Then find the ratio $\frac{I_1}{I_2}$.

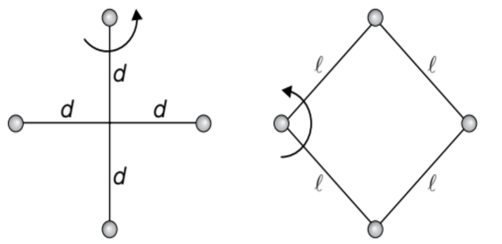
6. Find the difference between I_1 and I_2, where I_1 and I_2 moment of inertia of a rigid body mass m about an axis as shown in Figure.



7. Four particles with equal masses m are connected by rods of negligible mass. Find the moment of inertia, about the indicated axis, of

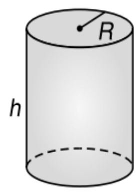
(a) the cross in Figure (a)

(b) the square in Figure (b)

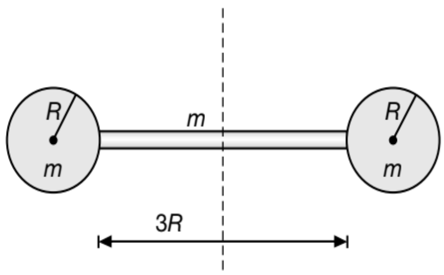


(a) (b)

8. A can is a hollow cylinder of radius R and height h. Its ends are sealed and it has no seams. The can is made from sheet of surface mass density σ. Calculate the moment of inertia about the central axis of symmetry?

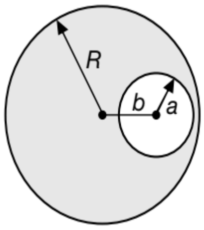


9. Two solid spheres of mass m and radius R are stuck to the ends of a thin rod of mass m and length $3R$. Find the moment of inertia of the system about the axis at the mid-point of the rod and perpendicular to it, as shown in Figure.

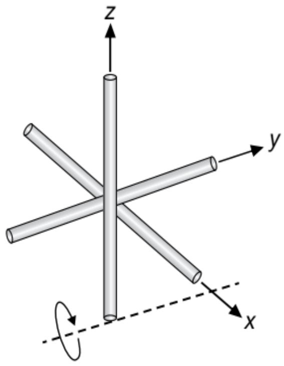


10. A uniform sphere of radius a has a concentric spherical cavity of radius b. Find the moment of inertia about a diameter. The mass of the object is M.

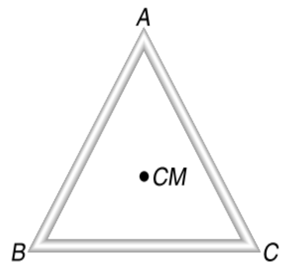
11. A uniform disk of mass M and radius R has a hole of radius a drilled through it, as in Figure. The center of the hole is at a distance b from the centre of the original disk. What is the moment of inertia of the disk about an axis through the center perpendicular to its plane?



12. Calculate the moment of inertia of a semi-circular disc about an axis passing through its centre of mass and perpendicular to the plane.
13. Three masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of side length a. Calculate the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1?
14. The radius of gyration of a uniform disc about a line perpendicular to the disc equals to its radius R. Find the distance of the line from the centre.
15. A circular lamina of radius R and centre O has a mass per unit area of kx^2, where x is the distance from O and k is a constant. If the mass of the lamina is M, find in terms of M and R, the moment of inertia of the lamina about an axis through O and perpendicular to the lamina.
16. Three identical thin rods, each of length L and mass m, are welded perpendicular to one another as shown in Figure. The assembly is rotated about an axis that passes through the end of one rod and is parallel to another. Find the moment of inertia of this structure.



17. Three rods each of mass m and length ℓ are joined together to form an equilateral triangle as shown in Figure. Calculate the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the plane of the triangle. Also, find the radius of gyration of system about said axis.



COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

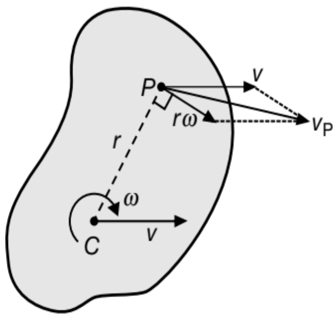
It has been observed that frequent attempts have been made by JEE Advanced paper setters to ask questions related to combined translational and rotational motion of a rigid body. The combined translational and rotational motion of a rigid body can be thought to be made up of two parts.

1. Pure translational motion in which centre of mass has velocity $v_{cm} = v$ and acceleration $a_{cm} = a$.
2. Pure rotational motion about an axis passing through the centre of mass with angular velocity ω and angular acceleration α.

In this type of motion, different particles of the rigid body will possess different linear velocities and linear accelerations. The following discussion is about finding the linear velocity and linear acceleration of any general particle P on the rigid body in such a motion.

LINEAR VELOCITY OF A GENERAL POINT *P* UNDER COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

Now consider a rigid body having centre of mass at C having velocity of centre of mass v and angular velocity ω as shown in Figure.



Velocity of point P w.r.t. centre of mass C is $\vec{v}_{PC}$. By definition

$$\vec{v}_{PC} = \vec{v}_P - \vec{v}_C$$

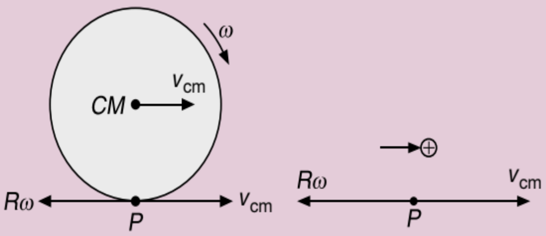
$\Rightarrow \quad \vec{v}_P = \vec{v}_{PC} + \vec{v}_C$

Now $|\vec{v}_C| = |\vec{v}_{CM}| = v$ and $|\vec{v}_{PC}| = r\omega$, perpendicular to the line CP. So, $|\vec{v}_P|$ is actually the magnitude of the resultant of $\vec{v}_{PC}$ and $\vec{v}_C$ as shown in Figure. If θ is the angle between $\vec{v}_{PC}$ and $\vec{v}_C$, then

$$|\vec{v}_P| = \sqrt{v^2 + r^2\omega^2 + 2v(r\omega)\cos\theta}$$

Conceptual Note(s)

For uniform pure rolling i.e. rolling without slipping, we must have the velocity of point of contact to be zero.



$$\vec{v}_P = \vec{v}_{CM} + \vec{v}_{tangential}$$

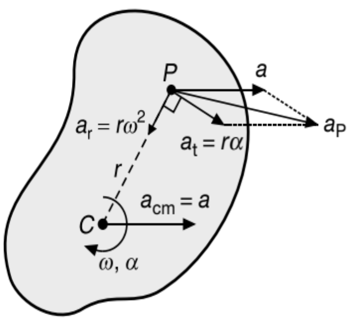
$$\Rightarrow \quad \vec{v}_P = \vec{v}_{CM} + (-R\omega) = 0$$

$$\Rightarrow \quad v_{CM} = R\omega$$

Hence, $(v_P)_{net} = v_{CM} - R\omega = 0$

LINEAR ACCELERATION OF A GENERAL POINT *P* UNDER COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

Now consider a rigid body having centre of mass at C having velocity of centre of mass v and angular velocity ω and angular acceleration α as shown in Figure.



Acceleration of point P w.r.t. the centre of mass C is $\vec{a}_{PC}$, given by

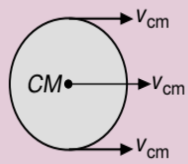
$$\vec{a}_{PC} = \vec{a}_P - \vec{a}_C$$

$$\Rightarrow \quad \vec{a}_P = \vec{a}_{PC} + \vec{a}_C$$

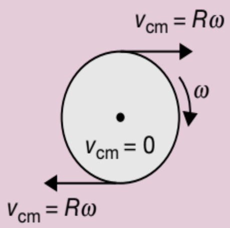
Now $|\vec{a}_C| = |\vec{a}_{CM}| = a$ and in the frame of the centre of mass, the point P has tangential acceleration $a_t = R\alpha$ perpendicular to the radius and the radial acceleration $a_r = r\omega^2$ along the radius towards the centre. *So, net acceleration of the point P is the vector resultant of* a, a_t *and* a_r. For different particles of the body, we observe that a, ω and α are the same, whereas r may be different, so different particles of the body possess different linear accelerations.

Conceptual Note(s)

(a) Linear Motion or Rectilinear Motion in which all particles move with the same translational velocity that equals the velocity of centre of mass. Also, angular velocity $\omega = 0$ for the case of pure translational motion as shown in Figure.

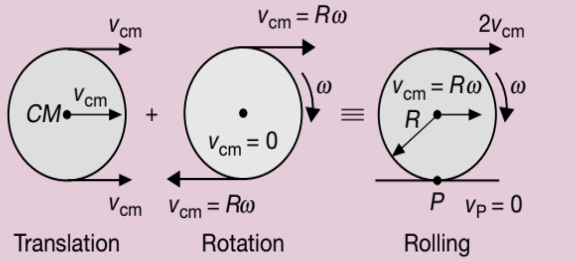


(b) Rotational Motion in which the velocity of centre of mass of the particles is zero because the diametrically opposite points have velocities equal in magnitude but opposite in direction as shown in Figure.

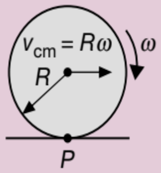


(c) Pure Rolling Motion in which the centre of mass has a velocity v_{cm} and the body has an angular velocity ω *such that velocity of the point of contact equals the velocity of the surface on which the body is rolling.*

Pure rolling can be regarded as the combined effect of translational motion and the rotational motion as shown in Figure.



If the surface is at rest on which the body is in pure rolling motion, then



$$v_{\text{point of contact}} = v_{\text{surface}} = 0$$

$$\Rightarrow\ v_{cm} - R\omega = 0$$

$$\Rightarrow\ v_{cm} = R\omega$$

Uniform pure rolling can also be taken as pure rotation of body about the point of contact, so

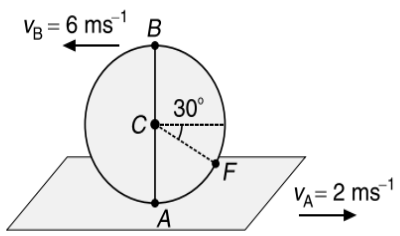
velocity of point of contact is $v_P = 0$

velocity of CM is $v_{cm} = R\omega$

velocity of topmost point is $v_Q = (2R)\omega$

ILLUSTRATION 7

Due to slipping, points A and B on the rim of the disk have the velocities shown. Determine the velocities of the centre point C and point F at this instant.

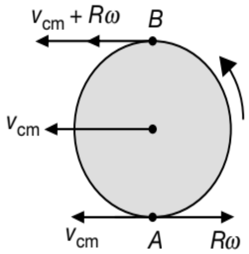


SOLUTION

Since $v_B > v_A$, so ω should be counterclockwise and v_{cm} should be leftwards. Hence, we have

$$v_B = v_{cm} + R\omega = 6 \quad \ldots(1)$$

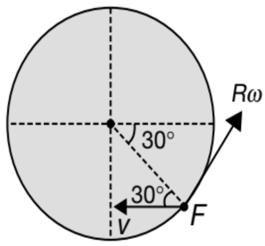
and $$v_A = R\omega - v_{cm} = 2 \quad \ldots(2)$$



Solving (1) and (2), we get

$$R\omega = 4\ \text{ms}^{-1} \text{ and } v_{cm} = 2\ \text{ms}^{-1}$$

So, velocity of point C, i.e., centre of mass is $2\ \text{ms}^{-1}$, leftwards



Now, for the point F, we have

$$v_F^2 = v_{cm}^2 + R^2\omega^2 + 2(v_{cm})(R\omega)\cos(90^\circ + 30^\circ)$$

$$\Rightarrow\ v_F^2 = 4 + 16 + 2(2)(4)\left(-\frac{1}{2}\right)$$

$$\Rightarrow\ v_F^2 = 4 + 16 - 8$$

$$\Rightarrow\ v_F = \sqrt{12} = 2\sqrt{3}\ \text{ms}^{-1}$$

Conceptual Note(s)

This problem will give the same result even when we would have taken v_{cm} rightwards. In that case equations would be

$$v_{cm} + R\omega = 2 \text{ and}$$

$$R\omega - v_{cm} = 6$$

Solving, we get

$$R\omega = 4\ \text{ms}^{-1} \text{ and } v_{cm} = -2\ \text{ms}^{-1}$$

The negative sign with v_{cm} indicates that it is directed leftwards.

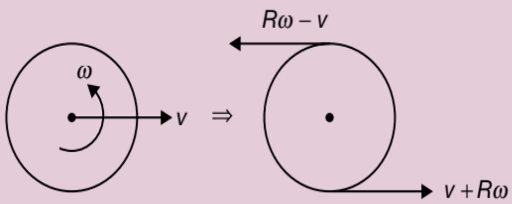
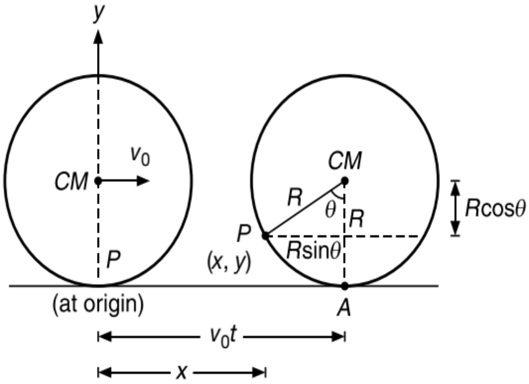


ILLUSTRATION 8

Consider a hoop of radius R having centre of mass velocity v_0. Let the hoop roll on a surface without sliding. Assuming a point P on the hoop to be at the origin initially. Find the velocity of this point as a function of time. Also find the acceleration of this point P. Prove that the trajectory followed by this point is a cycloid and the total distance travelled by this point is $8R$ (in one complete cycle).

SOLUTION

Since the hoop is rolling without slipping, therefore the arc length AP is equal to the displacement PA from the origin to point A i.e., v_0t



$$\overset{\frown}{AP} = v_0 t$$

$$\Rightarrow \quad R\theta = v_0 t$$

$$\Rightarrow \quad \theta = \frac{v_0 t}{R} \qquad \ldots(1)$$

Co-ordinates of P at t are (x, y) where

$$x = v_0 t - R\sin\theta$$

$$\Rightarrow \quad x = v_0 t - R\sin\left(\frac{v_0 t}{R}\right) \qquad \ldots(2)$$

$$y = R - R\cos\theta$$

$$\Rightarrow \quad y = R - R\cos\left(\frac{v_0 t}{R}\right) \qquad \ldots(3)$$

The (2) and (3) combination of x and y will give rise to the trajectory called **Cycloid**.

$$\text{Since, } v_x = \frac{dx}{dt} = v_0 - \frac{v_0}{R}\left[R\cos\left(\frac{v_0 t}{R}\right)\right]$$

$$\Rightarrow \quad v_x = v_0 - v_0\cos\left(\frac{v_0 t}{R}\right) \qquad \ldots(4)$$

$$\text{and} \quad v_y = \frac{dy}{dt} = v_0 \sin\left(\frac{v_0 t}{R}\right) \qquad \ldots(5)$$

$$\text{Since } \vec{v} = \vec{v}_p = v_x\hat{i} + v_y\hat{j}$$

$$\text{and} \quad |\vec{v}| = v_P = \sqrt{v_x^2 + v_y^2}$$

$$\Rightarrow \quad v_P = v_0\sqrt{1 + \cos^2\left(\frac{v_0 t}{R}\right) - 2\cos\left(\frac{v_0 t}{R}\right) + \sin^2\left(\frac{v_0 t}{R}\right)}$$

$$\Rightarrow \quad v_P = v_0\sqrt{2 - 2\cos\left(\frac{v_0 t}{R}\right)}$$

$$\Rightarrow \quad v_P = \sqrt{2}v_0\sqrt{1 - \cos\left(\frac{v_0 t}{R}\right)}$$

$$\Rightarrow \quad v_P = \sqrt{2}v_0\sqrt{2\sin^2\left(\frac{v_0 t}{2R}\right)}$$

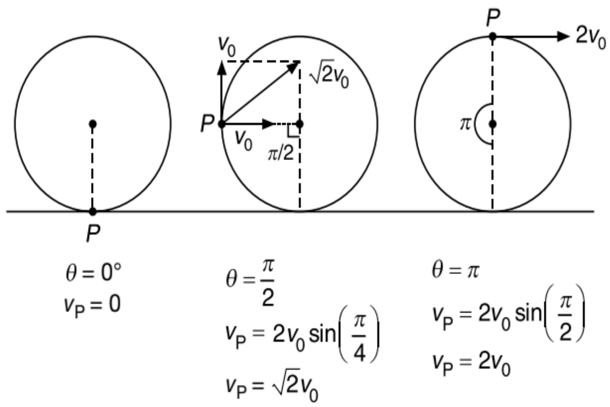
$$\Rightarrow \quad v_P = 2v_0\sin\left(\frac{v_0 t}{2R}\right)$$

$$\text{Since } \theta = \frac{v_0 t}{R}$$

$$v_P = 2v_0\sin\left(\frac{v_0 t}{2R}\right) = 2v_0\sin\left(\frac{\theta}{2}\right) \qquad \ldots(6)$$

Now let us calculate the acceleration of the point P. Since

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(v_0 - v_0\cos\left(\frac{v_0 t}{R}\right)\right)$$



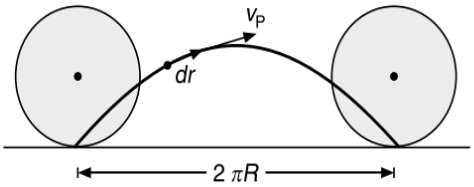
$$\Rightarrow \quad a_x = \frac{dv_x}{dt} = \frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right) \qquad \ldots(7)$$

Similarly

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}\left(v_0\sin\left(\frac{v_0 t}{R}\right)\right) = \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right) \qquad \ldots(8)$$

$$\text{Since, } a = \sqrt{a_x^2 + a_y^2}$$

$$\Rightarrow \quad a=\frac{v_0^2}{R} \qquad \ldots(9)$$



In time dt,

$$dr = v_P dt$$

$$\Rightarrow \quad dr = 2v_0 \sin\left(\frac{v_0 t}{2R}\right)dt$$

$$\Rightarrow \quad r = 2v_0 \int_0^{\frac{2\pi R}{v_0}} \sin\left(\frac{v_0 t}{2R}\right)dt = 2v_0\left[\frac{-\cos\left(\frac{v_0 t}{2R}\right)}{\left(\frac{v_0}{2R}\right)}\right]_0^{\frac{2\pi R}{v_0}}$$

$$\Rightarrow \quad r = -4R\left[\cos\left(\frac{v_0}{2R}\times\frac{2\pi R}{v_0}\right)-\cos 0\right]$$

$$\Rightarrow \quad r = -4R(-2) = 8R$$

$$\Rightarrow \quad r = 8R$$

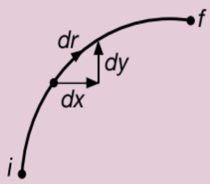
(a) Curve Rectification

$$\Rightarrow (dr)^2 = (dx)^2 + (dy)^2$$

$$\Rightarrow dr = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow dr = \sqrt{1+m^2}\,dx$$

$$\Rightarrow r = \int dr = \int\sqrt{1+m^2}\,dx = \int_i^f \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



(b) If at $t=0$, the point P lies at the bottom of the hoop, then

$$v_P = 2v_0 \sin\left(\frac{v_0 t}{2R}\right) = 2v_0 \sin\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad v_P = 2v_0 \sin\left(\frac{\omega_0 t}{2}\right)$$

where, θ is the angle between the point P at time t and the vertical line passing through the centre of the hoop.

(c) If at $t=0$, the point P lies at the top of the hoop, then

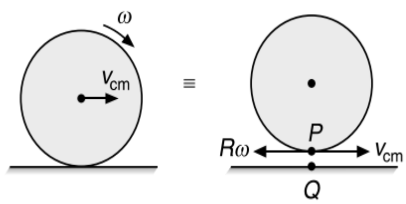
$$v_P = 2v_0 \cos\left(\frac{v_0 t}{2R}\right) = 2v_0 \cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad v_P = 2v_0 \cos\left(\frac{\omega_0 t}{2}\right)$$

where θ is the angle between the point P at time t and the vertical line passing through the centre of the hoop.

UNIFORM PURE ROLLING

Uniform Pure Rolling or simply "pure rolling" means that no relative motion exists at the point of contact between the body and the surface. Let a disc of radius R rolls without slipping on a horizontal stationary surface/ground. For the disc to roll without slipping, we must have



$$(v_{\text{net}})_P = v_{\text{surface}}$$

$$\Rightarrow \quad v_{\text{cm}} - R\omega = 0$$

$$\Rightarrow \quad v_{\text{cm}} = R\omega$$

So, $v_{\text{cm}} = R\omega$ is the condition for a body to be in pure rolling on a stationary horizontal surface/ground. It is sometimes simply called as ROLLING.

CASE OF FORWARD SLIPPING

If $v_P > v_Q$

$$\Rightarrow \quad v_{\text{cm}} - R\omega > 0$$

$$\Rightarrow \quad v_{\text{cm}} > R\omega$$

CASE OF BACKWARD SLIPPING (also called FORWARD ENGLISH)

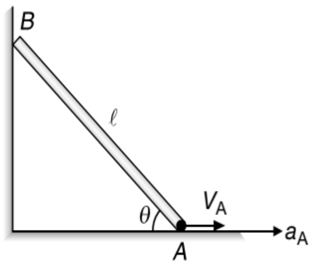
If $v_P < v_Q$

$$\Rightarrow \quad v_{cm} - R\omega < 0$$

$$\Rightarrow \quad v_{cm} < R\omega$$

ILLUSTRATION 9

A rod AB of length l is lying as shown in the Figure. At the instant shown, the point A has a speed V_A and acceleration a_A. Calculate the acceleration of point B and angular acceleration of the rod.



SOLUTION

By definition of relative velocity, we have

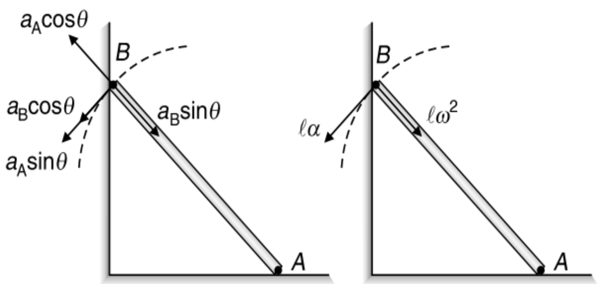
$$\vec{a}_{BA} = \vec{a}_{BG} - \vec{a}_{AG}$$

$$\Rightarrow \quad \vec{a}_{BG} = \vec{a}_{BA} + \vec{a}_{AG}$$

The angular velocity ω of the rod is

$$\omega = \frac{V_A}{l\sin\theta}$$

The components of the accelerations along the rod and perpendicular to the rod are shown in Figure.



So, we have

$$a_B \sin\theta - a_A \cos\theta = l\omega^2 \qquad \ldots(1)$$

and $a_A \sin\theta + a_B \cos\theta = l\alpha \qquad \ldots(2)$

$$\Rightarrow \quad a_B = \frac{a_A \cos\theta + l\omega^2}{\sin\theta} = \left(\frac{V_A^2}{l\sin^2\theta} + a_A \cos\theta\right)\frac{1}{\sin\theta} \ldots(3)$$

Substituting the value of a_B from equation (3) in equation (2), we get

$$\alpha = \frac{1}{l}\left[a_A \sin\theta + \left(\frac{V_A^2}{l\sin^2\theta} + a_A \cos\theta\right)\cot\theta\right]$$

$$l\omega^2 = \frac{V_A^2}{l\sin^2\theta}$$

In ground frame, net acceleration of point B will be the vector sum of all the three accelerations shown in Figure.

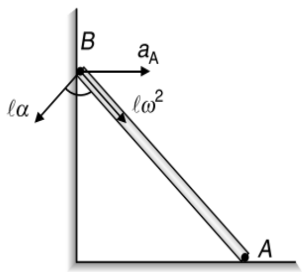
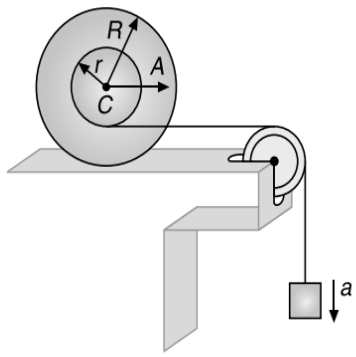


ILLUSTRATION 10

A bobbin having inner radius r and outer radius R is placed on a rough horizontal surface. A light string wrapped on its inner core connects a block with the bobbin as shown in Figure.



Now system is released from rest and bobbin moves on the horizontal surface without sliding. If the string does not slide from bobbin, calculate a/A.

SOLUTION

Since the bobbin is in pure rolling, point P should be instantaneous centre (IC) of zero velocity. So, for pure rolling acceleration of the point of contact is zero, hence we have

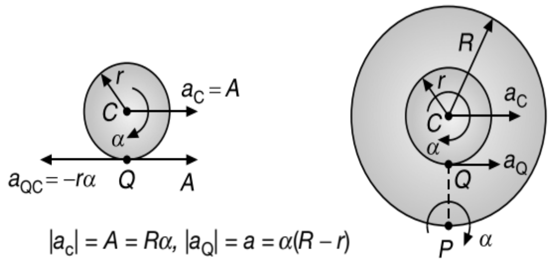
$$\vec{a}_P = \vec{0}$$

$$\Rightarrow \quad \vec{a}_P = \vec{a}_{PC} + \vec{a}_C = \vec{0}$$

$\Rightarrow \quad R\alpha - A = 0$

$\Rightarrow \quad A = R\alpha \qquad \ldots(1)$

Acceleration of the point Q with respect to the centre of mass C is shown in Figure.



$$\vec{a}_{QC} = \vec{a}_Q - \vec{a}_C$$

$$\Rightarrow \quad \vec{a}_Q = \vec{a}_{QC} + \vec{a}_C$$

where, $a_Q = a$, $a_{QC} = -r\alpha$ and $a_C = A = R\alpha$

$$a_Q = a = A - r = (R - r)\alpha$$

Since point Q is directly connected with block, it means the magnitude of acceleration of the point Q should be equal to the acceleration of the block

$$a_Q = a = \alpha(R - r) \qquad \ldots(2)$$

From Equation (1) and (2), we get

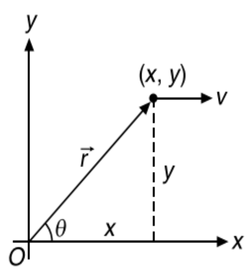
$$\frac{a}{A} = \frac{R - r}{R} = 1 - \frac{r}{R}$$

ILLUSTRATION 11

A particle of mass m is moving in x-y plane with a speed v, along x-axis. At a given instant of time, its position co-ordinates are (x, y). Calculate the angular velocity of the position vector of the particle and the angular acceleration of the particle with respect to origin.

SOLUTION

The position of x and velocity of the particle are shown in Figure.



As the particle moves along x-axis, its position vector turns clockwise, thus decreasing the angle θ. The time rate of change of this angle, i.e. $\frac{d\theta}{dt}$ equals the angular velocity. So,

$$\vec{\omega} = \left(\frac{d\theta}{dt}\right)\hat{k}$$

Since $y = x\tan\theta$

$$\Rightarrow \quad x = y\cot\theta$$

$$\Rightarrow \quad \frac{dx}{dt} = v = \left(-y\,\text{cosec}^2\theta\right)\frac{d\theta}{dt}$$

$$\left\{\because \frac{d}{d\theta}(\cot\theta) = -\text{cosec}^2\theta\right\}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = -\frac{v\sin^2\theta}{y} = -\frac{vy}{x^2 + y^2} \quad \left\{\because \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}\right\}$$

$$\Rightarrow \quad \vec{\omega} = -\left(\frac{vy}{x^2 + y^2}\right)\hat{k}$$

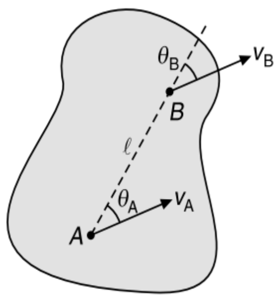
Also, angular acceleration $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

$$\Rightarrow \quad \vec{\alpha} = -vy\hat{k}\left[(-1)\left(x^2 + y^2\right)^{-2}\left(2x\frac{dx}{dt}\right)\right]$$

$$\Rightarrow \quad \vec{\alpha} = \frac{2xv^2y}{\left(x^2 + y^2\right)^2}\hat{k}$$

RIGID BODY CONSTRAINT

Consider a rigid body on which two points A and B having velocities $\vec{v}_A$ and $\vec{v}_B$ are taken as shown in Figure.

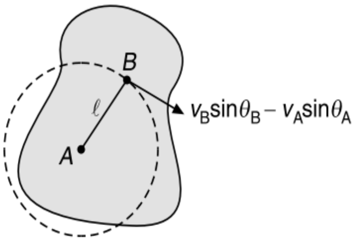


Since the body is rigid, so the distance between points A and B is constant and hence the component of velocity of A and B along the line joining A to B must be equal, i.e.

$$v_A\cos\theta_A = v_B\cos\theta_B$$

However, the perpendicular components may or not be equal.

If $v_A \sin\theta_A \neq v_B \sin\theta_B$ and if we observe the motion of the rigid body from the frame of point A, then the only possible way in which point B and all other points on the body can move is in a circle with A as the centre.



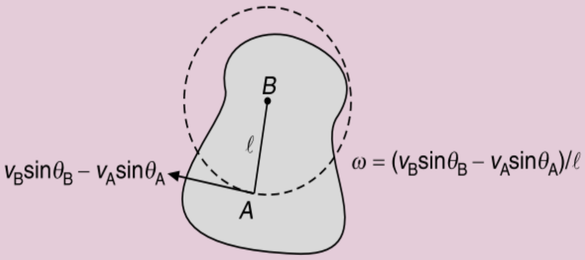
In the reference frame of point *A*

So, angular velocity of the line AB (or angular) velocity of the body is given by

$$\omega = \frac{v_B \sin\theta_B - v_A \sin\theta_A}{\ell}$$

Conceptual Note(s)

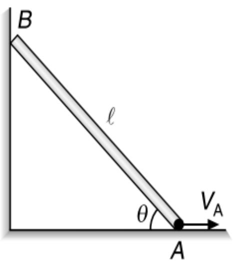
Angular velocity of the body means the rate of change of orientation of any line drawn on the body. Now let us try to find the angular velocity of the body from the frame of point B.



Thus, it can be concluded that angular velocity of the body remains same for all the points on the body.

ILLUSTRATION 12

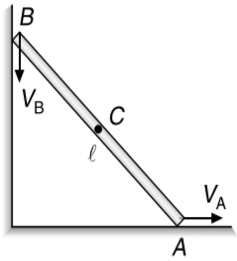
A rod AB of length l has its one end lying on the horizontal floor and the other end leaning against the vertical wall as shown in Figure.



If the point A is moving with a speed V_A, calculate the speed of point B and angular velocity of the rod at this instant.

SOLUTION

Let us consider the point B to move downwards with speed V_B. According to the rigid body constraint, the component of velocity of A and B along the AB must be equal. So, we have

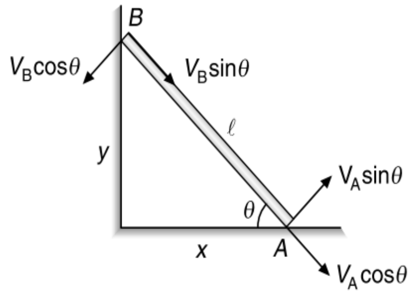


$$V_B \sin\theta = V_A \cos\theta$$

$$\Rightarrow \quad V_B = V_A \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad V_B = V_A \cot\theta \qquad \ldots(1)$$

The components of velocity of A and B perpendicular to the rod will contribute to the rotation of the body, so we have



$$\omega = \frac{(v_{\text{rel}})_\perp}{l} = \frac{V_B \cos\theta + V_A \sin\theta}{l}$$

Using equation (1), we get

$$\omega = \frac{\left(V_A \cos^2\theta + V_A \sin^2\theta\right)/l}{\sin\theta}$$

$$\Rightarrow \quad \omega = \frac{V_A}{l\sin\theta}$$

Also, we see that, $\tan\theta = \dfrac{y}{x}$

$$\Rightarrow \quad x = l\cos\theta$$

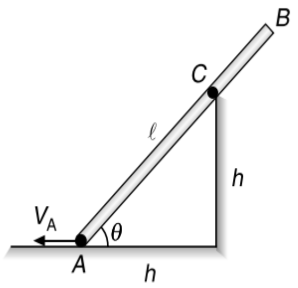
$$\Rightarrow \quad \frac{dx}{dt} = -l\sin\theta\left(\frac{d\theta}{dt}\right)$$

$$\Rightarrow \quad |\omega| = \frac{v_A}{l\sin\theta}$$

Both the methods fetch us same results.

ILLUSTRATION 13

A rod AB of length l is lying as shown in the Figure. The end A of the rod has speed V_A as shown in Figure.



Calculate the speed of point B and the angular velocity of the rod

SOLUTION

Since we observe that the velocity of the point C is along the rod, so by using the rigid body constraint, we get

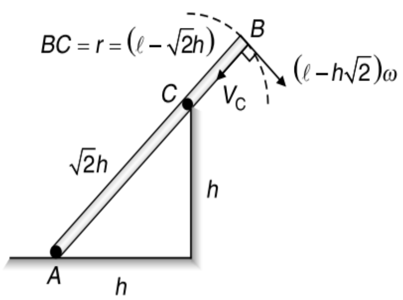
$$V_A \cos 45° = V_C$$

$$\Rightarrow \quad V_C = \frac{V_A}{\sqrt{2}}$$

Since $AC = h\sqrt{2}$, so the angular velocity ω is

$$\omega = \frac{V_A \sin 45°}{h\sqrt{2}} = \frac{V_A}{2h}$$

Once we know the angular velocity, then we can calculate the speed of point B by using the concept of relative motion and the rigid body constraint.



So, the linear tangential velocity of the point B is $r\omega = (l - h\sqrt{2})\omega$ and hence speed of point B is

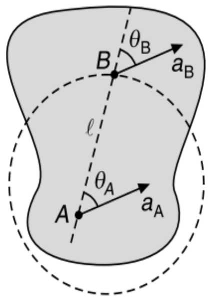
$$V_B = \sqrt{V_C^2 + (l - h\sqrt{2})^2 \omega^2}$$

$$\Rightarrow \quad V_B = V_A \sqrt{\frac{1}{2} + \left(\frac{l - h\sqrt{2}}{2h}\right)^2}$$

RIGID BODY CONSTRAINT FOR ACCELERATION

Just like we saw the constraint relation between the velocities of any two points on a rigid body, we can also find some relations between the acceleration, angular acceleration, velocity and angular velocity.

Consider two points A and B, separated by a distance l, having accelerations a_A and a_B as shown in Figure.



Since, B is moving in a circle with angular velocity ω when observed from reference frame attached to A. So, we have

$$a_A \cos\theta_A - a_B \cos\theta_B \neq 0$$

As a matter of fact, $a_A \cos\theta_A - a_B \cos\theta_B$ directed radially inverse provides centripetal acceleration or normal acceleration and hence

$$a_A \cos\theta_A - a_B \cos\theta_B = l\omega^2$$

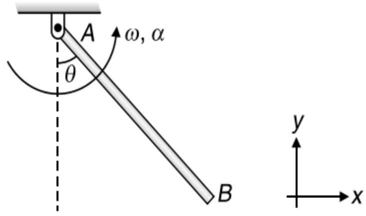
The components of accelerations perpendicular to the line joining A to B will provide the tangential acceleration α to the body, so tangential acceleration of B in frame of A is

$$|a_A \sin\theta_A - a_B \sin\theta_B| = l\alpha$$

Just like angular velocity, it can also be proved that angular acceleration of the body remains same in the frame of all the points on the rigid body.

ILLUSTRATION 14

The angular velocity and angular acceleration of the pivoted rod are ω and α respectively. Calculate the x and y components of acceleration of the end B of the rod.



SOLUTION

The acceleration of the point A is zero. The tangential and radial accelerations of the point B w.r.t. the point A are

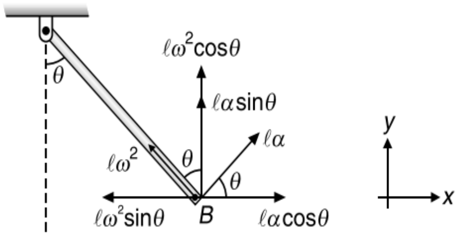
$$(a_{BA})_T = l\alpha \text{ and } (a_{BA})_R = l\omega^2$$

Since, by definition, we have

$$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A$$

$$\Rightarrow \quad \vec{a}_B = \vec{a}_{BA} + \vec{a}_A$$

Then resolving the vectors in x and y directions, we get

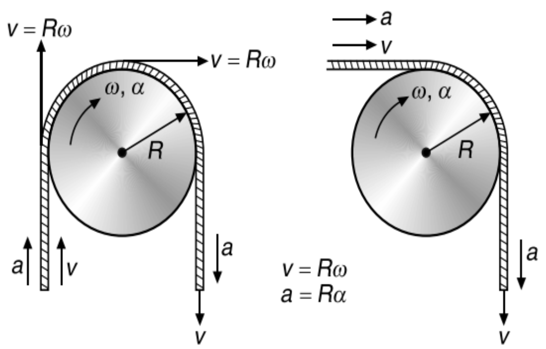


$$(a_B)_x = l\alpha\cos\theta - l\omega^2\sin\theta$$

and $$(a_B)_y = l\alpha\sin\theta + l\omega^2\cos\theta$$

ATWOOD'S MACHINE (SIMPLY PULLEY)

It is a simple pulley, hinged at the centre so as to turn freely without friction. The pulley has a mass M, radius R and it can be in the shape of a disc or a ring or a cylinder. Sufficient friction is present between the rope and the pulley surface so that the rope passing over its surface does not slip on the pulley surface. Due to this, the speed of the rope will be same as the speed of a point on the surface of the pulley as shown in Figure.



Since, $v = R\omega$

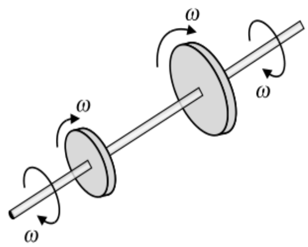
$$\Rightarrow \quad \frac{dv}{dt} = R\frac{d\omega}{dt}$$

$$\Rightarrow \quad a = R\alpha$$

So, acceleration of the ends of rope is related to angular acceleration of the pulley as $a = R\alpha$.

COAXIAL PULLEYS

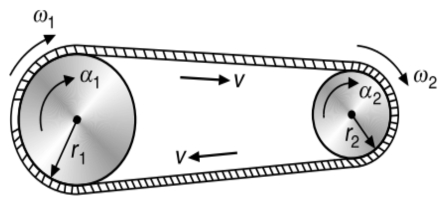
Consider an arrangement of two pulleys fixed on the same shaft (co-axial). When the shaft turns with angular speed ω, the two pulleys will also turn with angular speed ω. Consider the following Figure.



As the shaft turns, the two pulleys also turn with same angular speed ω.

DISCS CONNECTED BY A ROPE OR CHAIN

Consider two discs/wheels of radii r_1 and r_2 mounted on two separate fixed shafts turning with angular speeds ω_1 and ω_2. Let the two discs be connected to each other through a rope or a belt or a chain as shown in Figure.

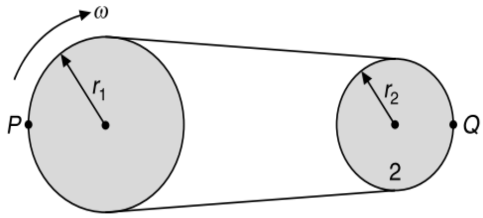


If the rope or belt or the chain does not slip over the two discs, the linear speed of any section of the rope is same as that of a point on the periphery of the two discs. So, we have

$$v = \omega_1 r_1 = \omega_2 r_2 \text{ and } a = \alpha_1 r_1 = \alpha_2 r_2$$

ILLUSTRATION 15

Two wheels 1 and 2 of radii r_1 and r_2 are connected by a belt as shown in Figure.

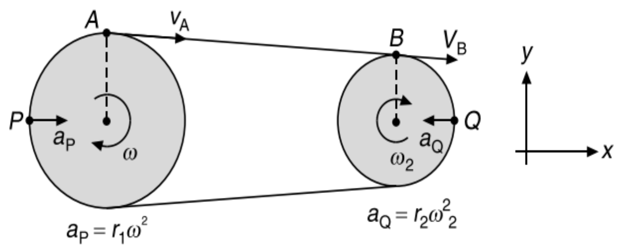


If the wheel 1 is rotating with uniform angular velocity ω and the belt does not slide over the wheels, calculate the speed of a point on the belt, the angular velocity of the wheel 2 and acceleration of the point P with respect to the point Q.

SOLUTION

Since the belt is not sliding, so we have

$$v_A = v_B$$



$$\Rightarrow \quad r_1\omega_1 = r_2\omega_2$$

$$\Rightarrow \quad \omega_2 = \frac{r_1\omega}{r_2}$$

Since ω is constant, so tangential acceleration of any point on the wheels is zero and hence we have

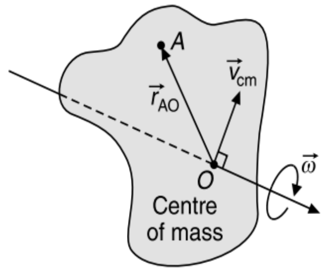
$$\vec{a}_P = r_1\omega^2(\hat{i}) \text{ and } \vec{a}_Q = r_2\omega_2^2(-\hat{i})$$

So, acceleration of point P w.r.t. point Q is

$$\vec{a}_{PQ} = \vec{a}_P - \vec{a}_Q = r_1\omega^2\left(1+\frac{r_1}{r_2}\right)\hat{i}$$

ACCELERATION AND VELOCITY OF ANY POINT ON A RIGID BODY

Consider a rigid body moving in space such that at any instant, the velocity of its centre of mass is $\vec{v}_{cm}$ and angular velocity is $\vec{\omega}$. To describe the motion of any point A with respect to a stationary point (or ground) we must understand that in a rigid body, the relative position of the points cannot change. This simply implies that, the relative velocity of point A (or any other point that lies on the body) w.r.t. centre of mass is tangential and is given by



$$\vec{v}_{A/cm} = \vec{\omega} \times \vec{r}_{AO}$$

$$\Rightarrow \quad \vec{v}_A - \vec{v}_{cm} = \vec{\omega} \times \vec{r}_{AO}$$

$$\Rightarrow \quad \vec{v}_A = \vec{v}_{cm} + \vec{\omega} \times \vec{r}_{AO}$$

Differentiating, we get

$$\vec{a}_A = \frac{d}{dt}\left(\vec{v}_{cm} + \vec{\omega}\times\vec{r}_{AO}\right) = \frac{d\vec{v}_{cm}}{dt} + \frac{d}{dt}\left(\vec{\omega}\times\vec{r}_{AO}\right)$$

$$\Rightarrow \quad \vec{a}_A = \vec{a}_{cm} + \frac{d\vec{\omega}}{dt}\times\vec{r}_{AO} + \vec{\omega}\times\frac{d\vec{r}_{AO}}{dt}$$

$$\Rightarrow \quad \vec{a}_A = \vec{a}_{cm} + \vec{\alpha}\times\vec{r}_{AO} + \vec{\omega}\times\vec{v}_{A/cm}$$

$$\Rightarrow \quad \vec{a}_A = \vec{a}_{cm} + \vec{\alpha}\times\vec{r}_{AO} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{AO}\right)$$

Also, we note that the velocity of any point A w.r.t. B is given by

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = \vec{\omega}\times\vec{r}_{AB}$$

where, $\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$ is the position vector of A w.r.t. B.

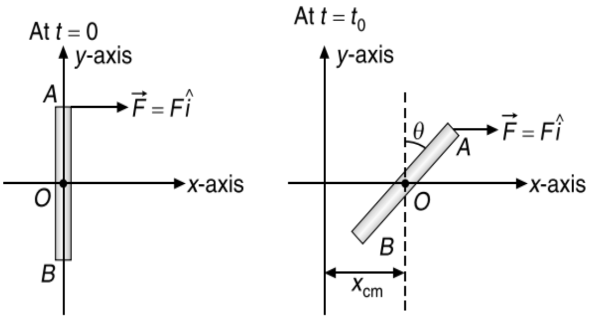
In general, we can say that the general motion of a body can be called as a combination of rotational motion and translational motion.

ILLUSTRATION 16

A thin uniform rod AB of mass m and length L is placed on a smooth horizontal table. A constant horizontal force of magnitude F starts acting on the rod at one of the ends AB. Initially, the force is perpendicular to the length of the rod. Taking the moment at which force starts acting as $t = 0$, calculate the distance moved by centre of mass of the rod in time t_0, magnitude of initial acceleration of the end A of the rod.

SOLUTION

Consider the rod to be placed in x-y plane (horizontal) with initial orientation along y-axis. Further, assuming that force acting on the rod is along the x-axis as shown in Figure.



The acceleration of centre of mass is given by

$$\vec{a}_{cm} = \frac{\vec{F}}{m} = \frac{F}{m}\hat{i} = \text{constant}$$

The displacement is given by

$$\Rightarrow \quad x_{cm} = \frac{1}{2}a_{cm}t_0^2 = \frac{1}{2}\frac{F}{m}t_0^2 \qquad \{\because a_{cm} = \text{constant}\}$$

The initial acceleration of end A of the rod is

$$\vec{a}_A = \vec{a}_{cm} + \vec{\alpha} \times \vec{r}_{AO} \qquad \{\because \omega = 0\}$$

$$\Rightarrow \quad \vec{a}_A = \frac{F}{m}\hat{i} + \left(-\frac{6F}{mL}\hat{k}\right) \times \left(\frac{L}{2}\hat{j}\right)$$

$$\Rightarrow \quad \vec{a}_A = \frac{4F}{m}\hat{i}$$

Test Your Concepts-II

Based on Rotational Kinematics, Combined Effect of Rotation and Translation Motion

(Solutions on page H.150)

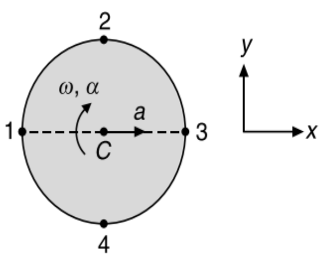
1. The angular position of a point on the rim of a rotating wheel is given by $\theta = 4t - 3t^2 + t^3$, where θ is in radian and t is in second

(a) Calculate the angular velocity at $t = 2$ s and $t = 4$ s.

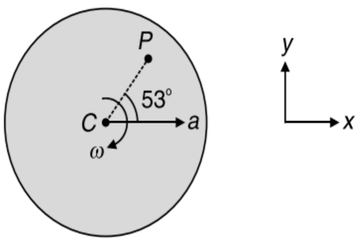
(b) Calculate the average angular acceleration for the time interval that begins at $t = 2$ s and ends at $t = 4$ s

(c) Calculate the instantaneous angular acceleration at the beginning and the end of this time interval.

2. A uniform disc of radius r spins with angular velocity ω and angular acceleration α. If the centre of mass of the disc has linear acceleration a, calculate the magnitude and direction of acceleration of the points 1, 2, 3 and 4.



3. Calculate the linear acceleration of the particle P at $t = 1$ s if $a = 2$ ms^{-2}, $\omega = (2t)$ rads^{-1} and $CP = 1$ m as shown in Figure. Express your result in terms of $\hat{i}$ and $\hat{j}$.



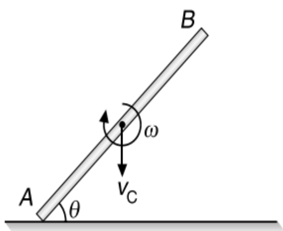
4. A disc starts rotating with constant angular acceleration of π rads^{-2} about a fixed axis perpendicular to its plane and through its centre.

(a) Find the angular velocity of the disc after 4 s.

(b) Find the angular displacement of the disc after 4 s and

(c) Find the number of turns accomplished by the disc in 4 s.

5. A rod AB of length l is supported against smooth horizontal surface. The rod is inclined at angle θ with horizontal. Now it released so that end A starts sliding on horizontal surface. Find the velocity of centre of mass of the rod in terms of its angular velocity ω.

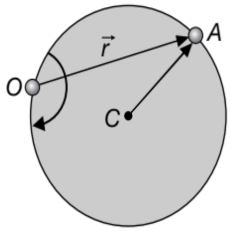


6. A flywheel of radius 20 cm starts from rest, and has a constant angular acceleration of 60 rads^{-2}. Find

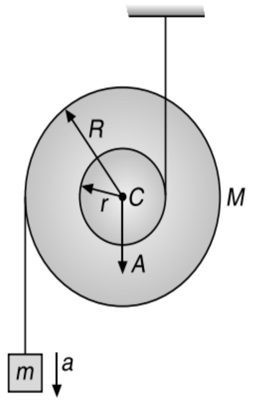
(a) the magnitude of the net linear acceleration of a point on the rim after 0.15 s

(b) the number of revolutions completed in 0.25 s

7. A particle A moves along a circle of radius $R = 10$ cm so that its radius vector $\vec{r}$ relative to O rotates with constant angular velocity $\omega_0 = 0.2\ \text{rads}^{-1}$. Find the modulus of the velocity of the particle and modulus and direction of its total acceleration.

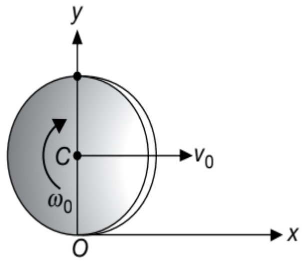


8. A bobbin with inner radius r and outer radius R is arranged with light strings and a block as shown in Figure.

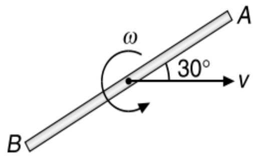


The string does not slide over the cylinder and when the system is released from rest, then the block and bobbin move down with accelerations a and A respectively. Calculate the ratio of the acceleration of the bobbin to the acceleration of the block.

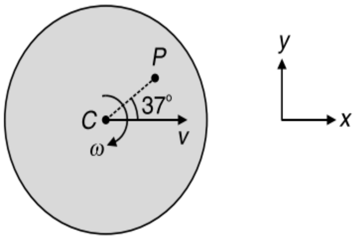
9. A disc of radius R start at time $t = 0$ moving along the positive x-axis with linear speed v_0 and angular speed ω_0. Find the x and y co-ordinates of the bottommost point at any time t.



10. A uniform rod of length l is spinning with an angular velocity $\omega = \dfrac{2v}{l}$ while its centre of mass moves with a velocity v. Calculate the velocity of the end A of the rod.

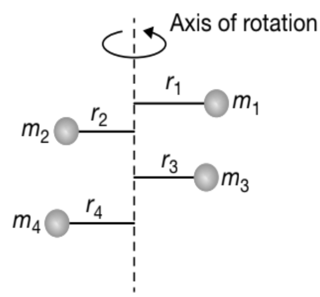


11. Calculate the linear velocity of particle P shown in Figure, if $v = 2\ \text{ms}^{-1}$, $\omega = 5\ \text{rads}^{-1}$ and $CP = 1\ \text{m}$. Express your result in terms of $\hat{i}$ and $\hat{j}$.



ROTATIONAL KINETIC ENERGY (R.K.E.)

Consider a system made of particles of masses m_1, m_2, m_3, m_4, ..., m_n placed at distances r_1, r_2, …, r_n from the axis of rotation. Let the particles rotate about the axis of rotation with uniform angular velocity ω such that the tangential velocities of the particles are v_1, v_2, …, v_n respectively. From above consideration we conclude that the system possesses kinetic energy due to rotational motion of the particles and hence this kinetic energy is called **rotational kinetic energy** denoted by K_R.



A discrete system of particles

$$\Rightarrow \quad K_{\text{rotational}} = K_R = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ... + \frac{1}{2}m_nv_n^2$$

But $v_1 = r_1\omega$, $v_2 = r_2\omega$, …, $v_n = r_n\omega$

$$\Rightarrow \quad K_R = \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2\right)\omega^2$$

$$\Rightarrow \quad K_R = \frac{1}{2} I\omega^2$$

For a system rotating about a point, we must keep in mind that the translational kinetic energy of a rigid body having mass M and centre of mass velocity v_{cm} is given by

$$K_{translational} = K_T = \frac{1}{2} M v_{cm}^2$$

and the rotational kinetic energy of the body about its centre of mass is

$$K_R = \frac{1}{2} I_{cm}\omega^2$$

So, the total energy of a body in rotation about an axis can be written as

$$K = K_R + K_T$$

ILLUSTRATION 17

A rod of mass m and length l is connected with a light rod of length l. The composite rod is made to rotate with angular velocity ω as shown in Figure.

Calculate the translational kinetic energy, rotational kinetic energy and total kinetic energy of rod.

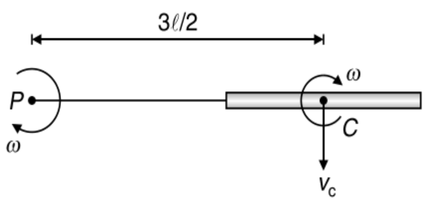
SOLUTION

Translational kinetic energy of rod,

$$K_{translational} = \frac{1}{2} m v_{cm}^2$$

Velocity of centre of mass is

$$v_{cm} = \omega\left(\frac{3}{2} l\right)$$



The translational kinetic energy of rod is

$$K_{translational} = K_T = \frac{1}{2} m\left(\frac{3\omega l}{2}\right)^2 = \frac{9}{8} m\omega^2 l^2$$

The rotational kinetic energy of rod is

$$K_{rotational} = \frac{1}{2} I_{cm}\omega^2 = \frac{1}{2}\left(\frac{ml^2}{12}\right)\omega^2 = \frac{1}{24} m\omega^2 l^2$$

Total kinetic energy of the rod is

$$K_{total} = K_{translational} + K_{rotational}$$

$$\Rightarrow \quad K_{total} = \frac{9}{8} m\omega^2 l^2 + \frac{1}{24} m\omega^2 l^2 = \frac{7}{6} m\omega^2 l^2$$

The total kinetic energy of the rod can also be thought to be the rotational kinetic energy of the rod about the fixed axis. Hence, we have

$$K_{total} = \frac{1}{2} I_P \omega^2$$

where I_P is the moment of inertial of the rod about the axis passing through the point P. According to parallel axis theorem, we have

$$I_P = I_{cm} + md^2 = I_{cm} + m\left(\frac{3}{2} l\right)^2$$

$$\Rightarrow \quad I_P = \frac{ml^2}{12} + \frac{9}{4} ml^2 = \frac{7}{3} ml^2$$

Hence total kinetic energy of the rod is

$$K_{total} = \frac{1}{2}\left(\frac{7}{3} ml^2\right)\omega^2 = \frac{7}{6} m\omega^2 l^2$$

MODIFIED WORK ENERGY THEOREM (MWET) AND CONSERVATION OF MECHANICAL ENERGY

If W_{ext} is the work done by external forces, W_{nc} is the work done by non-conservative forces, W_{ps} is the work done by pseudo forces and W_{int} is the work done by internal forces, then according to Modified Work Energy Theorem (MWET) studied earlier, we have

$$W_{ext} + W_{nc} + W_{ps} + W_{int} = \Delta U + \Delta K$$

However, in the absence of dissipative (non-conservative), external, pseudo forces, work done by thems is zero and *if work done by internal forces is zero*, then the total mechanical energy of a system is conserved, i.e.

$$\Delta K+\Delta U=0$$

$$\Rightarrow \quad K_f+U_f=K_i+U_i$$

$$\Rightarrow \quad (U+K)_{\text{initial}}=(U+K)_{\text{final}}$$

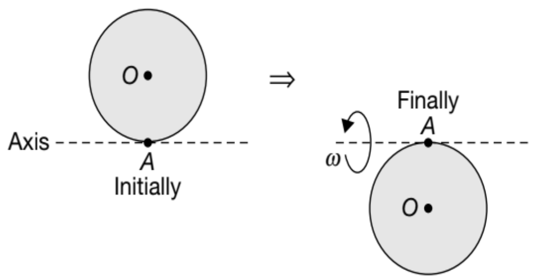
ILLUSTRATION 18

A uniform circular disc of mass m, radius R and centre O is free to rotate about a smooth, horizontal axis which is tangential to the disc at a point A. The disc is held in a vertical plane with A below O and is then slightly displaced from this position. Find the angular velocity of the disc when its plane is next vertical.

SOLUTION

By Law of Conservation of Mechanical Energy, we have

$$\begin{pmatrix}\text{Loss in Gravitational}\\ \text{Potential Energy of}\\ \text{Centre of Mass}\\ \text{of the Disc}\end{pmatrix}=\begin{pmatrix}\text{Gain in Rotational}\\ \text{Kinetic Energy of}\\ \text{Disc about the said}\\ \text{Axis of Rotation}\end{pmatrix}$$



$$\Rightarrow \quad mgh=\frac{1}{2}I\omega^2$$

$$\Rightarrow \quad mg(2R)=\frac{1}{2}\left(\frac{1}{4}mR^2+mR^2\right)\omega^2$$

$$\Rightarrow \quad \omega=4\sqrt{\frac{g}{5R}}$$

ILLUSTRATION 19

A uniform disk of mass M and radius R is pivoted so that it can rotate freely about a horizontal axis through its centre and normal to the plane of the disk. A small particle of mass m is attached to the rim of the disk at the top directly above the pivot. The system is given a gentle start and the disk begins to rotate.

(a) What is the angular velocity of the disk when the particle is at its lowest point?

(b) At this point, what force must be exerted on the particle by the disk to keep it on the disk?

SOLUTION

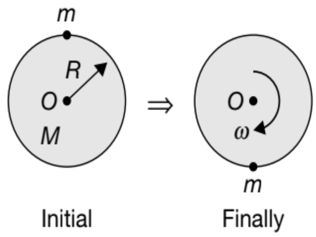
(a) By Law of Conservation of Mechanical Energy, we have

$$\begin{pmatrix}\text{Loss in}\\ \text{Gravitational}\\ \text{Potential Energy}\\ \text{of Particle}\end{pmatrix}=\begin{pmatrix}\text{Gain in}\\ \text{Rotational}\\ \text{Kinetic Energy}\\ \text{of disc + Particle}\end{pmatrix}$$

$$\Rightarrow \quad mg(2R)=\frac{1}{2}\left(\frac{1}{2}MR^2+mR^2\right)\omega^2$$

$$\Rightarrow \quad \omega=\sqrt{\frac{8mg}{(2m+M)R}}$$

(b) If F be the force exerted by disk on the particle (upwards), then



$$F-mg=mR\omega^2$$

$$\Rightarrow \quad F=mg+\frac{8m^2g}{(2m+M)}$$

$$\Rightarrow \quad F=\frac{mg(10m+M)}{(2m+M)}$$

ILLUSTRATION 20

ABC is a triangular framework of three uniform rods each of mass m and length $2l$. It is free to rotate in its own plane about a smooth horizontal axis through A normal to the plane ABC. If it is released from rest when AB is horizontal and C is above AB, find the maximum velocity of C in the subsequent motion.

SOLUTION

$$I_A=I_{AB}+I_{AC}+I_{BC}$$

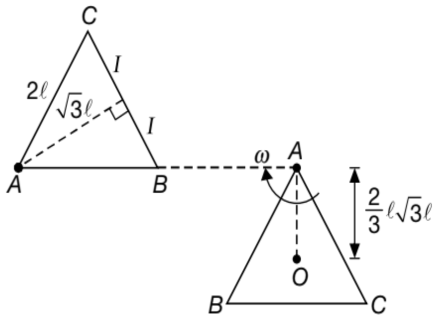
$$\Rightarrow \quad I_A=\frac{1}{3}m(2\ell)^2+\frac{1}{3}m(2\ell)^2+\left(\frac{1}{12}m(2\ell)^2+md^2\right)$$

where $d=\sqrt{3}\ell$

$$\Rightarrow \quad I_A = \frac{4}{3}m\ell^2 + \frac{4}{3}m\ell^2 + \left(\frac{1}{3}m\ell^2 + m\left(\ell\sqrt{3}\right)^2\right)$$

$$\Rightarrow \quad I_A = 6m\ell^2$$

Assume the horizontal line passing through A to be the zero Potential Energy Level (ZPEL) as shown in Figure.



Then initially CG is $\frac{1}{3}\left(\ell\sqrt{3}\right)$ above ZPEL and finally i.e., when C attains maximum velocity, the CG is $\frac{2}{3}\left(\ell\sqrt{3}\right)$ below ZPEL. So, in the process the CG has fallen through

$$h = \frac{2}{3}\left(\ell\sqrt{3}\right) + \frac{1}{3}\left(\ell\sqrt{3}\right) = \ell\sqrt{3}$$

Applying Law of Conservation of Mechanical Energy, we get

$$\begin{pmatrix}\text{Loss in} \\ \text{Gravitational} \\ \text{Potential Energy} \\ \text{of CG}\end{pmatrix} = \begin{pmatrix}\text{Gain in} \\ \text{Rotational} \\ \text{Kinetic Energy} \\ \text{of system}\end{pmatrix}$$

$$\Rightarrow \quad (3m)g\left(\ell\sqrt{3}\right) = \frac{1}{2}\left(6m\ell^2\right)\omega^2$$

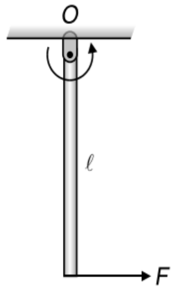
$$\Rightarrow \quad \omega = \sqrt{\frac{g\sqrt{3}}{\ell}}$$

So, velocity of C at this instant is

$$v_C = (2\ell)\omega = 2\sqrt{g\ell\sqrt{3}}$$

ILLUSTRATION 21

A uniform rod of mass m and length l is pivoted smoothly at O. A horizontal force acts at the bottom of the rod.

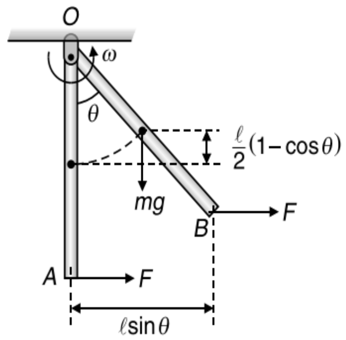


Calculate the angular velocity of the rod as the function of angle of rotation θ and the maximum angular displacement of the rod.

SOLUTION

According to modified work energy theorem, we have $W_{ext} = \Delta U + \Delta K$, where

$$F_{ext} = F\Delta r = Fl\sin\theta$$



ΔU is the rise in potential energy of the centre of mass of the rod, so

$$\Delta U = mg\left(\frac{l}{2}\right)(1-\cos\theta)$$

ΔK is the rotational kinetic energy of the rod about rotational axes passing through hinge and can also be called as the total kinetic energy of the rod, so

$$\Delta K = \frac{1}{2}I_O\omega^2$$

$$\Rightarrow \quad Fl\sin\theta = \frac{mgl}{2}(1-\cos\theta) + \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{6F\sin\theta}{ml} - \frac{3g}{l}(1-\cos\theta)} \quad \ldots(1)$$

At maximum angular displacement, the angular velocity of the rod is zero. Substituting $\omega = 0$ in equation (1), we get

$$0 = \sqrt{\frac{6F}{ml}\sin\theta - \frac{3g}{l}(1-\cos\theta)}$$

$$\Rightarrow \quad \frac{6F}{ml}\sin\theta = \frac{3g}{l}(1-\cos\theta)$$

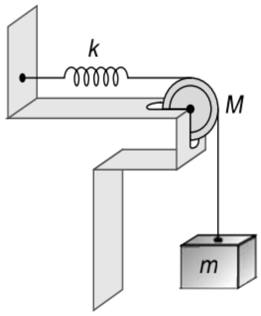
$$\Rightarrow \quad \frac{2F}{m}\left[2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right] = g\left(2\sin^2\left(\frac{\theta}{2}\right)\right)$$

$$\Rightarrow \quad \tan\left(\frac{\theta}{2}\right) = \frac{2F}{mg}$$

$$\Rightarrow \quad \theta = 2\tan^{-1}\left(\frac{2F}{mg}\right)$$

ILLUSTRATION 22

A block of mass $m = 4$ kg is attached to a spring of spring constant $(k = 32\text{ Nm}^{-1})$ by a rope that hangs over a pulley of mass $M = 8$ kg. If the system starts from rest with the spring unstretched, find the speed of the block after it falls 1 m. Treat the pulley as a disc, so $I = \frac{1}{2}MR^2$.



SOLUTION

Since the rim of the pulley moves at the same speed as the block, the speed of the block and the angular velocity of the pulley related by $v = \omega R$.

When the block falls by a distance x, its potential energy decreases, so $\Delta U_g = -mgx$, the potential energy of the spring increases, so $\Delta U_s = +\frac{1}{2}kx^2$, both the block and the pulley gain kinetic energy, so $\Delta K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Applying Law of Conservation of Mechanical Energy, we get

$$\Delta K + \Delta U = 0,$$

$$\Rightarrow \quad \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 + \frac{1}{2}kx^2 - mgx = 0$$

$$\Rightarrow \quad \frac{1}{2}\left(m + \frac{M}{2}\right)v^2 + \frac{1}{2}kx^2 - mgx = 0$$

Please note that in this problem, the radius R was not required. Substituting $m = 4$ kg, $M = 8$ kg, $k = 32\text{ Nm}^{-1}$ and $x = 1$ m, we get

$$\frac{1}{2}\left(4 + \frac{8}{2}\right)v^2 + \frac{1}{2}(32)(1)^2 - (4)(10)(1) = 0$$

$$\Rightarrow \quad 4v^2 + 16 - 40 = 0$$

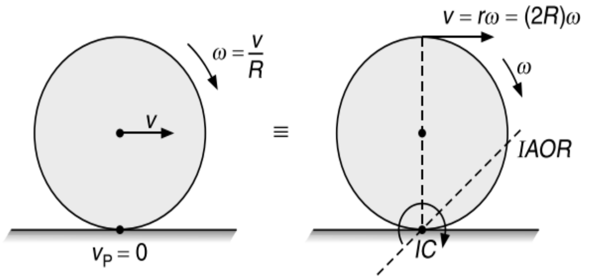
$$\Rightarrow \quad v = 2.4\text{ ms}^{-1}$$

TOTAL ENERGY OF A BODY IN PURE ROLLING

When a body rolls without slipping, then it possesses simultaneous translational motion (of the CM) and rotational motion of the entire body. So,

Total K.E. = Rotational K.E. + Translational K.E.

$$\Rightarrow \quad \text{Total K.E.} = E = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}mv_{CM}^2$$



Axis normal to the plane of rolling body and passing through *IC*

$$\Rightarrow \quad E = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$\Rightarrow \quad E = \frac{1}{2}(mk^2)\omega^2 + \frac{1}{2}mv^2$$

$$\Rightarrow \quad E = \frac{1}{2}(mk^2)\omega^2 + \frac{1}{2}m(R\omega)^2$$

$$\Rightarrow \quad E = \frac{1}{2}(mk^2 + mR^2)\omega^2 \qquad \ldots(1)$$

Version I

$$E = \frac{1}{2}(mk^2)\left(1 + \frac{R^2}{k^2}\right)\omega^2$$

$$\Rightarrow \quad E = \frac{1}{2}I\omega^2\left(1 + \frac{R^2}{k^2}\right)$$

Version II

$$E = \frac{1}{2}mk^2\left(1+\frac{R^2}{k^2}\right)\left(\frac{v^2}{R^2}\right)$$

$$\Rightarrow \quad E = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$$

Conceptual Note(s)

If K_R stands for rotational kinetic energy and K_T for translational kinetic energy, then we have

$K_R = \frac{1}{2}I_{cm}\omega^2$ and $K_T = \frac{1}{2}mv_{cm}^2$, then

(i) for a ring, $\frac{K_R}{K_T} = 1$

(ii) for a disc, $\frac{K_R}{K_T} = \frac{1}{2}$

(iii) for a cylinder, $\frac{K_R}{K_T} = \frac{1}{2}$

(iv) for a shell, $\frac{K_R}{K_T} = \frac{2}{3}$

(v) for a solid sphere, $\frac{K_R}{K_T} = \frac{2}{5}$

ILLUSTRATION 23

A hoop of mass m and radius R rolls without slipping with velocity v_0. Find its kinetic energy.

SOLUTION

METHOD I:

$$K = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2$$

$$K = \frac{1}{2}mv_o^2 + \frac{1}{2}(mR^2)\omega^2$$

Since $v_0 = \omega R$

$$\Rightarrow \quad K = \frac{1}{2}mv_o^2 + \frac{1}{2}mv_o^2$$

$$\Rightarrow \quad K = mv_o^2$$

METHOD II:

$$K = \frac{1}{2}I_0\omega^2 \text{ where } I_0 = I_c + mR^2$$

$$\Rightarrow \quad I_0 = mR^2 + mR^2 = 2mR^2$$

$$\Rightarrow \quad K = \frac{1}{2}(2mR^2)\omega^2$$

$$\Rightarrow \quad K = mR^2\omega^2 = mv_0^2$$

ILLUSTRATION 24

A carpet of mass M of inextensible material is rolled along its length in the form of a cylinder of radius R and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when negligibly small push is given to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius reduces to $\frac{R}{2}$.

SOLUTION

Let M be the initial mass of the carpet of radius R and length l (say), then

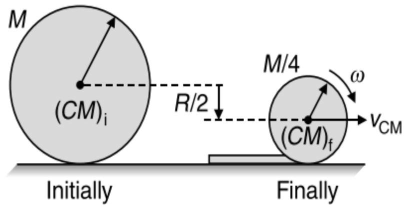
$$M = (\pi R^2 l)\rho$$

When the gentle push is given to the carpet and it unrolls such that its radius reduces to $R/2$, then the new mass of the rolling portion is

$$M' = \pi\left(\frac{R}{2}\right)^2 l\rho = \frac{M}{4}$$

By law of conservation of energy, we have

$$\left(\begin{array}{c}\text{Loss in}\\ \text{GPE of CM}\end{array}\right) = \left(\begin{array}{c}\text{Gain in}\\ \text{RKE of CM}\end{array}\right) + \left(\begin{array}{c}\text{Gain in}\\ \text{TKE of CM}\end{array}\right)$$



$$\Rightarrow \quad MgR - M'gR' = \left(\frac{1}{2}I'\omega^2\right) + \frac{1}{2}M'v_{cm}^2$$

$$\Rightarrow \quad \frac{7}{8}MgR = \frac{1}{2}\left(\frac{1}{2}M'R'^2\right)\left(\frac{v_{cm}}{R'}\right)^2 + \left(\frac{1}{2}\left(\frac{M}{4}\right)v_{cm}^2\right)$$

$$\Rightarrow \quad \frac{7}{8}MgR = \frac{1}{2}\left(\frac{M}{8}\right)v_{cm}^2 + \left(\frac{M}{8}\right)v_{cm}^2$$

$$\Rightarrow \quad \frac{7}{8}MgR = \left(\frac{M}{8}\right)\left(\frac{3}{2}\right)v_{cm}^2$$

$$\Rightarrow \quad v_{cm} = \sqrt{\frac{14}{3}gR}$$

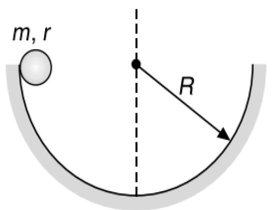
ILLUSTRATION 25

A small steel sphere of mass m and radius r rolls without slipping on the surface of a large hemisphere of radius $R(\gg r)$ whose axis of symmetry is vertical. It starts at the top from the rest.

(a) What is the kinetic energy at the bottom?
(b) What fraction is the rotational kinetic energy of the total kinetic energy?
(c) What fraction is the translational kinetic energy of the total kinetic energy?
(d) Calculate the normal force that the small sphere will exert on the hemisphere at its bottom. How the results will be affected if r is not very small as compared to R.

SOLUTION

Dotted line is the axis of symmetry, which is vertical.



(a) In pure rolling (on stationary ground) work done by friction is zero. Hence, mechanical energy remains conserved. So

K.E. at bottom = decrease in P.E. = $mg(R-r)$

(b) In pure rolling $\dfrac{K_R}{K_T}=\dfrac{2}{5}$ (for a sphere)

$$\Rightarrow \frac{K_R}{K_R+K_T}=\frac{2}{7}$$

(c) $\dfrac{K_T}{K_R+K_T}=\dfrac{5}{7}$

(d) $K_T=\dfrac{5}{7}$ (Total energy)

$$\Rightarrow \frac{1}{2}mv^2=\frac{5}{7}mg(R-r)$$

$$\Rightarrow \frac{mv^2}{R-r}=\frac{10}{7}mg$$

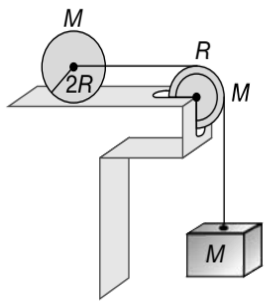
$$\Rightarrow N-mg=\frac{mv^2}{R-r}=\frac{10}{7}mg$$

$$\Rightarrow N=\frac{17}{7}mg$$

The normal reaction will still be $\dfrac{17}{7}mg$ even if r is not very small as compared to R.

ILLUSTRATION 26

A uniform solid cylinder of mass M and radius $2R$ rests on a horizontal table top. A string is attached by a yoke to a frictionless axle through the centre of the cylinder so that the cylinder can rotate about the axle. The string runs over a pulley in the shape of a disk of mass M and radius R that is mounted on a frictionless axle through its centre. A block of mass M is suspended from the free end of the string. The string does not slip over the pulley surface and the cylinder rolls without slipping on the table top. After the system is released from rest, what is the magnitude of the downward acceleration of the block?



SOLUTION

When the block moves down by h (say), in time t, then it loses potential energy. This potential energy lost by block is gained as

(a) translational kinetic energy, $\dfrac{1}{2}Mv^2$ by the block

(b) rotational kinetic energy, $\dfrac{1}{2}I_1\omega_1^2$ by the pulley disk

(c) rolling kinetic energy, $\dfrac{1}{2}Mv^2+\dfrac{1}{2}I_2\omega_2^2$ by the cylinder

So, by Law of Conservation of Energy, we get

$$Mgh=\frac{1}{2}Mv^2+\frac{1}{2}I_1\omega_1^2+\left(\frac{1}{2}Mv^2+\frac{1}{2}I_2\omega_2^2\right)$$

where $h=\dfrac{1}{2}at^2$, $v=at$, $\omega_1=\dfrac{v}{R}=\dfrac{at}{R}$, $\omega_2=\dfrac{v}{2R}=\dfrac{at}{2R}$

$$I_1=\frac{1}{2}MR^2 \text{ and } I_2=\frac{1}{2}M(2R)^2$$

So, we get

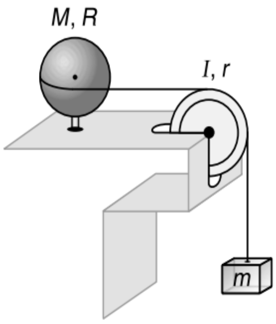
$$g\left(\frac{1}{2}at^2\right)=(at)^2+\frac{R^2}{4}\left(\frac{at}{R}\right)^2+\frac{(2R)^2}{4}\left(\frac{at}{2R}\right)^2$$

$$\Rightarrow \quad g=2a+\frac{a}{2}+\frac{a}{2}$$

$$\Rightarrow \quad a=\frac{g}{3}$$

ILLUSTRATION 27

A uniform spherical shell of mass M and radius R rotates about a vertical axis on frictionless bearing. A light cord passes around the equator of the shell, over a pulley of rotational inertia I and radius r and is attached to a small object of mass m that is otherwise free to fall under the influence of gravity. There is no friction of pulley's axle and the cord does not slip on the pulley. Calculate speed of the object after it has fallen a distance h from rest.



SOLUTION

By law of conservation of energy, we observe that the loss in gravitational potential energy of the block equals the sum of gain in kinetic energy of the block, gain in rotational kinetic energy of the pulley and the gain in rotational kinetic energy of the shell.

$$\Rightarrow \quad mgh=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2+\frac{1}{2}\left(\frac{2MR^2}{3}\right)(\omega')^2$$

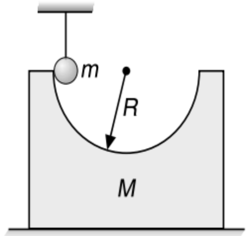
where, $\omega=\frac{v}{r}$ and $\omega'=\frac{v}{R}$

After substituting these values in the above equation and solving, we get

$$v=\sqrt{\frac{mgh}{\frac{m}{2}+\frac{I}{2r^2}+\frac{M}{3}}}$$

ILLUSTRATION 28

A semi-circular track of radius $R = 62.5$ cm is cut in a block. Mass of block, having track, is $M = 1$ kg and rests over a smooth horizontal floor. A cylinder of radius $r = 10$ cm and mass $m = 0.5$ kg is hanging by thread such that axes of cylinder and track are in same level and surface of cylinder is in contact with the track as shown in Figure.



When the thread is burnt, cylinder starts to move down the track. Sufficient friction exists between surface of cylinder and track, so that cylinder does not slip. Calculate velocity of axis of cylinder and velocity of the block when it reaches bottom of the track. Also find force applied by block on the floor at that moment $\left(g = 10\ \text{ms}^{-2}\right)$.

SOLUTION

Using Law of Conservation of Linear Momentum, we get

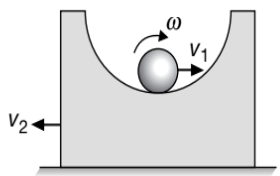
$$mv_1 = Mv_2 \qquad \ldots(1)$$

Applying Law of Conservation of Mechanical Energy, we get

$$mg(R-r)=\frac{1}{2}mv_1^2+\frac{1}{2}I\omega^2+\frac{1}{2}Mv_2^2 \qquad \ldots(2)$$

where, $I=\frac{1}{2}mr^2$ and $\omega=\frac{v_r}{r}$, where v_r is the velocity of cylinder axis relative to block

$$v_r = v_1 + v_2 \qquad \ldots(3)$$



Solving equations (1), (2) and (3) with given data, we get

$$v_1 = 2\ \text{ms}^{-1}$$

and $v_2 = 1.5\ \text{ms}^{-1}$

Further, we have

$$N - mg = \frac{mv_r^2}{R-r}$$

$$\Rightarrow \quad N = mg + \frac{mv_r^2}{R-r} = (0.5)(10) + \frac{(0.5)(3.5)^2}{0.525}$$

$\Rightarrow N = 16.67\ \text{N}$

Conceptual Note(s)

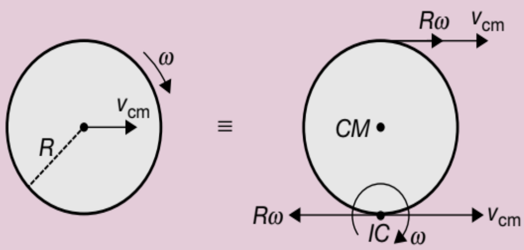
AN OBSERVATION

For a body in pure rolling motion, we have

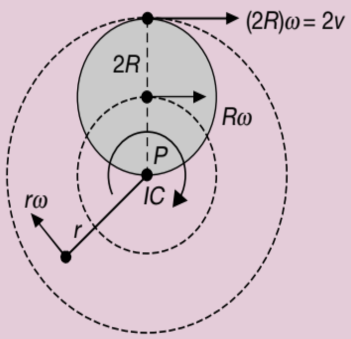
$$E = \frac{1}{2}(mk^2 + mR^2)\omega^2$$

Now $(mk^2 + mR^2)$ is actually moment of inertia of body (in combined effect of rotation and translation) about axis that passes through the point of contact P, which happens to be at rest (in case of pure rolling), so

$$E = \frac{1}{2}(I_{\text{about } P})\omega^2 = \frac{1}{2}(mk^2 + mR^2)\omega^2$$



This axis passing through P is always normal to the plane used to represent motion and the intersection of this axis with the plane is the location of instantaneous centre of zero velocity (IC).



If I_{IAOR} is the Moment of Inertia of body about **Instantaneous Axis of Rotation (IAOR)**, then

$$I_{IAOR} = I_{CG} + md^2 = mk^2 + mR^2$$

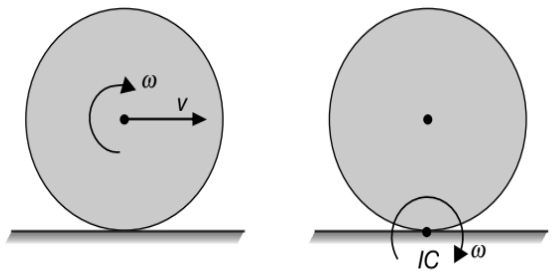
$$KE = \frac{1}{2}(I_{IAOR})\omega^2$$

The **Instantaneous Axis of Rotation (IAOR) passes through a point called the Instantaneous Centre of Zero Velocity (IC).**

So, from above we conclude that pure rolling is equivalent to the case of pure rotation about a new axis (called Instantaneous Axis of Rotation, IAOR) that passes through the point of contact (because it was the only point which is at rest in pure rolling).

INSTANTANEOUS AXIS OF ROTATION (IAOR)

The combined effect of translation of the centre of mass and rotation of a rigid body about an axis through the centre of mass is equivalent to a pure rotation with the same angular speed about an axis passing through a point of zero velocity. Such an axis is called the Instantaneous Axis of Rotation (IAOR).



So, $(\text{Rotation}) + (\text{Translation}) \equiv \begin{pmatrix} \text{Pure rotation} \\ \text{about } IAOR \\ \text{passing} \\ \text{through } IC \end{pmatrix}$

$$\Rightarrow \quad E = \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$$

$$\Rightarrow \quad E = \frac{1}{2}(mk^2 + mR^2)\omega^2 = \frac{1}{2}I_{IAOR}\omega^2$$

Problem Solving Technique(s)

WORD OF ADVICE

(a) Although the concept of IC is conveniently used to determine the velocity of any point in a body, **however generally, IC does not have zero acceleration and therefore, it should not be used for finding the acceleration of any point in the body.**

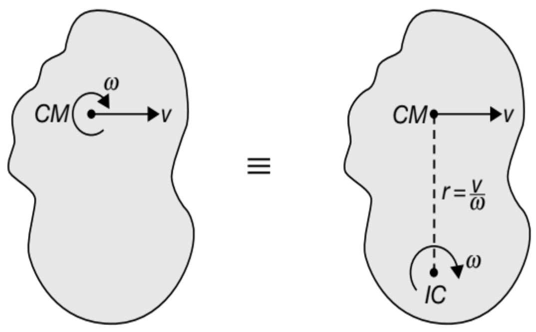
(b) When a body is subjected to general plane motion, the point determined as the instantaneous centre of zero velocity for the body can only be used for an instant of time. Since the body changes its position from one instant to the next, then for each position of the body a unique instantaneous centre must be determined. The locus of points which defines the IC during the body's motion is called a **centrode**. Thus, each point on the centrode acts as the IC for the body only for an instant of time.

LOCATION OF THE IC

The location of the IC is determined by using the fact that the relative position vector extending from the IC to a point is always perpendicular to the velocity of the point. Then the following three possibilities exist.

CASE-1

When the velocity of a point (generally the centre of mass) on the body and the angular velocity of the body are known.



Combined Effect of Rotation & Tranlation

Single Effect of Rotation about *IC*

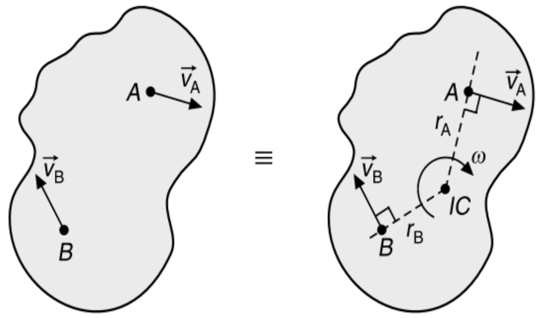
When v and ω are known, then IC is located along the line drawn perpendicular to $\vec{v}$ at CM, such that the distance from CM to IC is, $r = \dfrac{v}{\omega}$.

Note that IC lie on that side of CM which causes rotation about the IC, which is consistent with the direction of motion caused by $\vec{\omega}$ and $\vec{v}$.

CASE-2

When the lines of action of two non-parallel velocities are known.

Consider the body shown in Figure where the line of action of the velocities $\vec{v}_A$ and $\vec{v}_B$ are known.



STEP-1: Draw perpendiculars at A and B to these lines of action.

STEP-2: The point of intersection of these perpendiculars, locates the IC at the instant considered. If v_A and v_B originate at perpendicular distances r_A and r_B from IC, then

$$v_A = r_A\omega$$

and $$v_B = r_B\omega$$

CASE-3

When the magnitude and direction of two parallel velocities are known.

When the velocities of points A and B are parallel separated by perpendicular distance d and have known magnitudes v_A and v_B, then the location of the IC is determined by the following steps.

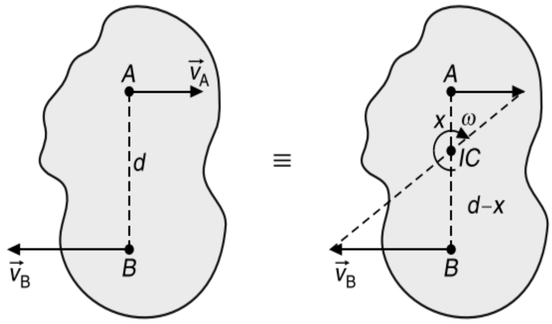
STEP-1: Always draw the velocities to the scale.

STEP-2: Join the points of origin of the two velocities i.e., tails of velocity vectors by a straight line.

STEP-3: Now, join the heads of the two velocity vectors by straight line.

STEP-4: IC is located at the intersection of the lines in STEP-2 and STEP-3.

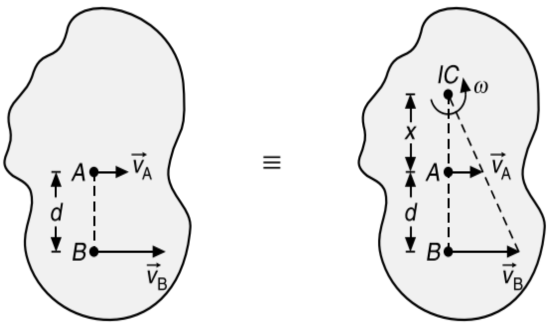
To locate position of IC in (a), we have



(a) When velocities are anti-parallel

$$v_A = x\omega \text{ and } v_B = (d - x)\omega$$

To locate position of IC in (b), we have



(b) When velocities are parallel

$$v_A = x\omega \text{ and } v_B = (d + x)\omega$$

As a special case, if the body is translating, $v_A = v_B$ and the IC would be located at infinity, in which case $\omega = 0$.

Conceptual Note(s)

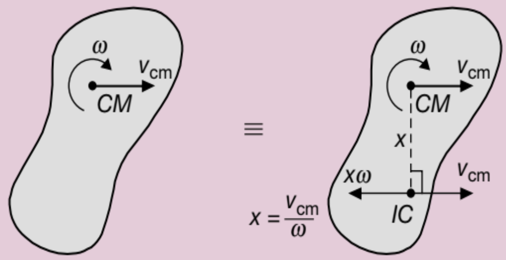
Q. Why do we use the concept of IAOR for bodies in combined rotation and translation?

A. When a body is under the combined influence of rotational and translational effect, then there exists a point (on the body or off the body) which happens to have zero net velocity, see diagram. **This makes us think about a point called Instantaneous Centre (IC) of zero velocity and the axis passing through IC, normal to plane of motion, is called Instantaneous Axis of Rotation (IAOR).** So, the combined effect of rotation and translation is just equivalent to a single effect of rotation about IAOR (as if the body is pinned to IC and is just rotating about IC.

So, $\begin{pmatrix}\text{Combined Effect}\\ \text{of Rotation}\\ \text{and Translation}\end{pmatrix} \equiv \begin{pmatrix}\text{Single Effect of}\\ \text{Rotation about}\\ \text{IC or IAOR}\end{pmatrix}$

Hence, if IC is at a distance x from centre of mass, then $(v_{IC})_{net} = 0$ i.e., $v_{cm} - x\omega = 0$

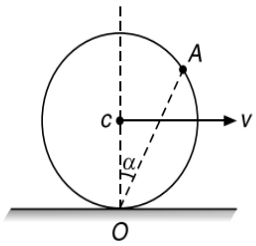
$$\Rightarrow \quad x = \frac{v_{cm}}{\omega}$$



IC lies in the portion or IC lies towards the portion that rotates opposite (i.e., has tangential velocity opposite) to v_{cm}.

ILLUSTRATION 29

A hoop of radius R rolls over a horizontal plane with a constant velocity v without slipping. Find the velocity of point A of the loop as shown in Figure.



SOLUTION

The velocity of the point A can be obtained by two methods.

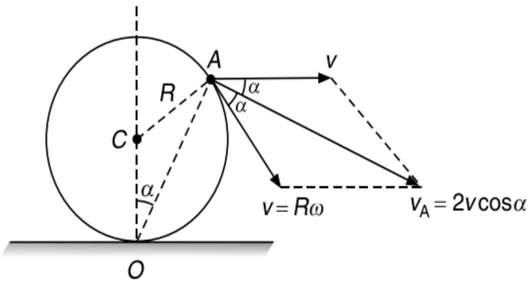
METHOD I:

Velocity of point A w.r.t. centre of mass C is $\vec{v}_{AC}$ given by

$$\vec{v}_{AC} = \vec{v}_A - \vec{v}_C$$

$$\Rightarrow \quad \vec{v}_A = \vec{v}_{AC} + \vec{v}_C$$

Now $|\vec{v}_C| = |\vec{v}_{CM}| = v$ and $|\vec{v}_{AC}| = R\omega$, perpendicular to the line CA. So, $|\vec{v}_A|$ is actually the magnitude of the resultant of $\vec{v}_{AC}$ and $\vec{v}_C$ as shown in Figure.



If θ is the angle between $\vec{v}_{PC}$ and $\vec{v}_C$, then $\theta = 2\alpha$.

$$\Rightarrow \quad |\vec{v}_A| = \sqrt{v^2 + R^2\omega^2 + 2v(R\omega)\cos(2\alpha)}$$

For pure rolling condition, $v = R\omega$

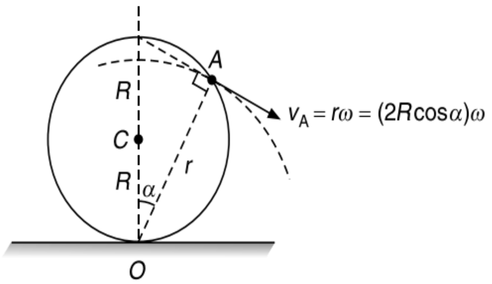
$$\Rightarrow \quad |\vec{v}_A| = \sqrt{v^2 + v^2 + 2v^2\cos(2\alpha)}$$

$$\Rightarrow \quad |\vec{v}_A| = v\sqrt{2(1+\cos 2\alpha)} = v\sqrt{2\times 2\cos^2\left(\frac{2\alpha}{2}\right)}$$

$$\Rightarrow \quad |\vec{v}_A| = 2v\cos\alpha$$

METHOD II:

A body in pure rolling can be considered to be in pure rotation about a point having zero velocity (in this case the point of contact), i.e. instantaneous centre (IC) of zero velocity. So, we have



$$v_A = r\omega = (2R\cos\alpha)\omega = 2(R\omega)\cos\alpha$$

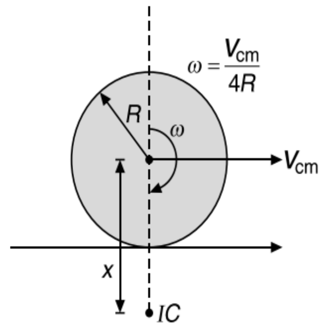
$$\Rightarrow \quad v_A = 2v\cos\alpha \qquad \{\because v_{cm} = v = R\omega\}$$

ILLUSTRATION 30

A wheel of radius R is moving on the ground. If v_{cm} be the velocity of centre of mass of the wheel and $\omega = \frac{v_{cm}}{4R}$ be the angular velocity of the wheel, then locate its instantaneous centre of zero velocity.

SOLUTION

Let the instantaneous centre of zero velocity be located at a distance x from the centre of the wheel as shown in Figure.



Since velocity of IC is zero, so we have $v_{IC} = 0$

$$\Rightarrow \quad v_{cm} - x\omega = 0$$

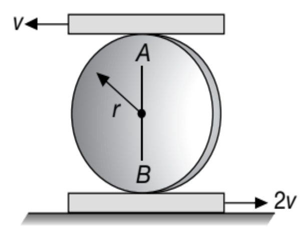
$$\Rightarrow \quad v_{cm} = x\left(\frac{v_{cm}}{4R}\right)$$

$$\Rightarrow \quad x = 4R$$

> **Problem Solving Technique(s)**
>
> If the body is placed on a surface which is in motion, then **for no slipping, the point of contact must move with the velocity of the surface.**
>
> $(v_P)_{net}$ = velocity of surface

ILLUSTRATION 31

The disc of mass m, radius r is confined to roll without slipping at A and B. If the plates have the velocities shown, determine the angular velocity of the disc. Also find the velocity of centre of mass of disc, the location of instantaneous centre of zero velocity and total energy of the disc.



SOLUTION

Let v_0 be the velocity of centre of mass and ω_0 the angular velocity of the disc as shown in Figure then, for no slipping at A and B, we have

$$(v_B)_{net} = 2v$$

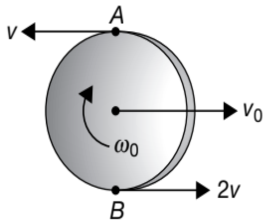
$$\Rightarrow \quad v_0 - r\omega_0 = 2v \qquad \ldots(1)$$

$$(v_A)_{net} = -v$$

$\Rightarrow \quad v_0 + r\omega_0 = -v$ ...(2)

Solving (1) and (2), we get

$$\omega_0 = -\frac{3v}{2r} \text{ and } v_0 = \frac{v}{2}$$



Also, we ca directly find the value of ω by using

$$\omega = \frac{(v_r)_\perp}{\perp \text{ separation between the velocities}}$$

$$\Rightarrow \quad \omega = \frac{3v}{2r}$$

Now let IC be located at a distance x from A or $(2r - x)$ from B, then

$$v = x\omega$$

$$\Rightarrow \quad x = \frac{v}{3v/2r} = \frac{2r}{3}, \text{ from } A$$

OR $\quad (2r - x) = \frac{4r}{3}$, from B

The total energy of the disc is

$$E = \frac{1}{2}mv_0^2 + \frac{1}{2}I\omega^2$$

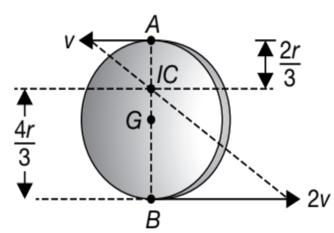
$$\Rightarrow \quad E = \frac{1}{2}m\left(\frac{v^2}{4}\right) + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{9v^2}{4r^2}\right)$$

$$\Rightarrow \quad E = \frac{mv^2}{8} + \frac{9mv^2}{16} = \frac{11}{16}mv^2$$

Alternatively, we have

$$E = \frac{1}{2}(I_{\text{IAOR}})\omega^2$$

where $I_{\text{IAOR}} = I_G + md^2$



where $I_G = \frac{1}{2}mr^2$ and $d = r - \frac{2r}{3} = \frac{r}{3}$

$$\Rightarrow \quad E = \frac{1}{2}\left(\frac{1}{2}mr^2 + \frac{1}{9}mr^2\right)\omega^2$$

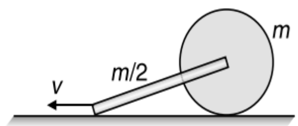
$$\Rightarrow \quad E = \frac{1}{2}\left(\frac{11}{18}mr^2\right)\left(\frac{3v}{2r}\right)^2$$

$$\Rightarrow \quad E = \frac{1}{2}\left(\frac{11}{18}mr^2\right)\left(\frac{9v^2}{4r^2}\right)$$

$$\Rightarrow \quad E = \frac{11}{16}mv^2$$

ILLUSTRATION 32

A uniform disc of mass m is fitted (pivoted smoothly) with a rod of mass $\frac{m}{2}$. If the bottom of the rod is pulled with a velocity v, it moves without changing its angle of orientation and the disc rolls without sliding. Calculate the kinetic energy of the rod disc system.



SOLUTION

Rolling can be considered as pure rotation about point of contact, so

$$K_{\text{rolling}} = \frac{1}{2}I_P\omega^2$$

$$\Rightarrow \quad K_{\text{rolling}} = \frac{1}{2}\left(\frac{mR^2}{2} + mR^2\right)\omega^2 = \frac{3}{4}mR^2\omega^2 \quad \text{...(1)}$$

The rod translates with the velocity v, hence velocity of centre of disc will also be v.

$$\Rightarrow \quad v = \omega R \quad \text{...(2)}$$

From (1) and (2), we get $K_{\text{rolling}} = \frac{3}{4}mv^2$

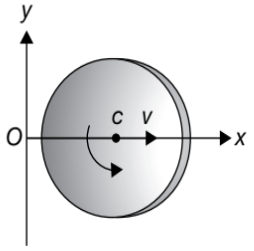
Kinetic energy of the rod is

$$K_{\text{rod}} = \frac{1}{2}\left(\frac{m}{2}\right)v^2 = \frac{mv^2}{4} \quad \text{...(3)}$$

$$\Rightarrow \quad K_{\text{total}} = K_{\text{rolling}} + K_{\text{rod}} = mv^2$$

ILLUSTRATION 33

A rotating disc moves in the positive direction of the x-axis. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation, if at the initial moment, the centre c of the disc was located at the point O after which it moved with constant velocity v while the disc started rotating counter-clockwise with a constant angular acceleration α. Assume the initial angular velocity to be zero.



SOLUTION

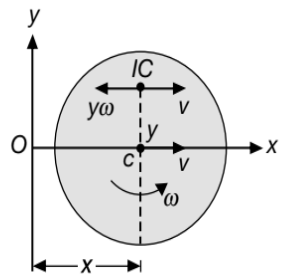
The position of disc at time t is $x = vt$

$$\Rightarrow \quad t = \frac{x}{v}$$

Since the disc is rotation with constant angular acceleration α, so

$$\omega = \omega_0 + \alpha t$$

$$\Rightarrow \quad \omega = \alpha t = \frac{\alpha x}{v} \qquad \{\because \omega_0 = 0\}$$



Now, if IC is located at a distance y from centre of disc, then we have

$$v_{IC} = 0$$

$$\Rightarrow \quad y\omega - v = 0$$

$$\Rightarrow \quad y = \frac{v}{\omega}$$

$$\Rightarrow \quad y = \frac{v}{\alpha x/v} = \frac{v^2}{\alpha x}$$

$$\Rightarrow \quad xy = \frac{v^2}{\alpha} = \text{constant}$$

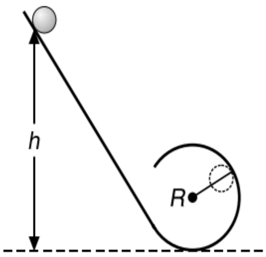
This is the desired x-y equation. This equation represents a rectangular hyperbola.

Test Your Concepts-III

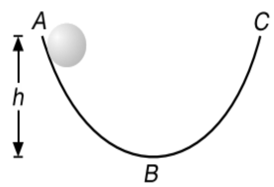
Based on Instantaneous Axis of Rotation, Pure Rolling and Conservation of Energy

(Solutions on page H.152)

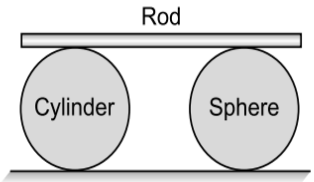
1. A uniform ball of radius r rolls without slipping along the loop-the-loop track in Figure. It starts from rest at height h above the bottom of the loop. If the ball is not to leave the track at the top of the loop, what is the least value h can have (in terms of the radius $R \gg r$ of the loop)? What would h have been if the ball were to slide along a frictionless track instead of rolling?



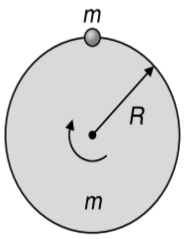
2. A solid ball rolls down a parabolic path ABC from a height h as shown in Figure. The portion AB of the path is rough while BC is smooth. How high will the ball climb in BC?



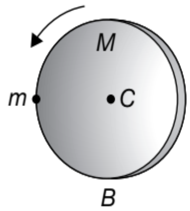
3. A rod of mass m is kept on a cylinder and sphere each of radius R. The masses of the sphere and cylinder are $m_1 = 4m$ and $m_2 = 5m$ respectively. If the speed of the rod is v, find the KE of the system. Assume that the surfaces do not slide relative to each other.



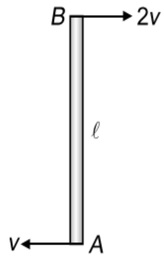
4. A uniform sphere of radius r starts rolling down without slipping from the top of another sphere of radius R. Find the angular velocity of the sphere after it leaves the surface of the larger sphere.
5. A bead of mass m is welded at the periphery of the smoothly pivoted disc mass m and radius R. Find the speed of the bead at its lowest position.



6. A uniform rod of length $2l$ and mass m is free to rotate in a vertical plane about a smooth fixed horizontal axis perpendicular to the rod through one end of the rod. Initially, the rod is held in horizontal position and is then released. Find the maximum angular velocity of the rod in the subsequent motion.
7. A uniform disc of mass M and radius R is pivoted about the horizontal axis normal to its plane and passing through its centre C. A point of mass m glued to the disc at its rim, as shown in Figure, is released from rest. Find the angular velocity of the disc when m reaches the point B.

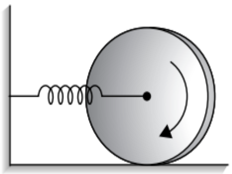


8. A uniform rod of mass m, length l is kept on a smooth horizontal surface. The ends A and B of the rod move with speeds v and $2v$ respectively perpendicular to the rod as shown in Figure.

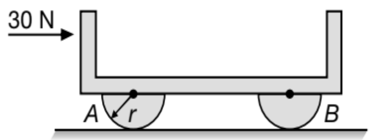


Calculate the angular velocity of the rod, linear velocity of centre of mass of the rod and the kinetic energy of the rod.

9. A uniform ring of radius a and mass m is free to rotate in a vertical plane about a fixed smooth axis which is perpendicular to the plane of the ring and passes through a point A on the ring. A particle of mass m is attached to the ring at B, where AB is a diameter. When the ring is hanging in a position of stable equilibrium the particle is struck a blow which gives it a velocity $3\sqrt{ga}$. Find the vertical height above A to which the particle rises.
10. A solid disk is rolling without slipping on a level surface at a constant speed of $2\ \text{ms}^{-1}$. How far can it roll up a 30° ramp before it stops? Take $g = 10\ \text{ms}^{-2}$.
11. A disc of mass m and radius R has a spring of constant k attached to its centre, the other end of the spring being fixed to a vertical wall (shown in Figure). If the disc rolls without slipping on a level floor, how far to the right does the centre of mass move, if initially the spring was unstretched and the angular speed of the disc was ω_0?



12. The 9 kg cradle is supported as shown by two uniform disks that roll without sliding at all surfaces of contact. The mass of each disk is $m = 6$ kg and the radius of each disk is $r = 80$ mm. Assuming that the system is initially at rest, find the velocity of the cradle after it has moved 250 mm.



TORQUE

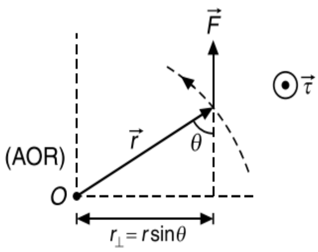
The rotational effect of force is called Torque. It is also called as the rotational analogue of force. Force acting on a body provides linear acceleration to it and correspondingly torque acting on a body provides angular acceleration to it.

It is **an axial vector** whose **direction** can be **found by** using the **Right-Hand Thumb Rule**. To find the direction of torque simply curl the fingers of the right hand in the sense of rotation in which the force tends to rotate the system about a given axis, then the direction of the thumb gives the direction of the torque.

Mathematically, the moment of force is called **torque i.e. torque is equal to the product of force and the perpendicular distance of the force from the axis of rotation (AOR).** Mathematically, if θ is angle between $\vec{r}$ and $\vec{F}$, then

$$\tau = Fr_{\perp} = F(r\sin\theta)$$

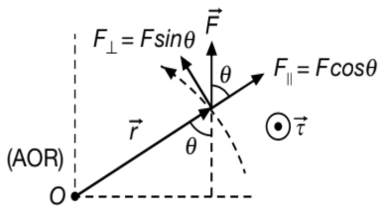
where $r_{\perp}$ is the perpendicular distance from the origin to the line of action of the force as shown in Figure. It is also called the lever arm.



The above definition of torque i.e. $\tau = rF\sin\theta$, may be also be interpreted as

$$\tau = r(F\sin\theta) = rF_{\perp}$$

The turning effect of a force about the origin is produced only by the perpendicular component $(F_{\perp} = F\sin\theta)$ as shown in Figure.



Vectorially,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

So, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, then

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \overset{\oplus}{\hat{i}} & \overset{\ominus}{\hat{j}} & \overset{\oplus}{\hat{k}} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

In the most simplest manner, if we know the sense of rotation (Clockwise or Anticlockwise) then, to find the direction of $\vec{\tau}$ we just curl the fingers of Right Hand along the sense of rotation and then the thumb gives direction of $\vec{\tau}$.

For a system with moment of inertia I having an angular acceleration $\vec{\alpha}$, we have

$$\vec{\tau} = I\vec{\alpha}$$

In Magnitude $\tau = I\alpha$ (Direction can be found by using Right Hand Thumb Rule). **This is also called Newton's Second Law of Rotational Motion.**

ILLUSTRATION 34

The position vector of point of application of force $\vec{F} = (\hat{i} + 2\hat{j} - 3\hat{k})$ N about O is $\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k})$ m. Calculate the torque of a force about a point O.

SOLUTION

Torque $\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}$

$$\Rightarrow \quad \vec{\tau} = \hat{i}(-9+2) + \hat{j}(-1+6) + \hat{k}(4-3)$$

$$\Rightarrow \quad \vec{\tau} = (-7\hat{i} + 5\hat{j} + \hat{k}) \text{ Nm}$$

Conceptual Note(s)

(a) Cases for zero torque

Since, $|\vec{\tau}| = rF\sin\theta$

So, for $|\vec{\tau}| = 0$, we have the following three cases:

CASE-1:

$F = 0$ i.e., no force acts on system

CASE-2:

$r = 0$ i.e., force acts at AOR

CASE-3:

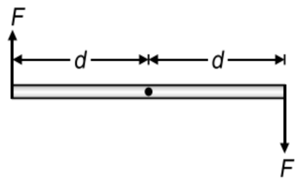
$F \neq 0$, $r \neq 0$ but angle between $\vec{F}$ and $\vec{r}$ is either $0°$ or $180°$ i.e., $\vec{F}$ acts along $\vec{r}$ or opposite to $\vec{r}$. **Such forces are called Radial Forces. So, torque due to Radial Forces is always zero.**

(b) SI unit of $|\vec{\tau}|$ is Nm (not joule)

(c) $\vec{\tau}=\vec{r}\times\vec{F}$, **so** $\vec{\tau}$ **is** $\perp$ **to** $\vec{r}$ **as well as** $\vec{F}$ i.e., $\vec{\tau}\cdot\vec{r}=0$ and $\vec{\tau}\cdot\vec{F}=0$

FORCE COUPLE

A pair of forces each of same magnitude and acting in opposite direction is called a force couple. The torque due to couple is the product of magnitude of either force and the perpendicular separation between the forces.



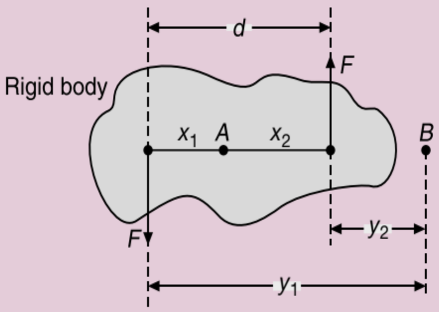
$$\begin{pmatrix}\text{Torque}\\ \text{due to}\\ \text{couple}\end{pmatrix}=\begin{pmatrix}\text{Magnitude}\\ \text{of one}\\ \text{force}\end{pmatrix}\times\begin{pmatrix}\perp\text{ Distance}\\ \text{between their}\\ \text{lines of action}\end{pmatrix}$$

$\Rightarrow \quad \tau=F(2d)$

Conceptual Note(s)

(a) A couple does not exert a net force on an object even though it exerts a torque.

(b) Net torque due to a force couple is same about any point.



Torque about A is $\tau_A=x_1F+x_2F$

$\Rightarrow \tau_A=F(x_1+x_2)=Fd$

Torque about B is $\tau_B=y_1F-y_2F$

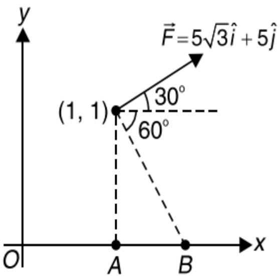
$\Rightarrow \tau_B=F(y_1-y_2)=Fd$

(c) If net force acting on a system is zero, torque is same about any point.

(d) A consequence is that, if $F_{net}=0$ and $\tau_{net}=0$ about one point, then $\tau_{net}=0$ about any point.

ILLUSTRATION 35

Calculate the torque about point O and A due to the force as shown in Figure.



SOLUTION

Torque about point O

$$\vec{\tau}=\vec{r}_0\times\vec{F},\ \vec{r}_0=\hat{i}+\hat{j},\ \vec{F}=5\sqrt{3}\hat{i}+5\hat{j}$$

$$\Rightarrow \quad \vec{\tau}=(\hat{i}+\hat{j})\times(5\sqrt{3}\hat{i}+5\hat{j})=5(1-\sqrt{3})\hat{k}$$

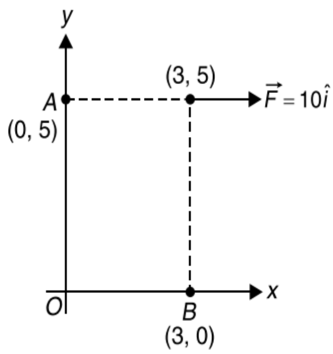
Torque about point A

$$\vec{\tau}=\vec{r}_a\times\vec{F},\ \vec{r}_a=\hat{j},\ \vec{F}=5\sqrt{3}\hat{i}+5\hat{j}$$

$$\Rightarrow \quad \vec{\tau}=\hat{j}\times(5\sqrt{3}\hat{i}+5\hat{j})=5(-\sqrt{3})\hat{k}$$

ILLUSTRATION 36

Calculate the torque about point A, O and B due to the force as shown in Figure.



SOLUTION

Torque about point A

$$\vec{\tau}_A=\vec{r}_A\times\vec{F},\ \vec{r}_A=3\hat{i},\ \vec{F}=10\hat{i}$$

$$\Rightarrow \quad \vec{\tau}_A=3\hat{i}\times10\hat{i}=0$$

Torque about point B

$$\vec{\tau}_B=\vec{r}_B\times\vec{F},\ \vec{r}_B=5\hat{j},\ \vec{F}=10\hat{i}$$

$$\Rightarrow \quad \vec{\tau}_B=5\hat{j}\times10\hat{i}=-50\hat{k}$$

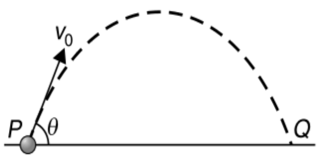
Torque about point O

$$\vec{\tau}_O = \vec{r}_O \times \vec{F},\ \vec{r}_O = 3\hat{i} + 5\hat{j},\ \vec{F} = 10\hat{i}$$

$$\Rightarrow \quad \vec{\tau}_O = (3\hat{i} + 5\hat{j}) \times 10\hat{i} = -50\hat{k}$$

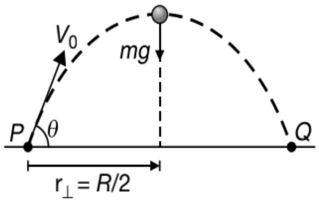
ILLUSTRATION 37

A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Calculate the torque about the point of projection acting on the particle when it is at its maximum height.



SOLUTION

When the particle reaches the maximum height as shown in Figure, then

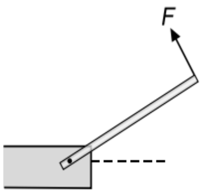


$$\tau = Fr_\perp = mg\left(\frac{R}{2}\right) = mg\left(\frac{v_0^2 \sin 2\theta}{2g}\right)$$

$$\Rightarrow \quad \tau = \frac{mv_0^2 \sin 2\theta}{2}$$

ILLUSTRATION 38

A uniform bar of mass m and length l is hinged at one end and held in the position shown by applying a constant force F at the other end, perpendicular to the rod.



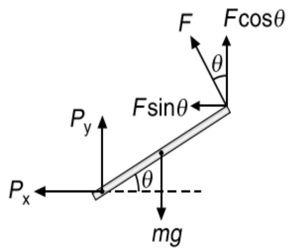
Calculate the magnitude and direction of the force exerted by the hinge on the rod.

SOLUTION

The two known forces that act on the rod are its weight mg acting vertically downward and the applied force F. Let the force applied by the hinge be having two components, the horizontal component P_x and the vertical component P_y. Then

$$P = \sqrt{P_x^2 + P_y^2}$$

Please note that the force due to the hinge can be assumed to be in any direction. The forces acting on the rod are drawn in the FBD shown in Figure.



Please note that, P_x and P_y have been assigned arbitrary directions. If the answers for P_x, P_y or both come out to be negative, then the direction will be opposite to the direction assumed.
For translatory equilibrium, we have

$$\Sigma F_x = 0 \text{ and } \Sigma F_y = 0$$

When $\Sigma F_x = 0$, we get

$$P_x + F\sin\theta = 0$$

$$\Rightarrow \quad P_x = -F\sin\theta \qquad \ldots(1)$$

Negative sign simply shows that P_x is directed to the right.
When $\Sigma F_y = 0$, we get

$$P_y + F\cos\theta - mg = 0$$

$$\Rightarrow \quad P_y = mg - F\cos\theta \qquad \ldots(2)$$

To calculate P_x, we must find the angle θ. For that, let us apply the condition for rotational equilibrium. For rotational equilibrium, we have

$$\Sigma\tau = 0 \qquad \text{(about any point)}$$

So, lets calculate the net torque acting on the rod about the hinge, then we get

$$-(mg)\left(\frac{l}{2}\cos\theta\right) + Fl = 0$$

$$\Rightarrow \quad F = \frac{mg}{2}\cos\theta \qquad \ldots(3)$$

From equations (1) and (3), we get

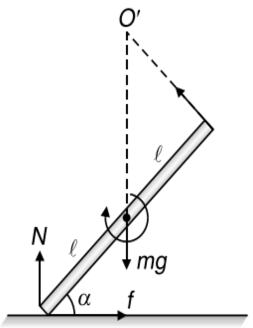
$$P_x = -F\sqrt{1 - \frac{4F^2}{m^2g^2}} = -\frac{F}{mg}\sqrt{m^2g^2 - 4F^2}$$

ILLUSTRATION 39

Calculate minimum coefficient of friction between a thin homogeneous rod and the floor for which a person can slowly lift the rod upwards without slipping on the floor by applying a force perpendicular to the rod.

SOLUTION

Consider the equilibrium of the rod when it is inclined at an angle α with the horizontal. The forces acting on the rod are shown in Figure.



Let us consider torque about the point of intersection O' of the force of gravity mg and the force F applied by the person perpendicular to the rod. Since the torque due to these forces about this point O' is zero, so doing this eliminates the unknown force F. At equilibrium, we have

$$(\Sigma\tau)_{\text{about } O'} = 0$$

Since, moment arm for N about O' is $l\cos\alpha$, while that for friction about O' is $\left(\dfrac{l}{\sin\alpha} + l\sin\alpha\right)$

$$\Rightarrow \quad Nl\cos\alpha - fl\left(\frac{1}{\sin\alpha} + \sin\alpha\right) = 0$$

$$\Rightarrow \quad f = N\left(\frac{\cos\alpha\sin\alpha}{1+\sin^2\alpha}\right) = N\left(\frac{\cos\alpha\sin\alpha}{2\sin^2\alpha + \cos^2\alpha}\right)$$

$$\Rightarrow \quad f = N\left(\frac{1}{2\tan\alpha\cot\alpha}\right)$$

Since, $f \le \mu N$

$$\Rightarrow \quad N\left(\frac{1}{2\tan\alpha\cot\alpha}\right) \le \mu N$$

$$\Rightarrow \quad \mu \ge \frac{1}{2\tan\alpha + \cot\alpha} \qquad \ldots(1)$$

For μ to be minimum, we must have

$$2\tan\alpha + \cot\alpha = \text{MAXIMUM}$$

$$\Rightarrow \quad \frac{d}{d\alpha}(2\tan\alpha + \cot\alpha) = 0$$

$$\Rightarrow \quad 2\sec^2\alpha - \operatorname{cosec}^2\alpha = 0$$

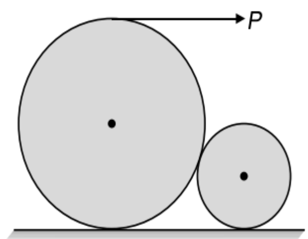
$$\Rightarrow \quad \tan\alpha = \frac{1}{\sqrt{2}}$$

Substituting this value in equation (1), we get

$$\mu_{\min} = \frac{1}{2\sqrt{2}}$$

ILLUSTRATION 40

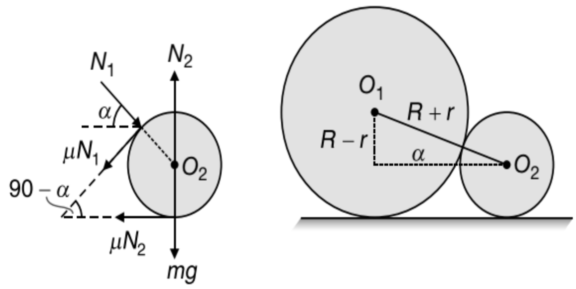
Consider two heavy right circular rollers of radii R and r respectively at rest on a rough horizontal plane as shown in Figure.



The larger roller has a string wound around it to which a horizontal force P can be applied. Assume the coefficient of friction μ has the same value for all surfaces in contact, calculate the necessary condition under which the larger roller can be pulled over the smaller one. Assume the smaller cylinder should neither roll nor slide.

SOLUTION

Let N_1 be the reaction between the two cylinders and N_2 be the normal reaction between the smaller cylinder and the horizontal surface. The forces acting on the smaller cylinder have been shown in Figure.



(a) (b)

At equilibrium, we have

$$\Sigma F_x = N_1\cos\alpha - \mu N_2 - \mu N_1\cos(90-\alpha) = 0 \quad \ldots(1)$$

Also, torque about O_2 (centre of smaller sphere) is zero, so we get

$$\sum \tau = \mu N_1 r - \mu N_2 r = 0$$

$$\Rightarrow \quad N_1 = N_2 \qquad \ldots(2)$$

Substituting equation (2) in equation (1), we get

$$\mu N_1 + \mu N_1 \sin\alpha = N_1 \cos\alpha$$

$$\Rightarrow \quad \mu(1+\sin\alpha) = \cos\alpha \qquad \ldots(3)$$

Also, we see that

$$\sin\alpha = \sqrt{\frac{R-r}{R+r}}$$

$$\Rightarrow \quad \cos\alpha = \sqrt{1-\sin^2\alpha} = \frac{2\sqrt{Rr}}{R+r}$$

Substituting these values in equation (3), we get

$$\mu\left(1+\frac{R-r}{R+r}\right) = \frac{2\sqrt{Rr}}{R+r}$$

$$\Rightarrow \quad \mu = \sqrt{\frac{r}{R}}$$

Hence the required condition is $\mu \geq \sqrt{\frac{r}{R}}$

ILLUSTRATION 41

A uniform rod of length L and mass M is pivoted freely at one end as shown in Figure.

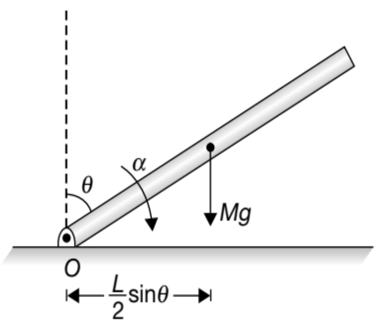
(a) Find the angular acceleration of the rod when it is at angle θ to the vertical.

(b) Assuming the rod to start from the vertical position, find the angular velocity as the function of θ.

(c) Find the tangential linear acceleration of the free end when the rod is horizontal.

SOLUTION

(a) Figure shows the rod at an angle θ to the vertical.



Net torque about the point O is

$$\tau_O = Mg\left(\frac{L}{2}\sin\theta\right)$$

Using the Second Law of Motion $\tau_O = I_O\alpha$

$$\frac{MgL}{2}\sin\theta = \frac{ML^2}{3}\alpha$$

$$\Rightarrow \quad \alpha = \frac{3g\sin\theta}{2L}$$

(b) From above, we have

$$\alpha = \frac{3g\sin\theta}{2L}$$

$$\Rightarrow \quad \omega\frac{d\omega}{d\theta} = \frac{3g\sin\theta}{2L}$$

$$\Rightarrow \quad \omega d\omega = \frac{3g}{2L}\sin\theta d\theta$$

Integrating within appropriate limits, we get

$$\int_0^{\omega} \omega d\omega = \frac{3g}{2L}\int_0^{\theta} \sin\theta d\theta$$

$$\Rightarrow \quad \frac{\omega^2}{2} = -\frac{3g}{2L}\cos\theta\Big|_0^{\theta} = \frac{3g}{2L}(1-\cos\theta)$$

$$\Rightarrow \quad \omega = \sqrt{\frac{3g}{L}(1-\cos\theta)}$$

The above result can also be obtained by using the Law of Conservation of Mechanical Energy, where we use,

$$\begin{pmatrix}\text{Loss in GPE} \\ \text{of CM of Rod}\end{pmatrix} = \begin{pmatrix}\text{Gain in RKE} \\ \text{of Rod}\end{pmatrix}$$

$$\Rightarrow \quad Mg\frac{L}{2}(1-\cos\theta) = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{3g}{L}(1-\cos\theta)}$$

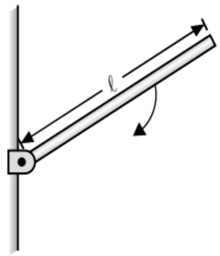
(c) When the rod is horizontal $\theta = \frac{\pi}{2}$, so $\alpha = \frac{3g}{2L}$. So, the tangential linear acceleration is

$$a_t = \alpha L = \frac{3g}{2}$$

This is greater than the acceleration of an object falling freely.

ILLUSTRATION 42

A thin rod of mass M and length l is pivoted at one end. Initially the rod makes an angle of $60°$ with the vertical and is then released. Calculate the magnitude and direction of the force on the pivot when the rod is horizontal.



SOLUTION

Let ω be the angular velocity of rod in horizontal position, then

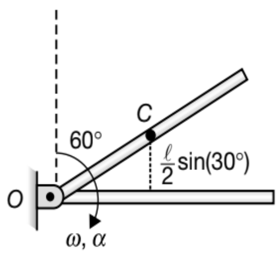
$$\alpha = \frac{\tau}{I} = \frac{(Mg)\frac{l}{2}}{\frac{Ml^2}{3}} = \frac{3}{2}\frac{g}{l} \quad \ldots(1)$$

By Law of Conservation of Energy

$$\begin{pmatrix}\text{Loss in GPE of}\\ \text{CM of Rod}\end{pmatrix} = \begin{pmatrix}\text{Gain in RKE}\\ \text{of the Rod}\end{pmatrix}$$

$$\Rightarrow \quad Mg\left(\frac{l}{2}\sin(30°)\right) = \frac{1}{2}I\omega^2$$

$$\Rightarrow \quad Mg\left(\frac{l}{4}\right) = \frac{1}{2}\left(\frac{1}{3}Ml^2\right)\omega^2$$

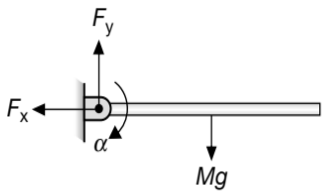


$$\Rightarrow \quad \omega = \sqrt{\frac{3g}{2l}} \quad \ldots(2)$$

$$F_x = M\left(\frac{l}{2}\right)\omega^2 = M\left(\frac{l}{2}\right)\left(\frac{3g}{2l}\right) = \frac{3}{4}Mg$$

$$Mg - F_y = Ma_t = M(\alpha)\left(\frac{l}{2}\right)$$

$$\Rightarrow \quad F_y = Mg - \frac{3}{4}Mg = \frac{Mg}{4}$$



So, $$F = \sqrt{F_x^2 + F_y^2} = \frac{\sqrt{10}}{4}Mg$$

$$\tan\alpha = \frac{F_y}{F_x} = \frac{\left(\frac{Mg}{4}\right)}{\left(\frac{3Mg}{4}\right)} = \frac{1}{3}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{1}{3}\right)$$

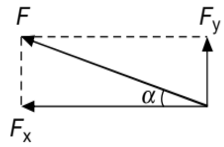
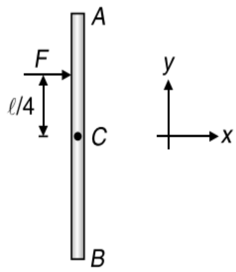


ILLUSTRATION 43

A thin uniform rod of mass m and length l is placed on smooth horizontal surface. It is acted upon by a force F as shown in Figure.

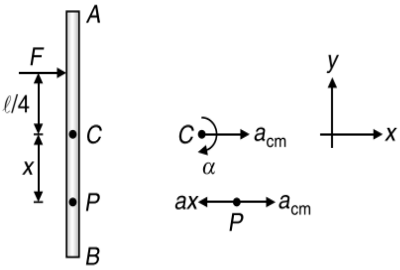


Calculate linear and angular acceleration of the rod. Find the location of a point on the rod which has zero acceleration.

SOLUTION

According to the problem, the external force is acting on rod along the x-direction, hence the centre of mass of the rod will accelerate along the x-direction.

Also, we observe that due to this force, a net torque will act on the rod in clockwise sense. Hence the motion of the rod will be combined rotational and translational.



According to Newton's Second Law for translational motion, we have

$$F = ma$$

$$\Rightarrow \quad a = \frac{F}{m} \quad \ldots(1)$$

According to Newton's Second Law for rotational motion, we have

$$\tau = I_{cm}\alpha$$

$$\Rightarrow \quad F\left(\frac{l}{4}\right) = \frac{ml^2}{12}\alpha$$

$$\Rightarrow \quad \alpha = \frac{3F}{ml} \quad \ldots(2)$$

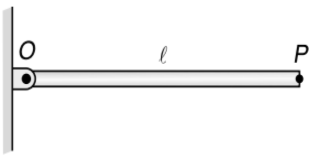
Point of zero acceleration on the rod should be between C and B, as the vectors due to acceleration due to translation and acceleration due to rotation act in opposite directions as shown in Figure.
At point P, $a_P = 0$

$$\Rightarrow \quad a_{cm} - \alpha x = \frac{F}{m} - \left(\frac{3F}{ml}\right)x = 0$$

$$\Rightarrow \quad x = \frac{l}{3}$$

ILLUSTRATION 44

A rod of mass m and length l (hinged at O) is released from the rest from horizontal position as shown in Figure.

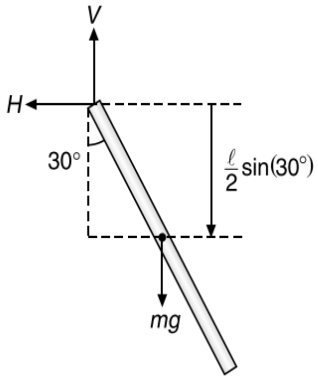


When the rod makes an angle of $30°$ with the vertical, calculate the angular acceleration, velocity of point P, acceleration of centre of mass and the horizontal component of force applied by hinge.

SOLUTION

Torque about hinged point is

$$\tau_{net} = \tau_{mg} = mg\left(\frac{l}{2}\sin 30°\right) = \frac{mgl}{4}$$



Since $\tau_{net} = I_O\alpha = \left(\frac{ml^2}{3}\right)\alpha$

$$\Rightarrow \quad \alpha = \frac{\tau}{I_O} = \frac{mgl/4}{ml^2/3} = \frac{3g}{4l} \quad \ldots(1)$$

By conservation of mechanical energy, the loss in gravitational potential energy of the centre of mass of the rod equals the gain in rotational kinetic energy of the rod.
Loss of potential energy of the centre of mass of the rod is

$$-\Delta U = mg\left(\frac{l}{2}\right)\cos 30° = mgl\frac{\sqrt{3}}{4}$$

Gain in rotational kinetic energy of the rod is

$$K_R = \frac{1}{2}I_O\omega^2 = \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$$\Rightarrow \quad \frac{mgl\sqrt{3}}{4} = \left(\frac{ml^2}{6}\right)\omega^2$$

$$\Rightarrow \quad \omega^2 = \frac{6g\sqrt{3}}{4l}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{6\sqrt{3}g}{4l}}$$

$$\Rightarrow \quad v_p = l\omega = \sqrt{\frac{6\sqrt{3}gl}{4}}$$

The tangential acceleration of the centre of mass of the rod is

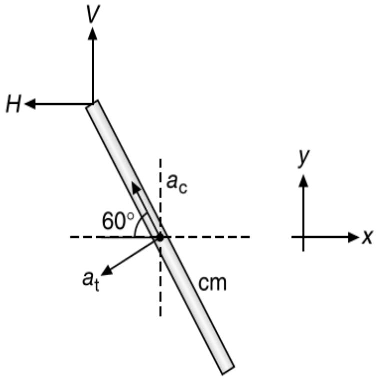
$$a_t = \frac{l}{2}\alpha = \left(\frac{l}{2}\right)\left(\frac{3g}{4l}\right) = \frac{3g}{8}$$

The centripetal acceleration of the centre of mass of the rod is

$$a_c = r\omega^2 = \left(\frac{l}{2}\right)\left(\frac{6\sqrt{3}g}{4l}\right) = \frac{3\sqrt{3}g}{4}$$

So, total acceleration of centre of mass of rod is

$$\Rightarrow \quad a_{\text{cm}} = \sqrt{a_c^2 + a_t^2} = \frac{g}{8}\sqrt{9+108} = \frac{g\sqrt{117}}{8}$$



To find the x and y components of $\vec{a}_{\text{cm}}$, we know that $\vec{a}_{\text{cm}} = \vec{a}_c + \vec{a}_t$, where

$$\vec{a}_c = a_c\left(-\cos 60°\hat{i} + \sin 60°\hat{j}\right)$$

$$\Rightarrow \quad \vec{a}_c = \frac{3\sqrt{3}g}{8}\left(-\hat{i} + \sqrt{3}\hat{j}\right)$$

and $\vec{a}_t = -a_t\left(\cos 30°\hat{i} + \sin 30°\hat{j}\right)$

$$\Rightarrow \quad \vec{a}_t = -\frac{3g}{16}\left(\sqrt{3}\hat{i} + \hat{j}\right)$$

$$\vec{a}_{\text{cm}} = \frac{3\sqrt{3}g}{8}\left(-\hat{i} + \sqrt{3}\hat{j}\right) - \frac{3g}{16}\left(\sqrt{3}\hat{i} + \hat{j}\right)$$

$$\Rightarrow \quad \vec{a}_{\text{cm}} = -\frac{1}{2}\left(a_c + a_t\sqrt{3}\right)\hat{i} + \frac{1}{2}\left(a_c\sqrt{3} - a_t\right)\hat{j}$$

$$\Rightarrow \quad \vec{a}_{\text{cm}} = -\frac{9\sqrt{3}g}{16}\hat{i} + \frac{15g}{16}\hat{j}$$

$$\Rightarrow \quad \left|\vec{a}_{\text{cm}}\right| = g\sqrt{\frac{243+225}{256}} = \frac{g\sqrt{117}}{8}$$

If horizontal component of force applied by the hinge is H and the vertical component force applied by the hinge is V, then from Newton's Second Law, we have

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}}$$

$$\Rightarrow \quad -H\hat{i} + V\hat{j} - mg\hat{j} = m\left(\frac{-9\sqrt{3}}{16}\hat{i} + \frac{15g}{8}\hat{j}\right)$$

$$\Rightarrow \quad -H\hat{i} + (V - mg)\hat{j} = \frac{-mg\sqrt{3}}{16}\hat{i} + \frac{15mg}{8}\hat{j}$$

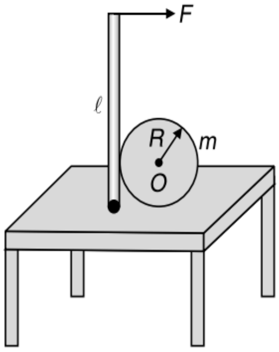
Comparing, we get

$$V - mg = \frac{15mg}{8},\ H = \frac{mg\sqrt{3}}{16}$$

$$\Rightarrow \quad V = \frac{23mg}{8},\ H = \frac{mg\sqrt{3}}{16}$$

ILLUSTRATION 45

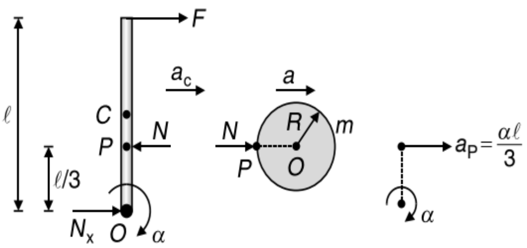
A uniform rod of mass m and length l which can rotate freely in vertical plane without friction, is hinged at its lower end on a table. A sphere of mass m and radius one third the length of the rod is placed in contact with the vertical rod and a horizontal force F is applied at the upper end of the rod as shown in Figure.



Calculate the acceleration of the sphere just after the force starts acting. Also, calculate the horizontal component of hinge reaction acting on the rod just after force F starts acting.

SOLUTION

The forces acting on the rod and the sphere are shown in Figure.



The torque equation for rod about O is

$$\tau = Fl - N\left(\frac{l}{3}\right) = I_O\alpha$$

$$\Rightarrow \quad Fl - \frac{Nl}{3} = \frac{ml^2}{3}\alpha$$

$$\Rightarrow \quad \frac{ml\alpha}{3} = F - \frac{N}{3} \quad \ldots(1)$$

The motion of the sphere is translational only, so the acceleration (a) of the sphere equals the acceleration of point P, i.e.

$$a = \frac{l\alpha}{3} \quad \ldots(2)$$

From equations (1) and (2), we get

$$ma = F - \frac{N}{3} \quad \ldots(3)$$

For sphere, $N = ma$ ...(4)

Substituting equation (3) in (4), we get

$$a = \frac{3F}{4m} \text{ and } N = \frac{3F}{4}$$

Acceleration of centre of mass of rod is

$$a_{\text{cm}} = \left(\frac{l}{2}\right)\alpha = \frac{3a}{2} = \frac{9F}{8m}$$

If the horizontal component of hinge reaction is N_x, then for the rod we have

$$F - \frac{3F}{4} + N_x = ma_{\text{cm}}$$

$$\Rightarrow \quad F - \frac{3F}{4} + N_x = m\left(\frac{9F}{8m}\right)$$

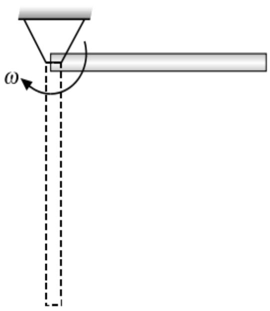
$$\Rightarrow \quad N_x = \frac{9F}{8} - \frac{F}{4} = \frac{7F}{8}$$

ILLUSTRATION 46

A rod of mass m and length l is hinged about its one of the ends. The rod is released from horizontal position. When the rod becomes vertical, find the angular speed of the rod and the hinge reaction

SOLUTION

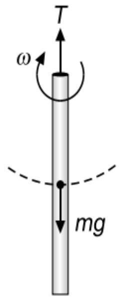
Applying conservation of mechanical energy, we see that the loss in gravitational potential energy equals gain in rotational kinetic energy.



$$\Rightarrow \quad mg\left(\frac{L}{2}\right) = \frac{1}{2}\left(\frac{mL^2}{3}\right)\omega^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{3g}{L}}$$

Since the centre of mass of rod is moving in a circular path of radius $\frac{L}{2}$, so we have



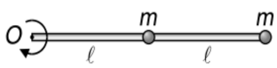
$$T - mg = m\omega^2 \frac{L}{2}$$

$$\Rightarrow \quad T = mg + m\left(\frac{L}{2}\right)\left(\frac{3g}{L}\right)$$

$$\Rightarrow \quad T = 4mg$$

ILLUSTRATION 47

A light rod is connected rigidly with two identical particles each of mass m. The free end of the rod is smoothly pivoted at O. The rod is released from rest from its horizontal position at $t = 0$ as shown in Figure.

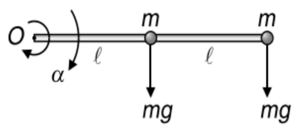


Calculate the initial angular acceleration of the rod and the initial reaction offered by the pivot.

SOLUTION

The net torque about O is

$$\tau_O = mgl + mg(2l) = 3mgl$$



This torque produces an angular acceleration α given by the relation

$$\alpha = \frac{\tau}{I_O}$$

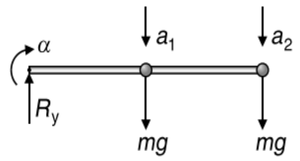
where, $I_O = ml^2 + m(2l)^2 = 5ml^2$

$$\Rightarrow \quad \alpha = \frac{3mgl}{5ml^2} = \frac{3g}{5l}$$

Initially just when the rod is released, then $\omega = 0$ but $\alpha = \frac{3g}{5l}$, so

$$R_x = ml\omega^2 + m(2l)\omega^2 = 0$$

The various forces acting on the rod are shown in Figure.



$$mg + mg - R_y = ma_1 + ma_2$$

where $a_1 = l\alpha$ and $a_2 = 2l\alpha$

$$\Rightarrow \quad R_y = m(2g - 3l\alpha)$$

Substituting $\alpha = \frac{3g}{5l}$, we get

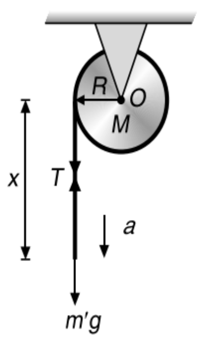
$$R_y = \frac{mg}{5}$$

ILLUSTRATION 48

A uniform cylinder of radius R and mass M can rotate freely about a stationary horizontal axis O. A thin cord of length l and mass m is wound on the cylinder in a single layer. Calculate the angular acceleration of the cylinder as a function of the length x of the hanging part of the cord. Assume that the wound part of cord has its centre of mass at the axis of the cylinder.

SOLUTION

At any instant, let m' be the mass of cord hanging from the cylinder as shown in Figure.



$$\Rightarrow \quad m' = \frac{m}{l}x$$

The mass of the cord wound on the perimeter of the cylinder will be

$$m'' = \frac{m}{l}(l - x)R^2$$

Let tension at the upper end of the cord in contact with the cylinder be T and its downward acceleration be a, then according to Newton's Second Law, we have

$$\left(\frac{m}{l}x\right)g - T = \left(\frac{m}{l}x\right)a \quad \ldots(1)$$

When the cord accelerates downwards, the cylinder will be rotating with an angular acceleration

$$\alpha = \frac{a}{R}$$

The tension in the string provides necessary torque to the cylinder to rotate, so we have

$$\tau = TR = I\alpha \quad \ldots(2)$$

where, I is the moment of inertia of the cylinder plus the part of cord wound on it, so

$$I = \frac{1}{2}MR^2 + m''R^2 = \frac{1}{2}MR^2 + \frac{m}{l}(l - x)R^2 \quad \ldots(3)$$

Substituting equation (3) in (2), we get

$$TR = \left(\frac{1}{2}MR^2 + \frac{m}{l}(l - x)R^2\right)\alpha$$

$$\Rightarrow \quad T = \left(\frac{MR}{2} + \frac{m}{l}(l - x)R\right)\alpha \quad \ldots(4)$$

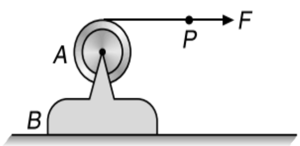
Substituting equation (4) in (1), we get

$$\left(\frac{m}{l}xg\right) - \left(\frac{MR}{2} + \frac{m}{l}(l - x)R\right)\alpha = \left(\frac{m}{l}x\right)R\alpha$$

$$\Rightarrow \quad \alpha = \frac{2mgx}{(2m + M)Rl}$$

ILLUSTRATION 49

A uniform solid cylinder A of mass m_1 can freely rotate about a horizontal axis fixed to a mount B of mass m_2. A constant horizontal force F is applied to end P of a light thread tightly wound on the cylinder as shown in Figure.



The friction between the mount and the supporting horizontal surface is supposed to be absent. Calculate the acceleration of this point P and the kinetic energy of this system t seconds after beginning of motion.

SOLUTION

Applying Newton's Second Law to the cylinder mount system, we get

$$F = (m_1 + m_2)a$$

$$\Rightarrow \quad a = \frac{F}{m_1 + m_2}$$

For cylinder, we have

$$\tau = FR = I\alpha = \left(\frac{1}{2}m_1R^2\right)\alpha$$

$$\Rightarrow \quad \alpha = \frac{2F}{m_1R}$$

Acceleration of point P is

$$a_P = a + R\alpha$$

$$\Rightarrow \quad a_P = \frac{F}{m_1 + m_2} + \frac{2F}{m_1}$$

$$\Rightarrow \quad a_P = \frac{F(3m_1 + 2m_2)}{m_1(m_1 + m_2)}$$

In t seconds, displacement of point P is

$$s = \frac{1}{2}a_Pt^2$$

The kinetic energy of system equals the work done by external force, so we get

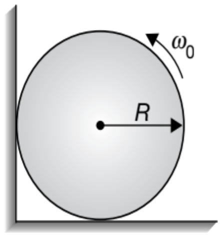
$$\Delta K = K - 0 = Fs$$

$$\Rightarrow \quad K = F\left[\frac{1}{2}\left(\frac{F(3m_1 + 2m_2)}{m_1(m_1 + m_2)}\right)t^2\right]$$

$$\Rightarrow \quad K = \frac{F^2t^2(3m_1 + 2m_2)}{2m_1(m_1 + m_2)}$$

ILLUSTRATION 50

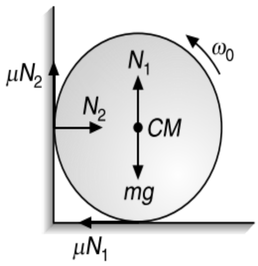
A uniform cylinder of mass m, radius R is spun about its axis to the angular velocity ω_0 and then placed into a corner, as shown in Figure.



The coefficient of kinetic friction between the corner walls and the cylinder is μ. Calculate the normal reaction imparted by the wall to the cylinder, by the ground to the cylinder. Also calculate the angular acceleration of the cylinder and the number of turns accomplish by the cylinder before stopping.

SOLUTION

Let us draw the free body diagram for the situation shown.



Since the centre of mass of cylinder does not accelerate, so

$$N_2 - \mu N_1 = 0$$

Also, $N_1 + \mu N_2 - mg = 0$

$$\Rightarrow \quad N_2 = \mu N_1 \qquad \ldots(1)$$

$$N_1 + \mu N_2 = mg \qquad \ldots(2)$$

From (1) and (2), we get

$$N_1 = \frac{mg}{1 + \mu^2} \text{ and } N_2 = \frac{\mu mg}{1 + \mu^2}$$

So, $\tau_{\text{cm}} = (\mu N_1)R + (\mu N_2)R$

$$\Rightarrow \quad \tau_{\text{cm}} = \mu mgR\left(\frac{1 + \mu}{1 + \mu^2}\right)$$

$$\Rightarrow \quad \left(\frac{1}{2}mR^2\right)\alpha = \mu mgR\left(\frac{1 + \mu}{1 + \mu^2}\right)$$

$$\Rightarrow \quad \alpha = \left(\frac{2\mu g}{R}\right)\left(\frac{1 + \mu}{1 + \mu^2}\right)$$

$$\Rightarrow \quad \alpha = \frac{2\mu(1 + \mu)g}{(1 + \mu^2)R}, \text{ clockwise sense}$$

Now $n_0 = \dfrac{\theta}{2\pi}$

Since, $\omega^2 - \omega_0^2 = 2\alpha\theta$

$$\Rightarrow \quad 0^2 - \omega_0^2 = 2\left(\frac{-2\mu(1+\mu)g}{(1+\mu^2)R}\right)\theta$$

$$\Rightarrow \quad \theta = \frac{(1+\mu^2)R\omega_0^2}{4\mu(1+\mu)g}$$

$$\Rightarrow \quad n_0 = \frac{(1+\mu^2)R\omega_0^2}{8\pi\mu(1+\mu)g}$$

GENERAL MOTION OF A RIGID BODY

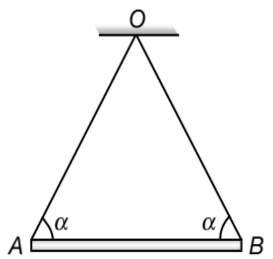
Till now, we have been learning ways to analyse the rotatory motion of a rigid body about a fixed axis of rotation. Now, let us describe the motion of a rigid body free to move in a plane. The basic laws of dynamics are applicable in this case also. You can use the following two results.

1. $\sum \vec{F}_{\text{net}} = m\vec{a}_{\text{cm}}$
2. $\sum \vec{\tau}_{\text{net}} = I\vec{\alpha}$

Please note that the second result must be applied about an axis taken through the centre of mass. This is so as the points on the body are accelerating and thus non-inertial. Although centre of mass also accelerates but the torque due to pseudo force about the centre of mass will be zero

ILLUSTRATION 51

A uniform rod AB of mass m and length $2l$ is suspended by two strings OA and OB of equal length attached to a fixed point O. The rod is at rest in a horizontal position and each string makes an angle α with the horizontal. If the string OB is cut, in what ratio the tension in OA will be instantaneously reduced?



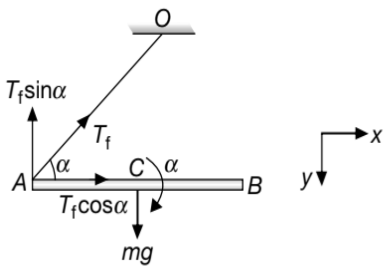
SOLUTION

Initially $2T_i \sin\alpha = mg$

$$\Rightarrow \quad T_i = \frac{mg}{2\sin\alpha} \qquad \ldots(1)$$

When the string OB is cut, let the tension in string OA be T_f, then equations of motion for centre of mass of rod are

$$a_x = \frac{T_f\cos\alpha}{m} \qquad \ldots(2)$$



$$a_y = \frac{mg - T_f\sin\alpha}{m} \qquad \ldots(3)$$

If α be the angular acceleration of the rod about its centre of mass C, then

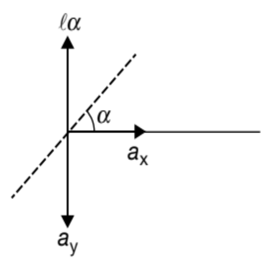
$$\alpha = \frac{(T_f\sin\alpha)l}{\dfrac{m(2l)^2}{12}} = \frac{3T_f\sin\alpha}{ml} \qquad \ldots(4)$$

At the instant OB is cut, point A moves perpendicular to OA, so acceleration of point A along AO should be zero, so

$$a_x\cos\alpha + l\alpha\sin\alpha = a_y\sin\alpha$$

Substituting the values, we get

$$\frac{T_f\cos^2\alpha}{m} + \frac{3T_f\sin^2\alpha}{m} = \frac{(mg - T_f\sin\alpha)\sin\alpha}{m}$$



Solving this equation, we get

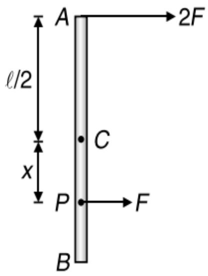
$$T_f = \frac{mg\sin\alpha}{\cos^2\alpha + 4\sin^2\alpha} \qquad \ldots(5)$$

From equations (1) and (5), we get

$$\frac{T_f}{T_i} = \frac{2\sin^2\alpha}{\cos^2\alpha + 4\sin^2\alpha} = \frac{2\sin^2\alpha}{1+3\sin^2\alpha}$$

ILLUSTRATION 52

A uniform rod of mass m and length l is acted upon by the forces F and $2F$ as shown in Figure.



Calculate the linear and angular acceleration of the rod. Also find the value of x for which the point P does not accelerate.

SOLUTION

Applying Newton's Second Law for translational motion, we get

$$2F + F = ma$$

$$\Rightarrow \quad a = \frac{3F}{m} \qquad \ldots(1)$$

Applying Newton's Second Law for rotational motion to get the torque equation about centre of mass, we have

$$2F\left(\frac{l}{2}\right) - Fx = \left(\frac{ml^2}{12}\right)\alpha$$

$$\Rightarrow \quad \alpha = \frac{12F(l-x)}{ml^2} \qquad \ldots(2)$$

If point P does not accelerate, then we have

$$\vec{a}_P = \vec{a}_{P,C} + \vec{a}_C = \vec{0}$$

$$\Rightarrow \quad a_P = 0 = -\alpha x + a$$

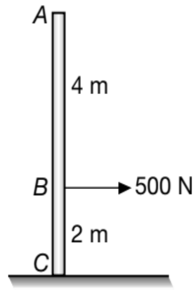
$$\Rightarrow \quad \alpha x = a$$

$$\Rightarrow \quad \frac{12F(l-x)x}{ml^2} = \frac{3F}{m}$$

$$\Rightarrow \quad x = \frac{l}{2}$$

ILLUSTRATION 53

The uniform 50 kg pole ABC is balanced in the vertical position. A 500 N horizontal force is suddenly applied at B. If the coefficient of kinetic friction between the pole and the ground is 0.3, determine the initial acceleration of point A. Take $g = 10\ \text{ms}^{-2}$.



SOLUTION

$$N = mg = 500$$

$$500 - \mu N = ma_x$$

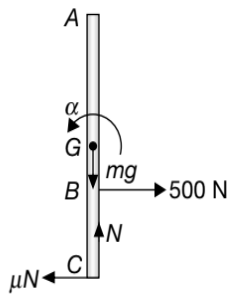
$$\Rightarrow \quad a_x = \frac{500 - \mu N}{m}$$

$$\Rightarrow \quad a_x = \frac{(500)-(0.3)(500)}{50} = 7\ \text{ms}^{-2}$$

Now, let us calculate torque due to forces about the centre of mass or centre of gravity G of the rod, then

$$\tau = (500)(1) - (0.3)(500)(3)$$

$$\Rightarrow \quad \tau = 500 - 450 = 50\ \text{Nm}$$



Since $\alpha = \dfrac{\tau}{I}$

$$\Rightarrow \quad \alpha = \frac{50}{\frac{1}{12}(50)(6)^2}$$

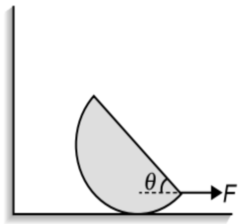
$$\Rightarrow \quad \alpha = \frac{1}{3}\ \text{rads}^{-2}$$

So, acceleration of point A is

$$a_A = a_x - 3\alpha = 7 - (3)\left(\frac{1}{3}\right) = 6\ \text{ms}^{-2}\ \text{(to the right)}$$

ILLUSTRATION 54

A hemisphere of radius r and mass M is pulled by means of a string (Figure) so that it moves with a uniform velocity. If μ is the coefficient of friction with the surface, find the angle of inclination of the hemisphere. Given that the centre of gravity of the hemisphere lies at a distance $\frac{3r}{8}$ from centre.

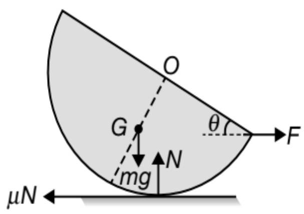


SOLUTION

The hemisphere is moving with uniform velocity. Hence, it is in equilibrium, both rotational as well as translational. So,

$$N = mg \quad \ldots(1)$$

$$F = \mu N \quad \ldots(2)$$



For rotational equilibrium about the bottom-most point,

$$\sum \tau_{\substack{\text{bottom-most}\\ \text{point}}} = 0$$

$$\Rightarrow \quad F(r - r\sin\theta) = mg\left(\frac{3r}{8}\sin\theta\right) \quad \ldots(3)$$

Solving equations (1), (2) and (3), we get

$$\sin\theta = \frac{8\mu}{3+8\mu}$$

ILLUSTRATION 55

A rough uniform rod, of mass m and length $4a$, is held on a horizontal table perpendicularly to an edge of the table, with a length $3a$ projecting horizontally over the edge. If the rod is released from rest and allowed to turn about the edge without slipping then find its angular speed after turning through an angle θ. Assuming that the rod has not started to slip, deduce an expression, in terms of θ, for the angular acceleration, and hence determine the reaction normal to the rod. Show that the rod begins to slip when $\tan\theta = \frac{4\mu}{13}$, where μ is the coefficient of friction.

SOLUTION

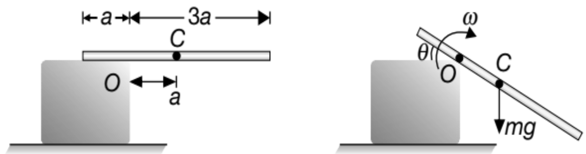
Let C be the centre of mass of rod then by Law of Conservation of Mechanical Energy, we have

$$\frac{1}{2}I_0\omega^2 = mg(a\sin\theta)$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{m(4a)^2}{12} + ma^2\right)\omega^2 = mga\sin\theta$$

$$\Rightarrow \quad \frac{7}{6}a\omega^2 = g\sin\theta$$

$$\Rightarrow \quad \omega = \sqrt{\frac{6g\sin\theta}{7a}}$$

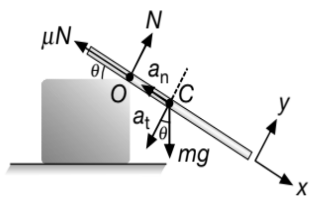


Since, $\alpha = \frac{\tau}{I} = \frac{mga\cos\theta}{\frac{7}{3}ma^2} = \frac{3g\cos\theta}{7a}$

Applying Newton's Second Law for translation motion along y-axis, we get

$$\Sigma F_y = ma_y$$

$$\Rightarrow \quad mg\cos\theta - N = ma_t$$



Since $a_t = r\alpha = a\alpha$

$$\Rightarrow \quad N = mg\cos\theta - ma_t = mg\cos\theta - m(a\alpha)$$

$$\Rightarrow \quad N = mg\cos\theta - \frac{3mg\cos\theta}{7} = \frac{4}{7}mg\cos\theta$$

Rod begins to slip when

$$\mu N - mg\sin\theta = ma_n$$

Since $a_n = r\omega^2 = a\omega^2$

$$\Rightarrow \quad \frac{4}{7}\mu mg\cos\theta - mg\sin\theta = m\left(a\omega^2\right)$$

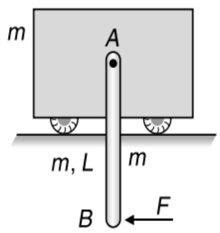
$$\Rightarrow \quad \frac{4}{7}\mu mg\cos\theta - mg\sin\theta = \frac{6mg\sin\theta}{7}$$

$$\Rightarrow \quad \frac{4}{7}\mu mg\cos\theta = \frac{13}{7}mg\sin\theta$$

$$\Rightarrow \quad \tan\theta = \frac{4\mu}{13}$$

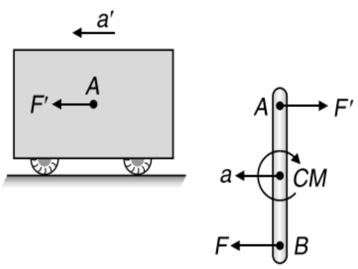
ILLUSTRATION 56

A uniform slender bar AB of mass m is suspended as shown from a small cart of the small cart of the same mass m. Neglecting the effect of friction, determine the accelerations of points A and B immediately after a horizontal force F has been applied at B.



SOLUTION

Let the acceleration of bar be a, cart be a' and α be the angular acceleration of the bar, then



for the bar, we have

$$F - F' = ma \qquad \ldots(1)$$

for the cart, we have

$$F' = ma' \qquad \ldots(2)$$

Applying the torque equation for rod about the centre of mass, we get

$$(F + F')\frac{L}{2} = \frac{ML^2}{12}\alpha$$

$$\Rightarrow \quad F + F' = \frac{ML\alpha}{6} \qquad \ldots(3)$$

The acceleration of point A on the rod will be same as the acceleration of the cart, so we have

$$\vec{a}_A = \vec{a}_{AC} + \vec{a}_C$$

$$\Rightarrow \quad -a' = \frac{\alpha L}{2} - a$$

$$\Rightarrow \quad a' = a - \frac{L\alpha}{2} \qquad \ldots(4)$$

Solving equations (1), (2), (3), and (4), we get

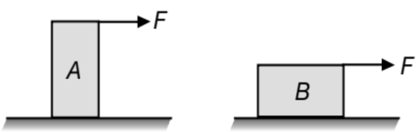
$$a = \frac{7F}{5m}, \ \alpha = \frac{18F}{5mL}$$

$$\Rightarrow \quad |a'| = |a_A| = a - \frac{L\alpha}{2} = -\frac{2F}{5m}$$

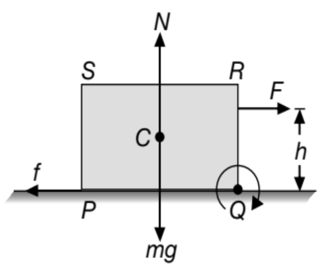
$$\Rightarrow \quad |a_B| = a + \frac{L\alpha}{2} = \frac{16F}{5m}$$

CONCEPT OF TOPPLING

In our practical life, we have observed that when a force F is applied on a block A of smaller width and greater height, it is more likely to topple down before sliding in comparison to another block B of broader base/bigger width and lower height where the chances of sliding are more compared to toppling.



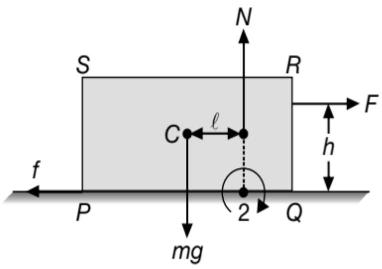
So, let us analyse these situations with the help of examples using basic concepts of torque. For this let us consider a block $PQRS$ having centre of mass at C.



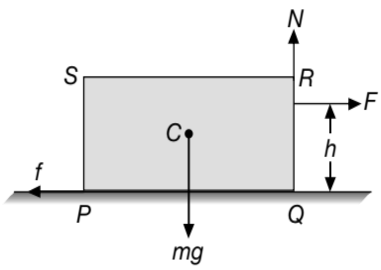
Let a force F be applied normally to face QR at height h above ground. Let the frictional force f be sufficient enough to prevent sliding. Now, if we say that the normal reaction also passes through C, then we have to face a problem that in spite of the fact that translational equilibrium exists i.e., $F = f$ and $N = mg$, we have an unbalanced CW torque due to

F about the point Q. This unbalanced CW torque has a tendency to topple the block about point Q. So, to cancel the effect of unbalanced torque due to F about the point Q, the normal reaction N shifts towards the right, say by l, such that

$$(mg)l = Fh$$



Now, if F or h (or both) are increased then correspondingly l also increases, however cannot go beyond the right side of the block (QR). So, in the extreme case the normal reaction passes through Q and beyond this the block will topple down.



Now, if F or h (or both) are further increased, then the block will topple down. It is due to this reason, that the block with broader base has less chances of toppling in comparison to a block of smaller base because the larger base has more margin to accommodate the shifting of the normal reaction.

Conceptual Note(s)

Why rolling is easy in comparison to sliding?

From arguments and reasons discussed, we observe that in case of a rolling body the normal reaction has a zero margin to shift. So, even if the body is in translational equilibrium $(F = f \text{ and } N = mg)$, still an unbalanced torque (due to F and f) is left behind that rotates the body in clockwise sense. The instant the body starts rolling, the force of friction gets adjusted in magnitude and direction such that pure rolling starts (if it is sufficient enough) or the body starts sliding if it is not sufficient enough).

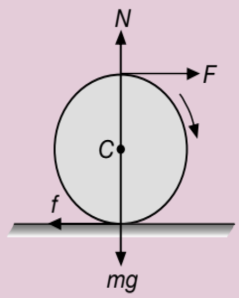


ILLUSTRATION 57

A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will

(a) slide before toppling

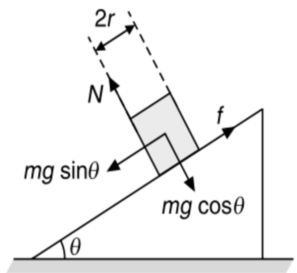
(b) topple before sliding.

SOLUTION

(a) The cylinder will slide if

$$mg\sin\theta > \mu mg\cos\theta$$

$$\Rightarrow \tan\theta > \mu \qquad \ldots(1)$$



The cylinder will topple if

$$(mg\sin\theta)\frac{h}{2} > (mg\cos\theta)r$$

$$\Rightarrow \tan\theta > \frac{2r}{h} \qquad \ldots(2)$$

Thus, the condition of sliding is $\tan\theta > \mu$ and condition of toppling is $\tan\theta > \frac{2r}{h}$. Hence, the cylinder will slide before toppling if

$$\mu < \frac{2r}{h}$$

(b) The cylinder will topple before sliding if

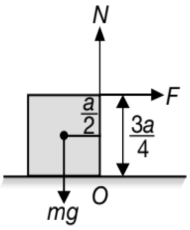
$$\mu > \frac{2r}{h}$$

ILLUSTRATION 58

A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height $\frac{3a}{4}$ above the base. What is the minimum value of F for which the cube begins to topple about an edge?

SOLUTION

In the limiting case normal reaction will pass through O. The cube will topple about O if torque of F exceeds the torque of mg.



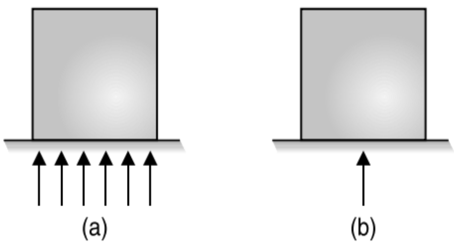
$$\Rightarrow \quad F\left(\frac{3a}{4}\right) > mg\left(\frac{a}{2}\right)$$

$$\Rightarrow \quad F > \frac{2}{3}mg$$

So, the minimum value of F is $\frac{2}{3}mg$

SHIFTING OF NORMAL REACTION AND TOPPLING

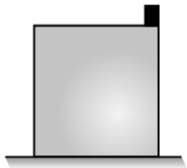
When a surface comes into contact with other surface then one surface applies normal reaction on other surface. The normal reaction does not act only at one point but it is distributed over the whole surface in contact. Consider a uniform block lying on a horizontal surface. Then the normal reaction is distributed uniformly over the surface as shown in figure (a). Now as the body is in translatory and rotatory equilibrium thus the net force and torque on the body should be zero.



To balance the torque and force of gravitational pull, normal reaction should have magnitude equal to the weight of the block and effectively normal reaction should pass through the centre of the block as shown in figure (b).

ILLUSTRATION 59

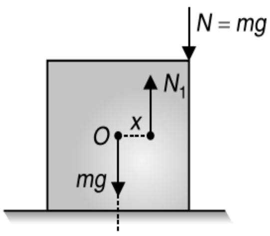
A uniform square block of side a and mass m is lying on a horizontal surface. A small block of mass m is placed on the top corner of the block. Find the distance by which normal reaction is shifted.



SOLUTION

For translational equilibrium of the system, we have

$$N_1 = 2mg$$



For rotational equilibrium of the system, we have

$$\tau_0 = \left(N_1 x - mg\frac{a}{2}\right) = 0$$

$$\Rightarrow \quad x = \frac{\left(mg\frac{a}{2}\right)}{2mg} = \frac{a}{4}$$

Here the normal reaction is still distributed but now its distribution is non uniform as shown in Figure.

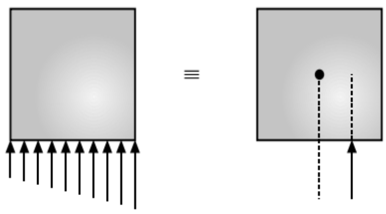


ILLUSTRATION 60

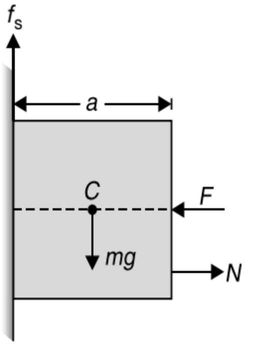
A cubical block of side a is held at rest, against a rough vertical wall by applying a force F acting along the centre. The mass of the block is m. Taking acceleration due to gravity as g, calculate the

(a) minimum coefficient of friction between block and the wall.

(b) torque by normal reaction about centre of mass.

SOLUTION

The situation is shown in the Figure.



Let C be the centre of mass of the block. For equilibrium of the block, we have

$$\Sigma F_x = 0 \quad \Rightarrow \quad F - N = 0 \qquad \ldots(1)$$

$$\Sigma F_y = 0 \quad \Rightarrow \quad f_s = mg \qquad \ldots(2)$$

$$\Sigma \tau = 0 \quad \Rightarrow \quad -f_s\left(\frac{a}{2}\right) + \tau_N = 0 \qquad \ldots(3)$$

While writing equation (3), we have taken clockwise torque to be negative.

Please note that the normal reaction N due to the wall on the block does not pass through the centre C.

$\Rightarrow \quad N = F$ and $f_s = mg$

(a) Since $f_s \le \mu N$

$\Rightarrow \quad mg \le \mu F$

$\Rightarrow \quad \mu \ge \dfrac{mg}{F}$

(b) $-f_s\left(\dfrac{a}{2}\right) + \tau_N = 0$

$\tau_N = f_s \times \dfrac{a}{2} = mg\dfrac{a}{2}$ (in counter clockwise sense)

TOPPLING

Consider a square plate shown in figure. A force is applied at the top as shown in figure. Now normal reaction will shift to balance the torque produced by F and friction about centre of mass. Now if force F increases then normal will shift further but shifting of normal has a limit. It cannot shift beyond point A. Now if F is further increased then normal cannot shift further to balance the torque. Hence now the block will start rotating about point A. This is called toppling.

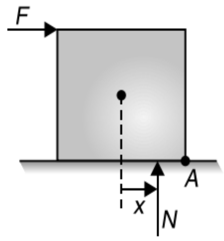


ILLUSTRATION 61

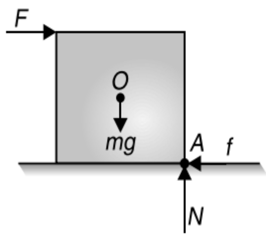
A square plate of side a and mass m is lying on a horizontal floor. A force F is applied at the top. Find the maximum force that can be applied on the square plate so that the plate does not topple about A.

SOLUTION

For translational equilibrium of the system, we have

$N = mg$

$f = F$



For rotational equilibrium of the system, we have

$$\tau_0 = 0 = F\frac{a}{2} + f\frac{a}{2} - N\frac{a}{2}$$

$$\Rightarrow \quad Fa = mg\frac{a}{2}$$

$$\Rightarrow \quad F = \frac{mg}{2}$$

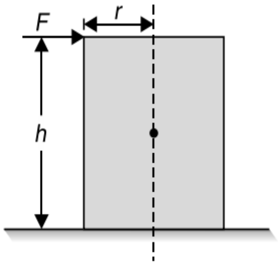
We can also think that when the body is in complete equilibrium, then net torque will be zero about any point taken. So net torque about the point A is zero.

$$\Rightarrow \quad Fa = mg\frac{a}{2}$$

$$\Rightarrow \quad F = \frac{mg}{2}$$

ILLUSTRATION 62

A force F is applied at the top most point of a cylinder of radius r and height h as shown in Figure.



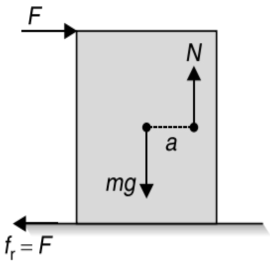
(a) If the cylinder remains at rest, then find distance of line of action of normal reaction from centre of mass.

(b) Find coefficient of friction required so that the cylinder topples before sliding.

SOLUTION

(a) Since body is at rest, it is in translational equilibrium

$$f_r = F$$



For rotational equilibrium,

$$\tau_{\text{net}} = 0 \quad \text{(about centre)}$$

$$\Rightarrow \frac{Fh}{2} + \frac{Fh}{2} - Na = 0 \quad \{\because f_r = F\}$$

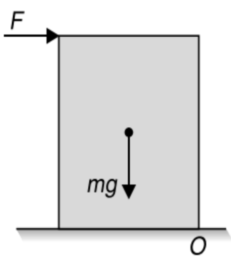
$$\Rightarrow a = \frac{Fh}{N} = \frac{Fh}{mg}$$

(b) For toppling about point O

$$\tau_F > \tau_{mg}$$

$$\Rightarrow Fh > mgr$$

$$\Rightarrow F > \frac{mgr}{h}$$



To avoid sliding, $F < \mu mg$

$$\Rightarrow \frac{mgr}{h} < F < \mu mg$$

$$\Rightarrow \frac{r}{h} < \mu$$

WORK DONE BY A TORQUE

If a torque $\vec{\tau}$ rotates a body through infinitesimal displacement $d\vec{\theta}$ then the infinitesimal work done is

$$dW = \vec{\tau} \cdot d\vec{\theta}$$

If $\vec{\tau}$ and $d\vec{\theta}$ are in the same direction, then

$$dW = \tau\, d\theta$$

$$\Rightarrow \quad W = \int dW = \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

If a constant torque τ acts on the body, then

$$W = \tau(\theta_2 - \theta_1)$$

$$\Rightarrow \quad W = \tau \Delta\theta$$

STRING CONSTANT (TORSIONAL CONSTANT) AND ENERGY STORED IN A STRING

If a string is given an angular displacement θ (measured in radian) then the torsion produced tries to restore it to its original configuration and this torque (torsion produced) is directly proportional to θ. Mathematically,

$$\tau \propto \theta$$

$$\Rightarrow \quad \tau = -C\theta$$

where, C is called the Torsional Constant or the String Constant of the string. *The negative sign shows that the torque is restoring in nature.*

If dW is the infinitesimal work done to give the string an angular displacement θ, then

$$dW = \tau\, d\theta \cos(180°) = -\tau\, d\theta$$

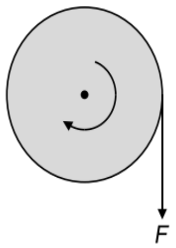
$$\Rightarrow \quad W = \int dW = \frac{1}{2}C\left(\theta_2^2 - \theta_1^2\right)$$

If $\theta_1 = 0$ and $\theta_2 = \theta$, then

$$W = \frac{1}{2}C\theta^2$$

ILLUSTRATION 63

A uniform cylinder of mass M and radius R, initially at rest is mounted so as to rotate freely about a horizontal axis that passes through the central axis of the cylinder. A constant force F acts on the cylinder as shown in Figure.



Calculate the angular velocity of the cylinder as it rotates by an angle of π.

SOLUTION

Work done by a constant force F is $W = \int \tau d\theta$, where

$$\tau = Fr_{\perp} = FR$$

$$\Rightarrow \quad W = \int FR d\theta = FR\int_0^{\pi} d\theta = FR\pi$$

According to Work-Energy theorem, we have

$$W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

$$\Rightarrow \quad \frac{1}{2}I\omega_f^2 = FR\pi \qquad \text{\{cylinder was initially at rest\}}$$

$$\Rightarrow \quad \omega_f = \sqrt{\frac{2FR\pi}{I}} = \sqrt{\frac{2FR\pi}{MR^2/2}} = \sqrt{\frac{4FR\pi}{MR^2}}$$

$$\Rightarrow \quad \omega_f = \sqrt{\frac{4F\pi}{MR}}$$

POWER

The rotational power or simply the **Power** is defined as the rate at which work is done by a torque. The SI unit of power is watt and here too $1\text{ W} = 1\text{ Js}^{-1}$

Average Power $P_{\text{av}} = \frac{W}{t} = \tau\left(\frac{\theta}{t}\right) = \tau\omega_{\text{av}}$

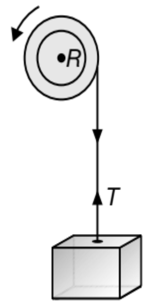
Instantaneous Power $P_{\text{ins}} = \frac{dW}{dt} = \vec{\tau}\cdot\frac{d\vec{\theta}}{dt} = \vec{\tau}\cdot\vec{\omega}$

ILLUSTRATION 64

A motor rotates a pulley of radius 25 cm at 20 rpm. A rope around the pulley lifts a 50 kg block, as shown in Figure. What is the power output of the motor?

SOLUTION

The tension in the rope is equal to the weight since there is no acceleration. Thus, $T = 500\text{ N}$.



Therefore, $\tau = TR = (500)(0.25) = 125\text{ Nm}$.

Angular velocity, $\omega = \frac{2\pi N}{60}$

$$\omega = \frac{2\pi(20)}{60} = \frac{2\pi}{3}\text{ rads}^{-1}$$

The power required is

$$P = \tau\omega = \left(125\text{ Nm}^{-1}\right)\left(\frac{2\pi}{3}\text{ rads}^{-1}\right) = 260\text{ W}$$

WORK-ENERGY PRINCIPLE

According to this theorem, work done by a torque equals the change in rotational kinetic energy.

i.e. $\quad W = \tau\theta = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$

So, in complete analogue to the Work Energy Theorem studied in Translational Motion, we can say that the net rotational work done by the forces is equal to the change in rotational kinetic energy of the body.

$$W_{\text{rot}} = \Delta K_{\text{rot}}$$

For a rolling body, we have, work done equals the change in total energy possessed by the rolling body. So,

$$W_{\text{total}} = \Delta K = \left(\frac{1}{2}mv^2 - \frac{1}{2}mu^2\right)\left(1 + \frac{k^2}{R^2}\right)$$

MODIFIED NEWTON'S SECOND LAW FOR FIXED AXIS ROTATION

Since torque is a rotational analogue of force, therefore, Newton's Second Law for rotational motion is given by

$$\tau_{\text{net}} = I_{\text{cm}}\alpha = I_{\text{cm}}\left(\frac{a_{\text{cm}}}{R}\right)$$

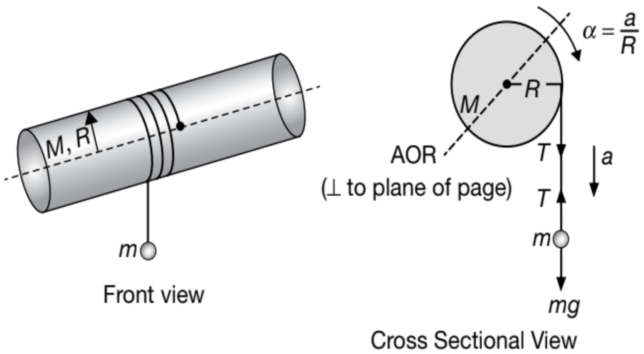
It is valid in two situations.

(a) The axis is fixed in position and direction.
(b) The axis passes through the centre of mass and is fixed in direction only.

The equation $\tau_{\text{cm}} = I_{\text{cm}}\alpha_{\text{cm}}$ is valid even if the centre of mass is accelerating.

MOTION OF A POINT MASS ATTACHED TO A CYLINDER WITH A THREAD

Consider a point mass m attached to a thread wound over a cylinder of radius R, mass M, moment of inertia $I\left(=\frac{1}{2}MR^2\right)$ and radius of gyration k. The point mass ascends down with acceleration a and the cylinder rotates while the thread unwinds. The tension in the thread provides necessary torque to the cylinder to rotate with angular acceleration α. So,



Front view
Cross Sectional View

$$\tau = TR\sin 90 = TR$$

Further $\tau = I\alpha$

$$\Rightarrow \quad TR = I\alpha$$

$$\Rightarrow \quad TR = I\left(\frac{a}{R}\right) \qquad \left\{\because \alpha = \frac{a}{R}\right\}$$

$$\Rightarrow \quad T = \frac{Ia}{R^2} \qquad \ldots(1)$$

Further,

$$mg - T = ma$$

$$\Rightarrow \quad mg - \frac{Ia}{R^2} = ma$$

$$\Rightarrow \quad mg = \left(\frac{I}{R^2} + m\right)a$$

$$\Rightarrow \quad a = \frac{g}{\left(1 + \frac{I}{mR^2}\right)} = \frac{g}{\left(1 + \frac{M}{2m}\right)}$$

Put this value of a in (1), we get

$$T = \frac{mg}{\left(1 + \frac{mR^2}{I}\right)} = \frac{mg}{\left(1 + \frac{2m}{M}\right)}$$

Special Case

If both the cylinder and the point mass possess equal mass, then $m = M$ and hence,

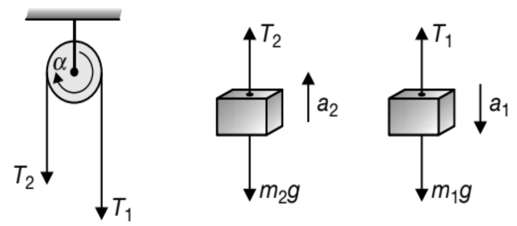
$$T = \frac{mg}{3} \text{ and } a = \frac{2g}{3}$$

ATWOOD'S MACHINE

The free body diagrams of the pulley and the blocks are shown in Figure. Note that tensions on two sides of the pulley are different. Why? Applying Newton's Second Law on the pulley, we get

$$\tau = T_1R - T_2R = (T_1 - T_2)R$$

Since, $\tau = I\alpha$



Therefore, $(T_1 - T_2)R = I\alpha$

$$\Rightarrow \quad T_1 - T_2 = \frac{Ia}{R^2} \qquad \ldots(1)$$

Applying Newton's Law on the blocks, we get

$$T_2 - m_2g = m_2a \qquad \ldots(2)$$

$$m_1g - T_1 = m_1a \qquad \ldots(3)$$

Solving equations (1), (2), (3), we get

$$a=\left(\frac{m_1-m_2}{m_1+m_2+\dfrac{I}{R^2}}\right)g$$

If the pulley had no mass, then

$$I=\frac{1}{2}MR^2\longrightarrow 0$$

$$a\cong\left(\frac{m_1-m_2}{m_1+m_2}\right)g$$

For pulley (generally assumed to be a disc or a cylinder)

$$I=\frac{1}{2}MR^2$$

$$\Rightarrow\quad a=\left(\frac{m_1-m_2}{m_1+m_2+\dfrac{M}{2}}\right)g$$

APPLICATION OF NEWTON'S SECOND LAW IN ROLLING MOTION

(a) **Ignoring Rotation,** write $F_{net}=Ma_{cm}$ for the object as if it were a point-mass.

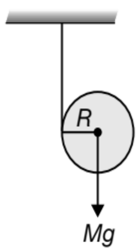
(b) **Ignoring Translation,** write $\tau=I_{cm}\alpha$ as if the object were only rotating about the centre of mass.

(c) Use of no slip condition i.e. $v_{cm}=R\omega$ or $a_{cm}=R\alpha$

(d) Solve the resulting equations simultaneously for any unknown.

ILLUSTRATION 65

A solid cylinder of mass M has a string wrapped several times around its circumference. The free end of string is attached to the ceiling and the cylinder is released from rest. Find the acceleration of the cylinder and the tension in the string.



SOLUTION

$$Mg-T=Ma \qquad \text{...(1)}$$

Also, $\tau=TR=I\alpha$

$$\Rightarrow\quad T=\frac{I\alpha}{R}\text{ where }I=\frac{1}{2}MR^2$$

$$\Rightarrow\quad T=\frac{MR\alpha}{2} \qquad \text{...(2)}$$

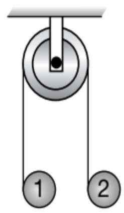
Condition of no slipping, $a=\alpha r$ ...(3)

Solving equation (1), (2) and (3) we obtain

$$a=\frac{2g}{3}\text{ and }T=\frac{Mg}{3}$$

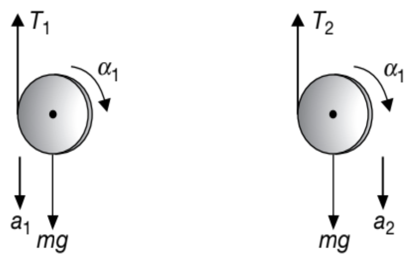
ILLUSTRATION 66

A thread is wound around two discs on either side. The pulley and the two discs have the same mass and radius. There is no slipping at the pulley and no friction at the hinge. Find out the accelerations of the two discs and the angular acceleration of the pulley.



SOLUTION

Let R be the radius of the discs and T_1 and T_2 be the tensions in the left and right segments of the rope.



Acceleration of disc 1,

$$a_1=\frac{mg-T_1}{m} \qquad \text{...(1)}$$

Acceleration of disc 2,

$$a_2=\frac{mg-T_2}{m} \qquad \text{...(2)}$$

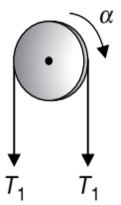
Angular acceleration of disc 1,

$$\alpha_1 = \frac{\tau}{I} = \frac{T_1 R}{\frac{1}{2}mR^2} = \frac{2T_1}{mR} \qquad \ldots(3)$$

Similarly, angular acceleration of disc 2,

$$\alpha_2 = \frac{2T_2}{mR} \qquad \ldots(4)$$

where, both α_1 and α_2 are clockwise.



Angular acceleration of pulley,

$$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2}mR^2} = \frac{2(T_2 - T_1)}{mR} \qquad \ldots(5)$$

For no slipping, $R\alpha_1 - a_1 = a_2 - R\alpha_2 = R\alpha \qquad \ldots(6)$

Solving these equations, we get

$$\alpha = 0 \text{ and } a_1 = a_2 = \frac{2g}{3}$$

OBSERVATION

Since, both the discs are in identical situation, $T_1 = T_2$ and $\alpha = 0$, i.e., each of the discs falls independently and identically.

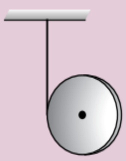
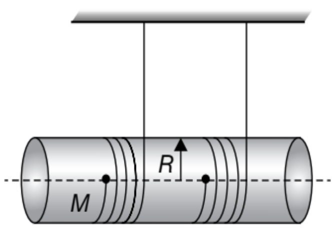


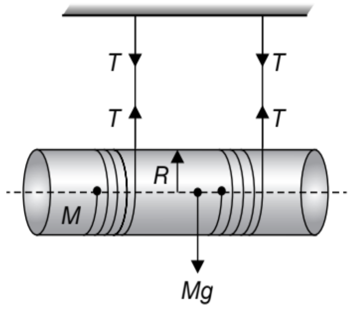
ILLUSTRATION 67

Find the acceleration and tension in the system shown. Assume that the cylinder remains straight and the strings are same.



SOLUTION

The forces acting on the system are shown in Figure.



Applying Newton's Second Law for rotational motion, we get

$$2(TR) = I\alpha$$

$$\Rightarrow \quad 2T = \frac{Ia}{R^2} \qquad \ldots(1)$$

Applying Newton's Second Law for translational motion, we get

$$Mg - 2T = Ma$$

$$\Rightarrow \quad Mg - \frac{Ia}{R^2} = Ma$$

$$\Rightarrow \quad a = \frac{Mg}{\left(M + \frac{I}{R^2}\right)}$$

Since, $I = \frac{1}{2}MR^2$

$$\Rightarrow \quad a = \frac{Mg}{M + \frac{M}{2}} = \frac{2g}{3}$$

Put this value of a in (1), we get

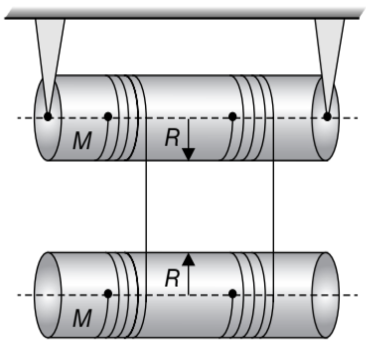
$$2T = \frac{I}{R^2}\left(\frac{2g}{3}\right)$$

$$\Rightarrow \quad 2T = \frac{M}{2}\left(\frac{2g}{3}\right) \qquad \left\{\because I = \frac{1}{2}MR^2\right\}$$

$$\Rightarrow \quad T = \frac{Mg}{6}$$

ILLUSTRATION 68

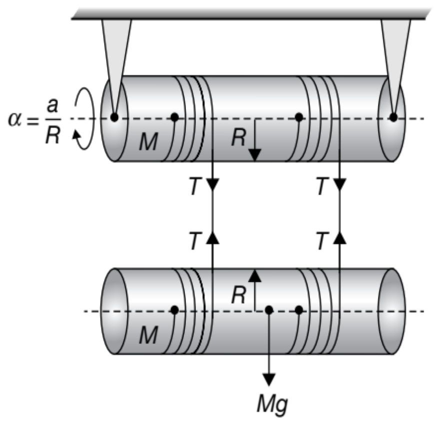
The arrangement shown in Figure consists of two identical uniform solid cylinders, each of mass M, radius R on which two light threads are wound symmetrically. Find the tension of each thread in the process of motion. The friction in the axle of the upper cylinder is assumed to be absent.



SOLUTION

If a be the acceleration at which the thread from the upper cylinder is released and A be the total acceleration of the lower cylinder, then

$$A = \begin{pmatrix}\text{Acceleration}\\ \text{of } CM\\ \text{of lower}\\ \text{cylinder}\end{pmatrix} + \begin{pmatrix}\text{Acceleration at}\\ \text{which thread}\\ \text{is being released}\\ \text{by upper cylinder}\end{pmatrix}$$



$\Rightarrow \quad A = R\alpha + R\alpha = 2a = 2R\alpha$

Applying Newton's Second Law for rotational motion, we get

$$\tau = (2T)R = I\alpha$$

$$\Rightarrow \quad 2T = \frac{Ia}{R^2} \qquad \ldots(1)$$

Applying Newton's Second Law for translational motion, we get

$$Mg - 2T = M(A) = M(2a)$$

$$\Rightarrow \quad Mg - \frac{Ia}{R^2} = 2Ma$$

$$\Rightarrow \quad Mg - \frac{M}{2}a = 2Ma$$

$$\Rightarrow \quad Mg = \left(2M + \frac{M}{2}\right)a$$

$$\Rightarrow \quad a = \frac{2g}{5}$$

$$\Rightarrow \quad A = 2a = \frac{4g}{5} \text{ and } 2T = \frac{M}{2}\left(\frac{2g}{5}\right)$$

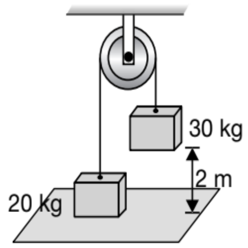
$$\Rightarrow \quad T = \frac{Mg}{10}$$

Test Your Concepts-IV

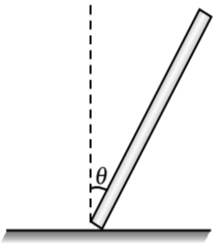
Based on Torque and Applications

(Solutions on page H.155)

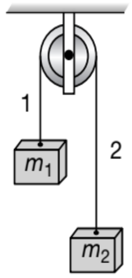
1. A uniform rod pivoted at its upper end hangs vertically. It is displaced through an angle of $60°$ and then released. Find the magnitude of the force acting on a particle of mass dm at the tip of the rod when the rod makes an angle of $37°$ with the vertical. Given, $\cos(37°) = \frac{4}{5}$ and $g = 10 \text{ ms}^{-2}$.
2. A thin horizontal uniform rod AB of mass m and length ℓ can rotate freely about a vertical fixed axis passing through its end A. At a certain moment the end B starts experiencing a constant force F which is always perpendicular to the original position of the stationary rod and directed in the horizontal plane. Find the angular velocity of the rod as a function of its rotation angle ϕ counted relative to the initial position.
3. The system in Figure is released from rest. The 30 kg body is 2 m above the floor and is connected through an ideal string passing over the pulley (a uniform disk with a radius of 10 cm and mass 5 kg) to another body of mass 20 kg. Find the speed of the 30 kg body just before it hits the floor and the angular speed of the pulley at that time, the tensions in the strings and the time it takes for the 30 kg body to reach the floor.



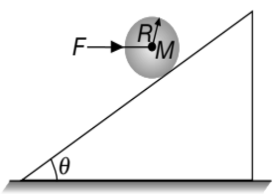
4. A slender rod of mass m and length ℓ is released from rest at $\theta = 0°$. Calculate the normal and frictional forces, which are exerted on the rod by the ground as it falls downward as a function of θ. Is it possible for the rod to slip as it falls? Give reason in support of your answer.



5. In the arrangement shown in Figure the mass of the uniform solid cylinder of radius R is equal to m and the masses of two bodies are equal to m_1 and m_2. The thread slipping and the friction in the axle of the cylinder are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions $\frac{T_1}{T_2}$ of the vertical sections of the thread in the process of motion.

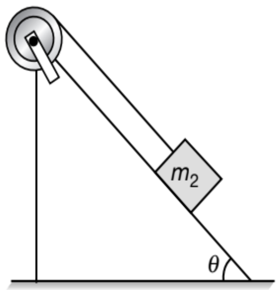


6. A horizontal force F is applied to the sphere shown in Figure. Calculate the
   (a) value of F required to hold the sphere in equilibrium.
   (b) frictional force of the incline on the sphere.



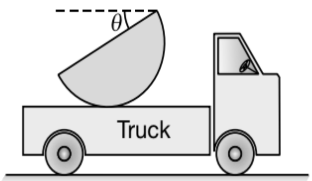
7. A uniform solid cylinder of mass m and radius a is free to rotate about its axis which is smooth and vertical. A light inextensible string is wrapped around the cylinder and its free end is pulled horizontally with a constant force $2mg$. Calculate the angular velocity of the cylinder when the free end of the string has moved through a distance $4a$.

8. A uniform cylinder of mass m_1 and radius R is pivoted on frictionless bearings. A string wrapped around the cylinder connects to a mass m_2, which is on a frictionless incline of angle θ, as shown in Figure.

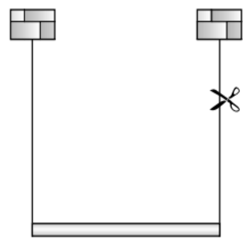


This system is released from rest with m_2 at a height h above the bottom of the incline. Find the acceleration of m_2, tension in the string, total energy of the system when m_2 is at height h, total energy when m_2 reaches bottom of the incline with speed v and the speed v.

9. A solid hemisphere rests on a truck as shown in Figure. If the coefficient of friction μ is just sufficient to prevent slipping, calculate the acceleration of the truck.

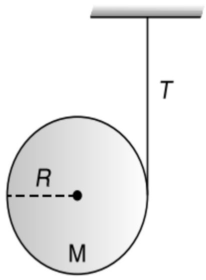


10. A uniform rod of mass m and length ℓ is held horizontally by two vertical strings of negligible mass, as shown in Figure. Immediately after the right string is cut, find the



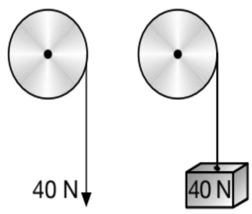
(a) linear acceleration of the free end of the rod.
(b) linear acceleration of the centre of mass of rod.
(c) tension in the left string.

11. A uniform cylinder of mass M and radius R has a string wrapped around it. The string is held fixed and the cylinder falls vertically, as shown in Figure. Calculate the acceleration of the cylinder and tension in the string.



12. A uniform solid cylinder of mass m and radius a starts rotating from rest freely about its axis of symmetry under the action of a constant torque $4mga$. Find the angular velocity of the cylinder at $t = 4s$ from the start.

13. A cord is wrapped around the rim of a solid cylinder of radius 0.25 m and a constant force of 40 N is exerted on the cord as shown in Figure.

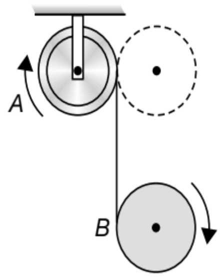


If the cylinder is mounted on frictionless bearings and moment of inertia of cylinder is $4\ \text{kgm}^2$, then

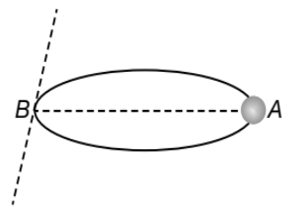
(a) use the work-energy principle to calculate the angular velocity of the cylinder after 5 m of cord have been unwound.
(b) if 40 N force is replaced by a 40 N weight, calculate the angular velocity of cylinder after 5 m of cord has unwound.

Take $g = 10\ \text{ms}^{-2}$.

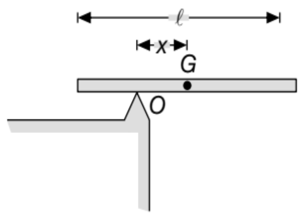
14. Two identical uniform discs A and B each of mass m and radius R are held, as shown in Figure with the help of a long massless string which is wrapped around the discs in opposite directions. Disc A is attached to the ceiling in such a way that it can rotate freely about its axis. The disc, B, initially held at same height as A, is then released to fall so that string unwinds from both the discs. Find the angular and linear accelerations of falling disc and tension in the string. Assume that string does not slip and motion is confined in the same vertical plane.



15. A ring of mass m and radius r has a particle of mass m attached to it at a point A. The ring can rotate about a smooth horizontal axis which is tangential to the ring at a point B diametrically opposite to A. The ring is released from rest when AB is horizontal. Find the angular velocity and the angular acceleration of the body when AB has turned through an angle $\frac{\pi}{3}$.



16. A uniform slender bar is released from rest in the horizontal position as shown in Figure. For what value of x the angular acceleration is maximum? Determine the corresponding angular acceleration α.



ACCELERATED PURE ROLLING

Till now we had been discussing uniform pure rolling where v and ω are constants. However, if an external force is applied to a rigid body, then the motion no longer is uniform pure rolling. Now for pure rolling on a stationary rough ground we must have

$$v = R\omega \quad \text{...(1)}$$

Taking derivative on both sides of (1) w.r.t. time t, we get

$$\frac{dv}{dt} = R\left(\frac{d\omega}{dt}\right)$$

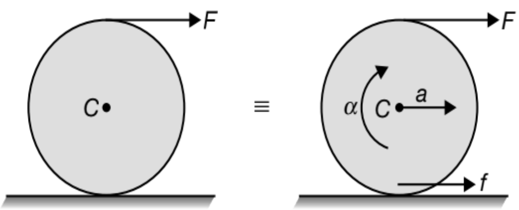
$$\Rightarrow \quad a = R\alpha \quad \text{...(2)}$$

So, in addition to $v = R\omega$ to be obeyed at every instant of time, the relation $a = R\alpha$ is also obeyed for pure rolling to take place and this is called Accelerated Pure Rolling.

Problem Solving Technique(s)

Here friction plays an important role in maintaining the pure rolling. The friction sometimes may act backward and sometimes may act forward or under certain conditions may also be zero, because the basic nature of friction is to self-adjust (up to a certain maximum limit) and it has a tendency to stop the relative motion between the two surfaces / bodies in contact.

Let us now have a more clear understanding of this concept by applying a force F to the top most point of a rigid body of mass M, radius R and moment of inertia I about an axis passing through its centre of mass, as shown in Figure.



Now, this applied force F produces in the body a/an

(a) linear acceleration (a) and

(b) angular acceleration (α)

Now three cases arise

(i) If $a = R\alpha$, then no need of friction and $f = 0$.

(ii) If $a < R\alpha$, then to support the linear motion and to oppose the angular motion, friction f acts forwards (along a).

(iii) If $a > R\alpha$, then to support the angular motion and to oppose the linear motion, friction f acts backwards (opposite to a).

So, we observe that the friction f can be zero, forward or backward depending upon value of I, M and R.

Applying Newton's Second Law for linear motion, we get

$$F + f = Ma \quad \text{...(3)}$$

Applying Newton's Second Law for rotational motion, we get

$$\tau = I\alpha$$

$$\Rightarrow \quad FR(CW) + fR(CCW) = I\alpha$$

$$\Rightarrow \quad FR - fR = I\alpha \quad \text{...(4)}$$

For pure rolling to take place, we have

$$a = R\alpha$$

$$\Rightarrow \quad FR - fR = I\left(\frac{a}{R}\right)$$

$$\Rightarrow \quad F - f = \left(\frac{I}{R^2}\right)a \quad \text{...(5)}$$

From (3) and (5), we get

$$f = \left(\frac{MR^2 - I}{MR^2 + I}\right)F \quad \text{...(6)}$$

From (6), we observe that if we have

(i) $I = MR^2$, i.e., a force F is applied at top of a ring, then $f = 0$ and the ring will roll without slipping.

(ii) $I < MR^2$ i.e., a force F is applied at top of a solid sphere, a shell, a cylinder or a disc, then f is positive i.e., f acts forwards.

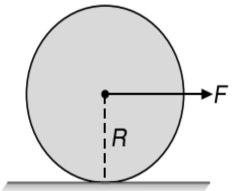
(iii) If $I > MR^2$ i.e., f acts backwards. Although for the axis passing through CM, **we cannot have** $I > MR^2$.

So, we conclude that f is either zero or forwards i.e., along F and so it supports linear motion.

ILLUSTRATION 69

A force F is applied at centre of a uniform round object of mass m radius R and moment of inertia about its centre of mass I_{cm}. Find (if $\frac{I_{cm}}{mR^2} = k$).

(a) Acceleration of centre of the round object if it rolls without slipping.
(b) Minimum coefficient of friction required so that the round object rolls without slipping.



SOLUTION

Equation for translatory motion

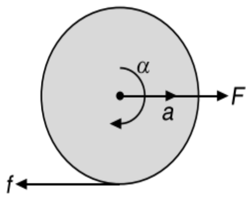
$$F - f = ma \quad \text{...(1)}$$

Equation for rotatory motion

$$fR = I_{cm}\alpha \quad \text{...(2)}$$

Condition for no slipping

$$a = R\alpha \quad \text{...(3)}$$



By solving the equations

$$F - \frac{I_{cm}a}{R^2} = ma$$

$$a = \frac{F}{m + \frac{I_{cm}}{R^2}} = \frac{F}{m\left[1 + \frac{I_{cm}}{mR^2}\right]} = \frac{F}{m(1+k)}$$

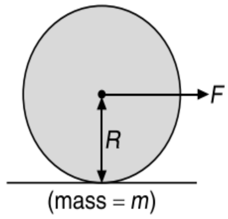
$$f = \frac{I_{cm}F}{mR^2(1+k)} = \frac{kF}{(1+k)}$$

Thus $\mu_{min} mg = \frac{kF}{1+k}$

$$\Rightarrow \quad \mu_{min} = \left(\frac{k}{1+k}\right)\frac{F}{mg}$$

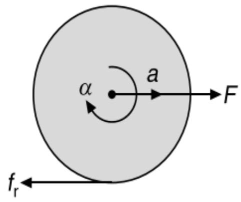
ILLUSTRATION 70

A force F is applied on a disc at its centre. Find acceleration of centre of mass in the case of pure rolling and also find minimum coefficient of friction required for pure rolling.



SOLUTION

Let the friction force acting on disc be f_r, acceleration of centre of mass is a and angular acceleration is α. We have three unknowns



(1) a **(2)** α **(3)** f_r

We require 3 equations to solve them

Translatory Motion

$$F - f_r = ma \quad \text{...(1)}$$

Rotatory Motion

$$rf_r = \frac{mr^2}{2}\alpha \quad \text{...(2)}$$

Condition of no slipping

$$a = r\alpha \quad \text{...(3)}$$

By solving (2) and (3)

$$f_r = \frac{ma}{2}, \; F = \frac{3ma}{2}$$

$$\Rightarrow \quad a = \frac{2F}{3m}, \; f_r = \frac{m}{2} \times \frac{2F}{3m}, \; f_r = \frac{F}{3}$$

Now, $f_r \le \mu N$

$$\Rightarrow \quad \frac{F}{3} \le \mu mg$$

$$\Rightarrow \quad \mu \ge \frac{F}{3mg}$$

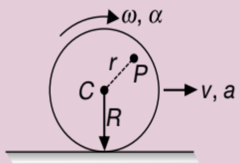
Problem Solving Technique(s)

(a) In accelerated pure rolling the velocity of the bottommost point is zero but despite the relation $a = R\alpha$, the acceleration of the bottommost point is not zero. Because acceleration of any point P can be given as

$$\vec{a}_P = \vec{a}_C + \vec{a}_{PC}$$

Here, $\vec{a}_{PC}$ has two components:

Tangential acceleration $a_t = r\alpha$ (which is perpendicular to CP) and radial or normal acceleration $a_n = r\omega^2$ (which is along PC)



Since, $\vec{a}_{PC} = \vec{a}_P - \vec{a}_C$

$\Rightarrow \vec{a}_P = \vec{a}_{PC} + \vec{a}_C$

$\Rightarrow \vec{a}_P = \vec{a}_C + (\vec{a}_{PC})_t + (\vec{a}_{PC})_n$

For the bottommost point, $\vec{a}_C + (\vec{a}_{PC})_t = 0$

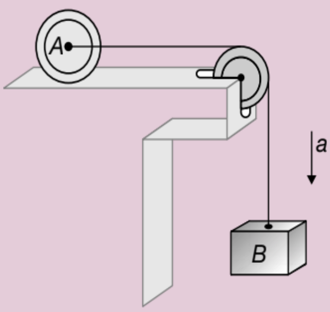
Because $\vec{a}_C = a$ {in forward direction}

and $(\vec{a}_{PC})_t = R\alpha$ {in backward direction}

and since, $a = R\alpha$ therefore, $\vec{a}_C + (\vec{a}_{PC})_t = 0$

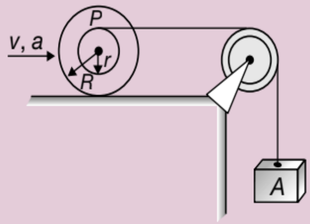
But $(\vec{a}_{PC})_n \neq 0$, because it is $R\omega^2$ towards centre. Thus, acceleration of bottommost point is $R\omega^2$ towards centre.

(b) If a_{cm} is acceleration of CM of A then acceleration of B is also $a_{CM} = a$.



(c) Similarly, in the problems like shown in Figure, it is wrong to say that acceleration of point P is equal to acceleration of block A. Although we can write,

$$a_A = a + r\alpha$$



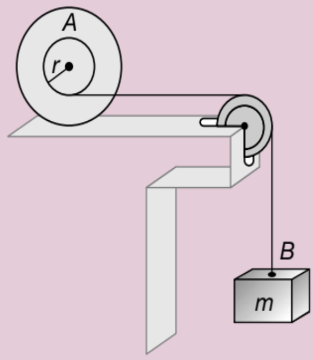
In case of pure rolling, problems can also be solved by using the energy conservation principle (provided no other dissipative forces are present). So, in this case we can also use the Law of Conservation of Energy to get

$$\left(\begin{array}{c}\text{Decrease}\\ \text{in GPE}\\ \text{of A}\end{array}\right) = \left(\begin{array}{c}\text{Increase}\\ \text{in KE}\\ \text{of A}\end{array}\right) + \left(\begin{array}{c}\text{Increase in}\\ \text{KE of Rolling}\\ \text{of Spool}\end{array}\right)$$

(d) If a_{CM} is acceleration of CM of spool, then acceleration of B is

$$a_B = \left(\begin{array}{c}\text{Acceleration}\\ \text{of CM}\\ \text{of spool}\end{array}\right) + \left(\begin{array}{c}\text{Acceleration at}\\ \text{which thread is}\\ \text{being released}\\ \text{from spool}\end{array}\right)$$

$$a_B = a_{CM} + r\alpha$$



If $a_{CM} = a$

$$a_B = a + r\alpha$$

In this particular case,

$$a_B = -a + r\alpha$$

{CM of spool moves backwards}

(e) In cases where pulley is having some mass and friction is sufficient enough to prevent slipping, the tension on two sides of the pulley will be different and rotational motion of the pulley is also to be considered.

(f) At a given instant the value of ω for a rigid body will be same for every point.

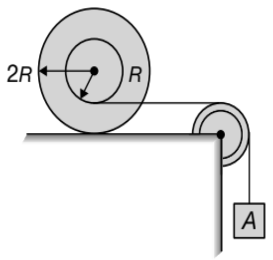
(g) The torque equation $(\tau = I\alpha)$ can be applied only about two points. These are,

(i) centre of mass

(ii) point about which body is in pure rotation.

ILLUSTRATION 71

A thin light thread is wound on a reel of mass 3 kg and moment of inertia $0.6\ \text{kgm}^2$. The inner radius is $R = 10$ cm and peripheral radius is $2R = 20$ cm. The reel is placed on a rough table and the friction is enough to prevent slipping. Find the acceleration of the centre of reel and of hanging mass of 1 kg.



SOLUTION

Let, a_1 = acceleration of centre of mass of reel

a_2 = acceleration of 1 kg block

α = angular acceleration of reel (clockwise)

T = tension in the string

and f = force of friction

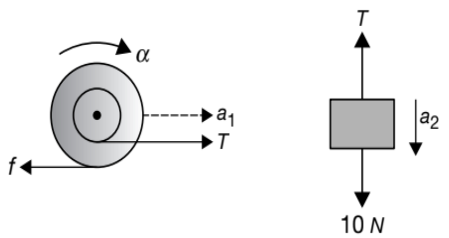
Free body diagram of reel is as shown where, only horizontal forces are taken.

Equations of motion for reel are

$$T - f = 3a_1 \quad \ldots(1)$$

$$\alpha = \frac{\tau}{I} = \frac{f(2R) - TR}{I} = \frac{0.2f - 0.1T}{0.6} = \frac{f}{3} - \frac{T}{6} \quad \ldots(2)$$

Free body diagram of mass is also shown here.



Equation of motion for mass is

$$10 - T = a_2 \quad \ldots(3)$$

For no slipping condition, we have

$$a_1 = 2R\alpha$$

Since $2R = 0.2$ m

$$\Rightarrow \quad a_1 = 0.2\alpha \quad \ldots(4)$$

and $a_2 = a_1 - R\alpha$

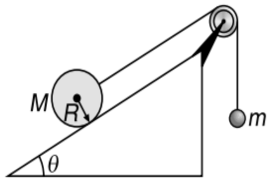
$$\Rightarrow \quad a_2 = a_1 - 0.1\alpha \quad \ldots(5)$$

Solving equations (1), (2), (3), (4) and (5), we get

$$a_1 = 0.27\ \text{ms}^{-2} \text{ and } a_2 = 0.135\ \text{ms}^{-2}$$

ILLUSTRATION 72

A uniform solid cylinder of mass M and radius R rolls a rough inclined plane with its axis perpendicular to the line of greatest slope as shown in Figure.



As the cylinder rolls it winds up a light string which passes over a light and smooth pulley and attached to a mass m, the part of the string between pulley and cylinder being parallel to the line of greatest slope. Prove that the tension in the string is

$$T = \left(\frac{(3 + 4\sin\theta)Mmg}{3M + 8m} \right)$$

SOLUTION

If cylinder rolls down at acceleration a, mass m goes up at acceleration $2a$ so we use

$$T - mg = m(2a) \quad \ldots(1)$$

$$Mg\sin\theta - T - f = Ma \quad \ldots(2)$$

$$fR - TR = \frac{1}{2}MR^2\left(\frac{a}{R}\right) \quad \ldots(3)$$

Adding equations (2) and (3), we get

$$Mg\sin\theta - 2T = \frac{3}{2}Ma$$

$$\Rightarrow \quad a = \frac{2}{3}g\sin\theta - \frac{4}{3}\frac{T}{M}$$

From equation (1), we use

$$T - mg = 2m\left(\frac{2}{3}g\sin\theta - \frac{4}{3}\frac{T}{M}\right)$$

$$\Rightarrow \quad T - mg = \frac{4}{3}mg\sin\theta - \frac{8m}{3M}T$$

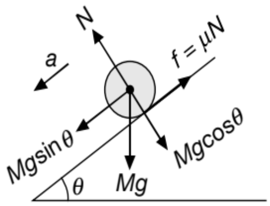
$$\Rightarrow \quad T\left(1+\frac{8m}{3M}\right) = mg\left(\frac{4}{3}\sin\theta + 1\right)$$

$$\Rightarrow \quad T = \frac{Mmg(4\sin\theta+3)}{3M+8m}$$

BODY ROLLING WITHOUT SLIPPING ON AN INCLINED PLANE

METHOD I: Using the concept of Newton's Second Law (for translation and rotation motion)

Consider a body of M radius R and moment of inertia I rolling without slipping on an inclined plane (making an angle θ with the horizontal). For the body to roll without slipping necessary friction (f) must be present. From the Figure we observe that all forces (other than friction) are acting along the radius and hence will not produce torque in the body.



However, friction being a tangential force will produce a torque τ given by

$$\tau = fR\sin 90$$

$$\Rightarrow \quad \tau = fR$$

Further

$$\tau = I\alpha$$

$$\Rightarrow \quad I\alpha = fR$$

$$\Rightarrow \quad I\frac{a}{R} = fR$$

$$\Rightarrow \quad f = \frac{Ia}{R^2} \qquad \ldots(1)$$

Further

$$Mg\sin\theta - f = Ma$$

$$\Rightarrow \quad Mg\sin\theta - \frac{Ia}{R^2} = Ma$$

$$\Rightarrow \quad Mg\sin\theta = \left(\frac{I}{R^2}+M\right)a$$

$$\Rightarrow \quad a = \frac{Mg\sin\theta}{\left(M+\frac{I}{R^2}\right)}$$

If k is the radius of gyration of the rolling body, then

$$I = Mk^2$$

$$\Rightarrow \quad a = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)} \qquad \ldots(2)$$

Put (2) in (1), we get

$$f = \frac{Mg\sin\theta}{\left(1+\frac{R^2}{k^2}\right)} \qquad \ldots(3)$$

METHOD II: Using the concept of IAOR

At the point of contact velocity is zero, so the net torque due to the various forces about the IAOR is

$$\tau = (Mg\sin\theta)R = I_{IAOR}\alpha$$

$$\Rightarrow \quad (\not{M}g\sin\theta)R = (\not{M}R^2 + \not{M}k^2)\frac{a}{R}$$

$$\Rightarrow \quad a = \frac{g\sin\theta R^2}{R^2+k^2}$$

$$\Rightarrow \quad a = \frac{g\sin\theta}{1+\frac{k^2}{R^2}}$$

$$\Rightarrow \quad f = \frac{Ia}{R^2} = \frac{(MR^2)}{R^2}\frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}$$

$$\Rightarrow \quad f = \frac{Mg\sin\theta}{\left(1+\frac{R^2}{k^2}\right)}$$

If μ be the coefficient of friction, then

$$\mu = \frac{f}{N}, \text{ where } N = Mg\cos\theta$$

Using (3), we get

$$\mu = \frac{\tan\theta}{\left(1+\frac{R^2}{k^2}\right)} \qquad \ldots(4)$$

CONDITION FOR A BODY TO ROLL WITHOUT SLIPPING

For a body to roll without slipping, the force of friction f calculated above must be less than or equal to the maximum value of friction i.e. $\mu Mg\cos\theta$.

$$\Rightarrow \quad \frac{Mg\sin\theta}{\left(1+\frac{R^2}{k^2}\right)} \le \mu Mg\cos\theta$$

$$\Rightarrow \quad \mu \ge \frac{\tan\theta}{\left(1+\frac{R^2}{k^2}\right)}$$

For various rolling bodies a, f and the condition to roll without slipping are shown in the table.

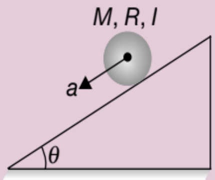
From table, we conclude

$$a_{\text{SPHERE}} > a_{\text{DISC}} = a_{\text{CYLINDER}} > a_{\text{SHELL}} > a_{\text{RING}}$$

Hence $t_{\text{SPHERE}} < t_{\text{DISC}} = t_{\text{CYLINDER}} < t_{\text{SHELL}} < t_{\text{RING}}$

i.e. the sphere reaches the first and the ring reaches the last down an incline (while rolling).

OBSERVATION



For a body rolling on a surface,

(a) if the surface is smooth, then, $a = g\sin\theta$ if surface is smooth

(b) if pure rolling takes place, then friction must be sufficient enough to prevent slipping, i.e. $\mu > \mu_{\text{min}}$ then

$$a = \frac{g\sin\theta}{\left(1+\frac{I}{MR^2}\right)} = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}$$

and $f = \dfrac{mg\sin\theta}{\left(1+\frac{R^2}{k^2}\right)}$

(c) if the surface is rough, but friction is insufficient to prevent slipping i.e. $\mu < \mu_{\text{min}}$ then forward slipping will take place and maximum friction will act in this case and then the acceleration of the body is $a = g\sin\theta - \mu g\cos\theta$

The various parameters for bodies of different shapes are given in the Table.

Condition to Roll Without Slipping	Ring $\left(\frac{k^2}{R^2}=1\right)$	Disc $\left(\frac{k^2}{R^2}=\frac{1}{2}\right)$	Cylinder $\left(\frac{k^2}{R^2}=\frac{1}{2}\right)$	Shell $\left(\frac{k^2}{R^2}=\frac{2}{3}\right)$	Sphere $\left(\frac{k^2}{R^2}=\frac{2}{5}\right)$
$a = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}$	$a = \frac{1}{2}g\sin\theta$	$a = \frac{2}{3}g\sin\theta$	$a = \frac{2}{3}g\sin\theta$	$a = \frac{3}{5}g\sin\theta$	$a = \frac{5}{7}g\sin\theta$
$f = \frac{Mg\sin\theta}{\left(1+\frac{R^2}{k^2}\right)}$	$f = \frac{1}{2}Mg\sin\theta$	$f = \frac{1}{3}Mg\sin\theta$	$f = \frac{1}{3}Mg\sin\theta$	$f = \frac{2}{5}Mg\sin\theta$	$f = \frac{2}{7}Mg\sin\theta$
Condition to Roll without Slipping is $\theta = \tan^{-1}\left[\mu\left(1+\frac{R^2}{k^2}\right)\right]$	$\theta = \tan^{-1}(2\mu)$	$\theta = \tan^{-1}(3\mu)$	$\theta = \tan^{-1}(3\mu)$	$\theta = \tan^{-1}\left(\frac{5\mu}{2}\right)$	$\theta = \tan^{-1}\left(\frac{7\mu}{2}\right)$

ILLUSTRATION 73

A shell is released from the top of an inclined plane of inclination θ. Calculate the minimum coefficient of friction between the shell and the plane to prevent sliding. Assuming the friction coefficient between the shell to be half the calculated minimum value, then calculate kinetic energy of the shell as it moves a length l down the incline.

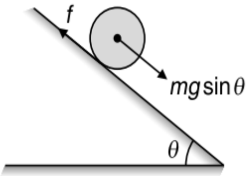
SOLUTION

For pure rolling of shell, we have

$$mg\sin\theta - f = ma_{cm}$$

where, $a_{cm} = \dfrac{g\sin\theta}{1+\left(k^2/R^2\right)}$

For hollow sphere $\dfrac{k^2}{R^2} = \dfrac{2}{3}$, so we get



$$mg\sin\theta - f = m\left(\frac{g\sin\theta}{1+(2/3)}\right)$$

$$\Rightarrow \quad f = \frac{2}{5}mg\sin\theta$$

For no sliding to take place, we must have

$$f < f_l\,(= \mu mg\cos\theta)$$

$$\Rightarrow \quad \frac{2}{5}mg\sin\theta < \mu mg\cos\theta$$

$$\Rightarrow \quad \mu > \frac{2}{5}\tan\theta$$

$$\Rightarrow \quad \mu_{min} = \frac{2}{5}\tan\theta$$

If $\mu = \dfrac{\mu_{min}}{2} = \dfrac{1}{5}\tan\theta$, then friction on shell is

$$f = \mu mg\cos\theta = \frac{1}{5}mg\sin\theta$$

Acceleration of shell is

$$a = g\sin\theta - \mu g\cos\theta = \frac{4}{5}g\sin\theta$$

Angular acceleration of shell is

$$\alpha = \frac{fR}{I} = \frac{\frac{1}{5}mgR\sin\theta}{\frac{2}{3}mR^2} = \frac{3}{10}\frac{g\sin\theta}{R}$$

Time of sliding is

$$t = \sqrt{\frac{2l}{a}} = \sqrt{\frac{5l}{2g\sin\theta}}$$

Speed attained by shell as it travels a distance l of inclined is

$$v = at$$

$$\Rightarrow \quad v = \sqrt{\frac{8}{5}gl\sin\theta}$$

Angular speed attained by shall as it travels a distance l

$$\omega = \alpha t$$

$$\Rightarrow \quad \omega = \frac{3g\sin\theta}{10R}\sqrt{\frac{5l}{2g\sin\theta}}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{6gl\sin\theta}{40R^2}}$$

Final kinetic energy of balls is

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

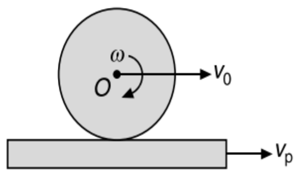
$$\Rightarrow \quad K = \frac{1}{2}m\left(\frac{8gl\sin\theta}{5}\right) + \frac{1}{2}\left(\frac{2}{3}mR^2\right)\left(\frac{9gl\sin\theta}{40R^2}\right)$$

$$\Rightarrow \quad K = \left(\frac{4}{5}+\frac{3}{40}\right)gl\sin\theta$$

$$\Rightarrow \quad K = \frac{7}{8}mgl\sin\theta$$

ROLLING WHEEL ON A MOVABLE PLANK

Till now we have seen round objects rolling on a fixed surface. Surfaces being either horizontal or inclined but fixed. But what if the surfaces themself move.



Consider a round body of radius R moving on a plank. Let the centre of the round object has velocity v_0 and the round object has angular velocity ω. The plank is moving with a speed v_p as shown. Now for pure rolling, the two points in contact should have same velocity. Thus

$$v_0 - R\omega = v_p$$

and $\quad a_0 - R\alpha = a_p$

where a_0 is acceleration of centre, α is angular acceleration and a_p is acceleration of plank.

For forward slipping, we have

$$v_0 - R\omega > v_p$$

$$a_0 - R\alpha > a_p$$

if v_0, ω and v_p satisfy the relation $v_0 - R\omega = v_p$.

For backward slipping, we have

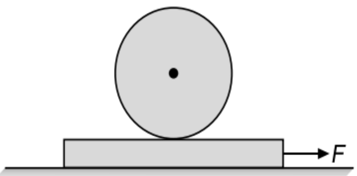
$$v_0 - R\omega < v_p$$

$$a_0 - R\alpha < a_p$$

if v_0, ω and v_p satisfy the relation $v_0 - R\omega = v_p$.

ILLUSTRATION 74

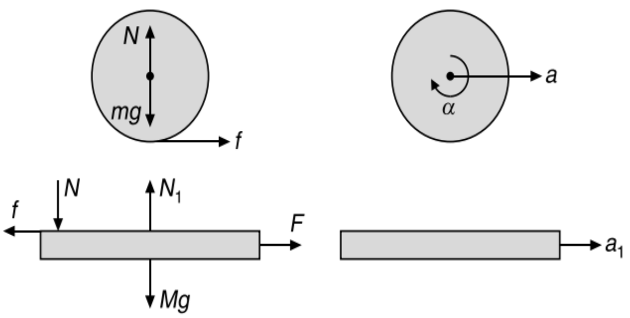
Consider a disc of mass m and radius R placed on a rough plank of mass M which in turn is placed on a smooth horizontal surface.



Now plank is pulled by a force F and disc starts to roll on the plank. If there is no friction anywhere then calculate the force of friction acting on the disc. Also calculate the angular acceleration of the disc.

SOLUTION

Let us apply Newton's laws on disc in horizontal direction



$$f = ma \qquad \ldots(1)$$

For plank, we have

$$F - f = Ma_1 \qquad \ldots(2)$$

Applying torque equation on disc about its centre of mass, we get

$$\tau = I\alpha$$

$$\Rightarrow \quad fR = \frac{mR^2}{2}\alpha \qquad \ldots(3)$$

For pure rolling, we have

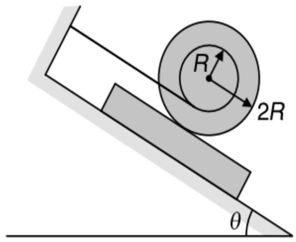
$$a_1 = a + R\alpha \qquad \ldots(4)$$

Solving the four equations, we get

$$f = \frac{Fm}{m+3M} \text{ and } \alpha = \frac{2F}{(m+3M)R}$$

ILLUSTRATION 75

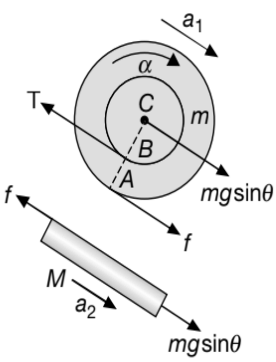
In the arrangement shown a spool of mass m, moment of inertia I is placed on a plank of mass M. There is no friction between the plank and the surface inclined at an angle θ. If friction between spool and plank is sufficient enough to prevent slipping, find the angular acceleration of the spool and the value of ratio $\left(\frac{m}{M}\right)$ for which plank will



(i) ascend
(ii) descend
(iii) will remain stationary

SOLUTION

The free body diagrams for spool and plank are shown here. According to Newton's Second Law, we have



for m, $mg\sin\theta + f - T = ma_1$ ...(1)

for M, $Mg\sin\theta - f = Ma_2$ ...(2)

Taking moment of forces about centre of spool, we get

$$T(R) - f(2R) = I\alpha \quad \text{...(3)}$$

For no slipping at point B, we have

$$a_1 - R\alpha = 0 \quad \text{...(4)}$$

For no slipping at point A, we have

$$a_1 - 2R\alpha = a_2 \quad \text{...(5)}$$

From equations (4) and (5), we get

$$a_2 = -R\alpha \quad \text{...(6)}$$

Solving above equations, we get

$$\alpha = \left(\frac{(m-M)g\sin\theta}{mR^2 + MR^2 + 1}\right)R$$

CASE-1:

For $m < M$ i.e., $\left(\frac{m}{M}\right) < 1$

$\alpha \to$ negative, $a_2 \to$ positive

So, the plank will descend.

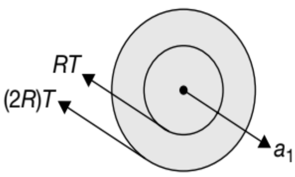
CASE-2:

For $m > M$ i.e., $\left(\frac{m}{M}\right) > 1$

α is positive, a_2 is negative

So, the plank will ascend.

CASE-3:



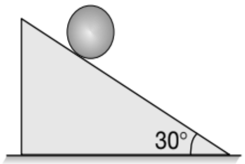
For $m = M$ i.e., $\frac{m}{M} = 1$,

$\alpha = 0$, $a_2 = 0$

So, the plank will remain stationary.

ILLUSTRATION 76

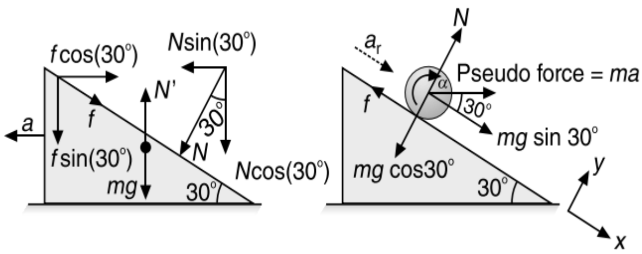
A sphere of mass 10 kg is placed on the inclined surface of a rough wedge of inclination 30°. If the mass of the wedge is also 10 kg, calculate the acceleration of the wedge when sphere is allowed to roll down without slipping. Neglect friction between ground and wedge. Take $g = 10\ \text{ms}^{-2}$.



SOLUTION

When the sphere rolls down without slipping on the wedge, the wedge exerts a normal reaction N on sphere. Consequently, the sphere exerts the same force N on wedge as shown. Furthermore, if friction f on the sphere due to the wedge is tangentially upwards, then friction on wedge due to sphere acts tangentially downwards as shown. Also N' be the normal exerted on wedge by horizontal frictionless ground and mg be the weight of the wedge. Due to these forces, if a be the acceleration of wedge towards left a_r be the acceleration of centre of mass of sphere down the plane relative to wedge. α be the angular acceleration of the sphere.

Free body diagrams for the wedge with respect to ground and the sphere with respect to wedge are shown in Figure.



For wedge, the equations of motion are

$$N' = N\cos(30°) + mg + f\sin(30°)$$

$$N\sin 30° - f\cos 30° = ma$$

$$\Rightarrow \quad \frac{N}{2} - \frac{\sqrt{3}f}{2} = 10a$$

$$\Rightarrow \quad N - \sqrt{3}f = 20a \quad \text{...(1)}$$

For sphere, the equations of motion are, $(m = 10\ \text{kg})$

$$\Sigma F_x = ma_r$$

$$\Rightarrow \quad mg\sin 30° + ma\cos 30° - f = ma_r$$

$$\Rightarrow \quad 50 + 5\sqrt{3}a - f = 10a_r \quad \text{...(2)}$$

Also, $\Sigma F_y = 0$

$\Rightarrow \quad N + ma\sin 30° = mg\cos 30°$

$\Rightarrow \quad N + 5a = 50\sqrt{3}$ ...(3)

Further, $\alpha = \dfrac{\tau}{I} = \dfrac{fR}{\frac{2}{5}mR^2} = \dfrac{f}{4R}$ ...(4)

For no slipping between the wedge and the sphere, we have

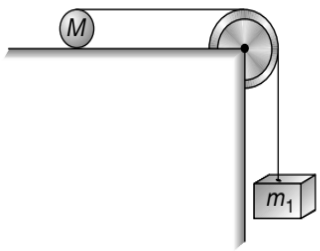
$a_r = R\alpha$ ...(5)

Solving equations (1), (2), (3), (4) and (5), we get

$a = 2.11\ \text{ms}^{-2}$

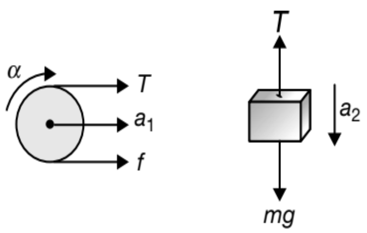
ILLUSTRATION 77

In the arrangement shown in Figure, the string is wrapped around a uniform cylinder which rolls without slipping. The other end of the string is passed over a light, frictionless pulley and is connected to a falling weight. Find the acceleration of the falling mass m in terms of only the mass of the cylinder M, the mass m and g.



SOLUTION

Let T be the tension in the string and f the force of (static) friction, between the cylinder and the surface. If a_1 be the acceleration of centre of mass of cylinder towards right a_2 be the downward acceleration of block m and α be the angular acceleration of cylinder (clockwise)



For block, $mg - T = ma_2$ ...(1)

For cylinder, $T + f = Ma_1$ ...(2)

$$\alpha = \frac{(T-f)R}{\frac{1}{2}MR^2} \quad ...(3)$$

Since the string attached the mass m to the highest point of the cylinder, so

$v_m = v_{CM} + R\omega$

Differentiating we get

$a_2 = a_1 + R\alpha$ ...(4)

For M to roll without slipping, we have

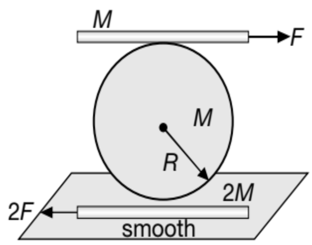
$a_1 = R\alpha$ ...(5)

Solving equations (1), (2), (3), (4) and (5), we get

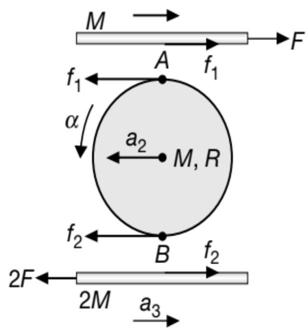
$$a_2 = \frac{8mg}{3M + 8m}$$

ILLUSTRATION 78

A cylinder of mass M, radius R is sandwiched between two planks having masses M and $2M$. Two constant horizontal forces F and $2F$ are applied on the planks as shown. Determine the acceleration of the centre of mass of cylinder, top plank and bottom plank if there is no slipping at the top and bottom of cylinder. Also find the friction between the planks and cylinder and the angular acceleration of the cylinder.



SOLUTION



Equations of motion

for plank M is, $F + f_1 = Ma_1$ ...(1)

for cylinder M is, $f_1 + f_2 = Ma_2$ ...(2)

for plank $2M$ is, $2F - f_2 = 2Ma_3$ ...(3)

Further for the cylinder, we have

$$\alpha = \frac{(f_1 - f_2)R}{\frac{1}{2}MR^2}$$

$$\Rightarrow \quad \alpha = \frac{2(f_1 - f_2)}{MR} \qquad \ldots(4)$$

For no slipping condition, at the points of contact A and B, we have

$$a_2 + R\alpha = -a_1 \qquad \ldots(5)$$

and $a_2 - R\alpha = a_3$ ...(6)

We have six unknowns, f_1, f_2, a_1, a_2, a_3 and α. Solving the above six equations, we get

$$a_1 = \frac{21F}{26M},\ a_2 = \frac{F}{26M},\ a_3 = \frac{23F}{26M}$$

$$f_1 = -\frac{5F}{26},\ f_2 = \frac{3F}{13} \text{ and } \alpha = -\frac{11F}{13MR}$$

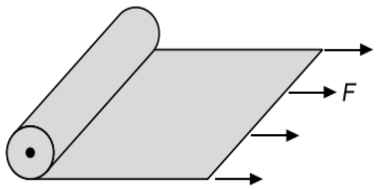
Negative sign with f_1 indicates its direction is opposite to the direction shown in Figure and negative sign with α shows it is in CW sense.

Test Your Concepts-V

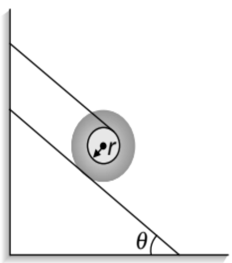
Based on Uniform and Accelerated Pure Rolling

(Solutions on page H.159)

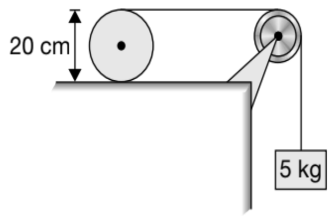
1. A heavy roll of wrapping paper of mass M, radius R, in the form of a solid cylinder is resting on a horizontal tabletop. If a horizontal force F is applied uniformly to the paper, as shown in Figure, find the linear acceleration of the centre of the roll and the angular acceleration around the centre of the roll. The coefficient of kinetic friction between the paper and the table is μ_K. Assume the roll of paper does slip on the surface.



2. A spool of thread of mass m is placed on an inclined smooth plane set at an angle θ to the horizontal. The free end of the thread is attached to the wall as shown in Figure. Find the acceleration of the centre of mass of the spool, if its moment of inertia about its axis is I and the radius of the wound thread layer is r.



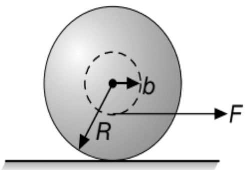
3. In Figure the cylinder of mass 10 kg and radius 10 cm has a tape wrapped round it. The pulley weighs 100 N and has a radius 5 cm. When the system is released, the 5 kg mass comes down and the cylinder rolls without slipping. Find the acceleration and velocity of the mass as a function of time.



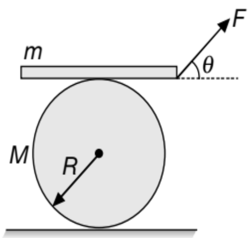
4. A lawn roller in the form of a thin-walled hollow cylinder of mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.
5. A solid cylindrical wheel of mass M and radius R is pulled by a force F applied to the centre of the wheel at 37° to the horizontal. If the wheel is to roll without slipping, what is the maximum value of $|F|$? The coefficients of static and kinetic friction are $\mu_S = 0.40$ and $\mu_K = 0.30$. $\sin(37°) = \frac{3}{5}$.
6. A sphere, a disc, and a hoop made of homogeneous materials have the same radius (10 cm) and mass (3 kg). They are released from rest at the top of a 30° incline and roll down without slipping through a vertical distance of 2 m.

(a) What are their speeds at the bottom?
(b) Find the frictional force f in each case.
(c) If they start together at $t=0$, at what time does each reach the bottom?

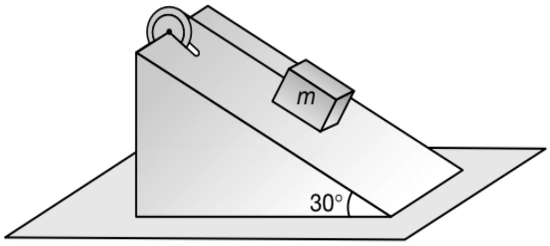
7. A Yo-Yo of mass M has an axle of radius b and a spool of radius R. Its moment of inertia can be taken to be $\frac{MR^2}{2}$. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force F as shown. The coefficient of friction between the Yo-Yo and the table is μ. What is the maximum value of F for which the Yo-Yo will roll without slipping?



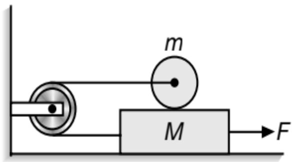
8. A cylinder of mass M and radius R is lying on a rough horizontal plane. It has a plank lying at its top as shown in Figure. A force F is applied on the plank such that the plank moves and causes the cylinder to roll. The plank always remains horizontal. There is no slipping at any point of contact. Find the acceleration of the cylinder and the frictional forces at the two contacts.



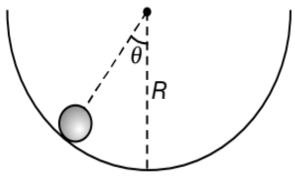
9. A block of mass $m=1\text{ kg}$ slides down the surface of a smooth incline as shown in Figure. The block is tied to a string which is wrapped around a disk capable of rotating about a horizontal axis. The disk has a mass $M=5\text{ kg}$ and a radius $R=0.2\text{ m}$. Initially the string is taut. If the mass is released, calculate its acceleration. Take $g=9.8\text{ ms}^{-2}$.



10. Find the acceleration of the cylinder of mass m and radius R and that of plank of mass M placed on smooth surface if pulled with a force F shown in Figure. Given that sufficient friction is present between cylinder and the plank surface to prevent sliding of cylinder.



11. A ball of mass m and radius r rolls along a circular path of radius R. Its speed at the bottom $(\theta=0)$ of the path is v_0. Find the tangential and the normal force exerted by the path on the ball as a function of θ.



ANGULAR MOMENTUM ($\vec{L}$)

The rotational effect of linear momentum is called Angular Momentum. When a body rotates about some point/axis, then the momentum associated with the body due to its rotation is called Angular Momentum. An external torque is required to change angular momentum just like an external force is required to change the linear momentum.

CASE-1: Angular Momentum of a Particle About Some Point

Angular Momentum, $\vec{L}$ of a particle about an arbitrary point O is the moment of linear momentum taken about that point.

$$L=\begin{pmatrix}\text{Linear}\\\text{Momentum}\end{pmatrix}\perp\begin{pmatrix}\text{Distance of}\\\vec{p}\text{ from }O\end{pmatrix}$$

$$\Rightarrow \quad L=pr_{\perp}$$

$\Rightarrow \quad L = pr\sin\theta$ {in magnitude}

where $r_\perp = r\sin\theta$ is called the moment arm.

Vectorially

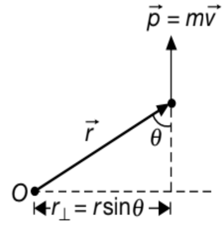
$$\vec{L} = \vec{r} \times \vec{p}$$

$$\Rightarrow \quad \vec{L} = m(\vec{r} \times \vec{v})$$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, then

$$\vec{L} = m\begin{vmatrix} \overset{\oplus}{\hat{i}} & \overset{\ominus}{\hat{j}} & \overset{\oplus}{\hat{k}} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

The direction of $\vec{L}$ is also found by Right Hand Thumb Rule.

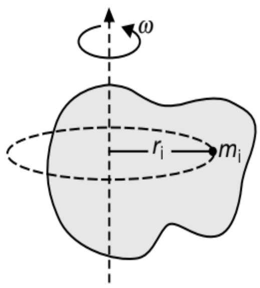


The SI unit of angular momentum is kgms^{-2}.

Please note that angular momentum is defined always with respect to a point.

CASE-2: Angular Momentum of a Rigid Body Rotating About a Fixed Axis

The total angular momentum $\vec{L}$ of a system of particles relative to a given origin is the sum of the angular momentum of the particles.



A rigid body rotating about a fixed axis

$$\vec{L} = \Sigma(\vec{r}_i \times \vec{p}_i)$$

Since $\vec{r}_i$ and $\vec{p}_i$ are perpendicular, so

$L = \Sigma r_i p_i$, where $p_i = m_i v_i = m_i r_i \omega$

$$\Rightarrow \quad L = \left(\Sigma m_i r_i^2\right)\omega$$

$\Rightarrow \quad L = I\omega$, where $I = \Sigma m_i r_i^2$

CASE-3: Angular Momentum of System in Combined Rotation and Translation

If a body is in combined rotation and translation like rolling and we are asked to find the angular momentum of the body about any fixed point or a reference point, then

$$\vec{L}_{\text{total}} = \vec{L}_{\text{CM}} + M\left(\vec{r}_{\text{CM}} \times \vec{v}_{\text{CM}}\right)$$

i.e., total angular momentum of system/body is equal to the sum of the angular momentum of the CM about that point and the angular momentum of the system about the CM.

So, for a rigid body undergoing linear and rotational motion, the total angular momentum may be split into two parts

(a) the orbital angular momentum, $\vec{L}_O$ and

(b) the spin angular momentum $\vec{L}_{\text{cm}}$.

The orbital angular momentum is the angular momentum of the centre of mass motion about an origin O in an inertial frame.

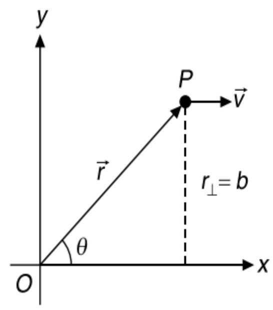
The spin angular momentum is the angular momentum relative to the centre of mass.

The orbital term treats the system as a point particle at the centre of mass, whereas the spin term is the sum of the angular momenta of the particles relative to the centre of mass. The total angular momentum relative to the origin O in an inertial frame is the sum of both the angular momenta i.e.,

$$\vec{L} = \vec{L}_{\text{Orbital}} + \vec{L}_{\text{Spin}} = \vec{L}_O + \vec{L}_{\text{cm}}$$

ILLUSTRATION 79

A particle of mass m is projected from origin O with speed u at an angle θ with positive x-axis. Positive y-axis is in vertically upward direction. Find the angular momentum of particle at any time t about O before the particle strikes the ground again.



SOLUTION

$$\vec{L} = m(\vec{r} \times \vec{v})$$

Here, $\vec{r}(t) = x\hat{i} + y\hat{j} = (u\cos\theta)t\hat{i} + \left(ut\sin\theta - \frac{1}{2}gt^2\right)\hat{j}$

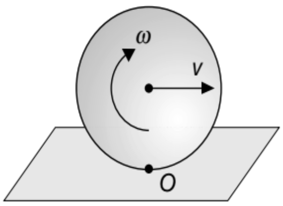
and $\vec{v}(t) = v_x\hat{i} + v_y\hat{j} = (u\cos\theta)\hat{i} + (u\sin\theta - gt)\hat{j}$

$$\Rightarrow \quad \vec{L} = m(\vec{r} \times \vec{v}) = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (u\cos\theta)t & (u\sin\theta)t - \frac{1}{2}gt^2 & 0 \\ u\cos\theta & u\sin\theta - gt & 0 \end{vmatrix}$$

$$\Rightarrow \quad \vec{L} = -\frac{1}{2}m(u\cos\theta)gt^2\hat{k}$$

ILLUSTRATION 80

A solid sphere of mass M and radius R rolls without slipping on a horizontal surface as shown in Figure. Find the total angular momentum of the sphere with respect to the origin O fixed on the ground.



SOLUTION

Let us assume the clockwise sense of rotation positive.

Orbital angular momentum about O is $L_O = MvR$

Spin angular momentum about centre of mass is

$$L_{cm} = I\omega = \frac{2}{5}MR^2\omega$$

The total angular momentum is

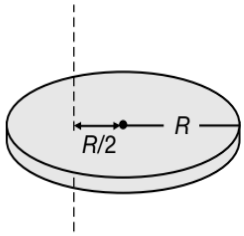
$$L = L_O + L_{cm} = MvR + \frac{2}{5}MvR$$

For pure rolling, we have $v = \omega R$

$$\Rightarrow \quad L = \frac{2}{5}MvR$$

ILLUSTRATION 81

A disc of mass M and radius R rotating at an angular velocity ω about an axis perpendicular to its plane at a distance $\frac{R}{2}$ from the centre, as shown in Figure. Calculate the angular momentum of the disc about the axis shown.



SOLUTION

The moment of inertia of the disc about the given axis may be found from the parallel axes theorem, equation $I = I_{cm} + Md^2$, where h is the distance between the given axis and a parallel axis through the centre of mass.

Here $h = \frac{R}{2}$, therefore, $I = \frac{1}{2}MR^2 + M\left(\frac{R}{2}\right)^2 = \frac{3}{4}MR^2$

The angular momentum is $L = I\omega = \frac{3}{4}MR^2\omega$

RELATION BETWEEN $\vec{L}$ AND $\vec{\tau}$

As force changes the linear momentum of a particle, torque changes the angular momentum of a particle.

Since, $\vec{L} = \vec{r} \times \vec{p}$ ...(1)

The rate of change of angular with time is found by taking the derivative on both sides of (1) with respect to time. So

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

Using, $\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$, we get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times \vec{p} \qquad \left\{\because \frac{d\vec{p}}{dt} = \vec{F} \text{ and } \frac{d\vec{r}}{dt} = \vec{v}\right\}$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + m(\vec{v} \times \vec{v}) \qquad \{\because \vec{p} = m\vec{v}\}$$

Since, $\vec{v} \times \vec{v} = \vec{0}$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

$$\Rightarrow \quad \vec{\tau} = \frac{d\vec{L}}{dt}$$

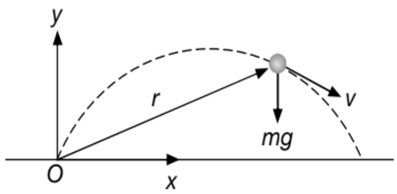
ILLUSTRATION 82

Show that the equation $\vec{\tau} = \frac{d\vec{L}}{dt}$, can be applied to the motion of a projectile.

SOLUTION

The change in angular momentum of the projectile is produced by the torque exerted by the force of gravity.

In Figure, we take the initial point as the origin. At a later time, $\vec{r} = x\hat{i} + y\hat{j}$.



Since the force on the particle is $\vec{F} = -mg\hat{j}$, the gravitational torque on it is

$$\vec{\tau} = \left(x\hat{i} + y\hat{j}\right) \times (-mg\hat{j}) = -mgx\hat{k}$$

The rate of change of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = m\vec{r} \times \frac{d\vec{v}}{dt}$$

But the acceleration is $\frac{d\vec{v}}{dt} = -g\hat{j}$. So,

$$\frac{d\vec{L}}{dt} = m\vec{r} \times \frac{d\vec{v}}{dt} = m\left(x\hat{i} + y\hat{j}\right) \times \left(-g\hat{j}\right) = -\left(mgx\right)\hat{k}$$

Hence the equation $\vec{\tau} = \frac{d\vec{L}}{dt}$ is applicable here.

ANGULAR IMPULSE

In complete analogy with the linear momentum, angular impulse is defined as

$$\Delta\vec{L} = \int \vec{\tau}_{\text{ext}} dt$$

Since we know that linear impulse $\vec{J}$ equals the change in linear momentum, so we have

$$\vec{J} = \vec{F}\Delta t = \Delta\vec{p} = m\left(\vec{v}_f - \vec{v}_i\right)$$

In one dimension, we can simply write this as

$$J = \Delta p = p_f - p_i = m\left(v_f - v_i\right)$$

If $v_i = 0$ and $v_f = v$, then we have $J = mv$

$$\Rightarrow \quad v = \frac{J}{m}$$

Just like translational the impulse momentum theorem in translational motion, we have angular impulse angular momentum theorem in rotation according to which angular impulse equals the change in angular momentum of the body and hence

$$\text{Angular Impulse (AI)} = \tau\Delta t = \Delta L$$

$$\Rightarrow \quad \text{AI} = \tau\Delta t = \Delta L = L_f - L_i = I\left(\omega_f - \omega_i\right)$$

Since, $\tau = Fr_\perp$

$$\Rightarrow \quad \text{AI} = \left(Fr_\perp\right)\Delta t = Jr_\perp \qquad \{\because F\Delta t = J\}$$

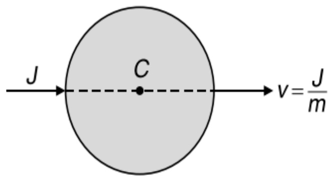
$$\Rightarrow \quad \text{AI} = Jr_\perp = L_f - L_i = I\left(\omega_f - \omega_i\right)$$

If $L_i = 0$, then $L_f = L = I\omega$

$$\Rightarrow \quad \text{AI} = Jr_\perp = I\omega$$

$$\Rightarrow \quad \omega = \frac{Jr_\perp}{I}$$

In Figure (a): A linear impulse J is applied at centre of mass C of the rigid body.

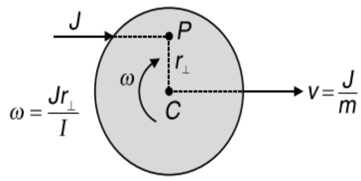


(a)

Just after hitting, it will have only translational motion and its linear velocity will be given by

$$v = \frac{J}{m}$$

In Figure (b): A linear impulse J is applied at point P, at a perpendicular distance $r_\perp = CP$.



(b)

Just after hitting it will have both translational and rotational motion. Its linear velocity v and angular velocity ω will be given by

$$v = \frac{J}{m} \text{ and } \omega = \frac{Jr_\perp}{I}$$

If $r_\perp$ is increased (keeping J to be constant) then v will remain same but ω will increase. So, the translational kinetic energy will have the same value but rotational kinetic energy will be more.

Problem Solving Technique(s)

Since, Angular Impulse (AI) $= \tau\Delta t = \Delta L$

Now, following three cases can be considered.

(a) If torque is constant, then angular impulse (AI) is obtained by directly multiplying this constant torque with the given time interval.

(b) If torque is a function of time, then angular impulse (AI) is given by

$$\text{Angular Impulse (AI)} = \int_{t_i}^{t_f} \tau dt$$

(c) If torque versus time graph is given, then angular impulse (AI) can be obtained by the area under that graph.

In all three cases, angular impulse is equal to the change in angular momentum.

ANGULAR IMPULSE-ANGULAR MOMENTUM THEOREM

This theorem is the rotational analogue of the Impulse Momentum theorem studied already in Newton's Laws of Motion. Using Newton's Second Law for Rotation Motion, we have

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

$$d\vec{L} = \vec{\tau} dt$$

$$\Rightarrow \quad \Delta\vec{L} = \vec{L}_f - \vec{L}_i = I\left(\vec{\omega} - \vec{\omega}_0\right) = \int \vec{\tau} dt$$

The net angular impulse acting on a rigid body is equal to the change in angular momentum of the body. This is called the **Angular Impulse Angular Momentum** theorem for rotational dynamics.

Please be careful to understand and see the hidden fundamental of the axis of rotation i.e. all $\vec{L}_f$, $\vec{L}_i$ and $\vec{\tau}$ must be about identical AOR to use the results as they are. Else suitable modifications have to be made in the results.

LAW OF CONSERVATION OF ANGULAR MOMENTUM

Since, by definition we know that

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad \ldots(1)$$

For no external torque acting on system, we have

$$\vec{\tau} = \vec{0}$$

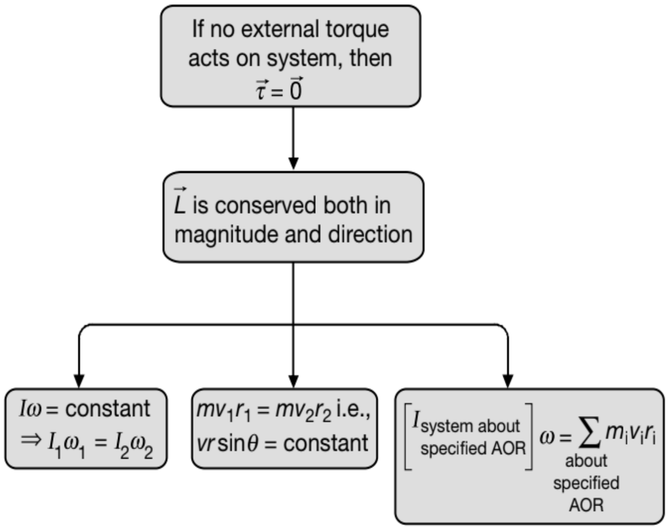
So, (1) gives

$$\vec{0} = \frac{d\vec{L}}{dt}$$

$\Rightarrow \quad \vec{L}$ = constant (both in magnitude and direction)

So, for no external torque acting on the system $\vec{L}$ is conserved both in magnitude and direction.

This law is the rotational analogue of the Law of Conservation of Linear Momentum.



Analogy between Rotational and Linear Dynamics.

Quantity	Linear	Rotational
1. Inertia	m	$\sum m_i r_i^2$ or $\int r^2 dm$
2. Newton's Second Law	$F_{\text{ext}} = ma$ $\vec{F}_{\text{ext}} = \frac{d\vec{p}}{dt}$	$\tau_{\text{ext}} = I\alpha$ $\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$

(Continued)

Quantity	Linear	Rotational
3. Work	$W_{lin} = \int \vec{F} \cdot d\vec{l}$	$W_{rot} = \int \tau d\theta$
4. Kinetic Energy	$K_{lin} = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$
5. Work Energy Theorem	$W_{lin} = \Delta K_{lin}$	$W_{rot} = \Delta K_{rot}$
6. Impulse	$I = \int F_{ext}dt = \Delta p$	$J = \int \tau_{ext}dt = \Delta L$
7. Momentum	$p = mv$	$L = I\omega$
8. Impulse Momentum Theorem	$\vec{I} = \Delta\vec{p}$	$\vec{J} = \Delta\vec{L}$
9. Power	$P = \vec{F} \cdot \vec{v}$	$P = \vec{\tau} \cdot \vec{\omega}$

ILLUSTRATION 83

A disc of moment of inertia $4\ \text{kgm}^2$ is spinning freely at $3\ \text{rads}^{-1}$. A second disc of moment of inertia $2\ \text{kgm}^2$ slides down the spindle and they rotate together.

(a) What is the angular velocity of the combination?
(b) What is the change in kinetic energy of the system?

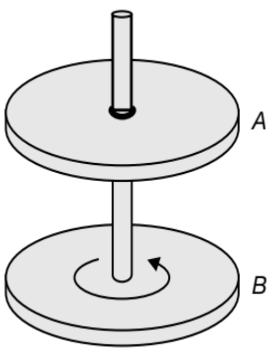
SOLUTION

(a) Since there are no external torques acting, we may apply the conservation of angular momentum.

$$I_f\omega_f = I_i\omega_i$$

$$(6\ \text{kgm}^2)\omega_f = (4\ \text{kgm}^2)(3\ \text{rads}^{-1})$$

Thus, $\omega_f = 2\ \text{rads}^{-2}$



(b) The kinetic energies before and after the collision are

$$K_i = \frac{1}{2}I_i\omega_i^2 = 18\ \text{J}$$

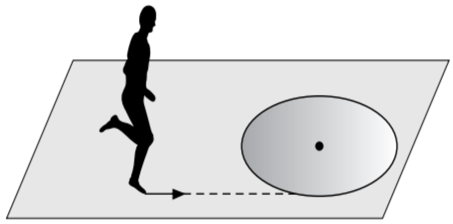
$$K_f = \frac{1}{2}I_f\omega_f^2 = 12\ \text{J}$$

The change is $\Delta K = K_f - K_i = -6\ \text{J}$.

In order for the two discs to spin together at the same rate, there had to be friction between them. The lost kinetic energy is converted with thermal energy.

ILLUSTRATION 84

A man of mass $m = 80$ kg runs at a speed $u = 4\ \text{ms}^{-1}$ along the tangent to a disc-shaped platform of mass $M = 160$ kg and radius $R = 2$ m. The platform is initially at rest but can rotate freely about an axis through its center. Take $I = \frac{1}{2}MR^2$.



Calculate the angular velocity of the platform after the man jumps on. If the man now walks to the centre, then calculate the new angular velocity. Treat the man as a point particle.

SOLUTION

Before trying to attempt this problem let us have a self-analysis done and answer the following questions.

Q. Can we apply the conservation of linear momentum?

A. No, it cannot be applied because the axle exerts an external force on the system i.e. man + platform.

Q. Can we apply the conservation of angular momentum?

A. Yes, since the axle does not exert any torque, we may use the conservation of angular momentum.

Q. Can we apply conservation of kinetic energy for the collision between the man and the platform?

A. No.

Let us consider the origin to be at the centre of platform. When the man runs in a straight line along the tangent, then his initial angular momentum about this origin is $L = r_{\perp}p$, where $r_{\perp} = R$ so

$$L_i = muR$$

After he jumps on, one must take into account his contribution mR^2 to the moment of inertia. The final angular momentum, $L = I\omega$, is,

$$L_f = \left(\frac{1}{2}MR^2 + mR^2\right)\omega$$

By Law of Conservation of Angular Momentum, we have $L_f = L_i$, wo

$$\omega = \frac{mu}{\left(\frac{M}{2}+m\right)R}$$

Substituting, $m = 80$ kg, $M = 160$ kg, $u = 4\text{ ms}^{-1}$, $R = 2$ m, we get

$$\omega = \frac{(80)(4)}{\left(\frac{160}{2}+80\right)2} = 1\text{ rads}^{-1}$$

When the man reaches the center, his contribution to the moment of inertia is zero. The final angular momentum and the initial momentum in this case are given by

$$L_i = \left(\frac{1}{2}MR^2 + mR^2\right)\omega_1 = 640\text{ kgm}^2\text{s}^{-1}$$

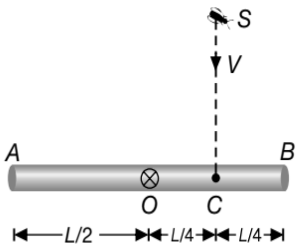
$$L_f = \left(\frac{MR^2}{2}\right)\omega_2 = 320\omega_2$$

Applying angular momentum conservation $L_f = L_i$ we get

$$\omega_2 = 2\text{ rads}^{-2}$$

ILLUSTRATION 85

A homogeneous rod AB of length $L = 1.8$ m and mass M is pivoted at the centre O in such a way that it can rotate freely in the vertical plane (shown in Figure). The rod is initially in the horizontal position. An insect S of the same mass M falls vertically with speed V on the point C, midway between the points O and B. Immediately after falling, the insect moves towards the end B such that the rod rotates with a constant angular velocity ω.

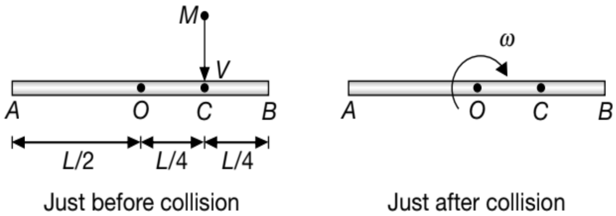


(a) Determine the angular velocity ω in terms of V and L.

(b) If the insect reaches the end B when the rod has turned through an angle of 90°, determine V.

SOLUTION

In this problem we will denote angular momentum by its standard symbol H because L has been used for length of the rod.



Angular momentum of the system (rod + insect) about the centre of the rod O will remain conserved just before collision and after collision i.e., $H_i = H_f$.

$$\Rightarrow \quad MV\frac{L}{4} = I\omega = \left[\frac{ML^2}{12} + M\left(\frac{L}{4}\right)^2\right]\omega$$

$$\Rightarrow \quad MV\frac{L}{4} = \frac{7}{48}ML^2\omega$$

$$\Rightarrow \quad \omega = \frac{12}{7}\frac{V}{L} \qquad \ldots(1)$$

Due to the torque of weight of insect about O, angular momentum of the system will not remain conserved (although angular velocity ω is constant). As the insect moves towards B, moment of inertia of the system increases, hence, the angular momentum of the system will increase.

Let at time t the insect be at a distance x from O and by then the rod has rotated through an angle θ. Then, angular momentum at that moment,

$$H = \left[\frac{ML^2}{12} + Mx^2\right]\omega$$

$$\Rightarrow \frac{dH}{dt} = 2M\omega x \frac{dx}{dt} \qquad \{\omega = \text{constant}\}$$

$$\Rightarrow \tau = 2M\omega x \frac{dx}{dt} \qquad \left\{\because \frac{dH}{dt} = \tau\right\}$$

$$\Rightarrow Mgx\cos\theta = 2M\omega x \frac{dx}{dt}$$

$$\Rightarrow dx = \left(\frac{g}{2\omega}\right)\cos\theta\, dt \qquad \{\because \theta = \omega t\}$$

$$\Rightarrow \frac{dx}{d\theta}\frac{d\theta}{dt} = \frac{g}{2\omega}\cos\theta$$

$$\Rightarrow \omega\frac{dx}{d\theta} = \frac{g}{2\omega}\cos\theta$$

$$\Rightarrow \int_{L/4}^{L/2} dx = \frac{g}{2\omega^2}\int_0^{\pi/2}\cos\theta\, d\theta$$

$$\Rightarrow x\Big|_{L/4}^{L/2} = \frac{g}{2\omega^2}\sin\theta\Big|_0^{\pi/2}$$

$$\Rightarrow \omega = \sqrt{\frac{2g}{L}}$$

Substituting in equation (1), we get

$$\sqrt{\frac{2g}{L}} = \frac{12}{7}\frac{V}{L}$$

$$\Rightarrow V = \frac{7}{12}\sqrt{2gL} = \frac{7}{12}\sqrt{2\times 10\times 1.8} = 3.5\ \text{ms}^{-1}$$

$$\Rightarrow V = 3.5\ \text{ms}^{-1}$$

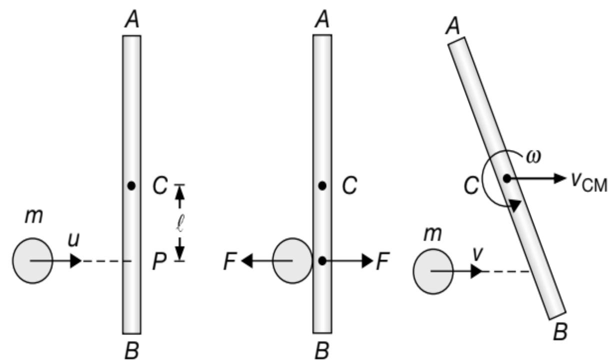
BALL ROD COLLISIONS

Consider a uniform rod AB of mass M, length L. Let a ball of mass m moving with initial velocity v, hit the rod at a point P (other than centre of mass of the rod). Now two cases arise.

CASE-1: When the Rod is not Hinged

For Ball

$$mv - mu = -Ft \qquad \ldots(1)$$



Just before impact   During impact   Just after impact

For Rod

$$mv_{cm} - 0 = Ft \qquad \ldots(2)$$

From (1) and (2), we get

$$mv - mu = -mv_{cm}$$

$$\Rightarrow mu = mv_{cm} + mv \qquad \ldots(3)$$

$$\begin{pmatrix}\text{Total Initial}\\ \text{Momentum of}\\ \text{Ball + Rod}\end{pmatrix} = \begin{pmatrix}\text{Total Final}\\ \text{Momentum of}\\ \text{Ball + Rod}\end{pmatrix}$$

The total linear momentum of Ball + Rod system is conserved.

Using Angular Impulse – Angular Momentum Theorem, we get

For Ball

$$(mv)l - (mu)l = -\tau t$$

$$\Rightarrow (mv)l - (mu)l = -(Fl)t \qquad \ldots(4)$$

For Rod

$$I\omega - 0 = (Fl)t \qquad \ldots(5)$$

From (4) and (5), we get

$$(mv)l - (mu)l = -I\omega$$

$$\Rightarrow (mu)l = I\omega + (mv)l \qquad \ldots(6)$$

$$\begin{pmatrix}\text{Total Initial}\\ \text{Angular}\\ \text{Momentum}\\ \text{of Ball + Rod}\end{pmatrix} = \begin{pmatrix}\text{Total Final}\\ \text{Angular}\\ \text{Momentum}\\ \text{of Ball + Rod}\end{pmatrix}$$

So, the total Angular Momentum of Ball + Rod is conserved.

So, if the rod is not hinged, then we observe both linear and angular momentum of Rod + Ball to be conserved.

Subcase-1(a)

If the collision is elastic, then we have

$$\left(\begin{array}{c}\text{Total Initial Energy}\\ \text{of Ball + Rod}\end{array}\right)=\left(\begin{array}{c}\text{Total Final Energy}\\ \text{of Ball + Rod}\end{array}\right)$$

$$\Rightarrow \quad \frac{1}{2}mu^2+0=\frac{1}{2}mv^2+\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\omega^2$$

Subcase-1(b)

If collision is inelastic

$$e=-\left[\frac{(v_2)_n-(v_1)_n}{(u_2)_n-(u_1)_n}\right]_{\text{at the point of impact}}$$

$$\Rightarrow \quad e=-\left[\frac{(v_{cm}+l\omega)-v}{0-u}\right]$$

(Net velocity of rod at P will be $v_{cm}+l\omega$, as it is under combined influence of translation and rotation.

Subcase-1(c)

For perfectly inelastic collision, we have

$$e=0$$

$$\Rightarrow \quad v=v_{cm}+l\omega$$

CASE-2: When the Rod is Hinged

For Ball

Using Impulse – Momentum Theorem, we get

$$mv-mu=-Ft \qquad \ldots(1)$$

Using Angular Impulse – Angular Momentum Theorem, about hinge, we get

$$(mv)l-(mu)l=-\tau t=-(Fl)t \qquad \ldots(2)$$

For Rod

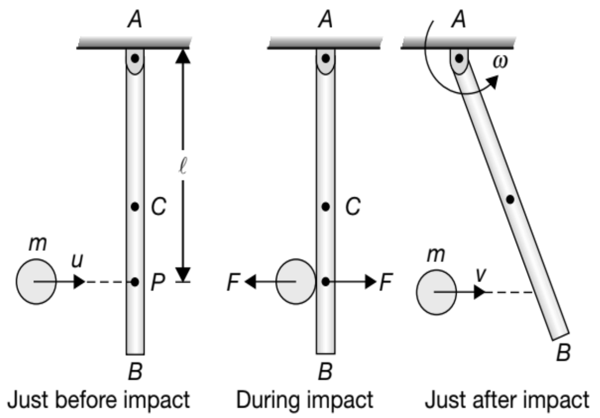
Using Angular Impulse – Angular Momentum Theorem, about hinge, we get

$$I\omega-0=(Fl)t \qquad \ldots(3)$$

From (2) and (3), we get

$$(mu)l=(mv)l+I\omega \qquad \{\text{about Hinge}\}$$

So, if the rod is hinged, then only angular momentum is conserved only about the hinge because τ due to the forces acting on the hinge about the hinge is zero.



Subcase-2(a)

If the collision is elastic, then we have

$$\frac{1}{2}mv^2=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2$$

Subcase-2(b)

If the collision is inelastic, then we have

$$e=-\left[\frac{(v_2)_n-(v_1)_n}{(u_2)_n-(u_1)_n}\right]_{\text{at the point of impact}}$$

$$\Rightarrow \quad e=-\left(\frac{l\omega-v}{0-u}\right)$$

Subcase-2(c)

If the collision is perfectly inelastic, then

$$e=0$$

$$\Rightarrow \quad v=l\omega$$

ILLUSTRATION 86

A rod AB of mass M and length L is lying on a horizontal frictionless surface. A particle of mass m travelling along the surface hits the end 'A' of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is elastic. After the collision the particle comes to rest.

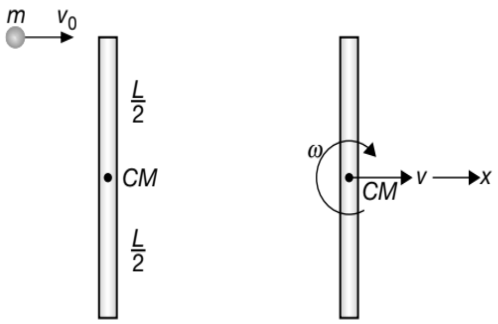
(a) Find the ratio $\frac{m}{M}$

(b) A point P on the rod is at rest immediately after collision. Find the distance AP.

(c) Find the linear speed of the point P after a time $\frac{\pi L}{3V_0}$ after the collision.

SOLUTION

(a) Let just after collision, velocity of centre of mass of rod is v and angular velocity about centre of mass is ω. Applying following three laws:



Before collision After collision

(i) External force on the system (rod + mass) in horizontal plane along x-axis is zero.

∴ Applying Conservation of Linear Momentum in x-direction.

$$mv_o = mv \quad \text{...(1)}$$

(ii) Net torque on the system about CM of rod is zero

∴ Applying Conservation of Angular Momentum about CM of rod, we get

$$mv_o\left(\frac{L}{2}\right) = I\omega$$

$$\Rightarrow mv_o\frac{L}{2} = \frac{ML^2}{12}\omega$$

$$\Rightarrow mv_o = \frac{ML\omega}{6} \quad \text{...(2)}$$

(iii) Since, the collision is elastic, kinetic energy is also conserved. So,

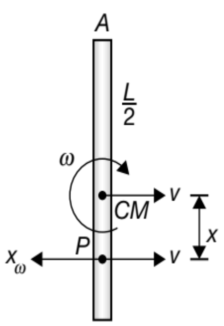
$$\frac{1}{2}mv_o^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$\Rightarrow mv_o = Mv^2 + \frac{ML^2}{12}\omega^2 \quad \text{...(3)}$$

From equations (1), (2) and (3), we get the following results

$$\frac{m}{M} = \frac{1}{4},\ v = \frac{mv_o}{M} \text{ and } \omega = \frac{6mv_o}{ML}$$

(b) Point P will be at rest if $x\omega = v$



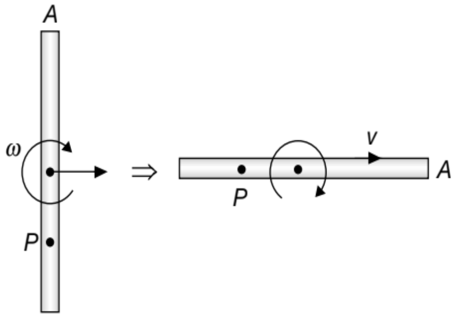
$$\Rightarrow x = \frac{v}{\omega} = \frac{\frac{mv_o}{M}}{\frac{6mv_o}{ML}}$$

$$\Rightarrow x = \frac{L}{6}$$

$$\Rightarrow AP = \frac{L}{2} + \frac{L}{6}$$

$$\Rightarrow AP = \frac{2}{3}L$$

(c) After time $t = \dfrac{\pi L}{3v_o}$



angle rotated by rod, $\theta = \omega t = \dfrac{6mv_o}{ML}\cdot\dfrac{\pi L}{3v_o}$

$$\Rightarrow \theta = 2\pi\left(\frac{m}{M}\right) = 2\pi\left(\frac{1}{4}\right)\frac{m}{M} = \frac{1}{4}$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

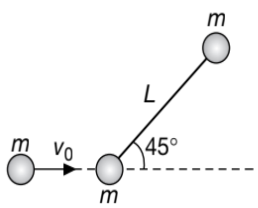
Therefore, situation will be as shown below:
So, resultant velocity of point P will be

$$|\vec{v}_P| = \sqrt{2}v = \sqrt{2}\left(\frac{m}{M}\right)v_o = \frac{\sqrt{2}}{4}v_o = \frac{v_o}{2\sqrt{2}}$$

$$\Rightarrow |\vec{v}_P| = \frac{v_o}{2\sqrt{2}}$$

ILLUSTRATION 87

A rigid massless rod of length L joins two particles each of mass m. The rod lies on a frictionless table, and is struck by a particle of mass m and velocity v_0, moving as shown. After the collision, the projectile moves straight back.



Find the angular velocity of the rod about its centre of mass after the collision, assuming that mechanical energy is conserved.

SOLUTION

Applying Law of Conservation of Linear Momentum, we get

$$mv_0 = 2mv_1 - mv_2$$

$$\Rightarrow \quad 2v_1 - v_2 = v_0 \qquad \ldots(1)$$

Applying Law of Conservation of Angular Momentum about centre of mass C of light rod and the two identical particles, we get

$$mv_0\left(\frac{L}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = 2m\left(\frac{L}{2}\right)^2\omega - mv_2\left(\frac{L}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad v_0 = \sqrt{2}L\omega - v_2 \qquad \ldots(2)$$

Since the mechanical energy is conserved so, the collision is elastic, hence $e = 1$ at point of impact along common normal direction

$$\Rightarrow \quad \begin{pmatrix}\text{Relative Speed}\\ \text{of Approach}\end{pmatrix} = \begin{pmatrix}\text{Relative Speed}\\ \text{of separation}\end{pmatrix}$$

$$\Rightarrow \quad v_0 = v_2 + v_1 + \left(\frac{L}{2}\omega\right)\left(\frac{1}{\sqrt{2}}\right)$$

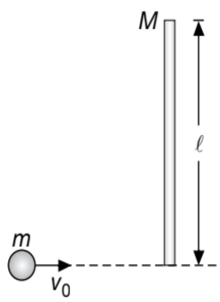
$$\Rightarrow \quad v_0 = v_1 + v_2 + \frac{\omega L}{2\sqrt{2}} \qquad \ldots(3)$$

Solving equations (1), (2) and (3), we get

$$\omega = \frac{4\sqrt{2}}{7}\frac{v_0}{L}$$

ILLUSTRATION 88

A boy of mass m runs on ice with velocity v_0 and steps on the end of a plank of length l and mass M which is perpendicular to his path.



(a) Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.

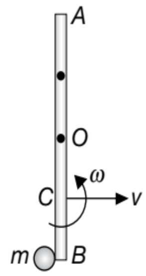
(b) One point on the plank is at rest immediately after the collision. Where is it?

SOLUTION

Let C be the centre of mass of boy plus plank. Let C be at a distance x from the end where the boy steps on the plank, then

$$x = \frac{m(0) + M\left(\frac{l}{2}\right)}{m+M}$$

$$\Rightarrow \quad BC = x = \left(\frac{M}{M+m}\right)\frac{l}{2}$$



So, distance of centre of mass C from the middle of the rod (O) is

$$OC = \left(\frac{l}{2} - x\right) = \left(\frac{m}{M+m}\right)\frac{l}{2}$$

Applying the Law of Conservation of Linear Momentum, we get

$$(M+m)v = mv_0$$

$$\Rightarrow \quad v = \left(\frac{m}{M+m}\right)v_0 \qquad \ldots(1)$$

Applying the Law of Conservation of Angular Momentum about point C, we get

$$mv_0(BC) = I_{\text{system}}\omega$$

$$\Rightarrow \quad \frac{mMv_0 l}{2(M+m)} = \left[m\left(\frac{M}{M+m}\right)^2\left(\frac{l}{4}\right)^2 + \frac{Ml^2}{12} + M\left(\frac{m}{M+m}\right)^2\left(\frac{l^2}{4}\right)\right]\omega$$

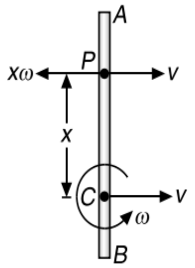
Substituting, $\dfrac{mv_0}{M+m} = v$ from equation (1), we get

$$\frac{v}{\omega}=\frac{l}{6}\left(\frac{4m+M}{M+m}\right)$$

Now, we have the plank divided in two portions

(a) White portion, below C till the end B.
(b) Grey shaded portion, above C till the end A.

Since $v\left(=\frac{mv_0}{M+m}\right)$ is actually the velocity of centre of mass of boy plus plank and so every point of the boy + plank system has a forward velocity v.
However, the **lower white portion** has a tangential velocity $r\omega$ where r is measured from C to B. This $r\omega$ is forward and so we cannot expect the resultant of v and $r\omega$ both forwards to be zero.



However, in the **upper grey portion**, all points move forward with velocity v but simultaneously the upper grey portion has a tangential velocity backwards. Now wherever (say the point P) the forward velocity v equals the backwards tangential velocity $x\omega$ (where x is the distance of point P from C), then at that point net velocity is zero, so

$$v=x\omega$$

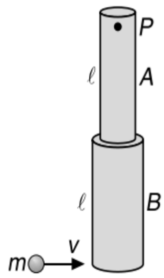
$$\Rightarrow \quad x=\frac{v}{\omega}$$

$$\Rightarrow \quad BP=BC+x$$

$$\Rightarrow \quad BP=\left(\frac{M}{M+m}\right)\left(\frac{l}{2}\right)+\frac{l}{6}\left(\frac{4m+M}{M+m}\right)=\frac{2l}{3}$$

ILLUSTRATION 89

Two uniform rods A and B of length 0.6 m each and of masses 0.01 kg and 0.02 kg respectively are rigidly joined end to end. The combination is pivoted at the lighter end, P as shown in Figure, such that it can freely rotate about point P in a vertical plane. A small object of mass 0.05 kg, moving horizontally, hits the lower end of the combination and sticks to it. What should be the velocity of the object so that the system could just be raised to the horizontal position?

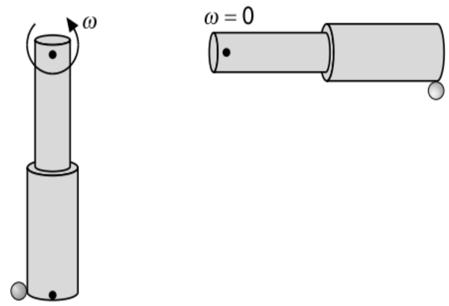


SOLUTION

System is free to rotate but not free to translate. During collision, net torque on the system (rod A + rod B + mass m) about point P i.e., hinge is zero, so angular momentum is conserved about P. If ω be the angular velocity of system just after collision, then

$$L_i=L_f$$

$$\Rightarrow \quad mv(2l)=I\omega$$



where, I is the moment of inertia of system about P, so

$$I=m(2l)^2+m_A\left(\frac{l^2}{3}\right)+m_B\left(\frac{l^2}{12}+\left(\frac{l}{2}+l\right)^2\right)$$

Given $l=0.6$ m, $m=0.05$ kg, $m_A=0.01$ kg and $m_B=0.02$ kg.

Substituting the values, we get

$$I=0.09\ \text{kgm}^2$$

Therefore, from equation (1), we get

$$\omega=\frac{2mvl}{I}=\frac{(2)(0.05)(v)(0.6)}{0.09}$$

$$\Rightarrow \quad \omega=0.67\,v \qquad \ldots(1)$$

Now, after collision, mechanical energy will be conserved.

Therefore, $\begin{pmatrix}\text{Decrease in}\\ \text{RKE}\end{pmatrix}=\begin{pmatrix}\text{Increase in}\\ \text{GPE of CM}\\ \text{of both Rods}\end{pmatrix}$

$$\Rightarrow \quad \frac{1}{2}I\omega^2 = mg(2l) + m_A g\left(\frac{l}{2}\right) + m_B g\left(l + \frac{l}{2}\right)$$

$$\Rightarrow \quad \omega^2 = \frac{gl(4m + m_A + 3m_B)}{I}$$

$$\Rightarrow \quad \omega^2 = \frac{(9.8)(0.6)(4\times 0.05 + 0.01 + 3\times 0.02)}{0.09} = 17.64$$

$$\Rightarrow \quad \omega = 4.2 \text{ rads}^{-1} \qquad \ldots(2)$$

Equating equations (1) and (2), we get

$$v = \frac{4.2}{0.67} \text{ ms}^{-1}$$

$$\Rightarrow \quad v = 6.3 \text{ ms}^{-1}$$

ILLUSTRATION 90

An ice cube of mass M and with sides of length a is sliding without friction across a countertop with a speed v_0 when it hits a ridge E at the edge of the counter (see the Figure). This collision causes the cube to tilt as shown. Show that the minimum value of v_0 needed for the cube to fall off the table is given by

$$v_0 = \sqrt{1.1ag}$$

Given that the moment of inertia of the cube about an axis passing through its centre of gravity and parallel to horizontal surface is $\frac{1}{6}Ma^2$

SOLUTION

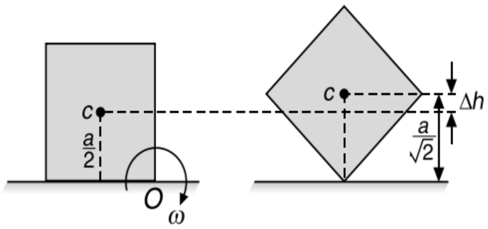
By Law of Conservation of Angular Momentum around the axis through E, we get

$$L_i = L_f$$

$$\Rightarrow \quad Mv_0\left(\frac{a}{2}\right) = I_0\omega = \left(\frac{Ma^2}{6} + M\left(\frac{a}{\sqrt{2}}\right)^2\right)\omega$$

$$\Rightarrow \quad Mv_0\left(\frac{a}{2}\right) = \frac{2}{3}Ma^2\omega$$

$$\Rightarrow \quad \omega = \frac{3}{4}\frac{v_0}{a}$$



Further by Law of Conservation of Energy, we have

$$\begin{pmatrix}\text{Loss in RKE} \\ \text{of Block about O}\end{pmatrix} = \begin{pmatrix}\text{Gain in GPE} \\ \text{of CM of Block}\end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{2}I\omega^2 = Mg(\Delta h)$$

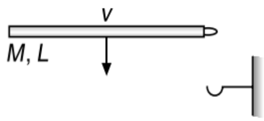
where $\Delta h = \frac{a}{\sqrt{2}} - \frac{a}{2}$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{2}{3}Ma^2\right)\left(\frac{3}{4}\frac{v_0}{a}\right)^2 = Mg\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

$$\Rightarrow \quad v_0 = \sqrt{1.1ag}$$

ILLUSTRATION 91

A rod of mass M and length L is falling vertically with speed v. Suddenly its one end gets stuck in a frictionless hook. Find the angular velocity of the rod just after its end gets stuck.



SOLUTION

Since all the impulse forces during the impact are applied by the hook so they don't produce any torque about the hook and hence angular momentum of the rod about the hook is conserved.

Initial angular momentum about hook is

$$H_i = mvr_\perp = Mv\left(\frac{L}{2}\right)$$

Final angular momentum about the hook is

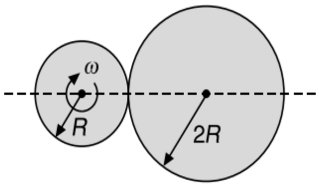
$$H_f = I\omega = \left(\frac{ML^2}{3}\right)\omega$$

$$\Rightarrow \quad \frac{MvL}{2} = \frac{ML^2}{3}\omega$$

$$\Rightarrow \quad \omega = \frac{3v}{2L}$$

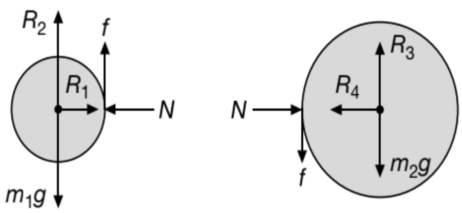
ILLUSTRATION 92

Two discs of radii R and $2R$ are pressed against each other. Initially disc with radius R is rotating with angular velocity ω and another disc was stationary. Both discs are hinged at their respective centres and free to rotate about them. Moment of inertia of smaller is I and bigger disc is $2I$ about their respective axis of rotation. Find the angular velocity of the bigger disc after long time.

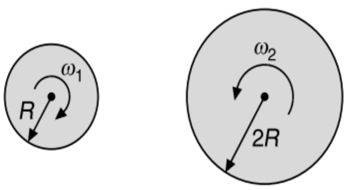


SOLUTION

Let R_1, R_2 be the horizontal and vertical components of the reaction offered by the first hinge to the pulley and R_3, R_4 be the horizontal and vertical components of the reaction offered by the second hinge to the pulley. The only force which is producing any torque about the centre of first disk is friction as shown in Figure.



Let the disc rotate with angular velocity ω_1 and ω_2 as shown in Figure.



Since v is same at the point of contact, so we have

$$\omega_1 R = \omega_2 (2R)$$

$$\Rightarrow \quad \omega_2 = \frac{\omega_1}{2}$$

Now, total angular impulse provided by the friction is equal to change in angular momentum of the disc. So, for disc 1, we get

$$\int fRdt = I(\omega - \omega_1)$$

For disc 2, we get

$$\int f(2R)dt = 2I\omega_2$$

$$\Rightarrow \quad 2I(\omega - \omega_1) = 2I\omega_2$$

$$\Rightarrow \quad \omega_1 + \omega_2 = \omega$$

$$\Rightarrow \quad 2\omega_2 + \omega_2 = \omega$$

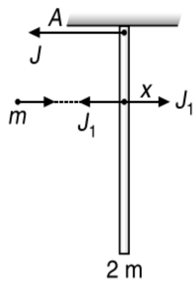
$$\Rightarrow \quad \omega_2 = \frac{\omega}{3}$$

ILLUSTRATION 93

A uniform rod AB of length $2l$ and mass $2m$ is suspended freely at A and hangs vertically at rest when a particle of mass m is fired horizontally with speed v to strike the rod at its mid-point. If the particle is brought to rest by the impact, calculate the impulsive reaction at A, the initial angular speed of the rod and the maximum angle the rod makes with the vertical in the subsequent motion.

SOLUTION

At the instant of collision, if impulsive reaction at A is J and impulse between particle and rod be J_1, then we have



by impulse-momentum theorem

$$J_1 = mv$$

and by angular impulse-angular momentum theorem

$$J_1 l = \frac{2m(2l)^2}{3}\omega$$

$$\Rightarrow \quad \omega = \frac{3J_1}{8ml} = \frac{3v}{8l}$$

For rod, we have

$$J_1 - J = 2m(l\omega)$$

$$\Rightarrow \quad J = J_1 - 2ml\left(\frac{3v}{8l}\right) = \frac{mv}{4}$$

If rod gets displaced by an angle before coming to rest, then by law of conservation of energy, we have

$$\frac{1}{2}I\omega^2 = mgh$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{(2m)(2l)}{3}\right)^2\left(\frac{3v}{8l}\right)^2 = 2mgl(1-\cos\phi)$$

$$\Rightarrow \quad \cos\phi = 1-\frac{3v^2}{32gl}$$

$$\Rightarrow \quad \phi = \cos^{-1}\left(1-\frac{3v^2}{32gl}\right)$$

ILLUSTRATION 94

A thin uniform square plate with side l and mass M can rotate freely about a stationary vertical axis coinciding with one of its sides. A small ball of mass m flying with velocity v at right angles to the plate strikes elastically at centre of the square plate. Calculate the velocity of the ball v' after the impact and the horizontal component of the force which the axis will exert on the plate after the impact.

SOLUTION

Applying conservation of angular momentum, we get

$$mv\left(\frac{l}{2}\right) = mv'\left(\frac{l}{2}\right)+\frac{Ml^2}{3}\omega$$

$$\Rightarrow \quad mv = mv'+\frac{2}{3}Ml\omega \qquad \ldots(1)$$

Since collision is elastic, so we have

$$v = \frac{\omega l}{2}-v' \qquad \ldots(2)$$

From equations (1) and (2), we get

$$mv = mv'+\frac{2}{3}M(v+v')(2)$$

$$\Rightarrow \quad mv-\frac{4}{3}Mv = \left(m+\frac{4}{3}M\right)v'$$

$$\Rightarrow \quad v' = \left(\frac{3m-4M}{3m+4M}\right)v$$

From, equation (2), we get

$$\omega = \frac{2}{l}(v+v') = \frac{2v}{l}\left(1+\frac{3m-4M}{3m+4M}\right)$$

$$\Rightarrow \quad \omega = \frac{12mv}{l(3m+4M)}$$

Force due to axis on the plate is

$$F = M\left(\frac{l}{2}\right)\omega^2 = \frac{72Mm^2v^2}{l(3m+4M)^2}$$

Test Your Concepts-VI

Based on Angular Momentum and Its Conservation

(Solutions on page H.162)

1. A small ball is suspended from a point O by a light thread of length ℓ. Then the ball is drawn aside so that the thread deviates through an angle θ from the vertical and set in motion in a horizontal direction at right angles to the vertical plane in which the thread is located. What is the initial velocity that has to be imparted to the ball so that it could deviate through the maximum angle of $\frac{\pi}{2}$ with the vertical in the process of motion?

2. A uniform rod of length L rests on a frictionless horizontal surface. The rod is pivoted about a fixed frictionless axis at one end. The rod is initially at rest. A bullet travelling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its centre and becomes embedded in it. The mass of the bullet is one-sixth the mass of the rod.

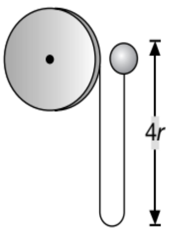
(a) What is the final angular velocity of the rod?

(b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?

3. A wheel of moment of inertia I and radius R is rotating about its axis at an angular speed ω_0. It picks up a stationary particle of mass m at its edge. Find the new angular speed of the wheel.

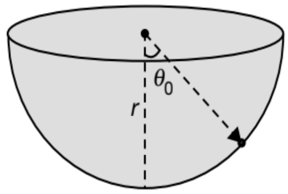
4. A uniform circular disc of mass m and radius a is rotating with constant angular velocity ω in a horizontal plane about a vertical axis through its centre A. A particle P of mass $2m$ is placed gently on the disc at a point distant $\frac{a}{2}$ from A. If the particle does not slip on the disc, find the new angular velocity of the rotating system.

5. A pulley in the form of a uniform disc of mass $2m$ and radius r, is free to rotate in a vertical plane about a fixed horizontal axis through its centre. A light inextensible string has one end fastened to a point on the rim of the pulley and is wrapped several times round the rim. The portion of string not wrapped round the pulley length $8r$ and carries a particle of mass m at its free end. The particle is held close to the rim of the pulley and level with its centre. If the particle is released from this position find the initial angular velocity of the pulley and the impulse of the sudden tension in the string when it becomes taut.



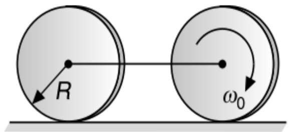
6. A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which a small body of mass m can slide without friction. A light thread running down through the hollow axle of the disc is tied to the body. Initially the body is located at the edge of the disc and the whole system is rotated with an angular velocity ω_0. Then, by means of a force F applied to the lower end of the thread the body is slowly pulled towards the axis of rotation. Find the
(a) angular velocity of the system in its final state.
(b) work performed by the force F.

7. A particle is projected horizontally along the interior of a smooth hemispherical bowl of radius r which is kept at rest. Find the minimum initial speed v_0 required for the particle to just reach the top of the bowl. The initial angular position of the particle is θ_0.



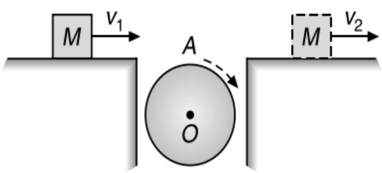
8. A uniform rod AB of length 2ℓ and mass m is rotating in a horizontal plane about a vertical axis through A, with angular velocity ω, when the midpoint of the rod strikes a fixed nail and is brought immediately to rest. Find the angular impulse exerted by the nail.

9. The assembly of two discs as shown in Figure is placed on a rough horizontal surface and the front disc is given an initial angular velocity ω_0. Determine the final linear and angular velocity when both the discs start rolling. It is given that friction is sufficient to sustain rolling in the rear wheel from the beginning of motion.



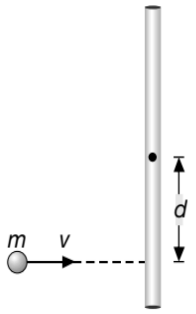
10. A man of mass m_1 stands on the edge of a horizontal uniform disc of mass m_2 and radius R which is capable of rotating freely about a stationary vertical axis passing through its centre. The man walks along the edge of the disc through an angle θ relative to the disc and then stops. Find the angle ϕ through which the disc turned by the time the man stopped.

11. The axis of a cylinder of radius R and moment of inertia about its axis I is fixed at centre O as shown in Figure. Its highest point A is in level with two plane horizontal surfaces. A block of mass M is initially moving to the right without friction with speed v_1. It passes over the cylinder to the dotted position. Calculate the speed v_2 in the dotted position and the angular velocity acquired by the cylinder if at the time of detaching from cylinder block stops slipping on it.



12. A uniform rod of mass m and length ℓ rests on a smooth horizontal surface. One of the ends of the rod is struck in a horizontal direction at right angles to the rod. As a result, the rod obtains velocity v_0. Find the force with which one half of the rod will act on the other in the process of motion.

13. A rod of mass M, length ℓ lies on horizontal table and is free to move on the table. A ball of mass m, moving perpendicularly to the rod at a distance d from its centre with speed v collides elastically with it as shown in Figure. What quantities are conserved in the collision? What must be the mass of the ball so that it remains at rest immediately after collision?



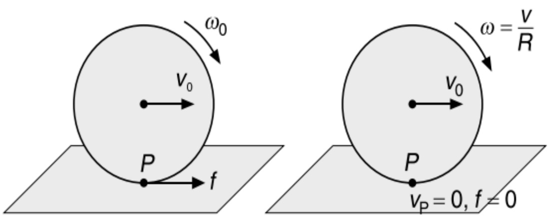
14. A uniform rod AB, of mass m and length $4a$, is smoothly pivoted at a point O of its length, where $AO = a$ and hangs at rest in equilibrium position with A above pivot O. A horizontal impulse of magnitude J is imparted to the rod at its centre of mass. Find the initial angular velocity of the rod. If the rod describes complete revolutions in the subsequent motion, find an inequality for J in terms of a, m and g.

15. A smooth rod rotates freely in a horizontal plane with the angular velocity ω_0 about a stationary vertical axis O, relative to which the rod's moment of inertia is I. A small ring the mass m is located on the rod close to the rotation axis and is tied to it by a thread. When the thread is burned, the ring starts sliding along the rod. Find the velocity v_r of the ring relative to the rod as a function of its distance r from the rotation axis.

ROLLING WITH SLIPPING ($v_0 > R\omega_0$)

CASE-1: When, $v_0 > R\omega_0$

Since $v_0 > R\omega_0$, so the point P has a tendency to slip forward and hence sliding friction (later on called as friction) acts backwards as shown.



Initially at $t = 0$ Finally at time t (Pure Rolling)

ROLE OF FRICTION

Since f is acting opposite to v_0, therefore it will have a tendency to decrease v_0. Also, when f acts backwards then it will provide a torque that will be acting in the clockwise sense and hence will have a tendency to increase ω_0, i.e., at some later time, t (say), we can have $v = r\omega$

Since, $v_0 > R\omega_0$ {at $t = 0$}

So, to attain $v = R\omega$ at some later time t, f must be directed such that the role of f is

(a) to decrease v_0 to v (say)

(b) to increase ω_0 to ω (say)

such that at t, we get $v = R\omega$ (condition for pure rolling)

From impulse momentum theorem and angular impulse angular momentum theorem, we get

$$mv - mv_0 = -ft \quad \ldots(1)$$

$$I\omega - I\omega_0 = +\tau t = (fR)t \quad \ldots(2)$$

But at t, we have

$$v = R\omega \quad \ldots(3)$$

So, we use (1), (2), (3) to get the desired results. Also, work done by friction equals change in KE, so

$$W_f = \underbrace{\left(\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\right)}_{\omega = v/R} - \underbrace{\left(\frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2\right)}_{\omega_0 \neq v/R} \quad \ldots(4)$$

Since, $f = \mu N = \mu mg$, retardation $a = \frac{\mu N}{m} = \mu g$, so distance travelled in time t is

$$s = v_0 t + \frac{1}{2}(-a)t^2 = v_0 t + \frac{1}{2}\left(-\frac{\mu mg}{m}\right)t^2 \qquad \ldots(5)$$

If θ is the total angle traversed, then

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \omega_0 t + \frac{1}{2}\left(\frac{\tau}{I}\right)t^2 \qquad \{\because \tau = I\alpha\}$$

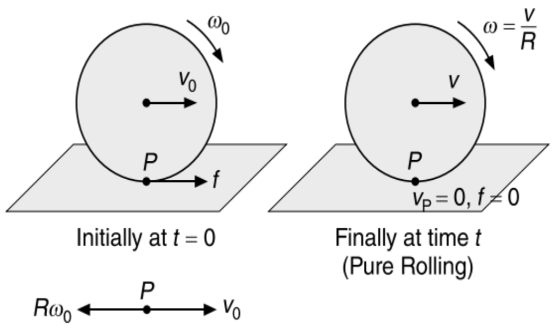
where $\tau = fR = I\alpha$

$$\Rightarrow \quad \alpha = \frac{fR}{I} = \frac{(\mu mg)}{I}R$$

So, $\theta = \omega_0 t + \frac{1}{2}\left(\frac{\mu mgR}{I}\right)t^2 \qquad \ldots(6)$

CASE-2: When, $v_0 < R\omega_0$

Since $v_0 < R\omega_0$, so the point P has a tendency to slip backward and hence sliding friction acts forward, as shown in Figure.



ROLE OF FRICTION

Since f is acting along v_0, so it will have a tendency to increase v_0. Also, when f acts forwards, along v_0, then it will provide a torque that will be acting in the counter-clockwise sense and hence will have a tendency to decrease ω_0, i.e., at some later time, t (say), we can have $v = r\omega$

Since, $v_0 < R\omega_0$ $\qquad$ {at $t = 0$}

So, to attain $v = R\omega$ at some later time t, f must be directed such that the role of f is

(a) to increase v_0 to v (say)
(b) to decrease ω_0 to ω (say)

Such that at t, we get $v = R\omega$ (condition for pure rolling)

From impulse momentum theorem and angular impulse angular momentum theorem, we get

$$mv - mv_0 = ft \qquad \ldots(1)$$

$$I\omega - I\omega_0 = -\tau t = -(fR)t \qquad \ldots(2)$$

But at t, we have

$$v = R\omega \qquad \ldots(3)$$

So, we use (1), (2), (3) to get the desired results. Also work done by friction equals change in KE, so

$$\Rightarrow \quad W_f = \underbrace{\left(\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\right)}_{\omega = v/R} - \underbrace{\left(\frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2\right)}_{\omega_0 \neq v/R} \qquad \ldots(4)$$

Since, $f = \mu N = \mu mg$ and $a = \frac{\mu N}{m} = \mu g$, so distance travelled in time t is

$$s = v_0 t + \frac{1}{2}(a)t^2 \qquad \{\because f \text{ accelerates } v_0\}$$

$$\Rightarrow \quad s = v_0 t + \frac{1}{2}\left(\frac{\mu mg}{m}\right)t^2 \qquad \ldots(5)$$

If θ is the angle traversed, then

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\Rightarrow \quad \theta = \omega_0 t + \frac{1}{2}\left(-\frac{\tau}{I}\right)t^2 \qquad \{\because \tau = I\alpha\}$$

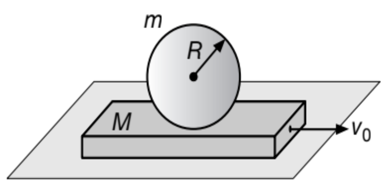
where, $\tau = fR = I\alpha$

$$\Rightarrow \quad \alpha = \frac{fR}{I} = \frac{(\mu mg)R}{I}$$

$$\Rightarrow \quad \theta = \omega_0 t - \frac{1}{2}\left(\frac{\mu mgR}{I}\right)t^2 \qquad \ldots(6)$$

ILLUSTRATION 95

A sphere of mass m and radius R is placed at rest on a plank of mass M which is placed on a smooth horizontal surface as shown in Figure. The coefficient of friction between the sphere and the plank is μ. At $t = 0$, a horizontal velocity v_0 is given to the plank. Find the time after which the sphere starts rolling.

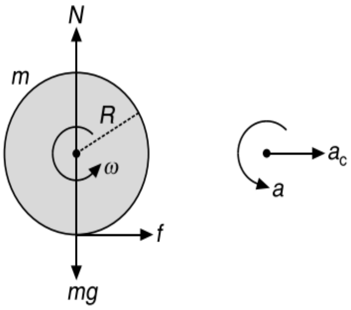


SOLUTION

For the Sphere:

$$a_c = \frac{f}{m} = \mu g$$

$$\alpha = \frac{\tau_c}{I_c} = \frac{fR}{\frac{2}{5}mR^2} = \frac{5}{2}\frac{\mu g}{R}$$



After time t

$$v_c = a_c t = \mu g t$$

$$\omega = \alpha t = \frac{5}{2}\frac{\mu g}{R}t$$

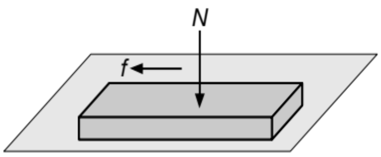
The velocity of the point of contact is

$$v = v_c + \omega R = \mu g t + \frac{5}{2}\mu g t = \frac{7}{2}\mu g t$$

For the Plank:

Retardation $a = \frac{f}{M} = \frac{\mu m g}{M}$

Instantaneous velocity $v = v_0 - \frac{\mu m g}{M}t$



Condition of pure rolling

$$v = \frac{7}{2}\mu g t = v_0 - \frac{\mu m g}{M}t$$

$$\Rightarrow \quad t = \frac{v_0}{\left(\frac{7}{2} + \frac{m}{M}\right)\mu g}$$

ILLUSTRATION 96

A wheel is held by a handle on its axle and given an initial angular velocity ω_0. The wheel is then placed in contact with the ground. At first the wheel remains stationary, spinning in place. After a short time it begins to move forward and eventually reaches the point where it rolls without slipping. Find the final velocity of the wheel in terms of the initial angular velocity ω_0. Take $I = \frac{MR^2}{2}$.

SOLUTION

We assume that wheel is initially rotating clockwise. Let the wheel starts rolling after a time t.
Then, Using Impulse Momentum Theorem

For translation:

$$\text{Impulse} = \Delta p = p_f - p_i$$

$$\Rightarrow \quad ft = Mv - 0 \qquad \ldots(1)$$

For rotation:

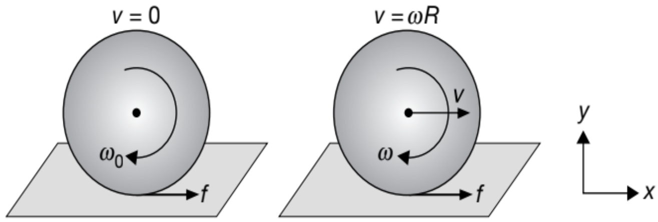
$$\text{Angular Impulse} = \Delta L = L_f - L_i$$

$$\Rightarrow \quad -fRt = I\omega - I\omega_0$$

Note that clockwise angular momentum is considered as positive.

Since $I = \frac{MR^2}{2}$

$$-(fR)t = \frac{MR^2}{2}(\omega - \omega_0) \qquad \ldots(2)$$



Initially at $t = 0$ Finally at time t

For the condition of pure rolling, we have

$$v = \omega R \qquad \ldots(3)$$

Using equations (1), (2) and (3), we get

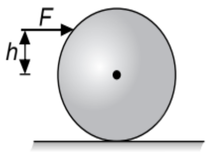
$$-M\omega R^2 = \frac{MR^2}{2}(\omega - \omega_0)$$

$$\Rightarrow \quad \omega = \frac{\omega_0}{3}$$

The linear velocity of the wheel is $v = R\omega = \frac{R\omega_0}{3}$

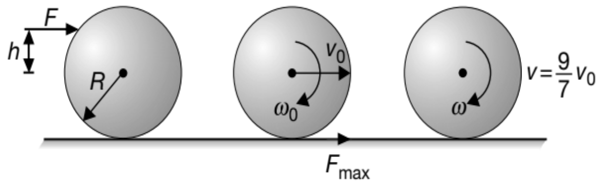
ILLUSTRATION 97

A billiard ball of radius R, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centre line as shown in Figure. The ball leaves the cue with a speed v_0 and because of its forward English (backward slipping) eventually acquires a final speed $\frac{9}{7}v_0$. Find h.



SOLUTION

Let ω_0 be the angular speed of the ball just after it leaves the cue. The maximum friction acts in forward direction till the slipping continues. Let v be the linear speed and ω the angular speed when slipping ceases, so



$$v = R\omega$$

$$\Rightarrow \quad \omega = \frac{v}{R}$$

Given, $v = \frac{9}{7}v_0$ ...(1)

$$\Rightarrow \quad \omega = \frac{9}{7}\frac{v_0}{R} \quad ...(2)$$

Applying impulse momentum theorem, we get

$$Fdt = mv_0 \quad ...(3)$$

Applying angular impulse angular momentum theorem, we get

$$\tau dt = I\omega_0$$

$$\Rightarrow \quad Fhdt = \frac{2}{5}mR^2\omega_0 \quad ...(4)$$

Angular momentum about bottommost point will remain conserved, so

$$L_i = L_f$$

$$\Rightarrow \quad I\omega_0 + mRv_0 = I\omega + mRv$$

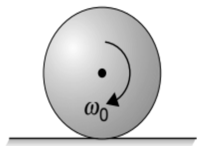
$$\Rightarrow \quad \frac{2}{5}mR^2\omega_0 + mRv_0 = \frac{2}{5}mR^2\left(\frac{9}{7}\frac{v_0}{R}\right) + \frac{9}{7}mRv_0 \quad ...(5)$$

Solving equations (3), (4) and (5) we get,

$$h = \frac{4}{5}R$$

ILLUSTRATION 98

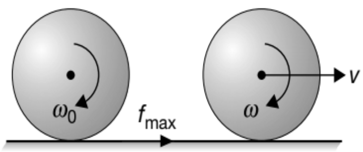
A solid sphere of radius r is gently placed on a rough horizontal ground with an initial angular speed ω_0 and no linear velocity. If the coefficient of friction is μ, find the time t when the slipping stops. In addition, state the linear velocity v and angular velocity ω at the end of slipping.



SOLUTION

METHOD I:

Let m be the mass of the sphere. According to the problem $v_0 < r\omega_0$, so it is a case of backward slipping and hence force of friction is in forward direction. Limiting friction will act in this case.



Linear acceleration $a = \frac{f}{m} = \frac{\mu mg}{m} = \mu g$

Angular retardation $\alpha = \frac{\tau}{I} = \frac{fr}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$

Slipping ceases when $v = r\omega$

$$\Rightarrow \quad (at) = r(\omega_0 - \alpha t)$$

$$\Rightarrow \quad \mu gt = r\left(\omega_0 - \frac{5}{2}\frac{\mu gt}{r}\right)$$

$$\Rightarrow \quad \frac{7}{2}\mu gt = r\omega_0$$

$$\Rightarrow \quad t = \frac{2}{7}\frac{r\omega_0}{\mu g}$$

$$\Rightarrow \quad v = at = \mu gt = \frac{2}{7}r\omega_0$$

and $\omega = \frac{v}{r} = \frac{2}{7}\omega_0$

METHOD II:

Net torque on the sphere about the bottommost point i.e., the point of contact is zero. Therefore, angular momentum of the sphere will remain conserved about the bottommost point.

$$L_i = L_f$$

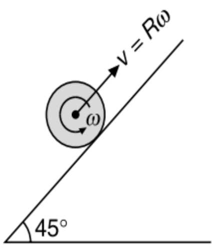
$$\Rightarrow \quad I\omega_0 = I\omega + mrv$$

$$\Rightarrow \quad \frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$$

$$\Rightarrow \quad \omega = \frac{2}{7}\omega_0 \text{ and } v = r\omega = \frac{2}{7}r\omega_0$$

ILLUSTRATION 99

A solid sphere of radius R is projected along an inclined plane as shown in Figure.

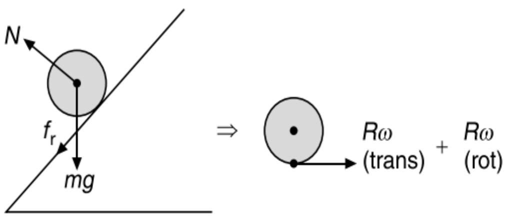


If coefficient of friction is $\mu = 0.5$ then calculate the time when linear velocity of the sphere becomes zero. Also calculate the angular velocity of the sphere at that instant.

SOLUTION

Velocity of lower most point is not zero, hence it is not the case of pure rolling and friction between the sphere and the incline is kinetic in nature. Since velocity of lower most point in forward direction therefore friction will act in backward direction.

Let retardation is a and angular retardation is α



Free body diagram of sphere

$$\Rightarrow \quad mg\sin\theta + \mu mg\cos\theta = ma$$

$$\Rightarrow \quad a = g(\sin\theta + \mu\cos\theta) = \frac{g}{\sqrt{2}}(1+0.5) = \frac{1.5g}{\sqrt{2}} \quad \ldots(1)$$

Now, τ due to friction is

$$\tau = R(\mu mg\cos\theta)$$

$$\Rightarrow \quad \alpha = \frac{(r\mu mg\cos\theta)}{\frac{2}{5}mR^2} = \frac{5g\mu\cos\theta}{2R}$$

$$\Rightarrow \quad \alpha = \frac{5g(0.5)(1/\sqrt{2})}{2R}$$

$$\Rightarrow \quad \alpha = \frac{5g}{4R\sqrt{2}} \quad \ldots(2)$$

For translatory motion

$$v' = v - at = R\omega - at$$

When $v' = 0$, then

$$t = \frac{R\omega}{a} = \frac{R\omega\sqrt{2}}{1.5g} = \frac{\sqrt{2}}{1.5}\left(\frac{R\omega}{g}\right)$$

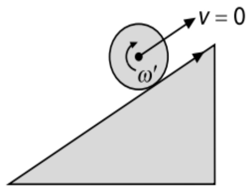
For rotatory motion, we have

$$\omega' = \omega - \alpha t$$

$$\Rightarrow \quad t = \frac{\omega}{\alpha} = \frac{\omega}{\frac{5g}{4R\sqrt{2}}} = \frac{4\sqrt{2}R\omega}{5g}$$

Linear velocity of sphere becomes zero before angular velocity become zero

Angular velocity at time t when linear velocity is zero.



$$\omega' = \omega - \left(\frac{5g}{4R\sqrt{2}}\right)\left(\frac{\sqrt{2}R\omega}{1.5g}\right)$$

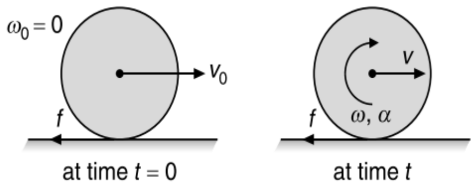
$$\Rightarrow \quad \omega' = \omega - \left(\frac{5}{4}\right)\left(\frac{\omega}{1.5}\right) = \frac{\omega}{6}$$

ILLUSTRATION 100

A uniform round object of mass m, radius R and moment of inertia about its centre of mass I_{cm} is thrown with speed v_0 without any rotation on a rough horizontal surface of coefficient of friction μ. Assuming $\frac{I_{cm}}{mR^2} = k$, calculate the time after which slipping stops. Also calculate the speed and angular speed of the object when the slipping stops.

SOLUTION

The direction of friction is such that it opposes the translational motion of object and provides torque that supports the rotational motion of the object as shown in Figure.



Applying Newton's Second Law for translational motion, we get

$$f = \mu N = \mu mg$$

$$\Rightarrow \quad a = \frac{f}{m} = \mu g$$

$$\Rightarrow \quad v = v_0 - at = v_0 - \mu gt \qquad \ldots(1)$$

Applying Newton's Second Law for rotational motion, we get

$$fR = I_{cm}\alpha$$

$$\Rightarrow \quad \alpha = \frac{\mu mgR}{I_{cm}}$$

$$\Rightarrow \quad \omega = \alpha t = \left(\frac{\mu mgR}{I_{cm}}\right)t$$

For pure rolling, we have

$$v = R\omega$$

$$\Rightarrow \quad v_0 - \mu gt = \frac{R\mu mgRt}{I_{cm}}$$

$$\Rightarrow \quad v_0 = \mu gt\left(1 + \frac{1}{k}\right) \qquad \left\{\because \frac{I_{cm}}{mR^2} = k\right\}$$

$$\Rightarrow \quad t = \frac{v_0 k}{\mu g(1+k)}$$

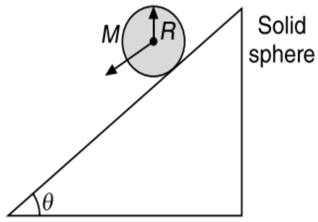
Substituting this value in equation (1), we get

$$v = v_0 - \mu g\frac{v_0 k}{(1+k)\mu g} = v_0\left(\frac{1+k-k}{1+k}\right)$$

$$\Rightarrow \quad v = \frac{v_0}{1+k} \text{ and } \omega = \frac{v}{R} = \frac{v_0}{R(1+k)}$$

ILLUSTRATION 101

A solid sphere starts to roll without slipping on a rough inclined plane as shown in Figure.

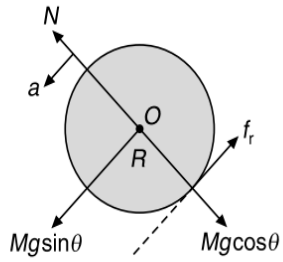


Calculate the friction force acting on the sphere, minimum coefficient of friction μ_0 for pure rolling of the sphere and the work done by friction if $\mu = \frac{\mu_0}{4}$ and the displacement of centre of mass is x.

SOLUTION

Acceleration of centre of mass of the sphere is

$$a = \frac{g\sin\theta}{1+\frac{2}{5}} = \frac{5g\sin\theta}{7}$$



Applying Newton's Second Law for translational motion, we get

$$Mg\sin\theta - f_r = Ma$$

$$\Rightarrow \quad f_r = Mg\sin\theta - \frac{5Mg\sin\theta}{7} = \frac{2Mg\sin\theta}{7}$$

Since, $f_r = \frac{2Mg\sin\theta}{7} \le \mu N$

$$\Rightarrow \quad \frac{2Mg\sin\theta}{7} \le \mu Mg\cos\theta$$

$$\Rightarrow \quad \mu \ge \frac{2}{7}\tan\theta$$

$$\Rightarrow \quad \mu_0 = \frac{2}{7}\tan\theta$$

If $\mu < \mu_{min}$, then there will be slipping and hence friction will be kinetic in nature, so we have

$$f_k = \mu Mg\cos\theta = \left(\frac{2}{7}\frac{\tan\theta}{4}\right)Mg\cos\theta$$

$$\Rightarrow \quad f_k = \frac{Mg\sin\theta}{14}$$

$$\Rightarrow \quad a = \frac{Mg\sin\theta - f_k}{M} = \frac{Mg\sin\theta - \dfrac{Mg\sin\theta}{14}}{M}$$

$$\Rightarrow \quad a = \frac{13g\sin\theta}{14}$$

$$\Rightarrow \quad \alpha = \frac{\mu MgR\cos\theta}{\frac{2}{5}MR^2} = \frac{5\mu g\cos\theta}{2R}$$

$$\Rightarrow \quad \alpha = \left(\frac{5g\cos\theta}{2R}\right)\left(\frac{2}{7}\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{4}\right) = \frac{5g\sin\theta}{28R}$$

$$\Rightarrow \quad \frac{a}{\alpha} = \frac{\dfrac{13g\sin\theta}{14}}{\dfrac{5g\sin\theta}{28R}} = \left(\frac{13}{14}\right)\left(\frac{28R}{5}\right)R = \frac{26R}{5}$$

Since, $\dfrac{x}{\theta} = \dfrac{a}{\alpha} = 5.2R$

$$\Rightarrow \quad \frac{x}{\theta} = 5.2R$$

Work done by friction is

$$W_{fr} = (W_{fr})_{\text{Rotation}} + (W_{fr})_{\text{Trans}}$$

$$\Rightarrow \quad W_{fr} = (\mu mg\cos\theta)R\theta + (\mu mg\cos\theta)x\cos(180^\circ)$$

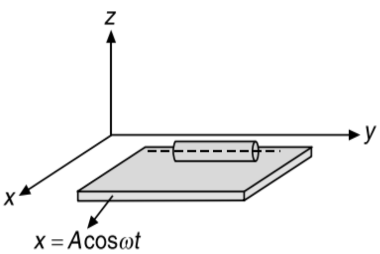
$$\Rightarrow \quad W_{fr} = \mu mg(R\theta - x)\cos\theta$$

$$\Rightarrow \quad W_{fr} = \mu mg\cos\theta\left(\frac{x}{5.2} - x\right) = -\frac{4.2}{5.2}(\mu mg\cos\theta)x$$

$$\Rightarrow \quad W_{fr} = -\frac{21}{26}\mu mgx\cos\theta$$

ILLUSTRATION 102

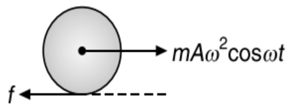
A cylinder of mass m and radius R is resting on a horizontal platform (which is parallel to xy plane) with its axis fixed along the y-axis and free to rotate about its axis as shown in Figure.



The platform is given a motion in the x-direction given by $x = A\cos(\omega t)$. There is no slipping between the cylinder and platform. Calculate the maximum torque acting on the cylinder during its motion.

SOLUTION

We must know that maximum torque will act on cylinder when it is ta the extreme positions of oscillatory motion. The cylinder moves under the influence of pseudo force with respect to platform as shown in Figure.



Thus, equation of motion of cylinder is

$$mA\omega^2 - f = ma_{\max} \quad \ldots(1)$$

and $$fR = \frac{1}{2}mR^2\left(\frac{a_{\max}}{R}\right) \quad \ldots(2)$$

Adding (1) and (2), we get

$$A\omega^2 = \frac{3}{2}a_{\max}$$

$$\Rightarrow \quad a_{\max} = \frac{2A\omega^2}{3}$$

Maximum angular acceleration is

$$\alpha_{\max} = \frac{a_{\max}}{R} = \frac{2A\omega^2}{3R}$$

Maximum torque on cylinder is

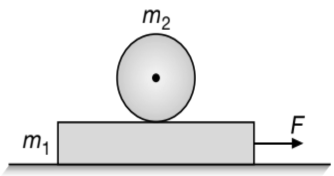
$$\tau_{\max} = I\alpha_{\max}$$

$$\Rightarrow \quad \tau_{\max} = \frac{1}{2}MR^2\left(\frac{2A\omega^2}{3R}\right)$$

$$\Rightarrow \quad \tau_{\max} = \frac{1}{3}mAR\omega^2$$

ILLUSTRATION 103

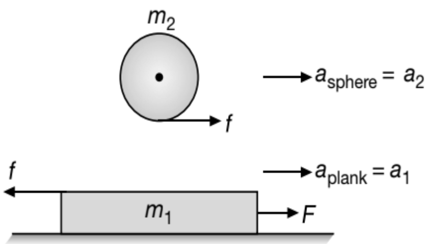
A plank of mass m_1 with a uniform sphere of mass m_2 placed on it rests on a smooth horizontal plane as shown in Figure. A constant horizontal force F is applied to the plank. Calculate the acceleration of the plank and the centre of the sphere if there is no sliding between the plank and the sphere.



SOLUTION

FROM FRAME ATTACHED TO GROUND

For the two bodies, equations of motion *from the ground reference frame* for accelerations a_1 and a_2 shown in Figure are



$$F - f = m_1 a_1 \quad \text{...(1)}$$

$$f = m_2 a_2 \quad \text{...(2)}$$

$$fR = -\frac{2}{5} m_2 R^2 \left(\frac{a_1 - a_2}{R} \right) \quad \text{...(3)}$$

From (1) and (2), we get

$$F = m_1 a_1 + m_2 a_2 \quad \text{...(4)}$$

From (2) and (3), we get

$$5m_2 a_2 = 2m_2 a_1 - 2m_2 a_2$$

$$\Rightarrow \quad 7a_2 = 2a_1 \quad \text{...(5)}$$

Using equation (4), we get

$$F = m_1 a_1 + m_2 \left(\frac{2a_1}{7} \right)$$

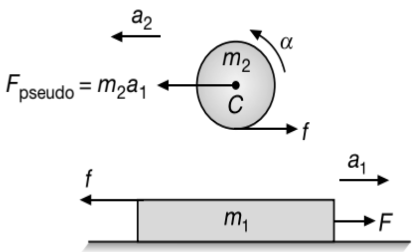
$$\Rightarrow \quad a_1 = \frac{7F}{7m_1 + 2m_2}$$

From equation (5), we get

$$a_2 = \frac{2F}{7m_1 + 2m_2}$$

FROM FRAME ATTACHED TO PLANK

Since the sphere does not slip over the plank surface, so it is the case of pure rolling and hence we can take friction on sphere in any direction, say towards right as shown in Figure.



Also note that since friction on sphere is acting towards right, so it must act on the plank towards left. Let plank moves toward right with an acceleration a_1, due to which the sphere will experience a pseudo force $m_2 a_1$ towards left, and hence it will roll towards left with respect to plank with an acceleration a_2. Since we have used the concept of pseudo force, so we must say that the acceleration a_2 must be with respect to the plank. Let the angular acceleration of the sphere during rolling be α, so we have

$$a_2 = R\alpha$$

For translational motion of plank, we have

$$F - f = m_1 a_1 \quad \text{...(1)}$$

For translational motion of sphere with respect to plank we have

$$m_2 a_1 - f = m_2 a_2 \quad \text{...(2)}$$

For rotational motion of sphere with respect to plank, we have

$$fR = I\alpha$$

$$\Rightarrow \quad fR = \left(\frac{2}{5} m_2 R^2 \right) \left(\frac{a_2}{R} \right)$$

$$\Rightarrow \quad f = \frac{2}{5} m_2 a_2 \quad \text{...(3)}$$

From equation (2) and (3), we get

$$m_2 a_1 - \frac{2}{5} m_2 a_2 = m_2 a_2$$

$$\Rightarrow \quad a_1 = \frac{7}{5} a_2 \quad \text{...(4)}$$

Using equations (1), (2) and (3), we get

$$F - \frac{2}{5} m_2 a_2 = m_1 a_1$$

$$\Rightarrow \quad F = \frac{2}{5} m_2 a_2 + m_1 a_1 = \frac{2}{5} m_2 \left(\frac{5}{7} a_1 \right) + m_1 a_1$$

$$\Rightarrow \quad a_1 = \frac{7F}{7m_1 + 2m_2}$$

From equation (4), we get

$$a_2 = \frac{5}{7}a_1 = \frac{5}{7}\left(\frac{7F}{7m_1 + 2m_2}\right)$$

$$\Rightarrow \quad a_2 = \frac{5F}{7m_1 + 2m_2}$$

Since, a_2 is acceleration of sphere relative to the plank, so net acceleration of the sphere is

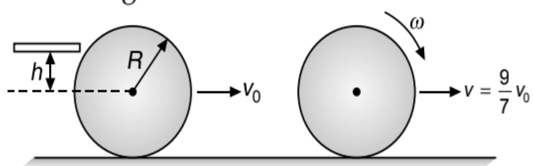
$$(a_{net})_{sphere} = a_1 - a_2 = \frac{2F}{7m_1 + 2m_2}$$

Please note that in the equation $a_2 = R\alpha$, a_2 is the acceleration of cylinder with respect to plank.

In problems where rolling of a body takes place in an inertial (ground) frame or a non-inertial (accelerated) frame, then in the equation $a = R\alpha$, the acceleration a must be relative, i.e. with respect to inertial (ground) frame or with respect to the non-inertial (accelerated) frame.

ILLUSTRATION 104

A billiard ball of mass m, radius R initially at rest is given a sharp horizontal impulse by a cue. The cue is held horizontally a distance h above the centre line as shown in Figure.



The ball leaves the cue with a speed v_0 and because of its forward English eventually acquires a final speed of $\frac{9}{7}v_0$, calculate the value of h.

SOLUTION

Let the impulse imparted by the cue to the ball be J and initial angular speed of ball be ω_0, then applying the impulse momentum theorem, we get

$$J = mv_0 \quad \text{...(1)}$$

Applying the angular impulse angular momentum theorem, we get

$$Jh = \frac{2}{5}mR^2\omega_0 \quad \text{...(2)}$$

From equations (1) and (2), we get

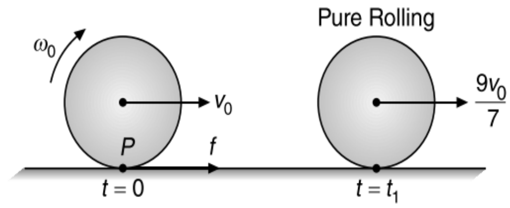
$$\omega_0 = \frac{5v_0 h}{2R^2}$$

Applying conservation of angular momentum about point of contact P on the ground, we get

$$L_{initial} = L_{final}$$

where, $L_{initial} = mv_0R + \left(\frac{2mR^2}{5}\right)\left(\frac{5v_0h}{2R^2}\right)$ and

$$L_{final} = m\left(\frac{9v_0}{7}\right)R + \frac{2}{5}mR^2\left(\frac{9v_0}{7R}\right)$$



$$\Rightarrow \quad mv_0R + \frac{2}{5}mR^2\left(\frac{5v_0h}{2R^2}\right) = m\left(\frac{9v_0}{7}\right)R + \frac{2}{5}mR^2\left(\frac{9v_0}{7R}\right)$$

$$\Rightarrow \quad \left(1 + \frac{h}{R}\right) = \left(\frac{9}{7} + \frac{18}{35}\right)$$

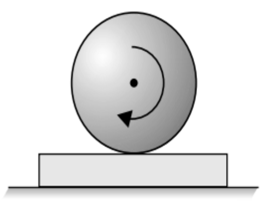
$$\Rightarrow \quad h = \left(\frac{63}{35} - 1\right)R = \frac{4}{5}R$$

Test Your Concepts-VII

Based on Rolling with Slipping

(Solutions on page H.165)

1. A horizontal plank having mass m lies on a smooth horizontal surface. A sphere of same mass and radius r is spun to an angular frequency ω_0 and gently placed on the plank as shown in Figure. If coefficient of friction between the plank and the sphere is μ, find the distance moved by the plank till the sphere starts pure rolling on the plank. The plank is long enough.

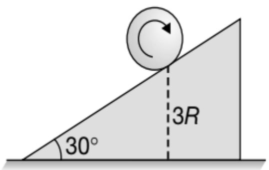


2. A disc of radius R and mass m is projected on to a horizontal floor with a backward spin such that its centre of mass speed is v_0 and angular velocity is ω_0. What must be the minimum value of ω_0 so that the disc eventually returns?

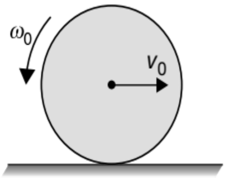
3. A solid cylinder of mass m and radius R is set in rotation about its axis with an angular velocity ω_0, then lowered with its lateral surface onto a horizontal plane and released. The coefficient of friction between the cylinder and the plane is μ. Calculate the time for which the cylinder will move with sliding and the total work done by friction.

4. A bowling ball is thrown straight down the alley. When it starts, its centre of mass has a speed v_0 and it is sliding without rotating. Determine how far the ball moves down the alley before it starts rolling without slipping. Express your answer in terms of v_0, g and μ_K.

5. A cylinder of mass m and radius R is rotated about its axis with angular velocity ω_0 (as shown in Figure). It is lowered on a rough inclined plane at an angle $30°$ with horizontal and having the coefficient of friction $\mu = \frac{1}{\sqrt{3}}$. The point of initial contact of cylinder and incline is at a height of $3R$ from horizontal. Calculate the total time taken by the cylinder to reach the bottom of incline.



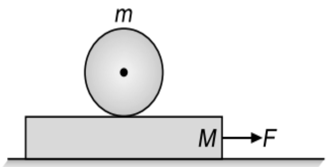
6. A sphere of radius R, mass M is projected along a rough horizontal surface (having coefficient of friction μ), with initial velocities shown in Figure.



If the final velocity of the sphere is zero, then calculate (in terms of v_0, R, ω_0, μ and M) the required magnitude of ω_0, the time t required by sphere to come to rest and the distance travelled by sphere before it stops.

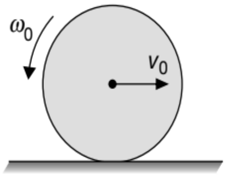
7. A bowling ball is thrown straight down the alley. When it starts, its centre of mass has a speed v_0 and it is sliding without rotating. Determine how far the ball moves down the alley before it starts rolling without slipping. Express your answer in terms of v_0, g and μ_K.

8. A plank of mass M is placed on a smooth surface over which a cylinder of mass m and radius R is placed as shown in Figure.



The plank is now pulled towards right with an external force F. If cylinder does not slip over the surface of plank, calculate the linear acceleration of plank and cylinder. Also calculate the angular acceleration of the cylinder.

9. A bowler projects a ball of mass M, radius R along an alley with forward velocity v_0 and a backspin ω_0. The coefficient of friction between the alley and the ball is μ. Calculate the time t at which the ball will start rolling with slipping and the speed of the ball at time t.



10. A uniform circular cylinder of mass m and radius r is given an initial angular velocity ω_0 and no initial translational velocity. It is placed in contact with a plane inclined at an angle α to the horizontal. If there is a coefficient of friction μ for sliding between the cylinder and plane, find the distance the cylinder moves up before sliding stops. Also calculate the maximum distance it travels up the plane. Assume $\mu > \tan\alpha$.

SOLVED PROBLEMS

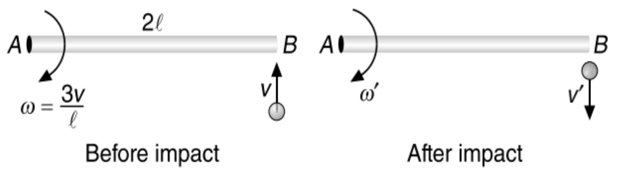
PROBLEM 1

A uniform rod AB of length $2l$ and mass m is turning freely in a horizontal plane about a vertical axis through A, with angular velocity $\frac{3v}{l}$. A particle P, also having mass m, is moving with constant velocity v in the same horizontal plane. The particle and the rod are moving towards each other and when AB is perpendicular to the path of the particle, P collides with point B of the rod. If the coefficient of restitution between the rod and the particle is $\frac{1}{2}$. Find the speed of the particle after impact.

SOLUTION

Angular momentum of the system about A is conserved, so

$$I\omega - mv(2l) = I\omega' + mv'(2l)$$



$$\Rightarrow \quad \frac{m(2l)^2}{3}\left(\frac{3v}{l}\right) - 2mvl = \frac{m(2l)^2}{3}\omega' + 2mv'l$$

$$\Rightarrow \quad 2mvl = \frac{4}{3}ml^2\omega' + 2mv'l$$

$$\Rightarrow \quad 2v = \frac{4}{3}l\omega' + 2v' \qquad \ldots(1)$$

$$\Rightarrow \quad 2v = \frac{4}{3}l\omega' + 2v'$$

At point B, we have

$$e = \frac{\text{Relative Velocity of Separation}}{\text{Relative Velocity of Approach}}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{v' - 2\omega' l}{v + \left(\frac{3v}{l}\right)(2l)} = \frac{v' - 2\omega' l}{7v}$$

$$\Rightarrow \quad 7v = 2v' - 4\omega' l$$

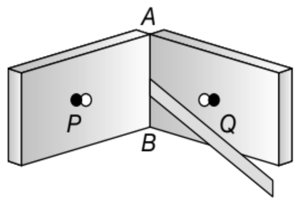
$$\Rightarrow \quad 4\omega' l = 2v' - 7v \qquad \ldots(2)$$

Solving equations (1) and (2), we get

$$v' = \frac{13}{8}v$$

PROBLEM 2

Two heavy metallic plates are joined together at $90°$ to each other. A laminar sheet of mass 30 kg is hinged at the line AB joining the two heavy metallic plates. The hinges are frictionless. The moment of inertia of the laminar sheet about an axis parallel to AB and passing through its centre of mass is 1.2 kgm^2. Two rubber obstacles P and Q are fixed, one on each metallic plate at a distance 0.5 m from the line AB. This distance is chosen so that the reaction due to the hinges on the laminar sheet is zero during the impact. Initially the laminar sheet hits one of the obstacles with an angular velocity 1 rads^{-1} and turns back. If the impulse on the sheet due to each obstacle is 6 Ns.



(a) find the location of the centre of mass of the laminar sheet from AB.

(b) at what angular velocity does the laminar sheet come back after the first impact?

(c) after how many impacts, does the laminar sheet come to rest?

SOLUTION

Let r be the perpendicular distance of centre of mass from the line AB and ω be the angular velocity of the sheet just after colliding with rubber obstacle for the first time.

The linear velocity of centre of mass before and after collision are

$$v_i = (r)(1\text{ rads}^{-1}) = r \text{ and } v_f = r\omega$$

where, $\vec{v}_i$ and $\vec{v}_f$ will be in opposite directions.

Now, by Impulse – Momentum Theorem, we have

$$\left(\begin{array}{c}\text{Linear Impulse}\\ \text{on CM}\end{array}\right) = \left(\begin{array}{c}\text{Change in Linear}\\ \text{Momentum of CM}\end{array}\right)$$

$$\Rightarrow \quad 6 = m\left(v_f + v_i\right) = 30\left(r + r\omega\right)$$

$$\Rightarrow \quad r\left(1+\omega\right) = \frac{1}{5} \qquad \ldots(1)$$

Similarly, by Angular Impulse – Angular Momentum Theorem, we have

$$\begin{pmatrix}\text{Angular Impulse}\\ \text{about } AB\end{pmatrix} = \begin{pmatrix}\text{Change in Angular}\\ \text{Momentum about } AB\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}\text{Angular}\\ \text{impulse}\end{pmatrix} = \begin{pmatrix}\text{Linear}\\ \text{impulse}\end{pmatrix} \times \begin{pmatrix}\text{Perpendicular}\\ \text{distance of}\\ \text{impulse from } AB\end{pmatrix}$$

$$\Rightarrow \quad 6\left(0.5m\right) = I_{AB}\left(\omega + 1\right)$$

(Initial angular velocity $= 1 \text{ rads}^{-1}$)

$$\Rightarrow \quad 3 = \left(I_{CM} + Mr^2\right)\left(1+\omega\right)$$

$$\Rightarrow \quad 3 = \left(1.2 + 30r^2\right)\left(1+\omega\right) \qquad \ldots(2)$$

Solving equations (1) and (2) for r, we get

$$r = 0.4 \text{ m} \quad \text{OR} \quad r = 0.1 \text{ m}$$

But at $r = 0.4$ m, ω comes out to be negative $\left(-0.5 \text{ rads}^{-1}\right)$ which is not acceptable. Therefore,

(a) $r =$ distance of CM from $AB = 0.1\ m$

(b) Substituting $r = 0.1m$ in equation (1), we get $\omega = 1 \text{ rads}^{-1}$ i.e., the angular velocity with which sheet comes back after the first impact is 1 rads^{-1}.

(c) Since, the sheet returns with same angular velocity of 1 rads^{-1}, the sheet will never come to rest.

PROBLEM 3

A uniform circular disc of mass M and radius a is pivoted at a point O on its circumference so that it can rotate about the tangent at O, which is horizontal, the centre of the disc describing a vertical circle of centre O in a plane perpendicular to the tangent. The point diametrically opposite to O is A and the disc is just displaced from rest when A is vertically above O. Find the angular velocity of the disc when A is vertically below O. At this instant a particle of mass M travelling with velocity u in the opposite direction of motion of the centre of the disc hits the disc at its centre and adheres to it. Find the angular velocity of the system immediately before and after the impact. If the disc just reaches its initial position show that

$$u = \left(3\sqrt{2} + \sqrt{5}\right)\sqrt{ag}$$

SOLUTION

(i) By Law of Conservation of Mechanical Energy, we get

$$Mg\left(2a\right) = \frac{1}{2}I\omega^2 \qquad \ldots(1)$$

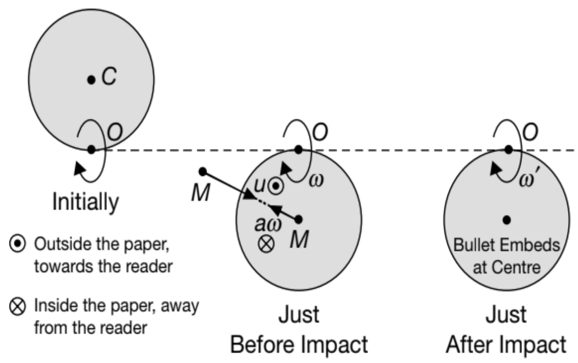
where, $I = \frac{1}{4}Ma^2 + Ma^2 = \frac{5}{4}Ma^2$

Substituting in equation (1), we get

$$2Mga = \frac{1}{2}\left(\frac{5}{4}Ma^2\right)\omega^2$$

$$\Rightarrow \omega = 4\sqrt{\frac{g}{5a}}$$

(ii) By Law of Conservation of Angular Momentum about an axis through O, we get



$$Mu\left(a\right) - \left(\frac{5}{4}Ma^2\right)\omega = \left(\frac{5}{4}Ma^2 + Ma^2\right)\omega'$$

$$\Rightarrow u - \frac{5}{4}\omega a = \frac{9}{4}\omega' a$$

$$\Rightarrow \omega' = \frac{4u - 5\omega a}{9a} = \frac{4}{9a}\left(u - \frac{5}{4}a\omega\right)$$

Substituting the value of ω we get

$$\omega' = \frac{4}{9a}\left(u - 5a\sqrt{\frac{g}{5a}}\right) = \frac{4}{9a}\left(u - \sqrt{5ag}\right)$$

(iii) Again, applying Law of Conservation of Mechanical Energy, we get

$$2Mg\left(2a\right) = \frac{1}{2}\left(\frac{5}{4}Ma^2 + Ma^2\right)\left(\omega'\right)^2$$

$$\Rightarrow 4Mga = \frac{9}{8}Ma^2\left(\omega'\right)^2$$

$$\Rightarrow 32g = 9a\left(\omega'\right)^2$$

$$\Rightarrow \omega' = \sqrt{\frac{32}{9}\frac{g}{a}} = 4\sqrt{\frac{2}{9}\frac{g}{a}}$$

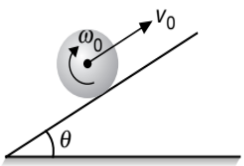
$$\Rightarrow 4\sqrt{\frac{2}{9}\frac{g}{a}} = \frac{4}{9a}\left(u - \sqrt{5ag}\right)$$

$$\Rightarrow u - \sqrt{5ag} = 9\sqrt{\frac{2}{9}ag}$$

$$\Rightarrow u = \left(3\sqrt{2} + \sqrt{5}\right)\sqrt{ag}$$

PROBLEM 4

A sphere of radius r is projected up an inclined plane for which $\mu = \left(\frac{1}{7}\right)\tan\theta$ with a velocity v_0 and initial angular velocity ω_0 such that, $v_0 > r\omega_0$. Prove that friction firstly acts down the incline and afterwards acts down the incline. Further prove that the total time of rise is $T = \dfrac{17v_0 + 4r\omega_0}{18g\sin\theta}$.



SOLUTION

Initially i.e., at $t = 0$, we have $v_0 > r\omega_0$, therefore, there is forward slipping. Hence, friction will be down the incline.

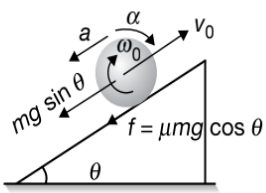
Once, $v = r\omega$, force of friction becomes up the incline, so

$$a = \frac{\mu mg\cos\theta + mg\sin\theta}{m} = \frac{8}{7}g\sin\theta$$

$$\left\{\because \mu = \frac{1}{7}\tan\theta\right\}$$

Also, $\alpha = \dfrac{\tau}{I} = \dfrac{\mu rmg\cos\theta}{\frac{2}{5}mr^2}$

$$\Rightarrow \alpha = \frac{5}{14}\frac{g\sin\theta}{r} \qquad \left\{\because \mu = \frac{1}{7}\tan\theta\right\}$$



Initially

This will last till pure rolling begins, so

$$v = r\omega$$

$$\Rightarrow \quad (v_0 - at_1) = r(\omega_0 + \alpha t_1)$$

Substituting the values, we get

$$t_1 = \frac{2}{3}\frac{(v_0 - r\omega_0)}{g\sin\theta}$$

Since $v = v_0 - at_1$

$$\Rightarrow \quad v = v_0 - \left(\frac{8}{7}g\sin\theta\right)\left(\frac{2}{3}\right)\left(\frac{v_0 - r\omega_0}{g\sin\theta}\right)$$

$$\Rightarrow \quad v = \frac{5}{21}v_0 + \frac{16}{21}r\omega_0$$

Once $v = r\omega$ i.e., the condition of pure rolling is obtained, then minimum value of μ required to maintain pure rolling is

$$\mu_{\min} = \frac{\tan\theta}{1 + \frac{mr^2}{I}} = \frac{\tan\theta}{1 + \frac{5}{2}} = \frac{\tan\theta}{3.5}$$

However, $\mu_{\text{given}} = \mu = \frac{1}{7}\tan\theta$ and since $\mu_{\text{given}} < \mu_{\min}$ so slipping will occur even after that and so the force of friction is upwards and maximum.

So, linear retardation is

$$a' = g\sin\theta - \mu g\cos\theta = \frac{6}{7}g\sin\theta$$

If t_2 be the further time taken by sphere to rise, then

$$0 = v - a't_2$$

$$\Rightarrow \quad t_2 = \frac{v}{a'} = \frac{\frac{5}{21}v_0 + \frac{16}{21}r\omega_0}{\frac{6}{7}g\sin\theta}$$

$$\Rightarrow \quad t_2 = \frac{5v_0 + 16r\omega_0}{18g\sin\theta}$$

So, we get, $T = t_1 + t_2 = \dfrac{17v_0 + 4r\omega_0}{18g\sin\theta}$

PROBLEM 5

A vertically oriented uniform rod of mass M and length l can rotate about its upper end. A horizontally flying bullet of mass m strikes the lower end of the rod and gets embedded in it as a result of which the rod swings through an angle α. Assuming that $m \ll M$, find the

(a) velocity of the flying bullet.

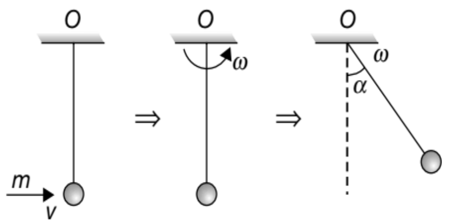
(b) momentum increment in the system "bullet-rod" during the impact and the cause that changes the momentum.

(c) distance x from the upper end of the rod the bullet must strike for the momentum of the system "bullet-rod" to remain constant during the impact.

SOLUTION

(a) Applying Law of Conservation of Angular Momentum about its upper end, we get

$$\sum mvr_{\perp} = \left(I_{\text{system}}\right)\omega$$



$$\Rightarrow \ (mv)l = \left(\frac{Ml^2}{3}\right)\omega \qquad \ldots(1)$$

$$\Rightarrow \ \omega = \frac{3m}{M}\frac{v}{l}$$

After the bullet gets embedded in the rod, the rod swings through an angle α, so by Law of Conservation of Energy we have

$$\begin{pmatrix}\text{Loss in}\\ \text{RKE of}\\ \text{Rod + Bullet}\end{pmatrix} = \begin{pmatrix}\text{Gain in}\\ \text{GPE of}\\ \text{CM of Rod}\end{pmatrix} + \begin{pmatrix}\text{Gain in}\\ \text{GPE of}\\ \text{Bullet}\end{pmatrix}$$

$$\Rightarrow \ \frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2 + ml^2\omega^2 = Mg\frac{l}{2}(1-\cos\alpha) + mgl(1-\cos\alpha)$$

Since $m \ll M$, so the underlined terms are neglected

$$\Rightarrow \ \frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2 = Mg\frac{l}{2}(1-\cos\alpha) \qquad \ldots(2)$$

Solving equations (1) and (2), we get

$$v = \frac{M}{m}\sqrt{\frac{2}{3}gl}\sin\left(\frac{\alpha}{2}\right)$$

(b) $\Delta P = P_f - P_i = M\left(\frac{l}{2}\omega\right) - mv$

$$\Rightarrow \ \Delta P = \frac{Ml}{2}\left(\frac{3m}{M}\right)\left(\frac{v}{l}\right) - mv = \frac{1}{2}mv$$

$$\Rightarrow \ \Delta P = \frac{1}{2}m\left(\frac{M}{m}\right)\sqrt{\frac{2}{3}gl}\sin\left(\frac{\alpha}{2}\right)$$

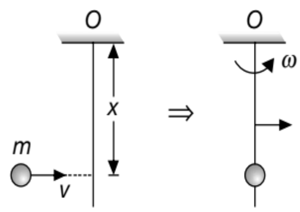
$$\Rightarrow \ \Delta P = M\sqrt{\frac{1}{6}gl}\sin\left(\frac{\alpha}{2}\right)$$

The force exerted by the support is responsible for the change in momentum.

(c) For Angular Momentum to be conserved, we have

$$L_i = L_f$$

$$\Rightarrow \ mvx = I\omega = \left(\frac{Ml^2}{3}\right)\omega \qquad \ldots(3)$$



For Linear Momentum to be conserved, we have

$$P_i = P_f$$

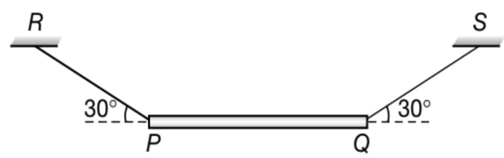
$$\Rightarrow \ mv = Mv_c = M\left(\frac{l}{2}\omega\right) \qquad \ldots(4)$$

From equations (3) and (4), we have

$$x = \frac{2}{3}l$$

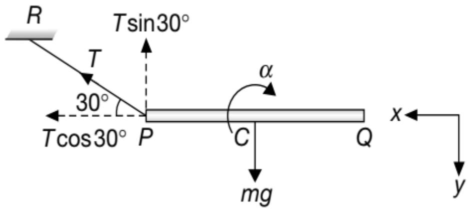
PROBLEM 6

A uniform rod of mass m and length l is suspended by the two wires at P and Q. The wire at Q suddenly breaks. Find the tension T in the wire at P immediately after the wire at Q is broken.



SOLUTION

Equation of motion for the centre of mass of rod are



$$a_x = \frac{T\cos 30°}{m} = \frac{\sqrt{3}T}{2m} \quad \ldots(1)$$

$$a_y = \frac{mg - T\sin 30°}{m} = g - \frac{T}{2m} \quad \ldots(2)$$

Angular acceleration of rod about its centre of mass C is

$$\alpha = \frac{\tau}{I} = \frac{(T\sin 30°)\frac{l}{2}}{\frac{ml^2}{12}} = \frac{3T}{ml} \quad \ldots(3)$$

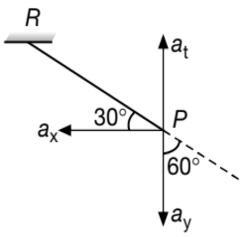
Acceleration of point P immediately after the wire Q breaks will be perpendicular to RP. Thus, along RP its component should be zero. Further

$$\vec{a}_{PC} = \vec{a}_P - \vec{a}_C$$

$$\Rightarrow \quad \vec{a}_P = \vec{a}_C + \vec{a}_{PC}$$

where, $\vec{a}_C$ has two components a_x and a_y.

Similarly $\vec{a}_{PC}$ has also two components a_t and a_n, where $a_n = 0$, because $a_n = \frac{l}{2}\omega^2$ and ω is zero at the instant wire Q breaks. Now, $a_t = \frac{l}{2}\alpha$ in the direction shown in Figure.



Since, acceleration of P along RP is zero so

$$a_t \sin(30°) + a_x \cos(30°) = a_y \cos(60°)$$

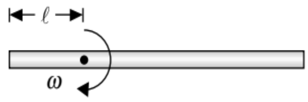
$$\Rightarrow \quad \frac{l}{4}\alpha + \frac{\sqrt{3}}{2}a_x = \frac{a_y}{2} \quad \ldots(4)$$

Solving above equations we get,

$$T = \frac{2}{7}mg$$

PROBLEM 7

A uniform rod of length $4l$ and mass m is free to rotate about a horizontal axis passing through a point distant l from its one end. When the rod is horizontal, its angular velocity is ω as shown in Figure. Calculate



(a) reaction of axis at this instant,

(b) acceleration of centre of mass of the rod at this instant,

(c) reaction of axis and acceleration of centre mass of the rod when rod becomes vertical for the first time.

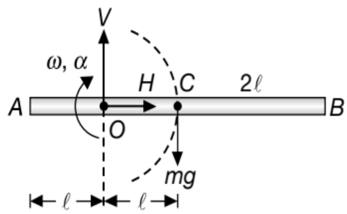
(d) minimum value of ω so that centre of rod can complete circular motion.

SOLUTION

(a) $\alpha = \dfrac{mgl}{\dfrac{m(4l)^2}{12} + ml^2} = \dfrac{3}{7}\dfrac{g}{l}$

If a_t be the tangential acceleration of the centre of mass of rod, then

$$a_t = l\alpha = \frac{3}{7}g \qquad \{\text{downwards}\}$$



Let V be the vertical reaction (upwards) at axis O, then

$$mg - V = ma_t = \frac{3mg}{7}$$

$$\Rightarrow \quad V = \frac{4}{7}mg \quad \ldots(1)$$

If H be the horizontal reaction (towards C) at axis, then

$$H = ma_n = ml\omega^2 \quad \ldots(2)$$

So, total reaction at axis is

$$N = \sqrt{H^2 + V^2} = \frac{4}{7}mg\sqrt{1 + \left(\frac{7l\omega^2}{4g}\right)^2}$$

(b) The acceleration of centre of mass of rod is

$$a_C = \sqrt{a_t^2 + a_n^2}$$

$$\Rightarrow a_C = \sqrt{\left(\frac{3g}{7}\right)^2 + \left(l\omega^2\right)^2}$$

(c) Let ω' be the angular speed of the rod when it becomes vertical for the first time. Then by Law of Conservation of Mechanical Energy, we have

$$\frac{1}{2}I\left(\omega'^2 - \omega^2\right) = mgl$$

$$\Rightarrow \omega'^2 = \omega^2 + \frac{2mgl}{I}$$

$$\Rightarrow \omega'^2 = \omega^2 + \frac{2mgl}{\frac{7}{3}ml^2} = \omega^2 + \frac{6g}{7l}$$

Acceleration of centre of mass at this instance is

$$a_C = l\omega'^2 = l\omega^2 + \frac{6g}{7}$$

Let V be the reaction (upwards) at axis at this instant, then,

$$V - mg = ma_C = ml\omega^2 + \frac{6mg}{7}$$

$$\Rightarrow V = \frac{13}{7}mg + ml\omega^2$$

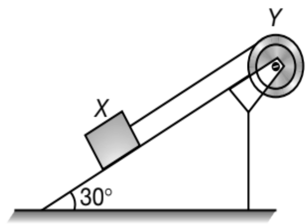
(d) By Law of Conservation of Mechanical Energy, we have

$$mgl = \frac{1}{2}I\omega_{\min}^2$$

$$\Rightarrow \omega_{\min} = \sqrt{\frac{2mgl}{I}} = \sqrt{\frac{2mgl}{\frac{7}{3}ml^2}} = \sqrt{\frac{6g}{7l}}$$

PROBLEM 8

A block X of mass 0.5 kg is held by a long massless string on a frictionless inclined plane of inclination $30°$ to the horizontal. The string is wound on a uniform solid cylindrical drum Y of mass 2 kg and of radius 0.2 m as shown in Figure.



The drum is given an initial angular velocity such that the block X starts moving up the plane.

(i) Find the tension in the string during the motion. Take $g = 9.8\ \text{ms}^{-2}$.

(ii) At a certain instant of time the magnitude of the angular velocity of Y is $10\ \text{rads}^{-1}$. Calculate the distance travelled by X from the instant of time until it comes to rest.

SOLUTION

Mass of block X is $m = 0.5$ kg

Mass of drum Y, $M = 2$ kg

Radius of drum, $R = 0.2$ m

Angle of inclined plane, $\theta = 30^o$

Let $'a'$ be the linear retardation of block X and α the angular retardation of drum Y. Then,

$$a = R\alpha$$

Since, $mg\sin 30^o - T = ma$ ...(1)

$$\Rightarrow \frac{mg}{2} - T = ma \quad ...(2)$$

Also $\alpha = \frac{\tau}{I} = \frac{TR}{\frac{1}{2}MR^2}$

$$\Rightarrow \alpha = \frac{2T}{MR} \quad ...(3)$$

Solving equations (1), (2) and (3) for T, we get

$$T = \frac{1}{2}\left[\frac{Mmg}{M+2m}\right]$$

Substituting the value, we get

$$T = \left(\frac{1}{2}\right)\left\{\frac{(2)(0.5)(9.8)}{2+(0.5)(2)}\right\} = 1.63\ \text{N}$$

$$\Rightarrow T = 1.63\ \text{N}$$

From equation (3), angular retardation of drum

$$\alpha = \frac{2T}{MR} = \frac{(2)(1.63)}{(2)(0.2)} = 8.15\ \text{rads}^{-2}$$

Linear retardation of block is

$$a = R\alpha = (0.2)(8.15) = 1.63\ \text{ms}^{-2}$$

At the moment when angular velocity of drum is

$$\omega_o = 10\ \text{rads}^{-1}$$

The linear velocity of block will be

$$v_o = \omega_o R = (10)(0.2) = 2 \text{ ms}^{-1}$$

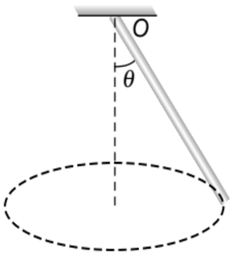
Now, the distance(s) travelled by the block until it comes to rest will be given by

$$s = \frac{v_o^2}{2a} \qquad \{\text{using } v^2 = v_o^2 - 2as \text{ with } v = 0\}$$

$$\Rightarrow s = \frac{(2)^2}{2(1.63)} m = 1.22 \text{ m}$$

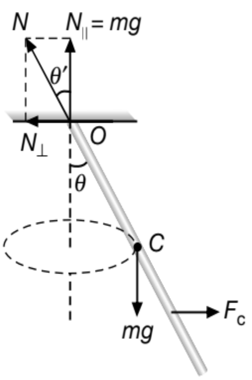
PROBLEM 9

A thin uniform rod of mass m and length l rotates with the constant angular velocity ω about the vertical axis passing through the rod's suspension point O. In doing so, the rod describes a conical surface with a half aperture angle θ. Find the angle θ as well as the magnitude and direction of the reaction force N at the point O.



SOLUTION

Let us consider the frame rotating about the vertical axis along with the rod. In this reference frame, the rod experiences not only the gravity mg and the reaction force N but also the centrifugal force F_c, directed radially outwards as shown in Figure.



Since the rod is observed to be in equilibrium in the given reference frame, the resultant torques of all forces relative to any point and the resultant of all the forces are equal to zero.

The gravity and the centrifugal forces are the only torque producing forces relative to the point O. So, if τ_C is the total torque produced due to centrifugal forces acting on the rod, then

$$\tau_{mg} = \tau_C$$

$$\Rightarrow \frac{mgl}{2}\sin\theta = \tau_c \qquad \ldots(1)$$

Let us now calculate τ_c by integration

The elementary torque due to the centrifugal force that acts on the infinitesimal rod element of length dx located at the distance x from the point O and having mass $dm\left(=\frac{m}{l}dx\right)$ is equal to,

$$d\tau_C = (dF_c)x\cos\theta$$

where $dF_C = (dm)(x\sin\theta)\omega^2 = \left(\frac{m}{l}dx\right)(x\sin\theta)\omega^2$

$$\Rightarrow d\tau_c = \frac{m\omega^2}{l}\sin\theta\cos\theta x^2 dx$$

Integrating this expression over the whole length of the rod, we get

$$\tau_c = \frac{m\omega^2 l^2}{3}\sin\theta\cos\theta \qquad \ldots(2)$$

From equation (1) and (2), we get

$$\cos\theta = \frac{3g}{2\omega^2 l} \qquad \ldots(3)$$

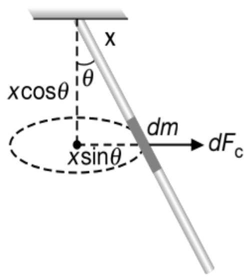
$$\Rightarrow \theta = \cos^{-1}\left(\frac{3g}{2\omega^2 l}\right)$$

Further, writing equations for centre of mass of the rod, we get

$$N_{\parallel} = mg$$

and $$N_{\perp} = ma_n = \frac{m\omega^2 l}{2}\sin\theta \qquad \ldots(4)$$

$$\Rightarrow N = \sqrt{N_{\parallel}^2 + N_{\perp}^2} = \sqrt{(mg)^2 + \left(\frac{m\omega^2 l}{2}\sin\theta\right)^2}$$



Substituting the value of θ from equation (3), we get

$$N=\frac{m\omega^2 l}{2}\sqrt{1+\frac{7g^2}{4\omega^4 l^2}}$$

For its direction, we observe that

$$\cos\theta'=\frac{mg}{N}$$

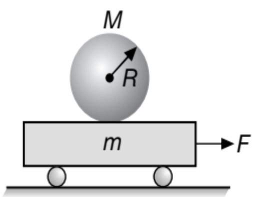
Substituting the values, we get

$$\cos\theta'=4\cos\theta\sqrt{9+7\cos^2\theta}$$

The centrifugal force F_c does not pass through point C but below it. (Think, why?)

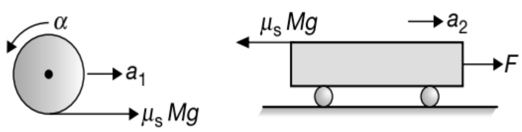
PROBLEM 10

Determine the maximum horizontal force F that may be applied to the plank of mass m for which the solid sphere does not slip as it begins to roll on the plank. The sphere has a mass M and radius R. The coefficient of static and kinetic friction between the sphere and the plank are μ_s and μ_k respectively.



SOLUTION

The free body diagrams for the sphere and the plank are shown in Figure.



For sphere: Linear acceleration

$$a_1=\frac{\mu_s Mg}{M}=\mu_s g \qquad \ldots(1)$$

Angular acceleration

$$\alpha=\frac{(\mu_s Mg)R}{\frac{2}{5}MR^2}=\frac{5}{2}\frac{\mu_s g}{R} \qquad \ldots(2)$$

For plank: Linear acceleration

$$a_2=\frac{F-\mu_s Mg}{m} \qquad \ldots(3)$$

For no slipping, we have

$$a_2=a_1+R\alpha \qquad \ldots(4)$$

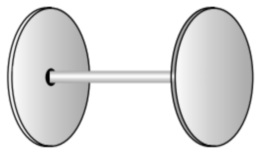
Solving the above four equations, we get

$$F=\mu_s g\left(M+\frac{7}{2}m\right)$$

Thus, maximum value of F can be $\mu_s g\left(M+\frac{7}{2}m\right)$

PROBLEM 11

Two thin circular discs of mass 2 kg and radius 10 cm each are joined by a rigid massless rod of length 20 cm. The axis of the rod is along the perpendicular to the planes of the disc through their centres. This object is kept on a truck in such a way that the axis of the object is horizontal and perpendicular to the direction of motion of the truck. Its friction with the floor of the truck is large enough so that the object can roll on the truck without slipping. Take X-axis as the direction of motion of the truck and Z-axis as the vertically upwards direction. If the truck has an acceleration 9 ms^{-2}, calculate



(i) the force of friction on each disc.

(ii) the magnitude and direction of the friction torque acting on each disc about the centre of mass O of the object. Express the torque in the vector form in terms of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ in X, Y and Z directions.

SOLUTION

Given mass of disc $m=2\text{ kg}$ and radius $R=0.1\text{ m}$

(i) FBD of any one disc is

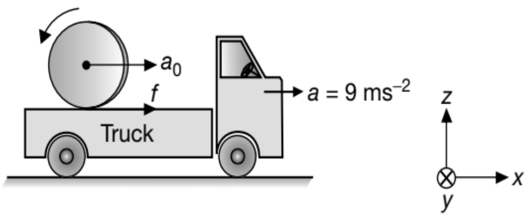
Frictional force on the disc should be in forward direction.

Let a_0 be the linear acceleration of centre of mass of disc and α the angular acceleration about its centre of mass. Then,

$$a_0=\frac{f}{m}=\frac{f}{2} \qquad \ldots(1)$$

$$\alpha=\frac{\tau}{I}=\frac{fR}{mR^2/2}=\frac{2f}{mR}=\frac{2f}{2(0.1)}=10f \qquad \ldots(2)$$

Since, there is no slipping between disc and truck.



Therefore,

$$\begin{pmatrix}\text{Acceleration of the}\\ \text{contact point}\end{pmatrix} = \begin{pmatrix}\text{Acceleration of}\\ \text{the truck}\end{pmatrix}$$

$\Rightarrow a_0 + R\alpha = a$

$\Rightarrow \left(\frac{f}{2}\right) + (0.1)(10f) = a$

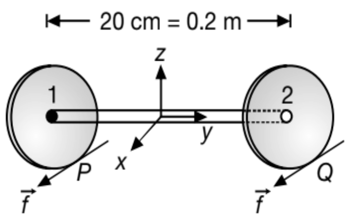
$\Rightarrow \frac{3}{2}f = a$

$\Rightarrow f = \frac{2a}{3} = \frac{2\times 9}{3}\ N$

$\Rightarrow f = 6\ N$

Since, this force is acting in positive x-direction, so in vector form,

$\vec{f} = (6\hat{i})N$



(ii) $\vec{\tau} = \vec{r}\times\vec{f}$

where, $\vec{f} = (6\hat{i})N$ (for both the discs)

$\vec{r}_P = \vec{r}_1 = -0.1\hat{j} - 0.1\hat{k}$

and $\vec{r}_Q = \vec{r}_2 = 0.1\hat{j} - 0.1\hat{k}$

Therefore, frictional torque on disk 1 about point O (centre of mass).

$\vec{\tau}_1 = \vec{r}_1\times\vec{f} = (-0.1\hat{j} - 0.1\hat{k})\times(6\hat{i})\text{Nm}$

$\vec{\tau}_1 = (0.6\hat{k} - 0.6\hat{j})$

$\Rightarrow \vec{\tau}_1 = 0.6(\hat{k} - \hat{j})$ Nm

and $|\vec{\tau}_1| = \sqrt{(0.6)^2 + (0.6)^2} = 0.85$ Nm

Similarly, $\vec{\tau}_2 = \vec{r}_2\times\vec{f} = 0.6(-\hat{j} - \hat{k})$

and $|\vec{\tau}_1| = |\vec{\tau}_2| = 0.85$ Nm

PROBLEM 12

Show that if a rod held at angle θ to the horizontal and released, its lower end will not slip if the friction coefficient between rod and ground is greater than $\frac{3}{2}\left(\frac{\sin 2\theta}{1+3\sin^2\theta}\right)$.

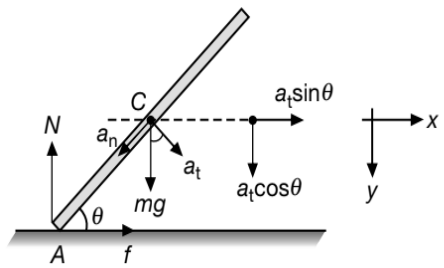
SOLUTION

Point A is momentarily at rest, so

$$\alpha = \frac{\tau}{I} = \frac{mg(l/2)\cos\theta}{ml^2/3} = \frac{3g\cos\theta}{2l}$$

If a_t be the tangential acceleration of the centre of mass of rod C, then

$$a_t = \frac{l}{2}\alpha = \frac{3}{4}g\cos\theta$$



Now, $\mu N = ma_x$ or $\mu N = ma_C\sin\theta$

$$\Rightarrow \mu N = \frac{3}{4}mg\sin\theta\cos\theta \qquad \ldots(1)$$

Further, $mg - N = ma_y$

$\Rightarrow N = mg - ma_t\cos\theta$

$$\Rightarrow N = mg - \frac{3}{4}mg\cos^2\theta \qquad \ldots(2)$$

Dividing equation (1) by (2), we get

$$\mu = \frac{\frac{3}{4}\sin\theta\cos\theta}{1-\frac{3}{4}\cos^2\theta} = \frac{3\sin\theta\cos\theta}{4-3\cos^2\theta}$$

$$\Rightarrow \quad \mu = \frac{3\sin\theta\cos\theta}{1+3\sin^2\theta} = \frac{3}{2}\left(\frac{2\sin\theta\cos\theta}{1+3\sin^2\theta}\right)$$

$$\Rightarrow \quad \mu = \frac{3}{2}\left(\frac{\sin(2\theta)}{1+3\sin^2\theta}\right)$$

PROBLEM 13

A uniform stick of length L, mass M hinged at one end is released from rest at an angle θ_0 with the vertical. When the angle with the vertical is θ, the hinge exerts a force F_r along the stick and F_t perpendicular to the stick. Calculate F_r and F_t.

SOLUTION

Let C be the centre of mass of the rod and ω be the angular speed of rod about point O at angle θ. By Law of Conservation of Mechanical Energy, we have

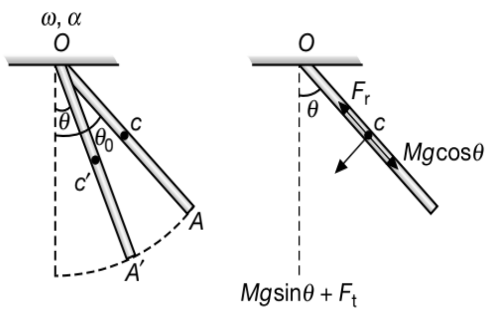
$$Mg\frac{L}{2}(\cos\theta - \cos\theta_0) = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

$$\Rightarrow \quad \omega^2 = \frac{3g}{L}(\cos\theta - \cos\theta_0) \quad \ldots(1)$$

$$\text{Since, } F_r - Mg\cos\theta = M\left(\frac{L}{2}\right)\omega^2 \quad \ldots(2)$$

From equations (1) and (2), we get

$$F_r = \frac{1}{2}Mg(5\cos\theta - 3\cos\theta_0)$$



Angular acceleration of rod at this instant,

$$\alpha = \frac{\tau}{I} = \frac{Mg\left(\frac{L}{2}\sin\theta\right)}{\frac{ML^2}{3}} = \frac{3}{2}\frac{g\sin\theta}{L}$$

Tangential acceleration of centre of mass C is

$$a_t = r\alpha = (\alpha)\left(\frac{L}{2}\right) = \frac{3}{4}g\sin\theta \quad \ldots(3)$$

$$\text{Now, } F_t + Mg\sin\theta = Ma_t \quad \ldots(4)$$

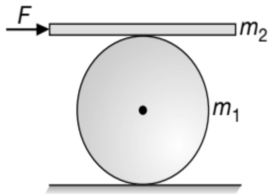
From equation (3) and (4), we get

$$F_t = -\frac{1}{4}Mg\sin\theta$$

Here negative sign implies that direction of F_t is opposite to the component $Mg\sin\theta$.

PROBLEM 14

A man pushes a cylinder of mass m_1 with the help of a plank of mass m_2 as shown. There is no slipping at any contact. The horizontal component of the force applied by the man is F, find



(a) the acceleration of the plank and the centre of mass of the cylinder, and

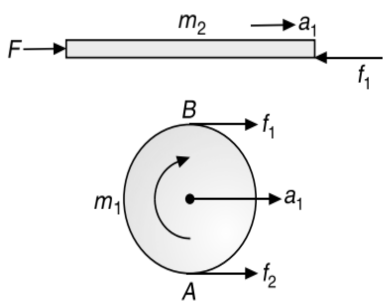
(b) the magnitudes and directions of frictional forces at contact points.

SOLUTION

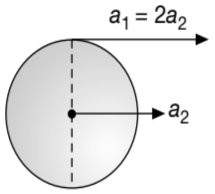
We can choose my arbitrary directions of frictional forces at different contacts. In the final answer the negative values will show the opposite directions.

Let the friction between plank and cylinder, i.e. at contact point B be f_1, the friction between cylinder and ground, i.e. at contact point A be f_2, acceleration of the plank be $a_p = a_1$, acceleration of centre of mass of cylinder be $a_c = a_2$ and angular acceleration of cylinder about its CM be α.

Directions of f_1 and f_2 are as shown in Figure.



Since, there is no slipping anywhere, so acceleration of plank equals the acceleration of top point of cylinder, so we have



$$a_1 = 2a_2 \quad \text{...(1)}$$

Also, $a_1 = \dfrac{F - f_1}{m_2}$ ...(2)

$$a_2 = \frac{f_1 + f_2}{m_1} \quad \text{...(3)}$$

If I be moment of inertia of cylinder about CM, then

$$\alpha = \frac{(f_1 - f_2)R}{I} = \frac{(f_1 - f_2)R}{m_1R^2/2}$$

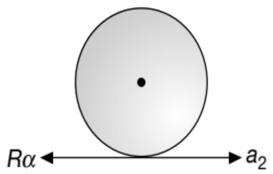
$$\Rightarrow \quad \alpha = \frac{2(f_1 - f_2)}{m_1R} \quad \text{...(4)}$$

$$\Rightarrow \quad a_2 = R\alpha = \frac{2(f_1 - f_2)}{m_1} \quad \text{...(5)}$$

Acceleration of bottom most point of cylinder is zero.

(a) Solving equations (1), (2), (3) and (5), we get

$$a_1 = \frac{8F}{3m_1 + 8m_2} \text{ and } a_2 = \frac{4F}{3m_1 + 8m_2}$$



(b) $f_B = f_1 = \dfrac{3m_1F}{3m_1 + 8m_2}$ and $f_A = f_2 = \dfrac{m_1F}{3m_1 + 8m_2}$

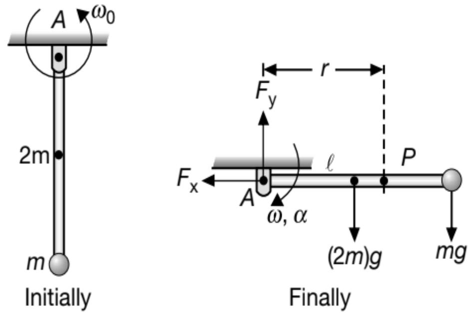
Since, all quantities are positive, they are correctly shown in the above Figures.

PROBLEM 15

A uniform rod AB of length $2l$ and mass $2m$ has a particle of mass m attached at B. The rod is free to rotate in a vertical plane about a horizontal axis through A. When the rod is hanging at rest, with B below A, it is given an angular velocity $\dfrac{7}{2}\sqrt{\dfrac{g}{l}}$. Find the reaction at the axis when the rod first becomes horizontal. Take $g = 10 \text{ ms}^{-2}$.

SOLUTION

When the rod first becomes horizontal, this situation is shown in Figure.



By Law of Conservation of Mechanical Energy, we have

$$\begin{pmatrix}\text{Loss in}\\ \text{RKE of}\\ \text{Particle + Rod}\end{pmatrix} = \begin{pmatrix}\text{Gain in}\\ \text{GPE of}\\ \text{CM of Rod}\end{pmatrix} + \begin{pmatrix}\text{Gain in}\\ \text{GPE of}\\ \text{Particle}\end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{2}I_A(\omega_0^2 - \omega^2) = (2m)gl + mg(2l)$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{2m(2l)^2}{3} + m(2l)^2\right)\left(\frac{49g}{4l} - \omega^2\right) = 4mgl$$

$$\Rightarrow \quad \omega^2 = \frac{221g}{20l}$$

In the horizontal position of the rod, we have

$$\tau = (2m)gl + mg(2l) = 4mgl$$

Since $\tau = I\alpha$

$$\Rightarrow \quad \alpha = \frac{\tau}{I} = \frac{2mgl + mg(2l)}{\left(\frac{2m(2l)^2}{3} + m(2l)^2\right)} = \frac{3}{5}\frac{g}{l}$$

Let the CM of Rod plus Particle be at point P, a distance r from A. Then

$$r = \frac{(2m)l + m(2l)}{3m} = \frac{4l}{3}$$

$$\Rightarrow \quad a_y = a_t = r\alpha = \left(\frac{4l}{3}\right)\left(\frac{3g}{5l}\right) = 0.8g$$

Since $3mg - F_y = (3m)a_t$

$$\Rightarrow \quad F_y = 3mg - 3m(0.8g)$$

$$\Rightarrow \quad F_y = 0.6mg \text{, upwards}$$

Also, $a_x = a_c = r\omega^2 = \left(\dfrac{4l}{3}\right)\left(\dfrac{221g}{20l}\right) = 14.73g$

Force exerted by hinge in x-direction is

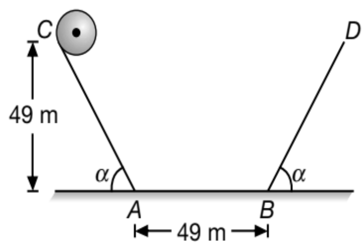
$$F_x = F_c = (3m)a_c = 44.2mg$$

So, total hinge force is given by

$$F = \sqrt{F_x^2 + F_y^2} = 442 \text{ mN}$$

PROBLEM 16

A solid sphere is released from a height of 49 m from horizontal on an inclined plane as shown in Figure. All the surfaces are smooth except AB which has $\mu = \frac{2}{3}$ and length of AB is 49 m. After passing over the rough surface, the sphere climbs to the right portion BD of incline.



(a) Find the height from horizontal to which it climbs on BD.

(b) Now the sphere returns and goes up the incline AC. Find the linear velocity of sphere at A during the left-out motion.

SOLUTION

From C to A we have, velocity at A is

$$v_A = \sqrt{2 \times g \times h} = \sqrt{2 \times 10 \times 49} = 31.3 \text{ ms}^{-1}$$

From A to B, let pure rolling starts after time t.

$$v = R\omega$$

$$\Rightarrow \quad v_A - at = R(\alpha t)$$

$$\Rightarrow \quad v_A - \mu g t = R\left(\frac{\mu m g R}{2mr^2/5}\right)t = 2.5\mu g t$$

$$\Rightarrow \quad t = \frac{v_A}{3.5\mu g} = \frac{31.3}{3.5 \times (2/3) \times 10} = 1.34 \text{ s}$$

Now, $s = v_A t - \frac{1}{2}(\mu g)t^2$

$$\Rightarrow \quad s = (31.3)(1.34) - \frac{1}{2}\left(\frac{2}{3}\right)(10) \times (1.34)^2$$

$$\Rightarrow \quad s = 36 \text{ m}$$

Since velocity $v = v_A - at$, so we have

$$v = v_A - \mu g t = 31.3 - \frac{2}{3} \times 10 \times 1.34 = 22.36 \text{ ms}^{-1}$$

The remaining distance of $49 - 36 = 13$ m the velocity is maintained at 22.36 ms^{-1}, because of pure rolling the friction is zero. So, the height raised above the horizontal is

$$H = \frac{v^2}{2g} = \frac{(22.36)^2}{2 \times 10} = 25 \text{ m}$$

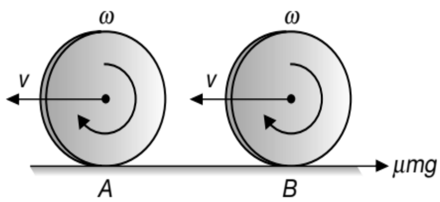
From B to A, let pure rolling start at time t', then

$$v' = R\omega'$$

where $\omega' = -\omega + \alpha t'$ and $v' = v - \mu g t'$

$$\Rightarrow \quad v - \mu g t' = -\omega R + \alpha R t'$$

$$\Rightarrow \quad t' = \frac{v + R\omega}{3.5\mu g} = \frac{2v}{3.5\mu g}$$



$$\Rightarrow \quad t' = \frac{2 \times 22.36}{3.5 \times (2/3) \times 10} = 1.91 \text{ s}$$

So, $v' = v - \mu g t$

$$\Rightarrow \quad v' = 22.36 - \frac{2}{3} \times 10 \times 1.91 = 9.62 \text{ ms}^{-1}$$

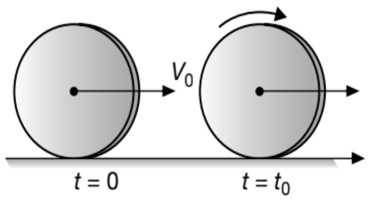
$$\Rightarrow \quad s' = 22.36 \times 1.91 - \frac{1}{2} \times \frac{2}{3} \times 10 \times (1.91)^2$$

$$\Rightarrow \quad s' = 30.53 \text{ m}$$

Since s' is less than 49 m, velocity of sphere at A during the remaining journey is 9.62 ms^{-1}.

PROBLEM 17

A uniform disc of mass m and radius R is projected horizontally with velocity V_0 on a rough horizontal floor, so that it starts off with a purely sliding motion at $t = 0$. After t_0 second, it acquires a purely rolling motion as shown in Figure.

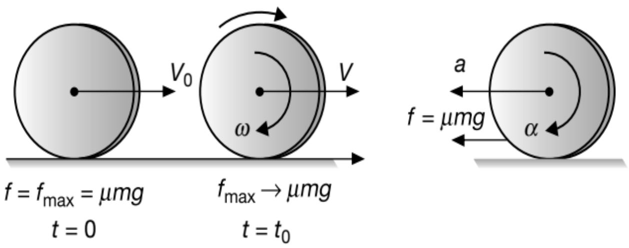


(a) Calculate the velocity of the centre of mass of the disc at t_0.

(b) Assuming the coefficient of friction to be μ, calculate t_0. Also calculate the work done by the frictional force as a function of time and the total work done by it over a time t much longer than t_0.

SOLUTION

Between the time $t = 0$ to $t = t_0$, there is forward sliding, so friction f is leftwards and maximum, i.e. μmg.
For time $t > t_0$, friction f will become zero, because now pure rolling has started i.e., there is no sliding (no relative motion) between the points of contact.



So, for time $t < t_0$, the linear retardation and the angular acceleration respectively are

$$a = \frac{f}{m} = \mu g \qquad \{\because f = \mu mg\}$$

$$\alpha = \frac{\tau}{I} = \frac{fR}{mR^2/2} = \frac{2\mu g}{R}$$

Now let V be the linear velocity and ω, the angular velocity of the disc at time $t = t_0$ then

$$V = V_0 - at_0 = V_0 - \mu g t_0 \qquad \ldots(1)$$

and $$\omega = \alpha t_0 = \frac{2\mu g t_0}{R} \qquad \ldots(2)$$

For pure rolling to take place

$$V = R\omega$$

i.e., $$V_0 - \mu g t_0 = 2\mu t_0$$

$$\Rightarrow \quad t_0 = \frac{V_0}{3\mu g}$$

Substituting in equation (1), we get

$$V = V_0 - \mu g\left(\frac{V_0}{3\mu g}\right)$$

$$\Rightarrow \quad V = \frac{2}{3}V_0$$

For $t \le t_0$, the linear velocity of disc at any time t is $V = V_0 - \mu g t$ and angular velocity is $\omega = \alpha t = \frac{2\mu g t}{R}$.

From work-energy theorem, work done by friction up to time t equals the change in kinetic energy, so we have

$$\Rightarrow \quad W = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2 - \frac{1}{2}mV_0^2$$

$$\Rightarrow \quad W = \frac{1}{2}m(V_0 - \mu g t)^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{2\mu g t}{2}\right)^2 - \frac{1}{2}mV_0^2$$

$$\Rightarrow \quad W = \frac{m}{2}\left(V_0^2 + \mu^2 g^2 t^2 - 2V_0\mu g t + 2\mu^2 g^2 t^2 - V_0^2\right)$$

$$\Rightarrow \quad W = \frac{\mu m g t}{2}(3\mu g t - 2V_0)$$

For $t > t_0$, friction force is zero i.e., work done in friction is zero. Hence the energy will be conserved. Therefore, total work done by friction over a time t much longer then t_0 is total work done up to time t_0 (because beyond the work done by friction is zero) which is equal to

$$W = \frac{m\mu g t_0}{2}(3\mu g t_0 - 2V_0)$$

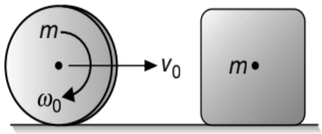
Substituting $t_0 = \frac{V_0}{3\mu g}$, we get

$$W = \frac{mV_0}{6}(V_0 - 2V_0)$$

$$\Rightarrow \quad W = -\frac{mV_o^2}{6}$$

PROBLEM 18

A solid cylinder of mass m is rolling on a rough horizontal surface with velocity v_0. It collides elastically with a cubical block of same mass at rest. The height of centre of mass of both the bodies is same. Assume that there is no friction between the cylinder and the block and coefficient of friction between all other surfaces is μ, find the

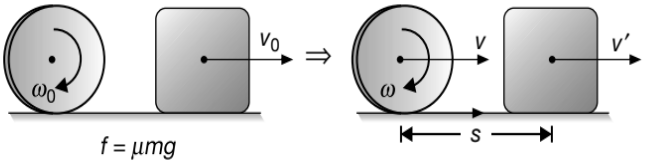


(a) velocity of block when cylinder starts pure rolling again.

(b) time after first collision at which the second collision will take place.

SOLUTION

In head on elastic collision between two equal masses velocities are exchanged.



Let the cylinder have radius R and let it start rolling after time t_1, then

$$v = R\omega \qquad \ldots(1)$$

where $v = at_1 = (\mu g)t_1$ and

$$\omega = \omega_0 - \alpha t = \omega_0 - \left(\frac{fR}{I}\right)t_1$$

$$\Rightarrow \quad \omega = \omega_0 - \left(\frac{2\mu g}{R}\right)t_1$$

So, from (1) i.e., $v = R\omega$, we get

$$\mu g t_1 = R\left(\omega_0 - \frac{2\mu g t_1}{R}\right)$$

$$\Rightarrow \quad \mu g t_1 = R\left(\frac{v_0}{R} - \frac{2\mu g t_1}{R}\right)$$

$$\Rightarrow \quad t_1 = \frac{v_0}{3\mu g}$$

Since, $v = \mu g t_1$

$$\Rightarrow \quad v = (\mu g)\left(\frac{v_0}{3\mu g}\right) = \frac{v_0}{3}$$

During this time t_1, the block moves forward under a retardation μg, so if v' be the velocity of block when cylinder starts pure rolling at t_1, then

$$v' = v_0 - \mu g t_1 = v_0 - \frac{v_0}{3} = \frac{2v_0}{3}$$

After the first collision, we have

$$v_r = v_0,\ a_r = -2\ \mu g$$

Since, $s = v_r t_1 + \frac{1}{2}a_r t_1^2$

$$\Rightarrow \quad s = \frac{v_0^2}{3\mu g} - \frac{v_0^2}{9\mu g} = \frac{2}{9}\frac{v_0^2}{\mu g}$$

After the cylinder starts pure rolling friction on cylinder becomes zero and $v_r = v' - v = \frac{v_0}{3}$, $a_r = -\mu g$

Let the second collision takes place after time t_2, then

$$-s = v_r t + \frac{1}{2}a_r t^2$$

Substituting the values, we get

$$-\frac{2}{9}\frac{v_0^2}{\mu g} = \frac{v_0}{3}t_2 - \frac{1}{2}\mu g t_2^2$$

$$\Rightarrow \quad t_2 = \left(\frac{\sqrt{5}+1}{3\mu g}\right)v_0$$

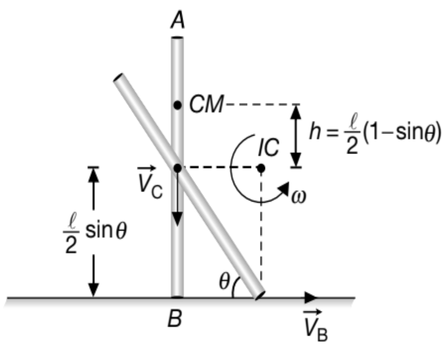
$$\Rightarrow \quad t = t_1 + t_2 = \frac{(\sqrt{5}+2)v_0}{3\mu g}$$

PROBLEM 19

A uniform thin rod of mass m and length l is standing on a smooth horizontal surface. A slight disturbance causes the lower end to slip on the smooth surface and the rod starts falling. Describe the trajectory of the centre of mass. Find the velocity of centre of mass of the rod at the instant when it makes an angle θ with horizontal. Also find the velocity of centre of mass when it touches the surface.

SOLUTION

Since the floor is smooth so the mechanical energy of the rod will remain conserved. Further, no horizontal force acts on the rod, hence the centre of mass moves vertically downwards in a straight line. Thus, velocities of centre of mass and the lower end B are in the directions shown in Figure.



The location of IC at this instant is found by drawing perpendiculars to $\vec{v}_C$ and $\vec{v}_B$ at respective points and IC is located at the point of intersection of these two perpendiculars. So, the rod may now be assumed to be in pure rotational motion about $IAOR$ passing through IC with angular speed ω.
Applying Law of Conservation of Mechanical Energy. we get

$$\begin{pmatrix}\text{Decrease in GPE}\\ \text{of CM of Rod}\end{pmatrix}=\begin{pmatrix}\text{Increase in RKE}\\ \text{of Rod about } IAOR\end{pmatrix}$$

$$\Rightarrow \quad mgh=\frac{1}{2}I_{IAOR}\omega^2$$

$$\Rightarrow \quad mg\frac{l}{2}(1-\sin\theta)=\frac{1}{2}\left(\frac{ml^2}{12}+\frac{ml^2}{4}\cos^2\theta\right)\omega^2$$

Solving this equation, we get

$$\omega=\sqrt{\frac{12g(1-\sin\theta)}{l(1+3\cos^2\theta)}}$$

Now, $|\vec{v}_c|=\left(\frac{l}{2}\cos\theta\right)\omega$

$$\Rightarrow \quad |\vec{v}_c|=\sqrt{\frac{3gl(1-\sin\theta)\cos^2\theta}{(1+3\cos^2\theta)}}$$

When the rod touches the surface, then $\theta=0°$, so

$$|\vec{v}_c|=\sqrt{\frac{3gl}{4}}$$

PROBLEM 20

A uniform rod AB of mass m and length $2a$ has a particle of mass m attached to the end B. The rod can rotate in a vertical plane about a smooth axis through A. If the body is slightly displaced from the position in which B is vertically above A, find the magnitude of the reaction at the axis, which is horizontal, when the rod has rotated through $\frac{\pi}{3}$ radians.

SOLUTION

By Law of Conservation of Energy, we get

$$\begin{pmatrix}\text{Gain in}\\ \text{RKE of}\\ \text{Rod}\end{pmatrix}=\begin{pmatrix}\text{Loss in}\\ \text{GPE of}\\ \text{CM of Rod}\end{pmatrix}+\begin{pmatrix}\text{Loss in}\\ \text{GPE of}\\ \text{Particle}\end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{m(2a)^2}{3}+m(2a)^2\right]\omega^2= mg(2a)(1-\cos 60°)+mg(a)(1-\cos 60°)$$

$$\Rightarrow \quad \frac{8}{3}ma^2\omega^2=\frac{3mga}{2}$$

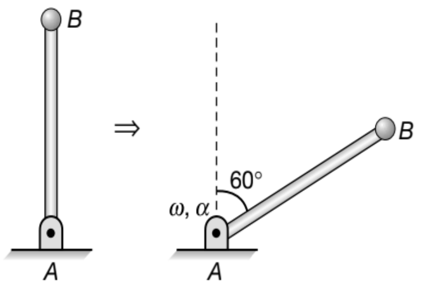
$$\Rightarrow \quad \omega^2=\frac{9}{16}\frac{g}{a} \qquad \ldots(1)$$

Now, $\alpha=\dfrac{\tau}{I}=\dfrac{mg(a\sin 60°)+mg(2a\sin 60°)}{\frac{16}{3}ma^2}$

$$\Rightarrow \quad \alpha=\frac{9\sqrt{3}}{32}\frac{g}{a} \qquad \ldots(2)$$

Now centre of mass of the system (point C) is at a distance of $\frac{3a}{2}$ from point A. Acceleration of C has two components a_t and a_n, where

$$a_t=\left(\frac{3a}{2}\right)\alpha=\frac{27\sqrt{3}}{64}g \text{ and } a_n=\left(\frac{3a}{2}\right)\omega^2=\frac{27}{32}g$$

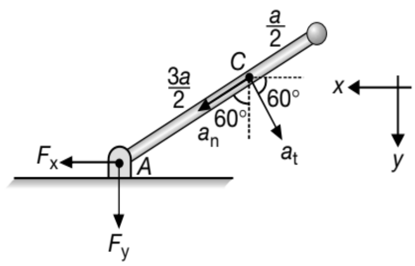


Force applied by hinge in x-direction

$$F_x=2ma_x=2m(a_n\sin 60°-a_t\cos 60°)$$

$$\Rightarrow \quad F_x=2m\left(\frac{27\sqrt{3}}{64}-\frac{27\sqrt{3}}{128}\right)g$$

$$\Rightarrow \quad F_x=\frac{54\sqrt{3}}{128}mg$$



Similarly, $F_y+(2m)g=(2m)a_y$

$$\Rightarrow \quad F_y=2m(a_t\sin 60°+a_n\cos 60°)-2mg$$

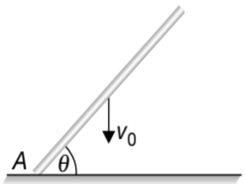
$$\Rightarrow \quad F_y=2m\left(\frac{81}{128}+\frac{27}{64}\right)g-2mg=\frac{270}{128}mg-2mg$$

$$\Rightarrow \quad F_y = \frac{14}{128} mg$$

So, $$F = \sqrt{F_x^2 + F_y^2} = \frac{mg}{32}\sqrt{559}$$

PROBLEM 21

A rod of length l forming an angle θ with the horizontal strikes a frictionless floor at A with its centre of mass velocity v_0 and no angular velocity. Assuming that the impact at A is perfectly elastic, find the angular velocity of the rod immediately after the impact.



SOLUTION

If v be the linear velocity of rod after impact (upwards), ω be the angular velocity of rod and J be the linear impulse at A during impact, then by Impulse – Momentum Theorem, we have

During impact   After impact

$$J = \Delta p = p_f - p_i$$

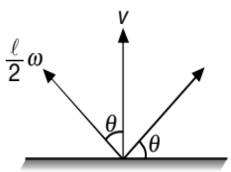
$$\Rightarrow \quad J = mv - (-mv_0)$$

$$\Rightarrow \quad J = m(v + v_0) \qquad \ldots(1)$$

Further by angular impulse angular momentum theorem, we have

$$J\left(\frac{l}{2}\cos\theta\right) = I\omega = \frac{ml^2}{12}\omega \qquad \ldots(2)$$

Since the collision is elastic, so at the point of impact, $e = 1$.



$$\Rightarrow \quad \begin{pmatrix}\text{Relative speed}\\ \text{of approach}\end{pmatrix} = \begin{pmatrix}\text{Relative speed}\\ \text{of separation}\end{pmatrix}$$

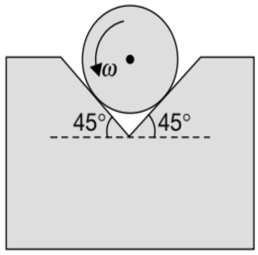
$$\Rightarrow \quad v_0 = v + \frac{l}{2}\omega\cos\theta \qquad \ldots(3)$$

Solving equations (1), (2) and (3), we get

$$\omega = \frac{6v_0\cos\theta}{l\left(1 + 3\cos^2\theta\right)}$$

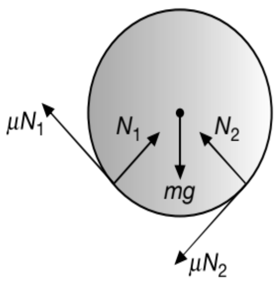
PROBLEM 22

A cylinder of mass M and radius R is rotated in a uniform V shaped groove with constant angular velocity ω. The coefficient of friction between the cylinder and each surface is μ. What torque must be applied to the cylinder to keep it rotating?



SOLUTION

The forces acting on the cylinder are shown in Figure.



For vertical equilibrium of cylinder, we have

$$N_1\cos 45° + N_2\cos 45° + \mu N_1\cos 45° = \mu N_2\cos 45° + Mg$$

$$\Rightarrow \quad N_1(1+\mu) + N_2(1-\mu) = \sqrt{2}Mg \qquad \ldots(1)$$

For horizontal equilibrium of cylinder,

$$N_1\sin 45° = N_2\sin 45° + \mu N_1\sin 45° + \mu N_2\sin 45°$$

$$\Rightarrow \quad N_1(1-\mu) - N_2(1+\mu) = 0 \qquad \ldots(2)$$

Solving these two equations, we get

$$2N_1\left(1+\mu^2\right) = \sqrt{2}Mg(1+\mu)$$

$$\Rightarrow \quad N_1 = \frac{\sqrt{2}Mg(1+\mu)}{2(1+\mu^2)}$$

From (2), we get

$$N_2 = \frac{\sqrt{2}Mg(1-\mu)}{2(1+\mu^2)}$$

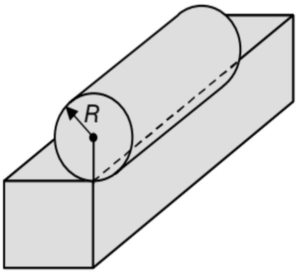
The required torque is,

$$\tau = \mu N_1 R + \mu N_2 R = \mu R(N_1 + N_2)$$

$$\Rightarrow \quad \tau = \frac{\sqrt{2}\mu MgR}{(1+\mu^2)}$$

PROBLEM 23

A rectangular rigid fixed block has a long horizontal edge. A solid homogeneous cylinder of radius R is placed horizontally at rest with its length parallel to the edge such that the axis of the cylinder and the edge of the block are in the same vertical plane as shown in Figure.

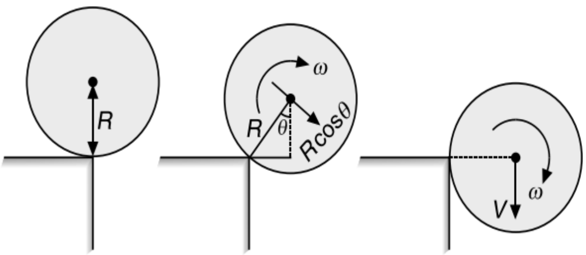


There is sufficient friction present at the edge so that a very small displacement causes the cylinder to roll off the edge without slipping. Determine:

(a) the angle θ_c through which the cylinder rotates before it leaves contact with the edge.
(b) the speed of the centre of mass of the cylinder before leaving contact with the edge, and
(c) the ratio of the translational to rotational kinetic energies of the cylinder when its centre of mass is in horizontal line with the edge.

SOLUTION

The cylinder rotates about the point of contact as shown in Figure.



Hence, the mechanical energy of the cylinder will be conserved i.e.,

$$(PE + KE) = (PE + KE)_2$$

$$\Rightarrow \quad mgR + 0 = mgR\cos\theta + \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

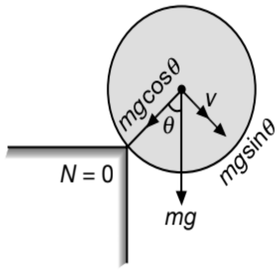
where, $I = \frac{1}{2}mR^2$ and for no slipping at point of contact, we have $\omega = \frac{v}{R}$

$$\Rightarrow \quad mgR = mgR\cos\theta + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v^2}{R^2}\right) + \frac{1}{2}mv^2$$

$$\Rightarrow \quad \frac{3}{4}v^2 = gR(1-\cos\theta)$$

$$\Rightarrow \quad v^2 = \frac{4}{3}gR(1-\cos\theta)$$

$$\Rightarrow \quad \frac{v^2}{R} = \frac{4}{3}g(1-\cos\theta) \qquad \ldots(1)$$



At the time of leaving contact, normal reaction $N = 0$ and $\theta = \theta_c$ hence,

$$mg\cos\theta = \frac{mv^2}{R}$$

$$\Rightarrow \quad \frac{v^2}{R} = g\cos\theta \qquad \ldots(2)$$

From equations (1) and (2),

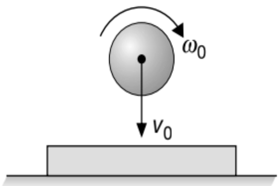
$$\frac{4}{3}g(1-\cos\theta_c) = g\cos\theta_c$$

$$\Rightarrow \quad \frac{7}{4}\cos\theta_c = 1$$

$$\Rightarrow \quad \cos\theta_c = \frac{4}{7}$$

PROBLEM 24

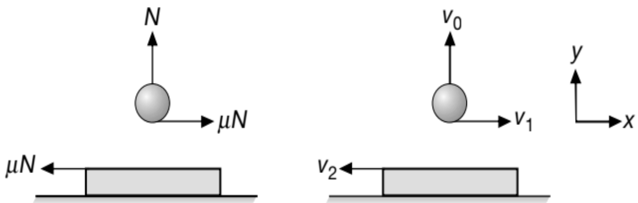
A spherical ball of radius r and mass m collides with a plank of mass M kept on a smooth horizontal surface. Before impact, the centre of the ball has a velocity v_0 and angular velocity ω_0 as shown in Figure.



The normal velocity is reversed with same magnitude and the ball stops rotating after the impact. Find the distance on the plank between first two impacts of the ball. The coefficient of friction between the ball and the plank is μ. Assume that plank is large enough.

SOLUTION

The forces during impact are shown in Figure.



Let the horizontal velocities of the ball and the plank be v_1 and v_2 in opposite directions as shown in Figure, then by Law of Conservation of Linear Momentum in horizontal direction, we get

$$mv_1 = Mv_2 \quad \text{...(1)}$$

According to Impulse Momentum Theorem,

$$\begin{pmatrix}\text{Linear Impulse}\\ \text{of the ball in}\\ \text{vertical direction}\end{pmatrix} = \begin{pmatrix}\text{Change in}\\ \text{Linear Momentum}\\ \text{in vertical direction}\end{pmatrix}$$

$$\Rightarrow \quad J = Ndt$$

$$\Rightarrow \quad J = 2mv_0 \quad \text{...(2)}$$

$$\begin{pmatrix}\text{Linear Impulse}\\ \text{on the ball in}\\ \text{horizontal}\\ \text{direction}\end{pmatrix} = \begin{pmatrix}\text{Change in}\\ \text{Linear Momentum}\\ \text{in horizontal}\\ \text{direction}\end{pmatrix}$$

$$\Rightarrow \quad \mu J = mv_1 \quad \text{...(3)}$$

According to Angular Impulse Angular Momentum Theorem, we have

$$\begin{pmatrix}\text{Angular Impulse}\\ \text{on the ball about}\\ \text{Centre of Mass}\end{pmatrix} = \begin{pmatrix}\text{Change in}\\ \text{Angular Momentum}\\ \text{about Centre of Mass}\end{pmatrix}$$

$$\Rightarrow \quad (\mu J)r = I\omega_0 = \left(\frac{2}{5}mr^2\right)\omega_0 \quad \text{...(4)}$$

Solving equations (1), (2), (3) and (4), we get

$$v_1 = \frac{2}{5}r\omega_0 \text{ and } v_2 = \frac{m}{M}\left(\frac{2}{5}r\omega_0\right)$$

Now, actual path of the ball is a projectile whose time of flight will be

$$T = \frac{2v_y}{g} = \frac{2v_0}{g}$$

Relative velocity of ball with respect to plank in horizontal direction is

$$v_r = v_1 + v_2 = \left(\frac{M+m}{M}\right)v_1 = \frac{2}{5}\left(\frac{M+m}{M}\right)r\omega_0$$

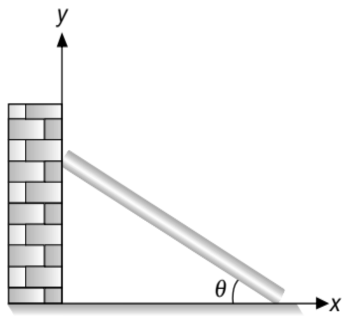
Therefore, the required distance is

$$s = v_r T$$

$$\Rightarrow \quad s = \frac{4}{5}\left(\frac{M+m}{M}\right)\frac{v_0 r\omega_0}{g}$$

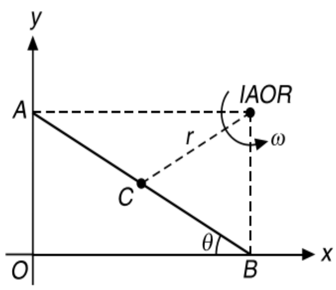
PROBLEM 25

A uniform rod of length L and weight W stands vertically touching a vertical wall (y-axis). When slightly displaced, its lower end begins to slide along the floor (x-axis). Obtain an expression for the angular velocity (ω) of the rod as a function of θ. Determine the distance moved by the lower end at which the rod no longer touches the vertical wall. Neglect friction everywhere.



SOLUTION

The position of instantaneous axis of rotation (IAOR) is shown in Figure.



$$C \equiv \left(\frac{l}{2}\cos\theta, \frac{l}{2}\sin\theta\right)$$

$$r = \frac{l}{2} = \text{half of the diagonal}$$

All surfaces are smooth. Therefore, mechanical energy will remain conserved. So,

$$\begin{pmatrix}\text{Decrease in GPE}\\ \text{of CM of Rod}\end{pmatrix} = \begin{pmatrix}\text{Increase in RKE}\\ \text{of Rod about } IAOR\end{pmatrix}.$$

$$\Rightarrow \quad mg\frac{l}{2}(1-\sin\theta) = \frac{1}{2}I\omega^2 \qquad \ldots(1)$$

where, $I = \dfrac{ml^2}{12} + mr^2$ {about IAOR}

$$\Rightarrow \quad I = \frac{ml^2}{12} + \frac{ml^2}{4} = \frac{ml^2}{3}$$

Substituting in equation (1), we get

$$mg\frac{l}{2}(1-\sin\theta) = \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{3g(1-\sin\theta)}{l}}$$

$$\Rightarrow \quad \omega^2 = \frac{3g(1-\sin\theta)}{l}$$

Differentiating w.r.t. time, we get

$$2\omega\left(\frac{d\omega}{dt}\right) = -\frac{3g}{l}\cos\theta\left(\frac{d\theta}{dt}\right) \qquad \ldots(2)$$

Since, $\dfrac{d\omega}{dt} = \alpha$ {the angular acceleration of rod}

and $\dfrac{d\theta}{dt} = \omega$ {the angular velocity of rod}

Substituting in equation (2), we have

$$\alpha = -\frac{3g}{2l}\cos\theta \qquad \ldots(3)$$

The x-coordinate of centre of mass at angle θ is

$$x = \frac{l}{2}\cos\theta$$

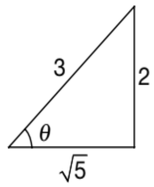
Differentiating twice w.r.t. time, we get

$$a_x = -\frac{\omega^2 l}{2}\cos\theta - \frac{\alpha l}{2}\sin\theta \qquad \ldots(4)$$

The rod will lose contact with the vertical wall at the instant when normal reaction by the wall (along x-axis) becomes zero. So,

$$F_x = 0$$

$$\Rightarrow \quad a_x = 0$$



Substitute, $a_x = 0$ in equation (4), we get

$$-\omega^2\cos\theta - \alpha\sin\theta = 0$$

$$\Rightarrow \quad -\frac{3g}{l}\cos\theta(1-\sin\theta) + \frac{3g}{2l}\sin\theta\cos\theta = 0$$

Solving this, we get

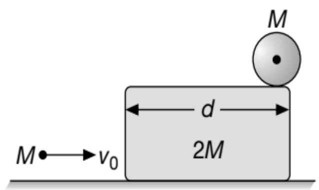
$$\sin\theta = \frac{2}{3}$$

At this moment the lower end has moved a distance d given by,

$$d = l\cos\theta = \frac{\sqrt{5}}{3}l$$

PROBLEM 26

A cylinder of mass M is placed on top of a block of mass $2M$. The friction coefficient between the block and the horizontal plane is zero and the friction coefficient between the cylinder and the top surface of block is μ. A small body of mass M travelling with a velocity v_0 hits the block horizontally such that the collision is completely elastic. Find the maximum value of v_0 for which the cylinder does start pure rolling before it topples from the top of the block.

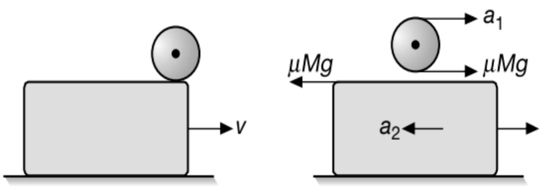


SOLUTION

Let v be the velocity of block after collision. Applying the equation of head on elastic collision between two particles, we get

$$v = \frac{2M}{M+2M}v_0 = \frac{2}{3}v_0$$

The forces acting on the cylinder and the block till pure rolling starts are shown in Figure.



$$a_1 = \frac{\mu Mg}{M} = \mu g$$

$$a_2 = \frac{\mu Mg}{2M} = \frac{\mu g}{2}$$

$$\alpha = \frac{\mu MgR}{MR^2/2} = \frac{2\mu g}{R}$$

Let the pure rolling starts after time t, then

$$v - a_2 t = a_1 t + (\alpha t)R$$

$$\Rightarrow \quad t = \frac{v}{a_1 + a_2 + \alpha R} = \frac{v}{\mu g + \frac{\mu g}{2} + 2\mu g}$$

$$\Rightarrow \quad t = \frac{2v}{7\mu g} = \frac{4v_0}{21\mu g}$$

Now, $d = \left(vt - \frac{1}{2}a_2 t^2\right) - \frac{1}{2}a_1 t^2$

$$\Rightarrow \quad d = vt - \frac{1}{2}(a_1 + a_2)t^2$$

$$\Rightarrow \quad d = \frac{8v_0^2}{63\mu g} - \frac{12v_0^2}{441\mu g}$$

$$\Rightarrow \quad 441\mu gd = 56v_0^2 - 12v_0^2$$

$$\Rightarrow \quad 44v_0^2 = 441\mu gd$$

$$\Rightarrow \quad v_0 = \frac{21}{2}\sqrt{\frac{\mu gd}{11}}$$

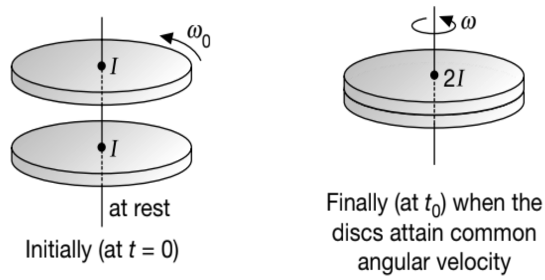
$$\Rightarrow \quad (v_0)_{\min} = \frac{21}{2}\sqrt{\frac{\mu gd}{11}}$$

PROBLEM 27

A uniform disc of radius r_0 lies on a smooth horizontal plane. A similar disc spinning with the angular velocity ω_0 is carefully lowered onto the first disc. How soon do both discs spin with the same angular-velocity if the friction coefficient between them is equal to μ.

SOLUTION

If I be the moment of inertia of each disc relative to common rotation axis as shown in Figure.

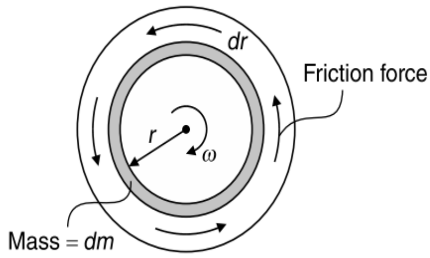


Then by law of conservation of angular momentum applied to the initial and the final situations, we get

$$I\omega_0 = 2I\omega$$

$$\Rightarrow \quad \omega = \frac{\omega_0}{2}$$

The angular velocity of each disc varies due to the torque τ of the friction forces. To calculate τ, let us take an elementary ring with radii r and $r + dr$ as shown in Figure.



The torque of the friction forces acting on the given ring is equal to

$$d\tau = fr_{\perp} = \mu(dN)r = [\mu(dm)g]r$$

$$\Rightarrow \quad d\tau = \mu\left(\frac{mg}{\pi r_0^2}\right)(2\pi r dr)r = \left(\frac{2\mu mg}{r_0^2}\right)r^2 dr$$

where, m is the mass of each disc. Integrating this with respect to r between 0 and r_0, we get

$$\tau = \frac{2}{3}\mu mg r_0 = \text{constant}$$

$$\Rightarrow \quad \alpha = \frac{\tau}{I} = \frac{(2\mu mg r_0/3)}{(mr_0^2/3)} = \frac{4\mu g}{3r_0} = \text{constant}$$

The angular speed of lower disc increases from zero to $\frac{\omega_0}{2}$ with constant angular acceleration α, so

$$\frac{\omega_0}{2} = 0 + \alpha t$$

$$\Rightarrow \quad t = \frac{\omega_0}{2\alpha} = \frac{3r_0\omega_0}{8\mu g}$$