CHAPTER 3 Vectors

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

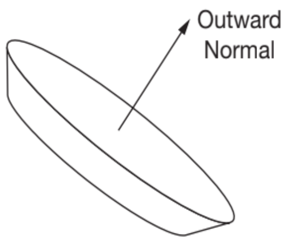
(a) Vector Fundamentals and Types
(b) Triangle, Parallelogram and Polygon Law of Vector Addition
(c) Position Vector and Vector Subtraction
(d) Vector Resolution
(e) Dot Product and Cross Product
(f) Scalar Triple Product and Vector Triple Product

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

Physical quantities having magnitude, direction and obeying laws of vector algebra are called **vectors**. Magnitude of a vector is also called its **modulus**.

EXAMPLE:
Displacement, velocity, acceleration, momentum, force, impulse, weight, thrust, torque, angular momentum, angular velocity.



The physical quantity 'AREA' is a vector with its direction along the outward normal to the surface.

Conceptual Note(s)

(a) "If you love hot pizzas, then it does not imply that everything hot is a pizza".
(b) Similarly if a physical quantity has magnitude and direction both, then it does not always imply that it is a vector. For it to be a vector the third condition of obeying laws of vector algebra has to be satisfied.

Example: The physical quantity CURRENT has both magnitude and direction but is still a scalar as it disobeys the laws of vector algebra.

ILLUSTRATION 1

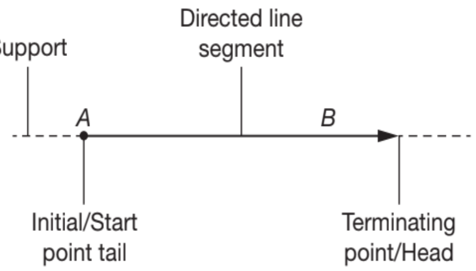
We can order events in time, and there is a sense of time, distinguishing past, present and future. Is therefore time a vector?

SOLUTION

Time always flow from past to present and then to future so a direction can be assigned to time. However as its direction is unique it is not to be specified. As direction is not to be specified so time cannot be a vector though it has direction. Also it does not obey laws of vector algebra.

GEOMETRICAL DEFINITION

A directed line segment is called a **vector,** denoted as $\overrightarrow{AB}$ (read as AB vector). In printing we can denote $\overrightarrow{AB}$ as $\boldsymbol{AB}$ (bold letters). Even if you have to write BA vector the arrow head above would always point from left to right i.e. $\overrightarrow{BA}$.



But mathematically $\overrightarrow{AB} = -\overrightarrow{BA}$

Also, let $\vec{A}$ and $\vec{B}$ be two vectors,

then $|\vec{A}| = A =$ magnitude of $\vec{A} =$ length of $\vec{A}$

$|\vec{B}| = B =$ magnitude of $\vec{B}$ = length of $\vec{B}$

Also note that if we have to compare vectors then we only compare their magnitudes. So,

$|\vec{A}| > |\vec{B}|$ and not $\vec{A} > \vec{B}$

Conceptual Note(s)

Every vector $(\overrightarrow{AB})$ has the following three characteristics

(a) Length: The length of $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$ or simply AB.

(b) Support: The line of unlimited length of which AB is segment is called the support of vector $\overrightarrow{AB}$.

(c) Sense: The sense of $\overrightarrow{AB}$ is from $A \to B$ and that of $\overrightarrow{BA}$ will be from $B \to A$. The sense of directed line segment is from its initial point to the terminal point.

TYPES OF VECTOR

Equal Vectors

Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal when they have equal magnitudes and same direction. Geometrically if head of one vector coincides will the head of other and so do the tails coincide then the vectors are said to be equal.

If $\vec{A} = \vec{B}$, then $A = B$, always.

But if, $A = B$ doesn't always imply $\vec{A} = \vec{B}$.

Problem Solving Technique(s)

A vector can always be transported parallel to its original direction.

Parallel Vector

Two vectors $\vec{A}$ and $\vec{B}$ are said to be parallel when

(a) both have same direction.

(b) magnitude of one is scalar multiple of magnitude of the other

EXAMPLE:

$\vec{B} = 2\vec{A}$, i.e., Magnitude of $\vec{B}$ is twice the magnitude of $\vec{A}$ and both have same direction.

In general if $\vec{A} = k\vec{B}$ (where k is any constant) i.e. magnitude of $\vec{A}$ is k times that of $\vec{B}$, then both are **parallel** if $k > 0$.

Anti-parallel Vectors

Two vectors $\vec{A}$ and $\vec{B}$ are said to be antiparallel when

(a) both have opposite direction.

(b) magnitude of one is a scalar multiple of the magnitude of the other.

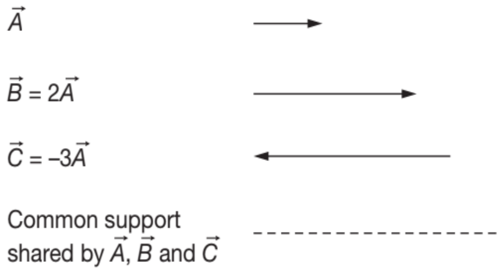
EXAMPLE:

$\vec{B} = -2\vec{A}$ Magnitude of $\vec{B}$ is twice the magnitude of $\vec{A}$ and both have opposite direction.

In general if $\vec{A} = k\vec{B}$ (where k is any constant) i.e. magnitude of $\vec{A}$ is k times that of $\vec{B}$, then both are **antiparallel** if $k > 0$.

Collinear Vectors

When the vectors under consideration can share the same support or have a common support then the considered vectors are collinear.



Zero Vector $(\vec{0})$

A vector having zero magnitude and arbitrary direction (not known to us) is a **zero vector**.

Properties of $\vec{0}$

(a) $\vec{A}+(-\vec{A})=\vec{0}$

(b) If $\vec{A}+\vec{B}=0 \Rightarrow \vec{A}=-\vec{B}$

(c) $k\,\vec{0}=\vec{0}$ [read as k times $\vec{0}$ equals $\vec{0}$]
{where, k is any scalar}

(d) $0\,\vec{A}=\vec{0}$ [read as zero times $\vec{A}$ equals zero vector]

(e) $\vec{A}+\vec{0}=\vec{0}+\vec{A}=\vec{A}$

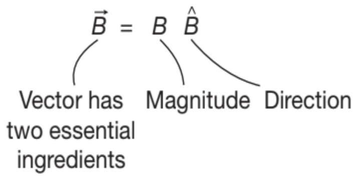
Unit Vector

A vector divided by its magnitude is a unit vector. Unit vector for $\vec{A}$ is $\hat{A}$ (read as A cap/A carat/A hat). Unit vector for $\vec{B}$ is $\hat{B}$.

$$|\hat{A}|=1,\ |\hat{B}|=1,\ |\hat{C}|=1,\ |\hat{x}|=1,\ |\hat{y}|=1$$

Since, $\hat{A}=\dfrac{\vec{A}}{A}$

$\Rightarrow \quad \vec{A}=A\hat{A}$



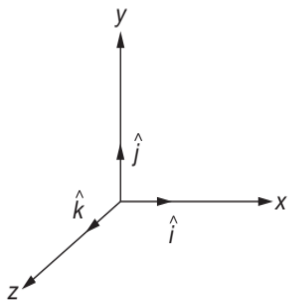
Thus, we conclude that unit vector gives us the direction.

Conceptual Note(s)

No units are to be attached (like newton (N), $msec^{-1}$, metre etc.) with a unit vector i.e., unit vector is dimensionless physical quantity.

Orthogonal Unit Vectors/Base Vectors

$\hat{i}$, $\hat{j}$ and $\hat{k}$ are called orthogonal unit vectors. These vectors must form a **Right Handed Triad**. It is a coordinate system such that when we Curl the fingers of right hand from x to y, then we must get the direction of z along thumb



from y to z, then we must get the direction of x along thumb
from z to x, then we must get the direction of y along thumb

$\hat{i}=\dfrac{\vec{x}}{x}$

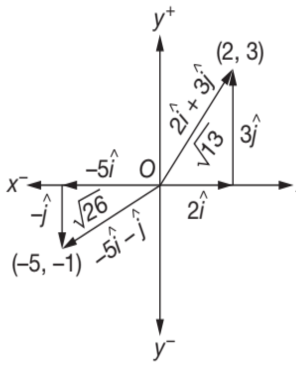
$\Rightarrow \quad \vec{x}=x\hat{i}$

$\hat{j}=\dfrac{\vec{y}}{y}$

$\Rightarrow \quad \vec{y}=y\hat{j}$

$\hat{k}=\dfrac{\vec{z}}{z}$

$\Rightarrow \quad \vec{z}=z\hat{k}$



Graphical representation of vectors

EXAMPLE:

(a) A vector of 3 unit along x axis is $\vec{x}=3\hat{i}$.

(b) A vector of magnitude 6 along $-x$ axis is $\vec{x}=6(-\hat{i})=-6\hat{i}$.

(c) A vector of magnitude 5 along $-z$ axis is $\vec{x}=5(-\hat{k})=-5\hat{k}$.

Also, $(0,0)\equiv 0\hat{i}+0\hat{j}\equiv\vec{0}$, $(2,3)\equiv 2\hat{i}+3\hat{j}$

$(-5,-1)\equiv -5\hat{i}-\hat{j}$.

Fixed Vector

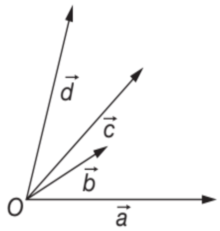
Fixed vector is that vector whose initial point or tail is fixed. Its also called **localised vector**.

EXAMPLE:
(a) Position vector
(b) Displacement vector

Free Vector

Free vector is that vector whose initial point or tail is not fixed. Its also known as **non localised vector**.

EXAMPLE:
Velocity vector of a particular moving along a straight line is a free vector.



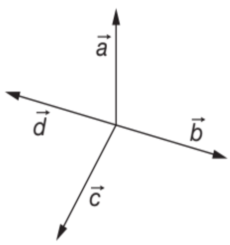
Negative Vector

A vector is said to be negative of a given vector if its magnitude is the same as that of the given vector but direction is reversed. e.g. The negative vector of a vector $\vec{a}$ is denoted by $-\vec{a}$.

If $\vec{b}$ is negative of $\vec{a}$, then $\vec{b}=-\vec{a}$.

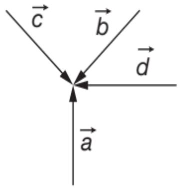
Co-initial Vectors

Co-initial vectors are those vectors which have the same initial point.



Co-terminus Vectors

Co-terminus vectors are those vectors which have common terminating point.



Coplanar Vector

Vectors are said to be coplanar if they occur in same or common plane. e.g. $\vec{A}$ and $\vec{B}$ and $\vec{C}$. Remember that any two vectors always lie in the same plane.

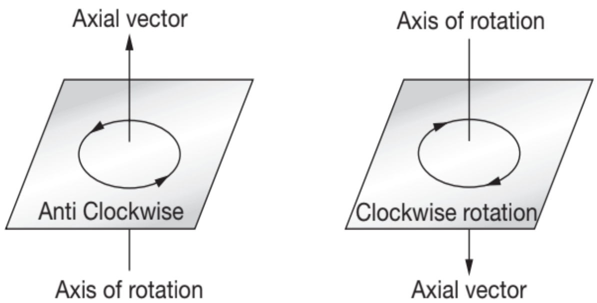
Polar Vector

Vectors producing straight line linear effect are called **polar vectors** e.g. force, momentum, velocity, displacement.

Axial Vectors/Pseudo Vectors: (Virtual, Imaginary)

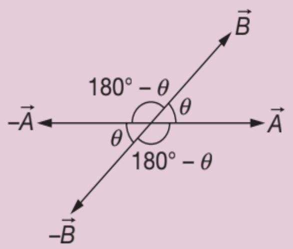
The rotational effect of a polar vector gives rise to a new vector called **axial vector** (acting along axis of rotation).

EXAMPLE:
Rotational effect of force (polar vector) is torque (an axial vector).
Rotational effect of linear momentum (a polar vector) is angular momentum (an axial vector).



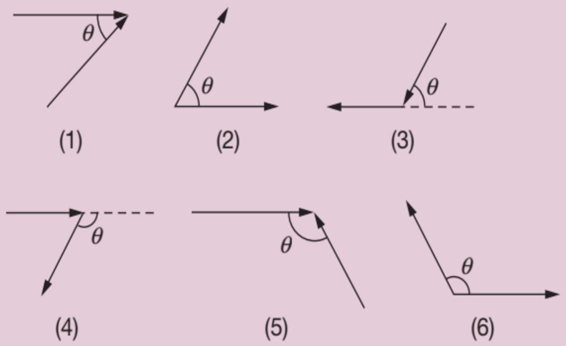
Conceptual Note(s)

(a) Whenever angle between two vectors is to be taken we must make sure that either their heads coincide or their tails coincide.



i.e., if two vectors have their heads coinciding or their tails coinciding, then internal angle is the angle between two vectors (whether acute or obtuse), as in (1), (2), (5) & (6).

If head coincides with tail then external supplementary angle (whether acute or obtuse) is the angle between the two vectors, as in (3) & (4).



(b) If angle between $\vec{A}$ and $\vec{B}$ is θ, then

(i) angle between $-\vec{A}$ and $\vec{B}$ is $(180° - \theta)$

(ii) angle between $\vec{A}$ and $-\vec{B}$ is $(180° - \theta)$

(iii) angle between $-\vec{A}$ and $-\vec{B}$ is θ

ILLUSTRATION 2

Let $\vec{A} = 2\hat{i} + \hat{j}$, $\vec{B} = 3\hat{j} - \hat{k}$ and $\vec{C} = 6\hat{i} - 2\hat{k}$, then find $\vec{A} - 2\vec{B} + 3\vec{C}$. Also calculate its magnitude.

SOLUTION

$$\vec{A} - 2\vec{B} + 3\vec{C} = (2\hat{i} + \hat{j}) - 2(3\hat{j} - \hat{k}) + 3(6\hat{i} - 2\hat{k})$$

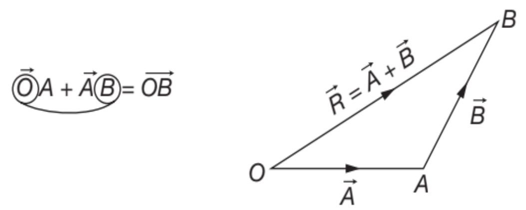
$$\vec{A} - 2\vec{B} + 3\vec{C} = 2\hat{i} + \hat{j} - 6\hat{j} + 2\hat{k} + 18\hat{i} - 6\hat{k}$$

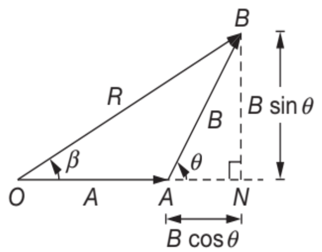
$$|\vec{A} - 2\vec{B} + 3\vec{C}| = \sqrt{(20)^2 + (-5)^2 + (-4)^2} = \sqrt{441} = 21$$

TRIANGLE LAW OF VECTOR ADDITION OF TWO VECTORS

If two non zero vectors are represented by the two sides of a triangle taken in same order then the resultant is given by the closing side of triangle in opposite order. i.e.,

$$\vec{R} = \vec{A} + \vec{B}$$





Magnitude of Resultant Vector

In ΔABN,

$\cos\theta = \dfrac{AN}{B}$	$\sin\theta = \dfrac{BN}{B}$
$\Rightarrow \quad AN = B\cos\theta$	$\Rightarrow \quad BN = B\sin\theta$

In ΔOBN, we have $OB^2 = ON^2 + BN^2$

$$\Rightarrow \quad R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

$$\Rightarrow \quad R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\Rightarrow \quad R^2 = A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$$

$$\Rightarrow \quad R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow \quad R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Direction of Resultant Vector

If θ is angle between $\vec{A}$ and $\vec{B}$, then

$$|\vec{B} + \vec{A}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}.$$

If the resultant, $\vec{R}$ makes an angle β with $\vec{A}$, then in ΔOBN, then

$$\tan\beta = \frac{BN}{ON} = \frac{BN}{OA+AN}$$

$$\tan\beta = \frac{B\sin\theta}{A+B\cos\theta}$$

Conceptual Note(s)

Resultant $\vec{R}$ of two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ is

$$R = |\vec{R}| = |\vec{A}+\vec{B}| = \sqrt{A^2+B^2+2AB\cos\theta}$$

and β is the angle made by the resultant $\vec{R}$ with $\vec{A}$, then $\tan\beta = \dfrac{B\sin\theta}{A+B\cos\theta}$

ILLUSTRATION 3

Can the magnitude of the resultant of two equal vectors be equal to the magnitude of either of the vectors? Explain your answer.

SOLUTION

Let the two vectors be $\vec{A}$ and $\vec{B}$ inclined at an angle θ, then

$$R^2 = A^2+B^2+2AB\cos\theta$$

According to the question, we have been given

$$R = A = B$$

$$\Rightarrow \quad A^2 = A^2+A^2+2A^2\cos\theta$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{2}$$

$$\Rightarrow \quad \theta = 120^\circ$$

So, this is possible, when two vectors of equal magnitude are inclined to each other at an angle of 120°.

ILLUSTRATION 4

The sum of magnitude of two forces (in newton) acting at a point is 18 and the magnitude of their resultant (in newton) is 12. If resultant is at 90° with force of smaller magnitude, what are magnitudes of forces?

SOLUTION

METHOD I

Let F_1 and F_2 be the two forces. Let one of them, F_1 (say) is of smaller magnitude.

Now $F_1+F_2 = 18$ ...(1)

$$R = \sqrt{F_1^2+F_2^2+2F_1F_2\cos\theta} = 12 \quad \text{...(2)}$$

$$\tan\alpha = \frac{F_2\sin\theta}{F_1+F_2\cos\theta} \to \infty \qquad \{\because \alpha = 90^\circ\}$$

$$\Rightarrow \quad F_1+F_2\cos\theta = 0 \quad \text{...(3)}$$

$$\Rightarrow \quad (18-F_2)+F_2\cos\theta = 0$$

$$\Rightarrow \quad F_2(1-\cos\theta) = 18 \quad \text{...(4)}$$

Subtracting equation (2) from equation (1), we get

$$2F_1F_2(1-\cos\theta) = (18)^2-(12)^2 = 180 \quad \text{...(5)}$$

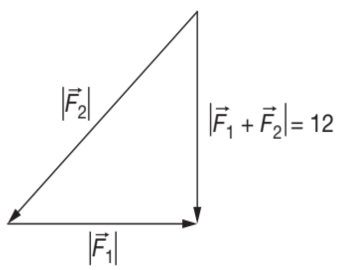
Dividing (5) by (4), we have

$$2F_1 = 10$$

$$\Rightarrow \quad F_1 = 5\text{ N and } F_2 = 13\text{ N}$$

METHOD II

Since, $(\vec{F}_1+\vec{F}_2)$ is resultant of $\vec{F}_1$ and $\vec{F}_2$, so from Triangle Law, we have drawn the diagram from which we get



$$F_2^2 = F_1^2+(12)^2$$

$$\Rightarrow \quad F_2^2-F_1^2 = 144$$

$$\Rightarrow \quad (F_1+F_2)(F_2-F_1) = 144 \quad \text{...(1)}$$

Since, $F_1+F_2 = 18$ ...(2)

So, (1) gives $F_2-F_1 = 8$ ...(3)

Solving (2) and (3), we get

$$F_1 = 5\text{ N and } F_2 = 13\text{ N}$$

ILLUSTRATION 5

The x and y components of vector $\vec{A}$ are 4 m and 6 m respectively. The x and y components of vector $\vec{A}+\vec{B}$ are 10 m and 9 m respectively. Calculate the

(a) x and y components of vector $\vec{B}$.

(b) length of vector $\vec{B}$.

(c) angle which $\vec{B}$ makes with x-axis.

SOLUTION

(a) $A_x = 4$ and $A_y = 6$ ...(1)

$$\vec{A}+\vec{B} = \left(A_x\hat{i} + A_y\hat{j}\right) + \left(B_x\hat{i} + B_y\hat{j}\right)$$

$$\Rightarrow \quad \vec{A}+\vec{B} = \left(A_x + B_x\right)\hat{i} + \left(A_y + B_y\right)\hat{j}$$

So, $A_x + B_x = 10$ and $A_y + B_y = 9$ ...(2)

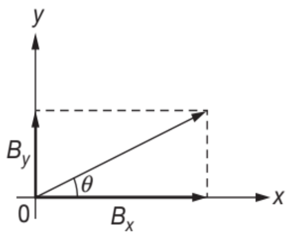
From equation sets (1) and (2), we get

$$B_x = 6 \text{ and } B_y = 3$$

So, $\vec{B} = 6\hat{i} + 3\hat{j}$

(b) $|\vec{B}| = \sqrt{6^2+3^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$ m

(c) Let $\vec{B}$ make an angle θ with x-axis, then from the diagram



$$\tan\theta = \frac{B_y}{B_x} = \frac{3}{6} = \frac{1}{2}$$

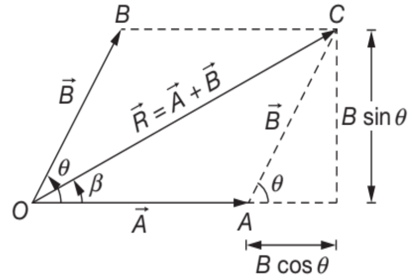
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{2}\right), \text{ with } x\text{-axis}$$

PARALLELOGRAM LAW OF VECTOR ADDITION OF TWO VECTORS

If two non zero vector are represented by the two adjacent sides of a parallelogram then the resultant is given by the diagonal of the parallelogram passing through the point of intersection of the two vectors.

Magnitude of Resultant Vector

Since, $R^2 = ON^2 + CN^2$



$$\Rightarrow \quad R^2 = (OA + AN)^2 + CN^2$$

$$\Rightarrow \quad R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow \quad \boxed{R = |\vec{R}| = |\vec{A}+\vec{B}| = \sqrt{A^2+B^2+2AB\cos\theta}}$$

Direction of Resultant Vector

$$\tan\beta = \frac{CN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}$$

Conceptual Note(s)

Triangle Law and Parallelogram Law, both are equivalent. Since we know that a vector can be transported parallel to its original direction, so their equivalency holds good and is shown below.

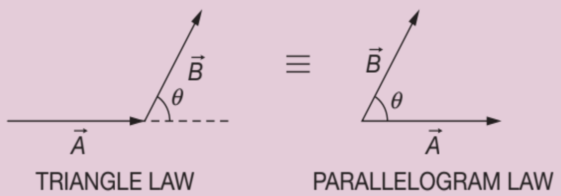


ILLUSTRATION 6

If $\vec{A} = 4\hat{i} - 3\hat{j}$ and $\vec{B} = 6\hat{i} + 8\hat{j}$ then find the magnitude and direction of $\vec{A}+\vec{B}$.

SOLUTION

$$\vec{A}+\vec{B} = 4\hat{i} - 3\hat{j} + 6\hat{i} + 8\hat{j} = 10\hat{i} + 5\hat{j}$$

$$|\vec{A}+\vec{B}| = \sqrt{(10)^2 + (5)^2} = 5\sqrt{5}$$

$$\tan\theta = \frac{5}{10} = \frac{1}{2}$$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{2}\right)$

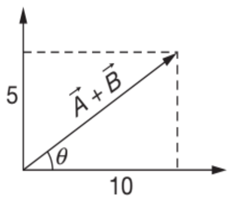
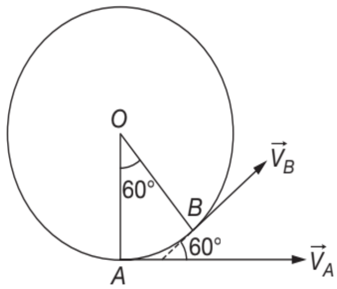


ILLUSTRATION 7

In figure, a particle is moving in a circle of radius r centred at O with constant speed v. What is the change in velocity in moving from A to B? Given $\angle AOB = 60°$.



SOLUTION

Change in velocity $= \Delta\vec{v} = \vec{v}_B - \vec{v}_A$

Since $|\vec{v}_B - \vec{v}_A| = \sqrt{v_B^2 + v_A^2 - 2v_A v_B \cos\theta}$

$\Rightarrow \quad |\vec{v}_B - \vec{v}_A| = \sqrt{v^2 + v^2 - 2v^2\cos(60°)}$

$\Rightarrow \quad |\vec{v}_B - \vec{v}_A| = \sqrt{v^2 + v^2 - 2v^2\left(\frac{1}{2}\right)} = v$

ILLUSTRATION 8

If the sum of two unit vectors is a unit vector, then find the magnitude of difference of these two vectors.

SOLUTION

Let $\hat{n}_1$ and $\hat{n}_2$ are the two unit vectors, then the sum and difference of these two vectors be represented by

$$\vec{n}_s = \hat{n}_1 + \hat{n}_2 \text{ and } \vec{n}_d = \hat{n}_1 - \hat{n}_2$$

$\Rightarrow \quad n_s^2 = n_1^2 + n_2^2 + 2n_1n_2\cos\theta = 1 + 1 + 2\cos\theta$

Since it is given that n_s is also a unit vector, therefore we have

$$1 = 1 + 1 + 2\cos\theta$$

$\Rightarrow \quad \cos\theta = -\frac{1}{2}$

$\Rightarrow \quad \theta = 120°$

The difference of the two vectors is given by

$$\vec{n}_d = \hat{n}_1 - \hat{n}_2$$

$\Rightarrow \quad n_d^2 = n_1^2 + n_2^2 - 2n_1n_2\cos\theta = 1 + 1 - 2\cos(120°)$

$\Rightarrow \quad n_d^2 = 2 - 2\left(-\frac{1}{2}\right) = 2 + 1 = 3$

$\Rightarrow \quad n_d = \sqrt{3}$

ILLUSTRATION 9

Consider two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ. If $|\vec{A}| = |\vec{B}| = R$, then prove that

(a) $|\vec{A}+\vec{B}| = 2R\cos\left(\frac{\theta}{2}\right)$

(b) $|\vec{A}-\vec{B}| = 2R\sin\left(\frac{\theta}{2}\right)$

SOLUTION

(a) Since $|\vec{A}| = |\vec{B}| = 1$

$\Rightarrow \quad |\vec{A}+\vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow \quad |\vec{A}+\vec{B}| = \sqrt{R^2 + R^2 + 2R^2\cos\theta} = \sqrt{2}R\sqrt{1+\cos\theta}$

Since $1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$

$\Rightarrow \quad |\vec{A}+\vec{B}| = 2R\cos\left(\frac{\theta}{2}\right)$

(b) Similarly,

$$|\vec{A}-\vec{B}| = \sqrt{R^2 + R^2 - 2R^2\cos\theta} = \sqrt{2}R\sqrt{1-\cos\theta}$$

Since $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

$\Rightarrow \quad |\vec{A}-\vec{B}| = 2R\sin\left(\frac{\theta}{2}\right)$

TRIANGLE INEQUALITY

Since a vector cannot have a resultant more than the maximum value and less than the least value of the resultant, so we have

$$R_{\min} \le R \le R_{\max}$$

$$\Rightarrow \quad A - B \le R \le A + B$$

If $A > B$, then $A - B \le |\vec{A} + \vec{B}| \le A + B$

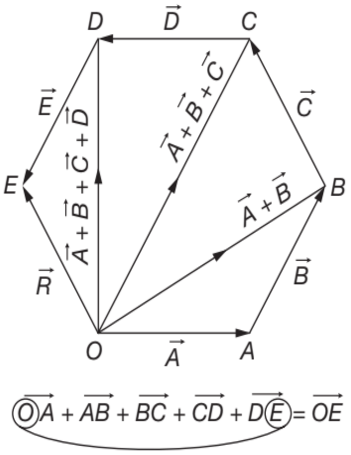
$$|\vec{A}| - |\vec{B}| \le |\vec{A} + \vec{B}| \le |\vec{A}| + |\vec{B}|$$

$$|\vec{A}| - |\vec{B}| \le |\vec{A} + \vec{B}| \ \& \ |\vec{A} + \vec{B}| \le |\vec{A}| + |\vec{B}|$$

{called Triangle Inequality}

POLYGON LAW OF VECTOR ADDITION

If a number of non zero vectors are represented by the $(n-1)$ sides of an n-sided polygon then the resultant



is given by the closing side or the n^{th} side of the polygon taken in opposite order. So,

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$$

PROPERTIES OF VECTOR ADDITION

(a) Vector Addition is Commutative

For any two vectors $\vec{a}$ and $\vec{b}$, we have

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(b) Vector Addition is Associative

For any three vector $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have

$$\vec{b} + (\vec{c} + \vec{a}) = \vec{c} + (\vec{a} + \vec{b}) = \vec{a} + (\vec{b} + \vec{c})$$

(c) Existence of Additive Identify

For every vector $\vec{a}$, we have

$$\vec{a} + \vec{0} = \vec{0} + \vec{a}$$

i.e., Adding $\vec{0}$ i.e., null vector to any vector (say $\vec{a}$) does not changes the magnitude of $\vec{a}$ as well as its direction, because $\vec{0}$ has zero magnitude and has an arbitrary/indeterminate direction.

(d) Existence of Additive Inverse

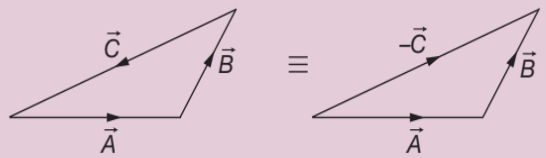
For a given vector $\vec{a}$, there exists a vector $(-\vec{a})$ such that

$$\vec{a} + (-\vec{a}) = \vec{0}$$

The vector $(-\vec{a})$ is called the additive inverse of $\vec{a}$ and vice versa.

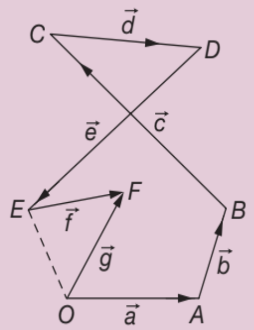
Problem Solving Technique(s)

(a) If the vectors form a closed n sided polygon with all the sides in the same order then the resultant is $\vec{0}$.



Addition of Vectors

(b) To find the sum of any number of vectors we must represent the vectors by the directed line segments with the terminal point of the previous vectors as the initial point of the next vector, then the lines segment joining the initial point of the first vector to the terminal



point of the last vector will represent the sum of the vectors. Such that

(c) If the terminal of the last vector coincides with the initial point of the first vector, then its sum will be zero.

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} + \vec{f} + \vec{g} = \vec{0}$$

MULTIPLICATION OF A VECTOR BY A SCALAR

If m by any scalar and $\vec{a}$ be any vector then $m\vec{a}$ is defined as a vector whose magnitude is $|m|$ times that of the magnitude of $\vec{a}$ and whose direction is same as of $\vec{a}$ if m is positive and will be opposite to that of $\vec{a}$ if m is negative.

PROPERTIES OF MULTIPLICATION OF VECTOR BY A SCALAR

(a) If $m = 0$ (ZERO), then $m\vec{a} = 0$ and not $\vec{0}$.

(b) If m and n are two scalars and $\vec{a}$ be any vector in space, then

$$m(n\vec{a}) = n(m\vec{a}) = mn(\vec{a})$$

(c) Vector multiplication is distributive over scalar addition

$$\vec{a}(m+n) = m\vec{a} + n\vec{a}$$

where, m, n are any two scalars and $\vec{a}$ be any vector in space.

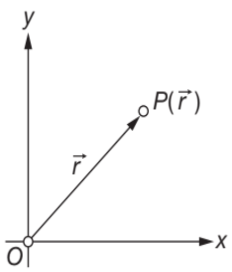
(d) Scalar multiplication is distributive over vector addition. Let $\vec{a}$ and $\vec{b}$ are any two vectors and m be any scalar, then

$$m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$$

POSITION VECTOR

If a point O is fixed as the origin is space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector of P with respect to O. So, position vector of a point P, from a convenient fixed origin O is a vector drawn from O to P.

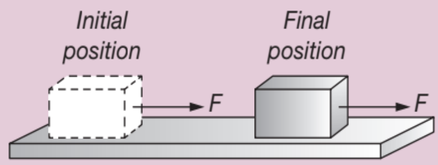
If it is said that P is any point in space, then it means that $\overrightarrow{OP}$ is a position vector or we can say that position vector of point P is $\vec{r}$ (say) with respect to some conveniently fixed origin O.



Conceptual Note(s)

A vector is an independent entity and can be transported any where keeping its Line of Action (LOA) parallel to the same straight line, magnitude and sense unchanged. i.e., A vector $\vec{a}$ will remain $\vec{a}$ till its magnitude and direction is retained.

EXAMPLE: In many examples, when a force F is applied on the block such that the block displaces from its original position, the force F applied on block remains same even after displacement w.r.t. the same frame of reference.



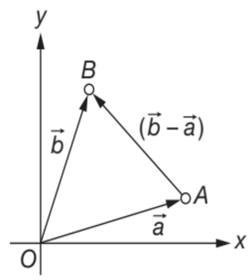
BUT

A position vector is a fixed quantity with its initial point as origin.

TO FIND $\overrightarrow{AB}$ IF THE POSITION VECTORS OF POINTS A AND B ARE KNOWN

Let $\vec{a}$ and $\vec{b}$ are position vectors of points A and B in plane (or space)

$$\therefore \quad \overrightarrow{OA} = \vec{a} \text{ and } \overrightarrow{OB} = \vec{b}$$



Now, in ΔOAB, according to Triangular Law of Vector addition, we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

(1) (2)

$$\Rightarrow \quad \overrightarrow{AB} = \begin{pmatrix}\text{Position Vector} \\ \text{of } B\end{pmatrix} - \begin{pmatrix}\text{Position Vector} \\ \text{of } A\end{pmatrix}$$

SUBTRACTION OF VECTORS

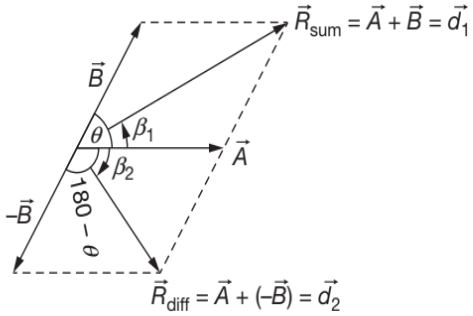
Since, $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$

So, from above we conclude that subtraction of vectors is actually finding the resultant of $\vec{A}$ and $-\vec{B}$ OR $\vec{B}$ and $-\vec{A}$.

Also, $|\vec{A}+\vec{B}|=\sqrt{A^2+B^2+2AB\cos\theta}$ and

$$\tan\beta_1=\frac{B\sin\theta}{A+B\cos\theta}$$

Since angle between $\vec{A}$ and $\vec{B}$ is θ, so angle between, $\vec{A}$ and $-\vec{B}$ is $(180-\theta)$.



$$\Rightarrow \quad |\vec{A}-\vec{B}|=\sqrt{A^2+B^2+2AB\cos(180-\theta)}$$

Since, $\cos(180-\theta)=-\cos\theta$

$$\Rightarrow \quad |\vec{A}-\vec{B}|=\sqrt{A^2+B^2-2AB\cos\theta}$$

$$\tan\beta_1=\frac{B\sin\theta}{A+B\cos\theta}$$

and $\tan\beta_2=\dfrac{B\sin(180-\theta)}{A+B\cos(180-\theta)}$

But $\sin(180-\theta)=\sin\theta$ and $\cos(180-\theta)=-\cos\theta$

$$\Rightarrow \quad \tan\beta_2=\frac{B\sin\theta}{A-B\cos\theta}$$

Significance of Vector Addition and Subtraction

If two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ then

$\vec{R}_{\text{sum}}=\vec{A}+\vec{B}=\vec{d_1}=$ Bigger Diagonal of parallelogram

$\vec{R}_{\text{diff}}=\vec{A}-\vec{B}=\vec{d_2}=$ Smaller Diagonal of parallelogram

So, $\vec{d_1}=\vec{A}+\vec{B}$ ...(1)

$\vec{d_2}=\vec{A}-\vec{B}$ ...(2)

Adding (1) & (2)

$$\vec{A}=\frac{\vec{d_1}+\vec{d_2}}{2}=\frac{1}{2}(\vec{d_1}+\vec{d_2})$$

Subtracting (1) & (2)

$$\vec{B}=\frac{\vec{d_1}-\vec{d_2}}{2}=\frac{1}{2}(\vec{d_1}-\vec{d_2})$$

RECTANGULAR COMPONENTS IN 2-D SPACE

Consider a vector $\vec{R}$ inclined to the x-axis at angle θ. If $\vec{R}_x$ and $\vec{R}_y$ are the components (projections) of $\vec{R}$ along x-axis and y-axis then by parallelogram law of vectors

$$\vec{R}=\vec{R}_x+\vec{R}_y \quad \ldots(1)$$

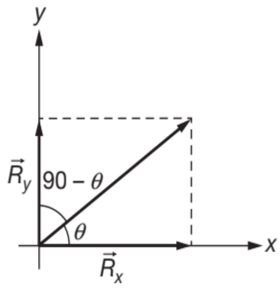
Since $\vec{R}_x=R_x\hat{i}$ and $\vec{R}_y=R_y\hat{j}$

$$\Rightarrow \quad \vec{R}=R_x\hat{i}+R_y\hat{j} \quad \ldots(2)$$

Further

$$R_x=R\cos\theta \quad \ldots(3)$$

$$R_y=R\sin\theta \quad \ldots(4)$$



Since, $\vec{R}=R_x\hat{i}+R_y\hat{j}$

$$\Rightarrow \quad \vec{R}=R(\cos\theta\hat{i}+\sin\theta\hat{j})$$

$$\Rightarrow \quad R\hat{R}=R(\cos\theta\hat{i}+\sin\theta\hat{j}) \qquad \{\because \vec{R}=R\hat{R}\}$$

$$\Rightarrow \quad \hat{R}=\cos\theta\hat{i}+\sin\theta\hat{j} \qquad \{|\hat{R}|=1\}$$

Squaring (3) and (4) and adding, we get

$$R_x^2+R_y^2=R^2\cos^2\theta+R^2\sin^2\theta$$

$$\Rightarrow \quad R_x^2 + R_y^2 = R^2$$

$$\Rightarrow \quad R = \sqrt{R_x^2 + R_y^2}$$

So, $\cos\theta = \dfrac{R_x}{R}$ {using (3)}

$$\Rightarrow \quad \cos\begin{pmatrix}\text{Angle which}\\ \vec{R}\text{ makes}\\ \text{with } x\text{-axis}\end{pmatrix} = \frac{R_x}{R} = \frac{\begin{pmatrix}\text{Component}\\ \text{of vector}\\ \text{along } x\text{-axis}\end{pmatrix}}{\begin{pmatrix}\text{Magnitude}\\ \text{of vector}\end{pmatrix}}$$

Similarly $\sin\theta = \dfrac{R_y}{R}$ {using (4)}

$$\Rightarrow \quad \cos(90-\theta) = \frac{R_y}{R}$$

$$\Rightarrow \quad \cos\begin{pmatrix}\text{Angle which}\\ \vec{R}\text{ makes}\\ \text{with } x\text{-axis}\end{pmatrix} = \frac{R_y}{R} = \frac{\begin{pmatrix}\text{Component}\\ \text{of vector}\\ \text{along } y\text{-axis}\end{pmatrix}}{\begin{pmatrix}\text{Magnitude}\\ \text{of vector}\end{pmatrix}}$$

If $\vec{R} = \vec{A} + \vec{B} + \vec{C}$

Then $\left.\begin{matrix} R_x = A_x + B_x + C_x \\ R_y = A_y + B_y + C_y \end{matrix}\right\}$ in 2 D space

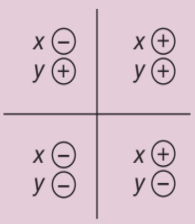
$$\tan\beta = \frac{R_y}{R_x} = \frac{A_y + B_y + C_y}{A_x + B_x + C_x}$$

where β is angle $\vec{R}$ makes with x-axis

Also, $\left.\begin{matrix} R_x = A_x + B_x + C_x \\ R_y = A_y + B_y + C_y \\ R_z = A_z + B_z + C_z \end{matrix}\right\}$ 3D space

Problem Solving Technique(s)

A vector inclined at an angle θ to x-axis in 1st and 4th Quadrant OR at an angle θ with $-x$ axis in 2nd and 3rd Quadrant is



$$\vec{r}_{1\text{st quad}} = \vec{r}_1 = r\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

$$\vec{r}_{2\text{nd quad}} = \vec{r}_2 = r\left(-\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

$$\vec{r}_{3\text{rd quad}} = \vec{r}_3 = -r\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

$$\vec{r}_{4\text{th quad}} = \vec{r}_4 = r\left(\cos\theta\hat{i} - \sin\theta\hat{j}\right)$$

ILLUSTRATION 10

Construct a vector of magnitude 6 unit making an angle of 30° with y-axis.

SOLUTION

30° with y-axis $\Rightarrow$ 60° with x-axis.

Since, $\vec{R} = R\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$

where, θ is the angle which $\vec{R}$ makes with x-axis

$$\Rightarrow \quad \vec{R} = 6\left(\cos 60\hat{i} + \sin 60\hat{j}\right)$$

$$\Rightarrow \quad \vec{R} = 3\left(1\hat{i} + \sqrt{3}\hat{j}\right)$$

$$\Rightarrow \quad \vec{R} = 3\hat{i} + 3\sqrt{3}\hat{j}$$

ILLUSTRATION 11

A particle undergoes three successive displacements in a plane. The first time it moves 4 m south-west the second time 5 m east and the third time 6 m in direction 60° north of east. Draw a vector diagram and determine total displacement of particle from starting point.

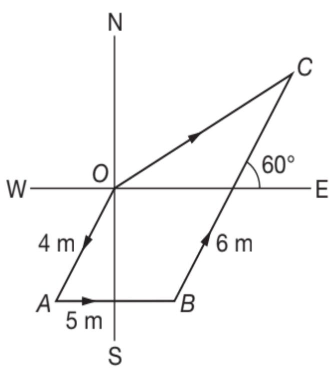
SOLUTION

$$\overrightarrow{OA} = 4\cos(180^\circ + 45^\circ)\hat{i} + 4\sin(180^\circ + 45^\circ)\hat{j}$$

$\overrightarrow{OA}$ makes $(180^\circ + 45^\circ)$ angle with positive x-axis and the same with positive y-axis so

x-component of $\overrightarrow{OA} = 4\cos(180^\circ + 45^\circ)$

y-component of $\overrightarrow{OA} = 4\sin(180^\circ + 45^\circ)$



$\Rightarrow \quad \overrightarrow{OA} = -2\sqrt{2}\hat{i} - 2\sqrt{2}\hat{j}$

$\overrightarrow{AB} = 5\cos 0°\hat{i} = 5\hat{i}$

$\overrightarrow{BC} = 6\cos 60°\hat{i} + 6\sin 60°\hat{j} = 3\hat{i} + 3\sqrt{3}\hat{j}$

$\Rightarrow \quad \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

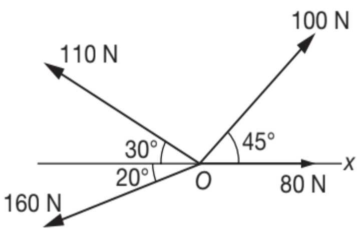
$\Rightarrow \quad \overrightarrow{OC} = \left(-2\sqrt{2}\hat{i} - 2\sqrt{2}\hat{j}\right) + 5\hat{i} + \left(3\hat{i} + 3\sqrt{3}\hat{j}\right)$

$\Rightarrow \quad \overrightarrow{OC} = \left(-2\sqrt{2} + 5 + 3\right)\hat{i} + \left(-2\sqrt{2} + 3\sqrt{3}\hat{j}\right)$

$\Rightarrow \quad \overrightarrow{OC} = +5\cdot 17\hat{i} + 2.37\hat{j}$

ILLUSTRATION 12

Four coplanar forces act on a body at point O as shown in diagram by use of rectangular component find direction and magnitude of resultant force.



SOLUTION

The vectors and their components are as follows

Magnitude of resultant vector	x component of resultant vector	y component of resultant vector
80	80	0
100	$110\cos 45° = 71$	$100\sin 45° = 71$
110	$-110\cos 30° = -95$	$110\sin 30° = 55$
160	$-160\cos 20° = -150$	$-160\sin 20° = -55$

$$R_x = 81 + 71 - 95 - 150 = -94 \text{ N}$$

$$R_y = 0 + 71 + 55 - 55 = 71 \text{ N}$$

Magnitude of resultant is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(94)^2 + (71)^2} = 118 \text{ N}$$

$$\tan\alpha = \frac{71}{94}$$

$\Rightarrow \quad \alpha = 37°$

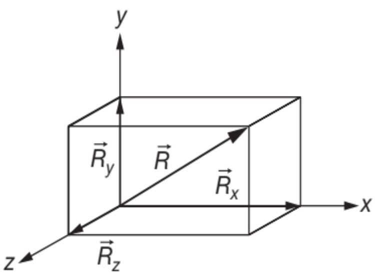
So resultant is 118 N at $180 - 37 = 143°$.

RECTANGULAR COMPONENTS IN 3-D SPACE

In 3-D space, we have

$$\vec{R} = \vec{R}_x + \vec{R}_y + \vec{R}_z$$

$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$



If $\vec{R}$ makes an angle α with x axis, β with y axis and γ with z axis, then

$\cos\alpha = \frac{R_x}{R}$	$\cos\beta = \frac{R_y}{R}$	$\cos\gamma = \frac{R_z}{R}$

$$\Rightarrow \quad \cos\alpha = \frac{R_x}{R} = \frac{R_x}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = l$$

$$\Rightarrow \quad \cos\beta = \frac{R_y}{R} = \frac{R_y}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = m$$

$$\Rightarrow \quad \cos\gamma = \frac{R_z}{R} = \frac{R_z}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = n$$

where l, m, n are called **Direction Cosines** of the vector $\vec{R}$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{R_x^2 + R_y^2 + R_z^2}{R_x^2 + R_y^2 + R_z^2} = 1$$

Conceptual Note(s)

(a) If l, m, n are called Direction Cosines of the vector, then $l^2+m^2+n^2=1$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

(b) In 3-D space a vector of magnitude r making an angle α with x-axis, β with y-axis and γ with z-axis can thus be written as

$$\vec{r}=r\left[(\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k}\right]$$

ILLUSTRATION 13

A bird moves with velocity of $20\ \text{ms}^{-1}$ in the direction making angle $60°$ with eastern line and $60°$ with vertically upward. Represent the velocity vector in rectangular form.

SOLUTION

Velocity vector $\vec{v}$ makes angle α, β and γ with x-, y- and z- axis respectively

$\therefore\quad \alpha=60°$ and $\gamma=60°$

Since, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\Rightarrow\quad \cos^2 60°+\cos^2\beta+\cos^2 60°=1$$

$$\Rightarrow\quad \left(\frac{1}{2}\right)^2+\cos^2\beta+\left(\frac{1}{2}\right)^2=1$$

$$\Rightarrow\quad \cos^2\beta=1-\frac{1}{2}$$

$$\Rightarrow\quad \cos\beta=\frac{1}{\sqrt{2}}$$

Since $\vec{v}=v\cos\alpha\hat{i}+v\cos\beta\hat{j}+v\cos\gamma\hat{k}$

$$\Rightarrow\quad \vec{v}=20\times\frac{1}{2}\hat{i}+20\times\frac{1}{\sqrt{2}}\hat{j}+20\times\frac{1}{2}\hat{k}$$

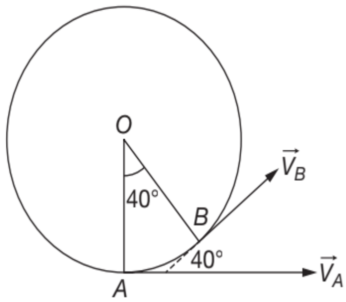
$$\Rightarrow\quad \vec{v}=10\hat{i}+10\sqrt{2}\hat{j}+10\hat{k}$$

Test Your Concepts-I

Based on Addition, Subtraction and Resolution

(Solutions on page H.41)

1. Can three vectors not in one plane give a zero resultant? Can four vectors do?
2. The x and y components of vector $\vec{A}$ are 4 m and 6 m respectively. The x and y components of vector $\vec{A}+\vec{B}$ are 10 m and 9 m respectively. Calculate for the vector $\vec{B}$ the following:
   (a) its x and y components,
   (b) its length and
   (c) the angle its makes with x-axis.
3. If $\vec{A}=4\hat{i}-3\hat{j}$ and $\vec{B}=6\hat{i}+8\hat{j}$, obtain the scalar magnitude and directions of $\vec{A}$, $\vec{B}$, $(\vec{A}+\vec{B})$, $(\vec{A}-\vec{B})$ and $(\vec{B}-\vec{A})$.
4. A particle has a displacement of 12 m towards east and 5 m towards north and 6 m vertically upwards. Find the magnitude of the sum of these displacements.
5. Show that if two vectors are equal in magnitude, their vector sum and difference are at right angles.
6. Establish the following vector inequalities:
   (a) $|\vec{a}-\vec{b}|\le|\vec{a}|+|\vec{b}|$
   (b) $|\vec{a}-\vec{b}|\ge|\vec{a}|-|\vec{b}|$
   When does the equality sign apply?
7. Find the magnitude and direction of the resultant of following forces acting on a particle: $\vec{F}_1=3\sqrt{2}$ Kgf due north-east, $\vec{F}_2=6\sqrt{2}$ Kgf due south-east and $\vec{F}_3=\sqrt{2}$ Kgf due north-west.
8. What does the statement $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|$ imply?
9. Two forces F_1 and F_2 acting at a point have a resultant R_1. If F_2 is doubled, the new resultant R_2 is at right angles to F_1. Prove that R_1 and F_2 have the same magnitude.
10. In figure, a particle is moving in a circle of radius r centred at O with constant speed v. What is the change in velocity in moving from A to B? Given $\angle AOB=40°$.



11. Prove that the resultant of two vectors of equal magnitude is equally inclined to either of the two vectors.

12. Consider two unit vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ. Prove that

(a) $|\vec{A}+\vec{B}| = 2\cos\left(\frac{\theta}{2}\right)$

(b) $|\vec{A}-\vec{B}| = 2\sin\left(\frac{\theta}{2}\right)$

13. Can the magnitude of the resultant of two equal vectors be equal to the magnitude of either of the vectors? Explain your answer.

14. Prove that the vector $\vec{r} = \hat{i}+\hat{j}+\hat{k}$ is equally inclined to all the three axis.

VECTOR MULTIPLICATION OF 2 VECTORS

MULTIPLICATION OF TWO VECTORS INCLINED AT AN ANGLE θ

SCALAR PRODUCT OR DOT PRODUCT

$\vec{A}\cdot\vec{B}$ (read as $\vec{A}$ dot $\vec{B}$)

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

VECTOR PRODUCT OR CROSS PRODUCT

$\vec{A}\times\vec{B}$ (read as $\vec{A}$ cross $\vec{B}$)

$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\hat{n}$

Where $\hat{n}$ is a unit vector giving the direction of $\vec{A}\times\vec{B}$ (or obviously pointing towards $\vec{A}\times\vec{B}$) and $\hat{n}$ is unit vector whose direction can be found by RIGHT HAND THUMB RULE (RHTR) according to which "Curl the fingers of RIGHT hand from 1st vector to 2nd vector (i.e., from $\vec{A}$ to $\vec{B}$ in case of $\vec{A}\times\vec{B}$ and $\vec{B}$ to $\vec{A}$ in case of $\vec{B}\times\vec{A}$), then the direction of the thumb gives direction of cross product or $\hat{n}$".

As a consequence of RHTR we conclude that it is a unit vector normal to the plane containing (or having) $\vec{A}$ and $\vec{B}$ or we can say that $\hat{n}$ is perpendicular to $\vec{A}$ and $\vec{B}$ simultaneously.

DOT PRODUCT

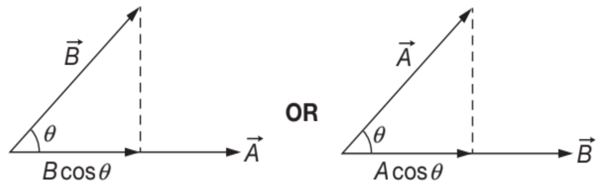
Mathematically, dot product is defined as the product of the magnitudes of the vectors and cosine of the angle between the vectors. So,

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

GEOMETRICAL INTERPRETATION

Since $\vec{A}\cdot\vec{B} = A(B\cos\theta)$

$\Rightarrow \quad \vec{A}\cdot\vec{B} = A(\text{projection of } \vec{B} \text{ along } \vec{A})$



$$\Rightarrow \quad B\cos\theta = \frac{\vec{A}\cdot\vec{B}}{A}$$

$$\vec{B}\cdot\vec{A} = B(A\cos\theta)$$

$$\Rightarrow \quad \vec{B}\cdot\vec{A} = B \times (\text{Projection of } A \text{ along } B)$$

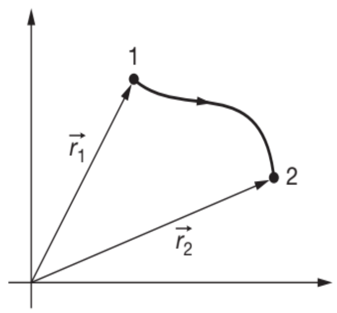
$$\Rightarrow \quad A\cos\theta = \frac{\vec{A}\cdot\vec{B}}{B}$$

PHYSICAL INTERPRETATION

Work done is dot product of force with displacement, so we have

$$W = \vec{F}\cdot\Delta\vec{r}$$

$$W = \vec{F}\cdot(\vec{r}_2 - \vec{r}_1) \quad \text{where } \Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$



Properties of Dot Product

(a) Dot product is commutative in nature

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

(b) Dot product is distributive with respect to sum

$$\vec{A}\cdot(\vec{B}\pm\vec{C}) = \vec{A}\cdot\vec{B} \pm \vec{A}\cdot\vec{C}$$

(c) If $\theta = 0°$ i.e. Vectors are parallel then $\vec{A}\cdot\vec{B} = AB\cos 0 = AB$ i.e., when the vectors are parallel $\vec{A}\cdot\vec{B}$ is just the simple product of the magnitudes of $\vec{A}$ and $\vec{B}$.

(d) Dot product of vector with itself is equal to the square of the magnitude of the vector

$$\vec{A}\cdot\vec{A} = (A)(A)\cos 0$$

$$\Rightarrow \quad \vec{A}\cdot\vec{A} = A^2$$

(e) If $\theta = 180°$ i.e., Vectors are antiparallel, then

$$\vec{A}\cdot\vec{B} = AB(-1) \qquad \{\because (\cos 180° = -1)\}$$

$$\vec{A}\cdot\vec{B} = -AB$$

i.e., If two vectors are antiparallel then their dot product equals the negative of simple product of magnitudes of vectors.

(f) If $\theta = 90°$ i.e., vectors are perpendicular

$$\vec{A}\cdot\vec{B} = AB(0) \qquad \vec{A}\cdot\vec{B} = 0 \text{ i.e.,}$$

Vectors perpendicular $\Leftrightarrow$ Dot product $= 0$

(g) $\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0$

Therefore, in general, $\hat{i}\cdot\hat{i} = 1$, $\hat{j}\cdot\hat{j} = 1$, $\hat{k}\cdot\hat{k} = 1$

(h) $\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90°$

$$\Rightarrow \quad \hat{i}\cdot\hat{j} = (1)(1)(0) = 0$$

Therefore, in general $\hat{i}\cdot\hat{j} = 0$, $\hat{j}\cdot\hat{k} = 0$, $\hat{k}\cdot\hat{i} = 0$ or $\hat{j}\cdot\hat{i} = 0$, $\hat{k}\cdot\hat{j} = 0$, $\hat{i}\cdot\hat{k} = 0$

(i) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ are any two vectors, then

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) +$$

$$A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

(j) Since $\vec{A}\cdot\vec{B} = AB\cos\theta$

and $-1 \le \cos\theta \le 1$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} \text{ is maximum at } \cos\theta = 1 (\text{i.e., } \theta = 0°)$$

$$\vec{A}\cdot\vec{B} \text{ is minimum at } \cos\theta = -1 (\text{i.e., } \theta = 180°)$$

$$\Rightarrow \quad (\vec{A}\cdot\vec{B})_{\text{min}} \le \vec{A}\cdot\vec{B} \le (\vec{A}\cdot\vec{B})_{\text{max}}$$

$$\Rightarrow \quad -AB \le \vec{A}\cdot\vec{B} \le AB$$

(k) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ and θ is $\angle$ between $\vec{A}$ & $\vec{B}$.

Then $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$ where, $\vec{A}\cdot\vec{B} = AB\cos\theta$

(l) $|\vec{A}+\vec{B}|^2=(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$

$|\vec{A}-\vec{B}|^2=(\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$

(m) $(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B})=\vec{A}\cdot\vec{A}-\vec{B}\cdot\vec{B}+\cancel{\vec{B}\cdot\vec{A}}-\cancel{\vec{A}\cdot\vec{B}}$

$\Rightarrow\ (\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B})=A^2-\vec{B}$

$\{\because\ \vec{A}\cdot\vec{A}=A^2 \text{ and } \vec{B}\cdot\vec{B}=B^2\}$

(n) Similarly

$|\vec{A}+\vec{B}+\vec{C}|^2=(\vec{A}+\vec{B}+\vec{C})\cdot(\vec{A}+\vec{B}+\vec{C})$

$\Rightarrow\ |\vec{A}+\vec{B}+\vec{C}|^2=\vec{A}\cdot\vec{A}+\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{A}+\vec{B}\cdot\vec{B}+\vec{B}\cdot\vec{C}+\vec{C}\cdot\vec{A}+\vec{C}\cdot\vec{B}+\vec{C}\cdot\vec{C}$

$\Rightarrow\ |\vec{A}+\vec{B}+\vec{C}|^2=A^2+B^2+C^2+2(\vec{A}\cdot\vec{B}+\vec{B}\cdot\vec{C}+\vec{C}\cdot\vec{A})$

(o) If $\vec{A}=A_z\hat{i}+A_y\hat{j}+A_z\hat{k}$, has l_1, m_1, n_1 as direction cosines

If $\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$, has l_2, m_2, n_2 as direction cosines

Also, $\cos\theta=\dfrac{A_xB_x+A_yB_y+A_zB_z}{AB}$

$\Rightarrow\ \cos\theta=\dfrac{A_xB_x}{AB}+\dfrac{A_yB_y}{AB}+\dfrac{A_zB_z}{AB}$

$\Rightarrow\ \cos\theta=\left(\dfrac{A_x}{A}\right)\left(\dfrac{B_x}{B}\right)+\left(\dfrac{A_y}{A}\right)\left(\dfrac{B_y}{B}\right)+\left(\dfrac{A_z}{A}\right)\left(\dfrac{B_z}{B}\right)$

$\Rightarrow\ \boxed{\cos\theta=l_1l_2+m_1m_2+n_1n_2}$

Conceptual Note(s)

(a) Angle between two vectors $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$ is

$$\cos\theta=\frac{A_xB_x+A_yB_y+A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\sqrt{B_x^2+B_y^2+B_z^2}}$$

(b) If $\langle l_1\ \ m_1\ \ n_1\rangle$ and $\langle l_2\ \ m_2\ \ n_2\rangle$ are direction cosines of $\vec{A}$ and $\vec{B}$, then $\cos\theta=l_1l_2+m_1m_2+n_1n_2$.

(c) If two vectors are perpendicular then

$l_1l_2+m_1m_2+n_1n_2=0$

(d) Also

$$\frac{d}{dt}(\vec{A}\cdot\vec{B})=\vec{A}\cdot\frac{d\vec{B}}{dt}+\vec{B}\cdot\frac{d\vec{A}}{dt}=\left(\frac{d\vec{A}}{dt}\right)\cdot\vec{B}+\left(\frac{d\vec{B}}{dt}\right)\cdot\vec{A}$$

ILLUSTRATION 14

Three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are such that $\vec{A}=\vec{B}+\vec{C}$ and their magnitude are 5, 4, 3. Find angle between $\vec{A}$ and $\vec{C}$.

SOLUTION

As $\vec{A}=\vec{B}+\vec{C}$

$\Rightarrow\ \vec{B}=\vec{A}-\vec{C}$

Now $\vec{B}\cdot\vec{B}=(\vec{A}-\vec{C})\cdot(\vec{A}-\vec{C})$

$B^2=A^2+C^2-2AC\cos\theta$

$\Rightarrow\ \cos\theta=\dfrac{A^2+C^2-B^2}{2AC}=\dfrac{5^2+3^2-4^2}{2\times5\times3}=\dfrac{18}{30}=0.6$

$\theta=\cos^{-1}(0.6)$

ILLUSTRATION 15

For what value of a are the vectors $\vec{A}=a\hat{i}-2\hat{j}+\hat{k}$ and $\vec{B}=2a\hat{i}+a\hat{j}-4\hat{k}$ perpendicular to each other?

SOLUTION

Now $\vec{A}$ and $\vec{B}$ are $\perp$ to each other

$\Rightarrow\ \vec{A}\cdot\vec{B}=0$

$\Rightarrow\ \vec{A}\cdot\vec{B}=(a\hat{i}-2\hat{j}+\hat{k})\cdot(2a\hat{i}+a\hat{j}-4\hat{k})$

$\Rightarrow\ \vec{A}\cdot\vec{B}=2a^2-2a-4=0$

$\{\because\ \hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1,\ \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0\}$

$\Rightarrow\ a^2-a-2=0$

$\Rightarrow\ a^2-2a+a-2=0$

$\Rightarrow\ a(a-2)+(a-2)=0$

$\Rightarrow\ (a+1)(a-2)=0$

$\Rightarrow\ a=-1,\ a=2$

ILLUSTRATION 16

Find the component of $\vec{a}=2\hat{i}+3\hat{j}$ along the direction of vector $\hat{i}+\hat{j}$.

SOLUTION

Given component of $\vec{a}$ in the direction of $\vec{b}$ is

$$\vec{r} = (a\cos\theta)\hat{b} = \left(\frac{\vec{a}\cdot\vec{b}}{b}\right)\hat{b} = \left(\frac{\vec{a}\cdot\vec{b}}{b^2}\right)\vec{b}$$

$$\Rightarrow \quad \vec{r} = \frac{(2\hat{i}+3\hat{j})\cdot(\hat{i}+\hat{j})}{\left|\sqrt{(1)^2+(1)^2}\right|^2}(\hat{i}+\hat{j}) = \frac{5}{2}(\hat{i}+\hat{j})$$

So, component vector of $\vec{a}$ along $\vec{b}$ is $\vec{r} = \frac{5}{2}(\hat{i}+\hat{j})$.

Test Your Concepts-II

Based on Dot Product

(Solutions on page H.43)

1. Express the scalar product of two vectors in terms of their rectangular components.
2. Two billiard balls are rolling on a flat table. One has the velocity components $v_x = 1\,\text{ms}^{-1}$, $v_y = \sqrt{3}\,\text{ms}^{-1}$ and the other has components $v'_x = 2\,\text{ms}^{-1}$ and $v'_y = 2\,\text{ms}^{-1}$. If both the balls start moving from the same point, what is the angle between their paths?
3. If $\vec{A} = 4\hat{i} + 6\hat{j} - 3\hat{k}$ and $\vec{B} = -2\hat{i} - 5\hat{j} + 7\hat{k}$, find the
   (a) direction cosines of $\vec{A}$ and $\vec{B}$.
   (b) angle between $\vec{A}$ and $\vec{B}$.
4. Show that the vectors $\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{C} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angle triangle.
5. Find the projection of $\vec{A} = 10\hat{i} + 8\hat{j} - 6\hat{k}$ along $\vec{r} = 5\hat{i} + 6\hat{j} + 9\hat{k}$.
6. Obtain the scalar product of the vectors $(6, 2, 3)$, $(2, -9, 6)$ and also the angle between them.
7. Show that, for a vector $\vec{u}$ of constant magnitude, we have $\vec{u}\cdot\frac{d\vec{u}}{dt} = 0$.
8. Forces acting on a particle have magnitudes 5 N, 3 N and 1 N and act in the directions of the vectors $6\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} - 2\hat{j} + 6\hat{k}$, $2\hat{i} - 3\hat{j} - 6\hat{k}$ respectively. These remain constant while the particle is displaced from the point $A(2, -1, -3)$ to $B(5, -1, -1)$. Find the work done by the forces, the unit of length being 1 m.

CROSS PRODUCT OR VECTOR PRODUCT

Mathematically,

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$$

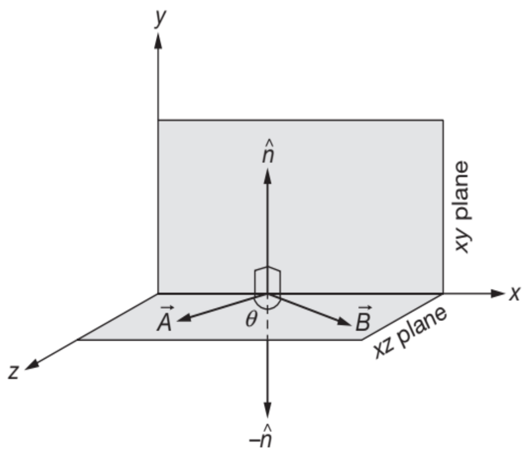
Direction of $\hat{n}$ is given by RHTR (stated earlier).

$\hat{n}$ indicates direction of $\vec{A}\times\vec{B}$ and $-\hat{n}$ indicates direction of $\vec{B}\times\vec{A}$.

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

So, cross product is Non-commutative in nature.

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$$



$\Rightarrow \quad |\vec{A}\times\vec{B}| = AB\sin\theta$

$\Rightarrow \quad |\vec{B}\times\vec{A}| = BA\sin\theta$

$\therefore \quad |\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$

Also, $\hat{n} = \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$, where, $\hat{n}$ indicates direction of $\vec{A}\times\vec{B}$

Conceptual Note(s)

(a) $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$

(b) $\hat{n}$ is a new vector perpendicular to $\vec{A}$ as well as $\vec{B}$.

$\hat{n}\cdot\vec{A} = 0$

{Perpendicular vectors have dot product=0}

$\hat{n}\cdot\vec{B} = 0$

Hence, $(\vec{A}\times\vec{B})$ is a new vector perpendicular to $\vec{A}$ as well as $\vec{B}$

(c) $\hat{n}$ is perpendicular to $\vec{A}$ as well as $\vec{B}$

(where $\hat{n}$ indicates direction of $\vec{A}\times\vec{B}$)

So, $\vec{A}\times\vec{B}$ is a new vector $\perp$ to $\vec{A}$ as well as $\vec{B}$.

$\Rightarrow \quad (\vec{A}\times\vec{B})\cdot\vec{A} = 0$

$\Rightarrow \quad (\vec{B}\times\vec{A})\cdot\vec{A} = 0$

$\Rightarrow \quad (\vec{A}\times\vec{B})\cdot\vec{B} = 0$

$\Rightarrow \quad (\vec{B}\times\vec{A})\cdot\vec{B} = 0$

GEOMETRICAL INTERPRETATION OF CROSS PRODUCT

Half of magnitude of cross product equals the area of triangle with adjacent sides $\vec{A}$ and $\vec{B}$.

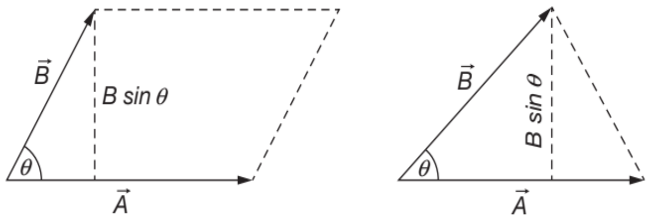
$$\text{Area } = \frac{1}{2}(A)(B\sin\theta) = \frac{1}{2}(AB\sin\theta)$$

$$\Rightarrow \quad \text{Area of triangle} = \frac{1}{2}|\vec{A}\times\vec{B}|$$

$$\begin{pmatrix}\text{Area of}\\ \text{parallelogram}\end{pmatrix} = (\text{Base})\times\begin{pmatrix}\text{perpendicular}\\ \text{distance between}\\ \text{parallel sides}\end{pmatrix}$$

$$\begin{pmatrix}\text{Area of}\\ \text{parallelogram}\end{pmatrix} = A(B\sin\theta) = AB\sin\theta$$

Also, Area of parallelogram $= |\vec{A}\times\vec{B}| = \dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$



Conceptual Note(s)

So, half of the modulus of cross product equals the area of the triangle with adjacent sides $\vec{A}$ and $\vec{B}$ and magnitude of cross product equals the area of the parallelogram with adjacent sides $\vec{A}$ and $\vec{B}$.

PHYSICAL INTERPRETATION

(a) The torque due to a force $\vec{F}$ acting at a point with position vector $\vec{r}$ about the origin is $\vec{\tau} = \vec{r}\times\vec{F}$. The torque due to a force $\vec{F}$ acting at a point with position vector $\vec{r}$ about another point having position vector $\vec{r}_0$ is $\vec{\tau} = (\vec{r}-\vec{r}_0)\times\vec{F}$.

τ (read as tau) i.e., Torque (in Physics)

$\vec{r}$ is the distance of point of application of force from axis of rotation (A.O.R.) and F is force.

τ is also called the MOMENT OF FORCE.

(b) $\vec{L} = \vec{r}\times\vec{p}$

$\vec{L} =$ Angular momentum also called MOMENT OF LINEAR MOMENTUM

$\vec{p} =$ Linear momentum $= m\vec{v}$

$\vec{p} = m\vec{v}$

Since, Mass is a scalar therefore momentum can be read as mass times velocity

$$\vec{L} = \vec{r}\times(m\vec{v}) = m(\vec{r}\times\vec{v}) = m\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ v_x & v_y & v_z\end{vmatrix}$$

(c) $\vec{v} = \vec{\omega}\times\vec{r}$, where $\vec{v}$ is the linear velocity of a particle moving in a circle of radius vector $\vec{r}$ with angular velocity $\vec{\omega}$.

(d) Magnetic force experienced by a charge q entering a magnetic field with a velocity $\vec{v}$ is given by

$$\vec{F}_{\text{magnetic}} = q(\vec{v}\times\vec{B})$$

Properties of Vector Product/Cross Product

(a) Vector Product is Non commutative i.e.

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

$$\Rightarrow\quad(\vec{A}\times\vec{B})+(\vec{B}\times\vec{A})=\vec{0}$$

i.e., cross product is position sensitive.

(b) Cross product is distributive with respect to sum i.e.

$$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

(c) If $\vec{A}$ and $\vec{B}$ are parallel i.e. $\theta=0°$, then

$$\vec{A}\times\vec{B}=AB\sin 0\hat{n}$$

$$\Rightarrow\quad\vec{A}\times\vec{B}=\vec{0}$$

Vectors parallel ⇔ cross-product equal to $\vec{0}$

(d) If $\vec{A}$ and $\vec{B}$ are antiparallel i.e. $\theta=180°$

$$\vec{A}\times\vec{B}=AB\sin(180°)\hat{n}$$

$$\Rightarrow\quad\vec{A}\times\vec{B}=\vec{0}\qquad\{\because\sin 180°=0\}$$

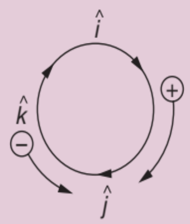
Vectors antiparallel ⇔ cross product equal to $\vec{0}$

(e) $\vec{A}\times\vec{A}=AA\sin 0\hat{n}$

$$\vec{A}\times\vec{A}=\vec{0}$$

i.e., cross product of vector with itself is $\vec{0}$

(f) $\hat{i}\times\hat{i}=\vec{0},\ \hat{j}\times\hat{j}=\vec{0},\ \hat{k}\times\hat{k}=\vec{0}$



(g) $\hat{i}\times\hat{j}=\hat{k},\ \hat{k}\times\hat{i}=\hat{j},\ \hat{j}\times\hat{k}=\hat{i}$

For a Right handed triad system, curl fingers of your right hand from

x to *y*, thumb gives direction of *z*
y to *z*, thumb gives direction of *x*
z to *x*, thumb gives direction of *y*

(h) If $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k},\ \vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$, then,

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}_{3\times3}$$

$$\Rightarrow\quad\vec{A}\times\vec{B}=\hat{i}\begin{vmatrix}A_y&A_z\\B_y&B_z\end{vmatrix}_{2\times2}-\hat{j}\begin{vmatrix}A_x&A_z\\B_x&B_z\end{vmatrix}_{2\times2}+\hat{k}\begin{vmatrix}A_x&A_y\\B_x&B_y\end{vmatrix}_{2\times2}$$

$$\Rightarrow\quad\vec{A}\times\vec{B}=\hat{i}(A_yB_z-B_yA_z)-\hat{j}(A_xB_z-A_zB_x)+\hat{k}(A_xB_y-B_xA_y)$$

Conceptual Note(s)

(a) $\vec{A}\times\vec{B}+\vec{C}\times\vec{A}\neq\vec{A}\times(\vec{B}+\vec{C})$

Instead, $\vec{A}\times\vec{B}+\vec{C}\times\vec{A}=\vec{A}\times\vec{B}-\vec{A}\times\vec{C}$

$$\Rightarrow\quad\vec{A}\times\vec{B}+\vec{C}\times\vec{A}=\vec{A}\times(\vec{B}-\vec{C})$$

Please note that Cross Product is always position sensitive. So be careful while changing the placement of vectors.

(b) Also we must know that

$$\frac{d}{dt}(\vec{A}\times\vec{B})=\vec{A}\times\frac{d\vec{B}}{dt}+\frac{d\vec{A}}{dt}\times\vec{B}=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

ILLUSTRATION 17

Find the area of triangle formed by tips of the vectors $\vec{a}=\hat{i}-\hat{j}-3\hat{k}$, $\vec{b}=4\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

SOLUTION

Let *ABC* be the triangle formed by the tips of given vectors. Then

$$\overrightarrow{BA}=\vec{a}-\vec{b}=(\hat{i}-\hat{j}-3\hat{k})-(4\hat{i}-3\hat{j}+\hat{k})=-3\hat{i}+2\hat{j}-4\hat{k}$$

$$\overrightarrow{BC}=\vec{c}-\vec{b}=(3\hat{i}-\hat{j}+2\hat{k})-(4\hat{i}-3\hat{j}+\hat{k})=-\hat{i}+2\hat{j}+\hat{k}$$

$$\Rightarrow\quad\overrightarrow{BA}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-3&2&-4\\-1&2&1\end{vmatrix}$$

$$\Rightarrow\quad\overrightarrow{BA}\times\overrightarrow{BC}=\hat{i}(2+8)-\hat{j}(-3-4)+\hat{k}(-6+2)$$

$$\Rightarrow\quad\overrightarrow{BA}\times\overrightarrow{BC}=10\hat{i}+7\hat{j}-4\hat{k}$$

$$\Rightarrow\quad|\overrightarrow{BA}\times\overrightarrow{BC}|=\sqrt{(10)^2+(7)^2+(-4)^2}$$

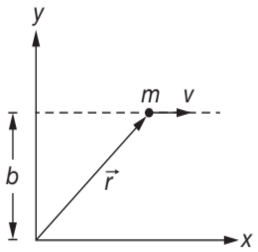
$$\Rightarrow\quad|\overrightarrow{BA}\times\overrightarrow{BC}|=\sqrt{165}=12.8$$

$$\Rightarrow\quad\text{Area of }\Delta ABC=\frac{1}{2}|\overrightarrow{BA}\times\overrightarrow{BC}|$$

$$\Rightarrow\quad\text{Area of }\Delta ABC=\frac{1}{2}\times12.8=6.4\text{ sq. unit}$$

ILLUSTRATION 18

If a particle of mass m is moving with constant velocity v parallel to x-axis in x-y plane as shown in figure, calculate its angular momentum with respect to origin at any time t.



SOLUTION

We know that

Angular momentum $\vec{L} = \vec{r} \times \vec{p}$

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

As motion is in x-y plane $\{\therefore z = 0 \text{ and } p_z = 0\}$

$$\vec{L} = \hat{k}\left(xp_y - yp_x\right)$$

Here $x = vt$

$$y = b \qquad p_x = mv \qquad p_y = 0$$

$$\vec{L} = \hat{k}[vt \times 0 - bmv] = -(mbv)\hat{k}$$

From Result we can conclude that, if motion is in x-y plane angular momentum is always directed along z-axis i.e., angular momentum is always perpendicular to plane of motion.

ILLUSTRATION 19

Find the value of a for which the vectors $3\hat{i} + 3\hat{j} + 9\hat{k}$ and $\hat{i} + a\hat{j} + 3\hat{k}$ are parallel.

SOLUTION

Let $\vec{A} = 3\hat{i} + 3\hat{j} + 9\hat{k}$

$$\vec{B} = \hat{i} + a\hat{j} + 3\hat{k}$$

Now $\vec{A} \parallel \vec{B}$ so we have

$$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z} = k$$

$$\Rightarrow \frac{3}{1} = \frac{3}{a} = \frac{9}{3} = k$$

$$\Rightarrow 9a = 9$$

$$\Rightarrow a = 1$$

CONFUSION?

Whether the vectors $\vec{A}$ and $\vec{B}$ are parallel or anti-parallel then in both the cases

$$\vec{A} \times \vec{B} = \vec{0}$$

HOW TO REMOVE THE CONFUSION?

If $\vec{A} = k\vec{B}(k > 0)$, then vectors are parallel and if $\vec{A} = -k\vec{B}(k > 0)$, then vectors are antiparallel.

OR

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

If $\dfrac{A_x}{B_x} = \dfrac{A_y}{B_y} = \dfrac{A_z}{B_z} = +k(k > 0)$, **then** $\vec{A}$ parallel to $\vec{B}$,

else if $\dfrac{A_x}{B_x} = \dfrac{A_y}{B_y} = \dfrac{A_z}{B_z} = -k(k > 0)$, **then** $\vec{A}$ anti-parallel to $\vec{B}$.

OR

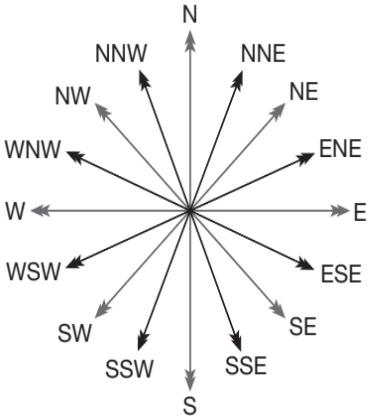
Find $\cos\theta$ i.e. angle between two vectors $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{AB}$

If $\cos\theta = 1$, **then** vectors are parallel

If $\cos\theta = -1$, **then** vectors are antiparallel

DIRECTIONS

The example below indicates the method to read and express directions.



NNE: North of North East.
ENE: East of North East.
ESE: East of South East.
SSE: South of South East.
SSW: South of South West.
WSW: West of South West.
WNW: West of North West.
NNW: North of North West.

(a) NW OR SW OR SE OR NE means $45°$ with either of the axis.

(b) $30°NW$ means $30°$ towards the north of west

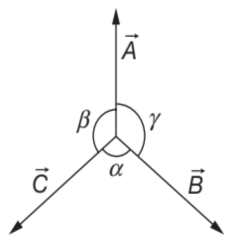
(c) $35°SW$ means $35°$ towards the south of west.

LAMI'S THEOREM

Statement

In any ΔABC with sides $\vec{a}$, $\vec{b}$, $\vec{c}$, if α, β and γ be the respective angles containing the sides, then

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$



OR

For any triangle the ratio of the sine of the angle containing the side to the length of the side is a constant.

Proof

For a triangle whose three sides are in the same order we establish the Lami's Theorem in the following manner. For the triangle shown all three sides are in same order, so

$$\vec{a}+\vec{b}+\vec{c}=\vec{0} \quad \ldots(1)$$

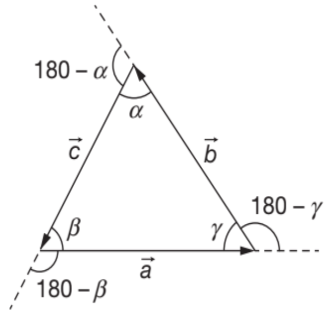
$$\Rightarrow \vec{a}+\vec{b}=-\vec{c} \quad \ldots(2)$$

Pre-multiplying both sides by $\vec{a}$

$$\vec{a}\times(\vec{a}+\vec{b})=-\vec{a}\times\vec{c}$$

$$\Rightarrow \vec{0}+\vec{a}\times\vec{b}=-\vec{a}\times\vec{c}$$

$$\Rightarrow \vec{a}\times\vec{b}=\vec{c}\times\vec{a} \quad \ldots(3)$$



Pre-multiplying both sides of (2) by $\vec{b}$

$$\vec{b}\times(\vec{a}+\vec{b})=\vec{b}\times\vec{c}$$

$$\Rightarrow \vec{b}\times\vec{a}+\vec{b}\times\vec{b}=-\vec{b}\times\vec{c}$$

$$\Rightarrow -\vec{a}\times\vec{b}=-\vec{b}\times\vec{c}$$

$$\Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c} \quad \ldots(4)$$

From (3) and (4), we get

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$$

Taking magnitude, we get

$$|\vec{a}\times\vec{b}|=|\vec{b}\times\vec{c}|=|\vec{c}\times\vec{a}|$$

$$\Rightarrow ab\sin(180-\gamma)=bc\sin(180-\alpha)=ca\sin(180-\beta)$$

$$\Rightarrow ab\sin\gamma=bc\sin\alpha=ca\sin\beta$$

Dividing throughout by abc, we have

$$\Rightarrow \frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

SCALAR TRIPLE PRODUCT (STP)

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, then the scalar triple product can be written as

$$(\vec{a}\times\vec{b})\cdot\vec{c} \text{ i.e., } (\vec{a} \text{ CROSS } \vec{b}) \text{ DOT } \vec{c}$$

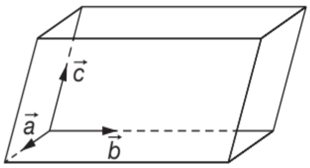
It can also be written as $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

Conceptual Note(s)

Scalar Triple Product (STP) is a scalar product and thus has no direction.

GEOMETRICAL INTERPRETATION OF SCALAR TRIPLE PRODUCT

Geometrically, the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of parallelopiped whose coterminus edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right handed system of vectors.



PROPERTIES OF SCALAR TRIPLE PRODUCT

(a) Cyclic Permutation of $\vec{a}$, $\vec{b}$ and $\vec{c}$ does not change the value of the scalar triple product

$$[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$$

(b) An antisymmetric or acyclic permutation changes sign only and not magnitude

$$[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{a}\ \vec{c}\ \vec{b}]=-[\vec{b}\ \vec{a}\ \vec{c}]$$

Conceptual Note(s)

The position of dot and cross can be interchanged keeping the cyclic order same. With such combinations, 12 different combinations are possible.

(c) Scalar Triple Product in Components Form

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, then, $\vec{a}=a_x\hat{i}+a_y\hat{j}+a_z\hat{k}$, $\vec{b}=b_x\hat{i}+b_y\hat{j}+b_z\hat{k}$ and $\vec{c}=c_x\hat{i}+c_y\hat{j}+c_z\hat{k}$

$$\Rightarrow\ (\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

(d) For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and scalar λ, we have

$$[\lambda\vec{a}\ \vec{b}\ \vec{c}]=[\vec{a}\ \lambda\vec{b}\ \vec{c}]=[\vec{a}\ \vec{b}\ \lambda\vec{c}]=\lambda[\vec{a}\ \vec{b}\ \vec{c}]$$

(e) If $[\vec{a}\ \vec{b}\ \vec{c}]=0$, then atleast two of the three vectors are collinear, equal or parallel.

(f) If $[\vec{a}\ \vec{b}\ \vec{c}]=0$, then the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

ILLUSTRATION 20

Prove that the four points $(4\hat{i}+5\hat{j}+\hat{k})$, $-(\hat{j}+\hat{k})$, $(3\hat{i}+9\hat{j}+4\hat{k})$ and $4(-\hat{i}+\hat{j}+\hat{k})$ are coplanar.

SOLUTION

For four points (i.e., three vectors) to be collinear, let us first find three vectors by taking one point as the origin. So, let the origin be at the first vector, then

$$\vec{A}=(-\hat{j}-\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})=-4\hat{i}-6\hat{j}-2\hat{k}$$

$$\vec{B}=(3\hat{i}+9\hat{j}+4\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})=-\hat{i}+4\hat{j}+3\hat{k}$$

and

$$\vec{C}=(-4\hat{i}+4\hat{j}+4\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})=-8\hat{i}-\hat{j}+3\hat{k}$$

For these to be coplanar, we must have $\vec{A}\cdot(\vec{B}\times\vec{C})=0$

$$\text{Now, } \vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$\Rightarrow\ \vec{A}\cdot(\vec{B}\times\vec{C})=-4(12+3)+6(-3+24)-2(1+32)$$

$$\Rightarrow\ \vec{A}\cdot(\vec{B}\times\vec{C})=-60+126-66$$

$$\Rightarrow\ \vec{A}\cdot(\vec{B}\times\vec{C})=0$$

Hence, we must say that the points or the vectors are coplanar.

VECTOR TRIPLE PRODUCT (VTP)

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, then the vector triple product is

$$(\vec{a}\times\vec{b})\times\vec{c} \text{ or } \vec{a}\times(\vec{b}\times\vec{c})$$

and is defined as

$$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$$

OR $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

i.e., $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Read as "$\vec{a}$ cross $\vec{b}$ cross $\vec{c}$ equals $\vec{a}$ dot $\vec{c}$ times $\vec{b}$ minus $\vec{a}$ dot $\vec{b}$ times $\vec{c}$".

Problems Solving Technique(s)

If $\vec{a}\times(\vec{b}\times\vec{c})$ can also be written as ❶ × (❷ × ❸), then it is defined as

❶ × (❷ × ❸) = (❶ · ❸)❷ − (❶ · ❷)❸

Conceptual Note(s)

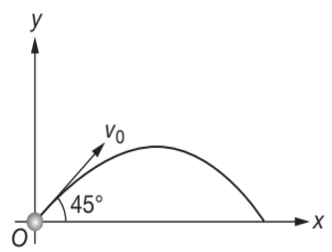
If $\vec{r}=\vec{a}\times(\vec{b}\times\vec{c})$, then $\vec{r}$ is a vector perpendicular to $\vec{a}$ and lies in the plane of $\vec{b}\times\vec{c}$.

Test Your Concepts-III

Based on Cross Product, Scalar and Vector Triple Product

(Solutions on page H.44)

1. If a force $\vec{F}=(3\hat{i}+5\hat{j}-2\hat{k})$ N acts at a point defined by $(7\hat{i}-2\hat{j}+5\hat{k})$ m, find the torque:
   (a) about the origin, and
   (b) about the point $(0,-1,0)$.
2. Express the vector product of two vectors in terms of their rectangular components.
3. Prove that $|\vec{a}\times\vec{b}|=\sqrt{a^2b^2-(\vec{a}\cdot\vec{b})^2}$
4. Show that $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=-2(\vec{a}\times\vec{b})$ and use this result to find the area of a parallelogram whose diagonals are $\hat{i}-2\hat{j}-3\hat{k}$ and $2\hat{i}-3\hat{j}+2\hat{k}$.
5. If $\vec{L}$ and $\vec{L}'$, are two length vectors, what physical quantity does $[\vec{L}\times\vec{L}']$ represent?
6. If $\vec{A}\times\vec{B}=\vec{0}$ and $\vec{A}\cdot\vec{B}=0$, does it imply that one of the vectors $\vec{A}$ or $\vec{B}$ must be a null vector?
7. The particle of mass m is projected at $t=0$ from a point O on the ground with speed v_0 at an angle 45° to the horizontal as shown in figure. Compute the magnitude and direction of the angular momentum of the particle about the point O at position $\vec{r}=\left(\frac{0.7v_0^2}{g}\right)\hat{i}+\left(\frac{0.2v_0^2}{g}\right)\hat{j}$ when the velocity of the particle is $\vec{v}=(0.7v_0)\hat{i}-(0.3v_0)\hat{j}$.



8. The force on a positively charged particle is given by $\vec{F}=q\vec{E}+q\vec{v}\times\vec{B}$. In a certain space there is a magnetic field B along z-axis and an electric field along y-axis. A positively charged particle is projected into this space. Find the direction and magnitude of minimum velocity so that it may pass on undeviated.
9. Find the moment of force $\vec{F}=\hat{i}+\hat{j}+\hat{k}$ acting at point $(-2,3,4)$ about the point $(1,2,3)$.
10. Prove that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$.

SOLVED PROBLEMS

PROBLEM 1

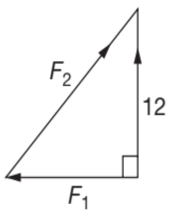
The sum of the magnitudes of two forces acting at a point is 18 and the magnitude of their resultant is 12. If the resultant is at 90° with the force of smaller magnitude, what are the magnitudes of forces?

SOLUTION

Let F_1 and F_2 be the two forces where $F_1 < F_2$

Here, $F_1 + F_2 = 18$ ...(1)

The statement to the question can be diagrammatically drawn as shown in Figure.



So, the diagram clearly shows that 12 is the resultant of $\vec{F}_1$ and $\vec{F}_2$ (taken in same order). Also we observe that 12 i.e., resultant is perpendicular to $\vec{F}_1$. So, from the diagram, by using Pythagoras theorem, we get

$$F_2^2 = 144 + F_1^2$$

$$\Rightarrow \quad F_2^2 - F_1^2 = 144$$

$$\Rightarrow \quad (F_2 - F_1)(F_2 + F_1) = 144$$

$$\Rightarrow \quad (F_2 - F_1)(18) = 144$$

$$\Rightarrow \quad F_2 - F_1 = 8 \qquad \ldots(2)$$

So, from (1) and (2), we get

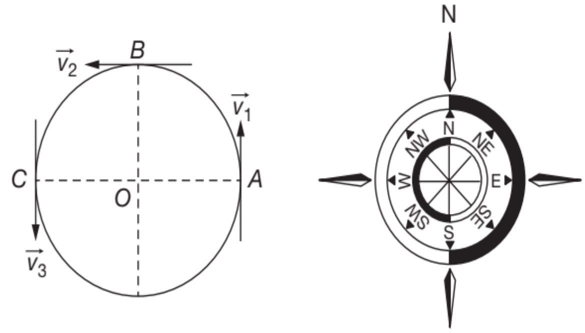
$$F_1 = 5 \text{ and } F_2 = 13$$

PROBLEM 2

A body is moving with uniform speed v in a horizontal circle in anticlockwise direction starting from A as shown in figure. Calculate the change in velocity in

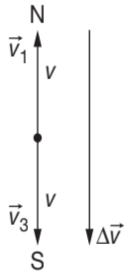
(a) half revolution

(b) first quarter revolution.



SOLUTION

Change in velocity $= \Delta\vec{v} = \vec{v}_f - \vec{v}_i$



(a) Now, for half revolution.

If $\vec{v}_3 = \vec{v}$ and $\vec{v}_1 = -\vec{v}$

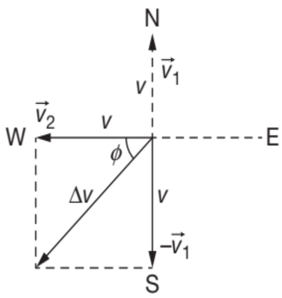
as their magnitudes are equal but directions opposite.

$$\Rightarrow \quad \Delta\vec{v} = \vec{v}_3 - \vec{v}_1 = \vec{v} - (-\vec{v}) = 2\vec{v}$$

$\Delta\vec{v} = 2v$ directed south.

(b) For, quarter revolution

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1 \text{ and } \theta = 90^\circ$$



$$\Delta\vec{v} = \sqrt{v^2 + v^2 + 2vv\cos 90^\circ} = \sqrt{2}v$$

$$\text{and } \phi = \tan^{-1}\left(\frac{v}{v}\right) = 45^\circ$$

So, $\Delta\vec{v} = (\sqrt{2})v$ south-west

PROBLEM 3

If vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitudes 8, 15 and 17 unit and $\vec{A} + \vec{B} = \vec{C}$, find the angle between $\vec{A}$ and $\vec{B}$.

SOLUTION

$$\vec{A} + \vec{B} = \vec{C}$$

Taking magnitude both sides

$$|\vec{A}+\vec{B}|=|\vec{C}|$$

{Magnitudes of equal vectors is also equal}

Squaring both sides, we get

$$|\vec{A}+\vec{B}|^2=|\vec{C}|^2$$

$$\Rightarrow (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})=\vec{C}\cdot\vec{C} \qquad \left\{\because \vec{A}\cdot\vec{A}=|\vec{A}|^2\right\}$$

$$\Rightarrow A^2+B^2+2AB\cos\theta=C^2$$

$$\Rightarrow \cos\theta=\frac{C^2-A^2-B^2}{2AB}$$

$$\Rightarrow \cos\theta=\frac{17^2-\left\{(15)^2+(8)^2\right\}}{2\times15\times8}$$

$$\Rightarrow \cos\theta=0$$

$$\Rightarrow \theta=90^\circ$$

PROBLEM 4

If the unit vectors $\hat{a}$ and $\hat{b}$ are inclined at angle θ prove that $|\vec{a}-\vec{b}|=2\sin\left(\frac{\theta}{2}\right)$.

SOLUTION

Since we know that,

$$|\hat{a}-\hat{b}|^2=|\hat{a}|^2+|\hat{b}|^2-2|\hat{a}||\hat{b}|\cos\theta$$

$$\Rightarrow |\hat{a}-\hat{b}|^2=1-\cos\theta-\cos\theta+1 \qquad \left\{\because |\hat{a}|=|\hat{b}|=1\right\}$$

$$\Rightarrow |\hat{a}-\hat{b}|^2=2-2\cos\theta$$

$$\Rightarrow |\hat{a}-\hat{b}|^2=2(1-\cos\theta)=2\times2\sin^2\left(\frac{\theta}{2}\right)=4\sin^2\left(\frac{\theta}{2}\right)$$

Taking square root both sides.

$$|\hat{a}-\hat{b}|=2\sin\left(\frac{\theta}{2}\right)$$

PROBLEM 5

(a) Find the area of the parallelogram determined by the vectors $\vec{a}=3\hat{i}+2\hat{j}$ and $\vec{b}=2\hat{j}-4\hat{k}$.

(b) Find the area of the triangle whose vertices are $(1,-1,-3)$, $(4,-3,1)$ and $(3,-1,2)$.

SOLUTION

(a) Vector area of parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b}=(3\hat{i}+2\hat{j})\times(2\hat{j}-4\hat{k})$$

$$\Rightarrow \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & 0\\ 0 & 2 & -4\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\vec{b}=\hat{i}[-8-0]-\hat{j}[-12-0]+\hat{k}[6-0]$$

$$\Rightarrow \vec{a}\times\vec{b}=-8\hat{i}+12\hat{j}+6\hat{k}$$

$\therefore$ Magnitude of the area of parallelogram

$$|\vec{a}\times\vec{b}|=|-8\hat{i}+12\hat{j}+6\hat{k}|$$

$$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{64+144+36}=2\sqrt{61}$$

(b) Position vectors of the vertices A, B and C of the triangle ABC are $\vec{a}=(\hat{i}-\hat{j}-3\hat{k})$,

$$\vec{b}=(4\hat{i}-3\hat{j}+\hat{k}) \text{ and } \vec{c}=(3\hat{i}-\hat{j}+2\hat{k})$$

$$\overrightarrow{BA}=\vec{a}-\vec{b}=-3\hat{i}+2\hat{j}-4\hat{k}$$

and $\overrightarrow{BC}=\vec{c}-\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$

Hence, vector area of triangle $=\frac{1}{2}(\overrightarrow{BA}\times\overrightarrow{BC})$

$$\Rightarrow \text{Area}=\frac{1}{2}(-3\hat{i}+2\hat{j}-4\hat{k})\times(-\hat{i}+2\hat{j}+\hat{k})$$

$$\Rightarrow \text{Area}=\frac{1}{2}(10\hat{i}+7\hat{j}-4\hat{k})$$

Magnitude of area of the triangle is

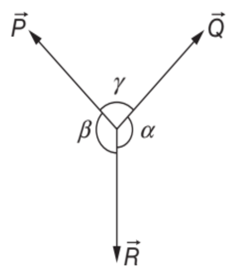
$$|\text{Area}|=\frac{1}{2}\sqrt{100+49+16}=\frac{1}{2}\sqrt{165} \text{ square units}$$

PROBLEM 6

A particle is in equilibrium under the action of three forces. Prove that each force bears a constant ratio with the sine of angle between the other two.

SOLUTION

Let $\vec{P}$, $\vec{Q}$ and $\vec{R}$ be the three forces acting at a point O. Since particle is in equilibrium.



$$\vec{P}+\vec{Q}+\vec{R}=0 \quad \ldots(1)$$

$$\Rightarrow \quad (\vec{P}+\vec{Q})=-\vec{R} \quad \ldots(2)$$

Taking cross product, with $\vec{P}$ on both sides of (2)

$$\vec{P}\times(\vec{P}+\vec{Q})=-\vec{P}\times\vec{R}$$

$$\Rightarrow \quad \vec{P}\times\vec{P}+\vec{P}\times\vec{Q}=\vec{R}\times\vec{P}$$

$$\Rightarrow \quad \vec{P}\times\vec{Q}=\vec{R}\times\vec{P} \quad \ldots(3)$$

Similarly taking cross product with $\vec{Q}$ on both sides of (3), we get

$$\vec{P}\times\vec{Q}=\vec{Q}\times\vec{R} \quad \ldots(4)$$

From (3) and (4)

$$\vec{P}\times\vec{Q}=\vec{Q}\times\vec{R}=\vec{R}\times\vec{P}$$

$$\Rightarrow \quad PQ\sin\gamma=QR\sin\alpha=RP\sin\beta$$

Dividing both sides by PQR, we get

$$\frac{PQ\sin\gamma}{PQR}=\frac{QR\sin\alpha}{PQR}=\frac{RP\sin\beta}{PQR}$$

$$\Rightarrow \quad \frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}$$

The Lami's Theorem is very useful in the chapters to come. So please understand the theorem carefully so that you can apply it at the place required.

PROBLEM 7

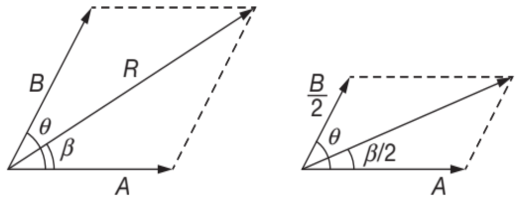
Two vectors of equal magnitude are inclined at an angle θ. When one of them is halved, the angle which the resultant makes with the other is also halved. Find θ.

SOLUTION

Let the two vectors be $\vec{A}$ and $\vec{B}$, inclined at an angle θ. Also we have been given that $|\vec{A}|=|\vec{B}|$. If β be the angle which the resultant (of $\vec{A}$ and $\vec{B}$) makes with $\vec{A}$, then

$$\tan\beta=\frac{B\sin\theta}{B+B\cos\theta}$$

$$\Rightarrow \quad \tan\beta=\frac{\sin\theta}{1+\cos\theta} \quad \ldots(1)$$



When B is halved, then still the angle between A and $\frac{B}{2}$ is θ. However, the new resultant is now inclined to A at an angle $\frac{\beta}{2}$. So, we get

$$\tan\left(\frac{\beta}{2}\right)=\frac{\frac{B}{2}\sin\theta}{A+\frac{B}{2}\cos\theta}$$

Since $|\vec{A}|=|\vec{B}|$, so

$$\tan\left(\frac{\beta}{2}\right)=\frac{\sin\theta}{2+\cos\theta} \quad \ldots(2)$$

Now, from concepts of trigonometric functions, we have

$$\tan\beta=\frac{2\tan\left(\frac{\beta}{2}\right)}{1-\tan^2\left(\frac{\beta}{2}\right)}$$

$$\Rightarrow \quad \frac{\sin\theta}{1+\cos\theta}=\frac{2\left(\frac{\sin\theta}{2+\cos\theta}\right)}{1-\frac{\sin^2\theta}{(2+\cos\theta)^2}}$$

$$\Rightarrow \quad \frac{\cancel{\sin\theta}}{1+\cos\theta}=\frac{2(\cancel{\sin\theta})(2+\cos\theta)}{4+\cos^2\theta+4\cos\theta-\sin^2\theta}$$

$$\Rightarrow \quad 4+\cos^2\theta+4\cos\theta-\sin^2\theta=(4+2\cos\theta)(1+\cos\theta)$$

$$\Rightarrow \quad 4+\cos^2\theta+4\cos\theta-\sin^2\theta=$$
$$4+2\cos^2\theta+6\cos\theta-\cos^2\theta-2\cos\theta-\sin^2\theta=0$$

$$\Rightarrow \quad -2\cos\theta-(\sin^2\theta+\cos^2\theta)=0$$

$$\Rightarrow \quad -2\cos\theta-1=0 \qquad \{\because \sin^2\theta+\cos^2\theta=1\}$$

$$\Rightarrow \quad \cos\theta=-\frac{1}{2}$$

$$\Rightarrow \quad \theta=120^\circ$$

PROBLEM 8

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

SOLUTION

Given that $|\vec{a}|=|\vec{b}|=|\vec{c}|$ and the three vectors are mutually perpendicular. So,

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$$

Let θ_1 be the angle between $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$. Then

$$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=|\vec{a}||\vec{a}+\vec{b}+\vec{c}|\cos\theta_1$$

$$\Rightarrow \quad \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=a|\vec{a}+\vec{b}+\vec{c}|\cos\theta_1$$

$$\Rightarrow \quad a^2=a|\vec{a}+\vec{b}+\vec{c}|\cos\theta_1$$

$$\Rightarrow \quad \cos\theta_1=\frac{a}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \ldots(1)$$

Similarly, if θ_2 be the angle between $\vec{b}$ and $\vec{a}+\vec{b}+\vec{c}$ and θ_3 is the angle between $\vec{c}$ and $\vec{a}+\vec{b}+\vec{c}$, then

$$\cos\theta_2=\frac{b}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \ldots(2)$$

$$\cos\theta_3=\frac{c}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \ldots(3)$$

Now, let us find the value of $|\vec{a}+\vec{b}+\vec{c}|$. Since we know that dot product of the vector with itself is equal to the square of its magnitude, so

$$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=|\vec{a}+\vec{b}+\vec{c}|^2$$

$$\Rightarrow \quad \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=|\vec{a}+\vec{b}+\vec{c}|^2$$

Since $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$, so we get

$$a^2+b^2+c^2=|\vec{a}+\vec{b}+\vec{c}|^2$$

Again $|\vec{a}|=|\vec{b}|=|\vec{c}|$, so $|\vec{a}+\vec{b}+\vec{c}|^2=3a^2$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}a \qquad \ldots(4)$$

From (1), (2), (3) and (4), we get

$$\cos\theta_1=\frac{a}{|\vec{a}+\vec{b}+\vec{c}|}=\frac{a}{\sqrt{3}a}=\frac{1}{\sqrt{3}}$$

$$\cos\theta_2=\frac{b}{|\vec{a}+\vec{b}+\vec{c}|}=\frac{b}{\sqrt{3}b}=\frac{1}{\sqrt{3}}$$

$$\cos\theta_3=\frac{c}{|\vec{a}+\vec{b}+\vec{c}|}=\frac{c}{\sqrt{3}c}=\frac{1}{\sqrt{3}}$$

Since $\cos\theta_1=\cos\theta_2=\cos\theta_3=\frac{1}{\sqrt{3}}$

$$\Rightarrow \quad \theta_1=\theta_2=\theta_3=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Hence the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

PROBLEM 9

For a vector $\vec{a}$ in 3-D space, find the value of the expression $|\hat{i}\times\vec{a}|^2+|\hat{j}\times\vec{a}|^2+|\hat{k}\times\vec{a}|^2$

SOLUTION

Let the vector $\vec{a}$ make angle α, β and γ with x, y and z axis respectively. Then

$$|\hat{i}\times\vec{a}|^2=|\hat{i}||\vec{a}|\sin^2\alpha$$

$$\Rightarrow \quad |\hat{i}\times\vec{a}|^2=a^2\sin^2\alpha$$

Similarly

$$|\hat{j}\times\vec{a}|^2=a^2\sin^2\beta \text{ and } |\hat{k}\times\vec{a}|^2=a^2\sin^2\gamma$$

Now $|\hat{i}\times\vec{a}|^2+|\hat{j}\times\vec{a}|^2+|\hat{k}\times\vec{a}|^2=$

$$a^2(\sin^2\alpha+\sin^2\beta+\sin^2\gamma) \qquad \ldots(1)$$

Now since

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

$$\Rightarrow \quad (1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma)=1$$

$$\Rightarrow \quad \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2 \qquad \ldots(2)$$

From (1) and (2) we get

$$|\hat{i}\times\vec{a}|^2+|\hat{j}\times\vec{a}|^2+|\hat{k}\times\vec{a}|^2=2a^2$$

PROBLEM 10

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. If $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=5$, then find the value of $(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$.

SOLUTION

Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \quad \vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow \quad \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}=-\vec{c}\cdot\vec{c}$

$\Rightarrow \quad \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}=-c^2 \quad \ldots(1)$

Similarly $\vec{a}+\vec{c}=-\vec{b}$

$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{b}=-b^2 \quad \ldots(2)$

Again $\vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow \quad \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}=-a^2 \quad \ldots(3)$

Adding (1), (2) and (3), we get

$$2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=-\left(a^2+b^2+c^2\right)$$

$$\Rightarrow \quad \left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=-\frac{1}{2}\left(3^2+4^2+5^2\right)=-25$$

$$\Rightarrow \quad \left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=-25$$