CHAPTER

4 Kinematics I

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Rest, Motion and Position
(b) Distance Displacement
(c) Average Speed and Average Velocity
(d) Instantaneous Speed and Instantaneous Velocity
(e) Average and Instantaneous Acceleration
(f) Uniformly Acceleration Motion
(g) Variable Accelerated Motion
(h) Graphical Interpretation and Graphs
(i) Vertical Motion Under Gravity
(j) Motion in a Plane
(k) Relative Motion in One Dimension
(l) Relative Motion in Two Dimensions
(m) Distance of Closest Approach between Moving Bodies
(n) River-Swimmer Problems
(o) Aeroplane-Wind Problems
(p) Rain-Man-Wind Problems

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

RECTILINEAR MOTION AND MOTION UNDER GRAVITY

INTRODUCTION TO CLASSICAL MECHANICS

The branch of Physics dealing with motion of particles or bodies in space and time is called **Mechanics.** As long as the velocity of the moving bodies is small in comparison to the velocity of light (c), the linear dimensions and the time intervals remain invariable in all Reference Frames (platform(s) from where motion is being observed), i.e., they do not depend on choice of reference frame. Mechanics dealing with such like motion (also called as Non-Relativistic motion) is called as **Classical Mechanics**. However, when the bodies move with speeds comparable to the speed of light (called as relativistic speeds), then the part of Physics dealing with such like motion(s) is called **Relativistic Mechanics**. An interesting fact about relativistic mechanics is that it is more general and reduces to classical mechanics for the case of small (non relativistic) velocities. In this chapter, we shall be describing and studying motion in terms of space and time while ignoring the causes that produce motion. This particular part of Classical Mechanics is called **Kinematics**. Furthermore, in this part of chapter, we shall be limiting ourselves to the motion in one dimension and two dimensions i.e., motion along a straight line, also called as **Rectilinear Motion** and **planar motion**. From our everyday experience, we observe that actually motion represents a continuous change in the position of an object. In Physics, we can divide motion into three categories

(a) Translational Motion (studying now).
EXAMPLE: A car moving down a highway.

(b) Rotational Motion (to study in Rotational Dynamics).
EXAMPLE: Spinning i.e., rotation of earth about its own axis.

(c) Vibrational Motion (to study in Simple Harmonic Motion).
EXAMPLE: Back and forth (or to and fro) motion of a pendulum (or a spring).

In the Chapters to came, we shall be discussing the branch of Classical Mechanics called **DYNAMICS**, where we shall be studying the motion along with the cause producing it.

CONCEPT OF POINT OBJECT (PARTICLE MODEL)

In our study of translational motion, we shall be using the concept of **Point Object** also called as the **Particle Model.** A body is considered as a point object depending upon the nature of the motion followed by the body. In general, an object is regarded a point-object when it travels large distances in comparison to its own size and dimensions.

Also, in planetary motion, the bodies under consideration can be regarded as point objects when distances of separation are very large.

EXAMPLE:
The planets revolving around the sun may be considered as point objects.

CONCEPT OF REFERENCE FRAME

A reference frame is a platform from where a physical phenomena, such as motion, is being observed. Reference frames are mainly of two types

(a) Inertial (Non-Accelerated frames)
(b) Non-Inertial (Accelerated frames)

The detailed discussion about both these frames follows in the next chapter. As of now, for this chapter, the reference frame is a frame (fixed or moving with constant velocity) i.e., non accelerating frame. The best convenient reference frame for this chapter would be the **Ground Frame**. Please note here, that I said "the best convenient" which has convenience, but not accuracy attached to it. For practical purposes, we regard earth as an inertial frame (though it is accelerating), hence I said convenient and not accurate.

State of Rest and Motion

A particle is said to be in the state of rest when it does not change its position w.r.t. surroundings with the passage of time.

A particle is said to be in the state of motion when it changes its position w.r.t. surroundings with the passage of time.

Rest and Motion, these two terms are not absolute i.e., complete in themselves. These terms are relative terms i.e., a body can be in the state of rest and motion simultaneously, depending upon the relative observer viewing motion.

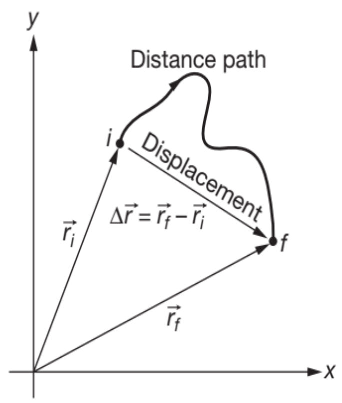
EXAMPLE:
Four persons sitting in the moving car are at rest w.r.t. an observer sitting in the car, whereas the same four persons are in motion w.r.t. a stationary observer viewing them from the ground.

Position of a Particle

The position of a particle is the location of the particle in space at a certain moment of time. In One-Dimensional Motion the position of the particle is denoted by x or y or z. However in 2-Dimensional and 3-Dimensional motion it is denoted by $\vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

DISTANCE AND DISPLACEMENT (RELATIVE POSITION VECTOR)

The length of the actual path followed between the initial and the final points in the motion is called **Distance**. Its SI unit is metre and cgs unit is cm.



The change in position vector of a particle going from initial position (say i) to final position (say f) is called **Displacement or Relative Position Vector of the particle**.

It is denoted by $\Delta\vec{r}$, such that

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

Conceptual Note(s)

In rectilinear motion of a particle, the displacement is denoted by $\Delta x = s = x_{\text{final}} - x_{\text{initial}}$

In 2-Dimensional or 3-Dimensional motion, the displacement is denoted by $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$

PROPERTIES OF DISPLACEMENT

(a) It is a vector quantity.

(b) It has units same as that of distance.

(c) It is independent of the choice of origin.

(d) It is unique (one and only one) for any kind of motion between two points.

(e) It is always concealing about the actual track followed by the particle's motion between any two points.

(f) It can be positive, negative and even be zero.

(g) The magnitude of the displacement is always less than or equal to the distance for particle's motion between two points i.e.,

$$0 \le |\text{Displacement}| \le \text{Distance}$$

(h) A body may have finite distance travelled for zero displacement.

Conceptual Note(s)

(a) For Rectilinear (straight line) motion, always make sure that the selected Cartesian Co-ordinate axis coincides with the line followed by the particle.

(b) In other words Displacement may also be defined as the vector drawn from the initial point to the final point in the motion of the particle.

(c) Please keep in mind that distance may or may not be equal to the magnitude of the displacement.

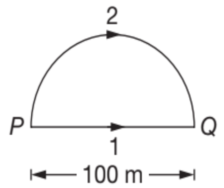
(d) In general, we have $0 \le \dfrac{|\text{Displacement}|}{\text{Distance}} \le 1$

ILLUSTRATION 1

Ram takes path 1 (straight line) to go from P to Q and Shyam takes path 2 (semicircle).

(a) Find the distance travelled by Ram and Shyam.

(b) Find the displacement of Ram and Shyam.



SOLUTION

(a) Distance travelled by Ram $= 100$ m

Distance travelled by Shyam $= \pi(50\text{ m}) = 50\pi$ m

(b) Displacement of Ram $= 100$ m

Displacement of Shyam $= 100$ m

ILLUSTRATION 2

The position x (in metre) of a particle varies with time t (in second) as $t = \sqrt{x} + 3$. Calculate the

(a) displacement of the particle from $t = 0$ to $t = 3$ s

(b) displacement of the particle from $t = 3$ to $t = 6$ s

(c) distance travelled and displacement from $t = 0$ to $t = 6$ s

SOLUTION

Since $t = \sqrt{x} + 3$

$$\Rightarrow \quad x = (t-3)^2 \qquad \ldots(1)$$

So, we get x as a function of t. Let us draw a table for x and t from $t = 0$ to $t = 6$. Then we get

t	0	1	2	3	4	5	6
x	9	4	1	0	1	4	9

This table gives us the answers to the three parts.

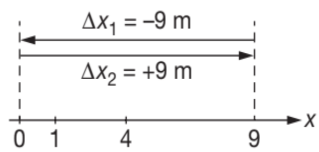
So, for (a), we have $\Delta x_1 = -9$ m

For (b), we have $\Delta x_2 = +9$ m

For (c), we have,

Distance $= |-9\text{ m}| + |9\text{ m}| = 18$ m and

Displacement = zero.



AVERAGE SPEED AND AVERAGE VELOCITY

Average speed of a particle, in a given interval of time is the ratio of the total distance travelled by the particle to the total time taken. It is a scalar quantity,

$$v_{av} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{s_1 + s_2 + s_3 + \ldots}{t_1 + t_2 + t_3 + \ldots}$$

Average velocity of the particle is the ratio of the displacement of the particle to the time taken. It is a vector quantity.

In one dimensional motion, we have

$$v_{av} = \frac{\text{Total Displacement}}{\text{Total Time}} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

In two or three dimensional motion, we have

$$\vec{v}_{av} = \frac{\text{Total Displacement}}{\text{Total Time}} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

Please note that, average speed may or may not be equal to the magnitude of average velocity. So,

Average Speed $\neq |\vec{v}_{av}|$ (sometimes may be equal)

CONCEPT OF AVERAGE SPEED

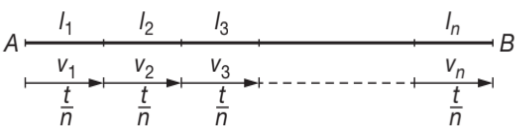
Average speed is simply defined as the total distance travelled by a body per unit total time taken to complete the motion.

$$\text{Average Speed} = \frac{\text{Total Distance Travelled}}{\text{Total Time Taken}}$$

For Interval Divided in Equal Time Parts

Let a particle go from A to B in a time t (say). Now if we say that for time $\frac{t}{n}$, the particle had a velocity v_1, for next $\frac{t}{n}$ it had a velocity v_2 and for the nth such equal time interval it had a velocity v_n (as shown). Let the corresponding distances travelled in each respective interval be l_1, l_2, l_3, …, l_n. Then

$$\text{Average Speed} = \frac{\text{Total Distance Travelled}}{\text{Total Time Taken}}$$



$$\Rightarrow \quad v_{av} = \frac{l_1 + l_2 + l_3 + \ldots + l_n}{\frac{t}{n} + \frac{t}{n} + \frac{t}{n} + \ldots + \frac{t}{n} (n \text{ times})}$$

$$\Rightarrow \quad v_{av} = \frac{v_1\left(\frac{t}{n}\right) + v_2\left(\frac{t}{n}\right) + v_3\left(\frac{t}{n}\right) + \ldots + v_n\left(\frac{t}{n}\right)}{t}$$

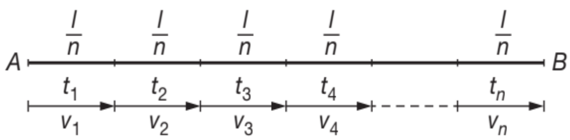
$$\Rightarrow \quad v_{av} = \frac{v_1 + v_2 + v_3 + \ldots + v_n}{n}$$

So, we observe that when an interval is divided into n equal time parts, then the "average speed is the simple average of the speeds in the respective intervals".

For Interval Divided in Equal Length Parts

Consider an interval of length l, divided equally in n equal length parts. Let a particle travel first length part with a velocity v_1, second with a velocity v_2 and so on the nth length part with a velocity v_n (as shown). Let t_1, t_2, …, t_n be the respective time taken by the particle to cover the respective interval. Then

$$\text{Average Speed} = \frac{\text{Total Distance Travelled}}{\text{Total Time Taken}}$$



$$\Rightarrow \quad v_{av} = \frac{\frac{l}{n} + \frac{l}{n} + \ldots n \text{ times}}{t_1 + t_2 + \ldots + t_n} = \frac{l}{t_1 + t_2 + \ldots + t_n}$$

$$\Rightarrow \quad v_{av} = \frac{l}{\frac{l}{nv_1} + \frac{l}{nv_2} + \ldots + \frac{l}{nv_n}}$$

$$\Rightarrow \quad \frac{1}{v_{av}} = \frac{\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n}}{n}$$

So, we observe that when an interval is divided into n equal length parts, then the **"reciprocal of the average speed is equal to the average of the reciprocals of the speeds in respective intervals".**

Problem Solving Technique(s)

So, we observe that

(a) when an interval is divided into n equal time parts, then the "average speed is the simple average of the speeds in the respective intervals".

(b) when an interval is divided into n equal length parts, then the "reciprocal of the average speed is equal to the average of the reciprocals of the speeds in respective intervals".

(c) Time Average Speed: When particle moves with different uniform speed v_1, v_2, v_3 ... etc. in different time intervals t_1, t_2, t_3, ... etc. respectively, its average speed over the total time of journey is given as

$$v_{av} = \frac{\text{Total distance covered}}{\text{Total time elapsed}}$$

$$v_{av} = \frac{d_1 + d_2 + d_3 + \ldots\ldots}{t_1 + t_2 + t_3 + \ldots\ldots} = \frac{v_1 t_1 + v_2 t_2 + v_3 t_3 + \ldots\ldots}{t_1 + t_2 + t_3 + \ldots\ldots}$$

(d) Distance Average Speed: When a particle describes different distances d_1, d_2, d_3, ... with different time intervals t_1, t_2, t_3, ... with speeds v_1, v_2, v_3, ... respectively, then the speed of particle averaged over the total distance can be given as

$$v_{av} = \frac{\text{Total distance covered}}{\text{Total time elapsed}} = \frac{d_1 + d_2 + d_3 + \ldots\ldots}{t_1 + t_2 + t_3 + \ldots\ldots}$$

$$v_{av} = \frac{d_1 + d_2 + d_3 + \ldots\ldots}{\frac{d_1}{v_1} + \frac{d_2}{v_2} + \frac{d_3}{v_3} + \ldots\ldots}$$

(e) If speed is continuously changing with time then

$$v_{av} = \frac{\int v\,dt}{\int dt}$$

(f) When a particle moves with a speed v_1 for half the time and with a speed v_2 for the remaining half of the time, then

$$v_{av} = \frac{v_1 + v_2}{2}.$$

(g) When a particle moves the first half of a distance with a speed v_1 and the second half of the distance with a speed v_2, then

$$v_{av} = \frac{2v_1 v_2}{v_1 + v_2}$$

(h) Similarly, when a particle covers one-third distance at speed v_1, next one third with a speed v_2 and the last one third at speed v_3, then

$$v_{av} = \frac{3v_1 v_2 v_3}{v_1 v_2 + v_2 v_3 + v_3 v_1}$$

ILLUSTRATION 3

Calculate the average speed and the average velocity in the following cases mentioned.

CASE-1: For a train that travels from one station to another at a uniform speed of 40 kmh^{-1} and returns to first station at a speed of 60 kmh^{-1}.

CASE-2: For a man who walks at a speed of 1 ms^{-1} for the first one minute and then runs at a speed of 3 ms^{-1} for the next one minute along a straight track.

CASE-3: For a man who walks 720 m at a uniform speed of 2 ms^{-1}, then runs at a uniform speed of 4 ms^{-1} for 5 minute and then again walks at a speed of 1 ms^{-1} for 3 minutes.

(Please consider all uniform speeds to be the average speeds in respective intervals).

SOLUTION

$$\text{Average Speed} = \frac{\text{Total Distance Travelled}}{\text{Total Time Taken}}$$

CASE-1: $v_{av} = \frac{2v_1 v_2}{v_1 + v_2} = \frac{(2)(40)(60)}{40 + 60} = 48\text{ kmh}^{-1}$

Average Velocity $= \vec{v}_{av} = \vec{0}$

{$\because$ train returns to its station}

CASE-2: $v_{av} = \frac{v_1 + v_2}{2} = \frac{1 + 3}{2} = 2\text{ ms}^{-1}$

CASE-3: $v_{av} = \frac{s_1 + s_2 + s_3}{t_1 + t_2 + t_3}$

where $s_1 = 720\text{ m}$ and

$$t_1 = \frac{s_1}{v_1} = 360\text{ s} = 6\text{ minute}$$

$$s_2 = (4)(5)(60) = 1200\text{ m},\ t_2 = 300\text{ s}$$

$$s_3 = (1)(3)(60) = 180 \text{ m},\ t_3 = 180 \text{ s}$$

$$\Rightarrow \quad v_{av} = \frac{720+1200+180}{360+300+180} = \frac{2100}{840}$$

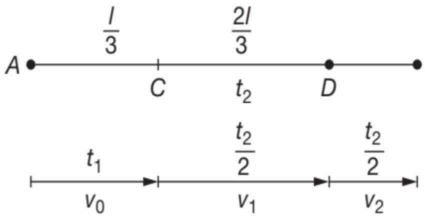
$$\Rightarrow \quad v_{av} = \frac{210}{84} = \frac{10}{4} = 2.5 \text{ ms}^{-1}$$

ILLUSTRATION 4

A particle traversed one third the distance with a velocity v_0. The remaining part of the distance was covered with velocity v_1 for half the time and with a velocity v_2 for the remaining half of time. Assuming motion to be rectilinear, find the mean velocity of the particle averaged over the whole time of motion.

SOLUTION

Let us first draw a pictorial representation to the problem.



For *AC*

$$\frac{l}{3} = v_0 t_1$$

$$\Rightarrow \quad t_1 = \frac{l}{3v_0} \qquad \ldots(1)$$

For *CB*

$$\frac{2l}{3} = CD + DB$$

$$\Rightarrow \quad \frac{2l}{3} = v_1\left(\frac{t_2}{2}\right) + v_2\left(\frac{t_2}{2}\right)$$

$$\Rightarrow \quad t_2 = \frac{4l}{3(v_1+v_2)} \qquad \ldots(2)$$

Since, average velocity is defined as

$$v_{av} = \frac{\text{Total Displacement}}{\text{Total Time}} = \frac{\frac{l}{3}+\frac{2l}{3}}{t_1+t_2}$$

$$\Rightarrow \quad v_{av} = \frac{3v_0(v_1+v_2)}{4v_0+v_1+v_2}$$

INSTANTANEOUS SPEED AND INSTANTANEOUS VELOCITY

The speed of a particle at a particular instant of time is called the **instantaneous speed**. It is a scalar quantity.

$$v_{\text{instantaneous}} = v_{\text{ins}} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x}$$

The velocity of a particle at a particular instant of time is called the **instantaneous velocity**, denoted by $\vec{v}$. So,

$$\vec{v}_{\text{instantaneous}} = \vec{v}_{\text{ins}} = \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t} = \frac{d\vec{x}}{dt} = \dot{\vec{x}}$$

(a) Please note that, in Physics, dot on a physical quantity indicates its derivative w.r.t. time e.g.

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}},\ \vec{v} = \frac{d\vec{x}}{dt} = \dot{\vec{x}},\ \vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}}$$

Also, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} = \ddot{\vec{x}}$

So, $\vec{a} = \dot{\vec{v}} = \ddot{\vec{x}}$

(b) The magnitude of the instantaneous velocity is always equal to the instantaneous speed i.e.,

$v = |\vec{v}| = \left|\frac{d\vec{x}}{dt}\right| \neq \frac{dx}{dt}$, because in general $d\vec{x} \neq dx$.

Acceleration ($\vec{a}$)

The rate of change of velocity with time is called acceleration.

(a) Average Acceleration $= \vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$

(b) Instantaneous Acceleration is

$$\vec{a}_{\text{ins}} = \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \dot{\vec{v}}$$

So, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}$

$$\Rightarrow \quad \vec{a} = \dot{\vec{v}} = \ddot{\vec{x}}$$

Conceptual Note(s)

(a) In 3-Dimensional space

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\left(\frac{dx}{dt}\right) + \hat{j}\left(\frac{dy}{dt}\right) + \hat{k}\left(\frac{dz}{dt}\right)$

$\Rightarrow \quad \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

and $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$

$\Rightarrow \quad \vec{a} = \hat{i}\left(\frac{d^2x}{dt^2}\right) + \hat{j}\left(\frac{d^2y}{dt^2}\right) + \hat{k}\left(\frac{d^2z}{dt^2}\right)$

$\Rightarrow \quad \vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

(b) $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of total acceleration. While $\frac{d|\vec{v}|}{dt}$ represents the time rate of change of speed (called the tangential acceleration, a component of total acceleration) as $|\vec{v}| = v$.

(c) These two are equal in case of one dimensional motion (without change in direction).

(d) In case of uniform circular motion speed remains constant while velocity changes.

Hence, $\frac{d|\vec{v}|}{dt} = 0$ while $\left|\frac{d\vec{v}}{dt}\right| \neq 0$

(e) $\frac{d|\vec{v}|}{dt} \neq 0$ implies that speed of particle is not constant. Velocity cannot remain constant if speed is changing. Hence, $\left|\frac{d\vec{v}}{dt}\right|$ cannot be zero in this case. So, it is not possible to have $\left|\frac{d\vec{v}}{dt}\right| = 0$ while $\frac{d|\vec{v}|}{dt} \neq 0$.

FACTORS AFFECTING ACCELERATION OF A BODY

The acceleration changes, when the velocity of the body changes. The change in velocity may be due to any of factors listed below.

(a) If the magnitude of velocity changes but direction remains the same.

(b) If the direction of the velocity changes but magnitude remains the same.

(c) If both magnitude and direction of the velocity change.

Conceptual Note(s)

(a) A particle moving with uniform velocity has zero acceleration i.e. it neither changes in magnitude nor its direction.

(b) A particle moves with uniform acceleration if rate of change of velocity is constant.

(c) A body is subjected to Retardation or Deceleration when acceleration acts opposite to velocity.

(d) In case of a body subjected to retardation, we take

(i) *a* as negative, if velocity is taken as positive.

(ii) *a* as positive, if velocity is taken as negative.

(e) In case of a body subjected to acceleration, we take

(i) *a* as positive, if velocity is taken as positive.

(ii) *a* as negative, if velocity is taken as negative.

(f) However, a body at rest will be in accelerated motion irrespective of the sign of acceleration.

(g) For a body moving in straight line with uniform acceleration the average acceleration and instantaneous value of acceleration have same value.

(h) If a particle has an acceleration a_1 for a time t_1 and an acceleration a_2 for a time t_2, then average acceleration is $a_{av} = \frac{a_1t_1 + a_2t_2}{t_1 + t_2}$

ILLUSTRATION 5

A body moves along a straight line. Its distance x from a point on its path at a time t after passing that point, is given by $x_t = 8t^2 - 3t^3$ where x_t is in meter and t in second. Find

(a) the instantaneous velocity at $t = 1$ s

(b) instant and position at which the body is at rest

(c) the acceleration at $t = 4$ s

(d) the average velocity and average speed during the interval $t = 0$ s to $t = 4$ s

SOLUTION

$$x = 8t^2 - 3t^3 \quad \ldots(1)$$

$$\Rightarrow \quad v = \frac{dx}{dt} = 16t - 9t^2 \quad \ldots(2)$$

$\Rightarrow \quad a = 16 - 18t$ ...(3)

(a) $v|_{t=1\text{ s}} = 16 - 9 = 7 \text{ ms}^{-1}$

(b) The body is at rest, when $v = 0$

$\Rightarrow \quad 16t - 9t^2 = 0$

$\Rightarrow \quad t = 0,\ t = \dfrac{16}{9} \text{ s}$

So, the particle is initially at rest. However at $t = 0$, $a = -18 \text{ ms}^{-2}$ which accelerates the particle to move.

$$x = 0 \text{ and } x = 8\left(\frac{16}{9}\right)^2 - 3\left(\frac{16}{9}\right)^3 = 8.43 \text{ m}$$

(c) $a|_{t=4\text{ s}} = 16 - 18(4) = -56 \text{ ms}^{-2}$

(d) Average Velocity $= \dfrac{x|_{t=4} - x|_{t=0}}{4 - 0}$

$\Rightarrow$ Average Velocity $= \dfrac{8(4)^2 - 3(4)^3 - 0}{4}$

$\Rightarrow$ Average Velocity $= -16 \text{ ms}^{-1}$

For calculating the average speed we must first calculate the zeros of velocity i.e., the times at which the particle is at rest (or momentarily at rest i.e., at the point of reversal of motion).
In this problem, we observe that

$v = 0$ at $t = 0$ (initially) and

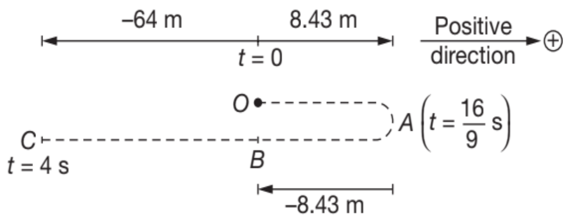
$v = 0$ at $t = \dfrac{16}{9}$ s {point of reversal of motion}

	Initial	Point of Zero Velocity	Final	Afterwards
t	0	$\frac{16}{9}$ s	4 s	$t > 4$
$x = 8t^2 - 3t^3$	0	8.43 m	–64 m	–
$v = 16t - 9t^2$	0	0	-80 ms^{-1}	–
$a = 16 - 18t$	16 ms^{-2}	-16 ms^{-2}	-56 ms^{-2}	–
Nature of Motion	Accelerating in Positive Direction	Accelerating in Negative Direction	Accelerating in Negative Direction	Accelerating

So, from the table, we observe that the particle reverses its direction of motion at $t = \dfrac{16}{9}$ s as shown. Hence average speed is

$$v_{av} = \frac{\text{Total Distance Covered}}{\text{Total Time Taken}}$$

$$\Rightarrow \quad v_{av} = \frac{8.43 + |-8.43| + |-64|}{4}$$



$\Rightarrow \quad v_{av} = 20.21 \text{ ms}^{-1}$

Alternatively,

$$\text{Average Speed} = \frac{\left|\int_0^{16/9} v\,dt\right| + \left|\int_{16/9}^{4} v\,dt\right|}{4} = 20.21 \text{ ms}^{-1}$$

CHECK YOURSELF!

ILLUSTRATION 6

A particle travels along a straight line with a velocity $v = (12 - 3t^2) \text{ ms}^{-1}$, where t is in seconds. When $t = 1$ s, the particle is located 10 m to the left of the origin. Calculate the

(a) acceleration when $t = 4$ s
(b) displacement from $t = 0$ to $t = 10$ s and
(c) distance the particle travels from $t = 0$ to $t = 10$ s.

SOLUTION

(a) $v = 12 - 3t^2$ ...(1)

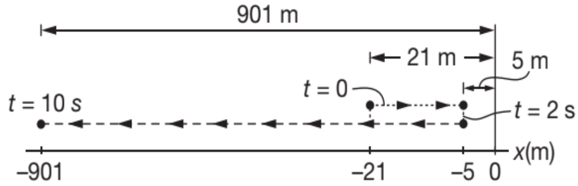
$$\Rightarrow \quad a = \frac{dv}{dt} = -6t\Big|_{t=4} = -24 \text{ ms}^{-2}$$

(b) Further $v = \dfrac{dx}{dt}$

$$\Rightarrow \quad \int_{-10}^{x} dx = \int_1^t v\,dt = \int_1^t (12 - 3t^2)\,dt$$

$\Rightarrow \quad x + 10 = 12t - t^3 - 11$

$\Rightarrow \quad x = 12t - t^3 - 21$



So, $x\Big|_{t=0} = -21\text{ m}$ and $x\Big|_{t=10} = -901\text{ m}$

$\Rightarrow \quad \Delta x = -901 - (-21) = -880\text{ m}$

(c) From equation (1), we have

$v = 0$ when $12 - 3t^2 = 0$

$\Rightarrow \quad t = 2\text{ s}$

So, $x\Big|_{t=2} = 12(2) - (2)^3 - 21 = -5\text{ m}$

$\Rightarrow$ Total Distance

$(x_{\text{total}}) = (21-5) + (901-5) = 912\text{ m}$

Test Your Concepts-I

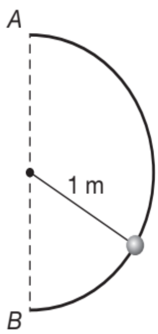
Based on Displacement, Velocity, Acceleration, Average Speed and Velocity

(Solutions on page H.65)

1. The acceleration of a particle as it moves along a straight line is given by $a = (2t - 1)\text{ ms}^{-2}$, where t is in seconds. If $x = 1\text{ m}$ and $v = 2\text{ ms}^{-1}$ when $t = 0$, determine the particle's velocity and position when $t = 6\text{ s}$. Also, determine the total distance the particle travels during this time period.

2. A particle moves in a straight line with a uniform acceleration a. Initial velocity of the particle is zero. Find the average velocity of the particle in first s metre.

3. In one second a particle goes from point A to point B moving in a semicircle (see figure). Find the magnitude of average velocity.



4. A particle is moving along a straight line such that its position from a fixed point is $x = (12 - 15t^2 + 5t^3)\text{ m}$, where t is in seconds. Determine the total distance travelled by the particle from $t = 1\text{ s}$ to $t = 3\text{ s}$. Also, find the average speed of the particle during this time interval.

5. The position (x) of a particle, in metre, moving along the x-axis depends on the time t, in seconds as $x = ct^2 - bt^3$, where $c = 3$ units and $b = 2$ units. Calculate the

(a) units of c and b.

(b) time taken by the particle to reach its maximum positive x value.

(c) the distance travelled and the displacement of the particle from $t = 0$ to $t = 4\text{ s}$.

(d) the velocity and acceleration at $t = 0$, 1, 2, 3 and 4 second.

6. The position of a particle along a straight line is given by $x = (t^3 - 9t^2 + 15t)\text{ m}$, here t is in second. Determine its maximum acceleration and maximum velocity during the time interval $0 \le t \le 10\text{ s}$.

7. At time $t = 0$, the position vector of a particle moving in the x-y plane is $5\hat{i}\text{ m}$. At time $t = 0.02\text{ s}$, its position vector has become $5.1\hat{i} + 0.4\hat{j}\text{ m}$. Determine the magnitude of the average velocity (v_{av}) during this interval and the angle θ made by the average velocity with the positive x-axis.

8. A particle travels along a straight line path such that in 4 s it moves from an initial position $x_A = -8\text{ m}$ to a position $x_B = +3\text{ m}$. Then in another 5 s it moves from x_B to $x_C = -6\text{ m}$. Determine the particle's average velocity and average speed during the 9 s time interval.

9. A particle moves along a horizontal path such that its velocity is given by $v = (3t^2 - 6t)$ ms^{-1}, where t is the time in seconds. If it is initially located at the origin O, determine the

(a) distance travelled by the particle during the time interval $t = 0$ to $t = 3.5$ s

(b) particle's average velocity and average speed during this time interval.

10. The position of a particle along a straight line is given by $x = (1.5t^3 - 13.5t^2 + 22.5t)$ m, where t is in seconds. Determine the position of the particle when $t = 6$ s and the total distance it travels during the 6 s time interval.

11. The velocity of a particle moving in a straight line decreases at the rate of 3 ms^{-1} per meter of displacement at an instant when the velocity is 10 ms^{-1}. Calculate the acceleration of the particle at this instant.

UNIFORMLY ACCELERATED MOTION SYSTEMS

Consider a particle moving in a straight line with constant acceleration a. If u is the initial velocity, v is the velocity at time t, s is the displacement $(= \Delta x)$ in time t and s_n^{th} is the displacement in the n^{th} second of motion, then equations governing the motion of such a particle are

$$v = u + at$$

$$s = \Delta x = ut + \frac{1}{2}at^2$$

$$v^2 - u^2 = 2as$$

$$s = \frac{u+v}{2}t$$

$$s_n^{\text{th}} = u + \frac{1}{2}a(2n-1)$$

For constant acceleration, $\vec{a}$, the equations of motion are written as

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$v^2 - u^2 = 2\vec{a}\cdot\vec{s} \quad \text{OR} \quad \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{u} = 2\vec{a}\cdot\vec{s}$$

$$\vec{s} = \left(\frac{\vec{u}+\vec{v}}{2}\right)t$$

where $\vec{u}$ = initial velocity of the particle

$\vec{v}$ = final velocity of the particle at time t

$\vec{s} = \Delta\vec{r}$ is the displacement of the particle

Conceptual Note(s)

(a) All the above equations of motion are to be applied only when the motion is uniformly accelerated.

(b) For applying the above equations greater care has to be taken about the direction of the vector quantities involved.

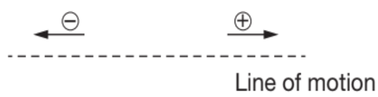
(c) Please note that this nth second has a duration of 1 second. So, s_{nth} is the distance travelled (or the displacement) in 1 sec, hence

$$[s_{nth}] = LT^{-1}$$

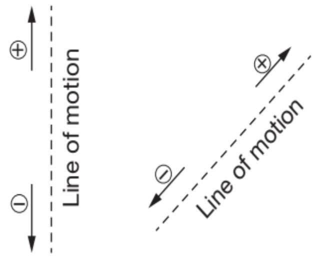
DIRECTIONS OF VECTORS IN STRAIGHT LINE MOTION

In straight line motion, all the vectors (position, displacement, velocity and acceleration) will have only one component (along the line of motion) and there will be only two possible directions for each vector. For example

(a) If a particle is moving in a horizontal line along x-axis, then, the two directions are right and left. Any vector directed towards right can be represented by a positive number and towards left can be represented by a negative number.



(b) For vertical or inclined motion, upward direction can be taken as positive and downward as negative.



For objects moving vertically near the surface of the earth, the only force acting on the particle is its weight (mg) i.e., the gravitational pull of the earth. Hence acceleration for this type of motion will always be $a=-g$ i.e., $a=-9.8 \text{ ms}^{-2}$ (negative sign, because the force and acceleration are directed downwards. If we select upward direction as positive).

Conceptual Note(s)

(a) If acceleration is in same direction as velocity, then speed of the particle increases.

(b) If acceleration is in opposite direction to the velocity then speed decreases i.e., the particle slows down. This situation is known as retardation.

ILLUSTRATION 7

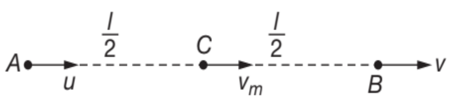
A point moving with constant acceleration from A to B in a straight line AB has velocities u and v at A and B respectively.

(a) Find its velocity at C, the midpoint of A and B.

(b) Find the ratio $\frac{v}{u}$, if time taken from A to C is twice the time to go from C to B.

SOLUTION

(a) Let v_m be the velocity at C, the midpoint of A and B.



For AC

$$v_m^2 - u^2 = 2a\left(\frac{l}{2}\right)$$

$$\Rightarrow \quad v_m^2 - u^2 = al \qquad \ldots(1)$$

For CB

$$v^2 - v_m^2 = 2a\left(\frac{l}{2}\right)$$

$$\Rightarrow \quad v^2 - v_m^2 = al \qquad \ldots(2)$$

Equating (1) and (2), we get

$$v_m = \sqrt{\frac{u^2+v^2}{2}} \qquad \ldots(3)$$

(b) Since $t_{A\to C} = 2t_{C\to B}$

Let $t_{C\to B} = t$, then $t_{A\to C} = 2t$

$$\Rightarrow \quad \frac{l}{2} = \left(\frac{u+v_m}{2}\right)2t \qquad \{\text{for } AC\} \qquad \ldots(4)$$

$$\Rightarrow \quad \frac{l}{2} = \left(\frac{v_m+v}{2}\right)t \qquad \{\text{for } CB\} \qquad \ldots(5)$$

$$\Rightarrow \quad u+v_m = \frac{v_m+v}{2}$$

$$\Rightarrow \quad 2u+2v_m = v_m+v$$

$$\Rightarrow \quad v_m = v-2u$$

$$\Rightarrow \quad v_m^2 = v^2+4u^2-4uv$$

Since, $v_m = \sqrt{\frac{u^2+v^2}{2}}$

$$\Rightarrow \quad \frac{u^2+v^2}{2} = v^2+4u^2-4uv$$

$$\Rightarrow \quad u^2+v^2 = 2v^2+8u^2-8uv$$

$$\Rightarrow \quad v^2-8uv+7u^2 = 0$$

$$\Rightarrow \quad v^2-7uv-uv+7u^2 = 0$$

$$\Rightarrow \quad v(v-7u)-u(v-7u) = 0$$

$$\Rightarrow \quad (v-u)(v-7u) = 0$$

As motion is an accelerated motion

So, $v-u \neq 0$

$$\Rightarrow \quad v = 7u$$

$$\Rightarrow \quad \frac{v}{u} = 7$$

ILLUSTRATION 8

A body starts with an initial velocity of $10\ \text{ms}^{-1}$ and moves along a straight line path with constant acceleration. When the velocity of the body is $50\ \text{ms}^{-1}$ the acceleration is reversed in direction. Find the velocity of the particle as it reaches the starting point.

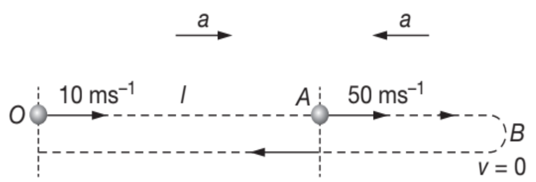
SOLUTION

For $OA(=l)$

$$(50)^2-(10)^2=2al$$

$$\Rightarrow\quad al=1200 \qquad \ldots(1)$$

Now, when the particle just attains the velocity of $50\ \text{ms}^{-1}$ at A, acceleration is reversed in direction i.e., it starts acting as retardation for $50\ \text{ms}^{-1}$ which will first be reduced to zero after travelling a distance AB and then the particle will again be accelerated from B to A with the same acceleration. So, velocity of the particle again at A will be $50\ \text{ms}^{-1}$, but in opposite direction.



From *A* to *O*

$$v^2-(50)^2=2al$$

$$\Rightarrow\quad v^2-2500=2(1200)$$

$$\Rightarrow\quad v^2=4900$$

$$\Rightarrow\quad v=70\ \text{ms}^{-1} \qquad \{\text{in the opposite direction}\}$$

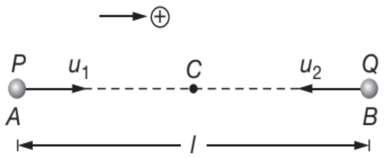
So, $v=70\ \text{ms}^{-1}$ from A to O

ILLUSTRATION 9

Two particles P and Q move in a straight line AB towards each other. P starts from A with a velocity u_1 and an acceleration a_1. Q starts from B with velocity u_2 and an acceleration a_2. They pass from each other at midpoint of AB and arrive at other ends of AB with equal velocities. Prove that $(u_1+u_2)(a_1-a_2)=8(a_1u_2-a_2u_1)$.

SOLUTION

Let the particle P reach C in time t, then the particle Q must reach C also in time t.



For *P*

$$\frac{l}{2}=u_1t+\frac{1}{2}a_1t^2 \qquad \ldots(1)$$

For *Q*

$$-\frac{l}{2}=-u_2t+\frac{1}{2}(-a_2)t^2$$

$$\Rightarrow\quad \frac{l}{2}=u_2t+\frac{1}{2}a_2t^2 \qquad \ldots(2)$$

From (1) and (2), we get (by subtracting)

$$(u_2-u_1)t+\frac{1}{2}(a_2-a_1)t^2=0$$

$$\Rightarrow\quad t=2\left(\frac{u_2-u_1}{a_1-a_2}\right) \qquad \ldots(3)$$

Now, substitute this value of t from (3) in any of equations (1) or (2), we get

$$\frac{l}{2}=u_1\left[2\left(\frac{u_2-u_1}{a_1-a_2}\right)\right]+\frac{1}{2}a_1\left[2\left(\frac{u_2-u_1}{a_1-a_2}\right)\right]^2$$

$$\Rightarrow\quad \frac{l}{2}=2\left(\frac{u_2-u_1}{a_1-a_2}\right)\left[u_1+a_1\left(\frac{u_2-u_1}{a_1-a_2}\right)\right]$$

$$\Rightarrow\quad l=\frac{4(u_2-u_1)}{(a_1-a_2)^2}\left(\cancel{u_1a_1}-u_1a_2+a_1u_2-\cancel{a_1u_1}\right)$$

$$\Rightarrow\quad l=\frac{4(u_2-u_1)}{(a_1-a_2)^2}(a_1u_2-a_2u_1) \qquad \ldots(4)$$

Again, after reading the question carefully, we observe that P reaches B and Q reaches A with equal velocities, say v.

Then, for P, we have

$$v^2-u_1^2=2a_1l \qquad \ldots(5)$$

For Q, we have

$$v^2-u_2^2=2a_2l \qquad \ldots(6)$$

$$\Rightarrow\quad u_2^2-u_1^2=2(a_1-a_2)l$$

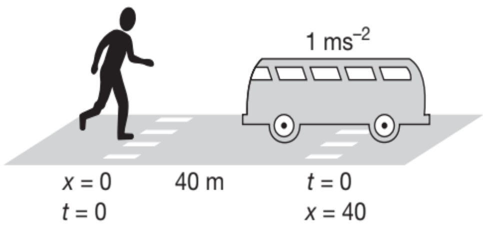
Substituting value of l, from (4), we get

$$u_2^2 - u_1^2 = 2(a_1 - a_2)\frac{4(u_2 - u_1)}{(a_1 - a_2)^2}(a_1u_2 - a_2u_1)$$

$$\Rightarrow \quad (u_1 + u_2)(a_1 - a_2) = 8(a_1u_2 - a_2u_1)$$

ILLUSTRATION 10

A man is standing 40 m behind the bus. Bus starts with $1\ ms^{-2}$ constant acceleration and also at the same instant the man starts moving with constant speed $9\ ms^{-1}$. Find the time taken by man to catch the bus.



SOLUTION

Let the man catches the bus at time t.

For bus

$$x = x_0 + ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad x = 40 + 0(t) + \frac{1}{2}(1)t^2$$

$$\Rightarrow \quad x = 40 + \frac{t^2}{2} \quad \ldots(1)$$

For man

$$x = 9t \quad \ldots(2)$$

From (1) and (2), we get

$$40 + \frac{t^2}{2} = 9t$$

$$\Rightarrow \quad t^2 - 18t + 80 = 0$$

$$\Rightarrow \quad t^2 - 10t - 8t + 80 = 0$$

$$\Rightarrow \quad (t-8)(t-10) = 0$$

$$\Rightarrow \quad t = 8\text{ s} \quad \text{OR} \quad t = 10\text{ s}$$

ILLUSTRATION 11

A car starts moving rectilinearly, first with an acceleration of $5\ ms^{-2}$ (initial velocity zero), then moves uniformly and finally decelerating at the same rate till it stops. The total time of journey is 25 s. The average speed during the interval is $72\ kmh^{-1}$. Calculate the time for which the car was in uniform motion.

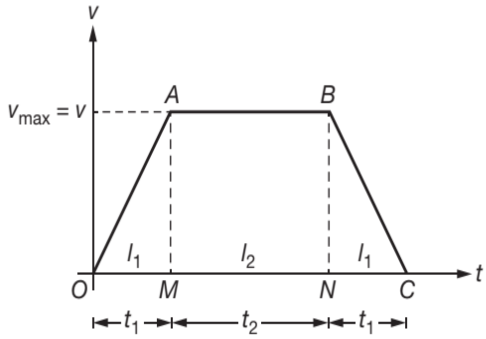
SOLUTION

Since the car accelerates from zero to v and decelerates from v to zero at the same rate of $5\ ms^{-2}$, so it must travel equal distances during the accelerated and the decelerated interval in equal times.

If t be the total time of journey, then

$$t = 25 = t_1 + t_2 + t_1$$

$$\Rightarrow \quad 25 = 2t_1 + t_2 \quad \ldots(1)$$



Furthermore, the average speed

$$v_{av} = 72\ kmh^{-1} = 20\ ms^{-1}$$

$$\Rightarrow \quad v_{av} = \frac{\text{Total Distance Travelled}}{\text{Total Time Taken}} = \frac{l_1 + l_2 + l_1}{25}$$

$$\Rightarrow \quad 2l_1 + l_2 = (20)(25) = 500$$

$$\Rightarrow \quad 2l_1 + l_2 = 500 \quad \ldots(2)$$

where $l_1 = \frac{5}{2}t_1^2 \quad \ldots(3)$

Also, for OA, we have

$$v = 5t_1 \quad \ldots(4)$$

For AB, we have

$$l_2 = vt_2 = 5t_1t_2 \quad \ldots(5)$$

$$\Rightarrow \quad 2\left(\frac{5}{2}t_1^2\right) + 5t_1t_2 = 500 \quad \text{\{substituting in (2)\}}$$

$$\Rightarrow \quad 5t_1^2 + 5t_1(25 - 2t_1) = 500$$

$$\Rightarrow \quad t_1^2 + t_1(25 - 2t_1) = 100$$

$$\Rightarrow \quad t_1^2 - 2t_1^2 + 25t_1 = 100$$

$\Rightarrow \quad t_1^2 - 25t_1 + 100 = 0$

$\Rightarrow \quad t_1^2 - 20t_1 - 5t_1 + 100 = 0$

$\Rightarrow \quad t_1(t_1 - 20) - 5(t_1 - 20) = 0$

$\Rightarrow \quad (t_1 - 5)(t_1 - 20) = 0$

$\Rightarrow \quad t_1 = 5 \text{ s} \quad \text{OR} \quad t_1 = 20 \text{ s}$

But if $t_1 = 20$ s, then total time becomes greater than 25 s which is impossible. So,

$t_1 = 5$ s

$\Rightarrow \quad t_2 = 15$ s

Hence the car was in uniform motion for 15 s.

ILLUSTRATION 12

A police inspector in a car is chasing a pickpocket an a straight road. The car is going at its maximum speed v (assumed uniform). The pickpocket rides on the motorcycle of a waiting friend when the car is at a distance d away and the motorcycle starts with a constant acceleration a. Show that the pick pocket will be caught if $v \geq \sqrt{2ad}$.

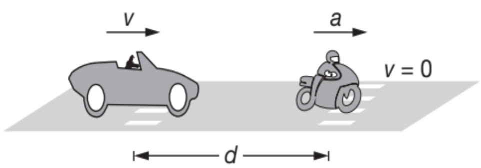
SOLUTION

Suppose the pickpocket is caught at a time t after motorcycle starts. The distance travelled by the motorcycle during this interval is

$$s = \frac{1}{2}at^2 \qquad \ldots(1)$$

During this interval the car travels a distance

$$s + d = vt \qquad \ldots(2)$$



From (1) and (2), we get

$$\frac{1}{2}at^2 + d = vt$$

$\Rightarrow \quad at^2 - 2vt + 2d = 0$

$\Rightarrow \quad t = \dfrac{v \pm \sqrt{v^2 - 2ad}}{a}$

The pickpocket will be caught if t is real and positive. This will be possible if

$v^2 \geq 2ad$

$\Rightarrow \quad v \geq \sqrt{2ad}$

REACTION TIME

When a situation demands our immediate action, it takes some time before we really respond. Reaction time is the time a person takes to observe, think and act.

ILLUSTRATION 13

A driver takes 0.20 s to apply the brakes after he sees a need for it. This is called the reaction time of the driver. If he is driving car at a speed of 5.4 kmh^{-1} and the brakes cause a deceleration of 6 ms^{-2}, find the distance travelled by the car after he sees the need to put the brakes.

SOLUTION

During the reaction time of 0.20 s, the car continues to move with a speed of 54 kmh^{-1} i.e. 15 ms^{-1}.
So, distance travelled by car during this time is

$s_1 = (15)(0.2) = 3$ m

Now, when brakes are applied, the distance s_2 travelled by the car is

$0^2 - (15)^2 = 2(-6)s_2$

$\Rightarrow \quad s_2 = \dfrac{225}{12} \text{ m} = 18.75 \text{ m}$

So, total distance covered by the car is

$s = s_1 + s_2 = 3 + 18.75$

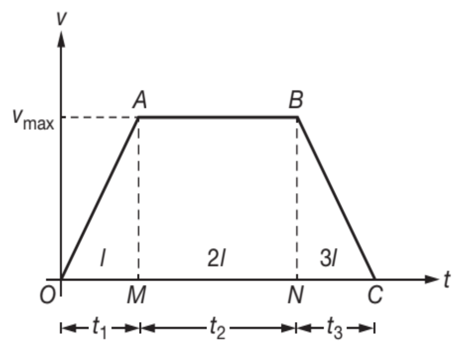
$\Rightarrow \quad s = 21.75$ m

Test Your Concepts-II

Based on Constant Acceleration

(Solutions on page H.67)

1. A particle moving in a straight line with constant acceleration travels a distance x, y and z during the pth, qth and rth second respectively. Prove that
$$(q-r)x+(r-p)y+(p-q)z=0$$
2. A car accelerates from rest at a constant rate α for some time, after which it decelerates at a constant rate β, to come to rest. If the total time elapsed is t seconds, evaluate
   (a) the maximum velocity reached and
   (b) the total distance travelled.
3. A particle starts moving from the position of rest under a constant acceleration. If it travels a distance x in t sec, what distance will it travel in next t sec?
4. In a car race, car A takes time t less than car B and passes the finishing point with a velocity v more than the velocity with which car B passes the point. Assuming that the cars start from rest and travel with constant accelerations a_1 and a_2, show that $v=t\sqrt{a_1a_2}$.
5. Two cars start off to race with velocities v_1 and v_2 and travel in a straight line with uniform accelerations a_1 and a_2. If the race ends in a dead heat, prove that the length of the course is
$$\frac{2(v_1-v_2)(v_1a_2-v_2a_1)}{(a_1-a_2)^2}$$
6. A car moving with constant acceleration covered the distance between two points 60 m apart in 6 s. Its speed as it passes the second point was 15 ms^{-1}.
   (a) What was the speed at the first point?
   (b) What was the acceleration?
   (c) At what prior distance from the first was the car at rest?
7. A train stopping at two stations 4 km apart takes 4 minute on the journey from one station to the other. Assuming that it first accelerates with a uniform acceleration x and then that of uniform retardation y, prove that $\frac{1}{x}+\frac{1}{y}=2$.
8. To stop a car, first you require a certain reaction time to begin braking and then the car slows under the constant braking deceleration. Suppose that the total distance moved by your car during these two phases is 56.7 m when its initial speed is 80.5 kmh^{-1} and 24.4 m when its initial speed is 48.3 kmh^{-1}. What is
   (a) your reaction time and
   (b) the magnitude of the deceleration?
9. A particle, starting from rest moves in a straight line with constant acceleration. After time t_0 the acceleration changes its direction without any change in its magnitude. Determine the time t from the beginning of motion in which the particle returns to the initial position.
10. The velocity v of a particle moving in a straight line varies with its displacement x as $v=(\sqrt{4+4x})$ ms^{-1}. Displacement of particle at time $t=0$ is $x=0$. Find displacement of particle at time $t=2$ s.
11. A particle starts from rest and traverses a distance l with uniform acceleration, then moves uniformly over a further distance $2l$ and finally it comes to rest after moving a further distance $3l$ under uniform retardation. Assuming the entire path to be a straight line find the ratio of the average speed over the journey to the maximum speed on its way.



12. An α particle travels along the inside of straight hollow tube, 2 m long, of a particle accelerator. Under uniform acceleration, how long is the particle in the tube if it enters at a speed of 1000 ms^{-1} and leaves at 9000 ms^{-1}. What is its acceleration during this interval?

13. A sports car passing a police check-point at 60 kmh^{-1} immediately started slowing down uniformly until its speed was 40 kmh^{-1}. It continued to move at this speed until it was passed by a police car 1 km from the check-point. This police car had started from rest at the check-point at the same instant as the sports car had passed the check-point. The police car had moved with constant acceleration until it had passed the sports car. Assuming that the time taken by the sports car in slowing down from 60 kmh^{-1} to 40 kmh^{-1} was equal to the time that it travelled at constant speed before passed by the police car, find
(a) the time taken by the police car to reach the sports car,
(b) the speed of the police car at the instant when it passed the sports car,
(c) the time measured from the check-point when the speeds of the two cars were equal.

14. Two railway stations A and B are 50 km apart and are served by electric trains which can decelerate at 5 kmh^{-1} per second, and accelerate 3 kmh^{-1} per second. The maximum speed is 90 kmh^{-1}. There are twelve intermediate stations all more than a km apart. Find the least time which can be taken to made the journey from A to B
(a) by a fast non-stop train and
(b) by a slow train which stops $\frac{1}{2}$ minute at every station.

15. A truck starts from rest with an acceleration of 1.5 ms^{-2} while a car 150 m behind starts from rest with an acceleration of 2 ms^{-2}. How long will it take before both the truck and car side by side, and how much distance is travelled by each?

16. Two cars travelling towards each other on a straight road at velocity 10 ms^{-1} and 12 ms^{-1} respectively. When they are 150 m apart, both drivers apply their brakes and each car decelerates at 2 ms^{-2} until it stops. How far apart will they be when they have both come to a stop?

17. Two motor cars start from A simultaneously and reach B after 2 hours. The first car travelled half the distance at a speed of 30 kmh^{-1} and the other half at a speed of 60 kmh^{-1}. The second car covered the entire distance with a constant acceleration. At what instant of time, the speeds of both the vehicles is the same? Will one of them overtake the other enroute?

18. A point travelling along a straight line traversed one third the distance with a velocity v_0. The remaining part of the distance was covered with velocity v_1 for half the time and with velocity v_2 for the other half of the time. Find the mean velocity of the point averaged over the whole time of motion.

19. A man runs at a speed of 4 ms^{-1} to overtake a standing bus. When he is 6 m behind the door $(\text{at } t=0)$, the bus moves forward and continues with a constant acceleration of 1.2 ms^{-2}. How long does it take for the man to go to the door? If he was initially 10 m behind the door, can he catch the bus?

20. Two cars A and B start off to a race on a straight path with initial velocities 8 ms^{-1} and 5 ms^{-1} respectively. Car A moves with uniform acceleration of 1 ms^{-2} and B moves with uniform acceleration of 1.1 ms^{-2}. If both the cars reach the winning post together, find the length of the track. Also find which of the two cars was ahead 10 s before the finish.

21. A car starts from rest and moves with a constant acceleration of 1.5 ms^{-2} until it achieves a velocity of 25 ms^{-1}. It then travels with constant velocity for 60 seconds. Determine the average speed and the total distance travelled.

22. A car is to be hoisted by elevator to the fourth floor of a parking garage, which is 14 m above the ground. If the elevator can accelerate at 0.2 ms^{-2}, decelerate at 0.1 ms^{-2} and reach maximum speed of 2.5 ms^{-1}, determine the shortest time to make the lift, starting from rest and ending at rest.

23. The driver of a car wishes to pass a truck that is travelling at a constant speed of 20 ms^{-1}. Initially, the car is also travelling at 20 ms^{-1}. Initially, the vehicles are separated by 25 m, and the car pulls back into the truck's lane after it is 25 m ahead of the truck. The car is 5 m long, and the truck is 20 m long. The car's acceleration is a constant 0.6 ms^{-2}.
(a) How much time is required for the car to pass the truck?
(b) What distance does the car travel during this time?
(c) What is the final speed of the car?

24. A train of length $l = 348$ m starts moving rectilinearly with constant acceleration $a = 3$ cms^{-2}. After 30 second a ball is dropped by the driver (event 1) and after 60 second another ball is dropped by the guard (event 2). How and at what constant velocity v should a driver drive his car parallel to the train so that he observes both the events to occur at the same point. Neglect the length of the car.

25. A particle moves rectilinearly with constant acceleration. The displacement, measured from a convenient fixed position, is 2 m at time $t = 0$ and is zero when $t = 10$ s. If the particle reverses its direction of motion at $t = 6$ s, determine the acceleration a and the velocity v when $t = 10$ s.

EQUATIONS OF MOTION FOR VARIABLE ACCELERATION

CASE-1: When acceleration a of the particle is a function of time

Since acceleration of a particle is a function of time, i.e., $a = f(t)$

$$\Rightarrow \quad \frac{dv}{dt} = f(t) \qquad \{\text{by definition}\}$$

$$\Rightarrow \quad dv = f(t)dt$$

Integrating both sides within suitable limits, we have

$$\int_u^v dv = \int_0^t f(t)dt$$

$$\Rightarrow \quad v = u + \int_0^t f(t)dt$$

CASE-2: When acceleration a of the particle is a function of distance

Since acceleration of a particle is a function of distance, i.e., $a = f(x)$

$$\Rightarrow \quad \frac{dv}{dt} = f(x) \qquad \{\text{by definition}\}$$

Multiply and divide the L.H.S. by dx and rearrange, we get

$$\frac{dv}{dx} \cdot \frac{dx}{dt} = f(x)$$

Integrating both sides within suitable limits, we have

$$\int_u^v v dv = \int_{x_0}^x f(x)dx$$

$$\Rightarrow \quad \frac{v^2}{2} - \frac{u^2}{2} = \int_{x_0}^x f(x)dx$$

$$\Rightarrow \quad v^2 = u^2 + 2\int_{x_0}^x f(x)dx$$

CASE-3: When acceleration a of the particle is a function of velocity

Since the particle is a function of velocity, i.e., $a = f(v)$

$$\Rightarrow \quad \frac{dv}{dt} = f(v) \qquad \{\text{by definition}\}$$

$$\Rightarrow \quad dt = \frac{dv}{f(v)}$$

Integrating both sides within suitable limits, we have

$$\int_0^t dt = \int_u^v \frac{dv}{f(v)}$$

$$\Rightarrow \quad t = \int_u^v \frac{dv}{f(v)}$$

In this case we shall get v as a function of time i.e., $v(t)$

Otherwise

$$v\frac{dv}{dx} = f(v)$$

Rearranging and integrating both sides within suitable limits, we have

$$\int_{x_0}^x dx = \int_u^v \frac{vdv}{f(v)}$$

$$\Rightarrow \quad x - x_0 = \int_u^v \frac{vdv}{f(v)}$$

$$\Rightarrow \quad x = x_0 + \int_u^v \frac{vdv}{f(v)}$$

In this case we shall get x as a function of velocity i.e., $x(v)$

ILLUSTRATION 14

An object starts from rest at $t = 0$ and accelerates at a rate given by $a = 6t$. What is

(a) its velocity and
(b) its displacement at any time t?

SOLUTION

As acceleration as a function of time is

$$a = 6t$$

$$\Rightarrow \quad \frac{dv}{dt} = 6t$$

$$\Rightarrow \quad dv = 6tdt$$

$$\Rightarrow \quad \int_0^v dv = 6\int_0^t t\,dt$$

$$\Rightarrow \quad v = 6\frac{t^2}{2} = 3t^2$$

$$\Rightarrow \quad v = 3t^2$$

Now $v = \dfrac{dx}{dt}$

$$\Rightarrow \quad \frac{dx}{dt} = 3t^2$$

$$\Rightarrow \quad dx = 3t^2 dt$$

$$\Rightarrow \quad \int_{x_1}^{x_2} dx = 3\int_0^t t\,dt$$

$$\Rightarrow \quad x_2 - x_1 = \Delta x = \left.\frac{3t^3}{3}\right|_0^t$$

$$\Rightarrow \quad x_2 - x_1 = \Delta x = t^3$$

$$\Rightarrow \quad \Delta x = t^3$$

ILLUSTRATION 15

For a particle moving along x-axis, velocity is given as a function of time as $v = 2t^2 + \sin t$. At $t = 0$, particle is at origin. Find the position and acceleration as a function of time. Also calculate initial speed and acceleration of the particle.

SOLUTION

$$v = 2t^2 + \sin t$$

$$\Rightarrow \quad \frac{dx}{dt} = 2t^2 + \sin t$$

$$\Rightarrow \quad \int_0^x dx = \int_0^t (2t^2 + \sin t)dt$$

$$\Rightarrow \quad x = \frac{2}{3}t^3 - \cos t + 1$$

Since $v = 2t^2 + \sin t$

$$\Rightarrow \quad a = \frac{dv}{dt} = 4t + \cos t$$

So, initial speed is $v|_{t=0} = u = 2(0)^2 + \sin(0) = 0$ and Initial Acceleration is $a|_{t=0} = 4(0) + \cos(0) = 1\ \text{ms}^{-2}$

ILLUSTRATION 16

A particle is subjected to an acceleration $a = \alpha t + \beta t^2$, where α and β are positive constants. The position and velocity of the particle at $t = 0$ are x_0 and v_0 respectively. Find the expression for the velocity (v) and position (x) of the particle at time t.

SOLUTION

Since $a = \alpha t + \beta t^2$ ...(1)

$$\Rightarrow \quad \frac{dv}{dt} = \alpha t + \beta t^2$$

$$\Rightarrow \quad dv = \alpha t dt + \beta t^2 dt$$

Integrating, we get

$$\Rightarrow \quad \int_{v_0}^v dv = \alpha\int_0^t tdt + \beta\int_0^t t^2 dt$$

$$\Rightarrow \quad v - v_0 = \alpha\frac{t^2}{2} + \beta\frac{t^3}{3}$$

$$\Rightarrow \quad v = v_0 + \frac{\alpha t^2}{2} + \frac{\beta t^3}{3} \qquad \text{...(2)}$$

Since $v = \frac{dx}{dt}$

$$\Rightarrow \frac{dx}{dt} = v_0 + \frac{\alpha t^2}{2} + \frac{\beta t^3}{3}$$

$$\Rightarrow dx = v_0 dt + \frac{\alpha t^2}{2} dt + \frac{\beta t^3}{3} dt$$

Integrating, we get

$$\Rightarrow \int_{x_0}^{x} dx = v_0 \int_0^t dt + \frac{\alpha}{2}\int_0^t t^2 dt + \frac{\beta}{3}\int_0^t t^3 dt$$

$$\Rightarrow x - x_0 = v_0 t + \frac{\alpha t^3}{6} + \frac{\beta t^4}{12}$$

$$\Rightarrow x = x_0 + v_0 t + \frac{\alpha t^3}{6} + \frac{\beta t^4}{12} \quad \ldots(3)$$

So, equations (2) and (3) give us the required expression for v and x.

ILLUSTRATION 17

For a particle moving along x-axis, acceleration is given as $a = x$. Find the position as a function of time if at $t = 0$, $x = 1$ m and $v = 1$ ms^{-1}

SOLUTION

$$a = x$$

Since, $a = \frac{dv}{dt} = v\frac{dv}{dx}$

$$\Rightarrow v\frac{dv}{dx} = x$$

$$\Rightarrow \int_0^v v\,dv = \int_0^x x\,dx$$

$$\Rightarrow \frac{v^2}{2} = \frac{x^2}{2}$$

$$\Rightarrow v = x$$

$$\Rightarrow \frac{dx}{dt} = x$$

$$\Rightarrow \int_1^x \frac{dx}{x} = \int_0^t dt$$

$$\Rightarrow \log_e x\big|_1^x = t$$

$$\Rightarrow \log_e x = t$$

$$\Rightarrow x = e^t$$

ILLUSTRATION 18

For a particle moving along x-axis, acceleration is given as $a = 2v^2$. If the speed of the particle is v_0 at $x = 0$, find speed as a function of x.

SOLUTION

$$a = 2v^2$$

$$\Rightarrow \frac{dv}{dt} = 2v^2$$

$$\Rightarrow \frac{dv}{dx} \times \frac{dx}{dt} = 2v^2$$

$$\Rightarrow v\frac{dv}{dx} = 2v^2$$

$$\Rightarrow \frac{dv}{dx} = 2v$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{v} = \int_0^x 2dx$$

$$\log_e v\big|_{v_0}^{v} = (2x)\big|_0^x$$

$$\Rightarrow \log_e\left(\frac{v}{v_0}\right) = 2x$$

$$\Rightarrow v = v_0 e^{2x} \qquad \{\because \text{ If } \log_e x = \alpha \Rightarrow x = e^{\alpha}\}$$

ILLUSTRATION 19

For a particle moving along x-axis, acceleration is given as $a = v$. Find the position as a function of time if at $t = 0$, $x = 0$ and $v = 1$ ms^{-1}.

SOLUTION

$$a = v$$

$$\Rightarrow \frac{dv}{dt} = v$$

$$\Rightarrow \int_1^v \frac{dv}{v} = \int_0^t dt$$

Since, $\int \frac{dv}{v} = \log_e v$

$$\Rightarrow \log_e v\big|_1^v = t\big|_0^t$$

$$\Rightarrow \log_e\left(\frac{v}{1}\right) = t$$

$$\Rightarrow v = e^t$$

$$\Rightarrow \quad \frac{dx}{dt} = e^t$$

$$\Rightarrow \quad dx = e^t dt$$

$$\Rightarrow \quad \int_0^x dx = \int_0^t e^t dt$$

$$\Rightarrow \quad x = e^t - e^0$$

$$\Rightarrow \quad x = e^t - 1$$

ILLUSTRATION 20

The motion of a body is given by the equation $\frac{dv(t)}{dt} = 6 - 3v(t)$, when $v(t)$ is the speed in ms^{-1} and t in seconds. If the body was at rest at $t = 0$; Then test the correctness of the following result

(a) the terminal speed is 2 ms^{-1}

(b) the magnitudes of initial acceleration is 6 ms^{-2}

(c) the speed varies with time as $v(t) = 2(1 - e^{-3t}) \text{ ms}^{-1}$

(d) the speed is 1 ms^{-1} when the acceleration is half the initial value

SOLUTION

(a) The terminal speed is the speed at which acceleration is zero i.e.,

$$\frac{dv(t)}{dt} = 0$$

$$\Rightarrow \quad v = 2 \text{ ms}^{-1}$$

(b) Since initially body is at rest hence

$$v(t) = v(0) = 0$$

$$\Rightarrow \quad \left.\frac{dv}{dt}\right|_{t=0} = 6 \text{ ms}^{-2}$$

(c) $\frac{dv}{6 - 3v} = dt$

Integrating both sides

$$\int_0^v \frac{dv}{6 - 3v} = \int_0^t dt$$

$$\Rightarrow \quad -\frac{1}{3}\log_e |6 - 3v| \Big|_0^v = t \Big|_0^t$$

$$\Rightarrow \quad \log_e\left(\frac{6 - 3v}{6}\right) = -3t$$

$$\Rightarrow \quad 6 - 3v = 6e^{-3t}$$

$$\Rightarrow \quad v = 2(1 - e^{-3t}) \text{ ms}^{-1}$$

(d) For acceleration to be half the initial value i.e.,

$$a = \frac{6}{2} = 3 \text{ ms}^{-2}$$

$$\Rightarrow \quad 3 = 6 - 3v(t)$$

$$\Rightarrow \quad v(t) = 1 \text{ ms}^{-1}$$

ILLUSTRATION 21

The velocity of a particle moving in the positive direction of x-axis varies as $v = \alpha\sqrt{x}$, where α is a positive constant. Assuming that at moment $t = 0$, the particle was located at the point $x = 0$. Find

(a) the time dependence of the velocity and the acceleration of the particle.

(b) the mean velocity of the particle averaged over the time that the particle takes to cover first s meters of the path.

SOLUTION

METHOD I

(a) $v = \alpha\sqrt{x}$

Since $a = \frac{dv}{dt} = \frac{d}{dt}(\alpha\sqrt{x})$

$$\Rightarrow \quad a = \frac{\alpha}{2\sqrt{x}}\frac{dx}{dt} = \frac{\alpha}{2\sqrt{x}}v \qquad \left\{\because \frac{dx}{dt} = v = \alpha\sqrt{x}\right\}$$

$$\Rightarrow \quad a = \left(\frac{\alpha}{2\sqrt{x}}\right)(\alpha\sqrt{x}) = \frac{\alpha^2}{2} = \text{constant}$$

Further, we have

$$a = \frac{dv}{dt}$$

$$\Rightarrow \quad a = \frac{dv}{dt} = \frac{\alpha^2}{2}$$

$$\Rightarrow \quad dv = \frac{\alpha^2}{2}dt$$

$$\Rightarrow \quad \int_0^v dv = \frac{\alpha^2}{2}\int_0^t dt$$

$$\Rightarrow \quad v = \left(\frac{\alpha^2}{2}\right)t$$

(b) Since $\langle v \rangle = \dfrac{\int_0^t v\,dt}{\int_0^t dt} = \dfrac{\text{Total Distance Travelled}}{\text{Total Time Taken}} = \dfrac{s}{t}$

To cover a distance s in time t with constant acceleration $a = \dfrac{\alpha^2}{2}$, we have

$$s = \frac{1}{2}\left(\frac{\alpha^2}{2}\right)t^2$$

$$\Rightarrow \quad t = \frac{2\sqrt{s}}{\alpha}$$

$$\Rightarrow \quad \langle v \rangle = v_{av} = \frac{s}{\left(\dfrac{2\sqrt{s}}{\alpha}\right)} = \frac{\alpha\sqrt{s}}{2}$$

METHOD II

Since $v = \alpha\sqrt{x}$, so we get

$$v^2 = \alpha^2 x$$

Comparing this equation with the equation of motion i.e.,

$$v^2 = u^2 + 2as$$

we get

$$u = 0 \text{ and } a = \frac{\alpha^2}{2}$$

Hence the motion is uniformly accelerated with zero initial velocity $(u = 0)$ and constant acceleration $\left(a = \dfrac{\alpha^2}{2}\right)$.

(a) Since $v = at$

$$\Rightarrow \quad v = \frac{\alpha^2 t}{2}$$

$$\Rightarrow \quad a = \frac{\alpha^2}{2} = \text{constant}$$

(b) $s = \dfrac{1}{2}at^2 = \dfrac{1}{2}\dfrac{\alpha^2}{2}t^2$

$$\Rightarrow \quad t = \frac{2\sqrt{s}}{\alpha} = \text{time taken to cover first } s \text{ metre}$$

$$\Rightarrow \quad v_{av} = \langle v \rangle = \frac{s}{t} = \frac{s}{\dfrac{2\sqrt{s}}{\alpha}}$$

$$\Rightarrow \quad v_{av} = \frac{\alpha\sqrt{s}}{2}$$

ILLUSTRATION 22

A particle having a velocity $v = v_0$ at $t = 0$ is decelerated at the rate $|a| = \alpha\sqrt{v}$, where α is a positive constant. After what time and distance will the particle will be at rest?

SOLUTION

$$v = v_0 \text{ at } t = 0 \text{ and } a = -\alpha\sqrt{v}$$

Time taken by particle to come to rest:

$$a = \frac{dv}{dt} = -\alpha\sqrt{v}$$

$$\Rightarrow \quad \int_{v_0}^{0} -\frac{dv}{\alpha\sqrt{v}} = \int_0^t dt$$

$$\Rightarrow \quad -\frac{1}{\alpha}\left|\frac{v^{-1/2+1}}{-1/2+1}\right|^0 = |t|_0^t$$

$$\Rightarrow \quad t = -\frac{1}{\alpha}\left(\frac{v^{1/2}}{1/2}\right)\Bigg|_{v_0}^{0}$$

$$\Rightarrow \quad t = -\frac{1}{\alpha}\left|0 - \frac{\sqrt{v_0}}{1/2}\right| = \frac{2}{\alpha}\sqrt{v_0}$$

Distance travelled before coming to rest:

$$a = v\frac{dv}{ds} = -\alpha\sqrt{v}$$

$$\Rightarrow \quad -\frac{vdv}{\alpha\sqrt{v}} = ds$$

$$\Rightarrow \quad -\int_{v_0}^{0} \frac{1}{\alpha}\sqrt{v}dv = \int_S^0 ds$$

$$\Rightarrow \quad -\frac{1}{\alpha}\left(\frac{v^{3/2}}{3/2}\right)\Bigg|_{v_0}^{0} = |s|_s^0$$

$$\Rightarrow \quad -\frac{1}{\alpha}\left[0-\frac{2}{3}|v_0|^{3/2}\right]=0-s$$

$$\Rightarrow \quad s=\frac{2}{3\alpha}v_0^{3/2}$$

ILLUSTRATION 23

The velocity of a particle travelling along a straight line is $v=v_0-kx$, where k is constant. If $x=0$ when $t=0$, determine the position and acceleration of the particle as a function of time.

SOLUTION

Since $v=\dfrac{dx}{dt}$

$$\Rightarrow \quad \int_0^t dt=\int_0^x \frac{dx}{v_0-kx}$$

$$\Rightarrow \quad t\Big|_0^t=-\frac{1}{k}\ln(v_0-kx)\Big|_0^x$$

$$\Rightarrow \quad t=\frac{1}{k}\ln\left(\frac{v_0}{v_0-kx}\right)$$

$$\Rightarrow \quad e^{kt}=\frac{v_0}{v_0-kx}$$

$$\Rightarrow \quad x=\frac{v_0}{k}\left(1-e^{-kt}\right)$$

Now, $v=\dfrac{dx}{dt}=\dfrac{d}{dt}\left[\dfrac{v_0}{k}\left(1-e^{-kt}\right)\right]$

$$\Rightarrow \quad v=v_0e^{-kt}$$

Since, we know that

$$a=\frac{dv}{dt}=\frac{d}{dt}\left(v_0e^{-kt}\right)$$

$$\Rightarrow \quad a=-kv_0e^{-kt}$$

Conceptual Note(s)

(a) For a particle having zero initial velocity if $v\propto t$, $s\propto t^2$ and $v^2\propto s$ then acceleration of particle must be constant i.e., particle is moving rectilinearly with uniform acceleration.

(b) For a particle having zero initial velocity if $s\propto t^{\alpha}$, where $\alpha>2$, then particle's acceleration increases with time.

(c) For a particle having zero initial velocity if $s\propto t^{\alpha}$, where $\alpha<0$, then particle's acceleration decreases with time.

Problem Solving Technique(s)

When acceleration of particle is not constant, we go for basic equations of velocity and acceleration, i.e.,

(a) $\vec{v}=\dfrac{d\vec{x}}{dt}$ or sometimes $\vec{v}=\dfrac{d\vec{r}}{dt}$

(b) $\vec{a}=\dfrac{d\vec{v}}{dt}$

(c) $d\vec{x}=d\vec{r}=\vec{v}dt$

(d) $d\vec{v}=\vec{a}dt$

For one dimensional motion, above relations can be written as under.

(a) $v=\dfrac{dx}{dt}$

(b) $a=\dfrac{dv}{dt}=v\dfrac{dv}{dx}$

(c) $dx=vdt$

(d) $dv=adt$ or $vdv=adx$

Test Your Concepts-III

Based on Variable Acceleration

(Solutions on page H.73)

1. A particle starts from rest and travels along a straight line with an acceleration $a=\left(30-\frac{v}{5}\right)$ ms^{-2} where v is in ms^{-1}. Determine the time when the velocity of the particle is $v=30$ ms^{-1}.
2. The acceleration (a) of a particle moving in a straight line varies with its displacement (s) as $a=2s$. The velocity of the particle is zero at zero displacement. Find the corresponding velocity-displacement equation.
3. The velocity of a particle travelling along a straight line is $v=v_0-kx$, where k is constant. Initially, it is observed that the particle is at the origin of the coordinate system, then determine the position and acceleration of the particle as a function of time.
4. The retardation of a particle moving in a straight line is proportional to its displacement (proportionality constant being 4). Find the total distance covered by the particle till it comes to rest. Given that velocity of particle is v_0 at zero displacement.
5. The acceleration (a) of a particle travelling along a straight line varies with distance x as $a=(8-2x)$ ms^{-2}, where x is in meters. Assuming that the particle starts from rest from the origin of the coordinate system, calculate the velocity of the particle at $x=2$ m and the position of the particle when the velocity is maximum. Also calculate the maximum velocity.
6. The acceleration of a particle travelling along a straight line is $a=\frac{k}{v}$, where k is a constant. If $x=0$, $v=v_0$ initially, determine the velocity of the particle as a function of time t.
7. The motion of a body falling from rest in a resisting medium is described by the equation given as $\frac{dv}{dt}=A-Bv$; where A and B are constants. Find the initial acceleration and the velocity at which the acceleration is zero. Find the velocity at any instant of time.
8. A particle is moving along a straight line such that its speed is defined as $v=\left(-4x^2\right)$ ms^{-1}, where x is in meters. If $x=2$ m when $t=0$, determine the velocity and acceleration as functions of time.
9. The acceleration of a particle is given by $a(t)=(3-2t)$.
   (a) Find the initial speed v_0 such that the particle will have the same x-coordinate at $t=5$ s as it had at $t=0$.
   (b) What will be the velocity at $t=5$ s?
10. The acceleration of a particle travelling along a straight line is $a=\left(5e^t\right)$ ms^{-2}, where t is in seconds. If $v=0$, $x=0$ when $t=0$, determine the velocity and displacement of the particle as a function of t.
11. A particle moves along a straight line with a velocity $v=(200x)$ mms^{-1}, where x is in millimeters. Determine the acceleration of the particle at $x=2000$ mm. How long does the particle take to reach this position if $x=500$ mm when $t=0$?
12. A particle has acceleration varying with time t as $\vec{a}=\left(2t\hat{i}+3t^2\hat{j}\right)$ ms^{-2}. If the particle is initially at rest, find the velocity of the particle at time $t=2$ s.
13. When a particle is projected vertically upwards with an initial velocity of v_0, it experiences an acceleration $a=-\left(g+kv^2\right)$, where g is the acceleration due to gravity, k is a constant and v is the velocity of the particle. Determine the maximum height reached by the particle.
14. If the velocity v of a particle moving along a straight line decreases linearly with its displacement from 20 ms^{-1} to a value approaching zero at $x=30$ m, determine the acceleration of the particle when $x=15$ m and show that the particle never reaches the 30 m displacement.
15. A particle is moving along a straight line with the acceleration $a=\left(12t-3\sqrt{t}\right)$ ms^{-2}, where t is in seconds. Determine the velocity and the position of the particle as a function of time. Assume that, initially the particle starts from rest from $x_0=15$ m.

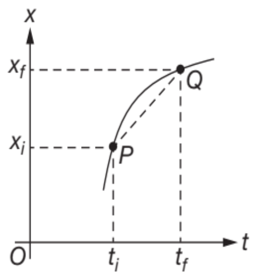
GRAPHICAL INTERPRETATION OF SOME QUANTITIES

Average Velocity

If a particle passes a point $P(x_i)$ at time $t = t_i$ and reaches $Q(x_f)$ at a later time instant $t = t_f$, its average velocity in the interval PQ is

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = (\text{Slope of chord } PQ)$$

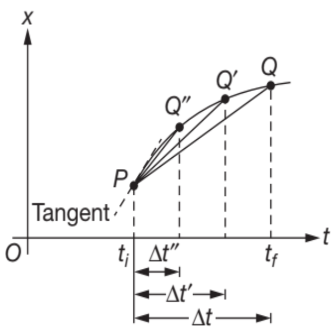
This expression suggests that the average velocity is equal to the slope of the line (chord) joining the points P and Q on the x-t graph.



Instantaneous Velocity

Consider the motion of the particle between the two points P and Q on the x-t graph shown. As the point Q is brought closer and closer to the point P, the time interval between PQ $(\Delta t, \Delta t', \Delta t'', \ldots\ldots..)$ get progressively smaller. The average velocity for each time interval is the slope of the appropriate dotted line $(PQ, PQ', PQ''\ldots..)$.

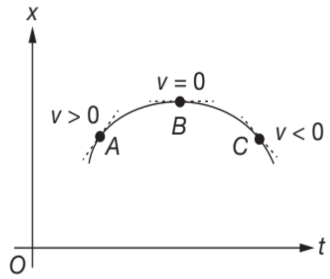
As the point Q approaches P, the time interval approaches zero, but at the same time the slope of the dotted line approaches that of the tangent to the curve at the point P.



As $\Delta t \to 0$, $v_{av}\left(= \frac{\Delta x}{\Delta t}\right) \to v_{\text{inst.}}$

Geometrically, as $\Delta t \to 0$, chord $PQ \to$ tangent at P.

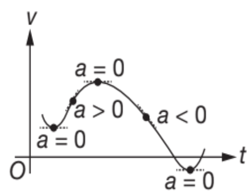
Hence the instantaneous velocity at P is the slope of the tangent at P in the x-t graph.



When the slope of the x-t graph is positive, v is positive (as at the point A in figure). At C, v is negative because the tangent has negative slope. The instantaneous velocity at point B (turning point) is zero as the slope is zero.

Instantaneous Acceleration

The derivative of velocity with respect to time is the slope of the tangent in velocity time $(v$-$t)$ graph.



MOTION WITH UNIFORM VELOCITY

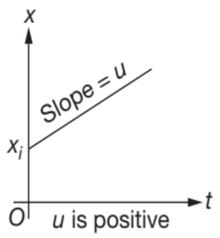
Consider a particle moving along x-axis with uniform velocity u starting from the point $x = x_i$ at $t = 0$.

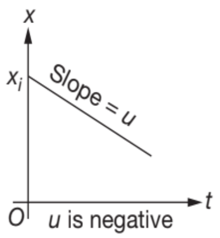
Equations of x, v, a are

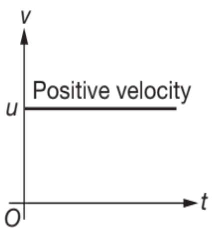
1. $x(t) = x_i + ut$ **2.** $v(t) = u$ **3.** $a(t) = 0$

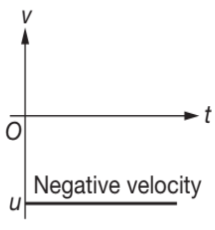
So, x-t graph is a straight line of slope u through x_i

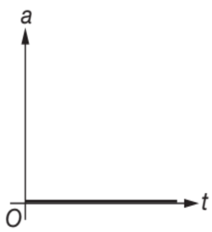
As velocity is constant, v-t graph is a horizontal line and a-t graph coincides with time axis because $a = 0$ at all-time instants.











Conceptual Note(s)

Graph Identification

(a) Straight Line Equation, Graph, Slope (+ve, -ve, zero slope)

If θ is the angle at which a straight line is inclined to the positive direction of *x*-axis and $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by *m*, is defined by $m = \tan\theta$.

If θ is $90°$, *m* does not exist, but the line is parallel to the *y*-axis.

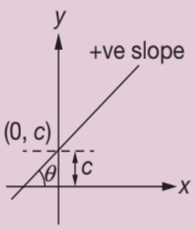
If $\theta = 0$, then $m = 0$ and the line is parallel to the *x*-axis.

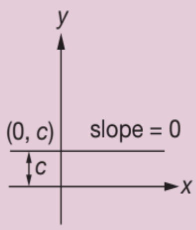
For a straight line, the average slope is equal to instantaneous slope, because a straight line has constant slope.

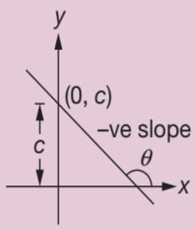
Equation of Line: Slope-Intercept Form

$y = mx + c$ is the equation of a straight line whose slope is *m* and which makes an intercept *c* on the *y*-axis i.e., the line cuts the *y*-axis at $(0, +c)$.

$$m = \text{slope} = \tan\theta = \frac{dy}{dx}$$

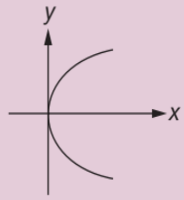




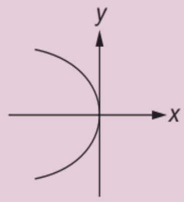


(b) Parabolic Curve-Equation and Graphs

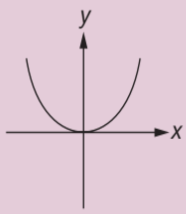
(i) $y^2 = kx$



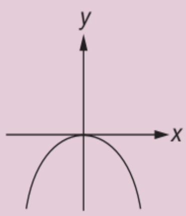
(ii) $y^2 = -kx$



(iii) $x^2 = ky$



(iv) $x^2 = -ky$



Where *k* is a positive constant.

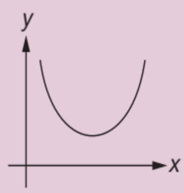
(c) Equation of Parabola

CASE-1:

$$y = ax^2 + bx + c$$

For $a > 0$

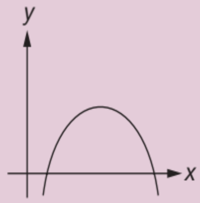
The nature of the parabola will be, like that of the nature of $x^2 = ky$



CASE-2:

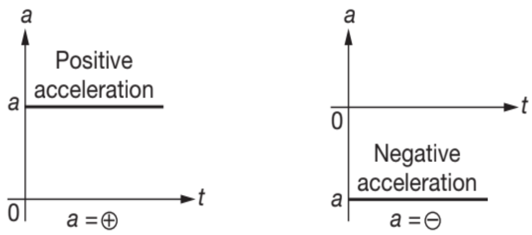
$a < 0$

The nature of the parabola will, like that of the nature of $x^2 = -ky$.

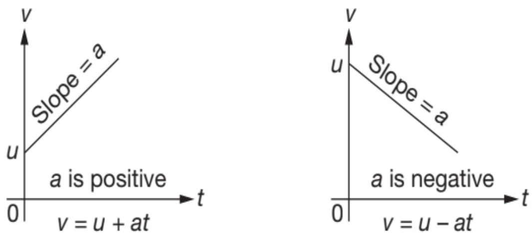


GRAPHS IN UNIFORMLY ACCELERATED MOTION ($a \neq 0$)

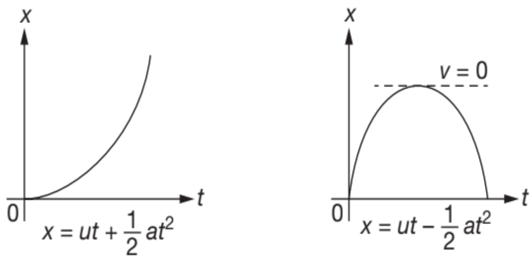
(a) If acceleration a is constant, then a-t graph is a horizontal line as shown.



(b) For constant acceleration, v is a linear polynomial in terms of t. Hence v-t graph is a straight line of slope a as shown.



(c) For constant acceleration, x is a quadratic polynomial in terms of t. Hence x-t graph is a parabola.



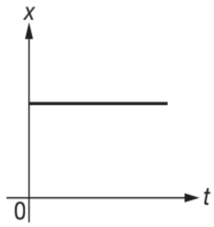
INTERPRETATION OF SOME MORE GRAPHS

Position vs Time Graph

(a) Zero Velocity: As position of particle is fixed at all the times, so the body is at rest.

$$\text{Slope } = \frac{dx}{dt} = \tan\theta = \tan 0^\circ = 0$$

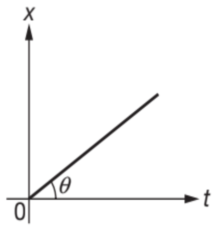
$\Rightarrow$ Velocity of particle is zero



(b) Uniform Velocity: Here $\tan\theta$ is constant so,

$$\tan\theta = \frac{dx}{dt} = \text{constant}$$

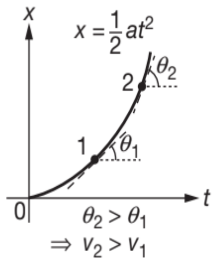
$\Rightarrow$ Velocity of particle is constant.



(c) Non Uniform Velocity (Increasing with Time): In this case, as time is increasing, θ is also increasing.

$$\Rightarrow \quad \frac{dx}{dt} = \tan\theta \text{ also increases}$$

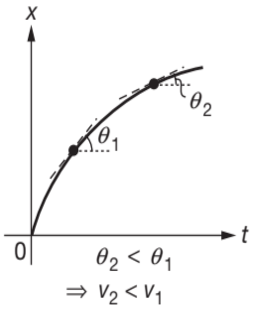
Hence, velocity of particle is increasing.



(d) Non Uniform Velocity (Decreasing with Time): In this case, as time increases, θ decreases.

$$\Rightarrow \quad \frac{dx}{dt} = \tan\theta \text{ also decreases}$$

Hence, velocity of particle is decreasing.



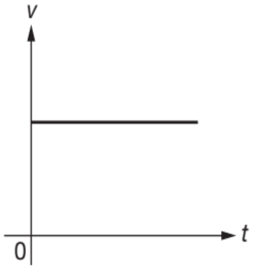
Velocity vs Time Graph

(a) Zero Acceleration: Velocity is constant

$$\Rightarrow \tan\theta = 0$$

$$\Rightarrow \frac{dv}{dt} = 0$$

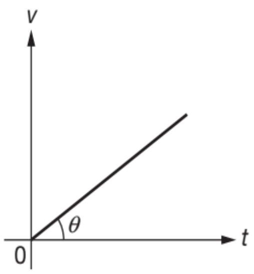
Hence, acceleration is zero i.e., v-t graph is a straight line parallel to time axis.



(b) Uniform Acceleration: $\tan\theta$ is constant

$$\Rightarrow \frac{dv}{dt} = \text{constant}$$

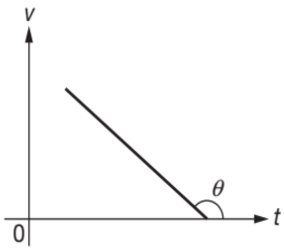
i.e., v-t graph is a straight line with positive slope.



(c) Uniform Retardation: Since $\theta > 90°$, so $\tan\theta$ is constant and negative.

$$\Rightarrow \frac{dv}{dt} = \text{negative constant}$$

i.e., v-t graph is a straight line with negative slope.

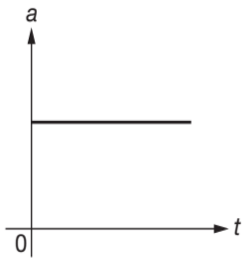


Acceleration vs Time Graph

(a) Constant Acceleration: $\tan\theta = 0$

$$\Rightarrow \frac{da}{dt} = 0$$

Hence acceleration is constant i.e., a-t graph is a straight line parallel to time axis.

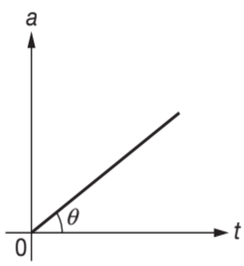


(b) Uniformly Increasing Acceleration: θ is constant.

$$0° < \theta < 90° \quad \Rightarrow \quad \tan\theta > 0$$

$$\Rightarrow \frac{da}{dt} = \tan\theta = \text{constant} > 0$$

Hence, acceleration is uniformly increasing with time i.e., a-t graph is a straight line with positive slope.



(c) Uniformly Decreasing Acceleration: Since $\theta > 90°$

$\Rightarrow \quad \tan\theta$ is constant and negative

$\Rightarrow \quad \dfrac{da}{dt} = $ negative constant

Hence, acceleration is uniformly decreasing with time i.e., a-t graph is a straight line with negative slope.

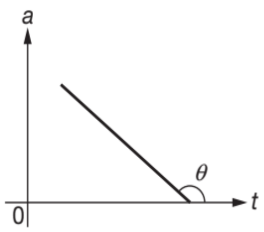
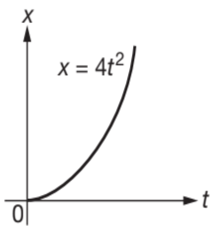


ILLUSTRATION 24

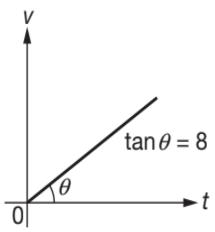
The displacement vs time graph of a particle moving along a straight line is shown in the figure. Draw velocity vs time and acceleration vs time graph.



SOLUTION

$$x = 4t^2$$

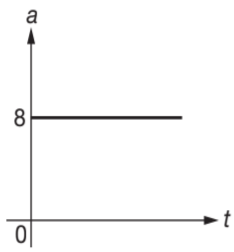
$$\Rightarrow \quad v = \frac{dx}{dt} = 8t$$



Hence, velocity-time graph is a straight line having slope i.e., $\tan\theta = 8$

$$a = \frac{dv}{dt} = 8$$

Hence, acceleration is constant throughout and is equal to $8\ \text{ms}^{-2}$.



Conceptual Note(s)

(a) For uniformly accelerated motion $(a \neq 0)$, x-t graph is a parabola (opening upwards if $a > 0$ and opening downwards if $a < 0$). The slope of tangent at any point of the parabola gives the velocity at that instant.

(b) For uniformly accelerated motion $(a \neq 0)$, v-t graph is a straight line whose slope gives the acceleration of the particle.

(c) In general, the slope of tangent in x-t graph is velocity and the slope of tangent in v-t graph is the acceleration.

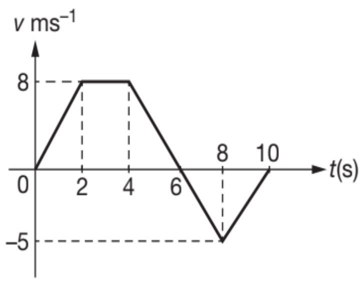
(d) The area under a-t graph gives the change in velocity.

(e) The area between the v-t graph gives the distance travelled by the particle, if we take all areas as positive.

(f) Area under v-t graph gives displacement, if areas below the t-axis are taken negative.

ILLUSTRATION 25

For a particle moving along x-axis, velocity-time graph is as shown in figure. Find the distance travelled and displacement of the particle?



SOLUTION

Distance travelled = Area under v-t graph (taking all areas as positive)

$$\begin{pmatrix}\text{Distance}\\ \text{travelled}\end{pmatrix}=\begin{pmatrix}\text{Area of}\\ \text{trapezium}\end{pmatrix}+\begin{pmatrix}\text{Area of}\\ \text{triangle}\end{pmatrix}$$

$$\Rightarrow\quad \text{Distance travelled}=\frac{1}{2}(2+6)(8)+\frac{1}{2}(4)(5)$$

$$\Rightarrow\quad \text{Distance travelled}=32+10=42\text{ m}$$

Displacement = Area under v-t graph (taking areas below time axis as negative)

$$\text{Displacement}=\begin{pmatrix}\text{Area of}\\ \text{trapezium}\end{pmatrix}-\begin{pmatrix}\text{Area of}\\ \text{triangle}\end{pmatrix}$$

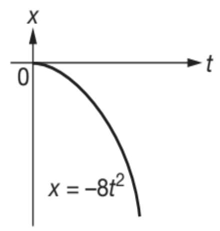
$$\Rightarrow\quad \text{Displacement}=\frac{1}{2}(2+6)(8)-\frac{1}{2}(4)(5)$$

$$\Rightarrow\quad \text{Displacement}=32-10=22\text{ m}$$

Hence, distance travelled is 42 m and displacement is 22 m

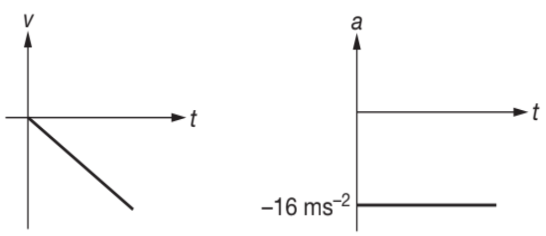
ILLUSTRATION 26

The displacement vs time graph of a particle moving along a straight line is shown in the figure. Draw velocity vs time and acceleration vs time graph.



SOLUTION

Taking upward direction as positive so, downward direction is taken as negative.



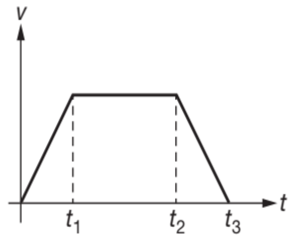
$$x=-8t^2$$

$$\Rightarrow\quad v=\frac{dx}{dt}=-16t$$

$$\Rightarrow\quad a=-16\text{ ms}^{-2}$$

ILLUSTRATION 27

Draw displacement-time and acceleration-time graph for the given velocity-time graph.



SOLUTION

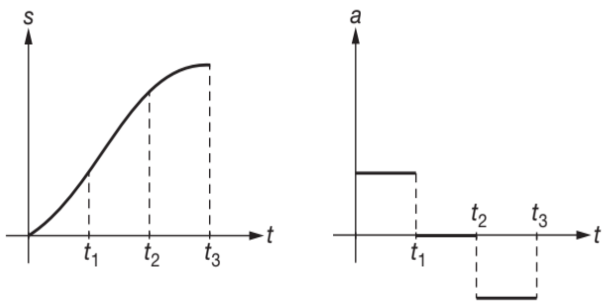


ILLUSTRATION 28

The speed of a train increases at a constant rate α from zero to v and then remains constant for an interval and finally decreases to zero at a constant rate β. If l is the total distance travelled by the train then find the total time taken to complete the journey. At what value of v is the time of travel shortest? What is the value of the shortest time?

SOLUTION

Let us first draw v-t graph showing the situation discussed.

For accelerated Motion (O to A)

$$v_{\max}=v=\alpha t_1$$

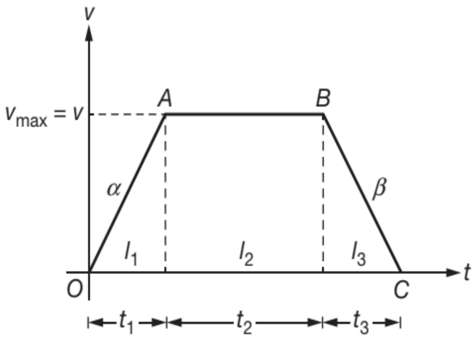
$$\Rightarrow\quad t_1=\frac{v}{\alpha}\qquad\ldots(1)$$

Also, $l_1 = \dfrac{v^2}{2\alpha}$ $\quad \{\because v^2 - 0^2 = 2\alpha l_1\}$ ...(2)

For Uniform Motion (*A* to *B*)

$$l_2 = vt_2$$

$$\Rightarrow t_2 = \frac{l_2}{v} \quad ...(3)$$



For Decelerated Motion (*B* to *C*)

$$0 = v + (-\beta)t_3$$

$$\Rightarrow t_3 = \frac{v}{\beta} \quad ...(4)$$

Also, $l_3 = \dfrac{v^2}{2\beta}$ $\quad \{\because 0^2 - v^2 = 2(-\beta)l_3\}$ ...(5)

If t be the total time of journey, then

$$t = t_1 + t_2 + t_3$$

$$\Rightarrow t = \frac{v}{\alpha} + \frac{l_2}{v} + \frac{v}{\beta}$$

$$\Rightarrow t = \frac{v}{\alpha} + \frac{l - (l_1 + l_3)}{v} + \frac{v}{\beta} \quad \{\because l_1 + l_2 + l_3 = l\}$$

$$\Rightarrow t = \frac{v}{\alpha} + \frac{l}{v} - \left(\frac{\frac{v^2}{2\alpha}}{v} + \frac{\frac{v^2}{2\beta}}{v}\right) + \frac{v}{\beta} \quad \{\text{using (2) and (5)}\}$$

$$\Rightarrow t = \frac{v}{\alpha} + \frac{l}{v} - \frac{v}{2\alpha} - \frac{v}{2\beta} + \frac{v}{\beta}$$

$$\Rightarrow t = \frac{l}{v} + \frac{v}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \quad ...(6)$$

For t to be MINIMUM, we must have

$$\frac{dt}{dv} = 0$$

$$\Rightarrow -\frac{l}{v^2} + \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = 0$$

$$\Rightarrow v = \sqrt{\frac{2l\alpha\beta}{\alpha + \beta}} \quad ...(7)$$

So, $t_{MIN} = \dfrac{l}{\sqrt{\dfrac{2l\alpha\beta}{\alpha+\beta}}} + \dfrac{1}{2}\sqrt{\dfrac{2l\alpha\beta}{\alpha+\beta}}\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)$

{Substitute (7) in (6)}

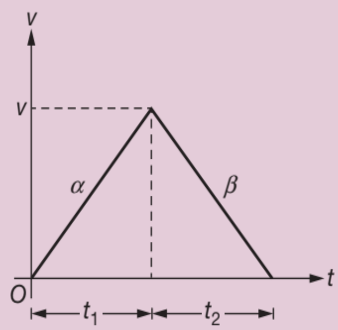
$$\Rightarrow t_{MIN} = \sqrt{\frac{l(\alpha+\beta)}{2\alpha\beta}} + \sqrt{\frac{l\alpha\beta}{2(\alpha+\beta)}}\left(\frac{\alpha+\beta}{\alpha\beta}\right)$$

$$\Rightarrow t_{MIN} = \sqrt{\frac{l(\alpha+\beta)}{2\alpha\beta}} + \sqrt{\frac{l(\alpha+\beta)}{2\alpha\beta}}$$

$$\Rightarrow t_{MIN} = 2\sqrt{\frac{l(\alpha+\beta)}{2\alpha\beta}} = \sqrt{\frac{2l(\alpha+\beta)}{\alpha\beta}}$$

$$\Rightarrow t_{MIN} = \sqrt{2l\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)}$$

Please note that this time equals the time of journey when the train just accelerated to attain a velocity v and immediately decelerated to zero velocity after covering a total distance l. Then



$$t_{MIN} = t_1 + t_2$$

$$\Rightarrow t_{MIN} = \frac{v}{\alpha} + \frac{v}{\beta}$$

$$\Rightarrow t_{MIN} = v\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \quad ...(1)$$

Also, $l = \dfrac{v^2}{2\alpha} + \dfrac{v^2}{2\beta}$

$$\Rightarrow \quad v = \sqrt{\frac{2l\alpha\beta}{\alpha+\beta}}$$

Substituting in (1), we get

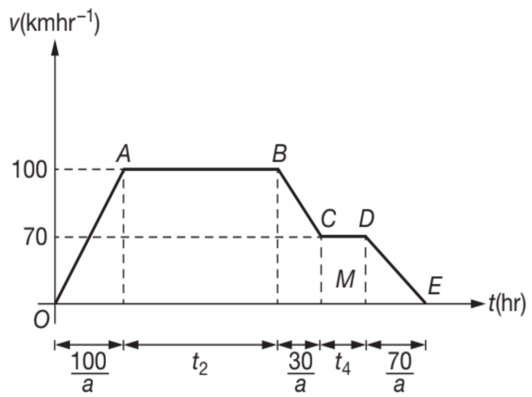
$$t_{MIN} = \sqrt{2l\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}$$

ILLUSTRATION 29

Two road rally checkpoints A and B are located on the same highway and are 12 km apart. The speed limits for the first 8 km and the last 4 km of the section of highway are 100 kmh^{-1} and 70 kmh^{-1}, respectively. Drivers must stop at each checkpoint and the specified time between points A and B is $\frac{25}{3}$ min. Knowing that a driver accelerates and decelerates at the same constant rate, determine the magnitude of his acceleration if he travels at the speed limit as much as possible.

SOLUTION

Total time taken by the driver to go from A to B is $\frac{25}{3}$ min. We must note here that a driver first accelerates with acceleration a (say) for time t_1, moves with uniform velocity of 100 kmh^{-1} for some time t_2 (say) and retards to a speed of 70 kmh^{-1} with deceleration a (again) for a time t_3 (say) to travel a total distance of 8 km. Just after he completes 8 km, he maintains his speed at 70 kmh^{-1} for a time t_4 (say) and then finally retards with deceleration a (again) for a time t_5 (say) to complete a further distance of 4 km. As per this discussion, the v-t plot for the situation is shown here.



Since $t_1+t_2+t_3+t_4+t_5 = \frac{25}{3}$ min

$$\Rightarrow \quad \frac{100}{a}+t_2+\left(\frac{100-70}{a}\right)+t_4+\left(\frac{70}{a}\right) = \frac{25}{3} \text{ min}$$

$$\Rightarrow \quad t_2+t_4+\frac{200}{a} = \frac{25}{3} \text{ min}$$

$$\Rightarrow \quad t_2+t_4+\frac{200}{a} = \frac{25}{180} \text{ hr} \qquad \ldots(1)$$

Further, area under the v-t graph gives displacement, so

$$\text{Area}(OABCM) = 8 \text{ km}$$

$$\frac{1}{2}\left(\frac{100}{a}\right)(100)+100t_2+\frac{1}{2}\left(\frac{30}{a}\right)(100+70) = 8$$

$$\Rightarrow \quad \frac{5000}{a}+100t_2+\frac{2550}{a} = 8$$

$$\Rightarrow \quad \frac{7550}{a}+100t_2 = 8 \qquad \ldots(2)$$

Also, $\text{Area}(CDE) = 4$ km

$$\Rightarrow \quad 70t_4+\frac{1}{2}\left(\frac{70}{a}\right)(70) = 4 \text{ km}$$

$$\Rightarrow \quad \frac{2450}{a}+70t_4 = 4 \qquad \ldots(3)$$

From (2), (3), we get

$$t_2 = \frac{1}{100}\left(8-\frac{7550}{a}\right) \text{ and } t_4 = \frac{1}{70}\left(4-\frac{2450}{a}\right)$$

Substituting in (1), we get

$$\frac{1}{100}\left(8-\frac{7550}{a}\right)+\frac{1}{70}\left(4-\frac{2450}{a}\right)+\frac{200}{a} = \frac{25}{180}$$

$$\Rightarrow \quad \frac{8}{100}-\frac{7550}{100a}+\frac{4}{70}-\frac{2450}{70a}+\frac{200}{a} = \frac{25}{180}$$

$$\frac{96}{700}-\frac{75.5}{a}-\frac{35}{a}+\frac{200}{a} = \frac{25}{180}$$

$$\Rightarrow \quad \frac{89.5}{a} = \frac{11}{6300}$$

$$\Rightarrow \quad a = 51259 \text{ kmhr}^{-2}$$

$$\Rightarrow \quad a = 3.96 \text{ ms}^{-2}$$

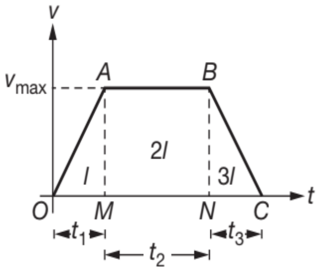
ILLUSTRATION 30

A particle starts from rest and traverses a distance l with uniform acceleration, then moves uniformly over a further distance $2l$ and finally comes to rest after moving a further distance $3l$ under uniform retardation. Assuming entire motion to be rectilinear motion, find the ratio of average speed over the journey to the maximum speed on its way.

SOLUTION

$$v_{av} = \frac{\text{Total Distance}}{\text{Total Time}}$$

$$\Rightarrow \quad v_{av} = \frac{l+2l+3l}{t_1+t_2+t_3}$$



For OA, $l = \frac{1}{2}t_1 v_{\max}$

$$\Rightarrow \quad t_1 = \frac{2l}{v_{\max}}$$

For AB, $2l = v_m t_2$

$$\Rightarrow \quad t_2 = \frac{2l}{v_{\max}}$$

For BC, $3l = \frac{1}{2}t_3 v_{\max}$

$$\Rightarrow \quad t_3 = \frac{6l}{v_{\max}}$$

$$\Rightarrow \quad v_{av} = \frac{6l}{\left(\frac{10l}{v_{\max}}\right)}$$

Hence, $\frac{v_{av}}{v_{\max}} = \frac{3}{5}$

ILLUSTRATION 31

A train starts from station A with uniform acceleration α for some distance and then goes with uniform retardation β for some more distance to come to rest at station B. The distance between station A and B is 4 km and the train takes 4 minute to complete this journey, then find the value of $\frac{1}{\alpha}+\frac{1}{\beta}$, where α and β are in $\text{km}(\text{min})^{-2}$.

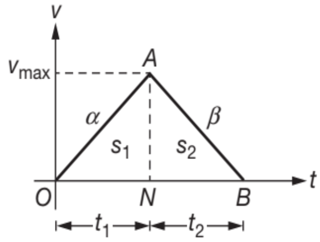
SOLUTION

Since, $s_1 + s_2 = 4\text{ km}$

and $t_1 + t_2 = 4\text{ min}$ {given}

Also, $t_1 = \frac{v_{\max}}{\alpha}$ and

$$t_2 = \frac{v_{\max}}{\beta}$$



Adding we get,

$$t_1 + t_2 = v_{\max}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) \quad \ldots(1)$$

Now $s_1 = \frac{1}{2}v_{\max}\, t_1$ ...(2)

and $s_2 = \frac{1}{2}v_{\max} t_2$ ...(3)

Adding (2) and (3) we get,

$$v_{\max} = 2\text{ km(min)}^{-1}$$

$$\therefore \quad \frac{1}{\alpha}+\frac{1}{\beta} = 2\text{ km}(\text{min})^{-2}$$

INTERPRETATION OF GRAPHS OF VARIOUS TYPES OF MOTION

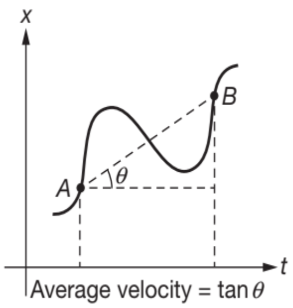
Interpretation of *x-t* Graph

(a) The line on the graph representing $x(t)$ is called a **World-Line**.

(b) A point on the World Line is called an **Event**.

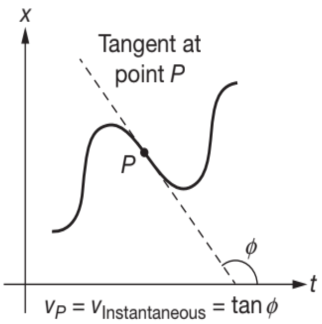
(c) The average slope of the World Line gives the average velocity. Just join the two points with a line and then find the angle which this line (joining the points) makes with the x-axis. If this angle is θ, then

$$\text{Average Velocity} = v_{av} = \tan\theta$$



(d) The slope of an event gives the instantaneous velocity. Just draw a tangent to the curve at a point P and then find the angle which this tangent makes with the x-axis. If this angle is ϕ, then the instantaneous velocity at the point P is

$$v_P = v_{\text{instantaneous}} = \tan\phi$$



Interpretation of *v*-*t* Graph

(a) The average slope of a curve in $\vec{v}(t)$ graph gives average acceleration.

(b) The instantaneous slope of a curve at a point in $\vec{v}(t)$ graph gives the instantaneous acceleration.

(c) The area under a curve in $\vec{v}(t)$ graph gives the displacement of the body.

(d) v-t graph can also be used to find the average velocity of the body, because

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{\text{Area under } v\text{-}t \text{ graph}}{\text{Time Interval}}$$

Interpretation of *a*-*t* Graph

(a) The area under a curve in a-t graph gives the change in the velocity of the body. So

$$\text{Area under } a\text{-}t \text{ graph} = \Delta v = v_f - v_i$$

(d) a-t graph can also be used to find the average acceleration of the body, because

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{\text{Area under } a\text{-}t \text{ graph}}{\text{Time Interval}}$$

Interpretation to Other Graphs

(a) Area under a curve in a-x graph $= \dfrac{v^2 - u^2}{2}$

(b) Slope of v^2-x graph gives twice the acceleration

i.e. $\dfrac{d}{dx}\left(v^2\right) = 2a$

(c) From v-x graph we can also find the instantaneous acceleration of a particle. Suppose we have to find the acceleration a_P at a point P on the v-x curve. To calculate a_P, we follow the steps given below.

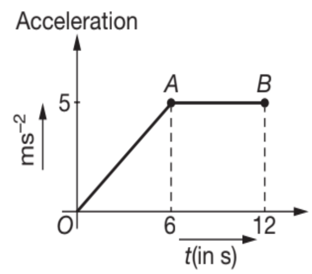
STEP-1: First find the velocity v_P of the particle at the point P.

STEP-2: Then calculate the instantaneous slope $\left(\dfrac{dv}{dx}\right)_P = \tan\theta$ of the curve at the same point.

STEP-3: Finally, the acceleration at the point P is the product of values calculated in Steps 1 and 2 i.e. $a_P = v_P\left(\dfrac{dv}{dx}\right)_P$.

ILLUSTRATION 32

For an airplane to take-off it accelerates according to the graph shown and takes 12 s to take-off from the rest position. Calculate the distance travelled by the airplane from $t = 0$ to $t = 12$ s.



SOLUTION

From O → A we have acceleration varying linearly with time hence

$$a - 0 = \frac{5}{6}(t - 0)$$

$$\Rightarrow \quad a = \frac{5}{6}t$$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{5}{6}t$$

$$\Rightarrow \quad \int_0^{v_1} dv = \frac{5}{6}\int_0^t t\,dt$$

$$\Rightarrow \quad v_1 = \frac{5}{12}t^2$$

Further $v_1 = \dfrac{dx_1}{dt} = \dfrac{5}{12}t^2$

$$\Rightarrow \quad \int_0^{x_1} dx_1 = \frac{5}{12}\int_0^6 t^2 dt$$

$$\Rightarrow \quad x_1 = \frac{5}{36}t^3\Big|_0^6$$

$$\Rightarrow \quad x_1 = \frac{5}{36}(6^3) = 30 \text{ m}$$

For A $\to$ B the acceleration is constant and have a value of 5 ms^{-2}. Now velocity at point A is

$$v_1 = \frac{5}{12}(36) = 15 \text{ ms}^{-1}$$

$$\Rightarrow \quad x_2 = v_1(6) + \frac{1}{2}(5)(6)^2$$

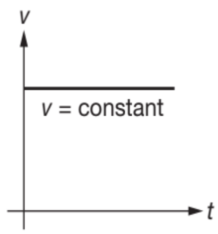
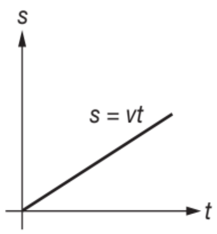
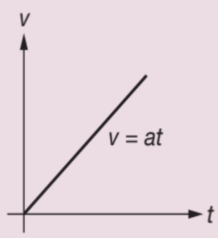
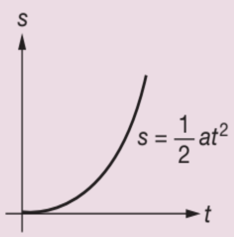
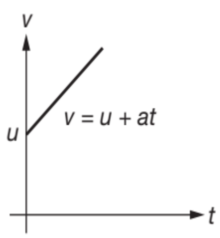
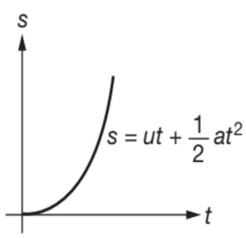
$$\Rightarrow \quad x_2 = 180 \text{ m}$$

So, total distance travelled is

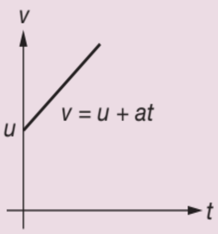
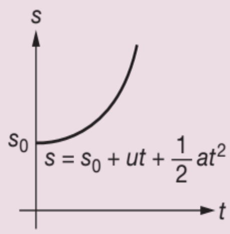
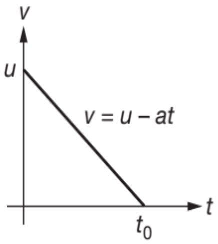
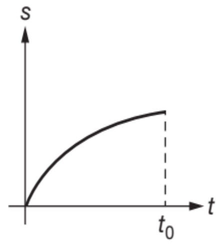
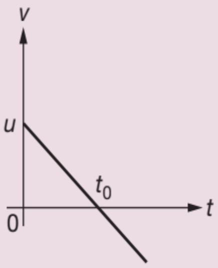
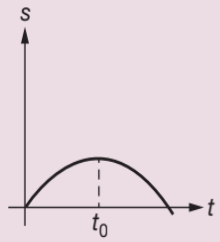
$$x = 30 + 180 = 210 \text{ m}$$

GRAPHS

Now, let us plot v-t and s-t graphs of some standard results. To draw the following graphs assume that the particle has got either a one-dimensional motion with uniform velocity or with constant acceleration.

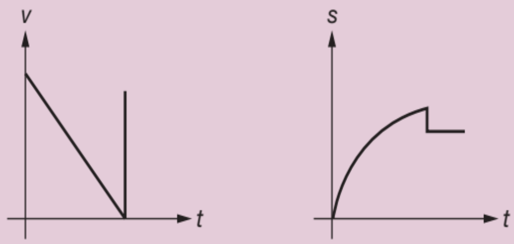
S.No.	Situation	*v-t* Graph	*s-t* Graph	Interpretation
1.	Uniform motion			(i) Slope of s-t graph i.e. v is constant. (ii) In s-t graph $s = 0$ at $t = 0$.
2.	Uniformly accelerated motion with $u = 0$ and $s = 0$ at $t = 0$			(i) $u = 0$, i.e., $v = 0$ at $t = 0$. (ii) $u = 0$, i.e., slope of s-t graph at $t = 0$, should be zero. (iii) a or slope of v-t graph is constant.
3.	Uniformly accelerated motion with $u \neq 0$ but $s = 0$ at $t = 0$			(i) $u \neq 0$, i.e., v or slope of s-t graph at $t = 0$ is not zero. (ii) v or slope of s-t graph gradually goes on increasing.

(Continued)

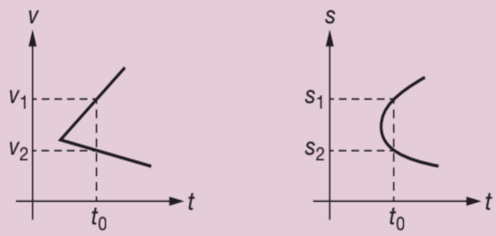
S.No.	Situation	*v-t* Graph	*s-t* Graph	Interpretation
4.	Uniformly accelerated motion with $u \neq 0$ and $s = s_0$ at $t = 0$			(i) $s = s_0$ at $t = 0$
5.	Uniformly retarded motion till velocity becomes zero			(i) Slope of *s-t* graph at $t = 0$ gives u. (ii) Slope of *s-t* graph at $t = t_0$ becomes zero. (iii) In this case u can't be zero.
6.	Uniformly retarded then accelerated in opposite direction.			(i) At time $t = t_0$, $v = 0$ or slope of *s-t* graph is zero. (ii) In *s-t* graph slope or velocity first decreases then increases with opposite sign.

Conceptual Note(s)

(a) Slopes of *v-t* or *s-t* graphs can never be infinite at any point, because infinite slope of *v-t* graph means infinite acceleration. Similarly, infinite slope of *s-t* graph means infinite velocity. Hence, the following graphs are not possible:



(b) At one time, two values of velocity or displacement are not possible. Hence, the following graphs are not acceptable:



(c) Different values of displacements in *s-t* graph corresponding to given *v-t* graph can be calculated just by calculating areas under *v-t* graph. There is no need of using equations like $v = u + at$, etc.

ILLUSTRATION 33

A particle starting from rest undergoes an acceleration that increases linearly with time. Estimate the particle's velocity n second after the start and the distance moved by the particle in these n second. Draw the graph showing the variation of acceleration, velocity and distance with time.

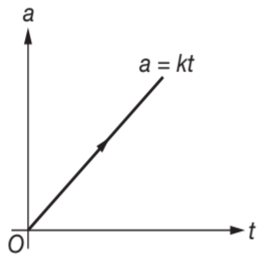
SOLUTION

$$a = kt$$

$$\Rightarrow \quad \frac{dv}{dt} = kt$$

$$\Rightarrow \quad dv = ktdt$$

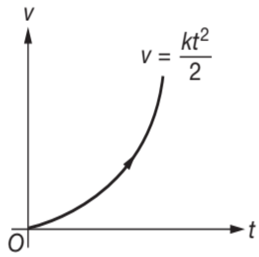
$$\Rightarrow \quad \int_0^v dv = k\int_0^t t\,dt$$



$$\Rightarrow \quad v = \frac{kt^2}{2} = \frac{kn^2}{2}$$

Further $v = \dfrac{dx}{dt}$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{kt^2}{2}$$



$$\Rightarrow \quad dx = \frac{kt^2}{2}dt$$

$$\Rightarrow \quad \int_0^x dx = \frac{k}{2}\int_0^t t^2 dt$$

$$\Rightarrow \quad x = \frac{kt^3}{3} = \frac{kn^3}{6}$$

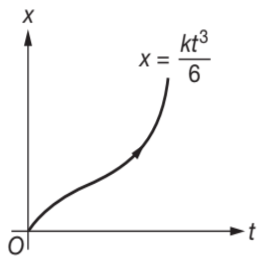
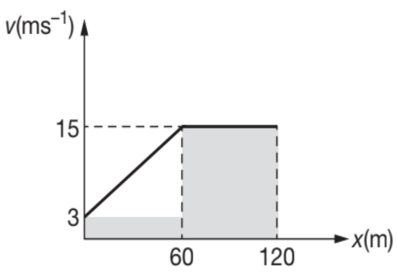


ILLUSTRATION 34

The velocity-displacement (v-x) graph describing the motion of a motorcycle is shown in figure. Construct the a-x graph of the motion and determine the time needed for the motorcycle to reach the position $x = 120$ m.



SOLUTION

For $0 < x \le 60$ m, we have

$$v = \frac{12}{60}x + 3 = 3 + \frac{x}{5}$$

$$\frac{dv}{dt} = \left(\frac{1}{5}\right)\frac{dx}{dt} = \frac{1}{5}(v) = \frac{1}{5}\left(3 + \frac{x}{5}\right) = \frac{3}{5} + \frac{x}{25} \qquad \ldots(1)$$

$$\Rightarrow \quad a = \frac{3}{5} + \frac{x}{25}$$

i.e., the a-x graph is a straight line, where at $x = 0$, $a = \frac{3}{5}\text{ ms}^{-2} = 0.6\text{ ms}^{-2}$ and at $s = 60$ m, $a = 3\text{ ms}^{-2}$

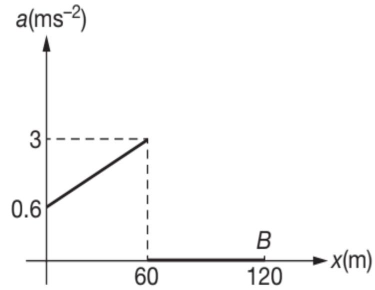
For $s > 60$ m, we have $v =$ constant

$$\Rightarrow \quad a = 0$$

The corresponding a-x graph is shown here

From equation (1), we have

$$\frac{dv}{dt} = \frac{v}{5}$$



$$\Rightarrow \quad \int_3^v \frac{dv}{v} = \frac{1}{5}\int_0^t dt$$

$$\Rightarrow \quad \ln\left(\frac{v}{3}\right) = \frac{t}{5}$$

$$\Rightarrow \quad v = 3e^{t/5}$$

$$\Rightarrow \quad \int_0^{60} dx = 3\int_0^{t_1} e^{t/5}dt \qquad \left\{\because v = \frac{dx}{dt}\right\}$$

$$\Rightarrow \quad 60 = 15\left(e^{t_1/5} - 1\right)$$

$$\Rightarrow \quad t_1 = 8\text{ s}$$

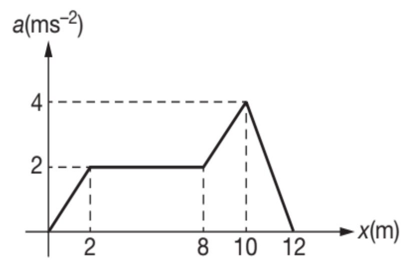
Time taken to travel next 60 m with speed 15 ms^{-1} will be t_2 given by

$$t_2 = \frac{60}{15} = 4\text{ s}$$

$\therefore$ Total time $= t_1 + t_2 = 8 + 4 = 12$ s

ILLUSTRATION 35

The acceleration-displacement $(a - x)$ graph of a particle moving in a straight line is as shown. Assume the particle to start from rest, find the velocity of the particle when displacement of the particle is, 12 m.



SOLUTION

Since $a = \frac{dv}{dt}$

$\Rightarrow \quad v\frac{dv}{dx} = a$

$\Rightarrow \quad vdv = adx$

$\Rightarrow \quad \int_0^v vdv = \int adx$

$\Rightarrow \quad \frac{v^2}{2} = \text{area under } a\text{-}x \text{ graph}$

$\Rightarrow \quad v = \sqrt{2(\text{area under } a\text{-}x \text{ graph})}$

The area A under the a-x graph is given by

$$A = \frac{1}{2}(2)(2) + (6)(2) + \frac{1}{2}(2+4)(2) + \frac{1}{2}(2)(4)$$

$\Rightarrow \quad A = 2 + 12 + 6 + 4 = 24 \text{ m}^2\text{s}^{-2}$

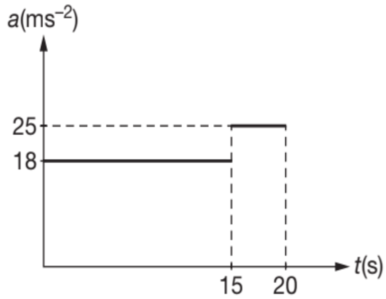
$\Rightarrow \quad v = \sqrt{2 \times 24} = 4\sqrt{3} \text{ ms}^{-1}$

Test Your Concepts-IV

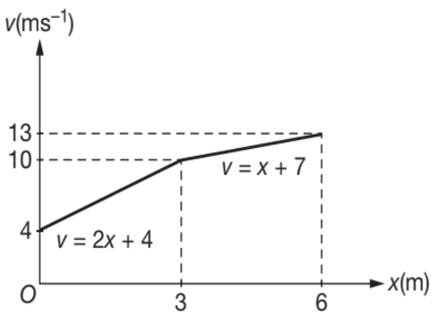
Based on Graph

(Solutions on page H.77)

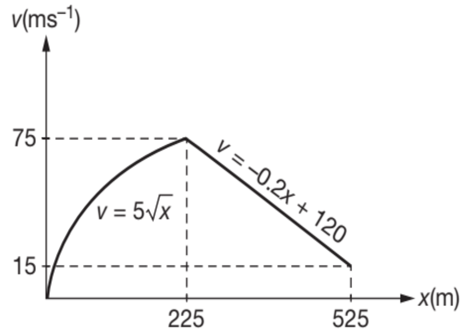
1. A two stage missile is fired vertically from rest with the acceleration shown. In 15 s the first stage burns out and the second stage ignites. Plot the v-t and s-t graphs which describe the two-stage motion of the missile for $0 \le t \le 20$ s.



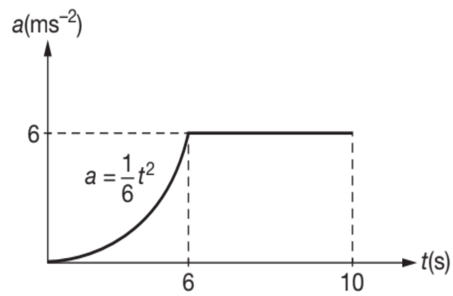
2. The particle moves rectilinearly with the velocity described by the graph. Construct the acceleration-displacement graph.



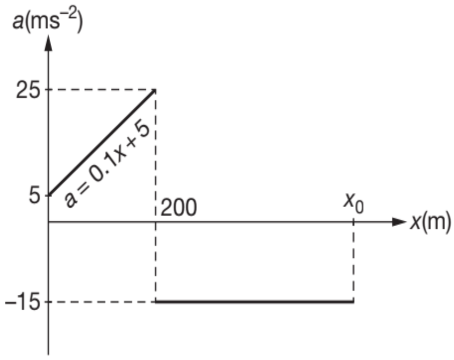
3. The car is moving along a straight road with the speed described by the v-x graph. Construct the a-x graph.



4. A motorcyclist starting from rest travels along a straight road and for $0 < t \le 10$ s, it has an acceleration as shown. Draw the v-t graph that describes the motion and find the distance travelled in 10 s.

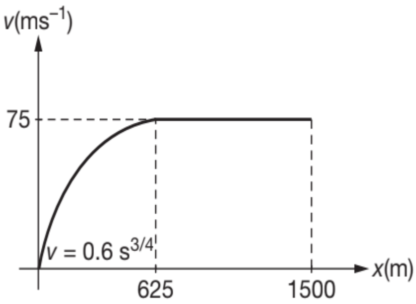


5. The dragster starts from rest and travels along a straight track with an acceleration-deceleration described by the graph. Construct the *v-x* graph for $0 \le x \le x_0$ and determine the distance x_0 travelled before the dragster again comes to rest.

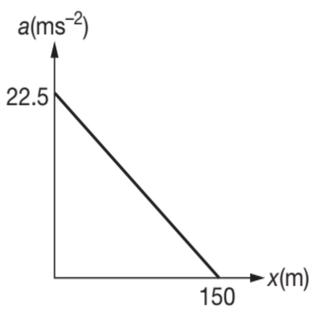


6. A particle travels along a curve defined by the equation $x = \left(t^3 - 3t^2 + 2t\right)$ m, where t is in seconds. Draw the *x-t*, *v-t* and *a-t* graphs for the particle for $0 \le t \le 3$ s.

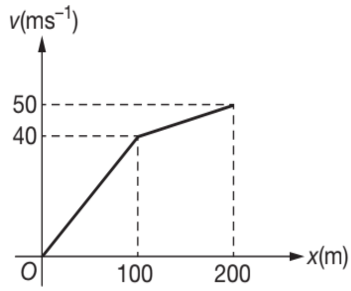
7. A truck is travelling along the straight line with a velocity described by the graph. Construct the *a-x* graph for $0 \le x \le 1500$ m.



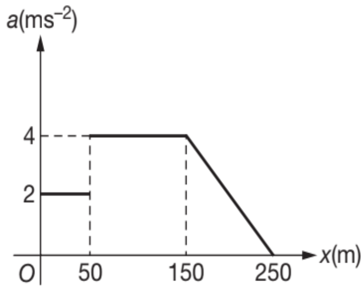
8. The jet plane starts from rest from $x = 0$ and is subjected to the acceleration shown. Determine the speed of the plane when it has travelled 60 m.



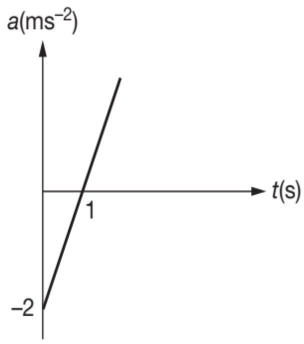
9. The *v-x* graph for an airplane travelling on a straight run-way is shown. Determine the acceleration of the plane at $x = 50$ m and $x = 150$ m. Draw the *a-x* graph.



10. Starting from rest from $x = 0$, a boat travels in a straight line with an acceleration as shown by the *a-x* graph. Determine the boat's speed at $x = 40$ m, 90 m and 200 m.



11. The acceleration of particle moving rectilinearly, varies with time as shown.

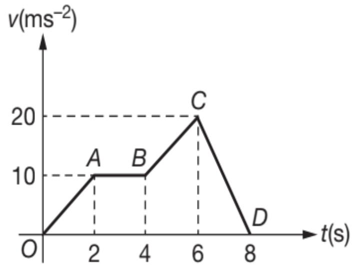


(a) Find an expression for velocity in terms of t.

(b) Calculate the displacement of the particle in the interval from $t = 2$ s to $t = 4$ s. Assume that $v = 0$ at $t = 0$.

12. Acceleration-time graph of a particle moving in a straight line is shown in figure. Velocity of particle at time $t = 0$ is 2 ms^{-1}. Find velocity at the end of fourth second.

13. Velocity-time graph of a particle moving in a straight line is shown in figure. Plot the corresponding displacement-time graph of the particle if at time $t = 0$, displacement $s = 0$.



VERTICAL MOTION UNDER GRAVITY

Vertical motion under gravity, is the case of motion of a particle moving rectilinearly under the influence of gravity. To solve problems involving motion of a particle under gravity, we make use of the following steps for the sake of convenience and accuracy.

STEP-1

Take initial direction of motion of the particle as positive.

FOR EXAMPLE

(a) If a particle is dropped from a tower, then we can take downward direction (the direction of initial motion) as positive.

(b) If a particle is thrown vertically upwards from a tower, then we can take upward direction (the direction of initial motion) as positive.

STEP-2

The quantities (i.e. u, v, a and s) along the initial direction of motion will be taken as positive whereas opposite to the initial direction of motion are taken as negative.

FOR EXAMPLE

(a) For the particle dropped from the tower of height h, we have taken downward direction (the direction of initial motion) as positive. So, in this case we have

$$u = 0,\ v = +v,\ a = +g \text{ and } s = +h$$

as all are in the downward direction.

(b) For the particle thrown vertically upwards from a tower of height h, we have taken upward direction (the direction of initial motion) as positive. So, in this case, since

- initial velocity u is upwards, so $u = +u$,
- acceleration due to gravity is downwards, so $a = -g$,
- particle eventually reaches the ground, so the displacement of the particle is downwards and hence $s = \Delta y = -h$.

STEP-3

Now apply the equations of rectilinear motion of the particle i.e.

$$v = u + at, \qquad s = ut + \frac{1}{2}at^2$$

$$v^2 - u^2 = 2as, \qquad s = \frac{u+v}{2}t$$

$$s_n^{th} = u + \frac{1}{2}a(2n-1)$$

and substituting the values as calculated in **Step 2**, to get the desired solution to the problems as discussed below.

FOR EXAMPLE

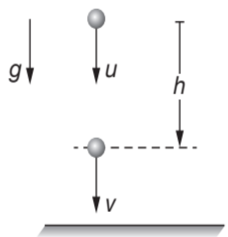
(a) For a body dropped $(u = 0)$ from a height h, the equation of motion are

$$v = gt, \qquad h = \frac{1}{2}gt^2, \qquad v = \sqrt{2gh}$$

Initial velocity in case of dropping is zero.

(b) For a body thrown downward with initial velocity u from a height h, the equations of motion are

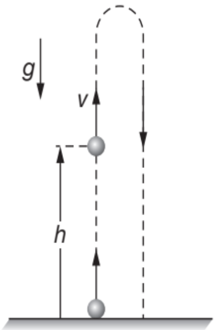
$$v = u + gt, \quad h = ut + \frac{1}{2}gt^2, \quad v = \sqrt{u^2 + 2gh}$$



Take downwards as positive

(c) For a body launched up from the ground, with initial velocity u, the equations of motion before the particle attains maximum height are

$$v = u - gt, \quad h = ut - \frac{1}{2}gt^2, \quad v = \sqrt{u^2 - 2gh}$$



Take upwards as positive

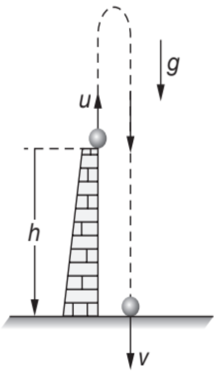
When the particle returns to the ground, then $s = 0$ and hence we have

$$0 = ut - \frac{1}{2}gt^2$$

$$\Rightarrow \quad t = \text{Time of Flight} = \frac{2u}{g}$$

(d) For a body launched up from a tower of height h, taking upward direction as positive, then we have

$$s = -h, \quad u = +u, \quad a = -g$$



So, the equations of motion are written as

$$-h = ut + \frac{1}{2}(-g)t^2$$

$$v = u + (-g)t$$

and

$$v^2 - u^2 = 2(-g)(-h)$$

ILLUSTRATION 36

A ball is thrown upwards from the ground with an initial speed of u. At two instants of time, having an interval of 6 s, the ball is at a height of 80 m from the ground. Find u. Take $g = 10\ \text{ms}^{-2}$.

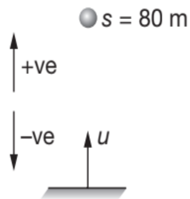
SOLUTION

METHOD I

Consider the upward direction as positive, then

$$u = +u\ \text{ms}^{-1},\ a = g = -10\ \text{ms}^{-2} \text{ and } s = +80\ \text{m}$$

Since, $s = ut + \frac{1}{2}at^2$, so we get



$$80 = ut - 5t^2$$

$$\Rightarrow \quad 5t^2 - ut + 80 = 0$$

$$\Rightarrow \quad t = \frac{u + \sqrt{u^2 - 1600}}{10}$$

and $\dfrac{u - \sqrt{u^2 - 1600}}{10}$

Now it is given that these two instants, when the ball is at 80 m from the ground have an interval of 6 s in between. So,

$$\frac{u + \sqrt{u^2 - 1600}}{10} - \frac{u - \sqrt{u^2 - 1600}}{10} = 6$$

$$\Rightarrow \quad \frac{\sqrt{u^2 - 1600}}{5} = 6$$

$$\Rightarrow \quad \sqrt{u^2 - 1600} = 30$$

$$\Rightarrow \quad u^2 - 1600 = 900$$

$$\Rightarrow \quad u^2 = 2500$$

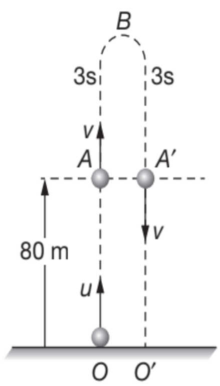
$$\Rightarrow \quad u = \pm 50\ \text{ms}^{-1}$$

$$\Rightarrow \quad |u| = 50\ \text{ms}^{-1}$$

METHOD II

Since $t_{A \to A'} = 6\text{s}$

$$\Rightarrow \quad t_{A \to B} = t_{B \to A'} = 3\text{s}$$



For A → B

$$0 = v + (-10)(3)$$

$$\Rightarrow \quad v = 30\ \text{ms}^{-1}$$

For O → A

$$v^2 - u^2 = 2(-10)(80)$$

$\Rightarrow \quad 900 - u^2 = -1600$

$\Rightarrow \quad u^2 = 900 + 1600$

$\Rightarrow \quad u = \sqrt{2500} = 50 \text{ ms}^{-1}$

ILLUSTRATION 37

A particle is dropped from height 100 m and another particle is projected vertically up with velocity 50 ms^{-1} from the ground along the same line. Find out the position where two particle will meet?

SOLUTION

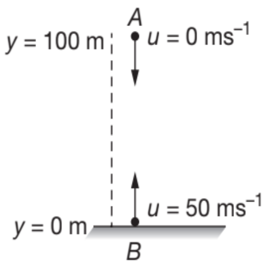
Let the upward direction be taken as positive.
Let the particles meet at a distance y from the ground.

For particle *A*

$y_0 = +100 \text{ m}$

$u = 0$

$a = -10 \text{ ms}^{-2}$



Since $y = y_0 + ut + \frac{1}{2}at^2$

$\Rightarrow \quad y = 100 + 0(t) - \frac{1}{2} \times 10 \times t^2$

$\Rightarrow \quad y = 100 - 5t^2 \qquad \ldots(1)$

For particle *B*

$y_0 = 0 \text{ m}$

$u = +50 \text{ ms}^{-1}$

$a = -10 \text{ ms}^{-2}$

$\Rightarrow \quad y = 50(t) - \frac{1}{2} \times 10 \times t^2$

$\Rightarrow \quad y = 50t - 5t^2 \qquad \ldots(2)$

According to the problem

$50t - 5t^2 = 100 - 5t^2$

$\Rightarrow \quad t = 2 \text{ s}$

Putting $t = 2 \text{ s}$ in equation (1), we get

$y = 100 - 20 = 80 \text{ m}$

Hence, the particles will meet at a height 80 m above the ground.

ILLUSTRATION 38

A particle is dropped from a tower is found to travel 45 m in the last second of its journey. Calculate the height of the tower.

SOLUTION

Let the total time of journey be n seconds. Then

$$s_{n^{\text{th}}} = u + \frac{a}{2}(2n - 1)$$

$\Rightarrow \quad 45 = 0 + \frac{10}{2}(2n - 1)$

$\Rightarrow \quad n = 5 \text{ s}$

So, 5^{th} second is the last second of motion. Hence height of tower is given by

$$h = \frac{1}{2}gt^2$$

$\Rightarrow \quad h = \frac{1}{2}(10)(5)^2$

$\Rightarrow \quad h = 125 \text{ m}$

ILLUSTRATION 39

A particle is projected vertically upwards. Prove that it will be at $\frac{3}{4}$ of its greatest height at time which are in the ratio $1:3$.

SOLUTION

For the particle projected upwards with an initial velocity u, the greatest height attained is

$$H = \frac{u^2}{2g}$$

Let t be the time when the particle is at a height

$$h = \frac{3H}{4} = \frac{3}{4}\left(\frac{u^2}{2g}\right)$$

Using the formula $s = ut + \frac{1}{2}gt^2$, we get

$$\frac{3}{4}\left(\frac{u^2}{2g}\right) = ut + \frac{1}{2}gt^2$$

$\Rightarrow \quad t^2 - \frac{2u}{g}t + \frac{6u^2}{8g^2} = 0$

Solving for t, we have

$$t = \frac{\frac{2u}{g} \pm \sqrt{\left(\frac{4u^2}{g^2} - \frac{3u^2}{g^2}\right)}}{2} = \frac{u}{g} \pm \frac{u}{2g}$$

Taking negative sign, $t_1 = \frac{u}{2g}$

Taking only positive sign, $t_2 = \frac{3u}{2g}$

$$\Rightarrow \quad \frac{t_1}{t_2} = \frac{\left(\frac{u}{2g}\right)}{\left(\frac{3u}{2g}\right)} = 1:3$$

ILLUSTRATION 40

A body projected vertically upwards from the top of a tower reaches the ground in t_1 second. If it is projected vertically downwards from the same point with same speed, it reaches the ground in t_2 second. If it is just dropped from the top, it reaches the ground in t second. Prove that $t = \sqrt{t_1 t_2}$.

SOLUTION

Let h be the height of the tower. Taking downward direction as positive, we get

for the body projected upward, $h = -ut_1 + \frac{1}{2}gt_1^2$ ...(1)

for the body projected downward, $h = ut_2 + \frac{1}{2}gt_2^2$ ...(2)

for the freely falling body, $h = 0 + \frac{1}{2}gt^2 = \frac{1}{2}gt^2$ ...(3)

Multiplying equation (1) by t_2, we get

$$ht_2 = -ut_1t_2 + \frac{1}{2}gt_1^2t_2$$

Multiplying equation (2) by t_1, we get

$$ht_1 = ut_1t_2 + \frac{1}{2}gt_2^2t_1$$

Adding (1) and (2), we get

$$h(t_1 + t_2) = \frac{1}{2}gt_1^2t_2 + \frac{1}{2}gt_2^2t_1 = \frac{1}{2}gt_1t_2(t_1 + t_2)$$

$$\Rightarrow \quad h = \frac{1}{2}gt_1t_2 \qquad ...(4)$$

$$\Rightarrow \quad \frac{1}{2}gt^2 = \frac{1}{2}gt_1t_2 \quad \left\{\because\ h = \frac{1}{2}gt^2 \text{ from equation (3)}\right\}$$

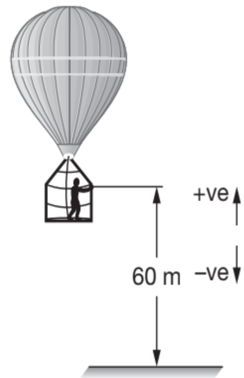
$$\Rightarrow \quad t^2 = t_1t_2$$

$$\Rightarrow \quad t = \sqrt{t_1t_2}$$

ILLUSTRATION 41

A stone is dropped from a balloon going up with a uniform velocity of $5\ \text{ms}^{-1}$. If the balloon was $60\ \text{m}$ high when the stone was dropped, find the height of balloon when the stone hits the ground. Take $g = 10\ \text{ms}^{-2}$.

SOLUTION



$$s = ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad -60 = 5(t) + \frac{1}{2}(-10)t^2$$

$$\Rightarrow \quad -60 = 5t - 5t^2$$

$$\Rightarrow \quad 5t^2 - 5t - 60 = 0$$

$$\Rightarrow \quad t^2 - t - 12 = 0$$

$$\Rightarrow \quad t^2 - 4t + 3t - 12 = 0$$

$$\Rightarrow \quad (t-4)(t+3) = 0$$

$$\Rightarrow \quad t = 4\ \text{s}$$

Height of balloon from ground at this instant is

$$h = 60 + (4)(5)$$

$$\Rightarrow \quad h = 80\ \text{m}$$

Conceptual Note(s)

As the particle is detached from the balloon it is having the same velocity as that of balloon, but it is moving under the influence of gravity, so $a = -g$ (Because, g is acting downwards and we have taken upward direction as positive).

ILLUSTRATION 42

A particle is projected vertically upwards from a point A on the ground. It takes t_1 second to reach a point B at a height h from A but still continues to move up. If it takes further t_2 second from B to ground again, then show that

(a) $h = \frac{1}{2}gt_1t_2$

(b) maximum height reached is $\frac{g(t_1 + t_2)^2}{8}$ and

(c) the velocity of the particle at a height $\frac{h}{2}$ is $\frac{g}{2}\left(t_1^2 + t_2^2\right)^{1/2}$.

SOLUTION

(a) Let the particle be projected upwards with a velocity u. Suppose t_1 and t_2 be the times taken for the motion from A to B and for the motion from B to C and then from C to A respectively.

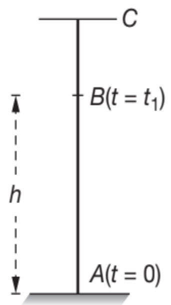
$$\therefore \text{ Time of flight } = (t_1 + t_2) = \left(\frac{2u}{g}\right)$$

$$\Rightarrow \quad u = \frac{g}{2}(t_1 + t_2)$$

For the motion AB,

$$h = ut_1 - \frac{1}{2}gt_1^2$$

$$\Rightarrow \quad h = \frac{g}{2}(t_1 + t_2)t_1 - \frac{1}{2}gt_1^2 = \frac{1}{2}gt_1t_2$$



(b) Maximum height reached

$$AC = H = \frac{u^2}{2g} = \frac{1}{2g}\left[\frac{g}{2}(t_1 + t_2)\right]^2$$

$$\Rightarrow \quad AC = \frac{g}{8}(t_1 + t_2)^2$$

(c) Let v be the velocity at a height $\frac{h}{2}$, then

$$v^2 = u^2 - 2g\left(\frac{h}{2}\right) = u^2 - gh$$

$$\Rightarrow \quad v^2 = \frac{g^2}{4}(t_1 + t_2)^2 - g \times \frac{1}{2}gt_1t_2$$

$$\Rightarrow \quad v^2 = \frac{g^2}{4}\left[(t_1 + t_2)^2 - 2t_1t_2\right]$$

$$\Rightarrow \quad v^2 = \frac{g^2}{4}\left(t_1^2 + t_2^2\right)$$

$$\Rightarrow \quad v = \frac{g}{2}\sqrt{t_1^2 + t_2^2}$$

ILLUSTRATION 43

A ball is dropped from a height of 80 m on a floor. At each collision with the floor, the ball loses one-tenth of its speed. Plot the speed-time graph of its motion between $t = 0$ to 11.2 s $\left(\text{Take } g = 10 \text{ ms}^{-2}\right)$.

SOLUTION

Just Before First Collision

Time taken by the ball to fall through a height of 80 m is obtained as follows:

$$u = 0$$

$$h = \frac{1}{2}gt^2$$

$$80 = \frac{1}{2} \times 10 \times t^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2 \times 80}{10}} = 4 \text{ s}$$

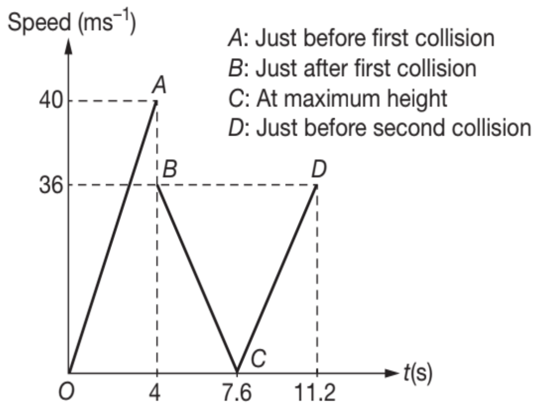
Now, $v(t) = gt$

$$\Rightarrow \quad v(4) = 10 \times 4 = 40 \text{ ms}^{-1}$$

From time $t = 0$ to $t = 4$ s, $v(t) = gt = 10t$

$$\Rightarrow \quad v(t) \propto t$$

In this duration speed increases linearly with time t from 0 to 40 ms^{-1} during the downward motion of the ball and this speed-time variation has been shown by straight line OA in figure.



Just After First Collision

At first collision with the floor

$$\text{Speed lost by ball} = \frac{1}{10}\times 40 = 4\ \text{ms}^{-1}$$

Thus, the ball rebounds with a speed of $40-4=36\ \text{ms}^{-1}$. For the further upward motion, the speed at any instant t is given by

$$v(t)=v(0)-gt=36-10\times t$$

Now, the speed decreases linearly with time and becomes zero after time

$$t=\frac{36}{10}=3.6\ \text{s}$$

Thus, the ball reaches the highest point again after time $t=4+3.6=7.6$ s from the start. Straight line BC represents the speed-time graph for this upward motion.

Just Before Second Collision

At highest point, speed of ball is zero. It again starts falling. At any instant t its speed is given by

$$v(t)=10t$$

Again, the speed of the ball increases linearly with time t from 0 to $36\ \text{ms}^{-1}$ (initial speed of the previous upward motion) in the next time-interval of 3.6 s. Total time taken from the start is $t=4+3.6+3.6=11.2$ s. This part of motion has been shown by straight line CD.

ILLUSTRATION 44

A ball is thrown upward with an initial velocity of $100\ \text{ms}^{-1}$.

(a) Calculate the time taken by the ball to return to the point of launch.

(b) Draw velocity-time graph for the ball and find from the graph
(i) the maximum height attained by the ball and
(ii) height of the ball after 15 s.

Take $g=10\ \text{ms}^{-2}$.

SOLUTION

(a) Taking upward direction as positive, we get

$$u=100\ \text{ms}^{-1},\ g=-10\ \text{ms}^{-2}$$

At highest point i.e., the point of reversal of velocity, $v=0$

Since $v=u+gt$

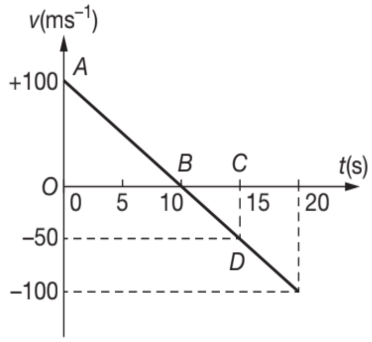
$$\Rightarrow\quad 0=100-10\times t$$

So, time taken to reach highest point, is

$$t=\frac{100}{10}=10\ \text{s}$$

The ball will return to the ground at $T=2t=20$ s.

(b) Corresponding velocity-time graph of the ball is shown in figure



(i) Maximum height (H) attained by the ball is equal to the Area of ΔAOB

$$\Rightarrow\quad H=\frac{1}{2}\times 10\times 100=500\ \text{m}$$

(ii) Height attained after 15 s is

$$h=\text{Area of }\Delta AOB+\text{Area of }\Delta BCD$$

$$\Rightarrow\quad h=500+\frac{1}{2}(15-10)\times(-50)$$

$$\Rightarrow\quad h=500-125=375\ \text{m}$$

ILLUSTRATION 45

A stone is dropped from the top of a cliff of height h. n second later, a second stone is projected downwards from the same cliff with a vertically downward velocity u. Show that the two stones will reach the bottom of the cliff together, if $8h(u-gn)^2=gn^2(2u-gn)^2$. What can you say about the limiting value of n.

SOLUTION

The time taken by the first stone to reach the bottom of the cliff is $\sqrt{\dfrac{2h}{g}}$

According to the given problem, the second stone is projected n second later. The two stones will reach the bottom together, if the second covers the same distance in time $t=\sqrt{\dfrac{2h}{g}}-n$

Since, $h = ut + \frac{1}{2}gt^2$

Hence, $h = u\left(\sqrt{\frac{2h}{g}} - n\right) + \frac{1}{2}g\left(\sqrt{\frac{2h}{g}} - n\right)^2$

$\Rightarrow \quad h = u\sqrt{\frac{2h}{g}} - un + \frac{1}{2}g\left(\frac{2h}{g} - 2n\sqrt{\frac{2h}{g}} + n^2\right)$

$\Rightarrow \quad h = (u - gn)\sqrt{\frac{2h}{g}} + \left(h - un + \frac{n^2 g}{2}\right)$

$\Rightarrow \quad 2h = (u - gn)\sqrt{\frac{2h}{g}} + \frac{n^2 g}{2} - un$

Solving it, we get

$$h = \frac{gn^2}{8}\left(\frac{(2u - gn)}{(u - gn)}\right)^2$$

$\Rightarrow \quad 8h(u - gn)^2 = gn^2(2u - gn)^2$

Hence, the two stones will reach the bottom together, if the above condition is satisfied. Also, since

$t = \sqrt{\frac{2h}{g}} - n$, so we must have $t \geq 0$

$\Rightarrow \quad \sqrt{\frac{2h}{g}} - n \geq 0$

$\Rightarrow \quad n \leq \sqrt{\frac{2h}{g}}$

So the limiting value of n is $n_{MAX} = \sqrt{\frac{2h}{g}}$

ILLUSTRATION 46

Two bodies are thrown vertically upward, with the same initial velocity of 98 ms^{-1} but 4 s apart. How long after the first one is thrown will they meet?

SOLUTION

Let y_{max} be the maximum height at which the velocity of first body reduces to zero.

Using the equation $v^2 = u^2 + 2gs$, we have

$$0 = (98)^2 - 2 \times (9.8) \times y_{max}$$

$\Rightarrow \quad y_{max} = \frac{(98)^2}{2 \times (9.8)} = 490 \text{ m}$

If t be the time taken by the body to reach the height 980 m, then from the equation $v = u + gt$, we have

$$0 = 98 - (9.8)t$$

$\Rightarrow \quad t = 10 \text{ s}$

Now the second body which was thrown after 4 sec has been moving upward for 6 sec. The velocity v acquired by this body is given by

$$v = 98 - (9.8) \times 6 = 39.2 \text{ ms}^{-1}$$

The height y reached by the body is

$$y = (98 \times 6) - \frac{1}{2}(9.8) \times (6)^2 \quad \left\{\because s = ut + \frac{1}{2}gt^2\right\}$$

$\Rightarrow \quad y = 588 - 176.4 = 411.6 \text{ m}$

At this moment, the distance between the first and second body is

$$\Delta y = 490 - 411.6 = 78.4 \text{ m}$$

Now the two bodies will meet during the downward journey of the first and the upward journey of the second. Let the two bodies meet after a time t measured from above moment. The first body is coming down (initial velocity zero) and let it covers a distance y in t second. The second body is moving upward and covers a distance $(78.4 - y)$ in t second. Using $y = ut + \frac{1}{2}gt^2$ for two bodies, we have

$$y = \frac{1}{2} \times 9.8 \times t^2$$

and $(78.4 - y) = (39.2)t - \frac{1}{2} \times 9.8 \times t^2$

Solving the two equations, we get

$$t = 2 \text{ s}$$

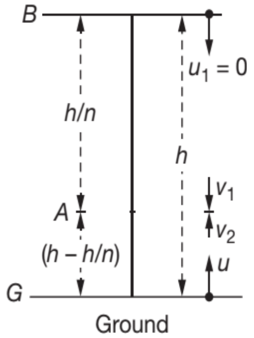
Hence the two bodies shall meet $10 + 2 = 12 \text{ s}$ after the first body is thrown.

ILLUSTRATION 47

A particle is dropped from the top of a tower h m high and at the same moment another particle is projected upwards from the bottom. They meet when the upper one has descended a distance $\frac{h}{n}$. Show that the velocities of the two when they meet are in the ratio $2:(n-2)$ and that the initial velocity of the particle projected up is $\sqrt{\frac{ngh}{2}}$.

SOLUTION

The situation is shown in figure.



Let the two particles meet at A after a time t

For first particle,

$$\frac{h}{n} = \frac{1}{2}gt^2 \qquad \ldots(1)$$

For second particle,

$$\left(h - \frac{h}{n}\right) = ut - \frac{1}{2}gt^2 \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$h = ut$$

$$\Rightarrow \quad t = \frac{h}{u} \qquad \ldots(3)$$

Substituting the value of t from equation (3) in equation (1), we get

$$\frac{h}{n} = \frac{1}{2}g\left(\frac{h}{u}\right)^2$$

$$\Rightarrow \quad u^2 = \frac{1}{2}ngh$$

$$\Rightarrow \quad u = \sqrt{\frac{ngh}{2}} \qquad \ldots(4)$$

Velocity v_1 of first particle at A is given by

$$v_1^2 = u_1^2 + 2g\left(\frac{h}{n}\right) = 0 + 2g\left(\frac{h}{n}\right) \qquad \ldots(5)$$

Velocity v_2 of second particle at A is given by

$$v_2^2 = u^2 - 2g\left(h - \frac{h}{n}\right)$$

$$\Rightarrow \quad v_2^2 = \frac{1}{2}ngh - 2gh\left(1 - \frac{1}{n}\right)$$

$$\Rightarrow \quad v_2^2 = gh\left[\frac{n}{2} - 2\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow \quad v_2^2 = gh\left(\frac{n}{2} - 2 + \frac{2}{n}\right) = gh\left(\frac{n^2 - 4n + 4}{2n}\right)$$

$$\Rightarrow \quad v_2^2 = gh\frac{(n-2)^2}{2n} \qquad \ldots(6)$$

Dividing equation (5) by equation (6), we get

$$\frac{v_1^2}{v_2^2} = \frac{\frac{2gh}{n}}{gh\frac{(n-2)^2}{2n}} = \frac{4gh}{gh(n-2)^2}$$

$$\Rightarrow \quad \frac{v_1}{v_2} = \frac{2}{n-2}$$

ILLUSTRATION 48

A parachutist after bailing out falls 50 m, without friction. When the parachute opens, he decelerates downwards with 2 ms^{-2}. He reaches the ground with a speed of 3 ms^{-1}.

(a) How long is the parachutist in the air?

(b) At what height did he bail out?

SOLUTION

For fall without friction, i.e., free fall $h = 50\text{ m}$, $u = 0$, $g = 9.8\text{ ms}^{-2}$ and $t = t_1$

Using $h = ut + \frac{1}{2}gt^2$, we have

$$50 = 0 + \frac{1}{2} \times 9.8 \times t_1^2$$

$$\Rightarrow \quad t_1 = \sqrt{\frac{50 \times 2}{9.8}} = 3.16\text{ s}$$

The velocity after falling through 50 m may be obtained by using the formula

$$v^2 = u^2 + 2gh$$

$$\Rightarrow \quad v^2 = 0 + 2 \times 9.8 \times 50$$

$$\Rightarrow \quad v = \sqrt{(2 \times 9.8 \times 50)}$$

$$\Rightarrow \quad v = 31.3\text{ ms}^{-1}$$

Taking downward direction as positive, the initial velocity is 31.3 ms^{-1} and final velocity is 3 ms^{-1}.

Acceleration $a = -2\ \text{ms}^{-2}$

Let time be t_2. Now using $v = u + at$, we have

$$3 = 31.3 - 2t_2$$

$$\Rightarrow\quad t_2 = \frac{28.3}{2} = 14.15\ \text{s}$$

If distance travelled be h, then $h = ut$ where $u = \frac{31.3+3}{2} = \frac{34.3}{2} = 17.15\ \text{ms}^{-1}$ and $t = 14.15$ s

$$\Rightarrow\quad h = 17.15 \times 14.15 = 242.7\ \text{m}$$

Now, total time $= t = 3.19 + 14.15 = 17.34$ s

and total height $= h = 50 + 242.7 = 292.7$ m

Test Your Concepts-V

Based on Motion Under Gravity

(Solutions on page H.81)

1. Can you think of examples where velocity of a particle is
   (a) in opposite direction to the acceleration
   (b) zero but its acceleration is not zero
   (c) perpendicular to the acceleration

2. As a body is projected to a high altitude above the earth's surface, the variation of the acceleration of gravity with respect to altitude y must be taken into account. Neglecting air resistance, this acceleration is determined from the formula $a = -g_0\left[\frac{R^2}{(R+y)^2}\right]$, where g_0 is the constant gravitational acceleration at sea level, R is the radius of the earth and the positive direction is measured upward. If $g_0 = 9.81\ \text{ms}^{-2}$ and $R = 6356$ km, determine the minimum initial velocity at which a projectile should be shot vertically from the earth's surface so that it does not return to the earth.

3. A particle is projected vertically upwards and t second after another particle is projected upwards with the same initial velocity. Prove that the particles will meet after a lapse of $\left(\frac{t}{2}+\frac{u}{g}\right)$ second from the instant of projection of the first particle. What are the velocities of the particles when they meet?

4. A ball is projected vertically upwards with a velocity of $100\ \text{ms}^{-1}$. Find the speed of the ball at half the maximum height. Take $g = 10\ \text{ms}^{-2}$.

5. A particle is projected vertically upwards from the ground at time $t = 0$ and reaches a height h at $t = T$. Show that the greatest height of the particle is $\frac{(gT^2+2h)^2}{8gT^2}$.

6. A rocket is fired vertically upwards with a net acceleration of $4\ \text{ms}^{-2}$ and initial velocity zero. At $t = 5$ s its fuel is finished and it decelerates with g. At the highest point its velocity becomes zero. Then it accelerates downwards with acceleration g and returns to ground. Plot velocity-time and displacement-time graphs for the complete journey. Take $g = 10\ \text{ms}^{-2}$.

7. A stone is dropped from the top of a tall cliff and n second later another stone is thrown vertically downwards with a velocity $u\ \text{ms}^{-1}$. How far below the top of the cliff will the second stone overtake the first?

8. A body of mass m is thrown straight up with a velocity u_0. Find the velocity u' with which the body comes down if the air drag equals cu^2 where c is a constant and u is the velocity of the body.

9. To test the quality of a tennis ball it is dropped onto the floor from a height of 4 m and it rebounds to a height of 2 m. If the ball is in contact with the floor for a duration of 12 ms. Calculate the average acceleration during that contact? Take $g = 9.8\ \text{ms}^{-2}$.

10. A rocket is fired vertically from rest and ascends with constant net vertical acceleration of $300\ \text{msec}^{-2}$ for 1 minute. Its fuel is then all used up and it continues as a free particle in the gravitational field of the earth. Find
   (a) maximum height reached
   (b) the total time elapsed from take-off until the rocket strikes the earth.

11. A football is kicked vertically upward from the ground and a student gazing out of the window sees it moving upwards past her at $5\ \text{ms}^{-1}$. The window is 15 m above the ground. Air resistance may be ignored.

(a) How high does the football go above ground?

(b) How much time does it take to go from the ground to its highest point?

Take $g = 10\ \text{ms}^{-2}$?

12. Two balls of same mass are shot upward one after another at an interval of 2 second along the same vertical line with the same initial velocity of $39.2\ \text{msec}^{-1}$. Find the height at which they collide. Take $g = 9.8\ \text{msec}^{-2}$.

13. Two bodies are projected vertically upwards from one point with the same initial velocity v_0. The second body is projected t_0 second after the first. After how much time will the bodies meet?

14. A stone is dropped from the top of a tower. When it crosses a point 5 m below the top, another stone is dropped from a point 25 m below the top. Both stones reach the bottom of the tower simultaneously. Find the height of the tower. Take $g = 10\ \text{ms}^{-2}$.

15. An elevator without a ceiling is ascending with a constant speed of $10\ \text{ms}^{-1}$. A boy on the elevator throws a ball directly upward, from a height of 2 m above the elevator floor, just as the elevator floor is 28 m above the ground. The initial speed of the ball with respect to the elevator is $20\ \text{ms}^{-1}$.

(a) What maximum height above the ground does the ball reach?

(b) How long does the ball take to return to the elevator floor?

Take $g = 10\ \text{ms}^{-2}$

16. A ball is projected vertically upwards with a velocity of $24.5\ \text{msec}^{-1}$ from the bottom of a tower. A boy, who is standing at the top of the tower is unable to catch the ball when it passes him in the upward direction. But the ball again reaches him after 3 second when it is falling and then he catches it. Find what was the velocity of the ball when the boy caught it and find also the height of the tower.

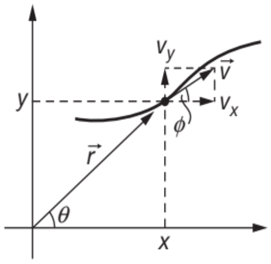
17. Two particles begin a free fall from rest from the same height, 1 s apart. How long after the first particle begins to fall will the two particles be 10 m apart?

Take $g = 10\ \text{ms}^{-2}$.

MOTION IN A PLANE AND RELATIVE VELOCITY

MOTION IN A PLANE: AN INTRODUCTION

Any type of planar motion can be resolved into two rectangular rectilinear motion i.e., two mutually perpendicular independent motions resolved along x and y axis (since x and y components do not have any dependence in each other). Consider a particle to be moving along a curve C in x-y plane. Let it be at a point $P(x, y)$ which has a position vector $\vec{r}$ at any particular instant of time t. Let this position vector make an angle θ with position x-axis as shown in Figure.



From Figure, we get

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Also, $r = \sqrt{r^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$

Let $\vec{v}$ be the velocity of the particle at point P and let ϕ be the angle made by $\vec{v}$ with x-axis. So, again we get

$$v_x = v\cos\phi$$

$$v_y = v\sin\phi$$

$$v = \sqrt{v_x^2 + v_y^2} \text{ and } \tan\phi = \frac{v_y}{v_x}$$

A similar and identical treatment can also be done for the acceleration components a_x and a_y. Further, the equations of motion in vector form are

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\vec{r} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$v^2 - u^2 = 2\vec{a}.\vec{r}$$

$$\vec{r} = \left(\frac{\vec{u} + \vec{v}}{2}\right)t$$

x-component of equations of motion	y-components of equations of motion
$v_x = u_x + a_x t$	$v_y = u_y + a_y t$
$x = u_x t + \frac{1}{2}a_x t^2$	$y = u_y t + \frac{1}{2}a_y t^2$
$v_x^2 - u_x^2 = 2a_x x$	$v_y^2 - u_y^2 = 2a_y y$
$x = \left(\frac{u_x + v_x}{2}\right)t$	$y = \left(\frac{u_y + v_y}{2}\right)t$

The most significant thing about these types of motion is that they are independent of each other.

ILLUSTRATION 49

Calculate the displacement, when a particle is displaced

(a) 5 m due North and then 12 m due East.

(b) from point 1 to 2 having position vector $\vec{r}_1 = \left(2\hat{i} - 3\hat{j} + 4\hat{k}\right)$ m and $\vec{r}_2 = \left(6\hat{i} - 6\hat{j} + 16\hat{k}\right)$ m.

Also find the magnitude in both the cases.

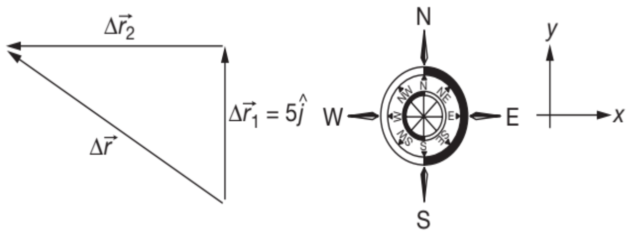
SOLUTION

(a) $\Delta\vec{r}_1 = 5\hat{j}$

$\Delta\vec{r}_2 = 12\left(-\hat{i}\right)$

$\Rightarrow \quad \Delta\vec{r} = \Delta\vec{r}_1 + \Delta\vec{r}_2 = \left(5\hat{j} - 12\hat{i}\right)$ m

$\Rightarrow \quad |\Delta\vec{r}| = \sqrt{25 + 144} = 13$ m



(b) $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{r}_f - \vec{r}_i$

$\Rightarrow \quad \Delta\vec{r} = \left(6\hat{i} - 6\hat{j} + 16\hat{k}\right) - \left(2\hat{i} - 3\hat{j} + 4\hat{k}\right)$

$\Rightarrow \quad \Delta\vec{r} = \left(4\hat{i} - 3\hat{j} + 12\hat{k}\right)$ m

$$\Rightarrow \quad |\Delta\vec{r}| = \sqrt{(4)^2+(3)^2+(12)^2} = 13 \text{ m}$$

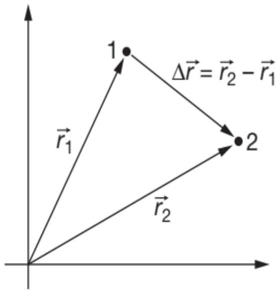


ILLUSTRATION 50

A particle travels along the parabolic path given by $y = bx^2$. If its component of velocity along the y-axis is $v_y = ct^2$, determine the x and y components of the particle's acceleration. Assume b and c to be positive constants.

SOLUTION

Since $v_y = ct^2$

$$\Rightarrow \quad \frac{dy}{dt} = ct^2$$

$$\Rightarrow \quad dy = ct^2 dt$$

$$\Rightarrow \quad \int_0^y dy = \int_0^t ct^2 dt$$

$$\Rightarrow \quad y = \frac{c}{3}t^3 \qquad \ldots(1)$$

Substituting y from (1) in $y = bx^2$, we get

$$\frac{c}{3}t^3 = bx^2$$

$$\Rightarrow \quad x = \sqrt{\frac{c}{3b}}t^{3/2}$$

Thus, the x component of the particle's velocity can be determined by taking the time derivative of x

$$\Rightarrow \quad v_x = \dot{x} = \frac{dx}{dt} = \frac{d}{dt}\left[\sqrt{\frac{c}{3b}}t^{3/2}\right] = \frac{3}{2}\sqrt{\frac{c}{3b}}t^{1/2}$$

Now again to calculate x and y components of acceleration, we use

$$a_x = \frac{dv_x}{dt} = \dot{v}_x \text{ and } a_y = \frac{dv_y}{dt} = \dot{v}_y$$

$$\Rightarrow \quad a_x = \dot{v}_x = \frac{d}{dt}\left(\frac{3}{2}\sqrt{\frac{c}{3b}}t^{1/2}\right) = \frac{3}{4}\sqrt{\frac{c}{3b}}t^{-1/2} = \frac{3}{4}\sqrt{\frac{c}{3b}}\frac{1}{\sqrt{t}}$$

$$\Rightarrow \quad a_y = \dot{v}_y = \frac{d}{dt}(ct^2) = 2ct$$

ILLUSTRATION 51

The position of a particle is $\vec{r} = (3t^3 - 2t)\hat{i} - (4\sqrt{t} + t)\hat{j} + (3t^2 - 2)\hat{k}$ m, where t is in seconds. Determine the magnitude of the particle's velocity and acceleration when $t = 2$ s.

SOLUTION

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(3t^3 - 2t)\hat{i} - (4\sqrt{t} + t)\hat{j} + (3t^2 - 2)\hat{k}\right]$$

$$\Rightarrow \quad \vec{v} = \left[(9t^2 - 2)\hat{i} - \left(\frac{2}{\sqrt{t}} + 1\right)\hat{j} + (6t)\hat{k}\right] \text{ ms}^{-1}$$

$$\Rightarrow \quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[(9t^2 - 2)\hat{i} - \left(\frac{2}{\sqrt{t}} + 1\right)\hat{j} + (6t)\hat{k}\right] \text{ ms}^{-1}$$

$$\Rightarrow \quad \vec{a} = \left[(18t)\hat{i} + t^{-3/2}\hat{j} + 6\hat{k}\right] \text{ ms}^{-2}$$

When $t = 2$ s,

$$\vec{v} = \left((9(2^2) - 2)\hat{i} - (2(2^{-1/2}) + 1)\hat{j} + 6(2)\hat{k}\right) \text{ ms}^{-1}$$

$$\Rightarrow \quad \vec{v} = \left(34\hat{i} - 2.414\hat{j} + 12\hat{k}\right) \text{ ms}^{-1}$$

So, the magnitude of the particle's velocity is

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{34^2 + (-2.414)^2 + 12^2}$$

$$\Rightarrow \quad |\vec{v}| = 36.1 \text{ ms}^{-1}$$

Acceleration

When $t = 2$ s,

$$\vec{a} = \left[18(2)\hat{i} + 2^{-3/2}\hat{j} + 6\hat{k}\right] \text{ ms}^{-2}$$

$$\Rightarrow \quad \vec{a} = \left[36\hat{i} + 0.3536\hat{j} + 6\hat{k}\right] \text{ ms}^{-2}$$

So, the magnitude of the particle's acceleration is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{36^2 + (0.3536)^2 + 6^2} = 36.5 \text{ ms}^{-2}$$

ILLUSTRATION 52

The velocity of a particle is $\vec{v} = \left[3\hat{i} + (6 - 2t)\hat{j}\right]$ ms^{-1}, where t is in seconds. If $\vec{r} = \vec{0}$ when $t = 0$, determine the displacement of the particle during the time interval $t = 1$ s to $t = 3$ s.

SOLUTION

The position $\vec{r}$ of the particle can be determined by integrating the kinematic equation $\vec{v}=\frac{d\vec{r}}{dt}$ (i.e., $d\vec{r}=\vec{v}dt$) and then using the initial condition i.e., at $t=0$, $\vec{r}=\vec{0}$ for solving the integral.

$$d\vec{r}=\vec{v}dt$$

$$\Rightarrow \int_0^r d\vec{r}=\int_0^t\left[3\hat{i}+(6-2t)\hat{j}\right]dt$$

$$\Rightarrow \vec{r}=\left[3t\hat{i}+\left(6t-t^2\right)\hat{j}\right]\text{m}$$

When $t=1$ s and 3 s,

$$\vec{r}\,|_{t=1\text{ s}}=3(1)\hat{i}+\left[6(1)-1^2\right]\hat{j}=\left(3\hat{i}+5\hat{j}\right)\text{ ms}^{-1}$$

$$\vec{r}\,|_{t=3\text{ s}}=3(3)\hat{i}+\left[6(3)-3^2\right]\hat{j}=\left(9\hat{i}+9\hat{j}\right)\text{ ms}^{-1}$$

Thus, the displacement of the particle is

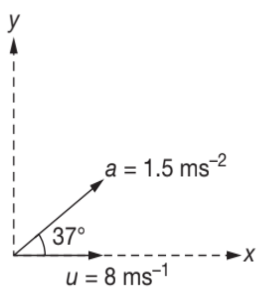
$$\Delta\vec{r}=\vec{r}\,|_{t=3\text{ s}}-\vec{r}\,|_{t=1\text{ s}}$$

$$\Rightarrow \Delta\vec{r}=\left(9\hat{i}+9\hat{j}\right)-\left(3\hat{i}+5\hat{j}\right)$$

$$\Rightarrow \Delta\vec{r}=\left(6\hat{i}+4\hat{j}\right)\text{ m}$$

ILLUSTRATION 53

A particle is moving in the x-y plane with constant acceleration of 1.5 ms^{-2} that makes an angle of 37° with the x-axis. At $t=0$ the particle is at the origin and its velocity is 8 ms^{-1} along the x-axis. Find the velocity and the position of the particle at $t=4$ s. Given $\sin(37°)=0.6$



SOLUTION

Let us first calculate the components of constant acceleration. If a_x and a_y be the respective acceleration components along x and y-axis then

$$a_x=\left(1.5\text{ ms}^{-2}\right)(\cos 37°)=\left(1.5\text{ ms}^{-2}\right)\times\frac{4}{5}=1.2\text{ ms}^{-2}$$

and $a_y=\left(1.5\text{ ms}^{-2}\right)\times\frac{3}{5}=0.9\text{ ms}^{-2}$

At $t=0$, $u_x=8\text{ ms}^{-1}$, $u_y=0$

$a_x=1.2\text{ ms}^{-2}$, $a_y=0.9\text{ ms}^{-2}$

$x=0$ and $y=0$

The x-component of the velocity at time $t=4$ s is given by

$$v_x=u_x+a_xt$$

$$\Rightarrow v_x=8\text{ ms}^{-1}+\left(1.2\text{ ms}^{-2}\right)(4\text{ s})$$

$$\Rightarrow v_x=8\text{ ms}^{-1}+4.8\text{ ms}^{-1}=12.8\text{ ms}^{-1}$$

The y-component of velocity at $t=4$ s is given by

$$v_y=u_y+a_yt$$

$$\Rightarrow v_y=0+\left(0.9\text{ ms}^{-2}\right)(4\text{ s})=3.6\text{ ms}^{-1}$$

The velocity of the particle at $t=4$ s is

$$v=\sqrt{v_x^2+v_y^2}=\sqrt{\left(12.8\text{ ms}^{-1}\right)^2+\left(3.6\text{ ms}^{-1}\right)^2}$$

$$v=13.3\text{ ms}^{-1}$$

Assume that the velocity makes an angle θ with the x-axis, then

$$\tan\theta=\frac{v_y}{v_x}=\frac{3.6\text{ ms}^{-1}}{12.8\text{ ms}^{-1}}=\frac{9}{32}$$

$$\Rightarrow \theta=\tan^{-1}\left(\frac{9}{32}\right)$$

The x-coordinate at $t=4$ s is given by

$$x=u_xt+\frac{1}{2}a_xt^2$$

$$\Rightarrow x=(8)(4)+\frac{1}{2}(1.2)(4)^2$$

$$\Rightarrow x=32\text{ m}+9.6\text{ m}=41.6\text{ m}$$

The y-coordinate at $t=4$ s is given by

$$y=u_yt+\frac{1}{2}a_yt^2$$

$$\Rightarrow y=\frac{1}{2}(0.9)(4)^2 \qquad \{u_y=0\}$$

$$\Rightarrow y=7.2\text{ m}$$

Thus, the coordinates of the particle at 4 s are (41.6 m, 7.2 m).

ILLUSTRATION 54

A particle is moving in a plane with velocity given by $\vec{v} = u_0\hat{i} + [a\omega\cos(\omega t)]\hat{j}$. If the particle is at the origin at $t = 0$,

(a) calculate the trajectory of the particle

(b) find its distance from the origin at time $\left(\dfrac{3\pi}{2\omega}\right)$.

SOLUTION

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$\Rightarrow v_x = \frac{dx}{dt} = u_0$$

$$\Rightarrow dx = u_0 dt$$

$$\Rightarrow \int_0^x dx = u_0\int_0^t dt$$

$$\Rightarrow x = u_0 t \qquad \ldots(1)$$

$$v_y = \frac{dy}{dt} = a\omega\cos(\omega t)$$

$$\Rightarrow \int_0^y dy = a\omega\int_0^t \cos(\omega t)\,dt$$

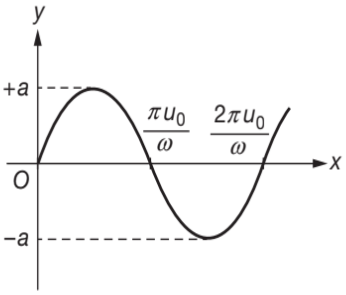
$$\Rightarrow y = a\omega\left(\frac{\sin(\omega t)}{\omega}\Bigg|_0^t\right)$$

$$\Rightarrow y = a\sin(\omega t) \qquad \ldots(2)$$

(a) From (1), $t = \dfrac{x}{u_0}$. Substituting in (2), we get

$$y = a\sin\left(\frac{\omega x}{u_0}\right) \qquad \{\text{Equation of Trajectory}\}$$

This is the desired trajectory and it is a sine curve as shown in figure.



(b) For $t = \left(\dfrac{3\pi}{2\omega}\right)$ from equations (1) and (2), we get

$$x = u_0\left(\frac{3\pi}{2\omega}\right) \text{ and } y = -a$$

So, distance from the origin at $t = \left(\dfrac{3\pi}{2\omega}\right)$ will be

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{3\pi u_0}{2\omega}\right)^2 + a^2}$$

$$\Rightarrow \quad r = \sqrt{a^2 + \frac{9\pi^2 u_0^2}{4\omega^2}}$$

ILLUSTRATION 55

A particle of mass 2 kg has a velocity of $2\ \text{ms}^{-1}$. A constant force of 4 N acts on the particle for 1 s in a direction perpendicular to its initial velocity. Find the velocity and displacement of the particle at $t = 1$ s.

SOLUTION

Let the velocity of the particle be along x-axis. Then

$$\vec{u} = 2\hat{i}$$

So, force on the particle is

$$\vec{F} = 4\hat{j}$$

$$\Rightarrow \quad \vec{a} = \frac{\vec{F}}{m} = 2\hat{j}$$

Since $\vec{a}$ = constant, so we can use

$$\vec{v} = \vec{u} + \vec{a}t \text{ and } \vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\Rightarrow \quad \vec{v} = 2\hat{i} + (2t)\hat{j}$$

$$\Rightarrow \quad \vec{v}\big|_{t=1\text{ s}} = 2\hat{i} + 2\hat{j}$$

So, the situation is shown in Figure (a).

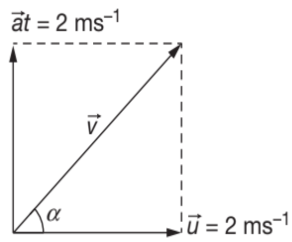


Figure (a)

If $\vec{v}$ makes an angle α with x-axis, then

$$\alpha = \tan^{-1}\frac{|\vec{a}t|}{|\vec{u}|} = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = 45°$$

Thus, velocity of the particle at the end of 1 s is $2\sqrt{2}$ ms^{-1} at an angle of 45° with its initial velocity. The situation is again shown in Figure (b).

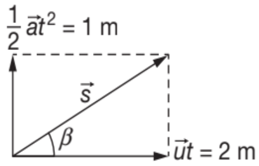


Figure (b)

Since $\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$

$$\Rightarrow \quad \vec{s} = (2t)\hat{i} + \frac{1}{2}(2)t^2\hat{j}$$

$$\Rightarrow \quad \vec{s}\big|_{t=1\text{ s}} = 2\hat{i} + \hat{j}$$

$$\Rightarrow \quad |\vec{s}| = \sqrt{5}\text{ m}$$

If β is the angle which $\vec{s}$ makes with x-axis, then

$$\beta = \tan^{-1}\left(\frac{\left|\frac{1}{2}\vec{a}t^2\right|}{|\vec{u}t|}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

So, displacement of the particle at the end of 1 s is $\sqrt{5}$ m at an angle of $\tan^{-1}\left(\frac{1}{2}\right)$ from its initial velocity.

ILLUSTRATION 56

A particle leaves the origin with an initial velocity $\vec{v} = (3\hat{i})$ ms^{-1} and a constant acceleration $\vec{a} = (-1\hat{i} - 0.5\hat{j})$ ms^{-2}. When the particle reaches its maximum x coordinate, what are

(a) its velocity and
(b) its position vector?

SOLUTION

(a) $u_x = 3$ ms^{-1}

$a_x = -1$ ms^{-2} and $a_y = -0.5$ ms^{-2}

Since $x = u_x t + \frac{1}{2}a_x t^2$

For x to be MAXIMUM, we have

$$\frac{dx}{dt} = 0$$

$$\Rightarrow \quad u_x + \frac{1}{2}a_x(2t) = 0$$

$$\Rightarrow \quad u_x + a_x t = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad t = -\frac{u_x}{a_x}$$

$$\Rightarrow \quad t = -\frac{3}{(-1)} = 3\text{ s}$$

Please note that this happens to be the time when the x motion of the particle is reversed i.e., this time is the time when the x component of velocity should be zero and equation (1) shows that. At this instant

$$v_x = 0$$

and $v_y = u_y + a_y t = 0 - 0.5 \times 3 = -1.5$ ms^{-1}

$$\Rightarrow \quad \vec{v} = (-1.5\hat{j})\text{ ms}^{-1}$$

(b) $x = u_x t + \frac{1}{2}a_x t^2 = 3 \times 3 + \frac{1}{2}(-1)(3)^2 = 4.5$ m

$y = u_y t + \frac{1}{2}a_y t^2 = 0 - \frac{1}{2}(0.5)(3)^2 = -2.25$ m

$$\Rightarrow \quad \vec{r} = (4.5\hat{i} - 2.25\hat{j})\text{ m}$$

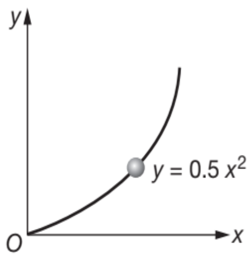
Test Your Concepts-VI

Based on Planar Motion

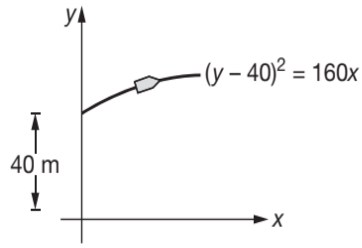
(Solutions on page H.85)

1. A particle starts from the origin at $t = 0$ with a velocity of $8\hat{j}$ ms^{-1} and moves in the xy plane with a constant acceleration of $(4\hat{i} + 2\hat{j})$ ms^{-2}. At the instant the particle's x coordinate is 32 m, what are
(a) its y-coordinate and
(b) its speed?

2. A particle starts from the origin of the coordinates along the path defined by the parabola $y = 0.5x^2$. If the component of velocity along the x-axis is $v_x = (5t)$ ms^{-1}, where t is in second, determine the particle's distance from the origin O and the magnitude of acceleration when $t = 1$ s.



3. A point moves the plane *x*-*y* according to the law $x = k\sin(\omega t)$ and $y = k[1 - \cos(\omega t)]$ where *k* and ω are positive constants. Find the distance traversed by the particle during time *t*.
4. A particle moves in the plane according to the law $x = kt,\ \ y = kt(1 - \alpha t)$, when *k* and α are positive constants, and *t* is the time. Find
   - **(a)** the equation of the particle's trajectory *y*(*x*)
   - **(b)** the velocity *v* and the acceleration *a* of the point as a function of time.
5. The speed of a particle moving in a plane is equal to the magnitude of its instantaneous velocity, $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$. Show that the rate of change of the speed is $\dfrac{dv}{dt} = \dfrac{\vec{a}\cdot\vec{v}}{|\vec{v}|}$.
6. As soon as a rocket reaches an altitude of 40 m it begins to travel along the parabolic path $(y - 40)^2 = 160x$, where the coordinates are measured in meters. Assuming that the component of velocity in the vertical direction is constant at $v_y = 180\ \text{ms}^{-1}$, determine the magnitudes of the rocket's velocity and acceleration when it reaches an altitude of 80 m.



7. Velocity and acceleration of a particle at time $t = 0$ are $\vec{u} = (2\hat{i} + 3\hat{j})\ \text{ms}^{-1}$ and $\vec{a} = (4\hat{i} + 2\hat{j})\ \text{ms}^{-2}$ respectively. Find the velocity and displacement of particle at $t = 2$ s.
8. A balloon is ascending at the rate $v = 12\ \text{kmh}^{-1}$ and is being carried horizontally by the wind blowing at $v_w = 20\ \text{kmh}^{-1}$. If a ballast bag is dropped from the balloon at the instant $h = 50$ m, determine the time required for it to strike the ground. Assume that the bag was released from the balloon with the same velocity as the balloon. Also, with what speed does the bag strike the ground? $\left(\text{Take } g = 10\ \text{ms}^{-2}\right)$.

RELATIVE MOTION

Motion is a combined property of the object under study as well as the observer. It is always relative because there is no such thing like absolute motion or absolute rest. *Motion is always defined with respect to an observer or a reference frame.*

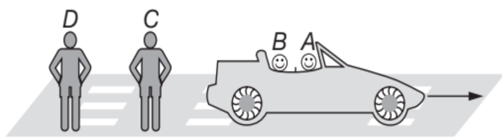
Reference Frame

The motion of a particle is described by using kinematical quantities such as velocity, acceleration. However, these quantities are dependent upon the state of motion as seen by the observer. So, to understand the concept of relative motion it becomes mandatory for us to talk about or introduce the concept of a "reference frame".

Reference frame is an axis system from which motion is observed along with a clock attached to the axis, to measure time. Reference frame can be stationary or moving.

In layman language a Reference Frame is a platform from where the observer observes motions and takes measurements with respect to it.

Suppose there are two persons A and B sitting in a car moving at constant speed. Two stationary persons C and D observe them from the ground.



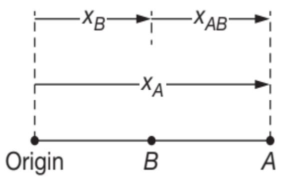
Here *B* appears to be moving for *C* and *D*, but at rest for A. Similarly C appears to be at rest for D but moving backward for A and B.

RELATIVE MOTION IN ONE DIMENSION

Relative Position

It is the position of a particle w.r.t. observer. In general if position of A w.r.t. origin is x_A and that of B w.r.t. origin is x_B, then position of A w.r.t B (i.e. B is now the new origin) is denoted by x_{AB} and is given by

$$x_{AB} = x_A - x_B$$



and v_{AB} = velocity of A w.r.t. $B = \dfrac{dx_{AB}}{dt} = \dfrac{d}{dt}(x_A - x_B)$

$$\Rightarrow \quad v_{AB} = \frac{dx_A}{dt} - \frac{dx_B}{dt} = v_A - v_B$$

So, for particles A and B moving in the same direction

$$v_{AB} = |v_A - v_B|$$

and for particles A and B moving in the opposite direction

$$v_{AB} = v_A + v_B$$

Relative Velocity

Relative velocity of a particle A with respect to B is defined as the velocity with which A appears to move if B is considered to be at rest. In other words, it is the velocity with which A appears to move as seen by B considering itself to be at rest.

Conceptual Note(s)

(a) All velocities are relative and have no significance unless observer is specified. However, when we say velocity of *A*, what we mean is, velocity of *A* w.r.t. ground which is assumed to be at rest.

(b) Velocity of an object w.r.t. itself is always zero.

(c) Velocity of *A* and *B* must always be measured from the same reference.

ILLUSTRATION 57

Two trains, each having a speed of 30 kmhr^{-1} are headed towards each other on the same straight track. A bird that can fly at 60 kmhr^{-1} flies off one train, when they are 60 km apart and heads directly for the other train. On reaching the other train, it flies directly back to the first and so on.

(a) How many trips can the bird make from one train to the other before they crash?

(b) What is the total distance the bird travels?

SOLUTION

The relative velocity of one train relative to the other is 60 kmhr^{-1} and as the distance between the trains is 60 km, the two trains will crash after 1 hr.

(a) Now, the velocity of bird with respect to train towards which it is moving will be $v = 90 \text{ kmhr}^{-1}$.

So, the time taken by bird for first trip is $t_1 = \left(\dfrac{60}{90}\right) = \dfrac{2}{3}$ hr and in this time the trains have moved towards each other $\left(\dfrac{2}{3}\right) \times 60 = 40$ km, so the remaining distance $= 60 - 40 = 20$ km.

So, the time taken by bird for second trip, $t_2 = \dfrac{20}{90} = \left(\dfrac{2}{3^2}\right)$ hr

Proceeding in the same way time taken by the bird for nth trip, $t_n = \left(\dfrac{2}{3^n}\right)$hr

Now, if the bird makes n trips till the train crashes,

$$t_1 + t_2 + t_3 + ... + t_n = 1 \text{ hr}$$

$$\Rightarrow \quad \frac{2}{3} + \frac{2}{3^2} + ... + \frac{2}{3^n} = 1 \text{ hr}$$

$$\Rightarrow \quad \frac{2}{3}\left[1 + \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^{n-1}}\right] = 1 \text{ hr}$$

$$\Rightarrow \quad \frac{2}{3}\left[\frac{1 - \left(\frac{1}{3}\right)^n}{1 - \left(\frac{1}{3}\right)}\right] = 1 \text{ hr}$$

$$\Rightarrow \quad 1 - \left(\frac{1}{3}\right)^n = 1$$

$$\Rightarrow \quad \left(\frac{1}{3}\right)^n = 0$$

$$\Rightarrow \quad n \to \infty$$

So, the bird will make infinite trips.

(b) As the speed of bird is 60 kmhr^{-1} and the two trains crash after 1 hr., so the total distance travelled by the bird is the distance travelled by the bird in 1 hr. i.e.,

$$d = 60\left(\frac{\text{km}}{\text{hr}}\right) \times 1 \text{ hr} = 60 \text{ km}$$

RELATIVE ACCELERATION

It is the rate at which relative velocity is changing

$$a_{AB} = \frac{dv_{AB}}{dt} = \frac{dv_A}{dt} - \frac{dv_B}{dt} = a_A - a_B$$

Equations of motion when relative acceleration a_{rel} is constant are

$$v_{rel} = u_{rel} + a_{rel}t$$

$$s_{rel} = u_{rel}t + \frac{1}{2}a_{rel}t^2$$

$$v_{rel}^2 = u_{rel}^2 + 2a_{rel}s_{rel}$$

where u_{rel} is the initial relative velocity, v_{rel} is the final relative velocity i.e. relative velocity at some instant of time t and s_{rel} is the relative displacement of the particles.

EQUATIONS OF MOTION IN RELATIVE VELOCITY FORM

The kinematical equations of motion are also modified as follows

$$v_r = u_r + a_r t$$

$$s_r = u_r t + \frac{1}{2}a_r t^2$$

$$v_r^2 - u_r^2 = 2a_r s_r$$

where u_r is the initial relative velocity
v_r is the final relative velocity at time t
a_r is the relative constant acceleration
s_r is the relative separation at time t

ILLUSTRATION 58

When two particles A and B are at point O, A is moving with a constant velocity 50 ms^{-1}, while B is not moving. But B possesses a constant acceleration of 10 ms^{-2}. After how much time they will be at a distance of 125 m?

SOLUTION

For particle A

$$u_A = 50 \text{ ms}^{-1}$$

$$a_A = 0 \text{ ms}^{-2}$$

For particle B

$$u_B = 0$$

$$a_B = 10 \text{ ms}^{-2}$$

So, initial velocity of A w.r.t. B

$$u_{AB} = u_A - u_B = 50 \text{ ms}^{-1}$$

and acceleration of A w.r.t. B

$$a_{AB} = a_A - a_B = -10 \text{ ms}^{-2}$$

The distance between A and B after time t is given by

$$s_{AB} = u_{AB}t + \frac{1}{2}a_{AB}t^2$$

$$\Rightarrow \quad 125 = 50t - 5t^2$$

$$\Rightarrow \quad t^2 - 10t + 25 = 0$$

$$\Rightarrow \quad (t-5)^2 = 0$$

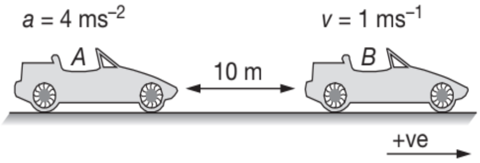
$$\Rightarrow \quad t = 5 \text{ s}$$

ILLUSTRATION 59

Car A and car B start moving simultaneously in the same direction along the line joining them. Car A with a constant acceleration $a = 4 \text{ ms}^{-2}$, while car B moves with a constant velocity $v = 1 \text{ ms}^{-1}$. At time $t = 0$, car A is 10 m behind car B. Find the time when car A overtakes car B.

SOLUTION

Given $u_A = 0$, $u_B = 1\ \text{ms}^{-1}$, $a_A = 4\ \text{ms}^{-2}$ and $a_B = 0$

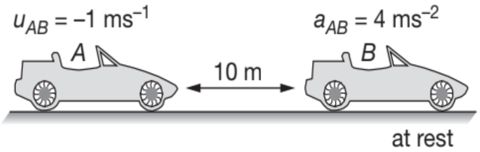


Assuming car B to be at rest, we have

$$u_{AB} = u_r = u_A - u_B = 0 - 1 = -1\ \text{ms}^{-1}$$

$$a_{AB} = a_r = a_A - a_B = 4 - 0 = 4\ \text{ms}^{-2}$$

Now, the problem can be assumed in simplified form as follows:



Substituting the proper values in equation

$$s_r = u_r t + \frac{1}{2} a_r t^2$$

$$\Rightarrow \quad 10 = -t + \frac{1}{2}(4)(t)^2$$

$$\Rightarrow \quad 2t^2 - t - 10 = 0$$

$$\Rightarrow \quad t = \frac{1 \pm \sqrt{1+80}}{4} = \frac{1 \pm \sqrt{81}}{4}$$

$$\Rightarrow \quad t = \frac{1 \pm 9}{4}$$

$$\Rightarrow \quad t = 2.5\ \text{s and } -2\ \text{s}$$

Ignoring the negative value, the desired time is 2.5 s

Problem Solving Technique(s)

This problem can also be solved without using the concept of relative motion as under,

At the time when A overtakes B,

$$s_A = s_B + 10$$

$$\Rightarrow \quad \frac{1}{2} \times 4 \times t^2 = 1 \times t + 10$$

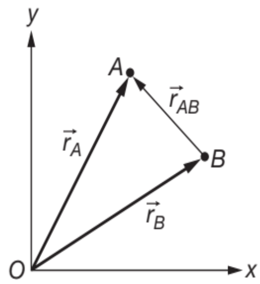
$$\Rightarrow \quad 2t^2 - t - 10 = 0$$

Which on solving gives $t = 2.5$ second and -2 second, the same as we found above.

RELATIVE MOTION IN TWO DIMENSION

Let $\vec{r}_A$ be the position of A with respect to O i.e. position vector of the point A is $\vec{r}_A$.

Similarly let $\vec{r}_B$ be the position of B with respect to O i.e. position vector of the point B is $\vec{r}_B$.



Then the position of A with respect to B is $\vec{r}_{AB}$ given by

$$\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$$

(The vector sum $\vec{r}_A - \vec{r}_B$ can be done by Δ law of addition or resolution method).

Now the velocity of A with respect to B is $\vec{v}_{AB}$, which is the rate at which position of A with respect to B changes i.e.

$$\vec{v}_{AB} = \frac{d}{dt}(\vec{r}_{AB})$$

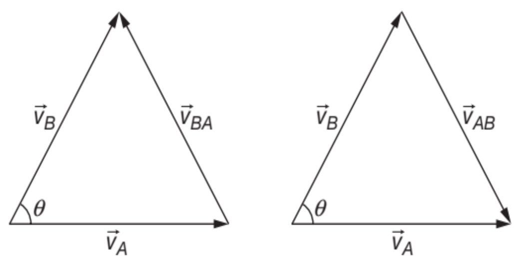
$$\Rightarrow \quad \frac{d}{dt}(\vec{r}_{AB}) = \frac{d}{dt}(\vec{r}_A) - \frac{d}{dt}(\vec{r}_B) = \frac{d\vec{r}_A}{dt} - \frac{d\vec{r}_B}{dt} = \vec{v}_A - \vec{v}_B$$

$$\Rightarrow \quad \vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

So, $\vec{v}_{AB} = \vec{v}_{A/B} = \vec{v}_A - \vec{v}_B =$ velocity of A relative to B

Similarly

$$\vec{v}_{BA} = \vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = \text{velocity of } B \text{ relative to } A$$



If $\vec{v}_A$ and $\vec{v}_B$ are inclined to each other at angle θ, then

$$|\vec{v}_{AB}| = |\vec{v}_{BA}| = \sqrt{v_A{}^2 + v_B{}^2 - 2v_2 v_B \cos\theta}$$

and $\vec{v}_{AB} = -\vec{v}_{BA}$

Problem Solving Technique(s)

(a) Since $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$

$$\Rightarrow \quad \vec{v}_{AB} = \vec{v}_A - \vec{v}_B + \vec{v}_P - \vec{v}_P$$

$$\Rightarrow \quad \vec{v}_{AB} = (\vec{v}_A - \vec{v}_P) - (\vec{v}_B - \vec{v}_P) = \vec{v}_{AP} - \vec{v}_{BP}$$

$$\Rightarrow \quad \vec{v}_{AB} = \vec{v}_A - \vec{v}_B = \vec{v}_{AP} - \vec{v}_{BP} = \vec{v}_{AO} - \vec{v}_{BO}$$

i.e., relative velocity is simplify independent of the velocity of the observer (O) analysing the motion, as long as both the particles A and B have their velocities specified w.r.t. the same observer.

By default $\vec{v}_A$ and $\vec{v}_B$ are specified with respect to the ground or with respect to a stationary observer at the ground.

(b) If $\Delta\vec{r}$ is the relative separation between any two particles at time Δt, then

$$\Delta\vec{r} = \vec{v}_r \Delta t$$

$$\Rightarrow \quad |\Delta\vec{r}| = \left(\sqrt{v_A{}^2 + v_B{}^2 - 2v_A v_B \cos\theta}\right)\Delta t$$

(c) A similar treatment is also given when we are asked to deal with the relative acceleration $\vec{a}_{AB}$ or $\vec{a}_{BA}$

$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B = \text{Acceleration of } A \text{ relative to } B$$

$$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = \text{Acceleration of } B \text{ relative to } A$$

Similarly

$$|\vec{a}_{AB}| = |\vec{a}_{BA}| = \sqrt{a_A^2 + a_B^2 - 2a_A a_B \cos\theta}$$

But here we must keep one thing in mind that if we are asked to calculate the acceleration of a ball falling freely with respect to another ball projected vertically upwards, then relative acceleration is zero (and not 2g). This is due to the fact that both the balls (either the ball falling freely or the one launched upward) move under the influence of gravity which acts vertically downwards for both.

Similarly, the acceleration of A with respect to B is $\vec{a}_{AB}$, which is the rate at which velocity of A with respect to B changes i.e.

$$\vec{a}_{AB} = \frac{d}{dt}(\vec{v}_{AB})$$

$$\Rightarrow \quad \frac{d}{dt}(\vec{v}_{AB}) = \frac{d}{dt}(\vec{v}_A) - \frac{d}{dt}(\vec{v}_B) = \frac{d\vec{v}_A}{dt} - \frac{d\vec{v}_B}{dt} = \vec{a}_A - \vec{a}_B$$

$$\Rightarrow \quad \vec{a}_{AB} = \vec{a}_A - \vec{a}_B$$

If $\vec{a}_{AB}$ is acceleration of A relative to B and $\vec{a}_{BA}$ is acceleration of B relative to A, then

$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B \text{ and } \vec{a}_{BA} = \vec{a}_B - \vec{a}_A$$

$$\Rightarrow \quad \vec{a}_{AB} = -\vec{a}_{BA}$$

RELATIVE MOTION FOR BODIES MOVING INDEPENDENTLY

For two bodies moving independently in the same direction, relative velocity is $v_r = v_A \sim v_B = |v_A - v_B|$ (~ sign indicates positive difference between v_A and v_B)

This relative velocity can be the relative velocity of approach or of separation depending upon the location of the bodies and the magnitudes of their velocities.

$v_r = v_A - v_B$, is the relative velocity of approach if $v_A > v_B$ and A is following (or approaching) B

Relative separation between A and B decreases with time

Whereas the same v_r becomes relative velocity of separation when B is behind A.

Relative separation between A and B increases with time

For two bodies moving independently in opposite direction

$$v_r = v_A + v_B$$

is the relative velocity of approach till the bodies do not meet and after they meet, the same value is the relative velocity of separation (or receding).

Bodies approaching and relative separation decreasing with time

Bodies receding and relative separation increasing with time

RELATIVE MOTION FOR BODIES MOVING DEPENDENTLY

Dependent motion is the case when a person is moving on a conveyer belt or on the surface of a moving carriage train or an escalator.

In such cases, we have to first understand that "if a velocity of the person walking on a moving surface is given, then this velocity is the velocity of the person with respect to the surface (i.e. his ground) on which he is moving and not with respect to the stationary ground."
Let's discuss and understand this through the following situations.

Situation 1

Consider a train T moving with a velocity v_T. Let a person P move with a velocity v (say) on this train along the direction of the train, then the velocity of the person with respect to his ground (which is the train) is v, so

$$v_{PT} = v \text{, where } v_{PT} = v_P - v_T$$

$$\Rightarrow \quad v_P = v_T + v_{PT}$$

which simply means that the velocity of the person with respect to a stationary observer A at the ground is

$$v_P = v_T + v$$

Situation 1a

If the observer A also starts moving on the ground in the direction of motion of the train, then he sees the person to be moving with a velocity

$$v_{PA} = (v_T + v) - v_A$$

Situation 1b

If the observer A also starts moving on the ground in the direction opposite to the motion of the train, then he sees the person to be moving with a velocity

$$v_{PA} = (v_T + v) + v_A$$

Situation 2

Similarly if the person P moves with a velocity v (say) on this train opposite to the direction of the train, then the velocity of the person with respect to his ground (which is the train) is v, so

$$v_{PT} = -v \text{, where } v_{PT} = v_P - v_T$$

(The direction of motion of train is taken as positive)

$$\Rightarrow \quad v_P = v_T + v_{PT}$$

which simply means that the velocity of the person with respect to a stationary observer at the ground is

$$v_P = v_T - v$$

Situation 2a

If the observer A also starts moving on the ground in the direction of motion of the train, then he sees the person to be moving with a velocity

$$v_{PA} = (v_T - v) - v_A$$

Situation 2b

If the observer A also starts moving on the ground in the direction opposite to the motion of the train, then he sees the person to be moving with a velocity

$$v_{PA} = (v_T - v) + v_A$$

Situation 3

If a person (P) is moving with a velocity v_P on a stationary escalator (E), then with respect to the ground (G), we have

$$v_{PG} = v_{PE} = v_P$$

Situation 3a

If a person (P) is moving with a velocity v up an escalator (E) which is also moving up with a velocity v_E, then with respect to the ground (G), we have

$$v_{PG} = v_{PE} + v_{EG}$$

Since, we must take a note here that $v_{PE} = v$, so velocity of the person with respect to ground or a stationary observer A on the ground is

$$v_P = v_E + v$$

Situation 3b

If a person (P) is moving with a velocity v down an escalator (E) which is moving up with a velocity v_E, then with respect to the ground (G), we have

$$v_{PG} = v_{PE} + v_{EG}$$

Taking direction of motion of the escalator as positive, we note here that $v_{PE} = -v$ and $v_{EG} = +v_E$, so velocity of the person with respect to ground or a stationary observer A on the ground is

$$v_P = v_E - v$$

ILLUSTRATION 60

Consider a collection of a large number of particles each with speed v. The direction of velocity is randomly distributed in the collection. Show that the magnitude of the relative velocity between a pair of particles averaged over all the pairs in the collection is greater that v.

SOLUTION

As shown in figure, let $\vec{v}_1$ and $\vec{v}_2$ be the velocities of any two particles and θ be the angle between them. As each particle has speed v, so

$$|\vec{v}_1| = |\vec{v}_2| = v$$

The magnitude of relative velocity $\vec{v}_{21}$ of particle 2 with respect to 1 is given by

$$v_{21} = \sqrt{|-\vec{v}_1|^2 + |\vec{v}_2|^2 + 2|-\vec{v}_1||\vec{v}_2|\cos(180° - \theta)}$$

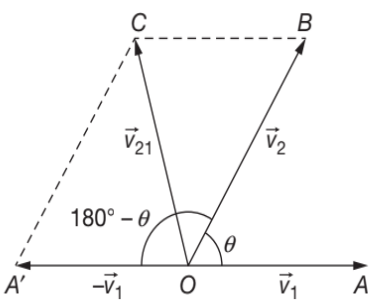
$$\Rightarrow \quad v_{21} = \sqrt{v^2 + v^2 - 2(v)(v)\cos\theta} = \sqrt{2v^2(1-\cos\theta)}$$

Since $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

$$\Rightarrow \quad v_{21} = \sqrt{2v^2 \times 2\sin^2\left(\frac{\theta}{2}\right)} = 2v\sin\left(\frac{\theta}{2}\right)$$

As the velocities of the particles are randomly distributed, so θ can vary from 0 to 2π. If $\langle v_{21} \rangle$ is the magnitude of the average velocity when averaged over all pairs, then

$$\langle v_{21} \rangle = \frac{\int_0^{2\pi} 2v\sin\left(\frac{\theta}{2}\right) d\theta}{\int_0^{2\pi} d\theta} = \frac{-2v\left[\frac{\cos\left(\frac{\theta}{2}\right)}{\left(\frac{1}{2}\right)}\right]_0^{2\pi}}{\theta\Big|_0^{2\pi}}$$



$$\Rightarrow \quad \langle v_{21} \rangle = \frac{-4v\left[\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}}{2\pi - 0} = -\frac{2v}{\pi}(\cos\pi - \cos 0)$$

$$\Rightarrow \quad \langle v_{21} \rangle = -\frac{2v}{\pi}(-1-1) = \frac{4v}{\pi} = 1.273v$$

$$\Rightarrow \quad \langle v_{21} \rangle > v$$

ILLUSTRATION 61

Two trains, each having a speed of 30 kmhr^{-1} are headed towards each other on the same straight track. A bird that can fly at 60 kmhr^{-1} flies off one train, when they are 60 km apart and heads directly for the other train. On reaching the other train, it flies directly back to the first and so on.

(a) How many trips can the bird make from one train to the other before they crash?

(b) What is the total distance the bird travels?

SOLUTION

The relative velocity of one train relative to the other is 60 kmhr^{-1} and as the distance between the trains is 60 km, the two trains will crash after 1 hr.

(a) Now, the velocity of bird with respect to train towards which it is moving will be $v = 90 \text{ kmhr}^{-1}$. So, the time taken by bird for first trip is $t_1 = \left(\frac{60}{90}\right) = \frac{2}{3}$ hr

and in this time the trains have moved towards each other $\left(\frac{2}{3}\right) \times 60 = 40$ km, so the remaining distance $= 60 - 40 = 20$ km.

So, the time taken by bird for second trip, $t_2 = \frac{20}{90} = \left(\frac{2}{3^2}\right)$ hr

Proceeding in the same way time taken by the bird for nth trip, $t_n = \left(\frac{2}{3^n}\right)$hr

Now, if the bird makes n trips till the train crashes,

$$t_1 + t_2 + t_3 + ... + t_n = 1 \text{ hr}$$

$$\Rightarrow \quad \frac{2}{3} + \frac{2}{3^2} + ... + \frac{2}{3^n} = 1 \text{ hr}$$

$$\Rightarrow \quad \frac{2}{3}\left[1+\frac{1}{3}+\frac{1}{3^2}+....+\frac{1}{3^{n-1}}\right]=1\text{ hr}$$

$$\Rightarrow \quad \frac{2}{3}\left[\frac{1-\left(\frac{1}{3}\right)^n}{1-\left(\frac{1}{3}\right)}\right]=1\text{ hr}$$

$$\Rightarrow \quad 1-\left(\frac{1}{3}\right)^n=1$$

$$\Rightarrow \quad \left(\frac{1}{3}\right)^n=0$$

$$\Rightarrow \quad n\to\infty$$

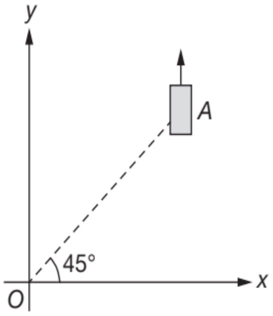
So, the bird will make infinite trips.

(b) As the speed of bird is 60 kmhr^{-1} and the two trains crash after 1 hr., so the total distance travelled by the bird is the distance travelled by the bird in 1 hr. i.e.,

$$d=60\left(\frac{\text{km}}{\text{hr}}\right)\times 1\text{ hr}=60\text{ km}$$

ILLUSTRATION 62

On a frictionless horizontal surface, assumed to be x-y plane, a small trolley A is moving along a straight line parallel to the y-axis (see figure) with a constant velocity of $(\sqrt{3}-1)\text{ ms}^{-1}$. At a particular instant, when the line OA makes an angle of $45°$ with the x-axis, a ball is thrown along the surface from the origin O. Its velocity makes an angle ϕ with the x-axis and it hits the trolley.

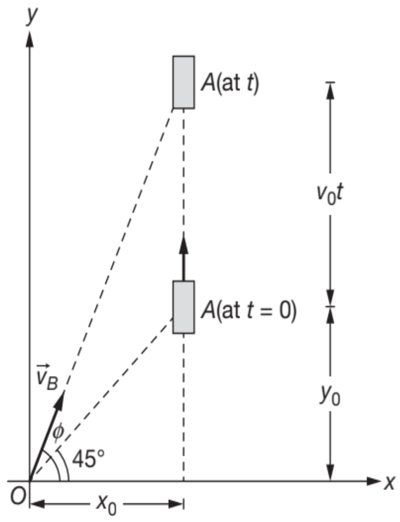


(a) The motion of the ball is observed from the frame of the trolley. Calculate the angle θ made by the velocity vector of the ball with the x-axis in this frame.

(b) Find the speed of the ball with respect to the surface, if $\phi=\frac{4\theta}{3}$.

SOLUTION

(a) Since the motion of the ball (say B) is observed from the frame of the trolley A, so in this problem we must deal with $\vec{v}_{BA}$ i.e., velocity of ball B with respect to the trolley A.



Now $\vec{v}_{BA}=\vec{v}_B-\vec{v}_A$ ...(1)

Since ϕ is the angle made by the velocity of the ball with the x-axis, to hit the trolley, so

$$\vec{v}_B=v_B\left(\cos\phi\hat{i}+\sin\phi\hat{j}\right) \quad ...(2)$$

Also, θ is the angle made by the velocity vector of the ball with the x-axis in the frame of the trolley B, so

$$\vec{v}_{BA}=v_{BA}\left(\cos\theta\hat{i}+\sin\theta\hat{j}\right) \quad ...(3)$$

Also, we are given that $\vec{v}_A=v_A\hat{j}=(\sqrt{3}-1)\hat{j}\text{ ms}^{-1}$

Substituting all in equation (1), we get

$$v_{BA}\left(\cos\theta\hat{i}+\sin\theta\hat{j}\right)=\left(v_B\cos\phi\right)\hat{i}+\left(v_B\sin\phi-v_A\right)\hat{j} \quad ...(4)$$

$$\Rightarrow \quad v_{BA}\cos\theta=v_B\cos\phi \quad ...(5)$$

and $v_{BA}\sin\theta=v_B\sin\phi-v_A$ ...(6)

So, $\tan\theta=\dfrac{v_B\sin\phi-v_A}{v_B\cos\phi}$ ...(7)

Now, let the ball B strike the trolley A at time t, then the y coordinate of the ball as well as the trolley have to be the same, so

$$(y)_{\text{Ball at } t} = (y)_{\text{Trolley at time } t}$$

$$\Rightarrow \quad (v_B \sin\phi)t = y_0 + v_0 t \qquad \ldots(8)$$

Also, we observe that

$$(v_B \cos\phi)t = x_0 \qquad \ldots(9)$$

Since we know observe that $x_0 = y_0$, so from (8) and (9), we get

$$(v_B \sin\phi)t = (v_B \cos\phi)t + v_0 t$$

$$\Rightarrow \quad v_B \sin\phi - v_0 = v_B \cos\phi \qquad \ldots(10)$$

Using (10) and (7), we get

$$\tan\theta = 1$$

$$\Rightarrow \quad \theta = 45^\circ$$

Also, we could have arrived the result without these calculations. Just keep in mind that for the ball (B) to hit the trolley A, relative velocity of the ball B with respect to the trolley A must be directed along OA. So, $\vec{v}_{BA}$ makes an angle of 45° with the x-axis. Hence $\theta = 45^\circ$.

(b) Since $\phi = \dfrac{4\theta}{3} = \dfrac{4(45^\circ)}{3} = 60^\circ$

So, from (7), we get

$$\tan 45^\circ = \frac{v_B \sin(60^\circ) - (\sqrt{3} - 1)}{v_B \cos(60^\circ)}$$

$$v_B\left(\frac{\sqrt{3}}{2}\right) - \sqrt{3} + 1 = \frac{v_B}{2}$$

$$\Rightarrow \quad v_B\left(\frac{\sqrt{3} - 1}{2}\right) = \sqrt{3} - 1$$

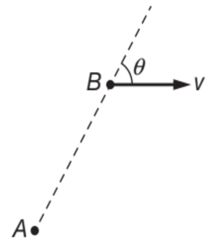
$$\Rightarrow \quad v_B = 2 \text{ ms}^{-1}$$

VELOCITY OF APPROACH/SEPARATION IN TWO DIMENSION

Relative velocity of approach of two bodies is the relative velocity of the bodies along the line joining the two bodies. If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.

ILLUSTRATION 63

Particle A is at rest and particle B is moving with constant velocity v as shown in the figure at $t = 0$. Find their velocity of separation.



SOLUTION

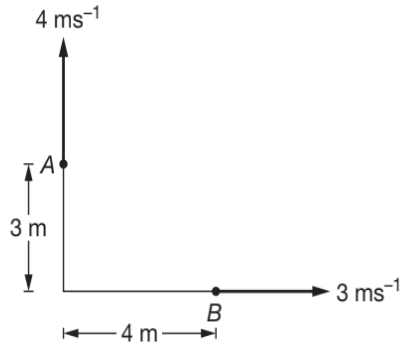
$$v_{BA} = v_B - v_A = v$$

$$v_{\text{sep}} = \text{component of } v_{BA} \text{ along line } AB = v\cos\theta$$

ILLUSTRATION 64

Particles A and B are moving as shown in the figure at $t = 0$. Find their velocity of separation

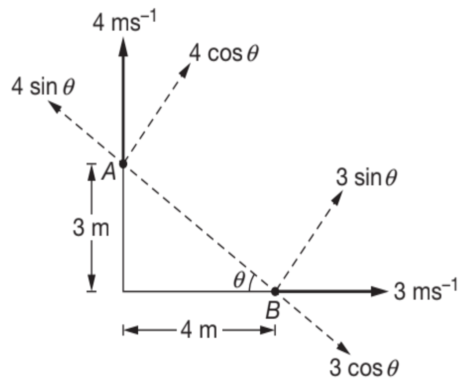
(a) at $t = 0$ **(b)** at $t = 1$ s



SOLUTION

(a) $\tan\theta = \dfrac{3}{4}$

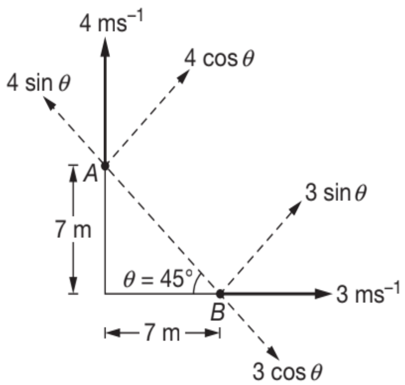
v_{sep} = relative velocity along line AB



$$\Rightarrow \quad v_{sep} = 3\cos\theta + 4\sin\theta$$

$$\Rightarrow \quad v_{sep} = 3\left(\frac{4}{5}\right) + 4\left(\frac{3}{5}\right) = \frac{24}{5} = 4.8 \text{ ms}^{-1}$$

(b) $\theta = 45°$



v_{sep} = relative velocity along line AB

$$\Rightarrow \quad v_{sep} = 3\cos\theta + 4\sin\theta$$

$$\Rightarrow \quad v_{sep} = 3\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right) = \frac{7}{\sqrt{2}} \text{ ms}^{-1}$$

ILLUSTRATION 65

Four persons K, L, M and N are initially at the four corners of a square of side a. Each person now moves with a uniform speed v in such a way that K moves always directly towards L, L directly towards M, M directly towards N and N directly towards K. Find the path-equation followed by the persons. Also calculate the time taken by them to meet.

SOLUTION

The symmetry of configuration implies that each person follows similar trajectories and perform similar motion. They, therefore meet at the same time at the centre O of the square. Since they are always directing their velocities to the next neighbours, therefore they must follow a curved path as shown in the figure. The persons are always located at the corners of the square which rotates and shrinks in size and ultimately collapses to zero.

Path Equation Followed by Persons

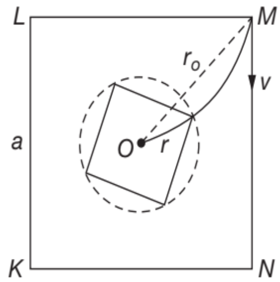
Let $\vec{r}$ be the position vector of point M at any instant t. The velocity $\vec{v}$ of the person M always makes fixed angle $\varphi = 45°$ with $\vec{r}$

In time dt, the person travels a distance $ds = vdt$ ...(1)

Further in ΔPQM,

$$\Rightarrow \quad (ds)^2 = (dr)^2 + (rd\theta)^2 \qquad ...(2)$$

where, dr = distance moved by particle along radial direction $rd\theta$ = distance moved by particle along tangential direction (normal to $\vec{r}$)



Further in ΔPQM,

$$\Rightarrow \quad \tan 45° = \frac{rd\theta}{dr}$$

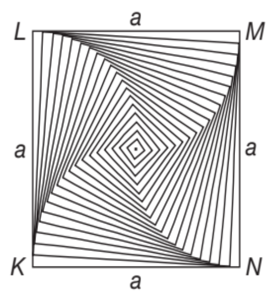
$$\Rightarrow \quad dr = rd\theta \qquad ...(3)$$

$$\Rightarrow \quad \int_{\frac{a}{\sqrt{2}}}^{r} \frac{dr}{r} = \int_{0}^{\theta} d\theta$$

$$\Rightarrow \quad \theta = \log_e r \Big|_{\frac{a}{\sqrt{2}}}^{r}$$

$$\Rightarrow \quad r \times \frac{\sqrt{2}}{a} = e^{\theta}$$

$$\Rightarrow \quad r = \frac{a}{\sqrt{2}} e^{\theta} \qquad \{\textbf{Equation of Trajectory}\}$$



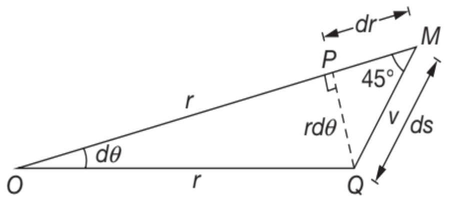
where θ is the angle measured from M along the line OM. Putting (3) in (2), generates

$$\Rightarrow \quad (ds)^2 = 2(dr)^2$$

$$\Rightarrow \quad ds = \sqrt{2}dr$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{2}\frac{dr}{dt}$$

$$\Rightarrow \quad v = \sqrt{2}\frac{dr}{dt}$$



$$\Rightarrow \quad \sqrt{2}\int_0^{\frac{a}{\sqrt{2}}} dr = v\int_0^T dt$$

$$\Rightarrow \quad \sqrt{2}\left(\frac{a}{\sqrt{2}} - 0\right) = v(T-0)$$

$$\Rightarrow \quad T = \frac{a}{v}$$

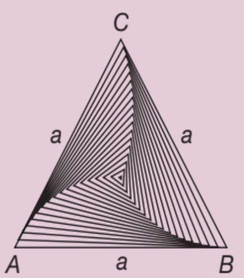
Conceptual Note(s)

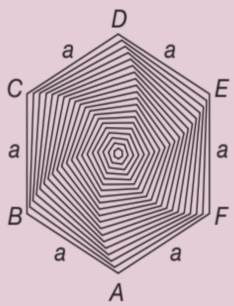
The above analysis can be applied to any other configuration where moving object makes a fixed angle ϕ with $\vec{r}$

Then equation of trajectory will be $r = r_0 e^{\frac{\theta}{\tan\phi}}$

Time taken by the objects to meet $T = \dfrac{r_0}{v\cos\phi}$

where, T is the time in which object moves from original radial distance r_0 to $r = 0$, with a uniform speed v.





e.g., For equilateral triangle $\phi = 30°$, $r_0 = \dfrac{a}{\sqrt{3}}$

$$\therefore \quad r = \frac{a}{\sqrt{3}}e^{\sqrt{3}\theta} \text{ and } T = \frac{a/\sqrt{3}}{v\cos 30} = \frac{2a}{3v}$$

e.g. For hexagon, $\phi = 60°$, $r_0 = a$

$$r = ae^{\frac{\theta}{\sqrt{3}}} \text{ and } T = \frac{a}{v\cos 60} = \frac{2a}{v}$$

Shortcut Method to Find Only the Time Taken

If the particles are located at the sides of an n sided symmetrical polygon with each side a and each particle moves towards the other, then

$$T = \frac{\text{Initial Separation}}{\text{Relative Velocity of Approach}}$$

$$\Rightarrow \quad T = \frac{a}{v - v\cos\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad T = \frac{a}{v\left[1 - \cos\left(\frac{2\pi}{n}\right)\right]}$$

$$\Rightarrow \quad T = \frac{a}{2v\sin^2\left(\frac{\pi}{n}\right)}$$

For triangle $n = 3 \quad \Rightarrow \quad T = \dfrac{2a}{3v}$

For square, $n = 4 \quad \Rightarrow \quad T = \dfrac{a}{v}$

For hexagon, $n = 6 \quad \Rightarrow \quad T = \dfrac{2a}{v}$

WHERE TO APPLY THE CONCEPT OF RELATIVE MOTION?

The concept of relative motion can be applied to two and three dimensional motion also. The problems involving the concept of relative motion mainly belong to the following categories.

(a) Distance of closest approach (i.e., minimum distance) between two moving bodies
(b) River-Boat problems or River-Swimmer problems
(c) Aeroplane-Wind problems
(d) Rain-Man problems
(e) Relative motion in the case of projectiles (discussed in projectile motion)

CATEGORY 1: DISTANCE OF CLOSEST APPROACH BETWEEN TWO MOVING BODIES

METHOD I (Calculus Method)

For calculating the distance of closest approach (i.e., the minimum distance between two moving bodies, we first calculate the distance r between them at any instant of time t in the light of some conditions given with the problem. Then we have to minimise r for calculating its minimum value. Mathematically, for r to be MINIMUM, we have

$$\frac{dr}{dt} = 0$$

So we find or calculate the time at which r attains a minimum value. Then this value of t is put in r to get $r_{\min}$.

METHOD II (Relative Velocity Method)

This method can be applied using the following steps.

STEP-1

First calculate $\vec{v}_{BA}$ i.e. velocity of B as seen by A (i.e. velocity of B as seen by A, when A considers itself to be at rest).

You can also calculate $\vec{v}_{AB}$ i.e. velocity of A as seen by B (i.e. velocity of A seen by B, when B considers itself to be at rest).

STEP-2

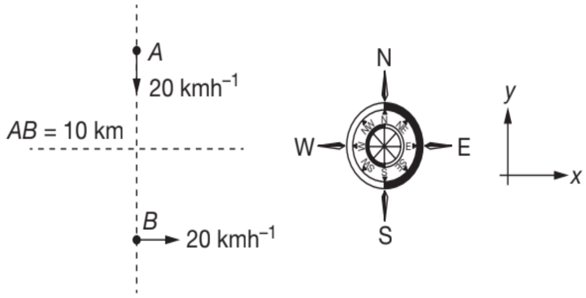
Find the direction of $\vec{v}_{BA}$ and then draw a line along the direction of $\vec{v}_{BA}$.

STEP-3

Then perpendicular distance between A (considering it to be at rest) and the line along which $\vec{v}_{BA}$ is directed is the distance of closest approach or the minimum separation between the two moving bodies. For this see Illustrative Example below.

ILLUSTRATION 66

Ship A is moving towards south with a speed of 20 kmh^{-1}. Another ship B is moving towards east with a speed of 20 kmh^{-1}. At a certain instant the ship B is due south of ship A and is at a distance of 10 km from ship A. Find the shortest distance between the ships and the time after which they are closest to each other.

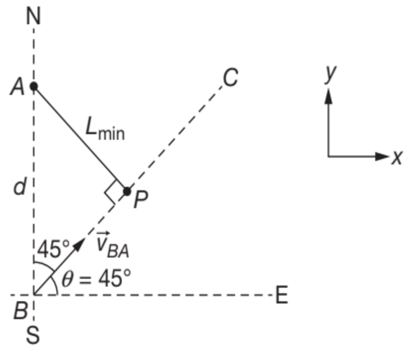


SOLUTION

If we analyse the positions of ships at different instants, we find that the separation decreases at first, becomes minimum at a particular instant of time and then increases.

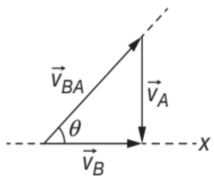
METHOD I: Using the Concept of Relative Velocity

Attach your reference frame with ship A and then analyse the motion of ship B. For you the ship A will be at rest and the ship B will appear to be moving with relative velocity $\vec{v}_{BA} = (20\hat{i} + 20\hat{j}) \text{ kmhr}^{-1}$ along line BC.



Motion of A relative to B

For A, B will appear to be moving with relative velocity $\vec{v}_{BA}$ of magnitude $20\sqrt{2} \text{ kmhr}^{-1}$ at an angle of $45°$ with east (i.e., $+x$ axis). There is a point P on the line of relative motion of B where the separation $AB(= L_{\min})$ becomes perpendicular to the line of relative motion. At this instant the separation is minimum.



For the triangle ABP,

$$\sin 45° = \frac{L_{\text{min}}}{d}$$

$$\Rightarrow \quad L_{\text{min}} = \frac{d}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2} \text{ km}$$

And the time taken is given by

$$|\vec{v}_{BA}| = \sqrt{|\vec{v}_A|^2 + |\vec{v}_B|^2} = 20\sqrt{2} \text{ kmh}^{-1}$$

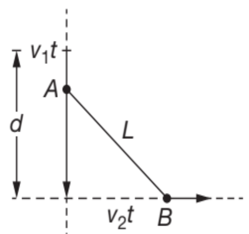
$$t = \frac{BP}{|\vec{v}_{BA}|} = \frac{d\cos 45°}{20\sqrt{2}} = \frac{10 \times \frac{1}{\sqrt{2}}}{20\sqrt{2}} = \frac{1}{4} \text{ hr} = 15 \text{ min}$$

$$\tan\theta = \frac{|\vec{v}_A|}{|\vec{v}_B|} = 1$$

$$\Rightarrow \quad \theta = 45°$$

METHOD II: Using Calculus

We have to find the time after which the separation is minimum, so let us write separation as a function of time and then minimize the function.



At time t the distances moved by A and B are v_1t and v_2t respectively.

Their separation L after time t is given by

$$L^2 = (d - v_1t)^2 + v_2^2t^2 \qquad \ldots(1)$$

Differentiating w.r.t. time

$$2L\frac{dL}{dt} = 2(d - v_1t)(-v_1) + 2v_2^2t$$

When the separation L is minimum, $\frac{dL}{dt} = 0$

$$\Rightarrow \quad (v_1^2 + v_2^2)t = v_1d$$

$$\Rightarrow \quad t = \frac{v_1d}{(v_1^2 + v_2^2)}$$

$$\Rightarrow \quad t = \frac{(10)(20)}{(20^2 + 20^2)}$$

$$\Rightarrow \quad t = \frac{1}{4} \text{ hr} = 15 \text{ min}$$

Now the minimum separation as given by equation (1) is

$$L_{\text{min}} = \sqrt{\left(10 - 20 \times \frac{1}{4}\right)^2 + 20^2 \times \frac{1}{4^2}}$$

$$\Rightarrow \quad L_{\text{min}} = 5\sqrt{2} \text{ km}$$

ILLUSTRATION 67

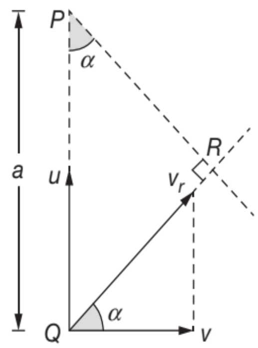
The distance between two moving particles P and Q at any time is a. If v_r be their relative velocity and if u and v be the components of v_r along and perpendicular to PQ, then show that their closest distance is $\frac{av}{v_r}$ and that the time that elapses before they arrive at their nearest distance is $\frac{au}{v_r^2}$.

SOLUTION

METHOD I

Assuming P to be at rest, particle Q is moving with velocity v_r in the direction shown in figure. Components of v_r along and perpendicular to PQ are u and v respectively. In the figure

$$\sin\alpha = \frac{u}{v_r}, \quad \cos\alpha = \frac{v}{v_r}$$



(i) The closest distance between the particles is PR. So,

$$s_{\text{min}} = PR = PQ\cos\alpha = (a)\left(\frac{v}{v_r}\right)$$

$$\Rightarrow \quad s_{\text{min}} = \frac{av}{v_r}$$

(ii) Time after which they arrive at their nearest distance is

$$t=\frac{QR}{v_r}=\frac{(PQ)\sin\alpha}{v_r}=\frac{(a)\left(\frac{u}{v_r}\right)}{v_r}=\frac{au}{v_r^2}$$

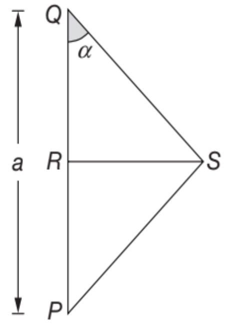
METHOD II

In time t, particle P will travel a distance $PR=ut$ relative to Q along PQ and a distance $RS=vt$ perpendicular to PQ. S is the position of P after time t relative to Q. The distance x between them after time t is given by

$$x^2=QR^2+RS^2=(a-ut)^2+v^2t^2$$

$$x^2=(u^2+v^2)t^2-2aut+a^2$$

$$x^2=v_r^2t^2-2aut+a^2 \qquad \{\because u^2+v^2=v_r^2\}$$



This can also be written as

$$x^2=v_r^2\left(t-\frac{au}{v_r^2}\right)^2+\frac{a^2v^2}{v_r^2} \qquad \ldots(1)$$

From equation (1) it is clear that x is minimum at $t=\frac{au}{v_r^2}$ and the minimum value of x is

$$x_{\min}=\frac{av}{v_r}$$

RELATIVE MOTION IN RIVER FLOW (ONE-DIMENSIONAL APPROACH)

If a swimmer (s) can swim relative to river (r) with velocity $\vec{v}_{sr}$ and river is flowing relative to ground with velocity $\vec{v}_r$, velocity of swimmer relative to the ground is $\vec{v}_s$ then

$$\vec{v}_{sr}=\vec{v}_s-\vec{v}_r$$

$$\Rightarrow \quad \vec{v}_s=\vec{v}_{sr}+\vec{v}_r$$

If $\vec{v}_r=\vec{0}$, then $\vec{v}_s=\vec{v}_{sr}$

Conceptual Note(s)

If $\vec{v}_r=\vec{0}$, then $\vec{v}_s=\vec{v}_{sr}$

In other words, velocity of man in still water = velocity of man w.r.t. river.

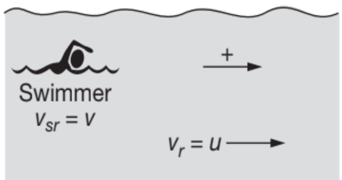
So, whenever a problem says that velocity of swimmer in still water is 5 kmhr^{-1}, then we must take $\vec{v}_{sr}=5$ kmhr^{-1}.

RIVER PROBLEM IN ONE DIMENSION

Velocity of river is u and velocity of man in still water is v.

CASE-1

Man swimming downstream (along the direction of river flow) In this case



Observer at ground sees the swimmer moving with velocity $v_s=(v+u)$

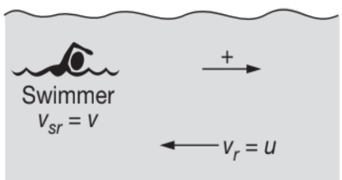
Velocity of river $v_r=+u$

Velocity of man w.r.t. river $v_{sr}=+v$

Now $v_s=v_{sr}+v_r=u+v$

CASE-2

Man swimming upstream (opposite to the direction of river flow). In this case



Observer at ground sees the swimmer moving with velocity $v_s=(v-u)$

Velocity of river $v_r = -u$

Velocity of man w.r.t. river $v_{sr} = +v$

Now $v_s = v_{sr} + v_r = (v - u)$

ILLUSTRATION 68

A swimmer capable of swimming with velocity v relative to water jumps in a flowing river having velocity u. The man swims a distance d downstream and returns to the original position. Find out the time taken in complete motion.

SOLUTION

$$\text{Total time} = \begin{pmatrix}\text{Time of}\\ \text{swimming}\\ \text{downstream}\end{pmatrix} + \begin{pmatrix}\text{Time of}\\ \text{swimming}\\ \text{upstream}\end{pmatrix}$$

$$t = t_{\text{down}} + t_{\text{up}} = \frac{d}{v+u} + \frac{d}{v-u} = \frac{2dv}{v^2 - u^2}$$

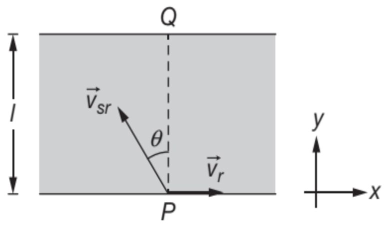
CATEGORY 2: RIVER-BOAT PROBLEMS OR RIVER-SWIMMER PROBLEMS

While solving problems related to river boat or river swimmer we come across the following terms

(a) Absolute velocity of swimmer/boatman $\vec{v}_s$ or $\vec{v}_b$ (absolute means velocity relative to ground)

(b) Absolute velocity of river current $(\vec{v}_r)$ and

(c) Velocity of swimmer/boatman with respect to the river $\vec{v}_{sr}$ or $\vec{v}_{br}$



Since, $\vec{v}_{sr} = \vec{v}_s - \vec{v}_r$

$$\Rightarrow \quad \vec{v}_s = \vec{v}_{sr} + \vec{v}_r \qquad \ldots(1)$$

Observing the situation shown in the diagram below, we arrive at some standard results and their special cases.

Consider a river of width l across which a swimmer wants to swim. Let $\vec{v}_{sr}$ be the relative velocity of swimmer with respect to river current and let $\vec{v}_{sr}$ make an angle θ with PQ shown in the figure. Let us assume the flow of river to be along the positive x-axis.

Since from (1), we get

$$\vec{v}_s = \vec{v}_{sr} + \vec{v}_r$$

$\Rightarrow (v_s)_x = (v_{sr})_x + (v_r)_x$	$\Rightarrow (v_s)_y = (v_{sr})_y + (v_r)_y$
$\Rightarrow (v_s)_x = (-v_{sr}\sin\theta) + v_r$	$\Rightarrow (v_s)_y = v_{sr}\cos\theta + 0$
$\Rightarrow (v_s)_x = v_r - v_{sr}\sin\theta$ ...(2)	$\Rightarrow (v_s)_y = v_{sr}\cos\theta$ ...(3)

If t is the time taken by the swimmer to cross the river, then

$$t = \frac{l}{(v_s)_y}$$

$$\Rightarrow \quad t = \frac{l}{v_{sr}\cos\theta} \qquad \ldots(4)$$

Further if x is the drift (displacement along x axis) when he reaches the opposite bank, then

$$x = (v_b)_x t$$

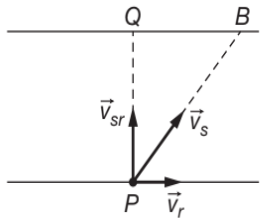
$$\Rightarrow \quad x = Drift = (v_r - v_{sr}\sin\theta)\frac{l}{v_{sr}\cos\theta} \qquad \ldots(5)$$

CONDITION FOR THE SWIMMER TO CROSS THE RIVER IN THE MINIMUM POSSIBLE TIME

Since $t = \dfrac{l}{v_{sr}\cos\theta}$

For t to be MINIMUM, $\cos\theta$ must be MAXIMUM i.e., $\cos\theta = 1$

$$\Rightarrow \quad \theta = 0^0$$



i.e., The relative velocity of swimmer with respect to river must be perpendicular to the river current.

As a result of this the swimmer will be drifted to the point B.

So,

$$AB = \text{Drift} = v_r t = \left(\frac{v_r}{v_{sr}}\right) l \qquad \ldots(6)$$

CONDITION FOR ZERO DRIFT OR CONDITION TO REACH THE OPPOSITE POINT

Since we calculated the drift value to be x given by

$$x = (v_r - v_{sr}\sin\theta)\frac{l}{v_{sr}\cos\theta}$$

For the above condition to be met, we must have

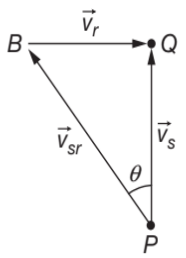
$$x = 0$$

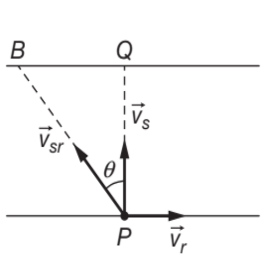
$$\Rightarrow \quad v_r = v_{sr}\sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{v_r}{v_{sr}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{v_r}{v_{sr}}\right) \qquad \ldots(7)$$

A diagramatic representation is given in support of the above mathematical argument.





Let $\vec{v}_{sr}$ be the velocity of swimmer relative to river current, $\vec{v}_r$ the river current, then the swimmer wants to swim in a manner such that he reaches the point just opposite to the point from where he started. For such a thing to happen he must have his relative velocity directed in a direction opposite to the river current at an angle θ to the line PQ (say).

Further since

$$\vec{v}_{sr} = \vec{v}_s - \vec{v}_r \quad \Rightarrow \quad \vec{v}_s = \vec{v}_{sr} + \vec{v}_r$$

So, $\sin\theta = \dfrac{BQ}{PB} = \dfrac{v_r}{v_{sr}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\dfrac{v_r}{v_{sr}}\right)$

So, we conclude that the swimmer reaches Q when he directs his velocity relative to river at an angle $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{sr}}\right)$, upstream from PQ.

Also we observe that if $v_r > v_{sr}$ then the swimmer would never reach the point Q.

Condition for Minimum Drift

For minimising the drift, we must have

$$\frac{dx}{d\theta} = 0$$

$$\Rightarrow \quad \frac{d}{d\theta}\left[(v_r - v_{sr}\sin\theta)\frac{l}{v_{sr}\cos\theta}\right] = 0$$

$$\Rightarrow \quad \frac{d}{d\theta}\left[\left(\frac{v_r}{v_{sr}}\sec\theta - \tan\theta\right)l\right] = 0$$

$$\Rightarrow \quad \frac{v_r}{v_{sr}}(\sec\theta\tan\theta) - \sec^2\theta = 0$$

$$\Rightarrow \quad \frac{v_r}{v_{sr}}(\tan\theta) - \sec\theta = 0$$

$$\Rightarrow \quad \frac{v_r}{v_{sr}}\frac{\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{v_{sr}}{v_r}\right)$$

If we are asked to calculate the angle with the horizontal, then we get

$$\text{Total angle} = \frac{\pi}{2} + \theta = \frac{\pi}{2} + \sin^{-1}\left(\frac{v_{sr}}{v_r}\right)$$

Also, we conclude that the drift x can be minimised only if $v_{sr} < v_r$.

ILLUSTRATION 69

A swimmer can swim at the rate of $5\ \text{kmh}^{-1}$ in still water. A 1 km wide river flows at the rate of $3\ \text{kmh}^{-1}$. The swimmer wishes to swim across the river directly opposite to the starting point.

(a) Along what direction must the swimmer swim?
(b) What should be his resultant velocity?
(c) How much time will he take to cross the river?

SOLUTION

(a) The velocity of swimmer with respect to river $v_{sr} = 5\text{ kmh}^{-1}$, this is greater than the river flow velocity, therefore, he can cross the river directly (along the shortest path). The angle of swim must be

$$\theta = \frac{\pi}{2} + \sin^{-1}\left(\frac{v_r}{v_{sr}}\right) = 90° + \sin^{-1}\left(\frac{v_r}{v_{sr}}\right)$$

$$\theta = 90° + \sin^{-1}\left(\frac{3}{5}\right) = 90° + 37°$$

$\theta = 127°$ w.r.t. the river flow or $37°$ w.r.t. perpendicular in backward direction

(b) Resultant velocity will be

$$v_s = \sqrt{v_{sr}^2 - v_r^2} = \sqrt{5^2 - 3^2} = 4\text{ kmh}^{-1}$$

Along the direction perpendicular to the river flow.

(c) Time taken to cross the

$$t = \frac{d}{\sqrt{v_{sr}^2 - v_r^2}} = \frac{1\text{ km}}{4\text{ kmh}^{-1}} = \frac{1}{4}\text{ h} = 15\text{ min}$$

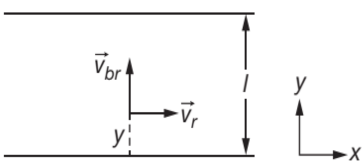
ILLUSTRATION 70

The current velocity of river grows in proportion to the distance from its bank and reaches the maximum value v_0 in the middle. Near the banks the velocity is zero. A boat is moving along the river in such a manner that it is always perpendicular to the current. The speed of the boat in still water is u. Find the distance through which the boat crossing the river will be carried away by the current if the width of the river is l. Also determine the trajectory of the boat.

SOLUTION

Given that $|\vec{v}_{br}| = v_y = \dfrac{dy}{dt} = u$ ...(1)

$$|\vec{v}_r| = v_x = \frac{dx}{dt} = \left(\frac{2v_0}{l}\right)y \quad ...(2)$$



From equations (1) and (2) we get

$$\frac{dy}{dx} = \frac{ul}{2v_0 y}$$

$$\Rightarrow \int_0^y y\,dy = \frac{ul}{2v_0}\int_0^x dx$$

$$\Rightarrow y^2 = \frac{ulx}{v_0}$$

At $y = \dfrac{l}{2}$, $x = \dfrac{lv_0}{4u}$

$$\Rightarrow x_{\text{net}} = 2x = \frac{lv_0}{2u}$$

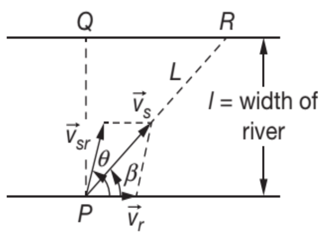
CONDITION WHEN THE BOATMAN CROSSES THE RIVER ALONG THE SHORTEST ROUTE

Here we have to discuss this condition in the light of two cases.

CASE-1: When the velocity of the swimmer with respect to river (v_{sr}) is greater than river current (v_r).

CASE-2: When the velocity of the swimmer with respect to river (v_{sr}) is less than river current (v_r).

For the sake of ease we re-assign symbols to v_{sr} and v_r as v and u respectively.



Let us further assume that $\vec{v}_{sr}$ (i.e., v) makes angle θ with the river current v_r (i.e., u) and $\vec{v}_s$ makes angle β with the river current. Then,

$$\tan\beta = \frac{v_{sr}\sin\theta}{v_r + v_{sr}\cos\theta} = \frac{v\sin\theta}{u + v\cos\theta} \quad ...(1)$$

Total length of the path is $PR = L$. Further

$$L = PR = \frac{l}{\sin\beta} = l\,\text{cosec}\,\beta$$

$$\Rightarrow \quad L = l\sqrt{1+\cot^2\beta}$$

$$\Rightarrow \quad L = l\sqrt{1+\left(\frac{u+v\cos\theta}{v\sin\theta}\right)^2}$$

$$\Rightarrow \quad L = \frac{l}{v\sin\theta}\sqrt{u^2\sin^2\theta+u^2+v^2\cos^2\theta+2uv\cos\theta}$$

$$\Rightarrow \quad L = \frac{l}{v\sin\theta}\sqrt{u^2+v^2+2uv\cos\theta}$$

For L to be MINIMUM or L^2 to be MINIMUM

$$\frac{d}{d\theta}(L^2)=0$$

$$\Rightarrow \quad \frac{d}{d\theta}\left[\frac{l^2}{v^2}\left(\frac{u^2+v^2+2uv\cos\theta}{\sin^2\theta}\right)\right]=0$$

$$\Rightarrow \quad \frac{\sin^2\theta(-2uv\sin\theta) - (u^2+v^2+2uv\cos\theta)(2\sin\theta\cos\theta)}{\sin^4\theta}=0$$

$$\Rightarrow \quad uv\sin^2\theta+(u^2+v^2+2uv\cos\theta)\cos\theta=0$$

$$\Rightarrow \quad uv(1-\cos^2\theta)+(u^2+v^2+2uv\cos\theta)\cos\theta=0$$

$$\Rightarrow \quad uv-uv\cos^2\theta+(u^2+v^2)\cos\theta+2uv\cos^2\theta=0$$

$$\Rightarrow \quad uv\cos^2\theta+(u^2+v^2)\cos\theta+uv=0$$

$$\Rightarrow \quad \cos\theta=-\frac{(u^2+v^2)\pm\sqrt{(u^2+v^2)^2-4u^2v^2}}{2uv}$$

$$\Rightarrow \quad \cos\theta=-\frac{(u^2+v^2)\pm\sqrt{(u^2+v^2)^2}}{2uv}$$

$$\Rightarrow \quad \cos\theta=-\frac{(u^2+v^2)\pm(u^2+v^2)}{2uv}$$

Either

$$\cos\theta=\frac{-u^2-v^2+u^2-v^2}{2uv}$$

$$\Rightarrow \quad \cos\theta=-\frac{v}{u} \qquad \ldots(2)$$

Equation (2) holds good only when $v<u$

Further put (2) and (3) one by one in (1), then

For $v<u$ we have

$$\cos\theta=-\frac{v}{u}$$

$$\Rightarrow \quad \tan\beta=\frac{v\sqrt{1-\cos^2\theta}}{u+v\cos\theta}$$

$$\Rightarrow \quad \tan\beta=\frac{v\sqrt{1-\frac{v^2}{u^2}}}{u+v\left(-\frac{v}{u}\right)}$$

$$\Rightarrow \quad \tan\beta=\frac{v}{\sqrt{u^2-v^2}}$$

$$\Rightarrow \quad \beta=\tan^{-1}\left(\frac{v}{\sqrt{u^2-v^2}}\right)$$

or

$$\cos\theta=\frac{-u^2-v^2-u^2+v^2}{2uv}$$

$$\Rightarrow \quad \cos\theta=-\frac{u}{v} \qquad \ldots(3)$$

Equation (3) holds good only when $v<u$

For $u<v$ we have

$$\cos\theta=-\frac{u}{v}$$

$$\Rightarrow \quad \tan\beta=\frac{v\sqrt{1-\cos^2\theta}}{u+v\cos\theta}$$

$$\Rightarrow \quad \tan\beta=\frac{v\sqrt{1-\frac{u^2}{v^2}}}{u+v\left(-\frac{u}{v}\right)}$$

$$\Rightarrow \quad \tan\beta\to\infty$$

$$\Rightarrow \quad \beta=\frac{\pi}{2}$$

So, we conclude that the direction of shortest route followed by the swimmer is at right angles to the river current when velocity of swimmer with respect to river current (v) is greater then river current (v_r) or u and for $v<u$ (the opposite case) it is in a direction $\beta=\tan^{-1}\left(\frac{v}{\sqrt{u^2-v^2}}\right)$

CATEGORY 3: AEROPLANE-WIND PROBLEMS

These problems proceed the same way as we have done for river-swimmer problems or river-boat problems. Here $\vec{v}_{sr}$ is just replaced by $\vec{v}_{aw}$ (velocity of aircraft with respect to wind), where $\vec{v}_{aw}=\vec{v}_a-\vec{v}_w$.

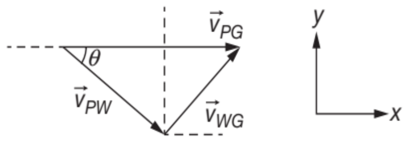
ILLUSTRATION 71

A plane moves in windy weather due east while the pilot points the plane somewhat south of east. The wind is blowing at 50 kmhr^{-1} directed 30° east of north, while the plane moves at 200 kmhr^{-1} relative to the wind. What is the velocity of the plane relative to the ground and what is the direction in which the pilot points the plane?

SOLUTION

Three vectors $\vec{v}_{PG}$, $\vec{v}_{PW}$ and $\vec{v}_{WG}$ are involved. We have to find the magnitude of $\vec{v}_{PG}$ and the direction of $\vec{v}_{PW}$ (i.e., θ). Let us first relate the three vectors using the concept of relative velocity. We can write

$$\begin{pmatrix}\text{Velocity of}\\ \text{plane}\\ \text{w.r.t. wind}\end{pmatrix}=\begin{pmatrix}\text{Velocity of}\\ \text{plane}\\ \text{w.r.t. ground}\end{pmatrix}-\begin{pmatrix}\text{Velocity of}\\ \text{wind}\\ \text{w.r.t. ground}\end{pmatrix}$$



$\Rightarrow \quad \vec{v}_{PW} = \vec{v}_{PG} - \vec{v}_{WG}$

$\Rightarrow \quad \vec{v}_{PG} = \vec{v}_{PW} + \vec{v}_{WG}$

Here we have two unknown quantities $(\vec{v}_{PG}$ and $\theta)$, so we should not relate the magnitudes of these vectors using triangle law. In such situations we resolve the vectors into components on the coordinate system and then solve the equations for both axes $(x$ and $y)$.

For the y components

$$v_{PG,y} = v_{PW,y} + v_{WG,y}$$

i.e., $0 = -(200)\sin\theta + (50\cos 30^\circ)$

Solving, we get

$$\sin\theta = \frac{\sqrt{3}}{8};\ \theta = \sin^{-1}(0.216) = 12^\circ$$

For the x components

$$v_{PG,x} = v_{PW,x} + v_{WG,x}$$

$$v_{PG} = (200\cos\theta) + (50\sin 30^\circ)$$

$$\Rightarrow \quad v_{PG} = 221 \text{ kmh}^{-1}$$

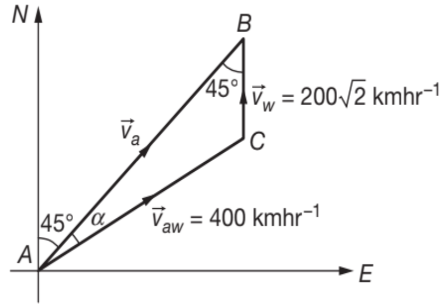
ILLUSTRATION 72

An aircraft flies at 400 kmhr^{-1} in still air. A wind of $200\sqrt{2} \text{ kmhr}^{-1}$ is blowing from the south. The pilot wishes to travel from A to a point B north east of A. Find the direction he must steer and time of his journey if $AB = 1000$ km.

SOLUTION

METHOD I

Given that $v_w = 200\sqrt{2} \text{ kmhr}^{-1}$, $v_{aw} = 400 \text{ kmhr}^{-1}$ and $\vec{v}_a$ should be along AB or in north-east direction. Thus, the direction of $\vec{v}_{aw}$ should be such as the resultant of $\vec{v}_w$ and $\vec{v}_{aw}$ is along AB or in north-east direction.



Let $\vec{v}_{aw}$ makes an angle α with AB as shown in figure. Applying sine law in triangle ABC, we get

$$\frac{AC}{\sin 45^\circ} = \frac{BC}{\sin\alpha}$$

$$\Rightarrow \quad \sin\alpha = \left(\frac{BC}{AC}\right)\sin 45^\circ = \left(\frac{200\sqrt{2}}{400}\right)\frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = 30^\circ$$

Therefore, the pilot should steer in a direction at an angle of $(45^\circ + \alpha)$ or 75° from north towards east.

Further, $\dfrac{|\vec{v}_a|}{\sin(180^\circ - 45^\circ - 30^\circ)} = \dfrac{400}{\sin 45^\circ}$

$$\Rightarrow \quad |\vec{v}_a| = \frac{\sin 105^\circ}{\sin 45^\circ} \times (400) \text{ kmh}^{-1}$$

$$\Rightarrow \quad |\vec{v}_a| = \left(\frac{\cos 15^\circ}{\sin 45^\circ}\right)(400) \text{ kmh}^{-1}$$

$$|\vec{v}_a| = \left(\frac{0.9659}{0.707}\right)(400) \text{ kmh}^{-1} = 546.47 \text{ kmh}^{-1}$$

So, the time of journey from A to B is

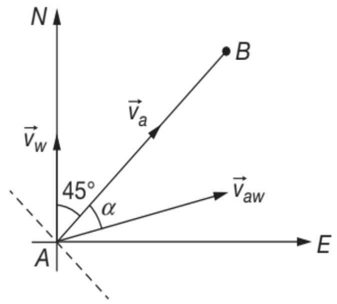
$$t = \frac{AB}{|\vec{v}_a|} = \frac{1000}{546.47} \text{ hr}$$

$$\Rightarrow \quad t = 1.83 \text{ hr}$$

METHOD II

Suppose a vector $\vec{C}$ is a vector sum of two vectors $\vec{A}$ and $\vec{B}$ and the direction of $\vec{C}$ is given to us. Let the vector $\vec{C}$ be directed along a line PQ. Then $\vec{A} + \vec{B} (= \vec{C})$ should be along the line PQ, i.e., the sum of components of $\vec{A}$ and $\vec{B}$ perpendicular to line PQ must be zero. Similarly, if $\vec{C} = \vec{A} - \vec{B}$ and $\vec{C}$ is directed along the line PQ, then the sum of components of $\vec{A}$ and $-\vec{B}$ perpendicular to line PQ must be zero.

For example, if $\vec{v}_a$ has to be along AB and we know that $\vec{v}_a = \vec{v}_{aw} + \vec{v}_w$. Therefore, sum of components of $\vec{v}_{aw}$ and $\vec{v}_w$ perpendicular to line AB (shown as dotted) should be zero.



So, $|\vec{v}_{aw}|\sin\alpha = |\vec{v}_w|\sin 45°$

$$\Rightarrow \quad \sin\alpha = \frac{|\vec{v}_w|}{|\vec{v}_{aw}|}\sin 45°$$

$$\Rightarrow \quad \sin\alpha = \left(\frac{200\sqrt{2}}{400}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = 30°$$

Now, $|\vec{v}_a| = |\vec{v}_{aw}|\cos\alpha + |\vec{v}_w|\cos 45°$

$$\Rightarrow \quad |\vec{v}_a| = (400)\cos 30° + (200\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad |\vec{v}_a| = (400)\frac{\sqrt{3}}{2} + 200$$

$$\Rightarrow \quad |\vec{v}_a| = 346.47 + 200 = 546.47 \text{ kmhr}^{-1}$$

Hence, the time of journey from A to B is given by

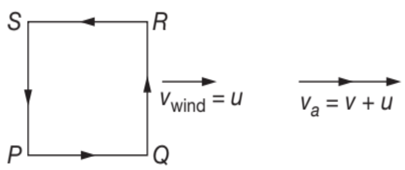
$$t = \frac{AB}{|\vec{v}_a|} = \frac{1000}{546.47} = 1.83 \text{ hr}$$

ILLUSTRATION 73

Find the time an aeroplane having velocity v, takes to fly around a square with side l if the wind is blowing at a velocity u along one side of the square.

SOLUTION

Velocity of aeroplane while flying from P to Q is

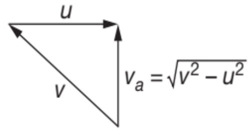


$$v_a = v + u$$

$$\Rightarrow \quad t_{PQ} = \frac{l}{v+u}$$

Velocity of aeroplane while flying from Q to R is

$$v_a = \sqrt{v^2 - u^2}$$

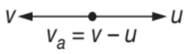


$$\Rightarrow \quad t_{QR} = \frac{l}{\sqrt{v^2 - u^2}}$$

Velocity of aeroplane while flying from R to S is

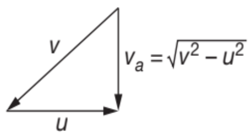
$$v_a = v - u$$

$$\Rightarrow \quad t_{RS} = \frac{l}{v-u}$$



Velocity of aeroplane while flying from S to P is

$$v_a = \sqrt{v^2 - u^2}$$



$$\Rightarrow \quad t_{SP} = \frac{l}{\sqrt{v^2 - u^2}}$$

Total time

$$t = t_{PQ} + t_{QR} + t_{RS} + t_{SP}$$

$$\Rightarrow \quad t = \frac{l}{v+u} + \frac{l}{\sqrt{v^2-u^2}} + \frac{l}{v-u} + \frac{l}{\sqrt{v^2-u^2}}$$

$$\Rightarrow \quad t = \frac{2a}{v^2 - u^2}\left(v + \sqrt{v^2 - u^2}\right)$$

CATEGORY 4: RAIN-MAN PROBLEMS

In such problems we come across the following terminology according to which

(a) $\vec{v}_r$ is the absolute velocity of rain.

(b) $\vec{v}_m$ is the absolute velocity of man/cyclist/motorist/observer.

(c) $\vec{v}_w$ is the absolute velocity of wind (if it is blowing) and

(d) $\vec{v}_{rm}$ is the velocity of rain with respect to man or the velocity of rain which appears to the man.

For dealing with such like problems, we have two options

Option A: When no wind is blowing.

$$\vec{v}_{rm} = \vec{v}_r - \vec{v}_m \quad \text{or} \quad \vec{v}_{rm} = \vec{v}_r + (-\vec{v}_m)$$

So, while dealing with the problems in which rain and man are there but no wind exists, then to calculate the direction of $\vec{v}_{rm}$, we simply reverse the direction of man's velocity $(-\vec{v}_m)$ and then find the resultant of $-\vec{v}_m$ and $\vec{v}_r$ i.e., $-\vec{v}_r + (-\vec{v}_m)$ to get the direction and magnitude of $\vec{v}_{rm}$. It's the Best Trick! Try following and see the results.

Option B: When wind is blowing.

$$\vec{v}_{rm} = (\vec{v}_r + \vec{v}_w) - \vec{v}_m \text{ or } \vec{v}_{rm} = \vec{v}_r + \vec{v}_w + (-\vec{v}_m)$$

So, while dealing with the problems involving rain, man and the blowing wind we first calculate the resultant of rain and wind i.e., net velocity of rain under the inference of wind $\vec{v}_r + \vec{v}_w = \vec{v}_{\text{netrain}}$. The man sees this rain falling with a velocity $\vec{v}_{rm} = \vec{v}_{\text{netrain}} - \vec{v}_m$

$$\Rightarrow \quad \vec{v}_{rm} = (\vec{v}_r + \vec{v}_w) - \vec{v}_m$$

$$\Rightarrow \quad \vec{v}_{rm} = (\vec{v}_r + \vec{v}_w) + (-\vec{v}_m)$$

Again, we reverse the direction of velocity of man and then find the resultant of $\vec{v}_{\text{netrain}}$ and $-\vec{v}_m$ to get $\vec{v}_{rm}$ with magnitude and direction.

So, to deal with problems involving rain, man and wind we just reverse the direction of $\vec{v}_m$ i.e., make it $-\vec{v}_m$ and then find the resultant of $\vec{v}_r$, $\vec{v}_w$ and $-\vec{v}_m$ i.e., $\vec{v}_r + \vec{v}_w + (-\vec{v}_m)$.

ILLUSTRATION 74

A standing man, observes rain falling with velocity of 20 ms^{-1} at an angle of $30°$ with the vertical.

(a) Find the velocity with which the man should move so that rain appears to fall vertically to him.

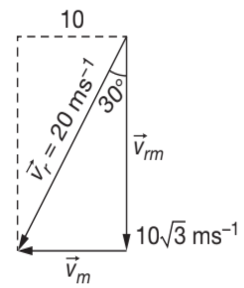
(b) Now if he further increases his speed, rain again appears to fall at $30°$ with the vertical. Find his new velocity.

SOLUTION

(a) $\vec{v}_m = -v\hat{i}$ (let)

$$\vec{v}_r = -10\hat{i} - 10\sqrt{3}\hat{j}$$

$$\vec{v}_{rm} = -(10 - v)\hat{i} - 10\sqrt{3}\hat{j}$$



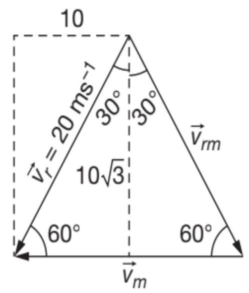
$\Rightarrow \quad -(10 - v) = 0$ (for vertical fall, horizontal component must be zero)

$\Rightarrow \quad v = 10 \text{ ms}^{-1}$

(b) $\vec{v}_r = -10\hat{i} - 10\sqrt{3}\hat{j}$

$$\vec{v}_m = -\vec{v}_x\hat{i}$$

$$\vec{v}_{rm} = -(10 - v_x)\hat{i} - 10\sqrt{3}\hat{j}$$



Angle with the vertical $= 30°$

$$\Rightarrow \quad \tan 30° = \frac{10 - v_x}{-10\sqrt{3}}$$

$$\Rightarrow \quad v_x = 20 \text{ ms}^{-1}$$

ILLUSTRATION 75

To a man walking at the rate of 3 kmh^{-1} the rain appears to fall vertically. When he increases his speed to 6 kmh^{-1} it appears to meet him at an angle of $45°$ will vertical. Find the speed of rain.

SOLUTION

Let $\hat{i}$ and $\hat{j}$ be the unit vectors along the horizontal and vertical directions respectively.

Let velocity of rain be $\vec{v}_r = a\hat{i} + b\hat{j}$ ...(1)

Then speed of rain will be

$$|\vec{v}_r| = \sqrt{a^2 + b^2} \quad \text{...(2)}$$

In the first case $\vec{v}_m$ = velocity of man $= 3\hat{i}$

$$\Rightarrow \quad \vec{v}_{rm} = \vec{v}_r - \vec{v}_m = (a-3)\hat{i} + b\hat{j}$$

It seems to be in vertical direction. Hence, the horizontal component must be zero.

$$\Rightarrow \quad a - 3 = 0$$

$$\Rightarrow \quad a = 3$$

In the second case $\vec{v}_m = 6\hat{i}$

$$\Rightarrow \quad \vec{v}_{rm} = (a-6)\hat{i} + b\hat{j} = -3\hat{i} + b\hat{j}$$

This seems to be at 45° with the vertical

$$\Rightarrow \quad \tan(45°) = \frac{|v_y|}{|v_x|} = \frac{|b|}{3}$$

$$\Rightarrow \quad |b| = 3$$

Therefore, from equation (2) speed of rain is

$$|\vec{v}_r| = \sqrt{(3)^2 + (3)^2} = 3\sqrt{2}\ \text{kmh}^{-1}$$

Test Your Concepts-VII

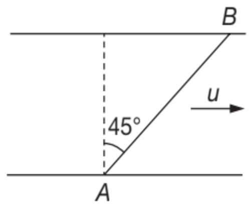
Based on Relative Velocity

(Solutions on page H.86)

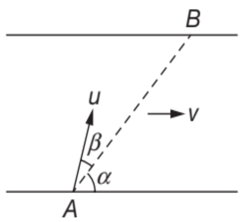
1. A river 400 m wide is flowing at a rate of 2 ms^{-1}. A boat is sailing at a velocity of 10 ms^{-1} with respect to the water, in a direction perpendicular to the river.
   (a) Find the time taken by the boat to reach the opposite bank.
   (b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?
2. Snow is falling vertically at a constant speed of 8 ms^{-1}. At what angle from the vertical do the snow flakes appear to be falling as viewed by the driver of a car travelling on a straight, level road with a speed of 50 kmh^{-1}?
3. Show that the direction of shortest route is at right angles to the river when the velocity of boat with respect to water v is greater than that of the river velocity u and in opposite case it is $\tan^{-1}\left(\frac{v}{\sqrt{u^2 - v^2}}\right)$.
4. A ship A is travelling due east at 10 $kmhr^{-1}$ and at 9 am is 30 km south-west of another ship B. If B travels at 15 $kmhr^{-1}$ so as to intercept A calculate the
   (a) direction in which B must travel.
   (b) time it takes when the interception takes place.
5. A motorboat going downstream overcame a raft at point A. One hour later it turned back and met the raft again at a distance 6 km from point A. Find the river velocity.
6. An elevator car whose floor to ceiling distance is equal to 2.7 m starts ascending with constant acceleration 1.2 ms^{-2}. 2 second after the start, a bolt begins falling from the ceiling of the car. Find the
   (a) time after which bolt hits the floor of the elevator.
   (b) net displacement and distance travelled by the bolt, with respect to earth.
   $\left(\text{Take } g = 9.8\ ms^{-2}\right)$
7. A particle is moving in a circle of radius r centred at O with constant speed v. Calculate the change in velocity in moving from A to B when $\angle AOB = 40°$.
8. Two ships A and B are 10 km apart on a line running south to north. Ship A farther north is streaming west at 20 $kmhr^{-1}$ and ship B is streaming north at 20 $kmhr^{-1}$. What is their distance of closest approach and how long do they take to reach it?
9. Car A has an acceleration of 2 ms^{-2} due east and car B, 4 ms^{-2} due north. What is the acceleration of car B with respect to car A?
10. A bullet train A starts from rest at $t = 0$ and travels along a straight track with a constant acceleration of 6 ms^{-2} until it reaches a speed of 80 ms^{-1}. Afterwards it maintains this speed. Also, when $t = 0$, another bullet train B located 6000 m down on a parallel track is travelling towards A at a constant speed of 60 ms^{-1}. Determine the distance travelled by train A when they pass each other.

11. A body is thrown up in a lift with a velocity u relative to the lift. If the time of flight is found to be t then find the acceleration of the lift.

12. A man wants to reach point B on the opposite bank of a river flowing at a speed as shown in the figure. What minimum speed relative to water should the man have so that the can reach point B? In which direction should he swim?



13. A launch plies between two points A and B on the opposite banks of a river always following the line AB. The distance S between points A and B is 1200 m. The velocity of the river current $v = 1.9\ \text{ms}^{-1}$ is constant over the entire width of the river. The line AB makes an angle $\alpha = 60°$ with the direction of the current. With what velocity u and at what angle β to the line AB should the launch move to cover the distance AB and back in a time $t = 5$ min? The angle β remains the same during the passage from A to B and from B to A.



14. A man wishes to cross a river of width 120 m by a motorboat. His rowing speed in still water is $3\ \text{ms}^{-1}$ and his maximum walking speed is $1\ \text{ms}^{-1}$. The river flows with velocity of $4\ \text{ms}^{-1}$.

(a) Find the path which he should take to get to the point directly opposite to his starting point in the shortest time.

(b) Also, find the time which he takes to reach his destination.

15. A ball is thrown vertically upward from the 12 m level in an elevator shaft with an initial velocity of $18\ \text{ms}^{-1}$. At the same instant an open platform type elevator passes the 5 m level, moving upward with a constant velocity of $2\ \text{ms}^{-1}$. Find

(a) when and where the ball will hit the elevator

(b) the relative velocity of the ball with respect to the elevator when the ball hits the elevator.

16. A swimmer heads directly across a river, swimming at $1.6\ \text{ms}^{-1}$ relative to still water. He arrives at a point 40 m downstream from the point directly across the river, which is 80 m wide.

(a) What is the speed of the river current?

(b) What is the swimmer's speed relative to the shore?

(c) In what direction should the swimmer head so as to arrive at the point directly opposite to the starting point?

17. A cyclist, riding at a speed V, overtakes a pedestrian who can move at a speed not greater than v, the two travelling along parallel tracks at a distance d apart. Show that if the cyclist rings his bell when at a distance less that $\frac{V}{v}d$, he may safely maintain his speed and keep to his course regardless of the behaviour of the pedestrian.

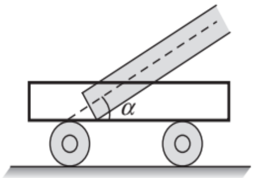
18. A swimmer wishes to cross a 500 m wide river flowing at $5\ \text{kmh}^{-1}$. His speed with respect to water is $3\ \text{kmh}^{-1}$.

(a) If he heads in a direction making an angle θ with the flow, find the time he takes to cross the river.

(b) Find the shortest possible time to cross the river.

(c) Assuming that the swimmer has to reach the other shore at the point directly opposite to his starting point. If he reaches the other shore somewhere else, he has to walk down to this point. Find the minimum distance that he has to walk.

19. A pipe which can be rotated in a vertical plane is mounted on a cart. The cart moves uniformly along a horizontal path with a velocity $v_1 = 2\ \text{ms}^{-1}$. At what angle α to the horizontal should the pipe be placed so that drops of rain falling vertically with a velocity $v_2 = 6\ \text{ms}^{-1}$ move parallel to the walls of the pipe without touching them.



20. Two mirrors, mounted vertically, are made to move towards each other with a speed v each. A particle that can bounce back between the two mirrors starts from one mirror when the mirrors are d apart. On reaching the second mirror, it bounces back and so on. If the particle keeps on travelling at a constant speed of 3 v, how many trips can it make before the mirrors run into each other? What total distance does it cover?

SOLVED PROBLEMS

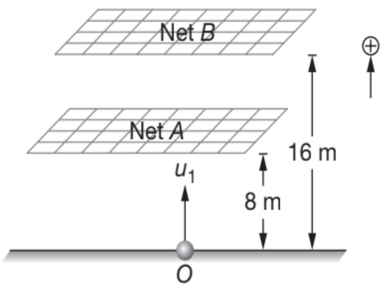
PROBLEM 1

A particle is projected vertically with a speed of $32.9\ \text{ms}^{-1}$ from a point O. It passes successively through two horizontal thin nets at height $8\ \text{m}$ and $16\ \text{m}$ respectively, above O. Find the greatest height above O that the particle attains net and its speed on reaching back to O. Assume that each thin net has a mesh which is broad enough to allow the ball to cross the net but witch half the velocity with which the ball enters.

SOLUTION

Take upward direction as positive. In the upward journey, let v_1 be the velocity of the particle just before it enters the net A. Then

$$v_1^2 = u_1^2 - 2gh = (39.2)^2 - 2\times 9.8\times 8$$



Solving it, we get $v_1 = 37.15\ \text{ms}^{-1}$

Velocity of the particle after crossing the net is $\dfrac{v_1}{2}$. So

$$\frac{v_1}{2} = 18.575\ \text{ms}^{-1}$$

Let v_2 be the velocity of the particle just before it enters the net B. Then

$$v_2^2 = (18.575)^2 - 2\times 9.8\times 8$$

Solving it, we get $v_2 = 13.72\ \text{ms}^{-1}$

Velocity of the particle after it crosses the net B is

$$v_2 = \frac{13.72}{2} = 6.86\ \text{ms}^{-1}$$

Let h be the height reached above the net B where velocity of particle becomes zero. Then

$$(0)^2 = (6.86)^2 - 2\times 9.8\times h$$

$$\Rightarrow\quad h = 2.4\ \text{m}$$

$\therefore$ Total height above O is $16 + 2.4 = 18.4\ \text{m}$

In the return journey, the velocity of the particle at the time of reaching net B is given by

$$v_3^2 = (0)^2 + 2gh$$

$$\Rightarrow\quad v_3 = \sqrt{2gh} = \sqrt{2\times 9.8\times 2.4}$$

Velocity after crossing net B is $\dfrac{1}{2}\sqrt{2\times 9.8\times 2.4}$

Velocity at net A is $\sqrt{\left(\dfrac{2\times 9.8\times 2.4}{4}\right) + 2\times 9.8\times 8}$

$$(v_A)_{\text{down}} = \sqrt{9.8\times 17.2}$$

Velocity after crossing the net A is $\dfrac{1}{2}\sqrt{9.8\times 17.2}$

Let the velocity at O be v_4. Then

$$v_4^2 = \frac{(9.8\times 17.2)}{4} + 2\times 9.8\times 8$$

$$\Rightarrow\quad v_4 = 14.1\ \text{ms}^{-1}$$

PROBLEM 2

Find the trajectory of the particle of mass m acted upon by a force $\vec{F} = \cos t\hat{i} + \sin t\hat{j}$. At $t = 0$, the position and velocity of particle are $x_0\hat{i}$ and $v_0\hat{j}$ respectively.

SOLUTION

Since, $\vec{F} = m\dfrac{d^2\vec{r}}{dt^2} = \cos t\hat{i} + \sin t\hat{j}$ {Given}

Integrating both sides w.r.t. t, we have

$$\int\left(m\frac{d^2\vec{r}}{dt^2}\right)dt = \int(\cos t\hat{i} + \sin t\hat{j})dt$$

$$\Rightarrow\quad m\frac{d\vec{r}}{dt} = \sin t\hat{i} + (-\cos t)\hat{j} + c_1 \qquad \ldots(1)$$

At time $t = 0$, we have ...(2)

$$mv_0\hat{j} = -\hat{j} + c_1 \qquad \{\text{putting } t = 0 \text{ in equation (1)}\}$$

$$\Rightarrow\quad c_1 = (mv_0 + 1)\hat{j}$$

Substituting value of c_1 in equation (1) we get

$$m\frac{d\vec{r}}{dt} = \sin t\hat{i} + (-\cos t)\hat{j} + (mv_0 + 1)\hat{j}$$

$$\Rightarrow \quad m\frac{d\vec{r}}{dt} = \sin t\hat{i} + (mv_0 + 1 - \cos t)\hat{j}$$

Integrating again both sides w.r.t. t, we have

$$\int \left(m\frac{d\vec{r}}{dt} \right) dt = \int (\sin t\hat{i} + (mv_0 + 1 - \cos t)\hat{j}) dt$$

$$\Rightarrow \quad m\vec{r} = (-\cos t)\hat{i} + (mv_0 t + t - \sin t)\hat{j} + c_2 \qquad \ldots(3)$$

Again at time $t = 0$, $\vec{r} = x_0\hat{i}$, thus equation (3) gives us

$$c_2 = (mx_0 + 1)\hat{i}$$

Substituting value of c_2 in equation (3), we get

$$m\vec{r} = (mx_0 + 1 - \cos t)\hat{i} + (mv_0 t + t - \sin t)\hat{j}$$

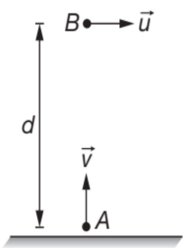
$$\Rightarrow \quad \vec{r} = \frac{1}{m}\left[(mx_0 + 1 - \cos t)\hat{i} + (mv_0 t + t - \sin t)\hat{j} \right] \qquad \ldots(4)$$

Comparing equation (4) with $\vec{r} = x\hat{i} + y\hat{j}$

$$\therefore \quad x = x_0 + \left(\frac{1 - \cos t}{m} \right) \text{ and } y = v_0 t + \left(\frac{t - \sin t}{m} \right)$$

PROBLEM 3

A sitting cat in a field suddenly sees a standing dog. To save its life the cat runs away in a straight horizontal line with constant velocity $\vec{u}$. Without any time lag the dog starts with velocity $\vec{v}$ constant in magnitude and always directed towards the cat to catch it. Initially, $\vec{v}$ and $\vec{u}$ are perpendicular and separation between the cat and the dog is d. Find the time after which the dog catches the cat.

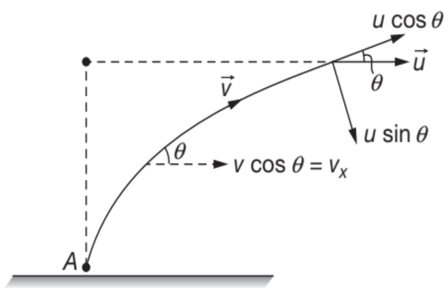


At $t = 0$

SOLUTION

Initially the particles are at A and B, a distance d apart and their velocities are perpendicular to each other.

The velocity $\vec{u}$ is constant in magnitude and direction while the velocity $\vec{v}$ changes continuously in direction and is always aimed at B. Let, at any instant t, $\vec{v}$



At any time t (say)

makes an angle θ with $\vec{u}$. Then resolving $\vec{u}$ into two components

(a) $u\cos\theta$, parallel to $\vec{v}$, and

(b) $u\sin\theta$, perpendicular to $\vec{v}$.

Therefore, relative velocity of approach of A towards B is

$$(v - u\cos\theta) = -\frac{dx}{dt}$$

(Negative sign indicates that distance between particles decreases with time)

$$\Rightarrow \quad -dx = (v - u\cos\theta)dt \qquad \ldots(1)$$

Integrating both sides of equation (1),

$$\Rightarrow \quad -\int_d^0 dx = \int_0^t (v - u\cos\theta)\, dt$$

$$\Rightarrow \quad d = \int_0^t v\, dt - \int_0^t u\cos\theta dt \qquad \ldots(2)$$

Since $u < v$, Therefore, $u\cos\theta < v$ and hence, they meet.

Also, the horizontal distance travelled by both of them is the same, when they meet.

Therefore, $\Delta x = \int v_x\, dt$

$$\Rightarrow \quad ut = \int_0^t v\cos\theta dt$$

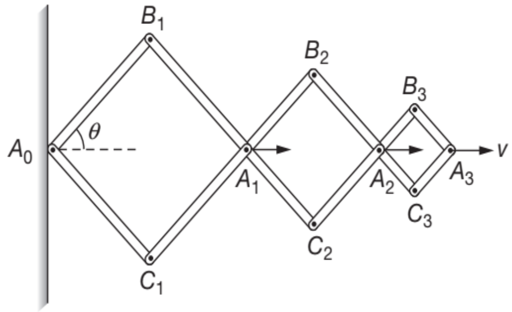
$$\Rightarrow \quad d = v\int_0^t dt - u\int_0^t \cos\theta\, dt$$

$$\Rightarrow \quad d = vt - u\left(\frac{ut}{v} \right)$$

$$\Rightarrow \quad t = \frac{vd}{v^2 - u^2}$$

PROBLEM 4

A hinged construction consists of three rhombus with ratio of sides $3:2:1$ (as shown in figure). Vertex A_3 moves in a horizontal direction with velocity v. Determine velocities of vertices A_1, A_2 and B_2 at instant when angle of construction is $45°$.



SOLUTION

Let A_0 be the Origin of the whole system shown in the diagram. Let

$$A_0B_1 = B_1A_1 = 3k$$

$$A_1B_2 = B_2A_2 = 2k$$

$$A_2B_3 = B_3A_3 = k$$

Hence, $A_0A_1 = 3K\cos\theta + 3K\cos\theta = 6K\cos\theta = x_{A_1}$

$$A_0A_2 = A_0A_1 + A_1A_2 = 6K\cos\theta + 4K\cos\theta$$

$$\Rightarrow \quad x_{A_2} = 10K\cos\theta$$

$$A_0A_3 = A_0A_2 + A_2A_3 = 10K\cos\theta + 2K\cos\theta$$

$$\Rightarrow \quad x_{A_3} = 12K\cos\theta$$

Similarly,

$$x_{B_2} = 6K\cos\theta + 2K\cos\theta = 8K\cos\theta,$$

and

$$y_{B_2} = 2K\sin\theta$$

According to Problem, $\dfrac{dx_{A_3}}{dt} = v$

$$\Rightarrow \quad -12K\sin\theta\frac{d\theta}{dt} = v$$

$$\Rightarrow \quad -K\sin\theta\frac{d\theta}{dt} = \frac{v}{12}$$

Therefore,

$$v_{A_2} = \frac{dx_{A_2}}{dt} = -10K\sin\theta\left(\frac{d\theta}{dt}\right) = 10\left(\frac{v}{12}\right) = \frac{5}{6}v$$

$$\Rightarrow \quad v_{A_1} = \frac{dx_{A_1}}{dt} = -6K\sin\theta\left(\frac{d\theta}{dt}\right) = 6\left(\frac{v}{12}\right) = \frac{v}{2}$$

For B_2:

$$\left(v_{B_2}\right)_x = \frac{dx_{B_2}}{dt} = -8K\sin\theta\left(\frac{d\theta}{dt}\right)$$

$$\Rightarrow \quad \left(v_{B_2}\right)_y = \frac{dy_{B_2}}{dt} = 2K\cos\theta\left(\frac{d\theta}{dt}\right)$$

Now, $\left(v_{B_2}\right)^2 = \left(v_{B_2}\right)_x^2 + \left(v_{B_2}\right)_y^2$

$$\Rightarrow \quad \left(v_{B_2}\right)^2 = 64\left(K\sin\theta\frac{d\theta}{dt}\right)^2 + 4\cos^2\theta\left(K\frac{d\theta}{dt}\right)^2$$

$$\Rightarrow \quad \left(v_{B_2}\right)^2 = 64\left(\frac{v}{12}\right)^2 + 4\cos^2\theta\left(K\frac{d\theta}{dt}\right)$$

$$\Rightarrow \quad \left(v_{B_2}\right)^2 = 64\left(\frac{v}{12}\right)^2 + 4\cos^2\theta\left(\left(\frac{v}{12}\right)^2\frac{1}{\sin^2\theta}\right)$$

$$\Rightarrow \quad \left(v_{B_2}\right)^2 = \left(\frac{v}{12}\right)^2\left(64 + 4\cot^2\theta\right)$$

$$\Rightarrow \quad v_{B_2} = \frac{v}{12}\sqrt{64 + 4\cot^2\theta}$$

Now when θ is $45°$, then

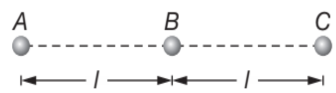
$$\Rightarrow \quad v_{B_2} = \frac{v}{12}\sqrt{68} = \frac{\sqrt{17}}{6}v$$

PROBLEM 5

At the initial moment three points A, B, and C are on a horizontal straight line at equal distance from one another. Point A begins to move vertically upwards with a constant velocity v and point C vertically downward without any initial velocity at a constant acceleration a. How should the point B move vertically for all the three points to be constantly on one straight line? The points begin to move simultaneously.

SOLUTION

Initially, at $t = 0$, all the particles are equidistant from each other.



Let at time t the particle A goes to A', B goes to B' and C goes to C'

$$AA' = vt$$

$$BB' = u_B t + \frac{1}{2}a_B t^2$$

$$CC' = \frac{1}{2}at^2$$

In similar triangles AOA', BOB' and COC' we have

$$\frac{vt}{l-x} = \frac{u_B t + \frac{1}{2}a_B t^2}{x} = \frac{\frac{1}{2}at^2}{l+x} \quad \ldots(1)$$

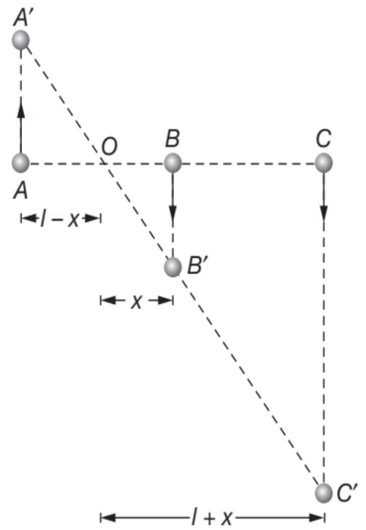
$$\Rightarrow \quad \frac{2v}{l-x} = \frac{at}{l+x}$$

$$\Rightarrow \quad x = \left(\frac{at-2v}{at+2v}\right)l \quad \ldots(2)$$

Put (2) in (1) we get

$$u_B t + \frac{1}{2}a_B t^2 = \left(\frac{-v}{2}\right)t + \frac{1}{2}\left(\frac{a}{2}\right)t^2$$

i.e., $u_B = -\dfrac{v}{2}$ and $a_B = \dfrac{a}{2}$



Hence, for all the particles to be on one straight line the particle at B must move with an initial velocity $\frac{v}{2}$ in the upward direction and an acceleration $\frac{a}{2}$ in the downward direction.

PROBLEM 6

Two boats, A and B, move away from a buoy anchored at middle of the river along mutually perpendicular straight lines. The boat A along the river and the boat B across the river. Having moved off an equal distance from the buoy the boats returned. Find the ratio of the times of motion of boats $\frac{\tau_A}{\tau_B}$ if the velocity of each boat w.r.t. water is $\eta = 1.2$ times greater than the stream velocity.

SOLUTION

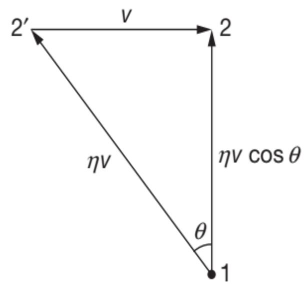
Let, d is the distance moved by boats, A and B, away from boys in direction of motion (or perpendicular to) stream velocity when they go from point 1 to 2.

v is the velocity of stream

Therefore, ηv is the velocity of the boat.

For Boat A:

$$\begin{pmatrix}\text{Total time}\\ \text{taken by}\\ \text{Boat A}\end{pmatrix} = \begin{pmatrix}\text{Time taken}\\ \text{downstream}\\ \text{by Boat A}\end{pmatrix} + \begin{pmatrix}\text{Time taken}\\ \text{upstream}\\ \text{by Boat A}\end{pmatrix}$$



$$\Rightarrow \quad t_A = t_{down} + t_{up}$$

$$\Rightarrow \quad t_A = \frac{d}{\eta v + v} + \frac{d}{\eta v - v}$$

$$\Rightarrow \quad t_A = \frac{2d\eta}{v(\eta^2 - 1)}$$

For Boat B: Time taken by Boat B in going from 1 to 2

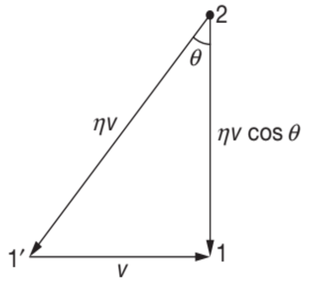
$$t_{1\to 2} = \frac{d}{\eta v \cos\theta} \quad \ldots(1)$$

and $\sin\theta = \dfrac{v}{\eta v} = \dfrac{1}{\eta}$

$$\Rightarrow \quad \cos\theta = \sqrt{1 - \frac{1}{\eta^2}}$$

Substituting respective values in (1),

$$\Rightarrow \quad t_{1\to 2} = \frac{d}{v\sqrt{\eta^2 - 1}}$$



Time taken by boat B to return from 2 to 1

$$\Rightarrow \quad t_{2\to1} = \frac{d}{v\sqrt{\eta^2 - 1}}$$

$$\Rightarrow \quad t_B = t_{1\to2} + t_{2\to1} = \frac{2d}{v\sqrt{\eta^2 - 1}}$$

$$\Rightarrow \quad \frac{t_A}{t_B} = \frac{\eta}{\sqrt{\eta^2 - 1}} = 1.8$$

PROBLEM 7

A particle of mass 10^{-2} kg is moving along the positive x-axis under the influence of force $F(x) = -\frac{k}{2x^2}$, $k = 10^{-2}$ Nm^2. At $t = 0$, it is at $x = 1$ m and its velocity is $v = 0$

(a) Find its velocity when it reaches $x = 0.5$ m.
(b) Time at which it reaches $x = 0.25$ m.

SOLUTION

(a) $F(x) = -\frac{k}{2x^2}$ {Given}

$$\Rightarrow \quad mv\frac{dv}{dx} = -\frac{k}{2x^2}$$

$$\Rightarrow \quad mvdv = -\frac{k}{2x^2}dx \qquad \ldots(1)$$

Integrating (1) within suitable limits

$$\Rightarrow \quad m\int_0^v vdv = -\frac{k}{2}\int_{1.0}^{0.5} x^{-2}dx$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = \frac{k}{2}\left[\frac{1}{0.5} - 1\right]$$

$$\Rightarrow \quad v^2 = +1$$

$$\Rightarrow \quad v = \pm 1 \text{ ms}^{-1}$$

Since the particle is going from $x = 1$ m to $x = 0.5$ m, hence, its velocity is directed along the negative x-axis.

$$\Rightarrow \quad \vec{v} = (-1)\hat{i}$$

(b) $v^2 = \frac{k}{m}\left[\frac{1}{x} - 1\right]$

$$\Rightarrow \quad -v = \sqrt{\frac{1-x}{x}} \text{ \{directed towards negative axis\}}$$

$$\Rightarrow \quad -\frac{dx}{dt} = \sqrt{\frac{1-x}{x}} \qquad \ldots(2)$$

Integrating relation (2) within suitable limits

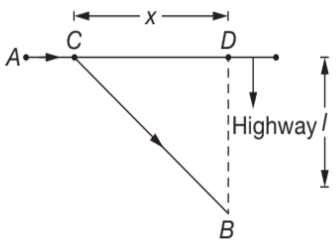
$$\Rightarrow \quad \int_0^t dt = -\int_1^{\frac{1}{4}} \sqrt{\frac{x}{1-x}}dx$$

Solving, by substituting $x = \sin^2\theta$, we get

$$t = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) \text{ s}$$

PROBLEM 8

From point A located on a highway one has to get by car as soon as possible to point B located in the field at a distance l from highway. It is known that car moves in the field η times slower than on highway. At what distance from D one must turn off the highway?



SOLUTION

Total Time = $t_{A\to C} + t_{C\to B}$

$$\Rightarrow \quad t = \frac{AC}{v} + \frac{CB}{\frac{v}{\eta}}$$

$$\Rightarrow \quad t = \frac{AD - x}{v} + \frac{\sqrt{x^2 + l^2}}{\frac{v}{\eta}}$$

The car has to reach B as soon as possible

$$\therefore \quad \frac{dt}{dx} = 0$$

$$\Rightarrow \quad t = -\frac{1}{v} + \frac{1}{\frac{v}{\eta}} \times \frac{2x}{2\sqrt{x^2 + l^2}}$$

$$\Rightarrow \quad \frac{\eta^2 x^2}{x^2 + l^2} = 1$$

$$\Rightarrow \quad x = \frac{l}{\sqrt{\eta^2 - 1}}$$

PROBLEM 9

A particle moves in x-y plane with velocity $\vec{v} = k_1\hat{i} + k_2 x\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along x and y axes, k_1 and k_2 are constants. Initially the particle is at origin (0, 0). Find

(a) equation of particles trajectory $y(x)$,

(b) curvature radius of trajectory of particle as a function of x.

SOLUTION

(a) $\vec{v} = k_1\hat{i} + k_2 x\hat{j}$ {Given}

$$\vec{v}_x = k_1\hat{i}$$

$$\frac{dx}{dt} = k_1$$

$$\int_0^x dx = \int_0^t k_1 dt$$

$$x = k_1 t \qquad \ldots(1)$$

$$\vec{v}_y = k_2 x\hat{j}$$

$$\frac{dy}{dt} = k_2 x = k_1 k_2 t$$

$$\int_0^y dy = k_1 k_2 \int_0^t t\,dt$$

$$y = \frac{k_1 k_2}{2} t^2 \qquad \ldots(2)$$

Eliminating t from the above set of two equations, we have

$$y = \left(\frac{k_2}{2k_1}\right) x^2 \qquad \ldots(3)$$

Equation (1) sounds much more like the equation of the parabola

(b) Let R be the radius of the parabolic equation,

$$\Rightarrow \quad R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)}$$

$$\Rightarrow \quad R = \frac{\left[1 + \left(\frac{k_2 x}{k_1}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{k_2}{k_1}\right)}$$

$$\Rightarrow \quad R = \frac{k_1}{k_2}\left[1 + \left(\frac{k_2 x}{k_1}\right)^2\right]^{\frac{3}{2}}$$

PROBLEM 10

A helicopter takes off along the vertical with an acceleration $a = 3\ \text{ms}^{-2}$ and zero initial velocity. In a certain time t_1, the pilot switches off the engine. At the point of take off, the sound dies away in time $t_2 = 30\ \text{s}$. Determine the velocity v of the helicopter at the moment when the engine is switched off; assuming that the velocity of sound, $c = 320\ \text{ms}^{-1}$.

SOLUTION

Since, $u = \text{ms}^{-1}$, $a = 3\ \text{ms}^{-2}$, $c = 320\ \text{ms}^{-1}$ and $t_2 = 30$ s {Given}

Velocity of helicopter at time t_1 is

$$v = 0 + 3t_1$$

$$\Rightarrow \quad v = 3t_1$$

Distance traversed by helicopter in time t_1 is

$$s = 0 + \frac{1}{2} \times 3 \times t_1^2$$

$$\Rightarrow \quad s = \frac{3}{2} t_1^2$$

Now, $t_2 = t_1 + \frac{s}{c}$ {Given}

$$\Rightarrow \quad 30 = t_1 + \frac{3}{2}\frac{t_1^2}{320}$$

$\Rightarrow \quad 3t_1^2 + 640\, t_1 - 19200 = 0$

$\Rightarrow \quad t_1 = \dfrac{-640 \pm \sqrt{(640)^2 + 4 \times 3 \times 19200}}{6}$

$\Rightarrow \quad t_1 = \dfrac{-640 \pm 800}{6}$

As time cannot be negative,

$\Rightarrow \quad t_1 = \dfrac{-640 + 800}{6} = \dfrac{160}{6} = \dfrac{80}{3}$ s

$\Rightarrow \quad v = 3t_1 = 3 \times \dfrac{80}{3}$

$\Rightarrow \quad v = 80 \text{ ms}^{-1}$

PROBLEM 11

A steel ball is dropped from a roof of a building. An observer standing in front of a window 1 m high notes that the ball takes 0.1 sec to fall from the top to the bottom of the window. The ball continues to fall and makes a complete elastic collision with the ground, so that it rebounds with the same velocity with which it strikes the ground. The ball reappears at the bottom of the window 2 second after passing the bottom of the window on the way down, find the height of the building (Take $g = 10 \text{ ms}^{-2}$)

SOLUTION

Let v_1 and v_2 be the velocities of a ball at the top and bottom of the windows respectively. Let h_1 be the height of the building above the top of the window and h_2 that below the bottom of window, then

Height of Building $= h_1 +$ Height of Window $+ h_2$

$\Rightarrow$ Height of Building $= (h_1 + 1 + h_2)$ m

For the path between the top and bottom of window,

We have $h = 1$ m, $t = 0.1$ s

$a = g = 10 \text{ ms}^{-2}$

$h = v_1 t + \dfrac{1}{2} g t^2$

$\Rightarrow \quad 1 = v_1 \times 0.1 + \dfrac{1}{2} \times (10) \times (0.1)^2$

$\Rightarrow \quad v_1 = 9.5 \text{ ms}^{-1}$

Also from equation $v = u + at$

$\Rightarrow \quad v_2 = v_1 + 10 \times 0.1 = 9.5 + 1 = 10.5 \text{ ms}^{-1}$

Also, from equation of motion $v^2 - u^2 = 2as$,

$\Rightarrow \quad v_1^2 = 0 + 2 \times 10 \times h_1$

$\Rightarrow \quad h_1 = \dfrac{v_1^2}{2 \times 10} = \dfrac{(9.5)^2}{2 \times 10} = 4.51$ m

$\Rightarrow$ Path above top of Window $= 4.51$ m

For the path below the bottom of window, the ball reappears at the bottom of window to second after passing the bottom on its way down. Hence, it takes a time of 1 s to fall from the bottom of window to the ground and rebounds to the same height in a further time of 1 s.

$\Rightarrow \quad h_2 = v_2 t + \dfrac{1}{2} g t^2$

$\Rightarrow \quad h_2 = 10.5 \times 1 + \dfrac{1}{2} \times 10 \times 1^2$

$\Rightarrow \quad h_2 = 10.5 + 5$

$\Rightarrow$ Path below the bottom of window $= 15.5$ m

Hence, Height of the Building is $H = 4.51 + 1 + 15.5$

$\Rightarrow \quad H = 21.01$ m

PROBLEM 12

A driver having a definite reaction time (i.e., the interval between the perception of a signal to stop and the application of brakes) is capable of stopping his car over a distance of 30 m on seeing a red traffic signal when the speed of the car is 72 kmh^{-1} and over a distance of 10 m when the speed of the car is 36 kmh^{-1}. Find the distance over which he can stop the car if it were running at a speed of 54 kmh^{-1}. Assume that his reaction time and the deceleration of the car remains same in all the three cases.

SOLUTION

Let t_0 is the reaction time, and a is the deceleration

During reaction time, the car travels at constant speed. If u is speed of car, the distance traversed in reaction time t_0 is ut_0, therefore for stopping distance s, the distance traversed with deceleration a is $(s - ut_0)$ and final velocity $= 0$

CASE-1

Initial speed $u_1 = 72 \text{ kmh}^{-1} = 20 \text{ ms}^{-1}$

$s_1 = 30$ m

$\Rightarrow \quad 0 = u_1^2 - 2a(30 - u_1 t_o)$

$\Rightarrow \quad u_1^2 = 2a(30 - 20u_1 t_o)$

$\Rightarrow \quad (20)^2 = 2a(30 - u_1 t_o)$ ...(1)

CASE-2

$u_2 = 36 \text{ kmh}^{-1} = 10 \text{ ms}^{-1}, \; s_2 = 10 \text{ m}$

$\Rightarrow \quad 0 = u_2^2 - 2a(10 - u_2 t_o)$

$\Rightarrow \quad u_2^2 = 2a(10 - u_2 t_o)$

$\Rightarrow \quad (10)^2 = 2a(10 - 10t_o)$ ...(2)

Solving (1) and (2),

$\Rightarrow \quad t_0 = 0.5 \text{ s}$ and $a = 10 \text{ ms}^{-2}$ {Retardation}

CASE-3

$u_3 = 54 \text{ kmh}^{-1} = 15 \text{ ms}^{-1}$

$s_3 = ?$

$\Rightarrow \quad 0 = (15)^2 - 2 \times 10 \times (s_3 - 15 \times 0.5)$

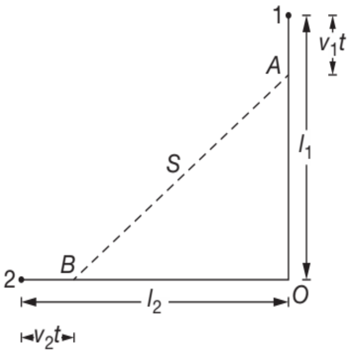
$\Rightarrow \quad s_3 = 18.75 \text{ m}$

PROBLEM 13

Two particles A and B move with constant velocity $\vec{v}_1$ and $\vec{v}_2$ along two mutually perpendicular straight lines towards intersection point O. At moment $t = 0$ particles were located at distances l_1 and l_2 respectively from O. How soon will the distance between particles be minimum and what is that minimum distance equal to?

SOLUTION

Let the separation between the particles be minimum at time t. Then



Since $OB = l_2 - v_2 t$ and $OA = l_1 - v_1 t$ and $AB^2 = OB^2 + OA^2$

$\Rightarrow \quad s^2 = (l_1 - v_1 t)^2 + (l_2 - v_2 t)^2$

For s to be minimum, $\frac{ds}{dt} = 0$ or $\frac{d}{dt}(s^2) = 0$

$\Rightarrow \quad 2s\frac{ds}{dt} = 2(l_1 - v_1 t)(-v_1) + 2(l_2 - v_2 t)(-v_2) = 0$

$\Rightarrow \quad -l_1 v_1 + v_1^2 t - l_2 v_2 + v_2^2 t = 0$

$\Rightarrow \quad t = \frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$

Minimum s is given by

$$s_{\text{min}}^2 = \left[l - v_1\left(\frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)\right]^2 + \left[l_2 - v_2\left(\frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)\right]^2$$

$\Rightarrow \quad s_{\text{min}}^2 = \frac{(l_1 v_2 - l_2 v_1)^2}{v_1^2 + v_2^2}$

$\Rightarrow \quad s_{\text{min}} = \frac{|l_1 v_2 - l_2 v_1|}{\sqrt{v_1^2 + v_2^2}}$

PROBLEM 14

Two bodies move in the same straight line at the same instant of time from the same origin. The first body moves with a constant velocity of 40 ms^{-1} and second starts with a constant acceleration of 4 ms^{-2}. Find the time t that elapses before the second catches the first body. Find also the greater distance between them prior to it and the time at which this occurs.

SOLUTION

Let the two bodies meet after a time t. The distance travelled by both is the same

The distance travelled by first body is

$s_1 = 40 \times t$

The distance travelled by second body is

$s_2 = 0 + \frac{1}{2}(4)t^2$

Since $s_1 = s_2$

$\Rightarrow \quad 40t = \frac{1}{2}(4)t^2$

$\Rightarrow \quad t = 20 \text{ s}$

The distance between the two bodies goes on increasing as long as the velocity of the second body remains less than $40\ \text{ms}^{-1}$. The distance between them will be greatest prior to their meeting when the velocity of the second body becomes $40\ \text{ms}^{-1}$. Let t_1 be the time when this happens. Then

$$40 = 0 + 4t_1 \qquad \{\because v = u + at\}$$

$$\Rightarrow \quad t_1 = 10\ \text{s}$$

So the distance between them will be greatest after $t_1 = 10\ \text{s}$. Let x_1 and x_2 be the distances moved by the two bodies in $t_1 = 10\ \text{s}$, then

$$x_1 = (40 \times 10) + \frac{1}{2}(0)(10)^2$$

$$\Rightarrow \quad x_1 = 400\ \text{m}$$

and $x_2 = 0 + \frac{1}{2}(4)(10)^2$

$$\Rightarrow \quad x_2 = 200\ \text{m}$$

Greatest distance between them is $\Delta x = x_1 - x_2$

$$\Rightarrow \quad \Delta x = 400 - 200 = 200\ \text{m}$$

PROBLEM 15

A motor cycle and a car start from rest at the same place at the same time and travel in the same direction. The cycle accelerates uniformly at $1\ \text{ms}^{-2}$ upto a speed of $36\ \text{kmh}^{-1}$ and the car at $0.5\ \text{ms}^{-2}$ upto a speed of $54\ \text{kmh}^{-1}$. Calculate the time and distance at which the car overtakes the cycle.

SOLUTION

When the car overtakes the motor cycle, the two have travelled the same distance in the same time. Let the time taken be t second while the total distance travelled be x metre

For Motor Cycle: Maximum speed attained is

$$v_1 = 36\ \text{kmh}^{-1} = 10\ \text{ms}^{-1}$$

Its acceleration is $a_1 = 1\ \text{ms}^{-2}$. Let t_1 be the time taken to reach the maximum speed, then using $v = u + at$, we get

$$10 = 0 + (1)(t_1)$$

$$\Rightarrow \quad t_1 = 10\ \text{s}$$

The distance travelled before reaching the maximum speed is

$$x_1 = 0 + \frac{1}{2}(1)t_1^2 = \frac{1}{2}(1)(10)^2 = 50\ \text{m} \qquad \ldots(1)$$

The time during which the motor cyclist moves with constant speed is $(t - 10)$ s

The distance travelled at constant speed is

$$x_2 = 10(t - 10) = (10t - 100)\ \text{m} \qquad \ldots(2)$$

Total distance $x = x_1 + x_2$

$$\Rightarrow \quad x = 50 + 10t - 100 = (10t - 50)\ \text{m} \qquad \ldots(3)$$

For Car: Maximum speed attained is

$$v_2 = 54\ \text{kmh}^{-1} = 15\ \text{ms}^{-1}$$

Its acceleration is $a_2 = 0.5\ \text{ms}^{-2}$. Let t_2 be the time taken to reach the maximum speed. Then

$$15 = 0 + (0.5)t_2$$

$$\Rightarrow \quad t_2 = \frac{15}{0.5} = 30\ \text{s}$$

The distance travelled before reaching the maximum speed is

$$x_3 = 0 + \frac{1}{2}(0.5)(30)^2 = 255\ \text{m} \qquad \ldots(4)$$

The time for which the car is moving at constant speed is $(t - 30)$ s

The distance travelled at constant speed

$$x_4 = 15(t - 30) = 15t - 450 \qquad \ldots(5)$$

Total distance travelled is $x' = x_3 + x_4$

$$\Rightarrow \quad x' = 225 + 15t - 450 = 15t - 225 \qquad \ldots(6)$$

Equating equations (3) and (6)

$$10t - 50 = 15t - 225$$

$$\Rightarrow \quad 5t = 175$$

$$\Rightarrow \quad t = 35\ \text{s}$$

Substituting this value of t either in equation (3) or in equation (6), we get

$$x = x' = (10)(35) - 50 = 300\ \text{m}$$

PROBLEM 16

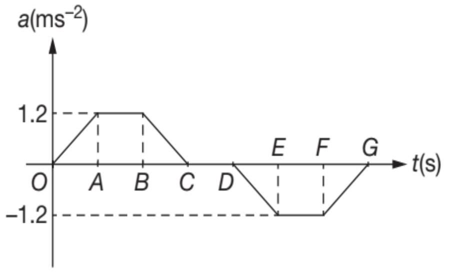
An airport shuttle train travels between two terminals that are 2.5 km apart. To maintain passenger comfort, the acceleration of the train is limited to $\pm 1.2\ \text{ms}^{-2}$ and the jerk, or rate of change of acceleration, is limited to $\pm 0.24\ \text{ms}^{-3}$. If the shuttle has a maximum speed of $30\ \text{kmh}^{-1}$, determine

(a) the shortest time for the shuttle to travel between the two terminals,

(b) the corresponding average velocity of the shuttle.

SOLUTION

(a) Corresponding a-t graph is as shown in figure



In the figure $t_{OA} = t_{BC} = t_{DE} = t_{FG} = \dfrac{1.2}{0.24} = 5$ s

Now let $t_{AB} = t_{EF} = t_1$ and $t_{CD} = t_2$

According to the problem, maximum speed is

$$v_{max} = v_C = 30 \text{ kmhr}^{-1} = \frac{25}{3} \text{ ms}^{-1}$$

Since $v_C =$ Area under a-t graph from O to C,

$$\Rightarrow \quad \frac{25}{3} = \frac{1}{2} \times 1.2(t_1 + t_1 + 10)$$

$$\Rightarrow \quad t_1 = \frac{35}{18} \text{ s}$$

From O to A, we have, $a = 0.24t$

$$\Rightarrow \quad \int_0^v dv = \int_0^t 0.24t\,dt$$

$$\Rightarrow \quad v = 0.12t^2$$

$$\Rightarrow \quad v_A = (0.12)(5)^2 = 3 \text{ ms}^{-1}$$

Again, $\displaystyle\int_0^{x_{OA}} dx = 0.12\int_0^5 t^2 dt$

$$\Rightarrow \quad x_{OA} = 5 \text{ m}$$

From A to B, we have

$a = 1.2 \text{ ms}^{-2} =$ constant

$$\Rightarrow \quad v_B = v_A + at_{AB} = 3 + (1.2)\left(\frac{35}{18}\right) = \frac{16}{3} \text{ ms}^{-1}$$

$$x_{AB} = v_A t_{AB} + \frac{1}{2}a(t_{AB})^2$$

$$\Rightarrow \quad x_{AB} = (3)\left(\frac{35}{18}\right) + \frac{1}{2}(1.2)\left(\frac{35}{18}\right)^2 \cong 8.1 \text{ m}$$

From B to C, we have $a = 1.2 - 0.24t$

$$\Rightarrow \quad \int_{16/3}^{v} dv = \int_0^t (1.2 - 0.24t)\,dt$$

$$\Rightarrow \quad v = \frac{16}{3} + 1.2t - 0.12t^2$$

Also $\displaystyle\int_0^{x_{BC}} dx = \int_0^5 \left(\frac{16}{3} + 1.2t - 0.12t^2\right)dt$

$$\Rightarrow \quad x_{BC} = \frac{80}{3} + 0.6(5)^2 - 0.04(5)^3 = 36.67 \text{ m}$$

$$\Rightarrow \quad x_{OC} = x_{DG} = (5 + 8.1 + 36.67) = 49.77 \text{ m}$$

$$\Rightarrow \quad x_{OC} = x_{DG} \cong 50 \text{ m}$$

$$\Rightarrow \quad x_{OC} + x_{DG} = 100 \text{ m}$$

The remaining distance $(2500 - 100)$ m or 2400 m must have been travelled with a constant speed of $\dfrac{25}{3}$ ms^{-1}. Hence

$$t_2 = \frac{2400}{25/3} = 288 \text{ s}$$

(a) Total time

$$T = 2\left(5 + \frac{35}{18} + 5\right) + 288 = 312 \text{ s} = 5.2 \text{ minute}$$

(b) Average velocity,

$$v_{av} = \frac{2.5}{\frac{5.2}{60}} \text{ kmhr}^{-1} = 28.8 \text{ kmhr}^{-1}$$

PROBLEM 17

An open lift is moving upwards with velocity 10 ms^{-1}. It has an upward acceleration of 2 ms^{-2}. A ball is projected upwards with velocity 20 ms^{-1} relative to ground. Find

(a) time when ball again meets the lift.
(b) displacement of lift and ball at that instant.
(c) distance travelled by the ball upto that instant.

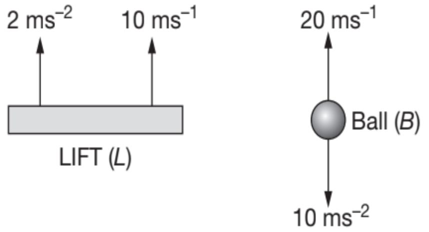
Take $g = 10 \text{ ms}^{-2}$

SOLUTION

(a) At the time when ball again meets the lift,

$$s_L = s_B$$

$$\Rightarrow \quad 10t + \frac{1}{2} \times 2 \times t^2 = 20t - \frac{1}{2} \times 10t^2$$



Solving, we get

$$t = 0 \text{ and } t = \frac{5}{3} \text{ s}$$

So, ball will again meet the lift after $\frac{5}{3}$ s

(b) At this instant

$$s_L = s_B = 10\left(\frac{5}{3}\right) + \frac{1}{2}(2)\left(\frac{5}{3}\right)^2 = \frac{175}{9} \text{ m} = 19.4 \text{ m}$$

(c) For the ball u and g are antiparallel. Therefore we will first find t_0, the time when its velocity becomes zero.

Since $v = u + at$

$$\Rightarrow \quad 0 = u + (-g)t_0$$

$$\Rightarrow \quad t_0 = \frac{u}{g} = \frac{20}{10} = 2 \text{ s}$$

Since $t\left(=\frac{5}{3} \text{ s}\right) < t_0$, so the distance and displacement are equal i.e.,

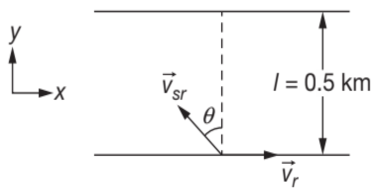
$$d = 19.4 \text{ m}$$

PROBLEM 18

A man can row a boat in still water at 3 kmh^{-1}. He can walk at a speed of 5 kmh^{-1} on the shore. The water in the river flows at 2 kmh^{-1}. If the man rows across the river and walks along the shore to reach the opposite point on the river bank, find the direction in which he should row the boat so that he could reach the opposite shore in the least possible time. Also calculate this time. The width of the river is 500 m.

SOLUTION

Suppose the boatman rows with velocity $\vec{v}_{sr}$ in the direction shown in figure



Given, $v_{sr} = 3 \text{ kmhr}^{-1}$, $v_r = 2 \text{ kmhr}^{-1}$

and v_w = walking speed $= 5 \text{ kmhr}^{-1}$

$$\vec{v}_s = \vec{v}_{sr} + \vec{v}_r$$

$$\Rightarrow \quad (v_s)_x = v_r - v_{sr}\sin\theta = 2 - 3\sin\theta$$

and $(v_s)_y = v_{sr}\cos\theta = 3\cos\theta$

Time taken to reach the other side

$$t_1 = \frac{l}{(v_s)_y} = \frac{0.5}{3\cos\theta} = \frac{1}{6\cos\theta}$$

Horizontal drift x is given by

$$x = (v_s)_x t_1 = (2 - 3\sin\theta)\frac{1}{6\cos\theta} = \frac{1}{3\cos\theta} - \frac{\tan\theta}{2}$$

Time to travel this distance by walking

$$t_2 = \frac{x}{v_w} = \frac{1}{15\cos\theta} - \frac{\tan\theta}{10} \quad \ldots(1)$$

$$\text{Total time } t = t_1 + t_2 = \frac{1}{10}\left(\frac{7}{3\cos\theta} - \tan\theta\right) \quad \ldots(2)$$

For time to be minimum, we have $\frac{dt}{d\theta} = 0$

$$\Rightarrow \quad \frac{7}{3}\sec\theta\tan\theta - \sec^2\theta = 0$$

$$\Rightarrow \quad \sin\theta = \frac{3}{7}$$

From equation (2), we get

$$t_{min} = \frac{1}{10}\left(\frac{7}{3} \times \frac{7}{\sqrt{40}} - \frac{3}{\sqrt{40}}\right)$$

$$\Rightarrow \quad t_{min} = 0.21 \text{ hour} = 756 \text{ s}$$

PROBLEM 19

The velocity of water current in a river changes with distance along the perpendicular to the river according to the function

$$u = \begin{cases} \frac{2u_0}{d}y & \text{for } 0 \le y \le \frac{d}{2} \\ \frac{2u_0}{d}(d - y) & \text{for } \frac{d}{2} \le y \le d \end{cases}$$

where u_0 is velocity at the mid-point of the river and d is width of the river. A boat travels from a point O on one bank of the river to the opposite bank and its steering angle is adjusted to keep its relative velocity perpendicular to the river current. Calculate the time in which the boat will reach the other bank. The velocity of the boat in still water is u_0.

SOLUTION

Denoting the river current by R and the boat by B. Let the boat steer making an angle θ with the river flow. Then

$$\vec{u}_R = u\hat{i}\,,\ \vec{u}_B = u_0\left(\cos\theta\hat{i} + \sin\theta\hat{i}\right)$$

$$\vec{u}_{BR} = \vec{u}_B - \vec{u}_R = \left(u_0\cos\theta - u\right)\hat{i} + u_0\sin\theta\hat{j}$$

As the boat is steered perpendicular to flow, so we have

$$\left(u_{BR}\right)_x = 0$$

$$\Rightarrow\quad u_0\cos\theta - u = 0$$

$$\Rightarrow\quad \cos\theta = \frac{u}{u_0}$$

$$\Rightarrow\quad \left|\vec{u}_{BR}\right| = u_0\sin\theta = \mu_0\sqrt{1-\frac{u^2}{u_0^2}} = \sqrt{u_0^2 - u^2}$$

$$\Rightarrow\quad \frac{dy}{dt} = \sqrt{u_0^2 - u^2}$$

$$\Rightarrow\quad \int dt = \int\frac{dy}{\sqrt{u_0^2 - u^2}}$$

$$\Rightarrow\quad t = \int_0^{\frac{d}{2}}\frac{dy}{\sqrt{u_0^2 - \frac{4u_0^2}{d^2}y^2}} + \int_{\frac{d}{2}}^{d}\frac{dy}{\sqrt{u_0^2 - \frac{4u_0^2}{d^2}\left(d-y\right)^2}}$$

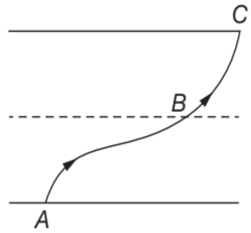
$$\Rightarrow\quad t = \frac{d}{2u_0}\int_0^{\frac{d}{2}}\frac{dy}{\sqrt{\left(\frac{d}{2}\right)^2 - y^2}} + \frac{d}{2u_0}\int_{\frac{d}{2}}^{d}\frac{dy}{\sqrt{\left(\frac{d}{2}\right)^2 - \left(d-y\right)^2}}$$

$$\Rightarrow\quad t = \frac{d}{2u_0}\left[\sin^{-1}\left(\frac{2y}{d}\right)\right]_0^{\frac{d}{2}} + \frac{d}{2u_0}\left[\sin^{-1}\left(\frac{2(d-y)}{d}\right)\right]_{\frac{d}{2}}^{d}$$

$$\Rightarrow\quad t = \frac{d}{2u_0}\left(\frac{\pi}{2} - 0\right) + \frac{d}{2u_0}\left[0 - \left(\frac{\pi}{2}\right)\right] = \frac{\pi d}{2u_0}$$

PROBLEM 20

The current velocity of a river grows in proportion to the distance from its bank and reaches the maximum value v_0 in the middle. Near the banks the velocity is zero. A boat is moving along the river in such a manner that it is always perpendicular to current and the speed of the boat in still water is u. Find the distance through which the boat crossing the river will be carried away by the current if the width of river is l. Also determine the trajectory of boat.



SOLUTION

Let the river flow or the current flow be along x-axis. At any instant t, let the boat be at a distance y from the bank, then

$$v_x = k_y \qquad \text{...(1)}$$

Also, we have been given in the problem, that

$$v_x = v_0 \text{ when } y = \frac{l}{2}$$

From (1), we get

$$v_0 = k\left(\frac{l}{2}\right)$$

$$\Rightarrow\quad k = \frac{2v_0}{l} \qquad \text{...(2)}$$

Since $y = ut$ ...(3)

So, equation (1) becomes

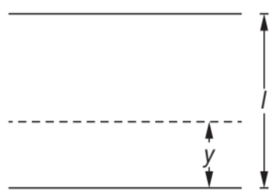
$$v_x = \left(\frac{2v_0}{l}\right)ut$$

$$\Rightarrow\quad \frac{dv_x}{dt} = a_x = \frac{2v_0 u}{l} = \text{constant}$$

$$\text{Since } x = u_x t + \frac{1}{2}a_x t^2$$

$$\Rightarrow\quad x = 0 + \frac{1}{2}\left(\frac{2v_0 u}{l}\right)t^2$$

$$\Rightarrow\quad x = \left(\frac{v_0 u}{l}\right)t^2 \qquad \text{...(4)}$$



From (3), $t = \frac{y}{u}$

$$\Rightarrow \quad x = \left(\frac{v_0 u}{lu^2}\right) y^2$$

$$\Rightarrow \quad x = \left(\frac{v_0}{lu}\right) y^2$$

$$\Rightarrow \quad y^2 = \left(\frac{lu}{v_0}\right) x \qquad \ldots(5)$$

This happens to be the equation of a parabola

Now, when $y = \frac{l}{2}$, we have, from (5)

$$\frac{l^2}{4} = \left(\frac{lu}{v_0}\right) x$$

$$\Rightarrow \quad x = \frac{lv_0}{4u}$$

Hence total drift is

$$\text{Drift} = 2x = \frac{2lv_0}{4u}$$

$$\Rightarrow \quad \text{Drift} = \frac{lv_0}{2u}$$