CHAPTER

6 Newton's Laws of Motion

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Newton's Laws of Motion
(b) Pseudo Force
(c) Friction
(d) Dynamics of Circular Motion

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

DYNAMICS: AN INTRODUCTION

In kinematics, we studied the motion of a particle, with emphasis on motion along a straight line and motion in a plane and simply described it in terms of vectors $\vec{r}$, $\vec{v}$ and $\vec{a}$. Now comes the time when we shall be discussing about the cause producing motion. This treatment, happens to be an aspect of mechanics, known as **dynamics**. In this chapter, we shall be primarily discussing the forces along with their respective nature and properties that account for the motion of a body. As done before, the bodies will be treated as if they were single particles. However, in the later chapters, we shall be extending the properties of this chapter to discuss the motion of a group of particles and extended bodies as well. So, this branch of Physics dealing with motion along with the cause producing motion is called **Dynamics**.

FORCE

Force is a pull or push which changes or tends to change the state of rest or of uniform motion or direction of motion of any object. Force is the interaction between the object and the source (providing the pull or push). It is a vector quantity. Its SI unit is newton (N) and cgs unit is dyne

$$1\text{ N} = 10^5\text{ dyne}$$

A force acting on a body

(a) may change only speed.
(b) may change only direction of motion.
(c) may change both the speed and direction of motion.
(d) may change size and shape of a body.

Unit of Force

SI unit of force is newton (N) and $1\text{ N} = 1\text{ kgms}^{-2}$
cgs unit of force is dyne (dyn) and $1\text{ dyne} = 1\text{ gcms}^{-2}$
Also, $1\text{ newton} = 10^5\text{ dyne}$.

The dimensional formula of force is $[MLT^{-2}]$

Another commonly used unit of force is **kilogram force (kgf).** It is the force with which a body of mass 1 kg is attracted towards the centre of the earth. So 1 kgf can also be thought as the weight corresponding to a body of mass 1 kg is the weight

$$1 \text{ kgf} = 9.8 \text{ N} \text{ and } 1 \text{ gf} = 980 \text{ dyne}$$

Conceptual Note(s)

A force is a vector quantity. Since we know that a vector quantity is a quantity which has both magnitude and direction. To fully describe the force acting upon an object, you must describe both the magnitude (size or numerical value) and the direction. Thus, 10 N, is not a full description of the force acting upon an object. In contrast, 10 N, downwards is a complete description of the force acting upon an object because, both the magnitude (10 N) and the direction (downwards) are given.

NEWTON'S LAWS OF MOTION

Newton gave three laws of motion, based on which motion associated with a particle can be explained easily.

(a) The First Law or The Law of Inertia

(b) The Second Law $\left(\vec{F} = m\vec{a}\right)$

(c) The Third Law or The Action - Reaction Law

NEWTON'S FIRST LAW OR LAW OF INERTIA

A body continues to maintain its state of rest or of uniform motion or of direction unless and until some external unbalanced force acts on it. If there is an interaction between the body and objects present in the environment, the effect may be to change the "natural" state of the body's motion.

EXAMPLES:

(a) When a bus suddenly starts/stops, the passengers tends to move backward/forward.

(b) A carpet is beaten with a stick to remove its dust.

(c) A coin kept on a cardboard on a glass tumbler, on striking hard the cardboard, the coin falls into the glass tumbler.

(d) Athlete runs some distance before making a jump.

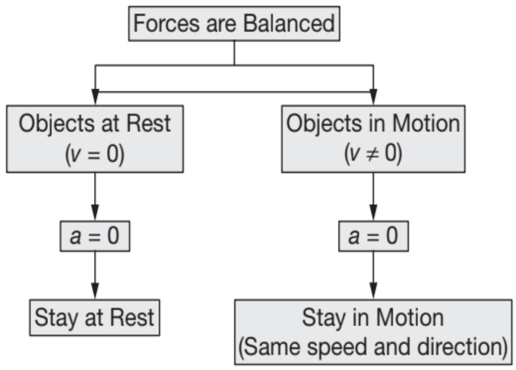
In other words, an object at rest tend to stay at rest and an object in motion tend to stay in motion with the same speed and in the same direction unless acted upon by an unbalanced force.

The behaviour of all objects can be described by saying that objects tend to keep on doing what they are doing (unless acted upon by an unbalanced force). There are two parts of the statement to Newton's First Law.

(a) one which predicts the **behaviour of stationary objects** and

(b) the other which predicts the **behaviour of moving objects**.

These two parts are summarized in the following diagram.



Conceptual Note(s)

It has been a common observation that every object **resists** any effort to change its velocity (both in magnitude and direction). This property expressing the degree of insusceptibility of a body to any change in its velocity is called the **property of inertia**. Different bodies reveal this property in different degrees. A measure of inertia is provided by the quantity called **mass**. A body possessing a greater mass has more inertia i.e., if equal forces are applied on two different bodies, the body with the greater mass possesses smaller acceleration.

MOMENTUM $(\vec{p})$

It is observed that more the mass of the body, the more is the momentum possessed by it. Similarly, the more the velocity possessed by a body, the more is its momentum. In simpler words, actually the momentum is a measure of the amount of motion possessed by a body. It is defined as the **product of mass and velocity** or **momentum** is just equal to the **mass times velocity**. The momentum is always directed along the velocity of the body whose momentum is being measured. Mathematically,

$$\vec{p} = m\vec{v}$$

It is a vector quantity with SI unit $kgms^{-1}$ and dimensional formula MLT^{-1}.

NEWTON'S SECOND LAW OF MOTION

The external force applied on a body is equal to the rate of change of momentum of a body.

$$\vec{F}_{ext} = \vec{F} = \frac{d\vec{p}}{dt}$$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

For constant mass system(s)

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

For variable mass system(s)

$$\vec{F} = \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt}$$

Special Case

If $\vec{v} = \text{constant}$, then $\frac{d\vec{v}}{dt} = \vec{0}$

$$\Rightarrow \quad |\vec{F}| = v\frac{dm}{dt}$$

$$\text{Thrust } = |\vec{F}| = \left|v\frac{dm}{dt}\right|$$

e.g., Rocket propulsion

ILLUSTRATION 1

The velocity of a particle of mass 2 kg is given by $\vec{v} = at\hat{i} + bt^2\hat{j}$. Find the force acting on the particle.

SOLUTION

From Second Law of Motion

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

$$\Rightarrow \quad F = 2\frac{d}{dt}\left(at\hat{i} + bt^2\hat{j}\right)$$

$$\Rightarrow \quad \vec{F} = 2a\hat{i} + 4bt\hat{j}$$

Conceptual Note(s)

(a) The Second Law is obviously consistent with the First Law as $F = 0$ implies $a = 0$.

(b) The Second Law of motion is a vector law. It is actually a combination of three equations, one for each component of the vectors.

$$F_x = \frac{dp_x}{dt} = ma_x$$

$$\Rightarrow \quad F_y = \frac{dp_y}{dt} = ma_y$$

$$\Rightarrow \quad F_z = \frac{dp_z}{dt} = ma_z$$

This means that if a force is not parallel to the velocity of the body, but makes some angle with it, it changes only the component of velocity along the direction of force. The component of velocity normal to the force remains unchanged.

(c) The Second Law of motion given above is strictly applicable to a single point mass. The force F in the law stand for the net external force on the particle and a stands for the acceleration of the particle. Any internal forces in the system are not to be included in F.

NEWTON'S THIRD LAW OF MOTION

To every action there is equal and opposite reaction **and both must act on two different bodies**. In other words, all interaction forces exist in pair, a pair which has two forces of equal magnitude, opposite direction and acting on different bodies along a straight line connecting them. Also the two forces comprising the pair have the same nature.

i.e., $\vec{F}_{AB} = -\vec{F}_{BA}$ ...(1)

$\vec{F}_{AB}$ = Force on body A due to B

$\vec{F}_{BA}$ = Force on body B due to A

From (1), we get

$$|\vec{F}_{AB}| = |\vec{F}_{BA}|$$

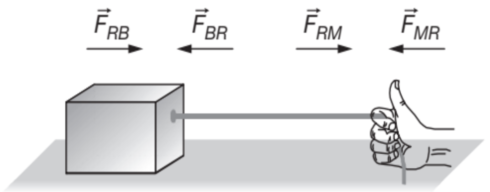
$$\Rightarrow \quad m_A a_A = m_B a_B$$

$$\Rightarrow \quad a_B = \frac{m_A a_A}{m_B}$$

If $m_A \ll m_B$, then $a_B \to 0$ (Think of apple-earth example)

EXAMPLE:

A block is being pulled by means of a rope. I have characterised some action-reaction pairs and have shown them in figures. Please see these figures carefully.



We also have, $\vec{F}_{BR} = -\vec{F}_{RB}$ and $\vec{F}_{RM} = -\vec{F}_{MR}$

Conceptual Note(s)

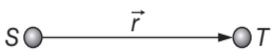
Action reaction forces always act on different bodies and their line of action is the same.

FUNDAMENTAL FORCES IN NATURE

All the forces observed in nature such as muscular force, tension, reaction, friction, elastic, weight, electric, magnetic, nuclear etc., can be explained in terms of only following four basic interactions.

Gravitational Force and Properties

The force of interaction which exists between two particles due to their masses is called gravitational force. The gravitational force between two masses m_1 (called the source mass S) and m_2 (called the test mass T) separated by a distance r is given by

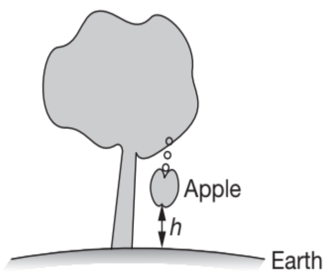


$$\vec{F} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

where, $\vec{r}$ is position vector of test particle T with respect to source particle S and G is universal gravitational constant whose value is given by $G = 6.67 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}$.

The negative sign tells about the attractive nature of the gravitational force as the test mass is always attracted towards the source mass (i.e. opposite to $\vec{r}$).

(i) It is the weakest force and is always attractive.
(ii) It is a long range force as it acts between any two particles situated at any distance in the universe.
(iii) Gravitational force between two bodies is independent of the presence of other bodies and it is also independent of the nature of intervening medium (i.e. medium present between the bodies).
(iv) It is a central force i.e. it acts along the line joining the centres of the two bodies.
(v) It is negligible for lighter bodies, however it dominates in case of planetary bodies and their motion.
(vi) Gravitons are exchange particles between two bodies and are responsible for the gravitational interaction between them.



Consider an apple to be falling freely as shown in figure. When it is at a height h, force between earth and apple is given by

$$F = \frac{GmM_e}{r^2} = \frac{GmM_e}{(R_e + h)^2}$$

where M_e is the mass of earth, R_e is the radius of earth. It acts towards earth's centre. Now rearranging above result, we get

$$F = m\left(\frac{GM_e}{R_e^2}\right)\left(\frac{R_e}{R_e + h}\right)^2$$

$$\Rightarrow \quad F = mg\left(\frac{R_e}{R_e + h}\right)^2 \qquad \left\{\because g = \frac{GM_e}{R_e^2}\right\}$$

For $h \ll R_e$, we have $\dfrac{R_e}{R_e + h} \simeq 1$

$\Rightarrow \quad F = mg$

This is the force exerted by earth on any particle of mass m near the earth surface.

The value of $g = 9.81 \text{ ms}^{-2} \simeq 10 \text{ ms}^{-2} \simeq \pi^2 \text{ ms}^{-2} \simeq 32 \text{ fts}^{-2}$. It is also called acceleration due to gravity near the surface of earth.

Electromagnetic Force

It includes the electrical and the magnetic forces which may be attractive or repulsive. Photons are exchange particles for electromagnetic interactions. So, force exerted by one particle on the other because of the electric charge on the particles is called electromagnetic force.

Following are the main characteristics of electromagnetic force

(a) These can be attractive or repulsive.
(b) These are long range forces.
(c) These depend on the nature of medium between the charged particles.
(d) All macroscopic forces (except gravitational) which we experience as push or pull or by contact are electromagnetic, i.e., tension in a rope, the force of friction, normal reaction, muscular force, and force experienced by a deformed spring are electromagnetic forces. These are manifestations of the electromagnetic attractions and repulsions between the atoms or molecules.

Nuclear Force

An attractive force, it is the force between two nucleons. It is the strongest force that binds all the nucleons in a tiny volume (corresponding to radius of order of 1 fermi) inspite of large electric repulsion between protons. Mesons are the exchange particles responsible for nuclear interactions. It acts within the nucleus that too upto a very small distance. Radioactivity, fission and fusion etc. result because of unbalancing of nuclear forces.

Weak Force

It exists between elementary particles and is responsible for the change in nucleus . Under its action a neutron can change into proton during negative beta decay by emitting an electron and a particle called antineutrino. It is not the same force as gravitational, electromagnetic or nuclear force. The range of weak force is very small, in fact much smaller than the size of a proton or a neutron.

It has been found that for two protons at a distance of 1 fermi

$$F_N : F_{EM} : F_W : F_G :: 1 : 10^{-2} : 10^{-7} : 10^{-38}$$

CLASSIFICATION OF FORCES ON THE BASIS OF CONTACT

On the basis of contact, forces can be classified as

Field Force

Force which acts on an object at a distance by the interaction of the object with the field produced by other object is called **Field Force or Action at a Distance Force**.

EXAMPLES:

(a) Gravitation force
(b) Electromagnetic force

Contact Force

Forces which are transmitted between bodies by short range atomic molecular interactions are called contact forces. When two objects come in contact they exert contact forces on each other.

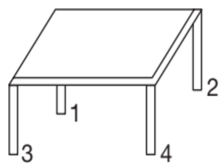
EXAMPLES:

Normal Force, Mechanical Force, Force of Friction

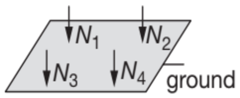
Normal force (N) or Normal Reaction (N)

It is the component of contact force perpendicular to the surface. It measures how strongly the surfaces in contact are pressed against each other. It is the electromagnetic force.

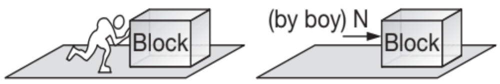
A table is placed on Earth as shown in figure.



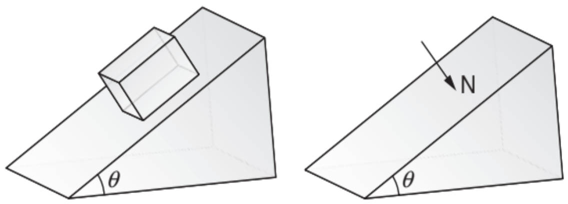
Here table presses the earth so normal force exerted by four legs of table on earth are as shown in figure.



Now a boy pushes a block kept on a frictionless surface. Here, force exerted by boy on block is electromagnetic interaction which arises due to similar charges appearing on finger and contact surface of block, it is normal force.



A block is kept on inclined surface. Component of its weight presses the surface perpendicularly due to which contact force acts between surface and block. Normal force exerted by block on the surface of inclined plane is shown in figure. Force acts perpendicular to the surface.



Frictional Force (*f*)

A contact force, it arises due to a contact between the surfaces of the two bodies and comes into being when there is a relative motion between the surfaces in contact. Actually, it is the component of the reaction force tangential to the surface on which the body is kept.

SYSTEM

Two or more than two objects which interact with each other form a system. The classification of forces on the basis of boundary of system are given.

Internal Forces

Forces acting each with in a system among its constituents.

External Forces

Forces exerted on the constituents of a system by the outside surroundings are called as external forces.

Real Force

Force which acts on an object due to other object is called as real force.

An isolated object (far away from all objects) does not experience any real force.

CONCEPT OF IMPULSE AND IMPULSE AS AREA UNDER $\vec{F}$-t GRAPH

Whenever a large force $(\vec{F})$ acts on a body for an extremely small time (say dt), then we introduce the concept of Impulse $(\vec{I})$. So, impulse is just defined as the product of the large force with the small time.

$\Rightarrow \quad d\vec{I} = \vec{F}dt$

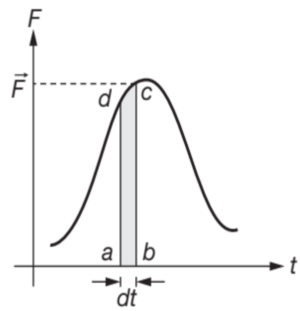
Impulse is also defined as the integral of force with respect to time.

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}dt$$

Since force is a vector and time is a scalar, the result of the integral in above equation is a vector. If the force is constant (both in magnitude and direction), it may be removed from the integral so that the integral is reduced to

$$I = F\int_{t_i}^{t_f} dt = F\left(t_f - t_i\right) = F\Delta t$$

Graphically, the impulse is the area between the force curve and the $F = 0$ axis, as shown in figure.



The SI unit of impulse is Ns.

If more than one force are acting on a particle, then the net impulse is given by the time integral of the net force.

$$\vec{I}_{\text{net}} = \int_{t_i}^{t_f} \vec{F}_{\text{net}} dt$$

For example: A cricketer hitting a ball to score a six. The bat hits the ball with the large force for a very short time.

If $\vec{F}$ be the large force acting on a body for a small time dt, then

$$d\vec{I} = \vec{F}dt$$

$$\Rightarrow \quad \vec{I} = \int d\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt$$

Since $|d\vec{I}| = \text{Area } abcd = \vec{F}dt$

$\Rightarrow \quad |\vec{I}| = \int \vec{F}dt =$ Area under a curve in a $\vec{F}$-t graph.

ILLUSTRATION 2

Find the impulse due to the force $\vec{F} = a\hat{i} + bt\hat{j}$, where $a = 2$ N and $b = 4$ Ns^{-1}, if this force acts from $t_i = 0$ to $t_f = 0.3$ s.

SOLUTION

$$\vec{I} = \int_{t_i}^{t_f} \vec{F}dt = \int_0^{0.3} \left(a\hat{i} + bt\hat{j}\right)dt$$

$$\Rightarrow \quad \vec{I} = a\hat{i}\int_0^{0.3} dt + b\hat{j}\int_0^{0.3} tdt$$

$$\Rightarrow \quad \vec{I} = at\hat{i}\Big|_0^{0.3} + \frac{bt^2}{2}\hat{j}\Big|_0^{0.3}$$

$$\Rightarrow \quad \vec{I} = (2)(0.3)\hat{i} + \frac{(4)(0.3)^2}{2}\hat{j}$$

$$\Rightarrow \quad \vec{I} = \left(0.6\hat{i} + 0.18\hat{j}\right) \text{ Ns}$$

ILLUSTRATION 3

A ball falling with velocity $\vec{v}_i = \left(-0.65\hat{i} - 0.35\hat{j}\right)$ ms^{-1} is subjected to a net impulse $\vec{I} = \left(0.6\hat{i} + 0.18\hat{j}\right)$ Ns^{-1}. If the ball has a mass of 275 g, calculate its velocity immediately following the impulse.

SOLUTION

Using Impulse - Momentum Theorem

$$m\vec{v}_f - m\vec{v}_i = \vec{I}$$

$$\Rightarrow \quad \vec{v}_f = \vec{v}_i + \frac{\vec{I}}{m}$$

Thus, $\vec{v}_f = -0.65\hat{i} - 0.35\hat{j} + \dfrac{0.6\hat{i} + 0.18\hat{j}}{0.275}$

$$\Rightarrow \quad \vec{v}_f = \left(-0.65\hat{i} - 0.35\hat{j}\right) + \left(2.18\hat{i} + 0.655\hat{j}\right)$$

$$\Rightarrow \quad \vec{v}_f = \left(1.53\hat{i} + 0.305\hat{j}\right) \text{ ms}^{-1}$$

ILLUSTRATION 4

An 150 g ball is thrown at 30 ms^{-1}. It is struck by a bat, which gives it a velocity of 40 ms^{-1} in the opposite direction. If the time of contact is 10^{-2} s, what is the average force on the ball?

SOLUTION

If we choose the original direction as +x-axis, then

$$\Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = m\left(-40\hat{i} - 30\hat{i}\right)$$

The average force is

$$F_{av} = \frac{\Delta p}{\Delta t} = \frac{-0.15 \text{ kg} \times 70\hat{i}}{10^{-2}} = -1050\hat{i} \text{ N}$$

Notice that this is much larger than the weight (1.5 N) of the ball.

IMPULSE – MOMENTUM THEOREM

According to Newton's Second Law, we have

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \quad d\vec{p} = \vec{F}dt$$

$$\Rightarrow \quad \int_i^f d\vec{p} = \int_i^f Fdt = \vec{I}$$

$$\Rightarrow \quad \vec{I} = \vec{p}_{\text{final}} - \vec{p}_{\text{initial}}$$

$\Rightarrow$ Impulse = Change in momentum

{called **Impulse Momentum Theorem**}

$$\Rightarrow \quad \vec{I} = m(\vec{v} - \vec{u})$$

Conceptual Note(s)

In the following cases shown, the change in momentum is calculated.

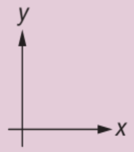
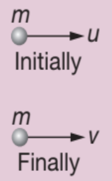
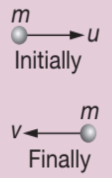
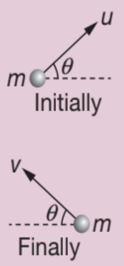
		Change in Momentum $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$
CASE-1		$\Delta p = m(v-u)$, along $+x$ direction
CASE-2		$\Delta p = -m(v+u)$ $\Rightarrow \lvert\Delta p\rvert = m(v+u)$, along $-x$ direction
CASE-3		$\Delta p_x = m(v_x - u_x)$ $\Rightarrow \Delta p_x = -m(v\cos\phi + u\cos\theta)$, along $-x$ direction $(\Delta p)_y = m(v_y - u_y)$ $\Rightarrow (\Delta p)_y = m(v\sin\phi - u\sin\theta)$, along $+y$ direction. So, $\Delta\vec{p} = (\Delta p_x)\hat{i} + (\Delta p_y)\hat{j}$

ILLUSTRATION 5

A box is put on a scale which is adjusted to read zero, when the box is empty. A stream of pebbles is then poured into the box from a height h above its bottom at the rate of n pebbles per second. Each pebble has a mass m. If the pebbles collide with the box such that they immediately come to rest after the collision. Find the reading at time t after which the pebbles begin to fill the box.

SOLUTION

Velocity (v) of each pebble just before striking the box is $v = \sqrt{2gh}$

$$\begin{pmatrix}\text{Reading}\\\text{of the}\\\text{scale at}\\\text{time } t\end{pmatrix} = \begin{pmatrix}\text{Sum of}\\\text{weights of}\\\text{pebbles}\\\text{collected}\\\text{in time } t\end{pmatrix} + \begin{pmatrix}\text{The additional}\\\text{force exerted}\\\text{by the pebbles}\\\text{due to a}\\\text{change in}\\\text{momentum}\end{pmatrix}$$

Total weight of pebbles in time t is $(mg)(nt)$

Change in momentum of 1 pebble is $m\left(\sqrt{2gh}\right)$

Change in momentum (Δp) of nt pebbles is

$$\Delta p = \left(m\sqrt{2gh}\right)(nt)$$

$$\Rightarrow \text{Additional force} = \frac{\Delta p}{t} = mn\sqrt{2gh}$$

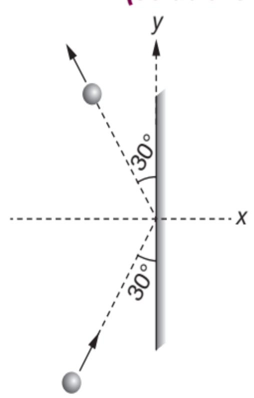
$$\Rightarrow \text{Total force} = mn\left(gt + \sqrt{2gh}\right)$$

Test Your Concepts-I

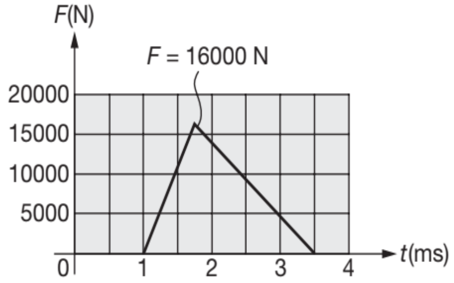
Based on Impulse Momentum

(Solutions on page H.193)

1. A steel ball of mass 3 kg strikes a wall with a speed of 10 ms^{-1} at an angle of 30° with the surface. It bounces off with the same speed and angle as shown. If the ball is in contact with the wall for 200 ms, calculate the average force exerted by the wall on the ball.



2. A ball of mass 150 g is dropped from a height of 20 m. It rebounds from the floor to reach a height of 5 m. What impulse was given to the ball by the floor? Take $g = 10\ \text{ms}^{-2}$.
3. An estimated force-time curve for a ball struck by a bat is shown in figure. From the curve, determine.
   (a) the impulse delivered to the ball
   (b) the average force exerted on the ball and
   (c) the peak force exerted on the ball.



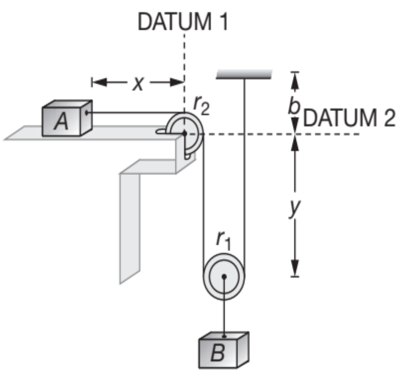
4. A garden hose is originally full of motionless water. What additional force is necessary to hold the nozzle stationary after the water flow is turned on, if the discharge rate is $0.6\ \text{kgs}^{-1}$ with a speed of $25\ \text{ms}^{-1}$?
5. A professional driver of mass 60 kg performs a dive from a platform 10 m above the water surface. Find the magnitude of the average impact force experienced by him if the impact time is 1 s on collision with the water surface. Assume that the velocity of the diver just after entering the water surface is $4\ \text{ms}^{-1}$. Take $g = 9.8\ \text{ms}^{-2}$.

CONSTRAINED MOTION OF CONNECTED PARTICLES

Sometimes it is observed that the motions of particles are interrelated because of the constraints imposed by interconnecting members. In such cases it becomes necessary for us to account for these constraints in order to determine the respective motions of the particles.

ONE DEGREE OF FREEDOM

Let us consider a simple system of two interconnected particles A and B shown in figure. It is quite evident by inspection that the horizontal motion of A is twice the vertical motion of B. However, we will use this example to illustrate the method of analysis which applies to more complex situations where the results cannot be easily obtained by inspection.



The motion of B is clearly the same as that of the center of its pulley, so we establish position coordinates y and x measured from a convenient fixed **datum** (a horizontal or a vertical fixed axis). The total length of the rope is

$$x + \frac{\pi r_2}{2} + 2y + \pi r_1 + b = L \qquad \ldots(1)$$

Since L, r_2, r_1, and b are all constant, so the first and second time derivatives of the equation (1) give

$$\dot{x} + 2\dot{y} = 0$$

$$\Rightarrow \quad v_A + 2v_B = 0 \qquad \ldots(2)$$

$$\Rightarrow \quad \dot{v}_A + 2\dot{v}_B = 0 \text{ or } \ddot{x} + 2\ddot{y} = 0$$

$$\Rightarrow \quad a_A + 2a_B = 0 \qquad \ldots(3)$$

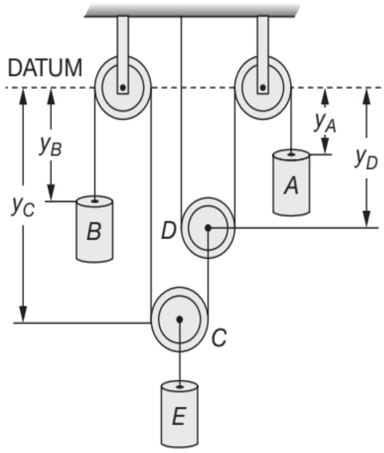
The velocity and acceleration constraint equations indicate that, for the coordinates selected, the velocity of A must have a sign which is opposite to that of the velocity of B, and similarly for the accelerations. The constraint equations are valid for the motion of the system in either direction. We emphasize that $v_A = \dot{x}$ is positive to the left and that $v_B = \dot{y}$ is positive down. The system shown is said to have one degree of freedom since only one variable, either x or y, is needed to specify the positions of all parts of the system.

TWO DEGREES OF FREEDOM

Consider a system, shown, having two degrees of freedom. Here the positions of the lower cylinder E and pulley C depend on the separate specifications of the two coordinates y_A and y_B. The lengths of the cables (measured from DATUM), attached to cylinders A and B can be written, respectively, as

$$y_A + 2y_D + \text{constant} = L_A \quad \ldots(1)$$

$$y_B + y_C + (y_C - y_D) + \text{constant} = L_B \quad \ldots(2)$$



and their time derivatives are

$$\dot{y}_A + 2\dot{y}_D = 0 \text{ and } \dot{y}_B + 2\dot{y}_C - \dot{y}_D = 0$$

$$\ddot{y}_A + 2\ddot{y}_D = 0 \text{ and } \ddot{y}_B + 2\ddot{y}_C - \ddot{y}_D = 0$$

Eliminating the terms in $\dot{y}_D$ and $\ddot{y}_D$ gives

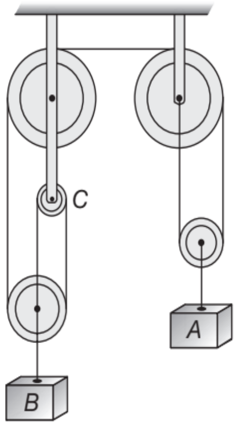
$$\dot{y}_A + 2\dot{y}_B + 4\dot{y}_C = 0 \text{ or } v_A + 2v_B + 4v_C = 0$$

$$\ddot{y}_A + 2\ddot{y}_B + 4\ddot{y}_C = 0 \text{ or } a_A + 2a_B + 4a_C = 0$$

It is clearly impossible for the signs of all three terms to be positive simultaneously. So, for example, if both A and B have downward (positive) velocities, then C will have an upward (negative) velocity.

ILLUSTRATION 6

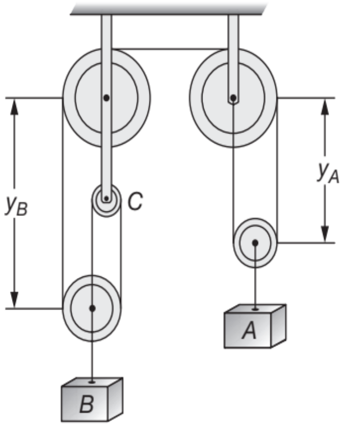
In the pulley configuration shown, cylinder A has a downward velocity of 0.3 ms^{-1}. Determine the velocity of B.



SOLUTION

The centers of the pulleys at A and B are located by the coordinates y_A and y_B measured from fixed positions. The total constant length of cable in the pulley system is

$$L = 3y_B + 2y_A + \text{constants}$$



where the constants account for the fixed lengths of cable in contact with the circumferences of the pulleys, and the constant vertical separation between the two upper left-hand pulleys. Differentiation with time gives

$$0 = 3\dot{y}_B + 2\dot{y}_A$$

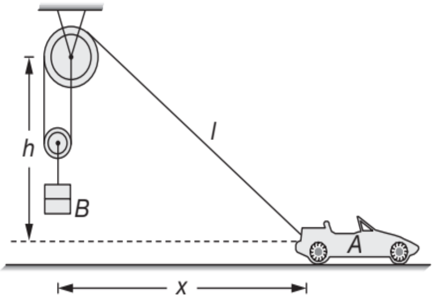
Substitution of $v_A = \dot{y}_A = 0.3$ ms^{-1} and $v_B = \dot{y}_B$ gives

$$0 = 3(v_B) + 2(0.3)$$

$$\Rightarrow \quad v_B = -0.2 \text{ ms}^{-1}$$

ILLUSTRATION 7

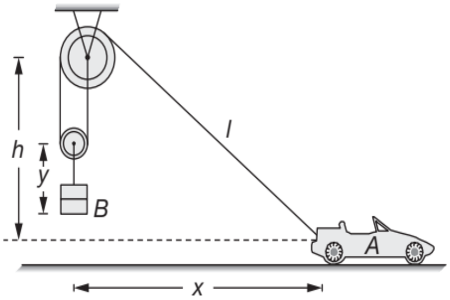
The car A is used to pull a load B with the pulley arrangement shown. If A has a forward velocity v_A, determine an expression for the upward velocity v_B of the load in terms of x.



SOLUTION

We designate the position of the car by the coordinate x and the position of the load by the coordinate y, both measured from a fixed reference. The total constant length of the cable is

$$L = 2(h-y)+l = 2(h-y)+\sqrt{h^2+x^2}$$



Differentiation with time yields

$$0 = -2\dot{y} + \frac{x\dot{x}}{\sqrt{h^2+x^2}}$$

Substituting $v_A = \dot{x}$ and $v_B = \dot{y}$ gives

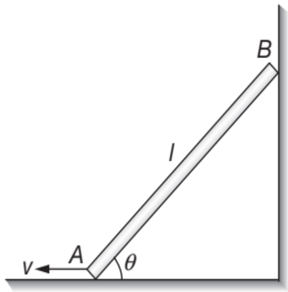
$$v_B = \frac{1}{2}\frac{xv_A}{\sqrt{h^2+x^2}}$$

SIMPLE CONSTRAINT MOTION OF BODIES AND PARTICLES IN TWO DIMENSIONS

Similar to projectile motion, there can be several two dimensional motions, in which the laws of motion can be separately applied to x and y directions and later on the developed relations can be linked for getting the required parameters. Sometime x and y directional motion or any two directions of the motion are related by some specific rule, we call such rules as constraint rules. These rules relate one direction of motion of an object with some other direction of the same object or some other object also.

ILLUSTRATION 8

Figure shows a rod of length l resting on a wall and the floor. Its lower end A is pulled towards left with a constant velocity u. Find the velocity of the other end B downward when the rod makes an angle θ with the horizontal.



SOLUTION

METHOD I

Here if the distance from the corner to the point A is x and that up to B is y. The velocity of point A can then be given by

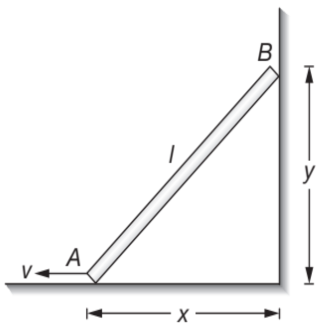
$$v = \frac{dx}{dt}$$

and that of B can be given as

$$v_B = \frac{dy}{dt}$$

Since, we have,

$$x^2 + y^2 = l^2$$



Differentiating with respect to t, we get

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$xv+yv_B=0$$

$$\Rightarrow \quad v_B=-\left(\frac{x}{y}\right)v$$

$$\Rightarrow \quad v_B=-v\cot\theta$$

{negative sign indicates, y decreasing with time}

METHOD II

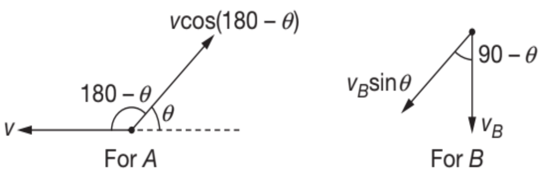
In cases when the relation between two points of a rigid body is required, we can make use of the fact that in a rigid body the distance between two points always remains same. Thus the relative velocity of one point of an object with respect to any other point of the same object in the direction of line joining them will always remain zero, as their separation always remains constant.

Here in above example the distance between the points A and B of the rod always remains constant, thus, the two points must have same velocity components in the direction of their line joining i.e., along the length of the rod.

If point B is moving down with velocity v_B, its component along the length of the rod is $v_B\sin\theta$. Similarly the velocity component of point A along the length of rod is $v\cos\theta$. Thus we have.

$$v_B\sin\theta=v\cos\theta$$

$$v_B=v\cot\theta$$

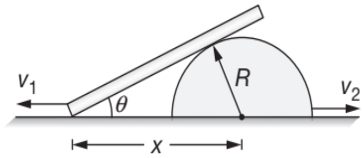


Conceptual Note(s)

In such type of problems, when velocity of one part of a body is given and that of other is required or in cases, when relation in two velocities is required, we first find the relation between the two displacements then differentiate with respect to time.

ILLUSTRATION 9

Figure shows a hemisphere and a supported rod. Hemisphere is moving in right direction with a uniform velocity v_2 and the end of rod which is in contact with ground is moving in left direction with a velocity v_1. Find the rate at which the angle θ is changing in terms of v_1, v_2, R and θ.



SOLUTION

Here x is the separation between centre of hemisphere and the end of rod. Rate of change of x is actually the relative velocity of end of rod and centre of hemisphere i.e., (v_1+v_2). We are required to find the rate of change of θ, i.e., $\frac{d\theta}{dt}$, knowing that $\frac{dx}{dt}=v_1+v_2$

Since, $x=R\operatorname{cosec}\theta$

Differentiating with respect to time we get

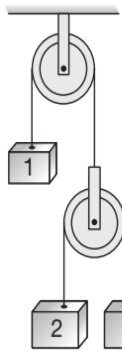
$$\frac{dx}{dt}=-R\operatorname{cosec}\theta\cot\theta\frac{d\theta}{dt}$$

$$\left\{\because \frac{d}{dt}(\operatorname{cosec}\theta)=-\operatorname{cosec}\theta\cot\theta\right\}$$

$$\Rightarrow \quad \frac{d\theta}{dt}=\frac{(v_1+v_2)\sin^2\theta}{R\cos\theta}$$

ILLUSTRATION 10

In the system shown, if a_1, a_2 and a_3 be the respective accelerations of 1, 2 and 3, then find a_1 in terms of a_2 and a_3.



SOLUTION

Let the datum pass from the centre of the fixed pulley. Since the points 1, 2, 3 and 4 are movable, so let their displacements at any instant from the datum be x_1, x_2, x_3 and x_4. We observe that

$$x_1 + x_4 = l_1 \qquad \ldots(1)$$

(length of first string between 1 and 4 is constant)

and $(x_2 - x_4) + (x_3 - x_4) = l_2$

(length of second string between 2 and 3 is constant)

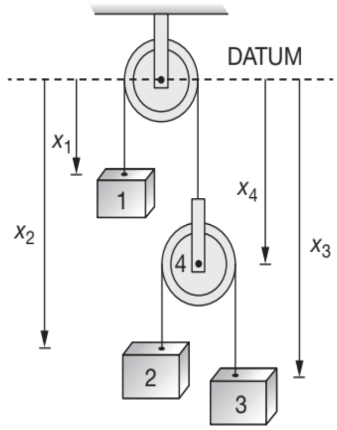
$$\Rightarrow \quad x_2 + x_3 - 2x_4 = l_2 \qquad \ldots(2)$$

Differentiating twice with respect to time, we get

$$\ddot{x}_1 + \ddot{x}_4 = 0 \text{ and } \ddot{x}_2 + \ddot{x}_3 - 2\ddot{x}_4 = 0$$

$$\Rightarrow \quad a_1 + a_4 = 0 \qquad \ldots(3)$$

$$\text{and} \quad a_2 + a_3 - 2a_4 = 0 \qquad \ldots(4)$$



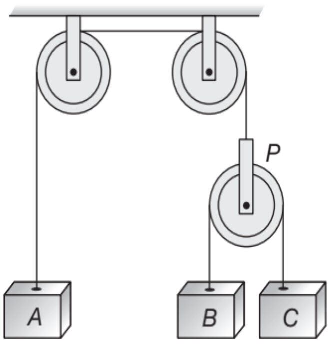
Now, since $a_4 = -a_1$, so we get {from equation (3)}

$$a_2 + a_3 + 2a_1 = 0$$

$$\Rightarrow \quad a_1 = \frac{a_2 + a_3}{2} \quad \text{(in magnitude)}$$

ILLUSTRATION 11

In the arrangement shown, the three blocks move with constant velocities. Knowing that the relative velocity of A with respect to C is 300 mms^{-1} upwards and that the relative velocity of B with respect to A is 200 mms^{-1} downwards, find the velocity of each block.



SOLUTION

Let v be the velocity of block A (upwards) and v_r be the velocity of block B with respect to moving pulley P (upwards).

Taking upward direction as positive, we have

$$v_A = +v,\ v_B = v_r - v \text{ and } v_C = -(v_r + v)$$

Now it is given that,

$$v_{AC} = +300 \text{ mms}^{-1}$$

$$\Rightarrow \quad v_A + v_C = 300 \text{ mms}^{-1} \qquad \ldots(1)$$

$$\Rightarrow \quad 2v + v_r = 300 \text{ mms}^{-1}$$

Similarly $v_{BA} = -200$ mms^{-1}

$$\Rightarrow \quad v_B - v_A = -200 \text{ mms}^{-1}$$

$$\Rightarrow \quad v_r - 2v = -200 \text{ mms}^{-1} \qquad \ldots(2)$$

Solving equations (1) and (2), we get

$$v_r = 50 \text{ mms}^{-1} \text{ and } v = 125 \text{ mms}^{-1}$$

So, $v_A = +v = 125$ mms^{-1} {upwards}

$$v_B = v_r - v = -75 \text{ mms}^{-1}$$

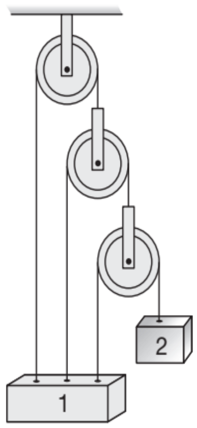
$$\Rightarrow \quad v_B = 75 \text{ mms}^{-1}\text{, downwards}$$

$$\text{and} \quad v_C = -v_r - v = -175 \text{ mms}^{-1}$$

$$\Rightarrow \quad v_C = 175 \text{ mms}^{-1}\text{, downwards.}$$

ILLUSTRATION 12

In the system shown, if a_0 be the acceleration of block 1, then find the acceleration of block 2.



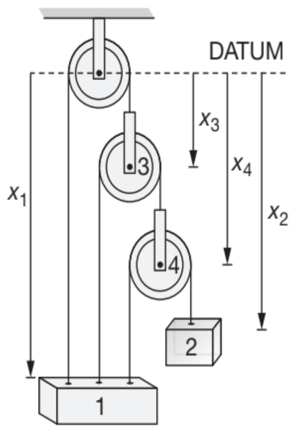
SOLUTION

Let the datum pass through the centre of the fixed pulley. Since the points 1, 2, 3 and 4 are movable, so let their displacements from the datum be x_1, x_2, x_3 and x_4. We observe that the length of the strings between 1 and 3, 1 and 4, 1 and 2 is constant. So,

$$x_1 + x_3 = l_1$$

$$(x_1 - x_3) + (x_4 - x_3) = l_2$$

$$(x_1 - x_4) + (x_2 - x_4) = l_3$$



On double differentiating with respect to time, we get

$$a_1 + a_3 = 0 \qquad \ldots(1)$$

$$a_1 + a_4 - 2a_3 = 0 \qquad \ldots(2)$$

$$a_1 + a_2 - 2a_4 = 0 \qquad \ldots(3)$$

Solving equations (1), (2) and (3), we get

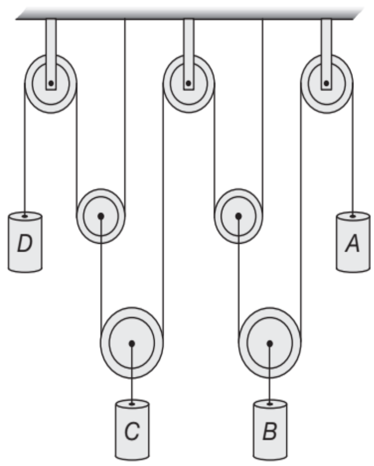
$$a_2 = -7a_1 = -7a_0$$

$$\Rightarrow \quad |a_2| = 7a_0$$

When 1 moves up by x, 3 goes down by x, 4 by $2x$ and 2 by $4x$, so that total distance moved down by 2 becomes $(x + 2x + 4x) = 7x$.

ILLUSTRATION 13

Determine the relationship which governs the velocities of the four cylinders. Express all velocities as positive down. How many degrees of freedom are there?



SOLUTION

$$L_1 = y_B + y_A + (y_B - y_1) + C_1$$

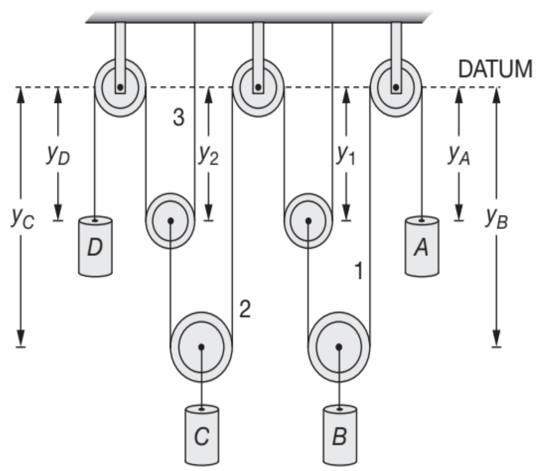
$$\Rightarrow \quad 2\dot{y}_B + \dot{y}_A - \dot{y}_1 = 0$$

$$L_2 = y_C + 2y_1 + (y_C - y_2) + C_2$$

$$\Rightarrow \quad 2\dot{y}_C + 2\dot{y}_1 - \dot{y}_2 = 0$$

$$L_3 = 2y_2 + y_D + C_3$$

$$\Rightarrow \quad 2\dot{y}_2 + \dot{y}_D = 0$$



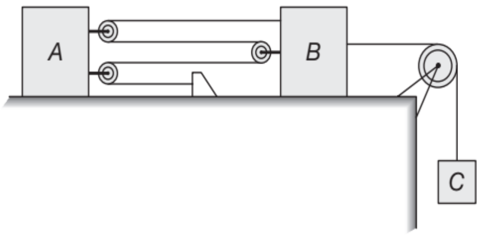
3 degrees of freedom

Eliminate $\dot{y}_1$ and $\dot{y}_2$, and get

$$4v_A + 8v_B + 4v_C + v_D = 0$$

ILLUSTRATION 14

Block C shown in figure is going down at acceleration $2\ \text{ms}^{-2}$. Find the acceleration of blocks A and B.



SOLUTION

The analysis is shown in figure. As block B and C are connected by a string there accelerations must be same hence we can directly state

$$a_B = 2\ \text{ms}^{-2}$$

Block A is also constrained to move with block B, with pulleys X, Y and Z. As shown in figure, we assume if block B and C moves by a distance y, A will move by x and due to this the parts (like length ab) of strings 1, 2, 3 and 4 which are passing over pulleys X and Y are slackened by a length $4x$. This will be tightened by the displacement of pulley Z along with the block B and the string ℓ which is attached to B at point d, by a distance y and this will pull the same string by $3y$ (like the length cd). Thus we have $4x = 3y$ and similarly we have the constrained relation for blocks A and B as.

$$a_A = \frac{3}{4}a_B = 1.5\ \text{ms}^{-2}$$

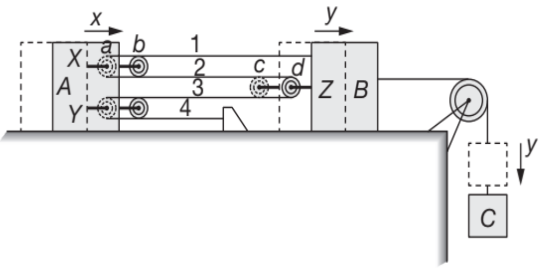
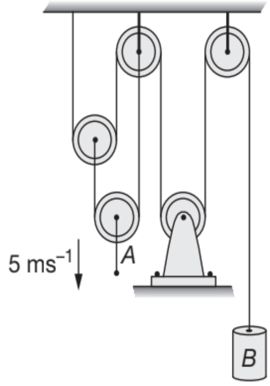


ILLUSTRATION 15

In the arrangement shown, if the end A of the rope moves downward with a speed of $5\ \text{ms}^{-1}$, determine the speed of cylinder B.



SOLUTION

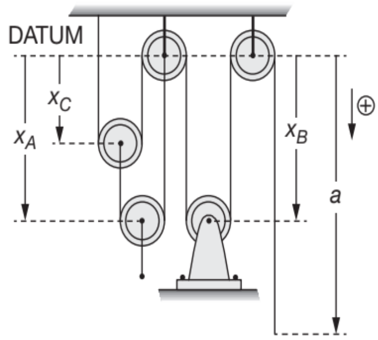
Let the datum be passing through both the fixed pulleys as shown. The length of the two ropes in terms of the position coordinates x_A, x_B and x_C are

$$x_B + 2a + 2x_C = l_1$$

$\Rightarrow \quad x_B + 2x_C = l_1 - 2a \qquad \ldots(1)$

and $\quad x_A + (x_A - x_C) = l_2$

$\Rightarrow \quad 2x_A - x_C = l_2 \qquad \ldots(2)$



Eliminating x_C from equations (1) and (2), we get

$x_B + 4x_A = l_1 - 2a + 2l_2 \qquad \ldots(3)$

Taking the time derivative of equation (3), we get

$$v_B + 4v_A = 0$$

Taking downward direction as positive, we get

$$v_A = 5\ \text{ms}^{-1}$$

$\Rightarrow \quad v_B + 4(5) = 0$

$\Rightarrow \quad v_B = -20\ \text{ms}^{-1} = 20\ \text{ms}^{-1}$, upwards

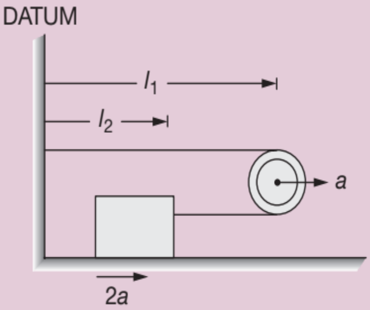
Problem Solving Technique(s)

For pulleys interconnected with strings, we can have the following methods to analyse constraint equations.

(a) Branch wise Analysis.

(b) Law of Conservation of Length of strings connecting the pulleys.

(c) If one end of a string passing over a moving pulley is fixed, then the acceleration of the other end (free or connected to something) is twice the acceleration of the moving pulley. Following situations show this to be made as a good rule for applications.

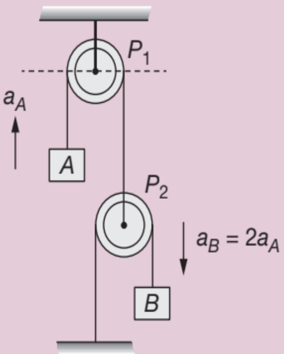


$$l_1 + (l_1 - l_2) = \text{constant}$$

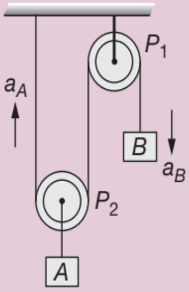
$$\Rightarrow \quad 2l_1 - l_2 = \text{constant}$$

$$\Rightarrow \quad 2\dot{l}_1 - \dot{l}_2 = 0 \quad \Rightarrow \quad \dot{l}_2 = 2\dot{l}_1$$

$$\Rightarrow \quad \ddot{l}_2 = 2\ddot{l}_1 \quad \Rightarrow \quad a_2 = 2a_1$$



$$a_B = 2(\text{Acc. of Pulley } P_2) = 2a_A$$



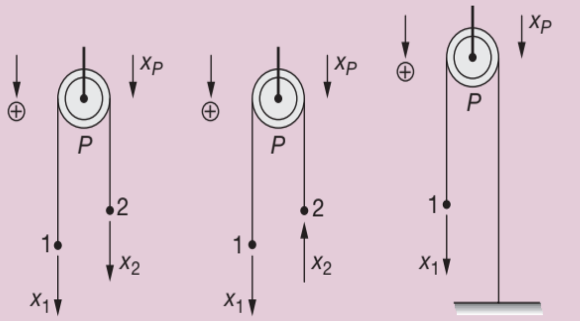
$$a_B = 2(\text{Acc. of Pulley } P_2)$$

$$\Rightarrow \quad a_B = 2a_A$$

(d) For a moving pulley, the displacement of the pulley equals the average of the displacements on the left and the right of the pulley.

$$x_P = \frac{x_1 + x_2}{2} \qquad x_P = \frac{x_1 - x_2}{2} \qquad x_P = \frac{x_1 + 0}{2}$$

$$\Rightarrow \quad x_1 = 2x_P \qquad \text{(As discussed in 1)}$$

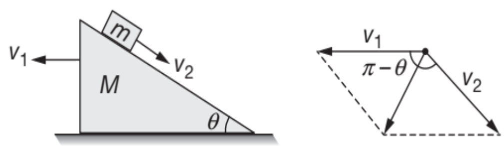


WEDGE CONSTRAINTS

Till now we have discussed the motion of blocks connected by strings governed by pulleys connected in several ways possible. Here we will discuss the relation between the motion of two or more bodies which are in contact and responsible for motion of bodies.

EXAMPLE 1:

First we consider a very simple case shown in Figure (a).



(a) (b)

Here a triangular block of mass M is free to move on ground and m is free to move on inclined surface of M. Here M is constrained to move only along horizontal ground and m is also constrained to move only along the inclined surface of M relative to it. Here if M is going toward left with speed v_1 (say), and if on its inclined surface m is going down with speed v_2, then we can state that the net speed of M is v_1 but m is also moving to the left along with M, thus, its net speed is given by vector sum of the two v_1 and v_2 as shown in Figure (b).

$$v_m = \sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos(\pi - \theta)}$$

$$v_m = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\theta}$$

Same can also be evaluated by using the velocity components in horizontal and vertical directions of small mass m. Here it is going along the incline with a velocity v_2 relative to M and it is also moving with M toward left with velocity v_1, thus we have

Horizontal velocity of m relative to ground is

$$v_x = v_2\cos\theta - v_1$$

Vertical velocity of m relative to ground is

$$v_y = v_2\sin\theta$$

Net velocity of m is

$$v_m = \sqrt{v_1^2 + v_y^2} = \sqrt{v_1^2 + v_y^2 - 2v_1v_2\cos\theta}$$

EXAMPLE 2:

Now consider the situation shown in figure. Here block A and B are constrained to move on their contact surface as well as horizontal ground and vertical wall. Here as B goes down, A will move to the left. If B goes down by a distance x, A will move toward left by a distance $x\cot\theta$. Thus if velocity and acceleration of B are v and a downward, velocity and acceleration A will be $v\cot\theta$ and $a\cot\theta$ toward left.

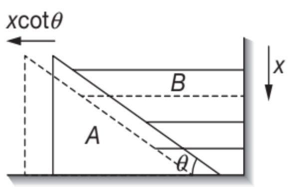
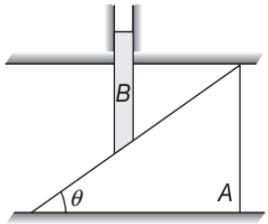


ILLUSTRATION 16

Find the relation among accelerations of wedge A and the rod B supported on wedge A. Rod B is restricted to move vertically by two fixed wall corners shown in figure.



SOLUTION

Here we can observe that the rod is restricted to move only in vertical direction and wedge can move along horizontal plane only. Here if wedge moves toward right by a distance x, figure shows that the rod moves vertically down by a distance $x\tan\theta$. Thus if wedge is moving toward right with an acceleration a_1, rod will go down with acceleration a_2, given as

$$a_2 = a_1\tan\theta$$

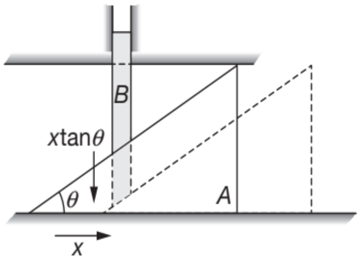
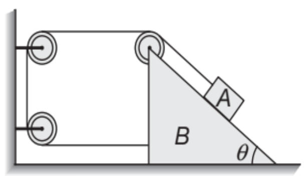
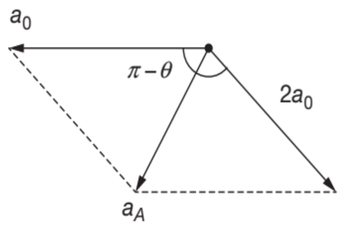


ILLUSTRATION 17

Figure shown a block A constrained to slide along the incline plane of the wedge B shown. Block A is attached with a string which passes through three ideal pulleys and connected to the wedge B. If wedge moves towards left with an acceleration a_0, find the acceleration of the block with respect to the wedge and the ground.



SOLUTION



When the wedge moves to the left towards the wall by x, then a portion $2x$ of the string is loosened which makes the block A to go down by $2x$. So, acceleration of the block with respect to the wedge is $2a_0$. However, with respect to the ground the situation is shown for

$$a_A=\sqrt{a_0^2+4a_0^2-4a_0^2\cos\theta}$$

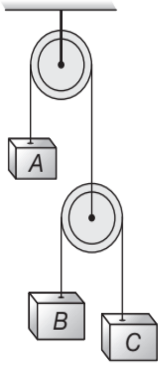
$$\Rightarrow \quad a_A=a_0\sqrt{5-4\cos\theta}$$

Test Your Concepts-II

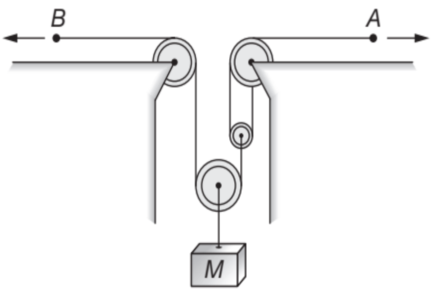
Based on Constraints

(Solutions on page H.193)

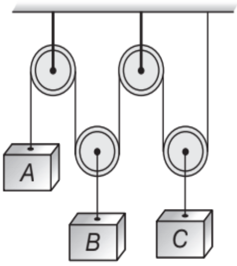
1. At certain moment of time, velocities of A and B both are 1 ms^{-1} upwards. Find the velocity of C at that moment.



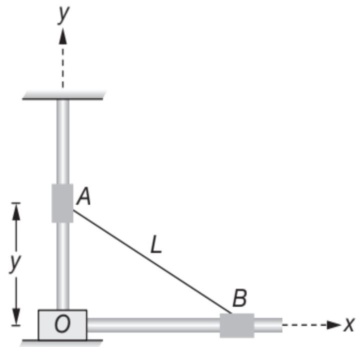
2. For the pulley system each of the cables at A and B is given a velocity of 2 ms^{-1} in the direction shown. Determine the upward velocity v of the load M.



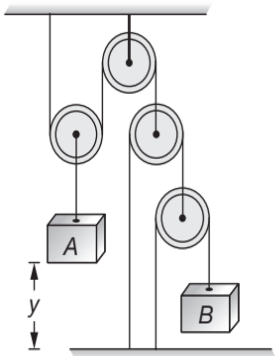
3. In the arrangement shown, if the block A of the pulley system is moving downward with a speed of 2 ms^{-1} while block C is moving up at 1 ms^{-1}, then determine the speed of block B.



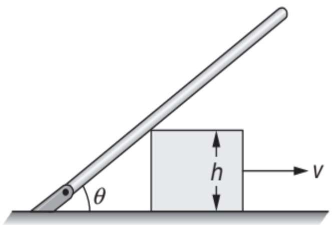
4. Collars A and B slide along the fixed right angled rods and are connected by a cord of length L. Determine the acceleration a_x of collar B as a function of y, if collar A is given a constant upward velocity v_A.



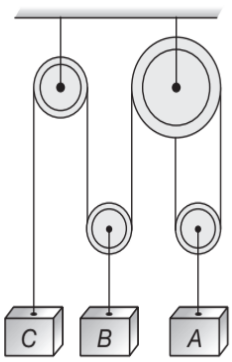
5. The vertical displacement of block A in meter is given by $y=\frac{t^2}{4}$, where t is in second. Calculate the downward acceleration a_B of block B.



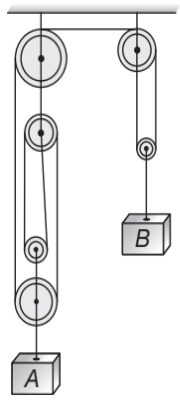
6. In the arrangement shown, the rod is freely pivoted at O and is in the contact with the block which moves on the horizontal frictionless ground. As the block is given a speed v forward, the rod rotates about O. Find the angular velocity of rod as a function of θ.



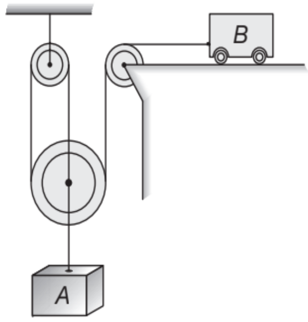
7. Find the relation which governs the accelerations of A, B and C, all measured positive down. Identify the number of degrees of freedom.



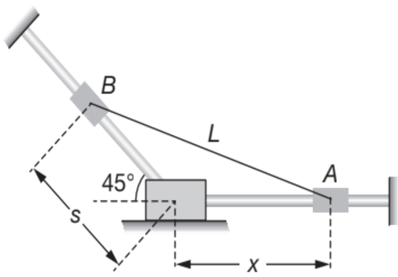
8. Cylinder B has a downward velocity in metre per second given by $v_B = \frac{t^2}{2} + \frac{t^3}{6}$, where t is in seconds. Calculate the acceleration of A when $t = 2$ s.



9. If block B has a leftward velocity of 1.2 ms^{-1}, determine the velocity of cylinder A.

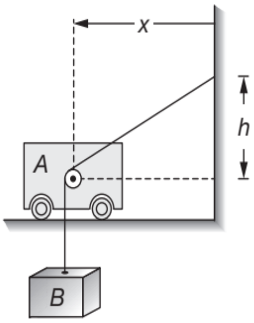


10. Collars A and B slide along the fixed rods and are connected by a cord of length L. If collar A has a velocity $v_A = \frac{dx}{dt}$ to the right, express the velocity $v_B = -\frac{ds}{dt}$ of B in terms of x, v_A, and s.

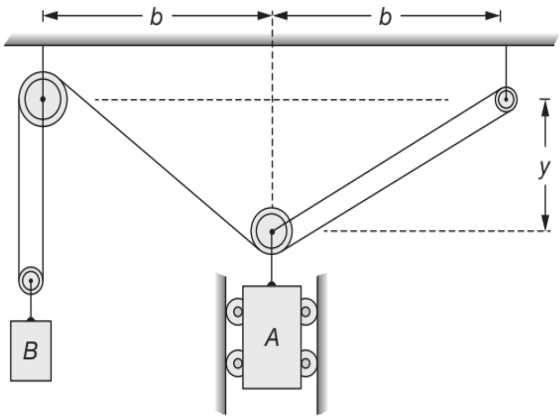


11. Neglect the diameter of the small pulley attached to body A and determine the magnitude of the total velocity of B in terms of the velocity v_A which body A has to the right. Assume that the cable between

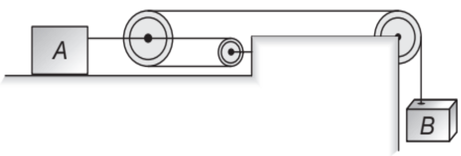
B and the pulley remains vertical and solve for a given value of *x*



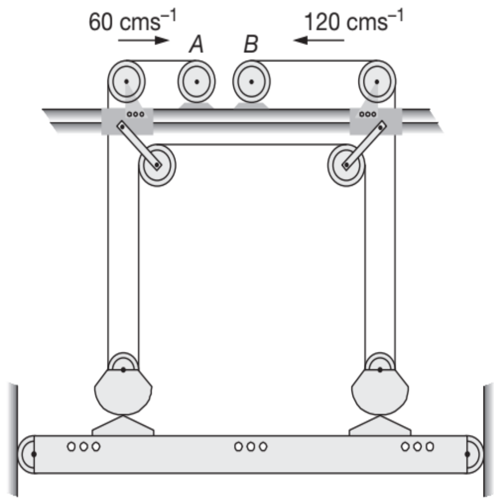
12. Establish the relationship between the velocity of *A* and the velocity of *B* for a given value of *y*. Neglect the diameters of the small pulley.



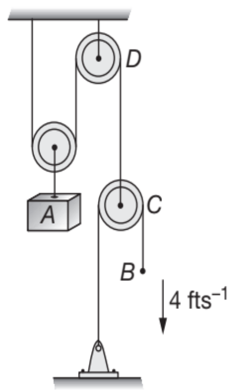
13. If block *A* has a velocity of $0.6\ \text{ms}^{-1}$ to the right, determine the velocity of cylinder *B*.



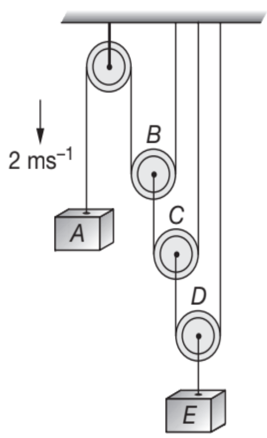
14. The crane is used to hoist the load. If the motors at *A* and *B* are drawing in the cable at a speed of $60\ \text{cms}^{-1}$ and $120\ \text{cms}^{-1}$, respectively, determine the speed of the load.



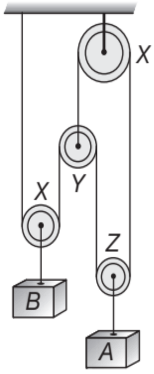
15. The cable at *B* is pulled downwards at $4\ \text{ms}^{-1}$ and the speed is decreasing at $2\ \text{ms}^{-2}$. Determine the velocity and acceleration of block *A* at this instant.



16. If the end of the cable at *A* is pulled down with a speed of $2\ \text{ms}^{-1}$, determine the speed at which block *E* rises.



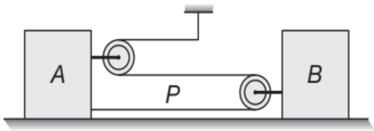
17. Figure shows a system of four pulleys with two masses A and B. Find the



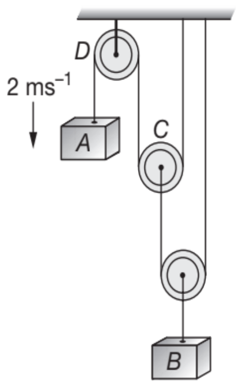
(a) speed of block A when the block B is going up at $1\ \text{ms}^{-1}$ and pulley Y is going up at $2\ \text{ms}^{-1}$.

(b) acceleration of block A if block B is going up at $3\ \text{ms}^{-2}$ and pulley Y is going down at $4\ \text{ms}^{-2}$.

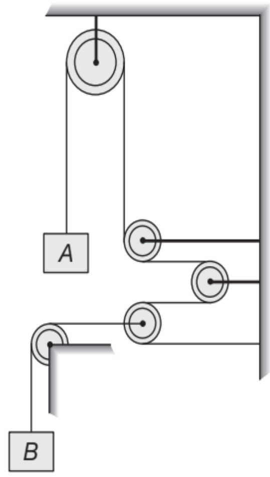
18. Block A shown in figure move by a distance 3 m toward left. Find the distance and direction in which the point P on string shown in figure is displaced.



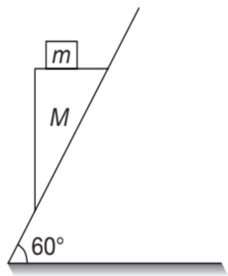
19. In the arrangement shown, if the end of the cable at A is pulled down with a speed of $2\ \text{ms}^{-1}$, determine the speed at which block B rises.



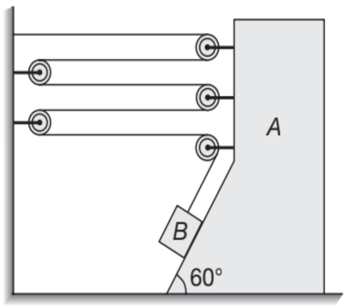
20. Find the velocity of the block A shown in figure, if B moves up with a velocity v_0.



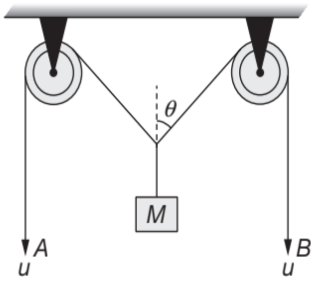
21. In the situation shown in figure, if mass M is going down along the incline at an acceleration of $5\ \text{ms}^{-2}$ and m is moving toward right relative to M horizontally with $3\ \text{ms}^{-2}$. Find the net acceleration of m.



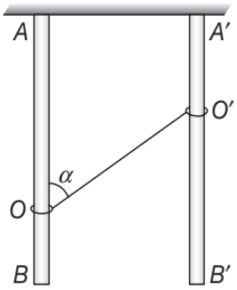
22. Find the acceleration of block B relative to the block A and relative to the ground, if the block A moves to the left with an acceleration a_0.



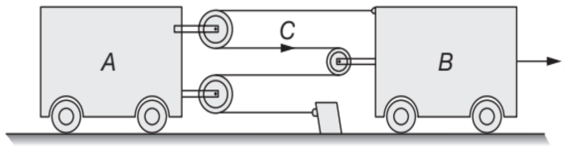
23. In the arrangement shown in figure, the ends A and B of an inextensible string move downwards with uniform speed u. Pulleys A and B are fixed. Find the speed with which the mass M moves upwards.



24. Two rings O and O' are put on two vertical stationary rods AB and $A'B'$, respectively as shown in figure. An inextensible string is fixed at point A' and on ring O and is passed through O'. Assuming that ring O' moves downwards at a constant speed v, find the velocity of the ring O in terms of α.



25. Under the action of force P, the constant acceleration of block B is $3\ ms^{-2}$ to the right. At the instant when the velocity of B is $2\ ms^{-1}$ to the right, determine the velocity of B relative to A, the acceleration of B relative to A and the absolute velocity of point C of the cable.



FREE BODY DIAGRAM (FBD)

A Free Body Diagram (FBD) consists of a diagrammatic representations of a single body or a subsystem of bodies isolated from its surroundings showing all the forces acting on it.

Steps for Drawing FBD

STEP-1: Identify the object or system and isolate it from other objects clearly specify its boundary.
STEP-2: First draw non-contact external force in the diagram. Generally it is weight.
STEP-3: Draw contact forces which act at the boundary of the object or system. Contact forces are normal, friction, tension and applied force.
STEP-4: In F.B.D., draw all the forces acting on the isolated body. Do not draw the forces which the body is exerting on others.

WEIGHT OF A BODY (*W*)

The weight $\vec{W}$ of a body is a force that pulls the body directly towards the earth. The force is due to gravitational attraction between two bodies. An object of mass m located at a point where the acceleration due to gravity has magnitude g will have a weight W whose magnitude is given by

$$W = mg$$

And it is always directed vertically downward (towards the centre of the earth). **Normally we assume that weight is measured in an inertial frame. If it is measured in a non-inertial frame, it is called apparent weight.**

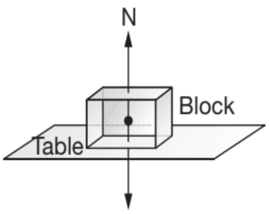
WEIGHT OF A BODY IS A NON-CONTACT FORCE. It acts on the body whether the body is in contact with a surface or not.

Weight is the force with which the earth attracts other bodies. It is also called the **force of gravity** or the **gravitational force**. It acts upon all the bodies near the earth. If they do not fall to the earth, then their motion is restricted by certain other bodies : a support, string, spring, etc. Bodies that restrict the motion of other bodies are called **constraints**. These bodies restrict the motion of the given bodies and hence oppose free motion.

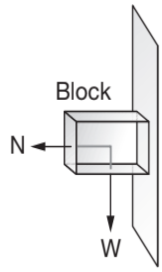
EXAMPLE:
The surface of the table is the constraint for all objects lying on it, the floor serves as the constraint for the table, etc.

NORMAL REACTION/ NORMAL CONTACT FORCE

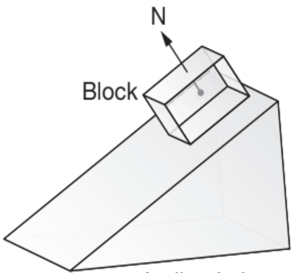
Normal Reaction (N) acts on a body when a body is lying on a surface or leaning against (supporting against) a surface. Normal reaction always acts normal to surface on which the body is kept or against which the body is leaning.



Normal reaction = N
Weight = W
(a)



(b)



(c)

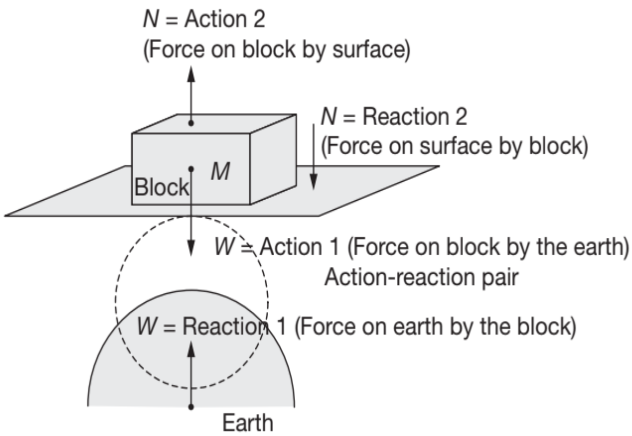
Conceptual Note(s)

Generally students think that the weight of the body, *W*, acts on the surface on which the body is kept. This is not so because WEIGHT IS A NON-CONTACT FORCE AND WILL ACT ON THE BODY IRRESPECTIVE OF THE FACT WHETHER THE BODY IS LYING ON THE SURFACE OR NOT.

However, when the body is kept on the surface, then the body will make its presence felt to the surface by exerting a force called normal reaction *N* on the surface.

CASE-1: Consider a block of mass M, weight W, place on a rigid horizontal surface. Then the forces are drawn as shown.

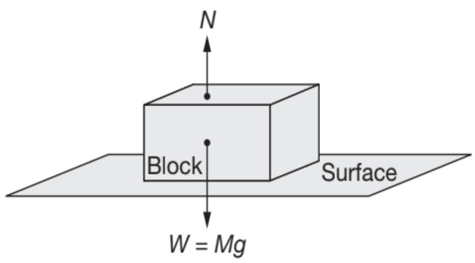
Out of these two action-reaction force pairs, the Reaction 1 (force on earth by the block) and Reaction 2 (force on surface by the block) do not contribute anything significant to the motion as the earth is massive and the surface is rigid and not moving, so our free body diagram can be redrawn as shown.



So, for the block to stay in equilibrium on the rigid surface, we have from Newton's Second Law,

$$N - Mg = 0$$

$$\Rightarrow \quad N = Mg$$



Word of Advice

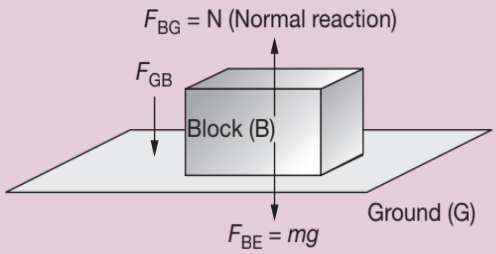
You may come across situations, when the forces are equal in magnitude and opposite in directions. Please **do not treat them as Action-Reaction pair**, because action and reaction must act on different bodies. So, $N = mg$ is not a consequence of Newton's Third Law. N and mg are not to be taken as action-reaction pair. Hence the equation $N = mg$ actually follows from Newton's Second Law and not from Newton's Third Law.
According to *NSL*,

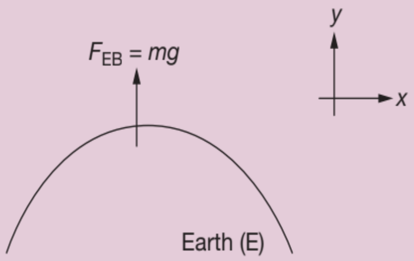
$$\sum \vec{F} = m\vec{a}$$

$\Rightarrow \quad \sum F_x = ma_x$

$\Rightarrow \quad \sum F_y = ma_y$

$\Rightarrow \quad \sum F_z = ma_z$





Since there is no motion along *y*-axis,

$$N(\text{up}) + mg(\text{down}) = m(0)$$

$$N - mg = 0$$

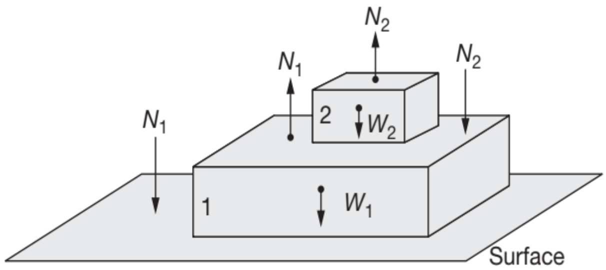
$$N = mg$$

So, this equation is a consequence of Newton's Second Law.

CASE-2: Now let us give CASE-1 an extension, where we place another the block 2 on block 1 (as in CASE-1), where we are designating the block placed on ground/surface as 1 and the block placed on 1 as 2 as shown. **Here, now we have not shown the earth in the diagram as the forces acting on it are negligible to produce any significant effect on it.**

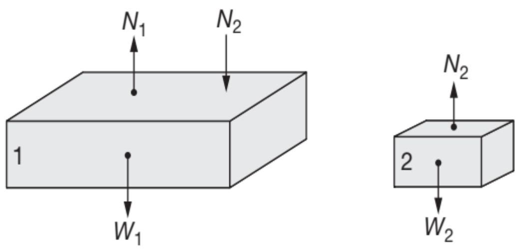
To understand the forces drawn, read the following stepwise arguments.

(a) First of all, both the blocks will be attracted towards the centre of the earth. This force of attraction, called the weight is shown as W_1 and W_2 in the figure.



(b) Now the block 1 is placed on the rigid surface, so the surface exerts a normal force N_1 on the block 1 upwards. Correspondingly the block 1 exerts a normal force N_1 (due to Newton's Third Law) downwards on the surface as shown. Actually till this point we have just rediscussed CASE-1.

(c) However, now the block 2 is placed on block 1, so the block 1 exerts a normal force N_2, upwards on block 2. Correspondingly the block 2 also exerts a normal force of equal magnitude but opposite direction on block 1 as shown.



FBD 1 FBD 2

Again using Newton's Second Law, we get

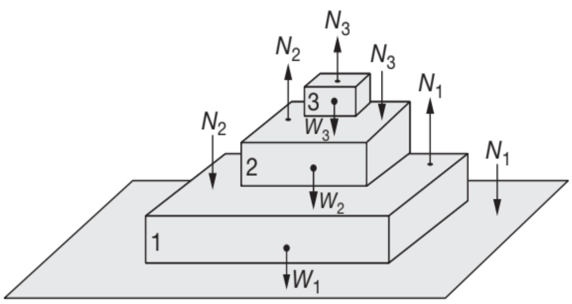
$$N_1 = N_2 + W_1 \quad \ldots(1)$$

$$N_2 = W_2 \quad \ldots(2)$$

So, from (1) and (2), we get

$$N_1 = W_1 + W_2$$

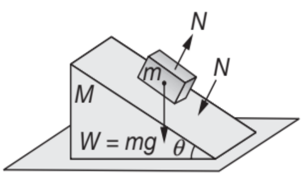
Similarly, for three blocks we can proceed in the same manner, to get $N_1 = (W_1 + W_2 + W_3)$.



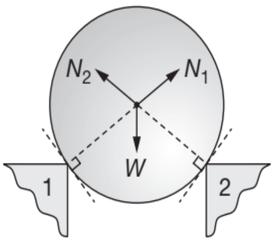
NORMAL REACTION FOR VARIOUS SITUATIONS

Normal reactions in various situations is shown in Figures given below.

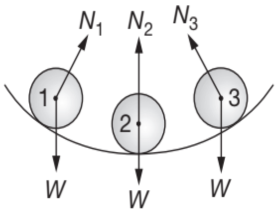
(a)



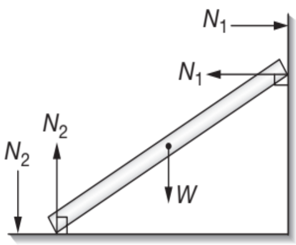
(b)



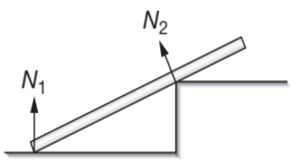
(c)



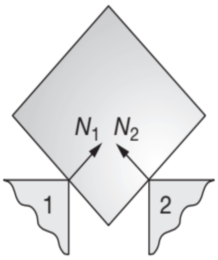
(d)



(e)



(f)

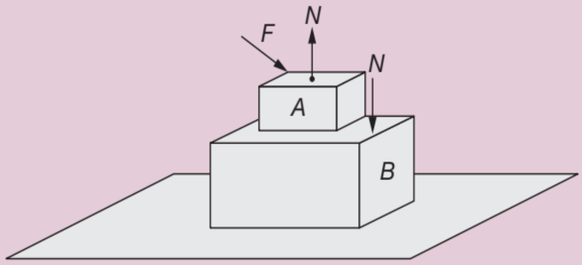


Conceptual Note(s)

Whenever a force acts on a body, it changes the motion of the body. As the motion of a body or bodies is concerned there can be two type of forces.

(a) **External Forces.** External forces are those which act from outside of the system, only action acts on the system, reaction of these forces are not utilized by the system.

(b) **Internal Forces.** Internal forces are those which are developed within the system bodies, hence, both action and reaction of these forces are in the system. If we consider a situation, shown in figure, box A is placed over box B, and a force F is applied on box A. Here system includes two blocks, A and B, and the force F which is acting from outside of the system is an external force and the normal contact force between the two blocks is the internal force of this system. Now will discuss properties and applications for some important forces.



TENSION IN A LIGHT STRING

The force exerted at any point in the light rope/string/wire/rod is called the **tension** at that point. We may measure the tension at any point in the light rope by cutting a suitable length from it and inserting a spring scale, then the tension is the reading of the scale.

The tension is same at all points in the rope only if the rope is unaccelerated and assumed to be massless.

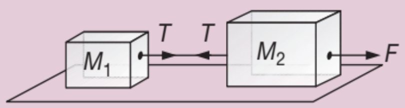
Whenever a thread or a rope or a wire is exposed to some kind of force, a tension (T) develops in it.

Conceptual Note(s)

(a) Tension in any branch of string must be shown as a pair.

(The pair is shown with two arrows facing each other).

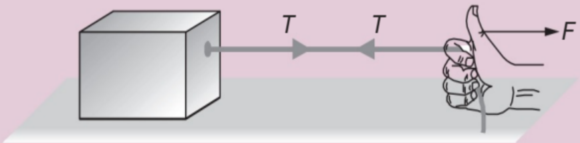
When two blocks (M_1 and M_2 say) connected by a string are pulled, then tension on M_1 acts towards M_2 and that on M_2 acts towards M_1 as shown in the figure.



(b) If a body A attached to an ideal light inextensible string, is pulled with a force F and it is placed on a horizontal frictionless surface, then

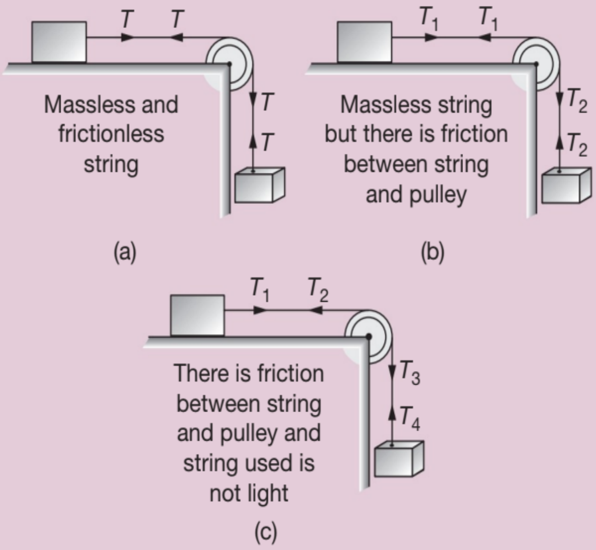
Tension in the string = Pull (F)

(c) Force of tension act on a body in the direction away from the point of contact or tied ends of the string. For example consider figure. A man pulls a box with a string. The tension in string acts on the box towards right or in the direction away from the tied point and on the man it is again away from it. The way of showing the direction of tension is shown in figure.



(d) If string is massless and frictionless, tension throughout the string remains constant as shown in figure (a). But if the string is massless and not frictionless, at every contact in the length of the string tension changes and if it is not light, tension at each point will be different depending on the acceleration of the string.

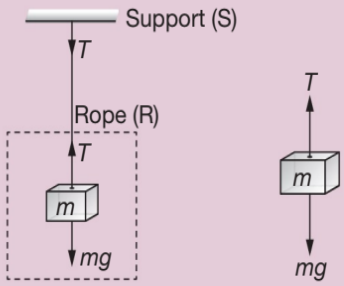
EXAMPLE 1: Consider the situation shown in figure (a). A box is tied to another mass with a string going over a pulley. If string is massless and there is no friction between the contact of string and pulley surface, tension throughout the string remains same as T and as there is no friction between pulley and string, string will not be able to rotate the pulley and it will slide on the surface of pulley. But if there is friction between surface of pulley and the string, due to friction, pulley will rotate on its axis as the string slides on it. In this case, due to friction between pulley and the string, tensions in string on two sides of the pulley will be different as shown in figure (b). If string has a mass, it will accelerate and tension at each point will be different on the string as shown in figure (c). How this tension can be obtained, we will explain in further sections.



EXAMPLE 2:

$$T - mg = 0$$

$$\Rightarrow \quad T = mg$$



EXAMPLE 3:

$$T_1 = m_1 g + T_2 \qquad \text{...(1)}$$

$$T_2 = m_2 g \qquad \text{...(2)}$$

From (1) and (2), we get

$$T_1 = (m_1 + m_2)g$$

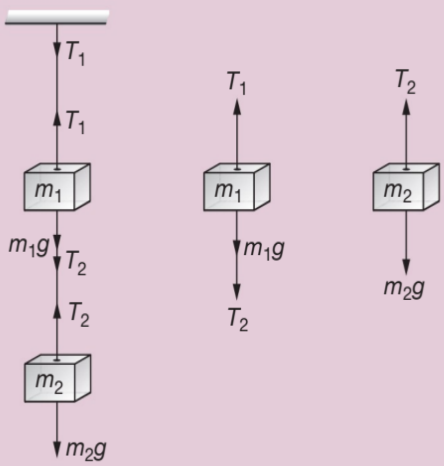
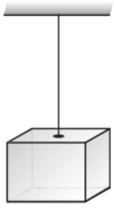


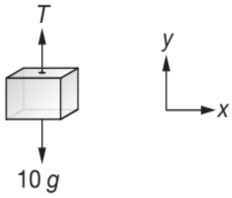
ILLUSTRATION 18

A block of mass 10 kg is suspended with string as shown in figure. Find tension in the string. $\left(g = 10\ \text{ms}^{-2}\right)$.



SOLUTION

F.B.D. of block



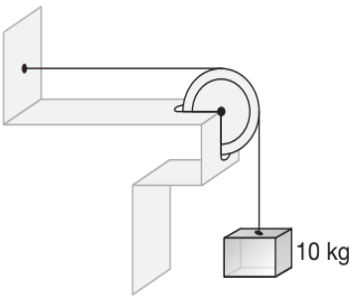
$$\Sigma F_y = 0$$

$$T - 10g = 0$$

$$\Rightarrow \quad T = 100\ \text{N}$$

ILLUSTRATION 19

Find magnitude of force exerted by string on pulley.

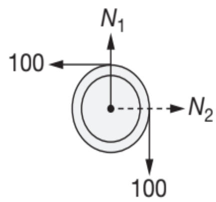


SOLUTION

F.B.D. of 10 kg block

$$T = 10g = 100\ \text{N}$$

F.B.D. of pulley



Since string is massless, so tension in both sides of string is same. Force exerted by string is

$$F = \sqrt{(100)^2 + (100)^2} = 100\sqrt{2}\ \text{N}$$

Conceptual Note(s)

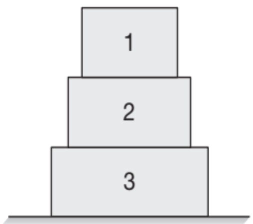
Since pulley is in equilibrium position, so net forces on it is zero. However force exerted by hinge on the pulley is $100\sqrt{2}$ N.

Test Your Concepts-III

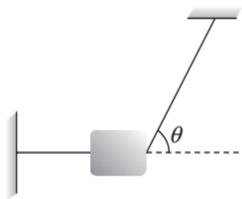
Based on FBD

(Solutions on page H.198)

1. Three blocks 1, 2 and 3 are placed one over the other as shown in figure. Draw free body diagrams of all the three blocks.

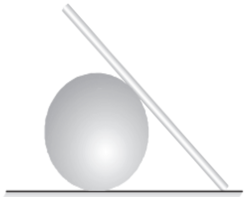


2. A block of mass *m* is attached with two strings as shown in figure. Draw the free body diagram of the block.



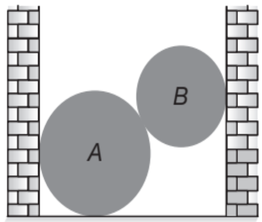
3. A spherical ball of radius R, weight W is resting on a V-groove as shown in figure. Draw its free body diagram.

4. The diagram shows a rough plank resting on a cylinder (shown cross-sectionally) with one end of the plank on rough ground. Neglect friction between plank and cylinder. Draw diagrams to show the



(a) forces acting on the cylinder.
(b) forces acting on the plank.

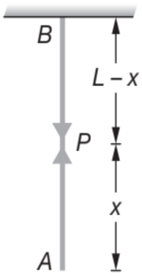
5. Two spheres A and B are placed between two vertical walls as shown in figure. Friction is absent everywhere. Draw the free body diagrams of both the spheres.



TENSION IN A ROPE HAVING UNIFORM MASS DISTRIBUTION

Now, if the rope/thread has a uniform mass, then the tension at a point P at distance x from the free end is equal to the weight of rope that hangs below P. So,

$$\begin{pmatrix}\text{Tension at} \\ \text{point } P\end{pmatrix} = \begin{pmatrix}\text{Weight of rope that} \\ \text{hangs below the point } P\end{pmatrix}$$



$$\Rightarrow \quad T_P = (\lambda x)g$$

where $\lambda = \dfrac{M}{L}$

M = total mass of rope
L = total length of rope and
λ = linear mass density

Tension at A

$$x = 0$$

$$\Rightarrow \quad T_A = 0$$

(we can think that no part of rope hangs below A, so $T_A = 0$)

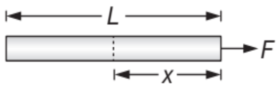
Tension at B

$$x = L$$

$$\Rightarrow \quad T_A = (\lambda L)g = Mg = \text{Total weight of rope.}$$

ILLUSTRATION 20

A horizontal force is applied on a uniform rod of length L kept on a frictionless surface. Find the tension in rod at a distance x from the end where force is applied.

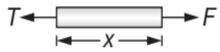


SOLUTION

Considering rod as a system, the acceleration of rod is

$$a = \frac{F}{m}$$

Now draw F.B.D. of rod having length x as shown in figure



According to Newton's Second Law, we have

$$\Sigma F = ma$$

$$\Rightarrow \quad F - T = \left[\frac{M}{L}\right]xa$$

$$\Rightarrow \quad T = F - \left(\frac{M}{L}x\right)\left(\frac{F}{M}\right)$$

$$\Rightarrow \quad T = F\left(1 - \frac{x}{L}\right)$$

ILLUSTRATION 21

For the situation given in figure, find the acceleration of each body for the following two cases.

(a) if the thread has a mass m.
(b) if the thread is massless.

It is given that masses of A and B are M_1 and M_2 respectively and the horizontal surface is smooth.

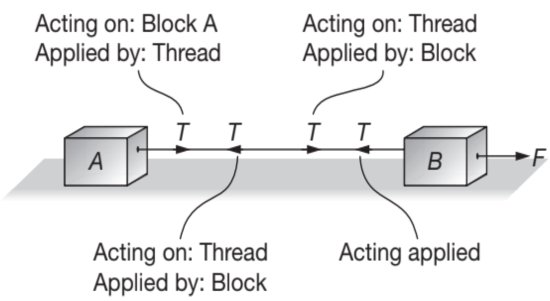
SOLUTION

(a) Suppose the two bodies and the thread are moving along the force F with an acceleration of a, as shown in figure. Then, we have,

$$T_1 = M_1 a \qquad \ldots(1)$$

$$T_2 - T_1 = ma \qquad \ldots(2)$$

$$F - T_2 = M_2 a \qquad \ldots(3)$$



Adding (1), (2) and (3), we get

$$F = (M_1 + M_2 + m)a$$

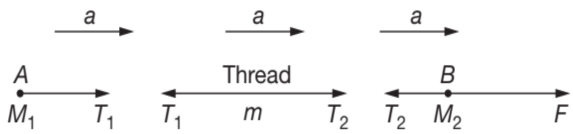
$$\Rightarrow \quad a = \frac{F}{M_1 + M_2 + m}$$

$$\Rightarrow \quad T_1 = M_1 a$$

$$T_1 = \frac{M_1 F}{M_1 + M_2 + m}$$

and $T_2 = F - M_2 a$

$$\Rightarrow \quad T_2 = \frac{(M_1 + m)F}{M_1 + M_2 + m}$$



(b) If thread is massless, then putting $m = 0$ (for any a) in equation (2), we get

$$T_1 = T_2$$

Again, after solving (1) and (3), we get

$$a = \frac{F}{M_1 + M_2}$$

and $T_1 = T_2 = \dfrac{M_1 F}{M_1 + M_2}$

Conceptual Note(s)

(a) Tensions at different points in a massless string is the same.

(b) Acceleration of each body could be calculated very easily if we would have assumed M_1, M_2 and thread as parts of a single system. In this way, action and reaction forces acting on different parts of the same system must get cancelled out and we would have to take care of only external forces. In this way, we can write.

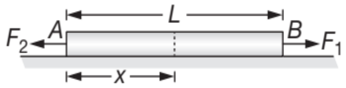
$$F_{ext} = M_{sys}a$$

$$\Rightarrow \quad a = \frac{F_{ext}}{M_{sys}} = \frac{F}{M_1 + M_2 + m}$$

(Put $m = 0$, if the thread is massless)

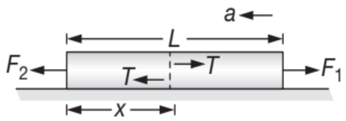
ILLUSTRATION 22

Two forces F_1 and $F_2(>F_1)$ are applied at the free ends of uniform rod kept on a horizontal frictionless surface. Find tension in rod at a distance x from end A.

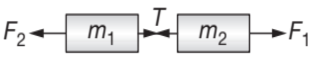


SOLUTION

$$a=\frac{F_2-F_1}{m}$$



Since, $T-F_1=m_2a$, where $m_2=\frac{m}{L}(L-x)$



$$\Rightarrow \quad T-F_1=\frac{m}{L}(L-x)\frac{F_2-F_1}{m}$$

$$\Rightarrow \quad T=F_1+\left(1-\frac{x}{L}\right)(F_2-F_1)$$

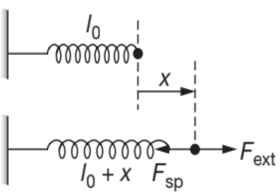
$$\Rightarrow \quad T=F+F_2-F_1-\frac{x}{L}(F_2-F_1)$$

$$\Rightarrow \quad T=F_2-(F_2-F_1)\frac{x}{L}$$

SPRING FORCE

Every spring opposes the attempts to change its length i.e. **every spring opposes the phenomenon of compression or extension.** This opposing or the resistive force called as the Restoring Force or the spring force F increases with change in length of the spring i.e., Extension or Compression (x).

When spring is in its natural length l_0, spring force (F_{sp}) is zero.



$$F_{sp}=0$$

$$F_{sp}=F=-kx$$

Spring force is given by $F_{sp}=-kx$, where x is the change in length of the spring (also called as compression or extension) and k is the Spring Constant or Force constant having SI unit Nm^{-1} and dimensional formula MT^{-2}. The spring constant (k) is also known as **force constant/spring factor/force factor.**

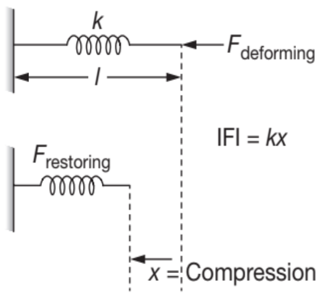
The spring constant k depends on geometry of the spring and on the material property. For us, it is important to know that the spring constant is inversely proportional to its natural length l, other things remaining the same i.e.,

$$k\propto\frac{1}{l} \qquad \text{\{everything else constant\}}$$

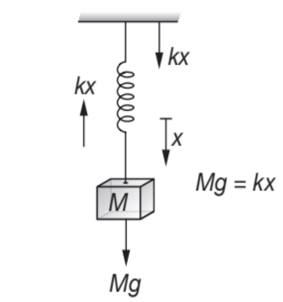
$$\Rightarrow \quad kl=\text{constant}$$

Therefore if you cut a spring into two parts whose length are in ratio $1:2$, their spring constants will be in ratio of $2:1$.

The negative sign in the above relation signifies that the spring force is always directed opposite to the compression or extension in the spring.



CASE-1: Horizontal Placement



CASE-2: Vertical Placement

As in case of rope, we will usually deal with a massless spring for which the force at each point is the same. Such springs are normally referred to as ideal.

Conceptual Note(s)

(a) For a spring of natural length ℓ, spring constant k, we have,

$$k\ell=\text{constant}$$

(b) No need to take the negative sign as the direction has already been taken.

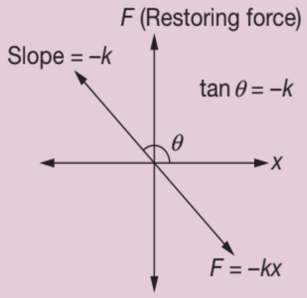
(c) The more the value of k, the more the elastic a spring is.

(d) Soft springs have small k in comparison to hard spring.

$$k_{soft} \ll k_{hard}$$

e.g., Spring used in a ball pen (soft spring). Spring used in the shocker of the car (hard spring).

(e) The graph showing variation of restoring force F with extension/compression x is shown here.



(f) If a spring with spring constant k is divided into n equal parts, the spring constant of each part is nk.

(g) If springs of spring constants k_1, k_2, k_3 ... are connected in series, then effective force constant

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots$$

(h) If springs of spring constants k_1, k_2, k_3 ... are connected in parallel, then effective spring constant.

$$k_{eff} = k_1 + k_2 + k_3 + \ldots$$

(i) SI unit of k is Nm^{-1} and $[k] = MT^{-2}$.

(j) When a spring is cut, then to calculate the force constant of each divided part we have

$$k\ell = k_1\ell_1 = k_2\ell_2 = \ldots = k_n\ell_n$$

ILLUSTRATION 23

A spring of force constant k and natural length l is cut into two parts of lengths $\frac{l}{3}$ and $\frac{2l}{3}$. Find the new force constant of the divided parts.

SOLUTION

Since, kl = constant

$$\Rightarrow \quad kl = k_1l_1 = k_2l_2 = \ldots\ldots.$$

So, $kl = k_1l_1$

$$\Rightarrow \quad kl = k_1\left(\frac{2l}{3}\right)$$

$$\Rightarrow \quad k_1 = \frac{3k}{2}$$

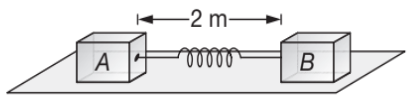
So, $kl = k_2l_2$

$$\Rightarrow \quad kl = k_2\left(\frac{l}{3}\right)$$

$$\Rightarrow \quad k_2 = 3k$$

ILLUSTRATION 24

Two blocks are connected by a spring of natural length 2 m. The force constant of spring is $200\ Nm^{-1}$. Find spring force in following situations.



(a) If block A and B both are displaced by 0.5 m in same direction.

(b) If block A and B both are displaced by 0.5 m in opposite direction.

SOLUTION

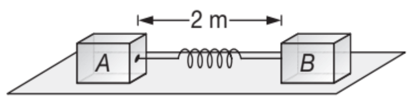
(a) Since both blocks are displaced by 0.5 m in same direction, so change in length of spring is zero. Hence, spring force is zero.

(b) In this case, change in length of spring is 1 m. So spring force is

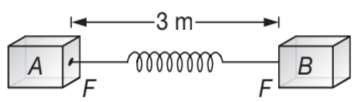
$$F = -Kx = -(200)(1)$$

$$\Rightarrow \quad F = -200\ N$$

Natural length



When spring is extended



When spring is compressed

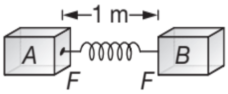
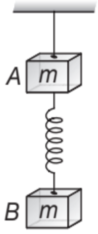


ILLUSTRATION 25

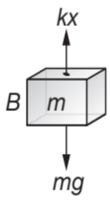
Two blocks A and B of same mass m attached with a light spring are suspended by a string as shown in figure. Find the acceleration of block A and B just after the string is cut.



SOLUTION

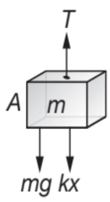
When block A and B are in equilibrium position, then

F.B.D. of B



$$kx = mg \qquad \ldots(1)$$

F.B.D. of A

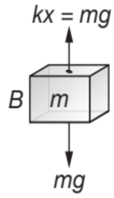


$$T = mg + kx \qquad \ldots(2)$$

$$\Rightarrow \quad T = 2mg$$

Now, when string is cut, tension T becomes zero. But spring does not change its shape just after cutting. So spring force acts on mass B, again draw F.B.D. of blocks A and B as shown in figure.

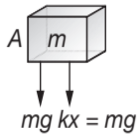
F.B.D. of B



$$kx - mg = ma_B$$

$$\Rightarrow \quad a_B = 0$$

F.B.D. of A



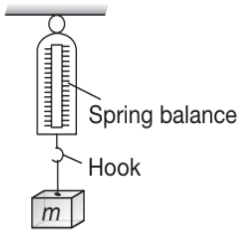
$$mg + kx = ma_A$$

$$\Rightarrow \quad 2mg = ma_A$$

$$\Rightarrow \quad a_A = 2g \quad \text{(downwards)}$$

SPRING BALANCE

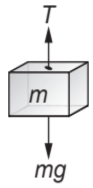
A spring balance does not measure the weight. It measures the force exerted by the object at the hook. Symbolically, it is represented as shown in figure. A block of mass m is suspended at hook.



When spring balance is in equilibrium, we draw the F.B.D. of mass m for calculating the reading of balance.

Unless and until mentioned, a spring balance is assumed to be light or having negligible mass.

F.B.D. of m



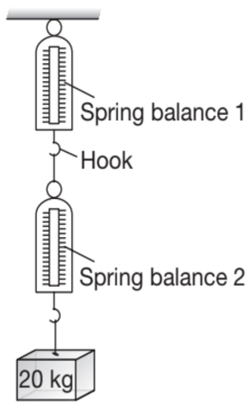
$$mg - T = 0$$

$$\Rightarrow \quad T = mg$$

Magnitude of T gives the reading of spring balance.

ILLUSTRATION 26

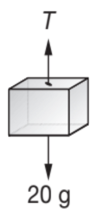
A block of mass 20 kg is suspended through two light spring balances as shown in figure. Calculate the



(a) reading of spring balance 1.
(b) reading of spring balance 2.

SOLUTION

For calculating the reading, first we draw F.B.D. of 20 kg block.



$$mg - T = 0$$

$$\Rightarrow \quad T = 20g = 200 \text{ N}$$

Since both balances are light so, both the scales will read 20 kg.

NEWTON'S SECOND LAW: REVISITED

We know from the first law, what happens when there is no unbalanced force on an object: its velocity remains constant. Now let us see What happens when there is an unbalanced force on an object? The Newton's second Law gives answer to this question, that is, net force acting on a body will produce an acceleration.

When there is a constant unbalanced force on an object, the object moves with constant acceleration. Furthermore, if the force varies, the acceleration varies in direct proportion with larger force producing larger acceleration. Twice the force produces twice the acceleration in the same mass.

The magnitude of the acceleration produced depends on the quantity of matter being pushed. The quantity of matter is referred to as the inertial mass.

Newton's Second Law states the relation between the net force and the inertial mass.

$$\sum \vec{F} = m\vec{a}$$

Note that the direction of acceleration is in the direction of the net force.

In terms of components

$$\sum F_x = ma_x,$$

$$\sum F_y = ma_y \text{ and}$$

$$\sum F_z = ma_z$$

Problem Solving Technique(s)

HOW TO MAKE FREE BODY DIAGRAMS (FBD): REVISITED

With the help of free body diagrams, it becomes easy for us to consider the motion of the object after taking the account the forces acting on it.

Rest is the case of motion where the net force acting on the body is zero. This is also the case of Equilibrium.

(a) Select a convenient co-ordinate axes.
(b) Isolate the object of interest.

(c) Draw all the forces that are acting on the isolated object keeping in mind that every force must be a part of action-reaction pair acting on two different bodies.
When drawing the FBD of the isolated object, **never take into account the forces that the isolated object is acting on others. Just draw the forces that are acting directly on the isolated object.**

(d) Apply Newton's Second Law such that we have

$$\sum \vec{F} = m\vec{a}$$

$$\Rightarrow \quad \sum F_x = ma_x, \sum F_y = ma_y, \sum F_z = ma_z$$

and obtain the equations required.

(e) If the body is at Rest or moving with a constant velocity, then the acceleration of the body is ZERO and this is also the CASE OF EQUILIBRIUM.

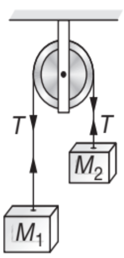
ATWOOD'S MACHINE

Masses M_1 and M_2 are tied to a string, which goes over a frictionless light pulley.
The string is light and inextensible.

(a) If $M_1 > M_2$ and they move with acceleration a, then

$$M_1 g - T = M_1 a$$

$$T - M_2 g = M_2 a$$



where T is the tension in the string. It gives

$$a = \left(\frac{M_1 - M_2}{M_1 + M_2}\right) g \text{ and } T = \left(\frac{2M_1 M_2}{M_1 + M_2}\right) g$$

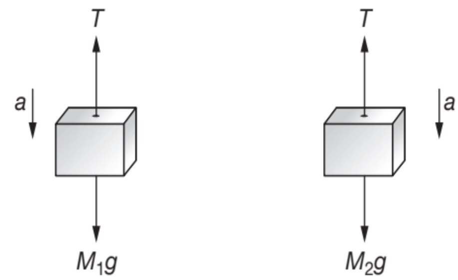
Thrust is given by $2T$. So,

$$\text{Thrust} = \left(\frac{4m_1 m_2}{m_1 + m_2}\right) g$$

(b) If the pulley begins to move with acceleration $\vec{a}_0$, then

$$a = \left(\frac{M_1 - M_2}{M_1 + M_2}\right)(\vec{g} - \vec{a}_0)$$

$$\text{and } T = \left(\frac{2M_1 M_2}{M_1 + M_2}\right)(\vec{g} - \vec{a}_0)$$



FBD M_1 FBD M_2

So, if pulley accelerates up, then

$$a = \left(\frac{M_1 - M_2}{M_1 + M_2}\right)(g + a_0)$$

$$\text{and } T = \left(\frac{2M_1 M_2}{M_1 + M_2}\right)(g + a_0)$$

and when it accelerates down, then

$$a = \left(\frac{M_1 - M}{M_1 + M_2}\right)(g - a_0)$$

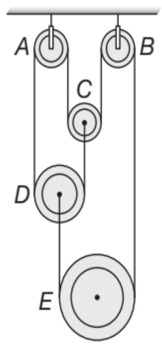
$$\text{and } T = \left(\frac{2M_1 M_2}{M_1 + M_2}\right)(g - a_0)$$

and thrust in the pulley is

$$\vec{T} = \frac{4M_1 M_2}{M_1 + M_2}(\vec{g} - \vec{a}_0)$$

ILLUSTRATION 27

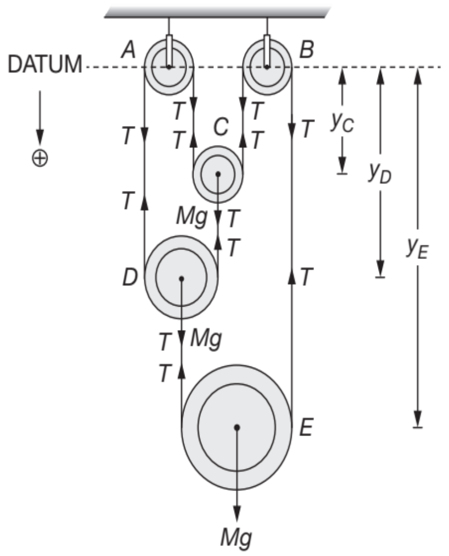
Consider identical pulleys A, B, C, D and E as shown in figure.



A and B are fixed while C, D and E are free to move. The pulleys are connected by light inextensible string. All the pulleys have identical masses say M each. At a particular instant, it is observed that C and D move down with velocity, $V_C = 2\ \text{ms}^{-1}$ and $V_D = 2\ \text{ms}^{-1}$. Calculate the velocity of pulley E at that instant. Also find the acceleration in the pulleys and the tension in the string.

SOLUTION

Let us first draw the fixed axis (called Datum) passing through the centre of pulleys A and B as shown in figure.



At any instant, the distance of the pulley C, D and E from the Datum be y_C, y_D and y_E respectively. Start from centre of C and go to centre, we observe a single thread

$\therefore$ Thread length = constant

$$\left(y_D - \cancel{y_C}\right) + \cancel{y_D} + \cancel{y_C} + y_C + y_E + \left(y_E - \cancel{y_D}\right) = \text{constant}$$

$$\Rightarrow \quad y_D + y_C + 2y_E = \text{constant} \qquad \ldots(1)$$

Take derivative w.r.t. time

$$\Rightarrow \quad \dot{y}_D + \dot{y}_C + 2\dot{y}_E = 0$$

$$\Rightarrow \quad V_D + V_C + 2V_E = 0 \qquad \ldots(2)$$

Now, take downwards as positive

$$V_C = +2\ \text{ms}^{-1}$$

$$V_D = +2\ \text{ms}^{-1}$$

$$\Rightarrow \quad 2 + 2 + 2V_E = 0$$

$$\Rightarrow \quad V_E = -2\ \text{ms}^{-1}$$

Since the pulleys are identical and the string is light and inextensible and connects from C to D as a single curve, therefore tension throughout the string will remain the same.

From (2) we get,

$$\dot{V}_D + \dot{V}_C + 2\dot{V}_E = 0$$

$$\Rightarrow \quad a_D + a_C + 2a_E = 0 \qquad \ldots(3)$$

FBD C

$$(Mg + T) - 2T = Ma_C \qquad \ldots(4)$$

$$Mg - T = Ma_C$$

FBD D

$$Mg + T - 2T = Ma_D$$

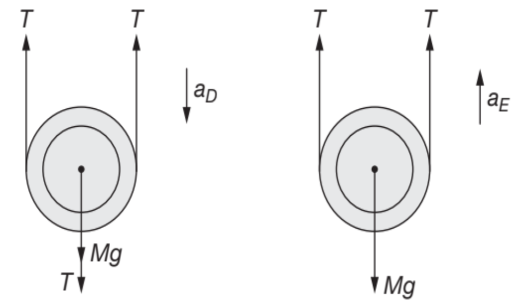
$$Mg - T = Ma_D \qquad \ldots(5)$$

From (4) and (5), we get

$$a_C = a_D \qquad \ldots(6)$$

From (6) and (3), we get

$$a_C = a_D = -a_E \qquad \ldots(7)$$



FBD E

$$2T - Mg = Ma_E \qquad \ldots(8)$$

From (4), we get

$$Mg - T = Ma \qquad \ldots(9)$$

From (8), we get

$$2T - Mg = Ma \qquad \ldots(10)$$

From (9) and (10)

$$(2 \times 9 + 10)$$

$$2Mg - Mg = 2Ma + Ma$$

$$\Rightarrow \quad a = \frac{g}{3}$$

Putting a in (9),

$$Mg - T = \frac{Mg}{3}$$

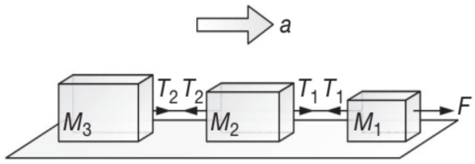
$\Rightarrow \quad 3Mg - 3T = Mg$

$\Rightarrow \quad 3T = 2Mg$

$\Rightarrow \quad T = \dfrac{2Mg}{3}$

MASSES CONNECTED WITH STRINGS

Three masses M_1, M_2, M_3 are connected with strings as shown in figure and lie on a frictionless surface.



They are pulled with a force F attached to M_1.

$$F - T_1 = M_1 a$$

$$T_1 - T_2 = M_2 a$$

$$T_2 = M_3 a$$

Such that the acceleration of system is

$$a = \frac{F}{M_1 + M_2 + M_3} \quad \text{and} \quad T_1 = \left(\frac{M_2 + M_3}{M_1 + M_2 + M_3}\right)F$$

$$T_2 = \left(\frac{M_3}{M_1 + M_2 + M_3}\right)F$$

MASSES ON A SMOOTH SURFACE IN CONTACT WITH EACH OTHER

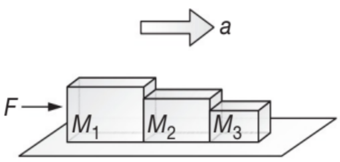
Three masses M_1, M_2, M_3 are placed on a smooth surface in contact with each other as shown in the figure. A force F pushes them and the three masses move with acceleration a, which is given by

$$a = \frac{F}{M_1 + M_2 + M_3}$$

$$F - F_2 = M_1 a$$

$$F_2 - F_3 = M_2 a$$

$$F_3 = M_3 a$$

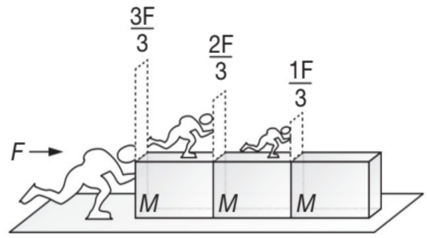


The forces acting are as follows

On M_1, the force is $F_1 = F$

On M_2, the force is $F_2 = \left(\dfrac{M_2 + M_3}{M_1 + M_2 + M_3}\right)F$

On M_3, the force is $F_3 = \left(\dfrac{M_3}{M_1 + M_2 + M_3}\right)F$



F_2 and F_3 are the forces of contact between M_1, M_2 and M_2, M_3 respectively.

If $M_1 = M_2 = M_3 = M(\text{say})$, then $a = \dfrac{F}{3M}$.

The forces acting are as shown in above figure.

Similarly we have seven identical masses then

$$a = \frac{F}{7M}$$

The forces acting are shown in above figure.

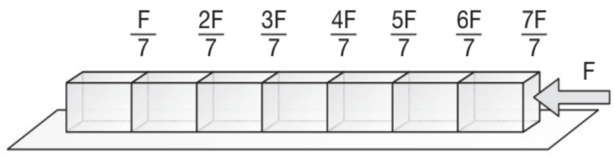
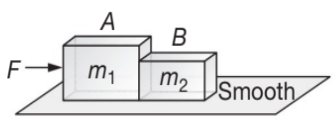


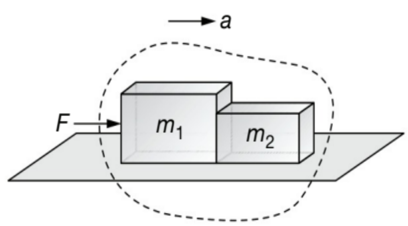
ILLUSTRATION 28

A force F is applied horizontally on mass m_1 as shown in figure. Find the contact force between m_1 and m_2.



SOLUTION

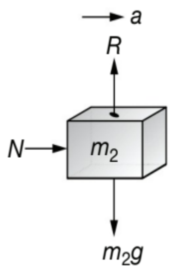
Considering both blocks as a system to find the common acceleration.



Common acceleration

$$a = \frac{F}{(m_1 + m_2)} \quad ...(1)$$

To find the contact force between A and B we draw F.B.D. of mass m_2.



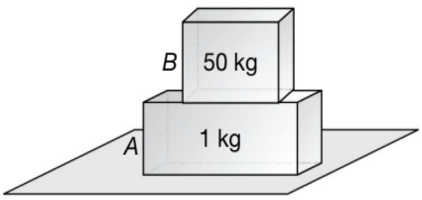
$$\Sigma F_x = ma_x$$

$$N = m_2 \cdot a$$

$$N = \frac{m_2 F}{(m_1 + m_2)}$$

ILLUSTRATION 29

A block of mass 50 kg is kept on another block of mass 1 kg as shown in figure. A horizontal force of 10 N is applied on the 1 kg block. Assuming all surfaces to be smooth and taking $g = 10\ \text{ms}^{-2}$. Calculate the

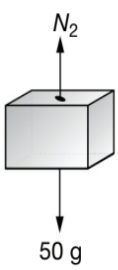


(a) acceleration of block A and B.

(b) force exerted by B on A.

SOLUTION

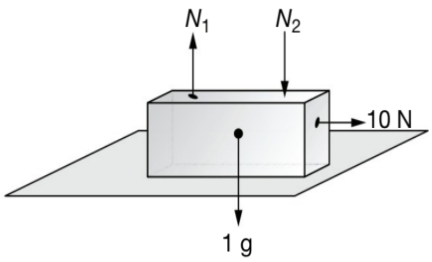
(a) F.B.D. of 50 kg block



$$N_2 = 50g = 500\ \text{N}$$

Along horizontal direction, there is no force, so $a_B = 0$.

(b) F.B.D. of 1 kg block



Along horizontal direction

$$10 = 1a_A$$

$$\Rightarrow \quad a_A = 10\ \text{ms}^{-2}$$

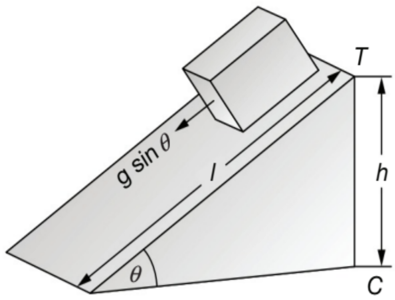
Along vertical direction

$$\Rightarrow \quad N_1 = N_2 + 1g$$

$$\Rightarrow \quad N_1 = 500 + 10 = 510\ \text{N}$$

BODY ON A SMOOTH INCLINED PLANE

Consider a body starting from rest moves along a smooth inclined plane of length l, height h and having angle of inclination θ.



(a) Its acceleration down the plane is $g \sin\theta$.

(b) Its velocity at the bottom of the inclined plane will be

$$\sqrt{2gh} = \sqrt{2gl \sin\theta}$$

(c) Time taken to reach the bottom will be

$$t = \sqrt{\frac{2l}{g \sin\theta}} = \sqrt{\frac{2h}{g \sin^2\theta}} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$$

(d) If angles of inclination are θ_1 and θ_2 keeping the height constant then $\frac{t_1}{t_2}=\frac{\sin\theta_2}{\sin\theta_1}$.

(e) If the angle of inclination is changed from θ_1 to θ_2 keeping the length constant then $\frac{t_1}{t_2}=\sqrt{\frac{\sin\theta_2}{\sin\theta_1}}$.

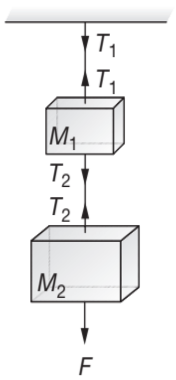
(f) The ratio of times to reach the bottom in the two cases are related to each other as

$$\left[\frac{t_1}{t_2}\right]_{\text{constant height}}=\left[\frac{t_1}{t_2}\right]^2_{\text{constant length}}$$

WHEN MASSES ARE SUSPENDED VERTICALLY FROM A RIGID SUPPORT

Two blocks of masses M_1 and M_2 are suspended vertically from a rigid support with the help of strings as shown in the figure. The mass M_2 is pulled down with a force F. The tension between the masses M_1 and M_2 will be

$$T_2=F+M_2g$$



Tension between the support and the mass M_1 will be

$$T_1=F+(M_1+M_2)g$$

Problem Solving Technique(s)

Whenever a light string is pulled then tension in the light string at all the points is equal to the pull.

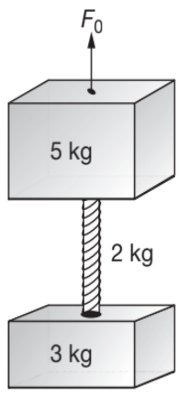
ILLUSTRATION 30

A 5 kg block has a rope of mass 2 kg attached to its underside and a 3 kg block is suspended from the other end of the rope. The whole system is accelerated upward at $2\ \text{ms}^{-2}$ by an external force F_0.

(a) What is F_0?

(b) What is the net force on rope?

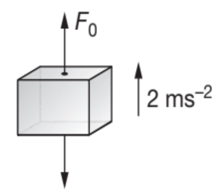
(c) What is the tension at middle point of the rope? $(g=10\ \text{ms}^{-2})$



SOLUTION

For calculating the value of F_0, consider two blocks with the rope as a single system.

(a) F.B.D. of whole system



$$F_0-100=10\times 2$$

$$\Rightarrow\quad F_0=120\text{ N} \qquad \ldots(1)$$

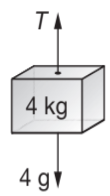
(b) According to Newton's Second Law, net force on rope

$$F=ma \quad \text{where } m=m_{\text{rope}}=2\text{ kg}$$

$$\Rightarrow\quad F=(2)(2)$$

$$\Rightarrow\quad F=4\text{ N} \qquad \ldots(2)$$

(c) For calculating tension at the middle point we draw F.B.D. of 3 kg block with half of the rope (mass 1 kg) as shown. So, total mass becomes 4 kg.



$$T-4g=4(2)$$

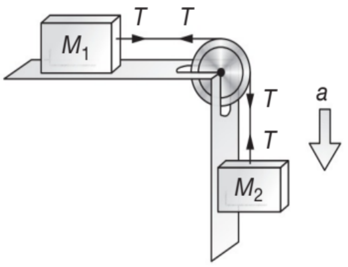
$$\Rightarrow\quad T=48\text{ N}$$

WHEN TWO MASSES ARE ATTACHED TO A STRING WHICH PASSES OVER A PULLEY ATTACHED TO THE EDGE OF A HORIZONTAL TABLE

Let the mass M_1 lies on the frictionless surface of the table (as shown in the figure). Let the tension in the string be T and the acceleration of the system be a. Then from Cause-Effect equations, we get

$M_2g - T = M_2a$ and $T = M_1a$, such that

$$a = \left(\frac{M_2}{M_1 + M_2}\right)g \text{ and } T = \left(\frac{M_1M_2}{M_1 + M_2}\right)g$$

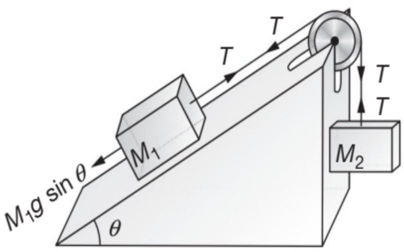


The system is accelerated for all values of M_1 and M_2 (as long as friction is assumed to be absent)

Thrust on pulley is $\sqrt{2}T$.

WHEN TWO MASSES ARE ATTACHED TO A STRING WHICH PASSES OVER A PULLEY ATTACHED TO THE EDGE OF AN INCLINED PLANE

Two masses M_1 and M_2 are attached to the ends of a string, which passes over a frictionless pulley at the top of the inclined plane of inclination θ. Let the tension in the string be T.



Two cases arise.

(a) When the mass M_1 moves upwards with acceleration a, then

$$a = \left(\frac{M_2 - M_1\sin\theta}{M_1 + M_2}\right)g$$

$$T = \left(\frac{M_1M_2(1+\sin\theta)}{M_1 + M_2}\right)g$$

(b) When the mass M_1 moves downwards with acceleration a, then

$$a = \left(\frac{M_1\sin\theta - M_2}{M_1 + M_2}\right)g$$

$$T = \left(\frac{M_1M_2(1+\sin\theta)}{M_1 + M_2}\right)g$$

Thrust F on the pulley in both the cases is the resultant of T and T inclined at an angle of $(90-\theta)$. So,

$$F = \sqrt{T^2 + T^2 + 2T^2\cos(90-\theta)}$$

$$F = \sqrt{2}T\sqrt{1+\sin\theta}$$

Since $\sqrt{1+\sin\theta} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)$

$$F = \sqrt{2}T\left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right]$$

From the above relations we can derive the following conclusions.

(i) System does not accelerate $(a = 0)$ when $M_2 = M_1\sin\theta$.

(ii) If the positions of the masses are reversed, the tension in the string remains unchanged. Also, the acceleration of the system remains unchanged.

(iii) If $M_1 = M_2 = M$ (say), then

$$a = \frac{1}{2}(1-\sin\theta)g$$

$$\Rightarrow \quad a = \frac{1}{2}\left\{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right\}^2 g$$

$$\Rightarrow \quad a = \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2\left(\frac{g}{2}\right) \text{ and}$$

$$T = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2\left(\frac{Mg}{2}\right)$$

WHEN TWO MASSES ARE ATTACHED TO A STRING WHICH PASSES OVER A PULLEY ATTACHED TO THE TOP OF A DOUBLE INCLINED PLANE

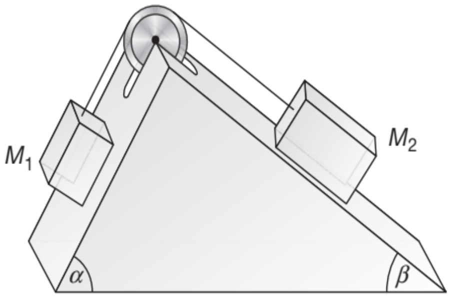
Two masses M_1 and M_2 are attached to the ends of a string over a pulley fixed to the top of a double inclined plane of angles of inclination α and β.

Let M_2 move downwards with acceleration a and the tension in the string be T. Then

$$a = \left(\frac{M_2\sin\beta - M_1\sin\alpha}{M_1 + M_2}\right)g$$

$$T=\frac{M_1M_2g}{(M_1+M_2)}[\sin\beta+\sin\alpha]$$

For system not to accelerate we must have $M_2\sin\beta=M_1\sin\alpha$.



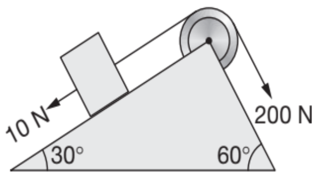
Thrust F on the pulley is the resultant of T and T inclined at an angle of $180-(\alpha+\beta)$. So,

$$\text{Thrust}=\sqrt{T^2+T^2+2T^2\cos\{180-(\alpha+\beta)\}}$$

$$\Rightarrow\quad F=\sqrt{2}T\sqrt{1-\cos(\alpha+\beta)}=2T\sin\left(\frac{\alpha+\beta}{2}\right)$$

ILLUSTRATION 31

A 10 kg block kept on an inclined plane is pulled by a string applying 200 N force. A 10 N force is also applied on 10 kg block as shown in figure. Calculate the



(a) tension in the string.

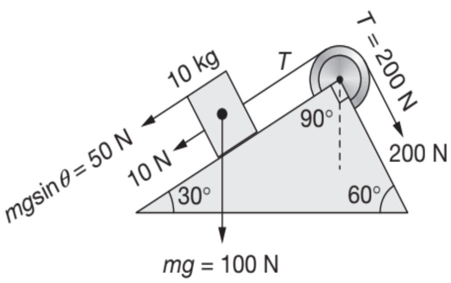
(b) acceleration of 10 kg block.

(c) net force on pulley exerted by string.

SOLUTION

(a) $T=200\text{ N}$

(b) $T-10-mg\sin\theta=ma$

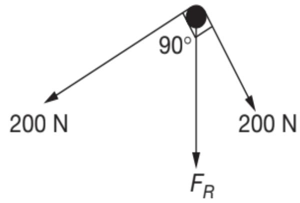


$$\Rightarrow\quad T-10-50=10a$$

$$\Rightarrow\quad 200-60=10a$$

$$\Rightarrow\quad a=\frac{140}{10}=14\text{ ms}^{-2}$$

(c)



$$\Rightarrow\quad F_R=\sqrt{(200)^2+(200)^2}$$

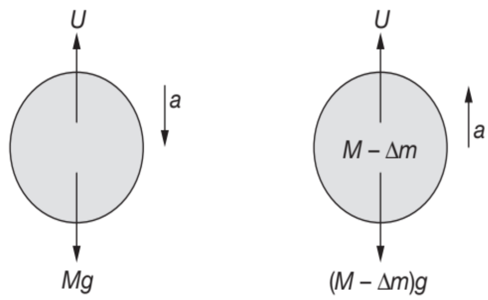
$$\Rightarrow\quad F_R=200\sqrt{2}\text{ N}$$

ILLUSTRATION 32

An aerostat of mass m starts coming down with a constant acceleration a. Determine the ballast mass to be dumped from the aerostat to reach the upward acceleration of the same magnitude. The air drag is to be neglected.

SOLUTION

$$Mg-U=Ma \qquad \ldots(1)$$



Let the mass Δm be dumped

$$U-(M-\Delta m)g=(M-\Delta m)a \qquad \ldots(2)$$

Adding (1) and (2),

$$Mg-(M-\Delta m)g=Ma+(M-\Delta m)a$$

$$\Delta mg=a(2m-\Delta M)$$

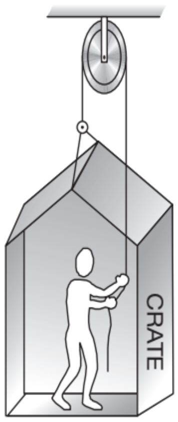
$$\Delta Mg=2ma-\Delta Ma$$

$$\Delta M(g+a)=2ma$$

$$\Delta M=\frac{2Ma}{g+a}$$

ILLUSTRATION 33

A painter of mass 100 kg is raising himself and the crate (such an arrangement is called Bosun's chair) on which he stands as shown. When he pulls the rope the force exerted by him on the crate's floor is 450 N. If the crate weighs 25 kg then, find the acceleration of the system and the tension in the rope.



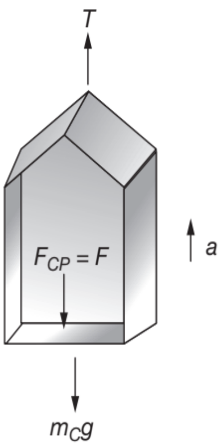
SOLUTION

$$T + F - m_P g = m_P a \quad \text{(For Painter)} \qquad \ldots(1)$$

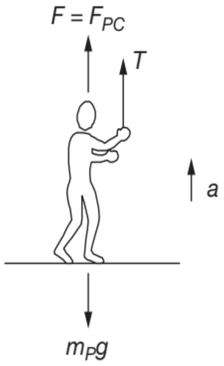
$$T - F - m_C g = m_C a \quad \text{(For crate)} \qquad \ldots(2)$$

Subtract (2) from (1), we get

$$2F + (m_C - m_P)g = (m_P - m_C)a$$



F.B.D. CRATE



F.B.D. PAINTER

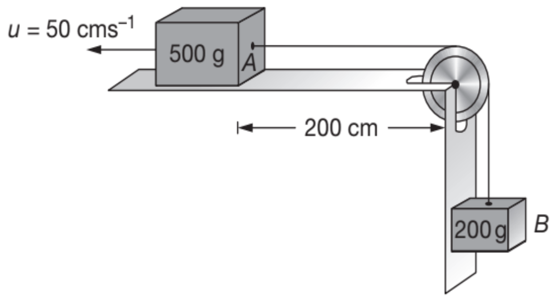
$$\Rightarrow \quad 900 + (25 - 100)(10) = 75a$$

$$\Rightarrow \quad 900 - 750 = 75a$$

$$\Rightarrow \quad a = 2\ \text{ms}^{-2},\ T = 750\ \text{N}$$

ILLUSTRATION 34

A block A of mass 500 g is placed on a smooth horizontal table with a light string attached to it. The string passes over a smooth pulley P to the right of A at the end of the table and connected to another block B of mass 200 g. The portion of the string over the table is parallel to the table surface and the portion beyond the pulley is vertical. Initially the block A is at a distance of 200 cm from pulley and is moving with a speed of 50 cms^{-1} to the left. Find the position and velocity of A at $t = 1$ s.



SOLUTION

$$T = M_1 g$$

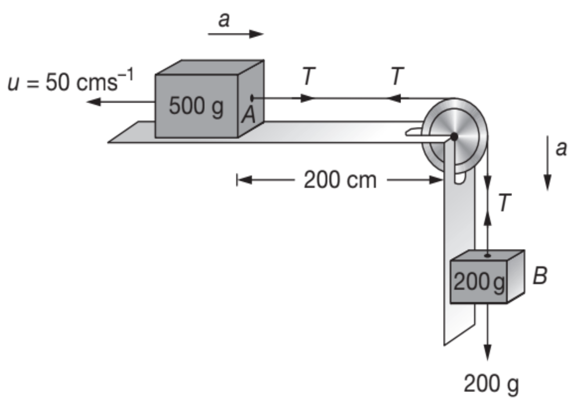
$$T = 500\ \text{g}$$

$$M_B g - T = M_B a \qquad \ldots(1)$$

$$T = M_A a \qquad \ldots(2)$$

$$a = \frac{M_B g}{M_A + M_B}$$

$$a = \left(\frac{200}{700}\right)(9.8)$$



$$a = 280\ \text{cms}^{-2}$$

$$s = ut + \frac{1}{2}at^2$$

$$s = (-50)(1) + \frac{1}{2}(280)(1)^2$$

$$s = 90 \text{ cm}$$

Distance covered $= 110$ cm

Using, $v = u + at$

$$v = -50 + (280)(1)$$

$$v = 280 - 50$$

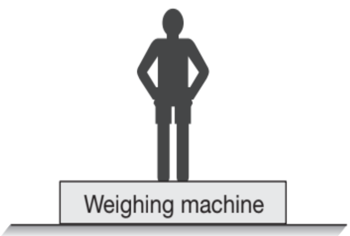
$$v = 230 \text{ cms}^{-1}$$

WEIGHING MACHINE

A weighing machine does not measure the weight but measures the force exerted by object at the upper surface of the machine (i.e. the surface on which the person is standing).

ILLUSTRATION 35

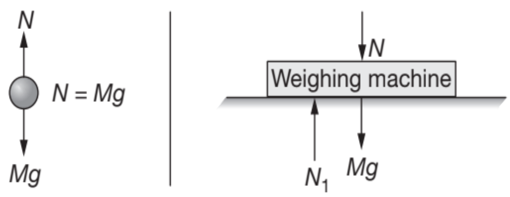
A man of mass 60 kg is standing on a weighing machine placed on ground. Calculate the reading of machine $(g = 10 \text{ ms}^{-2})$.



SOLUTION

For calculating the reading of weighing machine, we draw F.B.D. of man and machine separately. F.B.D. of man, F.B.D. of weighing machine.
Here force exerted by object on upper surface is N, so the reading of weighing machine is

$$N = Mg$$

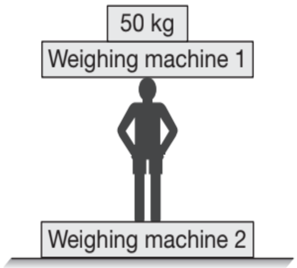


$\Rightarrow \quad N = 60 \times 10$

$\Rightarrow \quad N = 600$ N

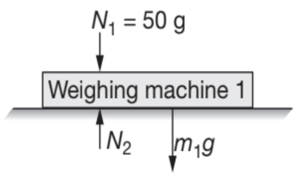
ILLUSTRATION 36

A man of mass 60 kg is standing on a weighing machine (2) of mass 5 kg placed on ground. Another same weighing machine is placed over man's head. A block of mass 50 kg is put on the weighing machine (1). Calculate the readings of weighing machines (1) and (2).



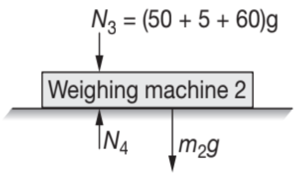
SOLUTION

Since a weighing machine measures the force applied at its upper surface, so we have the reading of weighing machine 1 given by



$$R_1 = N_1 = 50g = 500 \text{ N}$$

Similarly, we have the reading of weighing machine 2 given by



$$R_2 = N_3$$

$\Rightarrow \quad R_2 = (50 + 5 + 60)g$

$\Rightarrow \quad R_2 = 115 \times 10$

$\Rightarrow \quad R_2 = 1150$ N

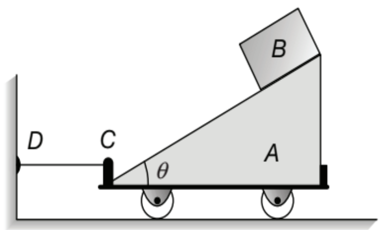
Test Your Concepts-IV

Based on Newton's Laws of Motion: Acceleration Systems

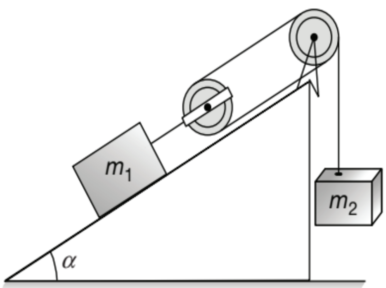
(Solutions on page H.199)

1. A weight of 200 kg hangs freely from the free end of a rope. The weight is hauled up vertically from rest by winding up the rope. The pull starts at 250 kgwt and diminishes uniformly at the rate of 1 kgwt per metre wound up. Find the velocity after 20 metre have been wound up. Neglect the weight of the rope $\left(\text{Take } g = 10 \text{ ms}^{-2}\right)$.

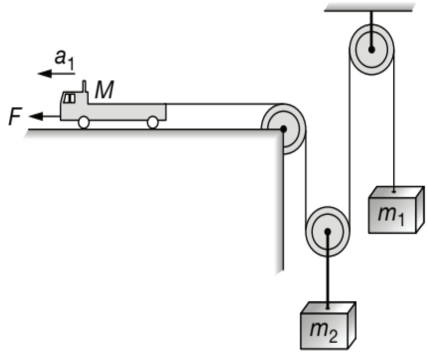
2. Block B has a mass m and is released from rest when it is on top of wedge A, which has a mass 3 m. Determine the tension in cord CD required to hold the wedge from moving while B is sliding down A. Neglect friction.



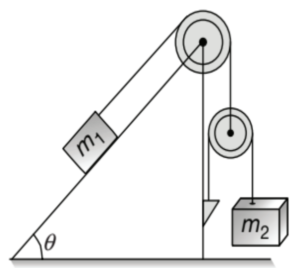
3. Find the acceleration of masses m_1 and m_2. The string and the pulley are massless and frictionless.



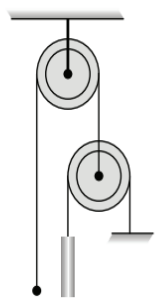
4. A toy truck of mass $M = 2$ kg is moving towards left with an acceleration a_1 due to a force $F = 20$ N as shown in figure. It is connected to a mass $m_1 = 1$ kg with a light and frictionless string, passing over a movable massless pulley, to which another mass $m_2 = 0.5$ kg is connected. Find the force acting on the truck towards right and the accelerations of mass m_1 and m_2. Take $g = 10 \text{ ms}^{-2}$.



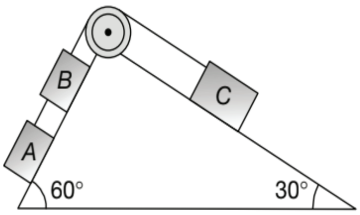
5. Find the acceleration of the body of mass m_2 in the arrangement shown in figure. If the mass m_2 is η times great as the mass m_1 and the angle that the inclined plane forms with the horizontal is equal to θ. The masses of the pulleys and threads, as well as the friction, are assumed to be negligible.



6. In the arrangement shown in figure the mass of the ball is η times as great as that of the rod. The length of the rod is l, the masses of the pulleys and the threads, as well as the friction, are negligible. The ball is set on the same level as the lower end of the rod and then released. How soon will the ball be opposite the upper end of the rod?

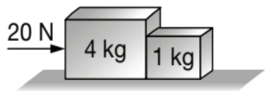


7. In the arrangement shown, mass of A, B and C are 1 kg, 3 kg and 2 kg, respectively, and friction is absent. Find the

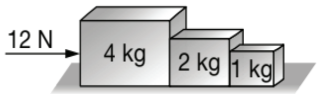


(a) acceleration of the system and
(b) tensions in the string.
Take $g = 10\ \text{ms}^{-2}$

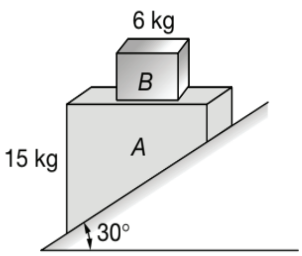
8. Two blocks of mass 4 kg and 1 kg are placed side by side on a smooth horizontal surface as shown in the figure. A horizontal force of 20 N is applied on 4 kg block. Find
(a) the acceleration of each block
(b) the normal reaction between two blocks.



9. Three blocks having mass 4 kg, 2 kg and 1 kg are placed side by side on a smooth surface as shown in figure. A horizontal force of 14 N is applied on 4 kg block. Find the net force on 2 kg block.



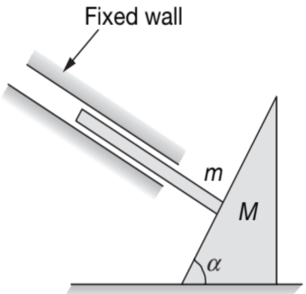
10. A 6 kg block B rests as shown on the upper surface of a 15 kg wedge A. Neglecting friction, determine the acceleration of A, the acceleration of B relative to A, immediately after the system is released from rest $\left(\text{Take } g = 10\ \text{ms}^{-2}\right)$.



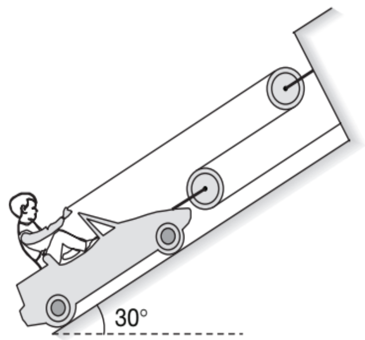
11. A heavy uniform chain of length $2l$, hangs over a small smooth fixed pulley, the length $l + c$ being at one side and $l - c$ at the other. If the end of the shorter portion be held and then let go, show that the chain will slip off the pulley in time given by

$$\sqrt{\frac{l}{g}}\log\left(\frac{l+\sqrt{l^2-c^2}}{c}\right)$$

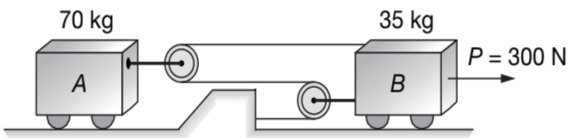
12. In the arrangement shown in the figure, the rod of mass m held by two smooth walls, remains always perpendicular to the surface of the wedge of mass M. Assuming all the surfaces are frictionless, find the acceleration of the rod and that of the wedge.



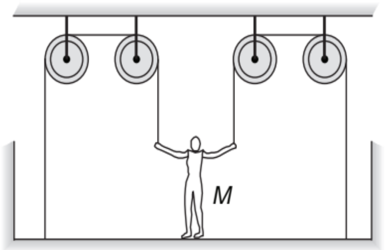
13. A man pulls himself up the 30° incline by the method shown. If the combined mass of the man and cart is 100 kg, determine the acceleration of the cart if the man exerts a pull of 250 N on the rope. Neglect all friction and the mass of the rope, pulleys and wheels.



14. Determine the accelerations of bodies A and B and the tension in the cable due to the application of the 300 N force. Neglect all friction and the masses of the pulleys.



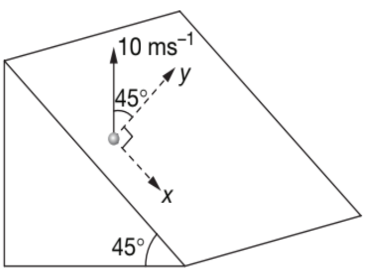
15. A painter of mass M stand on a crate of mass m and pulls himself up by two ropes which hang over pulley as shown. He pulls each rope with force F and moves upward with a uniform acceleration a. Find a, neglecting the fact that no one could do this for long time.



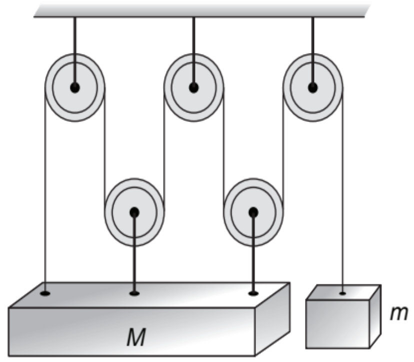
16. Two men A and B having masses M and $M+m$, start simultaneously from the ground and climb with uniform vertical accelerations up the free ends of a light inextensible rope which passes over a smooth pulley at a height h from the ground. If the lighter of the two men reaches the pulley in time t second, show that the heavier cannot get nearer to it than $\frac{m}{M+m}\left(\frac{gt^2}{2}+h\right)$.

17. A block is kept on the floor of an elevator at rest. The elevator starts descending with an acceleration of $a_0 = 12\ \text{ms}^{-2}$. Find the displacement of the block during the first 0.2 s after the start. Take $g = 10\ \text{ms}^{-2}$.

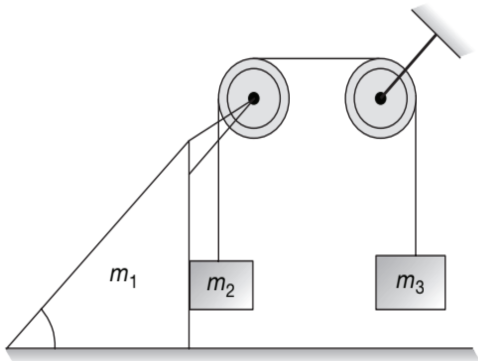
18. The small marble is projected with a velocity of $10\ \text{ms}^{-1}$ in a direction 45° from the horizontal y-direction on the smooth inclined plane. Calculate the magnitude v of its velocity after 2 second. Take $g = 10\ \text{ms}^{-2}$.



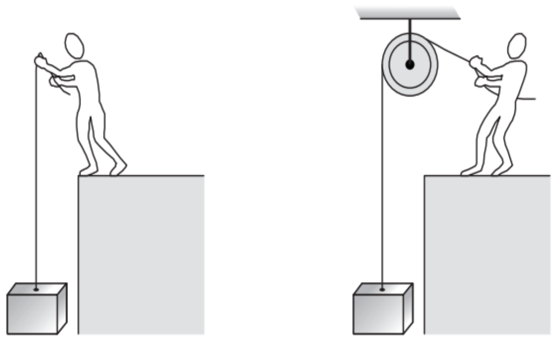
19. In the arrangement shown, the pulleys are light and frictionless and strings are light and inextensible. Find the acceleration of the two masses as shown in figure.



20. In the shown arrangement both pulleys and the string are massless and all the surfaces are frictionless. Find the acceleration of the wedge.



21. A man of 50 kg raises a block of mass 25 kg in two different ways as shown in figure. Calculate the action on the floor by the man in the two cases. If the floor yields to a normal force of 700 N, which mode should the man adopt to lift the block without the floor yielding? Take $g = 9.8\ \text{ms}^{-2}$.

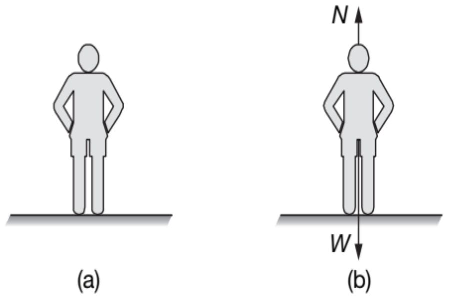


(a) (b)

EQUILIBRIUM OF COPLANAR FORCES

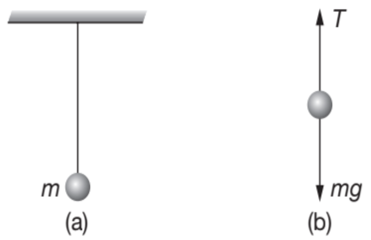
A body or a system is said to be in equilibrium if it does not tend to undergo any further change of its own. Any further change must be produced by external means (e.g., force).

The simplest kind of equilibrium situation is one where two forces act on a body. When you stand motionless, you experience the downward gravitational pull of the earth, your weight $\vec{W}$. The weight is balanced by an upward force exerted on you by the floor. This force is perpendicular to the floor and it is called the **normal force** $\vec{N}$. Note that although $\vec{N}$ and $\vec{W}$ are equal and opposite, they do not constitute an action-reaction pair.

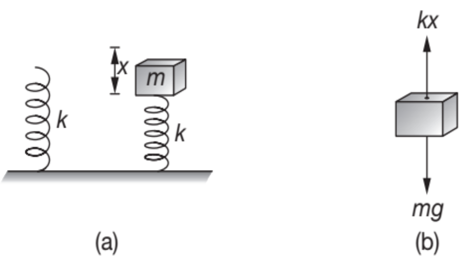


(a) A man standing on floor is in equilibrium.
(b) The free body diagram of the man gives $W = N$.

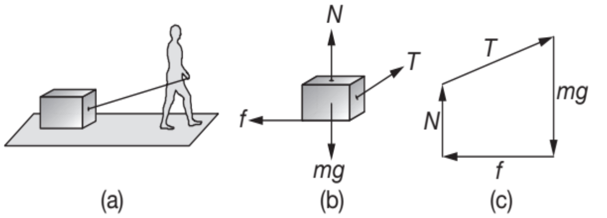
Some other examples of static equilibrium are shown in the following figures.



(a) A ball of mass m suspended from the ceiling with an inextensible string in equilibrium.
(b) The free body diagram of the ball gives $T = mg$.



(a) A block of mass m supported by a spring is in equilibrium.
(b) The free body diagram of the block gives $kx = mg$.



(a) A block of mass m is being pulled with a constant velocity on a horizontal surface.
(b) The free body diagram of the block.
(c) The block is in dynamic equilibrium because the vector sum of the forces is zero.

There is no net force $(\vec{F})$ and no net torque $(\vec{\tau})$ acting on the body, i.e., net force and torque acting on the body have to be zero.

Equilibrium	
Translational Equilibrium	**Rotational Equilibrium**
• $\sum \vec{F} = \vec{0}$	• $\sum \vec{\tau} = \vec{0}$
• $\sum F_x = 0$	• $\sum \tau_x = 0$
• $\sum F_y = 0$	• $\sum \tau_y = 0$
• $\sum F_z = 0$	• $\sum \tau_z = 0$

Conceptual Note(s)

(a) In the arrangement shown, for translational equilibrium to exist, we have

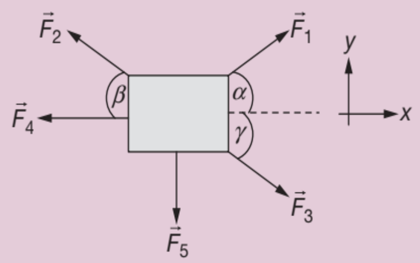
$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

$\sum F_x = 0$ gives

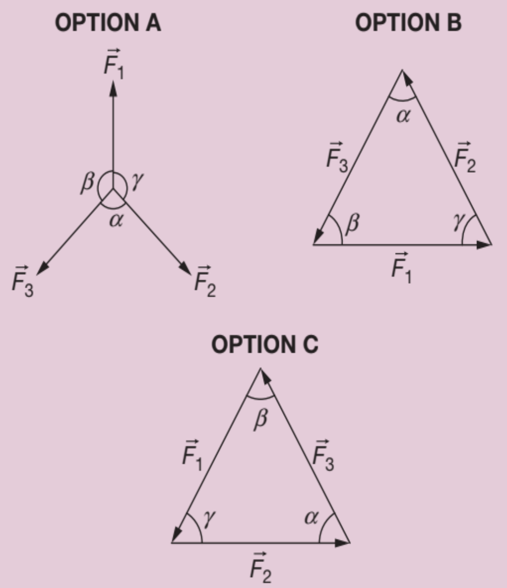
$$F_1 \cos\alpha + F_3 \cos\gamma - F_4 - F_2 \cos\beta = 0 \quad ...(1)$$

$\sum F_y = 0$ gives

$$F_1 \sin\alpha + F_2 \sin\beta - F_5 - F_3 \sin\gamma = 0 \quad ...(2)$$



(b) If three forces $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ happen to be in equilibrium, as shown (non-collinear lying in same plane)



$$\Rightarrow \quad \frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

$$\text{or} \quad \frac{\sin\alpha}{F_1} = \frac{\sin\beta}{F_2} = \frac{\sin\gamma}{F_3}$$

ROTATIONAL EQUILIBRIUM (LAW OF CONSERVATION OF MOMENTS OF FORCE)

$$\sum \vec{\tau} = \vec{0}$$

In an easy to use scalar form we must remember that

$$\sum F_i (r_i)_\perp = 0$$

where $(r_i)_\perp$ is the $\perp$ distance of the ith force F_i, from the point about which moments are to be taken.

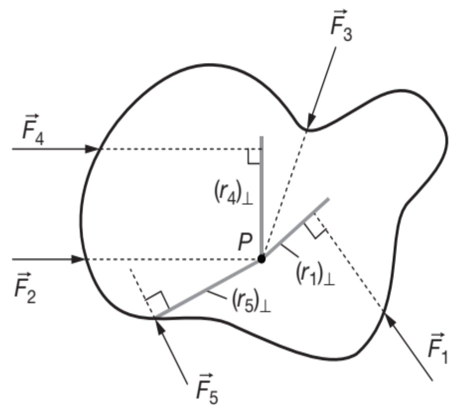
$$\Rightarrow \quad \sum_{CW} F_i (r_i)_\perp = \sum_{CCW} F_i (r_i)_\perp$$

Law of conservation of moments

$$\begin{pmatrix}\text{Total clockwise}\\ \text{moments of}\\ \text{force}\end{pmatrix} = \begin{pmatrix}\text{Total counter}\\ \text{clockwise moments}\\ \text{of force}\end{pmatrix}$$

EXAMPLE:

Consider a body on which forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, $\vec{F}_4$ and $\vec{F}_5$ are acting as shown. To check for the rotational equilibrium of the body, we shall be following the steps given.



STEP-1: Assume clockwise (*CW*) sense as positive.

STEP-2: Extend all the forces to check out which of them passes through the point about which moments have to be calculated (in this case *P*).

STEP-3: For the forces that pass through point *P* (like $\vec{F}_2$ and $\vec{F}_3$), $(r_2)_\perp = (r_3)_\perp = 0$.

So, the forces that pass through the point about which moments are to calculated will have no contribution to the moments of force.

STEP-4: For the left out forces (like $\vec{F}_1$, $\vec{F}_4$ and $\vec{F}_5$), let us calculate the moments taking each one by one as if the remaining others are absent.

For this we drop perpendiculars from the point *P* on the extended force lines. So that we get $\left((r_1)_\perp, (r_4)_\perp \text{ and } (r_5)_\perp\right)$.

STEP-5: Now for equilibrium, net moment of force is zero.

$$\Rightarrow \quad \underbrace{F_1(r_1)_\perp}_{CCW} + \underbrace{F_4(r_4)_\perp}_{CW} + \underbrace{F_5(r_5)_\perp}_{CW} = 0$$

Since *CW* sense is taken as positive, so we have

$$-F_1(r_1)_\perp + F_4(r_4)_\perp + F_5(r_5)_\perp = 0$$

$$\Rightarrow \quad F_1(r_1)_\perp = F_4(r_4)_\perp + F_5(r_5)_\perp$$

Conceptual Note(s)

(a) If all forces give *CW* moments, net moments cannot be zero.

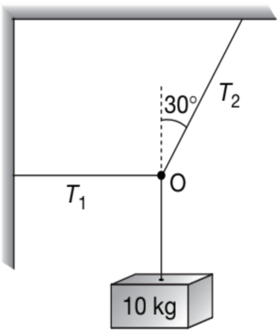
(b) If the net moment of force is non-zero, then net moment is given by

$$\tau = F_4(r_4)_\perp + F_5(r_5)_\perp - F_1(r_1)_\perp$$

If τ is ⊕, net moment is clockwise (CW)
If τ is ⊖, net moment is counter clockwise (CCW).

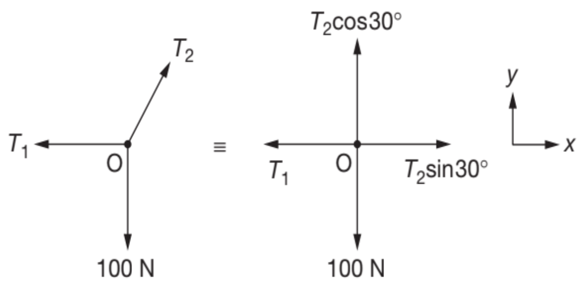
ILLUSTRATION 37

A block of mass 10 kg is suspended with two strings, as shown in the figure. Find the tensions T_1 and T_2 in the two strings.



SOLUTION

The free body diagram of the joint O is drawn as shown in the following figure.



Applying equations for equilibrium.

$$\sum F_x = 0$$

$$T_2 \sin 30° - T_1 = 0 \qquad \ldots(1)$$

$$\sum F_y = 0$$

$$T_2 \cos 30° - 100 = 0 \qquad \ldots(2)$$

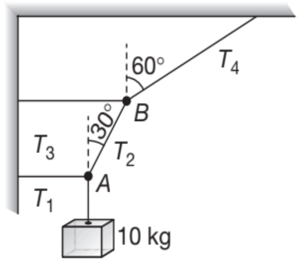
Thus, $T_2 = \dfrac{100}{\cos 30°} = \dfrac{200}{\sqrt{3}}$ N

Substituting the value of T_2 in equation (1), we get

$$T_1 = T_2 \sin 30° = \frac{100}{\sqrt{3}} \text{ N}$$

ILLUSTRATION 38

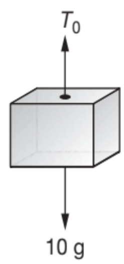
The system shown in figure is in equilibrium.



Find the magnitude of tension in each string T_1, T_2, T_3 and T_4. (Take $g = 10 \text{ ms}^{-2}$)

SOLUTION

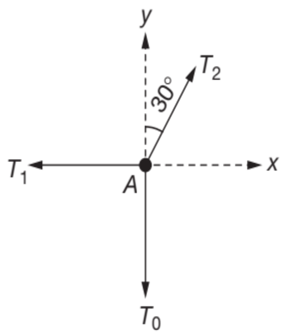
F.B.D. of 10 kg block



$$T_0 = 10g$$

$$\Rightarrow \quad T_0 = 100 \text{ N}$$

F.B.D. of point A



$$\Sigma F_y = 0$$

$$\Rightarrow \quad T_2 \cos(30°) = T_0 = 100 \text{ N}$$

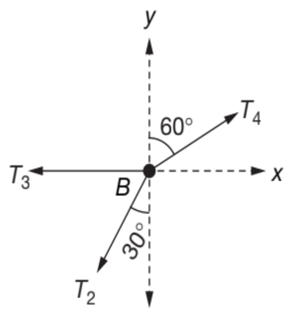
$$\Rightarrow \quad T_2 = \frac{200}{\sqrt{3}} \text{ N}$$

$$\Sigma F_x = 0$$

$$\Rightarrow \quad T_1 = T_2 \sin(30°)$$

$$\Rightarrow \quad T_1 = \left(\frac{200}{\sqrt{3}}\right)\left(\frac{1}{2}\right) = \frac{100}{\sqrt{3}} \text{ N}$$

F.B.D. of point B



$$\Sigma F_y = 0$$

$$\Rightarrow \quad T_4 \cos 60° = T_2 \cos 30°$$

$$\text{and} \quad \Sigma F_x = 0$$

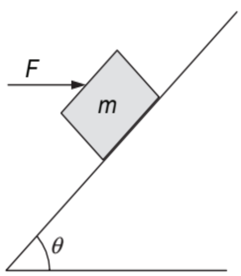
$$\Rightarrow \quad T_3 + T_2 \sin 30° = T_4 \sin 60°$$

$$\Rightarrow \quad T_3 = \frac{200}{\sqrt{3}} \text{ N and}$$

$$T_4 = 200 \text{ N}$$

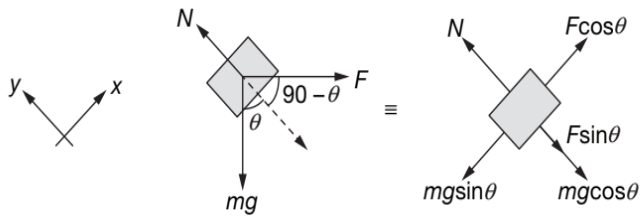
ILLUSTRATION 39

Find the magnitude of the horizontal force F required to keep the block of mass m stationary on the smooth inclined plane as shown in the figure.



SOLUTION

The forces acting on the block are shown in the free body diagram.



Applying equations for equilibrium,

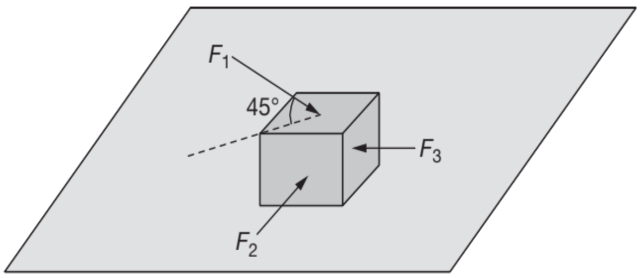
$$\sum F_x = 0$$

$$F \cos\theta - mg \sin\theta = 0$$

$$\Rightarrow \quad F = \frac{mg \sin\theta}{\cos\theta} = mg \tan\theta$$

ILLUSTRATION 40

A cubical block is experiencing three forces as shown in figure. Find the friction force acting on the block if it is at rest. Given that $F_1 = 30$ N; $F_2 = 50$ N and $F_3 = 42$ N.



SOLUTION

As shown in figure force F_1 is having three components, one along vertical $\left(= \frac{F_1}{\sqrt{2}}\right)$, other along force $F_2 \left(= \frac{F}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right)$ and one against $F_3 \left(= \frac{F}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right)$. Here the net horizontal force acting on the block is given by

$$F_{\text{net}} = \sqrt{\left(F_2 + \frac{F}{\sqrt{4}}\right)^2 + \left(F_3 - \frac{F}{\sqrt{4}}\right)^2}$$

$$\Rightarrow \quad F_{\text{net}} = \sqrt{(50+15)^2 + (42-15)^2} = 70.384 \text{ N}$$

As it is given that block is in static equilibrium, thus sum of all horizontal forces acting on it is must be zero. Sum of given external horizontal forces is 70.384 N and is in a direction $\theta = \tan^{-1}\left(\frac{65}{67}\right)$ from the direction of force F_3. We can state that friction force acting on block must be exactly opposing this force so as to keep the block in static equilibrium.

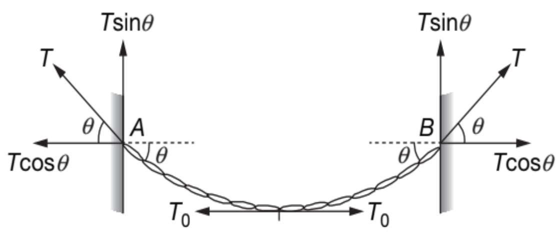
ILLUSTRATION 41

A chain of mass m is attached at two points A and B of two fixed walls as shown in figure. Due to its weight a sag is there in the chain such that at point A and B it makes an angle θ with the normal to the wall. Find the tension in the chain at : (Assume tension is always along the length of chain).

(a) point A and B.
(b) mid point of the chain

SOLUTION

(a) Let the tension in the chain at point A and B is T, so at these points chain will pull the wall hinges with the same force and wall hinges will also exert same force on chain in tangential direction as shown in figure.



Now for vertical equilibrium of chain we have

$$2T\sin\theta = mg$$

$$\Rightarrow \quad T = \frac{1}{2}mg\,\text{cosec}\,\theta$$

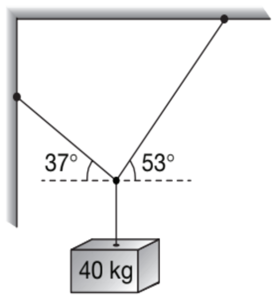
(b) Let tension at the mid point of chain be T_0. It must be along horizontal direction as at midpoint slope is zero. For horizontal equilibrium of chain we can state that at every point horizontal component of the tension in the chain must be equal as no other external force is acting on it in horizontal direction. Thus we have

$$T_0 = T\cos\theta$$

$$\Rightarrow \quad T_0 = \frac{1}{2}mg\cot\theta$$

ILLUSTRATION 42

The object in figure weighs 40 kg and hangs at rest. Find the tensions in the three cords that hold it.



SOLUTION

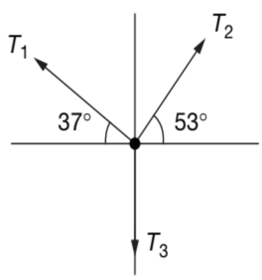
Because the object is at equilibrium, the vector sum of the forces acting directly on it must be zero. There are only two such forces, the tension in the lower cord and the pull of gravity, 400 N. Therefore, the tension in the lower cord must be 400 N. It is the tension that supports the object. Figure shows the junction where the three cords meet. As the system is in equilibrium, net sum of all the forces at the junction must be zero. For this we resolve the tensions in horizontal and perpendicular direction as

In horizontal direction

$$0.6T_2 - 0.8T_1 = 0 \qquad \ldots(1)$$

In vertical direction

$$0.6T_1 + 0.8T_2 - 400 = 0 \qquad \ldots(2)$$



On solving the above equations we get

$$T_1 = 240 \text{ N} \text{ and } T_2 = 320 \text{ N}$$

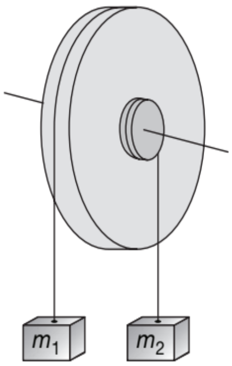
and the tension in third cord we already have

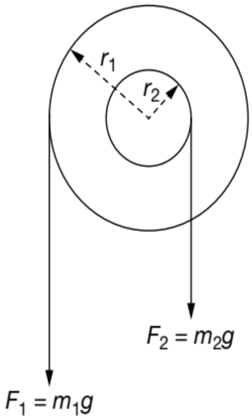
$$T_2 = 400 \text{ N}$$

EXAMPLES AND SITUATIONS FOR ROTATIONAL EQUILIBRIUM

To learn how forces and rotations are related, we can perform the experiment shown in figure. We see there a wheel that consists of two disks cemented together. It is free to rotate on a stationary axle that we call the axis, or pivot, of rotation. By hanging objects from the two cords, we can determine the turning effect of a force. The force F_2 tries to turn the wheel clockwise, while F_1 tries to turn the wheel counterclockwise. By experimenting with different radii r_1 and r_2 for the two disks, we find that the two turning effects balance whenever

$$F_1 r_1 = F_2 r_2$$

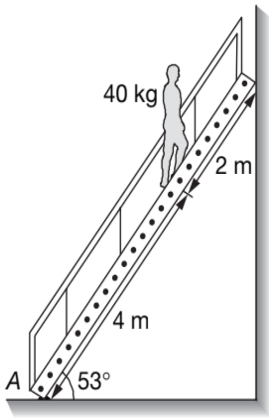




The above relation, product of force and the radii, is known as torque. Torque is the physical quantity which measures the turning effect of a force on a body. Its magnitude is given by the product of the force and the perpendicular distance from the axis of rotation or pivot. In above case it is simply the product of force and the radii.

ILLUSTRATION 43

The uniform 20 kg ladder hinged at the bottom in figure, leans against a smooth wall. If a 40 kg person stands on the ladder shown, how large are the forces at the wall the ground.



SOLUTION

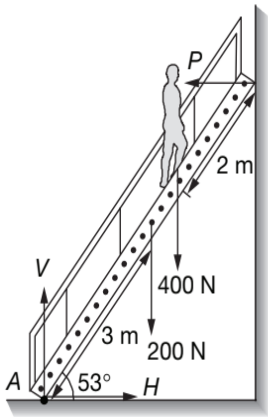
Consider figure, all the forces acting on the ladder are shown. For horizontal and vertical equilibrium, we use

$$H - P = 0 \quad \{\text{along } x\text{-direction}\}$$

$$V - 200 - 400 = 0 \quad \{\text{along } y\text{-direction}\}$$

On solving we get

$$V = 600 \text{ N}$$



For rotational equilibrium about point A, we have

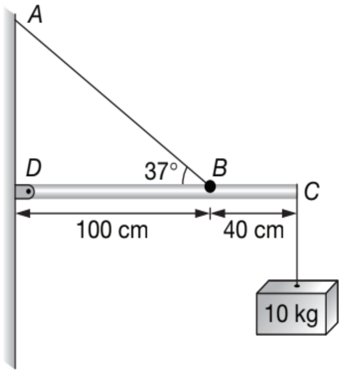
$$\sum \tau_A = 0$$

$$\Rightarrow \quad P(6)(0.8) - 200(3)(0.6) - 400(4)(0.6) = 0$$

$$\Rightarrow \quad P = H = 275 \text{ N}$$

ILLUSTRATION 44

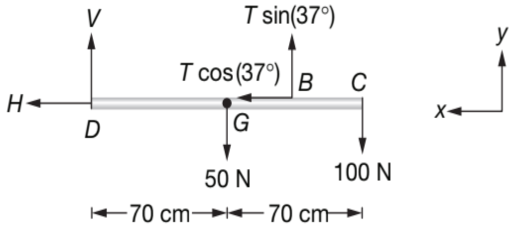
For the uniform 5 kg beam shown in figure, how large is the tension in the supporting cable AB and what are the components of the force exerted by the hinge on the beam at point D? Take $g = 10 \text{ ms}^{-2}$.



SOLUTION

The forces acting on the beam at difference locations are shown in figure. Note that the weight of the beam, 50 N, is taken as acting at the beam's centre of mass. Further, we have replaced the tension in the cable by its components. For equilibrium of the beam, we have

$$\Sigma\tau_D = 0,\ \Sigma F_x = 0 \text{ and } \sum F_y = 0$$



$\Rightarrow \quad T\sin(37°)\times 1 - 50\times 0.7 - 100\times 1.4 = 0 \quad \ldots(1)$

$\Rightarrow \quad H + T\cos(37°) = 0 \quad \ldots(2)$

$\Rightarrow \quad V + T\sin(37°) - 50 - 100 = 0 \quad \ldots(3)$

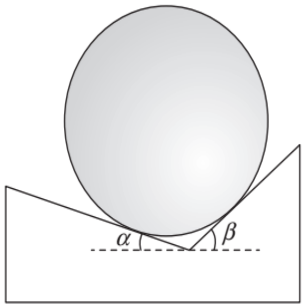
From (1), (2) and (3), we get

$$T = 292\text{ N},\ H = -234\text{ N and } V = -25.2\text{ N}$$

Here, V and H come with a negative sign. Which simply implies that the direction of vertical force from hinge on beam shown in figure is downwards and of horizontal force is towards the right.

ILLUSTRATION 45

A solid sphere of radius R and mass M is placed in a trough as shown in figure. The inner surfaces of the trough are frictionless. Determine the forces exerted by the trough on the sphere at the two contact points.



SOLUTION

Let the normal forces be denoted by F_1 and F_2. They make angles α and β with the vertical

$$\Sigma F_x = 0$$

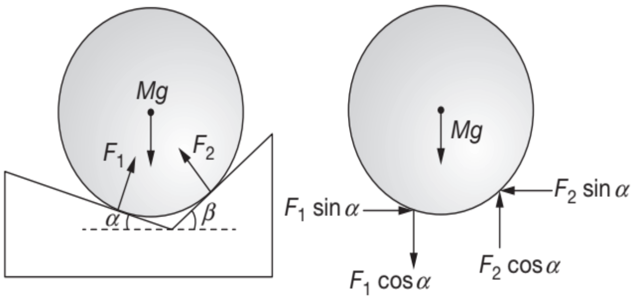
$\Rightarrow \quad F_1\sin\alpha - F_2\sin\beta = 0$

$\Rightarrow \quad \Sigma F_y = 0$

$\Rightarrow \quad F_1\cos\alpha - Mg + F_2\cos\beta = 0$

Substitute $F_2 = \dfrac{F\sin\alpha}{\sin\beta}$

$$F_1\cos\alpha + F_1\cos\beta\left(\frac{\sin\alpha}{\sin\beta}\right) = Mg$$

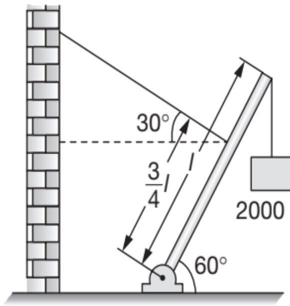


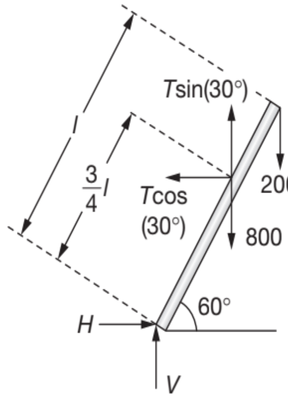
$\Rightarrow \quad F_1(\cos\alpha\sin\beta + \sin\alpha\cos\beta) = Mg\sin\beta$

$\Rightarrow \quad F_1 = Mg\dfrac{\sin\beta}{\sin(\alpha+\beta)}$ and $F_2 = Mg\dfrac{\sin\alpha}{\sin(\alpha+\beta)}$

ILLUSTRATION 46

A uniform rod of 800 N weight pivoted at the bottom, is supported by a cable as in figure. A 2000 N object hangs from its top. Find the tension in the cable and the components of the reaction force exerted by the floor on the rod. Take $\sqrt{3} = 1.7$.





SOLUTION

$$\sum \tau_{\text{pivot}} = 0 \text{ gives}$$

$$(T\cos 30°)\left(\frac{3l}{3}\sin 60°\right) + (T\sin 30°)\left(\frac{3l}{4}\cos 60°\right) = (2000)(l\cos 60°) + (800)\left(\frac{l}{2}\cos 60°\right)$$

$$\Rightarrow T\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3l}{4}\right)\left(\frac{\sqrt{3}}{2}\right) + T\left(\frac{1}{2}\right)\left(\frac{3l}{4}\right)\left(\frac{1}{2}\right) = (2000)(l)\left(\frac{1}{2}\right) + (800)\left(\frac{l}{2}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{9}{16}T + \frac{3T}{16} = 1000 + 200$$

$$\Rightarrow T = 1600 \text{ N}$$

Since, $\sum F_x = 0$, so we get

$$H = T\cos(30°) = 1360 \text{ N (towards right)}$$

Also, $\sum F_y = 0$, gives

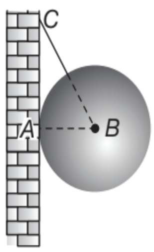
$$V = (2000 + 800) - T\sin(30°) = 2000 \text{ N (upwards)}$$

Test Your Concepts-V

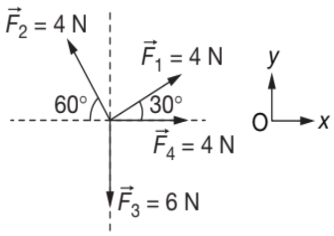
Based on Equilibrium

(Solutions on page H.204)

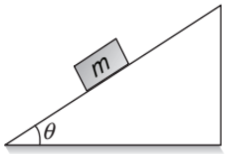
1. A point A on a sphere of mass M, radius a rests in contact with a smooth vertical wall and is supported by a string of length $2a$ which joins a point B on the sphere to a point C on the wall. Find tension in the string and the force exerted by the wall on the sphere.



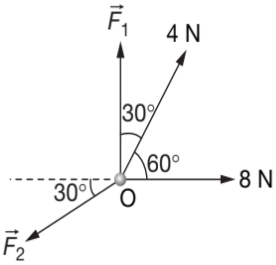
2. Write down the components of four forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ and $\vec{F}_4$ along x and directions as shown in figure. Also find the resultant of these forces.



3. A block of mass m is at rest on a rough wedge as shown in figure. What is the force exerted by the wedge on the block?

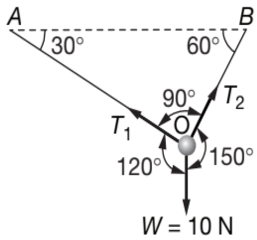


4. An object is in equilibrium under four concurrent forces in the directions shown in figure. Find the magnitude of $\vec{F}_1$ and $\vec{F}_2$.

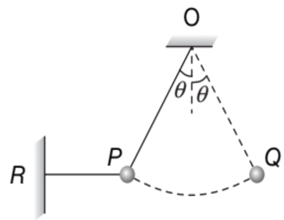


5. A force $\vec{F} = \vec{v} \times \vec{A}$ is exerted on a particle in addition to the force of gravity where $\vec{v}$ is the velocity of the particle and $\vec{A}$ is a constant vector in the horizontal direction. Find the minimum speed of projection of a particle of mass m so that it continues to move undeflected with the same constant velocity?

6. A ball of mass 1 kg hangs in equilibrium from two strings OA and OB as shown in figure. What are the tensions in strings OA and OB? Take $g = 10\ \text{ms}^{-2}$.



7. A ball of mass 1 kg is at rest in position P by means of two light strings OP and RP. The string RP is now cut and the ball swings to position Q. If $\theta = 45°$. Find the tensions in the strings in positions OP (when RP was not cut) and OQ (when RP was cut). Take $g = 10\ \text{ms}^{-2}$.



PSEUDO FORCE

FRAMES OF REFERENCE

Whenever any physical phenomena has to be observed, a suitable platform has to be taken by the observer. This platform from where a physical phenomena is being observed is called **the frame of reference.** Frames of reference are of two types

(a) Inertial: Non-Accelerated
(b) Non-Inertial: Accelerated

Inertial Frame

All non-accelerated frames (frames either at rest or moving with uniform velocity) are inertial frames.

Conceptual Note(s)

A frame of reference moving with constant velocity w.r.t. an inertial frame is also an Inertial Frame.

OR

A frame of reference at rest w.r.t. an inertial frame is also an Inertial Frame.

The Newton's Laws which we have already studied have been fabricated from an inertial frame of reference.

Non-inertial Frame

All accelerated frames are non-inertial frames.

Most of the students have a misconception that the Newton's Laws are not obeyed in the non-inertial frames. However this is not the case, because we have to introduce the concept of the **pseudo force** to keep the laws validated and acquire identical force equations irrespective of the frame from where the physical phenomenon is being observed.

Pseudo force is to be applied on a body whenever the body is lying in a non-inertial frame or the body is being observed from a non inertial frame.

Pseudo force is always directed opposite to the acceleration of the non inertial frame and has a value given by

$$F_{\text{pseudo}} = \begin{pmatrix}\text{Mass of}\\ \text{the block}\end{pmatrix}\begin{pmatrix}\text{Acceleration of}\\ \text{the frame}\end{pmatrix}$$

Pseudo force is a fictitious force because it has no physical origin, i.e., it is not caused by any of the basic interactions in the nature. Note that it has existence only in a non-inertial frame.

These forces are, then, simply a technique that permits us to apply $\vec{F} = m\vec{a}_0$ in the normal way to events if we insist on viewing the events from an accelerating reference frame.

In mechanics problems, then, we have two choices

(a) select an inertial frame as a reference frame and consider only "real" forces i.e., forces that we can associate with definite bodies (e.g. string, the earth, table etc.)
(b) select a non-inertial frame and consider ***both the real forces and pseudo forces****.*

Although we usually select the first alternative, but sometimes have to go for the second alternative, but both are completely equivalent and the choice is purely a matter of convenience.

Conceptual Note(s)

(a) All physical phenomenon should give identical results whether seen from an inertial frame or a non-inertial frame.
(b) All accelerated frames are non-inertial frames.
(c) Let us consider a block of mass M placed in an inertial frame. Let a force $\vec{F}$ be applied on the block. If $\vec{a}$ be the acceleration of the block, then

$$\vec{F} = M\vec{a} \quad \text{...(1)}$$

Suppose that the same block is now placed in a non-inertial frame and the same force $\vec{F}$ is applied on it. If $\vec{a}'$ be the acceleration of the block in non-inertial frame, then

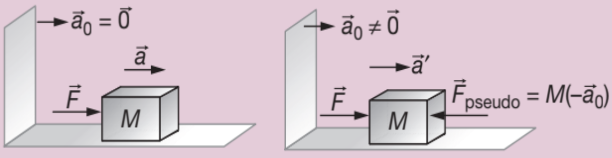
$$M\vec{a}' = \vec{F} - \vec{F}_{\text{pseudo}}$$

$$\Rightarrow \quad \vec{F} = M\vec{a}' + \vec{F}_{\text{pseudo}} \quad \text{...(2)}$$

So, we get from (1) and (2), that

$$M\vec{a} = M\vec{a}' + \vec{F}_{pseudo}$$

i.e., $\begin{pmatrix}\text{Force in}\\ \text{an inertial}\\ \text{frame}\end{pmatrix} = \begin{pmatrix}\text{Force in a}\\ \text{Non-inertial}\\ \text{frame}\end{pmatrix} + \begin{pmatrix}\text{Pseudo}\\ \text{Force}\end{pmatrix}$ ...(3)

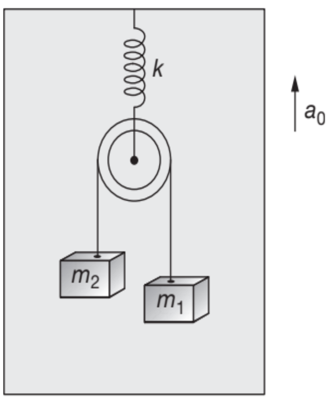


Clearly we observe that equation (3) establishes a relation between the phenomenon observed in an inertial frame and a non-inertial frame and the equivalency of both phenomenon with the help of pseudo force.

So donot develop a misconception that observers in different frames will get different result for an identical physical phenomenon. Just keep in mind that for the observer in an inertial frame Newton's Laws are valid as studied but for an observer in a non-inertial frame the Newton's Laws are being made suitably valid using the concept of Pseudo Force.

ILLUSTRATION 47

A pulley with two blocks system is attached to the ceiling of a lift with the help of a spring a force constant k. The lift is moving upward with an acceleration a_0. Find the deformation in the spring as observed by the inertial and non-inertial reference frame observer.



SOLUTION

We shall be discussing this problem taking into account two observers, one in an inertial frame and the other in the non-inertial frame. The observer in the inertial frame will taken into account all the real forces acting on the system, where as the observer in the non-inertial frame will take into account all the real + fictitious forces into account and both will arrive at the same results.

Observer in an Inertial Frame

Let us draw the free body diagram of the masses as seen by an observer standing on the ground or an inertial frame.

Applying Newton's Second Law

$$m_1 g - T = m_1 a_1 \quad \text{...(1)}$$

$$-m_2 g + T = m_2 a_2 \quad \text{...(2)}$$

Constraint relation

$$(x_0 - x_2) + (x_0 - x_1) = \text{constant}$$

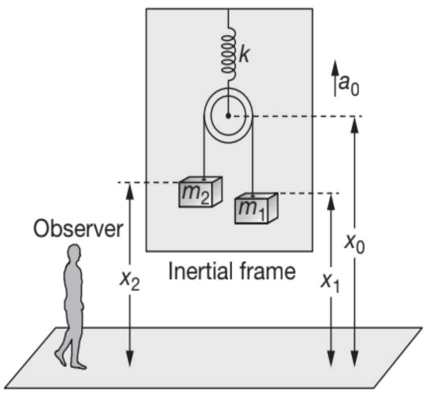
$$\Rightarrow \quad 2x_0 - x_1 - x_2 = \text{constant}$$

Differentiating twice w.r.t. time, we get

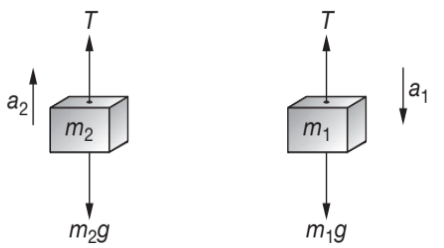
$$2\frac{d^2x_0}{dt^2} - \frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2} = 0$$

$$\Rightarrow \quad 2a_0 - (-a_1) - a_2 = 0$$

$$\Rightarrow \quad a_2 = 2a_0 + a_1 \quad \text{...(3)}$$



Observer in inertial frame



FBD of masses from ground/inertial frame

Solving equations (1), (2) and (3), we get

$$T=\left[\frac{2m_1m_2}{m_1+m_2}\right](g+a_0)$$

The stretching force on the spring is

$$F=2T$$

Using Hooke's Law

$$F=kx$$

where x is the deformation in the spring. Thus,

$$x=\frac{F}{k}=\left[\frac{4m_1m_2}{m_1+m_2}\right]\frac{(g+a_0)}{k}$$

Observer in a Non-inertial Frame

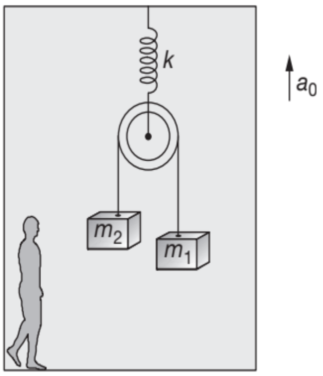
Let us draw the free body diagram of the masses as seen by an observer standing inside the cabin of the lift which happens to be a non-inertial frame due to the upward acceleration a_0 of the lift.

Relative to the centre of the pulley m_1 accelerates downward with a_{rel} and m_2 accelerates upward with a_{rel}.

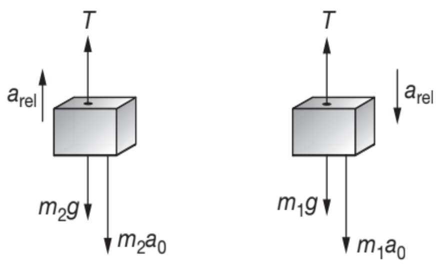
Applying Newton's Second Law

$$m_1g+m_1a_0-T=m_1a_{rel} \quad \ldots(4)$$

$$-m_2g-m_2a_0+T=m_2a_{rel} \quad \ldots(5)$$



Observer in non-inertial frame



FBD of masses from non-inertial frame

On adding equations

$$a_{rel}=\left(\frac{m_1-m_2}{m_2+m_2}\right)(g+a_0)$$

Substituting a_{rel} in equation (4)

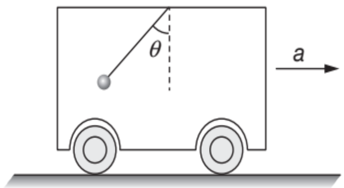
$$T=\left[\frac{2m_1m_2}{m_1+m_2}\right](g+a_0)$$

$$F=2T$$

$$x=\frac{F}{k}=\frac{2T}{k}$$

ILLUSTRATION 48

A bob of mass m is suspended from the ceiling of a train moving with an acceleration a as shown in figure. Find the angle θ in equilibrium position.

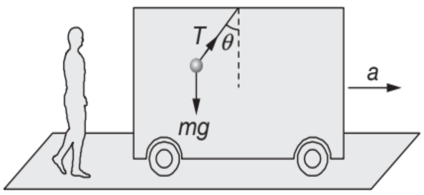


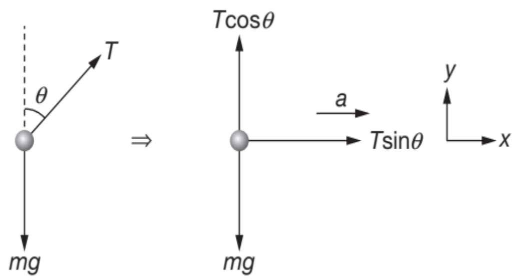
SOLUTION

This problem can also be solved by both the methods.

Inertial frame of reference (Ground)

F.B.D. of bob w.r.t. ground (only real forces):





FBD of bob w.r.t. ground

with respect to ground, bob is also moving with an acceleration a.

$\Rightarrow \quad \Sigma F_y = 0$

$\Rightarrow \quad T\cos\theta = mg$ ...(1)

and $\quad \Sigma F_x = ma$

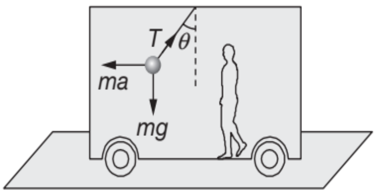
$\Rightarrow \quad T\sin\theta = ma$ ...(2)

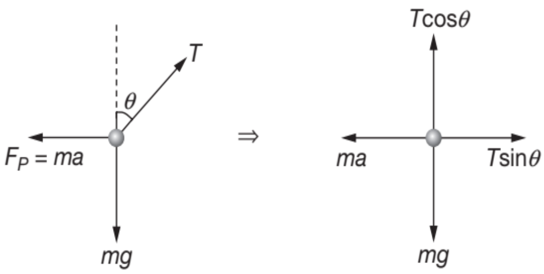
From equation (1) and (2), we get

$$\tan\theta = \frac{a}{g} \text{ or } \theta = \tan^{-1}\left(\frac{a}{g}\right)$$

Non-inertial frame of reference (Train)

F.B.D. of bob w.r.t. train. (real forces + pseudo force):





FBD of bob w.r.t. train

with respect to train, bob is in equilibrium

$\Rightarrow \quad \Sigma F_y = 0$

$\Rightarrow \quad T\cos\theta = mg$ ...(3)

and $\quad \Sigma F_x = 0$

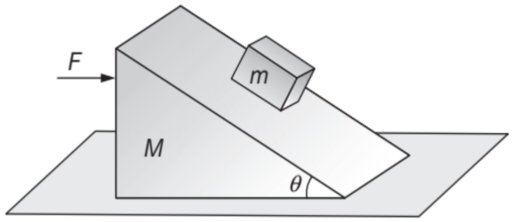
$\Rightarrow \quad T\sin\theta = ma$ ...(4)

From equation (3) and (4), we get the same result, i.e.,

$$\theta = \tan^{-1}\left(\frac{a}{g}\right)$$

ILLUSTRATION 49

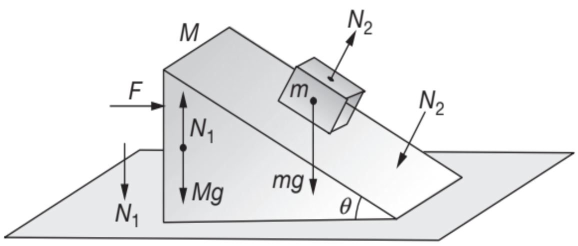
Figure shows a box of mass m is placed on a wedge of mass M on a smooth surface. How much force F is required to be applied on M so that during motion m remains at rest on its surface.



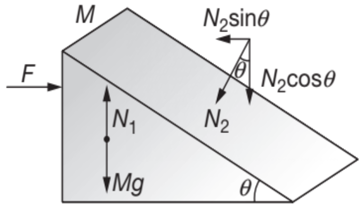
SOLUTION

The force acting on the bodies m and M are shown in figure along with free body diagrams of mass m and M. As the two bodies move together, we can find the acceleration of system towards right directly as

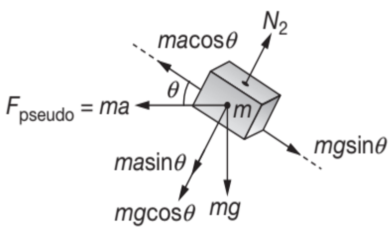
$$a = \frac{F}{m+M}$$



Forces acting on the system



FBD of M



FBD of m

Here the condition is, the small block of mass m should remain at rest on the incline surface of the wedge block. Look at the FBD of m in figure, the force acting on it towards left ma is the pseudo force on it as its reference frame is the wedge block. As wedge block is moving with an acceleration, we consider m relative to it. Now with respect to wedge block m is at rest or in equilibrium, we can balance all the forces along the tendency of motion of body (i.e., inclined plane) and perpendicular to it shown in FBD of it.

For m to be at rest, from FBD of m, along the plane

$$mg\sin\theta = ma\cos\theta$$

$$\Rightarrow \quad a = g\tan\theta$$

$$\Rightarrow \quad \frac{F}{m+M} = g\tan\theta$$

$$\Rightarrow \quad F = (m+M)g\tan\theta$$

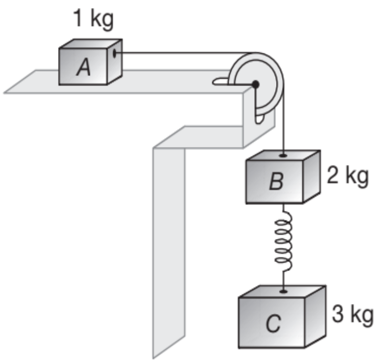
ILLUSTRATION 50

In the system shown in figure all surfaces are smooth, string is massless and inextensible. Find:

(a) acceleration of the system

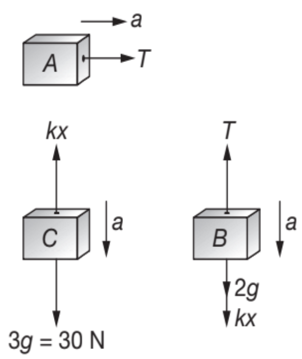
(b) tension in the string and

(c) extension in the spring if force constant of spring is $k = 50\ \text{Nm}^{-1}$ $\left(\text{Take } g = 10\ \text{ms}^{-2}\right)$.



SOLUTION

The free body diagrams for the masses is shown here



$$T = (1)a \quad \ldots(1)$$

$$2g + kx - T = 2a \quad \ldots(2)$$

$$3g - kx = 3a \quad \ldots(3)$$

(a) Adding (1), (2) and (3), we get

$$a = \frac{50}{6}\ \text{ms}^{-2}$$

(b) $T = ma = (1)\left(\frac{50}{6}\right) = \frac{50}{6}\ \text{N}$

(c) Substituting $a = \frac{50}{6}\ \text{ms}^{-2}$ in (3), we get

$$30 - kx = 3\left(\frac{50}{6}\right)$$

$$\Rightarrow \quad kx = 5$$

$$\Rightarrow \quad x = \frac{5}{50} = 0.1\ \text{m}$$

$$\Rightarrow \quad x = 10\ \text{cm}$$

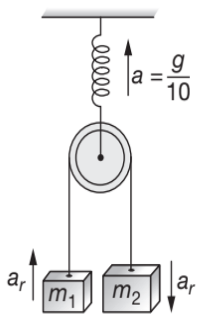
ILLUSTRATION 51

An Atwood's machine having two blocks of mass $m_1 = 1$ kg and $m_2 = 2$ kg is suspended from a spring balance attached to ceiling of a lift moving upwards with an acceleration of $\frac{g}{10}$. Determine the reading on the spring balance. Take $g = 10\ \text{ms}^{-2}$.

SOLUTION

The problem is represented diagrammatically in the figure. Let a_r be the relative acceleration of blocks with respect to pulley.

In the figure,



absolute acceleration of m_1 is $a_r + a = a_r + \frac{g}{10}$, upwards and absolute acceleration of m_2 is $a_r - a = a_r - \frac{g}{10}$, downwards

From Newton's Second Law, we have

$$T - m_1 g = m_1\left(a_r + \frac{g}{10}\right) \quad \ldots(1)$$

$$m_2 g - T = m_2\left(a_r - \frac{g}{10}\right) \quad \ldots(2)$$

Solving equations (1) and (2), with $m_1 = 1\text{ kg}$ and $m_2 = 2\text{ kg}$, we get

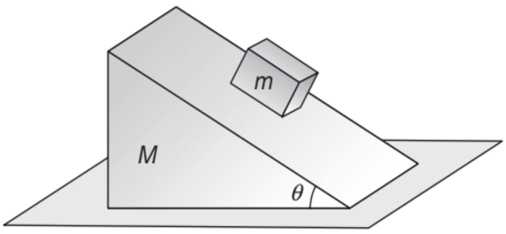
$$T = \frac{44}{3}\text{ N}$$

The reading of the spring balance is therefore

$$F = 2T = \frac{88}{3}\text{ N}$$

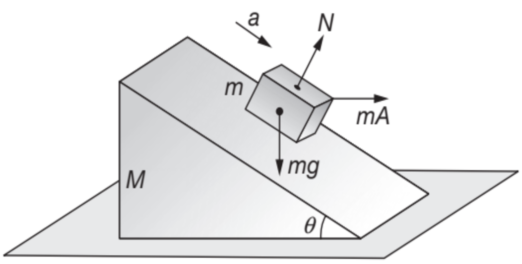
ILLUSTRATION 52

All the surfaces shown in figure are assumed to be frictionless. The block of mass m slides on the prism which in turn slides backward on the horizontal surface. Find the acceleration of the smaller block with respect to the prism.

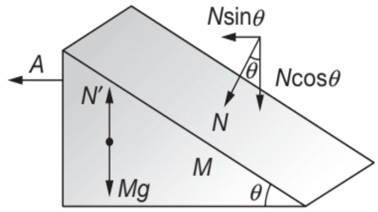


SOLUTION

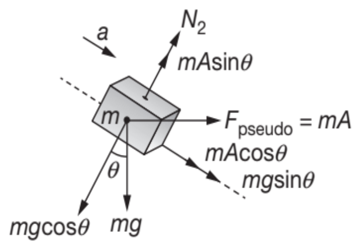
Suppose the acceleration of block with respect to prism is a down the plane and let the acceleration of the prism be A in the backward direction. Consider the motion of the smaller block from the frame of the prism. The forces acting on the system, free body diagram of M and m are shown here.



Forces acting on the system



FBD of M



FBD of m

The following three forces will act on the block in the directions shown in figure.

(a) Normal reaction N.
(b) Weight mg
(c) Pseudo force mA

The block slides down the plane. Component of the forces parallel to the incline gives

$$mA\cos\theta + mg\sin\theta = ma$$

$$\Rightarrow \quad a = A\cos\theta + g\sin\theta \quad \ldots(1)$$

Component of the forces perpendicular to the incline gives

$$N + mA\sin\theta = mg\cos\theta \quad \ldots(2)$$

Now, consider the motion of the prism from the ground frame. No pseudo force is needed as the frame used in inertial. The forces are:

(a) Mg downward
(b) N normal to the incline (by the block)
(c) N' upward (by the horizontal surface)

Horizontal components give,

$$N\sin\theta = MA$$

$$\Rightarrow \quad N = \frac{MA}{\sin\theta} \quad \ldots(3)$$

Putting in equation (2)

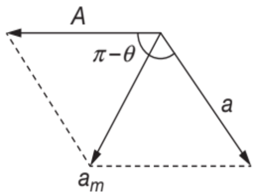
$$\frac{MA}{\sin\theta} + mA\sin\theta = mg\cos\theta$$

$$\Rightarrow \quad A = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

From equation (1),

$$a = \frac{mg\sin\theta\cos^2\theta}{M + m\sin^2\theta} + g\sin\theta = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$$

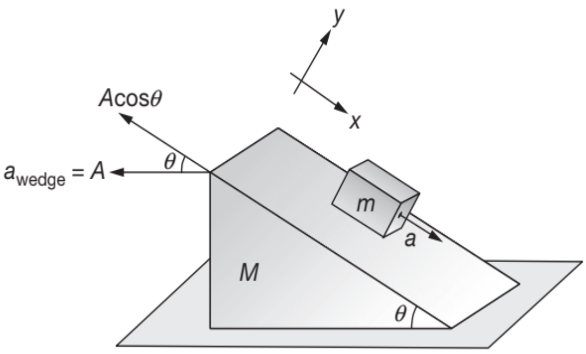
The net acceleration of m, is the vector addition of A and a, as m is also moving with A toward left along with M.
Net acceleration of m can be obtained by the vector sum shown in figure.



$$a_m = \sqrt{A^2 + a^2 - 2aA\cos\theta}$$

Inertial Method: Acceleration of wedge along x-axis is $A\cos\theta$, so net acceleration of block downwards along $+x$-axis is $(a - A\cos\theta)$.

Since block lies on frictionless incline, so it must accelerate down the incline with an acceleration $g\sin\theta$.



Observed from ground frame

Hence, $a - A\cos\theta = g\sin\theta$ ...(1)

From (1),

$$N\sin\theta = MA$$

$$N = \frac{MA}{\sin\theta}$$

Now in (4),

$$N + mA\sin\theta = mg\cos\theta$$

$$\frac{MA}{\sin\theta} + mA\sin\theta = mg\cos\theta$$

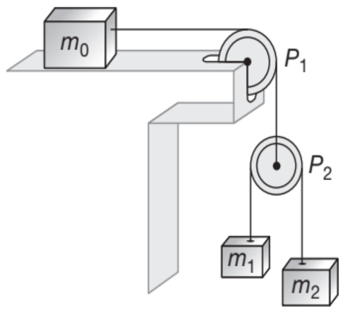
$$MA + mA\sin^2\theta = mg\cos\theta\sin\theta$$

$$A(M + m\sin^2\theta) = mg\cos\theta\sin\theta$$

$$A = \frac{mg\cos\theta\sin\theta}{M + m\sin^2\theta}$$

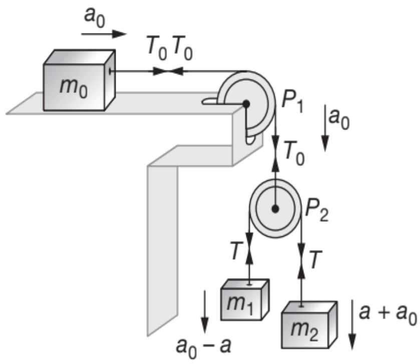
ILLUSTRATION 53

Three blocks of mass m_0, m_1 and m_2 are connected as shown in the figure. All the surfaces are frictionless and the string and the pulleys are light. Find the acceleration of m_0.



SOLUTION

Suppose the acceleration of m_0 is a_0 towards right. The acceleration of pulley P_2 will also be a_0 downwards, because the string connecting m_0 and P_2 is constant in length. Also the string connecting m_1 and m_2 has a constant length. This implies that the decrease in the separation between m_1 and P_2 equals the increase in the separation between m_2 and P_2. So, the upward acceleration of m_1 with respect to P_2 equals the downward acceleration of m_2 with respect to P_2. Let this acceleration be a.



As seen from ground frame

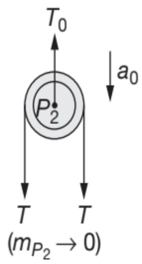
The acceleration of m_1 with respect to the ground is $a_0 - a$ (downward) and the acceleration of m_2 with respect to the ground is $a_0 + a$ (downward).

Let the tension be T_0 in the upper string and T in the lower string.

Consider the motion of the pulley P_2. The forces on this light pulley are

(a) T_0 upwards by the upper string and

(b) $2T$ downwards by the lower string



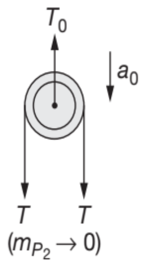
As the mass of the pulley is negligible,

$$2T - T_0 = 0 \text{ giving } T = \frac{T_0}{2} \qquad \ldots(1)$$

Motion of m_0:

In the horizontal direction, the equation is

$$T_0 = m_0 a_0 \qquad \ldots(2)$$



Motion of m_1:

$$m_1 g - \frac{T_0}{2} = m_1 (a_0 - a) \qquad \ldots(3)$$

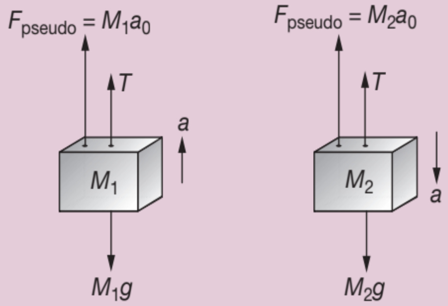
Motion of m_2:

$$m_2 g - \frac{T_0}{2} = m_2 (a_0 + a) \qquad \ldots(4)$$

Solving these four equations, we get

$$a_0 = \frac{g}{1 + \frac{m_0}{4}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$$

We can also do the problem w.r.t. an observer on pulley P_2, a non-inertial frame and get results identical to the ones obtain when the observer was in an inertial frame. Since P_2 acceleration down with a_0, so it is a non-inertial frame accelerating downwards.



For mass m_1

$$M_1 a_0 + T - M_1 g = M_1 a$$

$$\Rightarrow \quad T - M_1 g = M_1 (a - a_0) \qquad \ldots(5)$$

Equation (5) is identical as equation (3)

For mass m_2

$$M_2 g - (T + M_2 a_0) = M_2 a$$

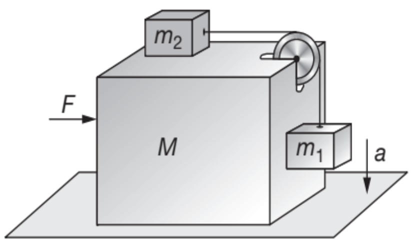
$$\Rightarrow \quad M_2 g - T = M_2 (a + a_0) \qquad \ldots(6)$$

Equation (6) is identical as equation (4)

Please be careful while solving the problem in earth frame you have to consider the acceleration of mass relative to its surface (accelerating) but write the equations in horizontal and vertical directions relative to earth (ground).

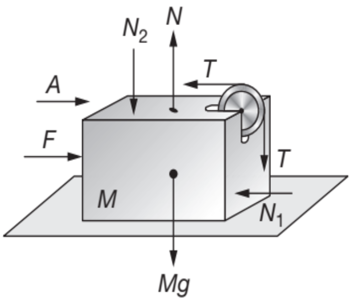
ILLUSTRATION 54

Figure shows a large block of mass M, supporting two small masses m_1 and m_2, connected by a light, frictionless thread. A force F is acting on M, such that the block m_1 is sliding down, with an acceleration a relative to M. Find the force F applied on M and also the acceleration of M. Assuming all surfaces are frictionless.

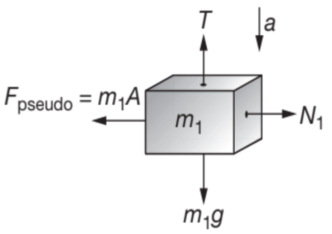


SOLUTION

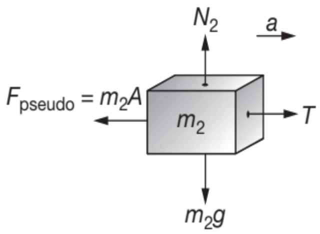
Let T be the tension in the string, N_1 be the normal reaction between M and m_1, N_2 be the normal reaction between M and m_2, N be the normal reaction between M and ground, A be the acceleration of M, then drawing the free body diagrams of different blocks and writing the equations of motion



FBD M



FBD m_1



FBD m_2

for mass M, we get

$$F - N_1 - T = MA \quad \ldots(1)$$

$$N - N_2 - T - Mg = 0 \quad \ldots(2)$$

for mass m_1, we get

$$m_1 g - T = m_1 a \quad \ldots(3)$$

$$N_1 - m_1 A = 0 \quad \ldots(4)$$

for mass m_2, we get

$$T - m_2 A = m_2 a \quad \ldots(5)$$

and $N_2 - m_2 g = 0 \quad \ldots(6)$

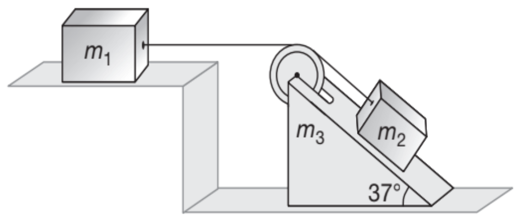
Solving these six equations, we get

$$A = \frac{m_1}{m_2} g - \left(1 + \frac{m_1}{m_2}\right) a$$

and $F = m_1 g + \dfrac{(M + m_1) m_1}{m_2}(g - a) - (M + 2m_1) a$

ILLUSTRATION 55

In the arrangement shown in figure, a wedge of mass $m_3 = 3.45$ kg is placed on a smooth horizontal surface. A small and light pulley is connected on its top edge. A light flexible thread passes over the pulley. Two blocks having mass $m_1 = 1.3$ kg and $m_2 = 1.5$ kg are connected at the ends of the thread, m_1 is on smooth horizontal surface and m_2 rests on inclined surface of the wedge.



If the whole system is released from rest, calculate

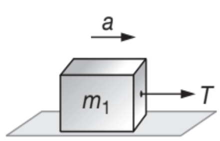
(a) the tension in the string and
(b) acceleration of m_1 and m_3.

All surfaces are smooth. Take $g = 10\ \text{ms}^{-2}$ and $\tan(37°) = \dfrac{3}{4}$

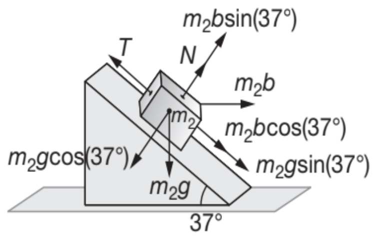
SOLUTION

Let, a be the acceleration of mass m_1 towards right and b be the acceleration of wedge towards left.
So, the acceleration of m_2 relative to m_3 is $(a+b)$ down the plane.

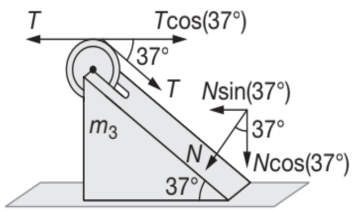
Let T be the tension in the thread and N, the normal reaction between m_2 and m_3. Drawing free body diagrams for m_1, m_2 and m_3 (see figure) and writing equations of motion,



FBD m_1



FBD m_2



FBD m_3

for mass m_1, we have

$$T = m_1 a \quad \text{...(1)}$$

for mass m_2, we have

along the plane

$$m_2 b\cos 37° + m_2 g\sin 37° - T = m_2(a+b) \quad \text{...(2)}$$

perpendicular to the plane

$$N + m_2 b\sin 37° - m_2 g\cos 37° = 0 \quad \text{...(3)}$$

for mass m_3, we have

$$T + N\sin 37° - T\cos 37° = m_3 b \quad \text{...(4)}$$

We have four unknowns T, N, a and b

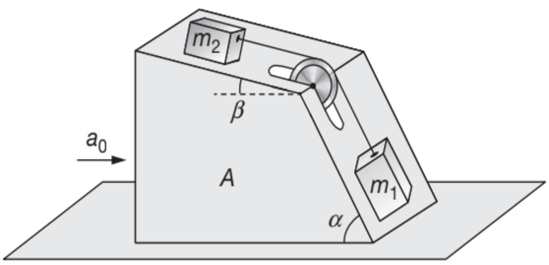
Since $\tan(37°)=\frac{3}{4}$, so $\sin(37°)=\frac{3}{5}$ and $\cos(37°)=\frac{4}{5}$

Substituting $m_1 = 1.3$ kg, $m_2 = 1.5$ kg, $m_3 = 3.45$ kg, $g = 10\ \text{ms}^{-2}$, we get

$$T = 3.9\ \text{N},\ a = 3\ \text{ms}^{-2} \text{ and } b = 2\ \text{ms}^{-2}$$

ILLUSTRATION 56

Two cubes of masses m_1 and m_2 lie on frictionless slopes of a block A which rests on a horizontal table. The cubes are connected by a string which passes over a pulley as shown in figure. If a_0 be the horizontal acceleration to which the whole system (block + masses) is subjected so that m_1 and m_2 do not move and T be the tension in the string in that situation then, find a_0 and T.

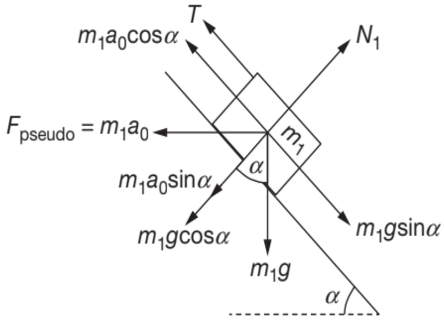


SOLUTION

For m_1

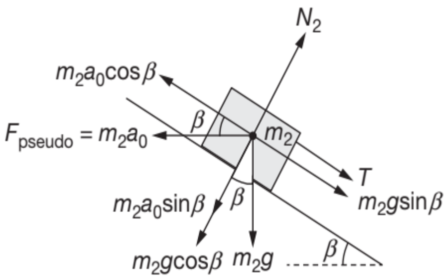
$$T + m_1 a_0 \cos\alpha = m_1 g\sin\alpha$$

$$\Rightarrow \quad T = m_1 g\sin\alpha - m_1 a_0\cos\alpha \quad \text{...(1)}$$



For m_2

$$T + m_2 g\sin\beta = m_2 a_0\cos\beta$$



$$\Rightarrow \quad T = m_2 a_0\cos\beta - m_2 g\sin\beta \quad \text{...(2)}$$

Equating (1) and (2), we get

$$m_1 g\sin\alpha - m_1 a_0\cos\alpha = m_2 a_0\cos\beta - m_2 g\sin\beta$$

$$\Rightarrow \quad (m_1\sin\alpha + m_2\sin\beta)g = (m_1\cos\alpha + m_2\cos\beta)a_0$$

$$\Rightarrow \quad a_0 = \left(\frac{m_1\sin\alpha + m_2\sin\beta}{m_1\cos\alpha + m_2\cos\beta}\right)g \quad \text{...(3)}$$

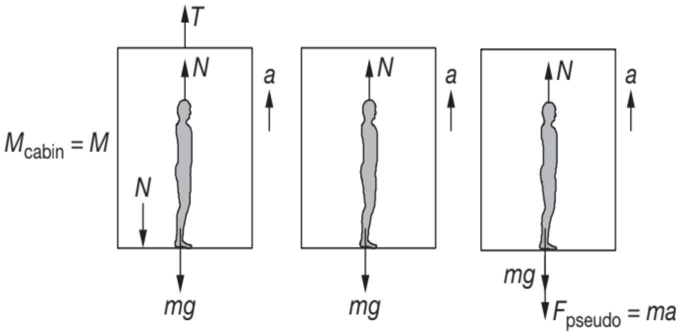
Substituting (3) either in (2) or in (1), we get

$$T = \left[\frac{m_1 m_2}{m_1\cos\alpha + m_2\cos\beta}\right] g\sin(\alpha-\beta) \quad \text{...(4)}$$

MAN IN A LIFT

Lift Accelerating Up/Retarding Down

Consider a man of mass m standing in a lift, accelerating upwards with an acceleration a, whose cabin has mass M. The force on man by the lift's floor is N, upwards. Then, the force on the lift by the man is also N, downwards. Let T be the tension in the cable supporting the cabin (lift + man), then the following two cases have been discussed both from the inertial and the non-inertial observer's view point. Force on man by the lift is N, upwards, so force on lift by man is also N, downwards.



Forces acting on man and lift | FBD man, inertial frame | FBD man, non-inertial frame

FBD Man (Inertial Frame):
According to the observer in the inertial frame, the man accelerates up, so

$$N - mg = ma$$

$$\Rightarrow \quad N = m(g + a)$$

FBD Man (Non inertial Frame attached to lift):
According to the non inertial frame, the man is in equilibrium, so

$$N = mg + ma$$

$$\Rightarrow \quad N = m(g + a)$$

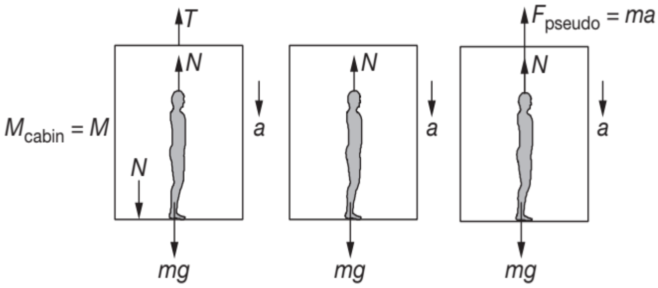
Effective weight or the apparent weight of the man in the lift accelerating up is greater than the actual weight mg. The tension in the cable supporting the lift is found by writing the combined equation for the cabin + man system. So, we have

$$T - (M + m)g = (M + m)a$$

$$\Rightarrow \quad T = (M + m)(g + a)$$

Lift Accelerating Down/Retarding Up

Consider a man of mass m standing in a lift, accelerating upwards with an acceleration a, whose cabin has mass M. The force on man by the lift's floor is N, upwards. Then, the force on the lift by the man is also N, downwards. Let T be the tension in the cable supporting the cabin (lift + man), then the following two cases have been discussed both from the inertial and the non-inertial observer's view point.



Forces acting on man and lift | FBD man, inertial frame | FBD man, non-inertial frame

FBD (Inertial):
According to the observer in the inertial frame, the man accelerates down, so

$$mg - N = ma$$

$$\Rightarrow \quad N = m(g - a)$$

FBD (Non-inertial):
According to the non inertial frame, the man is in equilibrium, so

$$mg - N - ma = 0$$

$$\Rightarrow \quad N = m(g - a)$$

Effective weight or the apparent weight of the man in the lift accelerating down is less than the actual weight mg. The tension in the cable supporting the lift is found by writing the combined equation for the cabin + man system. So, we have

$$(M + m)g - T = (M + m)a$$

$$\Rightarrow \quad T = (M + m)(g - a)$$

Problem Solving Technique(s)

(a) If M is the mass of the cabin including the person(s) in it, then tension in the cable of the lift when lift accelerates up is $T = M(g + a)$ and when it accelerates down is $T = M(g - a)$.

(b) A person of mass M climbs up a rope with acceleration a. The tension in the rope will be $T = M(g + a)$. For rope not to break, $T <$ Breaking Tension of Rope.

(c) If the person climbs down along the rope with acceleration a, the tension in the rope will be $T = M(g - a)$. For rope not to break, $T <$ Breaking Tension of Rope or $a < g$.

(d) When the person climbs up or down with uniform speed, tension in the string will be Mg.

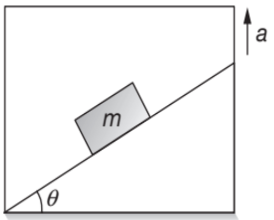
(e) When a man jumps with load on his head, the apparent weight of the load and the man is zero.

Test Your Concepts-VI

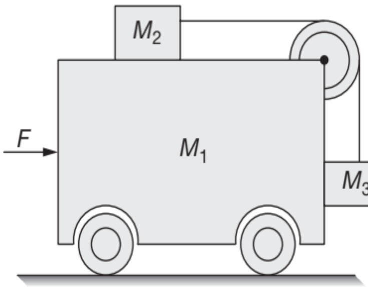
Based on Non-inertial Frames: Pseudo Force

(Solutions on page H.206)

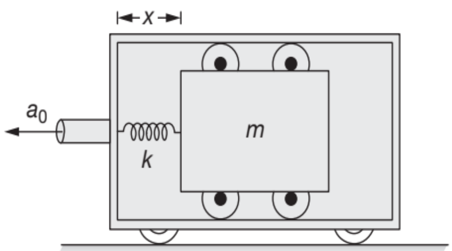
1. In the figure, a wedge is fixed to an elevator moving upwards with an acceleration a. A block of mass m is placed over the wedge. Find the acceleration of the block with respect to wedge. Neglect friction.



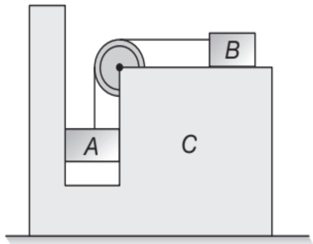
2. In the figure all surfaces are frictionless. What force F must by applied to M_1 to keep M_3 free from rising or falling?



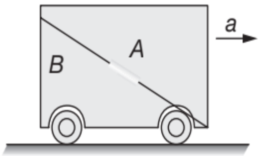
3. The block of mass m is attached to the frame by the spring of stiffness k and moves horizontally with negligible friction within the frame. The frame and block are initially at rest with $x = x_0$, the uncompressed length of the spring. If the frame is given a constant acceleration a_0, determine the maximum velocity $v_{max} = (v_{rel})_{max}$ of the block relative to the frame.



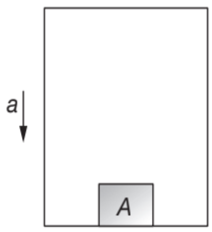
4. In the system shown in figure $m_A = 4\,m$, $m_B = 3\,m$ and $m_C = 8\,m$. Friction is absent everywhere. String is light and inextensible. If the system is released from rest find the acceleration of each block.



5. The collar A is free to slide along the smooth shaft B mounted in the frame. The plane of the frame is vertical. Determine the horizontal acceleration a of the frame necessary to maintain the collar in a fixed position on the shaft.



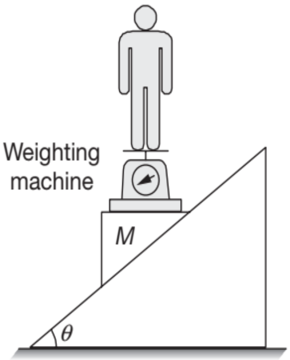
6. Calculate the acceleration a of the lift cabin shown in figure so that the block A of mass m exerts a force $\dfrac{mg}{4}$ on the floor of the cabin.



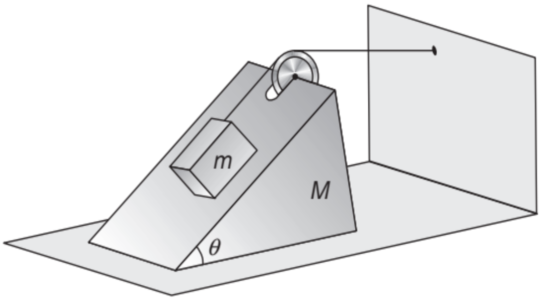
7. A plumb bob of mass 2 kg is hung from the ceiling of a car. The car moves on an inclined plane with constant velocity. If the angle of incline is $30°$. Find the angle made by the string with the normal to the ceiling. Also find the tension in the string. $\left(g = 10\text{ ms}^{-2}\right)$

8. Repeat both parts of the above question if the car moves with an acceleration $a_0 = \dfrac{g}{2}$ up the plane.

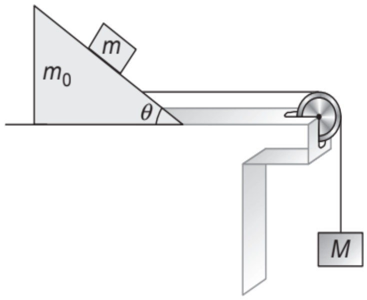
9. Find the weight shown by the weighing machine on which a man of mass m is standing at rest relative to it as shown in figure. Assume that the wedge of mass M is in free fall.



10. In the figure a bar of mass m is on the smooth inclined face of the wedge of mass M, the inclination to the horizontal being θ. The wedge is resting on a smooth horizontal plane. Assuming the pulley to be smooth and the string is light and inextensible. Find the acceleration of M, when M and m are always in contact.



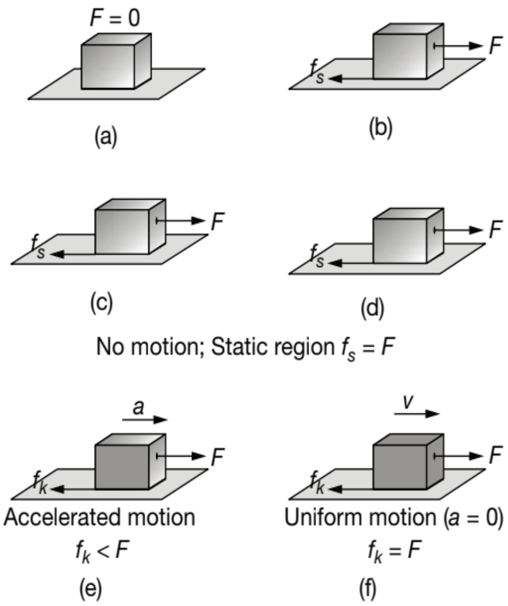
11. Find the mass M of the hanging block shown in figure, which will prevent the smaller block from slipping over the triangular block. All the surfaces are frictionless and the strings and the pulleys are light.



FRICTION

INTRODUCTION

Actually, whenever the surface of one body slides or has a tendency to slide on the surface of another then each body exerts a frictional force on the other, parallel to the surfaces. The frictional force on each body is in a direction opposite to its motion relative to the other body. Frictional forces automatically oppose the relative motion and never aid it. Even when there is no relative motion, frictional forces may exist between surfaces (but there must be a tendency of relative motion).



Consider a block at rest on a horizontal table as shown in figure. We find that the block will not move even though we apply a small force. We say that our applied force is balanced by an opposite frictional force exerted on the block by the table, acting along the surface of contact. As we increase the applied force we find that there is some definite value of the applied force at which the block just begins to move. Once motion has started, this same force (without increasing any further) produces accelerated motion. By reducing the force once motion has started, we find that it is possible to keep the block in uniform motion without acceleration; this force may be small, but it is never zero. Now, analyze the diagrams given in figure once again.

In Figure (a), no force is applied on the block due to which it does not have tendency to slide over the surface of the table and hence, the surface does not apply any opposing (i.e., frictional) force on the block.

In Figure (b), we have applied a small force on the block and its magnitude is being continuously increased. It is clear from the second, third and fourth figures, that as we are increasing external force, frictional force acting upon the block also increases and prevents the block from sliding over the surface of the table.

But at a certain instant friction reaches its **maximum value**, referred to as **limiting friction**, and now if we increase the external force, even slightly, the block will start sliding in the direction of the external force. We can also say that now the external force has sufficient magnitude to break molecular bonds between the surfaces.

Conclusion

Therefore, if the applied force is less than limiting friction, the block will not move and the frictional force will have the same magnitude as that of the external applied force. Since the body is at rest with respect to the surface on which it is placed so, this frictional force is called **static frictional force** and static friction being self adjusting, so static frictional force can have a magnitude lying between zero and limiting value of frictional force (which is the maximum value of static friction), or we can write

$$0 \le f_{static} \le f_{limiting}, \text{ where}$$

$f_{limiting} = (f_s)_{max}$ is the maximum value of static friction.

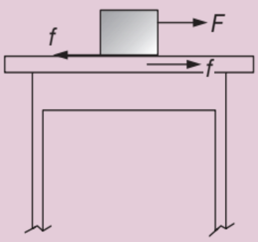
In Figure (d), frictional force has already reached its maximum value (i.e., its limiting value) and is just balancing the horizontal force F applied on the block. If F is increased further by even a negligible amount, the block starts sliding. Once the body has started sliding it is observed that the magnitude of friction (or we can say the magnitude of kinetic friction) is constant and is smaller than limiting friction (i.e., maximum value of static friction). We can write

$$f_{kinetic} < f_{limiting}$$

The frictional forces acting between surfaces at rest with respect to each other are called forces of **static friction**. The maximum force of static friction will be the same as the smallest force necessary to start motion. Once motion has started, the frictional forces acting between the surfaces usually decrease so that a smaller force is necessary to maintain uniform motion. The forces acting between surfaces in relative motion are called forces of **kinetic friction**.

Conceptual Note(s)

Note that in figure the block, of course exerts an equal and opposite frictional force on the table (as shown in figure), tending to drag it in the direction of the horizontal force we exert. This frictional force is due to the bonding of the molecules of the block and the table at the places where the surfaces are in very close contact. If we focus on the table only, we see that friction tends to move the table in the same direction in which force *F* tends to move the block. This observation leads us to two very important conclusions.



(a) friction does not always oppose motion (because it is friction only which is trying to move the table)

(b) friction always opposes relative motion between surfaces in contact (because it is trying to move the table in the same direction in which the block has a tendency to move).

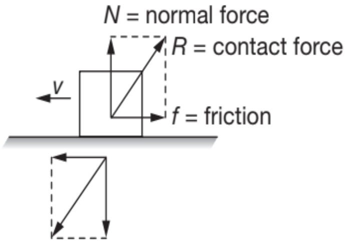
REASONS FOR FRICTION

(a) Frictional forces arise due to molecular interactions due to which a bonding between the molecules of the two surfaces or objects in contact comes into being due to which it becomes difficult to move one surface on the other.

(b) Inter locking of extended parts of one object into the extended parts of the other object.

CONTACT FORCE AND FRICTION

When two bodies are kept in contact then each body exerts a contact force on the other. The magnitudes of the contact forces acting on the two bodies are equal in magnitude and opposite in direction. So the contact forces obey Newton's Third Law.



The contact force acting on a particular body is not necessarily perpendicular to the contact surface. So, contact force can be resolved into two components.

(a) Perpendicular to the contact surface and

(b) Parallel to contact surface.

The perpendicular component is called the Normal Contact Force or Normal Reaction (generally written as N) and the parallel component is called Friction (generally written as f).

Therefore if R is contact force then

$$R = \sqrt{f^2 + N^2}$$

STATIC AND KINETIC FRICTION

The frictional force between two surfaces before the relative motion actually starts is called **static frictional force** or **static friction**, while the frictional force between two surfaces in contact and in relative motion is called **kinetic frictional force** or **kinetic friction**.

Static friction is a self-adjusting force and it adjusts both in magnitude and direction automatically. Its magnitude is always equal to external effective applied force, tending to cause the relative motion and its direction is always opposite to that of external applied force.

So, when a body is not in motion or is in equilibrium, then **force of static friction is equal to the component of the applied External force(s) tangential to the surface.**

$$\begin{pmatrix}\text{Force of} \\ \text{Static Friction}\end{pmatrix} = \begin{pmatrix}\text{Applied External} \\ \text{Force Tangential to the} \\ \text{surfaces in Contact}\end{pmatrix}$$

Conceptual Note(s)

Static Friction exists between the two surfaces when there is tendency of relative motion but no relative motion along the two contact surface.

For Example
Consider a bed inside a room. When we gently push the bed with a finger, the bed does not move. This means that the bed has a tendency to move in the direction of applied force but does not move as there exists static friction force acting in the direction opposite to the applied force.

Kinetic friction exists between two contact surfaces only when there is relative motion between the two contact surfaces. It stops acting when relative motion between two surfaces ceases.

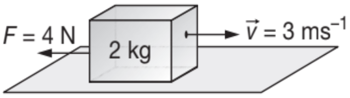
The direction of kinetic friction on an object is opposite to the relative velocity of the object with respect to the other object in contact considered.

Conceptual Note(s)

Note that, the direction of kinetic friction is not opposite to the force applied. It is opposite to the relative motion of the body considered which is in contact with the other surface.

ILLUSTRATION 57

Find the direction of kinetic force of friction.



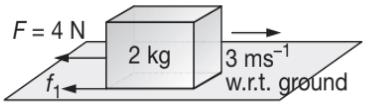
(a) f_1 on the block, exerted by the ground.

(b) f_2 on the ground, exerted by the block.

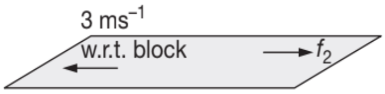
Also find the relation between f_1 and f_2

SOLUTION

(a) Since the block is moving to the right at the instant shown, so the kinetic friction on the block due to the surface is towards the left.



(b) Since friction force obeys Newton's Third Law, so the friction on the surface due to the block is to the right.

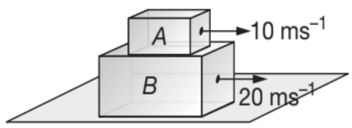


Since f_1 and f_2 obey Newton's Third Law, so they are equal in magnitude and opposite in direction. So $f_1 = f_2$.

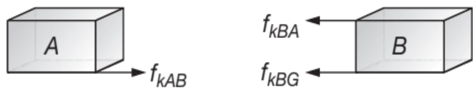
Also note that the direction of kinetic friction has nothing to do with applied force F.

ILLUSTRATION 58

All surfaces as shown in the figure are rough. Draw the friction force on A and B.



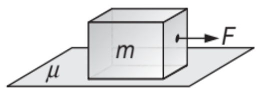
SOLUTION



Kinetic friction acts in such a way so as to reduce relative motion.

ILLUSTRATION 59

Calculate the acceleration of the block for different ranges of the applied force F.



SOLUTION

$$0 \le f \le \mu_S N$$

$$\Rightarrow \quad 0 \le f \le \mu_S (mg)$$

If the applied force $F \le \mu_S mg$, then

$$a = 0$$

and if the applied force $F > \mu mg$, then

$$a = \frac{F - \mu mg}{m}$$

LAWS OF FRICTION

In Static Region

The **maximum force of static friction** between any pair of dry unlubricated surfaces follows these two empirical laws.

(a) It is approximately independent of the area of contact, and
(b) it is proportional to the normal force.

$$f_s \propto N$$

The ratio of the magnitude of the maximum force of static friction to the magnitude of the normal force is called the **coefficient of static friction** (μ_s) for the surfaces involved.

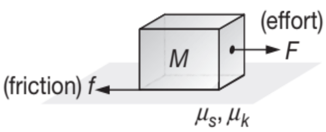
The magnitude of static friction is equal and opposite to the external force exerted, till the object at which force is exerted is at rest. This means it is a variable and self-adjusting force. However it has a maximum value called limiting friction. i.e.

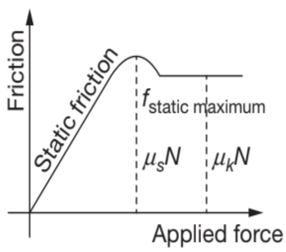
$$(f_s)_{\max} = f_{\text{limiting}} = f_l, \quad \text{where } f_{\max} = \mu_s N$$

The actual force of static friction may be smaller than $\mu_s N$ and its value depends on other forces acting on the body.

The magnitude of frictional force is equal to that required to keep the body at relative rest. So, we have $0 \le f_s \le f_l$.

The plot of force of friction with the applied force is shown here.





It is observed that

1. initially, when there is no applied force, then force of friction is zero.
2. as the applied force is increased and the body is still at rest, then the force of static friction is equal to the applied force.
3. for a particular maximum value of the applied force, the body is on the verge of motion or is just about to move. This maximum value of force of static friction is called the force of limiting friction.
4. when the applied force is increased beyond this maximum value, the body starts moving with respect to the surface on which it is kept and now the friction between the body and the surface is kinetic in nature.

Following table gives a rough estimate of the values of coefficient of static friction between certain pairs of materials. The actual value depends on the degree of smoothness and other environmental factors. For example, wood may be prepared at various degrees of smoothness and the friction coefficient will vary.

Material	μ_s	Material	μ_s
Steel and steel	0.58	Copper and copper	1.60
Steel and brass	0.35	Teflon and Teflon	0.04
Glass and glass	1.00	Wood and metal	0.40
Wood and wood	0.35	Rubber and rubber	1.16
Rubber tyre on dry concrete road	1.0	Rubber tyre on wet concrete road	0.7

In Kinetic Region

The force of kinetic friction, f_k, between dry unlubricated surfaces follows the same two laws as those of static friction.

(a) It is approximately independent of the area of contact and
(b) It is proportional to the normal force.

$$f_k \propto N$$

The force of kinetic friction is also reasonably independent of the relative speed with which the surfaces move over each other (provided it is neither too small, nor too large).

The ratio of the magnitude of the force of the kinetic friction to the magnitude of the normal force is called the **coefficient of kinetic friction** (μ_k). If f_k represents the magnitude of the force of kinetic friction, N the normal reaction and μ_k the coefficient of kinetic friction, then

$$f_k = \mu_k N$$

Conceptual Note(s)

(a) Note that the equation $f = \mu N$ is the relation between the magnitudes of the normal and frictional forces. These forces (frictional force and normal contact force) are always directed perpendicularly to each other. **So it would absolutely be incorrect to write $\vec{f} = \mu\vec{N}$.**

(b) Both μ_s and μ_k are dimensionless constants, each being the ratio of the magnitudes of the two forces (frictional force and normal contact force). Usually, for a given pair of surfaces $\mu_s > \mu_k$. The actual values of μ_s and μ_k depend on the nature of both the surfaces in contact. Both μ_s and μ_k can exceed unity, although commonly they are less than one.

(c) Coefficient of limiting friction is the ratio of the force of limiting friction $f_\ell\left(=(f_s)_{\text{max}}\right)$ to the normal reaction.

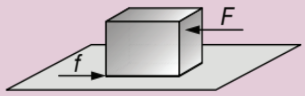
(d) If not mentioned then $\mu_s = \mu_k$ can be taken.

(e) Value of μ can be from 0 to ∞.

(f) **Static friction comes into play** whenever the surfaces in contact donot move with respect to each other
OR whenever a surface has a tendency to slip on another surface
OR whenever force applied on a body kept on a surface tends to move the body
and in all these cases the force of friction is opposite to the tendency of motion of the body due to the applied force.

(g) **Kinetic friction comes into play** whenever there is a relative motion between the two surfaces in contact and then the force of kinetic friction is always directed opposite to the direction of instantaneous relative velocity of the body with respect to the surface on which the body is kept.

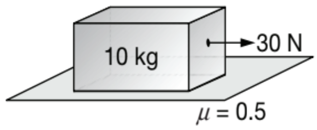
(h) Since force of friction is a contact force and hence it is shown tangentially to the surfaces in contact.



(i) For a given pair of surfaces, we have $\mu_s > \mu_k$, where μ_s and μ_k are proportionality constants called coefficients of static and kinetic friction. They are dimensionless quantities independent of shape and area of contact. Coefficient of friction μ is a property of the two surfaces in contact.

ILLUSTRATION 60

Calculate the acceleration of the block as shown in the figure.



SOLUTION

Since $F_{\text{app}} = 30\text{ N}$ and $f_{\text{limiting}} = (0.5)(100) = 50\text{ N}$

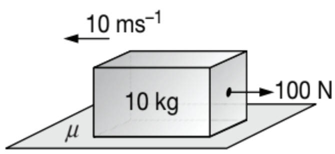
Since $F_{\text{app}} < f_{\text{limiting}}$

$\Rightarrow \quad a = 0$

and $\quad f_s = F_{\text{app}} = 30\text{ N}$

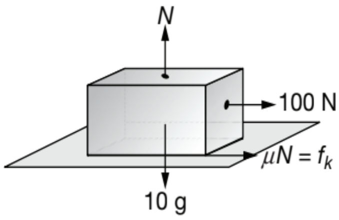
ILLUSTRATION 61

A block of mass 10 kg is given a velocity of 10 ms^{-1} and a force of 100 N in addition to friction force is also acting on the block. Calculate the retardation of the block?



SOLUTION

Since there is relative motion, so the kinetic friction will act to reduce this relative motion.



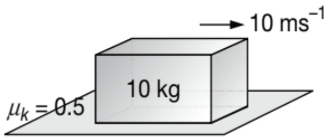
$$f_k = \mu N = (0.1)(10)(10) = 10 \text{ N}$$

$\Rightarrow \quad 100 + 10 = 10a$

$\Rightarrow \quad a = \dfrac{110}{10} = 11 \text{ ms}^{-2}$

ILLUSTRATION 62

Calculate the distance travelled by the block shown in the figure before it stops.



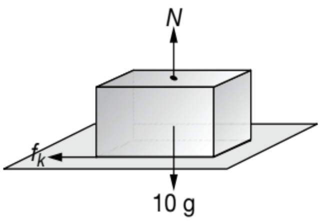
SOLUTION

Since, $f_k = \mu_k N$

When not mentioned, we have

$$\mu = \mu_s = \mu_k = 0.5$$

$\Rightarrow \quad f_k = (0.5)(100) = 50 \text{ N}$



Since, $N - 10g = 0$

$\Rightarrow \quad N = 100 \text{ N}$

Now, $F = ma$

$\Rightarrow \quad 50 = 10a$

$\Rightarrow \quad a = 5$

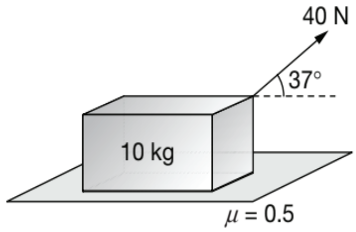
Since, $v^2 = u^2 + 2as$

$\Rightarrow \quad 0^2 = 10^2 + 2(-5)(5)$

$\Rightarrow \quad s = 10 \text{ m}$

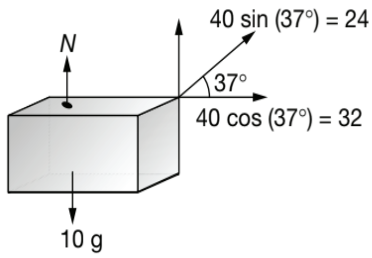
ILLUSTRATION 63

Calculate the acceleration of the block. Assume that initially the block is at rest.



SOLUTION

$N + 24 - 100 = 0$, for vertical direction



$\Rightarrow \quad N = 76 \text{ N}$

Now $0 \le f_s \le f_l$

$\Rightarrow \quad 0 \le f_s \le 76 \times 0.5$

$\quad 0 \le f \le 38 \text{ N}$

Since applied force is less than limiting force

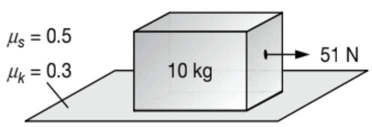
$\quad 32 < 38$

Hence $f = 32$

$\Rightarrow$ Acceleration of block is zero.

ILLUSTRATION 64

In the arrangement shown, calculate the acceleration of the block. Assume that initially the block is at rest.



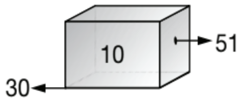
SOLUTION

Since, the limiting friction is given by

$$f_l = \mu N = 0.5(100)$$

$\Rightarrow \quad f_l = 50\text{ N} < F_{\text{app}}$

$\Rightarrow \quad 0 \le f_s \le \mu_s N$



$\Rightarrow \quad 0 \le f_s \le 50$

So, block will move and hence kinetic friction will come into play. So

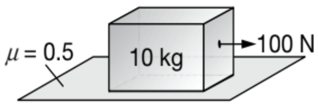
$$f_k = \mu_k N = 0.3 \times 100 = 30\text{ N}$$

$\Rightarrow \quad 51 - 30 = 10a$

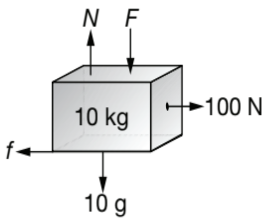
$\Rightarrow \quad a = 2.1\text{ ms}^{-2}$

ILLUSTRATION 65

Calculate the minimum force that must be applied on the block vertically downwards so that the block does not move.



SOLUTION



Since, $100 - f_s = 0$

$\Rightarrow \quad f_s = 100 \qquad \ldots(1)$

$F + 10g = N$

$\Rightarrow \quad N = 100 + F \qquad \ldots(2)$

Now $0 \le f_S \le \mu N$

$\Rightarrow \quad 100 \le 0.5N$

$\Rightarrow \quad 100 \le 0.5(100 + F)$

$\Rightarrow \quad 200 \le 100 + F$

$\Rightarrow \quad F \ge 100\text{ N}$

$\Rightarrow \quad$ Minimum force is $F_{\min} = 100\text{ N}$

Conceptual Note(s)

Direction of Static Friction Force

The static friction force on an object is opposite to its impending motion relative to the surface or tendency of motion of body w.r.t. the surface. The steps to be followed in determining the direction of static friction force on an object are

STEP-1: Draw the free body diagram with respect to the other object or the surface on which it is kept.

STEP-2: Also include pseudo force if contact surface is accelerating.

STEP-3: Find the resultant force.

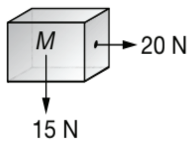
STEP-4: Calculate the component of this force tangential to the surfaces in contact i.e. parallel to the contact surfaces.

STEP-5: The direction of static friction is opposite to the tangential component of resultant contact force.

Here once again, we must note that the static friction is involved when there is no relative motion between two surfaces.

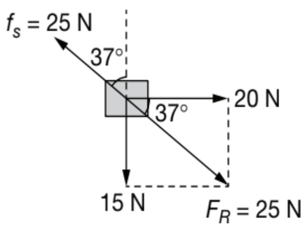
ILLUSTRATION 66

In the figure an object of mass M is kept on a rough table as seen from above. Forces are applied on it as shown. Find the direction of static friction if the object does not move.

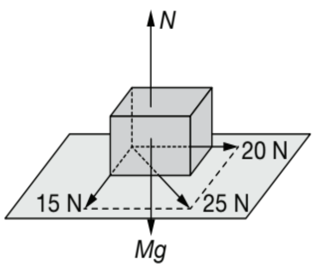


SOLUTION

In the above Illustration we first draw the free body diagram of find the resultant force.



TOP VIEW

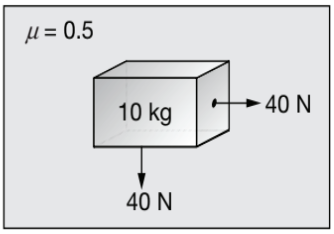


FRONT VIEW

As the object does not move so this is not a case of limiting friction. The direction of static friction is opposite to the direction of the resultant force F_R as shown in figure. Its magnitude is equal to 25 N.

ILLUSTRATION 67

A block of mass 10 kg is placed on a horizontal rough table surface. The top view of the block on the table is shown. Calculate the acceleration of the block, taking $\mu = 0.5$ and $g = 10\ \text{ms}^{-2}$.



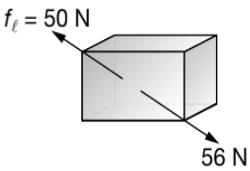
SOLUTION

The resultant force acting on the block is

$$F_R = \sqrt{40^2 + 40^2} = 40\sqrt{2}\ \text{N}$$

$$\Rightarrow \quad F_R = 56\ \text{N}$$

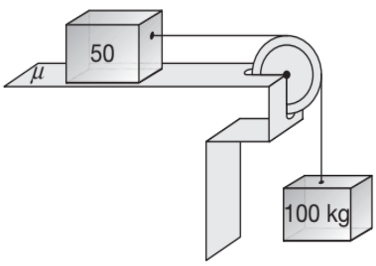
Since, $f_l = (0.5)(100) = 50\ \text{N} < F_R$



So, the block will move in the direction of resultant force with an acceleration $a = \dfrac{56-50}{10} = 0.6\ \text{ms}^{-2}$

ILLUSTRATION 68

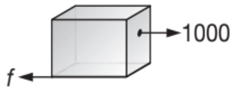
Find minimum μ so that the blocks remain stationary.



SOLUTION

$$T = 100g = 1000\ \text{N}$$

$\Rightarrow \quad f = 1000$ to keep the block stationary



Now $f_{max} = 1000$

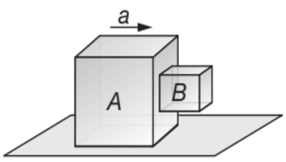
$$\mu N = 1000$$

$$\mu = 2$$

> Can μ be greater than 1?
>
> Yes $0 < \mu \le \infty$

ILLUSTRATION 69

In the arrangement shown, the coefficient of friction between A and B is 0.5. Calculate the minimum acceleration of block A so that the block B of mass 10 kg doesn't fall.



SOLUTION

Applying Newton's Law in horizontal direction, we get

$$N = 10a \qquad \ldots(1)$$

For block not to slip, we have

$$10g \le f_l$$

$$\Rightarrow \quad mg \le \mu N$$

$$\Rightarrow \quad \cancel{m}g \le \mu \cancel{m}a$$

$$\Rightarrow \quad a \ge \frac{g}{\mu}$$

$$\Rightarrow \quad a_{min} = \frac{g}{\mu} = 20\ \text{ms}^{-2}$$

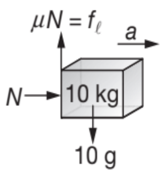
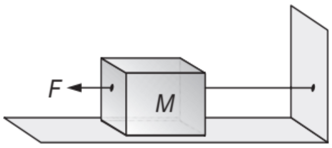


ILLUSTRATION 70

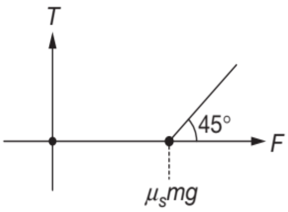
In the figure shown, the force F is gradually increased from zero.



Draw the graph between applied force F and tension T in the string. The coefficient of static friction between the block and the ground is μ.

SOLUTION

As the external force F is gradually increased from zero it is compensated by the static friction and the string bears no tension. So, till $F_{app} < f_{limiting}$, we have $T = 0$. So for $F_{app} > f_l$



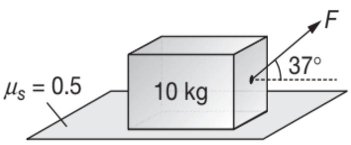
$$\Rightarrow \quad F_{app} > \mu mg$$

We have, $F = T + \mu mg$

$$\Rightarrow \quad T = F - \mu mg$$

ILLUSTRATION 71

In the arrangement shown, the force F is gradually increased from zero. Determine whether the block will first slide or lift up.

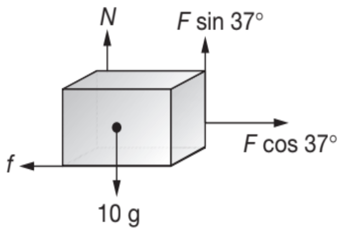


SOLUTION

We have to calculate the minimum magnitude of forces required both in horizontal and vertical direction either to slide or lift up the block. The block will first slide or lift up depends upon which minimum magnitude of force is lesser.

CASE-1: For block to start lifting up in vertical direction, we have

$$F\sin(37°) + N - Mg \geq 0$$



N becomes zero just before lifting, so

$$F_{lift} \geq \frac{10g}{3/5}$$

$$\Rightarrow \quad F_{lift} \geq \frac{500}{3}\text{ N}$$

CASE-2: For block to start sliding in horizontal direction, we have

$$F\cos(37°) \geq \mu_s N$$

Since $N = 10g - F\sin(37°)$

$$\Rightarrow \quad F\cos(37°) \geq 0.5\big(10g - F\sin(37°)\big)$$

Hence $F_{slide} \geq \dfrac{50}{\cos(37°) + 0.5\sin(37°)}$

$$\Rightarrow \quad F_{slide} \geq \frac{500}{11}\text{ N}$$

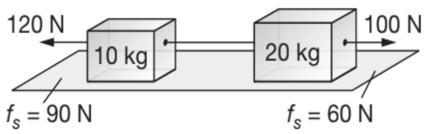
Since $F_{lift} \geq \dfrac{500}{3}$ N

$$\Rightarrow \quad F_{slide} < F_{lift}$$

Hence the block will begin to slide horizontally before it begins to lift up.

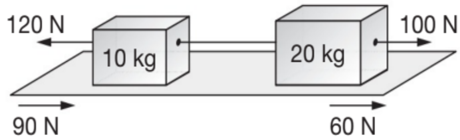
ILLUSTRATION 72

Calculate the tension in the string in situation as shown in the figure below. Forces 120 N and 100 N start acting when the system is at rest and the maximum value of static friction for 10 kg is 90 N and that for 20 kg is 60 N?



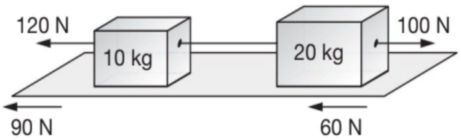
SOLUTION

(a) Let us assume that system moves towards left then as it is clear from *FBD*, net force in horizontal direction is towards right. Therefore the assumption is not valid.



Above assumption is not possible as net force on system comes towards right. Hence system is not moving towards left.

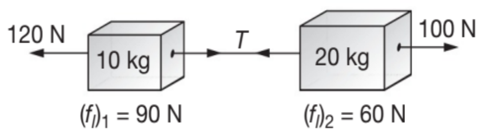
(b) Similarly let us assume that system moves towards right.

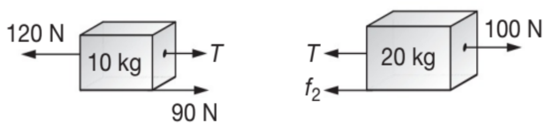


Above assumption is also not possible as net force on the system is towards left in this situation. Hence assumption is again not valid.

Therefore we conclude that the system remains stationary.

Assuming that the 10 kg block reaches limiting friction first then using FBD's





For 10 kg block to be in equilibrium, we have

$$120 = T + 90$$

$$\Rightarrow \quad T = 30 \text{ N}$$

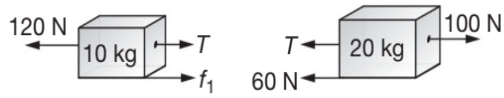
Also $T + f_2 = 100$

$$\Rightarrow \quad 30 + f_2 = 100$$

$$\Rightarrow \quad f_2 = 70 \text{ N}$$

Which is not possible as the limiting value is 60 N for this surface of block.

So, our assumption is wrong and hence now let us take the 20 kg block to be in limiting situation, hence



$$T + 60 = 100 \text{ N}$$

$$\Rightarrow \quad T = 40 \text{ N}$$

Also $f_1 + T = 120$ N

$$\Rightarrow \quad f_1 = 80 \text{ N} < (f_l)_1$$

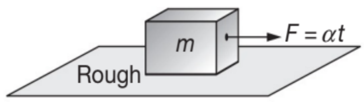
This is acceptable as static friction at this surface should be less than 90 N.

Hence the tension in the string is

$$T = 40 \text{ N}$$

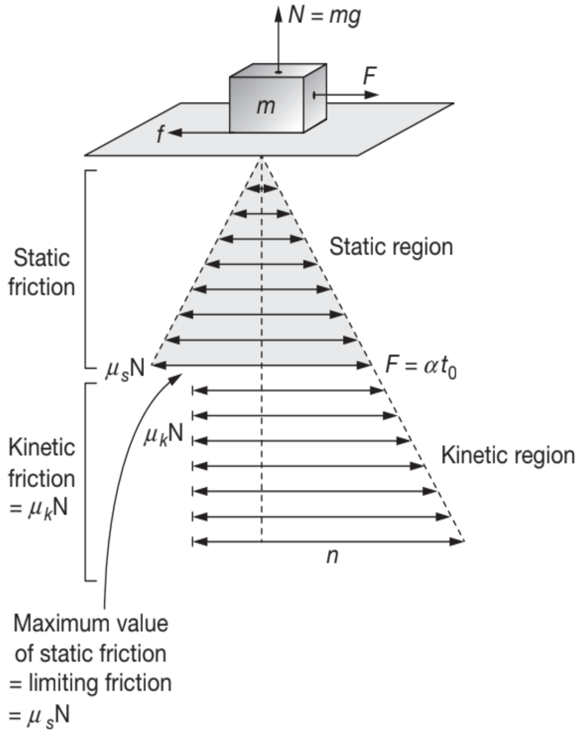
ILLUSTRATION 73

A block of mass m is placed on a rough horizontal surface, as shown in figure. At $t = 0$, a horizontal force $F = \alpha t$, where α is a positive constant, is applied on the block. If μ_s and μ_k be coefficients of static and kinetic frictions respectively, find the frictional force acting on the block as a function of time. Also plot the frictional force against the applied horizontal force F.



SOLUTION

The detailed analysis of the given situation is shown in figure. At $t = 0$, when $F = 0$, frictional force is also zero but as F starts increasing from zero, frictional force also increases and balances it or we can say that friction prevents the block from sliding over the horizontal surface. As time passes, F increases and f also increases.



In fact friction and F have the same magnitude until the block starts sliding. At some time t_0, static friction reaches its maximum value $\mu_s N$. After $t = t_0$, F will keep increasing but friction will not increase and hence the block can not remain in equilibrium and eventually it acquires motion along the direction of F. But as soon as the block starts sliding, the nature of the friction becomes kinetic and the magnitude of the frictional force comes down to $\mu_k N$ and thereafter remains constant as long as the block is in motion. If the block starts sliding immediately after $t = t_0$, then at $t = t_0$.

$$F = f$$

$$\Rightarrow \quad \alpha t_0 = \mu_s N$$

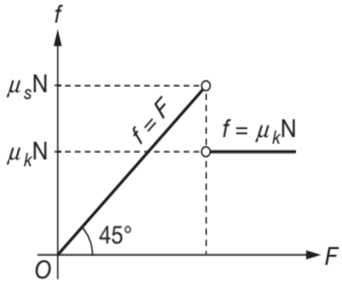
$$\Rightarrow \quad \alpha t_0 = \mu_s mg$$

$$\Rightarrow \quad t_0 = \frac{\mu_s mg}{\alpha}$$

Therefore, for the time interval $(0, t_0)$: $f = F(=\alpha t)$

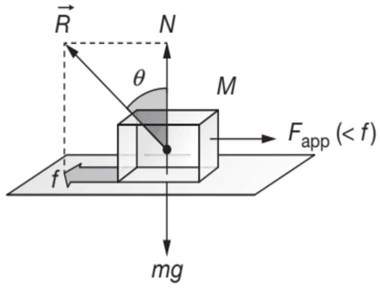
and for the time interval (t_0, ∞): $f = \mu_k N = \mu_k mg$

The plot of f against F is shown in figure.



COEFFICIENT OF FRICTION, LIMITING FRICTION AND ANGLE OF FRICTION

Consider a block resting on a rough horizontal surface. The forces acting on the block are its weight mg downwards and normal reaction N acting upward, such that $N = mg$.



Now suppose a force F_{app} is applied on the block to the right, then there will arise a frictional force f directed to the left (opposite to direction of applied force), which prevents the motion of the block. Let the resultant of $\vec{N}$ and $\vec{f}$ be $\vec{R}$ which makes an angle θ with normal reaction $\vec{N}$. Resolving $\vec{R}$, we get

$$R\cos\theta = N \text{ and } R\sin\theta = f \qquad \ldots(1)$$

For equilibrium $N = mg$ and $f = F_{\text{app}}$ ...(2)

If we increase the pull F_{app} continuously, the force of friction increases and a stage comes when the body is just on the state of moving. This state is called **limiting equilibrium**. Under this condition the frictional force is maximum and is equal to applied force.

THE COEFFICIENT OF FRICTION (μ)

It is defined as the ratio of limiting friction f to the normal reaction N between two surfaces in contact,

i.e., $\mu = \dfrac{f}{N}$

Limiting Friction

The maximum value of static frictional force exerted between two surfaces in contact parallel to surfaces for a given normal force between them, when the body is on the verge of motion, is called **limiting friction**.

Angle of Friction

Angle of friction (θ) is the angle which the resultant of force of static friction (f) and normal reaction (N) makes with the normal reaction.
From (1), we get

$$\tan\theta = \frac{f}{N}$$

Since $\mu = \dfrac{f}{N}$

$$\Rightarrow \quad \tan\theta = \mu$$

Hence coefficient of static friction is equal to tangent of the angle of friction.

RESULTANT FORCE EXERTED BY A SURFACE ON THE BLOCK

The Reaction Force is the resultant of force of friction and the normal reaction, so from the above figure, we have

$$R = \sqrt{f^2 + N^2}$$

$$\Rightarrow \quad R = \sqrt{(\mu mg)^2 + (mg)^2}$$

$$\Rightarrow \quad R = mg\sqrt{\mu^2 + 1}$$

when there is no friction $(\mu = 0)$, the reaction force R will be minimum

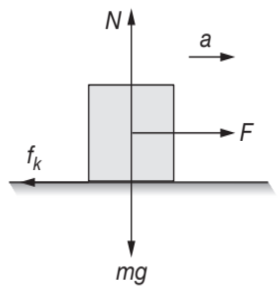
$$\Rightarrow \quad R = mg$$

Hence the range of Reaction Force R is

$$\Rightarrow \quad mg \le R \le mg\sqrt{\mu^2 + 1}$$

ACCELERATION OF BLOCK ON ROUGH HORIZONTAL SURFACE

When body is moving under application of force $F > f_l$, then kinetic friction f_k opposes its motion.



Let a be the acceleration of the body, then from the figure, we observe that

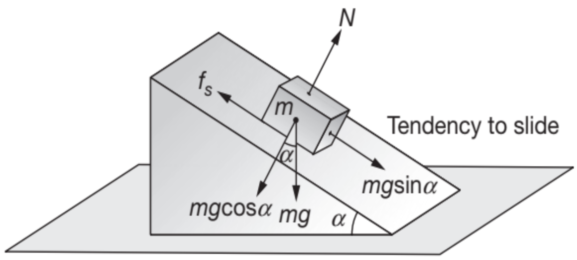
$$F - f_k = ma$$

So, $a = \dfrac{F - f_k}{m}$, for $F > f_l = \mu_s N = \mu_s(mg)$ and

$$a = 0, \text{ for } F \le f_l = \mu_s N = \mu_s(mg)$$

ANGLE OF REPOSE (α)

This is concerned with an inclined plane on which a block rests, exerting its weight on the plane.



The angle of repose α is the angle which an inclined plane makes with the horizontal such that a body placed on it is on the verge of motion (is just about to loose the state of rest).
Under this condition the forces acting on the block are:

(a) its weight mg, downward,

(b) normal reaction N, normal to plane,

(c) a force of friction f_s, parallel and tangential to plane upward.

Taking α as angle of inclination of the plane with the horizontal and resolving mg, parallel and perpendicular to inclined plane, then for equilibrium, we get

$$N = mg\cos\alpha \text{ and } f_s = mg\sin\alpha$$

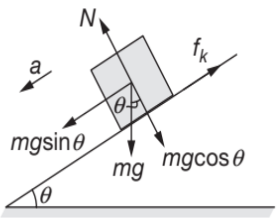
$$\Rightarrow \quad \tan\alpha = \frac{f_s}{N} = \mu_s \qquad \left\{\because \mu_s = \frac{f_s}{N}\right\}$$

Reaction force i.e. contact force is

$$R = \sqrt{f_s^2 + N^2} = \sqrt{(mg\sin\theta)^2 + (mg\cos\theta)^2} = mg$$

ACCELERATION OF BLOCK DOWN A ROUGH INCLINE

When angle of inclined plane (θ) is more than angle of repose (α), then the body placed on the inclined plane starts sliding down with an acceleration a. From the figure



$$mg\sin\theta - f_k = ma$$

$$\Rightarrow \quad ma = mg\sin\theta - \mu_k N$$

$$\Rightarrow \quad ma = mg\sin\theta - \mu_k mg\cos\theta$$

$\Rightarrow$ Acceleration, $a = g(\sin\theta - \mu_k\cos\theta)$

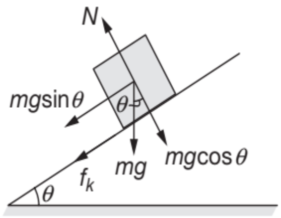
Conceptual Note(s)

For frictionless inclined plane $\mu = 0$, so we get $a = g\sin\theta$.

RETARDATION OF BLOCK MOVING UP A ROUGH INCLINE

When the angle of inclined plane (θ) is less than the angle of repose (α), then for the upward motion of the block, retardation is given by

$$\text{Retardation} = \frac{\text{Net retarding force}}{\text{Total mass}}$$



$$\Rightarrow \quad a = \frac{mg\sin\theta + f_k}{m}$$

$$\Rightarrow \quad a = \frac{mg\sin\theta + \mu mg\cos\theta}{m}$$

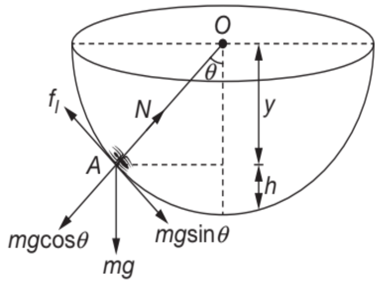
$\Rightarrow$ Retardation, $a = g(\sin\theta + \mu\cos\theta)$

Conceptual Note(s)

For frictionless inclined plane $\mu = 0$, so the retardation is given by $a = g\sin\theta$.

MAXIMUM HEIGHT (*H*) TO WHICH AN INSECT CAN CRAWL UP A ROUGH HEMISPHERICAL BOWL

The insect can crawl up the bowl, up to a certain height h only till the component of its weight along the bowl is balanced by limiting frictional force.



Let m be the mass of the insect, r be the radius of the bowl and μ be the coefficient of friction, then at the point A, we have

$$N = mg\cos\theta \qquad \ldots(1)$$

$$f_l = mg\sin\theta \qquad \ldots(2)$$

Dividing (2) by (1), we get

$$\tan\theta = \frac{f_l}{N} = \mu \qquad \left\{\because \mu = \frac{f_l}{N}\right\}$$

$$\Rightarrow \quad \frac{\sqrt{r^2-y^2}}{y}=\mu$$

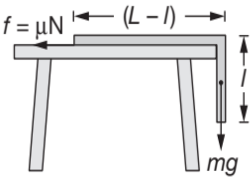
$$\Rightarrow \quad y=\frac{r}{\sqrt{1+\mu^2}}$$

Since, $h=r-y=r\left[1-\frac{1}{\sqrt{1+\mu^2}}\right]$

$$\Rightarrow \quad h=r\left[1-\frac{1}{\sqrt{1+\mu^2}}\right]$$

MAXIMUM LENGTH OF CHAIN THAT CAN HANG FROM THE TABLE WITHOUT FALLING FROM IT

Let a uniform chain of length L be placed on the table such that its length l is hanging over the edge of table without sliding. So, the weight of the hanging part of the chain must be balanced by the force of friction that is acting on the part of chain lying on the table



$$\Rightarrow \quad \mu N = mg$$

$$\Rightarrow \quad \mu\left[\lambda(L-l)g\right]=\lambda l g$$

where λ is mass per unit length of the chain

$$\Rightarrow \quad \mu(L-l)=l$$

$$\Rightarrow \quad l=\frac{\mu L}{\mu+1}$$

So percentage of length of chain hanging is $\frac{l}{L}\times 100\%$.

MINIMUM FORCE FOR MOTION ALONG HORIZONTAL SURFACE AND ITS DIRECTION

Consider a block of mass m on which a force F is applied such that it makes an angle θ with the horizontal. Then for the motion of the block to just begin (or for the block to be on the verge of motion), by resolving F in horizontal and vertical direction (as shown in figure), we get

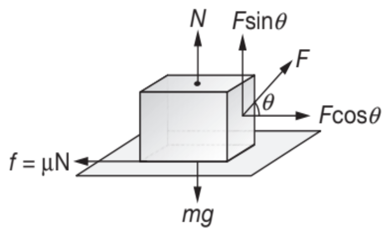
$$F\cos\theta \ge \mu N \text{ and} \quad \ldots(1)$$

$$N+F\sin\theta = mg$$

$$\Rightarrow \quad N = mg - F\sin\theta \quad \ldots(2)$$

Substituting (2) in (1), we get

$$F\cos\theta \ge \mu mg - \mu F\sin\theta$$



$$\Rightarrow \quad F \ge \frac{\mu mg}{\cos\theta+\mu\sin\theta} \quad \ldots(3)$$

Let $Z=\text{Denominator}=\cos\theta+\mu\sin\theta$

For F to be MINIMUM, Z must be MAXIMUM

$$\Rightarrow \quad \frac{dZ}{d\theta}=0$$

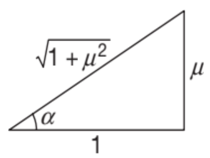
$$\Rightarrow \quad -\sin\theta+\mu\cos\theta=0$$

$$\Rightarrow \quad \tan\theta=\mu$$

$$\Rightarrow \quad \theta=\tan^{-1}\mu$$

So F will be minimum when its angle with the horizontal is equal to the angle of friction i.e. $\theta=\tan^{-1}\mu$.

Since $\tan\theta=\mu$, so from the figure, we get



$$\sin\theta=\frac{\mu}{\sqrt{1+\mu^2}} \text{ and } \cos\theta=\frac{1}{\sqrt{1+\mu^2}}$$

By substituting these value in equation (3), we get

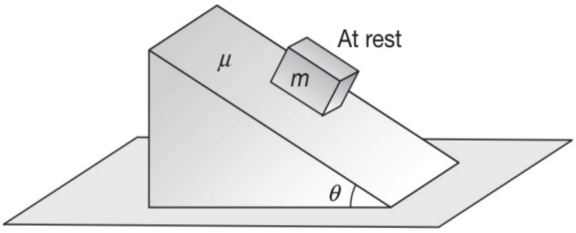
$$F \ge \frac{\mu mg}{\frac{1}{\sqrt{1+\mu^2}}+\frac{\mu^2}{\sqrt{1+\mu^2}}}$$

$$\Rightarrow \quad F \ge \frac{\mu mg\sqrt{1+\mu^2}}{1+\mu^2} \quad \Rightarrow \quad F \ge \frac{\mu mg}{\sqrt{1+\mu^2}}$$

$$\Rightarrow \quad F_{\min} = \frac{\mu mg}{\sqrt{1+\mu^2}}$$

ILLUSTRATION 74

A block of mass m, as shown in figure, is resting on a rough inclined plane with angle of inclination θ. If μ be the coefficient of friction between the block and the inclined surface, find:



(a) frictional force acting on the block

(b) the maximum angle of inclination, θ_0, for which the block stays in equilibrium

(c) if $\theta > \theta_0$, find the acceleration of the block.

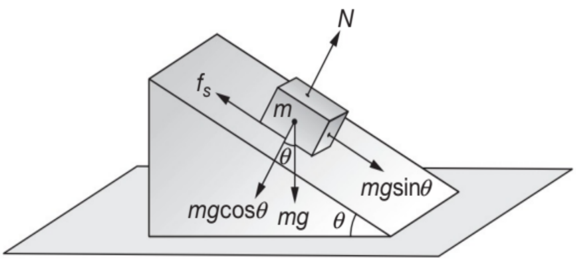
SOLUTION

The forces acting on the block and FBD of the block are shown in figure (a). The weight of the block, mg, has been resolved along the surface and the direction perpendicular to the inclined surface, as shown in figure (a). From figure (a) it is obvious that the component of the weight of the block parallel to the surface, $mg\sin\theta$, tries to slide the block down the inclined surface and therefore, the surface applies a frictional force opposite to the direction of $mg\sin\theta$.

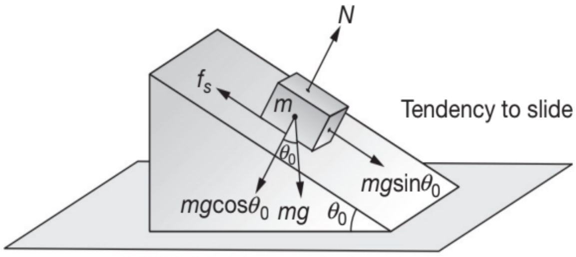
(a) Since, the block is in equilibrium, we have,

$$f = mg\sin\theta \quad \ldots(1)$$

and $N = mg\cos\theta \quad \ldots(2)$



(a)



(b)

(b) As the frictional force acting on the block is static in nature (therefore the block is at rest w.r.t. the inclined surface), it must not exceed its limiting value which is equal to μN. Therefore, from figure (b), we have

$$f \le \mu N$$

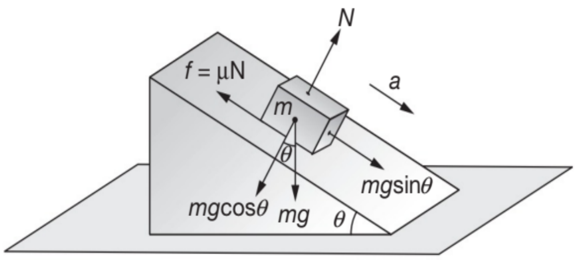
$$\Rightarrow \quad mg\sin\theta_0 \le \mu mg\cos\theta_0$$

$$\Rightarrow \quad \tan\theta_0 \le \mu$$

$$\Rightarrow \quad \theta_0 \le \tan^{-1}\mu \quad \ldots(3)$$

Therefore, the maximum angle of inclination θ_0 for which the block remains in equilibrium is $\tan^{-1}\mu$.

(c) If $\theta > \tan^{-1}\mu$, then obviously the block can not remain in equilibrium or we can also say that for this angle the limiting friction can not balance the component of the gravitational pull parallel to the surface. Therefore, the block will accelerate down the incline, as shown in the figure (c).



(c)

In this case the frictional force is kinetic in nature and hence its magnitude is constant and is equal to μN. Applying Newton's Second Law along the inclined surface, from figure (c), we get,

$$mg\sin\theta - f = ma$$

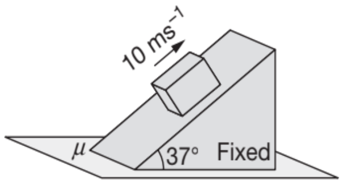
$\Rightarrow \quad mg\sin\theta - \mu N = ma$

$\Rightarrow \quad mg\sin\theta - \mu mg\cos\theta = ma$

$\Rightarrow \quad a = g(\sin\theta - \mu\cos\theta) \qquad \ldots(4)$

ILLUSTRATION 75

Find out the distance travelled by the block on incline before it stops. Initial velocity of the block is $10\ \text{ms}^{-1}$ and coefficient of friction between the block and incline is $\mu = 0.5$.



SOLUTION

$N = mg\cos(37°)$

Since $mg\sin(37°) + \mu N = ma$

$\Rightarrow \quad a = g[\sin(37°) + \mu\cos(37°)]$

$\Rightarrow \quad a = 10\left[\frac{3}{5} + 0.5\left(\frac{4}{5}\right)\right] = 10\ \text{ms}^{-2}$

$\Rightarrow \quad a = 10\ \text{ms}^{-2}$ down the incline

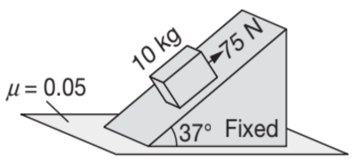
Since, $v^2 = u^2 + 2as$

$\Rightarrow \quad 0 = 10^2 + 2(-10)\ s$

$\Rightarrow \quad s = 5\ \text{m}$

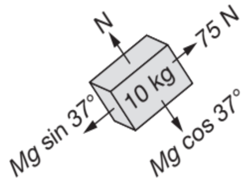
ILLUSTRATION 76

In the arrangement shown, calculate the acceleration of the block. Assume that the block is initially at rest. Calculate the force of friction acting between the block and incline. Also calculate the additional force (excess of 75 N) for which the block starts to move up the incline. Find the minimum force that can replace 75 N so that the block does not move.



SOLUTION

Let us first draw F.B.D. of the block excluding the friction.

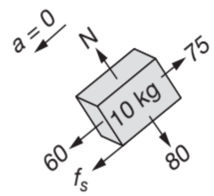


$N = 10g\cos(37°) = 80\ \text{N}$

Since the applied force on the block is

$Mg\sin(37°) = (100)\left(\frac{3}{5}\right) = 60\ \text{N} < 75\ \text{N}$

So, net force has a tendency to move the block up the incline due to which friction acts down the incline as shown. Also, $f_l = \mu N = 40\ \text{N}$



But $(F_{\text{applied}})_{\text{net}} = 75 - 60 = 15\ \text{N} < f_l$

So, block will not move, i.e., $a = 0$ and hence static friction acts between the block and the incline. Now the block is at rest or in equilibrium with respect to the incline, so we have

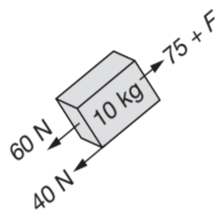
$f_s + 60 = 75$

$\Rightarrow \quad f_s = 15\ \text{N}$

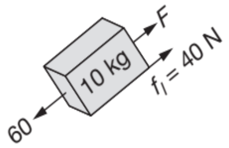
Let F be the additional force for which the block starts moving up the incline (i.e. still the block is in equilibrium or it does not move with respect to the incline), then we have

$75 + F = 60 + 40$

$\Rightarrow \quad F = 25\ \text{N}$



In this case the block has a tendency to move downwards. So the friction acts upwards.



$\Rightarrow \quad F + 40 = 60$

$\Rightarrow \quad F = 20\text{ N}$

Problem Solving Technique(s)

Block Over Block Problems

Method of Solving

STEP-1: Make free body diagram.

STEP-2: Calculate $f_\ell = \mu N$, i.e., the limiting value of friction between the required surface(s).

STEP-3: Denote static friction force by f_s (or f) because value of friction is not known. Then $f_s \le f_\ell$.

STEP-4: Calculate the static friction separately for the following two cases.

CASE-1: When the Blocks Move Together

STEP-5: Calculate acceleration, for the blocks moving together $a = \dfrac{F_{applied}}{\text{Total Mass}} = \dfrac{F}{m_1 + m_2}$.

STEP-6: Calculate f_s (or f) for above calculated value of a.

STEP-7(a): If $f_s < f_\ell$, which is true, then it means that the blocks will move together with acceleration $a = \dfrac{F}{m_1 + m_2}$.

CASE-2: When the Blocks Move Separately

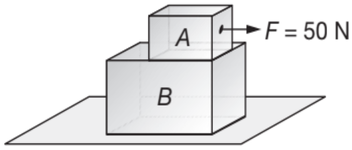
STEP-7(b): If $f_s > f_\ell$, which is impossible, so the blocks will move separately and kinetic friction $f_k = \mu_k N$ is involved.

STEP-8: Calculate acceleration of each block separately for this case.

STEP-9: Also keep in mind that for block A kept on block B (which is kept on a rough surface), the friction on B due to A, i.e. (f_{BA}) will be the part of external force applied on B and if this external force is less than friction between B and ground, then B will not move and friction between B and ground will be the applied force on block B, i.e., f_{BA}.

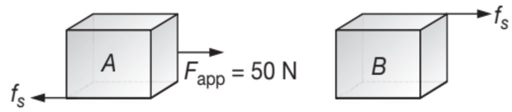
ILLUSTRATION 77

In the arrangement shown, calculate the acceleration of the two blocks each of mass 10 kg. The system is initially at rest and the friction coefficient between A and B is 0.5 whereas no friction exists between B and ground. Also calculate the maximum force applied on A for which the blocks move together.



SOLUTION

$$f_{\text{lim}} = \mu N = (0.5)(10)(10) = 50\text{ N}$$



Now let us assume that both blocks move together, then

$$a = a_A = a_B = \frac{50}{10+10} = 2.5\text{ ms}^{-2}$$

Let us now calculate the friction between A and B for $a_A = a_B = 2.5\text{ ms}^{-2}$. Since

$$f_s = m_B a_B = (10)(2.5) = 25\text{ N} < f_{\text{limiting}}$$

So, the blocks A and B will move together with an acceleration of 2.5 ms^{-2} and friction between A and B is $25\text{ N} < f_l (= 50\text{ N})$.

Let $F_{\max} = F$ be the maximum force for which the blocks move together. Then the friction between the blocks will be limiting i.e.

$$f_l = \mu(m_A g) = (0.5)(10)(10) = 50\text{ N}$$

So, we have

For A, $F_{\max} - 50 = 10a$ ...(1)

For B, $50 = 10a$ ...(2)

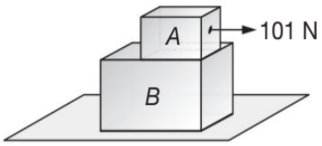
From (2), we get $a = 5\text{ ms}^{-2}$

Substituting in (1), we get

$$F_{\max} = 100\text{ N}$$

ILLUSTRATION 78

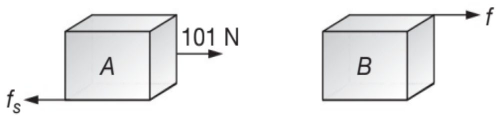
Find the acceleration of the two blocks A and B each of mass 10 kg. The system is initially at rest and the friction coefficient between A and B is 0.5 whereas the ground is smooth.



SOLUTION

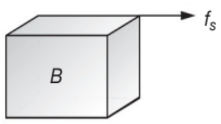
$$f_{\text{lim}} = 50\text{ N}$$

Also, $f_s \le f_l = 50\text{ N}$



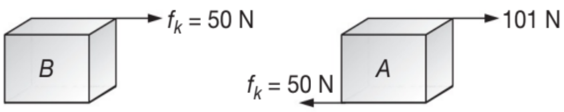
STEP-1: Assuming that blocks A and B move together, then $a = \dfrac{101}{20} = 5.05\text{ ms}^{-2}$

STEP-2: Calculating static friction on B due to A



$$f_s = 10 \times 5.05 = 50.5$$

Since, $f_s > f_{\text{lim}}$ which is not possible, so both blocks A and B move separately and so kinetic friction is involved.



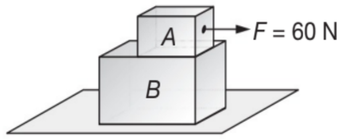
So, for A, $a_A = \dfrac{101-50}{10} = 5.1\text{ ms}^{-2}$ and

for B, $a_B = \dfrac{50}{10} = 5\text{ ms}^{-2}$

We observe that $a_A > a_B$ as force is applied on A.

ILLUSTRATION 79

Find the acceleration of the two blocks A (of mass 10 kg) and B (of mass 20 kg). The system is initially at rest and the friction coefficient between A and B is 0.5 whereas the ground is smooth. Also calculate the maximum force F applied horizontally on A so that blocks A and B move together.

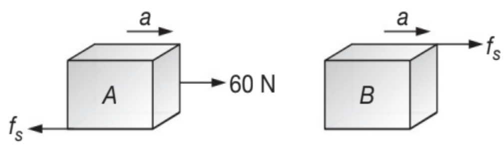


SOLUTION

$$f_{\text{lim}} = \mu m_A g = 50\text{ N}$$

Assuming that the blocks A and B move together, then

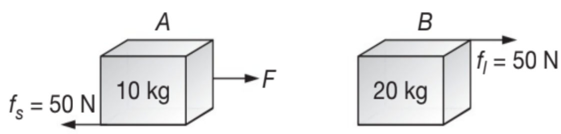
$$a_A = a_B = a = \frac{60}{10+20} = 2\text{ ms}^{-2}$$



Let f_s be static friction between A and B, then $f_s = m_B a = (20)(2) = 40\text{ N} < f_{\text{lim}}$

So the blocks move together with an acceleration of 2 ms^{-2} and friction between the blocks is static and 40 N.

The maximum force for which A and B move together corresponds to the situation when friction between both A and B is limiting i.e., $f_l = 50\text{ N}$.



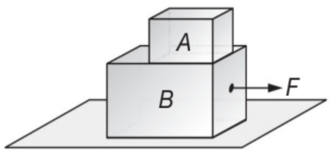
For A, $F - f_{\max} = 10a$ ...(1)

For B, $f_{\max} = 20a$ ...(2)

$\Rightarrow F = 75\text{ N}$

ILLUSTRATION 80

In the arrangement shown, the blocks A (of mass 10 kg) and B (of mass 20 kg) are initially at rest. Calculate the minimum value of F for which sliding starts between the two blocks. Assuming the friction coefficient between A and B to be 0.5 whereas the ground is smooth.

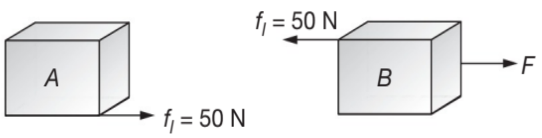


SOLUTION

When the sliding between A and B starts, then limiting friction is acting between A and B.

Since $f_l = \mu m_A g$

$\Rightarrow \quad f_l = 50\text{ N}$



For A, $\quad f_l = 10a$

$\Rightarrow \quad a = 5\text{ ms}^{-2}$

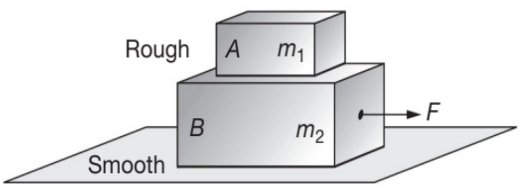
For B, $\quad F - 50 = 20a$

$\Rightarrow \quad F = 50 + 20(5) = 150\text{ N}$

$\Rightarrow \quad F_{\text{min}} = 150\text{ N}$

ILLUSTRATION 81

Block A is placed over block B which is placed over some smooth horizontal surface, as shown in figure. It is given that there is no friction between the horizontal surface and the lower block and appreciable friction exists between the two blocks. If block B is pulled by constant horizontal force, as shown in figure, then



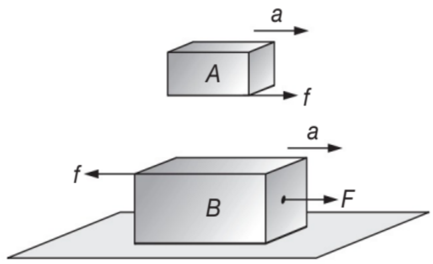
(a) find the acceleration of each block and the frictional force acting on each block, if the two blocks are moving together.

(b) if coefficient of friction between m_1 and m_2 be μ, what is the maximum value of F for which the two blocks will move together?

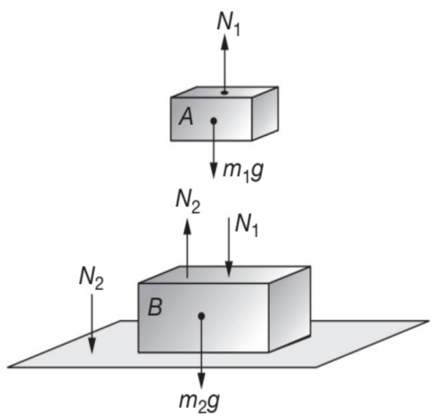
(c) if the force F is given by $F = \alpha t$ (α is a positive constant), find how the accelerations of the block A and of the block B depend on t, if the coefficient of friction between the blocks is equal to μ. Draw the approximate plot of these dependencies.

SOLUTION

(a) There is no friction between m_2 and ground. Also m_1 moves backwards relative to m_2. So, friction on m_1 is forward and the reaction pair of this friction will act on m_2 in the backward direction as shown.



(a)



(b)

For m_1: $\quad f = m_1 a \qquad \ldots(1)$

For m_2: $\quad F - f = m_2 a \qquad \ldots(2)$

Solving equations (1) and (2), we get

$$f = \frac{m_1 F}{m_1 + m_2} \text{ and } a = \frac{F}{m_1 + m_2}$$

(b) Since frictional force acting on the block A is static in nature, we can write,

$$f \le \mu N_1$$

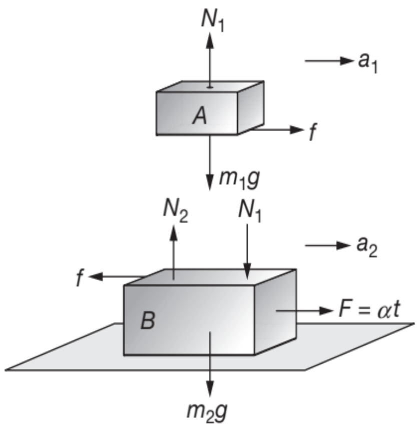
$$\Rightarrow \quad \frac{m_1 F}{m_1 + m_2} \le \mu m_1 g$$

$$\Rightarrow \quad F \le \mu(m_1 + m_2)g$$

$$\Rightarrow \quad F_{max} = \mu(m_1 + m_2)g$$

Therefore, the two bodies will move together only if the applied force F is not greater than $\mu(m_1 + m_2)g$.

(c) It is quite obvious from the previous discussion that from $t = 0$ to some instant, lets say t_0, the blocks will move together, that is, there will be no slipping between the blocks and for this duration of time friction will be static in nature. After t_0, there will be relative motion between the blocks and the friction would be kinetic in nature. The forces acting on the two blocks and their assumed accelerations are shown in figure.



For $t < t_0$:

$$a_1 = a_2 \qquad (= a\text{, say})$$

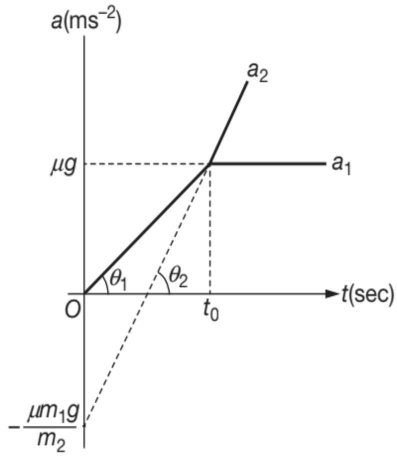
For m_1: $f = m_1 a$

For m_2: $F - f = m_2 a$

$$\Rightarrow \quad a = \frac{m_1 F}{m_1 + m_2}$$

$$\Rightarrow \quad a = \frac{\alpha m_2 t}{m_1 + m_2} \text{ and } f = m_1 a$$

$$\Rightarrow \quad f = \frac{\alpha m_1 m_2}{m_1 + m_2} t$$



At $t = t_0$:

$$f = \mu N_1$$

$$\Rightarrow \quad f = \frac{\alpha m_1 m_2}{m_1 + m_2} t_0 = \mu m_1 g$$

$$\Rightarrow \quad t_0 = \frac{\mu(m_1 + m_2)g}{\alpha m_2}$$

For $t > t_0$:

$$f = \mu N_1 = \mu m_1 g$$

$$\Rightarrow \quad a_1 = \frac{f}{m_1} = \frac{\mu m_1 g}{m_1} = \mu g$$

$$\text{and} \quad a_2 = \frac{F - f}{m_2} = \frac{\alpha t - \mu m_1 g}{m_2} = \frac{\alpha t}{m_2} - \frac{\mu m_1 g}{m_2}$$

It is clear from the obtained expressions for the accelerations of the blocks that till t_0, the bodies are moving together but at t_0, friction reaches its limiting value and thereafter acceleration of the upper block becomes constant and that of the lower block keeps on increasing due to increase in magnitude of F. Dependencies of a_1 and a_2 on time are shown in figure.

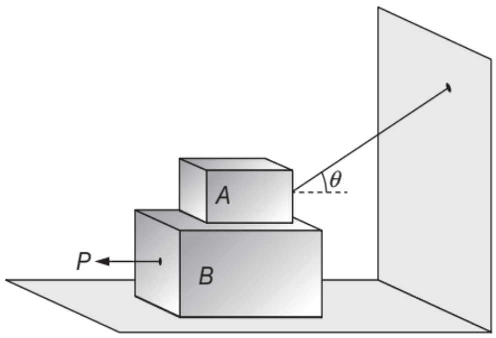
$$\tan\theta_1 = \frac{\alpha m_1}{m_1 + m_2} \text{ and } \tan\theta_2 = \frac{\alpha}{m_2}$$

Conceptual Note(s)

Now, in order to understand this result intuitively read the following. In part (a) we got $f = \frac{m_1 F}{m_1 + m_2}$ (for the case when the two bodies are moving together). Here you can see that $f \propto F$. It is quite obvious why we got $f \propto F$. When we increase F, acceleration of the block B increases and now to prevent relative motion between the block A and the block B, the acceleration of the block A also must increase. Here friction comes to the rescue. Friction increases its value to increase the acceleration of the block A and to decrease the acceleration of the block B and therefore, accelerations of the blocks are still equal. But if we keep on increasing F, then friction can increase only if its limiting value is not achieved. If we increase F when friction has already achieved its limiting value, then the acceleration of the lower block will increase but that of the upper block cannot increase because friction cannot increase anymore and hence acceleration of the lower block would be greater than that of the upper block. Now, there will be relative motion between the two blocks and the nature of the friction would be kinetic.

ILLUSTRATION 82

In the arrangement shown block A has weight $W_1 = 100$ N, block B has weight $W_2 = 200$ N and $\mu = 0.25$ for all surfaces in contact. A pull P is applied on W_2 such that it just slides under W_1. Find the tension in the string and the pull P, if $\theta = 45°$.

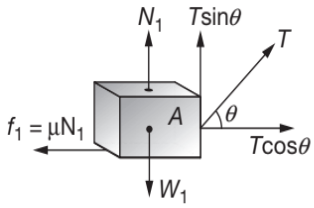


For all surfaces in contact, $\mu = 0.25$

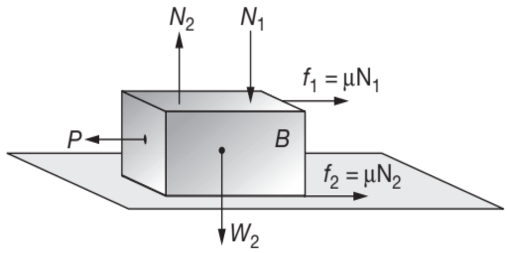
SOLUTION

For block A

Other than the routine forces shown in FBD, our main concern is to find the direction of friction acting on W_1. Since, W_1 is lying on W_2 (tendency to move along P) therefore, W_1 has the tendency to move opposite to P and hence force of friction of W_1 must be acting in the direction of P. The reaction pair of which will be acting of W_2 opposite to P.



FBD A



FBD B

For block B

Since W_2 is placed on ground, and hence will have a tendency to move along P therefore friction on W_2 due to ground will act opposite to P.

Apart from the other routine forces, f_1 is also acting on W_2 opposite to P (reaction pair of f_1 action on W_1). Now for W_1 to just move on W_2, i.e., condition for equilibrium to be just achieved, we have

For W_1

$$N_1 + T\sin\theta = W_1 \quad \ldots(1)$$

$$T\cos\theta = f_1 = \mu N_1 \quad \ldots(2)$$

For W_2

$$N_2 = N_1 + W_2 \quad \ldots(3)$$

$$P = f_1 + f_2 = \mu(N_1 + N_2) \quad \ldots(4)$$

Now, it is given that $W_1 = 100$ N, $W_2 = 200$ N, $\mu = 0.25$ and $\theta = 45°$. So from (1), (2), (3) and (4) we get

$$N_1 + \frac{T}{\sqrt{2}} = 100 \quad \ldots(5)$$

$$\frac{T}{\sqrt{2}} = (0.25)N_1 \quad \ldots(6)$$

$$N_2 = N_1 + 200 \quad \ldots(7)$$

$$P = 0.25(N_1 + N_2) \quad \ldots(8)$$

Substituting (6) in (5), we get

$$N_1 + 0.25N_1 = 100$$

$$\Rightarrow \quad 1.25N_1 = 100$$

$$\Rightarrow \quad N_1 = 80 \text{ N}$$

So, from (7), we get

$$N_2 = 280 \text{ N}$$

Substituting $N_1 = 80$ N in (6), we get

$$T = (0.25)\sqrt{2}(80)$$

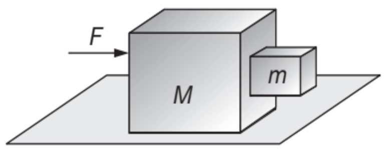
$$\Rightarrow \quad T = 20\sqrt{2} \text{ N}$$

From (8), we get

$$P = 0.25(80 + 280) = 90 \text{ N}$$

ILLUSTRATION 83

In the figure, the coefficient of friction between box (of mass M) and block (of mass m) is μ. Find the magnitude of horizontal force F required to keep the block stationary with respect to wedge.



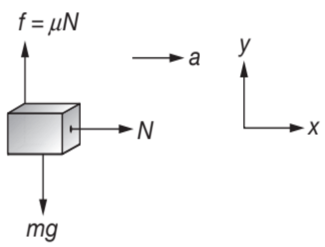
SOLUTION

Such problems can be solved with or without using the concept of pseudo force. Let us, solve the problem by both the methods.

a = Acceleration of (box + block) in horizontal direction

$$a = \frac{F}{M+m}$$

Inertial frame of reference (Ground): F.B.D. of block with respect to ground (only real forces have to applied) with respect to ground block is moving with an acceleration a. Therefore, $\Sigma F_y = 0$ and $\Sigma F_x = ma$.



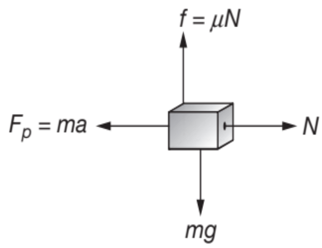
$$mg = \mu N \text{ and } N = ma$$

$$\Rightarrow \quad a = \frac{g}{\mu}$$

$$\Rightarrow \quad F = (M+m)a$$

$$\Rightarrow \quad F = (M+m)\frac{g}{\mu}$$

Non-inertial frame of reference (Box): F.B.D. of m with respect to box (real + one pseudo force)



With respect to wedge block is stationary.

$$\Rightarrow \quad \Sigma F_x = 0 = \Sigma F_y$$

$$\Rightarrow \quad mg = \mu N \text{ and } N = ma$$

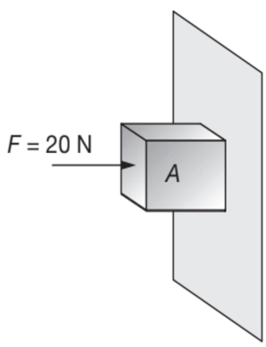
$$\Rightarrow \quad a = \frac{g}{\mu} \text{ and } F = (M+m)a$$

$$\Rightarrow \quad F = (M+m)\frac{g}{\mu}$$

From the above discussion, we can see that from both the methods results are same.

ILLUSTRATION 84

A block of mass 1 kg is pushed against a rough vertical wall with a force of 20 N, coefficient of static friction being $\frac{1}{4}$. Another horizontal force of 10 N is applied on the block in a direction parallel to the wall. Will the block move? If yes, in which direction? If no, find the frictional force exerted by the wall on the block. $(g = 10 \text{ ms}^{-2})$.



SOLUTION

Normal reaction on the block from the wall is

$$N = F = 20\text{ N}$$

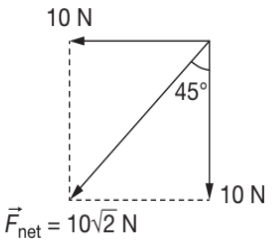
Therefore, limiting friction is given by

$$f_L = \mu N$$

$$\Rightarrow \quad f_L = \left(\frac{1}{4}\right)(20) = 5\text{ N}$$

Weight of the block is

$$W = mg = (1)(10) = 10\text{ N}$$



Since a horizontal force of 10 N is applied to the block so, the resultant of these two forces will be $10\sqrt{2}$ N in the direction shown in figure. Now, this resultant is greater than the limiting friction and so the block will move in the direction of $\vec{F}_{net}$ with an acceleration a given by

$$a = \frac{F_{net} - f_L}{m} = \frac{10\sqrt{2} - 5}{1} = 9.14\text{ ms}^{-2}$$

ILLUSTRATION 85

A small bar starts sliding down an inclined plane forming an angle α with the horizontal. The friction coefficient depends on the distance x covered as $\mu = kx$, where k is a constant. Find the distance covered by the bar till it stops and its maximum velocity over this distance.

SOLUTION

Equation of motion down the incline is given by

$$mg\sin\alpha - kxmg\cos\alpha = ma$$

$$\Rightarrow \quad a = g\sin\alpha - kxg\cos\alpha \qquad \ldots(1)$$

where a is the acceleration of bar

Since $a = v\dfrac{dv}{dx}$

$$\Rightarrow \quad \int_0^v v\,dv = \int_0^x (g\sin\alpha - kxg\cos\alpha)\,dx$$

$$\Rightarrow \quad v^2 = 2g\sin\alpha - gkx^2\cos\alpha \qquad \ldots(2)$$

$$\Rightarrow \quad v = \sqrt{(2x\sin\alpha - kx^2\cos\alpha)g} \qquad \ldots(3)$$

It can be seen that the velocity again becomes zero after covering a distance $x = \dfrac{2}{k}\tan\alpha$.

Therefore, the distance covered by the bar till it stops is $\dfrac{2}{k}\tan\alpha$.

Further, the maximum velocity of the bar will be when

$$\frac{dv}{dx} = 0$$

$$\Rightarrow \quad a = 0 \qquad \left\{\because\ a = v\frac{dv}{dx}\right\}$$

$$\Rightarrow \quad x = \frac{\tan\alpha}{k} \qquad \{\text{from equation (1)}\}$$

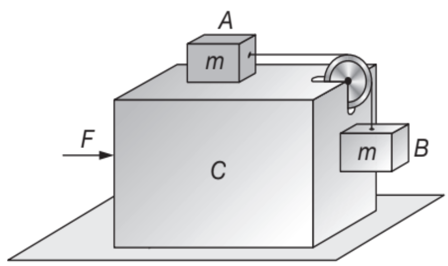
Substituting this in equation (3), we get the maximum velocity as

$$v_{max} = \sqrt{\left(\frac{2\sin\alpha\tan\alpha}{k} - \frac{\tan^2\alpha\cos\alpha}{k}\right)g}$$

$$\Rightarrow \quad v_{max} = \sqrt{\frac{g}{k}\tan\alpha\sin\alpha}$$

ILLUSTRATION 86

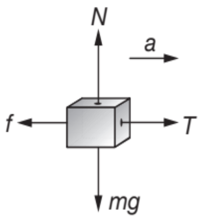
Consider the situation shown in figure. The horizontal surface below the bigger block is smooth. The coefficient of friction between the blocks is μ. Find the minimum and the maximum force F that can be applied in order to keep the smaller blocks at rest with respect to the bigger block.



SOLUTION

Suppose that the minimum force needed to prevent slipping between the blocks is F. Considering $A+B+C$ as the system, the acceleration of the system is

$$a = \frac{F}{M+2m} \quad \ldots(1)$$

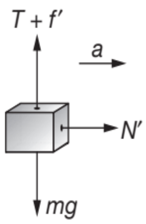


FBD of A

Now, consider the F.B.D. of A. The forces on A shown in figure are

(a) tension T by the string towards right,
(b) friction f by the block C towards left,
(c) weight mg downward and
(d) normal force N upwards

For vertical equilibrium $N = mg$. As the minimum force needed to prevent slipping is applied, the friction is limiting. Thus, $f = \mu N = \mu mg$



FBD of B

As the block moves towards right with an acceleration a,

$$T - f = ma$$

$$\Rightarrow \quad T - \mu mg = ma \quad \ldots(2)$$

Now, consider the F.B.D. of B. The forces on B shown in figure are

(a) tension T upwards,
(b) weight mg downward,
(c) normal force N' towards right and
(d) friction f' upwards.

As the block moves towards right with an acceleration a, so we have $N' = ma$. Since the friction is limiting, hence

$$f' = \mu N' = \mu(ma) \quad \ldots(3)$$

For vertical equilibrium

$$T + f' = mg \quad \ldots(4)$$

Solving these equations, we get

$$a_{\min} = \left(\frac{1-\mu}{1+\mu}\right)g$$

When a large force is applied the block A slips on C towards left and the block B slips on C in the upward direction. The friction on A is towards right and that on B is downwards. Solving as above, the acceleration in this case is $a_{\max} = \frac{1+\mu}{1-\mu}g$.

Thus, a lies between $\left(\frac{1-\mu}{1+\mu}\right)g$ and $\left(\frac{1+\mu}{1-\mu}\right)g$, i.e.,

$$\left(\frac{1-\mu}{1+\mu}\right)g \le a \le \left(\frac{1+\mu}{1-\mu}\right)g.$$

From equation (1) the force F lies between

$$\left(\frac{1-\mu}{1+\mu}\right)(M+2m)g \text{ and } \left(\frac{1+\mu}{1-\mu}\right)(M+2m)g$$

i.e., $$\left(\frac{1-\mu}{1+\mu}\right)(M+2m)g \le F \le \left(\frac{1+\mu}{1-\mu}\right)(M+2m)g.$$

Test Your Concepts-VII

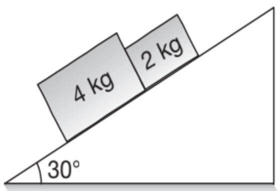
Based on Friction

(Solutions on page H.209)

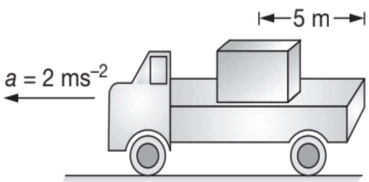
1. Calculate the height upto which an insect can crawl up a fixed bowl in the form of a hemisphere of radius r. Given, coefficient of friction is $\frac{1}{\sqrt{3}}$.

2. A particle of mass 2 kg rests on rough plane inclined at 30° to the horizontal and is just about to slip. Find the coefficient of friction between the plane and the particle.

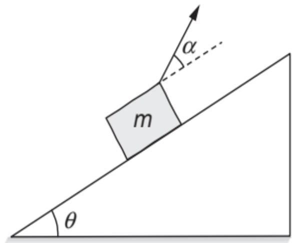
3. Figure shows two blocks in contact sliding down an inclined surface of inclination 30°. The friction coefficient between the block of mass 2 kg and the incline is $\mu_1 = 0.2$ and that between the block of mass 4 kg and the incline is $\mu_2 = 0.3$. Find the acceleration of 2 kg block. Take $g = 10\ \text{ms}^{-2}$.



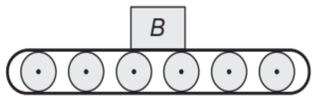
4. The rear side of a truck is open and a box of 40 kg mass is placed 5 m away from the open end as shown in figure. The coefficient of friction between the box and the surface below it is $\mu = 0.1$. On a straight road, the truck starts from rest and accelerates with $2\ \text{ms}^{-2}$, find the time when box falls off the truck. Take $g = 10\ \text{ms}^{-2}$.



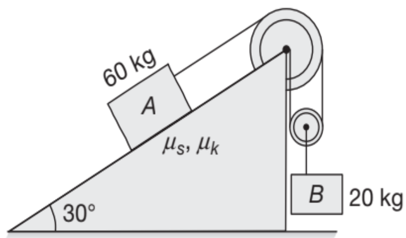
5. A block of mass m is pulled by means of a force F up an inclined plane forming an angle θ with the horizontal as shown in figure. The coefficient of friction is μ. Find the minimum force required for the block to move up and the angle α at that instant.



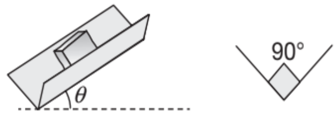
6. The conveyor belt is designed to transport packages of various weights. Each 10 kg package has a coefficient of kinetic friction $\mu_k = 0.15$. If the speed of the conveyor is $5\ \text{ms}^{-1}$ and then it suddenly stops, determine the distance the package will slide on the belt before it comes to rest.



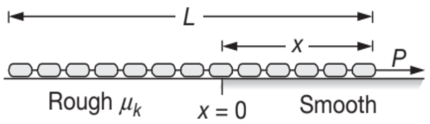
7. The system is released from rest with cable taut. Calculate the acceleration of each body and the tension T in the cable attached with A for $\mu_s = 0.25$ and $\mu_k = 0.2$. Neglect the small mass and friction of the pulleys. Take $g = 10\ \text{ms}^{-2}$.



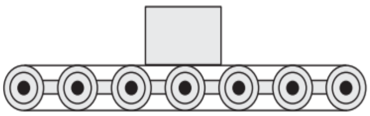
8. In figure, a crate slides down an inclined right-angled trough. The coefficient of kinetic friction between the crate and the trough is μ_k. What is the acceleration of the crate in terms of μ_k, θ and g?



9. A heavy chain with a mass per unit length ρ is pulled by the constant force F along a horizontal surface consisting of a smooth section and a rough section. The chain is initially at rest on the rough surface with $x=0$. If the coefficient of kinetic friction between the chain and the rough surface is μ_k, determine the velocity v of the chain when $x=L$. The force F is greater than $\mu_k \rho g L$ in order to initiate motion.

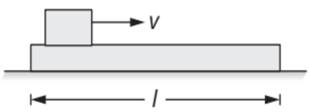


10. A package is at rest on a conveyor belt which is initially at rest. The belt is started and moves to the right for 1.3 s with a constant acceleration of $2\ ms^{-2}$. The belt then moves with a constant deceleration a_2 and comes to a stop after a total displacement of 2.2 m. Knowing that the coefficients of friction between the package and the belt are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine



(a) the deceleration a_2 of the belt,
(b) the displacement of the package relative to the belt as the belt comes to a stop.

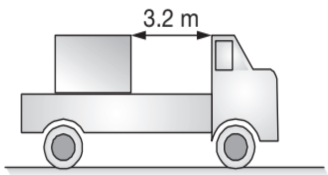
11. A small block of mass m is projected on a larger block of mass 10 m and length ℓ with a velocity v as shown in the figure. The coefficient of friction between the two blocks is μ_2 while that between the lower block and the ground is μ_1. Given that $\mu_2 > 11\mu_1$.



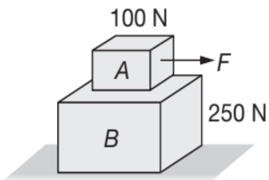
(a) Find the minimum value of v such that the mass m falls off the block of mass 10 m.
(b) If v has this minimum value, find the time taken by block m to do so.

12. A uniform rod is made to lean between a rough vertical wall and the rough ground. Show that the least angle at which the rod can be leaned without slipping is given by $\theta = \tan^{-1}\left(\frac{7}{4}\right)$, where coefficient of friction between the rod and the wall is $\mu_1 = \frac{1}{2}$ and that between the rod and the ground is $\mu_2 = \frac{1}{4}$.

13. Coefficients of friction between the flat bed of the truck and crate are $\mu_s = 0.8$ and $\mu_k = 0.7$. The coefficient of kinetic friction between the truck tires and the road surface is 0.9. If the truck stops from an initial speed of $15\ ms^{-1}$ with maximum braking (wheels skidding), determine where on the bed the crate finally comes to rest or the velocity v_{rel} relative to the truck with which the crate strikes the wall at the forward edge of the bed. (Take $g = 10\ ms^{-2}$).



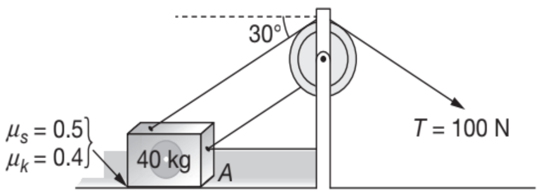
14. Block B rests on a smooth surface. If the coefficient of static friction between A and B is $\mu = 0.4$, determine the acceleration of each if



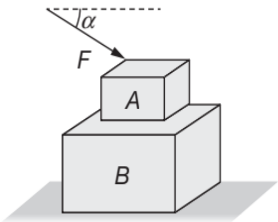
(a) $F = 30$ N and
(b) $F = 250$ N

Take $g = 10\ ms^{-2}$

15. In the arrangement shown, find the acceleration of block A for the instant depicted. Neglect the mass of the pulley. Take $g = 10\ ms^{-2}$.



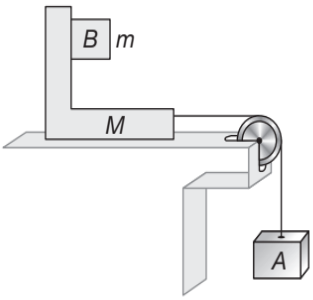
16. Two blocks *A* and *B* of mass 1 kg and 2 kg respectively are placed over a smooth horizontal surface as shown in figure. The coefficient of friction between blocks *A* and *B* is $\mu = \frac{1}{2}$. An external force of magnitude *F* is applied to the top block at an angle $\alpha = 30^\circ$ below the horizontal.



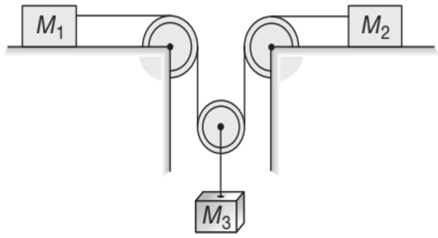
(a) If the two blocks move together, find their acceleration in terms of *F*.

(b) Find the maximum values of *F* so that both the blocks move together with same acceleration. $\left(g = 10\ \text{ms}^{-2}\right)$.

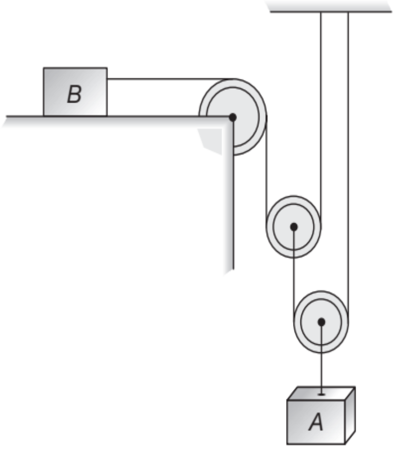
17. The figure shows an *L* shaped body of mass *M* placed on smooth horizontal surface. The block *A* is connected to the body by means of an inextensible string, which is passing over a smooth pulley of negligible mass. Another block *B* of mass *m* is placed against a vertical wall of the body. Find the minimum value of the mass of block *A* so that block *B* remains stationary relative to the wall. Coefficient of friction between the block *B* and the vertical wall is μ.



18. The system shown in figure uses massless pulley and rope. The coefficient of friction between the masses and horizontal surfaces is μ. Assume that M_1 and M_2 are sliding. Find the tension *T* in the rope.

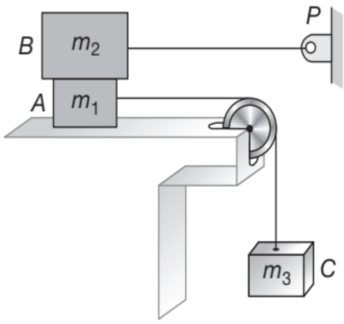


19. Determine the acceleration of the 5 kg block *A*. Neglect the mass of the pulley and cords. The block *B* has a mass of 10 kg. The coefficient of kinetic friction between block *B* and the surface is $\mu_k = 0.1$. $\left(\text{Take } g = 10\ \text{ms}^{-2}\right)$.



20. A worker wishes to pile a cone of sand onto a circular area in his yard. The radius of the circle is *r* and no sand is to spill onto the surrounding area. If μ is the static coefficient of friction between each layer of sand along the slope and the sand, show that the greatest volume of sand that can be stored in this manner is $\frac{1}{3}\pi\mu r^3$.

21. A block *A* of mass m_1 rests on a rough horizontal surface. The coefficient of friction between the block and the surface is μ. A uniform plank *B* of mass m_2 rests on *A*. *B* is prevented from moving by connecting it to a light rod and hinged at point *P*. The coefficient of friction between *A* and *B* is μ. Find the acceleration of blocks *A* and *C*.



22. A 4 m long ladder weighing 25 kg rests with its upper end against a smooth wall and lower end on rough ground. What should be the minimum coefficient of friction between the ground and the ladder for it to be inclined at 60° with the horizontal without slipping? Take $g = 10\ \text{ms}^{-2}$.

DYNAMICS OF CIRCULAR MOTION

CIRCULAR MOTION: AN INTRODUCTION

For a particle which moves in a plane such that its distance remains constant from a fixed or a moving point, then its motion is known as circular motion with respect to that fixed point or moving point. This point is called the centre of the circle and the distance of the particle from this point is called the radius.

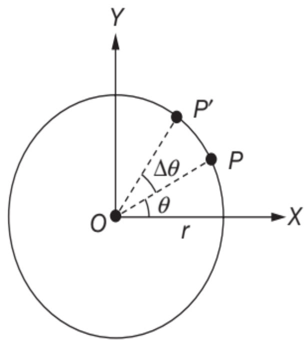
VARIABLES OF CIRCULAR MOTION

Angular Position (θ)

The angular position of a point in space is decided by specifying the

(i) origin and
(ii) reference line.

The angle made by the position vector w.r.t. origin, with the reference line is called angular position. Clearly, angular position depends on the choice of the origin as well as the reference line.



Circular motion is a two dimensional motion or motion in a plane.

Suppose a particle P is moving in a circle of radius r centred at O, then the angular position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle.

Angular Displacement ($\Delta\theta$)

Definition: Angle through which the position vector of the moving particle rotates in a given time interval is called its angular displacement. Angular displacement depends on origin, but it does not depends on the reference line. As the particle moves on above circle its angular position θ changes. Suppose that the point rotates through an angle $\Delta\theta$ in time Δt, then $\Delta\theta$ is angular displacement.

Average Angular Velocity

$$\omega_{av} = \frac{\text{Angular displacement}}{\text{Total time taken}}$$

$$\Rightarrow \quad \omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2. Since angular displacement is a scalar, average angular velocity is also a scalar.

Instantaneous Angular Velocity

It is the limit of average angular velocity as Δt approaches zero. i.e.,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Since infinitesimally small angular displacement $d\theta$ is a vector quantity, so instantaneous angular velocity ω is also a vector, whose direction is given by Right Hand Thumb Rule.

ILLUSTRATION 87

If angular displacement of a particle is given by $\theta = a - bt + ct^2$, then find its angular velocity.

SOLUTION

$$\omega = \frac{d\theta}{dt} = -b + 2ct$$

ILLUSTRATION 88

Is the angular velocity of rotation of hour hand of a watch greater or smaller than the angular velocity of Earth's rotation about its own axis?

SOLUTION

Hour hand completes one rotation in 12 hours while Earth completes one rotation in 24 hours. So, angular velocity of hour hand is double the angular velocity of Earth, because

$$\omega = \frac{2\pi}{T}$$

Conceptual Note(s)

(a) Angular displacement is a dimensionless quantity. Its SI unit is radian, some other units are degree and revolution.

$$2\pi \text{ rad} = 360° = 1 \text{ rev}$$

(b) Infinitesimally small angular displacement is a vector quantity, but finite angular displacement is a scalar, because while the addition of the infinitesimally small angular displacements is commutative, addition of finite angular displacement is not.

$$d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1 \text{ but } \theta_1 + \theta_2 \neq \theta_2 + \theta_1$$

(c) Direction of small angular displacement is decided by right hand thumb rule. When the fingers are directed along the motion of the point then thumb will represents the direction of angular displacement.

(d) Angular velocity has dimension formula $M^0L^0T^{-1}$ and SI unit rad s^{-1}.

(e) If a body makes n rotations in t seconds then average angular velocity in radian per second will be

$$\omega_{av} = \frac{2\pi n}{t}$$

(f) If T is the period and f the frequency of uniform circular motion, then

$$\omega_{av} = \frac{2\pi}{T} = 2\pi f$$

(g) For a rigid body, as all points will rotate through same angle in same time, angular velocity is a characteristic of the body as a whole, e.g. angular velocity of all points of earth about earth's axis is

$$\frac{2\pi}{24} \text{ rad hr}^{-1} = \frac{2\pi}{86400} \text{ rad s}^{-1}$$

(h) Average Angular Acceleration: Let ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2 respectively, then the average angular acceleration α_{av} is defined as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

(i) Instantaneous Angular Acceleration: It is the limit of average angular acceleration as Δt approaches zero, i.e.,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Since, $\omega = \frac{d\theta}{dt}$

$$\Rightarrow \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Also, $\alpha = \omega \frac{d\omega}{d\theta}$

(j) Both average and instantaneous angular acceleration are axial vectors with dimension $[T^{-2}]$ and unit rad s^{-2}.

(k) If $\alpha = 0$, circular motion is said to be uniform.

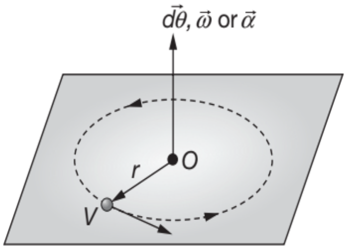
KINEMATICS OF CIRCULAR MOTION

Motion with Constant Angular Velocity

If a particle is moving in a circle with uniform angular velocity, then $\alpha = 0$ and hence $\theta = \omega t$.

Motion with Constant Angular Acceleration

Circular motion with constant angular acceleration is analogous to one dimensional translational motion with constant acceleration. Hence even here equation of motion have same form.



$$\omega = \omega_0 + \alpha t$$

$$\Rightarrow \quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\Rightarrow \quad \omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\Rightarrow \quad \theta = \left(\frac{\omega + \omega_0}{2}\right)t$$

$$\Rightarrow \quad \theta_{nth} = \omega_0 + \frac{\alpha}{2}(\theta_n - \theta_{n-1})$$

where ω_0 is the initial angular velocity,
ω is the final angular velocity,
α is the constant angular acceleration,
θ is the angle traversed from $t = 0$ to t and
θ_{nth} is the angle traversed in the nth second of motion.

ILLUSTRATION 89

A particle is moving with constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \frac{\pi}{2}$.

SOLUTION

Time taken to describe angle θ

$$t = \frac{\theta}{\omega} = \frac{\theta R}{v} = \frac{\pi R}{2v}$$

$$v_{av} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{\sqrt{2}R}{\pi R/2v} = \frac{2\sqrt{2}}{\pi}v$$

Instantaneous velocity $= v$

The ratio of average velocity to its instantaneous velocity is

$$\frac{v_{av}}{v_{ins}} = \frac{2\sqrt{2}}{\pi}$$

ILLUSTRATION 90

A fan is rotating with angular velocity 100 revs^{-1}. Then it is switched off. It takes 5 minutes to stop.

(a) Find the total number of revolution made before it stops. (Assume uniform angular retardation).
(b) Find the value of angular retardation.
(c) Find the average angular velocity during this interval.

SOLUTION

(a) $\theta = \left(\frac{\omega + \omega_0}{2}\right)t = \left(\frac{100 + 0}{2}\right) \times 5 \times 60$

$\Rightarrow \quad \theta = 15000$ revolution.

(b) $\omega = \omega_0 + \alpha t$

$\Rightarrow \quad 0 = 100 - \alpha(5 \times 60)$

$\Rightarrow \quad \alpha = \frac{1}{3} \text{ revs}^{-2}$

(c) $\omega_{av} = \frac{\text{Total angle of rotation}}{\text{Total time taken}}$

$\Rightarrow \quad \omega_{av} = \frac{15000}{50 \times 60}$

$\Rightarrow \quad \omega_{av} = 50 \text{ revs}^{-1}$

ILLUSTRATION 91

A fan rotating with $\omega = 100 \text{ rads}^{-1}$, is switched off. After $2n$ rotation its angular velocity becomes 50 rads^{-1}. Find the angular velocity of the fan after n rotations.

SOLUTION

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\Rightarrow \quad 50^2 = (100)^2 + 2\alpha(2\pi 2n) \qquad \ldots(1)$$

If angular velocity after n rotation is ω_n, then

$$\omega_n^2 = (100)^2 + 2\alpha(2\pi n) \qquad \ldots(2)$$

From equations (1) and (2), we get

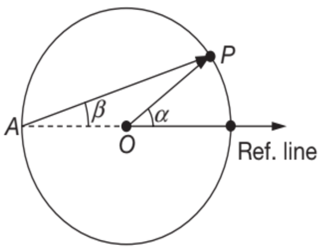
$$\frac{50^2 - 100^2}{\omega_n^2 - 100^2} = \frac{2\alpha(2\pi 2n)}{2\alpha 2\pi n} = 2$$

$$\Rightarrow \quad \omega_n^2 = \frac{50^2 + 100^2}{2}$$

$$\Rightarrow \quad \omega = 25\sqrt{10} \text{ rads}^{-1}$$

RELATIVE ANGULAR VELOCITY

Just as velocities are always relative, similarly angular velocity is also always relative. There is no such thing as absolute angular velocity. Angular velocity is defined with respect to origin, the point from which the position vector of the moving particle is drawn.



Consider a particle P moving along a circular path shown in the figure.

Here angular velocity of the particle P w.r.t. O (i.e. P is revolving with O as the centre) and the angular velocity of P w.r.t. A (i.e. P is revolving with A as the centre) will be different.

Angular velocity of a particle P w.r.t. O is

$$\omega_{PO} = \frac{d\alpha}{dt}$$

Angular velocity of a particle P w.r.t. A is

$$\omega_{PA} = \frac{d\beta}{dt}$$

Definition of Angular Velocity

Angular velocity of a particle A with respect to the other moving particle B is the rate at which position vector of A with respect to B rotates at that instant. (or it is simply, angular velocity of A with origin fixed at B). Angular velocity of A w.r.t. B, ω_{AB} is mathematically define as

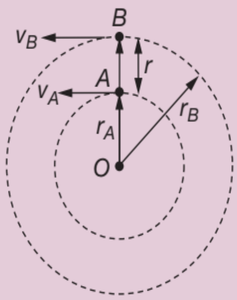
$$\omega_{AB} = \frac{\begin{pmatrix}\text{Component of relative} \\ \text{velocityof A w.r.t. B,} \\ \text{perpendicular to line}\end{pmatrix}}{\text{Separation between } A \text{ and } B}$$

$$\Rightarrow \quad \omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}}$$

Conceptual Note(s)

(a) If two particles are moving on two different concentric circles with different velocities then angular velocity of B as observed by A will depend on their positions and velocities. Consider the case when A and B are closest to each other moving in same direction as shown in figure. In this situation

$$(v_{AB})_{\perp} = v_B - v_A$$



Separation between A and B is

$$r_{BA} = r_B - r_A$$

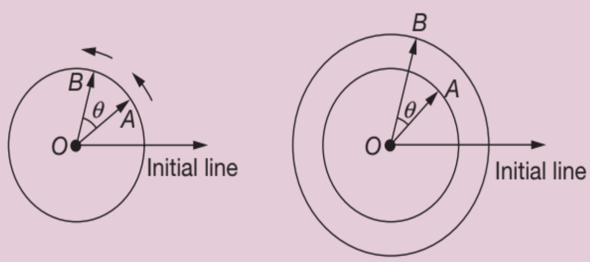
$$\text{So, } \omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}} = \frac{v_B - v_A}{r_B - r_A}$$

(b) If two particles are moving on the same circle or different coplanar concentric circles in same direction with different uniform angular speed ω_A and ω_B respectively, the rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is

$$\frac{d\theta}{dt} = \omega_B - \omega_A$$

So, the time taken by one to complete one revolution around O w.r.t. the other

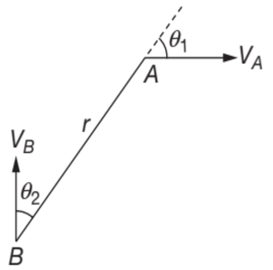
$$T = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{T_1 T_2}{T_1 - T_2}$$



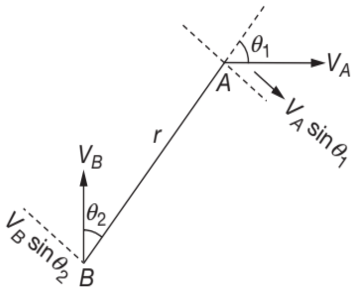
(c) $\omega_B - \omega_A$ is rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. This is not the angular velocity of B w.r.t. A. (which is rate at which line AB rotates).

ILLUSTRATION 92

Find the angular velocity of A with respect to B in the figure given below:



SOLUTION



Angular velocity of A with respect to B

$$\omega_{AB} = \frac{(V_{AB})_\perp}{r_{AB}}$$

From the figure drawn, we observe that

$$(V_{AB})_\perp = V_A \sin\theta_1 + V_B \sin\theta_2$$

and $r_{AB} = r$

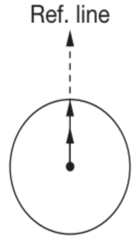
$$\Rightarrow \quad \omega_{AB} = \frac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$$

ILLUSTRATION 93

Find the time period of meeting of minute hand and second hand of a wall clock.

SOLUTION

For second and minute hand to meet again, we must have



$$\theta_{sec} - \theta_{min} = 2\pi$$

$$\Rightarrow \quad (\omega_{sec} - \omega_{min})t = 2\pi$$

Since, $\omega_{min} = \frac{2\pi}{60}$ radmin^{-1} and $\omega_{sec} = \frac{2\pi}{1}$ radmin^{-1}, so we have

$$2\pi\left(1 - \frac{1}{60}\right)t = 2\pi$$

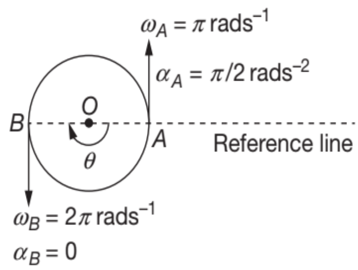
$$\Rightarrow \quad t = \frac{60}{59} \text{ min}$$

ILLUSTRATION 94

Two particles A and B move on a circle. Initially particle A and B are diagonally opposite to each other. Particle A move with angular velocity π rads^{-1}, angular acceleration $\frac{\pi}{2}$ rads^{-2} and particle B moves with constant angular velocity 2π rads^{-1}. Find the time after which both the particles A and B will collide.

SOLUTION

Suppose angle between OA and OB, be θ, then, rate of change of θ is called angular velocity.



With respect to the point A, we have

$$\omega_{BA} = \omega = \omega_B - \omega_A = 2\pi - \pi = \pi \text{ rads}^{-1}$$

$$\alpha_{BA} = \alpha = \alpha_B - \alpha_A = -\frac{\pi}{2} \text{ rads}^{-2}$$

If angular displacement is $\Delta\theta$, then

$$\Delta\theta = \omega t + \frac{1}{2}\alpha t^2$$

For A and B to collide, angular displacement is given by

$$\Delta\theta = \pi$$

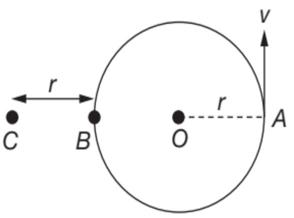
$$\Rightarrow \quad \pi = \pi t + \frac{1}{2}\left(\frac{-\pi}{2}\right)t^2$$

$$\Rightarrow \quad t^2 - 4t + 4 = 0$$

$$\Rightarrow \quad t = 2 \text{ sec}$$

ILLUSTRATION 95

A particle is moving with constant speed in a circle as shown, find the angular velocity of the particle A with respect to fixed point B and C if angular velocity with respect to O is ω.



SOLUTION

Angular velocity of A with respect to O (i.e. A is revolving with O as the centre) is

$$\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$$

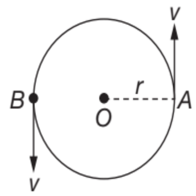
Similarly, we have

$$\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{v}{2r} = \frac{\omega}{2}$$

and $$\omega_{AC} = \frac{(v_{AC})_\perp}{r_{AC}} = \frac{v}{3r} = \frac{\omega}{3}$$

ILLUSTRATION 96

Particles A and B move with constant and equal speeds in a circle as shown, find the angular velocity of the particle A with respect to B, if angular velocity of particle A w.r.t. O is ω.



SOLUTION

Angular velocity of A with respect to O is

$$\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}} = \frac{v}{r} = \omega$$

Now, $\omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}}$

$$\Rightarrow \quad v_{AB} = 2v$$

Since v_{AB} is perpendicular to r_{AB}

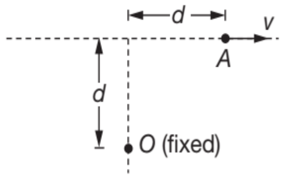
$$\Rightarrow \quad (v_{AB})_\perp = v_{AB} = 2v$$

$$r_{AB} = 2r$$

$$\Rightarrow \quad \omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{2v}{2r} = \omega$$

ILLUSTRATION 97

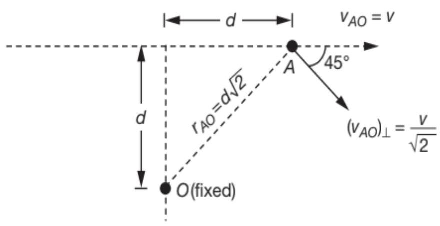
Find angular velocity of A with respect to O at the instant shown in the figure.



SOLUTION

Angular velocity of A with respect to O is

$$\omega_{AO} = \frac{(v_{AO})_\perp}{r_{AO}}$$



Since, $v_{AO} = v$

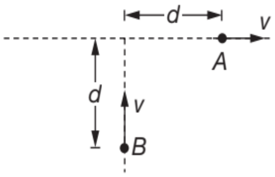
$$\Rightarrow \quad (v_{AO})_{\perp} = \frac{v}{\sqrt{2}}$$

and $r_{AO} = d\sqrt{2}$

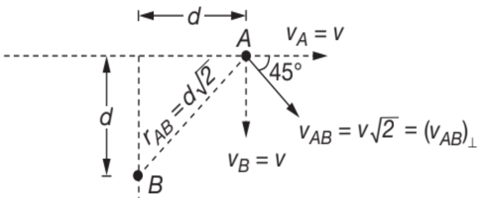
$$\Rightarrow \quad \omega_{AO} = \frac{(v_{AO})_{\perp}}{r_{AO}} = \frac{v/\sqrt{2}}{d\sqrt{2}} = \frac{v}{2d}$$

ILLUSTRATION 98

Find angular velocity of A with respect to B at the instant shown in the figure.



SOLUTION



Angular velocity of A with respect to B is

$$\omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}}$$

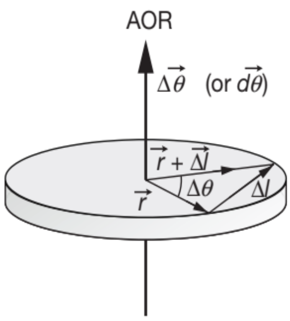
$$\Rightarrow \quad v_{AB} = \sqrt{2}v = (v_{AB})_{\perp}$$

$$\Rightarrow \quad r_{AB} = \sqrt{2}d$$

$$\Rightarrow \quad \omega_{AB} = \frac{(v_{AB})_{\perp}}{r_{AB}} = \frac{v\sqrt{2}}{d\sqrt{2}} = \frac{v}{d}$$

ANGULAR DISPLACEMENT ($d\vec{\theta}$): REVISITED

Finite angular displacement θ is NEVER a vector.



Infinitesimal (extremely small) angular displacement $d\vec{\theta}$ is a vector and

$$\Delta\vec{l} = \Delta\vec{\theta} \times \vec{r} \qquad \{\text{In magnitude } \Delta l = r\Delta\theta\}$$

For infinitesimal displacement

$$d\vec{l} = d\vec{\theta} \times \vec{r}$$

ANGULAR VELOCITY ($\vec{\omega}$): REVISITED

Rate of change of angular displacement is angular velocity.

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\vec{\theta}}{dt}$$

Since $d\vec{l} = d\vec{\theta} \times \vec{r}$

Divide both sides by dt

$$\Rightarrow \quad \frac{d\vec{l}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}$$

$$\Rightarrow \quad \vec{v} = \vec{\omega} \times \vec{r}$$

In magnitude

$$\Rightarrow \quad v = r\omega$$

Conceptual Note(s)

(a) Commonly used units for angular velocity ω are rpm, rph, rps, 1 degrees^{-1}.

1 revolution ≡ 2π radian

$$\text{rps} \xrightarrow{\times 2\pi} \text{rads}^{-1}$$

$$\text{rph} \xrightarrow{\times \frac{2\pi}{3600}} \text{rads}^{-1}$$

(b) Direction of ω

Direction of ω is given by Right Hand Thumb Rule (RHTR) according to which curl the fingers of right hand in the direction of rotation then thumb gives the direction of ω.

e.g.: The angular velocity for the second's hand of watch is inwards, perpendicular to the plane of watch.

ANGULAR ACCELERATION ($\vec{\alpha}$): REVISITED

Rate of change of angular velocity is angular acceleration

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$$

Since $\vec{v} = \vec{\omega} \times \vec{r}$

Differentiating both sides with respect to t, we get

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

Using formula

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$$

$$\Rightarrow \quad \vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

$$\Rightarrow \quad \vec{a} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r}$$

$$\Rightarrow \quad \vec{a} = \vec{a}_C + \vec{a}_T$$

where $\vec{a}_C$ is the centripetal acceleration also called as Radial acceleration and $\vec{a}_T$ is the tangential acceleration.

Mathematically, we have

$$\vec{a}_C = \vec{\omega} \times \vec{v}$$

$$\vec{a}_T = \vec{\alpha} \times \vec{r}$$

In magnitude, $a_C = v\omega = r\omega^2 = \frac{v^2}{r}$ and $a_T = r\alpha$

Also, $\vec{a}_C \perp \vec{a}_T$ and $\vec{a}_C = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$

RADIAL AND TANGENTIAL ACCELERATION

There are two types of acceleration in circular motion; Tangential acceleration and centripetal acceleration.

Tangential Acceleration (a_T)

Component of acceleration directed along tangent of circle is called tangential acceleration. It is responsible for changing the speed of the particle. It is defined as

$$a_T = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = \text{Rate of change of speed}$$

Also, $a_T = \alpha r$

Conceptual Note(s)

(a) In vector form $\vec{a}_T = \vec{\alpha} \times \vec{r}$

(b) Tangential acceleration is defined as the rate of change of speed (not velocity). So

$$a_T = \frac{d|\vec{v}|}{dt} \text{ or simply } a_T = \frac{dv}{dt}.$$

(c) If tangential acceleration is directed in direction of velocity then the speed of the particle increases.

(d) If tangential acceleration is directed opposite to velocity then the speed of the particle decreases.

Centripetal Acceleration (a_C)

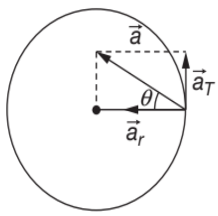
It is responsible for change in direction of velocity. In circular motion, there is always a centripetal acceleration.

Centripetal acceleration is always variable because it changes in direction. Centripetal acceleration is also called radial acceleration (a_r) or normal acceleration $(a_\perp)$. For a particle moving with speed v in a curved track of radius r, centripetal acceleration is given by

$$a_C = \frac{v^2}{r}$$

Total Acceleration

Total acceleration is vector sum of centripetal acceleration and tangential acceleration.



$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_C + \vec{a}_T$$

$$\Rightarrow \quad a = \sqrt{a_T^2 + a_C^2}$$

$$\Rightarrow \quad \tan\theta = \frac{a_C}{a_T}$$

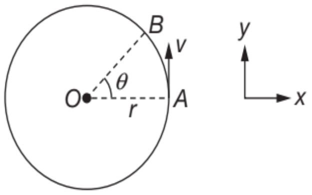
Conceptual Note(s)

(a) Differentiation of velocity $(\vec{v})$ gives total acceleration.

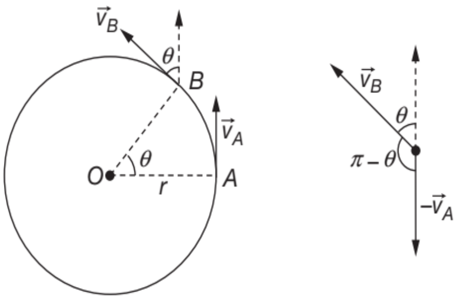
(b) $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d|\vec{v}|}{dt}$ are not same physical quantities, $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of rate of change of velocity, i.e., magnitude of total acceleration and $\frac{d|\vec{v}|}{dt}$ is a rate of change of speed, i.e., tangential acceleration.

CALCULATION OF CENTRIPETAL ACCELERATION

Consider a particle which moves in a circle with constant speed v as shown in Figure.



Change in velocity between the point A and B is



$$\Delta\vec{v} = \vec{v}_B - \vec{v}_A$$

Magnitude of change in velocity

$$|\Delta\vec{v}| = |\vec{v}_B - \vec{v}_A| = \sqrt{v_B^2 + v_A^2 + 2v_A v_B \cos(\pi - \theta)}$$

$$(v_A = v_B = v, \text{ since speed is same})$$

$$\Rightarrow \quad |\Delta\vec{v}| = 2v\sin\left(\frac{\theta}{2}\right)$$

Distance travelled by particle between A and $B = r\theta$

Hence time taken, $\Delta t = \frac{r\theta}{v}$

Net acceleration, $|\vec{a}_{net}| = \left|\frac{\Delta\vec{v}}{\Delta t}\right| = \frac{2v\sin\left(\frac{\theta}{2}\right)}{r\theta/v}$

$$\Rightarrow \quad |\vec{a}_{net}| = \frac{v^2}{r}\left[\frac{2\sin\left(\frac{\theta}{2}\right)}{\theta}\right]$$

If $\Delta t \to 0$, then θ is small, $\sin\left(\frac{\theta}{2}\right) = \frac{\theta}{2}$

$$\lim_{\Delta t \to 0}\left|\frac{\Delta\vec{v}}{\Delta t}\right| = \left|\frac{d\vec{v}}{dt}\right| = \frac{v^2}{r}$$

i.e., net acceleration is $\frac{v^2}{r}$ but speed is constant so that tangential acceleration

$$a_t = \frac{dv}{dt} = 0$$

$$\Rightarrow \quad a_{net} = a_r = \frac{v^2}{r}$$

Through we have derived the formula of centripetal acceleration under condition of constant speed, the same formula is applicable even when speed is variable.

Conceptual Note(s)

In vector form

$$\vec{a}_c = \vec{\omega} \times \vec{v}$$

ILLUSTRATION 99

A particle travels in a circle of radius 20 cm at a speed that uniformly increases. If the speed changes from $5\ ms^{-1}$ to $6\ ms^{-1}$ in 2 s, find the angular acceleration

SOLUTION

Since speed increases uniformly, average tangential acceleration is equal to instantaneous tangential acceleration

The instantaneous tangential acceleration is given by

$$a_T = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1}$$

$$\Rightarrow \quad a_T = \frac{6-5}{2}\ \text{ms}^{-2} = 0.5\ \text{ms}^{-2}$$

The angular acceleration is $\alpha = \frac{a_T}{r}$.

$$\Rightarrow \quad \alpha = \frac{0.5\ \text{ms}^{-2}}{20\ \text{cm}} = 2.5\ \text{rads}^{-2}$$

ILLUSTRATION 100

Find the magnitude of the acceleration of a particle moving in a circle of radius 10 cm with uniform speed completing the circle in $4s$.

SOLUTION

The distance covered in completing the circle is

$$2\pi r = 2\pi \times 10\ \text{cm}.$$

The linear speed is

$$v = \frac{2\pi r}{T} = \frac{2\pi \times 10\ \text{cm}}{4s} = 5\pi\ \text{cms}^{-1}$$

The acceleration is

$$a_C = \frac{v^2}{r} = \frac{(5\pi\ \text{cms}^{-1})^2}{10\ \text{cm}} = 2.5\pi^2\ \text{cms}^{-2}$$

ILLUSTRATION 101

A particle moves in a circle of radius 2 cm at a speed given by $v = 4t$, where v is in cms^{-1} and t is in seconds.

(a) Find the tangential acceleration at $t = 1$ s.
(b) Find total acceleration at $t = 1$ s.

SOLUTION

(a) Tangential acceleration

$$a_T = \frac{dv}{dt}$$

$$\Rightarrow \quad a_T = \frac{d}{dt}(4t) = 4\ \text{cms}^{-2}$$

$$\Rightarrow \quad a_C = \frac{v^2}{R} = \frac{(4)^2}{2} = 8\ \text{cms}^{-2}$$

$$\Rightarrow \quad a = \sqrt{a_T^2 + a_C^2} = \sqrt{(4)^2 + (8)^2}$$

$$\Rightarrow \quad a = 4\sqrt{5}\ \text{cms}^{-2}$$

ILLUSTRATION 102

A particle is moving with a constant angular acceleration of 4 rads^{-2} in a circular path. At time $t = 0$ particle was at rest. Find the time at which the magnitudes of centripetal acceleration and tangential acceleration are equal.

SOLUTION

$$a_t = \alpha R$$

$$\Rightarrow \quad v = 0 + \alpha R t$$

$$a_c = \frac{v^2}{R} = \frac{\alpha^2 R^2 t^2}{R}$$

$$\Rightarrow \quad |a_t| = |a_c|$$

$$\Rightarrow \quad \alpha R = \frac{\alpha^2 R^2 t^2}{R}$$

$$\Rightarrow \quad t^2 = \frac{1}{\alpha} = \frac{1}{4}$$

$$\Rightarrow \quad t = \frac{1}{2}\ \text{s}$$

DYNAMICS OF CIRCULAR MOTION

If there is no force acting on a body it will move in a straight line (with constant speed). Hence if a body is moving in a circular path or any curved path, there must be some force acting on the body.

If speed of body is constant, the net force acting on the body is along the inside normal to the path of the body and it is called centripetal force.

$$\text{Centripetal force } (F_C) = ma_C = \frac{mv^2}{r} = mr\omega^2$$

However if speed of the body varies then, in addition to above centripetal force which acts along inside normal, there is also a force acting along the tangent of the path of the body which is called tangential force.

$$\text{Tangential force } (F_T) = Ma_T = M\frac{dv}{dt} = Mr\alpha,$$

where α is the angular acceleration.

Conceptual Note(s)

Remember $\frac{mv^2}{r}$ is not a force itself. It is just the value of the net force acting along the inside normal (towards the centre) which is responsible for circular motion.

This force may be friction, normal, tension, spring force, gravitational force or a combination of them.

So to solve any problem in uniform circular motion we identify all the forces acting along the normal (towards centre), calculate their resultant and equate it to $\frac{mv^2}{r}$.

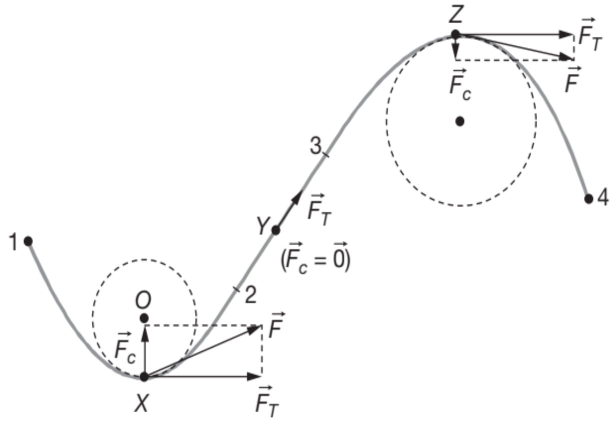
If circular motion is non-uniform then in addition to above step we also identify all the forces acting along the tangent to the circular path, calculate their resultant and equate it to $m\frac{dv}{dt}$ or $m\frac{d|\vec{v}|}{dt} = a_T$.

Some Common Examples of Centripetal Force are

SITUATION	CENTRIPETAL FORCE
A particle tied to a string and whirled in a horizontal circle.	Tension in the string.
Earth in orbit around the sun.	Gravitational force exerted by the sun.
An electron revolving around the nucleus in an atom.	Coulomb attraction exerted by the protons in the nucleus.
A charged particle describing a circular path in a magnetic field.	Magnetic force exerted by the agent that sets up the magnetic field.
Vehicle taking a turn on a level road.	Frictional force exerted by the road on the tyres.

MOTION OF A PARTICLE IN A CURVED TRACK AND RADIUS OF CURVATURE

Consider a curved track, 1234, having portions 12, 23 and 34 on which points X, Y and Z are taken respectively.



We must understand that when a single force acts on any particle it will just accelerate in the direction of force. So, for a particle to move in a curved track it must be experiencing two forces, as shown already,

(a) Centripetal force $\vec{F}_C$, acting radially inwards.

(b) Tangential force $\vec{F}_T$, acting tangentially.

So, at the point X, the curved track is due to the two forces $\vec{F}_C$ and $\vec{F}_T$ at that point.

At the point Y, $r \to \infty$, so $\vec{F}_C \to 0$, hence from 2 to 3, the particle just follows a straight track 23 under the influence of a single force $\vec{F}_T$.

At Z, story similar to the point X is repeated. So, we can say that for a particle moving in a curved track, net force and hence net acceleration are given by

$$F = \sqrt{F_C^2 + F_T^2} \qquad \{\because \vec{F}_C \perp \vec{F}_T\}$$

$$\Rightarrow \quad F = m\sqrt{a_C^2 + a_T^2}$$

$$\Rightarrow \quad F = m\sqrt{(r\omega^2)^2 + (r\alpha)^2}$$

$$\Rightarrow \quad F = mr\sqrt{\omega^4 + \alpha^2} \text{ and } a = \frac{F}{m} = r\sqrt{\omega^4 + \alpha^2}$$

Radius of Curvature

As observed, any curved track/path can be assumed to be made of a large number of circular arcs of variable radii. The radius of curvature at a point is the radius of the circular arc that suitably fits on the curve at that point.

Since $a_C = \frac{v^2}{r}$, where v is the tangential velocity and can be denoted by v_T.

$$\Rightarrow \quad a_C = \frac{v_T^2}{r}$$

$$\Rightarrow \quad r = \frac{v_T^2}{a_C} = \frac{v_\perp^2}{a_{||}}$$

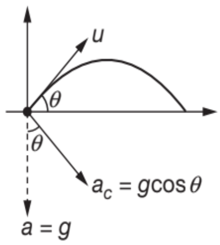
where $v_T = v_\perp$, i.e., tangential velocity equals the velocity of the particle **perpendicular to the radius** of the curve and hence v_T can also be denoted by $v_\perp$ (read as v perpendicular) and $a_C = a_{||}$, i.e., centripetal acceleration equals the acceleration of the particle **parallel to the radius** of the curve and hence a_C can also be denoted by $a_{||}$.

ILLUSTRATION 103

A particle of mass m is projected with speed u at an angle θ with the horizontal. Find the radius of curvature of the path traced out by the particle at the point of projection and also at the highest point of trajectory.

SOLUTION

At point of projection, we have



$$R = \frac{v_T^2}{a_C} = \frac{u^2}{g\cos\theta}$$

$$\Rightarrow \quad R = \frac{u^2}{g\cos\theta}$$

At highest point, we have

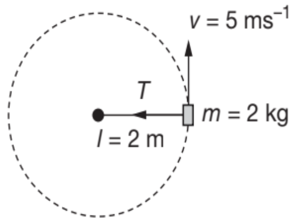
$$a_C = g, \; v = u\cos\theta$$

$$\Rightarrow \quad R = \frac{v_T^2}{a_C} = \frac{u^2\cos^2\theta}{g}$$

ILLUSTRATION 104

A block of mass 2 kg is tied to a string of length 2 m, the other end of which is fixed. The block is moved on a smooth horizontal table with constant speed $5\ \text{ms}^{-1}$. Find the tension in the string.

SOLUTION

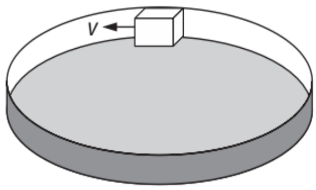


In this case, the centripetal force is provided by tension, so,

$$T = \frac{mv^2}{r} = \frac{2\times 5^2}{2} = 25\ \text{N}$$

ILLUSTRATION 105

A block of mass m moves with speed v against a smooth, fixed vertical circular groove of radius r kept on smooth horizontal surface.



Find the

(a) normal reaction of the floor on the block.

(b) normal reaction of the vertical wall on the block.

SOLUTION

Here centripetal force is provided by normal reaction of vertical wall

(a) normal reaction due to the floor

$$N_F = mg$$

(b) normal reaction due to the vertical wall

$$N_W = \frac{mv^2}{r}$$

ILLUSTRATION 106

A block of mass m is kept on the edge of a horizontal turn table of radius R, which is rotating with constant angular velocity ω (along with the block) about its axis. If coefficient of friction is μ, find the friction force between block and table if the block is at rest with respect to table.

SOLUTION

The centripetal force, in this case is provided by friction force

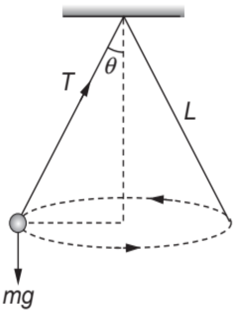
$$\text{Friction force} = \text{centripetal force} = mR\omega^2$$

ILLUSTRATION 107

A string breaks under a load of 50 kg. A mass of 1 kg is attached to one end of the string 10 m long and is rotated in horizontal circle. Calculate the greatest number of revolutions that the mass can make in one second without breaking the string.

SOLUTION

The situation is shown in the figure. Since we have



$$\omega = 2\pi n$$

$$T_{max} = 500\text{ N}$$

$$r = L\sin\theta$$

$$T\sin\theta = m\omega^2 r$$

$$\Rightarrow \quad T = m\omega^2 L$$

$$\Rightarrow \quad T_{max} = m\omega_{max}^2 L$$

$$\Rightarrow \quad T_{max} = m\left(2\pi n_{max}\right)^2 L$$

$$\Rightarrow \quad n_{max} = \frac{1}{2\pi}\sqrt{\frac{T_{max}}{mL}}$$

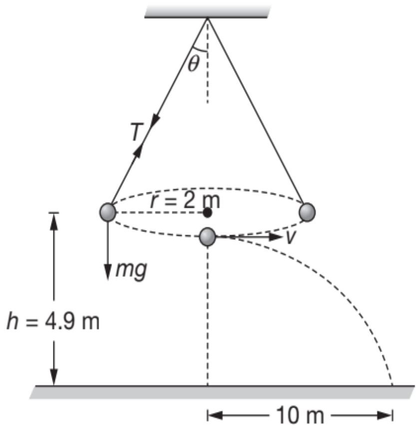
$$\Rightarrow \quad n_{max} = \frac{1}{2\pi}\sqrt{\frac{500}{1\times 10}}$$

$$\Rightarrow \quad n_{max} = \frac{\sqrt{50}}{2\pi}\text{ rps}$$

ILLUSTRATION 108

A boy whirls a stone in a horizontal circle of radius 2 m and at height 4.9 m above level ground. The string breaks and the stone files off horizontally and strikes the ground at a point which is 10 m away from the point on the ground directly below the point where the string had broken. What is the magnitude of the centripetal acceleration of the stone while in circular motion? $\left(g = 9.8\text{ ms}^{-2}\right)$.

SOLUTION



$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 4.9}{9.8}} = 1\text{ s}$$

$$\Rightarrow \quad v = \frac{10}{t} = 10\text{ ms}^{-1}$$

$$\Rightarrow \quad a = \frac{v^2}{R} = 50\text{ ms}^{-2}$$

ILLUSTRATION 109

A particle begins to move with a tangential acceleration of constant magnitude 0.6 ms^{-2} in a circular path. If it slips when its total acceleration becomes 1 ms^{-2}. Find the angle through which it would have turned before it starts to slip.

SOLUTION

$$a_{Net} = \sqrt{a_T^2 + a_C^2} \qquad \ldots(1)$$

Since, $\omega^2 = \omega_0^2 + 2\alpha\theta$ and $\omega_0 = 0$

$$\Rightarrow \quad \omega^2 = 2\alpha\theta$$

$$\Rightarrow \quad \omega^2 R = 2(\alpha R\theta)$$

$$\Rightarrow \quad a_C = \omega^2 R = 2a_T\theta$$

From (1), we get

$$\sqrt{0.36+(1.2\times\theta)^2}=1$$

$$\Rightarrow \quad 1-0.36=(1.2\theta)^2$$

$$\Rightarrow \quad \frac{0.8}{1.2}=\theta$$

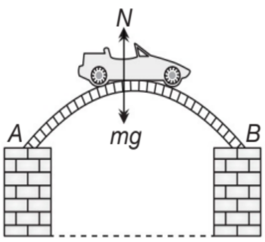
$$\Rightarrow \quad \theta=\frac{2}{3}\text{ radian}$$

ILLUSTRATION 110

Prove that a motor car moving over a convex bridge is lighter than the same car resting on the same bridge.

SOLUTION

The motion of the motor car over a convex bridge AB is the motion along the segment of a circle AB as shown in the figure.



The centripetal force is provided by the difference of weight mg of the car and the normal reaction N of the bridge.

$$\Rightarrow \quad mg-N=\frac{mv^2}{r}$$

$$\Rightarrow \quad N=mg-\frac{mv^2}{r}$$

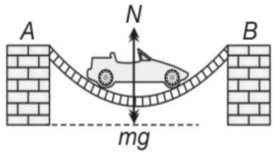
Since $N<mg$, so, the weight of the moving car is less than the weight of the stationary car.

ILLUSTRATION 111

Prove that a motor car moving over a concave bridge is heavier than the same car resting on the same bridge.

SOLUTION

The motion of the motor car over a concave bridge AB is the motion along the segment of a circle AB as shown in the figure.



The centripetal force is provided by the difference of normal reaction N of the bridge and weight mg of the car.

$$\Rightarrow \quad N-mg=\frac{mv^2}{r}$$

$$\Rightarrow \quad N=mg+\frac{mv^2}{r}$$

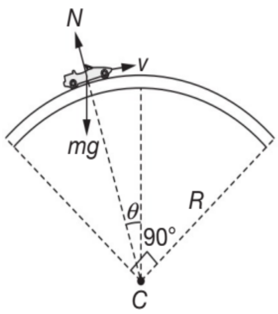
Since $N>mg$, so, the weight of the moving car is greater than the weight of the stationary car.

ILLUSTRATION 112

A car is moving with uniform speed over a circular bridge of radius R which subtends an angle of $90°$ at its centre. Find the minimum possible speed so that the car can cross the bridge without losing the contact anywhere.

SOLUTION

Let the car loses the contact at angle θ with the vertical, then $mg\cos\theta-N=\frac{mv^2}{R}$.



$$\Rightarrow \quad N=mg\cos\theta-\frac{mv^2}{R} \qquad \ldots(1)$$

For losing the contact $N=0$

$$\Rightarrow \quad v=\sqrt{Rg\cos\theta} \qquad \{\text{from }(1)\}$$

For minimum speed, cos θ should be minimum so that θ should be maximum

$$\theta_{max}=45°$$

$$\Rightarrow \quad \cos 45° = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad v_{min} = \sqrt{\frac{Rg}{\sqrt{2}}}$$

So, that if car cannot lose the contact at initial or final point, car cannot be lose the contact anywhere.

ILLUSTRATION 113

A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. If is found that the speed of the bob is v when the string makes an angle θ with the vertical. Find the tension in the string and the magnitude of net force on the bob at the instant.

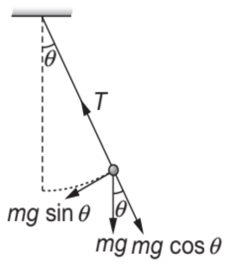
SOLUTION

(a) The forces acting on the bob are

(i) the tension T

(ii) the weight mg

As the bob moves in a circle of radius L with centre at O, A centripetal force of magnitude $\frac{mv^2}{L}$ is required towards O.



This force will be provided by the resultant of T and $mg\cos\theta$. Thus,

$$T - mg\cos\theta = \frac{mv^2}{L}$$

$$\Rightarrow \quad T = m\left(g\cos\theta + \frac{v^2}{L}\right)$$

(b) Since a_C is provided by $(T - mg\cos\theta)$ acting radially inwards, so $a_C = \frac{T - mg\cos\theta}{m} = \frac{v^2}{l}$.

$$a_{net} = \sqrt{a_T^2 + a_C^2} = \sqrt{(g\sin\alpha)^2 + \left(\frac{v^2}{l}\right)^2}$$

$$|\vec{F}_{net}| = ma_{net} = m\sqrt{g^2\sin^2\alpha + \frac{v^4}{L^2}}$$

CENTRIFUGAL FORCE

When a body is rotating in a circular path and if the centripetal force vanishes, then the body would leave the circular path. To an observer A who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release.

To another observer B, who is sharing the motion along the circular path (i.e., the observer B is also rotating with the body which is released, it appears to B, as if it has been thrown off along the radius away from the centre by some force called centrifugal force).

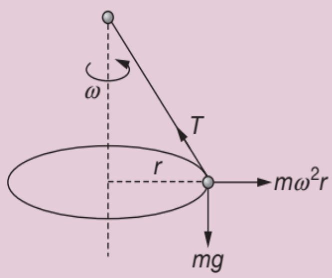
Its magnitude is equal to that of the centripetal force i.e.,

$$|F_{centrifugal}| = \frac{mv^2}{r} = mr\omega^2$$

Conceptual Note(s)

(a) Direction of centrifugal force, it is always directed radially outward.

(b) Centrifugal force is a fictitious force which has to be applied as a concept only in a rotating frame of reference to apply Newton's law of motion in that frame. FBD of ball w.r.t. non inertial frame rotating with the ball is shown here.



(c) Suppose we are working from a frame of reference that is rotating at a constant, angular velocity ω with respect to an inertial frame. If we analyse the dynamics of a particle of mass m kept at a distance r from the axis of rotation, we have to assume that a force $mr\omega^2$ react radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

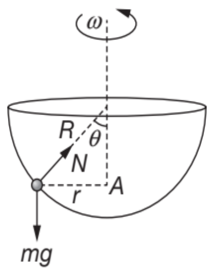
(d) While drawing FBD from rotating frame(s), we must draw the centrifugal force whereas while drawing FBD from ground or inertial frame we must not show the centripetal force. Instead we draw all forces and then the resultant of these forces acting towards the centre will provide the necessary centripetal force.

ILLUSTRATION 114

A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. It the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is θ. Find the angular speed at which the bowl is rotating.

SOLUTION

Let ω be the angular speed of rotation of the bowl. Two forces are acting on the ball.



(a) Normal reaction (N)

(b) Weight (mg)

The ball is rotating in a circle of radius $r(=R\sin\theta)$ with centre at A at an angular speed ω. Thus,

$$N\sin\theta = mr\omega^2 = mR\omega^2\sin\theta$$

$$\Rightarrow \quad N = mR\omega^2 \qquad \ldots(1)$$

and $$N\cos\theta = mg \qquad \ldots(2)$$

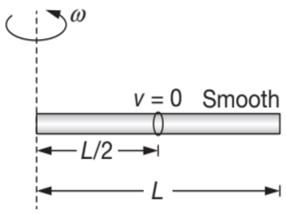
Dividing (1) by (2), we get

$$\frac{1}{\cos\theta} = \frac{\omega^2 R}{g}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{g}{R\cos\theta}}$$

ILLUSTRATION 115

A ring which can slide along the rod are kept at mid-point of a smooth rod of length L. The rod is rotated with constant angular velocity ω about vertical axis passing through its one end. Ring is released from mid-point. Find the velocity of the ring when it just leave the rod.



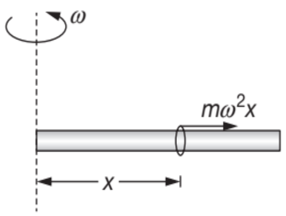
SOLUTION

Centrifugal force at distance x is

$$m\omega^2 x = ma$$

$$\Rightarrow \quad \omega^2 x = \frac{v\,dv}{dx}$$

Rearranging and integrating, we get



$$\int_{L/2}^{L} \omega^2 x\,dx = \int_0^v v\,dv$$

$$\Rightarrow \quad \omega^2\left(\frac{x^2}{2}\right)_{L/2}^{L} = \left(\frac{v^2}{2}\right)_0^v$$

$$\Rightarrow \quad \omega^2\left(\frac{L^2}{2} - \frac{L^2}{8}\right) = \frac{v^2}{2}$$

$$\Rightarrow \quad v = \frac{\sqrt{3}}{2}\omega L$$

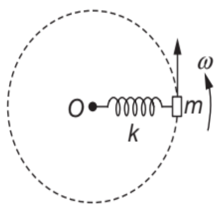
Velocity at time of leaving the rod is the resultant of tangential speed and the radial speed of the particle. So,

$$v' = \sqrt{(\omega L)^2 + \left(\frac{\sqrt{3}}{2}\omega L\right)^2}$$

$$\Rightarrow \quad v' = \frac{\sqrt{7}}{2}\omega L$$

ILLUSTRATION 116

A block of mass m is tied to a spring of spring constant k, natural length l and the other end of spring is fixed at O. If the block moves in a circular path on a smooth horizontal surface with constant angular velocity ω, find tension in the spring.



SOLUTION

Assume extension in the spring is x.
Here centripetal force is provided by spring force
Centripetal force

$$kx = m\omega^2(l + x)$$

$$\Rightarrow \quad x = \frac{m\omega^2 l}{k - m\omega^2}$$

$$\Rightarrow \quad \text{Tension} = kx = \frac{km\omega^2 l}{k - m\omega^2}$$

ILLUSTRATION 117

A block of mass m is kept on rough horizontal turn table at a distance r from centre of table. Coefficient of friction between turn table and block is μ. Now turn table starts rotating with uniform angular acceleration α.

(a) Find the time after which slipping occurs between block and turn table.

(b) Find angle made by friction force with velocity at the point of slipping.

SOLUTION

(a) $a_t = \alpha r$

Speed after t time

$$\frac{dv}{dt} = \alpha r$$

$$\Rightarrow \quad v = 0 + \alpha r t$$

Centripetal acceleration

$$a_c = \frac{v^2}{r} = \alpha^2 r t^2$$

Net acceleration $a_{net} = \sqrt{a_t^2 + a_c^2}$

$$\Rightarrow \quad a_{net} = \sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$$

Block just start slipping

$$\mu mg = ma_{net} = m\sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$$

$$\Rightarrow \quad t = \left(\frac{\mu^2 g^2 - \alpha^2 r^2}{\alpha^4 r^2}\right)^{1/4}$$

$$\Rightarrow \quad t = \left[\left(\frac{\mu g}{\alpha^2 r}\right)^2 - \left(\frac{1}{\alpha}\right)^2\right]^{1/4}$$

(b) $\tan\theta = \dfrac{a_c}{a_t}$

$$\Rightarrow \quad \tan\theta = \frac{\alpha^2 r t^2}{\alpha r}$$

$$\Rightarrow \quad \theta = \tan^{-1}(\alpha t^2)$$

ROTOR OR DEATH WELL

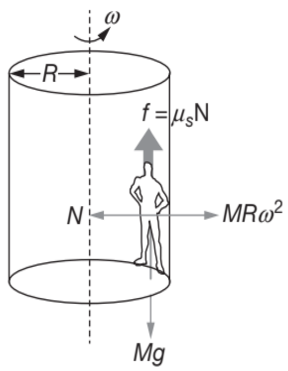
It is a hollow cylindrical room which is capable of rotating about a central vertical axis of cylinder. A person stands up against the wall. For some time the rotational speed of rotor increases and at a certain speed when the floor of the rotor room is removed the person in it does not fall but remains 'pinned up' against the wall of the rotor. The relation between rotational speed ω and the coefficient of friction between person and wall may be found as follows.

Let M be the mass of person and R the radius of cylindrical room. The forces acting on the person are

(a) Weight Mg (vertically downward)
(b) Normal reaction N (radially inwards)
(c) Frictional force $f = \mu_s N$ (upward)
(d) Centrifugal force $= mR\omega^2$ (radially outwards)

For vertical equilibrium

$$Mg = f \text{ or } Mg = \mu_s N \qquad \ldots(1)$$



The person will remain in contact with the wall only if

$$MR\omega^2 \geq N$$

$$\Rightarrow \quad MR\omega^2 \geq \frac{Mg}{\mu_s} \quad \text{or} \quad \omega^2 \geq \frac{g}{\mu_s R}$$

$$\Rightarrow \quad \omega \geq \sqrt{\frac{g}{\mu_s R}}$$

Minimum speed of rotor, $\omega_{\min} = \sqrt{\dfrac{g}{\mu_s R}}$

If v is the linear speed of rotor, then

$$v = R\omega = \sqrt{\frac{gR}{\mu_s}}$$

ILLUSTRATION 118

Two blocks of mass $m_1 = 10$ kg and $m_2 = 5$ kg connected to each other by a massless inextensible string of length 0.3 m are placed along a diameter of the turn table. The coefficient of friction between the table and m_1 is 0.5 while there is no friction between m_2 and the table. The table is rotating with an angular velocity of 10 rads^{-1} about a vertical axis passing through its centre O. The masses are placed along the diameter of the table on either side of the centre O such that the mass m_1 is at a distance of 0.124 m from O. The masses are observed to be at rest with respect to an observer on the turn table. $\left(g = 9.8 \text{ ms}^{-2}\right)$.

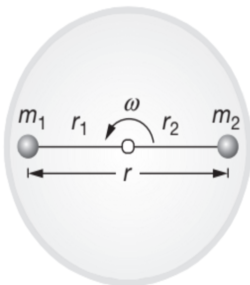
(a) Calculate the frictional force on m_1.
(b) What should be the minimum angular speed of the turn table so that the masses will slip from this position?
(c) How should the masses be placed with the string remaining taut so that there is no frictional force acting on the mass m_1?

SOLUTION

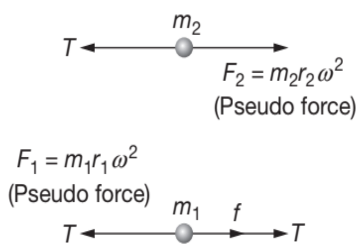
Given $m_1 = 10$ kg, $m_2 = 5$ kg, $\omega = 10$ rads^{-1}

$$r = 0.3 \text{ m}, \ r_1 = 0.124 \text{ m}$$

$$\Rightarrow \quad r_2 = r - r_1 = 0.176 \text{ m}$$



(a) Masses m_1 and m_2 are at rest with respect to rotating table. Let f be the friction between mass m_1 and table. Free body diagram of m_1 and m_2 with respect to table (non inertial frame of reference are shown in figure).



Equilibrium of m_2 gives

$$T = m_2 r_2 \omega^2 \qquad \ldots(1)$$

Since, $m_2 r_2 \omega^2 < m_1 r_1 \omega^2 \qquad \{m_2 r_2 < m_1 r_1\}$

Therefore, $m_1 r_1 \omega^2 > T$ and friction on m_1 will be inwards (towards centre)

Equilibrium of m_1 gives $f + T = m_1 r_1 \omega^2$ ...(2)

From equations (1) and (2), we get

$$f = m_1 r_1 \omega^2 - m_2 r_2 \omega^2$$

$$\Rightarrow \quad f = (m_1 r_1 - m_2 r_2)\omega^2 \quad ...(3)$$

$$\Rightarrow \quad f = (10 \times 0.124 - 5 \times 0.176)(100) \text{ Newton}$$

$$\Rightarrow \quad f = 36 \text{ N}$$

Therefore, frictional force on m_1 is 36 N (inwards).

(b) Since, from equation (3), we have

$$f = (m_1 r_1 - m_2 r_2)\omega^2$$

So, the masses will start slipping when this force is greater than f_{max}, i.e.,

$$(m_1 r_1 - m_2 r_2)\omega^2 > f_{max} > \mu m_1 g$$

Hence, the minimum value of ω is

$$\omega_{min} = \sqrt{\frac{\mu m_1 g}{m_1 r_1 - m_2 r_2}} = \sqrt{\frac{0.5 \times 10 \times 9.8}{10 \times 0.124 - 5 \times 0.176}}$$

$$\Rightarrow \quad \omega_{min} = 11.67 \text{ rads}^{-1}$$

(c) From equation (3), frictional force $f = 0$ when $m_1 r_1 = m_2 r_2$

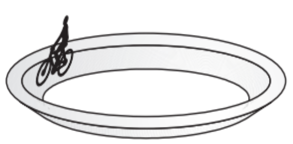
$$\Rightarrow \quad \frac{r_1}{r_2} = \frac{m_2}{m_1} = \frac{5}{10} = \frac{1}{2} \text{ and } r = r_1 + r_2 = 0.3 \text{ m}$$

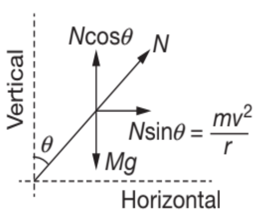
$$\Rightarrow \quad r_1 = 0.1 \text{ m and } r_2 = 0.2 \text{ m}$$

i.e., mass m_2 should be placed at 0.2 m and m_1 at 0.1 m from the centre O.

MOTION OF A CYCLIST

Let a cyclist moving on a circular path of radius r bend away from the vertical by an angle θ.





If R is the normal reaction of the ground, then R may be resolved into two components horizontal and vertical. The vertical component $R\cos\theta$ balances the weight mg of the cyclist and the horizontal component $R\sin\theta$ provides the necessary centripetal force to the cyclist to move in circular path i.e.,

$$R\sin\theta = \frac{mv^2}{r} \quad ...(1)$$

and $R\cos\theta = mg$ ...(2)

Dividing (1) by (2), we get

$$\tan\theta = \frac{v^2}{rg} \quad ...(3)$$

For less bending of cyclist, his speed v should be smaller and radius r of circular path should be greater. If μ is coefficient of friction, then for no skidding of cycle (or overturning of cyclist)

$$\mu \ge \tan\theta = \frac{v^2}{rg} \quad ...(4)$$

CIRCULAR TURNING ON ROADS

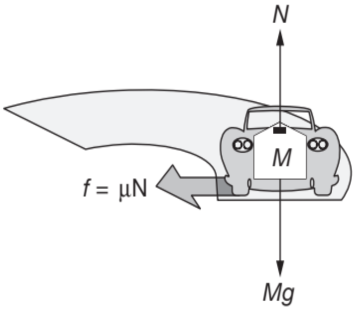
When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways.

(a) By friction only.
(b) By banking of roads only.
(c) By friction and banking of roads both.

In real life the necessary centripetal force is provided by friction and banking of roads both. Now let us write equations of motion in each of the three cases separately and see what are the constant in each case.

BY FRICTION ONLY: VEHICLE ON A LEVEL ROAD

When a vehicle goes around a curve, it has a tendency of skidding sideways i.e. away from the centre of the curve. Due to this tendency, the static friction f_s acts towards the centre and provides the necessary centripetal force for motion along the curve.



The forces acting on the vehicle are

(a) Weight Mg acting vertically downward

(b) Normal reaction N

(c) Static frictional force f_s

The static friction is self adjusting and if μ_s is coefficient of static friction, then $f_s \le \mu_s N$. So,

For vertical equilibrium, $N = Mg$

As frictional force provides the necessary centripetal force $f_s = \dfrac{Mv^2}{r}$

As $f_s \le \mu_s N$

$$\Rightarrow \quad \frac{Mv^2}{r} \le \mu_s N$$

$$\Rightarrow \quad \frac{Mv^2}{r} \le \mu_s Mg$$

$$\Rightarrow \quad v \le \sqrt{\mu_s rg}$$

So, maximum speed for no skidding is

$$v_{\max} = \sqrt{\mu_s rg}$$

ILLUSTRATION 119

A bend in a level road has a radius of 100 m. Calculate the maximum speed which a car turning this bend may have without skidding. Given: $\mu = 0.8$.

SOLUTION

$$v_{\max} = \sqrt{\mu rg} = \sqrt{0.8 \times 100 \times 10}$$

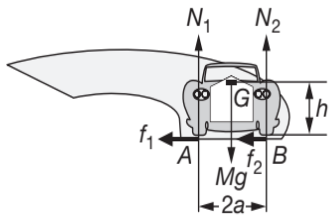
$$\Rightarrow \quad v_{\max} = \sqrt{0.8 \times 100 \times 10}$$

$$\Rightarrow \quad v_{\max} = \sqrt{800}$$

$$\Rightarrow \quad v_{\max} = 28 \text{ ms}^{-1}$$

MAXIMUM VELOCITY FOR SKIDDING AND OVERTURNING

Let A and B be inner and outer wheels of a vehicle moving on a circular track. The forces acting on the vehicle are



(a) normal reactions N_1 and N_2 vertically upward

(b) frictional forces $f_1 = \mu N_1$ and $f_2 = \mu N_2$

(c) Weight of vehicle Mg vertically downward

(d) centripetal force F (horizontally towards centre of turn)

For translational equilibrium

$$N_1 + N_2 = Mg \qquad \ldots(1)$$

and frictional force provides the necessary centripetal force

$$F = f_1 + f_2 = \mu N_1 + \mu N_2 \ge \frac{mv^2}{r}$$

$$\Rightarrow \quad \mu(N_1 + N_2) \ge \frac{mv^2}{r} \qquad \ldots(2)$$

where r is the radius of path.

Using (1), equation (2) gives

$$\Rightarrow \quad \mu Mg \ge \frac{mv^2}{r} \quad \text{(for no skidding)}$$

$$\Rightarrow \quad v \le \sqrt{\mu rg}$$

Thus maximum speed for no skidding is

$$v_{\max} = \sqrt{\mu rg} \qquad \ldots(3)$$

For No Overturning

If the wheels A and B are a distance $2a$ apart, then taking moments about G, we get $N_2 a = N_1 a + Fh$

where $F = \dfrac{mv^2}{r}$ given by (2)

The car tends to overturn when reaction N_1 the inner wheel is zero i.e. when inner wheel leaves contact with the ground $N_2 \cdot a \geq Fh$

If $N_1 = 0$, then from (1) $N_2 = Mg$

$$\Rightarrow \quad Mga \geq \frac{mv^2}{r}h$$

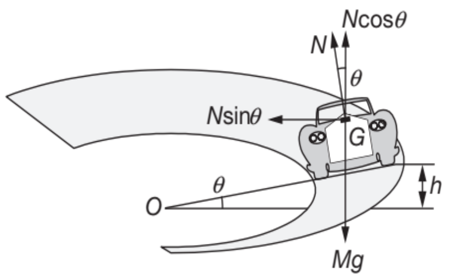
$$\Rightarrow \quad v \leq \sqrt{\frac{gra}{h}}$$

That is maximum speed for no overturning is

$$v_{\max} = \sqrt{\frac{gra}{h}}$$

BY BANKING OF ROADS/TRACKS

When a vehicle moves round a curve on the road with sufficient speed, then there is a tendency of overturning for the vehicle. To avoid this the road is given a slope rising outwards. The phenomenon is known as **banking**. Consider a vehicle on a road having a slope θ. N is the normal reaction of the ground. This may be resolved into two components : A vertical component $N\cos\theta$ which balances the weight of vehicle and a horizontal component $N\sin\theta$ which provides the necessary centripetal force i.e.,



$$N\sin\theta = \frac{mv^2}{r} \quad \ldots(1)$$

and $N\cos\theta = Mg \quad \ldots(2)$

Dividing equation (1) by (2), we get

$$\tan\theta = \frac{v^2}{rg}$$

This equation gives the angle of banking required.

Let $l(=OB)$ be the width of track and $h(=AB)$ be its height.

Assuming θ (the angle of banking to be small) then

$$\tan\theta \simeq \sin\theta = \frac{h}{l}$$

$$\Rightarrow \quad \frac{h}{l} = \frac{v^2}{rg}$$

$$\Rightarrow \quad h = \frac{v^2 l}{rg}$$

is the height through which outer part of the track has to be raised.

ILLUSTRATION 120

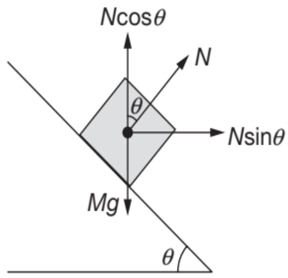
A circular track of radius 600 m is to be designed for cars at an average speed of 180 kmh^{-1}. What should be the angle of banking of the track?

SOLUTION

Let the angle of banking be θ. The forces on the car are (figure)

(a) weight of the car Mg downward and

(b) normal force N.



For proper banking, static frictional force is not needed. For vertical direction the acceleration is zero. So,

$$N\cos\theta = Mg \quad \ldots(1)$$

For horizontal direction, the acceleration is $\frac{v^2}{r}$ towards the centre, so that

$$N\sin\theta = \frac{Mv^2}{r} \quad \ldots(2)$$

From (1) and (2)

$$\tan\theta = \frac{v^2}{rg}$$

Substituting the values

$$\tan\theta = \frac{180 \text{ kmh}^{-2}}{600 \text{ m} \times 10 \text{ ms}^{-2}} = 0.4167$$

$$\Rightarrow \quad \theta = 22.6^\circ$$

BY FRICTION AND BANKING OF ROAD BOTH

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, i.e., weight (mg) is fixed both in magnitude and direction.

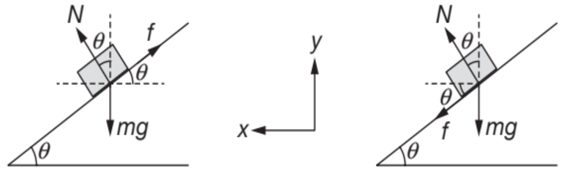


Figure (i) Figure (ii)

The direction of second force, i.e., normal reaction N is also fixed (perpendicular to road)

The direction of the third force i.e., friction f can be either inwards or outwards and its magnitude can be varied upto a maximum limit $(f_L = \mu N)$.

So the magnitude of normal reaction N, direction of friction and magnitude of friction f are so adjusted so that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre.

Since, m and r are also constant so, magnitude of normal reaction N, direction of friction and magnitude of friction f mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem.

Also, we observe that

(a) friction f will be outwards if the vehicle is at rest $v = 0$. Because in that case the component of weight $mg\sin\theta$ is balanced by f.

(b) friction f will be inwards if

$$v > \sqrt{rg\tan\theta}$$

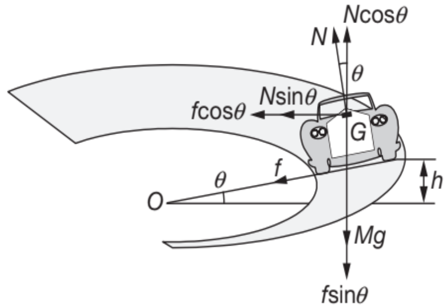
(c) friction f will be outwards if

$$v < \sqrt{rg\tan\theta} \text{ and}$$

(d) friction f will be zero if

$$v = \sqrt{rg\tan\theta}$$

(e) for maximum safe speed (shown in figure (ii)) and redrawn below



$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \quad \ldots(1)$$

$$N\cos\theta - f\sin\theta = mg \quad \ldots(2)$$

As maximum value of friction

$$f = \mu N$$

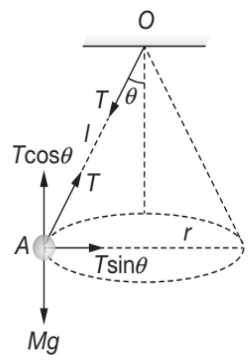
$$\Rightarrow \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

$$\Rightarrow v_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{(1 - \mu\tan\theta)}}$$

Similarly, $v_{min} = \sqrt{\frac{rg(\tan\theta - \mu)}{(1 + \mu\tan\theta)}}$

CONICAL PENDULUM

It consists of a string OA, whose upper end O is fixed and a bob is tied at the other free end. The bob is given a horizontal push a little through angular displacement θ and arranged such that the bob describes a horizontal circle with uniform angular velocity ω in such a way that the string always makes an angle θ with the vertical. As the string traces the surface of the cone, the arrangement is called **a conical pendulum**.



Let T be the tension in the string of length l and r the radius of circular path. The vertical component of tension T balances the weight of the bob and horizontal component provides the necessary centripetal force.

Thus

$$T\cos\theta = Mg \quad \ldots(1)$$

and $T\sin\theta = Mr\omega^2 \quad \ldots(2)$

Dividing (2) by (1), we get

$$\tan\theta = \frac{r\omega^2}{g} \text{ i.e., } \omega = \sqrt{\frac{g\tan\theta}{r}} \quad \ldots(3)$$

But $r = l\sin\theta$ and $\omega = \frac{2\pi}{t}$, t being period of completing one revolution

$$\Rightarrow \quad \frac{2\pi}{t} = \sqrt{\frac{g\tan\theta}{l\sin\theta}}$$

This gives

$$t = 2\pi\sqrt{\frac{l\sin\theta}{g\left(\frac{\sin\theta}{\cos\theta}\right)}}$$

$$\Rightarrow \quad t = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

ILLUSTRATION 121

A turn of radius 20 m is banked for the vehicle of mass 200 kg going at a speed of $10\ \text{ms}^{-1}$. Find the direction and magnitude of frictional force acting on a vehicle if it moves with a speed (a) $5\ \text{ms}^{-1}$ (b) $15\ \text{ms}^{-1}$. Take $g = 10\ \text{ms}^{-2}$ and assume that friction is sufficient to prevent slipping.

SOLUTION

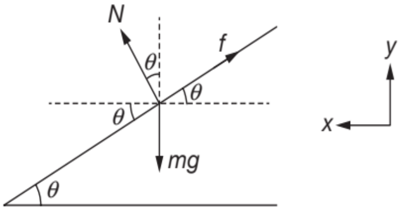
(a) The turn is banked for speed $v = 10\ \text{ms}^{-1}$. If θ is the angle of banking, then

$$\tan\theta = \frac{v^2}{rg} = \frac{(10)^2}{(20)(10)} = \frac{1}{2}$$

Now, as the speed is decreased, force of friction f acts outwards.

Using the equations

$$\Sigma F_x = \frac{mv^2}{r} \text{ and } \Sigma F_y = 0, \text{ we get}$$



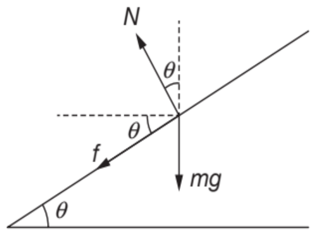
$$N\sin\theta - f\cos\theta = \frac{mv^2}{r} \quad \ldots(1)$$

$$N\cos\theta + f\sin\theta = mg \quad \ldots(2)$$

Substituting, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 5\ \text{ms}^{-1}$, $m = 200$ kg and $r = 20$ m, in equations (1) and (2), we get

$$f = 300\sqrt{5}\ \text{N} \quad \text{(outwards)}$$

(b) In the second case force of friction f will now act inwards.



Using $\Sigma F_x = \frac{mv^2}{r}$ and $\Sigma F_y = 0$, we get

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \quad \ldots(3)$$

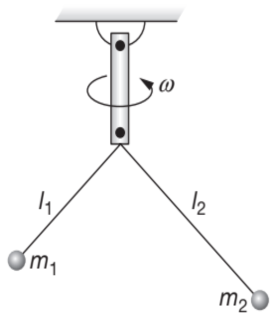
$$N\cos\theta - f\sin\theta = mg \quad \ldots(4)$$

Substituting $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 15\ \text{ms}^{-1}$, $m = 200$ kg and $r = 20$ m in equations (3) and (4), we get

$$f = 500\sqrt{5}\ \text{N} \quad \text{(inwards)}$$

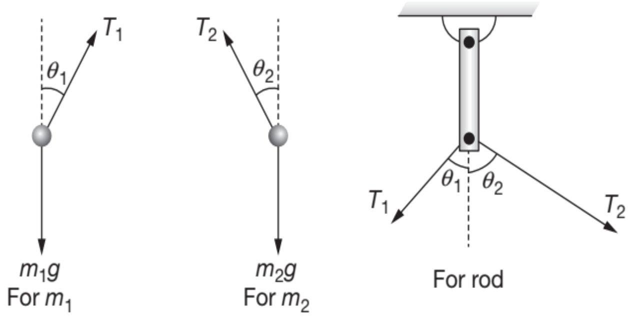
ILLUSTRATION 122

Two balls of mass m_1 and m_2 are suspended by two threads of lengths l_1 and l_2 at the end of a freely hanging rod. Determine the angular velocity ω at which the rod must be rotated about the vertical axis so that it remains vertical.



SOLUTION

Free body diagrams of two masses and the rod are as shown in figure.



Equations of motion are,

$$T_1 \sin\theta_1 = m_1\omega^2 l_1 \sin\theta_1 \quad \text{...(1)}$$

$$T_1 \cos\theta_1 = m_1 g \quad \text{...(2)}$$

$$T_2 \sin\theta_2 = m_2\omega^2 l_2 \sin\theta_2 \quad \text{...(3)}$$

$$T_2 \cos\theta_2 = m_2 g \quad \text{...(4)}$$

For the rod to remain in vertical position,

$$T_1 \sin\theta_1 = T_2 \sin\theta_2 \quad \text{...(5)}$$

Solving the above equations, we get

$$\omega^4 = \left(\frac{m_1^2 g^2 - m_2^2 g^2}{m_1^2 l_1^2 - m_2^2 l_2^2}\right)$$

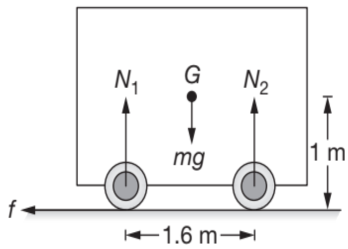
ILLUSTRATION 123

A vehicle whose wheel track is 1.6 m wide and whose centre of gravity is 1 m above the road, centred between the wheels, takes a curve whose radius is 50 m, on a level road. Taking $g = 10\ \text{ms}^{-2}$, find the speed at which the inner wheel would leave the road.

SOLUTION

The situation is shown in figure. Let N_1 and N_2 be the reactions at inner wheels and outer wheels respectively. f is the frictional force of the tracks. G represents the centre of gravity and mg, the weight of the vehicle acting downwards at centre of gravity. For vertical equilibrium,

$$N_1 + N_2 = mg \quad \text{...(1)}$$



For circular motion

$$f = \frac{mv^2}{r} \quad \text{...(2)}$$

For rotational equilibrium, net moment of all the forces about G should be zero. Hence,

$$f(1) + N_1\left(\frac{1.6}{2}\right) = N_2\left(\frac{1.6}{2}\right)$$

$$\Rightarrow \quad 0.8N_1 + f = 0.8N_2 \quad \text{...(3)}$$

When the inner wheel leaves the road, then $N_1 = 0$. Therefore from (3), we get

$$f = 0.8N_2 \quad \text{...(4)}$$

and from equation (1) we get

$$N_2 = mg \quad \text{...(5)}$$

Solving equations (2), (4) and (5), we get

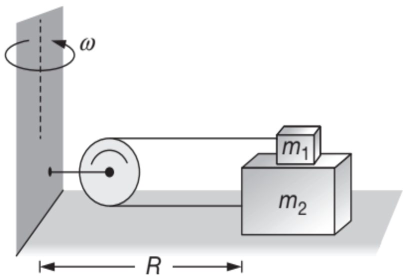
$$v = \sqrt{(0.8)\times(50)\times(10)} = 20\ \text{ms}^{-1}$$

Test Your Concepts-VIII

Based on Circular Motion

(Solutions on page H.214)

1. A circular table with smooth horizontal surface is rotating at an angular speed ω about its axis. A groove is made on the surface along a radius and a smooth ball of diameter equal to the width of the groove is gently placed inside the groove at a distance a from the centre. Find the speed of the particle with respect to the table when its distance from the centre becomes L.
2. A block of mass m moves on a horizontal circle against the wall of a cylindrical room of radius R. The floor of the room on which the block moves is smooth but the friction coefficient between the wall and the block is μ. The block is given an initial speed v_0. Find the
   - **(a)** normal force by the wall on the block
   - **(b)** frictional force by the wall
   - **(c)** tangential acceleration of the block.
   - **(d)** speed of block after one revolution.
3. If the system shown in the figure is rotated in a horizontal circle about the dotted axis, with angular velocity ω, find the minimum value of ω to start relative motion between the two blocks. Also calculate the tension in the string connecting m_1 and m_2 when slipping just starts between the blocks. The coefficient of friction between the two masses is 0.5, there is no friction between m_2 and ground, $R = 0.5$ m, $m_1 = 2$ kg, $m_2 = 1$ kg and $g = 10\text{ ms}^{-2}$. The dimensions of the blocks can be neglected.

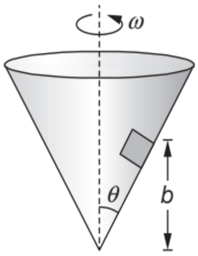


4. The coefficient of friction between a block of mass m placed on a horizontal ruler is μ. The ruler is fixed at one end and the block is at a distance L from the fixed end. The ruler is rotated about the fixed end in the horizontal plane through the fixed end.
   - **(a)** What can the maximum angular speed be for which the block does not slip?
   - **(b)** If the angular speed of the ruler is uniformly increased from zero at an angular acceleration α, at what angular speed will the block slip?
5. A bicyclist travels in a circle of radius 25 m at a constant speed of 9 ms^{-1}. The bicycle-rider mass is 85 kg. Calculate the magnitude of
   - **(a)** the force of friction on the bicycle from the road and
   - **(b)** the net force on the bicycle from the road.
6. A small bead can slide without friction on a circular hoop that is in a vertical plane and has a radius of 0.1 m. The hoop rotates at a constant rate of 3 revs^{-1} about a vertical diameter.
   - **(a)** Find the angle β at which the bead is in vertical equilibrium. (of course, it has a radial acceleration towards the axis).
   - **(b)** Is it possible for the bead to ride at the same elevation as the centre of the hoop? Take $g = 10\text{ ms}^{-2}$ and $\pi^2 = 10$.
7. A string has one end attached to the corner of a square board fixed on a smooth horizontal table and is wound round the square carrying a particle at its other end. The particle is projected with velocity u at right angles to the side of a square of size $a \times a$. If the length of the string is $4a$. Find the time that the string takes to unwrap itself form the squares, assuming that the speed of the particle remains the same throughout the motion.
8. A wheel of radius R rolls along the ground with velocity v. A pebble is carefully released on top of the wheel so that it is instantaneously at rest on the wheel.
   - **(a)** Show that the pebble will immediately fly off the wheel if $v > \sqrt{Rg}$.

(b) Show that in the case when $v < \sqrt{Rg}$ and the coefficient of friction is $\mu = 1$, the pebble starts to slide when it has rotated through an angle given by $\theta = \cos^{-1}\left(\frac{v^2}{\sqrt{2}Rg}\right) - \frac{\pi}{4}$.

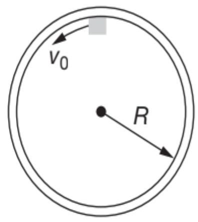
9. A turn of radius 20 m is banked for the vehicles going at a speed of 10 ms^{-1}. If the coefficient of static friction between the road and the tyre is 0.4, what are the possible speeds of a vehicle so that it neither slips down nor skids up? (Take $g = 10\ ms^{-2}$)

10. A small block of mass m is placed inside a hollow cone rotating about a vertical axis with angular velocity ω as shown in figure. The semi-vertex angle of the cone is θ and the coefficient of friction between the cone and the block is μ. If the block is to remain at a constant height b above the apex of the cone, as shown, what are the maximum and minimum values of ω?

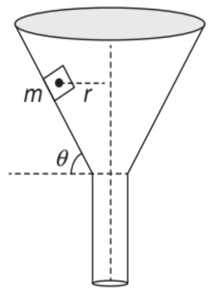


11. The width of the track of a car is 1.15 m and the centre of gravity is 0.6 m above the road and 0.075 m to the right of the centre line of the car. The road bends first to the right and then to the left, the radius of each curve being 40 m. If the coefficient of friction between the tyres and the road is 0.9 show that if the car were driven too fast, it would skid at the first bend and overturn at the second. Take $g = 9.8\ ms^{-2}$.

12. A block of mass m slides on a frictionless table. It is constrained to move inside a ring of radius R. At time $t = 0$, block is moving along the inside of the ring (i.e., in the tangential direction) with velocity v_0. The coefficient of friction between the block and the ring is μ. Find the speed of the block at time t.



13. A very small cube of mass m is placed on the inside of a funnel rotating about a vertical axis at a constant rate of n revolutions per sec. The wall of the funnel makes an angle θ with the horizontal. If the coefficient of static friction between the cube and funnel is μ and the centre of the cube is at a distance r from the axis of rotation, what are the largest and smallest values of n for which the block will not move with respect to the funnel?



14. A bicyclist travels in a circle of radius 25 m at a constant speed of 9 ms^{-1}. The bicycle rider mass is 85 kg. Calculate the magnitude of (a) the force of friction on the bicycle from the road and (b) the net force on the bicycle from the road.

SOLVED PROBLEMS

PROBLEM 1

Two small beads of weight W_1 and W_2 each capable of sliding freely on a smooth circular ring fixed in the vertical plane are connected by a light string. Show that in the position of equilibrium in which the string be straight and inclined at an angle θ to the horizontal we have

$$(W_1+W_2)\tan\theta=(W_1-W_2)\tan\frac{\alpha}{2}$$

Here, α is the angle subtended by the string at the centre.

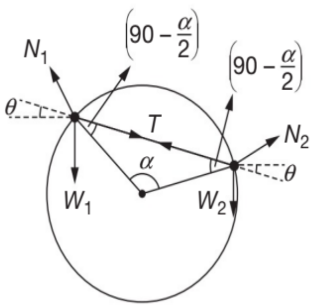
SOLUTION

Beads are in equilibrium under three concurrent forces. So applying Lami's theorem,

For Bead 1:

$$\frac{W_1}{\sin\left\{180-\left(90-\frac{\alpha}{2}\right)\right\}}=\frac{T}{\sin\left\{90+\theta+90-\frac{\alpha}{2}\right\}}$$

$$\Rightarrow\quad T=\frac{W_1\sin\left(\frac{\alpha}{2}-\theta\right)}{\cos\frac{\alpha}{2}} \quad \ldots(1)$$



For Bead 2:

$$\frac{W_2}{\sin\left\{180-\left(90-\frac{\alpha}{2}\right)\right\}}=\frac{T}{\sin\left\{90+90-\frac{\alpha}{2}-\theta\right\}}$$

$$\Rightarrow\quad T=\frac{W_2\sin\left(\frac{\alpha}{2}+\theta\right)}{\cos\frac{\alpha}{2}} \quad \ldots(2)$$

Equating equations (1) and (2), we get

$$W_1\sin\left(\frac{\alpha}{2}-\theta\right)=W_2\sin\left(\frac{\alpha}{2}+\theta\right)$$

Since $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

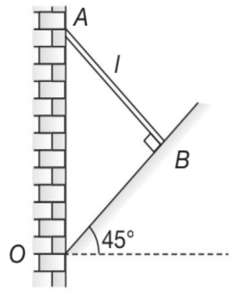
$$\Rightarrow\quad W_1\sin\left(\frac{\alpha}{2}\right)\cos\theta-W_1\sin\theta\cos\left(\frac{\alpha}{2}\right)=W_2\sin\left(\frac{\alpha}{2}\right)\cos\theta+W_2\sin\theta\cos\left(\frac{\alpha}{2}\right)$$

Dividing the equation by $\cos\theta\cos\left(\frac{\alpha}{2}\right)$, we get

$$(W_1+W_2)\tan\theta=(W_1-W_2)\tan\left(\frac{\alpha}{2}\right)$$

PROBLEM 2

At the bottom edge of a smooth vertical wall, an inclined plane is kept at an angle of 45°. A uniform ladder of length l and mass M rests on the inclined plane against the wall such that it is perpendicular to the incline.



(a) If the plane is also smooth, which way will the ladder slide.

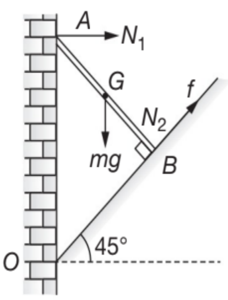
(b) What is the minimum coefficient of friction necessary so that the ladder does not slip on the incline.

SOLUTION

(a) N_2 and mg pass through G. N_1 has clockwise moment about G, so the ladder has a tendency to slip by rotating clockwise and the force of friction (f) at B is then up the plane.

(b) $\Sigma\tau_A=0$

$$\Rightarrow\quad fl=mg\left(\frac{l}{2}\sin(45^\circ)\right) \quad \ldots(1)$$



Also, for equilibrium $\Sigma F_V = 0$,

$$\Rightarrow \quad mg = N_2\cos(45°) + f\sin(45°) \qquad \ldots(2)$$

From equations (1) and (2), we get

$$N_2 = \frac{3mg}{2\sqrt{2}} \text{ and } f = \frac{mg}{2\sqrt{2}}$$

Since, we have, by definition

$$\mu_{\text{MIN}} = \frac{f}{N_2}$$

$$\Rightarrow \quad \mu_{\text{MIN}} = \frac{f}{N_3} = \frac{\left(\frac{mg}{2\sqrt{2}}\right)}{\left(\frac{3mg}{2\sqrt{2}}\right)} = \frac{1}{3}$$

PROBLEM 3

A uniform ladder of length L and mass m_1 rests against a frictionless wall. The ladder makes an angle θ with the horizontal.

(a) Find the horizontal and vertical forces the ground exerts on the base of the ladder when a fire-fighter of mass m_2 is a distance x from the bottom.

(b) If the ladder is just on the verge of slipping when the fire-fighter is a distance d from the bottom, what is the coefficient of static friction between ladder and ground?

SOLUTION

(a) $\Sigma F_x = f - N_w = 0 \qquad \ldots(1)$

$\Sigma F_y = N_g - m_1 g - m_2 g = 0 \qquad \ldots(2)$

$$\Sigma\tau_A = 0$$

$$-m_1 g\left(\frac{L}{2}\right)\cos\theta - m_2 g x\cos\theta + N_w L\sin\theta = 0$$

From the torque equation,

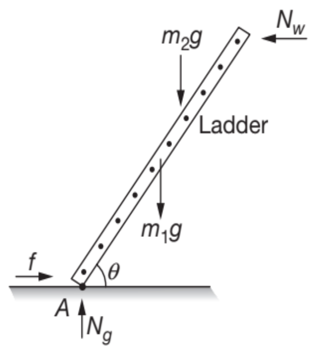
$$N_w = \left[\frac{1}{2}m_1 g + \left(\frac{x}{L}\right)m_2 g\right]\cot\theta$$

Then, from equation (1), we get

$$f = N_w = \left(\frac{1}{2}m_1 g + \left(\frac{x}{L}\right)m_2 g\right)\cot\theta$$

and from equation (2), we get

$$N_g = (m_1 + m_2)g$$



(b) If the ladder is on the verge of slipping when $x = d$, then

$$\mu = \frac{f|_{x=d}}{N_g} = \frac{\left(\frac{m_1}{2} + \frac{m_2 d}{L}\right)\cot\theta}{m_1 + m_2}$$

PROBLEM 4

A ladder of uniform density and mass m rests against a frictionless vertical wall, making an angle of $60°$ with the horizontal. The lower end rests on a flat surface where the coefficient of static friction is $\mu_s = \frac{1}{\sqrt{3}}$. A window cleaner with mass $M = 2m$ attempts to climb the ladder. What fraction of the length L of the ladder will the worker have reached when the ladder begins to slip?

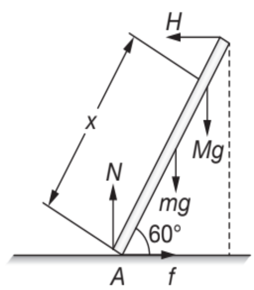
SOLUTION

$$N = (M+m)g \text{ and } H = f$$

So, $H_{\max} = f_{\max} = \mu_s(m+M)g$

$$\Sigma\tau_A = 0$$

$$\frac{mgL}{2}\cos 60° + Mgx\cos 60° - HL\sin 60° = 0$$



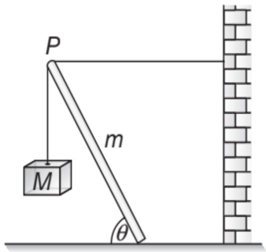
$$\Rightarrow \quad \frac{x}{L} = \frac{H\tan(60°)}{Mg} - \frac{m}{2M}$$

$$\Rightarrow \quad \frac{x}{L} = \frac{\mu_s(m+M)\tan(60°)}{M} - \frac{m}{2M}$$

$$\Rightarrow \quad \frac{x}{L} = \frac{3}{2}\mu_s\tan(60°) - \frac{1}{4} = 0.25$$

PROBLEM 5

A uniform beam of mass m is inclined at an angle θ to the horizontal. Its upper end produces a ninety degree bend in a very rough rope tied to a wall and its lower end rests on a rough floor.



(a) If the coefficient of static friction between beam and floor is μ_s, determine an expression for the maximum mass M that can be suspended from the top before the beam slips.

(b) Determine the magnitude of the reaction force at the floor and the magnitude of the force exerted by the beam on the rope at P in terms of m, M and μ_s.

SOLUTION

(a) For equilibrium, $\sum F_x = \sum F_y = 0$ and $\sum \tau = 0$ with pivot point at the contact on the floor.

Then $\sum F_x = T - \mu_s N = 0$ ...(1)

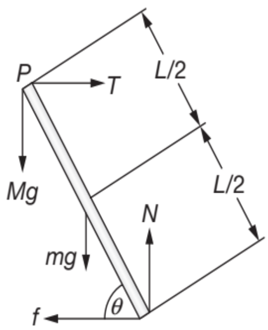
$\Sigma F_y = N - Mg - mg = 0$ ...(2)

$\Sigma \tau = 0$

$$Mg(L\cos\theta) + mg\left(\frac{L}{2}\cos\theta\right) - T(L\sin\theta) = 0$$ ...(3)

Solving equations (1), (2) and (3), we get

$$M = \frac{m}{2}\left(\frac{2\mu_s \sin\theta - \cos\theta}{\cos\theta - \mu_s \sin\theta}\right)$$



This answer is the maximum value for M if $\mu_s < \cot\theta$.

If $\mu_s \geq \cot\theta$, the mass M can increase without limit. It has no maximum value, and part (b) cannot be answered as stated either. In the case $\mu_s < \cot\theta$, we proceed.

(b) At the floor, we have the normal force in the y-direction and frictional force in the x-direction. The reaction force then is

$$R = \sqrt{N^2 + (\mu_s N)^2} = (M+m)g\sqrt{1+\mu_s^2}$$

At point P, the force of the beam on the rope is

$$F = \sqrt{T^2 + (Mg)^2} = g\sqrt{M^2 + \mu_s^2 (M+m)^2}$$

PROBLEM 6

At the moment $t = 0$, the force $F = kt$ is applied to a small body of mass m resting on a smooth horizontal plane (k is a constant). The direction of this force is always constant. Find the

(a) velocity of the body at the moment of its breaking off the plane.

(b) distance traversed by the body up to this moment.

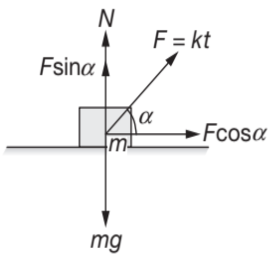
SOLUTION

(a) Along horizontal:

$$F\cos\alpha = kt\cos\alpha = m\frac{dv}{dt}$$ ...(1)

Along vertical:

$$N + kt\sin\alpha = mg$$



At break-off, we have

$$N = 0$$

$$\Rightarrow \quad t = t_0 = \frac{mg}{(k\sin\alpha)}$$

From (1), we get

$$mdv = kt\cos\alpha dt$$

Integrating both sides,

$$m\int_0^v dv = \int_0^t k\cos\alpha t dt$$

$$\Rightarrow \quad v(t) = \frac{kt^2\cos\alpha}{2m} \qquad \ldots(2)$$

At break-off, $t_0 = \dfrac{mg}{k\sin\alpha}$

$$\Rightarrow \quad v(t_0) = \frac{k\cos\alpha}{2m}\left(\frac{m^2g^2}{k^2\sin^2\alpha}\right)$$

$$\Rightarrow \quad v(t_0) = \left(\frac{mg^2\cos\alpha}{2k\sin^2\alpha}\right)$$

(b) From (2), we get

$$v(t) = \frac{dx}{dt} = \frac{kt^2\cos\alpha}{2m}$$

$$\Rightarrow \quad \int_0^x dx = \int_0^t \frac{k\cos\alpha}{2m}t^2 dt$$

$$\Rightarrow \quad x(t_0) = \frac{k\cos\alpha}{6m}\,\frac{m^3g^3}{k^3\sin^3\alpha}$$

$$\Rightarrow \quad x(t_0) = \frac{m^2g^3\cos\alpha}{6k^2\sin^3\alpha}$$

PROBLEM 7

A small disc A is placed on an inclined plane forming an angle α with the horizontal and is imparted an initial velocity v_0. Find how the velocity of the disc depends on the angle ϕ if the friction coefficient $\mu = \tan\alpha$ and at the initial moment $\phi_0 = \dfrac{\pi}{2}$.

SOLUTION

Let a be the acceleration of disc in the direction of its velocity (tangential acceleration). Then

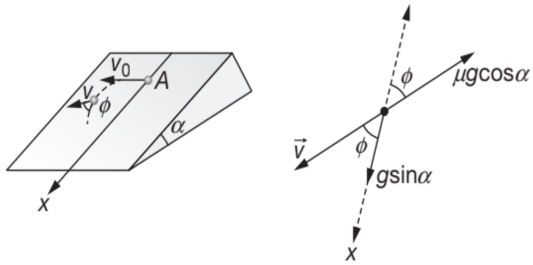
$$a = g\sin\alpha\cos\phi - \mu g\cos\alpha$$

$$\Rightarrow \quad a = g\sin\alpha(\cos\phi - 1) \qquad (\because\ \mu = \tan\alpha) \qquad \ldots(1)$$

Similarly, if a_x be the acceleration of disc along x-direction, then

$$a_x = g\sin\alpha - (\mu g\cos\alpha)\cos\phi$$

$$\Rightarrow \quad a_x = g\sin\alpha(1-\cos\phi) \qquad \ldots(2)$$



Adding equations (1) and (2), we get

$$a + a_x = 0$$

Integrating, we get

$$v + v_x = c$$

Here, c is a constant and $v_x = v\cos\phi$

$$\Rightarrow \quad v + v\cos\phi = c \qquad \ldots(3)$$

Since, at $\phi_0 = \dfrac{\pi}{2}$, we have $v = v_0$

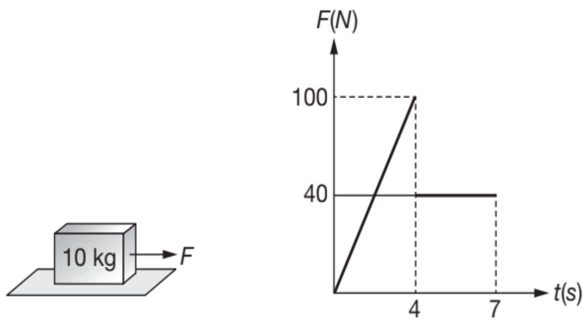
$$\Rightarrow \quad c = v_0$$

Substituting in equation (3), we get

$$v = \frac{v_0}{1+\cos\phi}$$

PROBLEM 8

The 10 kg block is resting on the horizontal surface when the force F is applied to it for 7 s. The variation of F with time is shown. Calculate the maximum velocity reached by the block and the total time t during which the block is in motion. The coefficients of static and kinetic friction are both 0.5. Take $g = 10\ \text{ms}^{-2}$.



SOLUTION

For $0 < t < 4$ s, the force applied on the block increases as a function of time and is given by

$$F_{app} = 25t \qquad \ldots(1)$$

The block will start moving when applied force equals the limiting value of friction. So, we get, at time t, (when the block starts moving)

$$F = \mu mg$$

$$\Rightarrow \quad 25t_1 = (0.5)(10)(10) = 50 \text{ N}$$

$$\Rightarrow \quad t_1 = 2 \text{ s}$$

Velocity is maximum at the end of 4 s

$$\Rightarrow \quad \frac{dv}{dt} = \frac{F - \mu mg}{m} = \frac{25t - 50}{10} = \frac{5}{2}t - 5$$

$$\Rightarrow \quad \int_0^{v_{max}} dv = \int_2^4 \left(\frac{5}{2}t - 5\right)dt$$

$$\Rightarrow \quad v_{max} = \left(\frac{5}{4}t^2 - 5t\right)\Bigg|_2^4 = \frac{5}{4}(4^2 - 2^2) - 5(4-2)$$

$$\Rightarrow \quad v_{max} = 15 - 10 = 5 \text{ ms}^{-2}$$

For $4s < t < 7s$

Net retardation is

$$a_1 = \frac{50-40}{10} = 1 \text{ ms}^{-2}$$

$$\Rightarrow \quad v = v_{max} - a_1 t_1 = 5 - (1)(3) = 2 \text{ ms}^{-1}$$

For $t > 7s$

Retardation $a_2 = \frac{50}{10} = 5 \text{ ms}^{-2}$

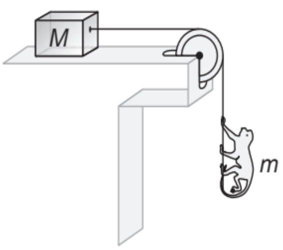
$$\Rightarrow \quad t = \frac{v}{a_2} = \frac{2}{5} = 0.4 \text{ s}$$

So, total time $T = (4-2) + (7-4) + (0.4)$

$$\Rightarrow \quad T = 5.4 \text{ s}$$

PROBLEM 9

A monkey of mass m clings to a rope slung over a fixed pulley. The opposite end of the rope is tied to a weight of mass M lying on a horizontal plate. The coefficient of friction between the weight and the plate is μ. Find the acceleration of weight and the tension of the rope for three cases.



(a) the monkey does not move with respect to the rope

(b) the monkey moves upwards with respect to the rope with an acceleration b.

(c) the monkey moves downwards with respect to the rope with an acceleration b.

SOLUTION

(a) Since monkey does not move w.r.t. rope so, the acceleration of block (or rope) and of monkey are equal. Hence,

$$T - \mu Mg = Ma \quad \ldots(1)$$

$$mg - T = ma \quad \ldots(2)$$

Solving equations (1) and (2), we get

$$a = \left(\frac{m - \mu M}{m + M}\right)g \text{ and } T = \frac{mMg(1+\mu)}{M+m}$$

(b) Monkey moves upwards w.r.t. rope with an acceleration b. Thus, its absolute acceleration downwards is $(a-b)$, where a is the acceleration of block (or rope). The equations of motion are,

$$T - \mu Mg = Ma \quad \ldots(3)$$

$$mg - T = m(a-b) \quad \ldots(4)$$

Solving equations (3) and (4), we get

$$a = \frac{m(g+b) - \mu Mg}{M+m}$$

and $T = \frac{mM[g(1+\mu)+b]}{M+m}$

(c) The absolute acceleration is $(a+b)$, so in this case equations of motion are

$$T - \mu Mg = Ma \quad \ldots(5)$$

$$mg - T = m(a+b) \quad \ldots(6)$$

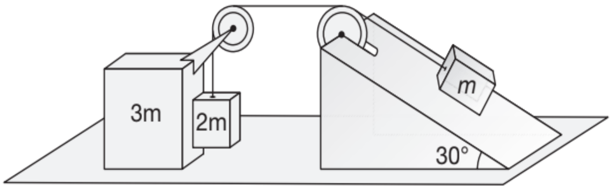
Solving equations (5) and (6), we get

$$a = \frac{m(g-b) - \mu Mg}{m+M}$$

and $T = \frac{mM[g(1+\mu)-b]}{m+M}$

PROBLEM 10

Find accelerations of m, $2m$ and $3m$ as shown in the figure. The wedge is fixed and friction is absent for all the contact surfaces.



SOLUTION

Drawing free body diagrams for $3m$, $2m$ and m and writing equations of motion, we get

$$T - N = 3ma_1 \quad \ldots(1)$$

$$N = 2ma_1 \quad \ldots(2)$$

$$2mg - T = 2ma_2 \quad \ldots(3)$$

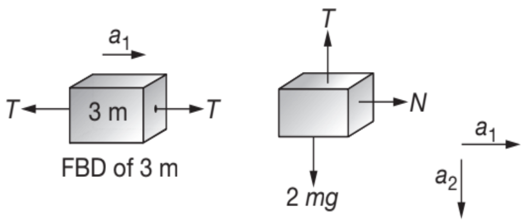
$$T - \frac{mg}{2} = ma_3 \quad \ldots(4)$$

Also from constraint equation, we have

$$a_2 = a_1 + a_3 \quad \ldots(5)$$

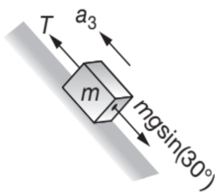
Solving the above five equations we get

$$a_1 = \frac{3}{17}g,\ a_2 = \frac{19}{34}g \text{ and } a_3 = \frac{13}{34}g$$



FBD of 3 m

FBD of 2 m



FBD of m

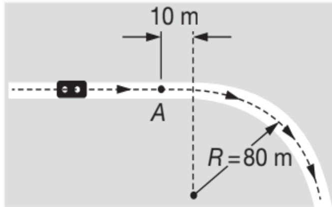
Acceleration of m is $a_3 = \frac{13}{34}g$

Acceleration of $2m$ is $\sqrt{a_1^2 + a_2^2} = \frac{\sqrt{397}}{34}g$ and

Acceleration of $3m$ is $a_1 = \frac{3}{17}g$

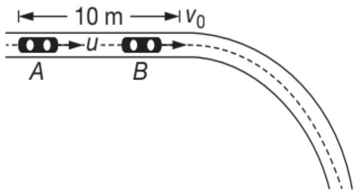
PROBLEM 11

Each tyre on the 1350 kg car can support a maximum friction force parallel to the road surface of 2500 N. This force limit is nearly constant over all possible rectilinear and curvilinear car motions and is attainable only if the car does not skid. Under this maximum braking, determine the total stopping distance s if the brakes are first applied at point A when the car speed is 25 ms^{-1} and if the car follows the center line of the road.



SOLUTION

Total force of friction available on four tyres is $f = 10,000$ N



So, $a_{max} = \frac{10,000}{1350} = 7.4 \text{ ms}^{-2}$

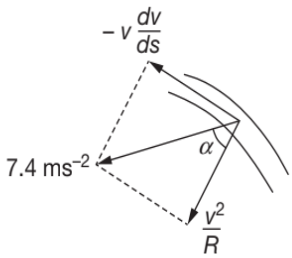
For straight track from A to B, we have

$$v_0^2 = (25)^2 - (2 \times 7.4 \times 10)$$

$$\Rightarrow \quad v_0 = 21.84 \text{ ms}^{-1}$$

For circular track, we have

$$a_{max} \cos\alpha = \frac{v^2}{R}$$



$$\Rightarrow \quad \frac{v^2}{80} = 7.4 \cos\alpha$$

$$\Rightarrow \quad \cos\alpha = \frac{v^2}{592}$$

Also, for circular path,

$$a_T = -a_{max} \sin\alpha$$

$$\Rightarrow \quad v\frac{dv}{ds} = -7.4\sin\alpha$$

$$\Rightarrow \quad v\frac{dv}{ds} = -7.4\sqrt{1-\frac{v^4}{(592)^2}} \quad \left\{\because \sin\alpha = \sqrt{1-\cos^2\alpha}\right\}$$

$$\Rightarrow \quad -\int_{v_0}^{0}\frac{vdv}{7.4\sqrt{1-\frac{v^4}{(592)^2}}} = \int_0^s ds\,,$$

where $v_0 = 21.84\ \text{ms}^{-1}$

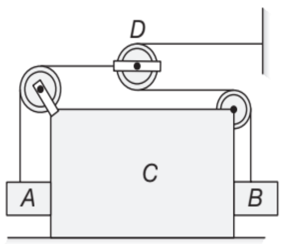
Solving this equation, we get

$s = 37.4$ m

Hence, total stopping distance is $10 + 37.4 = 47.4$ m

PROBLEM 12

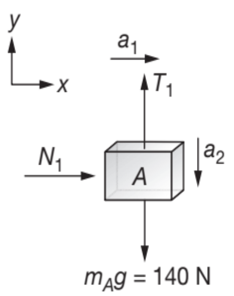
In the arrangement shown in figure, pulleys are light and smooth. Strings are light and inextensible. The masses of blocks A, B and C are 14 kg, 11 kg and 52 kg respectively. The block A can slide freely along a vertical rail fixed to left vertical face of block C. Assuming all the surfaces to be smooth, calculate magnitude of resultant acceleration of each of the blocks A, B and C. $\left(g = 10\ \text{ms}^{-2}\right)$.



SOLUTION

Let the block C have an acceleration a_1 in horizontal direction towards right. Then horizontal acceleration of A and B will be a_1.

Further, let the acceleration of block A, vertically downwards, be a_2 and the acceleration of block B, vertically upwards, be a_3.



Let, T_1 = tension in the string attached to A

T_2 = tension in the string attached to B

N_1 = force between A and C

and N_2 = normal reaction between B and C

Drawing free body diagram of A and then using Newtons Second Law, we get

$\Sigma F_x = ma_x$ and $\Sigma F_y = ma_y$

$\Rightarrow \quad N_1 = 14a_1 \qquad \ldots(1)$

$\Rightarrow \quad 140 - T_1 = 14a_2 \qquad \ldots(2)$

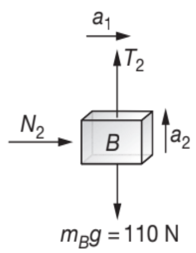
Drawing free body diagram of B and again applying Newton's Second Law, i.e., $\Sigma F_x = ma_x$ and $\Sigma F_y = ma_y$, we get

$N_2 = 11a_1 \qquad \ldots(3)$

$T_2 - 110 = 11a_3 \qquad \ldots(4)$

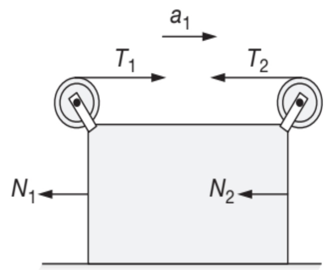
Since the pulley is light, so we get

$2T_2 = T_1 \qquad \ldots(5)$



Drawing the free body diagram of block C (showing only horizontal forces) and using $\Sigma F_x = ma_x$, we get

$T_1 - T_2 - N_1 - N_2 = 52a_1 \qquad \ldots(6)$



Further, by constraint relations, we can show that

$a_3 = 2a_2 \qquad \ldots(7)$

Now, we have seven unknowns, T_1, T_2, N_1, N_2, a_1, a_2 and a_3 to solve from seven equations. On solving, we get

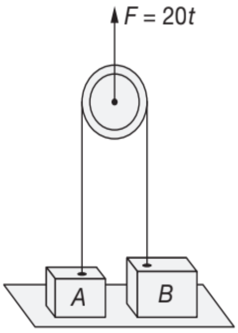
$a_1 = 1.03\ \text{ms}^{-2}, a_2 = -1.38\ \text{ms}^{-2}, a_3 = -2.76\ \text{ms}^{-2}$

Thus, acceleration of different blocks A, B and C will be as follows:

Block	Horizontal acceleration	Vertical acceleration	Resultant acceleration
A	$a_1 = 1.03\ \text{ms}^{-2}$	$1.38\ \text{ms}^{-2}\uparrow$	$1.72\ \text{ms}^{-2}$
B	$a_1 = 1.03\ \text{ms}^{-2}$	$2.76\ \text{ms}^{-2}\downarrow$	$2.95\ \text{ms}^{-2}$
C	$a_1 = 1.03\ \text{ms}^{-2}$	0	$1.03\ \text{ms}^{-2}$

PROBLEM 13

Two blocks A and B of mass 1 kg and 2 kg respectively are connected by a string, passing over a light frictionless pulley. Both the blocks are resting on a horizontal floor and the pulley is held such that string remains just taut. At the moment $t = 0$, a force $F = 20t$ newton starts acting on the pulley along vertically upward direction as shown in figure. Calculate



(a) velocity of A when B loses contact with the floor.

(b) height raised by the pulley upto that instant.

Take $g = 10\ \text{ms}^{-2}$.

SOLUTION

(a) If T be the tension in the string, then

$$F = 2T = 20t$$

$$\Rightarrow \quad T = 10t \text{ newton}$$

Let the block A lose its contact with the floor at time $t = t_1$ (say). This happens when the tension in string becomes equal to the weight of block A. So,

$$T = mg$$

$$\Rightarrow \quad 10t_1 = 1 \times 10$$

$$\Rightarrow \quad t_1 = 1\ \text{s} \qquad \ldots(1)$$

Similarly, for block B, we have

$$10t_2 = 2 \times 10$$

$$\Rightarrow \quad t_2 = 2\ \text{s} \qquad \ldots(2)$$

i.e., the block B loses contact with the floor after $t_2 = 2$ s.

For block A, at time t such that $t \geq t_1$ let a be its acceleration in upward direction. Then

$$10t - (1)(10) = (1)(a) = \left(\frac{dv}{dt}\right)$$

$$\Rightarrow \quad dv = 10(t-1)dt \qquad \ldots(3)$$

Integrating, we get

$$\int_0^v dv = 10\int_1^t (t-1)dt$$

$$\Rightarrow \quad v = 5t^2 - 10t + 5 \qquad \ldots(4)$$

Substituting $t = t_2 = 2$ s, we get

$$v = 20 - 20 + 5 = 5\ \text{ms}^{-1} \qquad \ldots(5)$$

(b) If y is the vertical displacement of block A at time $t(\geq t_1)$, then from equation (4), we have

$$dy = (5t^2 - 10t + 5)dt \qquad \ldots(6)$$

Integrating, we get

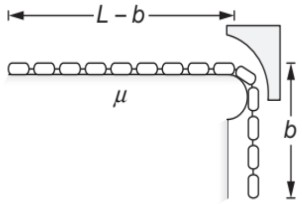
$$\int_{y=0}^{y=h} dy = \int_{t=1}^{t=2} (5t^2 - 10t + 5)dt$$

$$\Rightarrow \quad h = 5\left(\frac{t^3}{3}\right)\Bigg|_1^2 - 10\left(\frac{t^2}{2}\right)\Bigg|_1^2 + 5(t)\Bigg|_1^2 = \frac{5}{3}\ \text{m}$$

So, the height raised by pulley upto that instant is $\frac{h}{2} = \frac{5}{6}$ m

PROBLEM 14

The chain is released from rest with the length b of overhanging links just sufficient to initiate motion. The coefficients of static and kinetic friction between the links and the horizontal surface have essentially the same value μ. Determine the velocity v of the chain when the last link leaves the edge neglect any friction at the corner.



SOLUTION

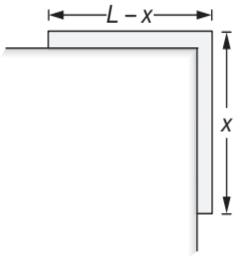
For equilibrium, we have

$$\begin{pmatrix}\text{Weight of the}\\ \text{hanging part}\end{pmatrix} = \begin{pmatrix}\text{Force of friction on the}\\ \text{part lying on the table}\end{pmatrix}$$

So, $m\left(\dfrac{b}{L}\right)g = \mu m\left(\dfrac{L-b}{L}\right)g$

$$\Rightarrow \quad \mu = \frac{b}{L-b} \qquad \ldots(1)$$

$$\Rightarrow \quad b = \left(\frac{\mu L}{1+\mu}\right)$$



Let the length of the chain hanging at any instant be x and if a is the acceleration of the chain, then we have

$$a = \frac{\left(m\dfrac{x}{L} - \mu m\dfrac{(L-x)}{L}\right)g}{m}$$

$$\Rightarrow \quad v\frac{dv}{dx} = \frac{g}{L}\left(x - \mu(L-x)\right) = \frac{g}{L}\left((1+\mu)x - \mu L\right)$$

$$\Rightarrow \quad \int_0^v v\,dv = \frac{g}{L}\int_b^L \left((1+\mu)x - \mu L\right)dx$$

$$\Rightarrow \quad \frac{v^2}{2} = \frac{g}{L}\left((1+\mu)\frac{x^2}{2} - \mu L x\right)\Bigg|_b^L$$

$$\Rightarrow \quad \frac{v^2}{2} = \frac{g}{L}\left((1+\mu)\frac{L^2}{2} - \mu L^2 - (1+\mu)\frac{b^2}{2} - \mu L b\right)$$

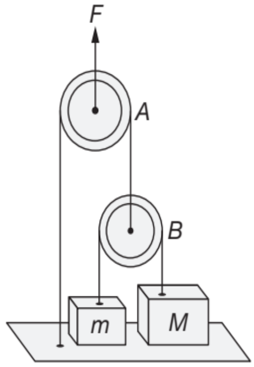
$$\Rightarrow \quad v^2 = \frac{g}{L}\left(L^2 + \mu L^2 - 2\mu L^2 - b^2 - \mu b^2 - 2\mu L b\right)$$

Substituting the value of b from equation (1), we get

$$v = \sqrt{\frac{gL}{1+\mu}}$$

PROBLEM 15

Two blocks of mass $m = 5$ kg and $M = 10$ kg are connected by a string passing over a pulley B as shown. Another string connects the centre of pulley B to the floor and passes over another pulley A as shown. An upward force F is applied at the centre of pulley A. Both the pulleys are massless.



Find the acceleration of blocks m and M if F is:

(a) 100 N

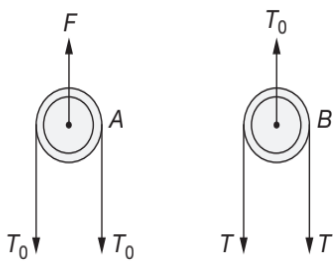
(b) 300 N

(c) 500 N $\left(\text{Take } g = 10\ \text{ms}^{-2}\right)$

SOLUTION

Let T_0 = tension in the string passing over A and

T = tension in the string passing over B



From free body diagram of pulleys A and B, we get

$$2T_0 = F \text{ and } 2T = T_0$$

$$\Rightarrow \quad T = \frac{F}{4}$$

(a) $T = \dfrac{F}{4} = 25$ N

The weights of blocks are

$$mg = 50 \text{ N and } Mg = 100 \text{ N}$$

As T happens to be less than mg and Mg both so, the blocks will remain stationary on the floor.

(b) $T = \frac{F}{4} = 75 \text{ N}$

As $T < Mg$ and $T > mg$, M will remain stationary on the floor, whereas m will move.

Acceleration of m is given by

$$a = \frac{T - mg}{m} = \frac{75 - 50}{5} = 5 \text{ ms}^{-2}$$

(c) $T = \frac{F}{4} = 125 \text{ N}$

As T now happens to be greater than mg and Mg so, both the blocks will accelerate upwards.

Acceleration of m is given by

$$a_1 = \frac{T - mg}{m} = \frac{125 - 50}{5} = 15 \text{ ms}^{-2}$$

Acceleration of M is given by

$$a_2 = \frac{T - Mg}{M} = \frac{125 - 100}{10} = 2.5 \text{ ms}^{-2}$$

PROBLEM 16

A body of mass m is thrown straight up with velocity v_0. Find the velocity v' with which the body comes down if the air drag equals kv^2, where k is a constant and v is the velocity of the body.

SOLUTION

When the body is projected upwards

Net acceleration $a_1 = -\left(g + \frac{kv^2}{m}\right)$

$$\Rightarrow \quad v\frac{dv}{ds} = -\left(g + \frac{kv^2}{m}\right)$$

$$\Rightarrow \quad \left(\frac{vdv}{g + \frac{kv^2}{m}}\right) = -ds$$

Integrating this expression, we get

$$\int_{v_0}^{0} \frac{vdv}{\left(g + \frac{kv^2}{m}\right)} = -\int_0^h ds \qquad \ldots(1)$$

where h is the maximum height attained by the body.

Integrating, we get

$$\Rightarrow \quad h = \frac{m}{2k}\ln\left(1 + \frac{kv_0^2}{mg}\right) \qquad \ldots(2)$$

When the body falls downwards

In this case, net acceleration $a_2 = g - \frac{kv^2}{m}$

$$\Rightarrow \quad \int_0^{v'} \frac{vdv}{\left(g - \frac{kv^2}{m}\right)} = \int_0^h ds$$

Integrating, we get

$$h = -\frac{m}{2k}\ln\left(1 - \frac{kv'^2}{mg}\right) \qquad \ldots(3)$$

Equating equations (2) and (3), we get

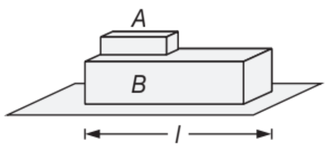
$$v' = \frac{v_0}{\sqrt{1 + \left(\frac{kv_0^2}{mg}\right)}}$$

PROBLEM 17

Figure shows a small block A of mass m kept at the left end of a plank B of mass $M = 2m$ and length l. The system can slide on a horizontal road. The system is started towards right with the initial velocity v. The friction coefficients between the road and the plank is $\frac{1}{2}$ and that between the plank and the block is $\frac{1}{4}$. Find

(a) the time elapsed before the block separates from the plank.

(b) displacement of block and plank relative to ground till that moment.



SOLUTION

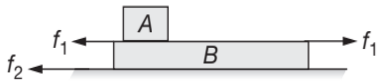
There will be relative motion between block and plank and plank and road. So at each surface limiting friction will act. The direction of friction forces at different surfaces are as shown in figure.

Here, $f_1 = \left(\frac{1}{4}\right)(mg)$

and $f_2 = \left(\frac{1}{2}\right)(m + 2m)g = \left(\frac{3}{2}\right)mg$

Retardation of A is

$$a_1 = \frac{f_1}{m} = \frac{g}{4}$$



and retardation of B is

$$a_2 = \frac{f_2 - f_1}{2m} = \frac{5}{8}g$$

Since, $a_2 > a_1$

Relative acceleration of A with respect to B is

$$a_r = a_2 - a_1 = \frac{3}{8}g$$

Initial velocity of both A and B is v. So there is no relative initial velocity. Hence,

(a) Applying $s = \frac{1}{2}at^2$

$$\Rightarrow \quad l = \frac{1}{2}a_r t^2 = \frac{3}{16}gt^2$$

$$\Rightarrow \quad t = 4\sqrt{\frac{l}{3g}}$$

(b) Displacement of block is given by

$$s_A = u_A t - \frac{1}{2}a_A t^2 = 4v\sqrt{\frac{l}{3g}} - \frac{1}{2}\left(\frac{g}{4}\right)\left(\frac{16l}{3g}\right)$$

$$\left\{a_A = a_1 = \frac{g}{4}\right\}$$

$$\Rightarrow \quad s_A = 4v\sqrt{\frac{l}{3g}} - \frac{2}{3}l$$

Displacement of plank is given by

$$s_B = u_B t - \frac{1}{2}a_B t^2 = 4v\sqrt{\frac{l}{3g}} - \frac{1}{2}\left(\frac{5}{8}g\right)\left(\frac{16l}{3g}\right)$$

$$\left\{a_B = a_2 = \frac{5}{8}g\right\}$$

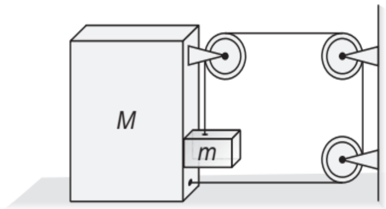
$$\Rightarrow \quad s_B = 4v\sqrt{\frac{l}{3g}} - \frac{5}{3}l$$

Here we observe that $s_A - s_B = l$.

This is a quite obvious result because the block A has moved a distance l relative to plank.

PROBLEM 18

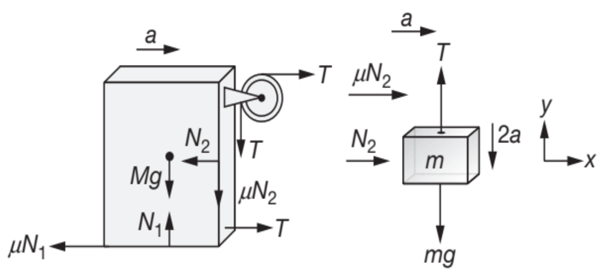
In the arrangement shown, $M = 2m$ and the coefficient of friction between all surfaces in contact is μ. Assuming the string to be light and inextensible and all the pulleys to be smooth find the acceleration of the bigger block M.



SOLUTION

From our knowledge of constraint relations we get that acceleration of block of mass m in vertical direction is two times the acceleration of block of mass M and in the horizontal direction, the acceleration of both blocks is equal.

So let a be the acceleration of block M in horizontal direction (towards right). Then acceleration of block m will be a in horizontal direction and $2a$ in vertical direction. Free body diagrams of both the blocks are shown in figure:



Let T be the tension in the string, N_1 be the normal reaction between bigger block and ground and N_2 be the normal reaction between both the blocks.

Using $\Sigma F_x = ma_x$ and $\Sigma F_y = ma_y$,

for block of mass $M(= 2m)$, we get

$$N_1 - \mu N_2 - 2mg - T = 0$$

$$\Rightarrow \quad N_1 = T + 2mg + \mu N_2 \quad \ldots(1)$$

and $\quad T + T - N_2 - \mu N_1 = 2ma$

$$\Rightarrow \quad 2T - N_2 - \mu N_1 = 2ma \quad \ldots(2)$$

for block of mass m, we get

$$N_2 = ma \quad \ldots(3)$$

and $mg - T - \mu N_2 = 2ma \quad \ldots(4)$

Solving these four equations for the four unknowns i.e., T, N_1, N_2 and a, we get

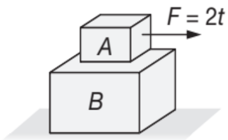
$$a = \left(\frac{2-3\mu}{7}\right)g$$

Hence, acceleration of the bigger block of mass M is

$$a = \left(\frac{2-3\mu}{7}\right)g.$$

PROBLEM 19

Two blocks A and B of mass 2 kg and 4 kg are placed one over the other as shown in figure. A time varying horizontal force $F = 2t$ is applied on the upper block as shown in figure. Here t is in second and F is in newton. Draw a graph showing accelerations of A and B on y-axis and time on x-axis. Coefficient of friction between A and B is $\mu = \frac{1}{2}$ and the horizontal surface over which B is placed is smooth. $\left(g = 10\ \text{ms}^{-2}\right)$.



SOLUTION

Limiting friction between A and B is

$$f_L = \mu m_A g = \left(\frac{1}{2}\right)(2)(10) = 10\ \text{N}$$

Block B moves due to friction only. Therefore, maximum acceleration of B can be

$$a_{max} = \frac{f_L}{m_B} = \frac{10}{4} = 2.5\ \text{ms}^{-2}$$

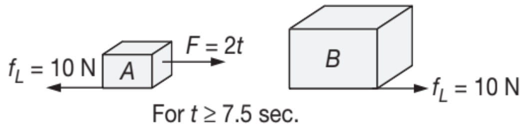
Thus, both the blocks move together with same acceleration till the common acceleration becomes $2.5\ \text{ms}^{-2}$, after that acceleration of B will become constant while that of A will go on increasing. To find the time when the acceleration of both the blocks becomes $2.5\ \text{ms}^{-2}$ (or when slipping will start between A and B) we will write:

$$2.5 = \frac{F}{(m_A + m_B)} = \frac{2t}{6}$$

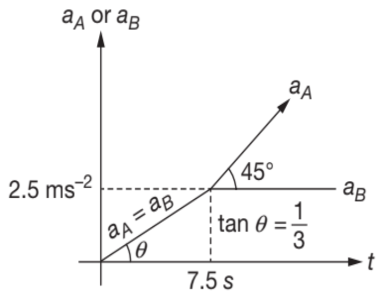
$$\Rightarrow \quad t = 7.5\ \text{s}$$

Hence, for $t \le 7.5$ s

$$a_A = a_B = \frac{F}{m_A + m_B} = \frac{2t}{6} = \frac{t}{3}$$



Thus, a_A versus t or a_B versus t graph is a straight line passing through origin of slope $\frac{1}{3}$.



For, $t \ge 7.5$ s

$$a_B = 2.5\ \text{ms}^{-2} = \text{constant}$$

and $a_A = \frac{F - f_L}{m_A}$

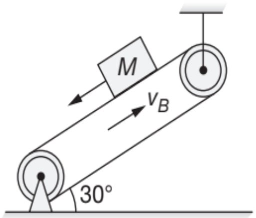
$$\Rightarrow \quad a_A = \frac{2t - 10}{2}$$

$$\Rightarrow \quad a_A = t - 5$$

Thus, a_A versus t graph is a straight line of slope 1 and intercept -5. While a_B versus t graph is a straight line parallel to t axis. The corresponding graph is as shown in figure.

PROBLEM 20

A conveyor belt is inclined at 30° with the horizontal. A block of mass $M = 1$ kg is kept on the belt as shown in figure. The frictional force, in newton, between the block and belt depends upon the relative speed of the body with respect to belt as $f = 0.4 v_{rel}$. The belt moves up at a constant speed vB while initially the block has a speed $v_M = 2\ \text{ms}^{-1}$ relative to ground in a direction down the belt.



(a) Find the speed of the belt in order for the block to come permanently at rest relative to ground.

(b) Using the speed of belt calculated in part (a) find the time when block M attains a speed $1\ \text{ms}^{-1}$ relative to ground $\left(g = 10\ \text{ms}^{-1}\right)$.

SOLUTION

(a) In equilibrium (in this case, permanently at rest), the net force on block should be zero.

Hence, $Mg\sin\theta = f$

$$\Rightarrow \quad (1)(10)\left(\frac{1}{2}\right) = 0.4(v_B + 0)$$

$$\Rightarrow \quad v_B = 12.5\ \text{ms}^{-1}$$

(b) Let at time t velocity of block be $v_M\ \text{ms}^{-1}$

$$\Rightarrow \quad M\frac{dv_M}{dt} = Mg\sin\theta - 0.4(v_B + v_M)$$

$$\Rightarrow \quad \frac{dv_M}{dt} = 5 - 0.4(v_M + 12.5) = -0.4v_M$$

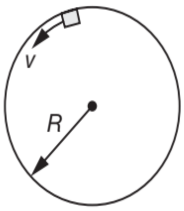
$$\Rightarrow \quad \int_2^1 \frac{dv_M}{v_M} = -0.4\int_0^t dt$$

$$\Rightarrow \quad 0.4t = \ln(2)$$

$$\Rightarrow \quad t = 2.5\ln(2) = 1.73\ \text{s}$$

PROBLEM 21

The small collar of mass m is given an initial velocity of magnitude v_0 on the horizontal circular track fabricated from a slender rod. If the coefficient of kinetic friction is μ_k, determine the distance travelled before the collar comes to rest.



SOLUTION

Net normal force acting on the collar is

$$N = m\sqrt{\left(\frac{v^2}{r}\right)^2 + g^2}$$

Since the frictional force depends upon the net normal force, so we have

$$f = \mu_K N = m\mu_K\sqrt{\left(\frac{v^2}{r}\right)^2 + g^2}$$

Now, $a_t = -\dfrac{f}{m} = -\mu_K\sqrt{\left(\dfrac{v^2}{r}\right)^2 + g^2}$

$$\Rightarrow \quad v\frac{dv}{ds} = -\mu_K\sqrt{\left(\frac{v^2}{r}\right)^2 + g^2}$$

$$\Rightarrow \quad -\int_{v_0}^{0}\frac{v\,dv}{\sqrt{v^4 + r^2g^2}} = \frac{\mu_k}{r}\int_0^s ds \qquad \ldots(1)$$

Substitute $v^2 = y$

$$\Rightarrow \quad 2v\,dv = dy$$

$$\Rightarrow \quad v\,dv = \frac{dy}{2}$$

$$\Rightarrow \quad \text{Integral } I = \frac{1}{2}\int\frac{dy}{\sqrt{y^2 + r^2g^2}}$$

Since $\displaystyle\int\frac{dx}{\sqrt{a^2 + x^2}} = \log_e\left(x + \sqrt{x^2 + a^2}\right)$

$$\Rightarrow \quad I = \frac{1}{2}\log_e\left(y + \sqrt{y^2 + r^2g^2}\right)$$

$$\Rightarrow \quad I = \frac{1}{2}\log_e\left(v^2 + \sqrt{v^4 + r^2g^2}\right) \qquad \{\because\ y = v^2\}$$

Substituting in (1), we get

$$-\frac{1}{2}\log\left(v^2 + \sqrt{v^4 + r^2g^2}\right)\Bigg|_{v_0}^{0} = \frac{\mu_k s}{r}$$

$$\Rightarrow \quad -\frac{1}{2}\left[\log_e(rg) - \log_e\left(v_0^2 + \sqrt{v_0^4 + r^2g^2}\right)\right] = \frac{\mu_k s}{r}$$

$$\Rightarrow \quad s = \frac{r}{2\mu_k}\log_e\left(\frac{v_0^2 + \sqrt{v_0^4 + r^2g^2}}{rg}\right)$$