ICON

CHAPTER 1

Work, Energy, Power and Conservation of Energy

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Explore the Concept of Work, Its Relation to Kinetic Energy

(b) Energy Associated with Motion of Objects

(c) Concept of Power

(d) Conservation of Energy

(e) Motion of Body in a Vertical Circle

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

WORK, ENERGY, POWER AND LAW OF CONSERVATION OF ENERGY

INTRODUCTION

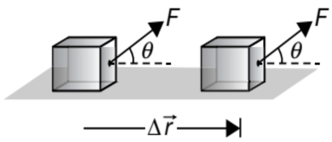
In Mechanics-I, we had analysed motion by using concepts such as position, velocity, acceleration, and force. However, after learning these concepts we may find difficulty in describing some types of motion by applying Newton's laws directly e.g. a block sliding down a curved slope may be difficult to describe using Newton's Laws. So, in this Chapter, we shall be analysing motion using the concepts of work and energy. Unlike force, which is a vector physical quantity, work and energy both are scalar physical quantities and they can be associated with particles as well as the systems of particles. These new concepts of work and energy will provide us with powerful methods to solve a wide variety of problems.

WORK DONE BY A CONSTANT FORCE

The work W done by a constant force F when the point of application of force undergoes a displacement Δr is defined as

$$W = F\Delta r\cos\theta \qquad \ldots(1)$$

where θ is the angle between $\vec{F}$ and $\Delta\vec{r}$ as indicated in Figure.



Only the component of $\vec{F}$ along $\Delta\vec{r}$, that is, $F\cos\theta$, contributes to the work done. Strictly speaking, the work is done by the source or agent that applies the force. Work is a scalar quantity and its SI unit is the joule (J). From equation (1), we see that

$$1\text{ J} = 1\text{ Nm}$$

From (1) we can also conclude that, work done is also defined as the dot product of force and its displacement as given by the following equation (2)

$$W = \vec{F}\cdot\Delta\vec{r} \qquad \ldots(2)$$

In terms of rectangular components, the two vectors are

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \text{ and } \Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$

Hence, equation (2) may be written as

$$W = \vec{F}\cdot\Delta\vec{r} = F_x\Delta x + F_y\Delta y + F_z\Delta z \quad \ldots(3)$$

The work done by a given force on a body depends only on the force, the displacement, and the angle between them. It does not depend on the velocity or the acceleration of the body, or on the presence of other forces.

Since the work is a scalar, its value also does not depend on the orientation of the coordinate axes. Since the magnitude of a displacement in a given time interval depends on the velocity of the frame of reference used to measure the displacement, the calculated **work also depends on the reference frame.**

Conceptual Note(s)

Work done by a force is defined as the dot product of the force $(\vec{F})$ and the displacement $(\Delta\vec{r})$ of the point of application of force with respect to the observer calculating the work.

Mathematically work done by a force is given by

$$W = \vec{F}\cdot\Delta\vec{r} = \vec{F}\cdot(\vec{r}_2 - \vec{r}_1)$$

If a number of forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$,... $\vec{F}_n$ are acting on a body and it shifts from position vector $\vec{r}_1$ to position vector $\vec{r}_2$, then

$$W = (\vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n)\cdot(\vec{r}_2 - \vec{r}_1)$$

ILLUSTRATION 1

A box is moved over a horizontal path by applying force $F = 80$ N at an angle $\theta = 60°$ to the horizontal. What is the work done during the displacement of the box over a distance of 0.5 km.

SOLUTION

By definition, $W = F\Delta r\cos\theta$

Here $F = 80$ N, $\Delta r = 0.5$ km $= 500$ m, $\theta = 60°$

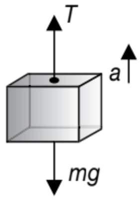
$\Rightarrow \quad W = (80)(500)\cos(60°) = 20$ kJ

ILLUSTRATION 2

A load of mass $m = 3000$ kg is lifted by a winch with an acceleration $a = 2$ ms^{-2}. Find the work done during the first one and a half seconds from the beginning of motion.

SOLUTION

The height to which the body is lifted during the first t second is $h = \frac{1}{2}at^2$



The tension in the rope is given by

$$T = mg + ma$$

Work done by tension is given by

$$W = Th\cos(0°) = m(g+a)\left(\frac{1}{2}at^2\right)$$

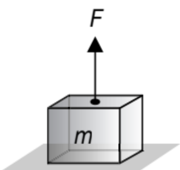
Here $m = 3000$ kg, $a = 2$ ms^{-2}, $g = 10$ ms^{-2}, $t = 1.5$ s

$\Rightarrow \quad W = (3000)(10+2)\left[\frac{1}{2}(2)(1.5)^2\right]$

$\Rightarrow \quad W = 81$ kJ

ILLUSTRATION 3

A block of mass $m = 4$ kg is pulled by a force $F = 20$ N upwards through a height $h = 2$ m as shown in Figure.



Calculate the work done on the block by the applied force F and its weight. Take $g = 10$ ms^{-2}.

SOLUTION

Weight of the block is $mg = (4)(10) = 40$ N

Work done by the applied force $W_F = Fh\cos(0°)$

The angle between force and displacement is $0°$, so

$$W_F = (20)(2)(1) = 40 \text{ J}$$

Similarly, work done by the weight of the block is

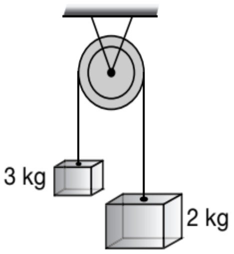
$$W_{mg} = (mg)(h)\cos(180^\circ)$$

$$\Rightarrow \quad W_{mg} = (40)(2)(-1) = -80 \text{ J}$$

Negative sign indicates that work is done against the gravitational pull of the earth.

ILLUSTRATION 4

Two unequal masses of 3 kg and 2 kg are attached at the two ends of a light inextensible string passing over a smooth pulley as shown in figure.



If the system is released from rest, find the work done by string on both the blocks in 2 s. Take $g = 10 \text{ ms}^{-2}$.

SOLUTION

The acceleration of the blocks in the system is

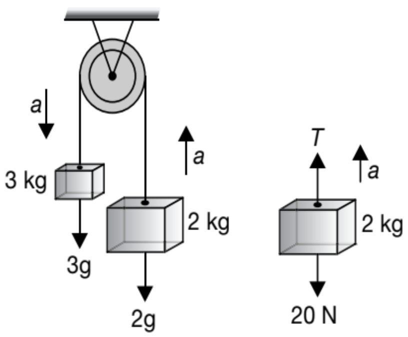
$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

$$\Rightarrow \quad a = \left(\frac{3-2}{3+2}\right)10 = 2 \text{ ms}^{-2}$$

Displacement of both the blocks in 1 s is

$$S = \frac{1}{2}at^2 = \frac{1}{2}(2)(2)^2 = 4 \text{ m}$$

Free body diagram of 2 kg block is shown in figure.



Using $\sum F = ma$, we get

$$T - 20 = 2a$$

$$\Rightarrow \quad T = 24 \text{ N}$$

Work done by string (tension) on 2 kg block in 2 s is

$$W_1 = (T)(S)\cos(0^\circ)$$

$$\Rightarrow \quad W_1 = (24)(4)(1) = 96 \text{ J}$$

Similarly, work done by string on 3 kg block in 2 s will be

$$W_2 = (T)(S)(\cos 180^\circ)$$

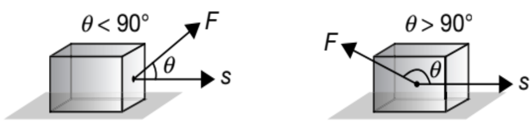
$$\Rightarrow \quad W_2 = (24)(4)(-1) = -96 \text{ J}$$

POSITIVE AND NEGATIVE WORK

Work done by a force may be positive or negative depending on the angle θ between the force and displacement.

When the angle θ is acute $(\theta < 90^\circ)$, then the component of force is parallel to the displacement and the work done is positive as shown in Figure (a).

When the angle θ is obtuse $(\theta > 90^\circ)$, then the component of force is antiparallel to the displacement and the work done by force is negative as shown in Figure (b).



(a) Positive work done by a force F

(b) Negative work done by a force F

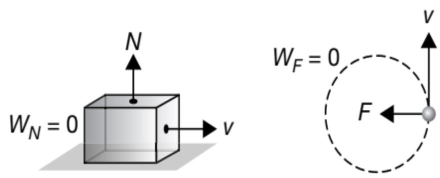
ZERO WORK DONE

Since we know that $W = F\Delta r\cos\theta$, so the work done by a force is zero when

(a) **EITHER** $F = 0$

(b) **OR** $\Delta r = 0$

(c) **OR** $\cos\theta = 0$ i.e. $\theta = 90^\circ$



(a) (b)

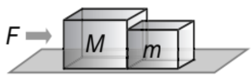
When the force and the displacement are perpendicular, the work done by the force is zero.

(a) the normal reaction N is perpendicular to displacement, therefore, $W_N = 0$.

(b) the centripetal force is perpendicular to displacement, thus $W_F = 0$.

ILLUSTRATION 5

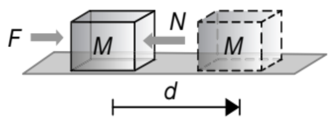
Two blocks having masses M and m are placed on a smooth horizontal floor and a horizontal force F is applied on the system as shown. Calculate the work done by normal reaction (between the blocks) on two blocks as the system moves to the right through a distance d.



SOLUTION

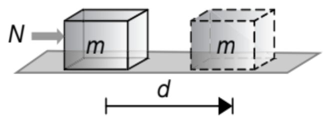
Work done by normal reaction acting on the block of mass M is

$$W_M = Nd\cos(180°) = -Nd$$



Work done by the normal reaction acting on the block of mass m is

$$W_m = Nd\cos(0°) = Nd$$



Total work done by normal reaction (acting between two blocks) on the system of two blocks is

$$W_{\text{net}} = \Delta W_m + \Delta W_m$$

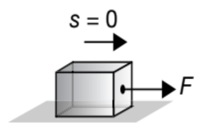
$$\Rightarrow \quad W_{\text{net}} = Nd + (-Nd)$$

$$\Rightarrow \quad W_{\text{net}} = 0$$

WORK DONE BY FRICTION

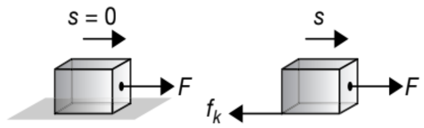
There is a misconception that the force of friction always does negative work. In reality, the work done by friction may be zero, positive or negative depending upon the situation.

In figure (a), when a block is pulled by a force F and the block does not move, the work done by friction is zero.



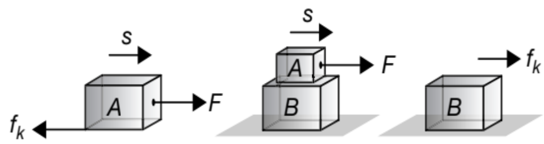
(a)

In figure (b), when a block is pulled by a force F on a stationary surface, the work done by the kinetic friction is negative.



(b)

In figure (c), block A is placed on the block B. When the block A is pulled with a force F, the friction force does negative work on block A and positive work on block B. The kinetic friction and displacement are oppositely directed in case of block A while in case of block B they are in the same direction.

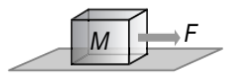


(c)

WORK DONE BY STATIC FRICTION

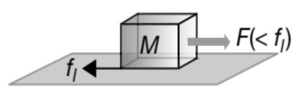
CASE-1:

Consider a block of mass M is placed on a rough horizontal surface. Let a pulling force F act horizontally on the block as shown in Figure.



When the applied force F is less than or equal to maximum possible frictional force between block and the surface i.e. limiting friction f_l, then the block will not slip on the surface and the work done by frictional force acting on the block will be zero because the displacement of block will be zero.

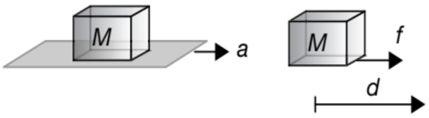
$$F \le f_l$$



CASE-2:

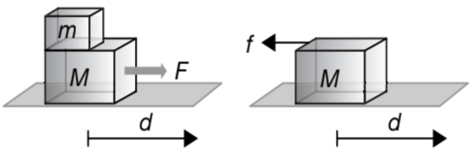
When the surface is accelerating towards the right with an acceleration a and the block does not slip on the surface, then the frictional force f (which may be static or limiting) on the block will be in the direction of motion of the block and work done by frictional force to move the block through a distance d is

$$W = fd\cos(0^\circ) = fd$$



CASE-3:

When another block of mass m is placed on the block of mass M placed on a smooth surface and the block of mass M is pulled by a force F through a distance d as shown in Figure.



In the case of no slipping between the two blocks, work done on the lower block by the frictional force f (which may be static or limiting) applied due to the upper block is

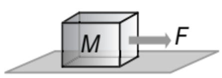
$$W = fd\cos(180^\circ) = -fd$$

Similarly, work done on the upper block by the frictional force f (which may be static or limiting) applied due to the lower block is

$$W = fd\cos(0^\circ) = fd$$

WORK DONE BY KINETIC FRICTION

Consider a block of mass M is placed on a rough horizontal surface. Let a pulling force F act horizontally on the block as shown in Figure.



When the applied force F is more than the limiting friction f_l, then the block will move on the surface and the friction between the block and surface will be kinetic given by

$$f_k = \mu_k N$$

This f_k will be directed opposite to the direction of motion of the block w.r.t. the surface.

When the block gets displaced by distance d, then the work done by kinetic friction acting on the block is given by

$$W = f_k N\cos(180^\circ) = -\mu_k Nd$$

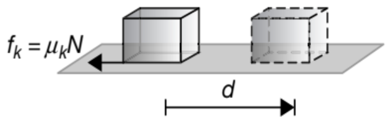
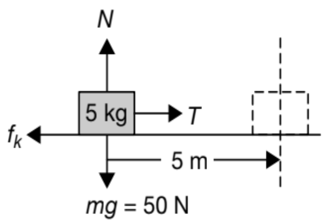


ILLUSTRATION 6

A block of mass 5 kg is placed on a rough horizontal ground with coefficient of friction 0.1. It is being pulled by means of a light horizontal rope, with a constant speed of 3 ms^{-1}. Calculate the work done on the block due to the tension in the rope, the force of friction and normal reaction, when the body moves through 5 m. Take $g = 10\ ms^{-2}$.

SOLUTION

The free body diagram (FBD) of the block is shown in the Figure.



Since the body moves with a constant speed, so its acceleration is zero.

$\Rightarrow \quad N = mg = 50\text{ N}$

$\Rightarrow \quad T = f_k = \mu N = 0.1 \times 50 = 5\text{ N}$

So, work done by tension T is

$$W_T = T \times 5 \times \cos(0^\circ) = 25\text{ J}$$

Work done by kinetic friction f_k is

$$W_{fk} = f_k \times 5 \times \cos(180^\circ) = -25\text{ J}$$

Work done by gravitational force mg is

$$W_{mg} = (mg)(5)\cos(90^\circ) = 0$$

Work done by normal reaction N is

$$W_N = (N)(5)\cos(90^\circ) = 0$$

Please note that the total work done is

$$W = 25 - 25 = 0$$

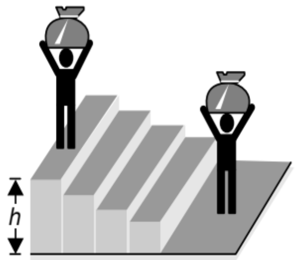
Since we observe that the block is moving with constant speed, therefore, there is no change in kinetic energy and hence in accordance with Work-Energy Theorem, work done by all forces acting on the block is zero.

DEPENDENCE OF WORK ON FRAME OF REFERENCE

Work as defined, depends on frame of reference also. When we change from one inertial reference frame to another inertial reference frame, the force does not change, while displacement may change, so the work done by a force will be different.

FOR EXAMPLE

(a) When a porter with a bag on his head moves up a staircase, work done by the upward lifting force relative to him will be zero.

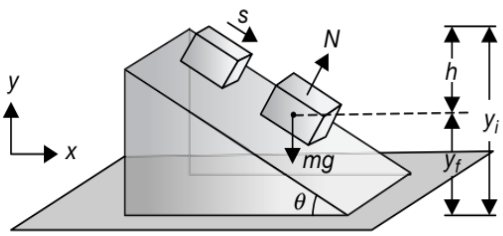


This is because, the displacement of the porter with respect to himself is zero. However, relative to a person on the ground, the work done by the porter will be mgh.

(b) Suppose a person is pushing a box in a moving train (in the same direction as the movement of train) by applying a force $\vec{F}$ on it. In the reference frame of the train, the work done by the force will be $\vec{F}\cdot\vec{r}$ where $\vec{r}$ is the displacement of the block with respect to the train. But in the reference frame of the earth the work will be $\vec{F}\cdot(\vec{r}+\vec{r}_0)$ where $\vec{r}_0$ is the displacement of the train with respect to the earth.

WORK DONE BY GRAVITY

Consider a block of mass m which slides down a smooth inclined plane of angle θ as shown in Figure.



Let us assume the coordinate axes as shown in the figure, to specify the components of the two vectors - although the value of work will not depend on the orientation of the axes.

Now, the force of gravity, $\vec{F}_g = -mg\hat{j}$

and the displacement is given by

$$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$

The work done by gravity is

$$W_g = \vec{F}_g\cdot\Delta\vec{r} = -mg\hat{j}\cdot\left(\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}\right)$$

$$\Rightarrow \quad W_g = -mg\Delta y \qquad \left\{\because\ \hat{j}\cdot\hat{i} = 0, \hat{j}\cdot\hat{j} = 1, \hat{j}\cdot\hat{k} = 0\right\}$$

Since $\Delta y = y_f - y_i = -h$

The work done by gravity is

$$W_g = -mg\left(y_f - y_i\right) = +mgh$$

If the block moves in the upward direction, then the work done by gravity is negative i.e., work is done against the gravitational pull and is given by

$$W_g = -mgh$$

Conceptual Note(s)

(a) The work done by the force of gravity depends only on the initial and final vertical coordinates, not on the path taken.

(b) The work done by gravity is zero for any path that returns to its initial point.

When several forces act on a body one may calculate the work done by each force individually. The net work done on the body is the algebraic sum of individual contributions.

$$W_{\text{net}} = \vec{F}_1\cdot\Delta\vec{r}_1 + \vec{F}_2\cdot\Delta\vec{r}_2 + \ldots\ldots\ldots\ldots + \vec{F}_n\cdot\Delta\vec{r}_n$$

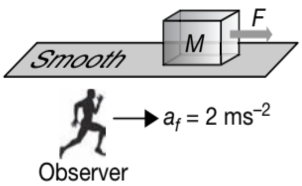
$$\Rightarrow \quad W_{\text{net}} = W_1 + W_2 + \ldots\ldots\ldots\ldots\ldots + W_n$$

WORK DONE BY PSEUDO FORCE

When a body is being displaced under the action of a force and we are observing this from a non-inertial frame, then to calculate the total work done, we have to also consider the work done by pseudo force(s).

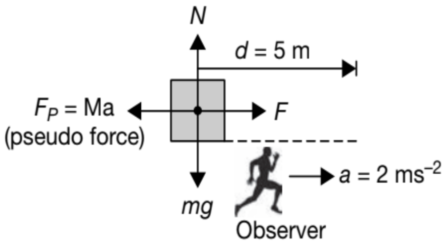
ILLUSTRATION 7

A block of mass $M = 5\text{ kg}$ is placed on a smooth horizontal surface and a force $F = 25\text{ N}$ starts acting on it parallel smooth to the plane. Calculate the net work done on the block as seen by an observer who is accelerating in the direction of the force with an acceleration of 2 ms^{-2} if the block gets displaced by 5 m in the direction of the force with respect to the observer.



SOLUTION

FBD of the block w.r.t. the observer is shown in Figure.



Work done by weight of the body and the normal reaction force will be zero.
Since, work done by the applied force F is

$$W_F = Fd\cos(0°) = Fd = (25)(5) = 125\text{ J}$$

and work done by the pseudo force F_{ps} is

$$W_{ps} = F_{ps}d\cos(180°) = -F_{ps}d$$

Since, $F_{ps} = ma = (5)(2) = 10\text{ N}$

$$\Rightarrow \quad W_{ps} = -F_{ps}d = -(10)(5) = -50\text{ J}$$

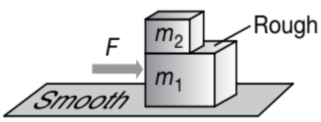
So, net work done W is given by

$$W = W_F + W_{ps}$$

$$\Rightarrow \quad W = 125 + (-50) = 75\text{ J}$$

ILLUSTRATION 8

Two blocks of masses m_1 and m_2 are placed one above the other. There is no friction between the lower block and ground. The lower block is being pushed by a constant horizontal force F as shown in Figure.



There is sufficient friction between the blocks so that they do not slip over each other. Draw the free body diagram of the upper block with respect to the ground frame and with respect to the frame attached to the lower block. Find the work done by various forces on the upper block in the two frames, as the arrangement moves through l.

SOLUTION

Since the two blocks move together, so

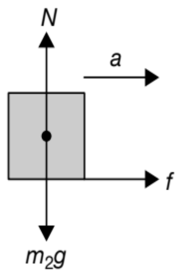
$$F = (m_1 + m_2)a$$

where a is their common acceleration.

$$\Rightarrow \quad a = \frac{F}{m_1 + m_2}$$

As long as the blocks move together, friction between the blocks will be static or limiting.

FBD of m_2 with respect to ground



$$N = m_2 g \text{ (equilibrium along vertical)}$$

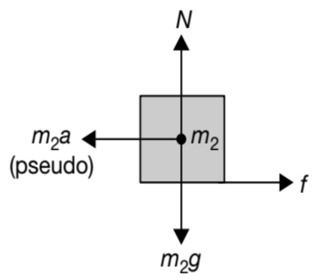
If f be the friction between the blocks, then

$$f = m_2 a \text{ (by Newton's 2}^{\text{nd}}\text{ law).}$$

$$W_N = 0,\ W_{m_2g} = 0 \text{ and}$$

$$W_{fr} = (m_2 a)l\cos 0° = \frac{m_2 Fl}{m_1 + m_2}$$

FBD of m_2 with respect to m_1



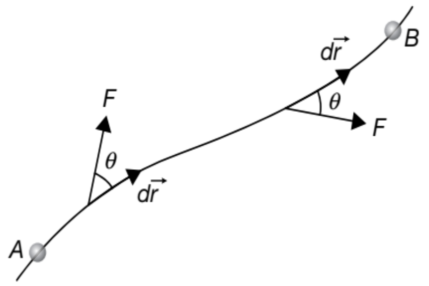
In the frame attached to m_1, the mass m_2 does not move. Therefore, work done by all the forces is zero.

WORK DONE BY A VARIABLE FORCE

When the magnitude and direction of a force vary in three dimensions, it can be expressed as a function of the position vector $\vec{F}(r)$, or in terms of the coordinates $\vec{F}(x,y,z)$. The work done by such a force in an infinitesimal displacement $d\vec{r}$ is

$$dW = \vec{F}\cdot d\vec{r}$$

The total work done in going from point A to point B as shown in the Figure is



A particle moves along a curved path subject to a variable force $\vec{F}$. The work done by the force in a displacement $d\vec{s}$ is $dW = \vec{F}\cdot d\vec{r}$.

$$W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B (F\cos\theta)\,dr$$

In terms of rectangular components,

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \text{ and } d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$\Rightarrow \quad W_{A\to B} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

ILLUSTRATION 9

A force $F = (5+2x+3x^2)$ acts on a particle in x-direction where F is in newton and x in metre. Find the work done by this force during a displacement from $x = 1$ m to $x = 2$ m.

SOLUTION

As the force is variable, we shall find the work done in a small displacement from x to $x+dx$ and then integrate it to find the total work. The work done in this small displacement is

$$dW = Fdx = (5+2x+3x^2)dx$$

Thus, $W = \int_1^2 dW = \int_1^2 (5+2x+3x^2)dx$

$$\Rightarrow \quad W = (5x + x^2 + x^3)\Big|_1^2 = 5(1) + (4-1) + (8-1)$$

$$\Rightarrow \quad W = 15 \text{ J}$$

ILLUSTRATION 10

A force varying with distance is given as $F = ae^{-bx}$ acts on a particle of mass m moving in a straight line. Find the work done on the particle in its displacement from origin to a distance d.

SOLUTION

Since the applied force varies with displacement, so the work done by the force is given by

$$W = \int_0^d F dx = \int_0^d ae^{-bx} dx$$

$$\Rightarrow \quad W = -\frac{a}{b}\left(e^{-bx}\right)\Big|_0^d = \frac{a}{b}\left(1 - e^{-bd}\right)$$

ILLUSTRATION 11

A force $F = -\dfrac{k}{x^2}(x \neq 0)$ acts on a particle in x-direction. Find the work done by this force in displacing the particle from $x = a$ to $x = +2a$, where k is a positive constant.

SOLUTION

$$W = \int F dx = \int_{+a}^{+2a}\left(\frac{-k}{x^2}\right)dx = \left(\frac{k}{x}\right)\Big|_{+a}^{+2a} = -\frac{k}{2a}$$

Problem Solving Technique(s)

It is important to note that work comes out to be negative which is quite obvious as the force acting on the particle is in negative x-direction $\left(F = -\dfrac{k}{x^2}\right)$ while displacement is along positive x-direction (from $x = a$ to $x = 2a$)

ILLUSTRATION 12

An object is displaced from point $A(2, 3, 4)$ m to a point $B(1, 2, 3)$ m under the influence of a constant force $\vec{F} = (2\hat{i} + 3\hat{j} + 4\hat{k})$ N. Calculate the work done by this force in the process.

SOLUTION

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \quad W = \int_{(2,3,4)\text{ m}}^{(1,2,3)\text{ m}} (2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\Rightarrow \quad W = (2x + 3y + 4z)\Big|_{(2,3,4)\text{ m}}^{(1,2,3)\text{ m}} = -9\text{ J}$$

ALTERNATE SOLUTION:

Since, $\vec{F} =$ constant, we can also use $W = \vec{F} \cdot \Delta\vec{r}$

Here, $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$

$$\Delta\vec{r} = (i + 2\hat{j} + 3\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}) = (-\hat{i} - \hat{j} - \hat{k})$$

ILLUSTRATION 13

An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})$ m to position vector $\vec{r}_2 = (4\hat{i} + 6\hat{j})$ m under a force $\vec{F} = (3x^2\hat{i} + 2y\hat{j})$ N. Calculate the work done by this force.

SOLUTION

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}, \text{ where } d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$\Rightarrow \quad W = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\hat{i} + 2y\hat{j}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$\Rightarrow \quad W = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2 dx + 2y dy)$$

$$\Rightarrow \quad W = \int_{(2,3)}^{(4,6)} d(x^3) + \int_{(2,3)}^{(4,6)} d(y^2) = (x^3 + y^2)\Big|_{(2,3)}^{(4,6)}$$

$$\Rightarrow \quad W = 83\text{ J}$$

ILLUSTRATION 14

An object is displaced from a point $A(0, 0, 0)$ to $B(1\text{ m}, 1\text{ m}, 1\text{ m})$ under a force $\vec{F} = (y\hat{i} + x\hat{j})$N. Calculate the work done by this force in this process.

SOLUTION

$$W = \int_A^B \vec{F} \cdot d\vec{r}, \text{ where } d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$\Rightarrow \quad W = \int_A^B (y\hat{i} + x\hat{j}) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$\Rightarrow \quad W = \int_A^B (ydx + xdy) = \int_A^B d(xy)$$

$$\Rightarrow \quad W = (xy)\Big|_{(0,0,0)}^{(1,1,1)}$$

$$\Rightarrow \quad W = 1\text{ J}$$

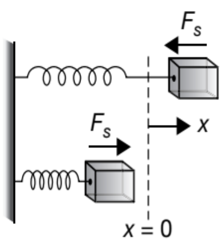
WORK DONE BY SPRING FORCE

If x be the displacement of the free end of the spring from its equilibrium position then, the restoring force (F_S) in the spring is given by Hooke's Law

$$F_S = -kx$$

where x is the extension or compression of the spring.

The negative sign signifies that the restoring force is always opposite to the extension $(x > 0)$ or the compression $(x < 0)$ in the spring. In simple words, the force tends to restore the system to its equilibrium position as shown in Figure.



The work done by the spring force for a displacement from x_i to x_f is given by

$$W_S = \int_{x_i}^{x_f} F_S dx \cos(180^\circ) = -\int_{x_i}^{x_f} kx dx$$

$$\Rightarrow \quad W_S = -\frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

Problem Solving Technique(s)

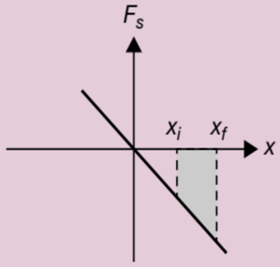
(a) The work done by a spring force is negative

(b) The work done by the spring force only depends on the initial and final points.

(c) The net work done by the spring force is zero for any path that returns to the initial point.

(d) The work done by the spring when the displacement of its free end changes from x_i to x_f is the area of the trapezoid.

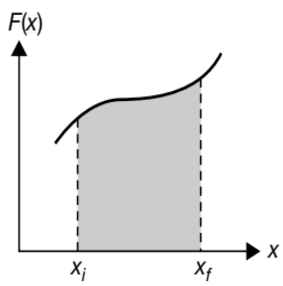
$$W_s = -\frac{1}{2}k\left(x_f^2 - x_i^2\right)$$



Graphically, the work done by the spring force in a displacement from x_i to x_f is the shaded area (as shown in the figure) which is the difference in the areas of two triangles.

WORK DONE AS AREA UNDER F-X GRAPH

In general, the work done by a variable force $F(x)$ from an initial point x_i to final point x_f is given by the **area under the force - displacement curve** as shown in the figure.



(a) The work done by a non-constant force is approximately equal to sum of the areas of the rectangles.

(b) The area under the curve is given by the integral

$$W = \int F(x)dx$$

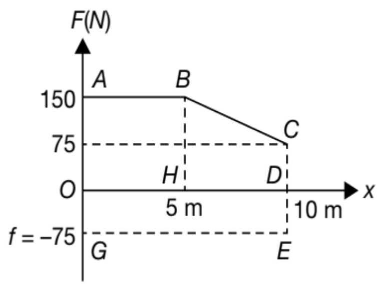
Area (work done) above the x-axis is taken as positive, and below the x-axis is taken as negative.

ILLUSTRATION 15

A gardener pushes a lawn roller on a rough surface. He applies a force of 150 N over a distance of 5 m. After some time, he gets tired and his applied force reduces to 75 N linearly with distance. The total distance moved by the gardener is 10 m. If the frictional force offered by the ground to the gardener is 75 N, then plot the graph of force applied by the gardener against the displacement. Also calculate the work done by the two forces over a distance of 10 m.

SOLUTION

The plot of force F applied by the gardener and the opposing frictional force $f = 75$ N vs displacement is shown in Figure.



At $x = 10$ m, $F = 75\text{ N}(\neq 0)$ and the frictional force is also $f = \left|\vec{f}\right| = 75$ N. So, the frictional force opposes motion and acts in a direction opposite to F. Taking the direction of F as positive, the frictional force f is therefore shown on the negative side of the force axis.

The work done by the gardener is

$$W_F = \begin{pmatrix}\text{Area of}\\ \text{rectangle}\\ ABHO\end{pmatrix} + \begin{pmatrix}\text{Area of}\\ \text{trapezium}\\ BCDH\end{pmatrix}$$

$$\Rightarrow \quad W_F = (150)(5) + \frac{1}{2}(150 + 75)(5)$$

$$\Rightarrow \quad W_F = 750 + 562.5 = 1312.5 \text{ J}$$

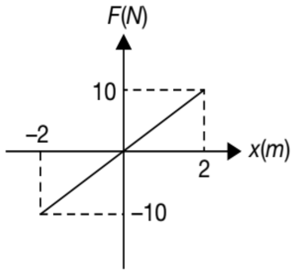
Work done by frictional force f is

$$W_f = \text{Area of rectangle } ODEG$$

$$\Rightarrow \quad W_f = (-75)\times 10 = -750 \text{ J}$$

ILLUSTRATION 16

A force F acting on a particle varies with the position x as shown in Figure.



Find the work done by this force in displacing the particle from

(a) $x = -2$ m to $x = 0$

(b) $x = 0$ to $x = 2$ m

SOLUTION

(a) From $x = -2$ m to $x = 0$, displacement of the particle is along positive x-direction while force acting on the particle is along negative x-direction. Therefore, work done is negative and given by the area under F-x graph.

$$\Rightarrow \quad W = -\frac{1}{2}(2)(10) = -10 \text{ J}$$

(b) From $x = 0$ to $x = 2$ m, displacement of particle and force acting on the particle both are along positive x-direction. Therefore, work done is positive and given by the area under F-x graph, or

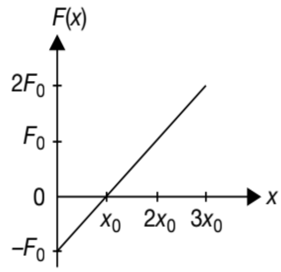
$$W = \frac{1}{2}(2)(10) = 10 \text{ J}$$

ILLUSTRATION 17

The force exerted on an object is $F = F_0\left(\frac{x}{x_0} - 1\right)$. Calculate the work done in moving the object from $x = 0$ to $x = 3x_0$ by plotting the $F(x)$ graph and then calculating the area under the graph. Also calculate the work by evaluating the line integral of force.

SOLUTION

The graph shows F as a function of x is shown in Figure.



Since, $F = F_0\left(\frac{x}{x_0} - 1\right)$, so the work done is negative as the object moves from $x = 0$ to $x = x_0$ and positive as it moves from $x = x_0$ to $x = 3x_0$. Since work done is the area under F-x graph so

$$W = -\frac{1}{2}F_0x_0 + \frac{1}{2}(2F_0)(2x_0) = \frac{3}{2}F_0x_0$$

The line integral of the force is the work done, so

$$W = \int_0^{3x_0} F_0\left(\frac{x}{x_0} - 1\right)dx$$

$$\Rightarrow \quad W = F_0\left(\frac{x^2}{2x_0} - x\right)\Bigg|_0^{3x_0} = \frac{3}{2}x_0F_0$$

Test Your Concepts-I

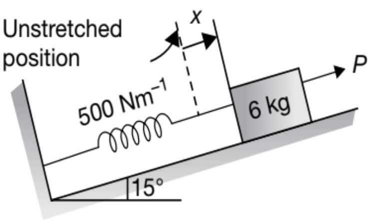
Based on Work Done by a Constant and Variable Force

(Solutions on page H.3)

1. A ball dropped from a height h strikes the ground with a speed of $k\sqrt{gh}$ where $k < \sqrt{2}$. Calculate the work done by air drag.
2. A block is pulled a distance ℓ_0 along a rough horizontal table by a horizontal string. If the tension in the string is T, the mass of the block is m, the normal reaction is N and frictional force is f. Find the work done by each of these forces.
3. A particle is pulled a distance ℓ up a rough plane inclined at an angle θ to the horizontal by a string

inclined at an angle ϕ to the plane $(\theta+\phi<90^\circ)$. If the tension in the string is T, the normal reaction between the particle and the plane is N, the frictional force is f and the mass of the particle is m. Find the work done by each of these forces.

4. The spring is unstretched at the position $x = 0$. Under the action of a force P, the cart moves from the initial position $x_1 = -150$ mm to the final position $x_2 = 80$ mm. Calculate the work done on the cart by

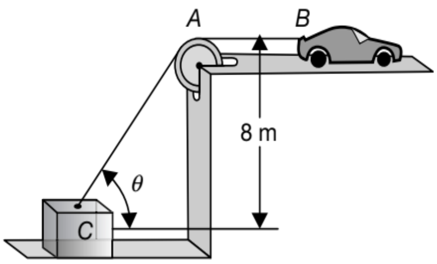


(a) the spring and
(b) its weight.

5. In a research apparatus, one of the forces exerted on a proton is $\vec{F} = -\alpha x^2\hat{i}$, where α has unit of Nm^{-2}. Calculate the work done by $\vec{F}$, when the proton undergoes a displacement along the straight line path

(a) from the point (0.1, 0)m to the point (0.1, 0.4)m
(b) along the straight-line path from the point (0.1, 0)m to the point (0.3, 0)m
(c) along the straight-line path from the point (0.3, 0)m to the point (0.1, 0)m

6. The 100 kg block is being dragged across the smooth surface by means of a car. If the towing cable passes over a small pulley at A, calculate the work done on the block when $\theta = 60^\circ$. The block is at rest when $\theta = 30^\circ$ and the car exerts a constant force $F = 500$ N on the cable.

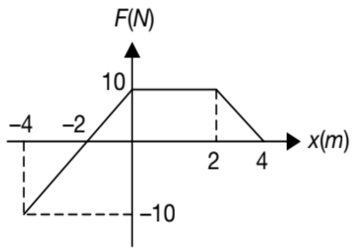


7. A force $\vec{F} = \left(3x^2\hat{i} + 2y\hat{j}\right)$N displaces an object from position vector $\vec{r}_1 = \left(2\hat{i} + 3\hat{j}\right)$m to $\vec{r}_2 = \left(4\hat{i} + 6\hat{j}\right)$ m. Calculate the work done by this force.

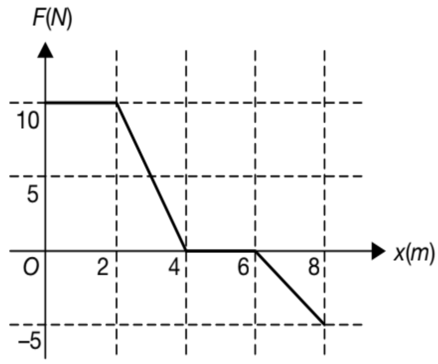
8. A cutting tool under microprocessor control has several forces acting on it. One force is $\vec{F} = -\alpha xy^2\hat{j}$, a force in the negative y-direction whose magnitude depends on the position of the tool. The constant is $\alpha = 1.5\ Nm^{-3}$. Consider the displacement of the tool from the origin to the point $(2, 2)$m.

(a) Calculate the work done, in joule, on the tool by $\vec{F}$ if this displacement is along the straight line $y = x$ that connects these two points.
(b) Calculate the work done on the tool by $\vec{F}$ if the tool is first moved out along the x-axis to the point (2, 0)m and then moved parallel to the y-axis to (2, 2)m.
(c) Compare the work done by $\vec{F}$ along these two paths. Is $\vec{F}$ conservative or non-conservative?

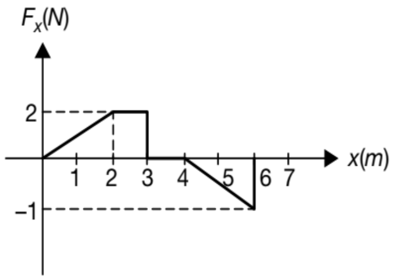
9. Force acting on a particle varies with displacement as shown in figure. Find the work done by this force on the particle from $x = -4$ m to $x = +4$ m.



10. A 5 kg block moves in a straight line on a horizontal frictionless surface under the influence of a force that varies with position as shown in the figure. Find the work done by this force as the block moves from the origin to $x = 8$ m.

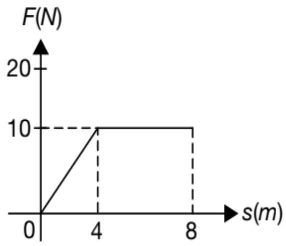


11. A child applies a force $\vec{F}$ parallel to the x-axis to a block moving on a horizontal surface. As the child controls the speed of the block, the x-component of the force varies with the x-coordinate of the block as shown in figure. Calculate the work done by the force $\vec{F}$ when the block moves



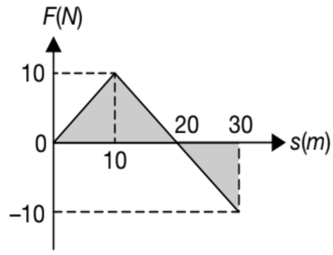
(a) from $x = 0$ to $x = 3$ m
(b) from $x = 3$ m to $x = 4$ m
(c) from $x = 4$ m to $x = 7$ m
(d) from $x = 0$ to $x = 7$ m

12. Force acting on a particle varies with displacement as shown in figure. Calculate the total work done by the force.



13. A force $\vec{F} = -k(y\hat{i} + x\hat{j})$, where k is a positive constant, acts on a particle moving in the xy plane. Starting from the origin, the particle is taken along the positive x-axis to the point $(a, 0)$ and then parallel to the y-axis to the point (a, a). Calculate the total work done by the force on the particle.

14. The force versus displacement graph for a body is shown in the Figure.



Calculate the work done by force to displace the body from zero to
(a) 10 m
(b) 20 m
(c) 30 m

15. A block of mass m is suspended from the ceiling of a lift by a light rope. The lift is going down with an acceleration $\frac{g}{2}$. Calculate the work done by various forces acting on the block as it goes down through h.

ENERGY

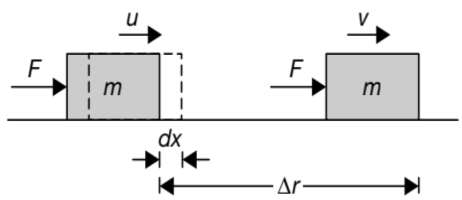
The capacity of the body to do work is called **Energy**. Energy can be of many types like kinetic energy, potential energy heat energy, light energy, sound energy, electrostatic energy, magnetic energy, atomic energy, nuclear energy etc. However, for this chapter we restrict ourselves to the study of **Mechanical Energy, which is the sum of kinetic energy and the potential energy.**

The Kinetic Energy is the energy possessed by a body by virtue of its motion.

The Potential Energy is the energy possessed by the body by virtue of its position or configuration.

CONCEPT OF KINETIC ENERGY AND WORK ENERGY THEOREM (OR THE CLASSICAL WORK ENERGY (CWE) THEOREM)

Consider a block of mass m undergoing a displacement Δr and a change in velocity under the action of a constant net force F, as shown in Figure.



Let the block the given an infinitesimal displacement dx under the influence of the force F. If dW is the work done, then

$$dW = F dx \cos(0^\circ) = F dx$$

$$\Rightarrow \quad dW = m\left(\frac{dv}{dt}\right)dx \qquad \left\{\because F = m\frac{dv}{dt}\right\}$$

$$\Rightarrow \quad dW = m\left(\frac{dx}{dt}\right)dv = mv dv \qquad \left\{\because \frac{dx}{dt} = v\right\}$$

$$\Rightarrow \quad W = \int dW = m\int_u^v v dv$$

$$\Rightarrow \quad W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = F\Delta r \qquad \ldots(1)$$

where, the terms

$\frac{1}{2}mv^2$ = Final Kinetic Energy of the block and

$\frac{1}{2}mu^2$ = Initial Kinetic Energy of the block

Also, from (1), we observe that the work done by an external force acting on the block equals the change in kinetic energy of the block, i.e.

$$(W_{\text{ext}})_{\text{total}} = \Delta K \qquad \{\text{Work Energy Theorem}\}$$

The quantity $K = \frac{1}{2}mv^2$ is a scalar and is called the **Kinetic Energy** of the particle which is the energy possessed by a particle by virtue of its motion.
Thus, the equation (1) takes the form

$$W = K_f - K_i = \Delta K \qquad \text{...(2)}$$

The work done by a force changes the kinetic energy of the particle. This is called the **Work-Energy Theorem.**

In general, **the net work done by the resultant of all the forces acting on the particle is equal to the change in kinetic energy of a particle.** So, we have

$$W_{\text{net}} = \Delta K \qquad \text{...(3)}$$

Conceptual Note(s)

(a) The kinetic energy of an object is a measure of the amount of work needed to increase its speed from zero to a given value.

(b) The kinetic energy of a particle is the work it can do on its surroundings in coming to rest.

(c) Since the velocity and displacement of a particle depend on the frame of reference, the numerical values of the work and the kinetic energy also depend on the frame.

ILLUSTRATION 18

The velocity v of a particle of mass m moving along x-axis is given by $v = b\sqrt{x}$, where b is a constant. Calculate the work done by the force acting on the particle during its motion from $x = 0$ to $x = 4$ m.

SOLUTION

Initial velocity (u) of the particle is the velocity at $x = 0$ i.e.

$$u = v|_{x=0} = b\sqrt{0} = 0$$

Final velocity (v) of the particle is the velocity at $x = 4$ i.e.

$$v = v|_{x=4} = b\sqrt{4} = 2b$$

According to Work-Energy Theorem, we have

$$W = \Delta K = K_f - K_i = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$\Rightarrow \quad W = \frac{1}{2}m(2b)^2 = 2mb^2$$

IMPORTANCE OF THE WORK ENERGY THEOREM

The work kinetic energy theorem does not represent a new independent law of mechanics. The theorem is useful for solving problems in which the work done can be easily computed and in which we are interested in finding the particle's speed at certain positions.

One method to do that is to apply Newton's Law of Motion and find out the acceleration of the particle from where we can get its speed. **However, this method is lengthier and particularly not useful when we are interested only in speed and not in acceleration of the particle.**

The other method is to use the Work-Energy Theorem because it has the advantage that it does away with all intermediate steps (i.e. what is happening during the motion). We have to simply calculate the sum of the works done by individual forces and equate that to the change in kinetic energy of the particle. This discussion also applies to the Conservation of Mechanical Energy principle which will follow the article on potential energy.

ILLUSTRATION 19

A particle of mass 0.01 kg travels along a curve with a velocity given by $4\hat{i} + 16\hat{k}$ ms^{-1}. After some time, its velocity becomes $8\hat{i} + 20\hat{j}$ ms^{-1} due to the action of a conservative force. Calculate the work done on the particle during this interval of time.

SOLUTION

Since, work done equals the change in kinetic energy, so

$$W = \Delta K = K_f - K_i = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

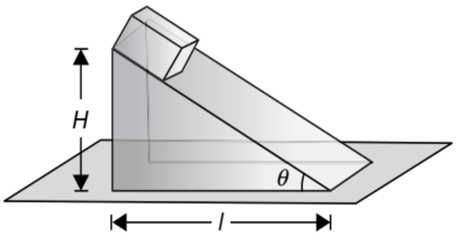
where, $u = \sqrt{4^2 + 16^2} = \sqrt{272}$ and

$$v = \sqrt{8^2 + 20^2} = \sqrt{464}$$

$$\Rightarrow \quad W = \frac{1}{2}m(v^2 - u^2) = \frac{1}{2} \times 0.01(464 - 272) = 0.96 \text{ J}$$

ILLUSTRATION 20

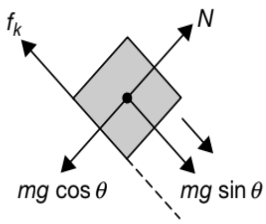
A small block of mass m is released from the top of a rough inclined plane as shown in Figure.



The coefficient of friction between the block and inclined plane is μ. Apply Work-Energy Theorem to calculate the speed of block as it passes the lowest point.

SOLUTION

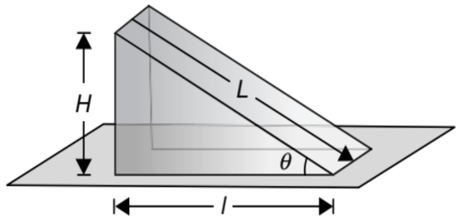
The FBD of the block is shown in Figure.



The initial speed is zero and let the final speed of the block as it passes the lowest point is v. According to Work-Energy Theorem, we have

$$W_{\text{Total}} = \Delta K = \frac{1}{2}mv^2 - 0$$

Let us now find W_{Total} by calculating the work done by each force.



Work done by normal reaction N is $W_N = 0$

Work done by $mg\cos\theta$ is $W_{mg\cos\theta} = 0$

Work done by $mg\sin\theta$ is

$$W_{mg\sin\theta} = (mg\sin\theta)(L)\cos(0°) = mgL\sin\theta$$

Since $\sin\theta = \dfrac{H}{L}$

$$\Rightarrow \quad W_{mg\sin\theta} = mg\left(\frac{H}{L}\right)(L) = mgH$$

Work done by kinetic friction f_k is

$$W_{f_k} = (\mu N)(L)\cos(180°) = -\mu mgL\cos\theta$$

Since $\cos\theta = \dfrac{l}{L}$

$$\Rightarrow \quad W_{f_k} = -\mu mg\left(\frac{l}{L}\right)(L) = -\mu mgl$$

So, the total work done W is given by

$$W = mgH - \mu mgl$$

Applying Work-Energy Theorem, we get

$$\frac{1}{2}mv^2 = mgH - \mu mgl$$

$$\Rightarrow \quad v = \sqrt{2g(H - \mu l)}$$

ILLUSTRATION 21

A running man has half the kinetic energy that a boy of half his mass has. The man speeds up by 1 ms^{-1} so as to have same kinetic energy as that of boy. Find the original speed of the man and the boy.

SOLUTION

Let mass of the man be M, then

$$(\text{K.E.})_{\text{man}} = \frac{1}{2}(\text{K.E.})_{\text{boy}}$$

$$\Rightarrow \quad \frac{1}{2}Mv_m^2 = \frac{1}{2}\left[\frac{1}{2}\left(\frac{M}{2}\right)v_b^2\right]$$

$$\Rightarrow \quad v_m = \frac{v_b}{2} \qquad \ldots(1)$$

Further, $\frac{1}{2}M(v_m+1)^2 = \frac{1}{2}\left(\frac{M}{2}\right)v_b^2$

$$\Rightarrow \quad v_m + 1 = \frac{v_b}{\sqrt{2}} \qquad \ldots(2)$$

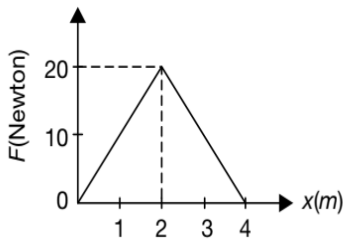
Solving (1) & (2), we get

$$v_b = 2(\sqrt{2}+1) = 4.82 \text{ ms}^{-1}$$

$$v_m = \sqrt{2}+1 = 2.41 \text{ ms}^{-1}$$

ILLUSTRATION 22

The graph between the resistive force F acting on a body and the distance covered by the body is shown in the figure. The mass of the body is 25 kg and initial velocity is 2 ms^{-1}. Find the kinetic energy when the distance covered by the body is 5 m.



SOLUTION

Initial kinetic energy of the body is

$$K_i = \frac{1}{2}mu^2 = \frac{1}{2}\times 25\times(2)^2 = 50 \text{ J}$$

Since Work done equals the change in kinetic energy, so

$$W = \Delta K = K_f - K_i$$

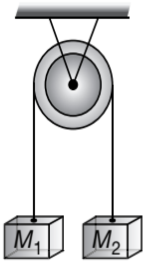
$$\Rightarrow \quad K_f = W + K_i$$

Now, work done against resistive force is equal to the negative of area under F-x graph, so

$$K_f = 50 - \frac{1}{2}\times 4\times 20 = 50 - 40 = 10 \text{ J}$$

ILLUSTRATION 23

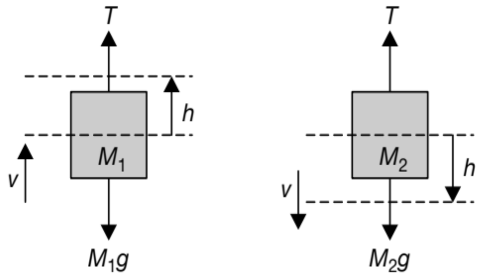
A system consists of two blocks having masses M_1 and M_2, (where $M_2 > M_1$) connected by a light inextensible string passing over a smooth pulley as shown in Figure.



Calculate the acceleration of the blocks by applying the Work-Energy Theorem.

SOLUTION

Since $M_2 > M_1$, so when the block M_2 moves down by a distance h, then the block M_1 will move up through the same distance h. The speed of M_1 and M_2 at this instant will be the same, say v as shown in figure.



Net work done on M_1 is

$$W_1 = Th\cos(0°) + M_1gh\cos(180°)$$

$$\Rightarrow \quad W_1 = (T - M_1g)h$$

Net work done on M_2 is

$$W_2 = M_2g\cos(0°) + Th\cos(180°)$$

$$\Rightarrow \quad W_2 = (M_2g - T)h$$

So, total work done is

$$W = W_1 + W_2 = (M_2 - M_1)gh$$

According to Work-Energy Theorem, we have

$$W = \Delta K$$

$$\Rightarrow \quad (M_2 - M_1)gh = \frac{1}{2}M_1v^2 + \frac{1}{2}M_2v^2$$

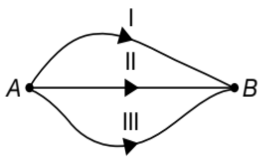
$$\Rightarrow \quad v^2 = 2\left(\frac{M_2 - M_1}{M_1 + M_2}\right)gh$$

Comparing with $v^2 = 2ah$, we get

$$a = \left(\frac{M_2 - M_1}{M_1 + M_2}\right)g$$

CONSERVATIVE AND NON-CONSERVATIVE FORCES

Statement I: Force is said to be conservative in nature when the work done by the force is independent of the path followed between any two points.

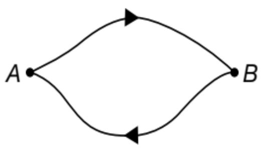


Mathematically, if for a force we have

$$\underset{\text{Path I}}{W_{A\to B}} = \underset{\text{Path II}}{W_{A\to B}} = \underset{\text{Path III}}{W_{A\to B}}$$

then F (the force) must be conservative in nature.

Statement II: Force is said to be conservative in nature when the work done by a force in a closed loop is zero.



Mathematically, if for a force we have

$$W_{A\to B} + W_{B\to A} = 0,$$

then F (the force) must be conservative in nature.

Statement III: Force is said to be conservative in nature when the line integral of the force i.e. $\int \vec{F}\cdot d\vec{l}$ (which is also the work done) is independent of the path followed between any two points.

Mathematically, if for a force we have

$$\int_{\text{Path I}} \vec{F}\cdot d\vec{l} = \int_{\text{Path II}} \vec{F}\cdot d\vec{l} = \int_{\text{Path III}} \vec{F}\cdot d\vec{l}$$

then $\vec{F}$ is a conservative force

Statement IV: Force is said to be conservative in nature when the line integral of a force over a closed path/loop is zero.

Mathematically, if for a force we have

$$\oint \vec{F}\cdot d\vec{l} = 0$$

then $\vec{F}$ is a conservative force.

EXAMPLE

(a) Electrostatic forces, gravitational forces, elastic forces, all constant forces are conservative forces.

(b) All forces which do not obey the above specified conditions are called **non-conservative forces**.

(c) Frictional force, air drag, viscous drag i.e. all dissipative forces.

POTENTIAL ENERGY

When we throw a ball upwards with an initial velocity, it rises to a certain height and becomes stationary for a moment.

What happens to the lost kinetic energy?

We know with our experience that the ball returns to our hands with a speed equal to its initial value (assuming air drag to be absent). The initial kinetic energy is somehow stored and is later fully recovered in the form of kinetic energy.

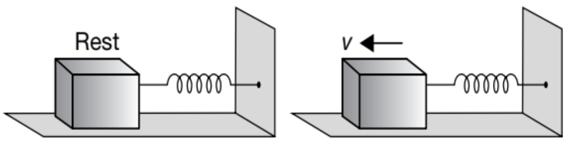
The ball must have some energy at the new height that it does not have at the previous level. That new energy by virtue of its position is called the **Potential Energy**.

Potential energy is the energy associated with the relative positions of two or more interacting particles.

Potential energy fits well to the idea of energy as the capacity to do work. The gravitational potential energy of an object raised off the ground can be used to compress or expand a spring or to lift another weight. As a coil spring unwinds, or a straight spring returns to its natural length, the stored elastic potential energy can be used to do work.

EXAMPLE

If a block is attached to a compressed spring, the elastic potential energy can be converted into kinetic energy of the block as shown in figure. The block gains kinetic energy when the compressed spring is released.

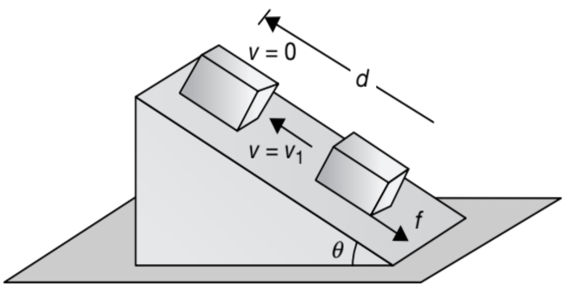


In the above discussion we have seen that in the case of gravity and elastic spring the kinetic energy imparted initially is stored as potential energy for a short time which is regained, later on. But this is not true in all cases.

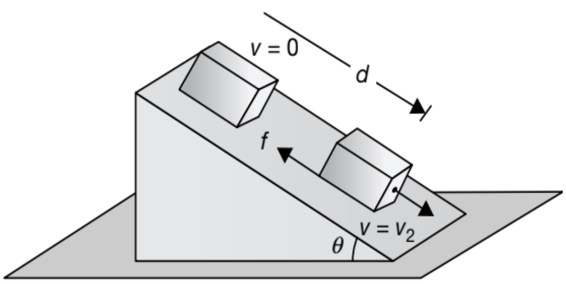
The forces, such as gravity and spring force, which do work in a reversible manner are called **Conservative Forces**. In contrast, the force, such as frictional force, which does work in an irreversible manner are called **Non-Conservative Forces**.

EXAMPLE

Consider block placed at rest on a rough horizontal surface. If we impart it some initial kinetic energy, it starts sliding on the surface, the frictional force does negative work on the block, decreasing its kinetic energy to zero. But it does not come back to our hand no matter how long we wait! The frictional force has used up the kinetic energy in a non - reversible way.



$W_g = -mgd\sin\theta$, $W_f = -fd$



$W_g = mgd\sin\theta$, $W_f = -fd$

A block slides up and down on a rough inclined plane. In the complete trip, the work done by gravity is zero whereas the work done by friction is negative.

THE POTENTIAL ENERGY IS DEFINED ONLY FOR CONSERVATIVE FORCES

The change in potential energy as a particle moves from point A to point B is equal to the negative of the work done by the associated conservative force

$$\Delta U = U_B - U_A = -W_C$$

Using definition of work

$$U_B - U_A = -\int_A^B \vec{F}_C \cdot d\vec{r}$$

From this equation, we see that, starting with potential energy U_A at point A, we obtain a unique value U_B at point B, because W_C has the same value for all paths. When a block slides along a rough floor, the work done by the force of friction on the block depends on the length of the path taken from point A to point B. **There is no unique value for the work done, so one cannot assign unique values for potential energy at each point. Hence, non-conservative force cannot have potential energy.**

When the forces within a system are conservative, external work done on the system is stored as potential energy and is fully recoverable.

Note that the potential energy is always defined with respect to a reference point.

CONSERVATIVE SYSTEM AND CONCEPT OF POTENTIAL ENERGY (U)

For a conservative system, since work done is independent of the path followed between any two points or it just depends upon the final state and initial state of the body. So, there must exist a certain scalar physical quantity such that the work done by this conservative force (W_c) is equal to the decrease in the value of this new scalar quantity. This scalar quantity is called the **Potential Energy,** U. So

$$W_c = -\Delta U = U_i - U_f$$

EXAMPLE

Just think of a body being dropped from a height h to the ground, then $U_{\text{initial}} = mgh$ and when it lands on the ground then $U_{\text{final}} = 0$. Also, the work done by the gravitational force (a conservative force) is equal to *mgh* and this happens to be the decrease in potential energy of the falling body.

So, we can simply conclude that the work done by a conservative force (W_c) is equal to the decrease in potential energy. Hence, we have

$$W_c = \begin{pmatrix} \text{Decrease in} \\ \text{Potential Energy} \end{pmatrix} = U_{\text{initial}} - U_{\text{final}} = -\Delta U$$

GRAVITATIONAL POTENTIAL ENERGY (NEAR THE EARTH'S SURFACE)

The work done by gravity on a particle of mass m whose vertical coordinate changes from y_A to y_B is

$$W_g = -mg\left(y_B - y_A\right)$$

Since, we know that $W_g = -\Delta U = -\left(U_B - U_A\right)$

Thus, gravitational potential energy at the point B near the surface of the earth is given by

$$U_B = U_A + mgh$$

If we assume potential energy at the point A to be zero, then potential energy at the point B is given by

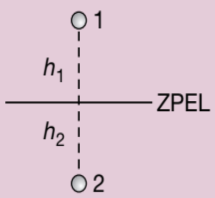
$$U_B = 0 + mgh = mgh$$

Conceptual Note(s)

(a) It is very important to **assign a Zero Potential Energy Level (ZPEL)** before we calculate the potential energy of a particle.

(b) Gravitational Potential Energy U depends upon the choice of **ZPEL** but Gravitational Potential Energy Difference ΔU is independent of the choice of **ZPEL.**

(c) If one point lies above **ZPEL** and other below **ZPEL**, then



$U_1 = +mgh_1$, $U_2 = -mgh_2$ with respect to the specified ZPEL. However,

$$U_1 - U_2 = \Delta U = mg\left(h_1 + h_2\right)$$

irrespective of the location of ZPEL.

SPRING POTENTIAL ENERGY OR ELASTIC POTENTIAL ENERGY

The work done by the spring force when the displacement of the free end changes from x_i to x_f is given by

$$W_S = -\frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

By definition $W_S = -\Delta U_S = U_i - U_f$

$$\Rightarrow \quad U_f = U_i + \frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

If we assume the potential energy stored in the spring at equilibrium is zero and all the extensions to be measured from equilibrium, then $x_i = 0$ and $U_i = 0$. Thus, final energy stored in the spring, when $x_f = x$ is

$$U_f = U_S = \frac{1}{2}kx^2$$

The potential energy function for an ideal spring is a parabolic function of extension or compression x as shown in figure.

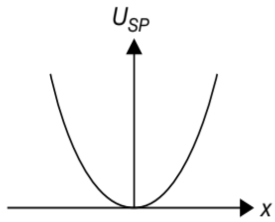
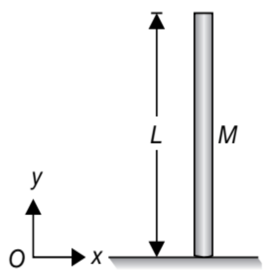


ILLUSTRATION 24

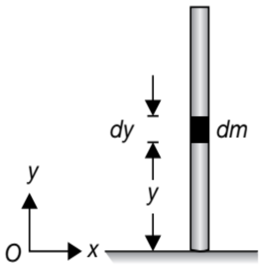
A uniform rod of mass M and length L is held vertically upright on a horizontal surface as shown in the figure.



Calculate the potential energy of the rod if the zero potential energy level is assumed at the horizontal surface.

SOLUTION

Since the parts of the rod are at different levels with respect to the horizontal surface, therefore, we have to use the integration to find its potential energy. Consider a small element of length dy at a height y from the horizontal as shown in Figure.



If dm is the mass of the element, then

$$dm = \frac{M}{L}dy$$

Its potential energy dU is given by

$$dU = (dm)gy = \frac{M}{L}gydy$$

Integrating, we get

$$U = \frac{Mg}{L}\int_0^L ydy = \frac{Mg}{L}\left(\frac{y^2}{2}\bigg|_0^L\right)$$

$$\Rightarrow \quad U = \frac{1}{2}MgL$$

Conceptual Note(s)

Note that the potential energy of the rod is equal to the product of Mg and height of the center of mass $\left(\frac{L}{2}\right)$ from the surface.

POWER

Power is defined as the rate at which work is done. If an amount of work ΔW is done in a time interval Δt, then the **average power** is defined to be

$$P_{av} = \frac{W}{t} = \frac{\Delta W}{\Delta t}$$

The SI unit of power is Js^{-1} which is given the name watt (W) in the honour of James Watt.

Thus, $1\text{ W} = 1\text{ Js}^{-1}$

Instantaneous power is the limiting value of P_{av} when $\Delta t \to 0$, so we have

$$P_{ins} = \frac{dW}{dt}$$

The work done by force F on a object that has an infinitesimal displacement $d\vec{r}$ is $dW = \vec{F}\cdot d\vec{r}$. The instantaneous power may be written as

$$P = \frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt}$$

$$\Rightarrow \quad P = \vec{F}\cdot\vec{v} \qquad \left\{\because \vec{v} = \frac{d\vec{r}}{dt}\right\}$$

Since the work and energy are closely related, a more general definition of power is the rate of energy transfer from one body to another, or the rate at which energy is transformed from one form to another.

$$P = \frac{dE}{dt}$$

ILLUSTRATION 25

A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration a_r is varying with time t as $a_r = k^2rt^2$, where k is a constant, Calculate the power delivered to the particle by the forces acting on it.

SOLUTION

Let the instantaneous speed of the particle be v, then centripetal acceleration is given by

$$a_r = \frac{v^2}{r}$$

According to the problem we have $a_r = k^2rt^2$, so we get

$$\Rightarrow \quad \frac{v^2}{r} = k^2rt^2$$

$$\Rightarrow \quad v = krt$$

The tangential acceleration is given by

$$a_t = \frac{dv}{dt} = kr$$

The tangential force is $F_t = ma_t = mkr$

Hence, power delivered is

$$P = F_t v = (mkr)(krt)$$

$$\Rightarrow \quad P = mk^2r^2t$$

ILLUSTRATION 26

An elevator that can carry a maximum load of 1500 kg (elevator + passengers) is moving up with a constant speed of $2\ \text{ms}^{-1}$. The frictional force opposing the motion is 3000 N. Calculate the minimum power delivered by the motor to the elevator in kilo watt and in horse power. Take $g = 10\ \text{ms}^{-2}$.

SOLUTION

Since the elevator is moving up with a constant speed of $2\ \text{ms}^{-1}$, so the net downward force on the elevator is

$$F = mg + f$$

$$\Rightarrow \quad F = (1500 \times 10) + 3000$$

$$\Rightarrow \quad F = 18000\ \text{N}$$

The motor must supply enough power to balance this force. If P be the power required for this purpose, then

$$P = Fv = (18000)(2) = 36000\ \text{W}$$

$$\Rightarrow \quad P = 36\ \text{kW}$$

Since $1\ \text{hp} = 746\ \text{W}$

$$\Rightarrow \quad P = \frac{36000}{746}\ \text{hp}$$

$$\Rightarrow \quad P = 48.3\ \text{hp}$$

ILLUSTRATION 27

A block of mass m is pulled by a constant power P placed on a rough horizontal plane. The friction coefficient between the block and surface is μ. Find the maximum velocity of the block.

SOLUTION

Power $P = Fv = \text{constant}$

$$\Rightarrow \quad F = \frac{P}{v}$$

$$\Rightarrow \quad F \propto \frac{1}{v}$$

as v increases, F decreases.

when $F = \mu mg$, net force on block becomes zero, i.e., it has maximum or terminal velocity, so

$$P = (\mu mg) v_{\max}$$

$$\Rightarrow \quad v_{\max} = \frac{P}{\mu mg}$$

ILLUSTRATION 28

A particle is moving along x-axis under the action of a force F which varies with its position x as $F \propto \frac{1}{\sqrt[4]{x}}$.

Calculate the variation of power due to this force with x.

SOLUTION

Since, $F \propto x^{-1/4}$

$$\Rightarrow \quad a \propto x^{-1/4}$$

$$\Rightarrow \quad a = kx^{-1/4} \quad \text{(where } k \text{ is a proportionality constant)}$$

Since, $a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$

$$\Rightarrow \quad v\frac{dv}{dx} = kx^{-1/4}$$

$$\Rightarrow \quad vdv = kx^{-1/4}dx$$

Integrating, we get

$$\int_0^v vdv = k\int_0^x x^{-1/4}dx$$

$$\Rightarrow \quad \frac{v^2}{2} = k\frac{x^{3/4}}{3/4}$$

$$\Rightarrow \quad v^2 \propto x^{3/4}$$

$$\Rightarrow \quad v \propto x^{3/8}$$

Since $P = Fv$

$$\Rightarrow \quad P \propto (x^{-1/4})(x^{3/8})$$

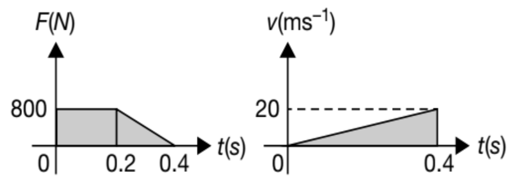
$$\Rightarrow \quad P \propto x^{1/8}$$

Test Your Concepts-II

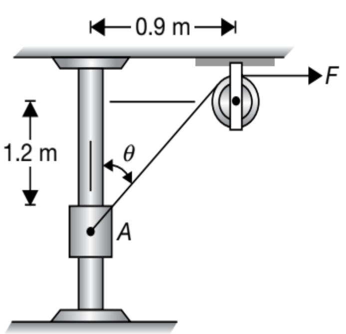
Based on Kinetic Energy, Potential Energy and Power

(Solutions on page H.5)

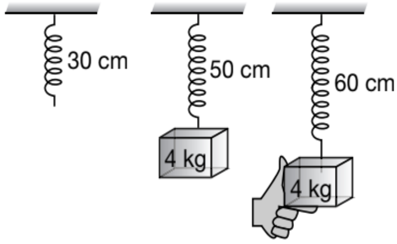
1. An engine working at a constant power P draws a load of mass m against a resistance force f. Find the maximum speed of the load and the time taken to attain half this speed.
2. An automobile of mass m accelerates, starting from rest. The engine supplies constant power P.
   (a) Find the velocity of the automobile as a function of time.
   (b) Find the position of the automobile as function of time.
3. A particle moving in a straight line is acted upon by a force which works at a constant rate and changes its velocity from u and v over a distance x. Prove that the time taken in it is $\dfrac{3(u+v)x}{2(u^2+v^2+uv)}$.
4. A baseball having a mass of 0.4 kg is thrown such that the force acting on it varies with time as shown in the first graph. Also, the velocity of the ball acting in the direction same as that of force varies with time as shown in the second graph. Determine the power applied as a function of time and the work done in $t = 0.4$ s.



5. The 50 N collar starts from rest at A and is lifted with a constant speed of $0.6\ \text{ms}^{-1}$ along the smooth rod. Determine the power developed by the force F at the instant shown.



6. A train has a constant speed of $40\ \text{ms}^{-1}$ on a level road against resistive force of magnitude 3×10^4 N. Find the power of engine.
7. A particle is moving along x-axis under the action of a force F which varies with its position x as $F \propto \dfrac{1}{\sqrt{x}}$. If the power due to this force is proportional to x^n. Calculate n.
8. A body of mass 5 kg is placed at origin. A force starts acting on the body given by $\vec{F} = (2+3x)\hat{i}$, where x is the distance of body from origin in meters. Calculate the speed of the body as it passes through $x = 5$ m.
9. A particle of mass 0.5 kg travels in a straight line with velocity $v = ax\sqrt{x}$ where $a = 5\ \text{m}^{-1/2}\text{s}^{-1}$. Calculate the work done by the all forces when the particle is displaced from $x = 0$ to $x = 2$ m.
10. A wind-powered generator converts wind energy into electrical energy. Assume that the generator converts a fixed fraction of wind energy interception by its blades into electrical energy. For wind speed v, the electrical power output will be proportional to v^n. Calculate n.
11. A pendulum bob has potential energy U_O when held taut in a horizontal position. The bob is allowed to fall until it is 30° away from the horizontal position, when it has potential energy U_A. It continues to fall until the string is vertical, when it has potential energy U_B. Calculate $\dfrac{U_O - U_A}{U_A - U_B}$
12. A spring acquires a potential energy of 30 J when stretched by a length of 20 cm. If the spring is now stretched further by 40 cm, calculate the additional amount of work required to do so.
13. Power of the engine of a motor boat is 50 HP. If resistance force F of water increases with speed v (in ms^{-1}) of the boat according to $F = 20v$ (in kN). Calculate the maximum speed of this motor boat in kmh^{-1}.
14. On attaching a 4 kg mass to a spring of length 30 cm, the spring stretches by 20 cm. Then the mass is pulled down until the length of the spring becomes 60 cm. Calculate the amount of elastic energy stored in the spring.



15. A brick of mass 5 kg and dimensions (in cm) $20 \times 10 \times 5$ lies on the ground on its largest face. If it is made to stand on its smallest face. Calculate the change in its gravitational potential energy.

16. A metre scale of mass 0.1 kg pivoted at the top end is turned from vertical position so that it is inclined at 30° with the horizontal in the vertical plane. Calculate the work done by gravity.

LAW OF CONSERVATION OF MECHANICAL ENERGY

Since, for a conservative system we have

$$W_c = \Delta K \qquad \ldots(1)$$

Also, we know that

$$W_c = -\Delta U \qquad \ldots(2)$$

Equating (1) and (2), we get

$$\Delta K = -\Delta U$$

$\Rightarrow \quad \Delta(K+U) = 0$

$\Rightarrow \quad K + U = \text{constant}$

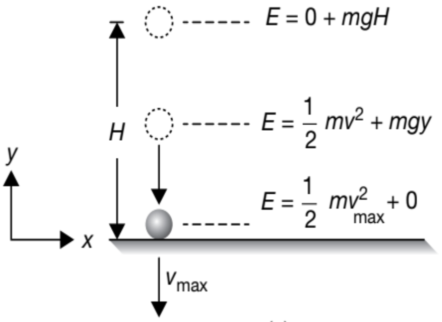
$\Rightarrow \quad K_i + U_i = K_f + U_f$

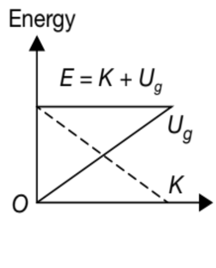
$\Rightarrow \quad \begin{pmatrix}\text{Total Initial} \\ \text{Mechanical Energy}\end{pmatrix} = \begin{pmatrix}\text{Total Final} \\ \text{Mechanical Energy}\end{pmatrix}$

Let us discuss few fundamental examples where we apply the Law of Conservation of Mechanical Energy to systems.

CASE-1: OBJECT FALLING FREELY

When an object falls from height H, its potential energy is converted to kinetic energy. At height H, the energy is $E = mgH$. Just as it lands, $E = \frac{1}{2}mv_{\text{max}}^2$.





(a) (b)

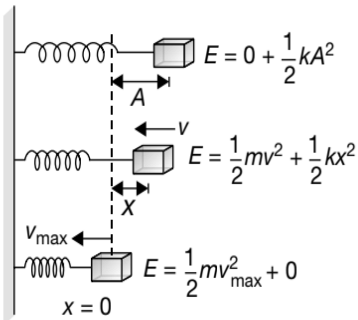
The potential energy and the kinetic energy vary linearly with vertical height y.

At any instant, the mechanical energy E remains constant. So,

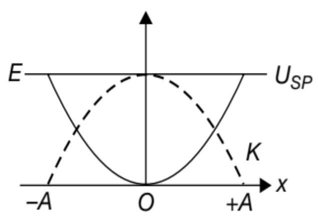
$$E = K + U = \frac{1}{2}mv^2 + mgy = \text{constant}$$

CASE-2: MASS ATTACHED TO A SPRING

When a mass m is attached to a spring of force constant k, then at maximum extension in the spring (called the amplitude), $KE = 0$ and $PE = \frac{1}{2}kA^2$.



(a)



(b)

At extension x, we have

$$KE = \frac{1}{2}mv^2 \text{ and } PE = \frac{1}{2}kx^2$$

At mean position, we have

$$PE = 0 \text{ and}$$

$$KE = \frac{1}{2}mv_{\max}^2$$

Here too, at any instant, the mechanical energy E remains constant. So, we have

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Problem Solving Technique(s)

To solve problems using the concept of Conservation of Mechanical Energy we can follow the series of steps mentioned below.

(a) Draw diagrams of the system showing initial and final configuration and assume a coordinate system.

(b) Specify the reference level for potential energy. **In case of spring**, it is advisable to assume zero potential energy at the natural length of the spring. **In case of gravity**, any convenient level can be chosen as reference frame.

(c) Looking at the initial configuration, **ask yourself What forms of energy are present in the system?**

- **(i)** if the particle is moving, include $\frac{1}{2}mv_i^2$
- **(ii)** if the particle is not located at the reference level, include mgy_i
- **(iii)** if the spring is stretched or compressed, include $\frac{1}{2}kx_i^2$

(d) Looking at the final configuration, **ask yourself What forms of energy are present?**

- **(i)** if the particle is moving, include $\frac{1}{2}mv_f^2$
- **(ii)** if the particle is not located at the reference, include mgy_f
- **(iii)** if the spring is stretched or compressed, include $\frac{1}{2}kx_f^2$

(e) Equate the initial and final total energies

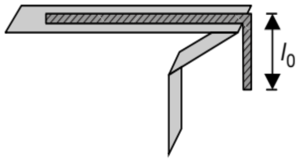
$$K_i + U_i = K_i + U_f$$

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}kx_f^2$$

(f) Solve for the unknown.

ILLUSTRATION 29

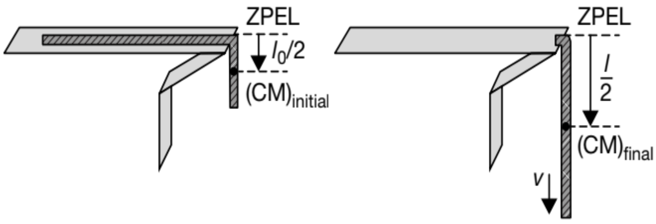
A rope of length $l = 80\text{ cm}$ and mass $m = 2\text{ kg}$ is hanging from the end of a plane so that the length l_0 of the vertical segment is 50 cm as shown in Figure.



The other end of the rope is fixed by a nail. At a certain instant, the nail is pushed out, what is the velocity of the rope at the moment it completely slides off the plane? Neglect the friction.

SOLUTION

We assume the Zero Potential Energy Level (ZPEL) at the horizontal plane. Assuming the rope to have a mass m and a length l. If λ is the mass per unit length of the rope then $\lambda = \frac{m}{l}$. The initial and final configuration of the rope are shown in Figure.



Initial kinetic energy of the rope is

$$K_i = 0$$

Initial potential energy of the hanging portion of the rope is obtained by locating the centre of mass (CM) of the hanging portion of the rope, which lies $\frac{l_0}{2}$ below the ZPEL.

$$U_i = 0 + \left(\frac{m}{l}l_0\right)g\left(-\frac{l_0}{2}\right) = -\frac{ml_0^2}{2l}g$$

Please note that, the part of rope lying over the table i.e. ZPEL, has zero potential energy.

Let v be the final velocity of the rope at the moment it completely slides off the plane. Then final kinetic energy of the rope is

$$K_f = \frac{1}{2}mv^2$$

Final potential energy of the rope is obtained by locating the centre of mass (CM) of the fully hanging rope, which lies $\frac{l}{2}$ below the ZPEL.

$$U_f = -mg\left(\frac{l}{2}\right)$$

Using the Law of Conservation of Energy, we get

$$K_f + U_f = K_i + U_i$$

$$\Rightarrow \quad \frac{1}{2}mv^2 - mg\left(\frac{l}{2}\right) = 0 - \frac{ml_0^2 g}{2l}$$

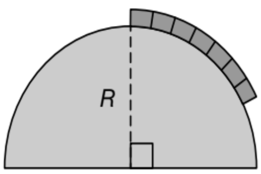
$$\Rightarrow \quad v = \sqrt{\frac{g}{l}\left(l^2 - l_o^2\right)}$$

Substituting $l = 0.8$ m, $l_0 = 0.5$ m and $g = 10$ ms^{-2}, we get

$$v = \sqrt{5}\ \text{ms}^{-1} = 2.24\ \text{ms}^{-1}$$

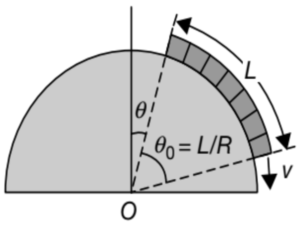
ILLUSTRATION 30

A uniform rope of mass m and length L is placed on the top of a smooth hemispherical surface. The rope is held at rest and released such that it slides down along the curvature as shown in Figure. Calculate the speed of the chain, as it moves through angle θ on the surface.

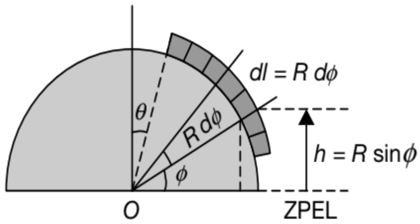


SOLUTION

Let θ_0 be the angle subtended by the rope at the centre of the sphere, then $\theta_0 = \frac{L}{R}$. As the rope slides down, it loses potential energy and gains kinetic energy. At any instant, the angular position of the rope is shown in the Figure.



Let us first calculate the gravitational potential energy of the rope. For that, consider a small element of length dl, having mass dm and subtending an angle $d\phi$ at the centre as shown in Figure.



Taking the Zero Potential Energy Level (ZPEL) at the bottom of the hemisphere, the gravitational potential energy for this infinitesimal element is given by

$$dU = (dm)gh$$

where, $h = R\sin\phi$ is the height of the element above ZPEL.

$$dm = \frac{m}{L}dl = \frac{m}{L}(Rd\phi)$$

$$\Rightarrow \quad dU = \frac{m}{L}gR^2 \sin\phi\, d\phi.$$

At the instant the upper end of the rope makes an angle θ with the vertical, the lower end of the rope makes an angle $\phi = \frac{\pi}{2} - (\theta + \theta_0)$ with the horizontal and at the same instant the upper end of the rope makes and angle $\phi = \frac{\pi}{2} - \theta$ with the horizontal.

So, to get the potential energy of the rope (with respect to ZPEL) at the instant mentioned, integrating by applying limits on ϕ from the lower end of the rope to the upper end of the rope, we get

$$U = \frac{m}{L}gR^2 \int_{\frac{\pi}{2}-(\theta+\theta_0)}^{\frac{\pi}{2}-\theta} \sin\phi\, d\phi$$

$$\Rightarrow \quad U = \frac{mgR^2}{L}(-\cos\phi)\Big|_{\frac{\pi}{2}-(\theta+\theta_0)}^{\frac{\pi}{2}-\theta}$$

$$\Rightarrow \quad U = \frac{mgR^2}{L}\left\{-\cos\left(\frac{\pi}{2}-\theta\right) + \cos\left(\frac{\pi}{2}-(\theta+\theta_0)\right)\right\}$$

$$\Rightarrow \quad U = \frac{mgR^2}{L}\left[\sin(\theta+\theta_0) - \sin\theta\right] \qquad \ldots(1)$$

PE of the rope, when it is just released, can be calculated by taking $\theta = 0$ in the above expression i.e. equation (1). Thus, we have

$$U_i = \frac{mgR^2}{L}\sin\theta_0$$

By Law of Conservation of Mechanical Energy, we have

$$(\text{Loss in GPE of Rope}) = (\text{Gain in KE of Rope})$$

$$\Rightarrow \quad U_i - U_f = K_f - K_i$$

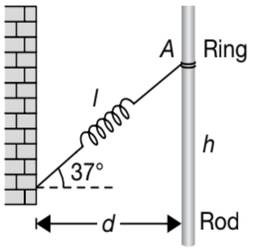
$$\Rightarrow \quad \frac{mgR^2}{L}\left[\sin\theta_0 - \left(\sin(\theta+\theta_0) - \sin\theta\right)\right] = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = \sqrt{\frac{2gR^2}{L}\left[\sin\theta_0 + \sin\theta - \sin(\theta+\theta_0)\right]}$$

where $\theta_0 = \dfrac{L}{R}$

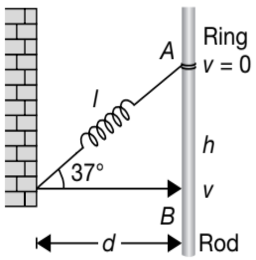
ILLUSTRATION 31

One end of a light spring of natural length d and spring constant k is fixed on a rigid wall and the other is attached to a smooth ring of mass m which can slide without friction on a vertical rod fixed at a distance d from the wall. Initially the spring makes an angle of $37°$ with the horizontal as shown in Figure.



When the system is released from rest, find the speed of the ring when the spring becomes horizontal. Take $\sin(37°) = \dfrac{3}{5}$.

SOLUTION



If l is the stretched length of the spring, then from figure, we have

$$\frac{d}{l} = \cos(37°) = \frac{4}{5}$$

$$\Rightarrow \quad l = \frac{5d}{4}$$

So, the extension in the spring is

$$x = l - d = \frac{5d}{4} - d = \frac{d}{4} \text{ and}$$

$$h = l\sin(37°) = \left(\frac{5d}{4}\right)\left(\frac{3}{5}\right) = \frac{3d}{4}$$

Now taking point B as reference level and applying Law of Conservation of Mechanical Energy between A and B,

$$(U+K)_{at\ A} = (U+K)_{at\ B}$$

At A, $h = \dfrac{3d}{4}$ and $x = \dfrac{d}{4}$

At B, $h = 0$ and $x = 0$

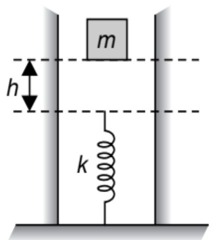
$$\Rightarrow \quad mgh + \frac{1}{2}kx^2 = \frac{1}{2}mv^2$$

$$\Rightarrow \quad \frac{3}{4}mgd + \frac{1}{2}k\left(\frac{d}{4}\right)^2 = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = d\sqrt{\frac{3g}{2d} + \frac{k}{16m}}$$

ILLUSTRATION 32

A small block of mass m is released from a height h above the free end of a light spring of spring constant k as shown in Figure. Calculate the



(a) speed of the block at the instant when the compression in the spring is x.
(b) maximum compression in the spring.
(c) compression in the spring when the block is in equilibrium.
(d) acceleration of the block in situations (a) and (b).

SOLUTION

(a) When the spring has been compressed by x, the block has moved down by a distance $(h+x)$. Let v be the speed of the block at this instant, then by Law of Conservation of Energy, we have

$$mg(h+x)=\frac{1}{2}kx^2+\frac{1}{2}mv^2 \qquad \ldots(1)$$

$$\Rightarrow \quad v=\sqrt{2g(h+x)-\frac{kx^2}{m}}$$

(b) When compression is maximum, speed of the block is zero, so we have

$$mg(h+x)=\frac{1}{2}kx^2$$

$$\Rightarrow \quad x^2-\left(\frac{2mg}{k}\right)x-\frac{2mgh}{k}=0$$

Solving the quadratic equation, we get

$$x_1=\frac{mg}{k}+\sqrt{\frac{m^2g^2}{k^2}+\frac{2mgh}{k}} \text{ and}$$

$$x_2=\frac{mg}{k}-\sqrt{\frac{m^2g^2}{k^2}+\frac{2mgh}{k}}$$

Rejecting the negative value x_2 of the maximum compression x_{max}, we get

$$x_{max}=x_1=\frac{mg}{k}+\sqrt{\frac{m^2g^2}{k^2}+\frac{2mgh}{k}}$$

(c) When the block is in equilibrium, net force on it is zero. If x_{eq} be the compression in the spring at equilibrium, then

$$mg=kx_{eq}$$

$$\Rightarrow \quad x=x_{eq}=\frac{mg}{k}$$

(d) At any instant, the forces acting on the block are
(i) Weight mg, acting downwards and
(ii) Spring force kx, acting upwards
If a be the acceleration of the block, then

$$a=\frac{mg-kx}{m}\text{, downwards}$$

For situation in (a), we have

$$a=g-\frac{k}{m}x\text{, downwards}$$

For situation in (b), we have

$$a=g-\frac{k}{m}x_{max}=g-\frac{k}{m}x_1$$

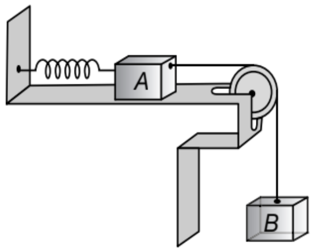
$$\Rightarrow \quad a=g-\frac{k}{m}\left(\frac{mg}{k}+\sqrt{\frac{m^2g^2}{k^2}+\frac{2mgh}{k}}\right)$$

$$\Rightarrow \quad a=-\sqrt{g^2+\frac{2kgh}{m}}$$

The negative sign shows that the acceleration is in upward direction.

ILLUSTRATION 33

Consider the situation shown in Figure, where the Mass of block A is m and that of block B is $2m$. The force constant of spring is K. Friction is absent everywhere.



The system is released from rest with the spring unstretched. Calculate

(a) the maximum extension of the spring x_m.
(b) the speed of block A when the extension in the spring is $x=\frac{x_m}{2}$.
(c) net acceleration of block B when extension in the spring is $x=\frac{x_m}{4}$.

SOLUTION

(a) At maximum extension in the spring, the blocks come momentarily to rest, so

$$v_A=v_B=0$$

Applying Conservation of Mechanical Energy, we have decrease in gravitational potential energy (GPE) of the block B must be equal to the increase in elastic potential energy (EPE) of the spring i.e.

$$\begin{pmatrix}\text{Decrease in GPE}\\ \text{of block } B\end{pmatrix}=\begin{pmatrix}\text{Increase in EPE}\\ \text{of spring}\end{pmatrix}$$

$$\Rightarrow \quad m_B g x_m = \frac{1}{2}Kx_m^2$$

$$\Rightarrow \quad 2mgx_m = \frac{1}{2}Kx_m^2$$

$$\Rightarrow \quad x_m = \frac{4mg}{K}$$

(b) At $x = \frac{x_m}{2} = \frac{2mg}{K}$, both the blocks move with the same speed, so $v_A = v_B = v$ (say).

Applying Law of Conservation of Mechanical Energy, we have

$$\begin{pmatrix}\text{Decrease}\\ \text{in GPE}\\ \text{of block}\end{pmatrix} = \begin{pmatrix}\text{Increase}\\ \text{in EPE}\\ \text{of spring}\end{pmatrix} + \begin{pmatrix}\text{Increase}\\ \text{in KE of}\\ \text{both blocks}\end{pmatrix}$$

$$\Rightarrow \quad m_B g x = \frac{1}{2}Kx^2 + \frac{1}{2}(m_A + m_B)v^2$$

$$\Rightarrow \quad 2mg\left(\frac{2mg}{K}\right) = \frac{1}{2}K\left(\frac{2mg}{K}\right)^2 + \frac{1}{2}(3m)v^2$$

$$\Rightarrow \quad v = 2g\sqrt{\frac{m}{3K}}$$

(c) At $x = \frac{x_m}{4} = \frac{mg}{K}$, net upward force on block B is Kx or mg and net downward force on block B is $2mg$

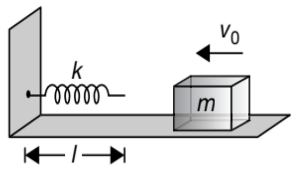
$$\Rightarrow \quad F_{net} = 2mg - mg = mg \qquad \text{(downwards)}$$

$$\Rightarrow \quad a = \frac{F_{net}}{m_A + m_B} = \frac{mg}{3m}$$

$$\Rightarrow \quad a = \frac{g}{3} \qquad \text{(downwards)}$$

ILLUSTRATION 34

A block of mass m is moving on a smooth horizontal surface with speed v_0 towards a vertical wall as shown. A light spring of spring constant k is attached to the wall such that the block compresses the spring as it moves closer to the wall. The natural length of the spring is l_0. Calculate the speed of the block when the spring has been compressed through x. Also calculate the maximum compression produced in the spring.



SOLUTION

By Law of Conservation of Energy, we have

$$\Delta U + \Delta K = 0$$

When the block collides with the spring then loss in kinetic energy of the block equals the gain in elastic potential energy of the spring. So, we have

$$\frac{1}{2}mv_0^2 - \frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

$$\Rightarrow \quad v = \sqrt{v_0^2 - \frac{k}{m}x^2}$$

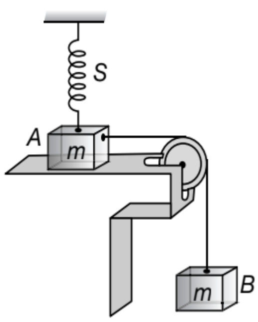
When compression in the spring is maximum i.e. $x = x_{max}$, then the block comes momentarily to rest i.e. $v = 0$ and we can say that the total kinetic energy of the block is converted as the elastic potential energy of the spring. Hence

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx_{max}^2$$

$$\Rightarrow \quad x_{max} = \sqrt{\frac{mv_0^2}{k}} = v_0\sqrt{\frac{m}{k}}$$

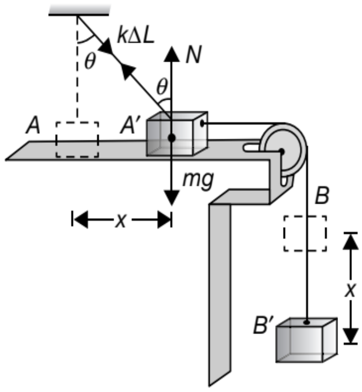
ILLUSTRATION 35

Two blocks A and B each having mass of 0.32 kg are connected by a light string passing over a smooth pulley as shown in Figure. The horizontal surface on which the block A slides is smooth. The block A is attached to a spring of force constant 40 Nm^{-1} whose other end is fixed to a support 0.40 m above the horizontal surface. Initially, when the system is released to move, the spring is vertical and unstretched. Find the velocity of the block A at the instant it breaks off the surface below it. $\left(\text{Take } g = 10\text{ ms}^{-2}\right)$.



SOLUTION

Let the block breaks off the surface when it travels a distance x as shown in figure.



So, the block A starts loosing contact with the surface below it at point A' after travelling a distance x. In this process the block B will shift from B to B' such that $BB' = AA' = x$ (as string is inextensible) and so there is a loss of gravitational potential energy of block B equal to mgx.

This energy is partly stored as elastic potential energy in the spring which is stretched by ΔL and partly appears as kinetic energy of blocks A and B. So, by Conservation of Mechanical Energy, we have

$$mgx = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}k(\Delta L)^2$$

$$\Rightarrow \quad v^2 = gx - \frac{k}{2m}(\Delta L)^2 \qquad \ldots(1)$$

Now, for vertical equilibrium of block A at A',

$$N + F\cos\theta = mg \qquad \ldots(2)$$

But for spring $F = k\Delta L$ and for breaking off, we have $N = 0$. So, the above equation reduces to

$$k\Delta L\cos\theta = mg \qquad \ldots(3)$$

where, $\Delta L = \frac{L}{\cos\theta} - L = L\left(\frac{1}{\cos\theta} - 1\right) \qquad \ldots(4)$

So, substituting the value of ΔL from Equations (3) in (4) and solving for $\cos\theta$, we get

$$\cos\theta = 1 - \frac{mg}{kL} = 1 - \frac{0.32\times 10}{40\times 0.40} = \frac{4}{5}$$

$$\Rightarrow \quad \Delta L = \left(\frac{L}{\cos\theta} - L\right) = \frac{0.4\times 5}{4} - 0.4 = 0.1 \text{ m and}$$

$$x = L\tan\theta = (0.4)\left(\frac{3}{4}\right) = 0.3 \text{ m}$$

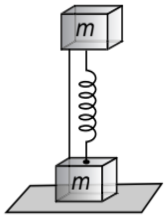
Substituting these values of ΔL and x in Equation (1), we get

$$v = \sqrt{10\times 0.3 - \frac{40\times(0.1)^2}{2\times 0.32}} = \sqrt{3 - 0.625} = \sqrt{2.375}$$

$$\Rightarrow \quad v = 1.54 \text{ ms}^{-1}$$

ILLUSTRATION 36

A system consists of two identical cubes, each of mass m, linked together by a compressed light spring of force constant k as shown in Figure.



The cubes are also connected by a thread which is burnt at a certain moment. At what values of initial compression x_0 of the spring, will the lower cube bounce up after the thread is burnt through?

SOLUTION

Let x be the elongation in the spring when it returns. Then by Law of Conservation of Mechanical Energy, we get

$$\frac{1}{2}kx_0^2 = \frac{1}{2}kx^2 + mg(x + x_0)$$

$$\Rightarrow \quad kx^2 + 2mgx - kx_0^2 + 2mgx_0 = 0$$

$$\Rightarrow \quad kx = -mg \pm (mg - kx_0)$$

$$\Rightarrow \quad kx = -kx_0 \text{ or } kx = kx_0 - 2mg$$

Since $kx = -kx_0$ gives $x = -x_0$ and this is not acceptable.

$$kx = kx_0 - 2mg$$

The lower cube will bounce up if $kx \geq mg$ (the weight of the lower cube)

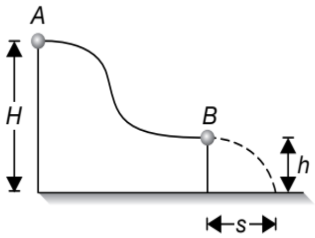
$$\Rightarrow \quad kx_0 - 2mg \geq mg$$

$$\Rightarrow \quad x_0 \geq \frac{3mg}{k}$$

Thus, for all values of x_0 greater than $\frac{3mg}{k}$ the lower cube will bounce up.

ILLUSTRATION 37

A small disc A slides down with initial velocity equal to zero from the top of a smooth hill of height H having a horizontal portion as shown in Figure.



What must be the height of the horizontal portion h to ensure the maximum distance s covered by the disc? What is it equal to?

SOLUTION

In get the velocity at point B, applying the Law of Conservation of Energy. So,

$$\begin{pmatrix}\text{Loss in Gravitational} \\ \text{Potential Energy}\end{pmatrix} = \begin{pmatrix}\text{Gain in} \\ \text{Kinetic Energy}\end{pmatrix}$$

$$\Rightarrow \quad mg(H-h) = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = \sqrt{2g(H-h)}$$

Since $y = u_y t + \frac{1}{2}a_y t^2$, where $u_y = 0$, $a_y = g$ and $y = h$, so $h = \frac{1}{2}gt^2$

$$\Rightarrow \quad t = \sqrt{\frac{2h}{g}}$$

Since $x = u_x t + \frac{1}{2}a_x t^2$, where $u_x = v$, $a_x = 0$ and $x = s$

$$\Rightarrow \quad s = v \times t = \sqrt{2g(H-h)} \times \sqrt{\frac{2h}{g}}$$

$$\Rightarrow \quad s = \sqrt{4h(H-h)} \qquad \ldots(1)$$

For maximum value of s or s^2, we have $\frac{d}{dh}(s^2) = 0$

$$\frac{d}{dh}(4Hh - 4h^2) = 0$$

$$\Rightarrow \quad 4H - 8h = 0$$

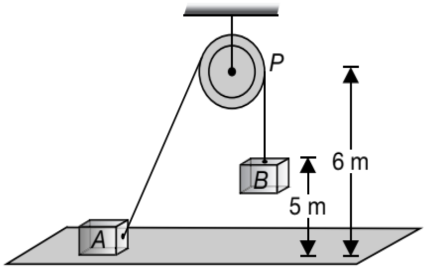
$$\Rightarrow \quad h = \frac{H}{2}$$

Substituting $h = \frac{H}{2}$, in Equation (1), we get

$$s = \sqrt{4\left(\frac{H}{2}\right)\left(H - \frac{H}{2}\right)} = \sqrt{H^2} = H$$

ILLUSTRATION 38

A block A of mass m is held at rest on a smooth horizontal floor. A light frictionless, small pulley is fixed at a height of 6 m from the floor. A light inextensible string of length 16 m, connected with A passes over the pulley and another identical block B is hung from the string. The initial height of B is 5 m from the floor as shown in Figure.



When the system is released from rest, B starts to move vertically downwards and A slides on the floor towards right.

(a) If at an instant, string makes an angle θ with the horizontal, calculate relation between velocity u of A and v of B.

(b) Calculate v when B strikes the floor.

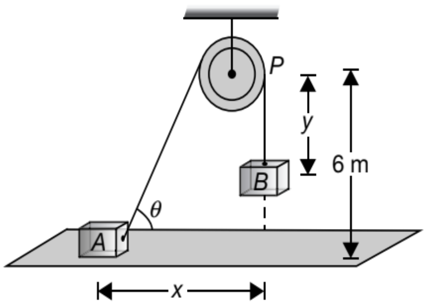
Take $g = 10\text{ ms}^{-2}$ and $\cos(37^\circ) = 0.8$

SOLUTION

(a) From the figure, we observe that

$$y + AP = 16$$

$$\Rightarrow \quad y + \sqrt{x^2 + 36} = 16$$



Differentiating with respect to time, we get

$$\frac{x}{\sqrt{(36+x^2)}}\frac{dx}{dt}+\frac{dy}{dt}=0$$

Here, $\frac{dy}{dt}=v=v_B$

As t increases, x decreases, hence

$$-\frac{dx}{dt}=u=v_A$$

$\Rightarrow$ $-u\cos\theta+v=0$

$\Rightarrow$ $u=v\sec\theta$

(b) When B strikes the floor, $y=6$ m, then

$$AP=16-6=10 \text{ m}$$

Now, $\sin\theta=\left(\frac{6}{10}\right)$

$\Rightarrow$ $\theta=37°$ $\qquad \{\because \sin^2(37°)+\cos^2(37°)=1\}$

Now, according to Law of Conservative of Energy, we have

$$\begin{pmatrix}\text{Loss in}\\ \text{PE of } B\end{pmatrix}=\begin{pmatrix}\text{Gain in}\\ \text{KE of } A\end{pmatrix}+\begin{pmatrix}\text{Gain in}\\ \text{KE of } B\end{pmatrix}$$

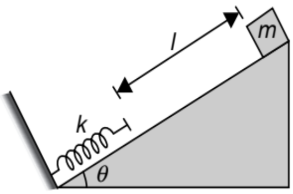
$\Rightarrow$ $mg(5)=\frac{1}{2}mu^2+\frac{1}{2}mv^2$

But $u=v\sec(37°)=\frac{5v}{4}$

Solving, we get $v=\left(\frac{40}{\sqrt{41}}\right)$ ms^{-1}

ILLUSTRATION 39

An ideal light spring S can be compressed 1 m by a force of 100 N. This spring is placed at the bottom of a frictionless inclined plane which makes an angle of 30° with the horizontal as shown in Figure.



A 10 kg mass is released from rest from the top of the incline and is brought to rest momentarily after compressing the spring 2 m. Calculate the distance through which the mass slides before coming to rest. Also calculate the speed of the mass just before it hits the spring.

SOLUTION

For the spring S, we have $F=kx_0$

Since it is given that, for $F=100$ N, the spring can be compressed by $x_0=1$ m, so, we have

$$k=\frac{F}{x_0}=100 \text{ Nm}^{-1}$$

If spring compresses through $x=2$ m by the mass, then by applying Law of Conservation of Energy, we have

$$\begin{pmatrix}\text{Loss in GPE}\\ \text{of the Body}\end{pmatrix}=\begin{pmatrix}\text{Gain in EPE}\\ \text{of the Spring}\end{pmatrix}$$

$\Rightarrow$ $mg\sin\theta(l+x)-\frac{1}{2}kx^2=0$

$\Rightarrow$ $(10)(10)(0.5)(l+2)-\frac{1}{2}(100)(2)^2=0$

$\Rightarrow$ $l+2-4=0$

$\Rightarrow$ $l=2$ m

So, the total distance through which the mass slides before coming to rest is

$$l+x=2+2=4 \text{ m}$$

When mass hits the spring, then by Law of Conservation of Energy, we have

$$\begin{pmatrix}\text{Loss in GPE}\\ \text{of the Body}\end{pmatrix}=\begin{pmatrix}\text{Gain in KE}\\ \text{of the Body}\end{pmatrix}$$

$\Rightarrow$ $\frac{1}{2}mv^2=mgh=mg(l\sin\theta)$

$\Rightarrow$ $v=\sqrt{2gl\sin\theta}$

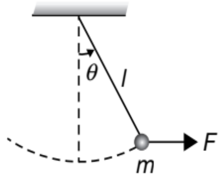
$\Rightarrow$ $v=\sqrt{20}$ ms^{-1}

Test Your Concepts-III

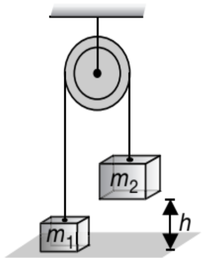
Based on Conservation of Energy and Work Energy Theorem

(Solutions on page H.8)

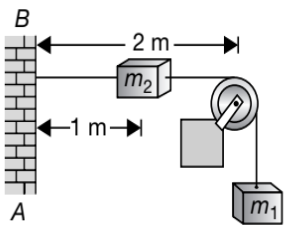
1. An object of mass m is tied to a string of length l and a variable force F is applied on it which brings the string gradually at angle θ with the vertical. Find the work done by the force F.



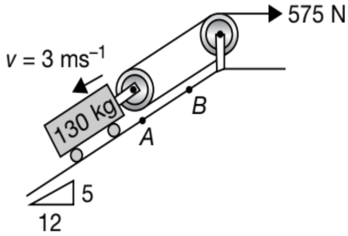
2. Two blocks with masses $m_1 = 3$ kg and $m_2 = 5$ kg are connected by a light string that slides over a frictionless pulley as shown in figure. Initially, m_2 is held 5 m off the floor while m_1 is on the floor. The system is then released. At what speed does m_2 hit the floor?



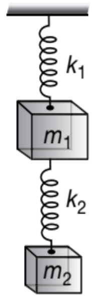
3. In the arrangement shown, calculate the speed of the bodies 1 and 2 having masses 0.5 kg and 2 kg respectively at the moment the block m_2 hits the wall AB. Assume that the bodies are released from rest. Take $g = 10$ ms^{-2}.



4. The 130 kg carriage has an initial velocity of 3 ms^{-1} down the incline at A, when a constant force of 575 N is applied to the hoisting cable as shown. Calculate the velocity of the carriage when it reaches B, a distance of 3 m above A on the incline. Show that in the absence of friction this velocity is independent of whether the initial velocity of the carriage at A was up or down the incline. Take $g = 10$ ms^{-2}.



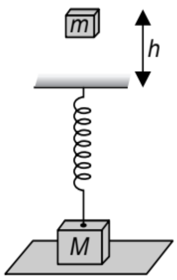
5. In the arrangement shown, we have $k_1 = 1500$ Nm^{-1}, $k_2 = 500$ Nm^{-1}, $m_1 = 2$ kg and $m_2 = 1$ kg. Calculate the



   (a) potential energy stored in the springs in equilibrium.
   (b) work done in slowly pulling down m_2 by 8 cm.

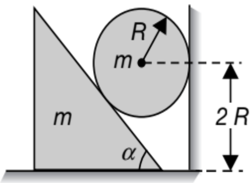
   Take $g = 10$ ms^{-2}

6. A block of mass m is dropped onto a spring of constant k from a height h. The second end of the spring is attached to another block of mass M as shown.

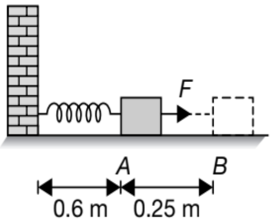


   Find the minimum value of h so that the block M bounces off the ground if the block of mass m sticks to the spring immediately after it comes into contact with it.

7. A sphere of mass m held at a height $2R$ between a wedge of same mass m and a rigid wall, is released. Assuming that all the surfaces are frictionless, find the speed of both the bodies when the sphere hits the ground.



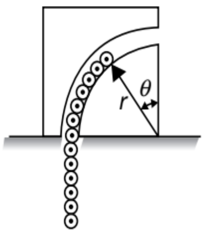
8. A 0.5 kg block attached to a spring with length 0.6 m and force constant $k = 40\ \text{Nm}^{-1}$ is at rest with the back of the block at point *A* on a frictionless, horizontal table. The mass of the spring is negligible. The block is pulled to the right along the surface with a constant horizontal force *F* = 20 N.



(a) What is the block's speed when the back of the block reaches point *B*, which is 0.25 m to the right of point *A*?

(b) When the back of the block reaches point *B*, the block is set free. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached?

9. The flexible bicycle type chain of length $\frac{\pi r}{2}$ and mass per unit length λ is released from rest with $\theta = 0°$ in the smooth circular channel and falls through the hole in the supporting surface. Determine the velocity *v* of the chain as the last link leaves the slot.



10. A helicopter lifts a 80 kg astronaut 15 m vertically from the ocean by means of a cable. The acceleration of the astronaut is $\frac{g}{10}$. Calculate the

(a) work done on the astronaut by the force from the helicopter

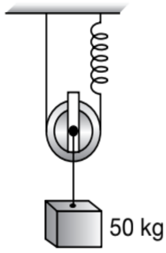
(b) work done on the astronaut by the gravitational force

(c) kinetic energy and

(d) speed of the astronaut just before reaching the helicopter.

Take $g = 10\ \text{ms}^{-2}$

11. The system is released from rest with the spring initially stretched 76 mm. Calculate the velocity *v* of the block of mass 50 kg after it has dropped 12 mm. The spring has a stiffness of $1000\ \text{Nm}^{-1}$. Neglect the mass of the small pulley.



12. A 1.5 kg block is initially at rest on a horizontal frictionless surface when a horizontal force in the positive direction of *x*-axis is applied to the block. The force is given by $\vec{F}(x) = (2.5 - x^2)\hat{i}$ N, where *x* is in metre and the initial position of the block is $x = 0$.

(a) What is the kinetic energy of the block as it passes through $x = 2$ m?

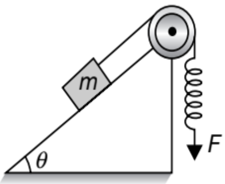
(b) What is the maximum kinetic energy of the block between $x = 0$ and $x = 2$ m?

13. A certain spring is found not to obey Hooke's Law. It exerts a restoring force $F_x(x) = -\alpha x - \beta x^2$ when stretched or compressed, where $\alpha = 80\ \text{Nm}^{-1}$ and $\beta = 24\ \text{Nm}^{-2}$. The mass of the spring is negligible.

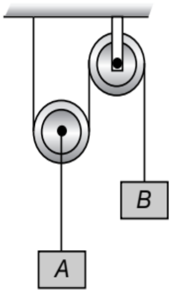
(a) Calculate the potential energy function *U*(*x*) for this spring. Let *U* = 0 when *x* = 0.

(b) An object with a mass of 2 kg on a frictionless horizontal surface is attached to this spring, pulled a distance 1 m to the right (the +*x* direction) to stretch the spring and released. Calculate the approximate speed of the object when it is 0.5 m to the right of the *x* = 0 equilibrium position?

14. A block rests on an inclined plane as shown in figure. A spring of force constant *k* to which it is attached via a pulley is being pulled downward with gradually increasing force. The value of μ_s is known. Calculate the potential energy *U* of the spring at the moment when the block begins to move.



15. Block A has a mass of 30 kg and block B has a mass of D 5 kg. Calculate the distance A must descend from rest before it obtains a speed of $2\ \text{ms}^{-1}$. Neglect the mass of the cord and pulleys. Assume friction to be absent. Take $g = 10\ \text{ms}^{-2}$.

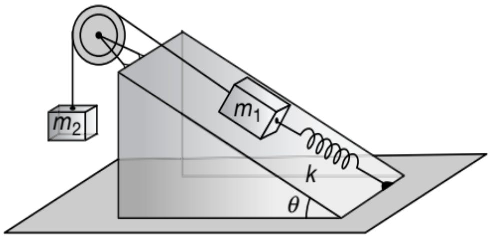


16. Two blocks with masses $m_1 = 2$ kg and $m_2 = 3$ kg hang on either side of a pulley as shown in figure. Block m_1 is on an incline $(\theta = 30°)$ and is attached to a spring whose stiffness constant is $40\ \text{Nm}^{-1}$. The system is released from rest with the spring in its natural length. Find

(a) the maximum extension of the spring

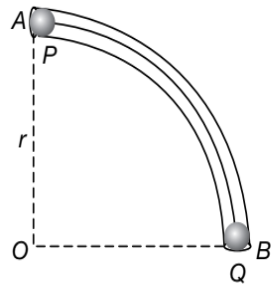
(b) the speed of m_1 when the extension is 0.5 m.

Ignore friction and mass of the pulley.



17. A block of mass m is pushed against a spring of spring constant k fixed at one end to a wall. The block can slide on a frictionless table. The natural length of the spring is L_0 and it is compressed to half its natural length when the block is released. Find the velocity of the block as a function of its distance x from the wall.

18. A smooth narrow tube in the form of an arc AB of a circle of centre O and radius r is fixed so that A is vertically above O and OB is horizontal. Two particles P and Q of mass m and $2m$ respectively are connected to each other with a light inextensible string of length $\left(\dfrac{\pi r}{2}\right)$ are placed inside the tube with P at A and Q at B and released from rest. Assuming the string remains taut during motion, calculate the speed of particles when P reaches B.



MODIFIED WORK-ENERGY THEOREM (MWET)

If external forces, pseudo forces, non-conservative forces and internal forces are present in the system and the work done by the respective forces be W_{ext}, W_{ps}, W_{nc} and W_{int} then according to ***Modified Work-Energy Theorem (MWET)***, we have

$$W_{ext} + W_{ps} + W_{nc} + W_{int} = \Delta U + \Delta K$$

Conceptual Note(s)

(a) Work done by internal forces may or may not be zero.

(b) If pseudo forces, non-conservative forces are absent, external forces are present and if work done by internal forces is zero, then we have

$$W_{ext} = \Delta U + \Delta K$$

(c) If external forces, pseudo forces are absent, non-conservative forces are present and if work done by internal forces is zero, then we have

$$W_{nc} = \Delta U + \Delta K$$

LAW OF CONSERVATION OF MECHANICAL ENERGY

If external forces, pseudo forces and non-conservative forces all are absent, then we have

$$\Delta U + \Delta K = 0$$

which simply implies that total mechanical energy $(U+K)$ is constant i.e.

$$(U+K)_{\text{initial}} = (U+K)_{\text{final}}$$

Since, for a conservative system we have

$$W_c = \Delta K \quad \ldots(1)$$

Also, we have

$$W_c = -\Delta U \quad \ldots(2)$$

Equating (1) and (2), we get

$$\Delta K = -\Delta U$$

$$\Rightarrow \quad \Delta(K+U) = 0$$

$$\Rightarrow \quad K+U = \text{constant}$$

$$\Rightarrow \quad K_i + U_i = K_f + U_f$$

$$\Rightarrow \quad \begin{pmatrix}\text{Total Initial}\\ \text{Mechanical Energy}\end{pmatrix} = \begin{pmatrix}\text{Total Final}\\ \text{Mechanical Energy}\end{pmatrix}$$

Since, any non-conservative system has got both conservative nature and a non-conservative nature, so using Work-Energy Theorem, we get

$$W = W_c + W_{nc} = \Delta K$$

and $W_c = -\Delta U$

$$\Rightarrow \quad -\Delta U + W_{nc} = \Delta K$$

$$\Rightarrow \quad W_{nc} = \Delta U + \Delta K$$

$$\Rightarrow \quad W_{nc} = (U_f - U_i) + (K_f - K_i)$$

$$\Rightarrow \quad W_{nc} = (U_f + K_f) - (U_i + K_i)$$

$$\Rightarrow \quad W_{nc} = \begin{pmatrix}\text{Total Final}\\ \text{Mechanical}\\ \text{Energy}\end{pmatrix} - \begin{pmatrix}\text{Total Initial}\\ \text{Mechanical}\\ \text{Energy}\end{pmatrix}$$

Conceptual Note(s)

So, to conclude we can say that the **Modified Work-Energy Theorem (MWET)**

$$W_{\text{ext}} + W_{\text{ps}} + W_{\text{nc}} + W_{\text{int}} = \Delta U + \Delta K$$

is a handy tool to solve problems.

WORK-ENERGY THEOREM FOR NON-CONSERVATIVE SYSTEM

For a non-conservative system, if W is the total work done (due to conservative and non-conservative nature of the system) then

$$W = W_c + W_{nc} = \Delta K = K_f - K_i$$

Conceptual Note(s)

A non-conservative system is bound to have a conservative part attached to it (It is impossible to have something that is purely impure like, purely impure milk or purely impure honey etc.). Hence for a non-conservative system we have

$$W_{\text{total}} = W_{\text{c}} + W_{\text{nc}} = \Delta K$$

Since, we know that the work done by a conservative force is given by

$$W_c = -\Delta U$$

$$\Rightarrow \quad -\Delta U + W_{nc} = \Delta K$$

$$\Rightarrow \quad W_{nc} = \Delta U + \Delta K$$

$$\Rightarrow \quad W_{nc} = (U_f - U_i) + (K_f - K_i)$$

$$\Rightarrow \quad W_{nc} = (U_f + K_f) - (U_i + K_i)$$

$$\Rightarrow \quad W_{nc} = \begin{pmatrix}\text{Total Final}\\ \text{Mechanical}\\ \text{Energy}\end{pmatrix} - \begin{pmatrix}\text{Total Initial}\\ \text{Mechanical}\\ \text{Energy}\end{pmatrix}$$

Conceptual Note(s)

(a) So, to conclude we can say that work done by a non-conservative force is the sum of change in potential energy and change in kinetic energy or work done by a non-conservative force equal the change in value of total mechanical energy.

(b) However, if we have a non-conservative system on which some external force is also acting, then, from Work Energy Theorem, we have

$$W_{\text{total}} = \Delta K$$

$$\Rightarrow \quad W_{\text{c}} + W_{\text{nc}} + W_{\text{ext}} = \Delta K \quad \ldots(1)$$

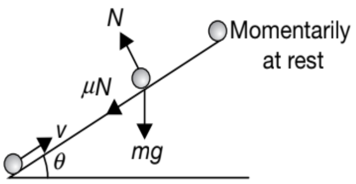
This formula happens to be the master key to be used for any kind of system.

ILLUSTRATION 40

A particle is launched straight up with a kinetic energy K up a rough inclined plane of inclination θ and coefficient of friction μ. Prove that the work done against friction before the particle first comes to rest is $\dfrac{\mu K\cos\theta}{\sin\theta+\mu\cos\theta}$.

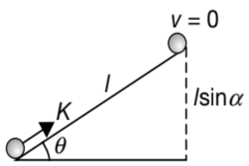
SOLUTION

Let us consider the mass of the particle to be m, the normal reaction on the particle exerted by the plane to be N, the initial velocity of the particle to be v and the distance travelled by the particle up the plane before it first comes to rest be l.



Since $W_{nc}=\Delta U+\Delta K$

$$\Rightarrow\quad -(\mu N)l = mgl\sin\theta+\left(\frac{1}{2}m(0)^2-\frac{1}{2}mu^2\right)$$



$$\Rightarrow\quad -\mu mgl\cos\theta = mgl\sin\theta - K$$

$$\Rightarrow\quad K = mgl\sin\theta+\mu mgl\cos\theta$$

$$\Rightarrow\quad K = mgl(\sin\theta+\mu\cos\theta)$$

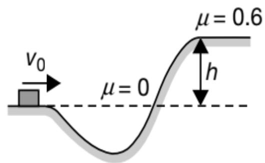
The work done W_f against friction force $f=\mu N=\mu mg\cos\theta$ is given by

$$W_f=\mu Nl=\mu mgl\cos\theta=\frac{\mu K\cos\theta}{\sin\theta+\mu\cos\theta}$$

Please note that when we write "*work done by friction force*" then we shall be applying the negative sign with the work done but when we write the "*work done against friction*", then we do not apply a negative sign with the work done.

ILLUSTRATION 41

A block slides along a track from one level to a higher level, by moving through an intermediate valley as shown in Figure. The track is frictionless until the block reaches the higher level, where a frictional force stops the block after it covers a distance d. If the coefficient of kinetic friction between the block and the rough surface is 0.6, the initial speed of the block is $v_0=6\text{ ms}^{-1}$, the height difference $h=1.1$ m, then calculate d. Take $g=10\text{ ms}^{-2}$.



SOLUTION

According to Modified Work-Energy Theorem, we have

$$W_{nc}=\Delta U+\Delta K$$

Applying this for the motion of block till it stops, we get

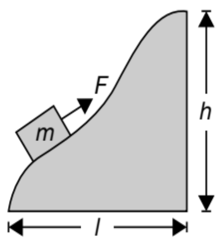
$$-\mu mgd = mgh-\frac{1}{2}mv_0^2$$

$$\Rightarrow\quad d=\frac{v_0^2-2gh}{2\mu g}$$

$$\Rightarrow\quad d=\frac{36-2\times10\times1.1}{2\times0.6\times10}=1.167\text{ m}$$

ILLUSTRATION 42

A body of mass m was slowly hauled up the hill as shown in the figure by a force F which at each point was directed along a tangent to the trajectory as shown in Figure.



Calculate the work performed by this force, if the height of the hill is h, the length of its base is l and the coefficient of friction is μ.

SOLUTION

Four forces are acting on the body

1. the external applied force (F)
2. weight (mg)

3. normal reaction (N)
4. friction (f)

Using Work-Energy Theorem

$$W_{\text{total}} = \Delta K$$

$$\Rightarrow \quad W_c + W_{nc} + W_{ext} = \Delta K$$

$$\Rightarrow \quad W_{nc} + W_{ext} = \Delta U + \Delta K$$

Since the body is hauled up slowly, so

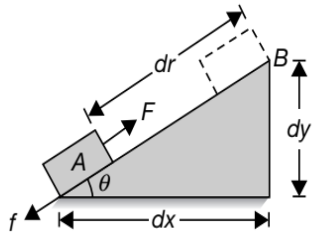
$$K_f = K_i = 0$$

$$\Rightarrow \quad \Delta K = 0$$

$$\Rightarrow \quad W_{nc} + W_{ext} = \Delta U \qquad \ldots(1)$$

Since, $\Delta U = -mgh$ and at all instants, the normal reaction is perpendicular to displacement at all points, so work done by the normal reaction is $W_N = 0$.

To calculate the work done by the non-conservative force i.e. friction, let us consider a very small displacement $d\vec{r}$ given to body. The **magnified view** of this is shown here in Figure.



Since, $f = \mu mg\cos\theta$

Also, $(dW_{AB})_f = -fds$

$$\Rightarrow \quad (dW_{AB})_f = -(\mu mg\cos\theta)ds$$

$$\Rightarrow \quad (dW_{AB})_f = -\mu mg(dl) \qquad \{\because ds\cos\theta = dl\}$$

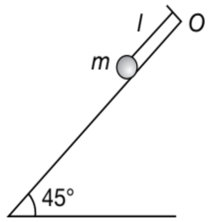
$$\Rightarrow \quad W_f = -\mu mg\int_0^l dl = -\mu mgl$$

Substituting these values in Equation (1), we get

$$W_F = mgh + \mu mgl$$

ILLUSTRATION 43

A particle of mass m is attached to one end of a light elastic string whose other end is fixed to a point O on an inclined plane having inclination angle $45°$ with the horizontal. The natural length of the string is l and its force constant is $\frac{mg}{l}$. The particle is held on the inclined plane so that the string lies just unstretched along a line of greatest slope and then released from rest as shown in Figure.



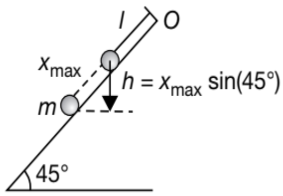
(a) If the inclined plane is smooth, then find the lowest position that can be reached by the particle. Also locate the equilibrium position.

(b) Assume that the inclined plane is rough enough with coefficient of friction between the particle and the plane to be μ. If the particle stops in first descend after covering a distance d along the greatest slope, then find d.

Please note that all the positions have to be measured with respect to the point O.

SOLUTION

(a) When the particle reaches the lowest position, then it momentarily comes to a stop. If $x_{\max}$ be the extension of the string when the particle is at its lowest position



By Law of Conservation of Energy, we have

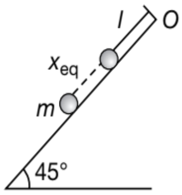
$$\begin{pmatrix}\text{Loss in GPE}\\ \text{of the particle}\end{pmatrix} = \begin{pmatrix}\text{Gain in EPE}\\ \text{of the string}\end{pmatrix}$$

$$\Rightarrow \quad mg(x_{\max}\sin 45°) = \frac{1}{2}\left(\frac{mg}{l}\right)x_{\max}^2$$

$$\Rightarrow \quad x_{\max} = \sqrt{2}l$$

So, the lowest position is located from O at a distance

$$l_1 = l + x_{\max} = l + \sqrt{2}l = (1+\sqrt{2})l$$



For equilibrium of the particle, the component of the weight acting down the incline is balanced by the elastic force.

$$\Rightarrow \quad mg\sin(45^\circ) = \left(\frac{mg}{l}\right)x_{eq}$$

$$\Rightarrow \quad x_{eq} = \frac{l}{\sqrt{2}}$$

So, the equilibrium position is located from O at a distance

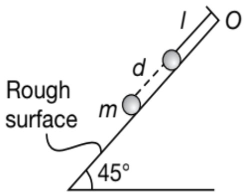
$$l_2 = l + x_{eq} = l + \frac{l}{\sqrt{2}} = l\left(1 + \frac{1}{\sqrt{2}}\right)$$

(b) According to Modified Work-Energy Theorem, we have

$$\Rightarrow \quad W_{nc} = \Delta U + \Delta K$$

$$\Rightarrow \quad W_{nc} = (\Delta U_g + \Delta U_e) + \Delta K$$

where ΔU_g is the change in gravitational potential energy of the mass and ΔU_e is the change in elastic potential energy of the string. Please note that the initial velocity and the final velocity of the mass both are zero and hence $\Delta K = 0$.



$$\Rightarrow \quad -\mu mgd\cos(45^\circ) = -mgd\sin(45^\circ) + \frac{1}{2}kd^2$$

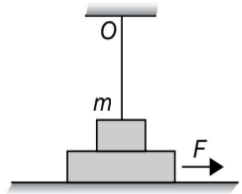
$$\Rightarrow \quad \frac{1}{2}\left(\frac{mg}{l}\right)d^2 + \mu mgd\cos(45^\circ) = mgd\sin(45^\circ)$$

$$\Rightarrow \quad \frac{d}{2l} + \frac{\mu}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad d = \sqrt{2}l(1-\mu)$$

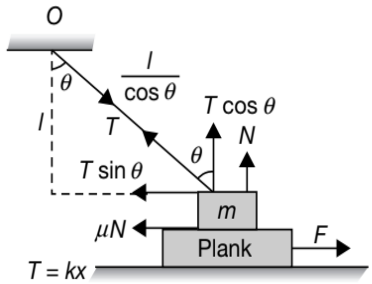
ILLUSTRATION 44

A horizontal plane supports a plank with a bar of mass 1 kg placed on it and attached by a light elastic non deformed cord of length 40 cm to a point O as shown in figure. The coefficient of friction between the bar and plank is 0.2. The plank is slowly shifted to the right until the bar starts sliding over it. It occurs at the moment when the cord deviates from the vertical by 30°. Calculate the work done (in millijoule) that has been performed by that moment due to friction force acting on the bar in the reference frame attached to the plane.



SOLUTION

The intermediate force diagram of the system, when the cord makes an angle $\theta(<30^\circ)$ is shown in Figure.



At this instant, the extension in cord is

$$x = \frac{l}{\cos\theta} - l = l(\sec\theta - 1) \qquad \ldots(1)$$

If the force constant of the cord is k, then the tension developed in the cord is given by $T = kx$

The bar will start sliding when

$$kx\sin\theta = f_l = \mu N \qquad \ldots(2)$$

where $N + kx\cos\theta = mg$

$$\Rightarrow \quad N = mg - kx\cos\theta$$

Substituting this value of N in equation (1), we get

$$kx\sin\theta = \mu(mg - kx\cos\theta)$$

$$\Rightarrow \quad kx(\sin\theta + \mu\cos\theta) = \mu mg$$

$$\Rightarrow \quad k = \frac{\mu mg}{x(\sin\theta + \mu\cos\theta)}$$

Substituting the value of x from equation (1), we get

$$\Rightarrow \quad k = \frac{\mu mg}{l(\sec\theta - 1)(\sin\theta + \mu\cos\theta)}$$

According to Modified Work-Energy Theorem (MWET), we have

$$W_{nc} = \Delta U + \Delta K$$

where, $\Delta U = \frac{1}{2}kx^2$ and $\Delta K = 0$

$$\Rightarrow \quad W_{nc} = \frac{1}{2}kx^2 = \frac{kx^2}{2}$$

Conceptual Note(s)

Alternatively, you can also think that in the process of shifting the plank slowly $(\Delta K = 0)$, the total work done against friction just increases the elastic potential energy of the cord by $\Delta U = \frac{1}{2}kx^2$. So, work done against friction is equal to the increment in potential energy of the spring

$$\Rightarrow \quad W_{nc} = \frac{\mu mgl^2(\sec\theta - 1)^2}{2l(\sec\theta - 1)(\sin\theta + \mu\cos\theta)}$$

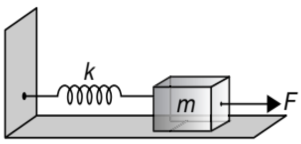
$$\Rightarrow \quad W = \frac{\mu mgl(1-\cos\theta)}{2\cos\theta(\sin\theta + \mu\cos\theta)}$$

Substituting the numerical data, we get

$$W = 0.09 \text{ J} = 90 \text{ mJ}$$

ILLUSTRATION 45

a block of mass m is resting on a smooth horizontal surface. It is connected to a rigid wall by a light spring of spring constant k. The spring is in its natural length and a constant horizontal force F starts acting on the block towards right as shown in Figure.



Calculate the speed of the block when extension in the spring is x and when the block passes the equilibrium position. Also calculate the maximum extension produced in the spring.

SOLUTION

According to the Modified Work-Energy Theorem (MWET), we have

$$W_{ext} = \Delta U + \Delta K \quad \ldots(1)$$

When the applied external force F displaces the block through x, then the spring also extends by x. If v is the velocity of the block at this instant then, we have

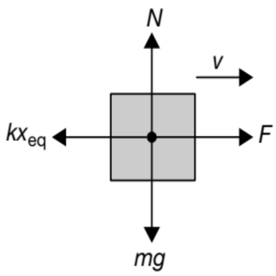
$$W_{ext} = Fx, \ \Delta U = \frac{1}{2}kx^2, \ \Delta K = \frac{1}{2}mv^2$$

Substituting these values in equation (1), we get

$$Fx = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = \sqrt{\frac{2Fx - kx^2}{m}} \quad \ldots(2)$$

When the block passes the equilibrium position, then net force on the block is zero as shown in Figure.



$$\Rightarrow \quad F = kx_{eq}$$

$$\Rightarrow \quad x_{eq} = \frac{F}{k}$$

Since from equation (2), we have

$$v = \sqrt{\frac{2Fx - kx^2}{m}}$$

Substituting $x_{eq} = \frac{F}{k}$ in this equation, we get

$$v = \sqrt{\frac{\frac{2F^2}{k} - \frac{F^2}{k}}{m}} = \frac{F}{\sqrt{mk}}$$

At the maximum extension, speed of the block becomes zero, so from equation (2), we get at $x = x_{max}$, $v = 0$.

$$\Rightarrow \quad \sqrt{\frac{2Fx_{max} - kx_{max}^2}{m}} = 0$$

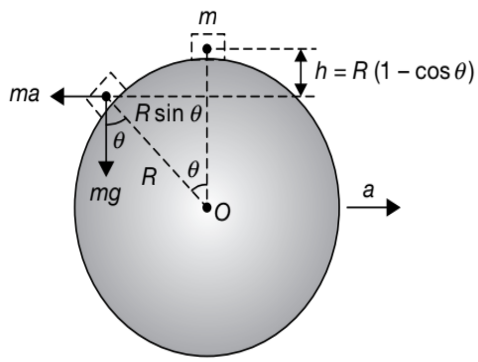
$$\Rightarrow \quad x_{max} = \frac{2F}{k}$$

ILLUSTRATION 46

A block is lying at the top of a smooth sphere of radius R. The sphere begins to accelerate with a constant horizontal acceleration a. Calculate the speed of block when it is at an angular position θ from the initial position relative to the sphere.

SOLUTION

First of all, we need to understand that since the block lies on the sphere, so let the speed of the particle be v with respect to the sphere (that happens to be the ground for the particle). However, the sphere is accelerating and hence it becomes a non-inertial frame for the particle due to which a pseudo force $F_{pseudo} = ma$ acts on the block opposite to the acceleration of the sphere. Now, due to this force the block is displaced to the left by $R\sin\theta$ as shown in Figure.



According to Modified Work-Energy Theorem (MWET), we have

$$W_{pseudo} = \Delta U + \Delta K$$

$$\Rightarrow \quad F_{ps}\Delta x = -mgh + \frac{1}{2}mv^2$$

$$\Rightarrow \quad maR\sin\theta = -mgR(1-\cos\theta) + \frac{1}{2}mv^2$$

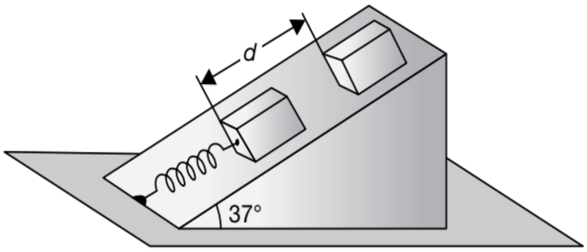
$$\Rightarrow \quad v = \sqrt{R\left(a\sin\theta + g(1-\cos\theta)\right)}$$

Test Your Concepts-IV

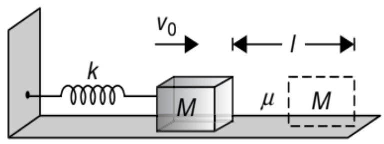
Based on Work Energy Theorem for Non-conservative Systems

(Solutions on page H.13)

1. A box of mass m is gently placed on a conveyor belt that moves at a constant speed v. The coefficient of kinetic friction is μ_k.
   (a) What is the work done by friction?
   (b) How far does the box move before reaching its final speed?
   (c) When the box reaches its final speed, how far has the belt moved?
2. A block of mass $m = 0.2$ kg is held against, but not attached to a spring $\left(k = 50 \text{ Nm}^{-1}\right)$ which is compressed by $x_0 = 20$ cm, as shown in figure. When the block is released, the block slides $d = 50$ cm up the rough incline before coming to rest. Find



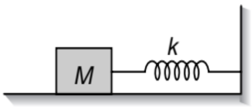
   (a) the force of friction
   (b) the speed of the block as it leaves the spring.
3. The block of mass M shown in figure initially has a velocity v_0 to the right and its position is such that the spring exerts no force on it, i.e., the spring is neither stretched nor compressed. The block moves to



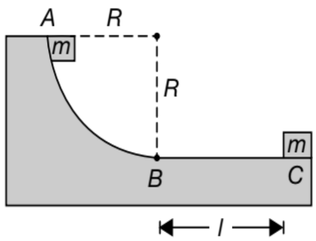
   the right a distance l before stopping in the dotted position shown. The spring constant is k and the coefficient of kinetic friction between block and the table is μ.
   (a) What is the work done on it by the friction force?
   (b) What is the work done on it by the spring force?
   (c) Are there other forces acting on the block and if so, what work do they do?

(d) What is the total work done on the block?
(e) Use the work-energy theorem to find the value of l in terms of M, v_0, μ, g and k

4. A block of mass M slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$ where b is a constant. Find the loss in mechanical energy when the block has first come momentarily to rest.



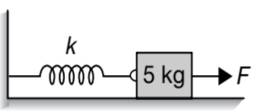
5. In a truck-loading station at a post office a small package of mass $m = 200$ g is released from rest at point A on a track that is one quarter of a circle with radius $R = 2$ m. The size of the package is much less than 1.6 m, so the package can be treated as a particle. It slides down the track and reaches point B with a speed of $4\ \text{ms}^{-1}$. From point B it slides on a level surface a distance of $l = 4$ m to point C, where it comes to rest.



(a) What is the coefficient of kinetic friction on the horizontal surface?
(b) How much work is done on the package by friction as it slides down the circular arc from A to B?

Take $g = 10\ \text{ms}^{-2}$

6. A 5 kg block is attached to an unstretched spring of constant $k = 2000\ \text{Nm}^{-1}$. The coefficients of static and kinetic friction between the block and the plane are 0.6 and 0.36, respectively. If a force F is slowly applied to the block until the tension in the spring reaches 100 N and then suddenly removed, determine



(a) the velocity of the block as it returns to its initial position,
(b) the maximum velocity achieved by the block.

Take $g = 10\ \text{ms}^{-2}$

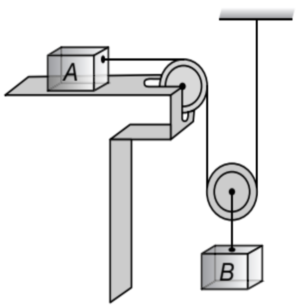
7. An object of mass 5 kg falls from rest through a vertical distance of 20 m and reaches a velocity of $10\ \text{ms}^{-1}$. Calculate the work done by push of the air on the object.

Take $g = 10\ \text{ms}^{-2}$

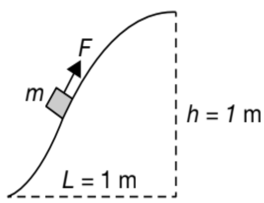
8. Two blocks of masses m_1 and m_2 connected by a light spring rest on a horizontal plane. The coefficient of friction between the blocks and the surface is equal to μ. What minimum constant force has to be applied in the horizontal direction to the block of mass m_1 in order to shift the other block?

9. A disc of mass 50 g slides with the zero initial velocity down an inclined plane set at an angle 30° to the horizontal. Having traversed a distance of 50 cm along the horizontal plane, the disc stops. Find the work performed by the friction forces over the whole distance, assuming the friction coefficient 0.15 for both inclined and horizontal planes. Take $g = 10\ \text{ms}^{-2}$.

10. In the arrangement shown in figure $m_A = 4$ kg and $m_B = 1$ kg. The system is released from rest and block B is found to have a speed $0.3\ \text{ms}^{-1}$ after it has descended through a distance of 1 m. Find the coefficient of friction between the block and the table is μ. Neglecting friction elsewhere and taking $g = 10\ \text{ms}^{-2}$.



11. A body of mass $m = 1$ kg was slowly hauled up a rough hill having friction coefficient 0.5 by a force F, which at each point was directed along a tangent to the trajectory. Calculate the work done against friction.



RELATION BETWEEN $\vec{F}_C$ AND U

Any conservative force $\vec{F}_C$ equals the negative gradient $(-\vec{\nabla})$ of the potential energy U. So,

$$\vec{F} = -\vec{\nabla}U = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)$$

Problem Solving Technique(s)

(a) Gradient is represented by the symbol $\vec{\nabla}$. (**Read as Del Operator** or Nabla Operator).

(b) Geometrically it gives slope of a scalar function in 3-D space.

(c) Gradient can only be applied on scalar functions and never on vectors.

(d) Once gradient is applied on a scalar function the result thus obtained is a vector.

(e) Mathematically, $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

So, $\vec{F} = -\vec{\nabla}U = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)$

where,

$$\frac{\partial U}{\partial x} = \left(\begin{array}{c}\text{Partial Derivative of } U \text{ w.r.t. } x \\ \text{i.e. Derivative of } U \text{ w.r.t. } x \\ \text{keeping } y \text{ and } z \text{ constant}\end{array}\right)$$

$$\frac{\partial U}{\partial y} = \left(\begin{array}{c}\text{Partial Derivative of } U \text{ w.r.t. } y \\ \text{i.e. Derivative of } U \text{ w.r.t. } y \\ \text{keeping } x \text{ and } z \text{ constant}\end{array}\right)$$

$$\frac{\partial U}{\partial z} = \left(\begin{array}{c}\text{Partial Derivative of } U \text{ w.r.t. } z \\ \text{i.e. Derivative of } U \text{ w.r.t. } z \\ \text{keeping } x \text{ and } y \text{ constant}\end{array}\right)$$

EXAMPLE

If $U = kxy$, k is a constant, then

$$\vec{F} = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)$$

Now, $\frac{\partial U}{\partial x} = ky$, $\frac{\partial U}{\partial y} = kx$, $\frac{\partial U}{\partial z} = 0$

So, $\vec{F} = -k\left(y\hat{i} + x\hat{j}\right)$ is the force corresponding to potential energy function $U = kxy$.

ILLUSTRATION 47

The potential energy function for a particle in a region of space is given by $U = 2x^2 + 3y^3 + 2z$, in joule. Calculate the force acting on the particle at the point $(1, 2, 3)$ m. Assume that x, y and z are expressed in metre.

SOLUTION

Since we know that,

$$\vec{F} = -\vec{\nabla}U = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)$$

$$\Rightarrow \quad \vec{F} = -\left(4x\hat{i} + 9y^2\hat{j} + 2\hat{k}\right)$$

$$\Rightarrow \quad \vec{F}\Big|_{\text{at }(1,2,3)\text{ m}} = -\left(4\hat{i} + 36\hat{j} + 2\hat{k}\right)\text{ N}$$

ILLUSTRATION 48

Force acting on a particle in a conservative force field is $\vec{F} = \left(2x\hat{i} + 3y^2\hat{j}\right)$. Find the potential energy function, if it is zero at origin.

SOLUTION

$$\int_{(0,0,0)}^{(x,y,z)} dU = -\int_{(0,0,0)}^{(x,y,z)} \vec{F}\cdot\vec{dr}$$

$$= -\int_{(0,0,0)}^{(x,y,z)} \left(2x\hat{i} + 3y^2\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$\Rightarrow \quad U(x, y, z) - U(0, 0, 0) = -\int_{(0,0,0)}^{(x,y,z)} \left(2xdx + 3y^2dy\right)$$

Since, we are given that $U(0, 0, 0) = 0$

$$\Rightarrow \quad U(x, y, z) = -(x^3 + y^3)$$

ILLUSTRATION 49

The potential energy function for the force between two atoms in a diatomic molecule can be expressed approximately as $U(r) = \frac{a}{r^{12}} - \frac{b}{r^6}$, where a and b are constants and r is the separation between the atoms.

(a) Determine the force function $F(r)$.

(b) Find the value of for which the molecule will be in the stable equilibrium.

SOLUTION

(a) The force between the two atoms is given by

$$F(r) = -\frac{dU}{dr}$$

$$\Rightarrow F(r) = \frac{12a}{r^{13}} - \frac{6b}{r^7}$$

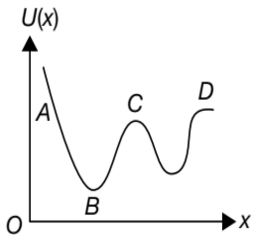
(b) For stable equilibrium $F(r) = 0$ and $\frac{d^2U}{dr^2} > 0$

Thus, $+\frac{12a}{r^{13}} - \frac{6b}{r^7} = 0$

$$\Rightarrow r = \left(\frac{2a}{b}\right)^{1/6}$$

POTENTIAL ENERGY CURVE

A graph plotted between the potential energy of a particle and its displacement from the centre of force is called potential energy curve. Figure shows a graph of potential energy function $U(x)$ for one dimensional motion.



NATURE OF FORCE

Since the negative gradient of the potential energy gives force i.e.

$$F = -\frac{dU}{dx}$$

(a) **Attractive Force:** On increasing x, if U increases, then $\frac{dU}{dx}$ = positive i.e. F is negative and hence force is attractive in nature. In graph this is represented in region BC.

(b) **Repulsive Force:** On increasing x, if U decreases, then $\frac{dU}{dx}$ = negative, i.e. F is positive in direction i.e. force is repulsive in nature. In graph this is represented in region AB.

(c) **Zero force (OR Equilibrium)**: On increasing x, if U does not change, then $\frac{dU}{dx} = 0$ i.e. F is zero

Points B, C and D represent the point of zero force or equilibrium is attained by the particle at these points.

TYPES OF EQUILIBRIUM

If net force acting on a particle is zero, it is said to be in equilibrium. Since the force acting on a particle is

$$F = -\frac{dU}{dx}$$

So, for equilibrium, we have

$$\frac{dU}{dx} = 0$$

However, equilibrium of particle can be of three types as explained.

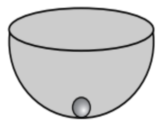
Stable

When a particle is displaced slightly from a position, then a force acting on it brings it back to the initial position, it is said to be in stable equilibrium position. In stable equilibrium, the potential energy is minimum i.e.

$$F = -\frac{dU}{dx} = 0 \text{ and } \frac{d^2U}{dx^2} = \text{positive}$$

i.e. rate of change of $\frac{dU}{dx}$ is positive.

EXAMPLE:



A marble placed at the bottom of a hemispherical bowl.

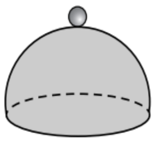
Unstable

When a particle is displaced slightly from a position, then a force acting on it tries to displace the particle further away from the equilibrium position, it is said to be in unstable equilibrium in which potential energy is maximum, i.e. when

$$F=-\frac{dU}{dx}=0 \text{ and } \frac{d^2U}{dx^2}=\text{negative}$$

i.e. rate of change of $\frac{dU}{dx}$ is negative.

EXAMPLE:



A marble balanced on top of a hemispherical bowl.

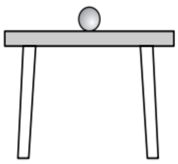
Neutral

When a particle is slightly displaced from a position then it does not experience any force acting on it and continues to be in equilibrium in the displaced position, it is said to be in neutral equilibrium in which potential energy is constant, i.e. when

$$F=-\frac{dU}{dx}=0 \text{ and } \frac{d^2U}{dx^2}=0$$

i.e. rate of change of $\frac{dU}{dx}$ is zero.

EXAMPLE:



A marble placed on horizontal table.

ILLUSTRATION 50

The potential energy of a conservative system is given by $U=ax^2-bx$, where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable or neutral.

SOLUTION

In a conservative field $F=-\frac{dU}{dx}$

$$\Rightarrow \quad F=-\frac{d}{dx}(ax^2-bx)=b-2ax$$

For equilibrium $F=0$

$$\Rightarrow \quad b-2ax=0$$

$$\Rightarrow \quad x=\frac{b}{2a}$$

From the given equation we can see that $\frac{d^2U}{dx^2}=2a$ (positive), i.e., U is minimum. Therefore, $x=\frac{b}{2a}$ is the stable equilibrium position.

ILLUSTRATION 51

The potential energy of a particle of mass 1 kg free to move along x-axis is given by $U(x)=\left(\frac{x^2}{2}-x\right)$ joule. If total mechanical energy of the particle is 2 J, then calculate the maximum speed of the particle. Ignore all other forces and assume that the conservative force corresponding to above potential energy is only acting on particle.

SOLUTION

The total mechanical energy of the particle at any instant is sum of kinetic and potential energy of the particle. So, we have

$$\text{KE}+\text{PE}=2\text{ J}$$

$$\Rightarrow \quad \frac{1}{2}mv^2+\left(\frac{x^2}{2}-x\right)=2$$

$$\Rightarrow \quad v^2=2x-x^2+4 \qquad \dots(1)$$

For maximum speed v, we have

$$\frac{d}{dx}(v^2)=0$$

$$\Rightarrow \quad 2-2x=0$$

$$\Rightarrow \quad x=1\text{ m}$$

So, the speed is maximum at $x=1\text{ m}$ and this maximum speed is obtained by substituting $x=1\text{ m}$ in equation (1).

$$v_{\max}^2=2-1+4$$

$$\Rightarrow \quad v_{\max}=\sqrt{5}\text{ ms}^{-1}$$

Test Your Concepts-V

Based on Relation Between Conservative Force and Potential Energy and Types of Equilibrium

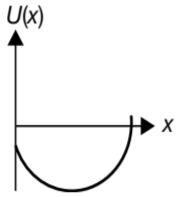
(Solutions on page H.15)

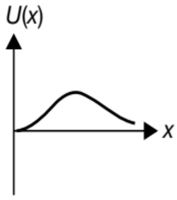
1. The potential energy function of a particle in a region of space is given by $U = (2xy + yz)$ J, where x, y and z are in metre. Calculate the force acting on the particle at a general point $P(x, y, z)$.
2. Suppose that the potential energy U in some region of space is given by

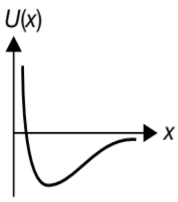
$$U(x, y, z) = U_0 \exp(-az)\cos(ax),$$

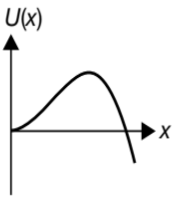
where a is a positive constant. Find the force corresponding to this potential energy.

3. Find the potential energy function $U(x, y)$ corresponding to a force $\vec{F} = 2axy\hat{i} + a(x^2 - y^2)\hat{j}$ where a is a constant. Assume that the potential energy at the origin is U_0.
4. A force is given by $\vec{F} = -4(y\hat{i} + x\hat{j})$. Calculate the potential energy at (2, 3) m assuming that the potential energy at the origin is –4 J.
5. A particle is moving along x-axis has potential energy $U = (2 - 20x + 5x^2)$ joule where x is in metre. The particle is released at $x = -3$ m. Calculate the maximum value of x.
6. A particle, which is constrained to move along x-axis, is subjected to a force in the same direction which varies with the distance x of the particle from the origin as $F(x) = -kx + ax^3$. Here, k and a are positive constant. For $x \geq 0$, the functional form of the potential energy $U(x)$ of the particle is

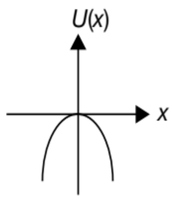
(A) 

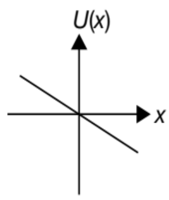
(B) 

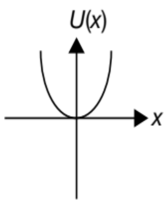
(C) 

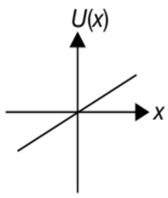
(D) 

7. A particle is placed at the origin and a force $F = kx$ is acting on it (where k is a positive constant). If $U(0) = 0$, the graph of $U(x)$ versus x will be : (where U is the potential energy function)

(A) 

(B) 

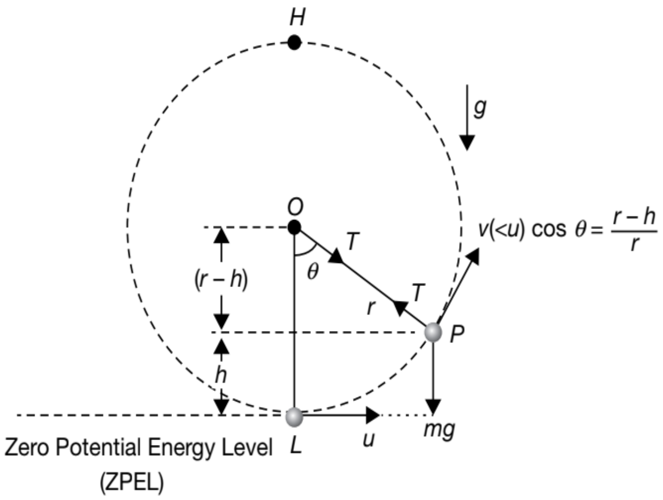
(C) 

(D) 

MOTION IN A VERTICAL CIRCLE

MOTION IN A VERTICAL CIRCLE

Consider a particle of mass m attached to a string of length r as shown in Figure.



Let the particle be whirled in a vertical circle (of radius r), then the speed of the particle is different at different points of the vertical circular path. Due to this, the centripetal force on the particle and hence the tension in the string changes continuously. Let the particle have speed u at the lowest point L, v at any point P and v_H at the highest point H. At the point P, the following forces act on the particle.

(i) the weight of the particle, mg, acting vertically downwards.

(ii) the tension T, in the string.

VELOCITY AT THE POINT P

Let us consider an intermediate point P (between L and H), such that it is at a vertical height h from the lowest point. If this point makes an angle θ with the initial position of the string, then we have

$$\cos\theta = \frac{r-h}{r}$$

Assuming the zero potential energy level (ZPEL) line to pass through L (as shown) and using the Law of Conservation of Mechanical Energy, we get

Version I	**Version II**
Loss in KE = Gain in PE	$(U+K)_L = (U+K)_P$
$\Rightarrow \quad \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgh$	$0 + \frac{1}{2}mu^2 = mgh + \frac{1}{2}mv^2$

So, from both we get

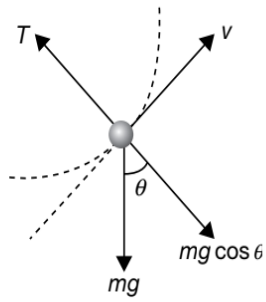
$$v^2 = u^2 - 2gh \qquad \ldots(1)$$

$$\Rightarrow \quad v = \sqrt{u^2 - 2gh} \qquad \ldots(2)$$

TENSION IN THE STRING AT ANY POINT P

The net force, $T - mg\cos\theta$, acting radially inwards is responsible for providing the centripetal force to the particle to move in a circle of radius r. So, we have

$$T - mg\cos\theta = \frac{mv^2}{r} \qquad \ldots(3)$$



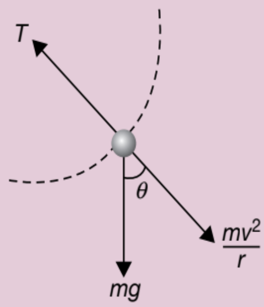
Inertial Frame

Since $\cos\theta = \frac{r-h}{r}$ and $v^2 = u^2 - 2gh$, so we get

$$T - mg\left(\frac{r-h}{r}\right) = \frac{m}{r}\left(u^2 - 2gh\right)$$

$$\Rightarrow \quad T = \frac{m}{r}\left(u^2 - 3gh + gr\right) \qquad \ldots(4)$$

We could have obtained the same result when observed from the frame attached to the particle i.e., a non-inertial frame as shown. From the frame attached to the particle it appears as if the particle is in equilibrium, so we have



$$T = mg\cos\theta + \frac{mv^2}{r} \qquad \{\text{same as (3)}\}$$

TENSION AT THE LOWEST POINT *L*

At the lowest point L, $h = 0$, so we get

$$T_L = \frac{m}{r}\left(u^2 + gr\right) \qquad \ldots(5)$$

TENSION AT THE HIGHEST POINT *H*

At the highest point H, $h = 2r$, so we get

$$T_H = \frac{m}{r}\left(u^2 - 5gr\right) \qquad \ldots(6)$$

So, from (5) and (6), we observe that

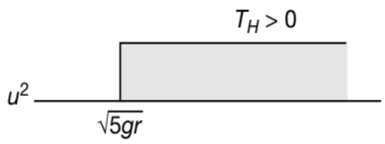
$$T_L - T_H = 6mg \qquad \ldots(7)$$

CONDITION FOR LOOPING THE LOOP

For "looping the loop" i.e., to just complete the loop, we must have

$$T_H \geq 0$$

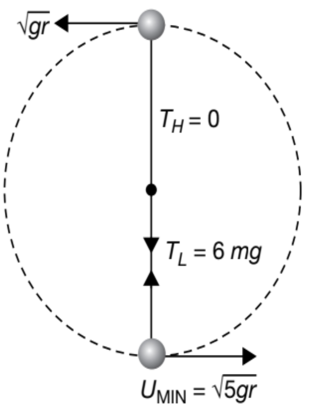
$$\Rightarrow \quad u^2 - 5gr \geq 0$$



$$\Rightarrow \quad u = u_{MIN} = \sqrt{5gr}$$

So, the minimum velocity (with which the particle should be launched from the lowest point) such that it just completes the circle is

$$u_{MIN} = \sqrt{5gr} \qquad \ldots(8)$$



Corresponding to this minimum velocity $u_{MIN} = \sqrt{5gr}$, velocity at maximum height H is obtained by putting $h = 2r$ in $v = \sqrt{u^2 - 2gh}$, so

$$v = \sqrt{5gr - 2g(2r)} = \sqrt{gr}$$

TENSION AND VELOCITY AT THE POINT *M* (MIDWAY BETWEEN *L* AND *H*)

At the point M, midway between L and H, we have $h = r$, so we get

$$v_M = \sqrt{u^2 - 2gr} \text{ and } T_M = \frac{m}{r}\left(u^2 - 2gr\right)$$

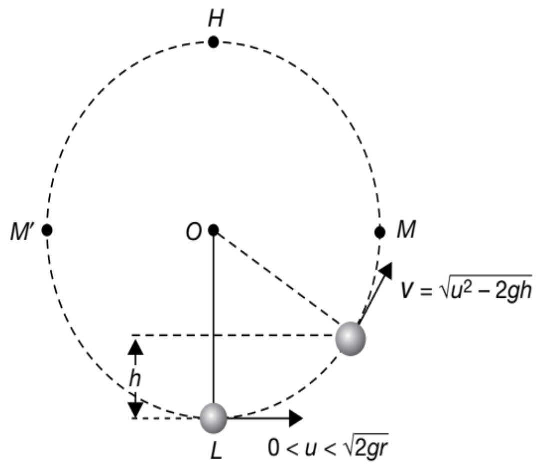
Now, from these equations, we **observe that the particle will reach** M only when

$$u^2 \geq 2gr$$

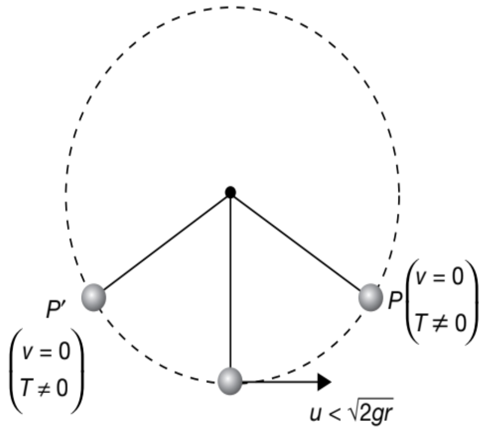
$$\Rightarrow \quad u \geq \sqrt{2gr}$$

FOR $0 < u < \sqrt{2gr}$ (BETWEEN L AND M)

For $u < \sqrt{2gr}$, the particle will not reach M and will oscillate between M and M'.



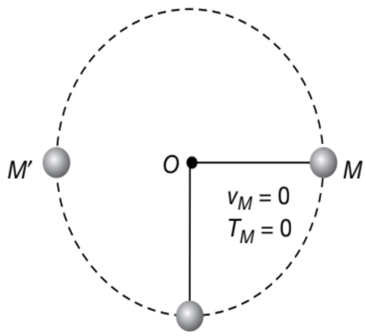
So, for $u < \sqrt{2gr}$, it will oscillate between P and P' with $v = 0$ at P (and P') and also everywhere between P and P' we observe $T \neq 0$.



i.e., for $u<\sqrt{2gr}$ we observe that v becomes zero earlier than T and hence the motion is reversed at P (or P') when $v=0$ i.e., the point of reversal of motion.

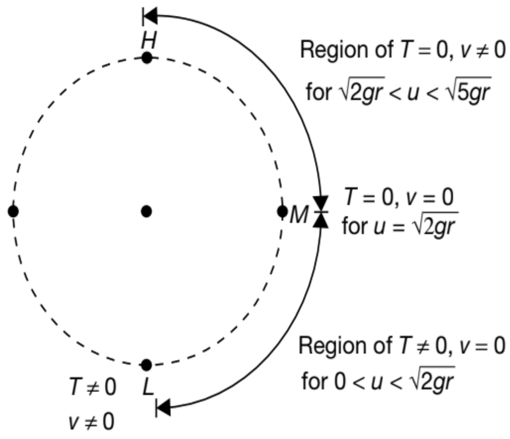
FOR $u=\sqrt{2gr}$

For $u=\sqrt{2gr}$, the particle will just be able to reach M (or M') i.e., it will still now oscillate between M and M'. Significantly, we observe that at M (or M') we have $v_M=0$ as well as $T_M=0$ i.e., $v_M=v_{M'}=0$ and $T_M=T_{M'}=0$

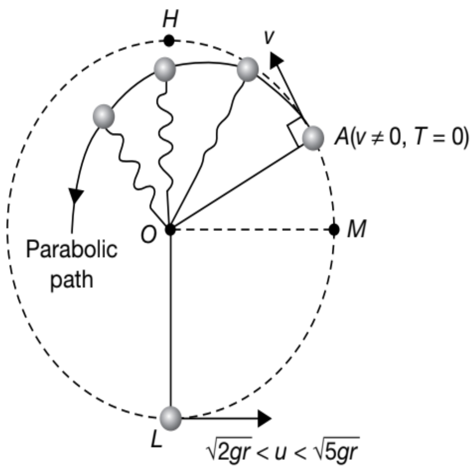


FOR $\sqrt{2gr}<u<\sqrt{5gr}$ (BETWEEN M AND H)

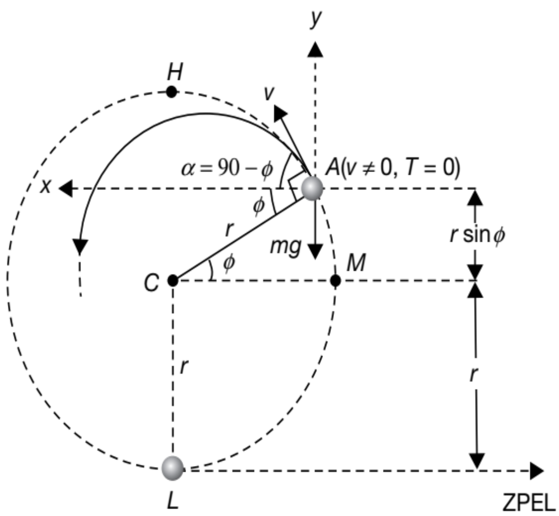
For this region we observe that T becomes zero before v (as discussed and concluded from the previous discussion). Hence for this region, we observe a point, say A, such that at this point A, tension in the string is zero (just as if it was absent) and the velocity of the particle at this point is non-zero.



Due to this non-zero velocity of the particle (and the string being absent, $T=0$) at A, the particle behaves like a projectile, launched from A (with a non-zero initial velocity) and is then free to move under the influence of gravity.



Let the origin be now fixed at A as shown.

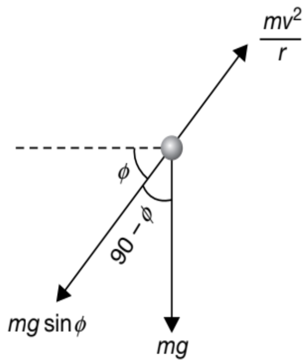


Applying Law of Conservation of Energy, we get

$$(U+K)_{\text{at }L}=(U+K)_{\text{at }A}$$

$$\Rightarrow \quad 0+\frac{1}{2}mu^2=mg(r+r\sin\phi)+\frac{1}{2}mv^2 \qquad \ldots(1)$$

Now, at A, we have $T=0$, so we get



$$mg\sin\phi=\frac{mv^2}{r}$$

$$\Rightarrow \quad v^2=gr\sin\phi \qquad \ldots(2)$$

Further for an oblique projectile launched from A, with initial velocity v at an angle of projection $\alpha(=90-\phi)$ we have

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha} \qquad \ldots(3)$$

$$\Rightarrow \quad y=x\tan(90-\phi)-\frac{gx^2}{2v^2\cos^2(90-\phi)}$$

$$\Rightarrow \quad y=x\cot\phi-\frac{gx^2}{2(gl\sin\phi)\sin^2\phi}$$

$$\Rightarrow \quad y=x\cot\phi-\frac{gx^2}{2gl\sin^3\phi} \qquad \ldots(4)$$

Now, if we asked to find the condition for the projectile to pass through (say),

(a) the point of suspension C, then we find the coordinates of $C(r\cos\phi,\ -r\sin\phi)$ and use equations (1), (2) and (3) (or (4)) to get the desired results.

(b) the lowest point L, then we find the coordinates of $L(r\cos\phi,-r(1+\sin\phi))$ and again make use of equations (1), (2) and (3) (or (4)) to get the desired conditions or results.

The following ILLUSTRATIONS make use of the fundamentals discussed above to have a smooth understanding of the concept.

ILLUSTRATION 52

A body weighing 0.4 kg is whirled in a vertical circle with a string making 2 revolutions per second. If the radius of the circle is 1.2 m. Find the tension

(a) at the top of the circle,

(b) at the bottom of the circle.

Given: $g=10\ \text{ms}^{-2}$ and $\pi=3.14$

SOLUTION

Angular velocity

$$\omega=2\ \text{rev/s}=4\pi\ \text{rad s}^{-1}=12.56\ \text{rad s}^{-1}$$

$$\{\because\ 1\ \text{rev}=2\pi\ \text{radian}\}$$

(a) At the top of the circle

$$T=\frac{mv^2}{r}-mg$$

$$\Rightarrow \quad T=mr\omega^2-mg=m(r\omega^2-g)$$

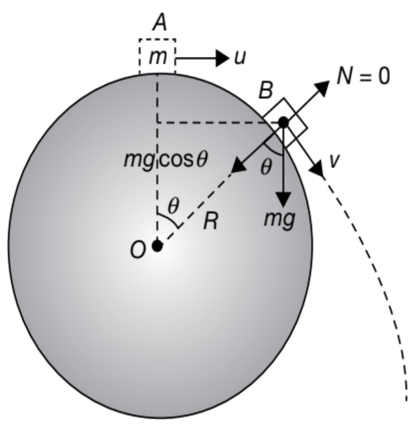
$$\Rightarrow \quad T=0.4(1.2\times12.56\times12.56-9.8)\ \text{N}=71.2\ \text{N}$$

(b) At the lowest point

$$T=m(r\omega^2+g)=80\ \text{N}$$

MOTION OF A BODY ON A SPHERICAL SURFACE

Consider a small body of mass m, placed at the top of a spherical fixed smooth surface of radius R. When the body is given a horizontal velocity u, then it moves some distance in circular path along the spherical surface and at some point B, it breaks off the surface i.e. loses contact with the surface $(N=0)$ and follows the projectile trajectory i.e. a parabolic path. As shown in Figure.



Let us first calculate the velocity of the body at point B, when it makes an angle θ with the vertical position. This can be done by applying the Law of Conservation of Mechanical Energy, according to which, we have

$$\begin{pmatrix}\text{Loss in GPE}\\ \text{of the Body}\end{pmatrix}=\begin{pmatrix}\text{Gain in KE}\\ \text{of the Body}\end{pmatrix}$$

$$\Rightarrow \quad mgR(1-\cos\theta)=\frac{1}{2}mv^2-\frac{1}{2}mu^2$$

$$\Rightarrow \quad \frac{1}{2}mv^2=\frac{1}{2}mu^2+mgR(1-\cos\theta)$$

$$\Rightarrow \quad v=\sqrt{u^2+2gR(1-\cos\theta)} \qquad \text{...(1)}$$

When the body breaks off the surface, then the force applied on the block by the sphere is zero i.e. $N=0$. At this point, the component of weight $mg\cos\theta$ acting towards the centre of the circle provides the necessary centripetal force to the body to move in a circular path, so we have

$$mg\cos\theta=\frac{mv^2}{R}$$

Substituting the value of v from Equation (1), we get

$$gR\cos\theta=u^2+2gR(1-\cos\theta)$$

$$\Rightarrow \quad 3gR\cos\theta=u^2+2gR$$

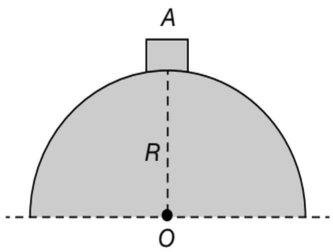
$$\Rightarrow \quad \cos\theta=\frac{u^2+2gR}{3gR}$$

If the horizontal velocity u i.e. the velocity of projection of the body is given, then we can get the angle θ with the vertical where the body leaves the spherical surface and starts moving as a projectile.

ILLUSTRATION 53

A block of mass m is released from the top of a frictionless fixed hemisphere as shown. Find

(a) the angle with the vertical where it breaks off.
(b) the velocity at the instant when it breaks off.
(c) the height where it breaks off.

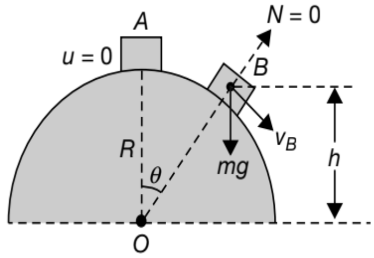


SOLUTION

At B, $N=0$

$$\Rightarrow \quad mg\cos\theta=\frac{mv_B^2}{R}$$

$$\Rightarrow \quad v_B=\sqrt{gR\cos\theta} \qquad \text{...(1)}$$



Now by equation of energy between A and B we have,

$$0+mgR=\frac{1}{2}mv_B^2+mgh$$

Substituting v_B from (1) and $h=R\cos\theta$, we get

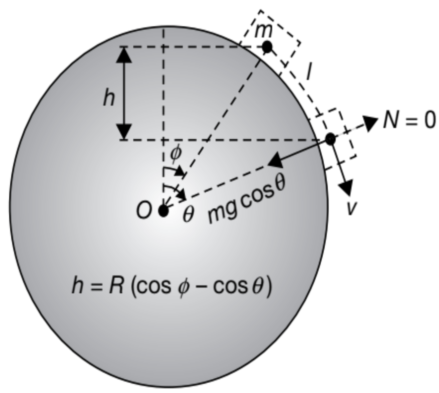
$$v_B=\sqrt{\frac{2}{3}gR} \text{ and } h=\frac{2R}{3} \text{ from the bottom}$$

ILLUSTRATION 54

A small block of mass m is placed on the outer surface of a smooth fixed sphere of radius R at a point where the radius makes an angle ϕ with the vertical. The block is given a small push such that it starts moving on the surface of the sphere. Calculate the distance travelled by the block before it leaves contact with the sphere.

SOLUTION

When the block starts falling along the circular path, then at the instant of breaking off the surface i.e. when the normal reaction force between the block and the sphere is zero. Let θ be the angle which radius makes with the vertical at the instant when block leaves the surface as shown in Figure.



The necessary centripetal force is provided to the block by the component of weight $mg\cos\theta$ i.e.

$$\frac{mv^2}{R} = mg\cos\theta$$

$$\Rightarrow \quad v = \sqrt{Rg\cos\theta} \qquad \ldots(1)$$

where v is the instantaneous velocity of the block at an angle θ from the vertical. As the block falls from rest through $h = R(\cos\phi - \cos\theta)$, then by Law of Conservation of Energy, we have

$$\begin{pmatrix}\text{Loss in GPE}\\ \text{of the Body}\end{pmatrix} = \begin{pmatrix}\text{Gain in KE}\\ \text{of the Body}\end{pmatrix}$$

$$\Rightarrow \quad mgh = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = \sqrt{2gh}$$

$$\Rightarrow \quad v = \sqrt{2gR(\cos\phi - \cos\theta)} \qquad \ldots(2)$$

Equating (1) and (2), we get

$$\sqrt{Rg\cos\theta} = \sqrt{2gR(\cos\phi - \cos\theta)}$$

$$\Rightarrow \quad 3\cos\theta = 2\cos\phi$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}\cos\phi\right)$$

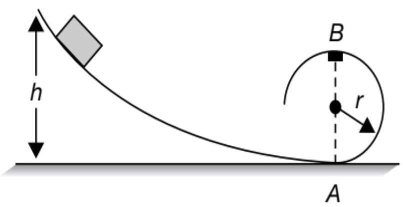
Thus distance l travelled by the box before leaving the contact with the sphere is

$$l = R(\theta - \phi)$$

$$\Rightarrow \quad l = R\left[\cos^{-1}\left(\frac{2}{3}\cos\phi\right) - \phi\right]$$

ILLUSTRATION 55

A block is released from the top of a smooth vertical track, which ends in a circle of radius r as shown in figure.



(a) Find the minimum value of h so that the block completes the circle.

(b) If $h = 3r$, find normal reaction when the block is at the points A and B.

(c) If $h = 2r$, find the velocity of the block when it loses the contact with the track.

SOLUTION

(a) For completing the circle, velocity at lowest point of circle (say A) is $\sqrt{5gr}$ from energy conservation

$$mgh = \frac{1}{2}m\left(\sqrt{5gr}\right)^2$$

$$\Rightarrow \quad h = \frac{5r}{2}$$

(b) $h = 3r$

From energy conservation velocity at point A and B are

$$mg(3r) = \frac{1}{2}mv_A^2$$

$$\Rightarrow \quad v_A = \sqrt{6gr}$$

$$\Rightarrow \quad mg(3r) = mg2r + \frac{1}{2}mv_B^2$$

$$\Rightarrow \quad v_B = \sqrt{2gr}$$

Therefore, normal reaction at A and B is

$$N_A - mg = \frac{mv_A^2}{r}$$

$$\Rightarrow \quad N_A = 7mg$$

$$\Rightarrow \quad N_B + mg = \frac{mv_B^2}{r}$$

$$\Rightarrow \quad N_B = mg$$

(c) $h = 2r$

It loses contact with the track when normal reaction is zero

$$\frac{mv^2}{r} = mg\cos\theta \qquad \ldots(1)$$

From energy conservation

$$mgh = mgr(1+\cos\theta) + \frac{1}{2}mv^2 \qquad \ldots(2)$$

From (1) and (2)

$$v = \sqrt{\frac{2g(h-r)}{3}} = \sqrt{\frac{2gr}{3}}$$

ILLUSTRATION 56

A heavy particle is suspended by a string of length l from a fixed point O. The particle is given a horizontal velocity v_0. The string slacks at some angle and the particle proceeds on a parabola. Find the value v_0 if the particle passes through the point of suspension.

SOLUTION

Let the string slacks at point B as shown in figure.

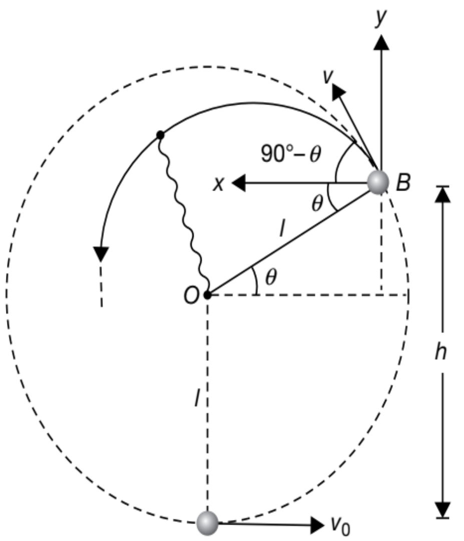
$$h = l + l\sin\theta = l(1+\sin\theta) \quad \ldots(1)$$

Applying Conservation of Mechanical Energy between points A and B, we get

$$v^2 = v_0^2 - 2gh \quad \ldots(2)$$

At point B, the string slacks, so we have

$$T = 0 \text{ and hence } mg\sin\theta = \frac{mv^2}{l}$$



$$\Rightarrow \quad v^2 = gl\sin\theta \quad \ldots(3)$$

After B, path of the particle is a projectile and it passes through O. Co-ordinate of point O with origin at B and x and y axes as shown in figure are:

$$(x, y) = (l\cos\theta, -l\sin\theta)$$

Angle of projection of the particle is $\alpha = 90° - \theta$ and the velocity of projection is v. So, substituting the above data in equation of an oblique projectile, i.e.,

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$$

we get,

$$-l\sin\theta = (l\cos\theta)\tan(90° - \theta) - \frac{g(l\cos\theta)^2}{2v^2\cos^2(90° - \theta)}$$

$$\Rightarrow \quad -l\sin\theta = \frac{l\cos^2\theta}{\sin\theta} - \frac{gl^2\cos^2\theta}{2v^2\sin^2\theta}$$

Substituting, $v^2 = gl\sin\theta$ from Equation (3), we get

$$l\sin\theta = \frac{gl^2\cos^2\theta}{2(gl\sin\theta)(\sin^2\theta)} - \frac{l\cos^2\theta}{\sin\theta}$$

$$\Rightarrow \quad 2\sin^4\theta = \cos^2\theta - 2\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad 2\sin^4\theta = (1 - \sin^2\theta) - 2\sin^2\theta(1 - \sin^2\theta)$$

$$\Rightarrow \quad 3\sin^2\theta = 1$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{3}}$$

From Equation (2), we get

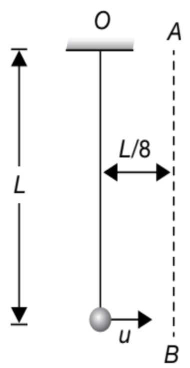
$$v_0^2 = v^2 + 2gh = (gl\sin\theta) + 2gl(1+\sin\theta)$$

$$\Rightarrow \quad v_0^2 = gl(2 + 3\sin\theta) = gl(2+\sqrt{3})$$

$$\Rightarrow \quad v_0 = \sqrt{gl(2+\sqrt{3})}$$

ILLUSTRATION 57

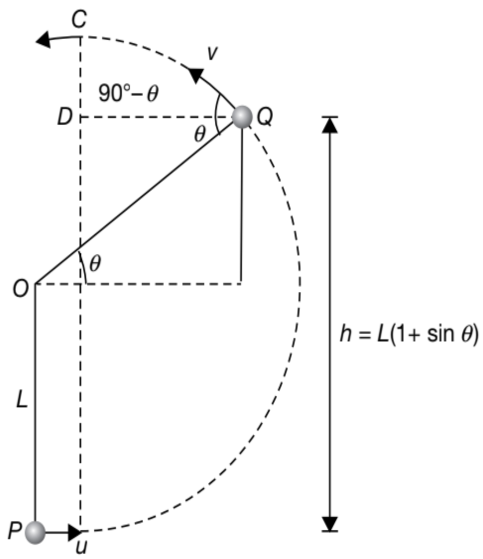
A particle is suspended vertically from a point O by an inextensible massless string of length L. A vertical line AB is at a distance of $\frac{L}{8}$ from O as shown.



The particle is given a horizontal velocity u. At some point, its motion ceases to be circular and eventually the object passes through the line AB. At the instant of crossing AB, its velocity is horizontal. Find u.

SOLUTION

Let the string slacks at point Q as shown in figure. From P to Q path is circular and beyond Q, path is parabolic. At point C, velocity of particle becomes horizontal, therefore, QD is half of the range of the projectile.



Now we have following equations:

(a) $T_Q = 0$

$$\Rightarrow\ mg\sin\theta = \frac{mv^2}{L} \qquad \ldots(1)$$

(b) $v^2 = u^2 - 2gh$

$$v^2 = u^2 - 2gL(1+\sin\theta) \qquad \ldots(2)$$

(c) $QD = \frac{1}{2}(\text{Range})$

$$\Rightarrow\ \left(L\cos\theta - \frac{L}{8}\right) = \frac{v^2\sin 2(90^\circ - \theta)}{2g}$$

$$\Rightarrow\ L\left(\cos\theta - \frac{1}{8}\right) = \frac{v^2\sin 2\theta}{2g} \qquad \ldots(3)$$

Equation (3) can be written as

$$\left(\cos\theta - \frac{1}{8}\right) = \left(\frac{v^2}{gL}\right)\sin\theta\cos\theta$$

From Equation (1), substituting value of $\left(\frac{v^2}{gL}\right) = \sin\theta$, we get

$$\left(\cos\theta - \frac{1}{8}\right) = \sin^2\theta\cos\theta = \left(1-\cos^2\theta\right)\cos\theta$$

$$\Rightarrow\ \cos\theta - \frac{1}{8} = \cos\theta - \cos^3\theta$$

$$\Rightarrow\ \cos^3\theta = \frac{1}{8}$$

$$\Rightarrow\ \cos\theta = \frac{1}{2}$$

$$\Rightarrow\ \theta = 60^\circ$$

From Equation (1), we get

$$v^2 = gL\sin\theta = gL\sin 60^\circ$$

$$\Rightarrow\ v^2 = \frac{\sqrt{3}}{2}gL$$

Substituting this value of v^2 and $\theta = 60^\circ$ in Equation (2), we get

$$u^2 = v^2 + 2gL(1+\sin\theta)$$

$$\Rightarrow\ u^2 = \frac{\sqrt{3}}{2}gL + 2gL\left(1+\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow\ u^2 = \frac{3\sqrt{3}}{2}gL + 2gL = gL\left(2+\frac{3\sqrt{3}}{2}\right)$$

$$\Rightarrow\ u = \sqrt{gL\left(2+\frac{3\sqrt{3}}{2}\right)} \approx 2.14\sqrt{gL}$$

Conceptual Note(s)

Since $u = 2.14\sqrt{gL} = \sqrt{4.6gL}$

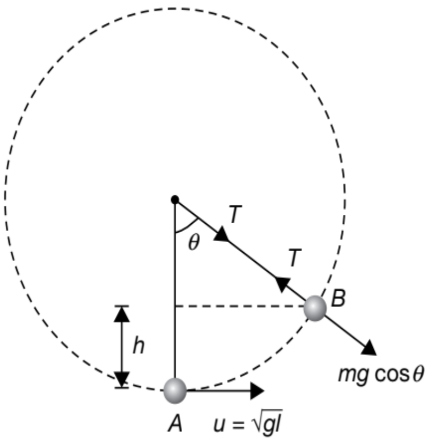
This value of u lies between $\sqrt{2gL}$ and $\sqrt{5gL}$

ILLUSTRATION 58

A heavy particle hanging from a fixed point by a light inextensible string of length l is projected horizontally with speed $\sqrt{gl}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of the motion when the tension in the string is equal to the weight of the particle.

SOLUTION

Let $T = mg$ at angle θ as shown in figure



$$h = l(1-\cos\theta) \quad \ldots(1)$$

Applying Conservation of Mechanical Energy between points A and B, we get

$$\frac{1}{2}m(u^2 - v^2) = mgh$$

Here, $u^2 = gl$ ...(2)

and v = speed of particle in position B

$$\Rightarrow \quad v^2 = u^2 - 2gh \quad \ldots(3)$$

Further, $T - mg\cos\theta = \dfrac{mv^2}{l}$

$$\Rightarrow \quad mg - mg\cos\theta = \frac{mv^2}{l} \quad (T = mg)$$

$$\Rightarrow \quad v^2 = gl(1-\cos\theta) \quad \ldots(4)$$

Substituting values of v^2, u^2 and h from Equations (4), (2) and (1) in Equation (3), we get

$$gl(1-\cos\theta) = gl - 2gl(1-\cos\theta)$$

$$\Rightarrow \quad \cos\theta = \frac{2}{3}$$

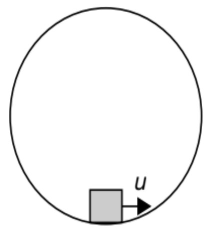
$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Substituting $\cos\theta = \dfrac{2}{3}$ in Equation (4), we get

$$v = \sqrt{\frac{gl}{3}}$$

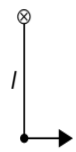
CONDITION FOR LOOPING THE LOOP IN SOME OTHER CASES

CASE-1: A mass moving on a smooth vertical circular track.



Mass moving along a smooth vertical circular loop. Condition for just looping the loop, normal at highest point $= 0$. By calculation similar to article (motion in vertical circle). Minimum horizontal velocity at lowest point is $u_{\min} = \sqrt{5gl}$.

CASE 2: A particle attached to a light rod rotated in vertical circle.

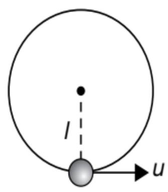


Condition for just looping the loop, velocity $v = 0$ at highest point (even if tension is zero, rod won't slack and a compressive force can appear in the rod).
By energy conservation, velocity at lowest point

$$\frac{1}{2}mu_{\min}^2 = mg(2l)$$

$$\Rightarrow \quad u_{\min} = \sqrt{4gl} \quad \text{\{for completing the circle\}}$$

CASE 3: A bead attached to a ring and rotated.

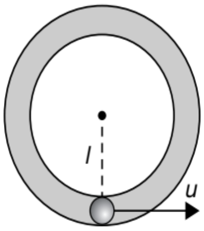


Condition for just looping the loop, velocity $v = 0$ at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward).
By energy conservation, velocity at lowest point

$$\frac{1}{2}mu_{\min}^2 = mg(2l)$$

$$\Rightarrow \quad u_{\min} = \sqrt{4gl} \quad \text{\{for completing the circle\}}$$

CASE 4: A block rotated between smooth surfaces of a pipe.



Condition for just looping the loop, velocity $v = 0$ at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward).

By energy conservation, we have

$$\frac{1}{2}mu_{\text{min}}^2 = mg(2l)$$

$$\Rightarrow \quad u_{\text{min}} = \sqrt{4gl} \qquad \text{\{for completing the circle\}}$$

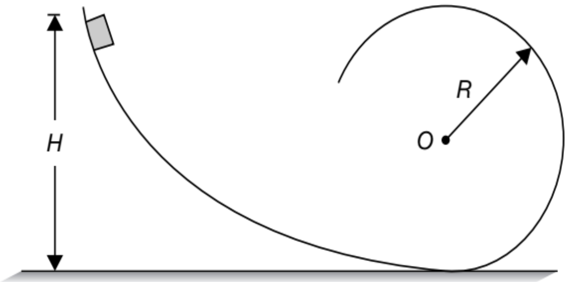
So, the minimum speed, at the top, required to perform a vertical loop is 15.65 ms^{-1}.

Test Your Concepts-VI

Based on Vertical Circle

(Solutions on page H.16)

1. A small body is allowed to slide on a frictionless track from rest position as shown in figure. What must be the minimum height in terms of *R*, so that body may successfully complete the loop.



2. A child revolves a stone of mass 0.5 kg tied to the end of a string of length 40 cm in a vertical circle. The speed of the stone at the lowest point of the circle is 3 ms^{-1}. Calculate the tension in the string at this point.

3. The bob of a simple pendulum of length $L = 2$m has a mass $m = 2$ kg and a speed $u = 1 \text{ ms}^{-1}$ when the string is at 37° to the vertical. Find the tension in the string at

(a) the lowest point in its swing

(b) the highest point

Given $\cos(37°) = 0.8$

4. The bob of a 2 m pendulum describes an arc of circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, calculate the velocity and the acceleration of the bob in that position.

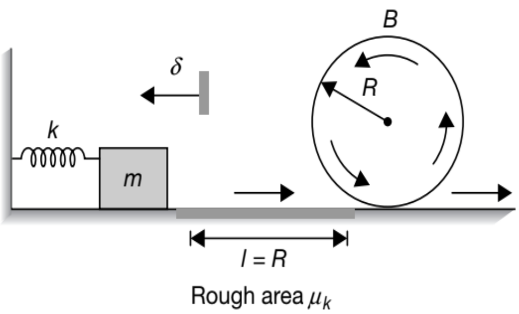
5. A small block with a mass of 0.8 kg is attached to a cord passing through a hole in a horizontal frictionless surface. The block is originally revolving at a distance of 0.4 m from the hole with a speed of 1 ms^{-1}. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.2 m. At this new distance the speed of the block is observed to be 4 ms^{-1}. Calculate the

(a) tension in the cord in the original situation when the block has speed $v = 1 \text{ ms}^{-1}$.

(b) tension in the cord in the final situation when the block has speed $v = 4 \text{ ms}^{-1}$.

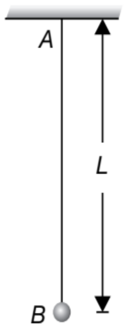
(c) work that had to be done by the person pulling the cord.

6. A spring of spring constant *k* is compressed and suddenly released, sending the particle of mass *m* sliding along the track shown. Determine the minimum spring compression δ for which the particle will not lose contact with looping the loop track. The sliding surface is smooth except for the rough portion of length *l* equal to radius of the loop, *R*, where the coefficient of kinetic friction is μ_k.

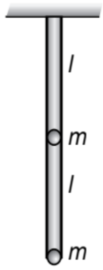


7. A particle is projected along the inside of a vertical hoop of radius r from its lowest point with such a velocity that it leaves the hoop and returns to the point of projection again. Find the velocity of projection and determine where the particle leaves the hoop.

8. A bob B of mass m of the pendulum AB is given an initial velocity $\sqrt{3gL}$ in the horizontal direction. Calculate the maximum height of the bob from the starting point if AB happens to be a very light
(a) rod.
(b) string.



9. Two point masses each of mass m are connected to the light rod of length l which is free to rotate in a vertical plane as shown. Calculate the minimum horizontal velocity is given to mass so that it completes the circular motion in vertical plane.



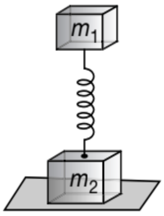
10. You may have seen in a circus a motorcyclist driving in vertical loops inside a death well (a hollow spherical chamber with holes, so that the cyclist does not drop down when he is at the uppermost point, with no support from below. What is the minimum speed required at the uppermost position to perform a vertical loop if the radius of the chamber is 25 m?

11. A stone tied to a string of length 5 m is whirled in a vertical circle with the other end of the string at the centre. At a certain instant of time, the stone is at the lowest position and has a speed of 10 ms^{-1}. Calculate the magnitude of the change in velocity as it reaches a position where the string is horizontal. $\left(\text{Take } g = 10 \text{ ms}^{-2}\right)$

SOLVED PROBLEMS

PROBLEM 1

Two plates of masses m_1 and m_2 are connected by a spring of force constant k. What force should be applied to the upper plate such that when the force is removed, the lower plate is just raised? Disregard the mass of the spring.



SOLUTION

Let F be the required force applied on the upper plate in the downward direction. If x_1 be the depression produced in the spring. Then for equilibrium, under the influence of

(a) the downward applied force F,

(b) upward spring force kx and

(c) the weight mg always acting downwards

$$\Rightarrow \quad F + m_1 g = kx_1 \qquad \ldots(1)$$

When released, let the spring extend by x_2. The lower mass will be lifted if

$$kx_2 \geq m_2 g \qquad \ldots(2)$$

Applying the Law of Conservation of Mechanical Energy between the initial and final positions of the spring, we get

$$\begin{pmatrix}\text{Energy}\\ \text{Gained}\\ \text{by Spring}\\ \text{due to}\\ \text{compression}\end{pmatrix} + \begin{pmatrix}\text{Loss}\\ \text{in}\\ \text{GPE}\\ \text{of } m_1\end{pmatrix} = \begin{pmatrix}\text{Energy}\\ \text{Gained}\\ \text{by}\\ \text{Spring}\\ \text{due to}\\ \text{extension}\end{pmatrix} + \begin{pmatrix}\text{Gain}\\ \text{in}\\ \text{GPE}\\ \text{of } m_1\end{pmatrix}$$

Since m_2 stays on the ground, initially and finally, so we have not taken any term that involves the potential energy of the mass m_2.

$$\Rightarrow \quad \frac{1}{2}kx_1^2 - m_1 g x_1 = \frac{1}{2}kx_2^2 + m_1 g x_2$$

$$\Rightarrow \quad \frac{1}{2}kx_1^2 = \frac{1}{2}kx_2^2 + m_1 g\left(x_1 + x_2\right)$$

$$\Rightarrow \quad k^2x_1^2 = k^2x_2^2 + \left(2m_1 g x_1 + 2m_1 g x_2\right)k$$

$$\Rightarrow \quad \left(kx_1 - m_1 g\right)^2 = \left(kx_2 + m_1 g\right)^2$$

$$\Rightarrow \quad kx_1 - m_1 g = \left(kx_2 + m_1 g\right)$$

$$\Rightarrow \quad F + m_1 g - m_1 g = \left(kx_2 + m_1 g\right) \qquad \{\text{using (1)}\}$$

$$\Rightarrow \quad F = kx_2 + m_1 g$$

$$\Rightarrow \quad kx_2 = F - m_1 g$$

For lifting the lower mass, $kx_2 \geq m_2 g$

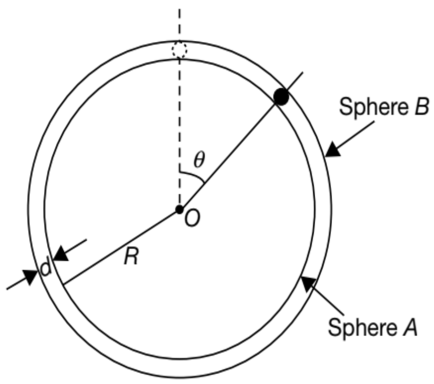
$$\Rightarrow \quad F - m_1 g \geq m_2 g$$

$$\Rightarrow \quad F \geq m_1 g + m_2 g$$

For just lifting, $F = \left(m_1 + m_2\right)g$

PROBLEM 2

A spherical ball of mass m is kept at the highest point in the space between two fixed, concentric spheres A and B (see figure). The smaller sphere A has a radius R and the space between the two spheres has a width d. The ball has a diameter very slightly less then d. All surfaces are frictionless. The ball is given a gentle push (towards the right in the figure). The angle made by the radius vector of the ball with the upward vertical is denoted by θ (shown in the figure).



(a) Express the total normal reaction force exerted by the spheres on the ball as a function of angle θ.

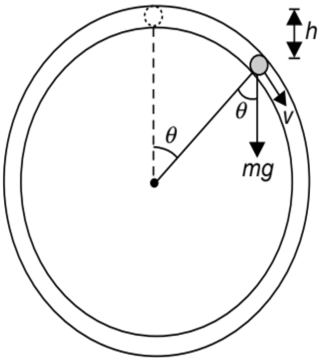
(b) Let N_A and N_B denote the magnitudes of the normal reaction forces on the ball exerted by the spheres A and B, respectively. Sketch the variations of N_A and N_B as functions of $\cos\theta$ in the range $0 \le \theta \le \pi$ by drawing two separate graphs, taking $\cos\theta$ on the horizontal axes.

SOLUTION

(a) $h = \left(R + \frac{d}{2}\right)(1 - \cos\theta)$

Velocity of ball at angle θ is

$$v^2 = 2gh = 2\left(R + \frac{d}{2}\right)(1 - \cos\theta)g \qquad \ldots(1)$$



Let N be the total normal reaction (away from centre) at angle θ. Then

$$mg\cos\theta - N = \frac{mv^2}{\left(R + \frac{d}{2}\right)}$$

Substituting value of v^2 from Equation (1), we get

$$mg\cos\theta - N = 2mg(1 - \cos\theta)$$

$$\Rightarrow N = mg(3\cos\theta - 2)$$

(b) The ball will lose contact with the inner sphere when

$$N = 0$$

$$\Rightarrow 3\cos\theta - 2 = 0$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

After this it makes contact with outer sphere and normal reaction starts acting towards the centre.

Thus, for $\theta \le \cos^{-1}\left(\frac{2}{3}\right)$

$$N_B = 0$$

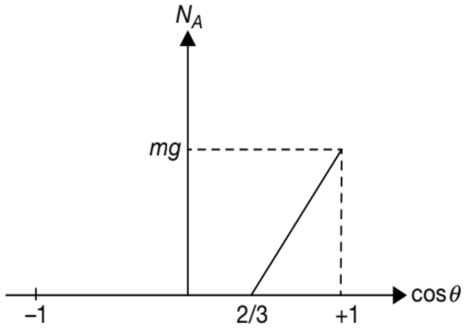
and $N_A = mg(3\cos\theta - 2)$

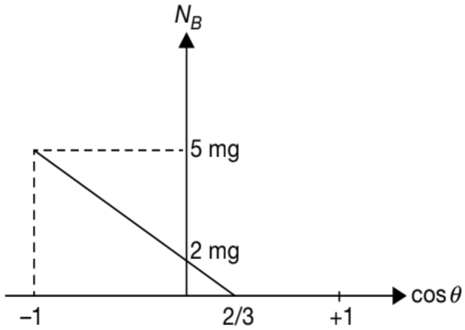
and for $\theta \ge \cos^{-1}\left(\frac{2}{3}\right)$

$$N_A = 0$$

and $N_B = mg(2 - 3\cos\theta)$

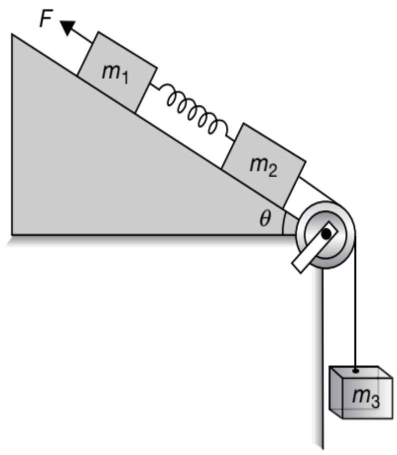
The corresponding graphs are as follows:





PROBLEM 3

Two blocks of mass $m_1 = 2$ kg and $m_2 = 4$ kg are attached by light ideal spring of force constant $k = 1000\ \text{Nm}^{-1}$. The system is kept on a smooth inclined plane inclined $30°$ with horizontal. A force $F = 15$ N is applied on m_1 and system is released from rest. The block m_2 is attached with a light string whose another end is connected with a mass $m_3 = 1$ kg. Assume initially the spring is at relaxed position and the system is released from rest. Find the maximum extension of the spring.



SOLUTION

At some instant of time let m_1 be displaced downwards by x_1 and m_2 be displaced downwards by x_2, with $x_2 > x_1$. Elongation of spring at this instant is then given by

$$x = x_2 - x_1$$

Writing equation of motion for m_1, m_2 and m_3, we have,

For m_1, we have

$$m_1 g \sin\theta + kx - F = m_1 a_1$$

$$\Rightarrow \quad a_1 = \frac{m_1 g \sin\theta + kx - F}{m_1} \quad \ldots(1)$$

For m_2, we have

$$T + m_2 g \sin\theta - kx = m_2 a_2 \quad \ldots(2)$$

For m_3, we have

$$m_3 g - T = m_3 a_2 \quad \ldots(3)$$

Adding (2) and (3), we get

$$a_2 = \frac{m_2 g \sin\theta - kx + m_3 g}{m_2 + m_3} \quad \ldots(4)$$

Acceleration of m_2 relative to m_1 is given by

$$a = a_2 - a_1$$

$$\Rightarrow \quad a = \frac{m_2 g \sin\theta - kx + m_3 g}{(m_2 + m_3)} - \frac{m_1 g \sin\theta + kx - F}{m_1}$$

Substituting the given values, we get

$$a = 8.5 - 700x$$

$$\Rightarrow \quad v\frac{dv}{dx} = 8.5 - 700x$$

$$\Rightarrow \quad \int_0^0 v\,dv = \int_0^{x_{max}} (8.5 - 700x)\,dx$$

$$\Rightarrow \quad x_{max} = 0.024 \text{ m}$$

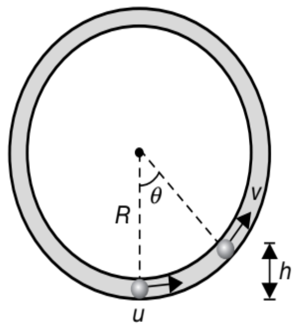
$$\Rightarrow \quad x_{max} = 2.4 \text{ cm}$$

PROBLEM 4

A smooth circular tube of radius R is fixed in a vertical plane. A particle is projected from its lowest point with a velocity just sufficient to carry it to the highest point. Calculate the time taken by the particle to reach the end of the horizontal diameter.

SOLUTION

Minimum velocity of particle at the lowest position to complete the circle should be $\sqrt{4gR}$ inside a tube. So, $u = \sqrt{4gR}$



At any instant, h has a function of θ is given by

Since, $h = R(1 - \cos\theta)$

$$\Rightarrow \quad v^2 = u^2 - 2gh$$

$$\Rightarrow \quad v^2 = 4gR - 2gR(1 - \cos\theta)$$

$$\Rightarrow \quad v^2 = 2gR(1 + \cos\theta)$$

$$\Rightarrow \quad v^2 = 2gR\left(2\cos^2\left(\frac{\theta}{2}\right)\right)$$

$$\Rightarrow \quad v = 2\sqrt{gR}\cos\left(\frac{\theta}{2}\right)$$

Since $v = \frac{ds}{dt}$

$$\Rightarrow \quad ds = v\,dt$$

Since, $ds = R\,d\theta$

$$\Rightarrow \quad R\,d\theta = 2\sqrt{gR}\cos\left(\frac{\theta}{2}\right) \cdot dt$$

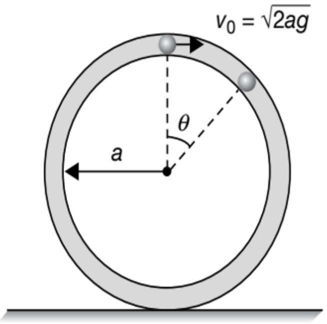
$$\Rightarrow \quad \int_0^t dt = \frac{1}{2}\sqrt{\frac{R}{g}} \int_0^{\pi/2} \sec\left(\frac{\theta}{2}\right) d\theta$$

$$\Rightarrow \quad t = \sqrt{\frac{R}{g}}\left[\ln\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right]_0^{\pi/2}$$

$$\Rightarrow \quad t = \sqrt{\frac{R}{g}}\ln\left(1 + \sqrt{2}\right)$$

PROBLEM 5

A heavy particle slides under gravity down the inside of a smooth vertical tube held in vertical plane. It starts from the highest point with velocity $\sqrt{2ag}$, where a is the radius of the circle. Find the angular position θ (as shown in figure) at which the vertical acceleration of the particle is maximum.



SOLUTION

At position θ, we have

$$v^2 = v_0^2 + 2gh$$

where $h = a(1-\cos\theta)$

$$\Rightarrow \quad v^2 = \left(\sqrt{2ag}\right)^2 + 2ag(1-\cos\theta)$$

$$\Rightarrow \quad v^2 = 2ag(2-\cos\theta) \qquad \ldots(1)$$

Also, we have

$$N + mg\cos\theta = \frac{mv^2}{a}$$

$$\Rightarrow \quad N + mg\cos\theta = 2mg(2-\cos\theta)$$

$$\Rightarrow \quad N = mg(4-3\cos\theta)$$

Net vertical force acting on the particle is

$$F = N\cos\theta + mg = mg\left(4\cos\theta - 3\cos^2\theta + 1\right)$$

This force (or acceleration) will be maximum when $\dfrac{dF}{d\theta} = 0$

$$\Rightarrow \quad -4\sin\theta + 6\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad \sin\theta(-4 + 6\cos\theta) = 0$$

Either $\sin\theta = 0$	OR
$\Rightarrow \theta = 0°$	$\cos\theta = \dfrac{2}{3}$
(Not Acceptable)	$\Rightarrow \theta = \cos^{-1}\left(\dfrac{2}{3}\right)$

Hence, the desired position is at $\theta = \cos^{-1}\left(\dfrac{2}{3}\right)$

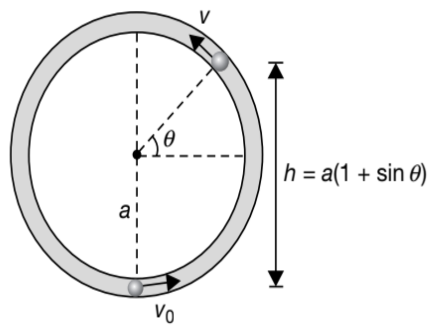
PROBLEM 6

A particle is constrained to move on a smooth vertical circular hoop of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Find the time after which the reaction between the particle and the hoop is zero.

SOLUTION

$$v_0 = \sqrt{4ga}$$

$$v^2 = v_0^2 - 2gh = \left(\sqrt{4ga}\right)^2 - 2ga(1+\sin\theta)$$



$$v^2 = 2ga(1-\sin\theta)$$

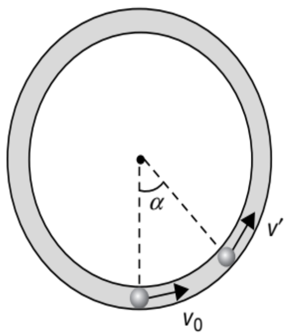
$$N = 0 \text{ at } mg\sin\theta = \frac{mv^2}{a}$$

$$\Rightarrow \quad mg\sin\theta = 2mg(1-\sin\theta)$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow \quad \theta = 41.8° \text{ (with the horizontal)}$$

Now, at angle α, $(v')^2 = v_0^2 - 2ga(1-\cos\alpha)$



$$\Rightarrow \quad (v')^2 = \left(\sqrt{4ga}\right)^2 - 2ga(1-\cos\alpha)$$

$$\Rightarrow \quad (v')^2 = 2ga(1+\cos\alpha) = 4ga\cos^2\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow \quad v' = 2\cos\left(\frac{\alpha}{2}\right)\sqrt{ga}$$

$$\Rightarrow \quad \omega = \frac{v'}{a} = 2\cos\left(\frac{\alpha}{2}\right)\sqrt{\frac{g}{a}}$$

$$\Rightarrow \quad \frac{d\alpha}{dt} = 2\cos\left(\frac{\alpha}{2}\right)\sqrt{\frac{g}{a}}$$

$$\Rightarrow \quad \int_{0^\circ}^{\frac{\pi}{2}+\theta} \sec\left(\frac{\alpha}{2}\right)d\alpha = 2\sqrt{\frac{g}{a}}\int_0^t dt$$

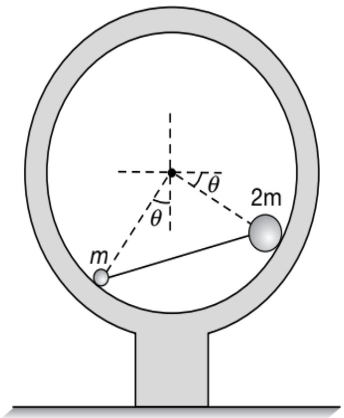
$$\Rightarrow \quad 2\sqrt{\frac{g}{a}}t = 2\ln\left[\tan\left(\frac{\pi}{4}+\frac{\alpha}{4}\right)\right]_{0^\circ}^{\pi/2+\theta}$$

$$\Rightarrow \quad \sqrt{\frac{g}{a}}t = \ln\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{8}+\frac{\theta}{4}\right)\right] - \ln\left[\tan\left(\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \quad t = 1.544\sqrt{\frac{g}{a}}$$

PROBLEM 7

The two particles of mass m and $2m$ respectively are connected by a rigid rod of negligible mass and slide with negligible friction in a circular path of radius r on the inside of the vertical circular ring. If the system is released from rest at $\theta = 0°$ determine



(a) the velocity v of the particles when the rod passes the horizontal position,

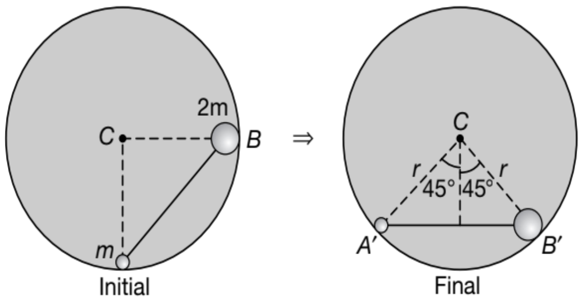
(b) the maximum velocity v_{max} of the particles and

(c) the maximum value of θ.

SOLUTION

(a) We observe that $\angle ACB = 90°$

From constraint relations we conclude that speeds of both the particles will be same.



By Law of Conservation of Mechanical Energy, we get

$$\begin{pmatrix}\text{Decrease in} \\ \text{Potential} \\ \text{Energy of} \\ \text{mass } 2m\end{pmatrix} = \begin{pmatrix}\text{Increase in} \\ \text{Potential} \\ \text{Energy of} \\ \text{mass } m\end{pmatrix} + \begin{pmatrix}\text{Increase in} \\ \text{Kinetic} \\ \text{Energies} \\ \text{of both}\end{pmatrix}$$

$$\Rightarrow \quad 2mg(h_B) = mg(h_A) + \frac{1}{2}(3m)(v^2)$$

$$\Rightarrow \quad 2g(h_B) = g(h_A) + \frac{3}{2}v^2 \qquad \ldots(1)$$

where $h_B = r\cos 45° = 0.707r$

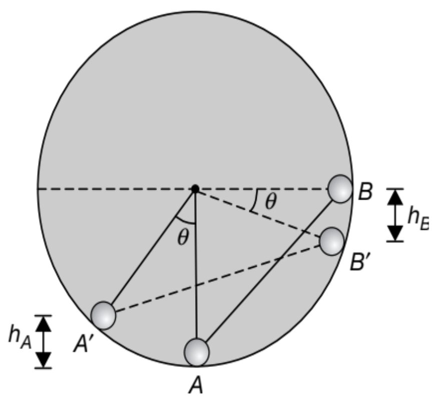
and $h_A = r(1-\cos 45°) = 0.293r$

Substituting the values, we get

$$(2)(0.707)gr = 0.293gr + 1.5v^2$$

$$\Rightarrow \quad v = 0.864\sqrt{gr}$$

(b) At any angle θ, we have



$$h_B = r\sin\theta$$

and $h_A = r(1-\cos\theta)$

Hence equation (1) can be written as

$$2g(r\sin\theta) = gr(1-\cos\theta) + \frac{3}{2}v^2$$

$$\Rightarrow \quad v^2 = \frac{2}{3}gr(2\sin\theta + \cos\theta - 1) \quad \ldots(2)$$

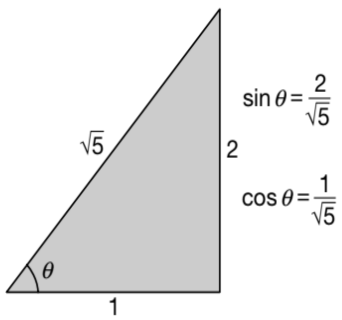
For v to be maximum, we have

$$\frac{d}{d\theta}(v^2)$$

$$\Rightarrow \quad 2\cos\theta - \sin\theta = 0$$

$$\Rightarrow \quad \tan\theta = 2$$

$$\Rightarrow \quad v_{max} = \sqrt{\frac{2}{3}gr(2\sin\theta + \cos\theta - 1)}$$



Substituting the values of θ we get

$$v_{max} = 0.908\sqrt{gr}$$

(c) At $\theta = \theta_{max}$, velocity of both the particles will momentarily be zero. Hence substituting $v^2 = 0$ in equation (2) we get

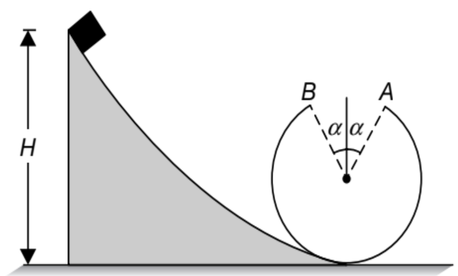
$$2\sin\theta + \cos\theta - 1 = 0$$

Solving this equation, we get

$$\theta_{max} \cong 127°$$

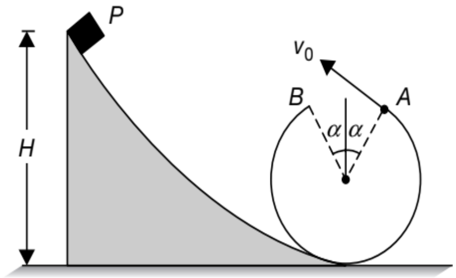
PROBLEM 8

A small object loops a vertical loop of radius R in which a symmetrical section of 2α has been removed. Find the maximum and minimum heights from which the object after loosing contact with the loop at point A and flying through the air, will reach point B. Find the corresponding angles of the section removed for which this is possible.



SOLUTION

Let v_0 be the speed of the object at point A. Between A and B path of the object is a parabola, with



$$AB = \text{Range}$$

$$\Rightarrow \quad 2R\sin\alpha = \frac{2v_0^2\sin\alpha\cos\alpha}{g}$$

$$\Rightarrow \quad v_0^2 = \frac{gR}{\cos\alpha} \quad \ldots(1)$$

By Law of Conservation of Mechanical Energy applied at P and A, we get

$$mgH = mgR(1+\cos\alpha) + \frac{1}{2}mv_0^2$$

$$\Rightarrow \quad \frac{H}{R} = 1 + \cos\alpha + \frac{v_0^2}{2gR}$$

Since $\frac{v_0^2}{gR} = \frac{1}{\cos\alpha}$ {from equation (1)}

$$\Rightarrow \quad \frac{H}{R} = 1 + \cos\alpha + \frac{1}{2\cos\alpha} = k \quad \{\text{say}\}$$

$$\Rightarrow \quad 2\cos^2\alpha - 2(k-1)\cos\alpha + 1 = 0$$

$$\Rightarrow \quad \cos^2\alpha - (k-1)\cos\alpha + \frac{1}{2} = 0$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2}\left(k - 1 \pm \sqrt{(k-1)^2 - 2}\right)$$

For roots to be real, the discriminant

$$(k-1)^2 - 2 \geq 0$$

$$\Rightarrow \quad k - 1 \geq \sqrt{2}$$

$$\Rightarrow \quad k \geq (1 + \sqrt{2}) \quad \ldots(2)$$

Also, we know that

$$\cos\alpha \leq 1$$

$$\Rightarrow \quad \frac{1}{2}\left((k-1) + \sqrt{(k-1)^2 - 2}\right) \leq 1$$

$\Rightarrow \quad (k-1)+\sqrt{(k-1)^2-2} \le 2$

$\Rightarrow \quad \sqrt{(k-1)^2-2} \le 2-(k-1)$

$\Rightarrow \quad (k-1)^2-2 \le (2-(k-1))^2$

$\Rightarrow \quad 4k \le 10$

$\Rightarrow \quad k \le 25$ ...(3)

Hence from (2) and (3), we get

$1+\sqrt{2} \le k \le 2.5$

$\Rightarrow \quad (1+\sqrt{2})R \le H \le 2.5R$ $\quad \left\{\text{as } k = \frac{H}{R}\right\}$

$\Rightarrow \quad 2.414R \le H \le 2.5R$

For the limiting values of cosines we have for, $k = 1+\sqrt{2}$

$\Rightarrow \quad k-1 = \sqrt{2}$

$\Rightarrow \quad \cos\alpha_1 = \frac{\sqrt{2}}{2}$

i.e., $\alpha_1 = 45°$ and for, $k = 2.5$

$\Rightarrow \quad k-1 = 1.5$

$\Rightarrow \quad \cos\alpha_2 = \frac{(1.5 \pm 0.5)}{2}$

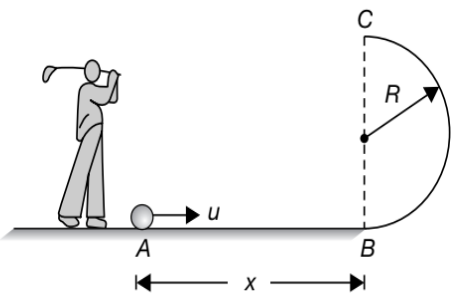
$\Rightarrow \quad \cos\alpha_2 = 0.5$ and $\cos\alpha_3 = 1$

$\Rightarrow \quad \alpha_2 = 60°$ and $\alpha_3 = 0°$

So, $45° \le \alpha \le 60°$

PROBLEM 9

A person rolls a small ball with speed u along the floor from point A. If $x = 3R$, determine the required speed u so that the ball returns to A after rolling on the circular surface in the vertical plane from B to C and becoming a projectile at C. What is the minimum value of x for which the game could be played if contact must be maintained at point C. Neglect friction.

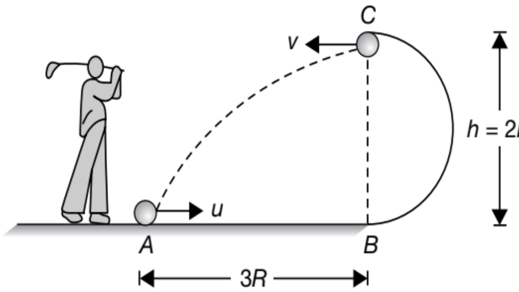


SOLUTION

(i) Let v be the velocity at the highest point, then

$$v^2 = u^2 - 2gh$$

$$\Rightarrow \quad v^2 = u^2 - 4gR \qquad \text{...(1)}$$



After point C, path of the ball becomes projectile with initial velocity in horizontal direction. Hence substituting in

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta} \qquad \text{...(2)}$$

So, we get

$$-2R = 3R\tan(0°) - \frac{g(3R)^2}{2(u^2-4gR)\cos^2(0°)}$$

$$\Rightarrow \quad 2R = \frac{9gR^2}{2(u^2-4gR)}$$

$$\Rightarrow \quad 4u^2 - 16gR = 9gR$$

$$\Rightarrow \quad u^2 = \frac{25}{4}gR$$

$$\Rightarrow \quad u = \frac{5}{2}\sqrt{gR} \qquad \left\{u > \sqrt{5gR}\right\}$$

(ii) Minimum value of v to maintain contact at C is $\sqrt{gR}$. Hence substituting $v = \sqrt{gR}$ in equation (2), we get

$$-2R = x\tan(0°) - \frac{gx^2}{2(gR)\cos^2(0°)}$$

$$\Rightarrow \quad 4gR^2 = gx^2$$

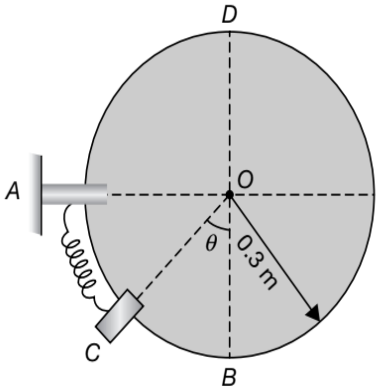
$$\Rightarrow \quad x = 2R$$

Hence minimum value of x is

$$x_{\min} = 2R$$

PROBLEM 10

A thin circular rod is supported in a vertical plane by a bracket at A. Attached to the bracket and loosely wound around the rod is a spring of constant $k = 40\ \text{Nm}^{-1}$ and underformed length equal to arc of the circle AB. A $0.2\ \text{kg}$ collar C, not attached to the spring, can slide without friction along the rod. The collar is released from rest at an angle θ with the vertical.



(a) Make the equation for minimum value of θ for which the collar will pass through D and reach point A.

(b) Determine the velocity of collar as it reaches point A for minimum value of θ. (Take $g = 10\ \text{ms}^{-2}$).

SOLUTION

(a) Compression of the spring in position C is

$$x = CB = R\theta = 0.3\theta \qquad \{R = 0.3\ \text{m}\}$$

The height difference between C and D is

$$h = R(1+\cos\theta) = 0.3(1+\cos\theta)$$

By Law of Conservation of Mechanical Energy, we get

$$\frac{1}{2}kx^2 = mgh$$

$$\Rightarrow \frac{1}{2}(40)(0.3\theta)^2 = (0.2)(10)(0.3)(1+\cos\theta)$$

$$\Rightarrow \theta^2 = \frac{1}{3}(1+\cos\theta)$$

$$\Rightarrow 3\theta^2 = 1+\cos\theta$$

(b) For the above angle, velocity of collar is zero at point D. The height difference between A and D is

$$h = R = 0.3\ \text{m}$$

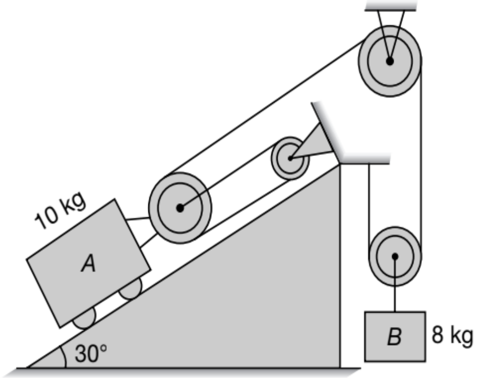
So, velocity of collar at point A is

$$v = \sqrt{2gh} = \sqrt{2(10)(0.3)}$$

$$\Rightarrow v = 2.45\ \text{ms}^{-1}$$

PROBLEM 11

If the system is released from rest, determine the speeds of both masses after B has moved 1 m. Neglect friction and masses of the pulleys. $(g = 10\ \text{ms}^{-2})$.



SOLUTION

From constraint relations we observe that

$$3v_A = 2v_B$$

$$\Rightarrow v_B = 1.5v_A$$

$$\Rightarrow v_A = \frac{2}{3}v_B$$

$$\Rightarrow s_A = \frac{2}{3}s_B$$

Now as the block B moves $\frac{1}{2}$ m vertically downwards, block A will move $\frac{2}{3}$ m along the plane or $\frac{2}{3}\sin(30^\circ)$ vertically upwards.

By Law of Conservation of Mechanical Energy, we get

$$\begin{pmatrix}\text{Decrease}\\ \text{in PE}\\ \text{of block } B\end{pmatrix} = \begin{pmatrix}\text{Increase}\\ \text{in PE}\\ \text{of block } A\end{pmatrix} + \begin{pmatrix}\text{Increase}\\ \text{in KE of both}\\ \text{the blocks}\end{pmatrix}$$

$$\Rightarrow (8)(g)(1) = (10)(g)\left(\frac{2}{3}\sin 30^\circ\right) + \frac{1}{2}(8)(v_B^2) + \frac{1}{2}(10)(v_A)^2$$

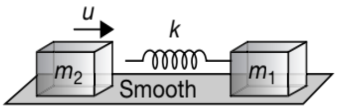
$$\Rightarrow 80 = \frac{100}{3} + \frac{1}{2}(8)(1.5v_A)^2 + \frac{1}{2}(10)v_A^2$$

$$\Rightarrow \quad \frac{140}{3} = 11v_A^2$$

$$\Rightarrow \quad v_A \cong 2 \text{ ms}^{-1} \text{ and } v_B \cong 3 \text{ ms}^{-1}$$

PROBLEM 12

A block of mass m_1 is resting on a smooth horizontal surface. A light spring of spring constant k is attached to the block. Another block of mass m_2 is moving with a speed u towards the stationary block as shown in Figure. Calculate the maximum compression in the spring.



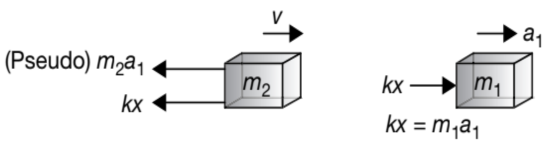
SOLUTION

Let us consider the motion of block m_2 as seen by m_1. As the block m_2 approaches m_1, it compresses the spring and the spring opposes the motion of m_2. When the speed of m_2 becomes zero relative to m_1, then compression in the spring is maximum.

At any instant, let the compression in the spring be x, then at this instant, the block m_1 has an acceleration a_1 given by

$$a_1 = \frac{kx}{m_1}, \text{ towards the right.}$$

The free body diagram of m_2 as seen from the reference frame attached to m_1 (i.e. a non-inertial frame) is shown in Figure.



Let speed of m_2 relative to m_1 be v. The force F acting on m_2 at this instant is given by

$$F = m_2a_1 + kx = m_2\left(\frac{kx}{m_1}\right) + kx$$

$$\Rightarrow \quad F = \left(\frac{m_1 + m_2}{m_1}\right)kx$$

Work done by this force on m_2 is given by

$$W = \int_0^x F dx \cos(180°) = -\int_0^x \left(\frac{m_1 + m_2}{m_1}\right)kx dx$$

$$\Rightarrow \quad W = -\left(\frac{m_1 + m_2}{m_1}\right)\left(\frac{1}{2}kx^2\right)$$

According to Work-Energy Theorem, we have

$$W = \Delta K = \frac{1}{2}m_2v^2 - \frac{1}{2}m_2u^2$$

$$\Rightarrow \quad \frac{1}{2}m_2v^2 - \frac{1}{2}m_2u^2 = -\left(\frac{m_1 + m_2}{m_1}\right)\left(\frac{1}{2}kx^2\right)$$

When $x = x_{max}$, then both the blocks move with the same velocity and hence $v = 0$.

$$\Rightarrow \quad -\frac{1}{2}m_2u^2 = -\left(\frac{m_1 + m_2}{m_1}\right)\left(\frac{1}{2}kx_{max}^2\right)$$

$$\Rightarrow \quad x_{max} = u\sqrt{\left(\frac{m_2m_1}{m_2 + m_1}\right)\frac{1}{k}}$$

where the term $\frac{m_2m_1}{m_2 + m_1}$ is called as the reduced mass μ of the system.

PROBLEM 13

A bungee jumper of mass 50 kg attached to a light elastic cord (bungee cord) of unstretched length L jumps from a tall bridge of height 100 m. The cord first straightens and then extends as the jumper falls. This prevents the jumper from hitting the water. Assume that the bungee cord behaves like a spring with stiffness constant 100 Nm^{-1}.

(a) Calculate the maximum allowed length L of the unstretched bungee cord to keep the jumper alive? (Assume that the stiffness constant of the cord is constant throughout the fall of the bungee jumper)

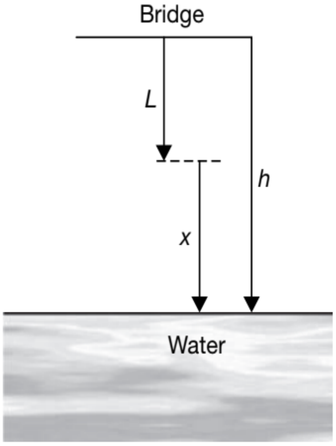
(b) Before the bungee jumper jumps off the bridge, his instructor of same mass verified the stiffness constant of the bungee cord by lowering himself very slowly from the bridge to the full extent of the cord and measured his remaining distance d from the water surface. Calculate d.

Assume air drag to be negligible, no heat is dissipated in the string, only gravitation and spring forces to be present.

SOLUTION

Given that, L is the unstretched length of the cord, $h = 100$ m, $k = 100 \text{ Nm}^{-1}$, $m = 50$ kg. Let x denote the distance through which the bungee jumper has fallen from the bridge

(a) We want to choose L so that the jumper comes to a stop just before touching the water surface. At this moment, the extension in the cord is $x = (h - L)$ and the jumper has fallen through h as shown in Figure.



So, by Law of Conservation of Mechanical Energy, we have loss in gravitational potential energy of the jumper equals the gain in elastic potential energy of the bungee cord.

$$\Rightarrow \; mgh = \frac{1}{2}kx^2$$

$$\Rightarrow \; mgh = \frac{1}{2}k(h-L)^2$$

$$\Rightarrow \; L = h - \sqrt{\frac{2mgh}{k}}$$

$$\Rightarrow \; L \approx 69 \text{ m}$$

(b) The bungee jumper will hang freely at the distance d above the water when the weight of the instructor is balanced by the spring force developed in the cord i.e. at equilibrium. If x_{eq} be the extension in the bungee cord at equilibrium, then we have

$$kx_{eq} = mg$$

where $x_{eq} = h - L - d$

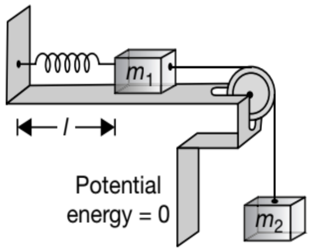
$$\Rightarrow \; k(h-L-d) = mg$$

$$\Rightarrow \; d = h - L - \frac{mg}{k}$$

$$\Rightarrow \; d \approx 26 \text{ m}$$

PROBLEM 14

In the arrangement shown in Figure, two blocks of masses $m_1 = M$ and $m_2 = 2M$ are taken. If $M = 20$ kg, the force constant of the spring (of natural length l_0) is $k = 10 \text{ kNm}^{-1}$, $g = 10 \text{ ms}^{-2}$, the frictional force between M and table is 80 N, the system is held at rest with $l = (l_0 + 10)$ cm and then released. Calculate the



(a) kinetic energy of the sliding block, when it has moved 2 cm from its point of release.

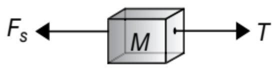
(b) kinetic energy of the sliding block m when it first slides back through the point where the spring is unstretched.

(c) maximum kinetic energy attained by the sliding block while it is sliding from its point of release to the point where the spring is unstretched.

SOLUTION

(a) *Before we arrive at any conclusion, we must first find whether the block of mass M moves to the right or to the left, when released.*

For that purpose, let us draw the FBD of the block M (imagining the friction to be absent) as shown in Figure.



As of now, we do not know the direction of the friction force too. So firstly, let us calculate the net force acting on M in the horizontal direction.

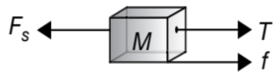
The spring force F_s acting initially on the block is

$$F_s = kx = 10000 \text{ Nm}^{-1} \times 0.1 \text{ m} = 1000 \text{ N}$$

The tension T acting on the block is

$$T = 2Mg = 2 \times 20 \text{ kg} \times 10 \text{ ms}^{-2} = 400 \text{ N}$$

Since, $F_s > T$, so we conclude that the block M goes to the left and hence the block $2M$ will go upwards. The friction on M will act to the right as shown in Figure.



So, the net force that moves the block of M to the right is $F_s - T - f = 1000 - 400 - 80 = 520$ N

According to the Modified Work-Energy Theorem (MWET) applied on the system, we have

$$W_{nc} = \Delta U + \Delta K \quad \ldots(1)$$

where ΔU is the sum of $\Delta U = \Delta U_e + \Delta U_g$

where, $W_{nc} = -(80)\left(\frac{2}{100}\right) = -1.6$ J

For the spring, $k = 10 \text{ kNm}^{-1} = 10000 \text{ Nm}^{-1}$

Initial extension in the spring is

$$x_1 = 10 \text{ cm} = 0.1 \text{ m}$$

So, the initial elastic energy stored in the spring is

$$(U_e)_i = \frac{1}{2}kx_1^2 = 50 \text{ J}$$

Final extension in the spring is

$$x_2 = (10 - 2) \text{ cm} = 0.08 \text{ m}$$

So, the final elastic energy stored in the spring is

$$(U_e)_f = \frac{1}{2}kx_2^2 = 32 \text{ J}$$

$$\Rightarrow \Delta U_e = 32 - 50 = -18 \text{ J}$$

The initial gravitational potential energy (GPE) is assumed to be zero at the initial position of mass $m_2 = 2M$. When the mass $2M$ moves up by 2 cm its GPE increases by

$$(\Delta U_g)_{2M} = m_2 gh = (40)(10)\left(\frac{2}{100}\right) = 8 \text{ J}$$

However, there is no change in the GPE of the mass m_1.

So, $\Delta U_g = 8$ J

The initial kinetic energy of both the blocks is zero, so

$$K_i = 0$$

Finally, if v is the speed of each block, then

$$K_f = \frac{1}{2}(M + 2M)v^2 = \frac{3}{2}Mv^2$$

Since, we know that

$$W_{nc} = (\Delta U_e + \Delta U_g) + \Delta K$$

$$\Rightarrow -1.6 = (-18 + 8) + \frac{3}{2}Mv^2$$

$$\Rightarrow \frac{3}{2}Mv^2 = 8.4$$

$$\Rightarrow \frac{1}{2}Mv^2 = \frac{8.4}{3} = 2.8 \text{ J}$$

(b) When the spring attains its natural length after the block $m_2 = 2M$ moves up by $x_1 = 0.1$ m i.e. when $l = l_0$, then the elastic potential energy stored in the spring is zero. So, $U_f = 0$

Since, we know that

$$W_{nc} = \Delta U + \Delta K$$

where, $\Delta U = \Delta U_e + \Delta U_g$

$$\Delta U_g = (2M)gx_1 = 2(200)(0.1) = 40 \text{ J}$$

$$\Delta U_e = 0 - \frac{1}{2}kx_1^2 = -50 \text{ J}$$

$$\Delta K = \frac{1}{2}(M + 2M)v^2 = \frac{3}{2}Mv^2 \text{ and}$$

$$W_{nc} = -fx_1 = -(80)(0.1) = -8 \text{ J}$$

Substituting the values in above equation, we get

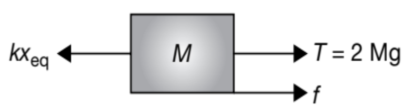
$$\frac{3}{2}Mv^2 + (40 - 50) = -8$$

$$\Rightarrow \frac{3}{2}Mv^2 = 2$$

$$\Rightarrow \frac{1}{2}Mv^2 = \frac{2}{3} \text{ J}$$

Hence, the kinetic energy of the sliding block when it first slides back through the point where the spring is unstretched is $\frac{2}{3}$ J

(c) Maximum kinetic energy is attained by the sliding block when it passes through the mean position (where net force on m_1 is zero), because at the mean position potential energy is minimum, so kinetic energy is maximum.



$$\Rightarrow kx_{eq} = 2Mg + f$$

$$\Rightarrow x_{eq} = \frac{2Mg + f}{k}$$

$$\Rightarrow\ x_{eq}=\frac{2(20)(10)+80}{k}$$

$$\Rightarrow\ x_{eq}=\frac{480}{10000}\text{ m}=0.048\text{ m}$$

Again applying, $W_{nc}=\Delta U+\Delta K$

where, $\Delta U=\Delta U_e+\Delta U_g$ and

$$\Delta U_g=-2Mg\left(0.1-x_{eq}\right)=-400(0.1-0.048)$$

$$\Rightarrow\ \Delta U_g=-400(0.052)=-20.8\text{ J}$$

$$\Delta U_e=\frac{1}{2}k\left(x_{eq}^2-x_1^2\right)$$

$$\Rightarrow\ \Delta U_e=\frac{1}{2}(10000)\left[(0.048)^2-(0.1)^2\right]$$

$$\Rightarrow\ \Delta U_e=-38.48\text{ J}$$

Since $W_{nc}=-f\left(0.1-x_{eq}\right)=-80(0.1-0.048)$

$$\Rightarrow\ W_{nc}=-4.16\text{ J and}$$

$$\Delta K=\frac{1}{2}(M+2M)v_{\max}^2=\frac{3}{2}v_{\max}^2$$

Substituting these values in $W_{nc}=\Delta U+\Delta K$, we get

$$-4.16=(-38.48-20.8)+3\left(\frac{1}{2}Mv_{\max}^2\right)$$

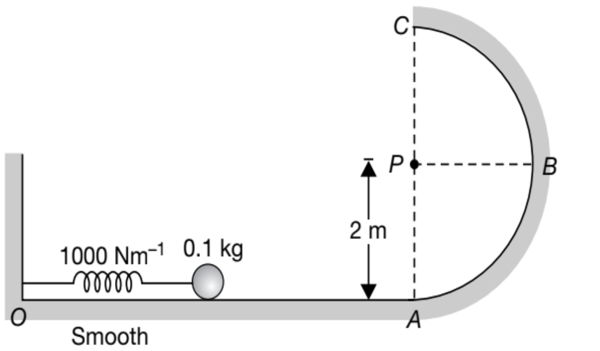
$$\Rightarrow\ 3\left(\frac{1}{2}Mv_{\max}^2\right)=55.12$$

$$\Rightarrow\ \frac{1}{2}Mv_{\max}^2=\frac{55.12}{3}\approx 18.4\text{ J}$$

Hence, the maximum kinetic energy attained by the sliding block while it is sliding from its point of release to the point where the spring is unstretched is 18.4 J

PROBLEM 15

A small ball of mass 0.1 kg is placed on a smooth plane surface OA which acquires a semi-circular shape ABC of radius 2 m. The ball just touches a light spring of stiffness 1000 Nm^{-1} The ball is pushed to the left so as to compress the spring by a distance x and is then released. This ball then starts moving towards the circular track ABC as shown in Figure. Take $g=10\text{ ms}^{-2}$.



(a) Calculate the minimum work done by external agent to push the ball to the left through 50 cm.

(b) If the ball is pushed to the left by 5 cm and released, then calculate the normal force on the ball just after crossing A and the maximum angle made by the ball with PA on the circular track when it comes to rest.

(c) Find the minimum distance $x_{\min}$ through which the ball should be pushed to the left and released so that it can reach upto the point C.

(d) If the ball is pushed to left through $0.7x_{\min}$, then calculate the reaction force between ball and track at point B and the maximum height attained by the ball above horizontal surface OA.

SOLUTION

(a) When the work done by external force to push the ball against the spring is minimum, there must be no kinetic energy associated with the ball. The work done is solely responsible for increasing the potential energy of the spring. So, we have

$$W=\frac{1}{2}kx^2$$

$$\Rightarrow\ W=\frac{1}{2}\times 1000\times\left(\frac{5}{100}\right)^2=1.25\text{ J}$$

(b) When the ball leaves the spring, it moves to the right with a speed v, such that the potential energy of the spring is converted to kinetic energy of the ball.

$$\Rightarrow\ \frac{1}{2}mv^2=\frac{1}{2}kx^2$$

$$\Rightarrow\ v=x\sqrt{\frac{k}{m}}$$

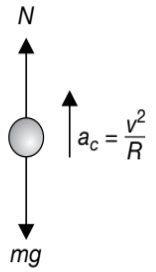
Since, $k=1000\text{ Nm}^{-1}$, $m=0.1\text{ kg}$, $x=5\text{ cm}$

$$\Rightarrow \; v = 0.05\times\sqrt{\frac{1000}{0.1}} = 5 \text{ ms}^{-1}$$

When the ball crosses the point A, then its path becomes circular and it experiences a centripetal acceleration

$$a_c = \frac{v^2}{r}$$

Applying Newton's Second Law, we get



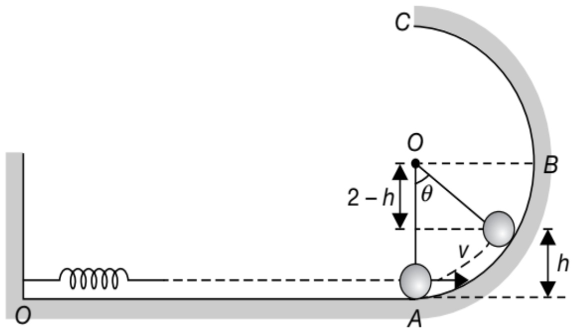
$$N - mg = ma_c$$

$$\Rightarrow \; N = mg + \frac{mv^2}{R}$$

$$\Rightarrow \; N = (0.1)(10) + \frac{(0.1)(5)^2}{2}$$

$$\Rightarrow \; N = 1 + 1.25 = 2.25 \text{ N}$$

As the ball rises on the track, its gravitational potential energy increases and kinetic energy decreases. When the ball comes to rest, then its kinetic energy becomes zero. Let this happen at a height h as shown in Figure.



Then by Law of Conservation of energy, we get

$$mgh + 0 = 0 + \frac{1}{2}mv^2$$

$$\Rightarrow \; h = \frac{v^2}{2g} = \frac{5^2}{2\times 10} = \frac{25}{20} = 1.25 \text{ m}$$

Since, $\cos\theta = \frac{2-h}{2} = \frac{2-1.25}{2} = \frac{0.75}{2} = \frac{3}{8}$

$$\Rightarrow \; \theta = \cos^{-1}\left(\frac{3}{8}\right)$$

(c) Let the spring be compressed by $x_{\min}$ and the speed of the ball after leaving contact with spring be v, then by Law of Conservation of Energy, we have

$$\frac{1}{2}mv^2 = \frac{1}{2}kx_{\min}^2$$

$$\Rightarrow \; v = x_{\min}\sqrt{\frac{k}{m}}$$

Now, to complete the circle, minimum speed required by the ball at A should be $\sqrt{5gR} = 10 \text{ ms}^{-1}$.

$$\Rightarrow \; x_{\min}\sqrt{\frac{k}{m}} = \sqrt{5gR}$$

$$\Rightarrow \; x_{\min} = 0.1 \text{ m}$$

(d) When the spring is compressed by $0.7x_{\min} = 0.07$ m, then the speed acquired by the ball while crossing the point A is

$$v = v_A = x\sqrt{\frac{k}{m}}$$

$$\Rightarrow \; v_A = 0.07\times\sqrt{\frac{1000}{0.1}} = 7 \text{ ms}^{-1}$$

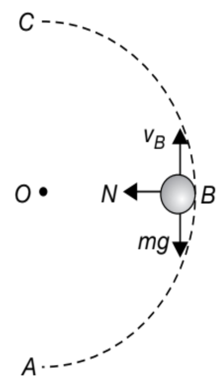
As the ball reaches B, its speed becomes v_B such that by Law of Conservation of Energy, we get

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_B^2 + mgR$$

$$\Rightarrow \; v_B = \sqrt{v^2 - 2gR} = \sqrt{49-40}$$

$$\Rightarrow \; v_B = 3 \text{ ms}^{-1}$$

The FBD of ball at B is shown in Figure.



The radial acceleration of the ball at this instant is

$$a_c = \frac{v_B^2}{R} = \frac{9}{2}\ \text{ms}^{-2}$$

Applying Newton's Second Law, we get

$$N = \frac{mv_B^2}{R} = \frac{(0.1)9}{2} = 0.45\ \text{N}$$

Now as we see that the speed of the ball at the lowest point A is $7\ \text{ms}^{-1}$ which is less than $\sqrt{5gR} = 10\ \text{ms}^{-1}$, so the ball will leave the circular track say at D and follow a parabolic path. Let OD make an angle θ with OC and let v_D be the speed of the ball at this position. If h be the height of the ball at D, then

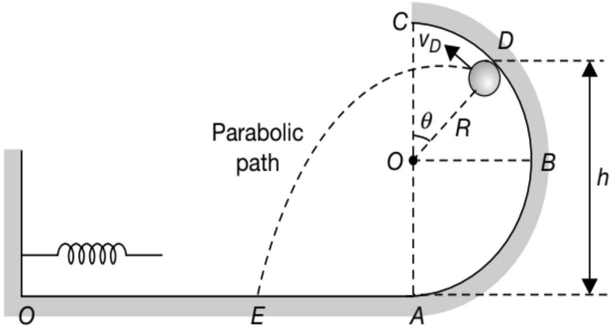
$$h = R(1+\cos\theta)$$

By Law of Conservation of Mechanical Energy, we get

$$\frac{1}{2}mv_A^2 - \frac{1}{2}mv_D^2 = mgR(1+\cos\theta)$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mv_D^2 + mgR(1+\cos\theta) \quad \ldots(1)$$

FBD of ball at D, is shown in Figure.



As ball leaves the track at D, normal reaction is zero and the component of weight i.e. $mg\cos\theta$ acting towards the centre O provides the necessary centripetal force to the particle. So, applying Newton's Second Law in radial direction, we get

$$mg\cos\theta = \frac{mv_D^2}{R}$$

$$\Rightarrow v_D^2 = gR\cos\theta \quad \ldots(2)$$

Substituting equation (2) in (1), we get

$$\frac{1}{2}mv^2 = \frac{1}{2}mgR\cos\theta + mgR(1+\cos\theta)$$

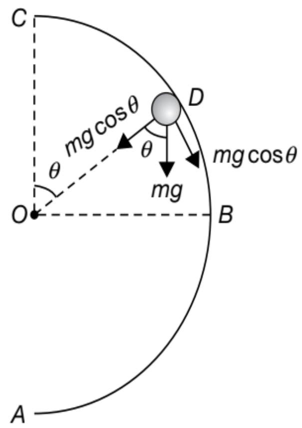
$$\Rightarrow v^2 - 2gR = 3gR\cos\theta$$

$$\Rightarrow \cos\theta = \frac{v^2-2gR}{3gR} = \frac{7^2-2(10)(2)}{3(10)(2)} = \frac{9}{60}$$

$$\Rightarrow \cos\theta = \frac{3}{20}$$

$$\Rightarrow h = R(1+\cos\theta) = R\left(1+\frac{3}{20}\right) = \frac{23R}{20}$$

$$\Rightarrow h = \frac{46}{20} = 2.3\ \text{m}$$



From point D, the ball moves in a parabolic path under the action of gravity alone. We can also think that the ball becomes an oblique projectile at the point D, where it is launched with a velocity $v_D = \sqrt{gR\cos\theta}$ making an angle of θ with the horizontal. The height H (from this point onwards) to which this oblique projectile rises further is given by

$$H = \frac{v_D^2\sin^2\theta}{2g}$$

$$\Rightarrow H = \frac{gR\cos\theta\sin^2\theta}{2g} = \frac{R}{2}(\cos\theta\sin^2\theta)$$

Since $\cos\theta = \frac{3}{20}$

$$\Rightarrow \sin^2\theta = 1-\cos^2\theta = 1-\frac{9}{400} = \frac{391}{400}$$

$$\Rightarrow H = \left(\frac{2}{2}\right)\left(\frac{3}{20}\right)\left(\frac{391}{400}\right) = \frac{1173}{8000}\ \text{m}$$

Maximum height h' above OA is

$$h' = 2.3 + \frac{1173}{8000} = 2.45\ \text{m}$$

PROBLEM 16

A small body of mass m is located on a rough horizontal plane at the point O. The body is given a horizontal velocity u. Calculate the power developed by the friction force during the whole time of motion, if the friction coefficient between the block and the surface is μ. Also calculate the maximum power developed by the friction force, if the friction coefficient varies with distance x from the point O as $\mu = \alpha x$, where α is a positive constant.

SOLUTION

The frictional force acting on the body

$$f_k = \mu mg$$

Retardation provided by this force is

$$a = \mu g$$

If t is the total time taken by the body to come to rest, then

$$0 = u + (-a)t$$

$$\Rightarrow \quad t = \frac{u}{\mu g}$$

Magnitude of change in kinetic energy due to friction is

$$|\Delta K| = \left|0 - \frac{1}{2}mu^2\right| = \frac{1}{2}mu^2$$

Power developed by friction during the whole time of motion, i.e. the average power is given by

$$\text{Average Power} = \frac{\text{Change in kinetic energy}}{\text{Total time of motion}}$$

$$\Rightarrow \quad P_{av} = \frac{|\Delta K|}{t}$$

$$\Rightarrow \quad P_{av} = \frac{\frac{1}{2}mu^2}{\left(\frac{u}{\mu g}\right)} = \frac{1}{2}\mu mug$$

When friction coefficient is $\mu = \alpha x$, the friction force on the body when it is at a distance x from the point O is

$$f_k = \alpha x mg$$

Retardation due to this force is

$$a = -\alpha g x$$

$$\Rightarrow \quad v\frac{dv}{dx} = -\alpha g x$$

$$\Rightarrow \quad vdv = -\alpha g x dx$$

Integrating the above expression to get the velocity at a distance x from point O, gives

$$\int_u^v vdv = -\int_0^x \alpha g x dx$$

$$\Rightarrow \quad \frac{v^2 - u^2}{2} = -\frac{\alpha g x^2}{2}$$

$$\Rightarrow \quad v = \sqrt{u^2 - \alpha g x^2}$$

Instantaneous power P due to friction force at a distance x from point O is

$$P = \vec{F}\cdot\vec{v}$$

$$\Rightarrow \quad P = P_{ins} = -\alpha mgx\sqrt{\left(u^2 - \alpha g x^2\right)} \qquad \ldots(1)$$

This power is maximum when $\frac{dP}{dx} = 0$

$$\Rightarrow \quad \frac{dP}{dx} = \frac{(\alpha mgx)(\alpha gx)}{\sqrt{u^2 - \alpha g x^2}} - \alpha mg\sqrt{u^2 - \alpha g x^2} = 0$$

$$\Rightarrow \quad x = \frac{u}{\sqrt{2\alpha g}} \qquad \ldots(2)$$

Equation (2) gives the value of x at which instantaneous power is maximum. Substituting the above value of x in Equation (1), we get maximum power to be

$$P_{max} = -\frac{\alpha mug}{\sqrt{2\alpha g}}\sqrt{u^2 - \frac{u^2}{2}}$$

$$\Rightarrow \quad P_{max} = -\frac{1}{2}mu^2\sqrt{\alpha g}$$

PROBLEM 17

A small box of mass m is kept on a fixed, smooth sphere of radius R at a position where the radius through the box makes an angle of $30°$ with the vertical. The box is released from this position. Calculate the force exerted by the sphere on the box just after the release. Also calculate the distance travelled by the box before it leaves contact with the sphere. Given that $\cos^{-1}(0.577) \approx 54°$

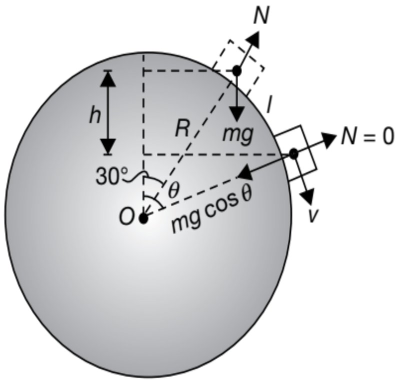
SOLUTION

At the instant of releasing the box, the speed of box is zero. Hence the box will push the sphere with a force that equals the component of its weight acing towards the centre of the sphere i.e. $mg\cos\theta$. So, we have

$$N = mg\cos(30^\circ)$$

$$\Rightarrow \quad N = \frac{\sqrt{3}}{2}mg$$

Let the box loose contact with the sphere at an angle θ from the vertical as shown in Figure.



At this instant its normal reaction becomes zero, and hence at this point, we have

$$\frac{mv^2}{R} = mg\cos\theta$$

$$\Rightarrow \quad v = \sqrt{gR\cos\theta} \qquad \ldots(1)$$

Velocity of particle (v) at this point can be given by applying the Law of Conservation of Energy (as it has fallen a distance h), according to which we have

$$\begin{pmatrix}\text{Loss in GPE}\\ \text{of the Body}\end{pmatrix} = \begin{pmatrix}\text{Gain in KE}\\ \text{of the Body}\end{pmatrix}$$

$$\Rightarrow \quad mgh = \frac{1}{2}mv^2$$

where, $h = R(\cos 30^\circ - \cos\theta) = R\left(\frac{\sqrt{3}}{2} - \cos\theta\right)$

$$\Rightarrow \quad v = \sqrt{2gh} = \sqrt{2gh\left(\frac{\sqrt{3}}{2} - \cos\theta\right)} \qquad \ldots(2)$$

Equating (1) and (2), we get

$$\cos\theta = \frac{\sqrt{3} - 2\cos\theta}{2}$$

$$\Rightarrow \quad 3\cos\theta = \sqrt{3}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \cos^{-1}(0.577)$$

$$\Rightarrow \quad \theta = 54^\circ$$

Angular displacement of the box before leaving the sphere is

$$\Delta\theta = 54^\circ - 30^\circ = 24^\circ$$

Since, $180^\circ = \pi$ radian

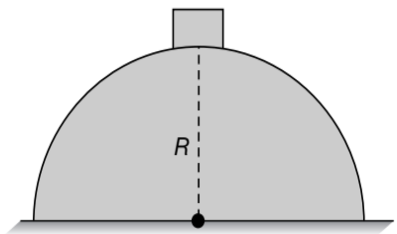
$$\Rightarrow \quad \Delta\theta = 24 \times \frac{\pi}{180} = \frac{2\pi}{15}$$

So, distance travelled l by the box is given by

$$l = R\Delta\theta = R\left(\frac{2\pi}{15}\right) = \frac{2\pi R}{15}$$

PROBLEM 18

A smooth surface hemisphere is fixed on a ground as shown in figure. From the topmost point of it, a small block starts sliding with no initial velocity. Calculate the distance s between the centre of base circle of hemisphere and the point where particle strikes the ground. Also find the time taken by the particle to strike the ground.



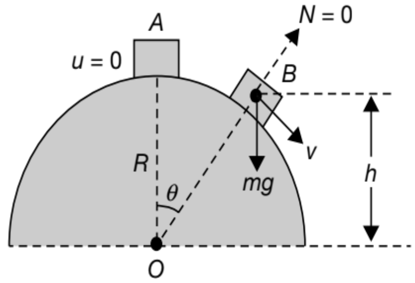
SOLUTION

Let the block leave the contact with hemisphere at point B, then at B, we have

$$N = 0$$

$$\Rightarrow \quad mg\cos\phi = \frac{mv_B^2}{R}$$

$$\Rightarrow \quad v_B = \sqrt{gR\cos\phi} \qquad \ldots(1)$$



Applying Law of Conservation of Energy between A and B we get,

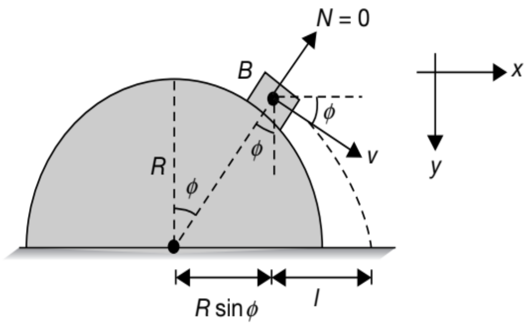
$$0 + mgR = \frac{1}{2}mv_B^2 + mgh$$

Substituting v_B from (1) and $h = R\cos\phi$, we get

$$v_B = \sqrt{\frac{2}{3}gR}$$

$$h = \frac{2R}{3} \text{ from the bottom and}$$

$$\cos\phi = \frac{2}{3}$$



After the block leaves contact with the hemisphere, it leaves the circular path and follows a parabolic path, because at the instant of leaving the hemisphere, the block has a velocity $v = \sqrt{\frac{2}{3}gR}$ and makes an angle ϕ below the horizontal.

If t is the time taken by the block to hit the ground, then considering the origin at the point B, we get

$$l = (v\cos\phi)t \quad \ldots(1)$$

$$R\cos\phi = (v\sin\phi)t + \frac{1}{2}gt^2 \quad \ldots(2)$$

$$\Rightarrow \quad t^2 + \left(\frac{2v\sin\phi}{g}\right)t - \frac{2R\cos\phi}{g} = 0 \quad \ldots(3)$$

where, $v = \sqrt{\frac{2}{3}gR}$, $\cos\phi = \frac{2}{3}$ and $\sin\phi = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

$$\Rightarrow \quad \frac{2v\sin\phi}{g} = \frac{2}{g}\sqrt{\frac{2gR}{3}}\frac{\sqrt{5}}{3} = \sqrt{\frac{40R}{27g}} = \frac{2}{3}\sqrt{\frac{10R}{3g}}$$

So, equation (3) becomes

$$t^2 + \left(\frac{2}{3}\sqrt{\frac{10R}{3g}}\right)t - \frac{4R}{3g} = 0$$

Solving this quadratic equation, we get

$$t = -\frac{1}{3}\sqrt{\frac{10R}{3g}} + \sqrt{\frac{10R}{27g} + \frac{4R}{3g}}$$

$$t = -\frac{1}{3}\sqrt{\frac{10R}{3g}} + \frac{1}{3}\sqrt{\frac{46R}{3g}}$$

$$\Rightarrow \quad t = \frac{1}{3}\sqrt{\frac{R}{3g}}\left(\sqrt{46} - \sqrt{10}\right)$$

So, the distance s is given by

$$s = R\sin\phi + l = R\sin\phi + (v\cos\phi)t$$

$$\Rightarrow \quad s = \frac{\sqrt{5}R}{3} + \left(\sqrt{\frac{2gR}{3}}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\sqrt{\frac{R}{3g}}\left(\sqrt{46} - \sqrt{10}\right)\right)$$

$$\Rightarrow \quad s = \frac{R}{3}\left(\sqrt{5} + \frac{4}{9}\left(\sqrt{23} - \sqrt{5}\right)\right)$$

$$\Rightarrow \quad s = \frac{R}{27}\left(4\sqrt{23} + 5\sqrt{5}\right)$$

Method-2 (to find s)

Speed of particle at point B, where it leaves the hemisphere is

$$v = \sqrt{\frac{2}{3}gR}$$

For an oblique projectile launched with a speed at an angle ϕ below the horizontal, the equation of trajectory is given by

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$$

Since the launch angle is below the horizontal, so $\theta = -\phi$

$$\Rightarrow \quad y = -x\tan\phi - \frac{gx^2}{2v^2\cos^2\phi} \quad \ldots(4)$$

Since, at the point where the block leaves the hemisphere, we have

$$\cos\phi = \frac{2}{3}$$

$$\Rightarrow \quad \sin\phi = \frac{\sqrt{5}}{3}$$

$$\Rightarrow \quad \tan\phi = \frac{\sqrt{5}}{2}$$

Substituting these values in equation (4), we get

$$y = -\frac{\sqrt{5}}{2}x - \frac{gx^2}{2\left(\frac{2gR}{3}\right)\left(\frac{4}{9}\right)}$$

$$\Rightarrow \quad y = \left(-\frac{\sqrt{5}}{2}\right)x - \left(\frac{27}{16R}\right)x^2 \qquad \ldots(5)$$

where, $y = -R\cos\phi = -\frac{2R}{3}$ and $x = l$

Substituting these values in equation (5), we get

$$-\frac{2}{3}R = -\frac{\sqrt{5}}{2}l - \frac{27l^2}{16R}$$

$$\Rightarrow \quad l^2 + \frac{8\sqrt{5}}{27}lR - \frac{32}{81}R^2 = 0$$

$$\Rightarrow \quad l = -\frac{4\sqrt{5}}{27}R + \frac{R}{2}\sqrt{\frac{64\times 5}{27\times 27} + \frac{4\times 32}{27\times 3}\times\frac{9}{9}}$$

$$\Rightarrow \quad l = -\frac{4\sqrt{5}R}{27} + \frac{4R}{27}\sqrt{5+18}$$

$$\Rightarrow \quad l = \frac{4R}{27}\left(\sqrt{23}-\sqrt{5}\right)$$

Distance s will be given by

$$s = R\sin\phi + l$$

$$\Rightarrow \quad s = \frac{\sqrt{5}}{3}R + \frac{4R}{27}\left(\sqrt{23}-\sqrt{5}\right)$$

$$\Rightarrow \quad s = \frac{R}{27}\left(5\sqrt{5} + 4\sqrt{23}\right)$$

PROBLEM 19

A particle of mass m approaches a region of force starting from $r \to +\infty$. The potential energy function in terms of distance r from the origin is given by,

$$U(r) = \begin{cases} \frac{K}{2a^3}\left(3a^2 - r^2\right), & \text{for } 0 \le r \le a \\ \frac{K}{r}, & \text{for } r \ge a \end{cases}$$

Calculate the force $F(r)$ acting on the particle and find out whether the force is repulsive or attractive in nature. Also, calculate the minimum velocity with which a particle should be launched from infinity towards the origin, such that it just crosses the origin.

Assuming that the particle is launched from infinity towards the origin with a speed $\sqrt{\frac{2K}{ma}}$, then describe the motion of the particle.

SOLUTION

Since, we know that $F = -\frac{dU}{dr}$

For $0 \le r \le a$, we have

$$U = \frac{K}{2a^3}\left(3a^2 - r^2\right)$$

$$\Rightarrow \quad F = -\frac{Kr}{a^3}$$

For, $r \ge a$, we have

$$U = \frac{K}{r}$$

$$\Rightarrow \quad F = \frac{k}{r^2}$$

In both the regions i.e. $0 \le r \le a$ and $r \ge a$, we see that on increasing r, the potential energy decreases. Hence force is repulsive

Let the particle be launched from infinity towards the origin with a velocity v_{min}. Then by Law of Conservation of Mechanical Energy, the total mechanical energy at infinity must be equal to the total mechanical energy at the origin O.

$$\Rightarrow \quad U_\infty + K_\infty = U_O + K_O$$

where $U_\infty = 0$, $K_\infty = \frac{1}{2}mv_{\text{min}}^2$, $U_O = \frac{3K}{2a}$, $K_O = 0$

$$\Rightarrow \quad 0 + \frac{1}{2}mv_{\text{min}}^2 = \frac{3K}{2a} + 0$$

$$\Rightarrow \quad v_{\text{min}} = \sqrt{\frac{3K}{ma}}$$

So, for the particle to just cross O, the minimum velocity it should be given at infinity is $v_{\min} = \sqrt{\frac{3K}{ma}}$

However, if the particle is imparted a speed less than the minimum speed, then instead of reaching O, the particle will go closest to O and then starts travelling back towards infinity.

Since $v = \sqrt{\frac{2K}{ma}} < v_{\min}$

Hence the particle will not be able to cross the origin.

PROBLEM 20

A car starts from rest in a circular flat road of radius R with an acceleration a. The friction coefficient between the road and the tyres is μ. Find the distance car will travel before it start skidding.

SOLUTION

During the circular motion, the car will skid when net force acting on it exceeds the limiting friction. Since, tangential acceleration of car is a, so the speed of car after travelling a distance s is given by

$$v = \sqrt{2as}$$

Corresponding to this speed v of the car, the centripetal acceleration is

$$a_C = \frac{v^2}{R} = \frac{2as}{R}$$

Net acceleration (a_{net}) of the car is

$$a_{\text{net}} = \sqrt{a_T^2 + a_C^2} = \sqrt{a^2 + \left(\frac{2as}{R}\right)^2}$$

Net force acting on car is given by

$$F_{\text{net}} = ma_{\text{net}} = m\sqrt{a^2 + \left(\frac{2as}{R}\right)^2}$$

For the car to skid, this net force must be greater than the limiting friction $f_l = \mu N = \mu mg$. Hence, we have

$$m\sqrt{a^2 + \frac{4a^2s^2}{R^2}} \geq \mu mg$$

$$\Rightarrow \quad s = \sqrt{\left(\mu^2 g^2 - a^2\right)\frac{R^2}{4a^2}}$$