CHAPTER 5

Kinematics II

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

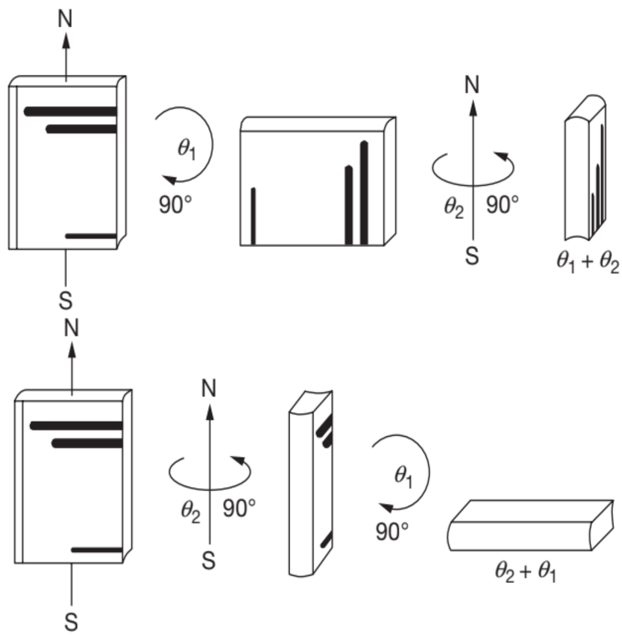
(a) Curvilinear Motion and Angular Parameters
(b) Angular, Centripetal and Tangential Acceleration
(c) Kinematics of Circular Motion
(d) Motion of Particle in a Curved Track
(e) Radius of Curvature
(f) Horizontal Projectile and Properties
(g) Oblique Projectile and Properties
(h) Relative Motion between two Projectiles
(i) Condition of Collision between two Projectiles
(j) Motion of Projectile Up and Down an Inclined Plane

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

CURVILINEAR MOTION

INTRODUCTION

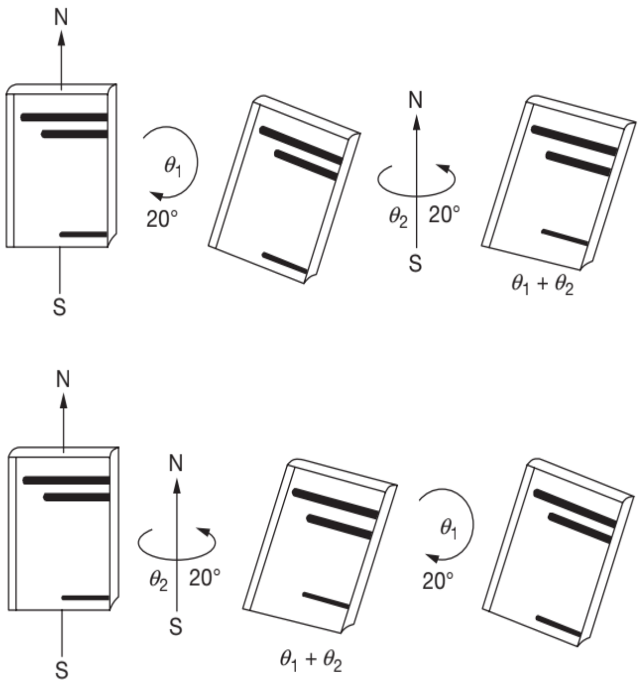
The linear displacement $(\vec{x})$, velocity $(\vec{v})$ and acceleration $(\vec{a})$ are vectors. Correspondingly, the angular counterparts may also be vectors, because in addition to the magnitude, we must also specify a direction for them, namely, the direction of **Axis of Rotation** in space. Let us first have our discussion for θ, which commonly is called Angular Displacement but is not. It should be called as angle traversed or just finite angle, because it is **not a vector**. So, **finite angle** θ **is not a vector**, because it does not add like vectors, as Illustrated by the successive rotation technique for the book, where we observe that $(\text{Rotation})_1 + (\text{Rotation})_2 \neq (\text{Rotation})_2 + (\text{Rotation})_1$.



A book rotated θ_1 (90° as shown about an axis at right angles to the page) and then θ_2 (90° as shown about a north-south axis) has a different final orientation that if rotated first through θ_2 and then θ_1. This property is called the non-commutivity of finite angles under addition: $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$.

This non-commutative property associated with finite rotation provides us a solid platform for calling it a non-vector. In the first case, we rotate the book through two successive rotations of $\frac{\pi}{2}$ each and we observe that $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$.

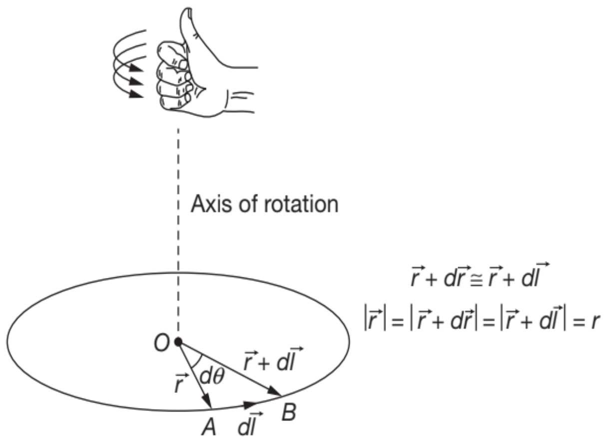
In the other case, if we rotate the book through successive rotations of $20°$, then too $\theta_1 + \theta_2$ would still differ from $\theta_2 + \theta_1$, but the difference would be smaller. As a matter of fact, as the two rotations are made smaller and smaller, the difference between the two sums disappears rapidly. So, if the infinitesimal angles are taken then we observe that the order of addition no longer affects the results. Hence **infinitesimal angular displacements** are **vector quantities**.



The lower group repeats the experiment for 20° displacements. We see here that $\theta_1 + \theta_2 \cong \theta_2 + \theta_1$. Also we observe that when θ_1, $\theta_2 \to 0$, the final positions approach each other. Finite angles under addition tend to commute as the angles become very small. Infinitesimal angles do commute under addition, making it possible to treat them as vectors.

ANGULAR DISPLACEMENT, ANGULAR VELOCITY, ANGULAR AND CENTRIPETAL ACCELERATION

From the above discussion, we have observed that, the finite angle is not a vector, so **cannot be called as Angular Displacement**. However, infinitesimal angle $d\theta$ is a vector and hence is **called as Angular Displacement.**



Consider a particle moving in a circle of radius r. Let the particle go from point A to B such that Arc $AB = dl$. If $d\theta$ is the angle subtended by the arc at the centre of the circle, then $dl = rd\theta$. Since $d\theta$ is very small, so we can also say

$$\overset{\frown}{AB} = d\vec{l} = d\vec{r}$$

So, in triangle OAB, we can have

$$\vec{r} + d\vec{r} = \vec{r} + d\vec{l}$$

Since, both $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the radius of the same circle, so

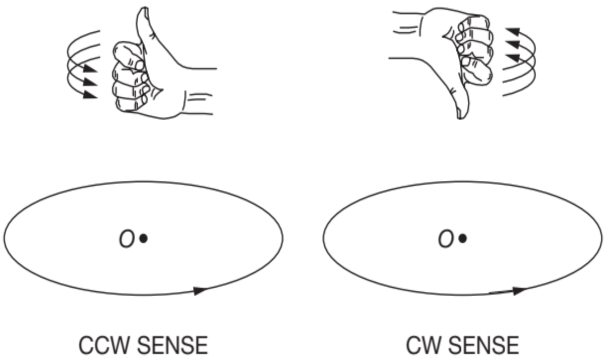
$$|\overrightarrow{OA}| = |\overrightarrow{OB}| = r$$

$$|\vec{r}| = |\vec{r} + d\vec{r}| = |\vec{r} + d\vec{l}| = r \quad \text{...(1)}$$

The direction of $d\vec{\theta}$ is given by Right Hand Thumb Rule, according to which, "curl the fingers of Right Hand in the sense of rotation, then thumb gives the direction of $d\vec{\theta}$ (angular displacement), $\vec{\omega}$ (angular velocity), $\vec{\alpha}$ (angular acceleration), $\vec{\tau}$ (torque) and $\vec{L}$ (angular momentum)".

So, in magnitude we have

$$dl = rd\theta$$



Direction of $d\vec{\theta}$, $\vec{\omega}$, $\vec{\alpha}$, $\vec{\tau}$ and $\vec{L}$ given by THUMB
But vectorially, we observe that

$$d\vec{l} = d\vec{\theta} \times \vec{r} \qquad \ldots(2)$$

Angle between $d\vec{\theta}$ and $\vec{r}$ is $\frac{\pi}{2}$, so

$$\left|d\vec{\ell}\right| = d\ell = r\,d\theta \qquad \left\{\because \sin\left(\frac{\pi}{2}\right) = 1\right\}$$

Dividing both sides of (1) by dt, we get

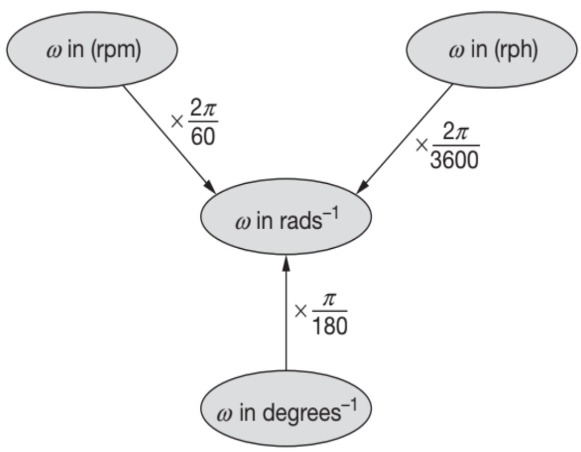
$$\frac{d\vec{l}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}$$

{Please note, it is dividing, not differentiating}

$$\Rightarrow \quad \vec{v} = \vec{\omega} \times \vec{r} \qquad \ldots(3)$$

where $\vec{\omega} = \frac{d\vec{\theta}}{dt}$ = Angular Velocity of the particle

$\vec{v} = \frac{d\vec{l}}{dt} = \frac{d\vec{r}}{dt}$ = Velocity of the particle



$$\omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

In magnitude,

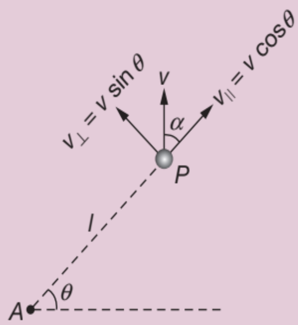
$$v = r\omega \qquad \ldots(4)$$

The $\vec{\omega}$, angular velocity of the particle is always to be measured in SI units of rads^{-1}. Generally, ω can also be expressed as rpm (revolution per minute), rph (revolution per hour), degrees^{-1}.

Conceptual Note(s)

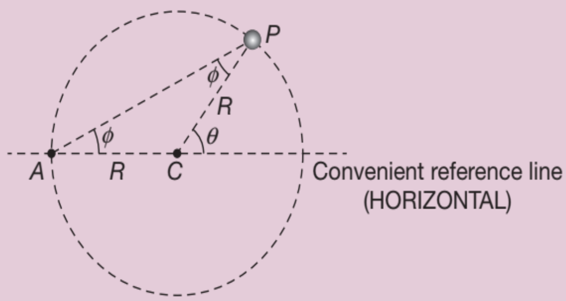
(a) Angular velocity is always to be defined with respect to the point from where the position vector of the point is drawn. Consider a particle having a velocity v, as shown. Let us calculate ω of the point P w.r.t. another arbitrarily chosen point A (say) or with respect to the observer at A is

$$\omega = \frac{v_\perp}{l} = \frac{v\sin\alpha}{l} = \frac{d\theta}{dt}$$



where θ is the angle made by AB at any instant with a convenient reference line (generally horizontal).

(b) Furthermore, the angular velocity of a particle can be different about different points. For that, let us consider a particle moving on the circumference of the circle, as shown.



Then the angular velocity of point P w.r.t. the observer at the centre of the circle C is

$$\omega_{PC} = \frac{d\theta}{dt}$$

Similarly, the angular velocity of the point P w.r.t. observer at the circumference of circle at A is

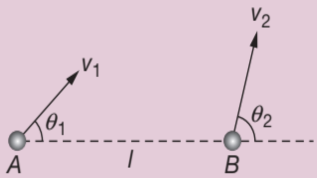
$$\omega_{PA} = \frac{d\phi}{dt}$$

In triangle ACP, since $\theta = 2\phi$

$$\Rightarrow \quad \omega_{PA} = \frac{d}{dt}\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(\frac{d\theta}{dt}\right) = \frac{1}{2}\omega_{PC}$$

$$\Rightarrow \quad \omega_{PC} = 2\omega_{PA}$$

(c) If two particles A and B are moving with velocities $\vec{v}_1$ and $\vec{v}_2$ in the directions shown in figure. Then $\frac{dl}{dt}$ = rate of change of distance between the two particles



$\frac{dl}{dt}$ = relative velocity between them along AB

$$\Rightarrow \quad \frac{dl}{dt} = v_2\cos\theta_2 - v_1\cos\theta_1$$

and $\omega_r = \omega_{BA}$ = angular velocity of B with respect to A or angular velocity of line AB, then

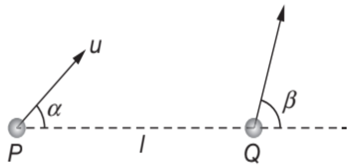
$$\omega_r = \omega_{BA} = \frac{|v_2\sin\theta_2 - v_1\sin\theta_1|}{l}$$

Magnitude of the relative angular velocity between them can also be given by

$$\omega_r = \frac{\text{Relative velocity perpendicular to } AB}{\text{Distance between them}}$$

ILLUSTRATION 1

The line joining P and Q is of constant length l and the velocities of P and Q are in directions inclined at angles α and β respectively with PQ. If u is the velocity of P, what is the angular speed of PQ?



SOLUTION

Let v be the velocity of Q. Then $u\cos\alpha = v\cos\beta$

Since l = constant

$$\Rightarrow \quad \frac{dl}{dt} = 0$$

$$\Rightarrow \quad v = \frac{u\cos\alpha}{\cos\beta}$$

Now angular speed of line PQ is given by

$$\omega = \frac{\text{Relative speed } \perp \text{ to } PQ}{l}$$

$$\Rightarrow \quad \omega = \frac{v\sin\beta - u\sin\alpha}{l} = \frac{\left(\frac{u\cos\alpha}{\cos\beta}\right)\sin\beta - u\sin\alpha}{l}$$

$$\Rightarrow \quad \omega = \frac{u(\tan\beta\cos\alpha - \sin\alpha)}{l}$$

ILLUSTRATION 2

Two points are moving with uniform velocities u and v along the perpendicular axes, OX and OY. The motion is directed toward O, the origin. When $t = 0$, they are at a distance a and b respectively from O.

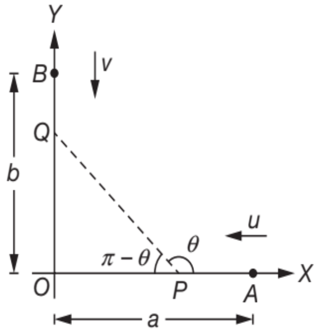
(a) Calculate the angular velocity of the line joining them at time t.

(b) Show that it is greatest, when $t = \frac{au + bv}{u^2 + v^2}$

SOLUTION

(a) At $t = 0$, the particles are at A and B respectively.

Let at time t, the particles are at P and Q respectively.



From figure, $OP = a - ut$ ...(1)

and $OQ = b - vt$ ...(2)

Let θ be the angle which the line PQ makes with the direction OX at time t. From ΔQOP.

$$\tan(\pi - \theta) = \frac{OQ}{OP} = \frac{b - vt}{a - ut}$$

$$\Rightarrow \quad -\tan\theta = \frac{b - vt}{a - ut}$$

$$\Rightarrow \quad \tan\theta = \left(\frac{b - vt}{ut - a}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{b - vt}{ut - a}\right) \quad \ldots(3)$$

Differentiating equation (3), we get

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{b - vt}{ut - a}\right)^2} \times \frac{(ut - a)(-v) - (b - vt)u}{(ut - a)^2}$$

$$\Rightarrow \quad \omega = \frac{d\theta}{dt} = \frac{av - bu}{(a - ut)^2 + (b - vt)^2}$$

(b) ω will be maximum when denominator is minimum, because numerator is constant.

Let, $(\text{Den})^r = S = (a - ut)^2 + (b - vt)^2$. For it to be MINIMUM, we have

$$\frac{dS}{dt} = 2(a - ut)(-u) + 2(b - vt)(-v) = 0$$

$$\Rightarrow \quad t = \frac{au + bv}{u^2 + v^2}$$

It can be seen that $\frac{d^2S}{dt^2} > 0$. Hence, S is minimum, or ω is maximum, at

$$t = \frac{au + bv}{u^2 + v^2}$$

ANGULAR, CENTRIPETAL, TANGENTIAL AND TOTAL ACCELERATION

Since we have now understood the concept of $\vec{\omega}$ so, let us now discuss about angular acceleration (α). Angular acceleration is the rate of change of angular velocity. Mathematically

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

The direction of $\vec{\alpha}$ is again given by Right Hand Thumb Rule, as discussed earlier.
If $\vec{a}$ be the acceleration of the particle, then

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Since $\vec{v} = \vec{\omega} \times \vec{r}$

$$\Rightarrow \quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

Using $\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$

$$\Rightarrow \quad \vec{a} = \frac{d\vec{v}}{dt} = \vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

$$\Rightarrow \quad \vec{a} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r} \quad \ldots(1)$$

The term $\vec{\omega} \times \vec{v}$ is the centripetal or radial acceleration of the particle and $\vec{\alpha} \times \vec{r}$ is the tangential acceleration of the particle. So,

$$\vec{a}_C = \vec{\omega} \times \vec{v} \quad \ldots(2)$$

$$\vec{a}_T = \vec{\alpha} \times \vec{r} \quad \ldots(3)$$

In magnitude, $a_C = v\omega$ and $a_T = r\alpha$

$$\Rightarrow \quad a_C = v\omega = r\omega^2 = \frac{v^2}{r} \quad \ldots(4)$$

We could have prove this vectorially, by using

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Substituting in (2), we get

$$\vec{a}_C = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Using $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$, we get

$$\vec{a}_C = (\vec{\omega} \cdot \vec{r})\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\vec{r}$$

Now, since $\vec{\omega} \perp \vec{r}$, so $\vec{\omega} \cdot \vec{r} = 0$

$$\Rightarrow \quad \vec{a}_C = 0 - \omega^2\vec{r} \qquad \{\because \vec{\omega} \cdot \vec{\omega} = \omega^2\}$$

$$\Rightarrow \quad \vec{a}_C = -\omega^2 \vec{r} \qquad \ldots(5)$$

$$\Rightarrow \quad |\vec{a}_C| = a_C = r\omega^2 \qquad \{\because \ |\vec{r}| = r\}$$

Please note that from (5), we conclude that the centripetal acceleration is directed radially inwards (see the negative sign justifying this).

Also, if we had been asked to give the expressions for centripetal force and tangential force, then we would have just multiplied the above expressions with the mass of the particle, say m, to get

$$\vec{F}_c = m\vec{a}_c = m(\vec{\omega}\times\vec{v}) = -\omega^2\vec{r} \text{ and } \vec{F}_T = m\vec{a}_T = m(\vec{\alpha}\times\vec{r})$$

Problem Solving Technique(s)

(a) For motion with uniform angular acceleration, we can use the following equations

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

$$\theta = \left(\frac{\omega+\omega_0}{2}\right)t$$

where, θ is the angle traversed (in radian) in time t (second)

ω_0 is the initial angular velocity (in rads^{-1}),

ω is the final angular velocity (in rads^{-1}) at time t and

α is the angular acceleration (in rads^{-2}).

(b) If α is not a constant, analyse the problem and feel free to use

$$\omega = \frac{d\theta}{dt} \text{ or } \alpha = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta}$$

(c) The SI unit of θ is radian, ω is rads^{-1} and α is rads^{-2}.

(d) $\omega = \frac{2\pi}{T} = 2\pi f$, where T is the period of revolution and f is the frequency and $f = \frac{1}{T}$.

(e) Here too, the graphs interpret analogous to their fellows in Linear Kinematics, like

Slope in the θ-t graph gives **angular velocity** (ω).

Slope in the ω-t graph gives **angular acceleration** (α).

Area under a curve in ω-t graph gives the **angle traversed** (θ).

ILLUSTRATION 3

A solid body rotates with deceleration about a stationary axis with an angular deceleration $|\alpha| = k\sqrt{\omega}$, where ω is its angular velocity and k is a positive constant. Find the mean angular velocity of the body averaged over the whole time of rotation if at the initial moment of time its angular velocity was equal to ω_0.

SOLUTION

$$|\alpha| = k\sqrt{\omega}$$

$$\Rightarrow \quad -\frac{d\omega}{dt} = k\sqrt{\omega}$$

$$\Rightarrow \quad \int_{\omega_0}^{\omega}\frac{d\omega}{\sqrt{\omega}} = -\int_0^t k\,dt$$

$$\Rightarrow \quad \left(2\sqrt{\omega}\right)\Big|_{\omega_0}^{\omega} = -kt$$

$$\Rightarrow \quad \sqrt{\omega_0} - \sqrt{\omega} = \frac{kt}{2}$$

$$\Rightarrow \quad \omega = \left(\sqrt{\omega_0} - \frac{kt}{2}\right)^2 \qquad \ldots(1)$$

The body will stop when

$$\sqrt{\omega_0} - \frac{kt}{2} = 0$$

$$\Rightarrow \quad t = \frac{2\sqrt{\omega_0}}{k}$$

Now average angular velocity over this time interval is

$$\langle\omega\rangle = \frac{\displaystyle\int_0^{\frac{2\sqrt{\omega_0}}{k}} \omega\,dt}{\displaystyle\int_0^{\frac{2\sqrt{\omega_0}}{k}} dt} = \frac{\displaystyle\int_0^{\frac{2\sqrt{\omega_0}}{k}} \left(\sqrt{\omega_0} - \frac{kt}{2}\right)^2 dt}{\displaystyle\int_0^{\frac{2\sqrt{\omega_0}}{k}} dt} = \frac{\omega_0}{3}$$

ILLUSTRATION 4

The speed (v) of a particle moving in a circle of radius R varies with distance s as $v = ks$, where k is a positive constant. Calculate the total acceleration of the particle.

SOLUTION

$$v = ks$$

Since the particle is moving in a circle, so total acceleration, a is

$$a = \sqrt{a_C^2 + a_T^2}$$

where $a_C = \frac{v^2}{R}$ and $a_T = \frac{dv}{dt}$

$$\Rightarrow \quad a_C = \frac{k^2s^2}{R} \text{ and } a_T = \frac{d}{dt}(ks) = k\left(\frac{ds}{dt}\right) = kv$$

$$\Rightarrow \quad a_C = \frac{k^2s^2}{R} \text{ and } a_T = k(ks) = k^2s \quad \left\{\because \frac{ds}{dt} = v = ks\right\}$$

$$\Rightarrow \quad a = \sqrt{\frac{k^4s^4}{R^2} + k^4s^2} = k^2s\sqrt{1 + \frac{s^2}{R^2}}$$

ILLUSTRATION 5

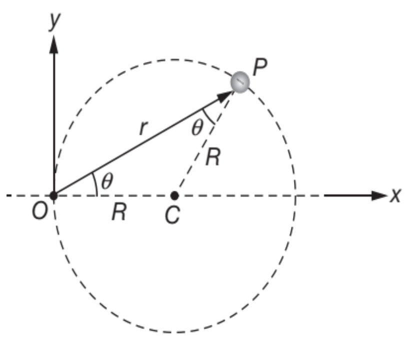
A particle P moves along a circle of radius R so that its radius vector $\vec{r}$, relative to the point O at the circumference rotates with constant angular velocity ω. Find the magnitude of the velocity of the particle and the direction of its total acceleration.

SOLUTION

In triangle OPC, we get from Lami's Theorem

$$\frac{r}{\sin(\pi - 2\theta)} = \frac{R}{\sin\theta}$$

$$\Rightarrow \quad \frac{r}{2\sin\theta\cos\theta} = \frac{R}{\sin\theta}$$



$$\Rightarrow \quad r = 2R\cos\theta \quad \ldots(1)$$

Further we observe that

$$\vec{r} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$\Rightarrow \quad \vec{r} = (2R\cos^2\theta)\hat{i} + (2R\sin\theta\cos\theta)\hat{j}$$

Since $\vec{v} = \frac{d\vec{r}}{dt}$

$$\Rightarrow \quad \vec{v} = -\left(4R\cos\theta\sin\theta\frac{d\theta}{dt}\right)\hat{i} + \left(2R\cos(2\theta)\frac{d\theta}{dt}\right)\hat{j}$$

Since $\frac{d\theta}{dt} = \omega$

$$\Rightarrow \quad \vec{v} = -2R\omega\left[-\sin(2\theta)\hat{i} + \cos(2\theta)\hat{j}\right] \quad \ldots(2)$$

$$\Rightarrow \quad |\vec{v}| = 2R\omega$$

Further, we know that $\vec{a} = \frac{d\vec{v}}{dt}$

$$\Rightarrow \quad \vec{a} = 4R\omega^2\left(\cos(2\theta)\hat{i} + \sin(2\theta)\hat{j}\right)$$

$$\Rightarrow \quad |\vec{a}| = 4R\omega^2$$

UNIT VECTORS ALONG THE RADIUS ($\hat{r}$) AND THE TANGENT ($\hat{t}$)

Consider a particle P moving in a circle of radius r centred at origin O. The angular position of the particle at some instant is say θ. Let us here define two unit vectors, one is $\hat{r}$ (called radial unit vector) which is along OP and the other is $\hat{t}$ (called the tangential unit vector) which is perpendicular to OP. Now, since

$$|\hat{r}| = |\hat{t}| = 1$$

We can write these two vectors as

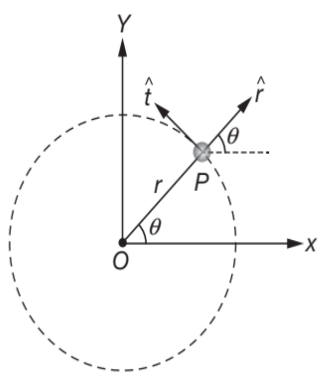
$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \text{ and } \hat{t} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

VELOCITY AND ACCELERATION OF PARTICLE IN CIRCULAR MOTION

The position vector of particle P at the instant shown in figure can be written as

$$\vec{r} = \overrightarrow{OP} = r\hat{r}$$

$$\Rightarrow \quad \vec{r} = r\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$



The velocity of the particle can be obtained by differentiating $\vec{r}$ with respect to time t So, we get

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{d}{dt}\left[r\left(\cos\theta\hat{i}+\sin\theta\hat{j}\right)\right]$$

$$\vec{v}=r\left[\left(-\sin\theta\frac{d\theta}{dt}\right)\hat{i}+\left(\cos\theta\frac{d\theta}{dt}\right)\hat{j}\right]$$

$$\Rightarrow\quad \vec{v}=r\omega\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)\qquad\left\{\because\frac{d\theta}{dt}=\omega\right\}\qquad\ldots(1)$$

$$\Rightarrow\quad \vec{v}=(r\omega)\hat{t}$$

Thus, we observe that velocity of the particle is $r\omega$ along $\hat{t}$ or in tangential direction. Acceleration of the particle can be obtained by differentiating equation (1) with respect to time t. So, we get

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left[r\omega\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)\right]$$

$$\Rightarrow\quad \vec{a}=r\left[\omega\frac{d}{dt}\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)+\left(\frac{d\omega}{dt}\right)\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)\right]$$

$$\Rightarrow\quad \vec{a}=r\omega\left[-\cos\theta\left(\frac{d\theta}{dt}\right)\hat{i}-\sin\theta\left(\frac{d\theta}{dt}\right)\hat{j}\right]+r\left(\frac{d\omega}{dt}\right)\hat{t}$$

$$\Rightarrow\quad \vec{a}=-r\omega^2\left[\cos\theta\hat{i}+\sin\theta\hat{j}\right]+r\left(\frac{d\omega}{dt}\right)\hat{t}$$

$$\Rightarrow\quad \vec{a}=-r\omega^2\hat{r}+\frac{dv}{dt}\hat{t}\qquad\ldots(2)$$

Thus, acceleration of a particle moving in a circle has two components one is along $\hat{t}$ (along tangent) and the other along $-\hat{r}$ (or towards centre). In equation (2), the first term is called the tangential acceleration. (a_t) and the second term is called radial or centripetal acceleration (a_r). Thus,

$$a_t=\frac{dv}{dt}=\text{rate of change of speed}$$

and $a_r=a_c=r\omega^2=\dfrac{v^2}{r}=v\omega\qquad\{\because v=r\omega\}$

Here, the two components are mutually perpendicular. Therefore, net acceleration of the particle will be

$$a=\sqrt{a_r^2+a_t^2}=\sqrt{a_c^2+a_T^2}$$

$$a=\sqrt{\left(r\omega^2\right)^2+\left(\frac{dv}{dt}\right)^2}=\sqrt{\left(\frac{v^2}{r}\right)^2+\left(\frac{dv}{dt}\right)^2}$$

Conceptual Note(s)

(a) In uniform circular motion, speed (v) of the particle is constant, i.e., $\frac{dv}{dt}=0$. Thus

$$a_t=0 \text{ and } a=a_r=r\omega^2$$

(b) In accelerated circular motion, $\frac{dv}{dt}=$ positive, i.e., a_t is along $\hat{t}$ or tangential acceleration of particle is parallel to velocity $\vec{v}$ because $\vec{v}=r\omega\hat{t}$ and $\vec{a}_t=\frac{dv}{dt}\hat{t}$.

(c) In decelerated circular motion, $\frac{dv}{dt}=$ negative and hence, tangential acceleration is anti-parallel to velocity $\vec{v}$.

ILLUSTRATION 6

The speed of a particle moving in a circle of radius $r=2$ m varies with time t as $v=t^2$, where t is in second and v in ms^{-1}. Find the radial, tangential and net acceleration at $t=2$ s.

SOLUTION

Linear speed of particle at $t=2$ s is $v=(2)^2=4\ ms^{-1}$

Radial acceleration $a_r=\dfrac{v^2}{r}=\dfrac{(4)^2}{2}=8\ ms^{-2}$

The tangential acceleration is

$$a_t=\frac{dv}{dt}=2t$$

So, tangential acceleration at $t=2$ s is

$$a_t=(2)(2)=4\ ms^{-2}$$

The net acceleration of particle at $t=2$ s is

$$a=\sqrt{(a_C)^2+(a_T)^2}=\sqrt{(8)^2+(4)^2}$$

$$\Rightarrow\quad a=\sqrt{80}\ ms^{-2}$$

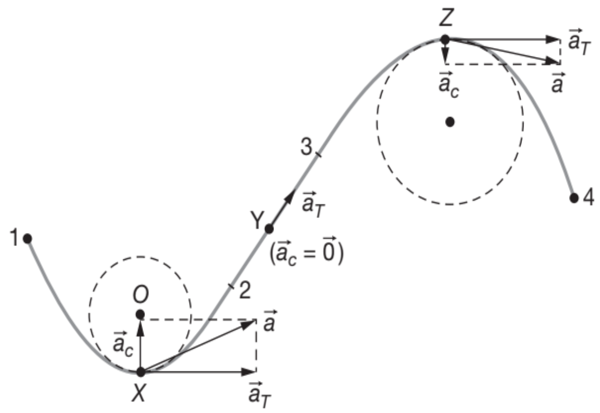
KINEMATICS OF MOTION OF PARTICLE IN A CURVED TRACK

Consider a curved track, 1234, having portions 12, 23 and 34 on which points X, Y and Z are taken respectively. The particle moving in a curved track has following accelerations.

(a) Centripetal acceleration $\vec{a}_C$, acting radially inwards.

(b) Tangential acceleration $\vec{a}_T$, acting tangentially.

So, at the points X and Z, on the curved track the particle has two accelerations $\vec{a}_C$ and $\vec{a}_T$.



At the point Y, $r \to \infty$, so $\vec{a}_C \to 0$, hence from 2 to 3, the particle just follows a straight track 23 under the influence of a single force $\vec{a}_T$.
So, we can say that for a particle moving in a curved track, net acceleration is given by

$$a = \sqrt{a_C^2 + a_T^2} \qquad \{\because \vec{a}_C \perp \vec{a}_T\}$$

$$\Rightarrow \quad a = \sqrt{(r\omega^2)^2 + (r\alpha)^2}$$

$$\Rightarrow \quad a = r\sqrt{\omega^4 + \alpha^2}$$

RADIUS OF CURVATURE

As observed, any curved track/path can be assumed to be made of a large number of circular arcs of variable radii. The radius of curvature at a point is the radius of the circular arc that suitably fits on the curve at that point.

Since $a_C = \dfrac{v^2}{r}$ where v is the tangential velocity and can be denoted by v_T.

$$\Rightarrow \quad a_C = \frac{v_T^2}{r}$$

$$\Rightarrow \quad r = \frac{v_T^2}{a_C} = \frac{v_\perp^2}{a_P}$$

where $v_T = v_\perp$ i.e., tangential velocity equals the velocity of the particle **perpendicular to the radius** of the curve and hence v_T can also be denoted by $v_\perp$ (read as v perpendicular) and $a_C = a_P$ i.e., centripetal acceleration equals the acceleration of the particle **parallel to the radius** of the curve and hence a_C can also be denoted by a_P.

Conceptual Note(s)

(a) On any curved path (not necessarily a circular one) the acceleration of the particle has two components a_T and a_N in two mutually perpendicular directions. Component of $\vec{a}$ along $\vec{v}$ is a_T and perpendicular to $\vec{v}$ is a_N or a_C. So, we have

$$|\vec{a}| = \sqrt{a_T^2 + a_C^2} = \sqrt{a_x^2 + a_y^2} \quad \{\text{For 2D motion}\}$$

where a_T = rate of change of speed $= \dfrac{dv}{dt} = \dfrac{d|\vec{v}|}{dt}$

$$\Rightarrow \quad a_T = \frac{d}{dt}\left(\sqrt{v_x^2 + v_y^2}\right) = \frac{v_x \dfrac{dv_x}{dt} + v_y \dfrac{dv_y}{dt}}{\sqrt{v_x^2 + v_y^2}}$$

$$\Rightarrow \quad a_T = \frac{v_x a_x + v_y a_y}{\sqrt{v_x^2 + v_y^2}} = \frac{\vec{a} \cdot \vec{v}}{v}$$

$\Rightarrow$ a_T = component of $\vec{a}$ along $\vec{v}$

(b) If a and a_T are known, a_C can easily be found using the relation

$$a = \sqrt{a_C^2 + a_T^2}$$

$$\Rightarrow \quad a_C = \sqrt{a^2 - a_T^2} = \sqrt{a_x^2 + a_y^2 - a_T^2}$$

$$\Rightarrow \quad a_C = \sqrt{\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2 - \left(\frac{dv}{dt}\right)^2}$$

ILLUSTRATION 7

A balloon starts rising from the earth's surface. The ascension rate is constant and equal to v_0. Due to the wind, the balloon gathers the horizontal velocity component $v_x = ky$, where k is a constant and y is the height of ascent. Find the dependence of the following quantities on y.

(a) The horizontal drift of the balloon $x(y)$.
(b) The total tangential acceleration of the balloon.
(c) The normal acceleration of the balloon.

SOLUTION

(a) Since the balloon is ascending at a constant rate. So,

$$v_0 = \frac{dy}{dt}$$

$$\Rightarrow \quad dy = v_0 dt$$

$$\Rightarrow \quad y = v_0 t$$

Also, we have

$$v_x = \frac{dx}{dt} = ky$$

$$\Rightarrow \quad dx = kydt = kv_0 t\,dt \qquad \{\because\ y = v_0 t\}$$

Integrating, we get

$$x = kv_0\left(\frac{t^2}{2}\right)$$

$$\Rightarrow \quad x = k\frac{v_0}{2}\left(\frac{y}{v_0}\right)^2$$

$$\Rightarrow \quad x = \frac{1}{2}\frac{ky^2}{v_0}$$

This is the trajectory of the particle which is a parabola.

(b) For finding the tangential and normal accelerations, we require an expression for the velocity as a function of height y. So, we have

$$v_y = v_0 \text{ and } v_x = ky$$

$$\Rightarrow \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + k^2y^2}$$

Therefore tangential acceleration,

$$a_T = \frac{dv}{dt} = \frac{k^2y}{\sqrt{v_0^2 + k^2y^2}}\frac{dy}{dt} = \frac{k^2yv_0}{\sqrt{v_0^2 + k^2y^2}}$$

$$\Rightarrow \quad a_T = \frac{k^2y}{\sqrt{1 + \dfrac{k^2y^2}{v_0^2}}}$$

Now, the total acceleration a is, given by

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{dv_y}{dt}\right)^2 + \left(\frac{dv_x}{dt}\right)^2}$$

$$\Rightarrow \quad a = \frac{dv_x}{dt} = k\frac{dy}{dt} = kv_0 \qquad \left\{\because\ \frac{dv_y}{dt} = 0\right\}$$

(c) To find the normal acceleration, since we know that

$$a^2 = a_N^2 + a_T^2$$

$$\Rightarrow \quad a_N = \sqrt{a^2 - a_T^2} = \frac{kv_0}{\sqrt{1 + \left(\dfrac{ky}{v_0}\right)^2}}$$

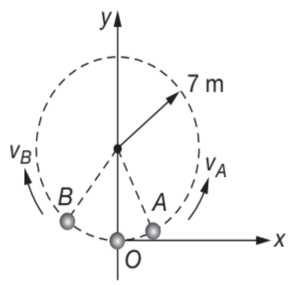
Test Your Concepts-I

Based on Curvilinear Motion

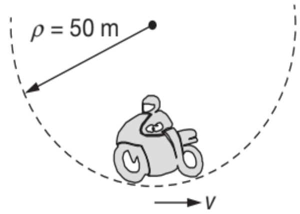
(Solutions on page H.143)

1. The speed of a particle moving in a plane is equal to the magnitude of its instantaneous velocity, $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$. Show that the rate of change of speed can be expressed as $\dfrac{dv}{dt} = \dfrac{\vec{v}\cdot\vec{a}}{v}$ and use this result to explain why $\dfrac{dv}{dt}$ is equal to a_T, the component of $\vec{a}$ that is parallel to $\vec{v}$.
2. If a point moves along a circle with constant speed, prove that its angular speed about any point on the circle is half of that about the centre.

3. Calculate the angular speed of a particle which moves in a circle of radius 0.25 m with a linear speed of 2 ms^{-1}.
4. A particle moves in a circle of radius 0.25 m at a speed that uniformly increases. Find the angular acceleration of particle if its speed changes from 2 ms^{-1} to 4 ms^{-1} in 4 s.
5. A car is travelling along a circular curve that has a radius of 50 m. If its speed is 16 ms^{-1} and is increasing uniformly at 8 ms^{-2}, determine the magnitude of its acceleration at this instant.
6. A boy whirls a stone in a horizontal circle of radius 0.5 m and at height 20 m above the level ground. The string breaks, and the stone flies off horizontally to strike the ground after travelling a horizontal distance of 10 m. Calculate the magnitude of the centripetal acceleration of the stone while in circular motion? $\left(\text{Take } g = 10 \text{ ms}^{-2}\right)$
7. Two particles A and B start from the origin O and travel in opposite directions along the circular path at constant speeds $v_A = 0.7$ ms^{-1} and $v_B = 1.5$ ms^{-1}, respectively. Determine the time when they collide and the magnitude of the acceleration of B just before this happens. $\left(\text{Take } \pi = \dfrac{22}{7}\right)$



8. Starting from rest, the motorcycle travels around the circular path of radius 50 m, at a speed $v = \dfrac{t^2}{5}$ ms^{-1}, where t is in seconds. Determine the magnitudes of the motorcycle's velocity and acceleration at the instant $t = 3$ s.



9. Determine the maximum constant speed a race car can have if the acceleration of the car cannot exceed 7.5 ms^{-2} while rounding a track having a radius of curvature of 200 m.
10. An automobile is travelling on a horizontal circular curve having a radius of 800 m. If the acceleration of the automobile is 5 ms^{-2}, determine the constant speed at which the automobile is travelling.
11. A car travels along a horizontal circular curved road that has a radius of 600 m. If the speed is uniformly increased at a rate of 2000 kmh^{-2}, determine the magnitude of the acceleration at the instant the speed of the car is 60 kmh^{-1}.
12. When designing a highway curve it is required that cars travelling at a constant speed of 25 ms^{-1} must not have an acceleration that exceeds 3 ms^{-2}. Determine the minimum radius of curvature of the curve.
13. At a given instant, a car travels along a circular curved road with a speed of 20 ms^{-1} while decreasing its speed at the rate of 3 ms^{-2}. If the magnitude of the car's acceleration is 5 ms^{-2}, determine the radius of curvature of the road.
14. The wheel of a truck moving with velocity 5 ms^{-1} throws up mud particles from its rim. The diameter of the wheel is 3 m. Find the maximum height from ground to which a particle can reach? (Assume no sliding between the wheel and road).

PROJECTILE MOTION

PROJECTILE MOTION: AN INTRODUCTION

A particle launched with an initial velocity in any arbitrary direction and then allowed to move freely under the gravitational influence of the earth $(\vec{g})$ is called a **projectile**.

For studying the projectile motion the following assumptions must be kept in mind.

Assumption 1: The surface of the earth is more or less flat.

Assumption 2: Acceleration due to gravity $(\vec{g})$ has a constant value throughout the entire motion.

Assumption 3: No air drag is present.

EXAMPLES:

(a) A cricket ball thrown by a bowler.

(b) A football kicked by the player.

(c) Food packet dropped from an airplane and many more examples can be thought of as projectiles from everyday life.

Conceptual Note(s)

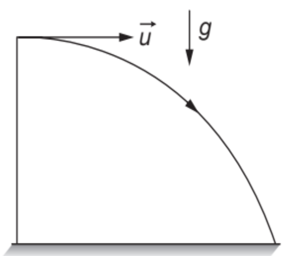
The path followed by a projectile is called its **trajectory**, which is a parabola.

TYPES OF PROJECTILE MOTION

We shall study three kinds of projectile motion.

Horizontal Projectile

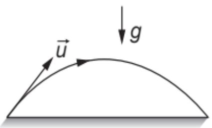
In this, the projectile is given an initial velocity directed along the horizontal and then it moves under the influence of gravity to follow a parabolic path.



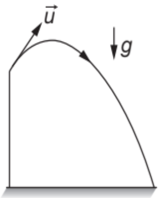
(a)

Oblique Projectile

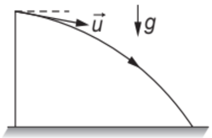
In this, the projectile is given an initial velocity making an angle θ with the horizontal (or the vertical) and it moves under the influence of gravity to follow a parabolic path.



(b)



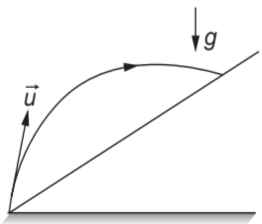
(c)



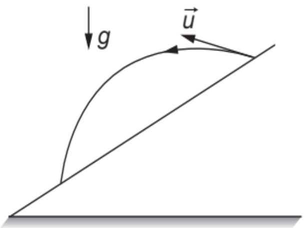
(d)

Projectile on an Inclined Plane

In this, the projectile is given an initial velocity making an angle with the horizontal on an inclined plane and it again moves under the influence of gravity to follow a parabolic path.



(e)



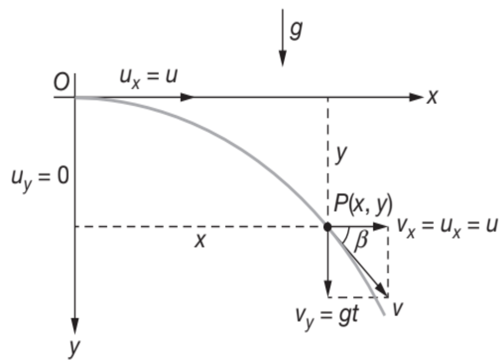
(f)

We now discuss in detail the equation of trajectory, velocity any point P and at particular instant and position.

HORIZONTAL PROJECTILE

Equation of Trajectory

Consider a point $P(x, y)$ at time t.



Horizontal Motion (HM)

Since acceleration due to gravity acts along the vertical and hence, has got no component along the horizontal i.e., $a_x = 0$.

So, horizontal motion is a non-accelerated motion with uniform velocity.

$$\Rightarrow \quad x = u_x t = ut \qquad \ldots(1)$$

Vertical Motion (VM)

Also $a_y = g$ and $u_y = 0$

Since, $y = u_y t + \frac{1}{2} a_y t^2$

$$\Rightarrow \quad y = 0 + \frac{1}{2} g t^2$$

$$\Rightarrow \quad y = \frac{1}{2} g t^2 \qquad \ldots(2)$$

From (1), we get $t = \frac{x}{u}$

Substituting in (2), we get

$$y = \left(\frac{g}{2u^2} \right) x^2$$

which is the equation of a parabola.

Velocity at Any Instant (t)

Horizontal Motion (HM)

Since horizontal motion is non-accelerated,

$$\Rightarrow \quad v_x = u_x = u$$

Vertical Motion (VM)

Also, $v_y = u_y + a_y t$

$$\Rightarrow \quad v_y = 0 + gt$$

$$\Rightarrow \quad v_y = gt$$

Since, we have

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

$$\Rightarrow \quad \vec{v} = u\hat{i} + (gt)\hat{j}$$

$$\Rightarrow \quad |\vec{v}| = v = \sqrt{u^2 + g^2 t^2}$$

If β is the angle made by $\vec{v}$ with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{gt}{u}$$

If h is the distance of the ground from the point of launch, T is the time taken to strike the ground and R is the range of the projectile when it hits the ground, then

$$h = \frac{1}{2} g T^2$$

$$\Rightarrow \quad T = \sqrt{\frac{2h}{g}} \text{ and } R = uT$$

$$\Rightarrow \quad R = u\sqrt{\frac{2h}{g}}$$

Deviation Suffered by a Horizontal Projectile in Time t

If gravity were absent, then the body launched with initial velocity u would continue to move horizontally forever. However in the presence of gravity it suffers a deviation, $\delta = y = \frac{1}{2} g t^2$. So,

$$\text{Deviation } = \delta = y = \frac{1}{2} g t^2 = \left(\frac{g}{2u^2} \right) x^2$$

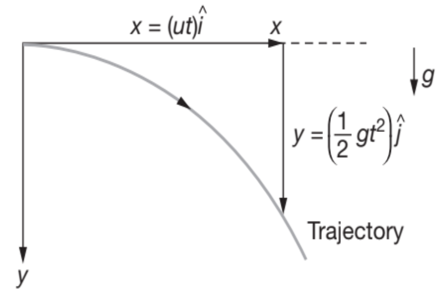


ILLUSTRATION 8

A projectile is fired horizontally with a velocity of 98 ms^{-1} from the top of a hill 490 m high. Find

(a) the time taken by the projectile to reach the ground

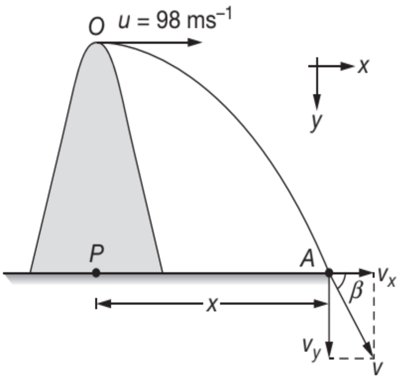
(b) the distance of the target from the hill and

(c) the velocity with which the projectile hits the ground. $\left(g = 9.8 \text{ ms}^{-2}\right)$.

SOLUTION

Here, it will be more convenient to choose x and y directions as shown in figure. So, let the downward direction be positive, then

$$u_x = 98 \text{ ms}^{-1},\ a_x = 0,\ u_y = 0 \text{ and } a_y = g$$



(a) At A, $y = 490$ m. So, applying

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\Rightarrow\quad 490 = 0 + \frac{1}{2}(9.8)t^2$$

$$\Rightarrow\quad t = 10 \text{ s}$$

(b) $PA = x = u_x t + \frac{1}{2}a_x t^2$

$$\Rightarrow\quad PA = (98)(10) + 0$$

$$\Rightarrow\quad PA = 980 \text{ m}$$

(c) $v_x = u_x = 98 \text{ ms}^{-1}$

$$v_y = u_y + a_y t = 0 + (9.8)(10) = 98 \text{ ms}^{-1}$$

$$\Rightarrow\quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{(98)^2 + (98)^2} = 98\sqrt{2} \text{ ms}^{-1}$$

and $\tan\beta = \dfrac{v_y}{v_x} = \dfrac{98}{98} = 1$

$$\Rightarrow\quad \beta = 45^\circ$$

Thus, the projectile hits the ground with a velocity $98\sqrt{2}$ ms^{-1} at an angle of $\beta = 45^\circ$ with horizontal as shown in figure.

ILLUSTRATION 9

Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located at one point and moved with velocities $u_1 = 3 \text{ ms}^{-1}$ and $u_2 = 4 \text{ ms}^{-1}$ horizontally in opposite directions. Find the distance between the particles when their velocity vectors become mutually perpendicular.

SOLUTION

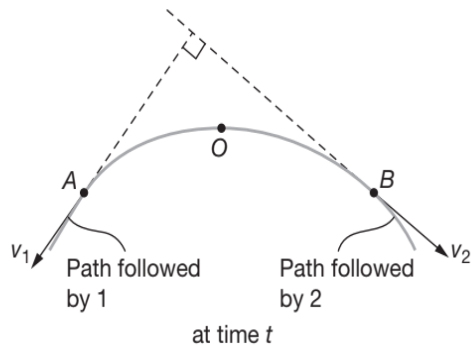
Let the velocities of the particles be at right angles to each other at time t. Then

$$\vec{v}_1 = \vec{u}_1 + \vec{g}t$$

$$\Rightarrow\quad \vec{v}_1 = 3\hat{i} + gt\hat{j}$$

$$\vec{v}_2 = \vec{u}_2 + \vec{g}t$$

$$\Rightarrow\quad \vec{v}_2 = -4\hat{i} + gt\hat{j}$$



For velocities to be mutually perpendicular, we have

$$\vec{v}_1 \cdot \vec{v}_2 = 0$$

$$\Rightarrow\quad -12 + g^2t^2 = 0$$

$$\Rightarrow\quad t = \sqrt{\frac{12}{g^2}} = \frac{2\sqrt{3}}{g} \text{ s}$$

Separation between the particles is

$$x = (u_1 + u_2)t = 7\frac{2\sqrt{3}}{g}$$

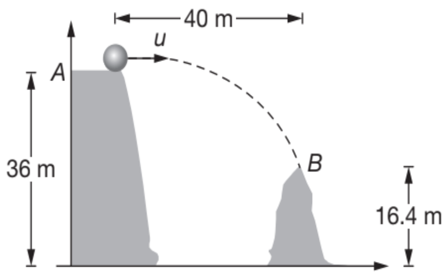
$$\Rightarrow\quad x = \frac{14\sqrt{3}}{g} \text{ m}$$

Test Your Concepts-II

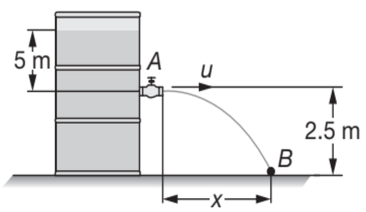
Based on Horizontal Projectile

(Solutions on page H.144)

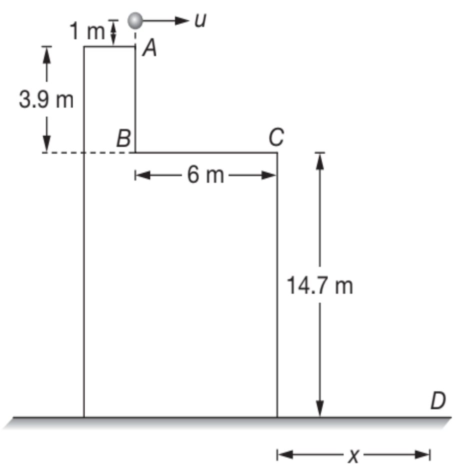
1. A body is thrown horizontally from the top of a tower and strikes the ground after three seconds at an angle of 45° with the horizontal. Find the height of the tower and the speed with which the body was projected. Take $g = 9.8\ \text{ms}^{-2}$.
2. With what minimum horizontal velocity u can a boy throw a ball at A and have it just clear the obstruction at B?



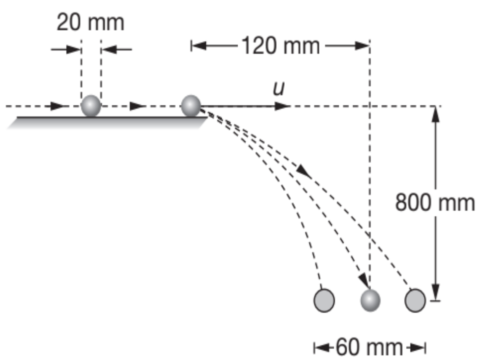
3. The velocity of the water jet discharging from the orifice (small hole in the tank) can be obtained from $u = \sqrt{2gh}$, where $h = 5$ m is the depth of the orifice from the free water surface. Determine the time for a particle of water leaving the orifice to reach point B and the horizontal distance x, where it hits the surface. Take $g = 10\ \text{ms}^{-2}$.



4. Determine the horizontal velocity u of a tennis ball launched from A at height of 2.25 m from the ground so that it just clears the net of height 1 m at B, a horizontal distance of 6.4 m from A. Also, find the horizontal distance from the net, where the ball strikes the ground. Take $g = 10\ \text{ms}^{-2}$.
5. A rock is thrown horizontally from top of a tower and hits the ground 4 s later. The line of sight from top of tower to the point where the rock hits the ground makes an angle of 30° with the horizontal. Calculate the horizontal launch velocity of the rock. $\left(\text{Take } g = 10\ \text{ms}^{-2}\right)$.
6. Calculate the minimum velocity u along the horizontal such that the ball just clears the point C. Assume that the ball is launched by a man who holds the ball at a distance 1 m above A. Also find x, where the ball strikes the ground. Take $g = 9.8\ \text{ms}^{-2}$.



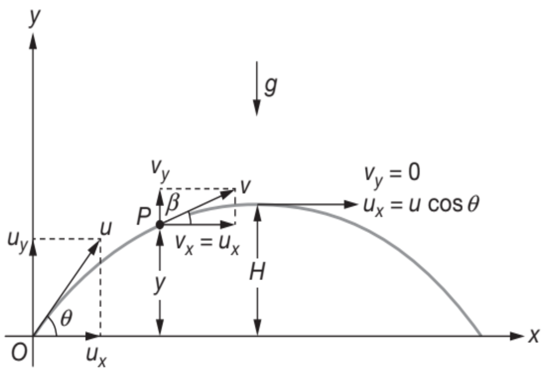
7. Ball bearings of diameter 20 mm leave the horizontal with a velocity of magnitude u and fall through the 60 mm diameter hole at a depth of 800 mm as shown. Calculate the permissible range of u which will enable the ball bearings to enter the hole. Take the dotted positions to represent the limiting conditions. $\left(\text{Take } g = 10\ \text{ms}^{-2}\right)$



OBLIQUE PROJECTILE

Let a projectile be launched with an initial velocity u making an angle θ with horizontal then

$$u_x = u\cos\theta \text{ and } u_y = u\sin\theta$$



Equation of Trajectory

Consider a point $P(x, y)$ at time t.

Horizontal Motion (HM)

Since horizontal motion is non-accelerated

i.e., $a_x = 0$.

Also,

$$x = u_x t + \frac{1}{2}a_x t^2$$

$$\Rightarrow \quad x = (u\cos\theta)t \qquad \ldots(1)$$

Vertical Motion (VM)

$$y = u_y t + \frac{1}{2}a_y t^2$$

If we take upwards as positive, then

$$y = (u\sin\theta)t + \frac{1}{2}(-g)t^2 \qquad \{\because a_y = -g\}$$

$$\Rightarrow \quad y = (u\sin\theta)t - \frac{1}{2}gt^2 \qquad \ldots(2)$$

From (1), $t = \dfrac{x}{u\cos\theta}$ and put in (2), we get

$$y = (u\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\Rightarrow \quad y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \qquad \ldots(3)$$

which is the equation of a parabola

Velocity at Any Instant (*t*)

Horizontal Motion (HM)

Since horizontal motion is non-accelerated. So,

$$v_x = u_x = u\cos\theta \qquad \ldots(4)$$

Vertical Motion (VM)

Since,

$$v_y = u_y + a_y t$$

$$\Rightarrow \quad v_y = u\sin\theta + (-g)t$$

$$\Rightarrow \quad v_y = u\sin\theta - gt \qquad \ldots(5)$$

From figure, we have

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$\Rightarrow \quad \vec{v} = (u\cos\theta)\hat{i} + (u\sin\theta - gt)\hat{j} \qquad \ldots(6)$$

$$\Rightarrow \quad |\vec{v}| = \sqrt{u^2 + g^2t^2 - 2ugt\sin\theta}$$

If β is the angle made by $\vec{v}$ with x-axis, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta} \qquad \ldots(7)$$

Velocity at Any Position at Vertical Height (*h*)

Horizontal Motion (HM)

Since horizontal motion is a non-accelerated motion, so

$$v_x = u_x = u\cos\theta \qquad \ldots(8)$$

Vertical Motion (VM)

$$v_y^2 - u_y^2 = 2a_y y$$

$$\Rightarrow \quad v^2\sin^2\beta = u^2\sin^2\theta + 2(-g)h$$

$$\Rightarrow \quad v_y = \sqrt{u^2\sin^2\theta - 2gh} \qquad \ldots(9)$$

Since,

$$\Rightarrow \quad \vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$\Rightarrow \quad \vec{v} = (u\cos\theta)\hat{i} + \left(\sqrt{u^2\sin^2\theta - 2gh}\right)\hat{j} \qquad \ldots(10)$$

$$\Rightarrow \quad |\vec{v}| = v = \sqrt{u^2 - 2gh}$$

If β is the angle made by $\vec{v}$ with x-axis, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{\sqrt{u^2\sin^2\theta - 2gh}}{u\cos\theta} \qquad \ldots(11)$$

Problem Solving Technique(s)

Projectile motion is a two dimensional motion with constant acceleration (g). So, we can use

$\vec{v}=\vec{u}+\vec{a}t$, $\vec{s}=\vec{u}t+\frac{1}{2}\vec{a}t^2$, etc. in projectile motion as well. Here

$$\vec{u}=u\cos\alpha\hat{i}+u\sin\alpha\hat{j} \text{ and } \vec{a}=-g\hat{j}$$

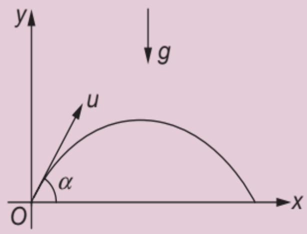
Now, suppose we want to find velocity at time t.

$$\vec{v}=\vec{u}+\vec{a}t=\left(u\cos\alpha\hat{i}+u\sin\alpha\hat{j}\right)-gt\hat{j}$$

$$\Rightarrow \vec{v}=u\cos\alpha\hat{i}+\left(u\sin\alpha-gt\right)\hat{j}$$

Similarly displacement at time t will be,

$$\vec{s}=\vec{u}t+\frac{1}{2}\vec{a}t^2$$



$$\vec{s}=\left(u\cos\alpha\hat{i}+u\sin\alpha\hat{j}\right)t-\frac{1}{2}gt^2\hat{j}$$

$$\vec{s}=ut\cos\alpha\hat{i}+\left(ut\sin\alpha-\frac{1}{2}gt^2\right)\hat{j}$$

Maximum Height (H)

Maximum vertical displacement of projection is called **Maximum Height**. At maximum height $v_y=0$.

Since, $v_y^2-u_y^2=2a_yy$

$$\Rightarrow 0^2-(u\sin\theta)^2=2(-g)H$$

$$\Rightarrow H=\frac{u^2\sin^2\theta}{2g}$$

Time of Flight (T)

Time taken by projectile to strike the ground. At $t=T$, $y=0$ (because projectile returns to the ground). Since

$$y=u_yt+\frac{1}{2}a_yt^2$$

$$\Rightarrow 0=(u\sin\theta)T+\frac{1}{2}(-g)T^2$$

$$\Rightarrow T=\frac{2u\sin\theta}{g} \qquad \{\because T\neq 0\}$$

Further, $T=t_{ascent}+t_{descent}$ and since no air drag exists, so,

$$t_{ascent}=t_{descent}$$

$$\Rightarrow t_{ascent}=t_{descent}=\frac{T}{2}=\frac{u\sin\theta}{g}$$

Range (R)

Maximum horizontal distance travelled is called **Range**.

At $t=T$, $x=R$

Since $x=u_xt+\frac{1}{2}a_xt^2$

and $a_x=0$ {∵ horizontal motion is non-accelerated}

$$\Rightarrow R=(u\cos\theta)T$$

$$\Rightarrow R=\frac{2}{g}(u\cos\theta)(u\sin\theta)$$

$$\Rightarrow R=\frac{2}{g}\begin{pmatrix}\text{Horizontal}\\\text{Component of}\\\text{Initial Velocity}\end{pmatrix}\begin{pmatrix}\text{Vertical}\\\text{Component of}\\\text{Initial Velocity}\end{pmatrix}$$

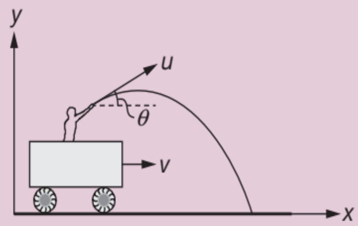
$$\Rightarrow R=\frac{u^2\sin(2\theta)}{g}$$

Problem Solving Technique(s)

When the ball is projected from a body as per the cases discussed.

CASE-1: Body moving in same direction

Consider that a man is sitting in a trolley and the trolley is moving with velocity v in positive x direction as shown. Let the man projects a ball in the direction of trolley.



In this case, we have

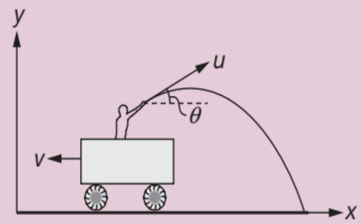
Horizontal component $u_x = u\cos\theta + v$

Vertical component $u_y = u\sin\theta$

$$\Rightarrow \quad R = \frac{2}{g}(u\cos\theta + v)(u\sin\theta)$$

CASE-2: Body moving in opposite direction

Consider that a man is sitting in a trolley and the trolley is moving with velocity v in negative x direction as shown. Let the man projects a ball in the direction of trolley.



In this case, we have

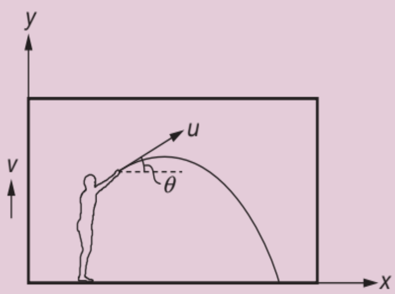
Horizontal component $u_x = u\cos\theta - v$

Vertical component $u_y = u\sin\theta$

$$\Rightarrow \quad R = \frac{2}{g}(u\cos\theta - v)(u\sin\theta)$$

CASE-3: Body moving up

Consider that a man is standing in a lift compartment which is moving up with velocity v as shown. Let the man project a ball in the direction shown.



In this case, we have

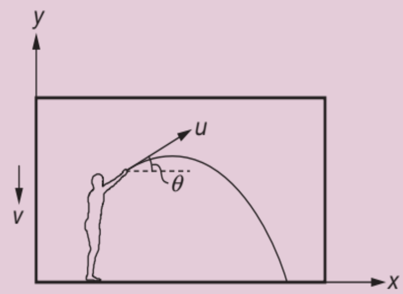
Horizontal component $u_x = u\cos\theta$

Vertical component $u_y = u\sin\theta + v$

$$\Rightarrow \quad R = \frac{2}{g}(u\cos\theta)(u\sin\theta + v)$$

CASE-4: Body moving down

Consider that a man is standing in a lift compartment which is moving down with velocity v as shown. Let the man project a ball in the direction shown.



In this case, we have

Horizontal component $u_x = u\cos\theta$

Vertical component $u_y = u\sin\theta - v$

$$\Rightarrow \quad R = \frac{2}{g}(u\cos\theta)(u\sin\theta - v)$$

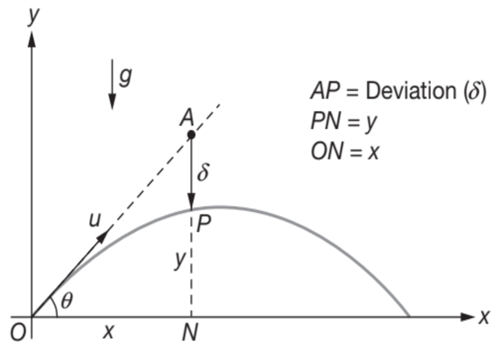
Deviation Suffered by an Oblique Projectile in Time t

Again we proceed with the same kind of thought process, where we assume gravity to be absent first. Then the body launched with initial velocity would continue to move forever along the line OA. However gravity is present, due to which it suffers a deviation δ from its actual track (in the absence of gravity). In triangle OAN.

$$\tan\theta = \frac{AN}{ON} = \frac{y+\delta}{x}$$

$$\Rightarrow \quad \delta = x\tan\theta - y \qquad \ldots(1)$$

Now since we know that $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$



$$\Rightarrow \quad \delta = (x\tan\theta) - \left(x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}\right)$$

$$\Rightarrow \quad \delta = \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

Since, $x = (u\cos\theta)t$

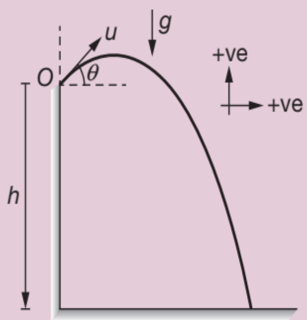
$$\Rightarrow \quad \delta = \frac{1}{2}gt^2$$

Hence deviation suffered by an oblique projectile in time t (or in terms of x, u and launch angle θ) is

$$\delta = \frac{1}{2}gt^2 = \frac{gx^2}{2u^2\cos^2\theta}$$

Problem Solving Technique(s)

CASE-1: Particle projected at an angle θ above the horizontal



The situation is shown in figure. In this case, we have

$$y = -h,\ u_x = u\cos\theta,\ u_y = u\sin\theta \text{ and } a_y = -g$$

For horizontal motion,

$$x = (u\cos\theta)t$$

For vertical motion,

$$-h = (u\sin\theta)t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad gt^2 - (2u\sin\theta)t - 2h = 0$$

$$\Rightarrow \quad t = \frac{u\sin\theta}{g} \pm \sqrt{\frac{u^2\sin^2\theta}{g^2} + \frac{2h}{g}}$$

CASE-2: Particle projected at an angle θ below the horizontal

The situation is shown in figure. In this case, we have

$$y = +h,\ u_x = u\cos\theta,\ u_y = +u\sin\theta \text{ and } a_y = +g$$

For horizontal motion,

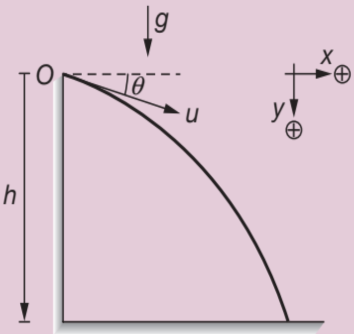
$$x = (u\cos\theta)t$$

For vertical motion,

$$+h = (u\sin\theta)t + \frac{1}{2}gt^2$$

$$\Rightarrow \quad gt^2 + (2u\sin\theta)t - 2h = 0$$

$$\Rightarrow \quad t = \frac{-u\sin\theta}{g} \pm \sqrt{\frac{u^2\sin^2\theta}{g^2} + \frac{2h}{g}}$$



Here negative root should be excluded otherwise t would be negative.

ILLUSTRATION 10

A stone is thrown from the top of a tower of height 50 m with a velocity of 30 ms^{-1} at an angle of 30° above the horizontal. Find

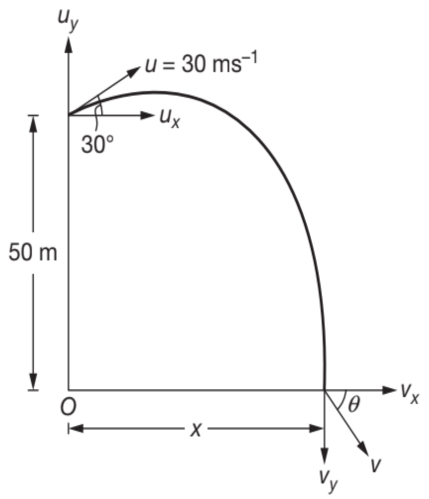
(a) the time during which the stone will be in air,
(b) the distance from the tower base to where the stone will hit the ground,
(c) the speed with which the stone will hit the ground,
(d) the angle formed by the trajectory of the stone with the horizontal at the point of hit.

SOLUTION

The situation is shown in figure

(a) Horizontal component of velocity

$$u_x = 30\cos(30°) = 25.98\ \text{ms}^{-1}$$



Vertical component of velocity

$$v_y = 30\sin(30°) = 15\ ms^{-1}$$

Let t be the time taken by the stone to reach the ground, i.e., the time during which the stone will be in air. Taking the upward direction as positive, we have

$$-50 = 15t - \frac{1}{2}\times 10\times t^2 \qquad \left\{\because\ h = ut + \frac{1}{2}gt^2\right\}$$

Solving for t, we get

$$t = 5\ s$$

(b) The distance x, where the stone will hit the ground

$$x = u_x t = 25.98\times 5 = 104.90\ m$$

(c) From figure, $v_x = u_x = 25.98\ ms^{-1}$

and $v_y = -15 + (10\times 5) = 35\ ms^{-1}$

Since $v = \sqrt{v_x^2 + v_y^2}$

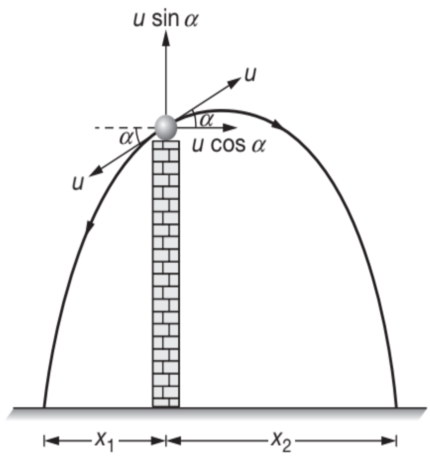
$$\Rightarrow\quad v = \sqrt{[(25.98)^2 + (35)^2]} = 43.6\ ms^{-1}$$

(d) $\tan\theta = \left(\dfrac{v_y}{v_x}\right)$

$$\Rightarrow\quad \theta = \tan^{-1}\left(\frac{35}{25.98}\right) \cong 53°$$

ILLUSTRATION 11

From a point at a height h above the horizontal ground a particle A is projected with a velocity u in an upward direction making an angle α with the horizon. Another particle B is projected with the same velocity u but in a downward direction exactly opposite to A. The two particles will strike the ground at a distance x apart. Find x.



SOLUTION

Taking upward direction as positive. Then

For particle A

Vertical displacement $= -h$ (downwards)

Vertical component of velocity $= u\sin\alpha$ (upwards)

and Acceleration $= -g$ (downwards)

$$\Rightarrow\quad -h = (u\sin\alpha)t_1 - \frac{1}{2}gt_1^2$$

$$\Rightarrow\quad \frac{1}{2}gt_1^2 - (u\sin\alpha)t_1 - h = 0$$

$$\Rightarrow\quad gt_1^2 - 2u\sin\alpha t_1 - 2h = 0$$

$$\Rightarrow\quad t_1 = \frac{2u\sin\alpha \pm \sqrt{4u^2\sin^2\alpha + 8gh}}{2g}$$

$$\Rightarrow\quad t = \frac{u\sin\alpha \pm \sqrt{u^2\sin^2\alpha + 2gh}}{g}$$

If we use negative sign, then time t_1 will be negative. So, dropping the negative sign, we get

$$t_1 = \frac{u\sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh}}{g}$$

For particle *B*

We have Vertical Displacement $= -h$ (downwards) Vertical Component of Velocity $= -u\sin\alpha$ (downwards) and Acceleration $= -g$ (downwards)

$$\Rightarrow \quad -h = -(u\sin\alpha)t_2 - \frac{1}{2}gt_2^2$$

$$\Rightarrow \quad h = (u\sin\alpha)t_2 + \frac{1}{2}gt_2^2$$

$$\Rightarrow \quad gt_2^2 + 2(u\sin\alpha)t_2 - 2h = 0$$

Solving, we get

$$\Rightarrow \quad t_2 = \frac{-u\sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh}}{g}$$

t_2 is the time taken by the particle to reach the ground

Now, distance travelled $x = x_1 - (-x_2) = x_1 + x_2$

$$\Rightarrow \quad x = (u\cos\alpha)t_1 + (u\cos\alpha)t_2$$

$$\Rightarrow \quad x = u\cos\alpha(t_1 + t_2)$$

$$\Rightarrow \quad x = u\cos\alpha\left(\frac{2}{g}\sqrt{u^2\sin^2\alpha + 2gh}\right)$$

$$\Rightarrow \quad x = \frac{2u\cos\alpha}{g}\sqrt{u^2\sin^2\alpha + 2gh}$$

ILLUSTRATION 12

A stone is projected from the ground with a velocity of 25 ms^{-1}. Two second later it just clears a wall 5 m high. Find the angle of projection of the stone, the greatest height reached. How far beyond the wall the stone will again hit the ground. $g = 10\text{ ms}^{-2}$.

SOLUTION

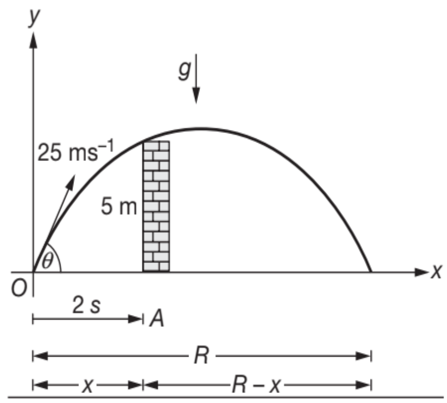
Since $y = u_y t + \frac{1}{2}a_y t^2$

$$\Rightarrow \quad 5 = (u\sin\theta)t + \frac{1}{2}(-g)t^2$$

$$\Rightarrow \quad 5 = (25\sin\theta)2 - \frac{1}{2}(10)(2)^2$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = 30^\circ$$



Greatest height reached is $H = \frac{u^2\sin^2\theta}{2g}$

$$\Rightarrow \quad H = \frac{(25)^2\left(\frac{1}{4}\right)}{2(10)} = 7.81\text{ m}$$

$$\text{Range} = R = \frac{u^2\sin(2\theta)}{g} = \frac{(25)^2\sin(60)}{10}$$

$$= \frac{(625)\sqrt{3}}{20} = 31.25\sqrt{3}\text{ m}$$

Since $x = (u\cos\theta)t = (25)\left(\frac{\sqrt{3}}{2}\right)(2) = 25\sqrt{3}\text{ m}$

So, $(R - x) = 6.25\sqrt{3}\text{ m} = 10.8\text{ m}$

ILLUSTRATION 13

Two seconds after the projection a projectile is travelling in a direction inclined at 30° to the horizon. After one more second it is travelling horizontally. Determine the magnitude and direction of its velocity.

SOLUTION

At $t = 2\text{ s}$

$$v\cos 30 = u\cos\theta \qquad \ldots(1)$$

Also,

$$v_y = u_y + a_y t$$

$$\Rightarrow \quad v\sin 30 = u\sin\theta - 20 \qquad \ldots(2)$$

After one more second, it is travelling horizontally i.e., it is at the maximum height. So, time of ascent is 3 s. Hence

$$\frac{u\sin\theta}{g}=3$$

$\Rightarrow\quad u\sin\theta=30 \qquad \ldots(3)$

From (2) and (3), we get

$$\frac{v}{2}=30-20$$

$\Rightarrow\quad v=20\text{ ms}^{-1}$

Substituting in (1), we get

$$u\cos\theta=10\sqrt{3} \qquad \ldots(4)$$

Squaring (3) and (4) and adding, we get

$$u^2=900+300$$

$\Rightarrow\quad u=\sqrt{1200}$

$\Rightarrow\quad u=20\sqrt{3}\text{ ms}^{-1}$

From (1), we get

$$10\sqrt{3}=20\sqrt{3}\cos\theta$$

$\Rightarrow\quad \theta=60^\circ$

So, the particle was launched with an initial velocity of $20\sqrt{3}\text{ ms}^{-1}$ making an angle of 60° with the horizontal.

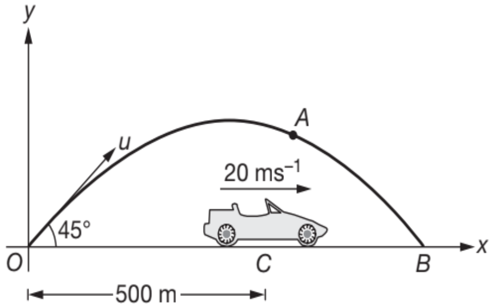
ILLUSTRATION 14

A gun kept on a horizontal straight road, is used to hit a car travelling along the same road away from the gun with a horizontal speed of 72 kmh^{-1}. The car is at a distance of 500 m from the gun when the gun is fired at an angle of 45° with the horizontal. Find the distance of the car from the gun when the shell hits it and the speed of the projection of the shell. (Take $g=9.8\text{ ms}^{-2}$ and $\sqrt{2}=1.41$)

SOLUTION

For the shell to hit the car, time taken by the shell to fly from $O\to A\to B$ must equal the time taken by the car to go from C to B with a uniform speed of 20 ms^{-1}. So,

$$t_{\substack{\text{car from}\\ C\to B}}=t_{\substack{\text{shell from}\\ O\to A\to B}}=T=\frac{2u\sin(45^\circ)}{g}=\frac{\sqrt{2}u}{g}$$



Also, we observe that when the shell hits the car at B, then

$$OC+CB=\text{Range}(R)$$

$\Rightarrow\quad 500+(20)T=\dfrac{u^2\sin(90^\circ)}{g}$

$\Rightarrow\quad 500+\dfrac{20\sqrt{2}u}{g}=\dfrac{u^2}{g}$

$\Rightarrow\quad u^2-20\sqrt{2}u-500g=0$

$\Rightarrow\quad u^2-20\sqrt{2}u-4900=0$

$\Rightarrow\quad u=\dfrac{20\sqrt{2}+\sqrt{800+4(4900)}}{2}$

$\Rightarrow\quad u=\dfrac{20\sqrt{2}+142.8}{2}=\dfrac{28.2+142.8}{2}=85.5\text{ ms}^{-1}$

When the shell hits the car, distance of car from the gun is equal to the range

$$R=\frac{u^2}{g}=746\text{ m}$$

ILLUSTRATION 15

The velocity of a projectile when it is at the greatest height is $\sqrt{\dfrac{2}{5}}$ times its velocity when it is at half of its greatest height. Determine its angle of projection.

SOLUTION

Let the particle be projected with velocity u at an angle θ with the horizontal. Horizontal component of its velocity at all heights will be $u\cos\theta$.

At the greatest height, the vertical component of velocity is zero, so the resultant velocity is

$v_1 = u\cos\theta$

At half the greatest height during upward motion, we have

$$v_y = \sqrt{u_y^2 - 2a_y y}$$

$$\Rightarrow \quad v_y = \sqrt{u^2\sin^2\theta - 2(g)\frac{u^2\sin^2\theta}{4g}}$$

$$\Rightarrow \quad v_y = \frac{u\sin\theta}{\sqrt{2}}$$

Hence, resultant velocity at half of the greatest height is

$$v_2 = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2\cos^2\theta + \frac{u^2\sin^2\theta}{2}}$$

Since, it is given that $\dfrac{v_1}{v_2} = \sqrt{\dfrac{2}{5}}$

$$\Rightarrow \quad \frac{v_1^2}{v_2^2} = \frac{u^2\cos^2\theta}{u^2\cos^2\theta + \dfrac{u^2\sin^2\theta}{2}} = \frac{2}{5}$$

$$\Rightarrow \quad \frac{1}{1 + \dfrac{1}{2}\tan^2\theta} = \frac{2}{5}$$

$$\Rightarrow \quad 2 + \tan^2\theta = 5$$

$$\Rightarrow \quad \tan^2\theta = 3$$

$$\Rightarrow \quad \tan\theta = \sqrt{3}$$

$$\Rightarrow \quad \theta = 60^\circ$$

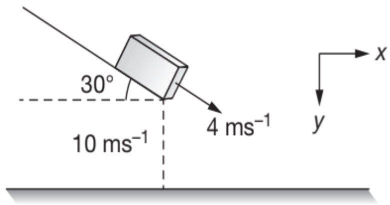
ILLUSTRATION 16

A person working on the roof of a house suddenly drops his hammer, which slides down the roof at a constant speed of 4 ms^{-1}. The roof makes an angle of 30° with the horizontal, and its lowest point is 10 m above the ground. What is the horizontal distance travelled by the hammer after it leaves the roof of the house before it hits the ground?

SOLUTION

$$u_x = 4\cos 30^\circ = 2\sqrt{3} \text{ ms}^{-1}$$

$$u_y = 4\sin 30^\circ = 2 \text{ ms}^{-1}$$



Since $y = u_y + \frac{1}{2}a_y t^2$

$$\Rightarrow \quad 10 = 2t + \frac{1}{2}(9.8)(t^2)$$

Solving these equations we get

$$t = 1.24 \text{ s}$$

and $x = u_x t = 4.29$ m

ILLUSTRATION 17

The range of the rifle bullet is 1000 m, when θ is the angle of projection. If the bullet is fired with the same angle from a car travelling horizontally at a speed of 36 kmh^{-1} towards the target, calculate the amount by which the range is increased.

SOLUTION

$$R = 1000 = \frac{2}{g}(u\cos\theta)(u\sin\theta) \qquad \ldots(1)$$

When the bullet is fired from a car travelling at a speed of $36 \text{ kmh}^{-1} (= 10 \text{ ms}^{-1})$ at the same angle, then the new horizontal component of the velocity of the bullet becomes

$$u_x' = u\cos\theta + 10$$

However, the y component remains unaltered. So, if R' is the new range, then

$$R' = \frac{2}{g}(u\cos\theta + 10)(u\sin\theta)$$

$$\Rightarrow \quad R' = \frac{2}{g}(u\cos\theta)(u\sin\theta) + \frac{20}{g}u\sin\theta$$

$$\Rightarrow \quad R' - R = \frac{20}{g}u\sin\theta \qquad \ldots(2)$$

From (1),

$$u=\sqrt{\frac{500g}{\sin\theta\cos\theta}}$$

$$\Rightarrow\quad R'-R=\frac{20}{g}\sqrt{\frac{500g}{\sin\theta\cos\theta}}\sin\theta$$

$$\Rightarrow\quad R'-R=20\sqrt{\frac{500}{g}(\tan\theta)}=20\sqrt{\frac{5000}{98}\tan\theta}$$

$$\Rightarrow\quad R'-R=20\sqrt{\frac{2500}{49}\tan\theta}$$

$$\Rightarrow\quad R'-R=\frac{1000}{7}\sqrt{\tan\theta}\text{ m}$$

ILLUSTRATION 18

It at any point on the path of a projectile its velocity be u and inclination be α. Show that the particle will move at right angles to the former direction after a time $t=\dfrac{u}{g\sin\alpha}$ when its velocity would be $v=u\cot\alpha$.

SOLUTION

The initial velocity of the particle is

$$\vec{u}=(u\cos\alpha)\hat{i}+(u\sin\alpha)\hat{j}\qquad\ldots(1)$$

At any time t, the velocity of the particle is $\vec{v}$. Then

$$\vec{v}=v_x\hat{i}+v_y\hat{j}$$

$$\Rightarrow\quad \vec{v}=(u\cos\alpha)\hat{i}+(u\sin\alpha-gt)\hat{j}\qquad\ldots(2)$$

Now, $\vec{v}\perp\vec{u}$, so we have

$$\vec{v}\cdot\vec{u}=0$$

$$\Rightarrow\quad u^2\cos^2\alpha+u^2\sin^2\alpha-ugt\sin\alpha=0$$

$$\Rightarrow\quad u^2-ugt\sin\alpha=0$$

$$\Rightarrow\quad t=\frac{u}{g\sin\alpha}$$

Further, from (2), we have

$$|\vec{v}|=\sqrt{u^2\cos^2\alpha+u^2\sin^2\alpha+g^2t^2-2ugt\sin\alpha}$$

$$\Rightarrow\quad |\vec{v}|=\sqrt{u^2+g^2t^2-2ugt\sin\alpha}$$

$$\Rightarrow\quad |\vec{v}|=\sqrt{u^2+g^2\frac{u^2}{g^2\sin^2\alpha}-2ug\left(\frac{u}{g\sin\alpha}\right)\sin\alpha}$$

$$\Rightarrow\quad |\vec{v}|=u\sqrt{1+\operatorname{cosec}^2\alpha-2}$$

$$\Rightarrow\quad |\vec{v}|=u\sqrt{\operatorname{cosec}^2\alpha-1}$$

$$\Rightarrow\quad |\vec{v}|=u\cot\alpha\qquad\{\because 1+\cot^2\alpha=\operatorname{cosec}^2\alpha\}$$

ILLUSTRATION 19

With what minimum speed must a particle be projected from origin so that it is able to pass through a given point $P(x,\ y)$?

SOLUTION

Let u and θ be the velocity and angle of projection respectively.
For the projectile to pass through $P(x,y)$

$$y=x\tan\theta-\frac{gx^2}{2u^2}(1+\tan^2\theta)$$

$$\Rightarrow\quad gx^2\tan^2\theta-2xu^2\tan\theta+(gx^2+2yu^2)=0$$

The projectile will pass through $P(x, y)$ if this equation (quadratic in $\tan\theta$) gives some real value of θ, i.e., its discriminant ≥ 0.

$$4x^2u^4-4gx^2(gx^2+2yu^2)\geq 0$$

$$u^4-2gyu^2-g^2x^2\geq 0$$

$$u^4-2gyu^2+g^2y^2\geq g^2x^2+g^2y^2$$

$$(u^2-gy)^2\geq(x^2+y^2)g^2$$

$$\Rightarrow\quad u\geq\sqrt{gy+g\sqrt{x^2+y^2}}$$

So, the minimum value of u is $u_{\text{MIN}}=\sqrt{gy+g\sqrt{x^2+y^2}}$.

ILLUSTRATION 20

Two shots are projected from a gun at the top of a hill with the same velocity u at angles of projection α and β respectively. If the shots strike the horizontal ground through the foot of the hill at the same point, show that the height h of the hill above the plane is given by

$$h=\frac{2u^2(1-\tan\alpha\tan\beta)}{g(\tan\alpha+\tan\beta)^2}$$

SOLUTION

Let R be the horizontal range for both. Taking the point of projection as origin, the point struck is (R, h). This point satisfies the equations of both the trajectories.

$$-h = R\tan\alpha - \frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha} \quad \ldots(1)$$

and $$-h = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta} \quad \ldots(2)$$

From these equations, we have

$$R\tan\alpha - \frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha} = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

$$\Rightarrow R(\tan\alpha - \tan\beta) = \frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha} - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

$$\Rightarrow \frac{2u^2}{g}(\tan\alpha - \tan\beta) = R(\sec^2\alpha - \sec^2\beta)$$

$$\Rightarrow \frac{2u^2}{g}\cdot(\tan\alpha - \tan\beta) = R\left[(1+\tan^2\alpha) - (1+\tan^2\beta)\right]$$

$$\Rightarrow \frac{2u^2}{g} = R(\tan\alpha + \tan\beta) \quad \ldots(3)$$

$$\Rightarrow R = \frac{2u^2}{g(\tan\alpha + \tan\beta)}$$

Substituting the value of R in equation (1), we get

$$h = R\left(\frac{gR}{2u^2\cos^2\alpha} - \tan\alpha\right)$$

$$\Rightarrow h = \frac{2u^2}{g(\tan\alpha+\tan\beta)}\left(\frac{\sec^2\alpha}{\tan\alpha+\tan\beta} - \tan\alpha\right)$$

$$\Rightarrow h = \frac{2u^2}{g(\tan\alpha+\tan\beta)^2}(\sec^2\alpha - \tan^2\alpha - \tan\alpha\tan\beta)$$

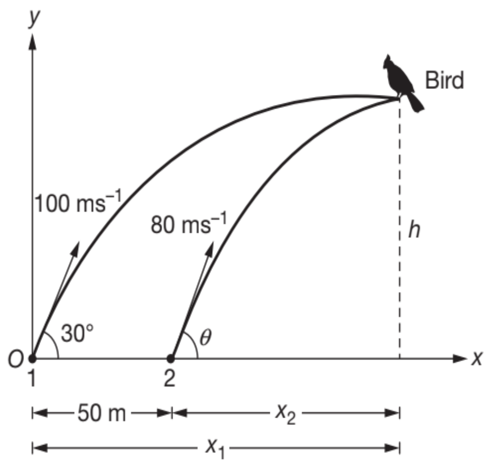
$$\Rightarrow h = \frac{2u^2}{g(\tan\alpha+\tan\beta)^2}(1 - \tan\alpha\tan\beta)$$

ILLUSTRATION 21

Two boys simultaneously aim their guns at a bird sitting on a tower. The first boy releases his shot with a speed of $100\ \text{ms}^{-1}$ at an angle of projection of $30°$. The second boy is ahead of the first by a distance of 50 m and releases his shot with a speed of $80\ \text{ms}^{-1}$. How must he aim his gun so that both the shots hit the bird simultaneously? What is the distance of the foot of the tower from the two boys and the height of the tower? With what velocities and when do the two shots hit the bird?

SOLUTION

The situation is shown in figure



For first shot,

horizontal displacement $= x_1$

vertical displacement $= h$

For second shot,

horizontal displacement $= x_2$

vertical displacement $= h$

Here, $x_1 = x_2 + 50 \quad \ldots(1)$

The two shots will hit the bird simultaneously at a particular instant t, if the horizontal and vertical displacements of the two shots are as shown.

For first shot,

$$h = (100\sin 30°)t - \frac{1}{2}gt^2 \quad \ldots(2)$$

For second shot,

$$h = (80\sin\theta)t - \frac{1}{2}gt^2 \quad \ldots(3)$$

$$\Rightarrow (100\sin 30°)t - \frac{1}{2}gt^2 = (80\sin\theta)t - \frac{1}{2}gt^2$$

$$\Rightarrow \sin\theta = \frac{100\sin 30°}{80} = \frac{50}{80} = 0.625$$

$$\Rightarrow \theta = \sin^{-1}(0.625) = 38°68'$$

Now, $x_1 = (100\cos 30^\circ)t$ and $x_2 = 80\cos(38^\circ 68')t$

Since $x_1 = x_2 + 50$

$\Rightarrow \quad (100\cos 30^\circ)t = 50 + 80\cos(38^\circ 68')t$

$\Rightarrow \quad t\left[100\left(\frac{\sqrt{3}}{2}\right) - 80 \times 0.7806\right] = 50$

$\Rightarrow \quad t[86.6025 - 62.4499] = 50$

$\Rightarrow \quad t = \frac{50}{24.1526} = 2.07 \text{ s}$

So, $x_1 = 100\cos 30^\circ \times 2.07 = 179.27 \text{ m}$ and

$x_2 = 179.27 - 50 = 129.27 \text{ m}$

Also, $h = 100\sin 30^\circ t - \frac{1}{2} \times 9.8 \times (2.07)^2$

$\Rightarrow \quad h = (103.5 - 20.996) = 82.504 \text{ m}$

ILLUSTRATION 22

A particle moves in the plane x-y with constant acceleration a directed along negative y-axis. The equation of motion of particle has the form $y = Ax - Bx^2$. A and B are positive constants. Find the velocity of the particle at the origin of the co-ordinates.

SOLUTION

METHOD I

Comparing the above given equation with the equation of the oblique projectile moving under the influence of the acceleration field $\vec{a}$.

$$\Rightarrow \quad y = x\tan\theta - \frac{ax^2}{2u^2\cos^2\theta}$$

{Equation for Oblique Projectile}

$$\Rightarrow \quad A = \tan\theta \text{ and } B = \frac{a}{2u^2\cos^2\theta}$$

$$\Rightarrow \quad A = \tan\theta \text{ and } \sec^2\theta = \frac{2u^2B}{a} \qquad \ldots(1)$$

We know that

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad \frac{2u^2B}{a} - A^2 = 1 \qquad \{\text{using (1)}\}$$

$$\Rightarrow \quad u = \sqrt{(1 + A^2)\frac{a}{2B}}$$

METHOD II

$$\frac{d^2y}{dt^2} = -a; \ \frac{d^2x}{dt^2} = 0 \qquad \{\text{Given}\}$$

$$\Rightarrow \quad y = Ax - Bx^2 \qquad \ldots(1)$$

Differentiating both sides of equation (1) w.r.t. t,

$$\Rightarrow \quad \frac{dy}{dt} = A\frac{dx}{dt} - 2Bx\frac{dx}{dt} \qquad \ldots(2)$$

Again differentiating both sides of equation (2) w.r.t. t,

$$\Rightarrow \quad \frac{d^2y}{dt^2} = A\frac{d^2x}{dt^2} - 2B\left[\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2}\right]$$

$$\Rightarrow \quad -a = 0 - 2B\left[\left(\frac{dx}{dt}\right)^2 + 0\right] \qquad \{\text{from given relations}\}$$

$$\Rightarrow \quad u_x^2 = \frac{a}{2B} \qquad \ldots(3)$$

Also from (2),

$$\Rightarrow \quad \left.\frac{dy}{dt}\right|_{x=0} = u_y = Au_x = A\sqrt{\frac{a}{2B}} \qquad \{\text{at Origin}\}$$

$$\Rightarrow \quad u_y = A\sqrt{\frac{a}{2B}} \qquad \{\because \text{ at Origin } x = 0\} \qquad \ldots(4)$$

Now, $u^2 = u_x^2 + u_y^2$

$$\Rightarrow \quad u^2 = \frac{a}{2B} + A^2\frac{a}{2B} \qquad \{\text{using (3) \& (4)}\}$$

$$\Rightarrow \quad u = \sqrt{(1 + A^2)\frac{a}{2B}}$$

RANGE, MAXIMUM HEIGHT AND TIME OF FLIGHT FOR COMPLIMENTARY ANGLES

For complimentary angles, say ϕ and $(90 - \phi)$, let the corresponding ranges, maximum heights and times of flight be R_ϕ, $R_{90-\phi}$, H_ϕ, $H_{90-\phi}$ and T_ϕ, $T_{90-\phi}$, then

$$R_\phi = \frac{2u^2\sin\phi\cos\phi}{g} \text{ and } R_{90-\phi} = \frac{2u^2\cos\phi\sin\phi}{g}$$

$$\Rightarrow \quad R_\phi = R_{90-\phi} = \frac{2u^2\sin\phi\cos\phi}{g} \qquad \ldots(1)$$

So, we must remember that for complimentary angles range is the same i.e.,

$R_{1^\circ} = R_{89^\circ}$, $R_{15^\circ} = R_{75^\circ}$ and so on

Further more,

$$H_\phi = \frac{u^2 \sin^2 \phi}{2g} \quad \ldots(2)$$

$$H_{90-\phi} = \frac{u^2 \sin^2(90-\phi)}{2g} = \frac{u^2 \cos^2 \phi}{2g} \quad \ldots(3)$$

From (2) and (3), we observe that

$$H_\phi + H_{90-\phi} = \frac{u^2}{2g} \quad \ldots(4)$$

Also, from equations (1), (2) and (3), we observe that

$$R_\phi = R_{90-\phi} = 4\sqrt{H_\phi H_{90-\phi}} \quad \ldots(5)$$

$$\Rightarrow \quad R_{1^\circ} = R_{89^\circ} = 4\sqrt{H_{1^\circ} H_{89^\circ}}$$

Finally,

$$T_\phi = \frac{2u\sin\phi}{g} \text{ and } T_{90-\phi} = \frac{2u\cos\phi}{g}$$

$$\Rightarrow \quad T_\phi T_{90-\phi} = \frac{4u^2 \sin\phi\cos\phi}{g^2} = \frac{2}{g}\left(\frac{2u^2\sin\phi\cos\phi}{g}\right)$$

$$\Rightarrow \quad T_\phi T_{90-\phi} = \frac{2R_\phi}{g} = \frac{2R_{90-\phi}}{g} \quad \ldots(6)$$

i.e., $T_{1^\circ} T_{89^\circ} = \dfrac{2R_{1^\circ}}{g} = \dfrac{2R_{89^\circ}}{g}$

Problem Solving Technique(s)

So, for complimentary angles ϕ and $90-\phi$, following points are worthnoting.

(a) $R_\phi = R_{90-\phi} = \dfrac{2u^2 \sin\phi\cos\phi}{g}$

(b) $H_\phi + H_{90-\phi} = \dfrac{u^2}{2g}$

(c) $R_\phi = R_{90-\phi} = 4\sqrt{H_\phi H_{90-\phi}}$

(d) $T_\phi T_{90-\phi} = \dfrac{2R_\phi}{g} = \dfrac{2R_{90-\phi}}{g}$

ILLUSTRATION 23

A shell bursts on contact with the ground and the fragments fly in all the directions with speeds upto 39.2 ms^{-1}. Show that a man 78.4 m away is in danger for $4\sqrt{2}$ s.

SOLUTION

Since $R = \dfrac{u^2 \sin(2\theta)}{g}$

$$\Rightarrow \quad 78.4 = \frac{(39.2)^2 \sin(2\theta)}{9.8}$$

$$\Rightarrow \quad \sin(2\theta) = \frac{1}{2}$$

$$\Rightarrow \quad 2\theta = 30^\circ \text{ or } 150^\circ$$

$$\Rightarrow \quad \theta = 15^\circ \text{ or } 75^\circ$$

i.e., the range will be same for these two complimentary angles of 15° and 75°. However, the time of flight will not be the same. Hence the person 78.4 m away will be in danger for a time t given by

$$t = t_{75^\circ} - t_{15^\circ}$$

$$\Rightarrow \quad t = \frac{2u}{g}(\sin(75^\circ) - \sin(15^\circ))$$

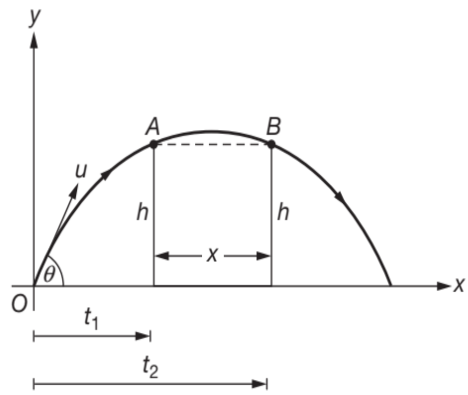
$$\Rightarrow \quad t = \frac{2(39.2)}{9.8}\left[2\sin\left(\frac{75^\circ+15^\circ}{2}\right)\sin\left(\frac{75^\circ-15^\circ}{2}\right)\right]$$

$$\Rightarrow \quad t = 16\sin(45^\circ)\sin(30^\circ)$$

$$\Rightarrow \quad t = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2} \text{ s}$$

TWO UNIQUE TIMES FOR WHICH PROJECTILE IS AT SAME HEIGHT

It has been observed that, there are two unique times at which the projectile is at same height $h(<H)$. Let us consider a projectile launched with initial velocity u, making an angle θ with the horizontal. Let the projectile be at a height $h(<H)$ at time t. Then we observe just by a qualitative argument that this is true for upward motion of the projectile and then for the downward motion, as shown.



Mathematically, at time t, we have

$$h=(u\sin\theta)t-\frac{1}{2}gt^2$$

$$\Rightarrow \quad t^2-\left(\frac{2u\sin\theta}{g}\right)t+\frac{2h}{g}=0 \qquad \ldots(1)$$

This equation happens to be a quadratic equation in t (with non-zero discriminant), hence has two unique roots t_1 and t_2. A very important observation which cannot be skipped here is that

$$t_1+t_2=\frac{2u\sin\theta}{g}=T \qquad \ldots(2)$$

$$t_1t_2=\frac{2h}{g} \qquad \ldots(3)$$

From the diagram we must understand that,

t_1 is the time taken by the projectile to fly from $O\to A$ and

t_2 is the time taken by the projectile to fly from $O\to A\to B$

So, time taken by the projectile to fly from A to B is

$$t=t_2-t_1 \qquad \ldots(4)$$

Since $(t_2-t_1)^2=(t_2+t_1)^2-4t_1t_2$

$$t=t_2-t_1=\sqrt{\frac{4u^2\sin^2\theta}{g^2}-\frac{8h}{g}} \qquad \ldots(5)$$

If, x be the horizontal separation between the two events, then

$$x=(u\cos\theta)t=(u\cos\theta)\sqrt{\frac{4u^2\sin^2\theta}{g^2}-\frac{8h}{g}} \qquad \ldots(6)$$

The following Illustration shows how to use equations (2), (3), (4), (5) and (6) to get the results efficiently.

ILLUSTRATION 24

A stone is projected from a point on the ground in such a direction so as to hit a bird on the top of a telegraph post of height h and then attain a maximum height of $2h$ above the ground. If at the instant of projection the bird were to fly away horizontally with a uniform speed then calculate the ratio between the horizontal velocity of the bird and the stone, if the stone still hits the bird while descending.

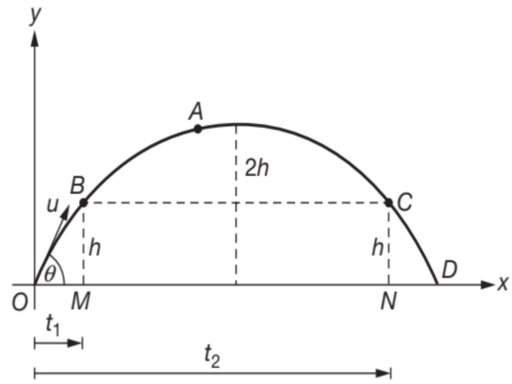
SOLUTION

$$H=2h=\frac{u^2\sin^2\theta}{2g}$$

$$\Rightarrow \quad u\sin\theta=2\sqrt{gh} \qquad \ldots(1)$$

Since $h=(u\sin\theta)t+\frac{1}{2}(-g)t^2$

$$\Rightarrow \quad h=2\sqrt{gh}\,t-\frac{1}{2}gt^2$$



$$\Rightarrow \quad t^2-4\sqrt{\frac{h}{g}}\,t+\frac{2h}{g}=0 \qquad \ldots(2)$$

This quadratic equation in t will have two roots, t_1 and t_2. So,

$$t_1=\frac{4\sqrt{\frac{h}{g}}-\sqrt{\frac{16h}{g}-\frac{8h}{g}}}{2}=\left(2-\sqrt{2}\right)\sqrt{\frac{h}{g}} \qquad \ldots(3)$$

$$t_2 = \frac{4\sqrt{\frac{h}{g}} + \sqrt{\frac{16h}{g} - \frac{8h}{g}}}{2} = \left(2+\sqrt{2}\right)\sqrt{\frac{h}{g}} \qquad \ldots(4)$$

So, time taken by the stone to fly from B to C is $(t_2 - t_1)$.

$$\Rightarrow \quad BC = (u\cos\theta)(t_2 - t_1) \qquad \ldots(5)$$

After reading the question carefully, we observe that at the instant of projection of stone the bird were to fly away horizontally with a uniform speed (say v_b) and the stone, during its downward motion (at C) still hits the bird. So, time taken by the bird to fly from B to C is the time taken by the stone to go from $O \to B \to A \to C$. Hence, for the bird, we have

$$BC = v_b t_2 \qquad \ldots(6)$$

From (5) and (6), we get

$$v_b t_2 = (u\cos\theta)(t_2 - t_1)$$

$$\Rightarrow \quad \frac{v_b}{u\cos\theta} = \frac{t_2 - t_1}{t_2} = \frac{2\sqrt{2}}{2+\sqrt{2}} = \frac{2}{\sqrt{2}+1}$$

RADIUS OF CURVATURE OF AN OBLIQUE PROJECTILE AT A POINT *P*

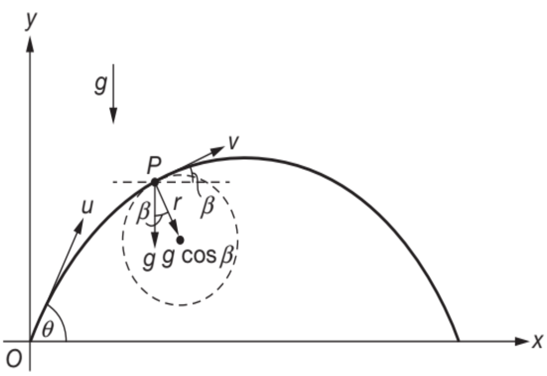
At Any Instant of Time (*t*)

Consider a projectile launched with initial velocity u, making an angle θ with the horizontal. Let the projectile be at a point P, at any instant t (say). Then we know that

$$r = \frac{v_T^2}{a_C} = \frac{v_\perp^2}{a_\parallel} = \frac{v^2}{g\cos\beta} \qquad \ldots(1)$$

where

$$v^2 = (u\sin\theta - gt)^2 + (u\cos\theta)^2 \qquad \text{\{studied earlier\}}$$



$$\Rightarrow \quad v^2 = u^2 + g^2t^2 - 2ugt\sin\theta$$

At Any Position at Vertical Height *h*(< *H*)

Here too everything remains the same i.e.,

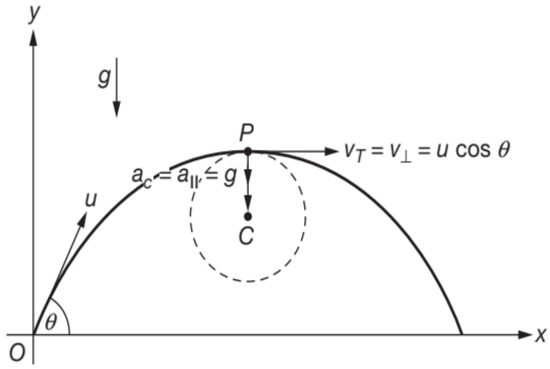
$$r = \frac{v_T^2}{a_C} = \frac{v_\perp^2}{a_\parallel} = \frac{v^2}{g\cos\beta}$$

But here, $v^2 = u^2 - 2gh$ {studied earlier}

At Maximum Height (*H*)

At the maximum height H, velocity is $u\cos\theta$.

$$\Rightarrow \quad v_T = v_\perp = u\cos\theta$$



Also, $a_C = a_\parallel = g$

$$\Rightarrow \quad r = \frac{v_T^2}{a_C} = \frac{v_\perp^2}{a_\parallel} = \frac{u^2\cos^2\theta}{g}$$

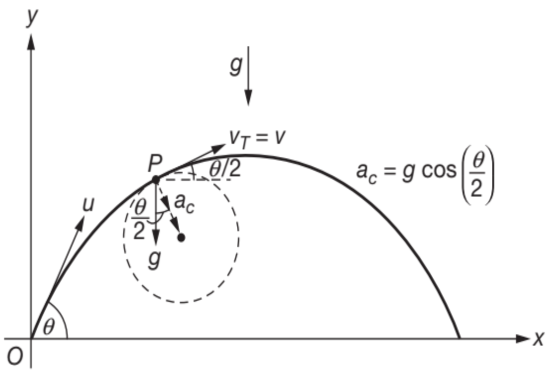
This result can also be obtained from cases discussed in (A) and (B), where we can put special values to get the desired result. (I hope this is a small exercise for you people).

ILLUSTRATION 25

A particle is projected with a speed u making an angle θ with the horizontal. Find the radius of curvature at the point where the particle velocity makes an angle $\frac{\theta}{2}$ with the horizontal. Assume the particle to move under gravity in the absence of any air drag.

SOLUTION

First of all, after reading the question carefully, we see that the particle will act as an oblique projectile following a parabolic path, as shown.



Since $r = \dfrac{v_T^2}{a_c} = \dfrac{v_\perp^2}{a_\parallel}$

$$\Rightarrow \quad r = \frac{v^2}{g\cos\left(\frac{\theta}{2}\right)} \qquad \ldots(1)$$

Since $v\cos\left(\dfrac{\theta}{2}\right) = u\cos\theta$

{∵ Horizontal Motion is Non-Accelerated}

$$\Rightarrow \quad v = \frac{u\cos\theta}{\cos\left(\frac{\theta}{2}\right)} \qquad \ldots(2)$$

Substituting (2) in (1), we get

$$r = \frac{u^2\cos^2\theta}{g\cos^3\left(\frac{\theta}{2}\right)}$$

EQUATION OF TRAJECTORY OF AN OBLIQUE PROJECTILE IN TERMS OF RANGE

Since, we know that for a projectile launched with initial velocity u, making an angle θ with the horizontal, the equation of trajectory is given by

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$\Rightarrow \quad y = x\tan\theta\left[1 - \frac{gx^2}{2u^2\cos^2\theta(x\tan\theta)}\right]$$

$$\Rightarrow \quad y = x\tan\theta\left[1 - \frac{x}{\left(\frac{2u^2\sin\theta\cos\theta}{g}\right)}\right]$$

Since $R = \dfrac{2u^2\sin\theta\cos\theta}{g}$

$$\Rightarrow \quad y = x\tan\theta\left(1 - \frac{x}{R}\right)$$

ILLUSTRATION 26

A ball is thrown from ground level so as to just clear a wall 4 m high at a distance of 4 m and falls at a distance of 14 m from the wall. Find the magnitude and direction of the velocity of the ball.

SOLUTION

As the ball strikes the ground at a distance 14 m from the wall, the range is $R = (4+14)\text{ m} = 18\text{ m}$.

Since $y = x\tan\theta\left(1 - \dfrac{x}{R}\right)$ …(1)

where, $x = 4\text{ m}$, $y = 4\text{ m}$ and $R = 18\text{ m}$.

$$\Rightarrow \quad 4 = 4\tan\theta\left(1 - \frac{4}{18}\right) = 4\tan\theta\left(\frac{7}{9}\right)$$

$$\Rightarrow \quad \tan\theta = \frac{9}{7}$$

$$\Rightarrow \quad \sin\theta = \frac{9}{\sqrt{130}} \text{ and } \cos\theta = \frac{7}{\sqrt{130}}$$

Again $R = \dfrac{2}{g}u^2\sin\theta\cos\theta$

$$\Rightarrow \quad R = \frac{2}{9.8}\times u^2\times\frac{9}{\sqrt{130}}\times\frac{7}{\sqrt{130}}$$

$$\Rightarrow \quad u^2 = \frac{18\times 9.8\times\sqrt{130}\times\sqrt{130}}{2\times 9\times 7}$$

$$\Rightarrow \quad u^2 = \frac{98\times 13}{7} = 182$$

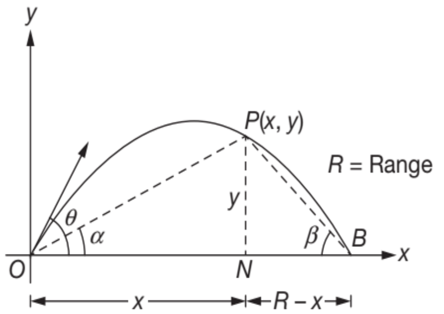
$$\Rightarrow \quad u = \sqrt{182}\text{ ms}^{-1} = 13.5\text{ ms}^{-1}$$

ILLUSTRATION 27

A particle is thrown over a triangle from one end of a horizontal base and after grazing the vertex falls on the other end of the base. If α and β be the base angles and θ the angle of projection, prove that $\tan\theta = \tan\alpha + \tan\beta$.

SOLUTION

The situation is shown in figure. In triangle NOP, $\tan\alpha = \frac{y}{x}$.



In triangle BPN, $\tan\beta = \frac{y}{R-x}$

$$\Rightarrow \quad \tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{R-x}$$

$$\Rightarrow \quad \tan\alpha + \tan\beta = \frac{yR}{x(R-x)} \qquad \ldots(1)$$

Since, the equation of trajectory is $y = x\tan\theta\left(1 - \frac{x}{R}\right)$

$$\Rightarrow \quad \tan\theta = \frac{yR}{x(R-x)} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$\tan\theta = \tan\alpha + \tan\beta$$

RELATIVE MOTION BETWEEN TWO PROJECTILES/MOTION OF ONE PROJECTILE AS SEEN FROM ANOTHER PROJECTILE

METHOD I

Since both the projectiles are moving under the influence of gravity, so acceleration of one projectile as seen from the other is

$$\vec{a} = \vec{g} - \vec{g} = \vec{0} \;\Rightarrow\; a_x\hat{i} + a_y\hat{j} = \vec{0}$$

$\Rightarrow \quad a_x = 0$ | $a_y = 0$

$\Rightarrow \quad \frac{dv_x}{dt} = 0$ | $\Rightarrow \quad \frac{dv_y}{dt} = 0$

$\Rightarrow \quad v_x = \text{constant} = k_1$ (say) | $\Rightarrow \quad v_y = \text{constant} = k_2$ (say)

$\Rightarrow \quad \frac{dx}{dt} = k_1$ | $\Rightarrow \quad \frac{dy}{dt} = k_2$

$\Rightarrow \quad dx = k_1 dt$ | $\Rightarrow \quad dy = k_2 dt$

$\Rightarrow \quad \int_0^x dx = k_1\int_0^t dt$ | $\Rightarrow \quad \int_0^y dy = k_2\int_0^t dt$

$\Rightarrow \quad x = k_1 t \quad \ldots(1)$ | $\Rightarrow \quad y = k_2 t \quad \ldots(2)$

$$\Rightarrow \quad \frac{y}{x} = \frac{k_2}{k_1} = k \qquad \text{(say)}$$

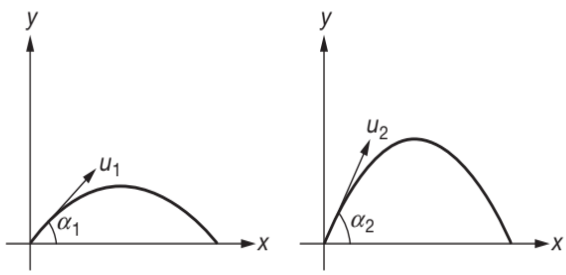
$\Rightarrow \quad y = kx$, equation of a straight line with slope $\frac{k_2}{k_1}$

So, motion of one projectile as seen from another projectile is a straight line passing through the origin (or the point of launch).

METHOD II

Let us now discuss the relative motion between two projectiles or the path observed by one projectile of the other. Suppose that two particles are projected from the ground with speeds u_1 and u_2 at angles α_1 and α_2 as shown in figure. Acceleration of both the particles is g downwards. So, relative acceleration between them is zero because

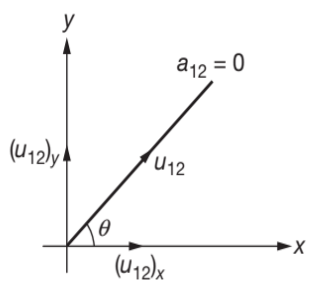
$$a_{12} = a_1 - a_2 = g - g = \text{zero}$$



i.e., the relative motion between the two particles is uniform. Now,

$$(u_1)_x = u_1\cos\alpha_1 \text{ and } (u_2)_x = u_2\cos\alpha_2$$

$$(u_1)_y = u_1\sin\alpha_1 \text{ and } (u_2)_y = u_2\sin\alpha_2$$



Therefore,

$$(u_{12})_x = (u_1)_x - (u_2)_x = u_1 \cos\alpha_1 - u_2 \cos\alpha_2$$

and $(u_{12})_y = (u_1)_y - (u_2)_y = u_1 \sin\alpha_1 - u_2 \sin\alpha_2$

$(u_{12})_x$ and $(u_{12})_y$ are the x and y components of relative velocity of 1 with respect to 2.

Hence, relative motion of 1 with respect to 2 is a straight line at an angle $\theta = \tan^{-1}\left[\frac{(u_{12})_y}{(u_{12})_x}\right]$ with positive x-axis.

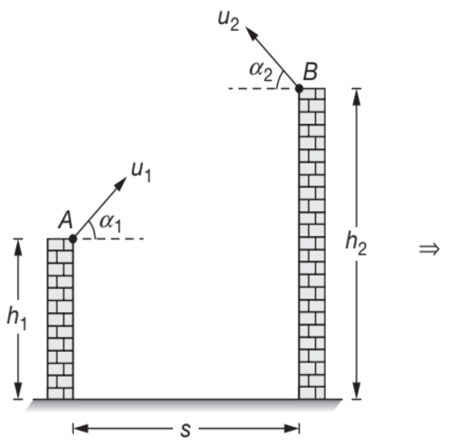
(a) If $(u_{12})_x = 0$, i.e., $u_1 \cos\alpha_1 = u_2 \cos\alpha_2$, then the relative motion is along y-axis or in vertical direction (as $\theta = 90°$).

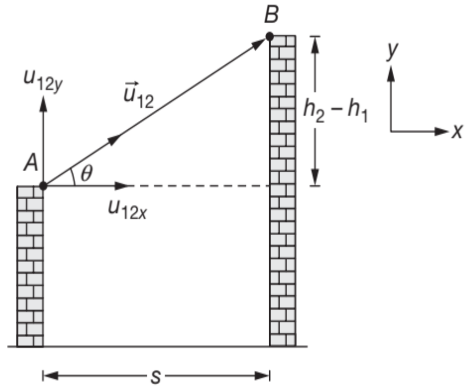
(b) Similarly, if $(u_{12})_y = 0$, i.e., $u_1 \sin\alpha_1 = u_2 \sin\alpha_2$, then the relative motion is along x-axis or in horizontal direction (as $\theta = 0°$).

CONDITION OF COLLISION BETWEEN TWO PROJECTILES

Now it is clear that relative motion between two projectiles is uniform and the path of one projectile as observed by the other is a straight line. Let the particles be projected simultaneously from two different heights h_1 and h_2 with speeds u_1 and u_2 in the directions shown in figure. Then the particles collide in air if the relative velocity of 1 with respect to $2(\vec{u}_{12})$ is along line AB or the relative velocity of 2 with respect to $1(\vec{u}_{21})$ is along the line BA. Thus,

$$\tan\theta = \frac{(u_{12})_y}{(u_{12})_x} = \left(\frac{h_2 - h_1}{s}\right)$$





Here, $(u_{12})_y = u_1 \sin\alpha_1 - u_2 \sin\alpha_2$ and

$$(u_{12})_x = (u_1 \cos\alpha_1) - (-u_2 \cos\alpha_2)$$

$$\Rightarrow \quad (u_{12})_x = u_1 \cos\alpha_1 + u_2 \cos\alpha_2$$

If both the particles are initially at the same level $(h_1 = h_2)$, then for collision

$$(u_{12})_y = 0 \text{ or } u_1 \sin\alpha_1 = u_2 \sin\alpha_2$$

The time of collision of the two particles will be

$$t = \frac{AB}{|\vec{u}_{12}|} = \frac{AB}{\sqrt{(u_{12})_x^2 + (u_{12})_y^2}}$$

Further, the above conditions are not merely sufficient for collision to takes place. For example, the time of collision discussed above should be less than the time of collision of either of the particles with the ground.

ILLUSTRATION 28

Two guns situated at the top of a hill of height 10 m, fire one shot each with the same speed $5\sqrt{3}$ ms^{-1} at some interval of time. One gun fires horizontally and the other fires upwards at an angle of $60°$ with the horizontal. The shots collide in air at a point P. Find

(a) the time interval between the firings and

(b) the coordinates of the point P. Take origin of coordinates system at the foot of the hill right below the muzzle and trajectory in x-y plane.

SOLUTION

Let the shot fired at $60°$ above horizontal reach point P in a time t. Then the shot fired horizontal must have taken a time $(t - \Delta t)$ to reach the point P, where Δt is the time log between firing of shots.

(a) Let they collide at the point $P(x, y)$, then

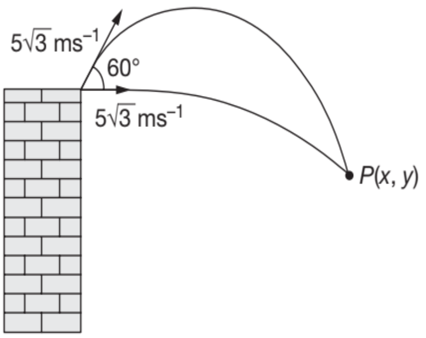
$$x = (5\sqrt{3}\cos 60)\ t = 5\sqrt{3}(t - \Delta t)$$

$$\Rightarrow\ t = 2t - 2\Delta t$$

$$\Rightarrow\ t = 2\Delta t \qquad \ldots(1)$$

$$\Rightarrow\ y = \left(-5\sqrt{3}\sin 60\right)\ t + \frac{1}{2}gt^2$$

$$\Rightarrow\ y = \frac{1}{2}g(t - \Delta t)^2$$



$$\Rightarrow\ -\frac{15}{2}t + 5t^2 = 5(t - \Delta t)^2$$

$$\Rightarrow\ -\frac{3}{2}t + t^2 = \frac{t^2}{4}$$

$$\Rightarrow\ -6t + 4t^2 = t^2$$

$$\Rightarrow\ 3t^2 - 6t = 0$$

$$\Rightarrow\ 3t(t - 2) = 0$$

$$\Rightarrow\ t = 2\ \text{s} \qquad \{\because t \neq 0\}$$

So, the interval between the firings is

$$\Delta t = \frac{t}{2} = 1\ \text{s}$$

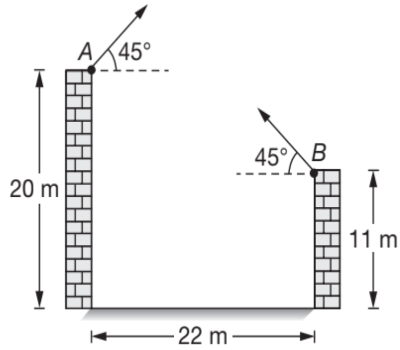
(b) $x = \left(5\sqrt{3}\cos 60\right)t = 5\sqrt{3}\ \text{m}$

$$\Rightarrow\ y = 5\ \text{m}$$

Hence the particles collide at $\left(5\sqrt{3}, 5\right)$ m.

ILLUSTRATION 29

Two particles are simultaneously thrown from the roofs of two high buildings as shown in figure. Their velocities are $v_A = 2\ \text{ms}^{-1}$ and $v_B = 14\ \text{ms}^{-1}$ respectively. Calculate the minimum distance between the particles in the process of their motion. Also find the time when they are at closest distance.



SOLUTION

Assuming A to be at rest

$$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = 0 \text{ as } \vec{a}_A = \vec{a}_B = g \text{ (downwards)}$$

Thus, the relative motion between them is uniform. Relative velocity of B with respect to A in vertical direction,

$$\left(u_{BA}\right)_V = u_B \sin 45° - u_A \sin 45° = (14 - 2)\frac{1}{\sqrt{2}}$$

$$\left(u_{BA}\right)_V = 6\sqrt{2}\ \text{ms}^{-1}$$

Relative velocity of B with respect to A in horizontal direction,

$$\left(u_{BA}\right)_H = u_B \cos 45° - \left(-u_A \cos 45°\right)$$

$$\left(u_{BA}\right)_H = (14 + 2)\frac{1}{\sqrt{2}} = 8\sqrt{2}\ \text{ms}^{-1}$$

Horizontal distance between A and B after time t

$$x = 22 - \left(\left(u_{BA}\right)_H\right)t = \left(22 - 8\sqrt{2}t\right)\text{m}$$

and vertical distance between A and B after time t is

$$y = 9 - \left(\left(u_{BA}\right)_V\right)t = \left(9 - 6\sqrt{2}t\right)\text{m}$$

Therefore, distance between them after time t is

$$r = \sqrt{x^2 + y^2}$$

$$\Rightarrow\ r^2 = x^2 + y^2 = \left(22 - 8\sqrt{2}t\right)^2 + \left(9 - 6\sqrt{2}t\right)^2$$

For r to be minimum $\frac{d}{dt}\left(r^2\right) = 0$

$$\Rightarrow\ 2\left(22 - 8\sqrt{2}t\right)\left(-8\sqrt{2}\right) + 2\left(9 - 6\sqrt{2}t\right)\left(-6\sqrt{2}\right) = 0$$

$$\Rightarrow\ 88 - 32\sqrt{2}t + 27 - 18\sqrt{2}t = 0$$

$$\Rightarrow \quad t = \frac{23}{10\sqrt{2}} \text{ s}$$

$$\Rightarrow \quad r_{min} = \sqrt{x^2 + y^2} \text{ at time } t = \frac{23}{10\sqrt{2}} \text{ s}$$

Substituting the values, we get

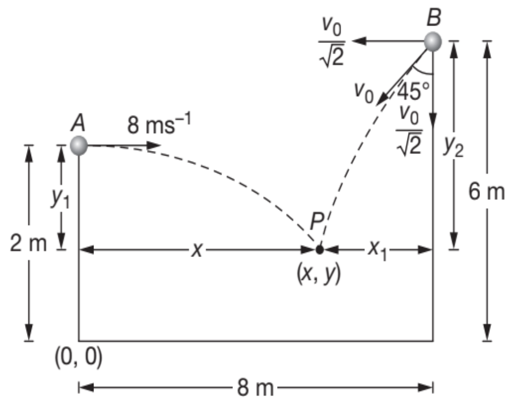
$$r_{min} = 6 \text{ m}$$

ILLUSTRATION 30

From points A and B, at the respective heights of 2 m and 6 m, two bodies are thrown simultaneously towards each other; one is thrown horizontally with a velocity of 8 ms^{-1} and the other, downward at an angle of $45°$ to the horizontal at an initial velocity such that the bodies collide in flight. The horizontal distance between points A and B equals 8 m. Calculate the initial velocity v_0 of the body thrown at an angle $45°$, the co-ordinates (x, y) of the point of collision, the time of flight t of the bodies before colliding and velocities v_A, v_B of the two bodies at the instant of collision. The trajectories lie in a single plane.

SOLUTION

The situation is shown here.



Let the two bodies collide after t second.

From figure, $x = 8 \times t$ and $x_1 = \left(\frac{v_0}{\sqrt{2}}\right) \times t$

$$\Rightarrow \quad x + x_1 = 8t + \left(\frac{v_0}{\sqrt{2}}\right)t$$

But given that $x + x_1 = 8$

$$\Rightarrow \quad 8 = 8t + \left(\frac{v_0}{\sqrt{2}}\right)t \quad \ldots(1)$$

Again $y_1 = \frac{1}{2}gt^2$

and $y = 2 - y_1 = 2 - \frac{1}{2}gt^2 \quad \ldots(2)$

Further, $y_2 = \left(\frac{v_0}{\sqrt{2}}\right)t + \frac{1}{2}gt^2$

$$\Rightarrow \quad y = 6 - \left(\frac{v_0}{\sqrt{2}}\right)t - \frac{1}{2}gt^2 \quad \ldots(3)$$

From equations (2) and (3), we get

$$2 - \frac{1}{2}gt^2 = 6 - \left(\frac{v_0}{\sqrt{2}}\right)t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad 4 = \left(\frac{v_0}{\sqrt{2}}\right)t \quad \ldots(4)$$

Substituting this value of t in equation (1), we get

$$8 = 8t + 4$$

$$\Rightarrow \quad t = 0.5 \text{ s}$$

From equation (4),

$$4 = \left(\frac{v_0}{\sqrt{2}}\right) \times 0.5$$

$$\Rightarrow \quad v_0 = 11.28 \text{ ms}^{-1}$$

Now, $x = 8t = 8 \times 0.5 = 4 \text{ m}$

$$y = 2 - \frac{1}{2}gt^2 = 0.775 \text{ m}$$

$$\Rightarrow \quad (x, y) = (4, 0.775)$$

At P, we have $v_A = \sqrt{v_{Ax}^2 + v_{Ay}^2}$

$$\Rightarrow \quad v_A = \sqrt{(8)^2 + (4.9)^2} = 9.4 \text{ ms}^{-1}$$

and $v_B = \sqrt{v_{Bx}^2 + v_{By}^2}$

$$\Rightarrow \quad v_B = \sqrt{\left(\frac{11.28}{\sqrt{2}}\right)^2 + \left(\frac{11.28}{\sqrt{2}} + \frac{9.8}{2}\right)^2}$$

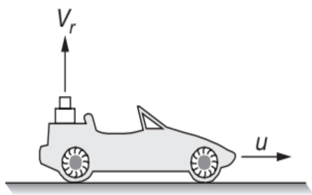
$$\Rightarrow \quad v_B = 15.2 \text{ ms}^{-1}$$

Test Your Concepts-III

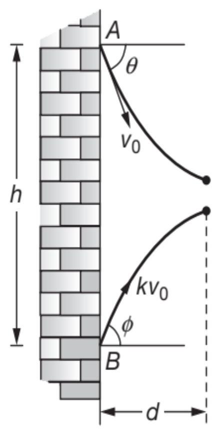
Based on Oblique Projectile

(Solutions on page H.146)

1. Find the angle of projection of a projectile for which the horizontal range and maximum height are equal.
2. Prove that the maximum horizontal range is four times the maximum height attained by the projectile.
3. Two projectiles are launched with same initial velocity at different launch angles so as to have identical range. Show that the sum of the maximum heights attained by them is independent of the launch angles.
4. Show that there are two values of time for which a projectile is at the same height. Also prove that the sum of these two times is equal to the time of flight. Also find the time lapse and the horizontal separation between the two events.
5. Two shots are fired simultaneously from the top and bottom of a vertical tower *AB* at angles α and β with horizontal respectively. Both shots strike at the same point *C* on the ground at distance *s* from the foot of the tower at the same time. Show that the height of the tower is $S(\tan\beta - \tan\alpha)$.
6. A ball is shot from the ground into the air. At a height of 9.1 m, its velocity is observed to be $\vec{v} = (7.6\hat{i} + 6.1\hat{j})$ ms^{-1}.
   - **(a)** To what maximum height does the ball rise?
   - **(b)** What total horizontal distance does the ball travel?
   - **(c)** Find the magnitude and direction of the ball's velocity just before it hits the ground?
7. The coach throws a baseball to a player with an initial speed of 20 ms^{-1} at an angle of 45° with the horizontal. At the moment the ball is thrown, the player is 50 m from the coach. At what speed and in what direction must the player run to catch the ball at the same height at which it was released?
8. A projectile is fired from the vertical tube mounted on the vehicle which is travelling at the constant speed $u = 30$ kmh^{-1}. The projectile leaves the tube with a velocity $v_r = 20$ ms^{-1} relative to the tube. If air resistance is neglected, show that the projectile will land on the vehicle at the tube location and calculate the distance *s* travelled by the vehicle during the flight of the projectile.



9. A stone is thrown from the top of a tower of height 50 m with a velocity of 30 ms^{-1} at an angle of 30° above the horizontal. Find the
   - **(a)** time during which the stone will be in air,
   - **(b)** distance from the tower base to where the stone will hit the ground,
   - **(c)** speed with which the stone will hit the ground. $(\text{Take } g = 10 \text{ ms}^{-2})$.
10. At the same instant two boys throw balls A and B from the window with a speed v_0 and kv_0, respectively, where k is a constant. Show that the balls with collide if $k = \dfrac{\cos\theta}{\cos\phi}$.

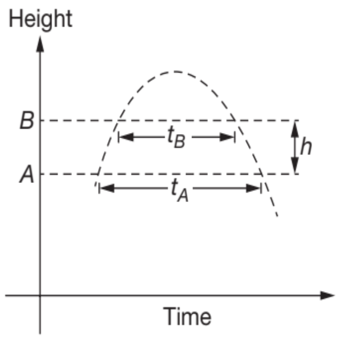


11. A projectile aimed at a mark which is in the horizontal plane through the point of projection falls *a* cm short of it when the elevation is α and goes *b* cm far when the elevation is β. Assume the speed of projection to be the same in all the cases, then show that the proper angle of projection is $\dfrac{1}{2}\sin^{-1}\left(\dfrac{a\sin(2\beta) + b\sin(2\alpha)}{a+b}\right)$.

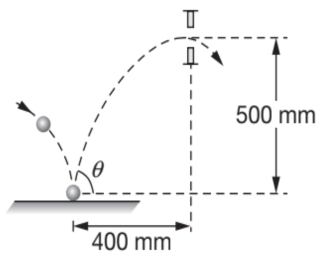
12. Two particles are simultaneously projected in the same vertical plane from the same point with velocities u and v at angles α and β with horizontal. Find the time that elapses when their velocities are parallel.

13. A projectile takes off with an initial velocity of $10\ \text{ms}^{-1}$ at an angle of elevation of 45°. It is just able to clear two hurdles of height 2 m each, separated from each other by a distance d. Calculate d. At what distance from the point of projection is the first hurdle placed? Take $g = 10\ \text{ms}^{-2}$.

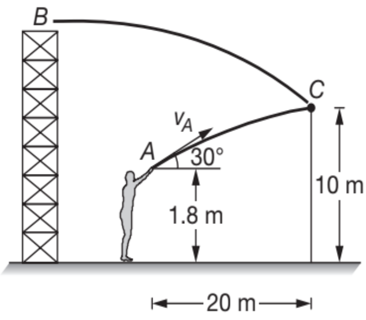
14. The acceleration of gravity can be measured by projecting a body upward and measuring the time it takes to pass two given points in both directions. Show that if the time the body takes to pass a horizontal line A in both directions is t_A and time to go by a second line B in both directions is t_B, then assuming that the acceleration is constant, its magnitude is $g = \dfrac{8h}{t_A^2 - t_B^2}$, where h is the height of the line B above line A.



15. To meet design criteria, small ball bearings must bounce through an opening of limited size at the top of their trajectory when rebounding from a heavy plate as shown. Calculate the angle θ made by the rebound velocity with the horizontal and the velocity v of the balls at the instant they pass through the opening.



16. A horizontal projectile is fired from a tower B. The shooter fires his gun from point A at an angle of 30°. Determine the muzzle speed of the bullet if it hits the projectile at C.



17. A particle is projected at an angle of elevation α and after t second it appears to have an elevation of β as seen from the point of projection. Find the initial velocity of projection.

18. A particle is projected with a velocity u at an angle α to the horizontal, in a vertical plane. At time t, it is moving in a direction making an angle β with the horizontal. Prove that $gt\cos\beta = u\sin(\alpha - \beta)$.

19. A ball is projected so as to just clear two walls, the first of height a at a distance b from the point of projection and the second of height b and at a distance a from point of projection. Show that the range on the horizontal plane is $R = \dfrac{a^2 + ab + b^2}{a + b}$.

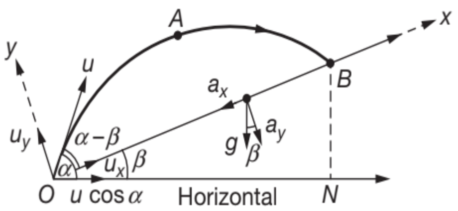
Also show that the angle of projection is given by $\tan^{-1}\left(\dfrac{a^2 + ab + b^2}{ab}\right)$.

20. The angular elevation of an enemy's position on a hill h m high is β. Show that, in order to shell it, the minimum initial velocity of the projectile must be $\sqrt{gh(1 + \operatorname{cosec}\beta)}$.

21. A gun is fired from a moving platform and ranges of the shot are observed to be R_1 and R_2 when the platform is moving forward and backward respectively with velocity v. Find the elevation of the gun α in terms of the given quantities.

MOTION OF A PROJECTILE UP AN INCLINED PLANE

Consider a projectile to be launched with initial velocity u making an angle α with horizontal on an inclined plane which makes angle $\beta(<\alpha)$ with the horizontal. Then u and g can be resolved into two components each (one along the incline and the other perpendicular to the incline)



$u_x = u\cos(\alpha-\beta)$
$u_y = u\sin(\alpha-\beta)$
$a_x = -g\sin\beta$
$a_y = -g\cos\beta$
OB = Range = R

(a) $u_x = u\cos(\alpha-\beta)$ along the incline along $+x$ axis.

(b) $u_y = u\sin(\alpha-\beta)$ along the incline along $+y$ axis.

(c) $|a_x| = g\sin\beta$ along $-x$ axis, acting as retardation to motion along the incline i.e., $a_x = -g\sin\beta$.

(d) $|a_y| = g\cos\beta$ along $-y$ axis, acting as retardation to motion perpendicular to incline i.e., $a_y = -g\cos\beta$.

Time of Flight (T) (Up the Incline)

Time taken by the particle to go from O to A to B. When the particle reaches B in a time $t=T$, then $y=0$.

Since, $y = u_y t + \frac{1}{2}a_y t^2$

$$\Rightarrow \quad 0 = \{u\sin(\alpha-\beta)\}T + \frac{1}{2}(-g\cos\beta)T^2$$

$$\Rightarrow \quad T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \qquad \{\because T \neq 0\}$$

Range of Projectile (R) (Up the Incline)

Since the particle goes from O to N with velocity component $u\cos\alpha$, uniformly with no acceleration. So,

$$ON = (u\cos\alpha)T$$

$$\Rightarrow \quad ON = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos\beta}$$

But $OB = \frac{ON}{\cos\beta}$

$$\Rightarrow \quad \text{Range} = R = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

We can also find R by the following method.

Since, $x = u_x t + \frac{1}{2}a_x t^2$ and at $t=T$, $x=R$

$$\Rightarrow \quad R = [u\cos(\alpha-\beta)]T + \frac{1}{2}(-g\sin\beta)T^2$$

Substituting $T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$, we get

$$R = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

Condition for Range to be Maximum (Up the Incline)

Since, $R = \frac{u^2}{g\cos^2\beta}[2\sin(\alpha-\beta)\cos\alpha]$

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\beta}[\sin(\alpha-\beta+\alpha)+\sin(\alpha-\beta-\alpha)]$$

$$\{\because 2\sin A\cos B = \sin(A+B)+\sin(A-B)\}$$

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta]$$

For R to be MAXIMUM, $\sin(2\alpha-\beta)$ is MAXIMUM i.e., 1

$$\Rightarrow \quad \sin(2\alpha-\beta) = 1$$

$$\Rightarrow \quad 2\alpha-\beta = \frac{\pi}{2}$$

$$\Rightarrow \quad \alpha = \frac{\pi}{4}+\frac{\beta}{2}$$

Maximum Range is given by

$$R_{max} = \frac{u^2}{g\cos^2\beta}(1-\sin\beta)$$

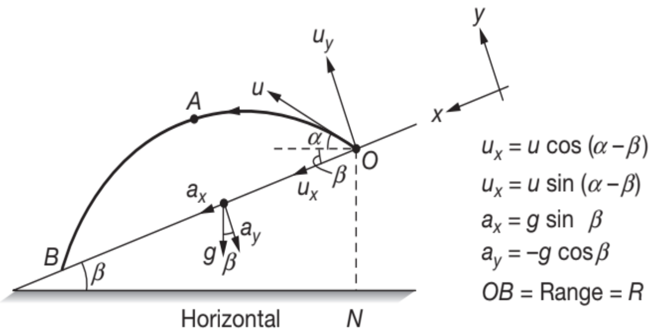
$$\Rightarrow \quad R_{max} = \frac{u^2}{g(1-\sin^2\beta)}(1-\sin\beta)$$

$$\Rightarrow \quad R_{max} = \frac{u^2}{g(1+\sin\beta)}$$

$$\Rightarrow \quad R_{\substack{\text{max. up}\\ \text{the incline}}} = \frac{R_{\text{max horizontal}}}{1+\sin\beta} \qquad \left\{\because R_{\text{max horizontal}} = \frac{u^2}{g}\right\}$$

MOTION OF A PROJECTILE DOWN AN INCLINED PLANE

Consider a projectile to be launched with initial velocity u making a angle α with horizontal down an inclined plane which makes angle β with the horizontal. Then u and g can be resolved into two components each (one along the incline and the other perpendicular to the incline)



(a) $u_x = u\cos(\alpha+\beta)$ along the incline along $+x$ axis.

(b) $u_y = u\sin(\alpha+\beta)$ along the incline along $+y$ axis.

(c) $|a_x| = g\sin\beta$ along $-x$ axis, acting as acceleration to the motion along the incline.

(d) $|a_y| = g\cos\beta$ along $-y$ axis, acting as retardation to the motion perpendicular to incline i.e., $a_y = -g\cos\beta$.

Time of Flight (T) (Down the Incline)

Time taken by the particle to go from O to A to B. When the particle reaches B in a time $t = T$, then $y = 0$.

Since, $y = u_y t + \frac{1}{2}a_y t^2$

$$\Rightarrow \quad 0 = \{u\sin(\alpha+\beta)\}T + \frac{1}{2}(-g\cos\beta)T^2$$

$$\Rightarrow \quad T = \frac{2u\sin(\alpha+\beta)}{g\cos\beta} \qquad \{\text{as } T \neq 0\}$$

Range of Projectile (R) (Down the Incline)

Since the particle goes from O to N with velocity component $u\cos\alpha$, uniformly with no acceleration. So,

$$ON = (u\cos\alpha)T$$

$$\Rightarrow \quad ON = \frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos\beta}$$

But $OB = \dfrac{ON}{\cos\beta}$

$$\Rightarrow \quad \text{Range} = R = \frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$$

We can also find R by the following method.

Since, $x = u_x t + \frac{1}{2}a_x t^2$ and

at $t = T$, $x = R$

$$\Rightarrow \quad R = [u\cos(\alpha+\beta)]T + \frac{1}{2}(g\sin\beta)T^2$$

Substituting $T = \dfrac{2u\sin(\alpha+\beta)}{g\cos\beta}$, we get

$$R = \frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$$

Condition for Range to be Maximum (Down the Incline)

Since, $R = \dfrac{u^2}{g\cos^2\beta}[2\sin(\alpha+\beta)\cos\alpha]$

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\beta}[\sin(\alpha+\beta+\alpha)+\sin(\alpha+\beta-\alpha)]$$

$$\{\because\ 2\sin A\cos B = \sin(A+B)+\sin(A-B)\}$$

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha+\beta)+\sin\beta]$$

For R to be MAXIMUM, $\sin(2\alpha+\beta)$ is MAXIMUM i.e. 1

$$\Rightarrow \quad \sin(2\alpha+\beta) = 1$$

$$\Rightarrow \quad 2\alpha+\beta = \frac{\pi}{2}$$

$$\Rightarrow \quad \alpha = \frac{\pi}{4} - \frac{\beta}{2}$$

Maximum range is given by

$$R_{\text{max}} = \frac{u^2}{g\cos^2\beta}(1+\sin\beta)$$

$$\Rightarrow \quad R_{\text{max}} = \frac{u^2}{g(1-\sin^2\beta)}(1+\sin\beta)$$

$$\Rightarrow \quad R_{\max} = \frac{u^2}{g(1-\sin\beta)}$$

Since $R_{\text{max horizontal}} = \dfrac{u^2}{g}$

$$\Rightarrow \quad R_{\substack{\text{max. down}\\ \text{the incline}}} = \frac{R_{\text{max horizontal}}}{1-\sin\beta}$$

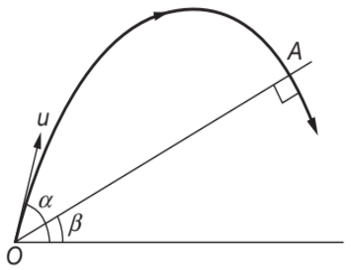
Condition for the Particle Launched Up the Incline to Strike it Normally

Let T be the time of flight from O to A, then

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \qquad \ldots(1)$$

Now we shall consider the motion of the particle along OA. Initial velocity along OA is

$$u_x = u\cos(\alpha-\beta)$$



Since the particle strikes the plane A at right angles, so the final velocity along OA is $v_x = 0$.

Acceleration due to gravity along OA is $a_x = -g\sin\beta$

Since $x = u_2 t + \dfrac{1}{2}a_2 t^2$

$$\Rightarrow \quad 0 = u\cos(\alpha-\beta) - (g\sin\beta)t \qquad \{\because v_x = u_x + a_x t\}$$

$$\Rightarrow \quad t = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$T = t$$

$$\Rightarrow \quad \frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

$$\Rightarrow \quad 2\tan(\alpha-\beta) = \cot\beta$$

$$\Rightarrow \quad 2\left(\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right) = \cot\beta$$

$$\Rightarrow \quad 2\tan\alpha - 2\tan\beta = \cot\beta + \tan\alpha$$

$$\Rightarrow \quad \tan\alpha = \cot\beta + 2\tan\beta \qquad \ldots(3)$$

Condition for the Particle Launched Up the Incline to Strike it Horizontally

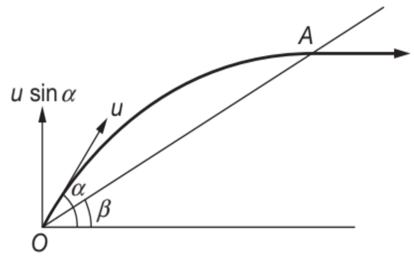
When the particle strikes the plane horizontally. Again in this case, time of flight is given by

$$t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

Since the particle strikes the plane horizontally, so vertical component of velocity at that instant is zero. Hence we get

$$0 = u\sin\alpha - gt$$

$$\Rightarrow \quad u\sin\alpha = g\left[\frac{2u\sin(\alpha-\beta)}{g\cos\beta}\right]$$



$$\Rightarrow \quad \sin\alpha\cos\beta = 2[\sin\alpha\cos\beta - \cos\alpha\sin\beta]$$

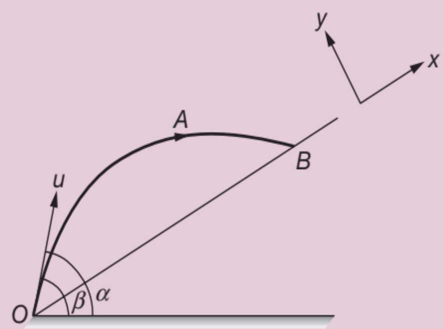
$$\Rightarrow \quad 2\cos\alpha\sin\beta = \sin\alpha\cos\beta$$

$$\Rightarrow \quad 2\tan\beta = \tan\alpha$$

Problem Solving Technique(s)

Following two points are important regarding the projectile motion in inclined planes:

(a) Time taken by the projectile to move from O to A is half the time of flight. Here A is a point where velocity of projectile is parallel to x-direction.



i.e., $t_{OA} = \frac{T}{2} = \frac{u\sin(\alpha-\beta)}{g\cos\beta}$

This can be proved as under,

At A, $v_y = 0 = u_y + a_y t_{OA}$

$$\Rightarrow \quad t_{OA} = -\frac{u_y}{a_y} = -\left[\frac{u\sin(\alpha-\beta)}{-g\cos\beta}\right] = \frac{u\sin(\alpha-\beta)}{g\cos\beta}$$

(b) At point B, where the projectile strikes the plane the y-component of its velocity (v_y) is just equal and opposite to the component with which it was projected from point O. i.e., $v_y = -u_y$ at B

This can be proved as:

$$v_y = u_y + a_y T$$

$$v_y = u\sin(\alpha-\beta) + (-g\cos\beta)\left[\frac{2u\sin(\alpha-\beta)}{g\cos\beta}\right]$$

$$v_y = -u\sin(\alpha-\beta) = -u_y$$

ILLUSTRATION 31

A ball starts falling with zero initial velocity on a smooth inclined plane which forms an angle α with the horizontal. Having fallen through a height h, the ball rebounds elastically off the inclined plane. At what distance from the impact point will the ball rebound for the second time?

SOLUTION

Resolving the component(s) of acceleration w.r.t. the inclined plane, we have

(a) $g\sin\alpha$, acceleration along the inclined plane, and

(b) $g\cos\alpha$, acceleration perpendicular to the inclined plane.

After falling freely through a height h, velocity at the point of striking is

$$v_0 = \sqrt{2gh} \qquad \ldots(1)$$

Further since the collision is perfectly elastic, hence, angle of rebound is also α.

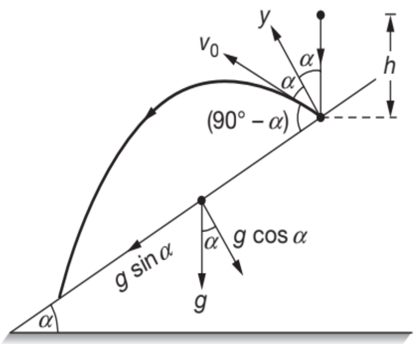
Motion Along the Inclined Plane

$$x = v\cos(90-\alpha)t + \frac{1}{2}(g\sin\alpha)t^2 \qquad \ldots(2)$$

Motion Perpendicular to the Inclined Plane

$$\Rightarrow \quad 0 = v_o\sin(90-\alpha)t + \frac{1}{2}(-g\cos\alpha)t^2$$

$$\Rightarrow \quad t = \frac{2v_o}{g} = \frac{2\sqrt{2gh}}{g}$$



Put value of t in (2) we get

$$x = v_o\sin\alpha\left(\frac{2v_o}{g}\right) + \frac{1}{2}g\sin\alpha\left(\frac{4v_o^2}{g^2}\right)$$

$$\Rightarrow \quad x = \frac{2v_o^2\sin\alpha}{g} + \frac{2v_o^2\sin\alpha}{g}$$

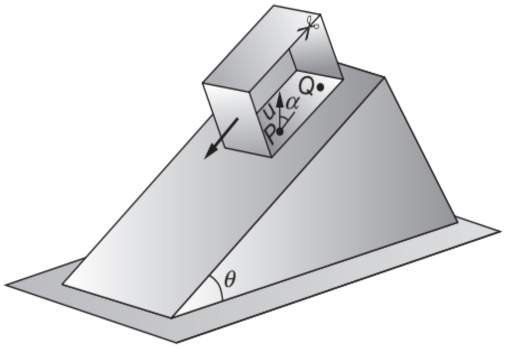
$$\Rightarrow \quad x = \frac{4v_o^2\sin\alpha}{g}$$

$$\Rightarrow \quad x = \frac{4\left(\sqrt{2gh}\right)^2\sin\alpha}{g} \qquad \{\text{using (1)}\}$$

$$\Rightarrow \quad x = 8h\sin\alpha$$

ILLUSTRATION 32

A large heavy box is sliding without friction down a smooth inclined plane of inclination θ. From a point P on bottom of box, a particle is projected inside the box. The initial speed of particle with respect to box is u and direction of projection makes and angle α with the bottom as shown in the figure.



(a) Find the distance along the bottom of box between point of projection P and point Q where the particle lands. (Assume that the particle does not hit any other surface of the box. Neglect air resistance).

(b) If the horizontal displacement of a particle as seen by an observer on ground is zero, find the speed of box with respect to ground at the instant the particle was projected.

SOLUTION

Lets consider reference frame x-y fixed with box with origin at one corner O and moving down with box. The acceleration of this frame down the incline is $g\sin\theta$.

Acceleration of the particle launched inside the box w.r.t. ground is g (vertically downwards) with components

(a) $g\sin\theta$; (along the negative x-axis) acting as retardation for particle's motion along the incline, and

(b) $g\cos\theta$; (along the negative y-axis) acting as retardation for particle's motion perpendicular to the incline.

Relative acceleration of the particle in reference frame along x-axis is

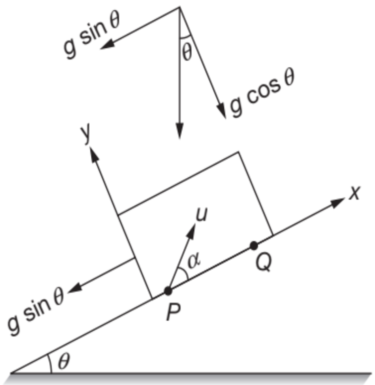
$$a_x = -g\sin\theta - (-g\sin\theta) = 0$$

Relative acceleration of the particle in reference frame along y-axis is

$$a_y = -g\cos\theta = 0$$

$$\Rightarrow \quad a_y = -g\cos\theta$$

Also the x component of particle's velocity in box $u_x = u\cos\alpha$ and the y component of the particle's velocity in box is $u_y = u\sin\alpha$.



When the particle (strikes) arrives at Q, net displacement perpendicular to the incline is zero.

$$\Rightarrow \quad 0 = (u\sin\alpha)\ t + \frac{1}{2}(-g\cos\theta)\ t^2$$

$$\Rightarrow \quad t\left[\left(\frac{1}{2}g\cos\theta\right)\ t - (u\sin\alpha)\right] = 0$$

Since $t \neq 0$

$$\Rightarrow \quad t = \frac{2u\sin\alpha}{g\cos\theta}$$

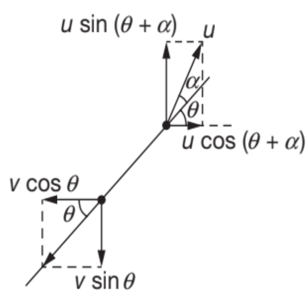
Since $a_x = 0$, so horizontal motion is non accelerated motion with uniform velocity, we have

$$PQ = (u\cos\alpha)t = \left(\frac{2u\sin\alpha}{g\cos\theta}\right)u\cos\alpha$$

$$\Rightarrow \quad PQ = \frac{u^2\sin 2\alpha}{g\cos\theta}$$

METHOD I

If the incline moves down with a velocity v and moves a distance QP, then it will appear to an observer on ground that horizontal displacement of particle is zero



For Box

$$PQ = vt + \frac{1}{2}(a_x)t^2$$

$$\Rightarrow \quad PQ = v\frac{2u\sin\alpha}{g\cos\theta} + \frac{1}{2}(g\sin\theta)\left(\frac{2u\sin\alpha}{g\cos\theta}\right)^2$$

$$\Rightarrow \quad \frac{2u\sin\alpha}{g\cos\theta}\left[v + \frac{u\sin\theta\sin\alpha}{\cos\theta}\right] = \frac{2u^2\sin\alpha\cos\alpha}{g\cos\theta}$$

{using part (a)}

$$\Rightarrow \quad v = u\left[\cos\alpha - \frac{\sin\theta\sin\alpha}{\cos\theta}\right]$$

$$\Rightarrow \quad v = u\frac{\cos(\theta+\alpha)}{\cos\theta}$$

METHOD II

For no horizontal displacement,

$$\Rightarrow \quad u\cos(\theta+\alpha) - v\cos\theta = 0$$

$$\Rightarrow \quad v = \frac{u\cos(\theta+\alpha)}{\cos\theta}$$

ILLUSTRATION 33

Two bodies are projected from the same point with equal speeds in such directions that they both strike the same point on a plane whose inclination is β. If α be the angle of projection of the first body with the horizontal show that the ratio of their times of flight is $\dfrac{\sin(\alpha-\beta)}{\cos\alpha}$.

SOLUTION

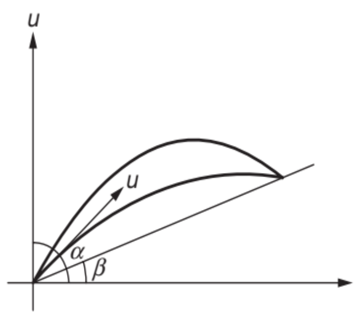
For the first body, we have

$$R = \frac{u^2}{g\cos^2\beta}\{\sin(2\alpha-\beta) - \sin\beta\}$$

Let α' be the angle of projection of the second body. Since, range of both the bodies is same. Therefore,

$$\sin(2\alpha-\beta) = \sin(2\alpha'-\beta)$$

$$\Rightarrow \quad 2\alpha' - \beta = \pi - (2\alpha-\beta)$$



$$\Rightarrow \quad \alpha' = \frac{\pi}{2} - (\alpha-\beta)$$

Now, $T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$ and $T' = \dfrac{2u\sin(\alpha'-\beta)}{g\cos\beta}$

$$\Rightarrow \quad \frac{T}{T'} = \frac{\sin(\alpha-\beta)}{\sin(\alpha'-\beta)} = \frac{\sin(\alpha-\beta)}{\sin\left(\frac{\pi}{2}-(\alpha-\beta)-\beta\right)}$$

$$\Rightarrow \quad \frac{T}{T'} = \frac{\sin(\alpha-\beta)}{\sin\left(\frac{\pi}{2}-\alpha\right)} = \frac{\sin(\alpha-\beta)}{\cos\alpha}$$

ILLUSTRATION 34

A particle projected with velocity u strikes at right angles a plane through the point of projection inclined at an angle β to the horizon. Show that the height of the point struck above the horizontal plane through the point of projection is $\dfrac{2u^2}{g}\left(\dfrac{\sin^2\beta}{1+3\sin^2\beta}\right)$ and that the time of flight up to that instant is, $t = \dfrac{2u}{g\sqrt{1+3\sin^2\beta}}$.

SOLUTION

$$\text{Time of flight } T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \qquad \ldots(1)$$

Also, we have, at B, at time t

$$v_x = 0$$

$$\Rightarrow \quad u_x + a_x t = 0$$

$$\Rightarrow \quad u\cos(\alpha-\beta) - g\sin\beta t = 0$$

$$\Rightarrow \quad t = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \qquad \ldots(2)$$

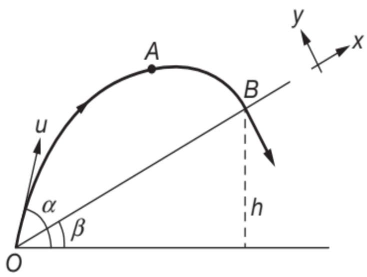
These two times being the same, can be equated. So, we get

$$T = t$$

$$\Rightarrow \quad \frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

$$\Rightarrow \quad 2\tan(\alpha-\beta) = \cot\beta$$

$$\Rightarrow \quad 2\left(\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right) = \cot\beta$$



$$\Rightarrow \quad 2\tan\alpha - 2\tan\beta = \cot\beta + \tan\alpha$$

$$\Rightarrow \quad \tan\alpha = 2\tan\beta + \cot\beta \qquad \ldots(3)$$

Further $OB = \text{Range} = R = \dfrac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$ ...(4)

and $h = R\sin\beta$

So, let us find R in terms of u and β only by eliminating α, after taking help from equation (3).

Since $\tan\alpha = 2\tan\beta + \cot\beta = 2\tan\beta + \dfrac{1}{\tan\beta}$

$$\Rightarrow \quad \tan\alpha = \frac{2\tan^2\beta + 1}{\tan\beta} = \frac{1+\sin^2\beta}{\sin\beta\cos\beta}$$

$$\Rightarrow \quad \cos\alpha = \frac{\sin\beta\cos\beta}{\sqrt{(1+\sin^2\beta)^2 + \sin^2\beta\cos^2\beta}}$$

$$\Rightarrow \quad \cos\alpha = \frac{\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}$$

$$\Rightarrow \quad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{\cos\beta}{\sqrt{1+3\sin^2\beta}}$$

Substituting these values in equation (4), we get

$$R = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

$$\Rightarrow \quad R = \frac{2u^2}{g\cos^2\beta}\left(\frac{\cos\beta}{\sqrt{1+3\sin^2\beta}}\right)\left(\frac{\sin\beta\cos\beta}{\sqrt{1+3\sin^2\beta}}\right)$$

$$\Rightarrow \quad R = \frac{2u^2\sin\beta}{g(1+3\sin^2\beta)}$$

So, $h = R\sin\beta = \dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$

Substituting value of $\sin(\alpha-\beta) = \dfrac{\cos\beta}{\sqrt{1+3\sin^2\beta}}$ in (1), we get

$$T = \frac{2u\cos\beta}{g\cos\beta\sqrt{1+3\sin^2\beta}} = \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

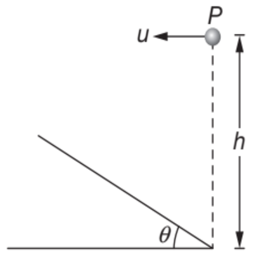
Test Your Concepts-IV

Based on Projectile on an Inclined Plane

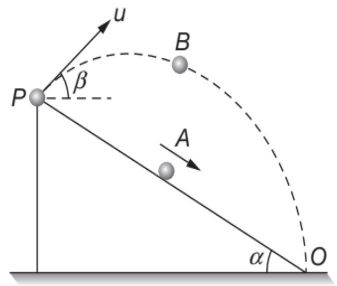
(Solutions on page H.150)

1. A heavy particle is projected from a point at the foot of a fixed plane, inclined at an angle 45° to the horizontal, in the vertical plane containing the line of greatest slope through the point. If $\phi(>45^\circ)$ is the inclination to the horizontal of the initial direction of projection, for what value of $\tan\phi$ will the particle strike the plane
   (a) horizontally
   (b) at right angle
2. Two parallel straight lines are inclined to the horizontal at an angle α. A particle is projected from a point mid way between them so as to graze one of the lines and strike the other normally. Show that if θ is the angle between the direction of projection and either of lines, then

   $$\tan\theta = (\sqrt{2}-1)\cot\alpha$$
3. A projectile is fired with a velocity u at right angles to the slope, which is inclined at an angle θ with the horizontal. Derive an expression for the distance R to the point of impact.
4. Determine the horizontal velocity u with which a stone must be projected horizontally from a point P, so that it may hit the inclined plane perpendicularly. The inclination of the plane with the horizontal is θ and point P is at a height h above the foot of the incline, as shown in the figure.



5. A perfectly elastic ball is thrown from the foot of a plane whose inclination to horizontal is β. If after striking the plane at a distance R from the point of projection it rebounds and retraces its former path, find the velocity of projection.
6. Particle A is released from a point P on a smooth inclined plane inclined at an angle α with the horizontal. At the same instant another particle B is projected with initial velocity u making an angle β with the horizontal. Both the particles meet again on the inclined plane. Find the relation between α and β.



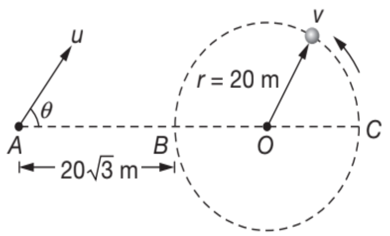
SOLVED PROBLEMS

PROBLEM 1

A particle is moving along a vertical circle of radius $r = 20$ m with a constant speed $v = 31.4\ \text{ms}^{-1}$ as shown in figure. Straight line ABC is horizontal and passes through the centre of the circle. A shell is fired from point A at the instant when the particle is at C. If distance AB is $20\sqrt{3}$ m and the shell collide with the particle at B, then prove

$$\tan\theta = \frac{(2n-1)^2}{\sqrt{3}}$$

where n is an integer. Further, show that smallest value of θ is 30°.



SOLUTION

At the time of firing of the shell, the particle was at C and the shell collides with it at B, therefore the number of revolutions completed by the particle is odd multiple of half, i.e., $\frac{(2n-1)}{2}$, where n is an integer.

Let t be the time period of revolution of the particle, then

$$t = \frac{2\pi r}{v} = \frac{2\times 3.14\times 20}{31.4} = 4\ \text{s}$$

If T be the time of flight of the shell, then this time T equals the time of $\left(\frac{2n-1}{2}\right)$ revolutions of the particle

$$\Rightarrow \quad T = \frac{(2n-1)}{2}\times 4 = 2(2n-1)\ \text{s}$$

For a projectile, the time of flight is given by

$$T = \frac{2u\sin\theta}{g}$$

So, $T = t$

$$\Rightarrow \quad \frac{2u\sin\theta}{g} = 2(2n-1) \qquad \ldots(1)$$

The range of the projectile is given by

$$R = \frac{2u^2\sin\theta\cos\theta}{g} = 20\sqrt{3} \qquad \ldots(2)$$

From (1) and (2), we get

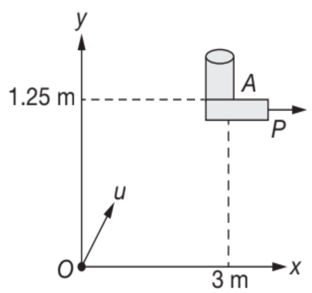
$$\tan\theta = \frac{(2n-1)^2}{\sqrt{3}}$$

For smallest θ, $n = 1$

$$\Rightarrow \quad \theta_{\min} = 30° = \frac{\pi}{3}\ \text{radian}$$

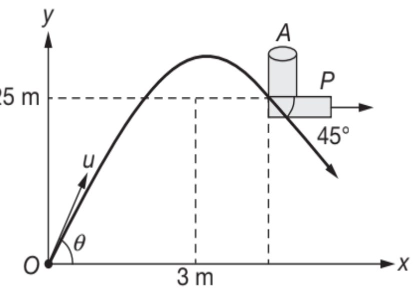
PROBLEM 2

An object A is kept fixed at the point $x = 3$ m and $y = 1.25$ m on a plank P raised above the ground. At time $t = 0$, the plank starts moving along the $+x$ direction with an acceleration $1.5\ \text{ms}^{-2}$. At the same instant a stone is projected from the origin with a velocity u as shown. A stationary person on the ground observes the stone hitting the object during its downward motion at an angle of 45° to the horizontal. All the motions are in the x-y plane. Find u and the time after which the stone hits the object. Take $g = 10\ \text{ms}^{-2}$.



SOLUTION

For stone to hit the object, A, at time t



$$x = (u\cos\theta)t = x_o + \frac{1}{2}at^2$$

$$\Rightarrow \quad (u\cos\theta)t = 3 + \frac{1}{2}(1.5)t^2 \qquad \ldots(1)$$

$$\Rightarrow \quad y = (u\sin\theta)t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad (u\sin\theta)t - \frac{1}{2}gt^2 = 1.25 \qquad \ldots(2)$$

Since stone hits A moving along $45°$ with horizontal during its downward motion, so

$$\tan(-45°) = \frac{u\sin\theta - gt}{u\cos\theta} = -1 \qquad \ldots(3)$$

$$\Rightarrow \quad u\sin\theta - gt = -u\cos\theta$$

$$\Rightarrow \quad (u\sin\theta)t - gt^2 = -(u\cos\theta)t \qquad \ldots(4)$$

Substituting values of $(u\sin\theta)t$ and $(u\cos\theta)t$ from (2) and (1) respectively and taking $g = 10 \text{ ms}^{-2}$ we get

$$1.25 + 5t^2 - 10t^2 = -3 - (0.75)t^2$$

$$\Rightarrow \quad (5 - 0.75)t^2 = (3 + 1.25)$$

$$\Rightarrow \quad t^2 = 1$$

$$\Rightarrow \quad t = 1 \text{ s}$$

So, from (1), $u\cos\theta = 3.75$ and $u\sin\theta = 6.25$

$$\Rightarrow \quad \vec{u} = u_x\hat{i} + u_y\hat{j}$$

$$\Rightarrow \quad \vec{u} = (3.75)\hat{i} + (6.25)\hat{j}$$

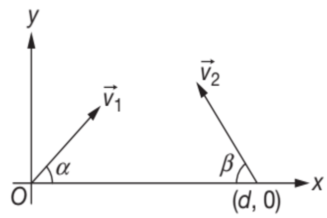
PROBLEM 3

Two particles A and B simultaneously leave 2 points O and A separated by a distance d, with velocities $\vec{v}_1$ and $\vec{v}_2$. The direction in which the particle travels forms an angle α and the second particle forms an angle β with the line OA. At what time, will the distance between the particles be minimum, also calculate this minimum distance between particles.

SOLUTION

METHOD I: FRAME FIXED w.r.t. EARTH

Let distance between the given particles be minimum at time t



For 1st particle, B

$$\Rightarrow \quad \vec{r}_1 = v_1\left(\cos\alpha\hat{i} + \sin\alpha\hat{j}\right)t$$

For 2nd particle, A

$$\Rightarrow \quad \vec{r}_2 = \left(d - v_2 t\cos\beta\right)\hat{i} + v_2 t\sin\beta\hat{j}$$

Separation between particles is $\vec{r} = \vec{r}_2 - \vec{r}_1$

$$|\vec{r}|^2 = r^2 = \left[d - \left(v_2\cos\beta + v_1\cos\alpha\right)t\right]^2 + \left[\left(v_2\sin\beta - v_1\sin\alpha\right)t\right]^2$$

For distance between particles to be minimum, $\frac{dr}{dt} = 0$

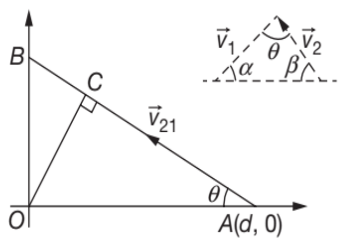
$$\Rightarrow \quad 2r\frac{dr}{dt} = 2\left[d - \left(v_2\cos\beta + v_1\cos\alpha\right)t\right] \times \left(-v\cos\beta - v_1\cos\alpha\right) + 2\left(v_2\sin\beta - v_1\sin\alpha\right)^2 t$$

$$\Rightarrow \quad t = \frac{d\left(v_2\cos\beta + v_1\cos\alpha\right)}{[v_1^2 + v_2^2 + 2v_1v_2\cos(\alpha+\beta)]}$$

$$\Rightarrow \quad r_{\min} = \left[\frac{d\left(v_2\sin\beta - v_1\sin\alpha\right)}{\sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos\left(\alpha+\beta\right)}}\right]$$

METHOD II: FRAME FIXED w.r.t. PARTICLE

Since both projectiles are moving under the influence of gravity having a constant value of $g = 9.8 \text{ ms}^{-2}$ acting vertically downwards for both. Hence, we arrive at the following conclusion(s)



(a) Relative vertical acceleration of one projectile w.r.t. other is zero.

(b) Relative horizontal acceleration of one projectile w.r.t. other is zero.

(c) As a consequence of (a) and (b), both the projectiles must follow a straight line motion along AC with uniform relative velocity $\vec{v}_{21}$, where

$$\vec{v}_{21} = \vec{v}_2 - \vec{v}_1 \quad \ldots(1)$$

Hence, minimum distance between the particles is OC i.e. perpendicular distance i.e. length of the perpendicular dropped from O on AC to meet at C.

Further, the angle between $\vec{v}_1$ and $\vec{v}_2$ is $180° - (\alpha + \beta)$. Therefore,

$$v_{21}^2 = v_1^2 + v_2^2 - 2v_1v_2 \cos\left[180° - (\alpha+\beta)\right]$$

$$\Rightarrow \quad v_{21}^2 = v_1^2 + v_2^2 + 2v_1v_2 \cos(\alpha+\beta) \quad \ldots(2)$$

Let $\vec{v}_{21}$ make an angle θ with the horizontal axis (i.e. x-axis)

$$\Rightarrow \quad \vec{v}_{21} = v_{21}\left(-\cos\theta\hat{i} + \sin\theta\hat{j}\right) \quad \ldots(3)$$

$$\Rightarrow \quad \vec{v}_2 = v_2\left(-\cos\beta\hat{i} + \sin\beta\hat{j}\right) \quad \ldots(4)$$

$$\Rightarrow \quad \vec{v}_1 = v_1\left(\cos\alpha\hat{i} + \sin\alpha\hat{j}\right) \quad \ldots(5)$$

Substituting (3), (4) & (5) in equation (1) and comparing we get

$$v_{21}\cos\theta = v_2\cos\beta + v_1\cos\alpha \quad \ldots(6)$$

$$v_{21}\sin\theta = v_2\sin\beta - v_1\sin\alpha \quad \ldots(7)$$

$$\Rightarrow \quad \cos\theta = \frac{v_2\cos\beta + v_1\cos\alpha}{v_{21}}$$

$$\Rightarrow \quad \cos\theta = \frac{v_2\cos\beta + v_1\cos\alpha}{\sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos(\alpha+\beta)}}$$

$$\Rightarrow \quad r_{\min} = OC = d\sin\theta = \frac{d(v_2\sin\beta - v_1\sin\alpha)}{\sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos(\alpha+\beta)}}$$

In right triangle OAC,

$$r_{\min} = OC = d\sin\theta = \frac{d(v_2\sin\beta - v_1\sin\alpha)}{\sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos(\alpha+\beta)}}$$

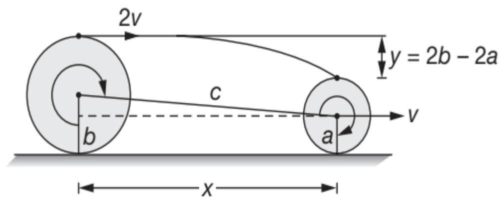
$$t = \frac{d\cos\theta}{v_{21}} = \frac{d(v_2\cos\beta + v_1\cos\alpha)}{[v_1^2 + v_2^2 + 2v_1v_2\cos(\alpha+\beta)]}$$

PROBLEM 4

The radii of the front and rear wheels of a carriage are a, b and c is the distance between the axle centres. A particle of dust driven from the highest point of the rear wheel is observed to alight on the highest point of the front wheel. Show that the velocity of the carriage is $\dfrac{\sqrt{(c+b-a)(c-b+a)g}}{4(b-a)}$.

SOLUTION

Velocity of projection of the dust particle from the top of the wheel is $2v$ horizontally.



For horizontal projectile

$$y = \frac{gx^2}{2v^2}$$

$$2b - 2a = \frac{g\left[c^2 - (b-a)^2\right]}{2v^2}$$

$$\Rightarrow \quad v = \sqrt{\frac{g(c+b-a)(c-b+a)}{4(b-a)}}$$

PROBLEM 5

A batsman hits a pitched cricket ball at a height of 1.2 m above the ground so that its angle of projection is 45° and its horizontal range is 110 m. The ball is lifted towards the left field line where a fence of 7.5 m is located 100 m from the position of the batsman. Will the ball clear the fence?

SOLUTION

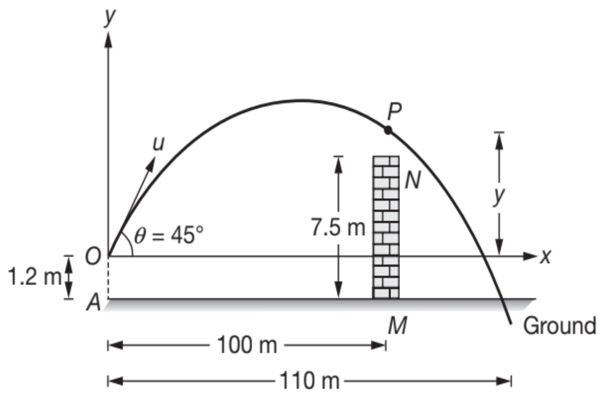
Lets firstly calculate initial velocity of launch of projectile. Considering origin at O we have

$$-1.2 = 110\tan(45°) - \frac{g(110)^2}{2u^2\cos^2(45°)}$$

$$\Rightarrow \quad u = 32.66 \text{ ms}^{-1}$$

Lets calculate the vertical distance of ball from x-axis at the position of fence. i.e. we are to find y for

$$x = 100 \text{ m}, \ \theta = 45°, \ u = 32.66 \text{ ms}^{-1}$$



$$y = 100\tan(45°) - \frac{9.8\times(100)^2}{2\times(32.66)^2\cos^2(45°)}$$

$$y = 100 - 91.87 = 8.13 \text{ m}$$

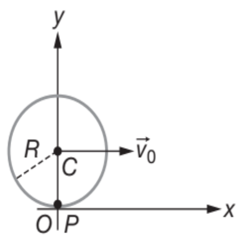
Total height of ball from the ground is

$$(1.2 + 8.13) = 9.33 \text{ m}$$

Since $9.33 \text{ m} > 7.5 \text{ m}$, therefore, the ball will clear the fence.

PROBLEM 6

A wheel of radius R rolls without slipping along the x-axis with constant speed v_0 (as shown in the figure). Investigate the motion of a point P on the rim of the wheel which starts from the origin O and find the total distance covered by the point between two successive moments at which it touches the surface.



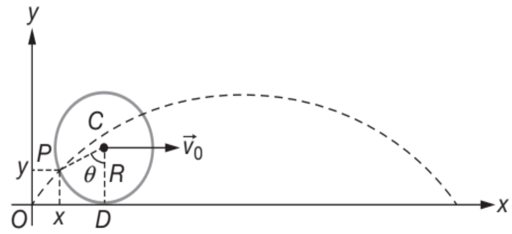
SOLUTION

After the time interval t, the center C of the wheel will have travelled a distance $v_0 t$ as shown, and since it rolls without slipping, the arc DP will also have the length $v_0 t$. Thus the angle DCP will be $\theta = \frac{v_0 t}{R}$, Then from the geometry of the figure, we can express the coordinates x and y of the point P as follows:

$$x = v_0 t - R\sin\left(\frac{v_0 t}{R}\right)$$

$$y = R - R\cos\left(\frac{v_0 t}{R}\right) \qquad \ldots(1)$$

With t as a parameter, these two equations define in rectangular coordinates the path of point P which is called a **CYCLOID**. In other words the two equations give the motion law for the point P on the rim of the wheel in the case of pure rolling with uniform motion.



Differentiating equation (1) with respect to time gives the velocity-time equations as follows:

$$v_x = v_0\left[1 - \cos\left(\frac{v_0 t}{R}\right)\right]$$

$$v_y = v_0 \sin\left(\frac{v_0 t}{R}\right) \qquad \ldots(2)$$

$$\Rightarrow \quad v = \sqrt{v_x^2 + v_y^2} = v_0\sqrt{2\left[1 - \cos\left(\frac{v_0 t}{R}\right)\right]}$$

$$\Rightarrow \quad v = 2v_0 \sin\left(\frac{v_0 t}{2R}\right) \qquad \ldots(3)$$

From this expression, we see that the maximum speed of point P is $2v_0$ when $t = \frac{\pi R}{v_0}$, that is, when the point P is at the top of its path.

Differentiating equation (2) again with respect to time, we obtain the acceleration-time equations

$$\Rightarrow \quad a_x = \frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right)$$

$$a_y = \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right) \qquad \ldots(4)$$

Therefore, $a = \sqrt{a_x^2 + a_y^2} = \frac{v_0^2}{R}$ $\qquad \ldots(5)$

Thus the point P has acceleration of constant magnitude always directed towards the centre C of the rolling wheel.

The total distance traversed by the point P between two successive moments at which it touches the surface.

$$\Rightarrow \quad s = \int v dt = \int_0^{\frac{2\pi R}{v_o}} \left[2v_o \sin\left(\frac{v_o t}{2R}\right)\right] dt$$

$$\Rightarrow \quad s = 8R$$

PROBLEM 7

A canon fires successively two shells with velocity u, the first at an angle θ_1 and second at an angle θ_2 with horizontal. Neglecting air drag. Find the time interval between firings leading to collision of shells.

SOLUTION

METHOD I

Let the time of flight before collision be t, i.e., let first shell reach $P(x, y)$ in time t.

Let second shell be fired with a delay say Δt.

So it falls short of time by Δt and has net time $(t - \Delta t)$ to be at P.

For First shell

$$x = (u \cos\theta_1)t \qquad \ldots(1)$$

For Second shell

$$x = (u\cos\theta_2)(t - \Delta t) \qquad \ldots(2)$$

$$\Rightarrow \quad \frac{t - \Delta t}{t} = \frac{\cos\theta_1}{\cos\theta_2}$$

$$\Rightarrow \quad \frac{\Delta t}{t} = \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_2} \qquad \ldots(3)$$

From (1), $t = \dfrac{x}{u\cos\theta_1}$

$$\Rightarrow \quad \frac{\Delta t(u\cos\theta_1)}{x} = \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_2}$$

$$\Rightarrow \quad \frac{u\Delta t}{x} = \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_1 \times \cos\theta_2} \qquad \ldots(4)$$

Also, $y = x\tan\theta_1 - \dfrac{gx^2}{2u^2\cos^2\theta_1}$ $\qquad \ldots(5)$

$$\Rightarrow \quad y = x\tan\theta_2 - \frac{gx^2}{2u^2\cos^2\theta_2} \qquad \ldots(6)$$

$$\Rightarrow \quad \tan\theta_1 - \tan\theta_2 = \frac{gx}{2u^2}\left[\frac{1}{\cos^2\theta_1} - \frac{1}{\cos^2\theta_2}\right]$$

{using (5) & (6)}

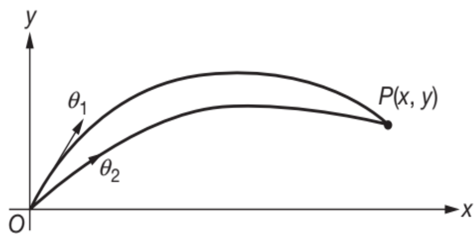
$$\Rightarrow \quad \frac{\sin\theta_1}{\cos\theta_1} - \frac{\sin\theta_2}{\cos\theta_2} = \frac{gx}{2u^2}\left[\frac{\cos^2\theta_2 - \cos^2\theta_1}{\cos^2\theta_2 \times \cos^2\theta_1}\right]$$

$$\Rightarrow \quad \frac{\sin(\theta_1 - \theta_2)}{\cos\theta_1 + \cos\theta_2} = \frac{gx}{2u^2}\left(\frac{u\Delta t}{x}\right)$$

$$\Rightarrow \quad \Delta t = \frac{2u}{g}\left[\frac{\sin(\theta_1 - \theta_2)}{\cos\theta_1 + \cos\theta_2}\right]$$

METHOD II

Since no friction is there, so let us assume both shells are fired at same instant. Let they pass through common point $P(x, y)$ at times t_1 and t_2 from the start, i.e., they miss each other by time $(t_1 - t_2)$ and hence, if this is the time lapse between firing of shells then they will collide at $P(x, y)$



For First Shell

$$x = (u\cos\theta_1)t_1 \qquad \ldots(1)$$

$$y = \left(u\sin\theta_1 t_1 - \frac{1}{2}gt_1^2\right) \qquad \ldots(2)$$

Also

For Second Shell

$$x = (u\cos\theta_2)t_2 \qquad \ldots(3)$$

$$u(\sin\theta_2 t_2 - \sin\theta_1 t_1) = \frac{1}{2}g(t_1 + t_2)(t_1 - t_2) \qquad \ldots(4)$$

Using relations (1) and (3), we get

$$u\left(\sin\theta_2 t_2 - \sin\theta_1 \frac{\cos\theta_2}{\cos\theta_1} t_2\right) = \frac{1}{2}g\left(\frac{\cos\theta_1 + \cos\theta_2}{\cos\theta_1}\right)t_2\Delta t \qquad \ldots(5)$$

Also using relations (3) and (4), we get

$$u\sin\theta_2 t_2 - \frac{1}{2}gt_2^2 = u\sin\theta_1 t_1 - \frac{1}{2}gt_1^2$$

$$\Rightarrow \quad u\left(\sin\theta_2 t_2 - \sin\theta_1 t_1\right) = \frac{1}{2}g\left(t_1+t_2\right)\left(t_1-t_2\right)$$

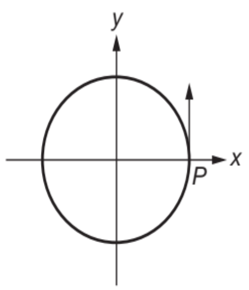
Using (5), we get

$$u\left(\sin\theta_2 t_2 - \sin\theta_1 \frac{\cos\theta_2}{\cos\theta_1}t_2\right) = \frac{1}{2}g\left(\frac{\cos\theta_1+\cos\theta_2}{\cos\theta_1}\right)t_2\Delta t$$

$$\Rightarrow \quad \Delta t = \frac{2u}{g}\left[\frac{\sin\left(\theta_1-\theta_2\right)}{\cos\theta_1+\cos\theta_2}\right]$$

PROBLEM 8

A train is moving with a constant speed of $10\ \text{ms}^{-1}$ in a circle of radius $\frac{16}{\pi}$ m. The plane of the circle lies in horizontal x-y plane. At time $t=0$ train is at point P and moving in counter-clockwise direction. At this instant a stone is thrown from the train with speed $10\ \text{ms}^{-1}$ relative to train towards negative x-axis at an angle of 37° with vertical z-axis. Find



(a) the velocity of particle relative to train at the highest point of its trajectory.

(b) the co-ordinates of points on the ground where it finally falls and that of the highest point of its trajectory.

Take $g = 10\ \text{ms}^{-2}$, $\sin(37^\circ) = 0.6$

SOLUTION

At $t=0$, $\vec{v}_T = \left(10\hat{j}\right)\ \text{ms}^{-1}$

$$\vec{v}_{ST} = 10\cos 37^\circ\hat{k} - 10\sin 37^\circ\hat{j} = \left(8\hat{k}-6\hat{i}\right)\ \text{ms}^{-1}$$

Since, $\vec{v}_{ST} = \vec{v}_S - \vec{v}_T$

$$\Rightarrow \quad \vec{v}_S = \vec{v}_{ST} + \vec{v}_T = \left(-6\hat{i}+10\hat{j}+8\hat{k}\right)\ \text{ms}^{-1}$$

(a) At highest point, vertical component $\left(\hat{k}\right)$ of $\vec{v}_S$ will become zero. Hence, velocity of particle at highest point will become $\left(-6\hat{i}+10\hat{j}\right)\ \text{ms}^{-1}$

(b) Time of flight,

$$T = \frac{2v_Z}{g} = \frac{2\times 8}{10} = 1.6\ \text{s}$$

$$\Rightarrow \quad y = (10)(1.6) = 16\ \text{m} \ \text{and} \ z = 0$$

Therefore coordinates of particle where it finally lands on the ground are $(-4.5\ \text{m}, 16\ \text{m}, 0)$

At highest point, we have

$$t = \frac{T}{2} = 0.8\ \text{s}$$

$$\Rightarrow \quad x = \frac{16}{\pi} - (6)(0.8) = 0.3\ \text{m}$$

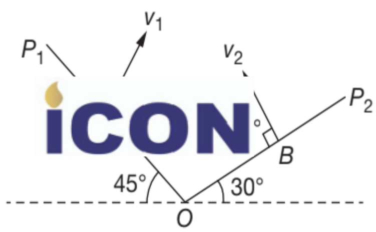
$$\Rightarrow \quad y = (10)(0.8) = 8\ \text{m}$$

and $z = \dfrac{v_Z^2}{2g} = \dfrac{(8)^2}{20} = 3.2\ \text{m}$

Therefore, coordinates at highest point are $(0.3\ \text{m}, 8\ \text{m}, 3.2\ \text{m})$.

PROBLEM 9

A particle is projected from an inclined plane OP_1 from A with velocity $v_1 = 8\ \text{ms}^{-1}$ at an angle 60° with horizontal. An another particle is projected at the same instant from B with velocity $16\ \text{ms}^{-1}$ and perpendicular to the plane OP_2 as shown in figure. After time $10\sqrt{3}$ s separation between them was minimum and found to be 70 m. Find the



(a) distance AB.

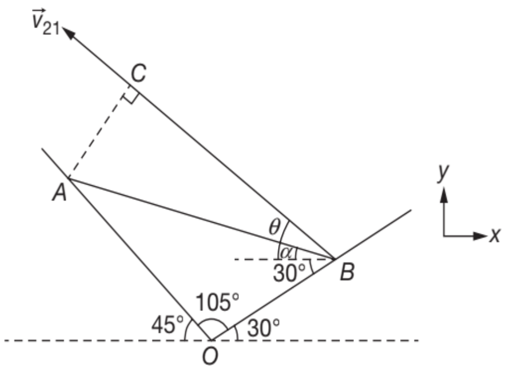
(b) height of A and B from O

SOLUTION

$$\left|\left(v_{21}\right)_x\right| = \left(v_1+v_2\right)\cos 60^\circ = 12\ \text{ms}^{-1}$$

$$\left|\left(v_{21}\right)_y\right| = \left(v_1+v_2\right)\sin 60^\circ = 4\sqrt{3}\ \text{ms}^{-1}$$

$$\Rightarrow \quad v_{21} = \sqrt{(12)^2+\left(4\sqrt{3}\right)^2} = 13.9\ \text{ms}^{-1}$$



and $\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{12}\right) = 30°$, $BC = (v_{21})t = 240$ m and

$AC = 70$ m {given}

(a) Hence using Pythagora's Theorem, we have

$$AB = \sqrt{(240)^2 + (70)^2} = 250 \text{ m}$$

(b) $\angle\alpha = \angle\theta - \angle ABC = 30° - \sin^{-1}\left(\frac{AC}{AB}\right)$

$$\Rightarrow \angle\alpha = 30° - \sin^{-1}\left(\frac{70}{250}\right) = 13.7°$$

Applying sine law in ΔABO, we get

$$\frac{AB}{\sin(105°)} = \frac{AO}{\sin(\alpha + 30°)} = \frac{BO}{\sin(180° - (\alpha + 30° + 105°))}$$

Solving this we get,

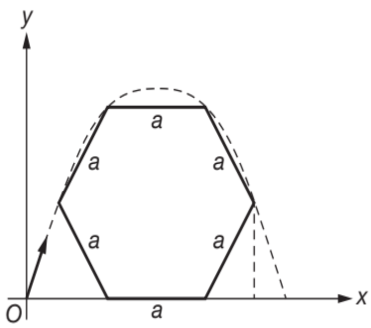
$AO = 181.2$ m and $BO = 134.6$ m

So, $h_A = AO \sin 45° = 128$ m and

$h_B = BO \sin 30° = 67.3$ m

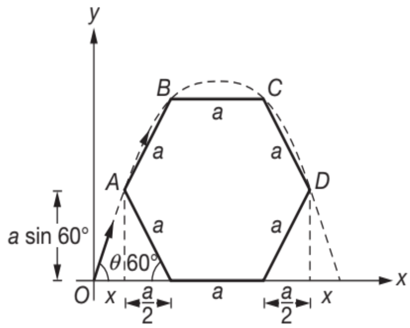
PROBLEM 10

A particle is launched such that it grazes the four vertices of a regular hexagon of side a as shown. Find the range of the particle.



SOLUTION

The coordinates of the points A, B, C and D are calculated here for convenience.



$$A\left(x, \frac{\sqrt{3}}{2}x\right)$$

$$B\left(x + \frac{a}{2}, \sqrt{3}a\right)$$

$$C\left(x + \frac{3a}{2}, \sqrt{3}a\right)$$

$$D\left(x + 2a, \frac{\sqrt{3}a}{2}\right)$$

Range, $R = x + \frac{a}{2} + a + \frac{a}{2} + x = 2a + 2x$

Also we know that

$$y = x\tan\theta\left(1 - \frac{x}{R}\right)$$

Since A lies on Trajectory, so

$$\frac{\sqrt{3}a}{2} = x\tan\theta\left(1 - \frac{x}{2a + 2x}\right) \quad \ldots(1)$$

Also, B lies on Trajectory, so

$$\sqrt{3}a = \left(x + \frac{a}{2}\right)\tan\theta\left(1 - \frac{\left(x + \frac{a}{2}\right)}{2a + 2x}\right) \quad \ldots(2)$$

Dividing (1) and (2), we get

$$\frac{\frac{\sqrt{3}}{2}a}{\sqrt{3}a} = \frac{x\tan\theta\left(1 - \frac{x}{2a + 2x}\right)}{\left(x + \frac{a}{2}\right)\tan\theta\left(1 - \frac{\left(x + \frac{a}{2}\right)}{2a + 2x}\right)}$$

$$\Rightarrow 2 = \frac{\left(x + \frac{a}{2}\right)}{x}\left(\frac{2a + 2x - x - \frac{a}{2}}{2a + 2x - x}\right)$$

$$\Rightarrow \quad 2=\left(\frac{2x+a}{2x}\right)\left(\frac{\frac{3a}{2}+x}{2a+x}\right)$$

$$\Rightarrow \quad 2=\frac{1}{4}\left(\frac{2x+a}{x}\right)\left(\frac{3a+2x}{2a+x}\right)$$

$$\Rightarrow \quad 8x(2a+x)=(2x+a)(3a+2x)$$

$$\Rightarrow \quad 16ax+8x^2=4x^2+6ax+2ax+3a^2$$

$$\Rightarrow \quad 4x^2+8ax-3a^2=0$$

$$\Rightarrow \quad x=-\frac{8a\pm\sqrt{64a^2+48a^2}}{8}$$

$$\Rightarrow \quad 8x=-8a\pm 4a\sqrt{4+3}$$

$$\Rightarrow \quad x=-a\pm\frac{a}{2}\sqrt{7}$$

$$\Rightarrow \quad x+a=\frac{a}{2}\sqrt{7}$$

$$\Rightarrow \quad 2x+2a=a\sqrt{7}$$

$$\Rightarrow \quad R=\sqrt{7}a$$

PROBLEM 11

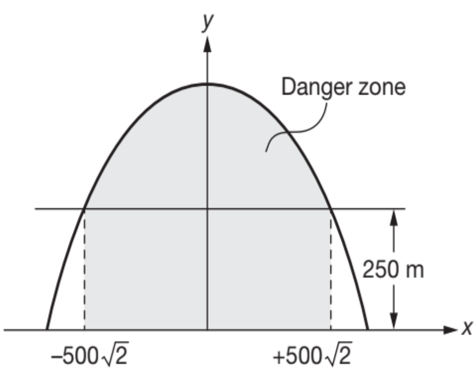
An enemy fighter jet is flying at a constant height of 250 m with a velocity of 500 ms^{-1}. The fighter jet passes over an anti-aircraft gun that can fire at any time and in any direction with a speed of 100 ms^{-1}. Determine the time interval during which the fighter jet is in danger of being hit by the gun bullets.

SOLUTION

The equation of trajectory of bullets is

$$y=x\tan\theta-\left(\frac{gx^2}{2u^2}\right)\sec^2\theta$$

$$\Rightarrow \quad y=x\tan\theta-\frac{gx^2}{2u^2}\left(1+\tan^2\theta\right) \qquad \ldots(1)$$



For a given value of x, maximum y can be determined from

$$\frac{dy}{d(\tan\theta)}=0$$

$$\Rightarrow \quad x-\frac{gx^2}{2u^2}(2\tan\theta)=0$$

$$\left\{\because \frac{d\left(\tan^2\theta\right)}{d(\tan\theta)}=2\tan\theta \text{ and } \frac{d(\tan\theta)}{d(\tan\theta)}=1\right\}$$

$$\Rightarrow \quad \tan\theta=\frac{u^2}{gx}$$

On substituting the expression for $\tan\theta$ in equation (1), we get

$$y_{\text{MAX}}=x\left(\frac{u^2}{gx}\right)-\frac{gx^2}{2u^2}\left(1+\frac{u^4}{g^2x^2}\right)$$

$$\Rightarrow \quad y_{\text{MAX}}=\frac{u^2}{2g}-\frac{gx^2}{2u^2}$$

The shell can hit an area defined by

$$y\le y_{\text{MAX}}$$

$$\Rightarrow \quad y\le\frac{u^2}{2g}-\frac{gx^2}{2u^2}$$

On substituting numerical values, $y=250$ m, $u=100\text{ ms}^{-1}$, $g=10\text{ ms}^{-2}$, we get

$$\frac{x^2}{2000}\le 250$$

$$\Rightarrow \quad x^2\le 500000$$

$$\Rightarrow \quad -500\sqrt{2}\le x\le 500\sqrt{2}$$

The fighter jet, can travel $1000\sqrt{2}$ m while it can be hit. So the plane is in danger for a period of

$$t=\frac{x}{u}=\frac{1000\sqrt{2}}{500}=2\sqrt{2}\text{ s.}$$