

Centre of Mass, Conservation of Linear Momentum and Collisions

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

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| (a) Newton's Second Law and Concept of Centre of Mass | (d) Work Energy Theorem for System of Particles and its Applications from Centre Of Mass Reference Frame (C-frame) |
| (b) Centre of Mass for Various Bodies and Configurations | (e) Conservation of Linear Momentum, Variable Mass Systems |
| (c) Concept of Motion of Centre of Mass | (f) Collisions: Head on and Oblique |

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

CENTRE OF MASS

INTRODUCTION

In the previous chapters, we had been discussing motion for the point objects. However, we come across mechanical systems which are actually not point objects. A mechanical system is either a group of particles (such as atoms of a gas enclosed in a container) or an extended object (such as an acrobat following a projectile motion after being fired from the human canon). The motion of a mechanical system can be easily described in terms of a special point called the **Centre of Mass** of the system. We shall also see that the centre of mass of the system moves as if the entire mass of the system were concentrated at that point. Furthermore, if a resultant force \vec{F}_{ext} acts on the system and the total mass of the system is M , then acceleration of the centre of mass of system is $\vec{a}_{\text{CM}} = \frac{\vec{F}_{\text{ext}}}{M}$. So, we observe that the system moves as

if the resultant force was being applied to a single particle of mass M , placed at the centre of mass. In simple words the centre of mass is the point from a chosen origin where the entire mass of an extended system can be assumed to be concentrated and instead of discussing about the motion of the particles of the system individually (to study the motion of the system) we can directly discuss about the motion of the centre of mass to get the same results.

CENTRE OF MASS AND NEWTON'S SECOND LAW

Consider a mechanical system formed by a collection of particles $m_1, m_2, m_3, \dots, m_n$. Let a force \vec{f}_1 be applied on particle 1, \vec{f}_2 on particle 2 and so on such that their respective accelerations are $\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots$,

\vec{a}_n . Now, according to Newton's Second Law, the net force \vec{F} acting on the mechanical system is given by

$$\vec{F} = \vec{f}_1 + \vec{f}_2 + \dots + \vec{f}_n$$

$$\Rightarrow \vec{F} = m_1\vec{a}_1 + m_2\vec{a}_2 + \dots + m_n\vec{a}_n \quad \dots(1)$$

If M be the total mass of the system and \vec{a} be the acceleration of the system, then

$$\vec{F} = M\vec{a} = m_1\vec{a}_1 + m_2\vec{a}_2 + \dots + m_n\vec{a}_n \quad \dots(2)$$

where, $M = m_1 + m_2 + \dots + m_n$

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be the respective velocities of the particles at an instant and \vec{v} be the velocity of the system at that instant. Then

$$\vec{a}_1 = \frac{d\vec{v}_1}{dt}, \vec{a}_2 = \frac{d\vec{v}_2}{dt}, \dots, \vec{a}_n = \frac{d\vec{v}_n}{dt} \text{ and } \vec{a} = \frac{d\vec{v}}{dt}$$

So, from equation (2), we get

$$M\left(\frac{d\vec{v}}{dt}\right) = m_1\left(\frac{d\vec{v}_1}{dt}\right) + m_2\left(\frac{d\vec{v}_2}{dt}\right) + \dots + m_n\left(\frac{d\vec{v}_n}{dt}\right)$$

$$\Rightarrow \frac{d}{dt}(M\vec{v}) = \frac{d}{dt}(m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_n\vec{v}_n)$$

Integrating both sides, we get

$$M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_n\vec{v}_n \quad \dots(3)$$

$$\Rightarrow \vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_n\vec{v}_n}{m_1 + m_2 + \dots + m_n} \quad \dots(4)$$

Further, if $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ be the respective position vectors of the particles (from the selected convenient origin) at any instant t and \vec{r} be the position vector of M , then

$$\vec{v}_1 = \frac{d\vec{r}_1}{dt}, \vec{v}_2 = \frac{d\vec{r}_2}{dt}, \dots, \vec{v}_n = \frac{d\vec{r}_n}{dt}, \vec{v} = \frac{d\vec{r}}{dt}$$

So, from (3), we get

$$M\left(\frac{d\vec{r}}{dt}\right) = m_1\left(\frac{d\vec{r}_1}{dt}\right) + m_2\left(\frac{d\vec{r}_2}{dt}\right) + \dots + m_n\frac{d\vec{r}_n}{dt}$$

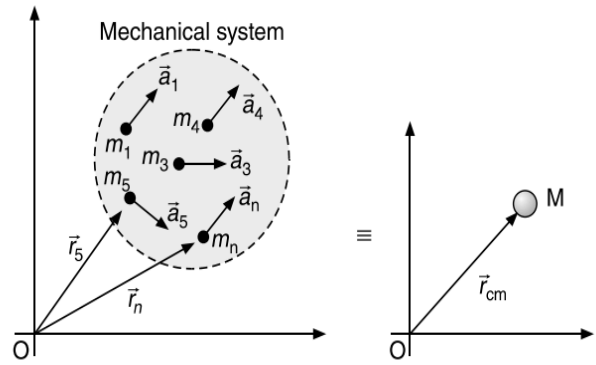
$$\Rightarrow \frac{d}{dt}(M\vec{r}) = \frac{d}{dt}(m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n)$$

Integrating, we get

$$M\vec{r} = m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n$$

$$\Rightarrow \vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n}{m_1 + m_2 + \dots + m_n}$$

This position vector of the combined mass M , is called as the centre of mass of the system of particles, denoted by \vec{r}_{cm} .



Furthermore, \vec{v}_{cm} is actually the velocity of centre of mass and \vec{a}_{cm} is the acceleration of the centre of mass. Hence, we have

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \dots + m_n\vec{a}_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_i m_i\vec{a}_i}{\sum_i m_i}$$

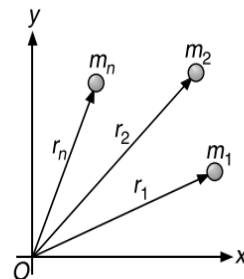
$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_n\vec{v}_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_i m_i\vec{v}_i}{\sum_i m_i}$$

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$$

These equations help us to reduce a many particle mechanical system to a single particle system (of mass M) placed at a position vector \vec{r}_{cm} .

DEFINITION OF CENTRE OF MASS

Consider a system of n discrete particles having position vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ as shown in Figure.



The position vector for the centre of mass \vec{r}_{cm} is

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n}{m_1 + m_2 + \dots + m_n}$$

$$\Rightarrow \vec{r}_{cm} = \frac{\sum m_i\vec{r}_i}{M} \quad \dots(1)$$

where, $M = \sum m_i$ is the total mass of the system.

The components of equation (1) are defined as

$$x_{\text{cm}} = \frac{\sum m_i x_i}{M}, \quad y_{\text{cm}} = \frac{\sum m_i y_i}{M} \quad \text{and} \quad z_{\text{cm}} = \frac{\sum m_i z_i}{M}$$

- (a) The location of the centre of mass is independent of the reference frame used to locate it.
- (b) The centre of mass of the system of particles depends only on the masses of the particles and the positions of the particles relative to one another.
- (c) The centre of mass has a tendency to shift towards the heavier side of the system.
- (d) The above formulae are equally applicable for collinear and non collinear systems.
- (e) However, it must be noted that to locate the position of the centre of mass an Origin of convenience has to be selected first.

ANY SYSTEM CAN BE THOUGHT OF AS A TWO PARTICLE SYSTEM

Dear Students, please remember that any particle system can be reduced to a two-particle system. As an example, let us consider a four particle system, then its CM is given by

$$\vec{r}_{\text{cm}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + m_4 \vec{r}_4}{m_1 + m_2 + m_3 + m_4} \quad \dots(1)$$

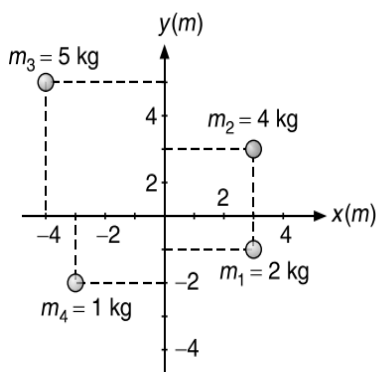
Now, if we want that particles 1, 3 should be a single particle and 2, 4 should be another single particle, then equation (1) can be re-written as

$$\vec{r}_{\text{cm}} = \frac{\left(\frac{m_1 \vec{r}_1 + m_3 \vec{r}_3}{m_1 + m_3} \right) (m_1 + m_3) + \left(\frac{m_2 \vec{r}_2 + m_4 \vec{r}_4}{m_2 + m_4} \right) (m_2 + m_4)}{(m_1 + m_3) + (m_2 + m_4)}$$

$$\Rightarrow \vec{r}_{\text{cm}} = \frac{(\vec{r}_{\text{cm of 1 \& 3}})(m_1 + m_3) + (\vec{r}_{\text{cm of 2 \& 4}})(m_2 + m_4)}{m_1 + m_2 + m_3 + m_4}$$

ILLUSTRATION 1

Find the center of mass of the four point masses as shown in figure.



SOLUTION

The total mass $M = 12$ kg, from equation, we have

$$x_{\text{cm}} = \frac{(2)(3) + (4)(3) + (5)(-4) + (1)(-3)}{2 + 4 + 5 + 1} = -\frac{5}{12} \text{ m}$$

$$y_{\text{cm}} = \frac{(2)(-1) + (4)(3) + (5)(4) + (1)(-2)}{2 + 4 + 5 + 1} = \frac{28}{12} \text{ m}$$

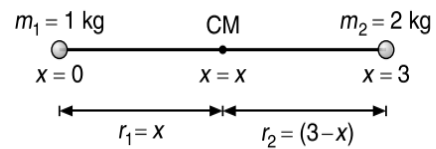
The position of the cm is $\vec{r}_{\text{cm}} = (-0.42\hat{i} + 2.3\hat{j}) \text{ m}$

ILLUSTRATION 2

Two particles of mass 1 kg and 2 kg are located at $x = 0$ and $x = 3$ m. Find the position of their centre of mass.

SOLUTION

Since, both the particles lie on x -axis, the cm will also lie on x -axis. Let the cm is located at $x = x$, then



$r_1 =$ distance of cm from the particle of mass 1 kg

$$\Rightarrow r_1 = x$$

and $r_2 =$ distance of cm from the particle of mass 2 kg

$$\Rightarrow r_2 = (3 - x)$$

Using $\frac{r_1}{r_2} = \frac{m_2}{m_1}$

$$\Rightarrow \frac{x}{3 - x} = \frac{2}{1}$$

$$\Rightarrow x = 2 \text{ m}$$

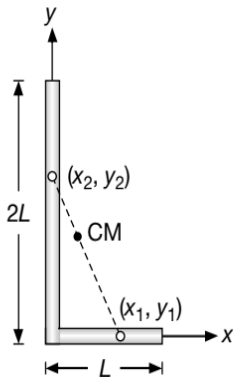
Thus, the cm of the two particles is located at $x = 2$ m

ILLUSTRATION 3

A thin rod of length $3L$ is bent at right angles at a distance L from one end. Locate the centre of mass of the arrangement with respect to the corner. Take $L = 1.2$ m.

SOLUTION

The centre of mass of each arm lies at its midpoint. The centre of mass of the two arms can be found by treating each arm as a point particle placed at its centre of mass. From the diagram we see that $x_1 = \frac{L}{2}$, $y_1 = 0$ and $x_2 = 0$, $y_2 = L$.



Since the distribution of the mass of the rod is uniform so the masses of the parts that are bent are m and $2m$ which have the respective lengths as L and $2L$. By definition, we get

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{M} = \frac{L}{6}$$

$$y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2}{M} = \frac{2L}{3}$$

The position of the centre of mass is given by

$$\vec{r}_{\text{cm}} = (-0.2\hat{i} + 0.8\hat{j}) \text{ m}$$

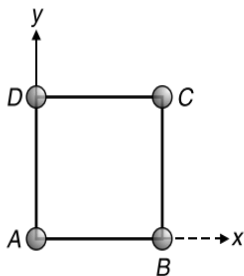
Note that the centre of mass may not necessarily have a physical mass present at it and may lie within or outside the body.

ILLUSTRATION 4

Four particles of mass 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A , B , C and D of a square of side 1 m. Find the position of centre of mass of the particles.

SOLUTION

Assuming A as the origin, AB as x -axis and AD as y -axis, we have



$$m_1 = 1 \text{ kg at } (x_1, y_1) = (0, 0)$$

$$m_2 = 2 \text{ kg at } (x_2, y_2) = (1, 0)$$

$$m_3 = 3 \text{ kg at } (x_3, y_3) = (1, 1)$$

$$\text{and } m_4 = 4 \text{ kg at } (x_4, y_4) = (0, 1)$$

Co-ordinates of their cm are

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$\Rightarrow x_{\text{cm}} = \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4}$$

$$\Rightarrow x_{\text{cm}} = \frac{5}{10} = \frac{1}{2} \text{ m} = 0.5 \text{ m}$$

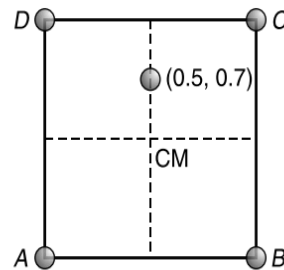
$$\text{Similarly, } y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

$$y_{\text{cm}} = \frac{(1)(0) + 2(0) + 3(1) + 4(1)}{1 + 2 + 3 + 4}$$

$$y_{\text{cm}} = \frac{7}{10} \text{ m} = 0.7 \text{ m}$$

$$\Rightarrow (x_{\text{cm}}, y_{\text{cm}}) = (0.5, 0.7) \text{ m}$$

Thus, position of cm of the four particles is as shown in Figure.



So, we see that the centre of mass of a system has a tendency to shift towards the heavier side of the system.

CENTRE OF MASS FOR THIN SHEETS AND LAMINAR BODIES

For thin sheets or lamina type (2-dimensional) bodies, mass is directly proportional to area A , so the formulae for finding the position of centre of mass is modified and is given by

$$\vec{r}_{\text{cm}} = \frac{A_1 \vec{r}_1 + A_2 \vec{r}_2 + \dots + A_n \vec{r}_n}{A_1 + A_2 + \dots + A_n}$$

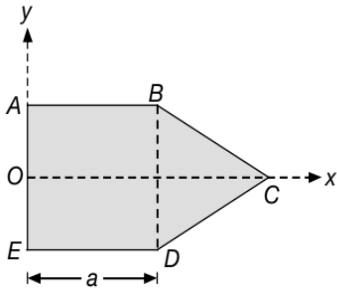
$$\Rightarrow x_{\text{cm}} = \frac{A_1 x_1 + A_2 x_2 + \dots + A_n x_n}{A_1 + A_2 + \dots + A_n}$$

$$\Rightarrow y_{\text{cm}} = \frac{A_1 y_1 + A_2 y_2 + \dots + A_n y_n}{A_1 + A_2 + \dots + A_n} \text{ and}$$

$$\Rightarrow z_{\text{cm}} = \frac{A_1 z_1 + A_2 z_2 + \dots + A_n z_n}{A_1 + A_2 + \dots + A_n}$$

ILLUSTRATION 5

A uniform lamina $ABCDE$ is made from a square $ABDE$ and an equilateral triangle BCD . Find the centre of mass of the lamina.



SOLUTION

Here, $A_1 =$ area of square $ABDE = (a)(a) = a^2$

$$A_2 = \text{area of triangle} = \frac{1}{2}(a)\left(\frac{\sqrt{3}a}{2}\right) = \frac{\sqrt{3}}{4}a^2$$

Coordinates of centre of mass of square is $(x_1, y_1) = \left(\frac{a}{2}, 0\right)$ and coordinates of centre of mass of the triangle is $(x_2, y_2) = \left(a + \frac{a}{2\sqrt{3}}, 0\right)$

$$\text{Now, } x_{\text{cm}} = \frac{A_1x_1 + A_2x_2}{A_1 + A_2}$$

$$\Rightarrow x_{\text{cm}} = \frac{(a^2)\left(\frac{a}{2}\right) + \left(\frac{\sqrt{3}a^2}{4}\right)\left(a + \frac{a}{2\sqrt{3}}\right)}{a^2 + \frac{\sqrt{3}}{4}a^2} = 0.74a$$

$$y_{\text{cm}} = 0 \text{ as } y_1 \text{ and } y_2 \text{ both are zero.}$$

Therefore, coordinates of cm of the given lamina are $(0.74a, 0)$.

CENTRE OF MASS FOR A SYSTEM HAVING A CAVITY

Consider a body of mass m_1 having its centre of mass at \vec{r}_1 from a fixed origin O (say). If a portion of mass m_2 having its centre of mass at \vec{r}_2 is removed from this body, then the centre of mass of the remaining body is given by

$$\vec{r}_{\text{cm}} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2} \quad \text{OR} \quad \vec{r}_{\text{cm}} = \frac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$$

$$\Rightarrow x_{\text{cm}} = \frac{m_1x_1 - m_2x_2}{m_1 - m_2} \quad \text{OR} \quad x_{\text{cm}} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

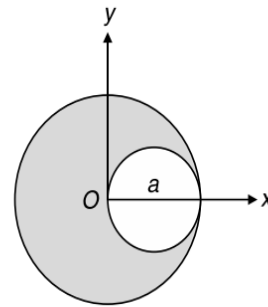
$$\Rightarrow y_{\text{cm}} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2} \quad \text{OR} \quad y_{\text{cm}} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

$$\Rightarrow z_{\text{cm}} = \frac{m_1z_1 - m_2z_2}{m_1 - m_2} \quad \text{OR} \quad z_{\text{cm}} = \frac{A_1z_1 - A_2z_2}{A_1 - A_2}$$

Please note that here A is the area of the corresponding portion if the body is in the shape of a lamina.

ILLUSTRATION 6

A disc of diameter a is removed from a uniform disc of radius a . Locate the position of centre of mass of the left out lamina shown in figure.



SOLUTION

Coordinates of centre of mass of large circle are $(x_1, y_1) = (0, 0)$ and coordinates of centre of mass of small circle are $(x_2, y_2) = \left(\frac{a}{2}, 0\right)$. Since

$$x_{\text{cm}} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

where, A_1 is area of complete circle, i.e. $A_1 = \pi a^2$ and

$$A_2 \text{ is the area of cavity, i.e. } A_2 = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$

$$\Rightarrow x_{\text{cm}} = \frac{-(\pi a^2/4)(a/2)}{\pi a^2 - (\pi a^2/4)} = -\frac{a/8}{3/4} = -\frac{a}{6}$$

and $y_{\text{cm}} = 0$ as y_1 and y_2 both are zero.

Therefore, coordinates of cm of the lamina shown in figure are $\left(-\frac{a}{6}, 0\right)$. Please note that Centre of Mass has a tendency to shift towards heavier side of system.

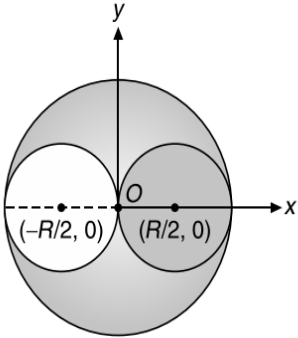
ILLUSTRATION 7

From a uniform disc of radius R , a disc of diameter R has been cut out from the left part and this removed part is then placed opposite to the cavity such that centres of the full disc, cavity and the removed disc

all lie in the same line and the removed disc just touches it cavity. Locate the centre of mass of the resulting disc assuming the origin to be at the centre of the full disc.

SOLUTION

Since the arrangement is symmetric about x -axis, so centre of mass of the system lies on the x -axis. Also, due to uniform thickness, mass is proportional to area.



The x -coordinate of centre of mass of the system is

$$x_{\text{cm}} = \frac{A_1 x_1 - A_2 x_2 + A_3 x_3}{A_1 - A_2 + A_3}$$

where, $A_1 = \pi R^2$ is the area of the full disc,

$$A_2 = A_3 = \pi \left(\frac{R}{2}\right)^2 = \frac{\pi R^2}{4} \text{ is the area of cavity}$$

x_1, x_2 and x_3 are the x coordinates of the centre of mass of full disc at origin, the removed disc at $\left(-\frac{R}{2}, 0\right)$ and the removed disc when placed opposite to the cavity tangentially at $\left(\frac{R}{2}, 0\right)$.

$$\Rightarrow x_{\text{cm}} = \frac{(\pi R^2)(0) - \left(\frac{\pi R^2}{4}\right)\left(-\frac{R}{2}\right) + \left(\frac{\pi R^2}{4}\right)\left(\frac{R}{2}\right)}{\pi R^2 - \frac{\pi R^2}{4} + \frac{\pi R^2}{4}}$$

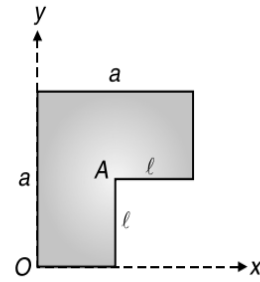
$$\Rightarrow x_{\text{cm}} = \frac{R}{4}$$

Hence the coordinate of centre of mass of system will be $\left(\frac{R}{4}, 0\right)$.

ILLUSTRATION 8

A square metal plate of side a from which a square plate of side $l (< a)$ has been cut as shown in the figure. Calculate the ratio $\left(\frac{a}{l}\right)$ so that the centre of

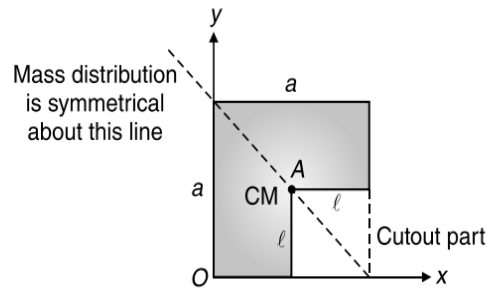
mass of the remaining L-shaped plate coincides with the point A .



SOLUTION

For centre of mass to lie at point A , we have

$$r_{\text{cm}} = (x_{\text{cm}}, y_{\text{cm}}) = (l, l)$$



$$\Rightarrow y_{\text{cm}} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{a^2 \left(\frac{a}{2}\right) - l^2 \left(\frac{l}{2}\right)}{a^2 - l^2} = l$$

$$\Rightarrow a^3 - l^3 = 2l(a^2 - l^2)$$

$$\Rightarrow (a-l)(a^2 + l^2 + al) = 2l(a+l)(a-l)$$

$$\Rightarrow a^2 - al - l^2 = 0 \quad \dots(1)$$

Solving equation (1), we get

$$\frac{a}{l} = \frac{\sqrt{5}-1}{2}$$

CENTRE OF MASS OF EXTENDED BODIES

A rigid body, such as a meter stick, can be thought of as a system of closely packed particles and hence it also has a centre of mass. *Since the number of particles in the body is very large and their spacing is extremely small, so we can treat the body as a continuous distribution of mass.* For continuous distribution of mass, the centre of mass is defined as

$$\vec{r}_{\text{cm}} = \frac{1}{M} \int \vec{r} dm$$

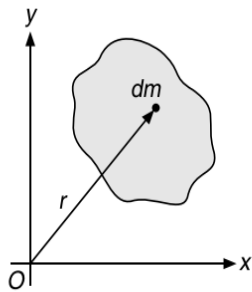
where \vec{r} is the position vector of the centre of mass of an infinitesimal mass element dm . The components of this equation are

$$x_{\text{cm}} = \frac{1}{M} \int x dm, \quad y_{\text{cm}} = \frac{1}{M} \int y dm \quad \text{and}$$

$$z_{\text{cm}} = \frac{1}{M} \int z dm$$

Often, we deal with homogeneous objects having a point, a line, or a plane of symmetry. Then the centre of mass will lie at the point, on the line, or in the plane of symmetry.

For example, the centre of mass of a homogeneous rod will lie at the centre of the rod, the centre of mass of a homogeneous sphere will be at the centre of the sphere, the centre of mass of a cone will be at the axis of the cone, etc.



In the above formulae we have taken an infinitesimal point element of mass dm placed at a distance dx or dy or dz from the selected convenient origin O at respective distance x or y or z from O .

Please note that, the same set of formulae cannot be extended to systems where we have to find the centre of mass of the extended system with the help of the elements that themselves are extended systems and are not point elements, like a disc where the infinitesimal elements are taken in the form of concentric rings which are not point like. In that case we modify the above formula as

$$\vec{r}_{\text{cm}} = \left[\begin{array}{l} \frac{1}{M} \int \vec{r} dm \quad \text{for infinitesimal elements} \\ \frac{1}{M} \int \left(\vec{r}_{\text{cm of the infinitesimal element}} \right) dm \quad \text{for extended elements} \end{array} \right]$$

ILLUSTRATION 9

If a homogeneous thin rod of mass m and l is given, from symmetry, we know that its centre of mass coincides with the geometrical centre (mid-point) of the rod. Prove this by calculation for the centre of mass of the rod.

SOLUTION

Let us choose the x -axis along the length of the rod and origin at the left end of the rod, as shown in figure. A differential mass element of the rod having mass dm and length dx is also shown in the figure at a distance x from the origin.

Mass of the element, dm , equals to the linear mass density (mass per unit length $\lambda = \frac{m}{l}$) times the length of the element, dx . Therefore,

$$dm = \lambda dx = \frac{m}{l} dx$$



If x_{cm} be the coordinate of the centre of mass of the rod, then

$$\begin{aligned} \Rightarrow x_{\text{cm}} &= \frac{\int x dm}{\int dm} = \frac{\int_0^l x \left(\frac{m}{l} \right) dx}{m} \quad \left\{ \because \int dm = m \right\} \\ \Rightarrow x_{\text{cm}} &= \frac{1}{l} \int_0^l x dx = \frac{1}{l} \left(\frac{l^2}{2} \right) \\ \Rightarrow x_{\text{cm}} &= \frac{l}{2} \end{aligned}$$

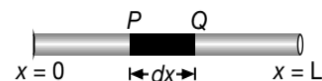
Hence, the centre of mass of the rod lies at the mid-point of the rod.

ILLUSTRATION 10

A rod of length L is placed along the x -axis between $x=0$ and $x=L$. The linear mass density, λ , of the rod varies with the distance x from the origin as $\lambda = \lambda_0 + ax$. Here, λ_0 and a are constants. Find the position of centre of mass of this rod.

SOLUTION

Mass of element PQ of length dx situated at $x = x$ as shown in figure is



$$dm = \lambda dx = (\lambda_0 + ax) dx$$

The cm of the element has coordinates $(x, 0, 0)$.
Therefore, x -coordinate of cm of the rod will be

$$x_{\text{cm}} = \frac{\int_0^L x dm}{\int_0^L dm} = \frac{\int_0^L x(\lambda dx)}{\int_0^L \lambda dx} = \frac{\int_0^L (x)(\lambda_0 + ax) dx}{\int_0^L (\lambda_0 + ax) dx}$$

$$\Rightarrow x_{\text{cm}} = \frac{\left(\frac{\lambda_0 x^2}{2} + \frac{ax^3}{3} \right) \Big|_0^L}{\left(\lambda_0 x + \frac{ax^2}{2} \right) \Big|_0^L} = \frac{3\lambda_0 L + 2aL^2}{6\lambda_0 + 3aL}$$

The result obtained can be rearranged to obtain

$$x_{\text{cm}} = \frac{L}{2} \left(\frac{3\lambda_0 + 2aL}{3\lambda_0 + \frac{3}{2}aL} \right)$$

In the above expression numerator of the term within square bracket is greater than the denominator.

Hence, $x_{\text{cm}} > \frac{L}{2}$

The y -coordinate of cm of the rod is

$$y_{\text{cm}} = \frac{\int y dm}{\int dm} = 0 \quad \{ \text{as } y = 0 \}$$

Similarly, $z_{\text{cm}} = 0$

Hence, the centre of mass of the rod lies at

$$\left(\frac{3aL + 2bL^2}{6a + 3bL}, 0, 0 \right)$$

ILLUSTRATION 11

Find the centre of mass of a thin semi-circular ring of radius R .

SOLUTION

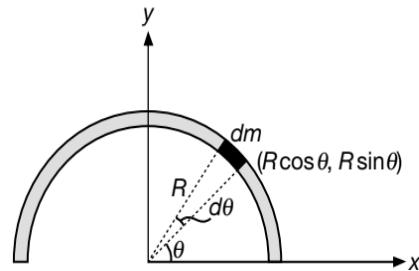
From symmetric distribution of the body we see that the centre of mass must lie along the y -axis, so $x_{\text{cm}} = 0$. In this case it is convenient to express the mass element in terms of the angle θ , measured in radian. The element, which subtends an angle $d\theta$ at the origin, has a length $Rd\theta$ and a mass $dm = Rd\theta$.

Coordinates of the infinitesimal element dm are

$$x = R \cos \theta \text{ and } y = R \sin \theta$$

So, after finding these coordinates for the infinitesimal element we use the fundamental formula to calculate the centre of mass according to which we have

$$x_{\text{cm}} = \frac{1}{M} \int x dm \text{ and } y_{\text{cm}} = \frac{1}{M} \int y dm$$



So, we get $x_{\text{cm}} = 0$ and

$$y_{\text{cm}} = \frac{1}{M} \int_0^\pi \lambda R^2 \sin \theta d\theta = \frac{\lambda R^2}{M} (-\cos \theta) \Big|_0^\pi = \frac{2\lambda R^2}{M}$$

The total mass of the ring is $M = \pi R \lambda$ and hence

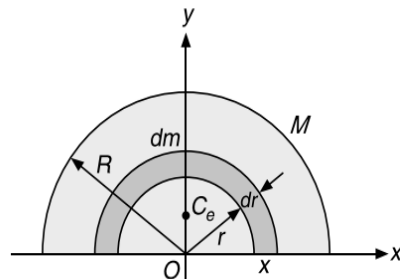
$$y_{\text{cm}} = \frac{2R}{\pi}$$

ILLUSTRATION 12

Locate the centre of mass of a homogeneous semi-circular disc of radius R and mass M .

SOLUTION

A semi-circular disc can be subdivided into large number of semi-circular ring elements, as shown in Figure.



According to result obtained in the last Illustration, all these elemental rings have centre of mass on the symmetrical axis of the disc (i.e., on the y -axis for chosen reference frame).

Now, if we replace all these elemental rings by point masses placed at their centre of mass position, then,

given disc reduces to a system of large number of particles distributed on the y -axis. So, to locate the centre of mass of the disc, centre of mass of these particles can be found and used equivalently.

Consider an infinitesimal semi-circular ring of radius r having thickness dr and mass dm , then

$$dm = \frac{M}{(\pi R^2/2)}(\pi r dr)$$

$$dm = \frac{2M}{R^2} r dr \quad \dots(1)$$

The centre of mass of the chosen extended element of radius r , thickness dr is at the point C_e , as shown in the same figure, then, coordinates of C_e can be given as

$$x = 0 \text{ and} \quad \dots(2)$$

$$y = \frac{2r}{\pi} \quad \dots(3)$$

Now, the chosen element can be treated as a point mass placed at the point C_e . If we consider all elemental rings shown in figure in a similar way, then, the coordinates of the centre of mass of the given semi-circular disc, X and Y , can be obtained by using equation. Therefore,

$$x_{cm} = \frac{\int x dm}{\int dm} = 0 \quad \{\text{using equation (2)}\}$$

$$\text{and } y_{cm} = \frac{\int y dm}{\int dm} = \frac{\int \left(\frac{2r}{\pi}\right) dm}{\int dm} \quad \{\text{using equation (3)}\}$$

$$\Rightarrow y_{cm} = \frac{\frac{2}{\pi} \int_0^R r \left(\frac{2M}{R^2}\right) (r dr)}{M} = \frac{4}{\pi R^2} \int_0^R r^2 dr$$

$$\Rightarrow y_{cm} = \left(\frac{4}{\pi R^2}\right) \left(\frac{R^3}{3}\right) = \frac{4R}{3\pi}$$

Hence, the centre of mass of the given semi-circular disc lies on the symmetrical axis of the disc at a distance of $\frac{4R}{3\pi}$ from its centre.

Centre of mass of a complete circular disc (uniform) lies at the centre of the disc.

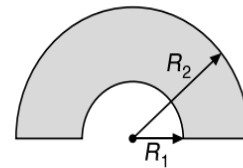
While choosing elements, you must take care the following:

- (a) the centre of mass of the element itself must be known. (If your chosen element is a point mass then its centre of mass coincides with its position.)
- (b) by using simple integration techniques your element must be able to cover the entire body.

$$\vec{r}_{cm} = \left[\begin{array}{l} \frac{1}{M} \int \vec{r} dm, \quad \text{for infinitesimal elements} \\ \frac{1}{M} \int \left(\vec{r}_{cm \text{ of the infinitesimal element}} \right) dm, \text{ for extended elements} \end{array} \right]$$

ILLUSTRATION 13

Locate the centre of mass of an annular semi-circular disc of inner radius R_1 and outer radius R_2 as shown in Figure.

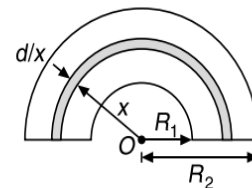


SOLUTION

Let σ be the mass per unit area of the disc. To locate the centre of mass of the annular disc, we consider a half ring of radius x width dx having mass dm , then we have

$$dm = \sigma dA = \sigma(\pi x dx)$$

Centre of mass of this infinitesimal half ring is at a distance $\frac{2x}{\pi}$ from O as shown in Figure.



$$\Rightarrow y_{cm} = \frac{1}{M} \int_{R_1}^{R_2} (\sigma \pi x dx) \frac{2x}{\pi} = \frac{2\sigma}{M} \int_{R_1}^{R_2} \sigma x^2 dx$$

$$\Rightarrow y_{\text{cm}} = \frac{2\sigma}{\sigma\pi(R_2^2 - R_1^2)/2} \left(\frac{x^3}{3} \right)_{R_1}^{R_2}$$

$$\Rightarrow y_{\text{cm}} = \frac{4(R_2^3 - R_1^3)}{3\pi(R_2^2 - R_1^2)}$$

We can also find the centre of mass of this object by considering it to be complete half disc of radius R_2 and a then removing a smaller half disc of radius R_1 from it. If y_{cm} be the centre of mass of this disc, then we have

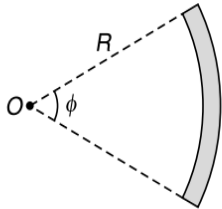
$$y_{\text{cm}} = \frac{m_{\text{full}}(y_{\text{cm}})_{\text{full}} - m_{\text{removed}}(y_{\text{cm}})_{\text{removed}}}{m_{\text{full}} - m_{\text{removed}}}$$

$$\Rightarrow y_{\text{cm}} = \frac{(\sigma\pi R_2^2)(4R_2/3\pi) - (\sigma\pi R_1^2)(4R_1/3\pi)}{\sigma(\pi R_2^2 - \pi R_1^2)/2}$$

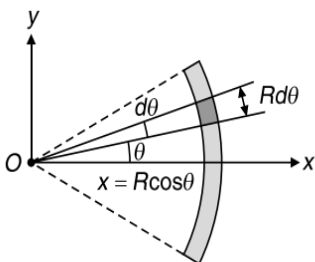
$$\Rightarrow y_{\text{cm}} = \frac{4}{3\pi} \left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2} \right)$$

ILLUSTRATION 14

Calculate the distance of centre of mass (from O) of the uniform circular arc subtending an angle ϕ at the centre O as shown in Figure.


SOLUTION

Let the mass of the arc be M . Consider an infinitesimal element of mass dm subtending an angle $d\theta$ at the centre of the arc as shown in Figure.



Its x coordinate is $R \cos \theta$ and its mass is

$$dm = \left(\frac{M}{\phi} \right) d\theta$$

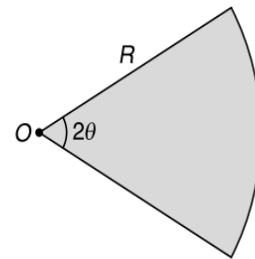
So, centre of mass of arc is given by

$$x_{\text{cm}} = \frac{1}{M} \int x dm = \frac{1}{M} \int_{-\phi/2}^{+\phi/2} (R \cos \theta) \left(\frac{M}{\phi} d\theta \right)$$

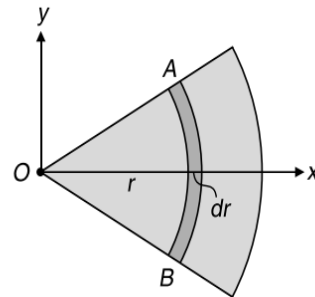
$$\Rightarrow x_{\text{cm}} = \frac{2R}{\phi} \sin \left(\frac{\phi}{2} \right)$$

ILLUSTRATION 15

Find the distance of centre of mass from point O of a thin uniform plate having the shape of a sector of a circular disc of radius R subtending an angle 2θ at the centre O as shown in Figure.


SOLUTION

Let M be the mass of the sector of the disc. Consider an elemental arc AB of radius r and width dr as shown in Figure.



If σ be the surface mass density of the disc, then the mass m of the arc AB is given by

$$dm = \sigma(2r\theta dr) = 2\sigma r\theta dr$$

As derived in the earlier ILLUSTRATION, due to symmetry the centre of mass of this arc must be on lie on the angle bisector i.e. on x -axis at distance

$$x = \frac{r \sin \theta}{\theta} \text{ from } O. \text{ The centre of mass of sector is}$$

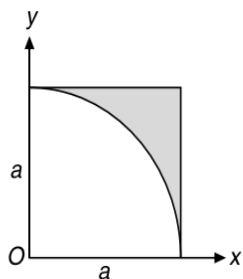
$$x_{\text{cm}} = \frac{1}{M} \int x dm, \text{ where } M = \sigma[R^2(2\theta)]$$

$$\Rightarrow x_{\text{cm}} = \frac{1}{M} \int_0^R \left(\frac{r \sin \theta}{\theta} \right) (2\sigma r\theta dr)$$

$$\begin{aligned} \Rightarrow x_{\text{cm}} &= \frac{1}{2\sigma R^2\theta} \left[\int_0^R \left(\frac{r \sin\theta}{\theta} \right) (2\sigma r \theta dr) \right] \\ \Rightarrow x_{\text{cm}} &= \frac{\sin\theta}{R^2\theta} \left(\int_0^R r^2 dr \right) = \left(\frac{\sin\theta}{R^2\theta} \right) \left(\frac{r^3}{3} \Big|_0^R \right) \\ \Rightarrow x_{\text{cm}} &= \left(\frac{\sin\theta}{R^2\theta} \right) \left(\frac{R^3}{3} \right) = \frac{R \sin\theta}{3\theta} \end{aligned}$$

ILLUSTRATION 16

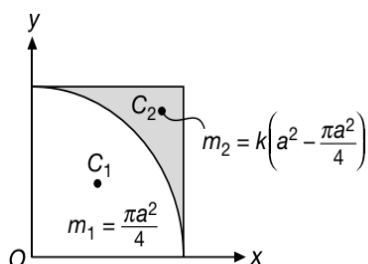
A quarter circular disc of radius a is removed from a square plate of side a as shown in Figure. Find the coordinates of centre of mass of the remaining part.



SOLUTION

The centre of mass of quarter disc taken out from the square plate lies at the point $C_1 \left(\frac{4a}{3\pi}, \frac{4a}{3\pi} \right)$. Let the

centre of mass of the remaining part be at $C_2(x, y)$ such that the combined centre of mass of the quarter disc and the remaining part must be located at centre of square, i.e. at $\left(\frac{a}{2}, \frac{a}{2} \right)$.



Since, we know that for the full square plate (made up of quarter circular disc and the remainder plate), the centre of mass is given by $\left(\frac{a}{2}, \frac{a}{2} \right)$, so we get

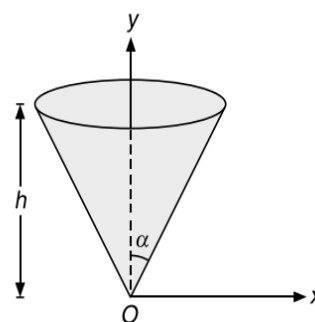
$$\begin{aligned} \frac{a}{2} &= \frac{m_{\text{quarter}}(x_{\text{cm}})_{\text{quarter}} + m_{\text{remainder}}(x_{\text{cm}})_{\text{remainder}}}{M} \\ \Rightarrow \frac{a}{2} &= \frac{1}{\sigma a^2} \left[\sigma \left(\frac{\pi a^2}{4} \right) \left(\frac{4a}{3\pi} \right) + (\sigma a^2) \left(\frac{4-\pi}{4} \right) x \right] \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{a}{2} &= \frac{a}{3} + \left(\frac{4-\pi}{4} \right) x \\ \Rightarrow x &= \frac{2a}{3(4-\pi)} \end{aligned}$$

Similarly, we get $y = \frac{2a}{3(4-\pi)}$

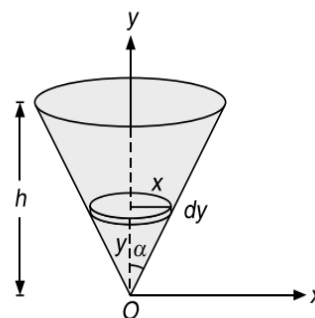
ILLUSTRATION 17

Find the centre of mass of a uniform solid cone of height h and semi-angle α , as shown in the figure.



SOLUTION

We assume the apex of the cone at the origin. It is clear that the centre of mass will lie along the y -axis. We divide the cone into disc of radius x and thickness dy as shown in Figure.



The volume of such a disc
 $dV = \pi x^2 dy = \pi (y \tan \alpha)^2 dy$

The mass of the disc is $dm = \rho dV$

Let us first find the total mass M of the cone

$$\begin{aligned} M &= \int dm = \pi \rho \tan^2 \alpha \int_0^h y^2 dy \\ \Rightarrow M &= (\pi \rho \tan^2 \alpha) \left(\frac{h^3}{3} \right) \end{aligned} \quad \dots(1)$$

The position of the centre of mass is given by

$$y_{\text{cm}} = \frac{1}{M} \int y dm = \frac{1}{M} \pi \rho \tan^2 \alpha \int_0^h y^3 dy$$

$$\Rightarrow y_{\text{cm}} = \frac{1}{M} \pi \rho \tan^2 \alpha \int_0^h y^3 dy$$

$$\Rightarrow y_{\text{cm}} = \left(\frac{1}{M} \pi \rho \tan^2 \alpha \right) \left(\frac{h^4}{4} \right) \quad \dots(2)$$

Equating equations (1) and (2), we get

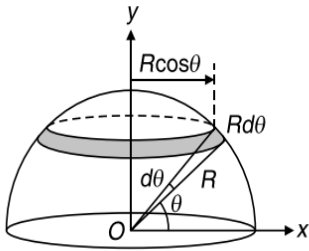
$$y_{\text{cm}} = \frac{3h}{4}$$

ILLUSTRATION 18

Locate the centre of mass of a uniform thin hemispherical shell of radius R from its centre O .

SOLUTION

Consider a hollow hemispherical shell of mass M and radius R . Let us consider an elemental circular strip of angular width $d\theta$ at an angular distance θ from the base of the hemisphere as shown in Figure.



This strip will have an area $dA = (2\pi R \cos \theta)(R d\theta)$

The mass dm of this strip is

$$dm = \sigma dA = \left(\frac{M}{2\pi R^2} \right) (2\pi R \cos \theta)(R d\theta)$$

Due to symmetry, we observe that centre of mass lies on y axis. So, $x_{\text{cm}} = 0$. Also,

$$y_{\text{cm}} = \frac{1}{M} \int \left(\begin{array}{l} \text{Mass of} \\ \text{Strip} \end{array} \right) \left(\begin{array}{l} y \text{ coordinate of} \\ \text{CM of strip} \end{array} \right)$$

Here y coordinate of this strip of mass dm can be taken as $R \sin \theta$, so we have

$$y_{\text{cm}} = \frac{1}{M} \int_0^{\pi/2} (R \sin \theta) dm$$

$$\Rightarrow y_{\text{cm}} = \frac{1}{M} \int_0^{\pi/2} \left(\frac{M}{2\pi R^2} 2\pi R^2 \cos \theta d\theta \right) R \sin \theta$$

$$\Rightarrow y_{\text{cm}} = R \int_0^{\pi/2} \sin \theta \cos \theta d\theta$$

$$\Rightarrow y_{\text{cm}} = \frac{R}{2} \int_0^{\pi/2} (2 \sin \theta \cos \theta) d\theta = \frac{R}{2} \int_0^{\pi/2} \sin(2\theta) d\theta$$

$$\Rightarrow y_{\text{cm}} = \frac{R}{2}$$

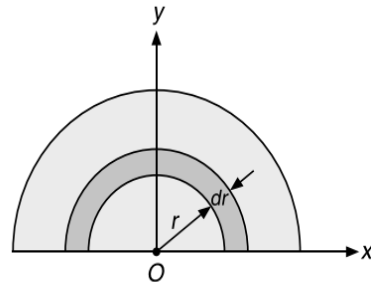
So, centre of mass of hemispherical shell lies at y axis at a distance $\frac{R}{2}$ from O i.e. centre of shell.

ILLUSTRATION 19

Locate the centre of mass of a uniform solid hemisphere of radius R .

SOLUTION

For a solid hemisphere, the mass is distributed uniformly over its volume. Consider an infinitesimal hemispherical shell of radius r thickness dr having mass dm as shown in Figure.



Also, we note that from symmetry, $x_{\text{CM}} = 0$

Now, centre of mass of a hemispherical shell is at $\frac{r}{2}$ from centre, so we have

$$y_{\text{cm}} = \frac{1}{M} \left[\int_0^R \left(\begin{array}{l} \text{Mass of} \\ \text{Shell} \end{array} \right) \left(\begin{array}{l} y \text{ coordinate} \\ \text{of CM of Shell} \end{array} \right) \right]$$

$$\Rightarrow y_{\text{cm}} = \frac{1}{M} \left[\int_0^R (y_{\text{cm shell}}) dm \right]$$

where, $dm = \rho dV = \left(\frac{M}{2\pi R^3/3} \right) (2\pi x^2 dx)$

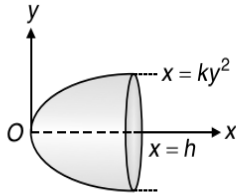
$$\Rightarrow dm = \left(\frac{3M}{R^3} \right) x^2 dx$$

$$\Rightarrow y_{\text{cm}} = \frac{1}{M} \left[\int_0^R \left(\frac{3M}{R^3} x^2 dx \right) \frac{x}{2} \right] = \frac{3}{2R^3} \left(\frac{x^4}{4} \Big|_0^R \right)$$

$$\Rightarrow y_{\text{cm}} = \frac{3R}{8}$$

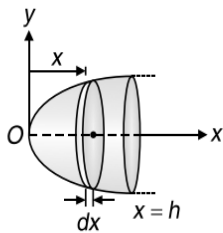
ILLUSTRATION 20

A solid generated by the rotation of a parabola is called a paraboloid. Find the location of centre of mass of such a paraboloid (from O) of uniform density formed by rotating a parabola $x = ky^2$ about x -axis. Assume that the height of object is h as shown in Figure.



SOLUTION

To find centre of mass of a paraboloid, let us consider an infinitesimal disc of radius y , width dx , mass dm at a distance x from origin as shown in Figure.



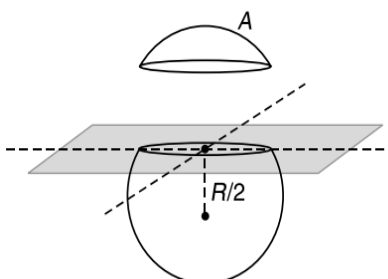
$$\Rightarrow dm = \rho(\pi y^2) dx = \pi \rho \left(\frac{x}{k} \right) dx = \frac{\pi \rho x dx}{k}$$

$$\Rightarrow x_{cm} = \frac{\int x dm}{M} = \frac{\int_0^h \pi \rho x^2 dx}{\int_0^h \pi \rho x dx} = \frac{x^3/3 \Big|_0^h}{x^2/2 \Big|_0^h}$$

$$\Rightarrow x_{cm} = \frac{h^3/3}{h^2/2} = \frac{2h}{3}$$

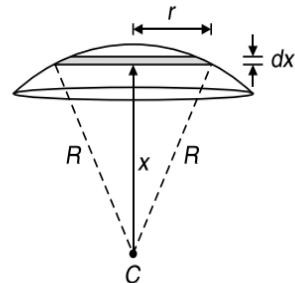
ILLUSTRATION 21

A small part A is cut from a solid sphere by a plane at a distance $\frac{R}{2}$ from its centre as shown in Figure. Find the location of centre of mass of object A from the centre of the sphere.



SOLUTION

To find centre of mass of object A , we consider a small elemental disc of width dx at a distance x from the centre C as shown in Figure.



Radius r of this elemental disc is

$$r = \sqrt{R^2 - x^2}$$

If ρ be the density of the sphere, then mass of the elemental disc will be

$$dm = \rho dV = \rho(\pi r^2 dx) = \pi \rho (R^2 - x^2) dx$$

Mass of object A can be obtained as

$$M = \int_{R/2}^R dm = \pi \rho \int_{R/2}^R (R^2 - x^2) dx$$

$$\Rightarrow M = \pi \rho \left(R^2 x - \frac{x^3}{3} \right) \Big|_{R/2}^R$$

$$\Rightarrow M = \pi \rho \left[\left(R^3 - \frac{R^3}{3} \right) - \left(\frac{R^3}{2} - \frac{R^3}{24} \right) \right] = \frac{5}{24} \pi \rho R^3$$

Since, $y_{cm} = \frac{1}{M} \int_{R/2}^R x dm$

$$\Rightarrow y_{cm} = \frac{24}{5 \pi \rho R^3} \int_{R/2}^R \rho \pi (R^2 - x^2) x dx$$

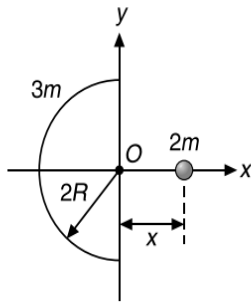
$$\Rightarrow y_{cm} = \frac{24}{5 R^3} \left(\frac{R^2 x^2}{2} - \frac{x^4}{4} \right) \Big|_{R/2}^R$$

$$\Rightarrow y_{cm} = \frac{12}{5 R^3} \left[\left(R^4 - \frac{R^4}{2} \right) - \left(\frac{R^4}{4} - \frac{R^4}{32} \right) \right]$$

$$\Rightarrow y_{cm} = \frac{27}{40} R$$

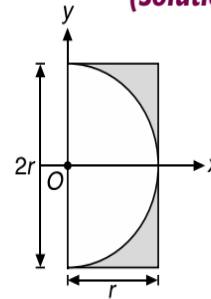
Test Your Concepts-I
Based on Centre of Mass

- There are two masses m_1 and m_2 placed at a distance ℓ apart, let the centre of mass of this system is at a point named C. If m_1 is displaced by ℓ_1 towards C and m_2 is displaced by ℓ_2 away from C. Find the distance, from C where new centre of mass will be located.
- Two particles having masses m_1 and m_2 are placed at a separation r between them. Let m_2 be displaced away from m_1 by a distance x . By what distance, m_1 should be displaced in order to keep the position of centre of mass unchanged.
- The position vector of three particles of mass $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg are $\vec{r}_1 = (\hat{i} + 4\hat{j} + \hat{k})m$, $\vec{r}_2 = (\hat{i} + \hat{j} + \hat{k})m$ and $\vec{r}_3 = (2\hat{i} - \hat{j} - 2\hat{k})m$ respectively. Find the position vector of their centre of mass.
- A ring of mass $3m$, radius $2R$ and a particle of mass $2m$ are kept as shown in Figure such that the centre of mass of the combination lies at the origin. Calculate x .

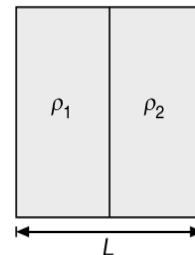


- Consider a rectangular plate of dimensions $a \times b$. If this plate is considered to be made up of four rectangles of dimensions $\frac{a}{2} \times \frac{b}{2}$ and we now remove one out of four rectangles. Find the position where the centre of mass of the remaining system will be.
- Determine the coordinate of centre of mass of a thin homogeneous plate having the form of a rectangle with sides r and $2r$ from which a semi-circle with a radius r is cut out.

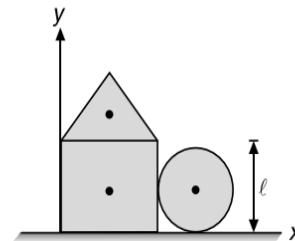
(Solutions on page H.69)



- The coordinates of a triangle ABC are A(1, 2), B(4, 6) and C(-3, -2). Three particles of masses 1 kg, 2 kg and m kg are placed the vertices of the triangle. If the coordinates of the centre of mass are $(\frac{3}{5}, 2)$, calculate the mass m .
- Half of the square plate shown in figure is made of a material of density ρ_1 and the other half of density ρ_2 . The length of the plate is L . Locate the centre of mass of the plate.

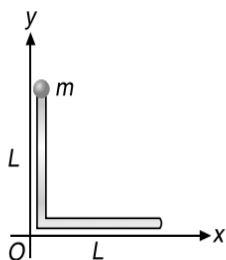


- A square sheet, a disc and an equilateral triangle having same surface mass densities are placed as shown in Figure. Find the coordinates of centre of mass of the three bodies.

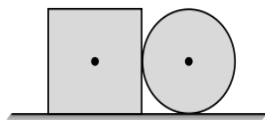


- The linear mass density of a thin rod of length ℓ varies with the distance x from one end as $\lambda = \lambda_0 \frac{x^2}{\ell^2}$. Find the position of centre of mass of rod.

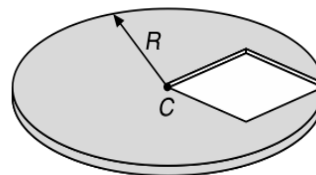
11. Find the centre of mass of the system of a L-shaped rod of mass m and a particle of mass m attached at the end of the rod as shown in Figure.



12. A uniform square plate of side a and a disc of diameter $2a$ having same mass per unit area are kept in contact as shown in Figure. Locate the position of the centre of mass of the system from the centre of square.



13. A square portion is removed from the disc as shown in Figure. Locate centre of mass of the remaining portion from the centre of the full disc.



14. A rod of length L , having negligible thickness, has a linear mass density that increases linearly with distance from one end of the rod from λ_1 to λ_2 at the other end. Locate the centre of mass of the rod from the lighter end.

VELOCITY AND ACCELERATION OF CENTER OF MASS

The instantaneous velocity of the centre of mass is given by $\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt}$. From definition, we have

$$\vec{v}_{cm} = \frac{1}{M} \sum \left(m_i \frac{d\vec{r}_i}{dt} \right) = \frac{1}{M} (\sum m_i \vec{v}_i)$$

The total momentum of a system of particles is equivalent to that of a single (imaginary) particle of mass $M = \sum m_i$ moving at the velocity of the centre of mass \vec{v}_{cm} . So, we have

$$\begin{aligned} \vec{p}_{system} &= \vec{p}_1 + \vec{p}_2 + \dots + \vec{p}_n \\ \Rightarrow \vec{p}_{system} &= m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n \\ \Rightarrow \vec{p}_{system} &= M \vec{v}_{cm} \end{aligned}$$

Using the concept of centre of mass, we may deal with the translational motion of extended objects or systems of particles, as if they were point particle with all the mass concentrated at the centre of mass. Differentiating equation, with respect to time, we get

$$a_{cm} = \frac{\sum m_i \vec{a}_i}{M} = \frac{\sum \vec{F}_i}{M}$$

Since the internal forces cancel in pairs, therefore,

$$\sum \vec{F}_i = \vec{F}_{ext}$$

$$\Rightarrow \vec{F}_{ext} = M \vec{a}_{cm}$$

The centre of mass accelerates as if it were a point particle of mass $M = \sum m_i$ and the net external force were applied at this point.

QUICK RECAP

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n}{m_1 + m_2 + \dots + m_n}$$

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \dots + m_n y_n}{m_1 + m_2 + \dots + m_n}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + \dots + m_n z_n}{m_1 + m_2 + \dots + m_n}$$

$$\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n}{m_1 + m_2 + \dots + m_n}$$

$$\vec{p}_{cm} = \vec{p}_1 + \vec{p}_2 + \dots + \vec{p}_n$$

$$\vec{a}_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + \dots + m_n \vec{a}_n}{m_1 + m_2 + \dots + m_n} \text{ and}$$

$$\vec{F}_{cm} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n$$



ILLUSTRATION 22

Two particles one of mass 1 kg and the other of 2 kg are projected simultaneously with the same speed from the roof of a tower, the one of mass 1 kg vertically upwards and the other vertically downwards. What is the acceleration of centre of mass of these two particles?

SOLUTION

In this Problem, students generally get three answers, i.e. g , $\frac{g}{3}$ and zero. However, the correct answer is g .

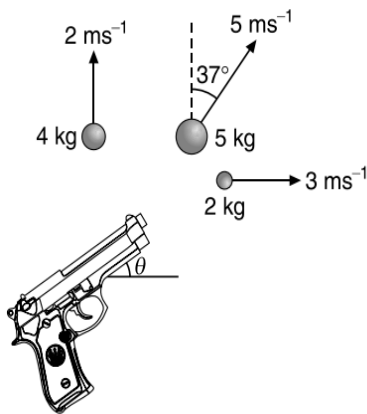
$$\text{Since, } \vec{a}_{\text{cm}} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2}$$

$$\text{where, } \vec{a}_1 = \vec{a}_2 = g \quad \{\text{downwards}\}$$

$$\Rightarrow \vec{a}_{\text{cm}} = \frac{(1)(g) + (2)(g)}{1 + 2} = g \quad \{\text{downwards}\}$$

ILLUSTRATION 23

A gun whose barrel is inclined at some angle to the horizontal fires three shells of mass 2 kg, 5 kg and 4 kg which move in different directions as shown in Figure.



If the centre of mass of the three shells moves along the direction in which the barrel points, then calculate the angle made by the gun's barrel with the horizontal. Take $\cos(53^\circ) = 3/5$.

SOLUTION

The x and y components of velocity of centre of mass of the shells are given by

$$(v_{\text{cm}})_x = \frac{(2)(3) + (5)(5 \sin 37^\circ) + (4)(0)}{4 + 5 + 2}$$

$$\Rightarrow (v_{\text{cm}})_x = \frac{6 + (25)(3/5)}{4 + 5 + 2} = \frac{21}{11} \text{ ms}^{-1}$$

$$(v_{\text{cm}})_y = \frac{(2)(0) + (5)(5 \cos 37^\circ) + (4)(2)}{4 + 5 + 2}$$

$$\Rightarrow (v_{\text{cm}})_y = \frac{(25)(4/5) + 8}{4 + 5 + 2} = \frac{28}{11} \text{ ms}^{-1}$$

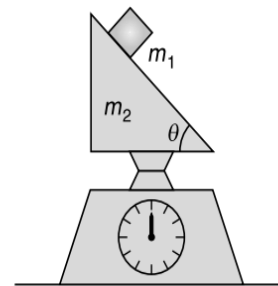
If θ be the angle which the gun's barrel makes with the horizontal, then we have

$$\tan \theta = \tan^{-1} \left[\frac{(v_{\text{cm}})_y}{(v_{\text{cm}})_x} \right] = \tan^{-1} \left(\frac{4}{3} \right)$$

$$\Rightarrow \theta = 53^\circ$$

ILLUSTRATION 24

A wedge of mass m_2 is placed at rest on a scale as shown in figure. A small block of mass m_1 slides down the frictionless incline of the wedge. Find the scale reading while the block slides.



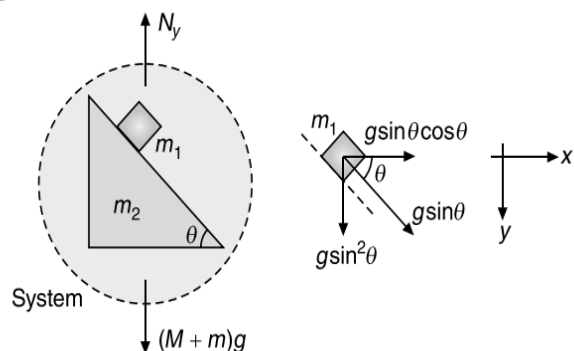
SOLUTION

Acceleration of m_1 in the vertical downward direction, i.e. along y direction will be

$$a_y = g \sin \theta (\sin \theta) = g \sin^2 \theta$$

$$\Rightarrow (a_{\text{cm}})_y = \frac{m_1 g \sin^2 \theta + 0}{m_1 + m_2} = \frac{m_1 g \sin^2 \theta}{m_1 + m_2}$$

The free body diagram of the system is shown in Figure.



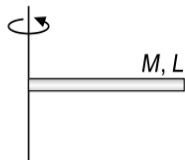
The scale reading will be the force exerted by the wedge on the measuring scale or vice versa. Applying

Newton's Second Law to the centre of mass of the system in the vertically downward direction, i.e. along the y direction, we get

$$\begin{aligned} (m_1 + m_2)g - N_y &= (m_1 + m_2)(a_{\text{cm}})_y \\ \Rightarrow N_y &= (m_1 + m_2)g - m_1g \sin^2 \theta \\ \Rightarrow N_y &= m_1g(1 - \sin^2 \theta) + m_2g \\ \Rightarrow N_y &= m_1g \cos^2 \theta + m_2g \\ \Rightarrow N_y &= (m_1 \cos^2 \theta + m_2)g \end{aligned}$$

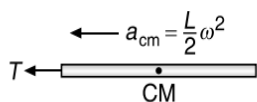
ILLUSTRATION 25

A uniform rod of mass M and length L is tied to a vertical shaft. It rotates in horizontal plane about the vertical axis at angular velocity ω . How much horizontal force does the shaft exert on the rod?



SOLUTION

Let T be the force applied horizontally on the rod. The acceleration of centre of mass is $\left(\frac{L}{2}\right)\omega^2$ because it is moving in a circle of radius $\frac{L}{2}$.



Using $\vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}}$, we get

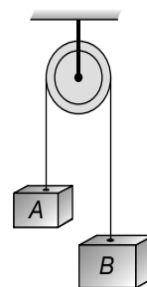
$$T = M\left(\frac{L}{2}\right)\omega^2$$

So, the horizontal force exerted is

$$T = Mr_{\text{cm}}\omega^2 = \frac{1}{2}ML\omega^2$$

ILLUSTRATION 26

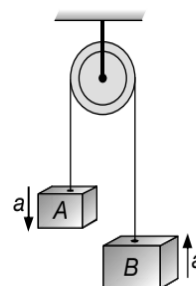
In the arrangement shown in figure, $m_A = 2$ kg and $m_B = 1$ kg. String is light and inextensible. Find the acceleration of centre of mass of both the blocks. Neglect friction everywhere.



SOLUTION

Net pulling force on the system is

$$(m_A - m_B)g = (2 - 1)g = g$$



Total mass being pulled is $m_A + m_B = 3$ kg

$$\Rightarrow a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{g}{3}$$

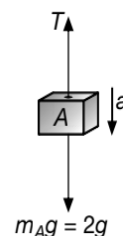
$$\text{Now, } \vec{a}_{\text{cm}} = \frac{m_A \vec{a}_A + m_B \vec{a}_B}{m_A + m_B}$$

$$\Rightarrow |\vec{a}_{\text{cm}}| = \frac{(2)(a) - (1)(a)}{1 + 2} = \frac{a}{3}$$

$$\Rightarrow |\vec{a}_{\text{cm}}| = \frac{g}{9}, \text{ downwards}$$

Alternate Method:

Free body diagram of block A is shown in figure.

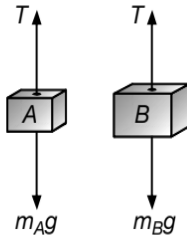


$$2g - T = m_A(a)$$

$$\Rightarrow T = 2g - m_A a$$

$$\Rightarrow T = 2g - (2)\left(\frac{g}{3}\right) = \frac{4g}{3}$$

Free body diagrams of A and B both are as shown in figure



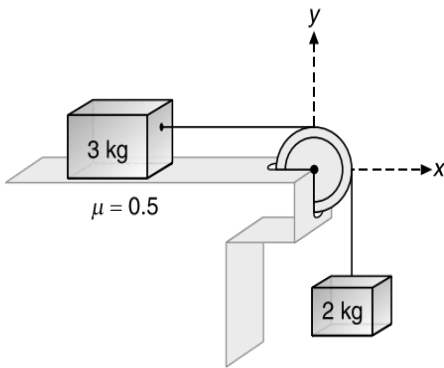
Since, $|\vec{a}_{cm}| = \frac{\text{Net force on both the blocks}}{m_A + m_B}$

$$\Rightarrow |\vec{a}_{cm}| = \frac{(m_A + m_B)g - 2T}{2 + 1} = \frac{3g - (8g/3)}{3}$$

$$\Rightarrow |\vec{a}_{cm}| = \frac{g}{9}, \text{ downwards}$$

ILLUSTRATION 27

In the arrangement shown in Figure, calculate the acceleration of centre of mass of the system. Take $g = 10 \text{ ms}^{-2}$.



SOLUTION

The limiting friction between the 3 kg block and the surface is

$$f_l = \mu N = (0.5)(3)(10) = 15 \text{ N}$$

The weight of the 2 kg block, i.e. 20 N provides the driving force to the system, due to which magnitude of acceleration of both masses is given by

$$a = \frac{20 - 15}{2 + 3} = \frac{5}{5} = 1 \text{ ms}^{-2}$$

So, for $m_1 = 3 \text{ kg}$, we have $\vec{a}_1 = 1\hat{i} \text{ ms}^{-2}$ and that for $m_2 = 2 \text{ kg}$, we have $\vec{a}_2 = -1\hat{j} \text{ ms}^{-2}$. Since

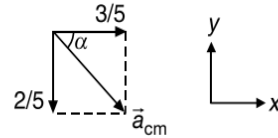
$$\vec{a}_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2}$$

$$\Rightarrow \vec{a}_{cm} = \frac{3\hat{i} - 2\hat{j}}{3 + 2} = \frac{3}{5}\hat{i} - \frac{2}{5}\hat{j}$$

Magnitude of acceleration of CM is given by

$$a_{cm} = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{\sqrt{13}}{5} \text{ ms}^{-2}$$

The direction of motion of centre of mass is calculated from the Figure shown.

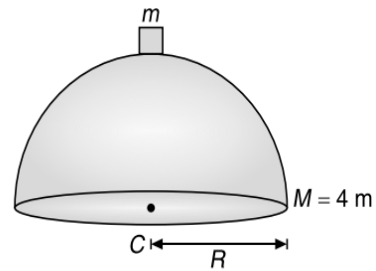


Since, $\tan \alpha = \frac{2/5}{3/5} = \frac{2}{3}$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{2}{3}\right), \text{ below the horizontal}$$

ILLUSTRATION 28

A small block of mass m starts sliding down from rest along the smooth surface of a fixed hollow hemisphere of same mass m . Calculate the distance of centre of mass of block and hemisphere from centre of hemisphere C when block m separates from the surface of hemisphere.



SOLUTION

When m starts from rest, it breaks off at an angular displacement of $\theta = \cos^{-1}\left(\frac{2}{3}\right)$ with the vertical. At this point, if the centre of mass of block and hemisphere is located at a point $P(x_{cm}, y_{cm})$ with origin at C, then we have

$$x_{cm} = \frac{m(\sqrt{5}R/3) + 4m(0)}{5m} = \frac{R}{3\sqrt{5}}$$

$$y_{cm} = \frac{m(2R/3) + (4m)(R/2)}{5m} = \frac{8}{15}R$$

So, the distance CP of centre of mass is

$$r_{\text{cm}} = \sqrt{x_{\text{cm}}^2 + y_{\text{cm}}^2}$$

$$\Rightarrow r_{\text{cm}} = R\sqrt{\frac{1}{45} + \frac{64}{225}} = R\sqrt{\frac{69}{225}} = R\sqrt{\frac{23}{75}}$$

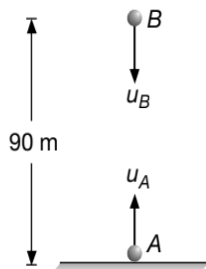
$$\Rightarrow h = \frac{900}{20} = 45 \text{ m}$$

Therefore, maximum height attained by the centre of mass is

$$H = 60 + 45 = 105 \text{ m}$$

ILLUSTRATION 29

Two particles A and B of mass 1 kg and 2 kg respectively are projected in the directions shown in figure with speeds $u_A = 200 \text{ ms}^{-1}$ and $u_B = 55 \text{ ms}^{-1}$. Initially they were 90 m apart. Find the maximum height attained by the centre of mass of the particles. Assume acceleration due to gravity to be constant during entire motion and take $g = 10 \text{ ms}^{-2}$.



SOLUTION

Let us first find location of centre of mass of system. Assume origin at A, we get

$$r_{\text{cm}} = \frac{(1)(0) + 2(90)}{3}$$

$$\Rightarrow r_{\text{cm}} = 60 \text{ cm from A}$$

i.e., cm is at height 60 m from the ground at time $t = 0$

Since $\vec{a}_A = \vec{a}_B = g$

$$\Rightarrow \vec{a}_{\text{cm}} = \frac{m_A \vec{a}_A + m_B \vec{a}_B}{m_A + m_B} = g = 10 \text{ ms}^{-2}, \text{ downwards}$$

$$\text{Also, } \vec{u}_{\text{cm}} = \frac{m_A \vec{u}_A + m_B \vec{u}_B}{m_A + m_B}$$

$$\Rightarrow \vec{u}_{\text{cm}} = \frac{(1)(200) - (2)(55)}{1 + 2} = 30 \text{ ms}^{-1}, \text{ upwards}$$

If h be the height attained by centre of mass beyond 60 m, then since

$$v_{\text{cm}}^2 = u_{\text{cm}}^2 + 2a_{\text{cm}}h$$

$$\Rightarrow 0 = (30)^2 - (2)(10)h$$

MOTION OF THE CENTRE OF MASS

Now we can discuss the physical importance of the centre of mass concept. Consider the motion of a group of particles m_1, m_2, \dots, m_n and whose total mass is $M = \sum m_i$ which is a constant. From the definition of the centre of mass, we have

$$M\vec{r}_{\text{cm}} = m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n$$

Differentiating this equation wrt time, we obtain

$$M\vec{v}_{\text{cm}} = m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_n\vec{v}_n$$

where $v_1 = \frac{d\vec{r}_1}{dt}$ is the velocity of the first particle, etc.,

and $v_{\text{cm}} = \frac{d\vec{r}_{\text{cm}}}{dt}$ is the velocity of the centre of mass.

Differentiating the above equation wrt time, we obtain

$$M\vec{a}_{\text{cm}} = m_1\vec{a}_1 + m_2\vec{a}_2 + \dots + m_n\vec{a}_n$$

where \vec{a}_1 is the acceleration of the first particle, etc., and \vec{a}_{cm} is the acceleration of the centre of mass. Now, from Newton's Second Law the force \vec{F}_1 acting on the first particle is given by $\vec{F}_1 = m_1\vec{a}_1$. Likewise, $\vec{F}_2 = m_2\vec{a}_2$, etc. We can then write above equation as

$$M\vec{a}_{\text{cm}} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n$$

Among all these forces the internal forces are exerted by the particles on each other. However, from Newton's Third Law, these internal forces will occur in equal and opposite pairs, so that they contribute nothing to the sum. The right hand sum in the above equation then represents the sum of only the external forces acting on all the particles (system). We can then rewrite the above equation as

$$M\vec{a}_{\text{cm}} = \sum \vec{F}_{\text{external}}$$

This states that the centre of mass of a system of particles moves as though all the mass of the system is concentrated at the centre of mass and all the external forces were applied at that point.

Problem Solving Technique(s)

In one such important situation when we have

$$\Sigma \vec{F}_{\text{external}} = \vec{0}$$

then, $\vec{v}_{\text{cm}} = \text{constant}$

Further if we have a two particle system initially at rest (or both its particle to be at rest), then as a consequence of the above result we have

$$(\vec{v}_{\text{cm}})_{\text{final}} = (\vec{v}_{\text{cm}})_{\text{initial}} = \vec{0}$$

$$\Rightarrow \vec{r}_{\text{cm}} = \text{constant}$$

$$\Rightarrow \Delta \vec{r}_{\text{cm}} = \vec{0}$$

For a two particle system, this result can be interpreted as

$$\Delta \left(\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \right) = \vec{0}$$

$$\Rightarrow m_1 \Delta \vec{r}_1 + m_2 \Delta \vec{r}_2 = \vec{0}$$

where, $\Delta \vec{r}_1$ and $\Delta \vec{r}_2$ are the displacements of the particles m_1 and m_2 with respect to the ground. The displacements $\Delta \vec{r}_1$ and $\Delta \vec{r}_2$ are in opposite directions.

DISPLACEMENT OF CENTRE OF MASS OF A SYSTEM OF PARTICLES

Consider a system of n point masses $m_1, m_2, m_3, \dots, m_n$ whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_n$, respectively. Then, the position vector of the centre of mass of the system is given by

$$\vec{r}_{\text{cm}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n}{m_1 + m_2 + \dots + m_n} \quad \dots(1)$$

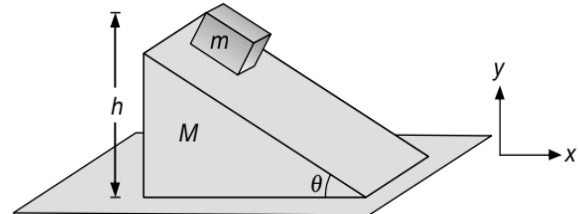
If $\Delta \vec{r}_1, \Delta \vec{r}_2, \Delta \vec{r}_3 \dots, \Delta \vec{r}_n$ be the displacement vectors of the particles of system, then the displacement vector of centre of mass of this system of particles can be directly given by

$$\Delta \vec{r}_{\text{cm}} = \frac{m_1 \Delta \vec{r}_1 + m_2 \Delta \vec{r}_2 + m_3 \Delta \vec{r}_3 + \dots + m_n \Delta \vec{r}_n}{m_1 + m_2 + m_3 + \dots + m_n} \quad \dots(2)$$

If net external force on the system in a particular direction is zero and initially the centre of mass of the system is at rest, then the centre of mass will not move along that particular direction even though some particles (or bodies) of the system may move along that direction.

ILLUSTRATION 30

A block of mass m is released from the top of a wedge of mass M as shown in figure. Find the displacement of wedge on the horizontal ground when the block reaches the bottom of the wedge. Neglect friction everywhere.



SOLUTION

Here the system is wedge + block. Net force on the system in horizontal direction (x -direction) is zero, therefore, the centre of mass of the system will not move in x -direction. Hence when block moves to the right the wedge moves to the left. So, we can apply,

$$m_R \Delta x_R = m_L \Delta x_L = x_L m_L \quad \dots(1)$$

Let x be the displacement of wedge to the left. Then

$$\Delta x_L = \text{displacement of wedge towards left} = x$$

$$m_L = \text{mass of wedge} = M$$

$$\Delta x_R = \text{displacement of block with respect to ground towards right} = h \cot \theta - x$$

and $m_R = \text{mass of block} = m$

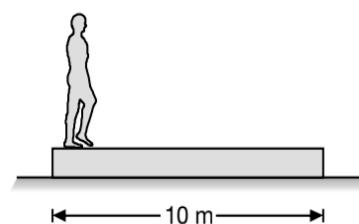
Substituting in equation (1), we get

$$m(h \cot \theta - x) = xM$$

$$\Rightarrow x = \frac{mh \cot \theta}{M + m}$$

ILLUSTRATION 31

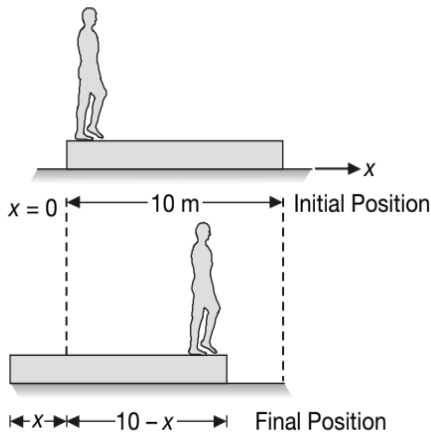
A wooden plank of mass 20 kg is resting on a smooth horizontal floor. A man of mass 60 kg starts moving from one end of the plank to the other end. The length of the plank is 10 m. Find the displacement of the plank over the floor when the man reaches the other end of the plank.



SOLUTION

Method I:

Here the system is man + plank. Net force on this system in horizontal direction is zero and initially the centre of mass of the system is at rest. Therefore, the centre of mass does not move in horizontal direction. Let x be the displacement of the Plank. Assuming the origin, i.e., $x = 0$ at the position shown in Figure



As we said earlier also, the centre of mass will not move in horizontal direction (x -axis). Therefore, for centre of mass to remain stationary.

$$x_i = x_f$$

$$\frac{(60)(0) + 20\left(\frac{10}{2}\right)}{60 + 20} = \frac{(60)(10 - x) + 20\left(\frac{10}{2} - x\right)}{60 + 20}$$

Please note that the centre of mass of the plank lies at its centre.

$$\Rightarrow \frac{5}{4} = \frac{6(10 - x) + 2\left(\frac{10}{2} - x\right)}{8} = \frac{60 - 6x + 10 - 2x}{8}$$

$$\Rightarrow 5 = 30 - 3x + 5 - x$$

$$\Rightarrow 4x = 30$$

$$\Rightarrow x = \frac{30}{4} \text{ m}$$

$$\Rightarrow x = 7.5 \text{ m}$$

Method II:

In this situation discussed above we can also apply

$$\Sigma m_R x_R = \Sigma m_L x_L \quad \dots(1)$$

where, $\Sigma m_R x_R$ is the summation of product of x and m of the particles (or bodies) which are moving towards right (R) and $\Sigma m_L x_L$ is the summation of product of x and m of the particles (or bodies) which are moving towards left (L).

However, remember the following two conditions while using the above equation.

- (i) This equation can be applied when centre of mass does not move in x -direction and no external force is acting along x -direction.
- (ii) In the above equation x is the displacement of particle relative to ground.

Here, displacement of plank towards left is $x_L = x$, mass of plank is $m_L = 20 \text{ kg}$, displacement of man relative to ground towards right is $x_R = 10 - x$ and mass of man is $m_R = 60 \text{ kg}$. From (1), we get

$$(10 - x)(60) = 20x$$

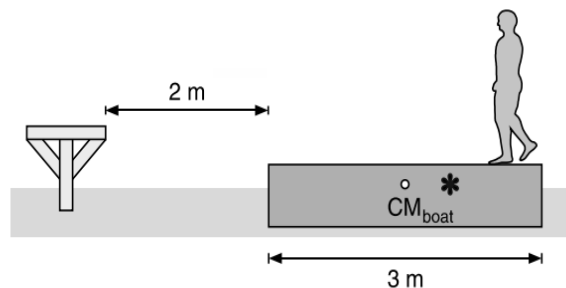
$$\Rightarrow x = 30 - 3x$$

$$\Rightarrow 4x = 30$$

$$\Rightarrow x = \frac{30}{4} = 7.5 \text{ m}$$

ILLUSTRATION 32

A man of mass $m_1 = 60 \text{ kg}$ is at the rear of a stationary boat of mass $m_2 = 40 \text{ kg}$ and length 3 m, which can move freely on the water, as shown in the Figure.



The front of the boat is 2 m from the dock. What happens when the man walks to the front? Treat the boat as a uniform object.

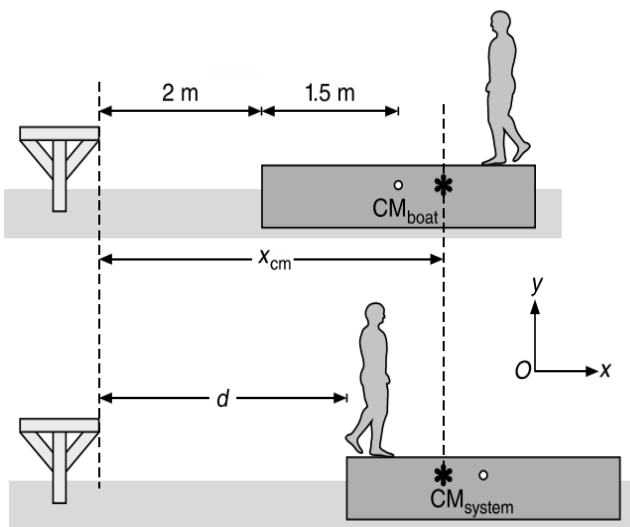
SOLUTION

Since there is no net external force, so x_{cm} is fixed. We treat the boat as a point particle whose mass is concentrated at its centre. In terms of the initial positions, the centre of mass is

$$(x_{cm})_{\text{initial}} = \frac{m_1 x_1 + m_2 x_2}{M} = \frac{(60)(5) + (40)(3.5)}{100}$$

$$\Rightarrow (x_{cm})_{\text{initial}} = 4.4 \text{ m} \quad \dots(1)$$

The initial position of the centre of mass is indicated by a star as shown in Figure.



When the person walks relative to the boat, the position of the center of mass of the system remains unchanged.

After the man walks to the front end or the opposite end of the boat, then let the front end be at a distance d from the dock.

In terms of the new positions,

$$(x_{\text{cm}})_{\text{final}} = \frac{m_1 d + m_2 (d + 1.5)}{100} \quad \dots(2)$$

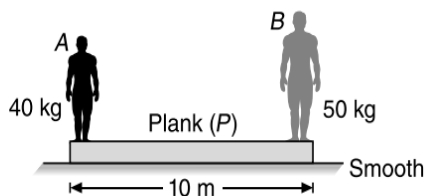
On equating (1) and (2), we get

$$d = 3.8 \text{ m}$$

The negative displacement of the man is accompanied by a positive displacement of the boat such that the centre of mass stays fixed. The total momentum of the system remains zero at all times.

ILLUSTRATION 33

Two men A and B are standing at the ends of a 10 kg plank of length 10 m. The plank is resting on a smooth horizontal surface. The system is initially at rest and masses are shown in Figure.



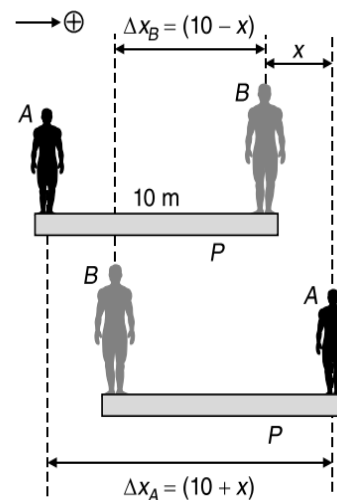
Find the displacement of plank when both A and B interchange their positions. Also find the displacement of the plank if both A and B move towards each other to finally meet at the middle of the plank.

SOLUTION

Let us consider man A , man B and plank as a system, then there is no external force acting on system in horizontal direction. Since the system is initially at rest, hence the displacement of centre of mass should also be zero.

$$\Delta x_{\text{cm}} = 0 = \frac{m_A \Delta x_A + m_B \Delta x_B + m_P \Delta x_P}{m_A + m_B + m_P} \quad \dots(1)$$

When both A and B exchange their position, then let the plank be displaced rightwards through x as shown in Figure.



So, displacement of A is

$$\Delta x_A = 10 + x, \text{ rightwards}$$

and displacement of B is

$$\Delta x_B = -(10 - x), \text{ leftwards}$$

From Equation (1), we get

$$\frac{40(10 + x) + 50(-10 + x) + 10(x)}{40 + 50 + 10} = 0$$

$$\Rightarrow -100 + 100x = 0$$

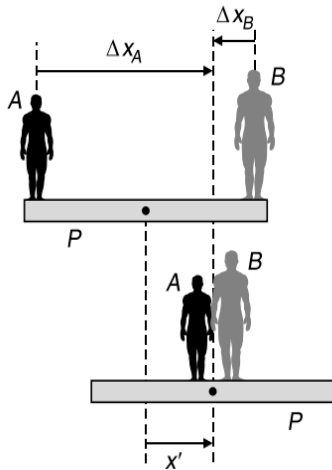
$$\Rightarrow x = 1 \text{ m}$$

Hence the plank will move 1 m towards right.

When both A and B move towards each other to finally meet at the middle of the plank, then too the displacement of the centre of mass will be zero, i.e.

$$\Delta x_{\text{cm}} = 0$$

Let the displacement of the plank in this case be x' rightwards as shown in Figure.



So, displacement of A is

$$\Delta x_A = 5 + x', \text{ rightwards}$$

and displacement of B is

$$\Delta x_B = -(5 - x'), \text{ leftwards}$$

$$\Rightarrow \Delta x_{\text{cm}} = \frac{40(5 + x') + 50(-5 + x') + 10x'}{40 + 50 + 10} = 0$$

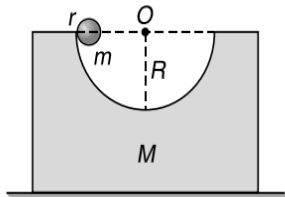
$$\Rightarrow -50 + 100x' = 0$$

$$\Rightarrow x' = \frac{1}{2} \text{ m}$$

Hence the displacement of the plank will be $\frac{1}{2}$ m towards right.

ILLUSTRATION 34

Calculate the displacement of M when the sphere reaches the bottom of the groove shown in Figure. Assume all surfaces to be frictionless.



SOLUTION

Since no force acts on the system in the horizontal direction, so the x position of the centre of mass of the system remains fixed. When the sphere reaches the bottom of the groove, let the mass M move towards the left by x , then displacement of the sphere will be $(R - r - x)$. Since, for centre of mass not to move, we have

$$m_1 \Delta x_1 = m_2 \Delta x_2$$

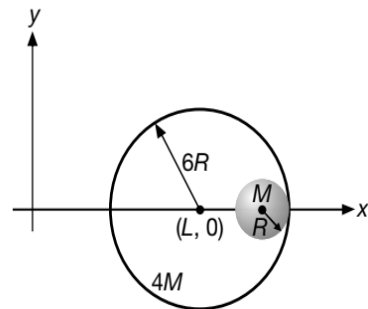
where Δx_1 and Δx_2 are the displacements of M and m with respect to ground and both displacements are in opposite directions.

$$\Rightarrow Mx = m(R - r - x)$$

$$\Rightarrow x = \frac{m(R - r)}{M + m}$$

ILLUSTRATION 35

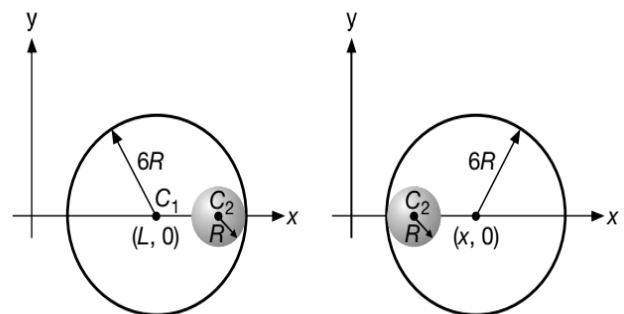
A small sphere of radius R is held against the inner surface of larger sphere of radius $6R$ (as shown in figure). The masses of large and small spheres are $4M$ and M respectively. This arrangement is placed on a horizontal table. There is no friction between any surfaces of contact. The small sphere is now released. Find the coordinates of the centre of the large sphere, when the smaller sphere reaches the other extreme position.



SOLUTION

Method I:

Since all the surfaces are smooth, no external force is acting on the system in horizontal direction. Therefore, the centre of mass of the system in horizontal direction will remain stationary.



$$x_i = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{(4M)(L) + M(L + 5R)}{4M + M}$$

$$\Rightarrow x_i = L + R \quad \dots(1)$$

Let, $(x, 0)$ be the coordinates of the centre of large sphere in final position. Then x -coordinates of cm in final position will be

$$x_f = \frac{(4M)(x) + M(x - 5R)}{4M + M} = (x - R) \quad \dots(2)$$

Equating equations (1) and (2), we get

$$x = L + 2R$$

Therefore, coordinates of large sphere, when the smaller sphere reaches the other extreme position, are $(L + 2R, 0)$.

Method II:

Since no force is acting along the x direction and initially the system is at rest, so the displacement of the centre of mass of the system along the x direction is zero due to which we get

$$m_1 \Delta x_1 = m_2 \Delta x_2$$

So, when M moves to left, i.e. to the other extreme, then $4M$ moves to right, say by x . Consequently M moves effectively to the left by $(10R - x)$ such that

$$M(10R - x) = (4M)x$$

$$\Rightarrow x = 2R$$

So, new coordinates of centre of big sphere are $(L + 2R, 0)$.

ILLUSTRATION 36

A projectile is fired at a speed of 100 ms^{-1} at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1:3, the smaller coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

SOLUTION

At the highest point, the projectile has horizontal velocity. The lighter part comes to rest. Hence, the heavier part will move with increased horizontal velocity. In vertical direction, both parts have zero velocity and undergo same acceleration. Thus, both will hit the ground together. As internal forces do not affect the motion of the centre of mass, the centre of mass hits the ground at the position where the original projectile would have landed. The range of the original projectile is,

$$x_{cm} = \frac{2u^2 \sin \theta \cos \theta}{g} = \frac{2(10^4)(3/5)(4/5)}{10} \text{ m}$$

$$\Rightarrow x_{cm} = 960 \text{ m}$$

The centre of mass will hit the ground at this position. As the smaller block comes to rest after breaking, it falls down vertically and hits the ground at half of the range, i.e., at $x = 480 \text{ m}$. If the heavier block hits the ground at x_2 , then

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = 960 \text{ m}$$

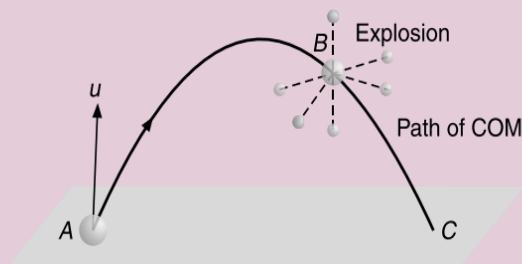
$$\Rightarrow 960 = \frac{(m)(480) + (3m)(x_2)}{(m + 3m)}$$

$$\Rightarrow x_2 = 1120 \text{ m}$$



Conceptual Note(s)

If a projectile explodes in mid-air, the path of the centre of mass remains unchanged. This is because, during explosion no external force (except gravity) acts on the centre of mass. The situation is shown in figure.



Path of centre of mass (cm) is ABC , even though the different parts travel in different directions after explosion.

ILLUSTRATION 37

A rope thrown over a pulley has a ladder with a man A on one of its ends and a counter balancing mass M on its other end. The man whose mass is m , climbs upwards by $\Delta \vec{r}$ relative to the ladder and then stops. Assuming the masses of the pulley and the rope to be negligible, pulley to be frictionless at the axis, calculate the displacement of the centre of mass of this system.

SOLUTION

Counter balancing mass M applied at the other end means that

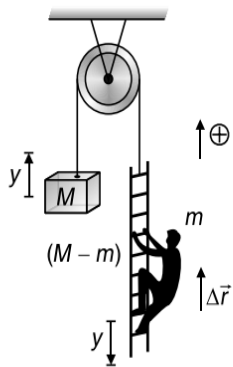
$$M = m + m_{\text{ladder}}$$

$$\Rightarrow m_{\text{ladder}} = M - m$$

$$m_{\text{pulley}} \approx m_{\text{rope}} \approx 0$$

Displacement of man with respect to ladder is

$$y_{m/l} = \Delta r$$



When man moves up by Δr , then let ladder move down by y and hence counter weight moves up by y . So, effective displacement of man is $(\Delta r - y)$.

The displacement of CM is given by

$$\Delta r_{\text{cm}} = \frac{m_l \Delta r_l + m_m \Delta r_m + m_c \Delta r_c}{m_l + m_m + m_c}$$

$$\Rightarrow \Delta r_{\text{cm}} = \frac{(M - m)y + m(\Delta r - y) + M(-y)}{M - m + m + M}$$

$$\Rightarrow \Delta r_{\text{cm}} = \frac{m\Delta r}{2M}, \text{ upwards}$$

ILLUSTRATION 38

A raft of mass M with a man of mass m aboard stays motionless on the surface of a lake. The man moves a distance l_0 relative to the raft with velocity v_0 and then stops. Assuming the water resistance to be negligible, find the displacement of the raft l relative to the shore. Also calculate the horizontal component of the force which the man acted on the raft during the motion.

SOLUTION

Since, centre of mass will remain stationary, so when man moves l_0 relative to the raft (say forward), then raft moves backwards, say by x . Hence effective displacement of man with respect to ground is $(l_0 - x)$.

$$\Rightarrow m(l_0 - x) = Mx$$

$$\Rightarrow x = \frac{ml_0}{m + M}, \text{ opposite to motion of man.}$$

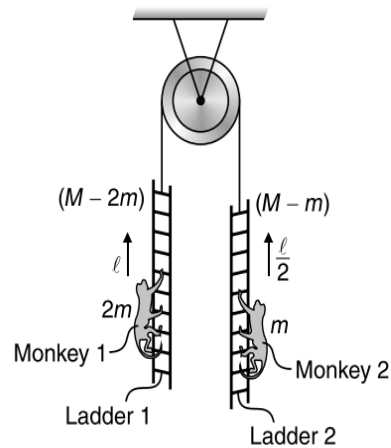
$$\text{Also, } v = \frac{dx}{dt} = \frac{d}{dt} \left(\frac{ml_0}{m + M} \right) = \left(\frac{m}{m + M} \right) \frac{dl_0}{dt}$$

$$\Rightarrow v = \left(\frac{m}{m + M} \right) v_0, \text{ opposite to velocity of man}$$

$$\Rightarrow F = M \frac{dv}{dt} = \left(\frac{Mm}{M + m} \right) \left(\frac{dv_0}{dt} \right)$$

ILLUSTRATION 39

Two ladders are hanging from ends of a light rope that passes over a light and smooth pulley. A monkey of mass $2m$ hangs near the bottom of one ladder whose mass is $M - 2m$. Another monkey of mass m hangs near the bottom of the other ladder whose mass is $M - m$ as shown in Figure.



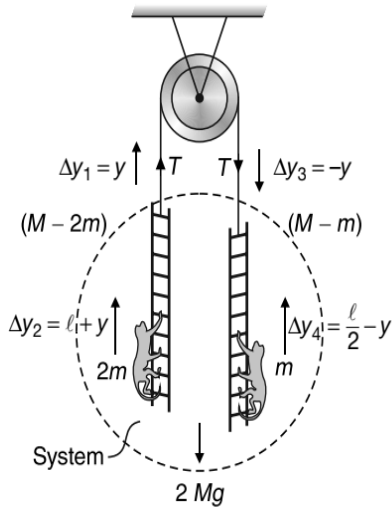
The monkey of mass $2m$ moves up a distance l with respect to the ladder. The monkey of mass m moves up a distance $l/2$ with respect to the ladder. Does the centre of mass of the system change? If so, then calculate the displacement of centre of mass of the system. Also calculate the displacement of ladder.

SOLUTION

In this problem let us consider both the monkeys and the ladders as a system. When the monkeys are at rest, the tension force in the string is balanced by the weight of the system. However, when a monkey starts moving up, then he pulls the respective ladder in downward direction due to which the tension in string increases. Hence, there will be a net upward external force acting on the system and the displacement of centre of mass of the system will be in upward direction.

Let the displacement of Ladder 1 in this process be y upwards, then the displacement of Ladder 2 will

be y downwards (because both the ladders are connected together by a single string). During this time the displacement of Monkey 1 will be $(l + y)$ and that of Monkey 2 will be $\left(\frac{l}{2} - y\right)$ as shown in Figure.



The displacement of the centre of mass of system is given by

$$\Delta y_{\text{cm}} = \frac{m_1 \Delta y_1 + m_2 \Delta y_2 + m_3 \Delta y_3 + m_4 \Delta y_4}{m_1 + m_2 + m_3 + m_4} \quad \dots(1)$$

where, $m_1 = M - 2m$ is the mass of the Ladder 1

$\Delta y_1 = y$ is the displacement of the Ladder 1

$m_2 = 2m$ is the mass of the Monkey 2

$\Delta y_2 = l + y$ is the displacement of the Monkey 2

$m_3 = M - m$ is the mass of Ladder 2

$\Delta y_3 = -y$ is the displacement of Ladder 2

$m_4 = m$ is the mass of Monkey 2

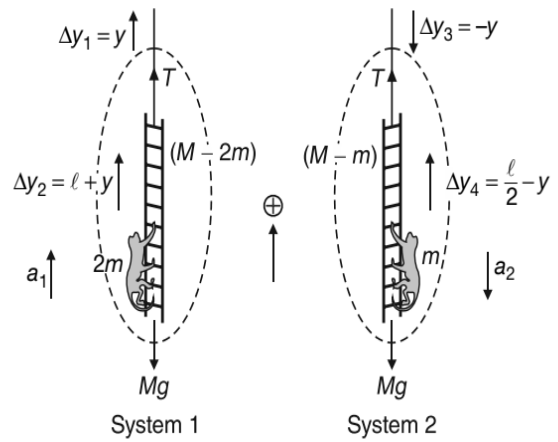
$\Delta y_4 = \frac{l}{2} - y$ is the displacement of Monkey 2

On substituting these values in equation (1), we get

$$\Delta y_{\text{cm}} = \frac{(M - 2m)y + 2m(l + y) - (M - m)y + m\left(\frac{l}{2} - y\right)}{(M - 2m) + 2m + (M - m) + m}$$

$$\Rightarrow \Delta y_{\text{cm}} = \frac{5ml}{4M}$$

To calculate the displacement of ladder, let us consider 'Ladder 1 and Monkey 1' as System 1 and 'Ladder 2 and Monkey 2' as System 2.



Since both the systems have equal mass and equal net external forces, and initially both are at rest, so the displacements of centre of mass of both the systems should be equal.

We can also think of this situation by using Newton's Second Law applied on Systems 1 and 2 separately.

For System 1, we have

$$T - Mg = Ma_1 \quad \dots(2)$$

For System 2, we have

$$Mg - T = Ma_2 \quad \dots(3)$$

Also, both systems are connected by inextensible string, so we have $a_1 = a_2 = a$ (say)

From equations (2) and (3), we get

$$a_1 = a_2 = a = 0$$

$$\Rightarrow a_{\text{cm}} = 0$$

Since initially the system is at rest, so the position of the centre of mass of the system remains fixed along the vertical direction.

$$\Rightarrow (\Delta y_{\text{cm}})_1 = (\Delta y_{\text{cm}})_2$$

$$\Rightarrow \frac{m_1 \Delta y_1 + m_2 \Delta y_2}{m_1 + m_2} = \frac{m_3 \Delta y_3 + m_4 \Delta y_4}{m_3 + m_4}$$

Taking the upward direction as positive, we get

$$\frac{y(M - 2m) + 2m(l + y)}{(M - 2m) + 2m} = \frac{-y(M - m) + m\left(\frac{l}{2} - y\right)}{(M - m) + m}$$

$$\Rightarrow (M - 2m)y + 2m(l + y) = -(M - m)y + m\left(\frac{l}{2} - y\right)$$

$$\Rightarrow My - 2my + 2ml + 2my = -My + my + \frac{ml}{2} - my$$

$$\Rightarrow My + 2ml = -My + \frac{ml}{2}$$

$$\Rightarrow 2My = -\frac{3ml}{2}$$

$$\Rightarrow y = -\frac{3ml}{4M}$$

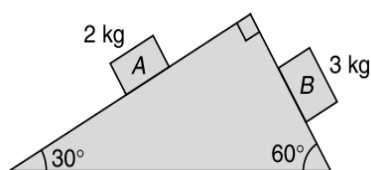
The negative sign indicates that displacement of the ladder is in the downward direction.

Test Your Concepts-II

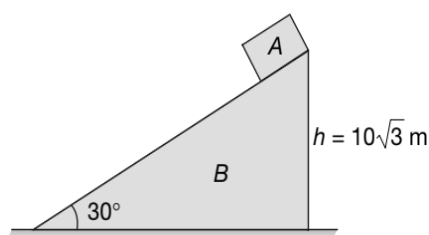
Based on Motion of Centre of Mass

(Solutions on page H.72)

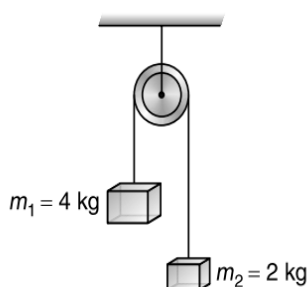
1. A cubical block of ice of mass m_1 and edge L is placed in a large tray of mass m_2 . If the ice melts, how far does the centre of mass of the system ice plus tray come down?
2. Figure shows a fixed wedge on which two blocks of masses 2 kg and 3 kg are placed on its smooth inclined surfaces. When the two blocks are released from rest, find the acceleration of centre of mass of the two blocks.



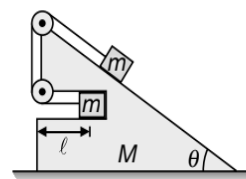
3. Block A has a mass of 5 kg and is placed on the top of smooth triangular block B having a mass of 25 kg. If the system is released from rest, determine the distance B moves, in metre, when A reaches the bottom. Neglect the size of block A.



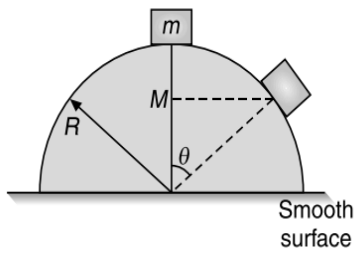
4. Two masses $m_1 = 4$ kg and $m_2 = 2$ kg are released from rest at $t = 0$ as shown in Figure. Calculate the distance travelled by centre of mass and its speed when $t = 9$ s.



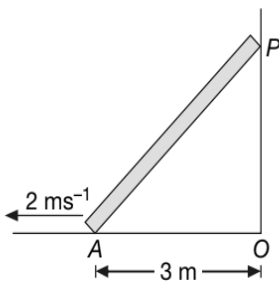
5. A dog of mass 10 kg stands on a stationary boat of mass 40 kg so that he is 20 m from the shore. He then walks 8 m on the boat towards the shore and halts. How far is he from the shore now? Assume that there is no friction between the boat and water.
6. Find the displacement of the wedge when m comes out of the wedge of mass $4m$. Assume friction to be absent everywhere and the base angle of the wedge to be 60° .



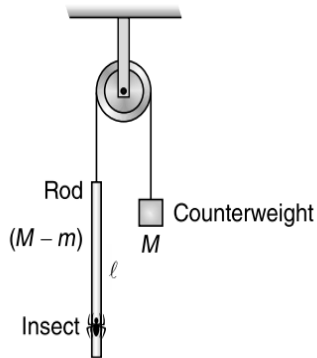
7. Two men A (mass M) and B (mass $2M$) are standing on the two opposite ends in a boat of mass $4M$ and length l on a frictionless water surface. Man A travels a distance of $\frac{l}{4}$ relative to the boat towards its centre and man B moves a distance $\frac{3l}{4}$ relative to boat and meet A. Find the distance travelled by the boat on water till A and B meet.
8. Two balls with masses $m_1 = 3$ kg and $m_2 = 5$ kg have initial velocities $v_1 = v_2 = 5$ ms⁻¹ the directions as shown in the figure. They collide at the origin.
 - (a) Find the velocity of the centre of mass 3 s before the collision
 - (b) Find the position of the centre of mass 2 s after the collision
9. A block is released on the convex surface of a hemispherical wedge as shown in figure. Calculate the displacement of the wedge when the block reaches the angular position θ .



10. A ladder AP of length 5 m inclined to a vertical wall is slipping over a horizontal surface with velocity of 2 ms^{-1} , when A is at distance 3 m from ground. Calculate the velocity of centre of mass at this moment.



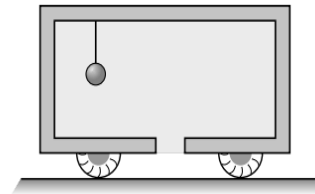
11. A rod of mass $M - m$ and length l carries an insect of mass m at its bottom end. The top end of the rod is connected with a string which passes over a smooth pulley and other end of the string is connected to a counter weight of mass M as shown in Figure.



Initially the insect is at rest. Consider insect, rod and counter weight as a system.

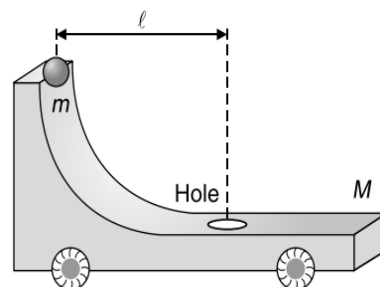
- Calculate the net external force acting on the system initially.
- If the insect starts crawling up on the rod with constant velocity, will the centre of mass of the system accelerate or not?
- If the insect moves up with acceleration, does the centre of mass of the system accelerate?
- Calculate the displacement of centre of mass of the system and that of the rod when insect reaches the other end of the rod.

12. At one instant, the centre of mass of a system of two particles is located on the x -axis at $x = 3 \text{ m}$ and has a velocity of $(6 \text{ ms}^{-1})\hat{j}$. One of the particles is at the origin, the other particle has a mass of 0.1 kg and is at rest on the x -axis at $x = 12 \text{ m}$.
- Calculate the mass of the particle at the origin.
 - Calculate the total momentum of this system.
 - Calculate the velocity of the particle at the origin.
13. A cart of mass M is at rest on a frictionless horizontal surface and a pendulum bob of mass m hangs from the roof of the cart. The string breaks, the bob falls on the floor, makes several collisions on the floor and finally lands in a small slot made on the floor. The horizontal distance between the string and the slot is L . Find the displacement of the cart during this process.

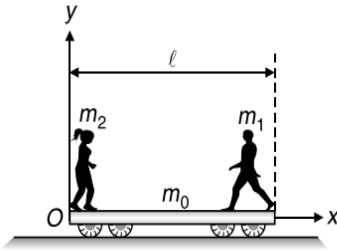


14. A stone is dropped at $t = 0$. A second stone, with twice the mass of the first, is dropped from the same point at $t = 200 \text{ ms}$.
- How far below the release point is the centre of mass of the two stones at $t = 400 \text{ ms}$? (Neither stone has yet reached the ground).
 - How fast is the centre of mass of the two-stone system moving at that time?
- Take $g = 10 \text{ ms}^{-2}$

15. Figure shows a flat car of mass M on a frictionless road. A small massless wedge is fitted on it as shown. A small ball of mass m is released from the top of the wedge, it slides over it and falls in the hole at distance l from the initial position of the ball. Calculate the distance that the flat car moves till the ball gets into the hole.

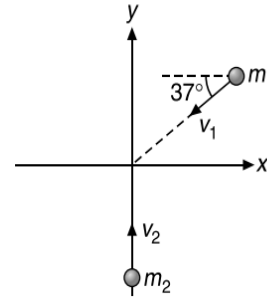


16. Initially a platform at rest at $x=0$ has a man of mass m_1 and a woman of mass m_2 standing at the extreme corners of the platform of mass m_0 and length l as shown in Figure.



The man and the woman start to move towards each other. If x_1 be the displacement of the man relative to the platform, then obtain an expression for the displacement s of the platform as function of x_1 when the man and woman meet.

17. Two balls with masses $m_1 = 3 \text{ kg}$ and $m_2 = 5 \text{ kg}$ have initial velocities $v_1 = v_2 = 5 \text{ ms}^{-1}$ in the directions shown in figure. They collide at the origin.



Calculate the velocity of the centre of mass 3 s before the collision. Also calculate the position of the centre of mass 2 s after the collision.

18. Two particles having masses m and nm start simultaneously from the intersection of two straight line with velocities nv and v respectively. It is observed that the path of their centre of mass is a straight line bisecting the angle θ between the given straight lines. Calculate the magnitude of the velocity of centre of mass.

CENTRE OF MASS REFERENCE FRAME OR CENTROIDAL FRAME OR C-FRAME OR ZERO MOMENTUM FRAME

A reference frame carried by the centre of mass of an isolated system of particles (i.e., a system not subjected to any external forces) is called as the **Centre of Mass Reference Frame or Centroidal Frame or C-frame or zero momentum frame**.

In the centre of mass reference frame or the centroidal frame,

- the position vector of centre of mass is zero, i.e. the origin is taken at the centre of mass.
- the sum of mass moments is zero, i.e. $\sum m_i \vec{r}_{ic} = \vec{0}$
- the velocity of centre of mass of system is zero.
- the linear momentum of centre of mass of system is also zero, i.e. $\sum m_i \vec{v}_{ic} = 0$. This means that neither the internal forces nor the external forces can change the momentum of the system relative to centre of mass.
- the acceleration of centre of mass is also zero.
- for a system of two particles, the momentum of the particles are equal in magnitude but opposite in direction.

KINETIC ENERGY OF A SYSTEM OF PARTICLES

Consider an n particle system having masses $m_1, m_2, m_3, \dots, m_n$ moving with respective velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$. If $\vec{v}_{cm} = \vec{v}_c$ be the velocity of the centre of mass, then with respect to the centre of mass the velocity of each particle is

$$\vec{v}_{1c} = \vec{v}_1 - \vec{v}_c$$

$$\vec{v}_{2c} = \vec{v}_2 - \vec{v}_c$$

$$\vec{v}_{3c} = \vec{v}_3 - \vec{v}_c \text{ and so on.}$$

For some i^{th} particle, we have

$$\vec{v}_{ic} = \vec{v}_i - \vec{v}_c$$

$$\Rightarrow \vec{v}_i = \vec{v}_{ic} + \vec{v}_c \quad \dots(1)$$

So, velocity of each particle can be written as sum of velocity of centre of mass \vec{v}_{cm} and the velocity of particle relative to centre of mass \vec{v}_i . The kinetic energy of this i^{th} particle is given by

$$K_i = \frac{1}{2} m_i \vec{v}_i^2 = \frac{1}{2} m_i (\vec{v}_{ic} + \vec{v}_c)^2$$

The total kinetic energy of the system is obtained by adding (or summing) these energies and hence, we get

$$K = \sum K_i = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i (\vec{v}_i \cdot \vec{v}_i)$$

$$\Rightarrow K = \frac{1}{2} \sum m_i (\vec{v}_{ic} + \vec{v}_c) \cdot (\vec{v}_{ic} + \vec{v}_c)$$

$$\Rightarrow K = \frac{1}{2} (\sum m_i v_c^2 + \sum m_i v_{ic}^2) + (\vec{v}_c \cdot \sum m_i \vec{v}_{ic})$$

In the last term we have removed $\vec{v}_c = \vec{v}_{cm}$ from the summation because it is same for each particle. The quantity $\sum m_i \vec{v}_{ic} = \vec{0}$, because the total momentum of the system relative to the centre of mass is zero.

$$K = \frac{1}{2} (\sum m_i) v_{cm}^2 + \frac{1}{2} \sum m_i v_{ic}^2 = \frac{1}{2} M v_{cm}^2 + K_{rel}$$

$$\Rightarrow K = K_{cm} + K_{rel}$$

where, M is the total mass, K_{cm} is the kinetic energy of the centre of mass of the system, K_{rel} is the kinetic energy of the particles relative to the centre of mass of the system which is also called as the Internal Kinetic Energy of the system. The term K_{rel} may involve translation, rotation or vibration relative to the centre of mass.

So, we conclude that the kinetic energy of a system of particles can be written as the sum of two terms,

- (a) the kinetic energy associated with the motion of centre of mass, $K_{cm} = \frac{1}{2} M v_{cm}^2$, where M is the total mass of the system and
- (b) the kinetic energy associated with the motion of the particles of the system relative to the centre of mass, $K_{rel} = \frac{1}{2} \sum m_i v_{ic}^2$ where, \vec{v}_{ic} is the velocity of the i th particle relative to the centre of mass.

When there is no external force, \vec{v}_{cm} is constant and the kinetic energy associated with bulk motion, i.e.

$K_{cm} = \frac{1}{2} M v_{cm}^2$ does not change. Only the relative kinetic energy (K_{rel}) can change in isolated system.

If a frame is attached to the centre of mass, then kinetic energy of the system is minimum, because $K_{cm} = 0$ and hence

$$K_{system} = K_{min} = K_{rel} = K_{internal}$$

This makes us conclude that the internal kinetic energy of the system does not depend upon the reference frame and is the internal property of the system.

KINETIC ENERGY AND MOMENTUM OF TWO PARTICLE SYSTEM WITH RESPECT TO CENTRE OF MASS

Consider a system of two masses m_1 and m_2 having velocities \vec{v}_1 and \vec{v}_2 . If \vec{v}_{cm} be the velocity of the centre of mass, then we have

$$\vec{v}_{cm} = \vec{v}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \quad \dots(1)$$

Velocity of m_1 with respect to centre of mass is

$$\vec{v}_{1c} = \vec{v}_1 - \vec{v}_c = \left(\frac{m_2}{m_1 + m_2} \right) (\vec{v}_1 - \vec{v}_2) \quad \dots(2)$$

Momentum of m_1 with respect to centre of mass is

$$\vec{p}_{1c} = m_1 \vec{v}_{1c} = \left(\frac{m_1 m_2}{m_1 + m_2} \right) (\vec{v}_1 - \vec{v}_2) = \mu (\vec{v}_1 - \vec{v}_2)$$

$$\Rightarrow \vec{p}_{1c} = \mu (\vec{v}_1 - \vec{v}_2) \quad \dots(3)$$

where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called as the reduced mass of a two particle system.

Velocity of m_2 with respect to the centre of mass is

$$\vec{v}_{2c} = \vec{v}_2 - \vec{v}_c = \left(\frac{m_1}{m_1 + m_2} \right) (\vec{v}_2 - \vec{v}_1) \quad \dots(4)$$

Momentum of m_2 with respect to centre of mass is

$$\vec{p}_{2c} = m_2 \vec{v}_{2c} = \left(\frac{m_1 m_2}{m_1 + m_2} \right) (\vec{v}_2 - \vec{v}_1) = \mu (\vec{v}_2 - \vec{v}_1)$$

$$\Rightarrow \vec{p}_{2c} = \mu (\vec{v}_2 - \vec{v}_1) \quad \dots(5)$$

So, from equations (3) and (5), we get

$$\vec{p}_{1c} + \vec{p}_{2c} = \vec{0}$$

Hence, we see that the total linear momentum of a two particle system with respect to the centre of mass is zero or we can also say that in the centre of mass reference frame, the two particles have momentum equal in magnitude but opposite in direction.

Similarly, we have the kinetic energy of the system with respect to the centre of mass given by

$$K_{cm} = \frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 v_{2c}^2 \quad \dots(6)$$

Substituting the values of \vec{v}_{1c} and \vec{v}_{2c} from equations (2) and (4) in equation (6), we get

$$K_{cm} = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (\vec{v}_1 - \vec{v}_2)^2 = \frac{1}{2} \mu |\vec{v}_{rel}|^2 \quad \dots(7)$$

where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system

and $\vec{v}_{\text{rel}} = \vec{v}_1 - \vec{v}_2$ is the relative velocity of the masses with respect to each other.

If velocities are in same direction, then equation (7) can be re-written as

$$K_{\text{cm}} = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (v_1 - v_2)^2 = \frac{1}{2} \mu |v_{\text{rel}}|^2$$

If velocities are in the opposite direction, then equation (7) can be re-written as

$$K_{\text{cm}} = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (v_1 + v_2)^2 = \frac{1}{2} \mu |v_{\text{rel}}|^2$$

ILLUSTRATION 40

A particle of mass $m_1 = 4 \text{ kg}$ moves at $5\hat{i} \text{ ms}^{-1}$, while $m_2 = 2 \text{ kg}$ moves at $2\hat{i} \text{ ms}^{-1}$. Find

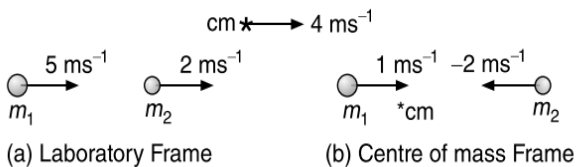
- (a) the kinetic energy of centre of mass and
- (b) the kinetic energy with respect to centre of mass.

SOLUTION

The velocity of the centre of mass is

$$v_{\text{cm}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{(4)(5) + (2)(2)}{4 + 2} = 4 \text{ ms}^{-1}$$

The velocities relative to centre of mass are shown in Figure.



(a) Laboratory Frame (b) Centre of mass Frame
The velocities of two particles and of their center of mass (a) relative to the laboratory and (b) relative to the frame in which the center of mass is at rest.

$$v_{1/\text{cm}} = v_1 - v_{\text{cm}} = 1 \text{ ms}^{-1}$$

$$v_{2/\text{cm}} = v_2 - v_{\text{cm}} = -2 \text{ ms}^{-1}$$

The two terms for the total kinetic energy are

(a) $K_{\text{cm}} = \frac{1}{2} (m_1 + m_2) v_{\text{cm}}^2 = 48 \text{ J}$

(b) $K_{\text{wrt cm}} = \frac{1}{2} m_1 (v_{1/\text{cm}})^2 + \frac{1}{2} m_2 (v_{2/\text{cm}})^2 = 6 \text{ J}$

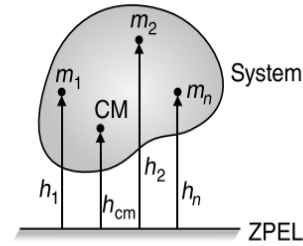
Please note that the kinetic energy with respect to the centre of mass is equal to

$$K_{\text{wrt cm}} = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (u_1 - u_2)^2$$

$$\Rightarrow K_{\text{wrt cm}} = \frac{1}{2} \left(\frac{(4)(2)}{4+2} \right) (5-3)^2 = 6 \text{ J}$$

GRAVITATIONAL POTENTIAL ENERGY OF A SYSTEM OF PARTICLES

Consider a system of particles having masses $m_1, m_2, m_3, \dots, m_n$ at heights or distances $h_1, h_2, h_3, \dots, h_n$ from the Zero Potential Energy Level (ZPEL) as shown in Figure.



For this system of discrete distribution of particles, let us take i^{th} particle of mass m_i situated at a height h_i from ZPEL. Then the gravitational potential energy of m_i is given by

$$U_i = m_i g h_i$$

The total potential energy of the system is obtained by summing up the potential energy of all particles. So, total potential energy of the system is

$$U = \Sigma U_i = \Sigma m_i g h_i = g \Sigma m_i h_i \quad \dots(1)$$

$$\Rightarrow U = Mg \left(\frac{\Sigma m_i h_i}{M} \right) = M g h_{\text{cm}}$$

where $h_{\text{cm}} = \frac{\Sigma m_i h_i}{M}$ is the height of centre of mass of the system from ZPEL.

WORK-ENERGY THEOREM FOR A SYSTEM OF PARTICLES

The work-energy theorem for a system of particles must include the works done by external as well as internal forces. So, we have

$$W_{\text{ext}} + W_{\text{int}} = \Delta K_{\text{cm}} + \Delta K_{\text{rel}}$$

Since all the basic interactions are conservative, one can always express internal work by the change of internal potential energy.

$$W_{\text{int}} = -\Delta U_{\text{int}}$$

$$W_{\text{ext}} = \Delta K_{\text{cm}} + (\Delta K_{\text{rel}} + \Delta U_{\text{int}}) = \Delta K_{\text{cm}} + \Delta E_{\text{int}}$$

So, from this equation, we conclude that external work done on a system of particles can change translational kinetic energy of centre of mass (CM) of system and can change the internal energy of the system (in whatever form possessed by it).

Please note that the internal energy may include the elastic potential energy, gravitational potential energy, thermal energy, electrostatic potential energy, electromagnetic energy, nuclear energy, chemical energy and so on.

WORK ENERGY THEOREM FROM THE CENTRE OF MASS REFERENCE FRAME

Since we know that most important tool for solving mechanics problems is the Work Energy Theorem, which can be applied from any frame of reference. However, when Work-Energy Theorem is applied from the centre of mass reference frame, then we have “the work done by all forces (internal or external) is equal to change in kinetic energy of the system calculated with respect to centre of mass”. Mathematically, we write

$$W_{\text{total}} = \Delta K_{\text{cm}}$$

$$\Rightarrow W_{\text{internal}} + W_{\text{external}} = \Delta K_{\text{cm}}$$

$$\Rightarrow W_{\text{internal}} + W_{\text{external}} = (K_{\text{cm}})_{\text{final}} - (K_{\text{cm}})_{\text{initial}}$$

For a two particle system, we can re-write

$$W_{\text{internal}} + W_{\text{external}} = \frac{1}{2} \mu (v_{\text{rel}}^2 - u_{\text{rel}}^2)$$

where, v_{rel} is the final relative velocity, u_{rel} is the initial relative velocity of the particles and μ is the reduced mass of the two particle system.

TOTAL WORK DONE BY PSEUDO FORCES IN CENTRE OF MASS REFERENCE FRAME OR CENTROIDAL FRAME

Let acceleration of centre of mass relative to an inertial frame is \vec{a}_c . Pseudo force on i th particle of mass m_i in centroidal frame is

$$\vec{F}_{\text{pseudo}} = m_i (-\vec{a}_c) = -m_i \vec{a}_c$$

Let displacement of i th particle in a time interval be $\Delta \vec{r}_i$ relative to the centroidal frame, then total work done by pseudo forces on all the particles in centroidal frame can now be expressed as

$$W = \vec{F}_{\text{pseudo}} \cdot \Delta \vec{r} = (-\Sigma m_i \vec{a}_c) \cdot \Delta \vec{r}_i = -\vec{a}_c \cdot \Sigma (m_i \Delta \vec{r}_i)$$

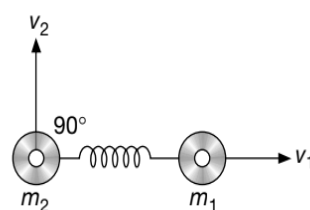
The displacement of the centre of mass in the centroidal frame is zero, so we get

$$W = -\vec{a}_c \cdot \vec{0} = 0$$

So, the total work done in centroidal frame on all the particles of a system by pseudo forces due to acceleration of centre of mass is zero.

ILLUSTRATION 41

Two small discs of masses m_1 and m_2 are connected by a weightless spring resting on a smooth horizontal plane. The discs are set in motion with initial velocities v_1 and v_2 whose directions are mutually perpendicular and in the same horizontal plane. Find the total initial energy E of this system with reference to the frame fixed to the centre of mass.



SOLUTION

Let \vec{v}_{cm} be the velocity of the centre of mass. Then the velocity of m_1 relative to the centre of mass is $\vec{v}_{1/\text{cm}} = \vec{v}_1 - \vec{v}_{\text{cm}}$ and that of m_2 is $\vec{v}_{2/\text{cm}} = \vec{v}_2 - \vec{v}_{\text{cm}}$.

$$E = \frac{1}{2} m_1 |\vec{v}_1 - \vec{v}_{\text{cm}}|^2 + \frac{1}{2} m_2 |\vec{v}_2 - \vec{v}_{\text{cm}}|^2 \quad \dots(1)$$

The velocity of the cm is given by

$$(m_1 + m_2) \vec{v}_{\text{cm}} = m_1 \vec{v}_1 + m_2 \vec{v}_2$$

$$\Rightarrow v_{\text{cm}} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$\Rightarrow \vec{v}_1 - \vec{v}_{\text{cm}} = \frac{m_2 (\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

$$\Rightarrow |\vec{v}_1 - \vec{v}_{\text{cm}}| = \frac{m_2}{m_1 + m_2} |\vec{v}_1 - \vec{v}_2|$$

Since \vec{v}_1 and \vec{v}_2 are perpendicular to each other, so

$$|\vec{v}_1 - \vec{v}_2| = \sqrt{v_1^2 + v_2^2}$$

$$\Rightarrow |\vec{v}_1 - \vec{v}_{\text{cm}}| = \frac{m_2}{m_1 + m_2} |\vec{v}_1 - \vec{v}_2| = \frac{m_2}{m_1 + m_2} \sqrt{v_1^2 + v_2^2}$$

$$\text{Similarly, } |\vec{v}_2 - \vec{v}_{\text{cm}}| = \frac{m_1}{m_1 + m_2} \sqrt{v_2^2 + v_1^2}$$

So, from equation (1), we get

$$E = \frac{1}{2} \left[\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2} \right] (v_1^2 + v_2^2)$$

$$\Rightarrow E = \frac{1}{2} \left[\frac{m_1 m_2 (m_2 + m_1)}{(m_1 + m_2)^2} \right] (v_1^2 + v_2^2)$$

$$\Rightarrow E = \frac{1}{2} \mu (v_1^2 + v_2^2), \text{ where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

The quantity μ is called the reduced mass of the system. In general, for an n particle system the reduced mass is defined as

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \dots + \frac{1}{m_n}$$

This character called as the REDUCED MASS has some unique properties.

(a) The concept of reduced mass is to be introduced for bonded systems where the particles forming the system share a bond (may be physical or chemical) between them and the motion of the system has to be studied with ease. Like the two masses connected with a spring, like CH_4 molecule or H_2O molecule, like a dumbbell like situation, like the double star system (to be studied in Gravitation and Satellites) and many more situations that will be encountered as we keep on studying further.

(b) This quantity has a value smaller than the lightest mass of the system and hence we have used the name reduced mass.

So, in bonded systems we can replace the entire mass of the system by a single mass called the reduced mass μ to make the study of such a system easier.

(c) In the event of a two particle system, if we have a massive and a light body bonded together, then the reduced mass of the system is approximately equal to the mass of the light body. This can be explained from the following argument.

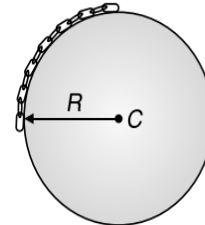
Since for a two particle system we have

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{mM}{m + M}$$

where, m is the mass of the light body and M is the mass of the massive body. Then in that case we have $m + M \approx M$ and hence the above expression for a two particle system reduces to $\mu \approx m$.

ILLUSTRATION 42

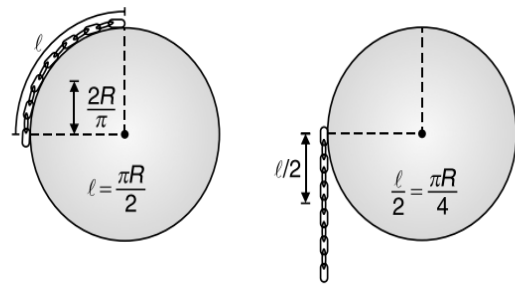
A chain AB of length equal to the quarter of a circle of radius R is placed on a smooth hemisphere as shown in figure. When it is released, it starts falling. Find the velocity of the chain when it falls of the end B from the hemisphere.



SOLUTION

As chain slips off the sphere, fall in its centre of mass height is

$$h = \frac{2R}{\pi} + \frac{\pi R}{4}$$



By work energy theorem, we use

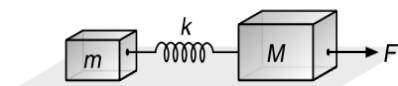
$$mgh = \frac{1}{2} mv^2$$

$$\Rightarrow v = \sqrt{2gh} = \sqrt{2g \left(\frac{2R}{\pi} + \frac{\pi R}{4} \right)}$$

$$\Rightarrow v = \sqrt{gR \left(\frac{4}{\pi} + \frac{\pi}{2} \right)}$$

ILLUSTRATION 43

A block of mass m is connected to another block of mass M by a massless spring of spring constant k . The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.



SOLUTION

Method I:

Let us solve the Problem in a frame of reference fixed to the centre of mass of the system. The centre of mass of the system (two blocks + spring) moves with an acceleration $a = \frac{F}{m+M}$. As this frame is accelerated with respect to the ground, we have to apply a pseudo force ma towards left on the block of mass m and Ma towards left on the block of mass M . The net external force on m is

$$F_1 = ma = \frac{mF}{m+M}, \text{ towards left}$$

and the net external force on M is

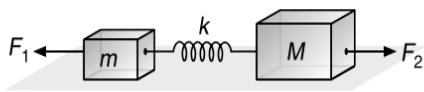
$$F_2 = F - Ma = F - \frac{MF}{m+M}$$

$$\Rightarrow F_2 = \frac{mF}{m+M}, \text{ towards right}$$

Please note that the centre of mass is at rest in this frame (i.e. centroidal frame) and when the blocks move in opposite directions then at maximum extension, the blocks come to instantaneous rest and hence their final relative velocity is zero.

Assuming the left block to be displaced through a distance x_1 and the right block through a distance x_2 from the initial positions, then total work done by the external forces F_1 and F_2 in this period are

$$W_{\text{ext}} = F_1 x_1 + F_2 x_2 = \frac{mF}{m+M} (x_1 + x_2)$$



Since work done by pseudo forces in the centroidal frame is zero, so this work done should be equal to the increase in the potential energy of the spring as there is no change in the kinetic energy.

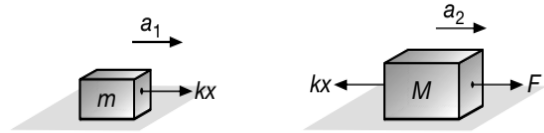
$$\Rightarrow \frac{mF}{m+M} (x_1 + x_2) = \frac{1}{2} k (x_1 + x_2)^2$$

$$\Rightarrow x_1 + x_2 = x_{\text{max}} = \frac{2mF}{k(m+M)}$$

Method II:

As a force F starts pulling M , it accelerates. This increases the length of the spring, developing a restoring force in the spring. Due to this restoring force, m also accelerates towards M . The length of the spring

will increase till the velocity of M with respect to m is directed away from m . When this velocity becomes zero, the elongation acquires a maximum value.



Let at any instant, x be the elongation in the spring. The forces and accelerations are shown here.

$$a_1 = \frac{kx}{m} \text{ and } a_2 = \frac{F - kx}{M}$$

The acceleration of M w.r.t. m is

$$a_r = \frac{F - kx}{M} - \frac{kx}{m}$$

If v_r be the relative velocity of M w.r.t. m , then

$$a_r = \frac{v_r dv_r}{dx}$$

where, x is elongation in the spring.

$$\Rightarrow \frac{v_r dv_r}{dx} = \frac{F}{M} - \left(\frac{k}{m} + \frac{k}{M} \right) x$$

$$\Rightarrow v_r dv_r = \frac{F}{M} dx - k \left(\frac{M+m}{mM} \right) dx$$

$$\Rightarrow \int_0^v v_r dv_r = \int_0^x \frac{F}{M} dx - \int_0^x \frac{k(m+M)}{mM} dx$$

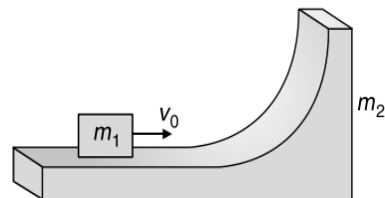
Initially, $v_r = 0$ and when elongation is maximum, then too $v_r = 0$. So, we have

$$\left(\frac{v_r^2}{2} \right) \Big|_0^0 = \frac{F}{M} (x) \Big|_0^{x_m} - \frac{k(m+M)}{Mm} \left(\frac{x^2}{2} \right) \Big|_0^{x_m}$$

$$\Rightarrow x_m = \frac{2mF}{m+M}$$

ILLUSTRATION 44

A block of mass m_1 is projected with a velocity v_0 so that it climbs onto the smooth wedge of mass m_2 . If the block does not leave the wedge, find the maximum height attained by the block.



SOLUTION

According to Work-Energy Theorem applied from CM frame, we have

$$W_{\text{internal}} + W_{\text{external}} = (K_{\text{cm}})_{\text{final}} - (K_{\text{cm}})_{\text{initial}} \dots (1)$$

Initial kinetic energy of system wrt CM is

$$(K_{\text{cm}})_{\text{initial}} = \frac{1}{2} \mu u_{\text{rel}}^2 = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (v_0 - 0)^2$$

$$\Rightarrow (K_{\text{cm}})_{\text{initial}} = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) v_0^2$$

When the block attains a maximum height, then the final relative velocity v_{rel} (between block and wedge) becomes zero, i.e. $v_{\text{rel}} = 0$, so the final kinetic energy of system wrt CM is also zero.

$$\Rightarrow (K_{\text{cm}})_{\text{final}} = \frac{1}{2} \mu v_{\text{rel}}^2 = 0$$

In this case, $W_{\text{int}} = 0$ but $W_{\text{ext}} = -m_1 g h$

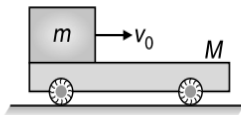
From equation (1), we get

$$-m_1 g h + 0 = -\frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) v_0^2$$

$$\Rightarrow h = \frac{v_0^2}{2 \left(1 + \frac{m_1}{m_2} \right) g}$$

ILLUSTRATION 45

A block of mass m is pushed with a velocity v_0 along the surface of a trolley car of mass M . If the horizontal ground is smooth and the coefficient of kinetic friction between the block and plank is k , then calculate the minimum distance of relative sliding between the block and plank.



SOLUTION

According to Work-Energy Theorem applied from CM frame, we have

$$W_{\text{internal}} + W_{\text{external}} = (K_{\text{cm}})_{\text{final}} - (K_{\text{cm}})_{\text{initial}} \dots (1)$$

Initial kinetic energy of system wrt CM is

$$(K_{\text{cm}})_{\text{initial}} = \frac{1}{2} \mu u_{\text{rel}}^2 = \frac{1}{2} \left(\frac{mM}{m+M} \right) (v_0 - 0)^2$$

$$\Rightarrow (K_{\text{cm}})_{\text{initial}} = \frac{1}{2} \left(\frac{mM}{m+M} \right) v_0^2$$

Finally, when both move together, then the final relative velocity, i.e. $v_{\text{rel}} = 0$, so the final kinetic energy of system wrt CM is zero.

$$\Rightarrow (K_{\text{cm}})_{\text{final}} = \frac{1}{2} \mu v_{\text{rel}}^2 = 0$$

Due to friction, work done is

$$W_{\text{int}} = f_k x \cos(180^\circ) = -kmgx \quad \{ \because \mu_k = k \}$$

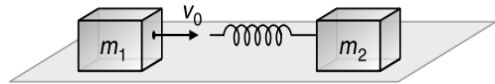
So, from equation (1), we get

$$0 + (-kmgx) = 0 - \frac{1}{2} \left(\frac{mM}{m+M} \right) v_0^2$$

$$\Rightarrow x = \frac{v_0^2}{2 \left(1 + \frac{m}{M} \right) kg}$$

ILLUSTRATION 46

Two smooth blocks of masses m_1 and m_2 attached with an ideal spring of stiffness k and kept on a horizontal surface. If m_1 is projected with a horizontal velocity v_0 , find the maximum compression of the spring by applying the Work-Energy Theorem with respect to the centre of mass frame.



SOLUTION

When seen from the frame attached to the centre of mass, we observe that the linear momentum of system from CM frame is zero and no external force is acting on the system. According to Work-Energy Theorem applied from CM frame, we have

$$W_{\text{internal}} + W_{\text{external}} = (K_{\text{cm}})_{\text{final}} - (K_{\text{cm}})_{\text{initial}} \dots (1)$$

Initial kinetic energy of system wrt centre of mass is

$$(K_{\text{cm}})_{\text{initial}} = \frac{1}{2} \mu u_{\text{rel}}^2 = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (v_0 - 0)^2$$

$$\Rightarrow (K_{\text{cm}})_{\text{initial}} = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) v_0^2$$

Finally, both the blocks move with a common velocity, so the final relative velocity

$$v_{\text{rel}} = 0$$

Hence, the final kinetic energy of system wrt centre of mass is zero.

$$\Rightarrow (K_{\text{cm}})_{\text{final}} = \frac{1}{2} \mu v_{\text{rel}}^2 = 0$$

Since, $W_{\text{int}} = W_{\text{spring}} = -\Delta U = -\frac{1}{2} kx_{\text{max}}^2$

Using equation (1), we get

$$0 + \left(-\frac{1}{2} kx^2\right) = 0 - \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2}\right) v_0^2$$

$$\Rightarrow x = v_0 \sqrt{\frac{1}{k} \left(\frac{m_1 m_2}{m_1 + m_2}\right)} = v_0 \sqrt{\frac{\mu}{k}}$$

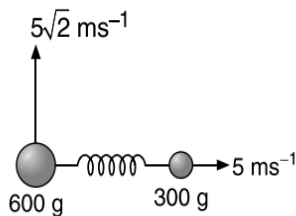


Test Your Concepts-III

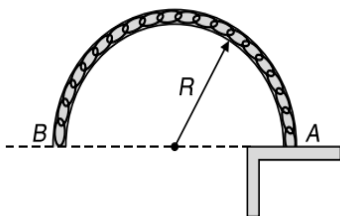
Based on Work Energy Theorem for System of Particles and its Applications from Centre of Mass Reference Frame

(Solutions on page H.76)

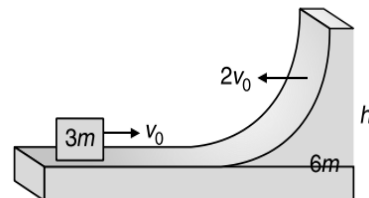
- Two small balls of masses 600 g and 300 g are connected by a light spring resting on a smooth horizontal plane. The balls are set in motion with initial velocities $5\sqrt{3} \text{ ms}^{-1}$ and 5 ms^{-1} whose initial directions are mutually perpendicular and in the same horizontal plane. Calculate the total initial kinetic energy of this system with reference to the centroidal frame.



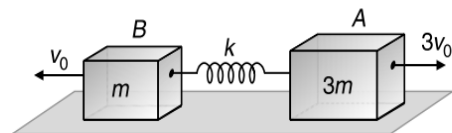
- A heavy, flexible, uniform chain lies in a smooth semi-circular tube AB of radius R . The cross-sectional radius of the tube is much smaller than the radius of the tube (R). Assuming a slight disturbance to start the chain in motion, find the velocity with which it will emerge from the end B of the tube.



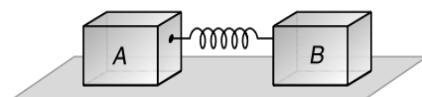
- A block of mass $3m$ is projected to the right with a speed v_0 onto a smooth wedge of mass $6m$ which is simultaneously projected due to the left with a speed $2v_0$. If the particle attains the highest point of the wedge, calculate h .



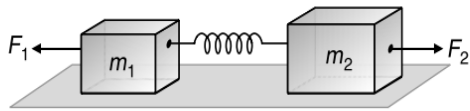
- The block A and B of masses m and $3m$ are connected by an ideal spring of force constant k . Now both the blocks are given impulse in opposite direction so that they started moving with velocities v_0 and $3v_0$ as shown in Figure. Find the maximum stretch of spring.



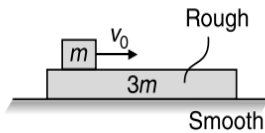
- Two blocks A and B having masses $6m$ and $3m$ respectively lying on a smooth horizontal surface are connected by light spring of spring constant k . If the spring is stretched through x and then released, calculate the relative velocity of the blocks when the spring comes to its natural length.



- Two constant forces F_1 and F_2 act on two smooth blocks having masses m_1 and m_2 interconnected by a light spring of stiffness k placed on a smooth horizontal surface as shown in Figure. Calculate the maximum elongation in the spring using work energy theorem from the centre of mass reference frame.



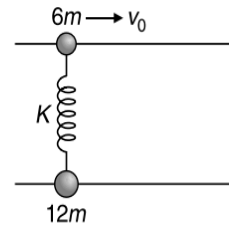
7. A small block of mass m is projected with velocity v_0 on a very long rough plank of mass $3m$ placed on a smooth horizontal surface as shown in Figure. Calculate the work done by friction when slipping stops.



8. A uniform chain is held on a frictionless table with one third of its length hanging over the edge. If the chain has a length l and a mass m , how much

work is required to pull the hanging part back slowly on the table?

9. Two beads having masses $6m$ and $12m$ are connected with a light spring. The beads can slide over two frictionless parallel horizontal rails as shown in Figure.



The lighter bead is given an initial velocity v_0 in horizontal direction. Calculate the maximum extension produced in the spring.

CONSERVATION OF LINEAR MOMENTUM AND COLLISIONS

CONSERVATION OF LINEAR MOMENTUM

According to Newton's Second Law, we have

$$\vec{F}_{\text{ext}} = \vec{F} = \frac{d\vec{p}}{dt}$$

If we observe that $\vec{F}_{\text{ext}} = \vec{0}$

$$\Rightarrow \frac{d\vec{p}}{dt} = \vec{0}$$

$\Rightarrow \vec{p} = \text{constant}$ (both in magnitude and direction)

$$\Rightarrow \sum \vec{p}_{\text{initial}} = \sum \vec{p}_{\text{final}}$$

So, total initial momentum = final momentum (if $\vec{F}_{\text{ext}} = \vec{0}$)

LAW OF CONSERVATION OF LINEAR MOMENTUM

So, in the absence of an external force, the linear momentum of a particle (or the body) remains constant. This is called the **Law of Conservation of Linear Momentum**. The law can be extended to a system of particles or to the centre of mass of a system of particles.

If net force (or the vector sum of all the forces) on a system of particles is zero, the vector sum of linear momentum of all the particles remain conserved. So, when

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots + \vec{F}_n = \vec{0}$$

$$\Rightarrow \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \dots + \vec{P}_n = \text{constant}$$

The same is the case for the centre of mass of a system of particles, i.e., when

$$\vec{F}_{\text{cm}} = \vec{0}, \vec{P}_{\text{cm}} = \text{constant}$$

Thus, the law of conservation of linear momentum can be applied to a single particle, to a system of particles or even to the centre of mass of the particles.

Please note that the law of conservation of linear momentum enables us to solve a number of problems which cannot be solved by a straight application of the relation $\vec{F} = m\vec{a}$.

EXAMPLE:

Suppose a particle of mass m initially at rest, suddenly explodes into two fragments of masses m_1 and m_2 which fly apart with velocities \vec{v}_1 and \vec{v}_2 respectively. Obviously, the forces resulting in the explosion of the particle must be internal forces, as no external force has been applied. In the absence of the external forces, therefore, the momentum must remain conserved and we should have

$$m\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2$$

Since, the particle was initially at rest, so $\vec{v} = \vec{0}$ and therefore,

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

$$\Rightarrow \vec{v}_1 = -\frac{m_2}{m_1}\vec{v}_2$$

$$\Rightarrow \frac{|\vec{v}_1|}{|\vec{v}_2|} = \frac{m_2}{m_1}$$

Showing at once that the velocities of the two fragments must be inversely proportional to their masses and in opposite directions along the same line. This result could not possibly be arrived at from the relation $\vec{F} = m\vec{a}$, since we know nothing about the forces that were acting during the explosion. Nor, could we derive it by using the Law of Conservation of Energy.

Also, note that partial conservation of momentum along either of x , y or z direction does not guarantee total conservation. However, total conservation guarantees partial conservation.

FORCE			MOMENTUM			
F_x	F_y	F_z	p_x	p_y	p_z	\vec{p}
0	0	0	C	C	C	C
0	$\neq 0$	$\neq 0$	C	NC	NC	NC
$\neq 0$	0	$\neq 0$	NC	C	NC	NC
$\neq 0$	$\neq 0$	0	NC	NC	C	NC
= 0	= 0	$\neq 0$	C	C	C	NC

Conserved (C), Not Conserved (NC)

Problem Solving Technique(s)

While solving problems, we will come across situations where a force is acting on a particle/system. However, if on careful observation, we find that along a particular direction, the component of the external force is zero, then along this direction, momentum is conserved. However, total momentum is not conserved.

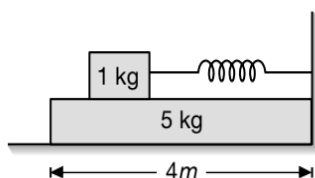
EXAMPLE: Take the case of oblique projectile where the only force acting on the particle is its own weight which has no component along the horizontal. So, momentum of the projectile is conserved along horizontal direction. However, total momentum is not conserved.

However, in situations where some external force is acting on the system, then momentum is not conserved and instead we can also make use of the Impulse-Momentum Theorem to get the desired results.

$$m\vec{v} - m\vec{u} = \int_0^t \vec{F} dt$$

ILLUSTRATION 47

A plank of mass 5 kg is placed on a frictionless horizontal plane. Further a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. The system is now released from the rest. What is the velocity of the plank when block leaves the plank? (The stiffness constant of spring is 100 Nm^{-1}).



SOLUTION

Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively. If M is mass of plank, m is mass of block, then by Law of Conservation of Energy, we get

$$\frac{1}{2} kx^2 = \frac{1}{2} mu^2 + \frac{1}{2} Mv^2$$

$$\Rightarrow 100 = u^2 + 5v^2 \quad \dots(1)$$

By Law of Conservation of Momentum, we get

$$mu + Mv = 0$$

$$\Rightarrow u = -5v \quad \dots(2)$$

Solving equations (1) and (2), we have

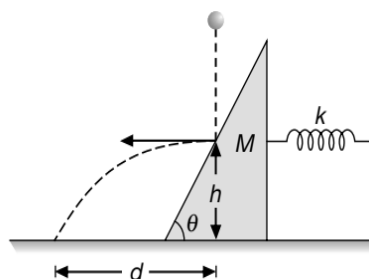
$$30v^2 = 100$$

$$\Rightarrow v = \sqrt{\frac{10}{3}} \text{ ms}^{-1}$$

From this moment until block falls, both plank and block keep their velocity constant. So, when block falls, velocity of plank is $\sqrt{\frac{10}{3}} \text{ ms}^{-1}$.

ILLUSTRATION 48

A ball of mass m when dropped from certain height strikes a wedge kept on smooth horizontal surface and move horizontally just after impact as shown in diagram. If the ball strikes the ground at a distance d from its initial line of fall, determine the amplitude of oscillation of wedge after being hit by the ball.



SOLUTION

$$\text{Since, } h = \frac{1}{2} gt^2$$

$$\Rightarrow t = \sqrt{\frac{2h}{g}}$$

Let v_1 be the horizontal velocity of ball after collision. Then

$$d = v_1 t$$

$$\Rightarrow v_1 = \frac{d}{t} = d \sqrt{\frac{g}{2h}}$$

Suppose, v_2 be the velocity of wedge just after collision. Then from conservation of linear momentum, we have,

$$Mv_2 = mv_1$$

$$\Rightarrow v_2 = \frac{m}{M}v_1 = \frac{md}{M}\sqrt{\frac{g}{2h}}$$

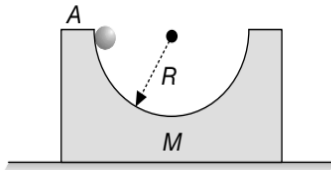
By Law of Conservation of Energy, we have

$$\frac{1}{2}kA^2 = \frac{1}{2}Mv_2^2$$

$$\Rightarrow \text{Amplitude, } A = v_2\sqrt{\frac{M}{k}} = \frac{md}{M}\sqrt{\frac{gM}{2hk}}$$

ILLUSTRATION 49

A block of mass M with a semi-circular track of radius R rests on a horizontal frictionless surface shown in figure. A uniform cylinder of radius r and mass m is released from rest at the point A . The cylinder slips on the semi-circular frictionless track. How far has the block moved when the cylinder reaches the bottom of the track? How fast is the block moving when the cylinder reaches the bottom of the track?



SOLUTION

Applying $x_R m_R = x_L m_L$ we have

$$m(R-r-x) = Mx$$

Here x = displacement of block towards left

$$\Rightarrow x = \frac{m(R-r)}{M+m}$$

By Law of Conservation of Energy, we get

$$\left(\begin{array}{c} \text{Loss in Gravitation} \\ \text{Potential Energy} \\ \text{of Cylinder} \end{array} \right) = \left(\begin{array}{c} \text{Gain in} \\ \text{K.E. of} \\ \text{Cylinder} \end{array} \right) + \left(\begin{array}{c} \text{Gain in} \\ \text{K.E. of} \\ \text{Block} \end{array} \right)$$

$$\Rightarrow mg(R-r) = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 \quad \dots(1)$$

Also, by Law of Conservation of Linear Momentum, along the horizontal we have

$$mv = MV \quad \dots(2)$$

So, from (1) and (2), we get

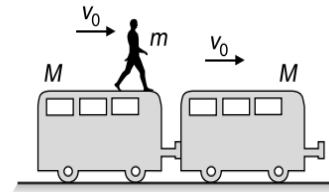
$$V = \frac{m}{M}v = m\sqrt{\frac{2g(R-r)}{M(M+m)}}$$

ILLUSTRATION 50

Two identical buggies move one after the other due to inertia (without friction) with the same velocity v_0 . A man of mass m jumps into the front buggy from the rear buggy with a velocity u relative to his buggy. Knowing that the mass of each buggy is equal to M , find the velocities with which the buggies will move after that.

SOLUTION

If we take rear 'buggy + man' as system. There is no external force acting on system in horizontal direction. Hence linear momentum of the system should be conserved. Let v_R be the velocity of rear buggy just after man of mass m jumps to the front buggy, then velocity of man just after jumping will be $(v_R + u)$.



Initial momentum of rear buggy is

$$p_i = (M+m)v_0$$

When the man jumped into the front buggy with velocity u (relative to rear buggy), then momentum of rear buggy along with the man (in air) is

$$p_f = m(u+v_R) + Mv_R$$

By Law of Conservation of Momentum, we have

$$m(u+v_R) + Mv_R = (M+m)v_0$$

$$\Rightarrow v_R = v_0 - \frac{mu}{(M+m)}$$

Similarly, initial momentum of front buggy along with the man (in air) is

$$p_i = Mv_0 + m(u+v_R)$$

Final momentum of front buggy is

$$p_f = (M+m)v_F$$

By Law of Conservation of Momentum, we have

$$\Rightarrow Mv_0 + m(u+v_R) = (M+m)v_F$$

$$\Rightarrow v_F = \left(\frac{M}{M+m} \right)v_0 + \left(\frac{m}{M+m} \right)(u+v_R)$$

Substituting the value of v_R , we get

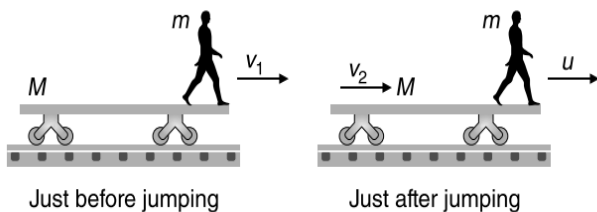
$$v_F = v_0 + \frac{Mmu}{(M+m)^2}$$

ILLUSTRATION 51

A railroad car of mass M with a man of mass m is moving with a velocity v_1 . The man jumps in the direction of motion of car with a velocity u with respect to the car. Find the final velocities of the man and that of the car after jump.

SOLUTION

Taking 'man and car' as a system, we observe that there is no external force acting on this system along the horizontal direction. Due to this, the linear momentum of the system is conserved along the horizontal direction. After the man jumps, the car attains a velocity v_2 in the same direction, which is less than v_1 , due to backward push of the man for jumping. After the jump, the man attains a velocity $(u + v_2)$ in the direction of motion of car with respect to ground.



Applying the Law of Conservation of Linear Momentum, we get

$$(M+m)v_1 = Mv_2 + m(u+v_2)$$

So, velocity of car after jump is

$$v_2 = \frac{(M+m)v_1 - mu}{M+m}$$

and velocity of man after jump is

$$u + v_2 = \frac{(M+m)v_1 + Mu}{M+m}$$

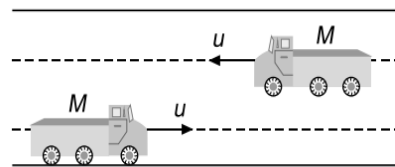
ILLUSTRATION 52

Two trucks each of mass M each (including the content it carries) are moving in opposite direction on adjacent parallel tracks with same velocity u . One truck is carrying potatoes and other is carrying onions. Each bag of potatoes has a mass m_1 and that of onions has a mass m_2 . When trucks get close to each other while passing, the drivers exchange a bag each from their

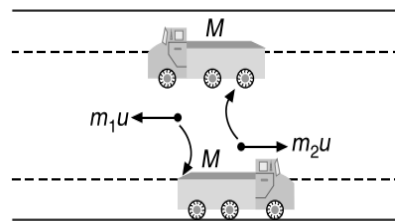
truck by throwing it towards the other one. Find the final velocity of each truck after the exchange of bags.

SOLUTION

At the time of exchange of bags, momentum in the direction of individual motion remains conserved. The situation is shown in Figure.



When a bag of potatoes having mass m_1 is thrown from the first truck in a direction perpendicular to its motion, towards the second truck, then during the throw, the bag has a velocity u in the direction of motion of the first truck as shown in figure.



When a bag of onions having mass m_2 is thrown from the second truck in a direction perpendicular to its motion, towards the first truck, then during the throw, the bag has a velocity u in the direction of motion of the second truck.

Similarly, when the second truck driver throws the onion bag of mass m_2 , towards the first truck, it brings a momentum m_2u in the direction of the second truck.

Applying Conservation of Momentum on first truck carrying potatoes, we get

$$(M - m_1)u - m_2u = (M - m_1 + m_2)v_1$$

So, velocity of first truck after exchange is

$$v_1 = \frac{Mu - m_1u - m_2u}{M - m_1 + m_2}$$

Applying Conservation of Momentum on second truck carrying onions, we get

$$(M - m_2)u - m_1u = (M - m_2 + m_1)v_2$$

So, velocity of second truck after exchange is

$$v_2 = \frac{Mu - m_1u - m_2u}{M - m_2 + m_1}$$

INSTANTANEOUS IMPULSE

There are many occasions when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (i.e., impulse) can be known by measuring the initial and final momenta. Thus, we can write

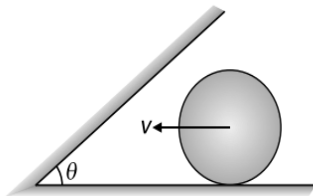
$$\vec{J} = \int \vec{F} dt = \Delta \vec{p} = \vec{p}_f - \vec{p}_i$$

Regarding the impulse it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under force-time ($F-t$) graph in the same time interval.

ILLUSTRATION 53

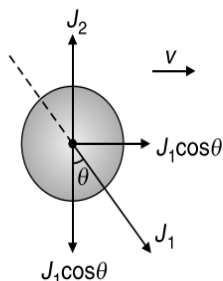
A sphere of mass m slides with velocity v on a frictionless surface towards a smooth inclined wall as shown in Figure. If the collision with the wall is perfectly elastic, find

- the impulse imparted by the wall on the sphere,
- the impulse imparted by the floor on the sphere.



SOLUTION

As the collision is elastic, so the sphere will rebound with the same speed after collision. The impulses acting on the sphere are shown in Figure.



$$J_1 \cos \theta = J_2 \quad \dots(1)$$

$$-mv + J_1 \sin \theta = mv \quad \dots(2)$$

Solving equation (1) and (2), we get

$$J_2 = 2mv \cot \theta$$

$$J_1 = 2mv \operatorname{cosec} \theta$$

IMPULSIVE TENSIONS

When a string jerks, equal and opposite tensions act suddenly at each end. Consequently, equal and opposite impulses act on the objects to which the two ends of the string are attached. There are two cases to be considered.

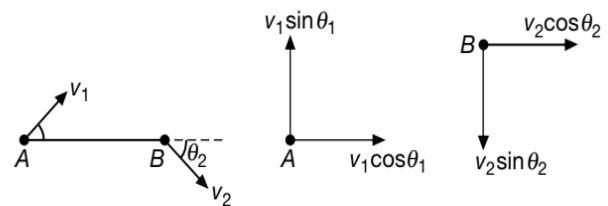
CASE-1: One end of the string is fixed

The impulse which acts at the fixed end of the string cannot change the momentum of the fixed object there. The object attached to the free end however will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string where no impulsive forces act.

CASE-2: Both ends of string attached to movable objects

In this case equal and opposite impulses act on the two objects, producing equal and opposite changes in momentum. The total momentum of the system therefore remains constant, although the momentum of each individual object is changed in the direction of the string. Perpendicular to the string however, no impulse acts and the momentum of each particle in this direction is unchanged.

The velocities of two objects moving at the ends of a taut string are not independent. The relationship between them is understood by taking a taut string AB and particles A, B which are moving with velocities as shown in the Figure. The important components of velocity are those along the string AB .



If $v_1 \cos \theta_1 > v_2 \cos \theta_2$ the string is not taut and

If $v_2 \cos \theta_2 > v_1 \cos \theta_1$ the string is snapped i.e., it breaks

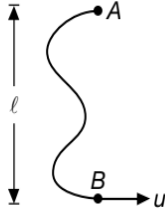
Hence, for the string to remain taut and unbroken, we must have

$$v_1 \cos \theta_1 = v_2 \cos \theta_2$$

Hence, we conclude that the two ends of a taut string have equal velocity components in the direction of the string.

ILLUSTRATION 54

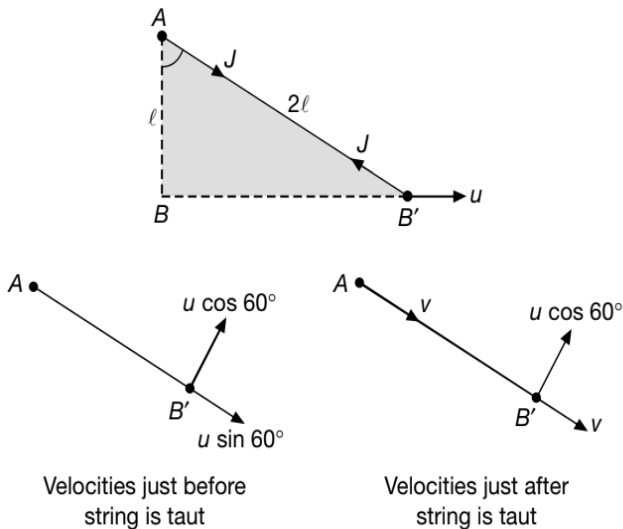
Two particles A and B of equal mass m each are attached by a string of length $2l$ and initially placed over a smooth horizontal table in the position shown in figure. Particle B is projected across the table with speed u perpendicular to AB as shown in figure.



Find the velocities of each particle after the string becomes taut and the magnitude of the impulsive tension.

SOLUTION

When the string becomes taut, both particles begin to move with equal velocity component v along the direction AB'



Perpendicular to AB' there is no impulse on either particle, velocity components in this direction are therefore unchanged.

Applying the Law of Conservation of Momentum in the direction AB' , we get

$$0 + mu \sin(60^\circ) = mv + mv$$

$$\Rightarrow v = \frac{\sqrt{3}}{4} u$$

So, just after the string becomes taut, we have

(a) Velocity of mass A is $v_A = \frac{\sqrt{3}}{4} u$ along AB'

(b) Velocity of mass B is

$$v_B = \sqrt{(u \cos 60^\circ)^2 + \left(\frac{\sqrt{3}}{4} u\right)^2} = \frac{\sqrt{7}}{4} u$$

In a direction inclined to AB' at $\tan^{-1}\left(\frac{u \cos 60^\circ}{v}\right)$, i.e., at $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

The magnitude of impulsive tension (J) is found by calculating the change in momentum of one of the particles.

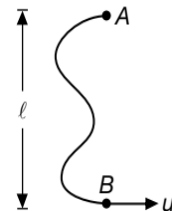
For the mass A , in the direction AB' , we have

$$J = \Delta p_A = mv - 0$$

$$\Rightarrow J = \frac{\sqrt{3}}{4} mu$$

ILLUSTRATION 55

A string AB of length $2l$ is fixed at A to a point on a smooth horizontal table. A particle of mass m attached to B is initially at a distance l from A as shown in figure.

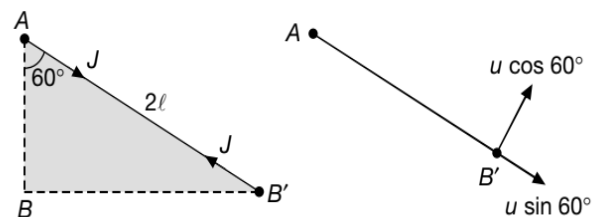


The particle is projected horizontally with speed u at right angles to AB . Find the impulsive tension J in the string when it becomes taut and the velocity of the particle immediately afterwards.

SOLUTION

When the string becomes taut $AB' = 2l$

$$\Rightarrow \angle B'AB = 60^\circ$$



Just before the string becomes taut, the components of the velocity of particle parallel and perpendicular

to AB' are $u \sin(60^\circ)$ and $u \cos(60^\circ)$ respectively. When the string becomes taut the length of AB is fixed and particle cannot travel further along the direction AB' . So, after the string becomes taut, the velocity of the particle is, therefore, perpendicular to AB' (as it cannot move any further along AB'). Since, we know that Impulse = Change in Momentum, so

(a) Along $B'A$, we have

$$J = 0 - (-mu \sin(60^\circ))$$

$$\Rightarrow J = \frac{\sqrt{3}}{2} mu$$

(b) Perpendicular to $B'A$ (no impulsive component)

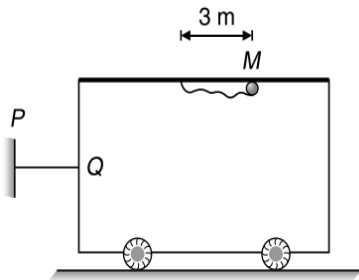
$$\Rightarrow 0 = mv - mu \cos(60^\circ)$$

$$\Rightarrow v = u \cos(60^\circ) = \frac{u}{2}$$

Hence, the velocity of the particle just after the string becomes taut is $\frac{u}{2}$, perpendicular to the string.

ILLUSTRATION 56

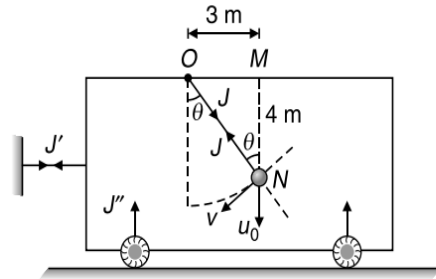
A particle of mass $\sqrt{5}$ kg is attached to a string of length 5 m attached to a fixed point O . The particle is released from the position shown in Figure.



Calculate the impulse developed in the string when it becomes taut, the velocity of the particle just after the string becomes taut and the impulse developed in this string PQ at this instant.

SOLUTION

When ball is released from rest from M then it falls to N , a distance 4 m below M , due to which an impulse will be developed in the string ON as shown in Figure.



Velocity of ball just before string becomes taut is

$$u = \sqrt{2 \times 10 \times 4} = \sqrt{80} \text{ ms}^{-1} = 4\sqrt{5} \text{ ms}^{-1}$$

Velocity of ball in direction perpendicular to the impulse will remain the same just before and after and the impulse is developed.

$$v = u \sin \theta = (4\sqrt{5}) \left(\frac{3}{5} \right) = \frac{12}{\sqrt{5}} \text{ ms}^{-1}$$

$$\Rightarrow J = mu \cos \theta = \sqrt{5} (4\sqrt{5}) \left(\frac{4}{5} \right) = 16 \text{ kgms}^{-1}$$

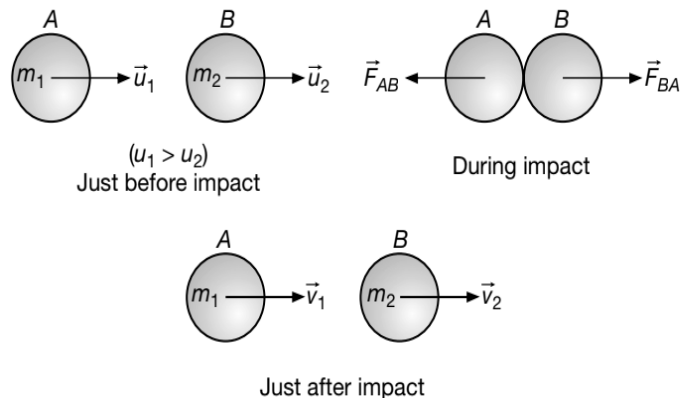
$$\Rightarrow J' = J \sin \theta = (16) \left(\frac{3}{5} \right) = \frac{48}{5} \text{ kgms}^{-1}$$

and impulse imparted by earth is

$$J'' = J \cos \theta = (16) \left(\frac{4}{5} \right) = \frac{64}{5} \text{ kgms}^{-1}$$

CONSERVATION OF LINEAR MOMENTUM FOR A TWO PARTICLE SYSTEM

Consider two bodies A and B having masses m_1 and m_2 moving with initial velocities \vec{u}_1 and \vec{u}_2 (as shown).



Let the bodies collide such that during impact a force \vec{F}_{AB} acts on A due to B and a force \vec{F}_{BA} acts on B due to A . Then by Newton's Third Law

$$\vec{F}_{AB} = -\vec{F}_{BA} \quad \dots(1)$$

Let \vec{v}_1 and \vec{v}_2 be the velocities of the two bodies just after impact. Let us now calculate the change in momentum of A and B i.e., $\Delta\vec{p}_A$ and $\Delta\vec{p}_B$.

For A, we have

$$\Delta\vec{p}_A = m_1\vec{v}_1 - m_1\vec{u}_1 = \int_0^t \vec{F}_{AB} dt \quad \dots(2)$$

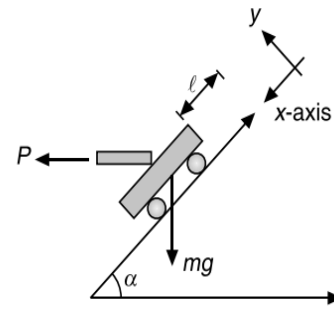
For B, we have

$$\Delta\vec{p}_B = m_2\vec{v}_2 - m_2\vec{u}_2 = \int_0^t \vec{F}_{BA} dt \quad \dots(3)$$

For A+B system, by adding (2) and (3), we get

$$m_1\vec{v}_1 - m_1\vec{u}_1 + m_2\vec{v}_2 - m_2\vec{u}_2 = \int_0^t \vec{F}_{AB} dt + \int_0^t \vec{F}_{BA} dt$$

$$\Rightarrow m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$



$$v^2 - 0^2 = 2(g \sin \alpha)l \quad \{ \because a = g \sin \alpha \}$$

$$\Rightarrow v = \sqrt{2gl \sin \alpha}$$

Initial momentum along x-axis is $(p_i)_x = mv$

Final momentum along x-axis is $(p_f)_x = P \cos \alpha$

Change in momentum along x-axis i.e., $\Delta p_x = (P \cos \alpha - mv)$

Since, Force = Rate of Change of Momentum

$$\Rightarrow mg \sin \alpha = \frac{(P \cos \alpha - mv)}{t}$$

$$\Rightarrow t = \frac{P \cos \alpha - mv}{mg \sin \alpha} = \frac{P \cos \alpha - m\sqrt{2gl \sin \alpha}}{mg \sin \alpha}$$

Please note that the normal reaction will have no component along x-axis.

Conceptual Note(s)

CHOICE OF A SYSTEM

We observe that momentum is not conserved for A (because of force \vec{F}_{AB} acting on it) and also not conserved for B (because of force \vec{F}_{BA} acting on it). However, for A+B **system net external force acting is zero**, hence momentum of (A+B) system is conserved.

ILLUSTRATION 57

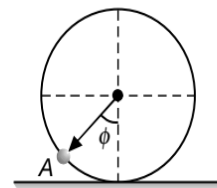
A cannon of mass m starts sliding freely down a smooth inclined plane at an angle α to the horizontal. After the cannon covered the distance l , a shot was fired, the shell leaving the cannon in the horizontal direction with a momentum P . As a consequence, the cannon is stopped. Assuming the mass of the shell to be negligible, as compared to that of the cannon, determine the duration of the shot.

SOLUTION

To start with, let us calculate the velocity of the cannon when the shot was fired.

ILLUSTRATION 58

A thin hoop of mass M and radius r is placed on a horizontal plane. At the initial instant, the hoop is at rest. A small washer of mass m with zero initial velocity slides from the upper point of the hoop along a smooth groove in the inner surface of the hoop. Determine the velocity u of the centre of the hoop at the moment when the washer is at a certain point A of the hoop, whose radius vector forms an angle ϕ with the vertical (figure). The friction between the hoop and the plane should be neglected.



SOLUTION

Let v_r be the velocity of washer relative to centre of hoop and u be the velocity of centre of hoop. Applying Law of Conservation of Linear Momentum along the horizontal, we get

$$m(v_r \cos \phi - u) = Mu \quad \dots(1)$$

Applying Law of Conservation of Mechanical Energy, we get

$$\left(\begin{array}{c} \text{Loss in} \\ \text{GPE of} \\ \text{Washer} \end{array} \right) = \left(\begin{array}{c} \text{Gain in} \\ \text{KE of} \\ \text{Hoop} \end{array} \right) + \left(\begin{array}{c} \text{Gain in} \\ \text{KE of} \\ \text{Washer} \end{array} \right)$$

$$\Rightarrow mgr(1 + \cos \phi) = \frac{M}{2}u^2 + \frac{m}{2}(v_r^2 + u^2 - 2uv_r \cos \phi) \quad \dots(2)$$

Solving equations (1) and (2), we get

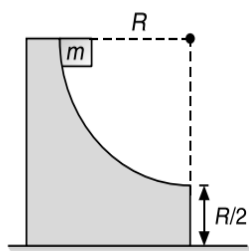
$$u = m \cos \phi \sqrt{\frac{2gr(1 + \cos \phi)}{(M + m)(M + m \sin^2 \phi)}}$$

Test Your Concepts-IV

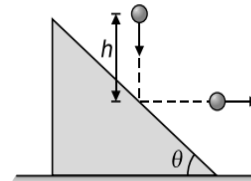
Based on Conservation of Linear Momentum

(Solutions on page H.78)

1. A man of mass m_1 is standing on a platform of mass m_2 kept on a smooth horizontal surface. The man starts moving on the platform with a velocity v_r relative to the platform. Find the recoil velocity of platform.
2. A railroad car of mass M is at rest on a frictionless track with a man of mass m standing at its edge. If the man jumps off from the car towards right with an initial velocity u , with respect to the car, find the velocity of the car after its jump.
3. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M , as shown in figure. M rests on a table and both blocks move without friction. The blocks are initially at rest and m starts from the top of the path. Find the horizontal distance from the bottom of block when cube hits the table.



4. A particle of mass m is released from height h on an inclined plane. If after collision with the plane, the particle starts moving horizontally as shown in Figure.



Calculate the velocity of particle just after collision, impulse provided by the plane to the particle and loss in kinetic energy of the particle due to impact.

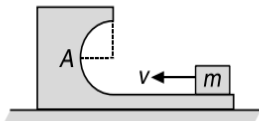
5. A 50 N projectile is fired from ground level with an initial velocity of 24 ms^{-1} making an angle of 60° with the horizontal. When it reaches its highest point, it explodes into two 25 N fragments. If one fragment is seen to travel vertically upward, and after they fall, they are 45 m apart, calculate the speed of each fragment just after the explosion. Neglect the size of the gun.
6. A projectile is launched at an angle θ_0 with velocity v_0 . At the highest point of its path, it explodes in two parts having mass ratio 1 : 3. If the lighter mass retraces its path, then calculate velocity of the heavier part just after the explosion.
7. A wagon of mass M can move without friction along horizontal rails. A simple pendulum consisting of a sphere of mass m is suspended from the ceiling of the wagon by a string of length ℓ . At the initial moment the wagon and the pendulum are at rest and the string is deflected

through an angle α from the vertical. Find the velocity of the wagon when the pendulum passes through its mean position.

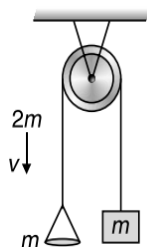
8. Two particles, each of mass m , are connected by a light inextensible string of length $2l$. Initially they lie on a smooth horizontal table at points A and B distant l apart. The particle at A is projected across the table with velocity u . Find the speed with which the second particle begins to move, if the direction of u is
- along the line BA ,
 - at an angle of 120° with AB
 - perpendicular to AB .

In each case also calculate (in terms of m and u) the impulsive tension in the sting.

9. Figure shows a small block of mass m which is started with a speed v on the horizontal part of the bigger block of mass M placed on a horizontal floor. The curved part of the surface shown is semi-circular. All the surfaces are frictionless. Find the speed of the bigger block when the smaller block reaches the point A of the surface.



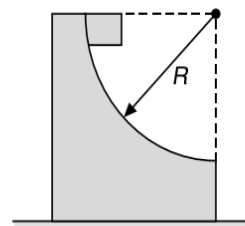
10. A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially, the system is at rest, then a particle of mass $2m$ falls on the pan and sticks to it. If the particle strikes the pan with a speed v , calculate the speed with which the system moves just after the collision.



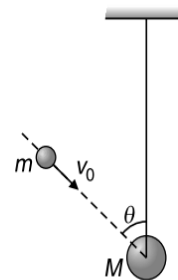
11. Two men, each of mass m stand on the edge of a stationary buggy of mass M . Assuming the friction to be negligible, calculate the velocity of the buggy after both men jump off with the same horizontal velocity u relative to the buggy,

- simultaneously
- one after the other

12. A shell of mass m is fired from a gun of mass km which can recoil freely on a horizontal plane, the elevation of the gun is 45° . Find the ratio of the energy of the shell to that of the gun.
13. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M as shown in figure. M rests on a table and both blocks move without friction. The blocks are initially at rest and m starts from top of the path. Find the velocity v of the cube as it leaves the block.

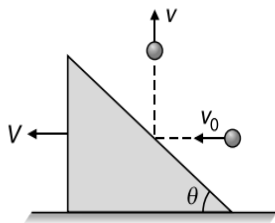


14. A bob of mass M is suspended by an inextensible string. A particle of mass m , moving with velocity v_0 strikes the bob as shown in Figure.



If the particle comes to rest just after collision, then calculate impulse due to tension in the string, velocity of the bob just after collision.

15. A heavy mass M resting on the ground is attached to a small mass m via massless inextensible string passing over a pulley. The string connected to M is loose. The smaller mass falls freely through a height h and the string becomes taut. Find the time from this instant when the heavier mass again makes contact with the ground. Also obtain the loss in kinetic energy when M is jerked into motion.
16. A right-angled wedge of mass M and base angle θ is lying on a smooth horizontal surface. A small ball of mass m , moving horizontally with velocity v_0 collides with wedge on inclined surface as shown in Figure.

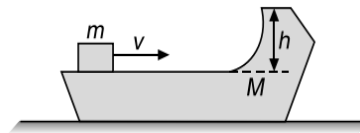


If after collision the ball starts moving in vertical, upward direction, then calculate the impulse due to normal reaction applied by wedge on the ball, impulse due to normal reaction applied by ground on wedge, velocity of the ball and wedge just after collision. If coefficient of friction between ground and wedge is μ , then calculate impulse due to frictional force.

17. A bullet of mass 0.25 kg is fired with velocity 302 ms^{-1} into a block of wood of mass $m_1 = 37.5 \text{ kg}$. It gets embedded into it. The block m_1 is resting

on a long block m_2 of mass $m_2 = 12.5 \text{ kg}$ and the horizontal surface on which it is placed is smooth. The coefficient of friction between m_1 and m_2 of 0.5. Find the displacement of m_1 on m_2 and the common velocity of m_1 and m_2 .

18. A body of mass M with a small box of mass m placed on it rests on a smooth horizontal surface. The box is set in motion in the horizontal direction with a velocity u as shown in Figure. Calculate the height relative to the initial level to which the box will rise after breaking off from the body M . Assume all surfaces to be smooth.



VARIABLE MASS

In our discussion of the conservation of linear momentum, we have so far dealt with systems whose mass remains constant. We now consider those systems whose mass is variable, i.e., those in which mass enters or leaves the system. A typical case is that of the rocket from which hot gases keep on escaping, thereby continuously decreasing its mass.

In such problems we apply a thrust force (\vec{F}_t) to the main mass in addition to the all other forces acting on it. This thrust force is given by,

$$\vec{F}_t = \vec{v}_{\text{rel}} \left(\frac{dm}{dt} \right)$$

where, \vec{v}_{rel} is the velocity of the mass gained or **mass ejected relative to the main mass**. In case of rocket this is also called the **exhaust velocity** of the gases, also $\frac{dm}{dt}$ is the rate at which mass is increasing or decreasing.

The expression for the thrust force can be derived from the Law of Conservation of Linear Momentum in the absence of any external forces on a system.

Let at some time t , the mass of the body be m and its velocity be \vec{v} . After some infinitesimal time say $t + dt$ its mass becomes $(m - dm)$ and velocity becomes $\vec{v} + d\vec{v}$. Let the mass dm be ejected with relative velocity \vec{v}_r . Absolute velocity of mass dm is therefore $(\vec{v}_r + \vec{v} + d\vec{v})$. If no external forces are acting on the system, then linear momentum of the system will remain conserved, i.e.,

$$\begin{aligned} \Rightarrow \vec{p}_i &= \vec{p}_f \\ \Rightarrow m\vec{v} &= (m - dm)(\vec{v} + d\vec{v}) + dm(\vec{v}_r + \vec{v} + d\vec{v}) \\ \Rightarrow m\vec{v} &= m\vec{v} + md\vec{v} - \vec{v}dm - dmd\vec{v} + \\ &\quad \vec{v}dm + \vec{v}_r dm + dmd\vec{v} \\ \Rightarrow md\vec{v} &= -\vec{v}_r dm \\ \Rightarrow m \left(\frac{d\vec{v}}{dt} \right) &= \vec{v}_r \left(-\frac{dm}{dt} \right) \end{aligned}$$

where, $m \left(\frac{d\vec{v}}{dt} \right) = \text{Thrust Force } (\vec{F}_t)$

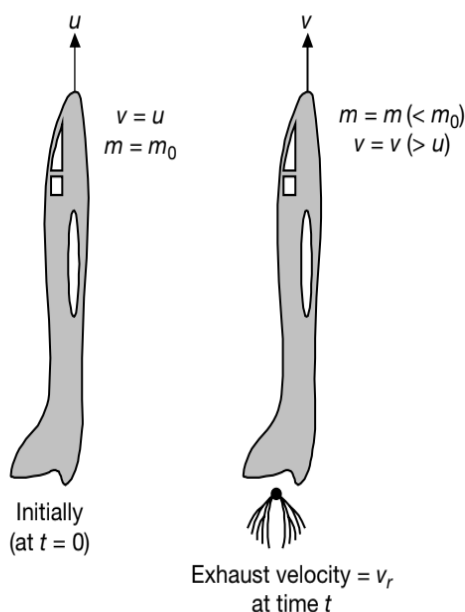
and $-\frac{dm}{dt} = \text{Rate at which mass is being ejected from system}$

Problem Solving Technique(s)

- (a) Make a list of all the forces acting on the main mass and apply them on it.
- (b) Apply an additional thrust force \vec{F}_t on the mass, the magnitude of which is $\left| \vec{v}_r \left(\pm \frac{dm}{dt} \right) \right|$.
- (c) The direction of this thrust force \vec{F}_t is given by the direction of \vec{v}_r **in case the mass is increasing** else it is the direction of $-\vec{v}_r$ if the mass is decreasing.
- (d) Calculate the net force on the mass and apply $\vec{F}_{net} = m \frac{d\vec{v}}{dt}$ (m = mass at that particular instant).
- (e) Integrate it within proper limits to find velocity at any time t .

ROCKET PROPULSION

Let m_0 be the mass of the rocket initially (i.e., at time $t=0$) and m be its mass at any later time t . Let v be its velocity at time t let the velocity of the rocket be u at $t=0$.



Further, let $-\frac{dm}{dt}$ be the mass of the gas ejected per unit time and v_r be the exhaust velocity of the gases.

Usually $\left(-\frac{dm}{dt} \right)$ and v_r are kept constant throughout the journey of the rocket. Few equations which can be used in the problems of rocket propulsion. At time $t = t$. The forces that act on the rocket, at time t (say) are

- (a) Thrust force on the rocket due to ejection of gases is

$$F_t = v_r \left(-\frac{dm}{dt} \right) \quad \{\text{upwards}\}$$

- (b) Weight of the rocket is

$$W = mg \quad \{\text{downwards}\}$$

- (c) So, net force acting on the rocket is

$$F_{net} = F_t - W \quad \{\text{upwards}\}$$

$$\Rightarrow F_{net} = v_r \left(-\frac{dm}{dt} \right) - mg$$

- (d) Net acceleration of the rocket

$$a = \frac{F}{m}$$

Since, $F_{net} = m \frac{dv}{dt}$

$$\Rightarrow m \frac{dv}{dt} = -v_r \frac{dm}{dt} - mg$$

$$\Rightarrow \frac{dv}{dt} = \frac{v_r}{m} \left(-\frac{dm}{dt} \right) - g$$

$$\Rightarrow dv = v_r \left(-\frac{dm}{m} \right) - g dt$$

$$\Rightarrow \int_u^v dv = v_r \int_{m_0}^m -\frac{dm}{m} - g \int_0^t dt$$

$$\Rightarrow v - u = v_r \ln \left(\frac{m_0}{m} \right) - gt$$

$$\Rightarrow v = (u - gt) + v_r \ln \left(\frac{m_0}{m} \right)$$



Conceptual Note(s)

- (a) $F_t = v_r \left(-\frac{dm}{dt} \right)$ is upwards, as v_r is downwards

and $\frac{dm}{dt}$ is negative.

- (b) If gravity is ignored and initial velocity of the rocket $u=0$, then we get $v = v_r \ln \left(\frac{m_0}{m} \right)$.

- (c) There can be many cases in which the mass of system varies with time. In all the cases of mass variation, we use momentum conservation at an intermediate instant from time t to $t + dt$.

ILLUSTRATION 59

A rocket, with an initial mass of 1000 kg, is launched vertically upwards from rest under gravity. The rocket burns fuel at the rate of 10 kg s^{-1} . The burnt matter is ejected vertically downwards with a speed of 2000 ms^{-1} relative to the rocket. If burning ceases after one minute, find the maximum velocity of the rocket. (Assume g to be constant and have a value 10 ms^{-2}).

SOLUTION

Since, we know that

$$v = (u - gt) + v_r \ln\left(\frac{m_0}{m}\right)$$

According to the problem, we have $u = 0$, $t = 60 \text{ s}$, $g = 10 \text{ ms}^{-2}$, $v_r = 2000 \text{ ms}^{-1}$, $m_0 = 1000 \text{ kg}$

and $m = 1000 - 10 \times 60 = 400 \text{ kg}$

$$\Rightarrow v = 0 - 600 + 2000 \ln\left(\frac{1000}{400}\right)$$

$$\Rightarrow v = 2000 \ln(2.5) - 600$$

The maximum velocity of the rocket is

$$v = 200(10 \ln 2.5 - 3) = 1232.6 \text{ ms}^{-1}$$

ILLUSTRATION 60

A double stage rocket has an initial mass M_1 . Gas is exhausted from the rocket at a constant rate of $\frac{dm}{dt}$ and with an exhaust velocity u relative to the rocket. When the mass of the rocket reaches the value M , the first stage of mass m of which fuel is exhausted is disengaged from the rocket. Then rocket continues to the second stage at the same rate and exhaust velocity as in the first stage, until it reaches a mass M_2 . Calculate the rocket velocity at the end of the first and second stage (if the rocket had started from rest). Also calculate the final velocity of a single stage rocket of same initial mass M_1 and having same amount of fuel. Is this velocity greater or less than final velocity of the double stage rocket?

SOLUTION

The rocket has constant exhaust velocity given by

$\frac{dm}{dt}$ during its motion. Since, we know that

$$m \frac{dv}{dt} = -v_r \frac{dm}{dt}$$

$$\Rightarrow \int_0^{v_1} dv = -u \int_{M_1}^M \frac{dm}{m}$$

$$\Rightarrow v_1 = u \ln\left(\frac{M_1}{M}\right)$$

After a mass m is disengaged, the rocket mass changes to $(M - m)$ at the start of second stage and hence we have

$$\int_{v_1}^{v_2} dv = -u \int_{M-m}^{M_2} \frac{dm}{m}$$

$$\Rightarrow v_2 - v_1 = u \ln\left(\frac{M - m}{M_2}\right)$$

$$\Rightarrow v_2 = u \left[\ln\left(\frac{M_1}{M}\right) + \ln\left(\frac{M - m}{M_2}\right) \right]$$

$$\Rightarrow v_2 = u \ln\left[\frac{M_1(M - m)}{MM_2}\right]$$

However, if there is only single stage of mass M_1 , then the final mass after exhaust of fuel will be $(M_2 + m)$. Hence, we have

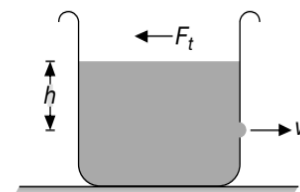
$$\int_0^{v_f} dv = -u \int_{M_1}^{M_2+m} \frac{dm}{m}$$

$$\Rightarrow v_f = u \ln\left(\frac{M_1}{M_2 + m}\right)$$

So, we observe that $v_2 > v_f$.

LIQUID COMING OUT OF ORIFICE IN A BEAKER

A liquid of density ρ is filled in a container as shown in Figure.



The liquid comes out from the container through a orifice of area a at a depth h below the free surface of the liquid with a velocity v . This exerts a thrust force in the container in the backward direction. This thrust force is given by

$$F_t = v_r \left(-\frac{dm}{dt} \right)$$

Here, $v_r = v$ {in forward direction}

and $\left(-\frac{dm}{dt} \right) = \rho av$

Since, $\left(\frac{dV}{dt} \right) = \text{Volume of liquid flowing per second}$

$$\Rightarrow \left(\frac{dV}{dt} \right) = av$$

$$\Rightarrow \left(-\frac{dm}{dt} \right) = \rho \left(\frac{dV}{dt} \right) = \rho av$$

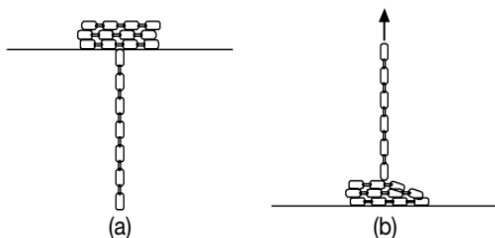
$$\Rightarrow F_t = v(\rho av)$$

$$\Rightarrow F_t = \rho av^2 \quad \text{\{in backward direction\}}$$

Further, we will see in the chapter of fluid mechanics that $v = \sqrt{2gh}$.

CHAIN FALLING THROUGH A HOLE OR CHAIN BEING LIFTED VERTICALLY UP

Suppose, a chain of mass per unit length λ begins to fall through a hole in the ceiling as shown in figure (a) or the end of the chain piled on the platform is lifted vertically as in figure (b).



In both the cases, due to increase of mass in the portion of the chain which is moving with a velocity v at certain moment of time a thrust force acts on this part of the chain which is given by

$$F_t = v_r \left(\frac{dm}{dt} \right)$$

Here, $v_r = v$ and $\frac{dm}{dt} = \lambda v$

Here, v_r is upwards in case (a) and downwards in case (b). Thus,

$$F_t = \lambda v^2$$

The direction of F_t is upwards in case (a) and downwards in case (b).

ILLUSTRATION 61

A uniform chain of mass m and length l hangs on a thread and touches the surface of a table by its lower end. Find the force exerted by the table on the chain when half of its length has fallen on the table. The fallen part does not form heap.

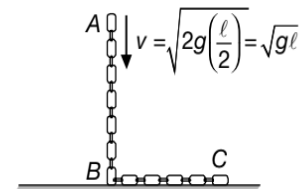


SOLUTION

The force exerted by the chain on the table consists of two parts,

- (a) Weight of the portion BC of the chain lying on the table,

$$W = \frac{mg}{2} \quad \text{\{downwards\}}$$



- (b) Thrust force exerted on table by the falling part of the chain, $F_t = \lambda v^2$

where, $\lambda = \text{mass per unit length of chain} = \frac{m}{l}$

$$\text{and } v^2 = (\sqrt{gl})^2 = gl$$

$$\Rightarrow F_t = \left(\frac{m}{l} \right) (gl) = mg \quad \text{\{downwards\}}$$

So, net force exerted by the chain on the table is

$$F = W + F_t = \frac{mg}{2} + mg = \frac{3}{2}mg \quad \text{\{downwards\}}$$

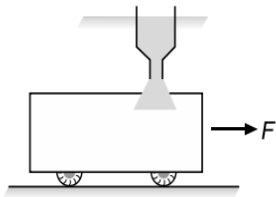
So, from Newton's Third Law the force exerted by the table on the chain will be $\frac{3}{2}mg$ (vertically upwards).

Conceptual Note(s)

Here the thrust force (F_t) applied by the chain on the table will be vertically downwards. Since $F_t = v_r \left(\frac{dm}{dt} \right)$, and in this expression v_r is downwards and $\frac{dm}{dt}$ is positive. Hence, F_t will act downwards.

ILLUSTRATION 62

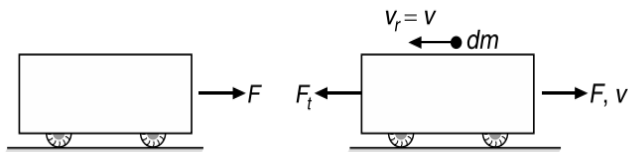
A flat car of mass m_0 starts moving to the right due to a constant horizontal force F . Sand spills on the flat car from a stationary hopper. The rate of loading is constant and equals to $\mu \text{ kgs}^{-1}$. Find the time dependence of the velocity and the acceleration of the flat car in the process of loading. The friction is negligibly small.



SOLUTION

Initial velocity of the flat car is zero
i.e., at $t = 0$, $v = 0$

Let v be its velocity at time t and m its mass at that instant i.e.,



at time t , $v = v$ and $m = m_0 + \mu t$ {backwards}

Here, $v_r = v$ and $\frac{dm}{dt} = \mu$

Thrust force, $F_t = v_r \frac{dm}{dt} = \mu v$ {backwards}

Net force on the flat car at time t is $F_{\text{net}} = F - F_t$

$$\Rightarrow m \frac{dv}{dt} = F - \mu v \quad \dots(1)$$

$$\Rightarrow (m_0 + \mu t) \frac{dv}{dt} = F - \mu v$$

$$\begin{aligned} \Rightarrow \int_0^v \frac{dv}{F - \mu v} &= \int_0^t \frac{dt}{m_0 + \mu t} \\ \Rightarrow -\frac{1}{\mu} (\ln(F - \mu v)) \Big|_0^v &= \frac{1}{\mu} (\ln(m_0 + \mu t)) \Big|_0^t \\ \Rightarrow \ln\left(\frac{F}{F - \mu v}\right) &= \ln\left(\frac{m_0 + \mu t}{m_0}\right) \end{aligned}$$

Taking antilog both sides, we get

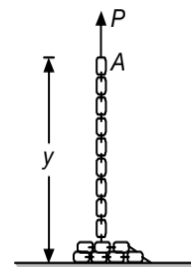
$$\begin{aligned} \frac{F}{F - \mu v} &= \frac{m_0 + \mu t}{m_0} \\ \Rightarrow v &= \frac{Ft}{m_0 + \mu t} \end{aligned}$$

Since $\frac{dv}{dt} = a$, i.e. acceleration of flat car at time t , so from equation (1), we get

$$\begin{aligned} \frac{dv}{dt} &= \frac{F - \mu v}{m} = \frac{F - \frac{F\mu t}{m_0 + \mu t}}{m_0 + \mu t} \\ \Rightarrow a &= \frac{Fm_0}{(m_0 + \mu t)^2} \end{aligned}$$

ILLUSTRATION 63

A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v_0 , express in terms of the length y of chain which is off the floor at any given instant.



- the magnitude of the force P applied to end A .
- energy lost during the lifting of the chain.

SOLUTION

- When the chain is raised with a constant upward speed, then net force on it is zero. So,

$P = \text{Weight of length } y \text{ of chain} + \text{Thrust force}$

$$\Rightarrow P = \left(\frac{m}{l}\right)gy + \rho v_0^2 \quad \left\{ \text{here } \rho = \frac{m}{l} \right\}$$

$$\Rightarrow P = \frac{m}{l}(gy + v_0^2)$$

$$(b) \left(\begin{array}{c} \text{Energy Lost} \\ \text{During} \\ \text{Lifting} \end{array} \right) = \left(\begin{array}{c} \text{Work done} \\ \text{by Applied} \\ \text{Force} \end{array} \right) - \left(\begin{array}{c} \text{Increase in} \\ \text{Mechanical} \\ \text{Energy of} \\ \text{Chain} \end{array} \right)$$

$$\Rightarrow -\Delta E = Py - \left(\frac{m}{l}y\right)g\left(\frac{y}{2}\right) - \frac{1}{2}\left(\frac{m}{l}y\right)v_0^2$$

$$\Rightarrow -\Delta E = \frac{my}{l}(gy + v_0^2) - \frac{mgy^2}{2l} - \frac{myv_0^2}{2l}$$

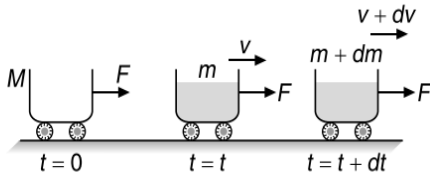
$$\Rightarrow -\Delta E = \frac{1}{2}\left(\frac{mgy^2}{l} + \frac{myv_0^2}{l}\right) = \frac{my}{2l}(gy + v_0^2)$$

ILLUSTRATION 64

A tank car of mass M is at rest on a road. At $t = 0$, a force F starts acting on the tank car and also the rain fall starts, in vertical direction, as shown in figure. The rain is falling with a velocity v_r with respect to earth and the rate of collection of water in the tank is $r \text{ kgs}^{-1}$. Find the velocity of the tank-car as a function of time t .

SOLUTION

The situation at time t and also at time $t + dt$ is shown in Figure.



At time t let mass of the car be m and its velocity be v . In the duration dt , a further mass dm is added and in this duration car gains a speed $v + dv$.

By Impulse Momentum Theorem, we have

$$Fdt = (m + dm)(v + dv) - mv$$

$$\Rightarrow Fdt = vdm + mdv$$

where, $m = M + rt$ and $dm = rdt$

$$\Rightarrow Fdt = v(rdt) + (M + rt)dv$$

$$\Rightarrow (M + rt)dv = (F - rv)dt$$

$$\Rightarrow \frac{dv}{F - rv} = \frac{dt}{M + rt}$$

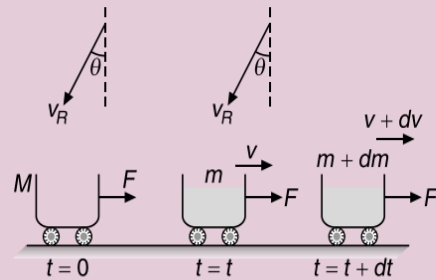
Integrating, we get

$$\int_0^v \frac{dv}{F - rv} = \int_0^t \frac{dt}{M + rt}$$

$$\Rightarrow \ln\left(\frac{F}{F - rv}\right) = \ln\left(\frac{M + rt}{M}\right)$$

$$\Rightarrow v = \frac{Frt}{Mr + r^2t}$$

Dear Students, this example can be modified if we consider rain to be falling at an angle θ with the vertical in the direction against the velocity of the car as shown in figure.



So, in this case we will add the momentum imparted by the rain to the car in the opposite direction in time dt . So, we get

$$Fdt = (m + dm)(v + dv) - [mv - (v_R \sin\theta)dm]$$

Now we can rearrange the terms of dv and dt according to the conditions given in the problem and then integrate it to get the velocity of car as a function of time.

ILLUSTRATION 65

A particle of mass M_0 initially at rest starts moving under action of a constant force $F\hat{i}$. It encounters the resistance of a stream of fine dust moving with velocity $-v_0\hat{i}$ which deposits on it at a constant rate k . Calculate the distance travelled by the particle when its mass becomes M .

SOLUTION

Applying Impulse Momentum Theorem to the particle, we get

$$Fdt = (m + dm)(v + dv) - (M_0v - v_0kdt)$$

$$\Rightarrow (F - kv_0)dt = kvdt + (M_0 + kt)dv$$

$$\Rightarrow (F - kv_0 - kv)dt = (M_0 + kt)dv$$

$$\Rightarrow \int_0^v \frac{dv}{F - kv_0 - kv} = \int_0^t \frac{dt}{M_0 + kt}$$

$$\Rightarrow \ln\left(\frac{F - kv_0 - kv}{F - kv_0}\right) = \ln\left(\frac{M_0}{M_0 + kt}\right)$$

$$\Rightarrow \frac{F - kv_0 - kv}{F - kv_0} = \frac{M_0}{M_0 + kt}$$

$$\Rightarrow F - kv_0 - kv = \frac{(F - kv_0)M_0}{M_0 + kt}$$

Since, $M = M_0 + kt$... (1)

$$\Rightarrow kv = (F - kv_0) - \frac{(F - kv_0)M_0}{M}$$

$$\Rightarrow v = \frac{1}{k}(F - kv_0)\left(1 - \frac{M_0}{M}\right)$$

$$\Rightarrow \frac{dx}{dt} = \left(\frac{F - kv_0}{k}\right)\left(1 - \frac{M_0}{M}\right)$$
 ... (2)

From equation (1), we have $M = M_0 + kt$
 $\Rightarrow dM = kdt$

$$\Rightarrow dt = \frac{dM}{k}$$

So, equation (2) becomes

$$\frac{kdx}{dM} = \left(\frac{F - kv_0}{k}\right)\left(1 - \frac{M_0}{M}\right)$$

$$\Rightarrow dx = \left(\frac{F - kv_0}{k^2}\right)\left(1 - \frac{M_0}{M}\right)dM$$

Integrating, we get

$$l = \int_0^l dx = \left(\frac{F - kv_0}{k^2}\right) \int_{M_0}^M \left(1 - \frac{M_0}{M}\right) dM$$

$$\Rightarrow l = \frac{F - kv_0}{k^2} \left[(M - M_0) - M_0 \ln\left(\frac{M}{M_0}\right) \right]$$

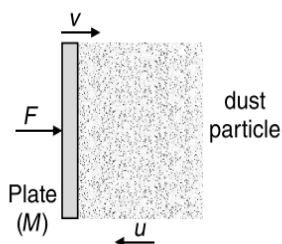
$$\Rightarrow l = \frac{F - kv_0}{k^2} \left[M - M_0 \left(1 + \ln\left(\frac{M}{M_0}\right)\right) \right]$$

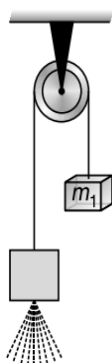
Test Your Concepts-V

Based on Variable Mass Systems

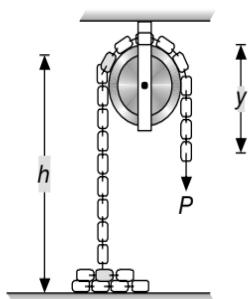
(Solutions on page H.83)

- A cart loaded with sand moves along a horizontal floor due to a constant force F coinciding in direction with the cart's velocity. However, in this process the sand spills through a hole in the bottom with a constant rate $\mu \text{ kgs}^{-1}$. Find the acceleration and velocity of the cart at the moment t , if at the initial moment $t = 0$ the cart with loaded sand had the mass m_0 and its velocity was equal to zero. Friction is to be neglected.
- A plate of mass M is moved with constant velocity v against dust particles moving with velocity u in opposite direction as shown. The density of the dust is $\rho \text{ kgm}^{-3}$ and plate area is A . Calculate the force F required to keep the plate moving uniformly.
- (a) A rocket set for vertical firing weighs 50 kg and contains 450 kg of fuel. It can have a maximum exhaust velocity of 2 kms^{-1} . What should be its minimum rate of fuel consumption (i) to just lift it off the launching pad. (ii) to give it an acceleration of 20 ms^{-2} .
 (b) What will be the speed of the rocket when the rate of consumption of fuel is 10 kgs^{-1} after the entire fuel is consumed? Take $g = 10 \text{ ms}^{-2}$.
- A mass m_1 is connected by a very light cable passing over a frictionless pulley to a container of water, whose mass is m_0 at $t = 0$. If the container ejects water in downward direction at a constant rate of $b \text{ kgs}^{-1}$ with a velocity v_0 relative to the container, then calculate the acceleration of m_1 as a function of time.





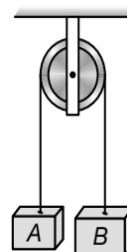
5. Determine the force P required to give the open link chain a constant velocity $v = \frac{dy}{dt}$. The chain has a total length L and mass per unit length λ . Neglect the small size and mass of the pulley and any friction in the pulley.



6. A rocket with initial mass M is launched by emitting gas with velocity v_0 relative to the rocket downwards. The mass of the gas emitted per second is k ,

where k is a constant such that $kv_0 < Mg$. Calculate the time when rocket starts to lift.

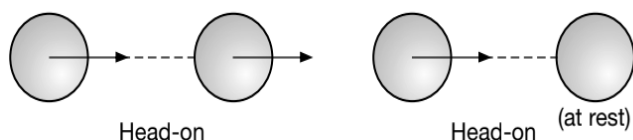
7. Find the mass of the rocket as a function of time, if it moves with a constant acceleration a , in absence of external forces. The gas escapes with a constant velocity u relative to the rocket and its mass initially was m_0 .
8. A cart loaded with sand moves along a horizontal floor due to a constant force F directed along the cart's velocity. During this process, sand spills through a hole in the bottom at a constant rate of $\mu \text{ kgs}^{-1}$. Calculate acceleration and velocity of the cart at time t , if initially the cart loaded with sand had mass m_0 and zero velocity. Neglect friction.
9. Two blocks A and B each masses of mass M_0 are hanging with the help of a light string, which passes over a massless pulley. Mass of A starts leaking out at a rate of $\mu \text{ kgs}^{-1}$ with velocity v_0 with respect to the mass A . Find the velocity of block B as a function of time.



TYPES OF COLLISIONS

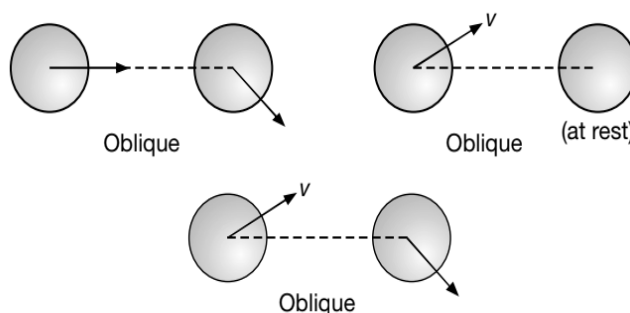
Head on Collision

Whenever the colliding bodies have their velocities along the line joining their centres, then the collision is said to be head-on.



Oblique Collision

Whenever the colliding bodies do not have their velocities along the line joining their centres, then the collision is said to be oblique.



WORD OF ADVICE

- (a) These diagrams are drawn **just before** or **just after** the collision.
- (b) Based on energy, we divide collisions into three categories

- (i) **Elastic collision:** In such type of collisions, momentum and kinetic energy, both are conserved.
- (ii) **Inelastic collision:** In such type of collisions, momentum is conserved. However, kinetic energy is not conserved ($k_f < k_i$).
- (iii) **Super-elastic collision:** In such types of collisions, momentum is conserved. However, the final kinetic energy is greater than initial kinetic energy ($k_f > k_i$). As an example, a bomb initially at rest ($KE = 0$) on exploding, has its different fragments moving in different directions with some finite kinetic energy such that total final momentum of the fragments is also zero (as a system). However, final kinetic energy is greater than initial kinetic energy (which was zero).

COEFFICIENT OF RESTITUTION (e)

The coefficient of restitution is only defined for elastic and inelastic collisions. It is the measure of the degree of elasticity of the collision and is defined as the ratio of the impulse due to reformation and impulse due to deformation of either body.

$$e = \frac{\text{Impulse due to Reformation}}{\text{Impulse due to Deformation}} = \frac{\int F_r dt}{\int F_d dt}$$

It can also be defined as the ratio of the relative velocity of separation just after the collision to the relative velocity of approach just before collision.

$$e = \frac{\left(\begin{array}{c} \text{Relative velocity of separation} \\ \text{just after collision} \end{array} \right)}{\left(\begin{array}{c} \text{Relative velocity of approach} \\ \text{just after collision} \end{array} \right)}$$

$$\Rightarrow e = \left(\frac{v_2 - v_1}{u_1 - u_2} \right)$$

$$\Rightarrow e = - \left(\frac{v_2 - v_1}{u_2 - u_1} \right) = - \left(\frac{v_1 - v_2}{u_1 - u_2} \right)$$

In general, we have, $0 \leq e \leq 1$

Conceptual Note(s)

- (a) For elastic (or perfectly elastic) collisions, $e = 1$.
- (b) For inelastic collisions, $0 \leq e < 1$.
- (c) For perfectly inelastic collisions, $e = 0$.
In this collision, bodies move independently before collision, but after collision stick to each other and move as are single body.
 $\Rightarrow ([e] = M^0 L^0 T^0)$

WORD OF ADVICE

The definition of 'e' is applicable only for elastic and inelastic collisions.

However, if we could have applied this definition of 'e' to a super elastic collision, then $e > 1$.

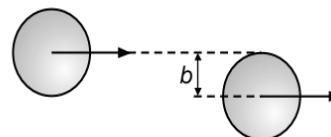
So, can 'e' be greater than 1?

If the definition of 'e' can be applied to a super elastic collision, then 'e' can be greater than 1.

IMPACT PARAMETER (b)

It is the perpendicular distance between the velocities of the two colliding bodies.

$$[b] = M^0 L^1 T^0$$



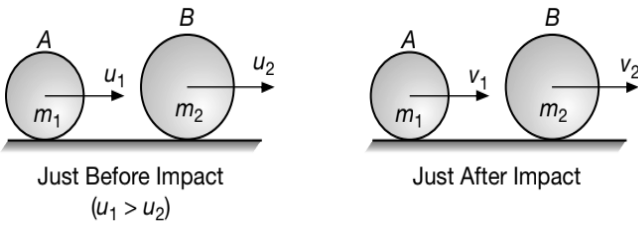
For, $b = 0$, then collision is head-on and for

$0 < b < (r_1 + r_2)$, the collision is oblique

TYPES OF COLLISIONS

- (a) Elastic Head-On: $e = 1$ and $b = 0$
- (b) Inelastic Head-On: $0 < e < 1$ and $b = 0$
- (c) Perfectly Inelastic Head-On: $e = 0$ and $b = 0$
- (d) Inelastic Oblique: $0 < e < 1$ and $0 < b < (r_1 + r_2)$
- (e) Elastic Oblique: $e = 1$ and $0 < b < (r_1 + r_2)$
- (f) Perfectly Inelastic: $e = 0$ and $0 < b < (r_1 + r_2)$
- (g) Super-Elastic Collisions: ($K_{\text{final}} > K_{\text{initial}}$)

HEAD-ON ELASTIC COLLISION



By Law of Conservation of Linear Momentum, we have

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \quad \dots(1)$$

($\vec{u}_1, \vec{u}_2, \vec{v}_1$ and \vec{v}_2 all are in same direction)

By Law of Conservation of Kinetic Energy, we have

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad \dots(2)$$

$$\text{From (1), } m_1 (u_1 - v_1) = m_2 (v_2 - u_2) \quad \dots(3)$$

$$\text{From (2), } m_1 (u_1^2 - v_1^2) = m_2 (v_2^2 - u_2^2) \quad \dots(4)$$

Divide (4) by (3), we get

$$u_1 + v_1 = u_2 + v_2 \quad \dots(5)$$

Rearranging (5), we get

$$u_1 - u_2 = v_2 - v_1 \quad \dots(6)$$

$$\left(\begin{array}{l} \text{Relative velocity} \\ \text{of approach just} \\ \text{before collision} \end{array} \right) = \left(\begin{array}{l} \text{Relative velocity} \\ \text{of approach just} \\ \text{after collision} \end{array} \right)$$

From (6) and by definition of e , we get

$$e = - \left(\frac{v_2 - v_1}{u_2 - u_1} \right) = 1$$

From (5), we get

$$v_2 = u_1 + v_1 - u_2 \quad \dots(7)$$

Put (7) in (1), we get

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 (u_1 + v_1 - u_2)$$

Rearranging, we get

$$(m_1 + m_2) v_1 = (m_1 - m_2) u_1 + 2m_2 u_2$$

$$\Rightarrow v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 + \left(\frac{2m_2}{m_1 + m_2} \right) u_2$$

Simply replace subscript 2 by 1 and 1 by 2,

$$v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) u_2$$

SPECIAL CASES

CASE-1: Colliding Bodies of equal masses

When colliding bodies are of equal masses, then

$$m_1 = m_2 = m, \text{ then } v_1 = u_2 \text{ and } v_2 = u_1$$

$m_1 = m_2$	Velocity Just		Momentum Just		Kinetic Energy Just	
	Before Impact	After Impact	Before Impact	After Impact	Before Impact	After Impact
Ball 1	u_1	u_2	mu_1	mu_2	$\frac{1}{2} mu_1^2$	$\frac{1}{2} mu_2^2$
Ball 2	u_2	u_1	mu_2	mu_1	$\frac{1}{2} mu_2^2$	$\frac{1}{2} mu_1^2$
Conclusion	Velocity Exchange		Momentum Exchange		Kinetic Energy Exchange	

i.e. for a perfectly elastic head on collision, the bodies exchange their velocity momentum and kinetic energy just after impact.

CASE-2: Target body at rest

If $u_2 = 0$, i.e. second body at rest, then

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 \text{ and } v_2 = \left(\frac{2m_1}{m_2 + m_2} \right) v_2$$

CASE-3: Colliding Bodies of equal masses and target body at rest

If $m_1 = m_2 = m$ (say) and $u_2 = 0$

i.e., a body collides with an identical body at rest, then, we get

$$v_1 = 0 \text{ and } v_2 = u_1$$

$m_1 = m_2$ and $u_2 = 0$	Velocity Just		Momentum Just		Kinetic Energy Just	
	Before Impact	After Impact	Before Impact	After Impact	Before Impact	After Impact
Ball 1	u_1	0	mu_1	0	$\frac{1}{2} mu_1^2$	0
Ball 2	0	u_1	0	mu_1	0	$\frac{1}{2} mu_1^2$
Conclusion	100% Transfer of Velocity		100% Transfer of Momentum		100% Transfer of Kinetic Energy	

So, we observe that in an elastic head-on collision if one body is at rest and the two bodies have equal mass then after collision hundred percent (100%)

transfer of velocity/momentum/kinetic energy takes place.

CASE-4: Massive body collides with a light body at rest

When a massive body collides with a light body at rest i.e., $m_1 \gg m_2$ and $u_2 = 0$

$$\text{Since } v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 \text{ and } v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1$$

So, for $m_1 \gg m_2$, we have $m_1 - m_2 \approx m_1$

and $m_1 + m_2 \approx m_1$

$$\Rightarrow v_1 \approx u_1 \text{ and } v_2 \approx 2u_1$$

So, when a massive body collides with a light body at rest (elastic head-on) then, massive body continues to move with the same velocity and the light body moves with twice the velocity of massive body.

CASE-5: Light body collides with the massive body at rest

When a light body collides with the massive body at rest, then $m_2 \gg m_1$ and $u_2 = 0$

$$\text{Since } v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 \text{ and } v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1$$

So, for $m_2 \gg m_1$, we have

$$m_1 - m_2 \approx -m_2 \text{ and } m_1 + m_2 \approx m_2$$

$$\Rightarrow v_1 \approx -u_1 \text{ (rebound) and } v_2 \approx 2 \left(\frac{m_1}{m_2} \right) u_1$$

Since $m_1 \ll m_2$, i.e. $\frac{m_1}{m_2}$ is negligible

$$\Rightarrow v_2 \rightarrow 0$$

So, when a light body collides with massive body at rest, then the light body rebounds and massive body continues to be at rest.

CASE-6: Massive body collides with a light body

When massive body collides with a light body, then $m_1 \gg m_2$ and $u_2 \neq 0$. Since, we have

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 + \left(\frac{2m_2}{m_1 + m_2} \right) u_2$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) u_2$$

For $m_1 \gg m_2$, $m_1 - m_2 \approx m_1$ and $m_1 + m_2 \approx m_1$

$$\Rightarrow v_1 \approx 2u_2 - u_1 \text{ and } v_2 \approx u_2$$

ILLUSTRATION 66

In an elastic head-on collision, if the second body is at rest, then calculate

- fraction of kinetic energy lost by m_1
- fraction of kinetic energy gained by m_2
- fraction of kinetic energy retained by m_1
- the condition for which the fraction of kinetic energy lost by m_1 is the maximum.

SOLUTION

$$u_2 = 0, v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 \text{ and } v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1$$

- (a) Fraction of K.E. lost by m_1 is

$$f_{\text{lost by } m_1} = \frac{(k_1)_i - (k_1)_f}{(k_1)_i} = \frac{\frac{1}{2}m_1u_1^2 - \frac{1}{2}m_1v_1^2}{\frac{1}{2}m_1u_1^2}$$

$$\Rightarrow f_{\text{lost by } m_1} = 1 - \left(\frac{v_1}{u_1} \right)^2$$

$$\Rightarrow f_{\text{lost by } m_1} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2$$

$$\Rightarrow f_{\text{lost by } m_1} = \frac{4m_1m_2}{(m_1 + m_2)^2}$$

- (b) Since the collision is ideal, So, fraction of K.E. gained by m_2 is equal to the fraction of K.E. lost by m_1
Hence,

$$f_{\text{gained by } m_2} = f_{\text{lost of } m_1} = \frac{4m_1m_2}{(m_1 + m_2)^2}$$

- (c) Fraction retained by m_1 is $(1 - f_{\text{lost of } m_1})$

$$\Rightarrow f_{\text{retained by } m_1} = 1 - \frac{4m_1m_2}{(m_1 + m_2)^2} = \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2}$$

- (d) $f_{\text{lost by } m_1} = \text{MAX} = 1$

$$\Rightarrow \frac{4m_1m_2}{(m_1 + m_2)^2} = 1$$

$$\Rightarrow (m_1 - m_2)^2 = 0$$

$$\Rightarrow m_1 = m_2$$

ILLUSTRATION 67

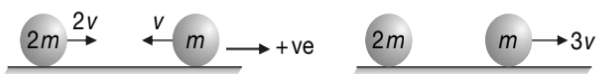
Two particles of mass m and $2m$ moving in opposite directions collide elastically with velocities v and $2v$. Find their velocities after collision.

SOLUTION

$$\text{Since, } v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 + \left(\frac{2m_2}{m_1 + m_2} \right) u_2$$

$$\text{and } v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) u_2$$

Here, $v_1 = -v$, $v_2 = 2v$, $m_1 = m$ and $m_2 = 2m$



Substituting these values, we get

$$v_1' = \left(\frac{m - 2m}{m + 2m} \right) (-v) + \left(\frac{4m}{m + 2m} \right) (2v)$$

$$\Rightarrow v_1' = \frac{v}{3} + \frac{8v}{3} = 3v$$

$$\text{and } v_2' = \left(\frac{2m - m}{m + 2m} \right) (2v) + \left(\frac{2m}{m + 2m} \right) (-v)$$

$$\Rightarrow v_2' = \frac{2}{3}v - \frac{2}{3}v = 0$$

i.e., the second particle (of mass $2m$) comes to a rest while the first (of mass m) moves with velocity $3v$ in the direction shown in figure.

ILLUSTRATION 68

Two pendulum bobs of mass m and $2m$ collide elastically at the lowest point in their motion. If both the balls are released from a height H above the lowest point, to what heights do they rise for the first time after collision?

SOLUTION

Given, $m_1 = m$, $m_2 = 2m$, $v_1 = -\sqrt{2gH}$ and $v_2 = \sqrt{2gH}$

Since, the collision is elastic.

$$\Rightarrow v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 + \left(\frac{2m_2}{m_1 + m_2} \right) u_2$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) u_2$$

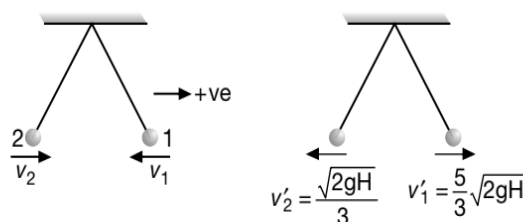
$$v_1' = \left(\frac{m - 2m}{m + 2m} \right) (-\sqrt{2gH}) + \left(\frac{4m}{m + 2m} \right) \sqrt{2gH}$$

$$v_1' = \frac{\sqrt{2gH}}{3} + \frac{4\sqrt{2gH}}{3} = \frac{5}{3}\sqrt{2gH}$$

$$\text{and } v_2' = \left(\frac{2m - m}{m + 2m} \right) (\sqrt{2gH}) + \left(\frac{2m}{m + 2m} \right) (-\sqrt{2gH})$$

$$v_2' = \frac{\sqrt{2gH}}{3} - \frac{2\sqrt{2gH}}{3} = -\frac{\sqrt{2gH}}{3}$$

i.e., the velocities of the balls after the collision are as shown in figure.



Therefore, the heights to which the balls rise after the collision are

$$h_1 = \frac{(v_1')^2}{2g} \quad \left\{ \text{using } v^2 = u^2 - 2gh \right\}$$

$$\Rightarrow h_1 = \frac{\left(\frac{5}{3}\sqrt{2gH} \right)^2}{2g}$$

$$\Rightarrow h_1 = \frac{25}{9}H \text{ and } h_2 = \frac{(v_2')^2}{2g}$$

$$\Rightarrow h_2 = \frac{\left(\frac{\sqrt{2gH}}{3} \right)^2}{2g} = \frac{H}{9}$$

Since the collision is elastic, mechanical energy of both the balls will remain conserved, or

$$E_i = E_f$$

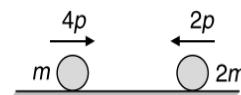
$$\Rightarrow (m + 2m)gH = mgh_1 + 2mgh_2$$

$$\Rightarrow 3mgH = (mg)\left(\frac{25}{9}H\right) + (2mg)\left(\frac{H}{9}\right)$$

$$\Rightarrow 3mgH = 3mgH$$

ILLUSTRATION 69

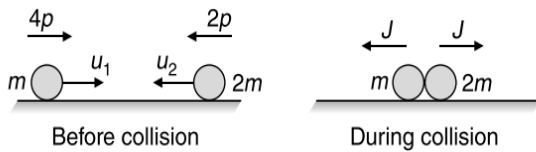
Two balls of masses m and $2m$ having momenta $4p$ and $2p$ respectively collide as shown in Figure.



During collision, the value of linear impulse between them is J . Calculate the coefficient of restitution e in terms of J and p . Find the condition for which collision is elastic and the condition for which it is perfectly inelastic.

SOLUTION

Before the collision, the directions of linear momenta and during the collision, the directions of linear impulses on the two colliding bodies are shown in Figure.



Linear impulse is equal to the change in linear momentum, so for ball m , we have

$$-J = p_f - p_i = p_f - 4p$$

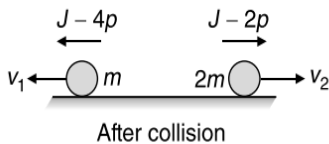
$$\Rightarrow p_f = -(J - 4p), \text{ i.e. leftwards}$$

for ball $2m$, we have

$$J = p_f - (-2p) = p_f + 2p$$

$$\Rightarrow p_f = J - 2p, \text{ i.e. rightwards}$$

Hence, momenta of the balls after collision are shown in Figure.



Let u_1, u_2 be the velocities of balls before collision and v_1, v_2 be the velocities of balls after collision.

$$\text{Since, } v = \frac{\text{linear momentum}}{\text{mass}} = \frac{p}{m}$$

Also, by the definition of the coefficient of restitution, we have

$$e = \frac{\text{Relative Velocity of Separation}}{\text{Relative Velocity of Approach}}$$

$$\Rightarrow e = \frac{v_1 + v_2}{u_1 + u_2} = \frac{\frac{J - 4p}{m} + \frac{J - 2p}{2m}}{\frac{4p}{m} + \frac{2p}{2m}}$$

$$\Rightarrow e = \frac{3J - 10p}{10p} = \frac{3J}{10p} - 1$$

For elastic collision, $e = 1$

$$\Rightarrow \frac{3J}{10p} = 2$$

$$\Rightarrow J = \frac{20}{3}p$$

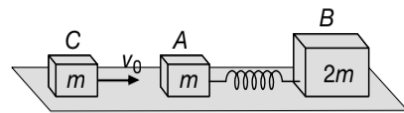
For perfectly inelastic collision, $e = 0$

$$\Rightarrow \frac{3J}{10p} = 1$$

$$\Rightarrow J = \frac{10}{3}p$$

ILLUSTRATION 70

Two blocks A and B of masses m and $2m$ respectively are placed on a smooth floor. They are connected by a spring. A third block C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A , as shown in figure. At a certain instant of time t_0 after collision, it is found that the instantaneous velocities of A and B are the same. Further, at this instant the compression of the spring is found to be x_0 . Determine (i) the common velocity of A and B at time t_0 and (ii) the spring constant.



SOLUTION

Initially, the blocks A and B are at rest and C is moving with velocity v_0 to the right.

As masses of C and A are same and the collision is elastic the body C transfers its whole momentum mv_0 to body A and as a result the body C stops and A starts moving with velocity v_0 to the right. At this instant the spring is uncompressed and the body B is still at rest.

The momentum of the system at this instant $= mv_0$
Now, the spring is compressed and the body B comes in motion. After time t_0 , the compression of the spring is x_0 and common velocity of A and B is v (say). As external force on the system is zero, the Law of Conservation of Linear Momentum gives

$$mv_0 = mv + (2m)v$$

$$\Rightarrow v = \frac{v_0}{3}$$

The Law of Conservation of Energy gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}kx_0^2$$

$$\Rightarrow \frac{1}{2}mv_0^2 = \frac{3}{2}mv^2 + \frac{1}{2}kx_0^2$$

$$\frac{1}{2}mv_0^2 = \frac{3}{2}m\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}kx_0^2$$

$$\Rightarrow \frac{1}{2}kx_0^2 = \frac{1}{2}mv_0^2 - \frac{1}{6}mv_0^2$$

$$\Rightarrow \frac{1}{2}kx_0^2 = \frac{1}{3}mv_0^2$$

$$\Rightarrow k = \frac{2}{3} \frac{mv_0^2}{x_0^2}$$

INELASTIC HEAD-ON

$$m_1u_2 + m_2u_2 = m_1v_1 + m_2v_2 \quad \dots(1)$$

Since collision is inelastic, so we use

$$e = -\left(\frac{v_2 - v_1}{u_2 - u_1}\right)$$

$$v_2 - v_1 = eu_1 - eu_2$$

$$v_2 = eu_1 - eu_2 + v_1 \quad \dots(2)$$

Putting (2) in (1), we get

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2(ev_1 - ev_2 + v_1)$$

$$m_1v_1 + m_2v_2 = m_1v_1 + m_2ev_1 - m_2ev_2 + m_2v_1$$

$$m_1u_1 + m_2u_2 + m_2ev_2 - m_2ev_1 = m_1v_1 + m_2v_1$$

$$v_1 = \frac{u_1(m_1 - m_2e)}{m_1 + m_2} + \left[\frac{(1+e)m_2}{m_1 + m_2}\right]v_2$$

$$v_2 = \left[\frac{(1+e)m_1}{m_1 + m_2}\right]u_1 + \left[\frac{m_2 - em_1}{m_1 + m_2}\right]u_2$$

CHECK POINT

Put $e = 1$ to get the results matching with those of perfectly elastic collision.

(Loss in KE) = (Total Initial KE) – (Total Final KE)

$$\text{Loss} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$\text{Loss} = \frac{1}{2} \underbrace{\left(\frac{m_1m_2}{m_1 + m_2}\right)}_{\substack{\text{Reduced Mass} \\ \text{(discussed in CM)}}} (1 - e^2)(u_1 - u_2)^2$$

CHECK POINT

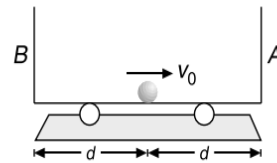
If $e = 1$, we observe

$$\text{Loss} = 0$$

which is the case of a perfectly elastic collision.

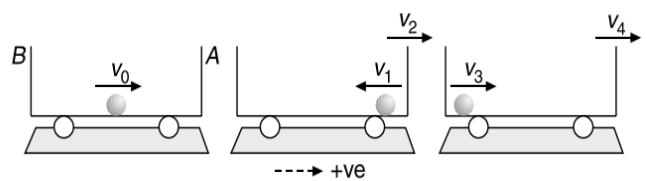
ILLUSTRATION 71

A ball of negligible size and mass m is given a velocity v_0 on the centre of the cart which has a mass M and is originally at rest. If the coefficient of restitution between the ball and walls A and B is e , determine the velocity of the ball and the cart just after the ball strikes A . Also, determine the total time needed for the ball to strike A , rebound, then strike B and rebound and then return to the centre of the cart. Neglect friction.



SOLUTION

Applying conservation of linear momentum and



$$e = \frac{\text{relative speed of separation}}{\text{relative speed of approach}}, \text{ we get}$$

$$\Rightarrow mv_0 = Mv_2 - mv_1 \quad \dots(1)$$

$$\Rightarrow v_1 + v_2 = ev_0 \quad \dots(2)$$

Solving these two equations, we get,

$$v_1 = \left(\frac{eM - m}{M + m}\right)v_0, \quad v_2 = m\left(\frac{e + 1}{M + m}\right)v_0$$

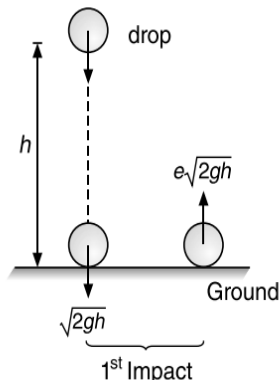
The desired time is

$$t = \frac{d}{v_0} + \frac{2d}{ev_0} + \frac{d}{e^2v_0}$$

$$\Rightarrow t = \frac{d}{v_0} \left(1 + \frac{2}{e} + \frac{1}{e^2}\right)$$

BALL GROUND INELASTIC HEAD-ON COLLISION

For ball wall collision or ball ground inelastic head-on collision, we have



$$e = -\left(\frac{v_2 - v_1}{u_2 - u_1}\right) = -\left(\frac{0 - v_1}{0 - \sqrt{2gh}}\right) = -\frac{v_1}{\sqrt{2gh}}$$

$$\Rightarrow v_1 = -e\sqrt{2gh}$$

So, for a ball colliding inelastically head-on with the ground we observe that for the ball

$$\left(\text{Speed just after Collision}\right)_{n\text{-line}} = e \left(\text{Speed just before Collision}\right)_{n\text{-line}}$$

ILLUSTRATION 72

A ball of mass m is dropped from a height h to hit the ground and then rebound to a certain height. The process carries on and the ball continues to hit the ground time and again. The coefficient of restitution between the ground and the ball is e . Find the average speed of the ball, the total momentum imparted to the floor by the ball, the total distance travelled by the ball before coming to rest and the total momentum imparted to the floor by the ball.

SOLUTION

Velocity after 1st impact is $v_1 = e\sqrt{2gh}$

Velocity just before second impact is $v_1 = e\sqrt{2gh}$

Velocity just after second impact is

$$v_2 = ev_1 = e^2\sqrt{2gh}$$

Height after first impact is $h_1 = \frac{v_1^2}{2g} = e^2h$

Total distance just before second impact is $2h_1$

Time taken between 1st and 2nd impact is

$$2t_1 = 2\left(\frac{v_1}{g}\right) = \frac{2e\sqrt{2gh}}{g} = 2e\sqrt{\frac{2h}{g}}$$

Time taken between 2nd and 3rd impact is

$$2t_2 = 2\left(\frac{2v_1}{g}\right) = \frac{2e^2\sqrt{2gh}}{g} = 2e^2\sqrt{\frac{2h}{g}}$$

After every impact, the ball loses some momentum which is gained by the ground. If Δp_1 is the change in momentum after first impact.

$$|\Delta p_1| = |m(-e\sqrt{2gh} - \sqrt{2gh})|$$

$$\Rightarrow |\Delta p_1| = m\sqrt{2gh}(1+e)$$

If Δp_2 is the change of momentum of ball after second impact, then

$$|\Delta p_2| = |m(-e^2\sqrt{2gh} - e\sqrt{2gh})|$$

$$\Rightarrow |\Delta p_2| = e\Delta p_1 = em\sqrt{2gh}(1+e)$$

Similarly, $|\Delta p_3| = e^2m\sqrt{2gh}(1+e)$

Total momentum imparted by the ball to the floor is

$$\Delta p = \Delta p_1 + \Delta p_2 + \Delta p_3 \dots$$

$$\Rightarrow \Delta p = m\sqrt{2gh}(1+e)(1+e+e^2+e^3+\dots)$$

$$\Rightarrow \Delta p = m\sqrt{2gh}\left(\frac{1+e}{1-e}\right)$$

Total time taken by the ball to come to rest is

$$t = \sqrt{\frac{2h}{g}} + 2e\sqrt{\frac{2h}{g}} + 2e^2\sqrt{\frac{2h}{g}} + 2e^3\sqrt{\frac{2h}{g}} + \dots$$

$$\Rightarrow t = \sqrt{\frac{2h}{g}} + 2e\sqrt{\frac{2h}{g}}(1+e+e^2+\dots)$$

$$\Rightarrow t = \sqrt{\frac{2h}{g}} + 2e\sqrt{\frac{2h}{g}}\left(\frac{1}{1-e}\right)$$

$$\Rightarrow t = \sqrt{\frac{2h}{g}}\left[1 + \frac{2e}{1-e}\right] = \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}$$

Total distance travelled by the ball before coming to rest is

$$s = h + 2e^2h + 2e^4h + 2e^6h \dots$$

$$\Rightarrow s = h + 2e^2h(1+e^2+e^4+\dots)$$

$$\Rightarrow s = h + \frac{2e^2h}{1-e^2} = h \left(1 + \frac{2e^2}{1-e^2} \right)$$

$$\Rightarrow s = h \left(\frac{1+e^2}{1-e^2} \right)$$

The average speed v_{av} of the ball is given by

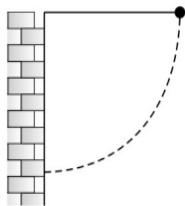
$$v_{av} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

$$\Rightarrow v_{av} = \frac{h \left(\frac{1+e^2}{1-e^2} \right)}{\sqrt{\frac{2h}{g} \left(\frac{1+e}{1-e} \right)}}$$

$$\Rightarrow v_{av} = \sqrt{\frac{gh}{2} \left(\frac{1+e^2}{1-e^2} \right)}$$

ILLUSTRATION 73

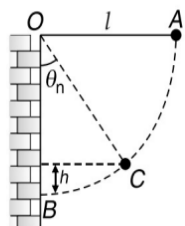
A simple pendulum is suspended from a peg on a vertical wall. The pendulum is pulled away from the wall to a horizontal position (see figure) and released.



The ball hits the wall, the coefficient of restitution being $\frac{2}{\sqrt{5}}$. What is the minimum number of collisions after which the amplitude of oscillation becomes less than 60 degrees?

SOLUTION

As shown in figure initially when the bob is at A, its potential energy is mgl .



When the bob is released and it strikes the wall at B, its potential energy mgl is converted into its kinetic energy. If v be the velocity with which the bob strikes the wall, then

$$mgl = \frac{1}{2}mv^2 \text{ or } v = \sqrt{2gl} \quad \dots(1)$$

Speed of the bob after rebounding (first time)

$$v_1 = e\sqrt{2gl} \quad \dots(2)$$

The speed after second rebound is

$$v_2 = e^2\sqrt{2gl}$$

In general, after n rebounds, the speed of the bob is

$$v_n = e^n\sqrt{2gl} \quad \dots(3)$$

Let the bob rises to a height h after n rebounds. Applying the Law of Conservation of Energy, we have

$$\frac{1}{2}mv_n^2 = mgh$$

$$\Rightarrow h = \frac{v_n^2}{2g} = \frac{(e^{2n})(2gl)}{2g} = le^{2n}$$

$$\Rightarrow h = l \left(\frac{2}{\sqrt{5}} \right)^{2n} = l \left(\frac{4}{5} \right)^n \quad \dots(4)$$

If θ_n be the angle after n collisions, then

$$h = l - l \cos \theta_n = l(1 - \cos \theta_n) \quad \dots(5)$$

From equations (4) and (5), we have

$$\left(\frac{4}{5} \right)^n l = l(1 - \cos \theta_n)$$

$$\Rightarrow \left(\frac{4}{5} \right)^n = (1 - \cos \theta_n)$$

For θ_n to be less than 60° , i.e., $\cos \theta_n$ is greater than $\frac{1}{2}$, i.e., $(1 - \cos \theta_n)$ is less than $\frac{1}{2}$, we have

$$\left(\frac{4}{5} \right)^n < \left(\frac{1}{2} \right)$$

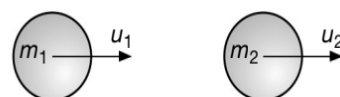
This condition is satisfied for $n = 4$

So, required number of collisions is 4.

PERFECTLY INELASTIC HEAD-ON COLLISION

CASE-1: Bodies moving in same direction

When the bodies are moving in the same direction as shown in Figure, then by conservation of linear momentum, we have



$$m_1 u_1 + m_2 u_2 = m_1 v_{\text{comb}} + m_2 v_{\text{comb}}$$

$$\Rightarrow m_1 u_1 + m_2 u_2 = v_{\text{comb}} (m_1 + m_2)$$

$$\Rightarrow v_{\text{comb}} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

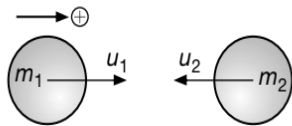
Also, loss in kinetic energy is

$$-\Delta K = K_i - K_f$$

$$\Rightarrow -\Delta K = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (u_1 - u_2)^2$$

CASE-2: Bodies moving in opposite direction

When the bodies are moving in the opposite direction as shown in Figure, then by conservation of linear momentum, we have



$$m_1 u_1 + m_2 (-u_2) = m_1 v_{\text{comb}} + m_2 v_{\text{comb}}$$

$$\Rightarrow m_1 u_1 - m_2 u_2 = v_{\text{comb}} (m_1 + m_2)$$

$$\Rightarrow v_{\text{comb}} = \frac{m_1 u_1 - m_2 u_2}{m_1 + m_2}$$

OPTION-I

If $v_{\text{comb}} = \oplus$, then

v_{comb} is along u_1

OPTION-II

If $v_{\text{comb}} = \ominus$, then

v_{comb} is along u_2

Also, loss in kinetic energy is

$$-\Delta K = K_i - K_f$$

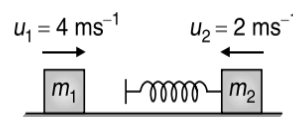
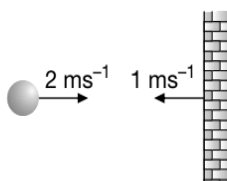
$$\Rightarrow -\Delta K = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) (u_1 + u_2)^2$$

Test Your Concepts-VI

Based on Head-On Collisions

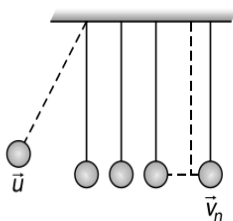
(Solutions on page H.85)

- A ball of mass m moving at a speed u makes a head on collision with an identical ball at rest. The kinetic energy of the balls after the collision is $3/4$ th of the original. Find the coefficient of restitution.
- A ball of mass 4 kg moving with a velocity of 12 ms^{-1} impinges directly on another ball of mass 8 kg moving with velocity of 4 ms^{-1} in the same direction. Find their velocities after impact and calculate the loss of KE due to impact if $e = 0.5$.
- A ball is moving with velocity 2 ms^{-1} towards a heavy wall moving towards the ball with speed 1 ms^{-1} as shown in figure. Assuming collision to be elastic, find the velocity of ball immediately after the collision.
- A particle A of mass m moving on a smooth horizontal surface collides with a stationary particle B of mass $2m$ directly, situated at a distance d from a wall. The coefficient of restitution both between A and B , between B and the wall is $e = \frac{1}{4}$. Calculate the distance from the wall where they collide again. Assume that the entire motion takes place along a straight line perpendicular to the wall.
- A block of mass $m_1 = 2 \text{ kg}$ initially moving to right with a speed of $u_1 = 4 \text{ ms}^{-1}$ on a frictionless, horizontal track, collides with a light spring attached to a second block of mass $m_2 = 1 \text{ kg}$ initially moving to left with a speed of $u_2 = 2 \text{ ms}^{-1}$ as shown in Figure.



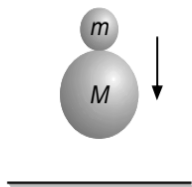
Calculate the velocities of the blocks when spring again comes to original length.

- A small block of mass m moving with a speed v_0 collides elastically with an identical block kept at rest. Now the second block after travelling a distance d collides elastically with a heavy wall moving towards it with velocity u . The coefficient of friction between the blocks and the ground is μ . Find the maximum value of u for which no further collision takes place between the blocks.
- n identical spheres of mass m are suspended with wires of equal length. The spheres are almost in contact with each other. Sphere 1 is pulled aside and released as shown in Figure.

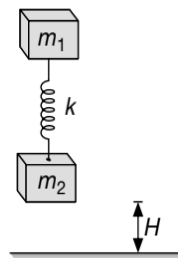


If sphere 1 strikes sphere 2 with velocity u , find an expression for velocity v_n of the n th sphere immediately after being struck by the one adjacent to it. Assume that the coefficient of restitution for each impact is e .

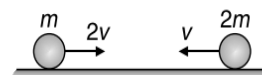
- A small ball of mass m is placed on top of a super ball of mass M and the two balls are dropped to the floor from height h . How high does the small ball rise after the collision? Assume that the collisions with the superball are elastic and that $m \ll M$.



- Three balls having masses m_1 , m_2 and m_3 are lying in a straight line. The first ball moving with a certain velocity strikes the second ball directly and itself comes to rest. The second ball collides with the third and also comes to rest. If e is the coefficient of restitution for each ball, find the mass of ball m_2 in terms of m_1 and m_3 .
- A spring mass system is held at rest with the spring relaxed at a height H above the ground. Determine the minimum value of H so that the system has a tendency to rebound after hitting the ground. Given that the coefficient of restitution between m_2 and ground is zero.



- Two particles of masses m and $2m$ moving in opposite directions collide elastically with velocity $2v$ and v , respectively. Find their velocities after collision.



- A light spring of spring constant k is kept compressed between two blocks of masses m and M on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions. The spring loses contact with the block when it acquires natural length. If the spring was initially compressed through a distance x , find the final speeds of the two blocks.

OBLIQUE COLLISION

Physics of Oblique Collision

Before dealing with oblique collisions, we must keep in mind the following steps so that the problems can be solved with ease:

STEP-1: Draw a **common tangent line (t -line)** to the colliding surfaces at the point of impact. This line is called t -line.

STEP-2: Perpendicular to the common tangent line (t -line), draw the line of impact (also called n -line). The impact forces will be acting only along n -line (and have no component along t -line. **In other words, the impact forces are acting along the line joining the centres of the two bodies.**

STEP-3: Since impact forces have no components along t -line, hence the **individual momentum** or **individual velocity** of the particle must be conserved along t -line.

For A:

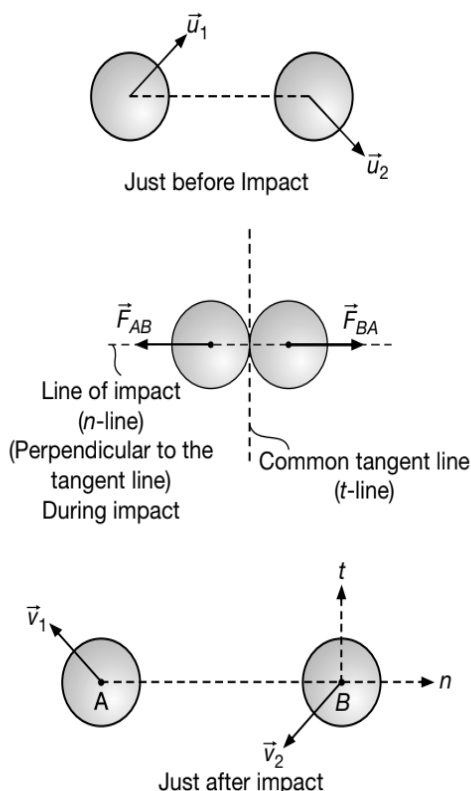
$$[(p_A)_i]_t = [(p_A)_f]_t$$

$$\Rightarrow (u_1)_t = (v_1)_t$$

For B:

$$[(p_B)_i]_t = [(p_B)_f]_t$$

$$\Rightarrow (u_2)_t = (v_2)_t$$



STEP-4: The **total momentum** of the system ($A + B$ system) remains conserved along n -line.

For A (along n -line)

$$\Delta \vec{p}_A = \int \vec{F}_{AB} dt$$

$$\Rightarrow m_1 [(v_1)_n - (u_1)_n] = \int \vec{F}_{AB} dt \quad \dots(1)$$

For B (along n -line)

$$\Delta \vec{p}_B = \int \vec{F}_{BA} dt$$

$$\Rightarrow m_2 [(v_2)_n - (u_2)_n] = \int \vec{F}_{BA} dt \quad \dots(2)$$

Since $\vec{F}_{AB} = -\vec{F}_{BA}$ {from Newton's Third Law}

$$\Rightarrow m_1 (v_1)_n - m_1 (u_1)_n = -m_2 (v_2)_n + m_2 (u_2)_n$$

$$\Rightarrow m_1 (\bar{v}_1)_n + m_2 (\bar{v}_2)_n = m_1 (\bar{v}_1)_n + m_2 (\bar{v}_2)_n$$

$$\left(\begin{array}{l} \text{Total initial} \\ \text{momentum of} \\ A + B \text{ system} \end{array} \right)_{\text{along } n\text{-line}} = \left(\begin{array}{l} \text{Total final} \\ \text{momentum of} \\ A + B \text{ system} \end{array} \right)_{\text{along } n\text{-line}}$$

STEP-5: For elastic oblique collisions,

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

STEP-6: For inelastic oblique collision (instead of STEP-5), if e be the coefficient of restitution at the point of impact, then

$$e = - \left[\frac{(v_2)_n - (v_1)_n}{(u_2)_n - (u_1)_n} \right]$$



Conceptual Note(s)

To summarise, we observe that we select two axis very smartly, the t -axis and the n -axis. Along t -axis (or t -line) no component of force acts, so individual momentum of particles is constant along t -axis. Along n -axis (or n -line) the net force on collective system is zero, so total momentum of system is conserved along n -line.

ILLUSTRATION 74

A ball of mass m hits a floor with a speed v_0 making an angle of incidence θ with the normal. The coefficient of restitution is e . Find the speed of the reflected ball and the angle of reflection ϕ of the ball.

SOLUTION

Since impact force has got no component along t -line, because it always acts along n -line, so the component of velocity v_0 along common tangent direction $v_0 \sin \alpha$ will remain unchanged. Let v be the velocity of ball just after impact such that angle of reflection is ϕ with the vertical. So,

$$v \sin \phi = v_0 \sin \theta \quad \dots(1)$$

Also, we know that

$$e = - \left[\frac{(v_2)_n - (v_1)_n}{(u_2)_n - (u_1)_n} \right]$$

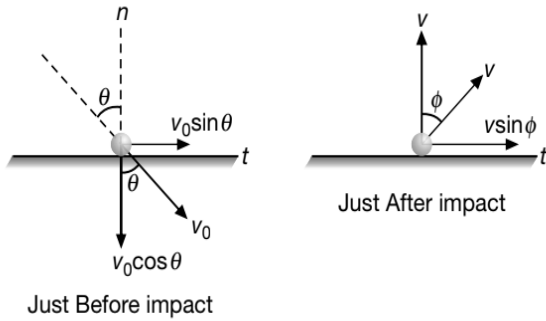
Now think of **1 as Ball** and **2 as Ground**

Since ground is at rest finally as well as initially, so $(v_2)_n = (u_2)_n = 0$

$$\Rightarrow e = -\left(\frac{0 - v \cos \phi}{0 - (-v_0 \cos \theta)}\right)$$

$$\Rightarrow e = \frac{v \cos \phi}{v_0 \cos \theta}$$

$$\Rightarrow v \cos \phi = e(v_0 \cos \theta) \quad \dots(2)$$



To calculate the speed of ball after impact, squaring (1) and (2) and adding, to get

$$v^2 = v_0^2 \sin^2 \theta + e^2 (v_0 \cos \theta)^2$$

$$\Rightarrow v = v_0 \sqrt{\sin^2 \theta + e^2 \cos^2 \theta}$$

To calculate the angle of reflection, divide (1) by (2), to get

$$\tan \phi = \frac{\tan \theta}{e}$$

$$\Rightarrow \phi = \tan^{-1} \left(\frac{\tan \theta}{e} \right)$$

ILLUSTRATION 75

A sphere, of mass m , impinges obliquely on a sphere, of mass M , which is at rest. Show that, if $m = eM$, the directions of motion of the spheres after impact are at right angles.

SOLUTION

Let v_1 and v_2 be the components of velocity of m before collision, then we have

	Mass m		Mass M	
	t -line	n -line	t -line	n -line
Before collision	v_1	v_2	0	0
After collision	v_1	$v_3 = 0$	0	v_4

$$v_3 = \left(\frac{M - eM}{m + eM}\right)v_2 = 0 \quad \{\because m = 2m\}$$

$$\text{and } v_4 = \left(\frac{m + eM}{m + M}\right)v_2 \neq 0$$

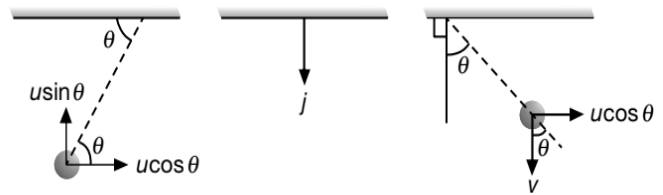
So, after collision m is along t -line while M along n -line and since t -line is perpendicular to n -line, so after impact both spheres move at right angles to each other.

ILLUSTRATION 76

The coefficient of restitution between a snooker ball and the side cushion is $\frac{1}{3}$. If the ball hits the cushion and then rebounds at right angles to its original direction, show that the angles made with the side cushion by the direction of motion before and after impact are 60° and 30° respectively.

SOLUTION

Let the original speed be u , in a direction making an angle θ with the side cushion.



Using the law of restitution

$$v = \frac{1}{3}(u \sin \theta)$$

$$\text{After impact, } \tan \theta' = \frac{u \cos \theta}{v} = \frac{3 \cos \theta}{\sin \theta}$$

$$\Rightarrow \tan^2 \theta' = 3$$

$$\Rightarrow \tan \theta' = \sqrt{3}$$

$$\Rightarrow \theta' = 60^\circ$$

Therefore, the directions of motion before and after impact are at 60° and 30° to the cushion.

ILLUSTRATION 77

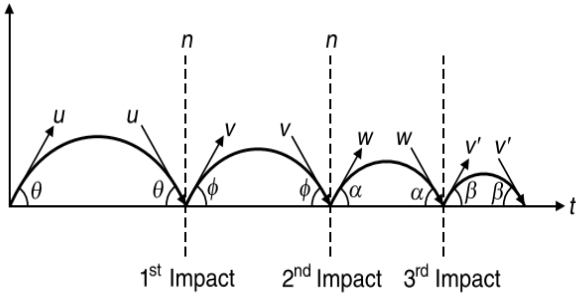
A projectile is launched with initial velocity u making angle θ with the horizontal. Assuming the coefficient of restitution between the projectile and the ground to be e , find the following:

- (a) the total horizontal distance travelled before it comes to rest.

- (b) the total time lapsed before it comes to rest.
 (c) the average horizontal velocity during the tenure.
 (d) the launch angle of the projectile just after n^{th} impact.

SOLUTION

About t -line, individual velocity of particle is conserved, so we have



$$v \cos \phi = u \cos \theta \quad \dots(1)$$

$$w \cos \alpha = v \cos \phi = u \cos \theta \quad \dots(2)$$

About n -line

$$\left(\begin{array}{c} \text{Speed After} \\ \text{Collision} \end{array} \right)_{\text{along } n\text{-line}} = e \left(\begin{array}{c} \text{Speed Before} \\ \text{Collision} \end{array} \right)_{\text{along } n\text{-line}}$$

After 1st Impact

$$v \sin \phi = e(u \sin \theta) \quad \dots(3)$$

After 2nd Impact

$$w \sin \alpha = e(v \sin \phi) = e^2(u \sin \theta) \quad \dots(4)$$

After 3rd Impact

$$v' \sin \beta = e(w \sin \alpha) = e^2(v \sin \phi) = e^3(u \sin \theta) \dots(5)$$

(a) Now, $R_1 = \frac{2}{g}(u \cos \theta)(u \sin \theta) = R$

and $R_2 = \frac{2}{g}(v \cos \phi)(v \sin \phi)$

$$\Rightarrow R_2 = \frac{2}{g}(u \cos \theta)(eu \sin \theta) = eR$$

Similarly, $R_3 = e^2R$

Total horizontal distance travelled by the projectile before it comes to rest is

$$x = R + eR + e^2R + e^3R + \dots$$

$$\Rightarrow x = R(1 + e + e^2 + \dots)$$

$$\Rightarrow x = \frac{R}{1 - e}$$

$$\Rightarrow x = \frac{2u^2 \sin(2\theta)}{g(1 - e)}$$

Total time lapsed is

$$T = \frac{2u \sin \theta}{g} + \frac{2v \sin \phi}{g} + \frac{2w \sin \alpha}{g} + \dots$$

$$\Rightarrow T = \frac{2u \sin \theta}{g}(1 + e + e^2 + \dots)$$

$$\Rightarrow T = \frac{2u \sin \theta}{g} \left(\frac{1}{1 - e} \right)$$

(c) Average Horizontal Velocities

$$(v_{av})_x = \frac{\text{Total Horizontal Displacement}}{\text{Total Time Taken}}$$

$$\Rightarrow (v_{av})_x = \frac{x}{t} = \frac{\left(\frac{2u^2 \sin(2\theta)}{g(1 - e)} \right)}{\left(\frac{2u \sin \theta}{g(1 - e)} \right)}$$

Since $\sin(2\theta) = 2 \sin \theta \cos \theta$

$$\Rightarrow (v_{av})_x = u \cos \theta$$

(d) From (3) and (1), we get

$$\tan \phi = e \tan \theta$$

From (4) and (2), we get

$$\tan \alpha = e \tan \phi = e^2 \tan \theta$$

So, after n^{th} impact, we get

$$\tan(\theta_n) = e^n \tan \theta$$

$$\Rightarrow \theta_n = \tan^{-1}(e^n \tan \theta)$$

Conceptual Note(s)

In an oblique collision between two bodies A and B if body B is at rest, then after impact, the body at rest i.e., B will move along the n -line (or the line of impact). Because, for the body B , at rest, we have, along t -line

$$(v_B)_t = (u_B)_t = 0$$

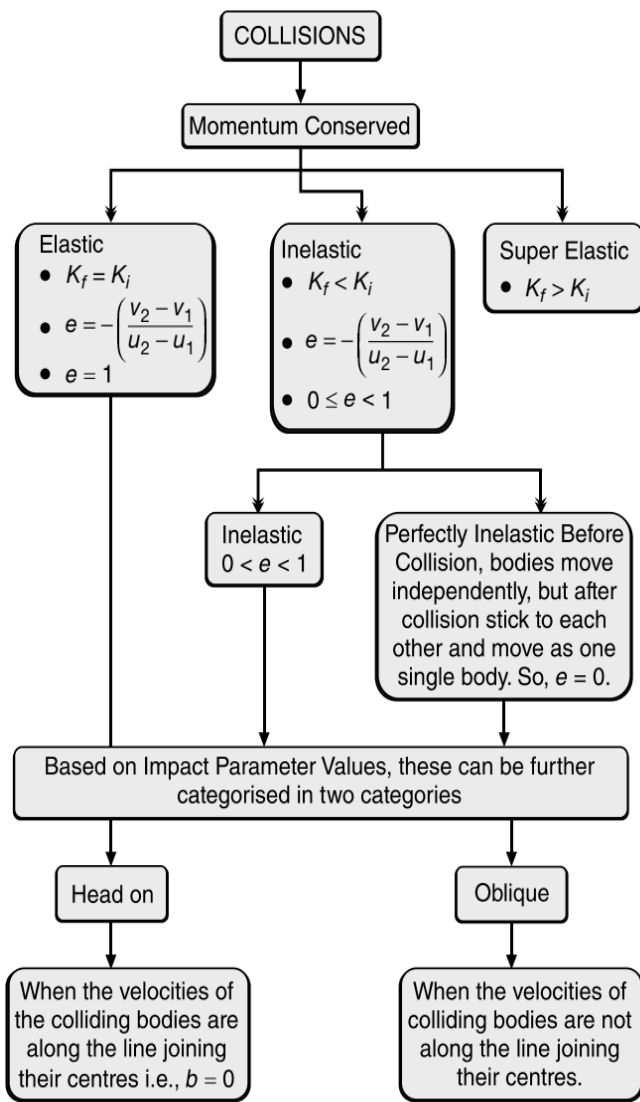
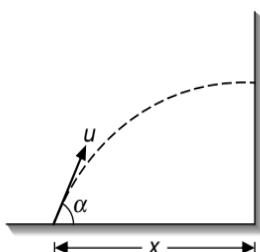


ILLUSTRATION 78

A ball is projected from a given point with velocity u at some angle with the horizontal and after hitting a vertical wall returns to the same point. If the distance of the point from the wall is x , then prove that $x < \frac{eu^2}{(1+e)g}$, where e is the coefficient of restitution.

SOLUTION

The situation discussed in the problem is shown in Figure.



$$\text{Since, } \frac{x}{u \cos \alpha} + \frac{x}{eu \cos \alpha} = T = \frac{2u \sin \alpha}{g}$$

$$\Rightarrow x = \frac{eu^2 \sin 2\alpha}{(1+e)g}$$

At $\alpha = 45^\circ$, $\sin(2\alpha) = 1$, so the value of x is maximum and is given by

$$x_{\max} = \frac{eu^2}{(1+e)g}$$

Hence, all other values of α will yield a value of x that happens to be less than $x_{\max} = \frac{eu^2}{(1+e)g}$.

$$\Rightarrow x < x_{\max}$$

$$\Rightarrow x < \frac{eu^2}{(1+e)g}$$

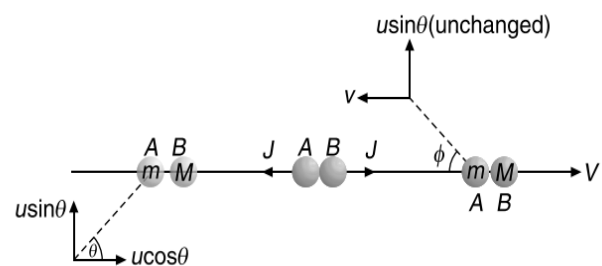
$$\Rightarrow x_{\max} = \frac{eu^2}{(1+e)g} \text{ at } 2\alpha = 90^\circ$$

ILLUSTRATION 79

Two smooth spheres A and B , of equal radius but masses m and M , are free to move on a horizontal table. A is projected with speed u towards B which is at rest. On impact, the line joining their centres is inclined at an angle θ to the velocity of A before impact. If e is the coefficient of restitution between the spheres, find the speed with which B begins to move. If A 's path after impact is perpendicular to its path before impact, show that $\tan^2 \theta = \frac{eM - m}{M + m}$.

SOLUTION

When B is struck by the impulse J , it begins to move in the direction of J as shown in the diagram.



Along the line of centres, we apply

(a) Conservation of Linear Momentum,
i.e., $mu \cos \theta = MV - mv$... (1)

(b) Law of restitution,
i.e., $eu \cos \theta = V + v$... (2)

Solving equations (1) and (2), we get

$$v = \frac{(eM - m)u \cos \theta}{(M + m)}$$

and $V = \frac{(1 + e)mu \cos \theta}{M + m}$

Hence, $\tan \phi = \frac{u \sin \theta}{v} = \frac{(M + m) \tan \theta}{(eM - m)}$

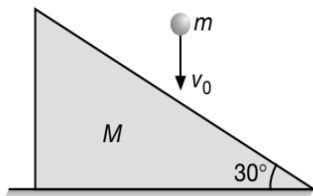
But the paths of A before and after impact are at right angles, therefore, $\cot \phi = \tan \theta$

$$\Rightarrow \frac{(eM - m)}{(M + m) \tan \theta} = \tan \theta$$

$$\Rightarrow \tan^2 \theta = \frac{eM - m}{(M + m)}$$

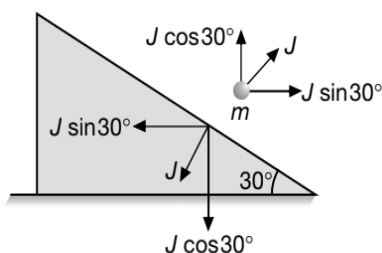
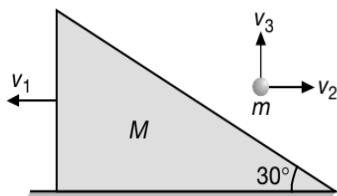
ILLUSTRATION 80

A ball of mass $m = 1 \text{ kg}$ falling vertically with a velocity $v_0 = 2 \text{ ms}^{-1}$ strikes a wedge of mass $M = 2 \text{ kg}$ kept on a smooth, horizontal surface as shown in figure. The coefficient of restitution between the ball and the wedge is $e = \frac{1}{2}$. Find the velocity of the wedge and the ball immediately after collision.



SOLUTION

Let, J be the impulse between ball and wedge during collision and v_1 , v_2 and v_3 be the **components of velocity** of the wedge and the ball in horizontal and vertical directions respectively as shown in Figure.



Since, Impulse = Change in Momentum

$$\Rightarrow J \sin(30^\circ) = Mv_1 = mv_2, \text{ along horizontal}$$

$$\Rightarrow \frac{J}{2} = 2v_1 = v_2 \quad \dots(1)$$

Also, $J \cos(30^\circ) = m(v_3 + v_0)$, along vertical

$$\Rightarrow \frac{\sqrt{3}}{2} J = (v_3 + 2) \quad \dots(2)$$

Since $e = -\frac{(v_2)_n - (v_1)_n}{(u_2)_n - (u_1)_n}$

$$\left(\begin{array}{l} \text{Relative Velocity} \\ \text{of Separation just} \\ \text{after Impact} \end{array} \right)_{n \text{ line}} = e \left(\begin{array}{l} \text{Relative Velocity} \\ \text{of Approach just} \\ \text{before Impact} \end{array} \right)_{n \text{ line}}$$

$$\Rightarrow (v_1 + v_2) \sin(30^\circ) + v_3 \cos(30^\circ) = \frac{1}{2}(v_0 \cos(30^\circ))$$

$$\Rightarrow v_1 + v_2 + \sqrt{3}v_3 = \sqrt{3} \quad \dots(3)$$

Solving equations (1), (2) and (3), we get

$$v_1 = \frac{1}{\sqrt{3}} \text{ ms}^{-1}, v_2 = \frac{2}{\sqrt{3}} \text{ ms}^{-1} \text{ and } v_3 = 0$$

Thus, velocities of wedge and ball are

$$v_1 = \frac{1}{\sqrt{3}} \text{ ms}^{-1} \text{ and } v_2 = \frac{2}{\sqrt{3}} \text{ ms}^{-1} \text{ in horizontal}$$

direction as shown in Figure.

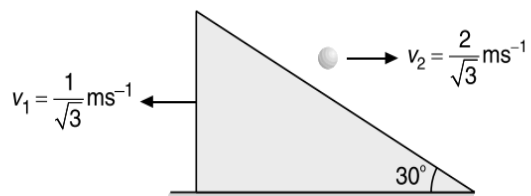
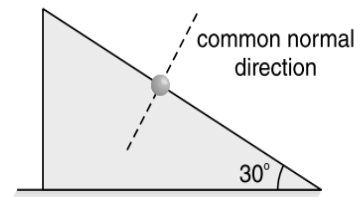


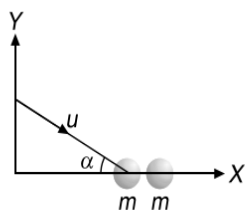
ILLUSTRATION 81

A ball moving translationally collides elastically with another stationary ball of the same mass. At the moment of impact the angle between the straight line passing through the centres of the balls and the direction of the initial motion of the striking ball is equal to $\alpha = 45^\circ$. Assuming the balls to be smooth, find the fraction η of the kinetic energy of the striking

ball that turned into potential energy at the moment of the maximum deformation.

SOLUTION

Let a ball of mass m with velocity u collides with stationary ball as shown in figure.



Applying the Law of Conservation of Momentum along x -axis, we have

$$mu \cos \alpha = mv_{1x} + mv_{2x}$$

At the moment of maximum deformation

$$v_{1x} = v_{2x}$$

$$\Rightarrow u \cos \alpha = v_{1x} + v_{1x} = 2v_{1x}$$

or
$$v_{1x} = \frac{u \cos \alpha}{2}$$

$$\text{Initial kinetic energy, } K_{\text{initial}} = \frac{1}{2} mu^2$$

$$\text{Final kinetic energy, } K_{\text{final}} = \frac{1}{2} mv_{1x}^2 + \frac{1}{2} mv_{2x}^2 + \frac{1}{2} mv_{1y}^2$$

$$\Rightarrow K_{\text{final}} = 2 \left(\frac{1}{2} mv_{1x}^2 \right) + \frac{1}{2} mv_{1y}^2$$

$$\Rightarrow K_{\text{final}} = \frac{mu^2 \cos^2 \alpha}{4} + \frac{mu^2 \sin^2 \alpha}{2}$$

$$\Rightarrow K_{\text{final}} = \frac{mu^2}{4} \cos^2 45^\circ + \frac{mu^2}{2} \sin^2 45^\circ$$

$$\Rightarrow K_{\text{final}} = \frac{mu^2}{8} + \frac{mu^2}{4} = \frac{3mu^2}{8}$$

Therefore, potential energy at the moment of maximum deformation is

$$U = \frac{mu^2}{2} - \frac{3mu^2}{8} = \frac{mu^2}{8}$$

So, the desired fraction η is given by

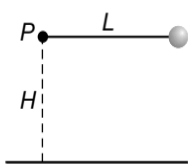
$$\eta = \frac{mu^2/8}{mu^2/2} = \frac{1}{4} = 0.25$$

Test Your Concepts-VII

Based on Oblique Collisions

(Solutions on page H.88)

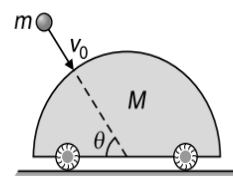
- A point P is fixed at a height H above a perfectly inelastic smooth horizontal plane. A light inextensible string of length $L (> H)$ has one end attached to P and other is attached to a heavy particle. The particle is held at the level of P with string just taut and released from rest. Find the height of the particle above the plane when it is next instantaneously at rest.



- A smooth sphere of mass m is moving on a horizontal plane with a velocity $3\hat{i} + \hat{j}$ when it collides with a vertical wall which is parallel to the vector \hat{j} . If the coefficient of restitution between the sphere and the wall is $\frac{1}{2}$, find the velocity of the sphere

after impact, the loss in kinetic energy caused by the impact and the impulse \vec{J} that acts on the sphere.

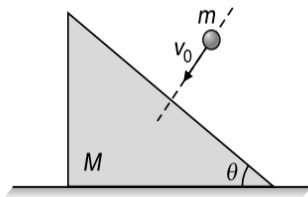
- A putty ball of mass m strikes a smooth stationary wedge of mass M with a velocity v_0 making an angle θ with the horizontal as shown in Figure.



If the collision is perfectly inelastic, calculate the velocity of the wedge just after collision and the loss in kinetic energy of the system.

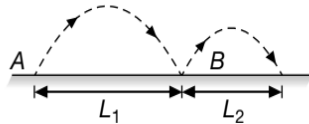
- A sphere of mass m is moving with a velocity $4\hat{i} - \hat{j}$ when it hits a wall and rebounds with velocity $\hat{i} + 3\hat{j}$. Calculate the impulse it receives. Also calculate the coefficient of restitution between the sphere and the wall.

5. A ball of mass m , moving with a velocity u along x -axis, strikes another ball of mass $2m$ kept at rest. The first ball comes to rest after collision and the other breaks into two equal pieces. One of the pieces starts moving along y -axis with a speed v . What will be the speed of the other piece?
6. A ball of mass m normally hits a wedge of mass M lying at rest on a smooth horizontal surface as shown in Figure.

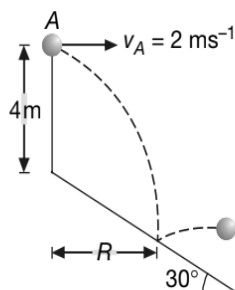


If v_0 be the velocity with which the ball hits the wedge, e be the coefficient of restitution between ball and the wedge, then calculate the velocity of wedge and ball just after collision.

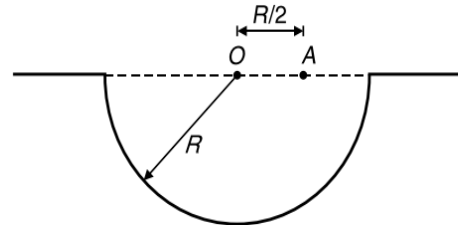
7. A projectile is launched from point A has a horizontal range L_1 as shown. If the coefficient of restitution at B is e , determine the distance L_2 .



8. A 0.5 kg ball is thrown horizontally from point A with velocity $v_A = 2 \text{ ms}^{-1}$. Determine the horizontal distance R where the ball strikes the smooth inclined plane. If the coefficient of restitution is $e = 0.6$, determine the speed at which it bounces from the plane. Take $g = 9.8 \text{ ms}^{-2}$.



9. A semi cylindrical frictionless track of radius R is centred at the point O . A particle is released from point A such that $OA = R/2$ as shown in Figure.

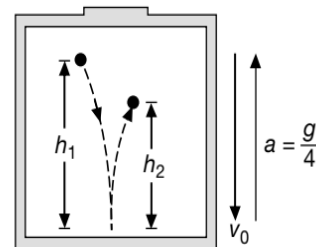


Calculate the coefficient of restitution e , if after collision with the track, the particle moves along the track. Also calculate e if the velocity of particle becomes horizontal just after collision with the track.

10. A ball is released from rest relative to the elevator at a distance h_1 above the floor. The speed of the elevator at the instant ball is released is v_0 . Determine the bounce height h_2 of the ball

- (a) if v_0 is constant and
 (b) if an upward elevator acceleration $a = \frac{g}{4}$ begins at the instant the ball is released.

The coefficient of restitution for the impact is e .



11. A ball is projected from the ground with speed u at an angle α with horizontal. It collides with a wall at a distance a from the point of projection and returns to its original position. Find the coefficient of restitution between the ball and the wall.

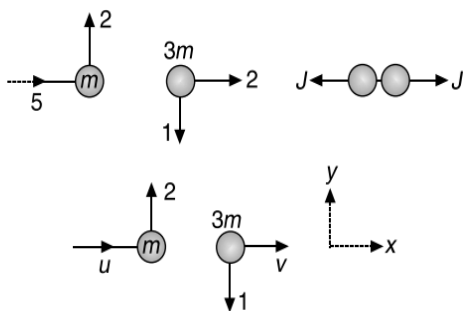
SOLVED PROBLEMS

PROBLEM 1

Two smooth spheres, A and B , of equal radius, lie on a horizontal table. A is of mass m and B is of mass $3m$. The spheres are projected towards each other with velocity vectors $5\hat{i} + 2\hat{j}$ and $2\hat{i} - \hat{j}$ respectively and when they collide the line joining their centres is parallel to the vector \hat{i} . If the coefficient of restitution between A and B is $\frac{1}{3}$, find their velocities after impact and the loss in kinetic energy caused by the collision. Find also the magnitude of the impulses that act at the instant of impact.

SOLUTION

Since, during impact, the line joining the centres is parallel to the vector \hat{i} so, the velocity components of A and B perpendicular to \hat{i} stay unchanged by the impact.



Applying Law of Conservation of Linear Momentum and the Law of Restitution, we get

$$(m)(5) + (3m)(2) = mu + 3mv \quad \dots(1)$$

$$\text{and } \frac{1}{3}(5 - 2) = v - u \quad \dots(2)$$

Solving equations (1) and (2), we get

$$u = 2 \text{ and } v = 3$$

The velocities of A and B after impact are therefore, $2\hat{i} + 2\hat{j}$ and $3\hat{i} - \hat{j}$ respectively

Before impact, the kinetic energy of A and B is

$$(K_A)_i = \frac{1}{2}m(5^2 + 2^2) = \frac{29}{2}m$$

$$(K_B)_i = \frac{1}{2}(3m)(2^2 + 1^2) = \frac{15}{2}m$$

After impact, the kinetic energy of A and B is

$$(K_A)_f = \frac{1}{2}m(2^2 + 2^2) = 4m$$

$$(K_B)_f = \frac{1}{2}(3m)(3^2 + 1^2) = 15m$$

Therefore, the loss in K.E. at impact is

$$-\Delta K = (K_A + K_B)_i - (K_A + K_B)_f$$

$$\Rightarrow -\Delta K = \left(\frac{29}{2}m + \frac{15}{2}m\right) - (4m + 15m) = 3m$$

To find the value of J , we consider the change in momentum along \hat{i} for any one sphere.

For sphere B , we have

$$J = 3m(3 - 2)$$

$$\Rightarrow J = 3m$$

PROBLEM 2

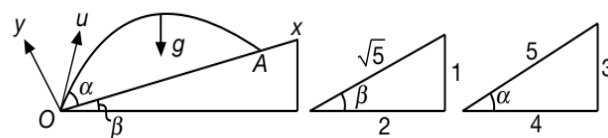
A particle P is projected from a point O on a smooth plane inclined to the horizontal at $\tan^{-1}\left(\frac{1}{2}\right)$. The

particle is projected at $\tan^{-1}\left(\frac{3}{4}\right)$ to the plane. It hits

the plane at higher point A and rebounds. Find the possible range of the coefficient of restitution e if the particle P continues to move up the plane after the impact.

SOLUTION

Time of flight of the particle is



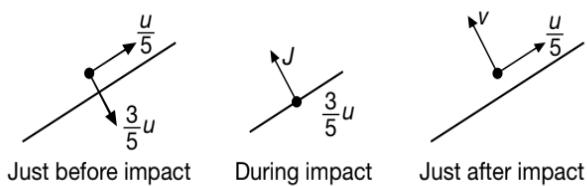
$$T = \frac{2u_y}{g \cos \beta} = \frac{2u \sin \alpha}{g \cos \beta} = \frac{(2u)\left(\frac{3}{5}\right)}{(g)\left(\frac{2}{\sqrt{5}}\right)} = \frac{3u}{\sqrt{5}g}$$

At A , $v_x = u_x + a_x T$ and $v_y = u_y + a_y T$

Substituting values, we get

$$v_x = \frac{u}{5}$$

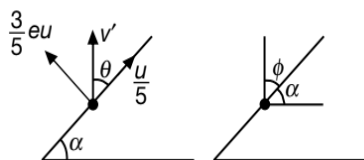
$$v_y = -\frac{3}{5}u$$



From the Figure, we get $u = e \left(\frac{3}{5}u \right)$

Just after impact, the resultant velocity v' of P is inclined to the plane at an angle θ , where

$$\tan \theta = \frac{\left(\frac{3}{5}\right)eu}{\left(\frac{u}{5}\right)} = 3e$$



P continues to move up the plane if,

$$\theta < \phi, \text{ where } \phi = \frac{\pi}{2} - \beta$$

$$\Rightarrow \tan \theta < \tan \left(\frac{\pi}{2} - \beta \right)$$

$$\Rightarrow \tan \theta < \cot \beta$$

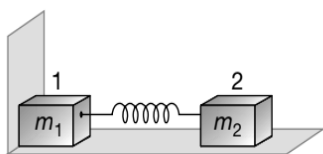
$$\Rightarrow 3e < 2$$

$$\Rightarrow e < \frac{2}{3}$$

Since, we also know that $e \geq 0$, so the possible range of e is $0 \leq e < \frac{2}{3}$

PROBLEM 3

Two bars of masses m_1 and m_2 connected by a light spring of force constant k at rest on a smooth horizontal plane. Bar 2 is shifted a small distance x to the left and then released. Find the velocity of the centre of mass of the system after bar 1 breaks off the wall.



SOLUTION

When bar 1 breaks off the wall, its velocity is zero. Let v_2 be the velocity of bar 2. Then the velocity of the cm is given by

$$(m_1 + m_2)v_{cm} = m_1(0) + m_2v_2$$

$$\Rightarrow v_{cm} = \frac{m_2v_2}{m_1 + m_2}$$

Using the Law of Conservation of Energy

$$\frac{1}{2}kx^2 = \frac{1}{2}m_2v_2^2$$

$$\Rightarrow v_2 = x\sqrt{\frac{k}{m_2}}$$

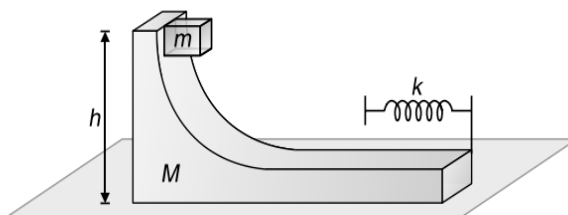
$$\Rightarrow v_{cm} = \frac{m_2x}{m_1 + m_2}\sqrt{\frac{k}{m_2}} = \frac{x}{m_1 + m_2}\sqrt{km_2}$$

Since the system is free from external force, the acceleration of its cm is zero. So, the velocity of the cm is always

$$v_{cm} = \frac{x}{m_1 + m_2}\sqrt{km_2}$$

PROBLEM 4

A block of mass m is released from rest from a height h onto a smooth sledge of mass M fitted with an ideal spring of stiffness k . Calculate the velocity of the block and sledge just before the block touches the spring. Also calculate the maximum compression in the spring.



SOLUTION

Since no external force is acting along the horizontal direction, so the linear momentum is conserved along the horizontal direction. Let the velocity of block and wedge be v and V respectively up to the case when the block reaches on horizontal surface on the wedge.

$$mv = MV \quad \dots(1)$$

Mechanical energy is also conserved in this case

$$\Delta K + \Delta U = 0$$

$$\Rightarrow \left(\frac{1}{2}mv^2 + \frac{1}{2}MV^2 - 0 \right) + (-mgh) = 0 \quad \dots(2)$$

From equations (1) and (2), we get

$$\frac{1}{2}mv^2 + \frac{1}{2}M\left(\frac{m}{M}v\right)^2 = mgh$$

$$\Rightarrow v = \sqrt{\frac{2Mgh}{M+m}} \text{ and } V = \sqrt{\frac{2m^2gh}{M(M+m)}}$$

We can also get the same result by applying the work energy theorem from centre of mass reference frame

$$W_{\text{gravity}} = \Delta K_{\text{cm}}$$

$$\Rightarrow mgh = \frac{1}{2}\left(\frac{mM}{m+M}\right)(v+V)^2$$

Also, by conservation of linear momentum, we have

$$mv = MV$$

$$\Rightarrow v = \frac{MV}{m}$$

$$\Rightarrow mgh = \frac{1}{2}\left(\frac{mM}{m+M}\right)V^2\left(\frac{M}{m}+1\right)^2$$

$$\Rightarrow V = \sqrt{\frac{2m^2gh}{M(m+M)}}$$

Since initial linear momentum of the system is zero and there is no external force on the system, hence at the instant of maximum compression of spring, the linear momentum of the system should also be zero. So, the final kinetic energy of system is also zero. By conservation of mechanical energy, we have

$$\Delta U + \Delta K = 0 \quad \dots(3)$$

where, $\Delta K = K_f - K_i = 0$

So, equation (3), becomes

$$\Delta U_{\text{gravity}} + \Delta U_{\text{spring}} = 0$$

$$\Rightarrow -mgh + \frac{1}{2}kx^2 = 0$$

$$\Rightarrow x = \sqrt{\frac{2mgh}{k}}$$

PROBLEM 5

A uniform thin rod of mass M and length L is standing vertically along the y -axis on a smooth horizontal surface, with its lower end at the origin $(0, 0)$. A slight disturbance at time $t = 0$ causes the lower end to slip on the smooth surface along the positive x -axis, and the rod starts falling.

- What is the path followed by the centre of mass of the rod during its fall?
- Find the equation of trajectory of a point P on the rod located at a distance r from the lower end. What is the shape of the path of this point?

SOLUTION

- As the floor is smooth, no horizontal forces act on the rod.

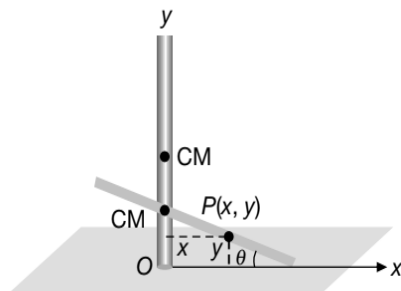
So, centre of mass moves vertically downwards or the path followed by centre of mass is a vertical line along y -axis, i.e., $x = 0$

- At any instant for a point P on the rod.

$$x = \left[\left(\frac{L}{2}\right) - r\right] \cos \theta \text{ and } y = r \sin \theta$$

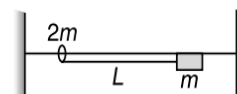
So, eliminating θ between these, we get trajectory of the point P as

$$\frac{x^2}{\left[\left(\frac{L}{2}\right) - r\right]^2} + \frac{y^2}{r^2} = 1 \text{ which is an ellipse.}$$



PROBLEM 6

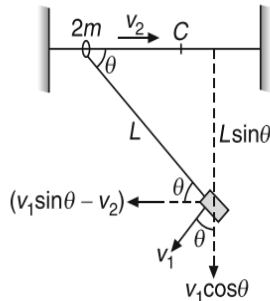
A bead can slide on a smooth straight wire and a particle of mass m attached to the bead by a light string of length l . The particle is held in contact with the wire with string taut and as shown in figure and then let fall. If the bead has a mass $2m$. Then when the string makes an angle θ with the wire, find the speed of the bead and the distance it slides up to this instant.



SOLUTION

When the system is released from rest, m starts falling and due to tension in string bead starts sliding to right. Since no external force is acting on system

along the horizontal direction, so centre of mass of the system will not displace in horizontal direction. Consider the situation when string makes an angle θ with the wire as shown in Figure.



Let centre of mass of system be at position C , then initially the distance of mass m from C is $\frac{2L}{3}$ and when it is at angle θ from the wire its horizontal distance from C is $\frac{2}{3}L\cos\theta$ and similarly that for bead are $\frac{L}{3}$ and $\frac{L}{3}\cos\theta$ respectively. So, the distance it slides is given by

$$d_{\text{bead}} = \frac{L}{3} - \frac{L}{3}\cos\theta = \frac{L}{3}(1 - \cos\theta)$$

If at this instant the speed of block and bead are v_1 (relative to bead) and v_2 then using work energy theorem we get

$$mgL\sin\theta = \frac{1}{2}m(v_1^2 + v_2^2 - 2v_1v_2\sin\theta) + \frac{1}{2}(2m)v_2^2 \dots(1)$$

Using momentum conservation in horizontal direction as

$$\begin{aligned} \Sigma F_{\text{horizontal}} &= 0 \\ m(v_1\sin\theta - v_2) &= (2m)v_2 \dots(2) \end{aligned}$$

Solving equations (1) and (2), we get

$$\Rightarrow v_1 = \frac{3v_2}{\sin\theta}$$

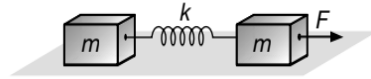
From equation (1), we get

$$2gL\sin\theta = v_2^2 \left(1 + \frac{9}{\sin^2\theta} - 6 \right)$$

$$\Rightarrow v_2 = \sqrt{\frac{2gL\sin^3\theta}{9 - 5\sin^2\theta}}$$

PROBLEM 7

Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure.



- Find the displacement of the centre of mass at time t .
- If the extension of the spring is x_0 at time t , find the displacement of the two blocks at this instant.

SOLUTION

- The acceleration of the centre of mass is

$$a_{\text{cm}} = \frac{F}{2m}$$

The displacement of the centre of mass at time t will be

$$x = \frac{1}{2}a_{\text{cm}}t^2 = \frac{Ft^2}{4m}$$

- Suppose the displacement of the first block is x_1 and that of the second is x_2 . Then,

$$\begin{aligned} x &= \frac{mx_1 + mx_2}{2m} \\ \Rightarrow \frac{Ft^2}{4m} &= \frac{x_1 + x_2}{2} \\ \Rightarrow x_1 + x_2 &= \frac{Ft^2}{2m} \dots(1) \end{aligned}$$

Further, the extension of the spring is $x_1 - x_2$.

$$\Rightarrow x_1 - x_2 = x_0 \dots(2)$$

From equations (1) and (2), we get

$$x_1 = \frac{1}{2} \left(\frac{Ft^2}{2m} + x_0 \right) \text{ and } x_2 = \frac{1}{2} \left(\frac{Ft^2}{2m} - x_0 \right)$$

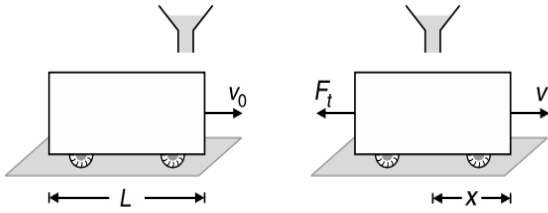
PROBLEM 8

A railroad car of length L and mass m_0 when empty is moving freely on a horizontal track while being loaded with sand from a stationary chute at a rate $\frac{dm}{dt} = \lambda$. Assuming that the car was approaching the chute at a speed v_0 , determine

- the mass of the car and its load after the car has cleared the chute,
- the speed of the car at that time.

SOLUTION

(a) At time t , $m = (m_0 + \lambda t)$



$$\text{Thrust force } F_t = v_r \frac{dm}{dt} = \lambda v \quad \{\text{backwards}\}$$

$$\Rightarrow a_t = \frac{F_t}{m} = \frac{\lambda v}{(m_0 + \lambda t)}$$

$$\Rightarrow \left(-\frac{dv}{dt}\right) = \left(\frac{\lambda v}{m_0 + \lambda t}\right)$$

$$\Rightarrow \frac{1}{\lambda} \int_{v_0}^v \frac{dv}{v} = -\int_0^t \frac{dt}{m_0 + \lambda t}$$

$$\Rightarrow \frac{1}{\lambda} \ln\left(\frac{v}{v_0}\right) = \frac{-1}{\lambda} \ln\left(\frac{m_0 + \lambda t}{m_0}\right)$$

$$\Rightarrow \frac{v}{v_0} = \left(\frac{m_0}{m_0 + \lambda t}\right)$$

$$\Rightarrow v = v_0 \left(\frac{m_0}{m_0 + \lambda t}\right) \quad \dots(1)$$

$$\Rightarrow \frac{dx}{dt} = v_0 \left(\frac{m_0}{m_0 + \lambda t}\right)$$

$$\Rightarrow \int_0^L dx = \int_0^t v_0 \left(\frac{m_0}{m_0 + \lambda t}\right) dt$$

$$\Rightarrow L = \frac{m_0 v_0}{\lambda} (\ln(m_0 + \lambda t)) \Big|_0^t$$

$$\Rightarrow L = \frac{m_0 v_0}{\lambda} \ln\left(\frac{m_0 + \lambda t}{m_0}\right)$$

So, mass of car and its load after the car has cleared the chute is

$$(m_0 + \lambda t) = m_0 e^{\frac{\lambda L}{m_0 v_0}}$$

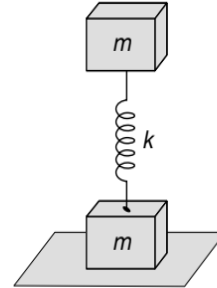
(b) From equation (1) speed of car at that moment is

$$v = \frac{m_0 v_0}{\frac{\lambda L}{m_0 v_0}} = v_0 e^{-\frac{\lambda L}{m_0 v_0}}$$

$$m_0 e^{\frac{\lambda L}{m_0 v_0}}$$

PROBLEM 9

A system consists of two identical cubes, each of mass m , linked together by the compressed light spring of spring constant k . The cubes are connected by a thread which is burnt at a certain moment. Find



- (a) at what values of x_0 , the initial compression of the spring, the lower cube will bounce up after the thread has been burned through;
- (b) to what maximum height will centre of mass of this system rise if the initial compression of the spring $\Delta l = \frac{7mg}{k}$.

SOLUTION

(a) Let x be the elongation in the spring when it returns. Then by Law of Conservation of Mechanical Energy, we get

$$\frac{1}{2} k x_0^2 = \frac{1}{2} k x^2 + mg(x + x_0)$$

$$\Rightarrow kx^2 + 2mgx - kx_0^2 + 2mgx_0 = 0$$

$$\Rightarrow kx = -mg \pm (mg - kx_0)$$

$$\Rightarrow kx = -kx_0 \text{ or } kx = kx_0 - 2mg$$

Since $kx = -kx_0$ gives $x = -x_0$ and this is not acceptable.

$$kx = kx_0 - 2mg$$

The lower cube will bounce up if $kx \geq mg$ (the weight of the lower cube)

$$\Rightarrow kx_0 - 2mg \geq mg$$

$$\Rightarrow x_0 \geq \frac{3mg}{k}$$

Thus, for all values of x_0 greater than $\frac{3mg}{k}$ the lower cube will bounce up.

- (b) If the initial compression, $\Delta l = \frac{7mg}{k}$ is greater than $\frac{3mg}{k}$, the lower mass m also moves up.

Let v be the velocity of the upper block at the position when the lower block just bounces off the floor, then by Law of Conservation of Energy, we get

$$\frac{1}{2}mv^2 = \frac{1}{2}k[(\Delta l)^2 - x^2] - mg(\Delta l + x)$$

$$\Rightarrow v^2 = 32 \frac{mg^2}{k} \quad \left\{ \because x = \frac{mg}{k} \right\}$$

Hence, the velocity of the centre of mass, at the moment lower block just bounces off the floor is

$$v_c = \frac{mv + 0}{2m} = \frac{v}{2}$$

If the centre of mass further rises to x' , then by Law of Conservation of Energy

$$\frac{1}{2}(2m)v_c^2 = (2m)gx'$$

$$\Rightarrow x' = \frac{v_c^2}{2g} = \frac{v^2}{8g} = \frac{1}{8g} \left(\frac{32mg^2}{k} \right) = \frac{4mg}{k}$$

$\{a_{cm} = g\}$

Before the break off of the lower mass, centre of mass has already moved up by a distance

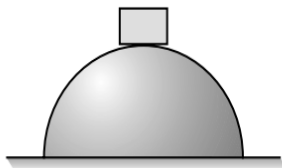
$$\frac{(\Delta l + x)m + 0}{2m} = \frac{4mg}{k} \quad \left\{ \Delta l = \frac{7mg}{k} \right\}$$

Hence, the maximum height attained by centre of mass of the system

$$h = \frac{4mg}{k} + \frac{4mg}{k} = \frac{8mg}{k}$$

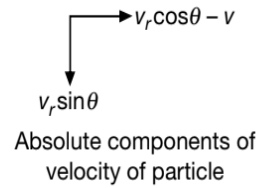
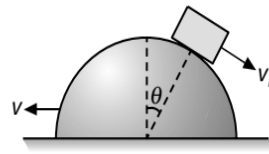
PROBLEM 10

A particle of mass m is placed on top of a smooth hemisphere also of mass m which is placed on a smooth horizontal surface. If the particle begins to slide down due to a negligible small impulse, show that it will lose contact with the hemisphere when the radial line through it makes an angle θ given by the equation $\cos^3 \theta - 6 \cos \theta + 4 = 0$.



SOLUTION

Let v_r be the velocity of particle as it leaves contact with the sphere ($N = 0$) and v the horizontal velocity of sphere at this instant.



Applying Conservation of Linear Momentum in horizontal direction we get,

$$mv = m(v_r \cos \theta - v)$$

$$\Rightarrow 2v = v_r \cos \theta \quad \dots(1)$$

Conservation of Mechanical Energy gives,

$$mgr(1 - \cos \theta) = \frac{1}{2}mv^2 + \frac{1}{2}m(v_r^2 + v^2 - 2vv_r \cos \theta)$$

$$\Rightarrow gr(1 - \cos \theta) = v^2 + \frac{v_r^2}{2} - vv_r \cos \theta \quad \dots(2)$$

Equation of Law of Motion gives,

$$mg \cos \theta = \frac{mv_r^2}{r}$$

$$\Rightarrow gr = \frac{v_r^2}{\cos \theta} \quad \dots(3)$$

Solving equations (1), (2) and (3), we get

$$\cos^3 \theta - 6 \cos \theta + 4 = 0$$

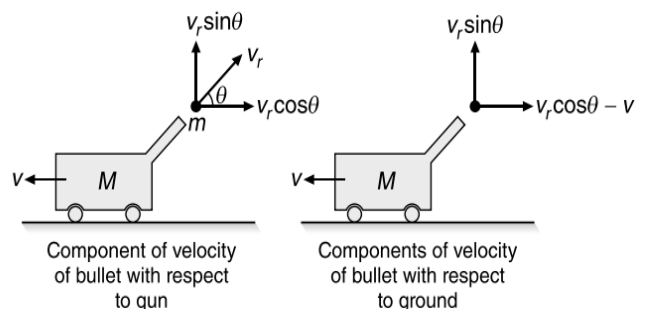
PROBLEM 11

A gun of mass M (including the carriage) fires a shot of mass m . The gun along with the carriage is kept on a smooth horizontal surface. The muzzle speed of the bullet v_r is constant. Find

- the elevation of the gun with horizontal at which maximum range of bullet with respect to the ground is obtained.
- the maximum range of the bullet.

SOLUTION

Muzzle velocity v_r is given to be constant



From Conservation of Linear Momentum in horizontal direction we have,

$$Mv = m(v_r \cos \theta - v)$$

$$\Rightarrow v = \frac{mv_r \cos \theta}{M+m} \quad \dots(1)$$

Further, range of bullet on horizontal ground

$$R = \frac{2v_r \sin \theta}{g} (v_r \cos \theta - v)$$

$$R = \frac{2v_r \sin \theta}{g} \left(v_r \cos \theta - \frac{mv_r \cos \theta}{M+m} \right)$$

$$R = \frac{2Mv_r^2 \sin \theta \cos \theta}{(M+m)g}$$

$$\Rightarrow R = \left(\frac{M}{M+m} \right) \frac{v_r^2 \sin 2\theta}{g} \quad \dots(2)$$

(a) From equation (2) we see that maximum range is at $\theta = 45^\circ$

(b) At $\theta = 45^\circ$, $R_{\max} = \left(\frac{M}{M+m} \right) \frac{v_r^2}{g}$

PROBLEM 12

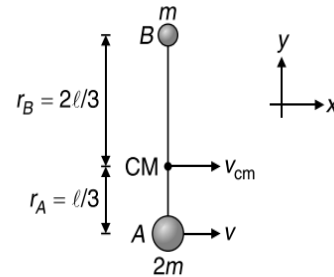
Two small spheres A and B of masses $2m$ and m respectively, are connected with an inextensible string of length l lie on a smooth horizontal plane. The sphere A is given a velocity of v along the horizontal plane perpendicular to line AB as shown in Figure. Calculate the tension in string during their subsequent motion by analysing the problem from the centre of mass reference frame.



SOLUTION

Since no external force is acting on the system of particles (i.e. the two spheres), so velocity of centre of mass of system remains unchanged and hence centre of mass of system will move along the x -direction and both spheres will move in circular path about centre of mass of the system.

In the centre of mass reference frame, the particles A and B move along circular path with same angular velocity, say ω .



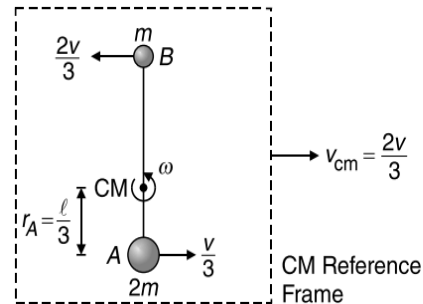
Velocity of centre of mass is given by

$$v_{\text{cm}} = \frac{m(0) + (2m)v}{3m} = \frac{2v}{3}$$

Distance of centre of mass from the sphere A is equal to the radius of the circle in which the sphere A revolves, so we have

$$r_A = \frac{(2m)(0) + ml}{2m + m} = \frac{l}{3}$$

In the centre of mass reference frame, the system is shown in the Figure.



Velocity of mass $2m$ with respect to centre of mass is given by

$$v_2 = v - \frac{2v}{3} = \frac{v}{3}$$

Hence, the angular velocity of particle A is given by

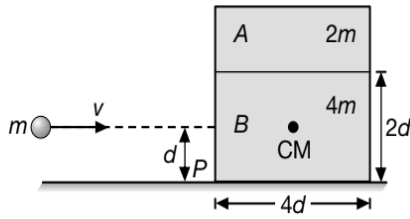
$$\omega = \frac{v/3}{l/3} = \frac{v}{l}$$

$$\Rightarrow T = (2m)r\omega^2 = (2m)\left(\frac{l}{3}\right)\left(\frac{v^2}{l^2}\right)$$

$$\Rightarrow T = \frac{2mv^2}{3l}$$

PROBLEM 13

A block A of mass $2m$ is placed on another block B of mass $4m$ which in turn is placed on a fixed table. The two blocks have the same length $4d$ and they are placed as shown in figure.



The coefficient of friction (both static and kinetic) between the block B and the table is μ . There is no friction between the two blocks. A small object of mass m moving horizontally along a line passing through the centre of mass of the block B and perpendicular to its face with a speed v collides elastically with the block B at a height d above the table.

- (a) What is the minimum value of v (call it v_0), required to make the block A topple.
 (b) If $v = 2v_0$, find the distance (from the point P) at which the mass m falls on the table after collision.

SOLUTION

If v_1 and v_2 are the velocities of object of mass m and block of mass $4m$, just after collision then by conservation of momentum,

$$mv = mv_1 + 4mv_2,$$

$$\text{i.e., } v = v_1 + 4v_2 \quad \dots(1)$$

Further, as collision is elastic

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}4mv_2^2,$$

$$\text{i.e., } v^2 = v_1^2 + 4v_2^2 \quad \dots(2)$$

Solving, these two equations we get either

$$v_2 = 0 \text{ or } v_2 = \frac{2}{5}v$$

$$\text{Therefore, } v_2 = \frac{2}{5}v$$

Substituting in equation (1)

$$v_1 = -\frac{3}{5}v$$

when $v_2 = 0$, $v_1 = v_2$, but it is physically unacceptable.

- (a) Now, after collision the block B will start moving with velocity v_2 to the right. Since, there is no friction between blocks A and B the upper block A will stay at its position and will topple if B moves a distance s such that

$$s > 2d \quad \dots(3)$$

However, the motion of B is retarded by frictional force $f = \mu(4m + 2m)g$ between table and its lower surface. So, the distance moved by B till it stops

$$0 = v_2^2 - 2\left(\frac{6\mu mg}{4m}\right)s,$$

$$\text{i.e., } s = \frac{v_2^2}{3\mu g}$$

Substituting this value of s in equation (3), we find that for toppling of A

$$v_2^2 > 6\mu g d$$

$$\Rightarrow \frac{2}{5}v > \sqrt{6\mu g d} \quad \left\{ \text{as } v_2 = \frac{2v}{5} \right\}$$

$$\text{i.e., } v > \frac{5}{2}\sqrt{6\mu g d}$$

$$\Rightarrow v_{\min} = v_0 = \frac{5}{2}\sqrt{6\mu g d}$$

- (b) If $v = 2v_0 = 5\sqrt{6\mu g d}$, the object will rebound with speed

$$v_1 = \frac{3}{5}v = 3\sqrt{6\mu g d}$$

and as time taken by it to fall down

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2d}{g}} \quad \left\{ \text{as } h = d \right\}$$

The horizontal distance moved by it to the left of P in this time

$$x = v_1 t = 6d\sqrt{3\mu}$$

- (a) Toppling will take place if line of action of weight does not pass through the base area in contact.
 (b) v_1 and v_2 can also be obtained by using the equations of head on elastic collision

$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_1 + \left(\frac{2m_2}{m_1 + m_2} \right) v_2$$

$$\text{and } v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_2 + \left(\frac{2m_1}{m_1 + m_2} \right) v_1$$

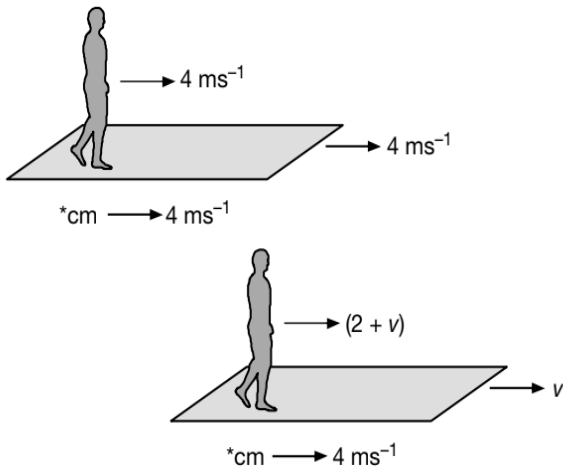
PROBLEM 14

A 75 kg man stands at the rear end of a platform of mass 25 kg and length 4 m, which moves initially at $4\hat{i} \text{ ms}^{-1}$ over a frictionless surface. At $t = 0$, he walks

at 2 ms^{-1} relative to the platform and then stops at the front end. During the period of walking, find the displacement of the platform, the man and the centre of mass

SOLUTION

Initially the man, the platform, and the centre of mass have the same velocity, $4\hat{i} \text{ ms}^{-1}$, as shown in the Figure.



- (a) A person at the rear of a platform moving at 4 ms^{-1} .
- (b) The person walks at 2 ms^{-1} relative to the platform. The velocity of the center of mass does not change.

When he begins to walk forward, his increase in momentum must be compensated for by a decrease in the platform's momentum. Let us say that the velocity of the platform relative to the ground while he is walking i.e.

$$\vec{v}_{PG} = v_p \hat{i}$$

The velocity of man relative to the ground is then

$$\vec{v}_{MG} = \vec{v}_{MP} + \vec{v}_{PG} = (2 + v_p)(4\hat{i})$$

From the conservation of momentum, we have

$$(75 + 25)4 = 75(2 + v_p) + 25v_p$$

Thus, the velocity of the platform is $v_p = 2.5 \text{ ms}^{-1}$ and the velocity of the man is $v_M = 4.5 \text{ ms}^{-1}$.

Since the velocity of man relative to platform is 2 ms^{-1} , therefore, it takes $t = \frac{4 \text{ m}}{2 \text{ ms}^{-1}} = 2 \text{ s}$ to walk from the rear to the front.

Displacement of platform is

$$x_p = (v_p)t = (2.5)(2) = 5 \text{ ms}^{-1}$$

Displacement of the man is

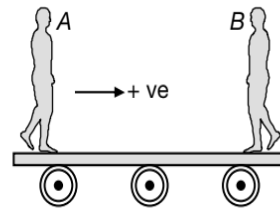
$$x_M = (v_M)t = (4.5)(2) = 9 \text{ m}$$

Displacement of the centre of mass

$$x_{cm} = (v_{cm})t = (4)(2) = 8 \text{ m}$$

PROBLEM 15

Two persons A and B , each of mass m are standing at the two extremes of rail-road car of mass M . Person A jumps to the left with a horizontal speed u with respect to the car. Thereafter, the person B jumps to the right, again with the same horizontal speed u with respect to the car. Find the velocity of the car after both the persons have jumped off.



SOLUTION

Initial velocity of car = 0

Let v_1 be the velocity of car (with person B) to the right, when person A jumps off the car. Then by Law of Conservation of Linear Momentum applied relative to earth, we get

$$0 = (m + M)v_1 + m(v_1 - u)$$

$$\Rightarrow v_1 = \frac{mu}{M + 2m} \quad \dots(1)$$

Let v_2 be the velocity of the car to the right when the second person B jumps off the car to the right with velocity u relative to car. Then again applying Law of Conservation of Linear Momentum, we get

$$(M + m)v_1 = Mv_2 + m(u + v_2)$$

$$\Rightarrow v_2 = v_1 - \frac{mu}{(M + m)}$$

Substituting the value of v_1 from equation (1), we get,

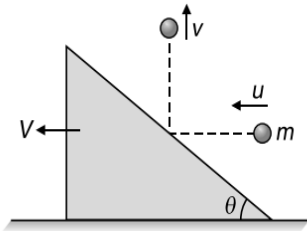
$$v_2 = \frac{mu}{M + 2m} - \frac{mu}{(M + m)}$$

$$\Rightarrow v_2 = \frac{-m^2u}{(M + 2m)(M + m)} \quad \{\text{towards right}\}$$

OR
$$v_2 = \frac{m^2u}{(M + 2m)(M + m)} \quad \{\text{towards left}\}$$

PROBLEM 16

A ball of mass m moving horizontally with velocity u hits a wedge of mass M . The wedge is situated on a smooth horizontal surface. If after striking with wedge the ball starts moving in vertical direction and the wedge starts moving in horizontal plane as shown in Figure.



Calculate the velocity V of wedge, the velocity (v) with which the ball moves in vertical direction, the impulse imparted by the ball to the wedge and the coefficient of restitution e .

SOLUTION

Since no external force is acting on the system along the horizontal direction, so the linear momentum is conserved along the horizontal direction. So, we have

$$mu = MV$$

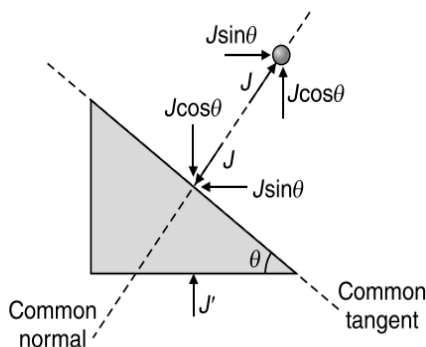
$$\Rightarrow V = \frac{mu}{M} \quad \dots(1)$$

There is no impulse on the ball in the direction parallel to incline, so the velocity of ball along the incline, i.e. along the common tangent or the t -line is constant

$$\Rightarrow u \cos \theta = v \sin \theta$$

$$\Rightarrow v = u \cot \theta \quad \dots(2)$$

We can also find the values of v and V by using the impulse-momentum theorem. When ball hits the wedge, an impulse is generated (between ball and wedge) along the direction perpendicular to the incline, i.e. along the line of impact or the n -line.



For ball, we have

$$mu - J \sin \theta = 0$$

$$\Rightarrow J \sin \theta = mu \quad \dots(3)$$

Also, $J \cos \theta = mv$

$$\dots(4)$$

From equation (3) and (4), we get

$$v = u \cot \theta$$

From equation (3), we have

For the wedge, we have

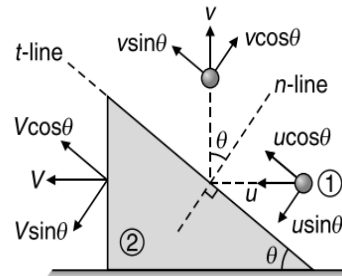
$$J \sin \theta = MV$$

$$\Rightarrow V = \frac{J \sin \theta}{M}$$

Using equation (3), we get

$$V = \frac{mu}{M}$$

The velocities of ball, wedge before and after the collision along the t -line and the n -line are shown in Figure.



The coefficient of restitution is given by

$$e = \frac{V \sin \theta - (-v \cos \theta)}{u \sin \theta - 0}$$

$$\Rightarrow e = \frac{m \sin^2 \theta + M \cos^2 \theta}{M \sin^2 \theta}$$

$$\Rightarrow e = \frac{m}{M} + \cot^2 \theta$$

PROBLEM 17

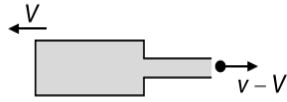
A gun of mass M fires a shell of mass m and recoils horizontally. If the shell travels along the barrel with speed v . Find the speed with which the barrel begins to recoil when

- (a) the barrel is horizontal
- (b) the barrel is inclined at an angle 30° to the horizontal.

In each case find the constant force required to bring the gun to rest in $t = 2$ s.

SOLUTION

- (a) The speed of the shell is $v - V$ as it leaves the barrel, because the barrel is recoiling with speed V . Before firing the shell, the gun is at rest and the total momentum is zero.



Using conservation of linear momentum (in the direction of the shell's motion)

$$0 = m(v - V) - MV$$

$$\Rightarrow (M + m)V = mv$$

Therefore, the initial velocity of recoil is $\frac{mv}{M + m}$

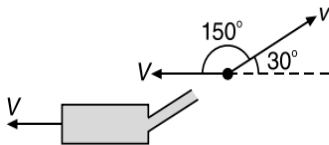
If a constant force F_1 brings the gun to rest, it must exert in 2 second, an impulse equal to the initial momentum of the gun, so

$$2F_1 = M\left(\frac{mv}{M + m}\right)$$

So, the force required is $F_1 = \frac{Mmv}{2(M + m)}$

- (b) This time the shell leaves the barrel with a velocity which is the resultant of two components inclined at 150° .

That momentum is not conserved in the vertical direction because the impulse exerted by the ground on the gun is an external impulse which does change the total momentum.



Applying Law of Conservation of Linear Momentum in the direction of recoil, we get

$$0 = MV + m(V - v \cos 30^\circ)$$

$$\Rightarrow \frac{1}{2}mv\sqrt{3} = (M + m)V$$

$$\Rightarrow V = \frac{mv\sqrt{3}}{2(M + m)}$$

and the force, F_2 , required to stop the gun in two second is found by using Impulse - Momentum Theorem

$$2F_2 = M\left(\frac{mv\sqrt{3}}{2(M + m)}\right)$$

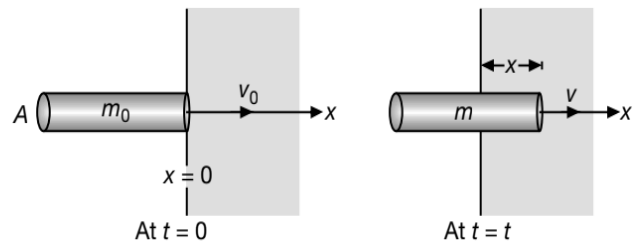
So, the required force is $F_2 = \frac{Mmv\sqrt{3}}{4(M + m)}$

PROBLEM 18

A cylindrical solid of mass 10^{-2} kg and cross-sectional area 10^{-4} m² is moving parallel to its axis (the X-axis) with a uniform speed of 10^3 ms⁻¹ in the positive direction. At $t = 0$, its front face passes the plane $x = 0$. The region to the right of this plane is filled with stationary dust particles of uniform density 10^{-3} kgm⁻³. When a dust particle collides with the face of the cylinder, it sticks to its surface. Assuming that the dimensions of the cylinder remains practically unchanged, and that the dust sticks only to the front face of the cylinder, find the x-coordinate of the front of the cylinder at $t = 150$ s.

SOLUTION

Given, $m_0 = 10^{-2}$ kg, $A = 10^{-4}$ m², $v_0 = 10^3$ ms⁻¹ and $\rho_{\text{dust}} = \rho = 10^{-3}$ kgm⁻³



At any instant mass of cylinder is $m = m_0 +$ mass of dust collected so far $= m_0 + Axp_{\text{dust}}$

$$\Rightarrow m = m_0 + Axp$$

The linear momentum at $t = 0$ is

$$p_0 = m_0v_0$$

and momentum at t is

$$p_t = mv = (m_0 + Axp)v$$

By Law of Conservation of Linear Momentum, we have

$$p_0 = p_t$$

$$\Rightarrow m_0v_0 = (m_0 + Axp)v$$

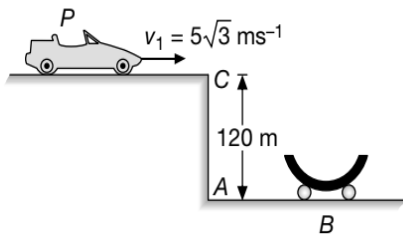
$$\begin{aligned} \Rightarrow m_0 v_0 &= (m_0 + A\rho) \frac{dx}{dt} \\ \Rightarrow (m_0 + A\rho) dx &= m_0 v_0 dt \\ \Rightarrow \int_0^x (m_0 + A\rho) dx &= m_0 v_0 \int_0^{150} dt \\ \Rightarrow \left(m_0 x + A\rho \frac{x^2}{2} \right)_0^x &= (m_0 v_0 t)_0^{150} \\ \Rightarrow m_0 x + A\rho \frac{x^2}{2} &= 150 m_0 v_0 \end{aligned}$$

Solving this quadratic equation and Substituting the values of m_0 , A , ρ and v_0 , we get a quadratic in x , which on solving yields a positive value of x as 10^5 m.

$$\Rightarrow x = 10^5 \text{ m}$$

PROBLEM 19

A car P is moving with a uniform speed of $5\sqrt{3} \text{ ms}^{-1}$ towards a carriage of mass 9 kg at rest kept on the rails at a point B as shown in figure.



The height AC is 120 m . Cannon balls of 1 kg are fired from the car with an initial velocity 100 ms^{-1} at an angle 30° with the horizontal. The first cannon ball hits the stationary carriage after a time t_0 and sticks to it. Determine t_0 . At t_0 , the second cannon ball is fired. Assume that the resistive force between the rails and the carriage is constant and ignore the vertical motion of the carriage throughout. If the second cannon ball also hits and sticks to the carriage. What will be the horizontal velocity of the carriage just after the second impact? Take $g = 10 \text{ ms}^{-2}$.

SOLUTION

Unless and until it is mentioned in the question, the velocity is taken relative to ground, so velocity of the cannon ball is relative to ground is 100 ms^{-1} .

Horizontal component of its velocity,

$$u_x = u \cos 30^\circ = 100 \left(\frac{\sqrt{3}}{2} \right) \text{ ms}^{-1} = 50\sqrt{3} \text{ ms}^{-1}$$

and vertical component of its velocity,

$$u_y = u \sin 30^\circ = 100 \left(\frac{1}{2} \right) \text{ ms}^{-1} = 50 \text{ ms}^{-1}$$

Vertical displacement of the ball when it strikes the carriage is -120 m or

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$\Rightarrow -120 = (50t) + \left(\frac{1}{2} \right) (-10t^2)$$

$$\Rightarrow t^2 - 10t - 24 = 0$$

$$\Rightarrow t = 12 \text{ s or } -2 \text{ s}$$

Ignoring the negative time, we get

$$t_0 = 12 \text{ s}$$

When it strikes the carriage, its horizontal component of velocity is still $50\sqrt{3} \text{ ms}^{-1}$. It sticks to the carriage. Let v_2 be the velocity of (carriage-cannon ball) system after collision. Then applying Conservation of Linear Momentum in horizontal direction

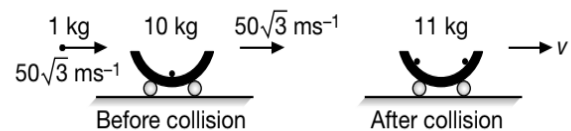
$$m_{\text{ball}} u_x = (m_{\text{ball}} + m_{\text{carriage}}) v_2$$

$$\Rightarrow (1 \text{ kg})(50\sqrt{3} \text{ ms}^{-1}) = (10 \text{ kg})(v_2)$$

$$\Rightarrow v_2 = v_{\text{carriage}} = 5\sqrt{3} \text{ ms}^{-1}$$

The second cannon ball is fired when the first cannon ball strikes the carriage, i.e., after 12 second. In these 12 second the car will move forward a distance of $12v_1$ or $60\sqrt{3} \text{ m}$. This ball will strike the carriage only when the carriage also covers the same distance of $60\sqrt{3} \text{ m}$ in next 12 s. This is possible only when the resistive force is zero, because then only

$$v_{\text{car}} = v_{\text{carriage}} = 5\sqrt{3} \text{ ms}^{-1}$$



Hence at the time of second collision, the horizontal component of velocity of cannon ball is $u_x = 50\sqrt{3} \text{ ms}^{-1}$ and horizontal velocity of carriage and first cannon ball is $5\sqrt{3} \text{ ms}^{-1}$.

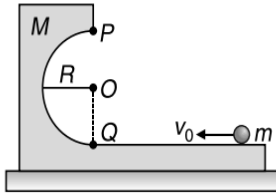
If v be the desired velocity of carriage after second collision. Then Conservation of Linear Momentum in horizontal direction gives

$$11v = (1)(50\sqrt{3}) + (10)(5\sqrt{3}) = 100\sqrt{3}$$

$$\Rightarrow v = \frac{100\sqrt{3}}{11} \text{ ms}^{-1} \approx 15.75 \text{ ms}^{-1}$$

PROBLEM 20

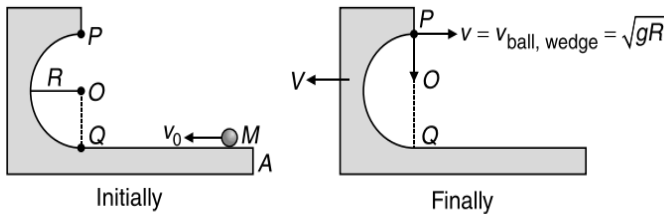
A small ball of mass m is projected with a minimum horizontal velocity v_0 on a smooth wedge of mass M so that it will reach the highest point P of the wedge as shown in Figure. Calculate v_0 .



SOLUTION

For ball to reach the point P , its velocity with respect to wedge should be

$$v = \sqrt{gR}$$



Applying work energy theorem from the centre of mass reference frame at A and the highest point P , we get

$$W_{\text{ext}} + W_{\text{int}} = W_{\text{ext}} + 0 = \Delta K_{\text{cm}}$$

$$\Rightarrow W_{\text{gravity}} = \Delta K_{\text{cm}}$$

$$\Rightarrow -mg(2R) = \frac{1}{2}\mu(v_{\text{rel}}^2 - u_{\text{rel}}^2), \text{ where } \mu = \frac{mM}{m+M}$$

$$\Rightarrow -2mgR = \frac{1}{2}\mu(V+v)^2 - \frac{1}{2}\mu v_0^2 \quad \dots(1)$$

Since, no external forces are acting on the system in horizontal direction, so linear momentum of the system is be conserved. Hence, we have

$$mv_0 = m(V - v) + MV$$

$$\Rightarrow V = \frac{m(v_0 + v)}{m + M} \quad \dots(2)$$

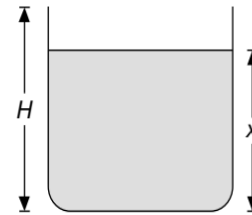
From equations (1) and (2), we get

$$v_0 = \sqrt{\left(5 + \frac{4m}{M}\right)gR}$$

PROBLEM 21

A liquid has been poured into a cylindrical vessel of mass M (the mass of the vessel bottom can be ignored) and height H . The linear density of the liquid, which is, the ratio of the mass of the liquid column to its height is denoted by α . Calculate the height x of the column of liquid at which the common centre of mass of the liquid plus the vessel is in the lowest position.

Assume the centre of mass of vessel to be at $\frac{H}{2}$ from ground.



SOLUTION

Distance of centre of mass of system (vessel plus liquid) from the bottom.

$$x_{\text{cm}} = \frac{M\left(\frac{H}{2}\right) + (\alpha x)\frac{x}{2}}{M + (\alpha x)}$$

For x_{cm} to be minimum,

$$\frac{dx_{\text{cm}}}{dx} = 0$$

$$\Rightarrow (M + \alpha x)(\alpha x) - \left(M\frac{H}{2} + \frac{\alpha x^2}{2}\right)(\alpha) = 0$$

$$\Rightarrow M\alpha x + \alpha^2 x^2 - \frac{MH\alpha}{2} - \frac{\alpha^2 x^2}{2} = 0$$

$$\Rightarrow \alpha x^2 + 2Mx - MH = 0$$

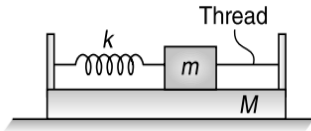
$$\Rightarrow x = \frac{-2M \pm \sqrt{4M^2 + 4MH\alpha}}{2\alpha}$$

$$\Rightarrow x = \frac{\sqrt{M(M + H\alpha)} - M}{\alpha}$$



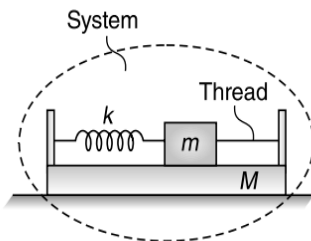
PROBLEM 22

A spring is connected with plank and other end of spring is connected with a block of mass m . Initially spring is stretched by a distance x_0 and block is connected with a thread which is connected to another end of the plank as shown in Figure. If thread is cut, calculate the maximum speed of the plank. Assume friction to be absent everywhere.



SOLUTION

Assume the system to be made of spring, plank and block as shown in Figure.



No external force acts on the system in horizontal direction.

Since no external force is acting on the system along the horizontal direction. Both linear momentum and mechanical energy will be conserved for the system. Let speed of block be v and maximum speed of the plank be V . Applying conservation of linear momentum at this situation, we get

$$mv = MV \quad \dots(1)$$

Applying conservation of mechanical energy, we get

$$\Delta K + \Delta U = 0$$

$$\left[\left(\frac{1}{2}MV^2 + \frac{1}{2}mv^2 \right) - 0 \right] + \left[0 - \frac{1}{2}kx_0^2 \right] = 0 \quad \dots(2)$$

From equation (1), we have

$$v = \frac{MV}{m}$$

Substituting this value in equation (2), we get

$$\left[\frac{1}{2}MV^2 + \frac{1}{2}m \left(\frac{MV}{m} \right)^2 \right] = \frac{1}{2}kx_0^2$$

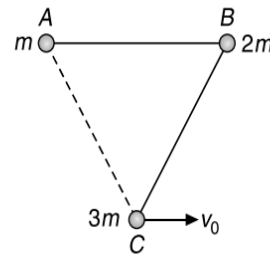
$$\Rightarrow \frac{1}{2} \left(M + \frac{M^2}{m} \right) V^2 = \frac{1}{2}kx_0^2$$

$$\Rightarrow V^2 = \frac{mkx_0^2}{Mm + M^2}$$

$$\Rightarrow V = x_0 \sqrt{\frac{mk}{M(m+M)}}$$

PROBLEM 23

Three particles A , B and C of mass m , $2m$ and $3m$ respectively lie on a smooth horizontal table at the vertices of an equilateral triangle. A and B as well as B and C are connected by light inextensible strings. C is given a velocity v_0 parallel to AB as shown in Figure.



Show that A eventually begins to move with a velocity $\frac{2v_0}{19}$.

SOLUTION

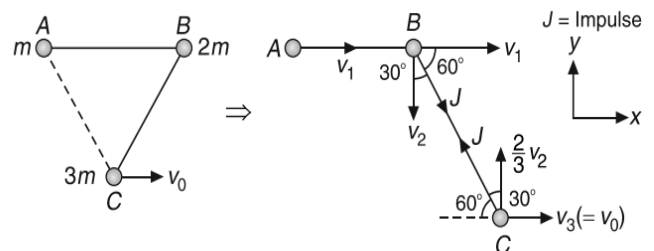
For B-C, system, by Law of Conservation of Momentum along y -direction, we have

$$0 = 3mv_{Cy} + 2mv_{By}$$

$$\Rightarrow v_{Cy} = -\frac{2}{3}v_{By}$$

If v_1 and $-v_2$ be the x and y components of final velocity of B , then $v_{By} = -v_2$

$$\Rightarrow v_{Cy} = \frac{2}{3}v_2$$



Now, applying Law of Conservation of Momentum along x -direction, we get

$$3mv_0 = mv_1 + 2mv_1 + 3mv_3$$

$$\Rightarrow v_0 = v_1 + v_3 \quad \dots(1)$$

Components of velocities along BC for particles B and C will be equal. Hence,

$$v_1 \cos 60^\circ + v_2 \cos 30^\circ = v_3 \cos 60^\circ - \frac{2}{3}v_2 \cos 30^\circ$$

$$\Rightarrow \frac{v_1}{2} + \frac{\sqrt{3}v_2}{2} = \frac{v_3}{2} - \frac{2\sqrt{3}}{6}v_2$$

$$\Rightarrow v_1 - v_3 + \frac{5}{\sqrt{3}}v_2 = 0 \quad \dots(2)$$

Now, since Impulse = Change in Momentum, so for C , we have

$$\text{Along } x\text{-axis, } J \cos(60^\circ) = 3m(v_0 - v_3)$$

$$\Rightarrow \frac{J}{2} = 3m(v_0 - v_3) \quad \dots(3)$$

$$\text{Along } y\text{-axis, } J \cos(30^\circ) = 3m\left(\frac{2}{3}v_2\right)$$

$$\Rightarrow \frac{\sqrt{3}}{2}J = 2mv_2 \quad \dots(4)$$

Solving equations (1), (2), (3) and (4) we get

$$v_1 = \frac{2}{19}v_0$$