

# Heat and Thermodynamics

## Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

- |   |   |
|---|---|
| (a) Thermometry and Thermal Expansion   | (i) Adiabatic Processes                   |
| (b) Calorimetry   | (j) Polytropic Processes                  |
| (c) Kinetic Theory of Gases (KTG) and Ideal Gas Equation                                    | (k) Cyclic Processes                      |
| (d) Concept of Internal Energy, Degrees of Freedom and Molar Specific Heats for Ideal Gases | (l) Heat Engine and Refrigerator          |
| (e) Work Done and First Law of Thermodynamics (FLT D)                                       | (m) Conduction and Convection             |
| (f) Isochoric Processes   | (n) Radiation and its Properties          |
| (g) Isobaric Processes  | (o) Stefan's Law                          |
| (h) Isothermal Processes  | (p) Newton's Law of Cooling               |
|   | (q) Solar Constant and Temperature of Sun |
|   | (r) Wien's Law                            |

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

## THERMOMETRY, THERMAL EXPANSION AND CALORIMETRY

### HEAT

A form of energy producing in us the sensation of warmth and which is responsible for the change in thermal conditions of the body. It can also be thought as a form of energy, which flows due to the maintenance of an appropriate temperature difference between the two bodies.

### THERMAL EQUILIBRIUM

A system is said to be in the state of thermal equilibrium if the macroscopic variables that characterize the system remain constant with time.

Consider an ideal gas enclosed within a rigid closed container with fixed values of pressure, volume, temperature, mass and composition which remains constant with time. If the container is insulated from its surroundings, the container is in a state of thermodynamic equilibrium or thermal equilibrium.



### Conceptual Note(s)

Equilibrium state of energy thermodynamic system is completely described by specific values of some macroscopic variables or state variables. The relation between the state variables is called the equation of state. The thermodynamic state variables are of two types (a) extensive and (b) intensive.

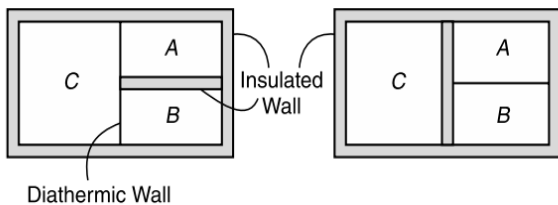
- (a) Extensive variables depend on the quantities of system e.g., volume, mass etc.
- (b) Intensive variables are independent of quantity of system e.g., pressure, density, etc.

### ZEROTH LAW AND CONCEPT OF TEMPERATURE

Consider two systems  $A$  and  $B$  separated by an adiabatic wall (an insulating wall that does not allow flow of energy (heat) through it), while each is in contact with a third sys-

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tem C by a conducting wall or diathermic wall. The status of the systems changes until both A and B come to thermal equilibrium with respect to C. Now if the internal walls are interchanged, the state of the system will not change further. This means that A and B are also in thermal equilibrium with each other. This observation is the basis of **Zerth Law of Thermodynamics**.



So, finally according to this Law, if two systems A and B are separately in thermal equilibrium with a third system C, then A and B must be in thermal equilibrium with each other. Hence to conclude there must exist certain scalar physical quantity which must be identical for all the systems in thermal equilibrium. This quantity (a scalar) is called **Temperature**.

Hence for the systems A, B and C in thermal equilibrium

$$T_A = T_B = T_C$$

It may be defined as that physical quantity which determines the degree of hotness of a body and the direction of heat flow.

### THERMOMETRY AND THERMOMETERS

The branch dealing with measurement of temperature is called **Thermometry** and the devices used to measure temperature are called **Thermometers**. Any physical property, say X of the substance, which is a function of temperature, has to be taken in account for making these devices.

For a substance if X is a temperature dependent property varying linearly with temperature, then

$$\text{Temperature } T = \left( \frac{X_T - X_0}{X_{100} - X_0} \right) \times 100 \text{ } ^\circ\text{C}$$

### TEMPERATURE SCALES

The centigrade ( $^\circ\text{C}$ ), Fahrenheit ( $^\circ\text{F}$ ), Kelvin (K), Reaumer (R), Rankine ( $R_n$ ) are commonly used temperature scales. However, the following three scales are used very often while expressing the temperatures.

All these scales have Lower Fixed Point (L.F.P.) as melting point of ice and Upper Fixed Point (U.F.P.) as boiling point of water or the steam point. All of these scales are having (U.F.P. – L.F.P.) number of equal divisions.

The ice point is defined as the equilibrium temperature of a mixture of ice and water at a pressure of one atmosphere and the steam point is defined as the equilibrium temperature of water and steam at a pressure of one atmosphere.

### The Celsius Scale

Devised by Anders Celsius in the year 1710, the interval between the lower fixed point and the upper fixed points is divided into 100 equal parts. Each division of the scale is called one degree centigrade or one degree Celsius i.e.,  $1 \text{ } ^\circ\text{C}$ .

At normal pressure, the melting point of ice is  $0 \text{ } ^\circ\text{C}$  and is called the lower fixed point of the Celsius scale.

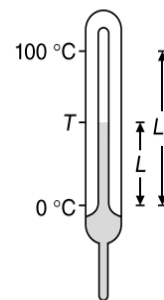
At normal pressure, the boiling point of water is  $100 \text{ } ^\circ\text{C}$  and is called the upper fixed point of the Celsius scale.

Assuming the cross-section of the thermometer capillary to be uniform and rate of expansion of the liquid with change in temperature to be constant, then we divide the distance between ice point and steam point into 100 equal parts and each part is then called as one degree. At any temperature the level of the liquid to the nearest mark can be easily compared. This scale was originally known as the centigrade scale because it has one hundred divisions between the principal reference marks.

A Celsius scale thermometer at temperature T, at which the liquid is extended to a distance L beyond the zero position is shown in Figure.

The temperature T is given by

$$T = \left( \frac{L}{L_0} \right) \times 100$$



Although the Celsius temperature scale is widely used, there is no logic behind choosing the ice point to be  $0^\circ$  and the steam point to be  $100^\circ$ .

### The Fahrenheit Scale

This scale was devised by Gabriel Fahrenheit in the year 1717. The interval between the lower and the upper fixed points is divided into 180 equal parts. Each division of this scale is called one degree Fahrenheit ( $1 \text{ } ^\circ\text{F}$ ). On this scale, the melting point of ice at normal pressure is  $32 \text{ } ^\circ\text{F}$ . This is the lower fixed point. The boiling point of water at normal pressure is taken as  $212 \text{ } ^\circ\text{F}$ . This is the upper fixed point.

It is easy to transform temperature from one system into temperature on the other system. The general transformation formula used for this is

$$\frac{T_{\text{on X Scale}} - (\text{LFP})_{\text{on X Scale}}}{(\text{UFP})_{\text{on X Scale}} - (\text{LFP})_{\text{on X Scale}}} = \text{constant}$$

The above formula, when used to transform temperature  $T_C$  on Celsius scale to its value  $T_F$  on Fahrenheit scale gives

$$\frac{T_C - 0}{100 - 0} = \frac{T_F - 32}{212 - 32}$$

$$\Rightarrow T_F = \frac{9}{5} T_C + 32$$

### The Kelvin Scale

However, one temperature scale has more fundamental choice of zero point. This was given by Lord Kelvin from the study of gases. Kelvin scale uses intervals equal to those of the Celsius degree but with zero set at the lowest theoretical temperature that a gas can reach. The scale is based on the fact that a gas at 0 °C will lose 1/273.15 of its volume for a 1 °C drop in temperature. If this reduction in volume were to continue with decreasing temperature and if the gas did not liquefy, the volume would become zero at -273.15 °C. This is a temperature called absolute zero. The temperature scale based on this zero is the kelvin temperature scale. Since,

$$\frac{T_{\text{on X Scale}} - (\text{LFP})_{\text{on X Scale}}}{(\text{UFP})_{\text{on X Scale}} - (\text{LFP})_{\text{on X Scale}}} = \text{constant}$$

So, the relation between temperatures measured on Celsius scale  $T_C$  and Kelvin Scale  $T_K$  is given by

$$\frac{T_C - 0}{100 - 0} = \frac{T_K - 273.15}{373.15 - 273.15}$$

$$\Rightarrow T_K = T_C + 273.15$$

This temperature in Kelvin scale is used with a unit of Kelvin (K) and it is not written with a degree sign. The temperature on Kelvin is also called absolute temperature.

### The Reaumer Scale

This scale was devised by R. A. Reaumer in the year 1730. The interval between the lower and the upper fixed point is divided into 80 equal parts. Each division is called one degree Reaumer (1 °R). On this scale, the melting point of ice at normal pressure 0 °R. This is lower fixed point. The boiling point of water at normal pressure is 80 °R. This is the upper fixed point.

### TEMPERATURE RELATION BETWEEN SCALES

All these temperatures (given in the table) are related to each other by the following relationship

Temperature Scale	°C	°F	K	°R	$R_N$	Any Scale
L.F.P.	0	32	273	0	492	$T_L$
U.F.P.	100	212	373	80	672	$T_U$
Number of Divisions (N)	100	180	100	80	180	$T_U - T_L$

$$\frac{T_C - 0}{100 - 0} = \frac{T_F - 32}{212 - 32} = \frac{T - 273}{373 - 273} = \frac{T_R - 0}{80 - 0} = \frac{T_{R_n} - 492}{672 - 492} = \frac{T - T_L}{T_U - T_L}$$

$$\Rightarrow \frac{T_C - 0}{100} = \frac{T_F - 32}{180} = \frac{T - 273}{100} = \frac{T_R - 0}{80} = \frac{T_{R_n} - 492}{180} = \frac{T - T_L}{T_U - T_L}$$

The temperature increments on these scales are related to each other as

$$\frac{\Delta T_C}{100} = \frac{\Delta T_F}{180} = \frac{\Delta T}{100}$$

### Problem Solving Technique(s)

- (a) Rise in temperature is same for Celsius and Kelvin scale.  
 (b) Whenever, ratio of the temperatures has to be taken do not forget to convert the temperatures to the absolute Kelvin scale. Like for example if we are asked to take the ratio of the temperatures say  $T_1 = 10$  °C and

$$T_2 = 5$$
 °C, then the ratio  $\frac{T_1}{T_2} \neq 2$ .

Instead of that the ratio is

$$\frac{T_1}{T_2} = \frac{10 + 273}{5 + 273} = \frac{283}{278} = 1.02$$

and this ratio happens to be very close to 1 (not 2). So please take care while expressing the temperature ratios.

- (c) In both the Fahrenheit and Celsius temperature scale, the assignment of the zero point is arbitrary and temperatures below these zero points are achievable.

### ILLUSTRATION 1

An iron piece is heated from 30 °C to 90 °C. Find the change in its temperature on the Fahrenheit and the kelvin scale.

#### SOLUTION

Since, we have seen that  $\frac{\Delta T_C}{100} = \frac{\Delta T_F}{180} = \frac{\Delta T}{100}$

$$\Rightarrow \Delta T_F = \frac{9}{5} \Delta T_C = \frac{9}{5} (90 - 30) = 108$$
 °F

Further, rise in temperature for both the celcius and the kelvin scale is the same, so  $\Delta T = \Delta T_C = 60$  K

### ILLUSTRATION 2

Express a temperature of 60 °F in degree celcius and in kelvin scale.

#### SOLUTION

Since, we know that  $\frac{T_F - 32}{180} = \frac{T_C - 0}{100} = \frac{T - 273}{100}$

$$\Rightarrow T_C = \frac{100}{180} (F - 32) = \frac{5}{9} (60 - 32) = 15.55$$
 °C

Since,  $T = T_C + 273 = 15.55 + 273 = 288.55$  K

## TYPES OF THERMOMETERS (OPTIONAL READING)

### Liquid Thermometer

The functioning of these thermometers is based on the principle of expansion of liquids on heating e.g., mercury thermometer, alcohol thermometer etc. Mercury is preferably used in thermometers since it possesses low specific heat and high conductivity. The range of Hg thermometer is  $-30\text{ }^{\circ}\text{C}$  to  $357\text{ }^{\circ}\text{C}$ .

### Gas Thermometer

These thermometers are based on Gay Lussac's Law according to which temperature of the gas ( $T$ ) is directly proportional to its pressure ( $P$ ) at constant volume ( $V$ ) i.e.,  $P \propto T$ .

If  $P_0$  be the pressure of the gas at  $0\text{ }^{\circ}\text{C}$ ,  $P_{100}$  be the pressure of the gas at  $100\text{ }^{\circ}\text{C}$  and  $P_t$  be the pressure of the gas at  $t\text{ }^{\circ}\text{C}$ , all at constant volume, then

$$t = \left( \frac{P_t - P_0}{P_{100} - P_0} \right) \times 100\text{ }^{\circ}\text{C}$$

Gas thermometer measure temperature ranging from  $-268\text{ }^{\circ}\text{C}$  to  $1500\text{ }^{\circ}\text{C}$

### Resistance Thermometers

These thermometers employ the variation of resistance of metals with the change in temperature. If  $R_t$  and  $R_0$  are resistances of a metal at  $t\text{ }^{\circ}\text{C}$  and  $0\text{ }^{\circ}\text{C}$  then experimentally it has been checked that  $R_t = R_0(1 + \alpha t)$  where  $\alpha$  is called **Temperature coefficient of resistance** and has unit  $(^{\circ}\text{C})^{-1}$  or  $\text{K}^{-1}$ .

Platinum is used in resistance thermometers as it has high melting point and variation in resistance for pure platinum is quite large and is uniform throughout the range  $-200\text{ }^{\circ}\text{C}$  to  $1200\text{ }^{\circ}\text{C}$ . The unknown temperature  $t$  is calculated from the equation

$$t = \left( \frac{R_t - R_0}{R_{100} - R_0} \right) \times 100\text{ }^{\circ}\text{C}$$

But if we do not know  $R_0$  we calculate the resistance of metal  $R_1$  and  $R_2$  at temperatures  $t_1$  and  $t_2$  respectively.

$$\Rightarrow R_1 = R_0(1 + \alpha t_1) \quad \dots(1)$$

$$\Rightarrow R_2 = R_0(1 + \alpha t_2) \quad \dots(2)$$

Dividing (1) and (2) and rearranging, we get

$$\alpha = \frac{R_2 - R_1}{R_1 t_2 - R_2 t_1}$$

### Thermocouple Thermometers

These types of thermometers are based on phenomenon of **Seebeck effect**.

According to this effect "when the distinct metals are joined (together called **thermocouple**) and a temperature

difference is maintained across its two junctions then an e.m.f. is developed in the thermocouple". This developed e.m.f. is called **thermo-emf** and is a function of temperature.

$$\xi = At + Bt^2 \text{ (in volt)}$$

where  $A$  and  $B$  are constants and depend on the nature of metals selected to form a thermocouple. Thermocouple thermometers have range varying from  $-200\text{ }^{\circ}\text{C}$  to  $1600\text{ }^{\circ}\text{C}$  with different thermocouples.

### Total Radiation Pyrometers

Pyrometers are capable of measuring the temperatures (however high) irrespective of the distance of source from the pyrometer. These are based on **Stefan's Law** according to which "the radiant energy emitted per second per unit area ( $E$ ) of a black body is directly proportional to the fourth power of absolute temperature ( $T$ ) i.e.  $E \propto T^4$

$$\Rightarrow E = \sigma T^4$$

where  $\sigma$  is the constant of proportionality called **Stefan's Constant** whose value is  $5.67 \times 10^{-8} \text{ Wm}^{-2} \text{ K}^{-4}$ . The minimum temperature it measures is  $800\text{ }^{\circ}\text{C}$ .

### Disappearing Filament Pyrometers

These are based on the principle that the filament of the bulb when seen through a red filter disappears when its temperature is equal to temperature of distant object emitting radiation. Its temperature measuring range varies from  $600\text{ }^{\circ}\text{C}$  to  $2700\text{ }^{\circ}\text{C}$ .

### Vapour Pressure Thermometer

This thermometer is based on the fact that the saturated vapour pressure of a liquid depends upon its temperature and is related to temperature as

$$\log P = a + bT - \frac{c}{T}$$

where  $a$ ,  $b$  and  $c$  are constants and  $T$  is temperature on absolute scale. On using liquid Helium, the minimum attainable temperature using this device is  $0.71\text{ K}$ . Making use of other liquid vapours it can measure temperatures upto  $122\text{ K}$ .

### Magnetic Thermometer

It is used in measuring temperatures close to  $0\text{ K}$ .

### Problem Solving Technique(s)

The International Practical Temperature scale was developed in 1927 to provide a temperature scale which had an easy utilisation for practical purposes i.e., calibration of industrial and scientific instruments and was the best possible approximation to Kelvin scale.

This fixed point in thermometry is the triple point of water i.e.,  $273.16\text{ K}$

## THERMAL EXPANSION

It has been a general observation that heating a body is accompanied by an increase in the size of the body, be its length, breadth, thickness (obviously area and volume too). This phenomenon is called the **Phenomenon of Thermal Expansion**.

On heating a body if its length increases, we call it linear expansion, if its surface area increases, we call it **Superficial Expansion** or **Areal Expansion**, if its volume increases, we call it **Volume Expansion** or **Cubical Expansion**.

The coefficients of linear expansion, superficial expansion and volume expansion are denoted respectively by  $\alpha$ ,  $\beta$  and  $\gamma$ .

If  $L_0$ ,  $A_0$  and  $V_0$  be the length, surface area and volume respectively at temperature  $T_0$  and  $L$ ,  $A$  and  $V$  be the respective quantities at temperature  $(T_0 + \Delta T)$ , where  $\Delta T$  is the rise in temperature, then the average values of  $\alpha$ ,  $\beta$  and  $\gamma$  are

$$\langle \alpha \rangle = \frac{1}{L_0} \left( \frac{\Delta L}{\Delta T} \right), \langle \beta \rangle = \frac{1}{A_0} \left( \frac{\Delta A}{\Delta T} \right), \langle \gamma \rangle = \frac{1}{V_0} \left( \frac{\Delta V}{\Delta T} \right)$$

Their instantaneous values can be found by taking the limit  $\Delta T \rightarrow 0$ .

$$\alpha = \frac{1}{L_0} \lim_{\Delta T \rightarrow 0} \left( \frac{\Delta L}{\Delta T} \right) = \frac{1}{L_0} \left( \frac{dL}{dT} \right)$$

$$\beta = \frac{1}{A_0} \lim_{\Delta T \rightarrow 0} \left( \frac{\Delta A}{\Delta T} \right) = \frac{1}{A_0} \left( \frac{dA}{dT} \right)$$

$$\gamma = \frac{1}{V_0} \lim_{\Delta T \rightarrow 0} \left( \frac{\Delta V}{\Delta T} \right) = \frac{1}{V_0} \left( \frac{dV}{dT} \right)$$

For most substances, values of  $\alpha$ ,  $\beta$  and  $\gamma$  are independent of temperature as long as temperature variations are not large. For large temperature variations they start varying as a function of temperature.

If the length of a rod is  $L_0$  at  $0^\circ\text{C}$  and  $L_T$  at  $T^\circ\text{C}$ , then for small  $\alpha$  assumed to be a constant over the given temperature interval, we have

$$\alpha = \frac{L_T - L_0}{L_0 T}$$

$$\Rightarrow L_T = L_0 (1 + \alpha T)$$

This equation the most used one for solving problems related to linear expansion.

From above we also conclude that the coefficient of thermal expansion is defined as the change in dimension per unit original dimension for a unit rise of temperature. Unit of  $\alpha$ ,  $\beta$ ,  $\gamma$  is  $(^\circ\text{C})^{-1}$  or  $\text{K}^{-1}$ .

For the normal variations in the temperature, we have

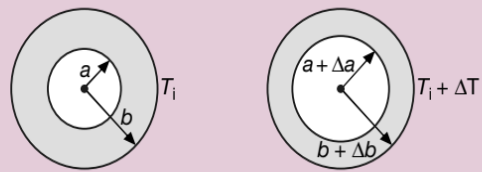
$$\frac{\alpha}{1} = \frac{\beta}{2} = \frac{\gamma}{3}$$

So, we can conclude that  $\beta = 2\alpha$  and  $\gamma = 3\alpha$ .

### Problem Solving Technique(s)

- (a) The values of  $\alpha$ ,  $\beta$  and  $\gamma$  are identical for both Celsius scale and Kelvin scale. This is due to the fact that rise in temperature is same on both the scales.
- (b) **If a solid object has a hole in it, what happens to the size of the hole, when the temperature of the object increases?**

A common misconception is that if the object expands, the hole will shrink because material expands into the hole. But the truth is that if the object expands, the hole will expand too, because every linear dimension of an object changes in the same way when the temperature changes.



### ILLUSTRATION 3

In an aluminium sheet, there is a hole of diameter 2 m and the sheet is mounted horizontally on a stand. On this hole, an iron sphere of diameter 2.004 m is placed. Initial temperature of this system is  $25^\circ\text{C}$ . When the arrangement is heated, calculate the temperature at which the iron sphere will fall down through the hole in sheet. The coefficients of linear expansion for aluminium and iron are  $2.4 \times 10^{-5}$  and  $1.2 \times 10^{-5}$  respectively.

### SOLUTION

Since coefficient of linear expansion for aluminium is more than that of iron, so it expands faster than iron. On heating the arrangement, at some higher temperature, say  $T$ , when the diameter of hole becomes exactly equal to that of the iron sphere, then the sphere will pass through the hole. So,

$$(\text{Diameter of Hole})_{Al} = (\text{Diameter of Sphere})_{iron}$$

$$\Rightarrow 2[1 + \alpha_{Al}(T - 25)] = 2.004[1 + \alpha_{iron}(T - 25)]$$

$$\Rightarrow 2\alpha_{Al}(T - 25) = 0.004 + 2.004\alpha_{iron}(T - 25)$$

$$\Rightarrow T = \left( \frac{0.004}{2\alpha_{Al} - 2.004\alpha_{iron}} + 25 \right) ^\circ\text{C}$$

$$\Rightarrow T = \frac{0.004}{2 \times 2.4 \times 10^{-5} - 2.004 \times 1.2 \times 10^{-5}} + 25$$

$$\Rightarrow T = 191.7^\circ\text{C}$$

### ILLUSTRATION 4

A rod  $AB$  of length  $l$  is pivoted at an end  $A$  and freely rotated in a horizontal plane at an angular speed  $\omega$  about a vertical axis passing through  $A$ . If coefficient of linear

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expansion of material of rod is  $\alpha$ , find the percentage change in its angular velocity if temperature of system is increased by  $\Delta T$  without mechanically disturbing the system.

### SOLUTION

Since there is no mechanical disturbance, so no torque acts on the system and hence angular momentum is constant.

$$I\omega = \text{constant}$$

$$\Rightarrow \Delta(I\omega) = 0$$

$$\Rightarrow I\Delta\omega + \omega\Delta I = 0$$

$$\Rightarrow \frac{\Delta\omega}{\omega} = -\frac{\Delta I}{I} \quad \dots(1)$$

where,  $I = \frac{Ml^2}{3}$  is the moment of inertia of the rod about the specified axis of rotation.

On heating the rod, its length will change, so its moment of inertia also changes.

The fractional change in the length of the rod is

$$\frac{\Delta l}{l} = \alpha\Delta T \quad \{\text{by definition}\}$$

Also, we have  $I \propto l^2$

$$\Rightarrow \frac{\Delta I}{I} = 2\frac{\Delta l}{l} \quad \{\text{for small increments}\}$$

$$\Rightarrow \frac{\Delta I}{I} = 2\alpha\Delta T \quad \dots(2)$$

From equations (1) and (2), we get

$$\frac{\Delta\omega}{\omega} = -\frac{\Delta I}{I} = -2\frac{\Delta l}{l} = -2\alpha\Delta T$$

So, percentage change in angular velocity of rod due to heating is

$$\left| \frac{\Delta\omega}{\omega} \right| \times 100\% = \frac{\omega - \omega'}{\omega} \times 100\%$$

$$\Rightarrow \left| \frac{\Delta\omega}{\omega} \right| \times 100\% = (2\alpha\Delta T) \times 100\%$$

### ILLUSTRATION 5

Two rods of lengths  $l_1$  and  $l_2$  are made of materials having coefficients of linear expansion  $\alpha_1$  and  $\alpha_2$  respectively. For what relation between above values, the difference in the lengths of the two rods is independent of the variation in temperature?

### SOLUTION

The difference in length is

$$l_1 - l_2 = \text{constant} \quad \{\text{assuming } l_1 > l_2\}$$

As  $l_1 - l_2$  is independent of temperature, so we have

$$\Delta(l_1 - l_2) = 0$$

$$\Rightarrow \Delta l_1 - \Delta l_2 = 0$$

Since, we know that  $\Delta l = l\alpha\Delta T$

$$\Rightarrow l_1\alpha_1\Delta T - l_2\alpha_2\Delta T = 0$$

$$\Rightarrow l_1\alpha_1 = l_2\alpha_2$$

### ILLUSTRATION 6

An iron wire and a copper wire are marked 100 m at 20 °C. Calculate the difference in lengths of the wires at 60 °C. Given that the coefficients of linear expansion for iron and copper are  $1.2 \times 10^{-5} \text{ K}^{-1}$  and  $1.7 \times 10^{-5} \text{ K}^{-1}$ .

### SOLUTION

For iron wire, we have

$$l_{1i} = l_{0i}(1 + \alpha_i t_1) \quad \dots(1)$$

$$\text{and } l_{2i} = l_{0i}(1 + \alpha_i t_2) \quad \dots(2)$$

The elongation in the iron wire is

$$l_{2i} - l_{1i} = l_{0i}\alpha_i(t_2 - t_1) \quad \dots(3)$$

From equation (1), substituting  $l_{0i} = \frac{l_{1i}}{(1 + \alpha_i t_1)}$  in equation (3), we get elongation in iron wire to be

$$l_{2i} - l_{1i} = \frac{l_{1i}\alpha_i(t_2 - t_1)}{(1 + \alpha_i t_1)} \quad \dots(4)$$

Similarly, elongation in copper wire is given by

$$l_{2c} - l_{1c} = \frac{l_{1c}\alpha_c(t_2 - t_1)}{(1 + \alpha_c t_1)} \quad \dots(5)$$

where,  $l_{1i} = l_{1c} = l_1 = 100 \text{ m}$

Subtracting (4) from (5), we get

$$l_{2c} - l_{2i} = l_1 \frac{(\alpha_c - \alpha_i)(t_2 - t_1)}{(1 + \alpha_c t_1)(1 + \alpha_i t_1)} = 19.9 \text{ mm}$$

For low values of temperature  $t$  i.e., when  $\alpha t < 1$ , it is not necessary to reduce  $l_1$  and  $l_2$  to  $l_{01}$  and  $l_{02}$  at  $t = 0$  °C. To a sufficiently high degree of accuracy, we can assume that  $\Delta l = l\alpha\Delta t$ . Under this assumption, the problem can be solved in a simpler and quicker way. Since

$$\Delta l_i = l_{1i}\alpha_i\Delta T \text{ and } \Delta l_c = l_{1c}\alpha_c\Delta T$$

where,  $l_{1i} = l_{1c} = l_1$ . So, we have

$$\Delta l = \Delta l_c - \Delta l_i = l_c(t_2 - t_1)(\alpha_c - \alpha_i) = 20 \text{ mm}$$

It can be seen that the deviation from a more exact value of 19.9 mm amounts to 0.1 mm, which corresponds to a relative error of

$$\text{Relative Error} = \frac{0.1}{19.9} \times 100\% = 0.5\%$$

### ILLUSTRATION 7

A steel ball initially at a pressure of  $10^5$  Pa is heated from  $20^\circ\text{C}$  to  $120^\circ\text{C}$  keeping its volume constant. Find the final pressure inside the ball. Given that coefficient of linear expansion of steel is  $1.1 \times 10^{-5} \text{ }^\circ\text{C}^{-1}$  and Bulk modulus of steel is  $1.6 \times 10^{11} \text{ Nm}^{-2}$ .

### SOLUTION

On increasing temperature of ball by  $100^\circ\text{C}$  (from  $20^\circ\text{C}$  to  $120^\circ\text{C}$ ), the thermal expansion in its volume can be given as

$$\Delta V = V \gamma_s \Delta T = V(3\alpha_s) \Delta T \quad \dots(1)$$

Since it is given that no change in volume is allowed, so the increment in volume due to thermal expansion is cancelled due to elastic compression by external pressure. The Bulk's modulus of a material is defined as

$$B = \frac{\Delta P}{\Delta V/V}$$

The excess pressure inside the ball at  $120^\circ\text{C}$  to keep its volume constant during the heating process is

$$\Delta P = B \left( \frac{\Delta V}{V} \right) = B(3\alpha_s \Delta T)$$

$$\Rightarrow \Delta P = (1.6 \times 10^{11}) [3(1.1 \times 10^{-5})(100)]$$

$$\Rightarrow \Delta P = 5.28 \times 10^8 \text{ Nm}^{-2} = 5.28 \times 10^8 \text{ Pa}$$

### ILLUSTRATION 8

A metal rod  $A$  of length 25 cm expands by 0.05 cm, when its temperature is raised from  $0^\circ\text{C}$  to  $100^\circ\text{C}$ . Another rod  $B$  of a different metal of length 40 cm expands by 0.04 cm for the same rise in temperature. A third rod  $C$  of length 50 cm made up of pieces of rods  $A$  and  $B$  placed end to end expands by 0.03 cm on heating from  $0^\circ\text{C}$  to  $50^\circ\text{C}$ . Find the lengths of each portion of the composite rod.

### SOLUTION

For Rod  $A$ , we have  $0.050 = 25\alpha_A(100)$

$$\Rightarrow \alpha_A = 2 \times 10^{-5} (\text{ }^\circ\text{C})^{-1} \quad \dots(1)$$

Also, for Rod  $B$ , we have  $0.040 = 40\alpha_B(100)$

$$\Rightarrow \alpha_B = 10^{-5} (\text{ }^\circ\text{C})^{-1} \quad \dots(2)$$

Now, a third rod of length 50 cm is made from  $A$  and  $B$ , so

$$l_A + l_B = 50 \text{ cm} \quad \dots(3)$$

$$\Rightarrow \Delta l_A + \Delta l_B = 0.03 \text{ cm}$$

$$\Rightarrow l_A \alpha_A (50) + l_B \alpha_B (50) = 0.03$$

$$\Rightarrow l_A (2 \times 10^{-5})(50) + l_B (10^{-5})(50) = 0.03$$

$$\Rightarrow 2l_A + l_B = \frac{0.03}{50 \times 10^{-5}}$$

$$\Rightarrow 2l_A + l_B = \frac{3 \times 10^{-2} \times 10^5}{50} = 60 \text{ cm} \quad \dots(4)$$

Subtracting (3) from (4), we get

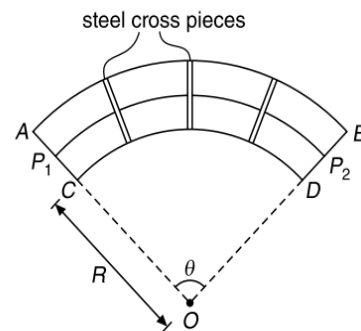
$$l_A = 10 \text{ cm and } l_B = 40 \text{ cm}$$

### ILLUSTRATION 9

Two straight thin bars, one of brass and the other of steel are joined together side by side by short steel cross-pieces one cm long at  $0^\circ\text{C}$ . When heated to  $100^\circ\text{C}$ , the composite bar becomes bent into the arc of a circle. If  $\alpha$  for brass is  $19 \times 10^{-6} (\text{ }^\circ\text{C})^{-1}$  and that for steel is  $11 \times 10^{-6} (\text{ }^\circ\text{C})^{-1}$ , then calculate the radius of this circle.

### SOLUTION

Since the expansion of brass is more than steel, hence the combination will bend with brass bar  $AB$  on the outside and steel wire  $CD$  on the inside as shown in Figure.



Let  $OC = R$  and  $\angle COD = \theta$

Length of steel cross pieces at  $100^\circ\text{C}$  is

$$l_{\text{cross piece}} = 1(1 + 11 \times 10^{-6} \times 100) = 1.0011 \text{ cm}$$

So,  $OA = (R + 1.0011) \text{ cm}$

$$\text{Also, we see that } CD = R\theta \quad \dots(1)$$

$$\text{and } AB = (R + 1.0011)\theta \quad \dots(2)$$

$$\text{Since, } AB = L_0(1 + 19 \times 10^{-6} \times 100) \quad \dots(3)$$

$$\text{and } CD = L_0(1 + 11 \times 10^{-6} \times 100) \quad \dots(4)$$

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From equation (1), (2), (3) and (4), we get

$$\frac{AB}{CD} = \frac{R+1.0011}{R} = \frac{1+(19 \times 10^{-6})(100)}{1+(11 \times 10^{-6})(100)}$$

$$\Rightarrow 1 + \frac{1.001}{R} = \left( \frac{1+19 \times 10^{-4}}{1+11 \times 10^{-4}} \right) \left( \frac{1.0011}{R} \right) = \frac{8 \times 10^{-4}}{1+11 \times 10^{-4}}$$

$$\Rightarrow R = (1.0011) \left( \frac{1+11 \times 10^{-4}}{8 \times 10^{-4}} \right) = 1252.8 \text{ cm}$$

### ILLUSTRATION 10

If a solid has coefficients of linear expansion  $\alpha_x$ ,  $\alpha_y$  and  $\alpha_z$  for three mutually perpendicular directions in a solid, what is the coefficient of volume expansion  $\gamma$  for the solid?

### SOLUTION

Consider a cube with edges parallel to  $x$ ,  $y$  and  $z$  of dimension  $l_0$  at  $T = 0$ . After a change in temperature  $\Delta T = (T - 0)$ , the dimensions change to  $l_x = l_0(1 + \alpha_x T)$ ,  $l_y = l_0(1 + \alpha_y T)$  and  $l_z = l_0(1 + \alpha_z T)$ .

The volume of the cuboid is given by

$$V = l_0^3 (1 + \alpha_x T)(1 + \alpha_y T)(1 + \alpha_z T)$$

$$\Rightarrow V \approx V_0 [1 + (\alpha_x + \alpha_y + \alpha_z)T] \quad (\because V_0 = l_0^3)$$

So, coefficient of volume expansion is given by

$$\gamma = \alpha_x + \alpha_y + \alpha_z$$

### ILLUSTRATION 11

A glass window is to be fitted in an aluminium frame. The temperature on the working day is  $40^\circ\text{C}$  and the glass piece is a rectangle of sides 30 cm by 20 cm. Find the size of the aluminium frame so that in winters glass does not experience any stress when temperature drops to  $0^\circ\text{C}$ . Given that coefficient of linear expansion for glass is  $9 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$  and that for aluminium is  $2.4 \times 10^{-5} \text{ }^\circ\text{C}^{-1}$ .

### SOLUTION

At  $40^\circ\text{C}$ , the dimensions of glass in winters are

$$l = 30(1 - 9 \times 10^{-6} \times 40) = 29.989 \text{ cm}$$

$$b = 20(1 - 9 \times 10^{-6} \times 40) = 19.993 \text{ cm}$$

So, dimensions of aluminium frame at  $40^\circ\text{C}$  are

$$l_1 = 29.989 \times (1 + 2.4 \times 10^{-5} \times 40) \text{ cm} = 30.018 \text{ cm}$$

$$b_1 = 19.993(1 + 2.4 \times 10^{-5} \times 40) \text{ cm} = 20.012 \text{ cm}$$

### ILLUSTRATION 12

An isosceles triangle is formed with a rod of length  $l_1$  and coefficient of linear expansion  $\alpha_1$  for the base and two thin rods each of length  $l_2$  and coefficient of linear expansion

$\alpha_2$  for the two pieces, if the distance between the apex and the midpoint of the base remain unchanged as the temperature is varied show that  $\frac{l_1}{l_2} = 2\sqrt{\frac{\alpha_2}{\alpha_1}}$ .

### SOLUTION

According to the problem, we have  $h = \text{constant}$

$$\Rightarrow h^2 = l_2^2 - \frac{l_1^2}{4} = \text{constant}$$

$$\Rightarrow \Delta(l_2^2) - \Delta\left(\frac{l_1^2}{4}\right) = 0$$

$$\Rightarrow 2l_2 \Delta l_2 - \frac{1}{4} 2l_1 \Delta l_1 = 0$$

Since, by definition,  $\Delta l = l \alpha \Delta T$

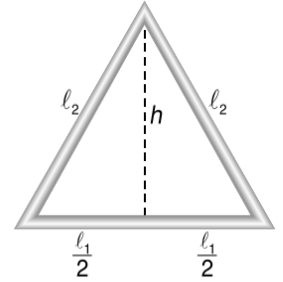
$$\Rightarrow \Delta l_2 = l_2 \alpha_2 \Delta T \text{ and } \Delta l_1 = l_1 \alpha_1 \Delta T$$

$$\Rightarrow 2l_2 (l_2 \alpha_2 \Delta T) = \frac{1}{4} (2l_1) (l_1 \alpha_1 \Delta T)$$

$$\Rightarrow l_2^2 \alpha_2 = \frac{1}{4} l_1^2 \alpha_1$$

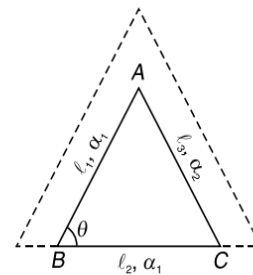
$$\Rightarrow \frac{l_1^2}{l_2^2} = 4 \left( \frac{\alpha_2}{\alpha_1} \right)$$

$$\Rightarrow \frac{l_1}{l_2} = 2\sqrt{\frac{\alpha_2}{\alpha_1}}$$



### ILLUSTRATION 13

Three rods A, B and C for an equilateral triangle at  $0^\circ\text{C}$ . Rods AB and BC have same coefficient of expansion  $\alpha_1$  and rod AC has  $\alpha_2$ . If temperature of the system is increased to  $T^\circ\text{C}$ , what is the change in angle  $\theta$  formed by rods AB and BC?



### SOLUTION

After increase of temperature, the triangle will not be equilateral. Let the lengths of the three rods be  $l_1$ ,  $l_2$  and  $l_3$ . From Cosine Law, we have

$$\cos \theta = \frac{l_1^2 + l_2^2 - l_3^2}{2l_1 l_2}$$

$$\Rightarrow 2l_1 l_2 \cos \theta = l_1^2 + l_2^2 - l_3^2 \quad \dots(1)$$

Differentiating equation (1), we get

$$2l_1 \cos \theta dl_2 + 2l_2 \cos \theta dl_1 - 2l_1 l_2 \sin \theta d\theta = 2l_1 dl_1 + 2l_2 dl_2 - 2l_3 dl_3 \quad \dots(2)$$

The changes in length of respective rods due to temperature rise are

$$\begin{aligned} dl_1 &= l_1 \alpha_1 \Delta T \\ dl_2 &= l_2 \alpha_1 \Delta T \\ dl_3 &= l_3 \alpha_2 \Delta T \end{aligned} \quad \dots(3)$$

For an equilateral triangle, we have  $l_1 = l_2 = l_3 = \ell$  and  $\theta = 60^\circ$ . So, from equations (2) and (3), we get

$$\begin{aligned} 2\ell^2 \alpha_1 \Delta T \cos 60^\circ + 2\ell^2 \alpha_1 \Delta T \cos 60^\circ - 2\ell^2 \sin 60^\circ d\theta &= 2\ell^2 \alpha_1 \Delta T + 2\ell^2 \alpha_1 \Delta T - 2\ell^2 \alpha_2 \Delta T \\ \Rightarrow \frac{\alpha_1}{2} \Delta T + \frac{\alpha_1}{2} \Delta T - \frac{\sqrt{3}}{2} d\theta &= \alpha_1 \Delta T + \alpha_1 \Delta T - \alpha_2 \Delta T \\ \Rightarrow \alpha_1 \Delta T - \frac{\sqrt{3}}{2} d\theta &= 2\alpha_1 \Delta T - \alpha_2 \Delta T \\ \Rightarrow (\alpha_2 - \alpha_1) \Delta T &= \frac{\sqrt{3}}{2} d\theta \\ \Rightarrow d\theta &= \frac{2}{\sqrt{3}} (\alpha_2 - \alpha_1) T \quad \{\because \Delta T = T\} \end{aligned}$$

### EFFECT OF TEMPERATURE ON PENDULUM CLOCKS

A pendulum clock consists of a metal bar attached with a bob at one end and fixed at the other end. The length of pendulum depends on temperature, and hence time period of clock depends on temperature. Let pendulum clock read correct time when its length is  $l_0$ . Its time period

$$\begin{aligned} \text{is } t_0 &= 2\pi \sqrt{\frac{l_0}{g}}. \text{ Now suppose that the temperature is raised} \\ \text{by } \Delta T, \text{ then new time period becomes } t' &= 2\pi \sqrt{\frac{l}{g}} \\ \Rightarrow \frac{t'}{t} &= \sqrt{\frac{l}{l_0}} \end{aligned} \quad \dots(1)$$

$$\begin{aligned} \text{Since } l &= l_0 (1 + \alpha \Delta T), \text{ so } \frac{l}{l_0} = 1 + \alpha \Delta T \\ \Rightarrow \frac{t}{t_0} &= (1 + \alpha \Delta T)^{1/2} \quad \{\text{from (1)}\} \end{aligned}$$

For small  $\alpha \Delta T$ , we have  $(1 + \alpha \Delta T)^{1/2} \approx 1 + \frac{1}{2} \alpha \Delta T$

$$\begin{aligned} \Rightarrow \frac{t}{t_0} &= 1 + \frac{1}{2} \alpha \Delta T \\ \Rightarrow \frac{t}{t_0} - 1 &= \frac{t - t_0}{t_0} = \frac{1}{2} \alpha \Delta T \end{aligned}$$

$$\text{So, fractional loss in time } \frac{\Delta t}{t_0} = \frac{1}{2} \alpha \Delta T$$

Number of seconds in one day is 86400 s

$$\text{So, time lost per day is } \Delta t = \frac{1}{2} \alpha \Delta T \times 86400 \text{ s}$$

#### ILLUSTRATION 14

A pendulum gives correct time at  $20^\circ \text{C}$  at a place where  $g = 9.8 \text{ ms}^{-2}$ . The pendulum consists of a light steel rod connected to a heavy ball. If it is taken to a different place where  $g = 9.788 \text{ ms}^{-2}$ . At what temperature will it give correct time? Coefficient of linear expansion of steel is  $12 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$ .

#### SOLUTION

Since  $t = 2\pi \sqrt{\frac{l}{g}}$  and for a pendulum to keep correct time, its period must be 2 s.

$$\Rightarrow 2 = 2\pi \sqrt{\frac{l_0}{9.8}} \quad \dots(1)$$

$$\Rightarrow 2 = 2\pi \sqrt{\frac{l_0 [1 + 12 \times 10^{-6} \times (T - 20)]}{9.788}} \quad \dots(2)$$

Equating (1) and (2), we get

$$\begin{aligned} \frac{l_0}{9.8} &= \frac{l_0}{9.788} [1 + 12 \times 10^{-6} \times (T - 20)] \\ \Rightarrow \frac{9.788}{9.8} &= 1 + 12 \times 10^{-6} \times (T - 20) \\ \Rightarrow \frac{9.788}{9.8} - 1 &= 12 \times 10^{-6} \times (T - 20) \\ \Rightarrow -\frac{0.012}{9.8} &= 12 \times 10^{-6} \times (T - 20) \\ \Rightarrow -\frac{12 \times 10^{-3}}{9.8} &= 12 \times 10^{-6} (T - 20) \\ \Rightarrow T - 20 &= -102 \\ \Rightarrow T &= -82^\circ \text{C} \end{aligned}$$

#### ILLUSTRATION 15

A clock with a metallic pendulum is 5 seconds fast each day at a temperature of  $15^\circ \text{C}$  and 10 seconds slow each day at a temperature of  $30^\circ \text{C}$ . Find coefficient of linear expansion for the metal.

#### SOLUTION

$$\text{The time lost per day is } \Delta t = \frac{1}{2} \alpha \Delta T \times 86400 \text{ s}$$

Let  $T_0$  be the graduation temperature of clock.

At  $15^\circ \text{C}$ , the clock is gaining time, so  $\Delta t = +5 \text{ s}$ .

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$$\Rightarrow 5 = \frac{1}{2} \alpha (T_0 - 15) \times 86400 \quad \dots(1)$$

At 30 °C, the clock is losing time, so  $\Delta t = -10$  s

$$\Rightarrow -10 = \frac{1}{2} \alpha (T_0 - 30) \times 86400 \quad \dots(2)$$

Dividing equation (2) by (1), we get

$$2(T_0 - 15) = (30 - T_0)$$

$$\Rightarrow T_0 = 20 \text{ }^\circ\text{C}$$

Substituting in equation (1), we get

$$5 = \frac{1}{2} \alpha (20 - 15) \times 86400$$

$$\Rightarrow \alpha = 2.31 \times 10^{-5} \text{ }^\circ\text{C}^{-1}$$

### ILLUSTRATION 16

A pendulum clock and a digital clock both are synchronized to keep correct time at temperature 20 °C in the morning on 1<sup>st</sup> March, 2003. At 12:00 noon temperature increases to 40 °C and remains constant for three months. Now on 1<sup>st</sup> June, 2003, at 12:00 noon temperature drops to 10 °C and remains constant for a very long duration. Find the date and time on which both the clocks will again be synchronized for a moment.

### SOLUTION

Since the digital clock (which is ideal) always keeps correct time. On increasing temperature on 1<sup>st</sup> March 12:00 noon, the pendulum clock slows down and start losing time. If  $\alpha$  be the coefficient of linear expansion of the material of pendulum, then the fractional loss in time is

$$\frac{\Delta t}{t} = \frac{1}{2} \alpha \Delta T$$

In three months of March, April and May, we have 92 days, so the loss in time during these 92 days is given by

$$\Delta t_{92 \text{ days}} = \frac{1}{2} \alpha (40 - 20) (92 \times 86400) \text{ s} \quad \dots(1)$$

Now, on 1<sup>st</sup> June 12:00 noon, the temperature drops to 10 °C which is 10° less than the temperature at which clock keeps correct time. So, now the clock starts gaining time.

If after  $N$  days it gains exactly the time lost during previous three months i.e., 92 days then it shows right time at that moment.

Time gained by the clock in  $N$  days is

$$\Delta t_{N \text{ days}} = \frac{1}{2} \alpha (20 - 10) (N \times 86400) \text{ s} \quad \dots(2)$$

Equating equations (1) and (2), we get

$$|\Delta t_{92 \text{ days}}| = |\Delta t_{N \text{ days}}|$$

$$\Rightarrow \frac{1}{2} \alpha \times 20 \times 92 \times 86400 = \frac{1}{2} \alpha \times 10 \times N \times 86400$$

$$\Rightarrow N = 184 \text{ days}$$

So, after 184 days from 1<sup>st</sup> June 2003, the pendulum clock will show correct time and both the clocks will be in synchronization for a moment and after 184 days means the date is 2<sup>nd</sup> December 2003 and time is 12:00 noon.

### ILLUSTRATION 17

A clock with a metallic pendulum gains 6 seconds each day when the temperature is 20 °C and loses 6 second when the temperature is 40 °C. Find the coefficient of linear expansion of the metal.

### SOLUTION

For simple pendulum,  $t = 2\pi \sqrt{\frac{l}{g}}$  i.e.,  $T \propto l^{\frac{1}{2}}$

$$\Rightarrow \frac{\Delta t}{t} = \frac{1}{2} \frac{\Delta l}{l} = \frac{1}{2} \alpha \Delta T$$

Assuming that the clock gives correct time at temperature  $T_0$ , then we have

$$\frac{6}{24 \times 3600} = \frac{1}{2} \alpha (T_0 - 20) \quad \dots(1)$$

$$\text{and } \frac{(-6)}{24 \times 3600} = \frac{1}{2} \alpha (T_0 - 40) \quad \dots(2)$$

From equations (1) and (2), we get  $T_0 = 30 \text{ }^\circ\text{C}$

$$\Rightarrow \alpha = 1.4 \times 10^{-5} \text{ }^\circ\text{C}^{-1}$$

## ERROR IN METAL SCALE DUE TO EXPANSION OR CONTRACTION

A metal scale expands with rise in temperature. Let a metal scale gives correct length  $L_0$  at a certain temperature say  $T_1$  °C. When the temperature of the scale is greater than  $T_1$  °C, then the distance between any two scale divisions increases in the ratio  $1 : (1 + \alpha T)$ , where  $\alpha$  is the coefficient of linear expansion of the material. If the reading of the scale is  $L$  cm, then the actual reading will be  $L(1 + \alpha T)$  cm and hence a correction of  $+L\alpha T$  cm should be applied. Here  $L$  is the reading of scale at higher temperature.

When the scale is below  $T_1$  °C, the distance between any two divisions contracts. If the scale reading be  $L$  cm, then the actual distance will be  $L(1 - \alpha T)$  and a correction  $-L\alpha T$  should be applied. Here  $L$  is the length at lower temperature.

### Problem Solving Technique(s)

It is clear from the above analysis that at higher temperature, the reading of the scale is lower than the true value in the ratio  $1 : (1 + \alpha T)$  while at lower temperature, the scale reading is higher than the true distance in the ratio  $(1 + \alpha T) : 1$ .

**ILLUSTRATION 18**

A meter scale is made of steel and measures correct length at 16 °C. What will be the percentage error if the scale is used

- (a) on a summer day when temperature is 46 °C and
- (b) on a winter day when the temperature is 6 °C?

Coefficient of linear expansion of steel is  $11 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$ .

**SOLUTION**

- (a) On a summer day the scale will measure less than the actual. Hence, the percentage error is

$$\% \text{ error} = -\left(\frac{\Delta l}{l}\right) \times 100$$

$$\Rightarrow \% \text{ error} = -(\alpha \Delta \theta) \times 100$$

$$\Rightarrow \% \text{ error} = -(11 \times 10^{-6})(46 - 16) \times 100$$

$$\Rightarrow \% \text{ error} = -0.033\%$$

- (b) On a winter day the scale will measure more than the actual, hence the percentage error is

$$\% \text{ error} = +(\alpha \Delta \theta) \times 100$$

$$\Rightarrow \% \text{ error} = +(11 \times 10^{-6})(16 - 6) \times 100$$

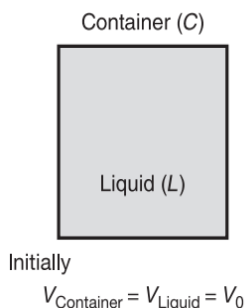
$$\Rightarrow \% \text{ error} = +0.011\%$$

**THERMAL EXPANSION IN LIQUIDS**

Thermal expansion in liquids is also similar to that of volume expansion in solids and is governed by the similar relationship which is  $V = V_0(1 + \gamma \Delta T)$ .  $\gamma$  for liquids is generally higher than that for solids.

Since liquids are always to be heated along with a container that contains them, so initially on heating the system (liquid + container), the level of liquid in container falls (since container expands more because it initially absorbs more heat compared to liquid which initially expands less). However, finally the liquid starts rising due to its faster expansion.

Consider a container (C) of volume  $V_0$ , coefficient of volume expansion  $\gamma_C$  to be filled up to the brim with a liquid (obviously of volume  $V_0$ ), coefficient of volume expansion  $\gamma_L$  as shown in Figure.



When the complete arrangement is heated, then let the rise in temperature be  $\Delta T$ . If  $V_L$  and  $V_C$  be the new respective volumes of the liquid and the container, then we have

For liquid,  $V_L = V_0(1 + \gamma_L \Delta T)$  ... (1)

For container,  $V_C = V_0(1 + \gamma_C \Delta T)$  ... (2)

So, volume of liquid that may overflow due to heating is

$$\Delta V = V_L - V_C = V_0(\gamma_L - \gamma_C)\Delta T \quad \dots(3)$$

This expression can be written as

$$\Delta V = V_0 \gamma_{\text{apparent}} \Delta T, \text{ where } \gamma_{\text{apparent}} = \gamma_{\text{app}} = \gamma_L - \gamma_C$$

- (a) If  $\gamma_L > \gamma_C$ , then  $\gamma_{\text{app}} > 0$  i.e., the level of liquid in the beaker will rise.
- (b) If  $\gamma_L < \gamma_C$ , then  $\gamma_{\text{app}} < 0$  i.e., the level of liquid in the beaker will fall.
- (a) If  $\gamma_L = \gamma_C$ , then  $\gamma_{\text{app}} = 0$  i.e., the level of liquid in the beaker will remain the same.

**W Conceptual Note(s)**

Liquids generally increase in volume with increasing temperature and have volume expansion coefficients about ten times greater than these of solid. Water is an exception to this rule. From 0 °C to 4 °C water contracts and beyond 4 °C it expands. Hence density of water reaches a maximum value of  $1000 \text{ kgm}^{-3}$  at 4 °C.

**ILLUSTRATION 19**

A long horizontal glass capillary tube open at both ends contains a mercury thread 1 m long at 0 °C. A scale is etched on the glass tube. This scale is correct at 0 °C. Find the length of mercury thread, as shown by this scale at 100 °C. Also find original length of thread at 100 °C. Given,  $\gamma_{\text{mercury}} = 18.2 \times 10^{-5} \text{ K}^{-1}$  and  $\alpha_{\text{glass}} = 9 \times 10^{-6} \text{ K}^{-1}$ .

**SOLUTION**

Let  $V_0$  and  $V_t$  are volumes of mercury at 0 °C and  $t$  °C,  $A_0$  and  $A_t$  are area of cross section of at 0 °C and at  $t$  °C. Then

$$V_0 = A_0 l_0 \quad \text{and} \quad V_t = A_t l_t$$

Since,  $V_t = V_0(1 + \gamma_{Hg} t)$

$$\Rightarrow A_t l_t = A_0 l_0 (1 + \gamma_{Hg} t)$$

$$\Rightarrow A_0 l_t (1 + 2\alpha_g t) = A_0 l_0 (1 + \gamma_{Hg} t)$$

$$\Rightarrow l_t = l_0 \left( \frac{1 + \gamma_{Hg} t}{1 + 2\alpha_g t} \right)$$

Expanding and neglecting negligible terms, we get

$$l_t = l_0 [1 + (\gamma_{Hg} - 2\alpha_g) t]$$

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$$\Rightarrow l_{100} = (1)[1 + (18.2 - 1.8) \times 10^{-5} \times 100]$$

$$\Rightarrow l_{100} = 1.0164 \text{ m}$$

Let  $L$  be the required reading of thread on glass scale at temperature  $t$ . Then this section of the glass (scale) has length  $L$  at  $0^\circ\text{C}$  and  $l_t$  at  $t^\circ\text{C}$ . So,

$$l_t = L(1 + \alpha_g t)$$

$$\Rightarrow L = \frac{l_0 [1 + (\gamma_{\text{Hg}} - 2\alpha_g)t]}{1 + \alpha_g t} = l_0 [1 + (\gamma_{\text{Hg}} - 3\alpha_g)t]$$

Substituting the values, we get

$$L = 1.0155 \text{ m}$$

### ILLUSTRATION 20

A glass flask with volume  $200 \text{ cm}^3$  is filled to the brim with mercury at  $20^\circ\text{C}$ . How much mercury overflows when the temperature of the system is raised to  $100^\circ\text{C}$ ? The coefficient of linear expansion of the glass is  $0.40 \times 10^{-5} \text{ K}^{-1}$ . Cubical expansion of mercury =  $18 \times 10^{-5} \text{ K}^{-1}$ .

### SOLUTION

The coefficient of volume expansion for the glass is

$$\gamma_{\text{glass}} = 3\alpha_{\text{glass}} = 1.2 \times 10^{-5} \text{ K}^{-1}$$

The glass flask of volume  $200 \text{ cm}^3$  is filled to the brim with mercury, so we have

$$\Delta V = V_0 \gamma_{\text{app}} \Delta T = V_0 (\gamma_{\text{Hg}} - \gamma_{\text{glass}}) \Delta T$$

$$\Rightarrow \Delta V = (200)[(18 - 1.2) \times 10^{-5}](100 - 20) \approx 2.7 \text{ cc}$$

### ILLUSTRATION 21

The coefficient of apparent expansion of a liquid in a copper vessel is  $C$  and in a silver vessel  $S$ . The coefficient of volume expansion of copper is  $\gamma_C$ . Find the coefficient of linear expansion of silver.

### SOLUTION

Apparent coefficient of volume expansion for liquid is

$$\gamma_{\text{app}} = \gamma_L - \gamma_S$$

$$\Rightarrow \gamma_L = \gamma_{\text{app}} + \gamma_S$$

where  $\gamma_S$  is coefficient of volume expansion for solid vessel.

Since,  $\gamma_{\text{app}}$  for liquid in copper vessel is  $C$ , so when liquid is placed in copper vessel then

$$\gamma_L = C + \gamma_{\text{copper}} \quad \dots(1)$$

Also,  $\gamma_{\text{app}}$  for liquid in silver vessel is  $S$ , so when liquid is placed in silver vessel then

$$\gamma_L = S + \gamma_{\text{silver}} \quad \dots(2)$$

From equation (1) and (2), we get

$$C + \gamma_{\text{copper}} = S + \gamma_{\text{silver}}$$

$$\Rightarrow \gamma_{\text{silver}} = C + \gamma_{\text{copper}} - S$$

$$\Rightarrow \alpha_{\text{silver}} = \frac{\gamma_{\text{silver}}}{3} = \frac{C + \gamma_{\text{copper}} - S}{3}$$

### ILLUSTRATION 22

A glass beaker holds exactly 1 litre at  $0^\circ\text{C}$ . What is its volume at  $50^\circ\text{C}$ ? If the beaker is filled with mercury at  $0^\circ\text{C}$ , what volume of mercury overflows when the temperature is  $50^\circ\text{C}$ ?

Given,  $\alpha_g = 8.3 \times 10^{-6} (\text{C}^\circ)^{-1}$  and  $\gamma_{\text{Hg}} = 1.82 \times 10^{-4} (\text{C}^\circ)^{-1}$

### SOLUTION

The volume of beaker after the temperature change is,

$$V_{\text{beaker}} = V_0 (1 + 3\alpha_g \Delta T)$$

$$\Rightarrow V_{\text{beaker}} = (1)[1 + 3 \times 8.3 \times 10^{-6} \times 50]$$

$$\Rightarrow V_{\text{beaker}} = 1.001 \text{ litre}$$

Volume of mercury at  $50^\circ\text{C}$  is

$$V_{\text{mercury}} = V_0 (1 + \gamma_{\text{Hg}} \Delta T)$$

$$\Rightarrow V_{\text{mercury}} = (1)[1 + 1.82 \times 10^{-4} \times 50]$$

$$\Rightarrow V_{\text{mercury}} = 1.009 \text{ litre}$$

The overflow is thus given by

$$\Delta V = 1.009 - 1.001 = 0.008 \text{ litre} = 8 \text{ ml}$$

### ILLUSTRATION 23

A glass bulb whose volume at  $0^\circ\text{C}$  equal  $10 \text{ cc}$  is filled with mercury. It is then joined to stem of diameter  $0.2 \text{ cm}$  at  $0^\circ\text{C}$ . What length of the stem will the liquid occupy at  $60^\circ\text{C}$ ?

Given, coefficient of volume expansion of mercury is  $18 \times 10^{-5} \text{ C}^\circ^{-1}$  and coefficient of linear expansion of glass is  $9 \times 10^{-6} \text{ C}^\circ^{-1}$ .

### SOLUTION

Volume of Hg at  $60^\circ\text{C}$  is  $V_{60} = V_0 (1 + 60\gamma_{\text{Hg}})$

Volume of glass bulb at  $60^\circ\text{C}$  is  $V_0 (1 + 60\gamma_g)$



So, volume of Hg that will pass from the bulb into stem is  $\Delta V = V_0 (\gamma_{\text{Hg}} - \gamma_g) 60$ . The cross section of stem at  $60^\circ\text{C}$

is  $A_{60} = A_0(1 + 60\beta_g)$  and let  $l$  be the length of mercury column in the stem at  $60^\circ\text{C}$ . Then

$$l = \frac{V_0(\gamma_{Hg} - \gamma_g)60}{A_0(1 + 60\beta_g)}$$

where,  $A_0 = \pi d^2/4 = \pi(0.2)^2/4 = 0.0314 \text{ cm}^2$

$$\Rightarrow l = \frac{(10)(18 - 2.7) \times 10^{-5} \times 60}{(0.0314)(1 + 60 \times 1.8 \times 10^{-5})} = 2.92 \text{ cm}$$

### ILLUSTRATION 24

A one litre closed flask contains some mercury. It is found that at different temperatures the volume of air inside the flask remains the same. What is the volume of mercury in flask? Given coefficient of linear expansion of glass  $= 9 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$ . Coefficient of volume expansion of Hg  $= 1.8 \times 10^{-4} \text{ }^\circ\text{C}^{-1}$ .

### SOLUTION

Since the volume of air in flask remains same. This is possible only when the expansion of flask is exactly the same as the expansion of mercury in the flask. So, coefficient of cubical expansion of glass is

$$\gamma_g = 3\alpha = 3 \times 9 \times 10^{-6} = 27 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$$

When the volume of air in flask is constant, then expansion of flask equals the expansion of mercury. Assuming  $x$  to be the volume of mercury in the flask, then

$$(1000)\gamma_g\Delta T = x\gamma_m\Delta T$$

$$\Rightarrow x = V \left( \frac{\gamma_g}{\gamma_m} \right) = 1000 \left( \frac{27 \times 10^{-6}}{1.8 \times 10^{-4}} \right) = 150 \text{ cm}^3$$

## EXPANSION OF GASES

On heating, gases expand much greater than solids or liquids and equal volume of different gases expand equally when heated by the same amount. The change in pressure ( $P$ ) and temperature ( $T$ ) may change the volume of given mass of gas. It is important to note that

- while dealing with the coefficient of thermal expansion of a gas the pressure is assumed constant.
- all gases have same coefficient of volume expansion i.e.  $\gamma_V$  with volume variation given by  $V = V_0(1 + \gamma_V T)$  and the pressure coefficient of the gas is given by  $\gamma_P = \frac{1}{273}$  with pressure variation given by  $P = P_0(1 + \gamma_P T)$ .
- the coefficient of volume expansion for an ideal gas at constant pressure is equal to the reciprocal of the absolute temperature  $T$ . This can be shown by starting with the ideal gas equation

$$PV = nRT \quad \dots(1)$$

$$\Rightarrow d(PV) = d(nRT)$$

$$\Rightarrow PdV = nRdT$$

$$\Rightarrow \gamma = \frac{1}{V} \left( \frac{dV}{dT} \right) = \frac{1}{V} \left( \frac{nR}{P} \right) = \frac{nR}{PV} = \frac{1}{T} \quad \{\because \text{of (1)}\}$$

## CHANGE IN DENSITY OF SOLIDS AND LIQUIDS WITH TEMPERATURE

Since on heating, the mass of the bodies remains same and hence we conclude that heating of bodies is accompanied by an increase in volume and decrease in density (note this for water which has an anomalous behaviour from  $0^\circ\text{C}$  to  $4^\circ\text{C}$  and beyond  $4^\circ\text{C}$ )

$$M = V_0\rho_0 = V\rho$$

Since,  $V = V_0(1 + \gamma\Delta T)$

$$\Rightarrow \frac{\rho}{\rho_0} = \frac{V_0}{V} = \frac{1}{1 + \gamma\Delta T} = (1 + \gamma\Delta T)^{-1}$$

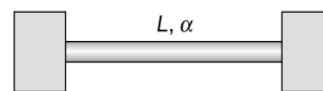
For  $\gamma\Delta T \ll 1$ , we have

$$\rho \approx \rho_0(1 - \gamma\Delta T)$$

where,  $\rho$  is density at temperature  $T^\circ\text{C}$  and  $\rho_0$  is density at  $T_0^\circ\text{C}$ .

## THERMAL STRESS IN A ROD FIXED BETWEEN TWO RIGID SUPPORTS

Consider a rod of length  $L_0$  with Young's modulus  $Y$  which is supplied heat such that temperature increases from  $T_0$  to  $(T_0 + \Delta T)$  thus producing a stress in the rod, then thermal stress is  $F/A$  and thermal strain is  $\Delta L/L_0 = \alpha\Delta T$ .



Rod held by rigid supports

$$\Rightarrow Y = \frac{\text{Thermal stress}}{\text{Thermal strain}} = \frac{F/A}{\Delta L/L_0}$$

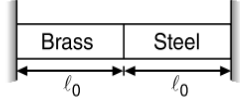
$$\Rightarrow \text{Thermal Stress} = \frac{F}{A} = Y \left( \frac{\Delta L}{L_0} \right) = Y\alpha\Delta T$$

### Problem Solving Technique(s)

Force which comes into play on heating a body is  $F = YA\alpha\Delta T$ .

**ILLUSTRATION 25**

Two rods, one of brass and other of steel of each of length  $l_0$  having equal cross-sectional area are joined end to end and are supported between two rigid vertical walls as shown in Figure.



Initially the rods are unstrained. If the temperature of system is raised by  $\Delta T$ , calculate the displacement of the junction of two rods, if the coefficients of linear expansion and Young's modulus of brass and steel are  $\alpha_b$ ,  $\alpha_s$  ( $\alpha_b > \alpha_s$ ),  $Y_b$  and  $Y_s$  respectively.

**SOLUTION**

Since  $\alpha_b > \alpha_s$ , so the junction of the two rods is displaced towards right. Due to their elastic nature, each rod exerts an elastic restoring force on the other due to which the brass rod is expanded but less as compared to free expansion and steel is compressed due to the stress developed between the two rods.

Let  $x$  be the displacement of the interface.

**For Brass Rod**

Final length due to free thermal expansion only is  $(l_0 + l_0\alpha_b\Delta T)$ . However, the actual final length of the brass rod is  $(l_0 + x)$  i.e., slightly less than the length of brass rod in free expansion.

Hence strain developed in the brass rod is

$$\left(\frac{\Delta l}{l_0}\right)_{\text{brass}} = \frac{(l_0 + l_0\alpha_b\Delta T) - (l_0 + x)}{l_0} = \frac{l_0\alpha_b\Delta T - x}{l_0}$$

**For Steel Rod**

Final length due to free thermal expansion only is  $(l_0 + l_0\alpha_s\Delta T)$ . However, the actual final length of the steel rod is  $(l_0 - x)$  i.e., slightly less than the length of steel rod in free expansion.

Hence strain developed in the brass rod is

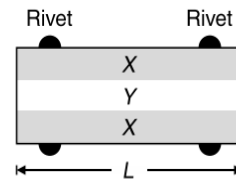
$$\left(\frac{\Delta l}{l_0}\right)_{\text{steel}} = \frac{(l_0 + l_0\alpha_s\Delta T) - (l_0 - x)}{l_0} = \frac{l_0\alpha_s\Delta T + x}{l_0}$$

However, the two rods are in contact, so stress developed in the two rods is same, so

$$\begin{aligned} (\text{Stress})_{\text{brass}} &= (\text{Stress})_{\text{steel}} \\ \Rightarrow Y_b \left(\frac{\Delta l}{l_0}\right)_{\text{brass}} &= Y_s \left(\frac{\Delta l}{l_0}\right)_{\text{steel}} \\ \Rightarrow Y_b \left(\frac{\alpha_b l_0 \Delta T - x}{l_0}\right) &= Y_s \left(\frac{\alpha_s l_0 \Delta T + x}{l_0}\right) \\ \Rightarrow x &= \frac{(Y_b \alpha_b - Y_s \alpha_s) l_0 \Delta T}{Y_b + Y_s} \end{aligned}$$

**ILLUSTRATION 26**

Two rods of material X sandwich another rod of material Y as shown in Figure.



At temperature  $T$ , the three rods are in a state of zero strain have length  $L$  and are riveted to each other. If the temperature of the system increases to  $T + \Delta T$ , find the final length of the system of the three rods. Given that the coefficients of linear expansions of the rods are  $\alpha_x$  and  $\alpha_y$  ( $\alpha_x > \alpha_y$ ), Young's modulus for material X is  $Y_x$  and that for material Y is  $Y_y$ .

**SOLUTION**

Natural elongation in rod  $x$  and  $y$  respectively are

$$l_x = L(1 + \alpha_x \Delta T) \text{ and } l_y = L(1 + \alpha_y \Delta T)$$

Due to pivots, if final length is  $L_f$ , then strain developed in rods  $x$  and  $y$  is

$$(\text{Strain})_x = \frac{L_f - l_x}{l_x} \approx \frac{L_f - l_x}{L}$$

$$(\text{Strain})_y = \frac{l_y - L_f}{l_y} \approx \frac{l_y - L_f}{L}$$

There we can use  $l_x \approx l_y \approx L$  as numerator is very small difference. Also, we note that stress on  $y$  is twice that of  $x$ , so

$$\begin{aligned} Y_y (\text{Strain})_y &= 2Y_x (\text{Strain})_x \\ \Rightarrow Y_y (L(1 + \alpha_y \Delta T) - L_f) &= 2Y_x (L_f - L(1 + \alpha_x \Delta T)) \\ \Rightarrow Y_y L + \alpha_y Y_y L \Delta T - L_f Y_y &= 2Y_x L_f - 2Y_x L - 2Y_x \alpha_x L \Delta T \\ \Rightarrow L_f &= L \left[ 1 + \left( \frac{\alpha_y Y_y + 2\alpha_x Y_x}{Y_y + 2Y_x} \right) \Delta T \right] \end{aligned}$$

**CORRECTION FOR BAROMETRIC READING**

The brass scale of a barometer is usually calibrated at  $0^\circ\text{C}$ . A barometer is usually read at room temperature (say  $t^\circ\text{C}$ ). Therefore, corrections due to expansions of brass-scale and mercury must be taken into account.

**CORRECTION FOR EXPANSION OF BRASS SCALE**

If  $H$  is true height at  $0^\circ\text{C}$ , then reading at  $t^\circ\text{C}$

$$H_t = H(1 + \alpha t) \quad \dots(1)$$

$\alpha$  being linear coefficient of expansion for brass.

### CORRECTION FOR EXPANSION OF MERCURY

If  $\rho_t$  is density of mercury at  $t^\circ\text{C}$ , then pressure is

$$H_t \rho_t g = H(1 + \alpha t) \rho_t g$$

If  $\rho_0$  is density of mercury at  $0^\circ\text{C}$ , then corresponding height of mercury at  $0^\circ\text{C}$ ,  $H_0$  is given by

$$H_0 \rho_0 g = H_t \rho_t g$$

$$H_0 \rho_0 g = H(1 + \alpha t) \rho_t g$$

But  $\rho_t = \frac{\rho_0}{1 + \gamma t}$  {  $\gamma$  being cubical expansion of mercury }

$$\Rightarrow H_0 \rho_0 g = H(1 + \alpha t) \left( \frac{\rho_0}{1 + \gamma t} \right) g$$

$$\Rightarrow H_0 = H \left( \frac{1 + \alpha t}{1 + \gamma t} \right) \text{ or } H_0 = H \{ 1 - (\gamma - \alpha)t \}$$

### EFFECT OF TEMPERATURE ON UPTHURST

When a solid body is completely immersed in a liquid its apparent weight gets decreased due to an upthrust  $F$  acting on it by the liquid. The apparent weight is given by

$$W_{\text{app}} = W - F \quad \dots(1)$$

where  $F = \text{upthrust} = V_s \rho_L g$

where  $V_s = \text{volume of solid body}$  and  $\rho_L = \text{density of liquid}$

Now, as the temperature is increased  $V_s$  increases while  $\rho_L$  decreases. So,  $F$  may increase or decrease (or may remain constant also) depending upon the situation that which factor dominates on the other. We can write

$$F \propto V_s \rho_L$$

$$\Rightarrow \frac{F'}{F} = \frac{V'_s}{V_s} \cdot \frac{\rho'_L}{\rho_L} = \left( \frac{V_s + \Delta V_s}{V_s} \right) \left( \frac{1}{1 + \gamma_L \Delta T} \right)$$

$$\Rightarrow \frac{F'}{F} = \left( \frac{V_s + \gamma_s V_s \Delta T}{V_s} \right) \left( \frac{1}{1 + \gamma_L \Delta T} \right)$$

$$\Rightarrow F' = F \left( \frac{1 + \gamma_s \Delta T}{1 + \gamma_L \Delta T} \right)$$

Now, if  $\gamma_s > \gamma_L$ ,  $F' > F$

$$\Rightarrow W'_{\text{app}} < W_{\text{app}} \quad \{ \because \text{of Equation (1)} \}$$

and vice versa and if  $\gamma_s = \gamma_L$ ,  $F' = F$  or  $W'_{\text{app}} = W_{\text{app}}$

#### ILLUSTRATION 27

A solid body floats in liquid at temperature  $0^\circ\text{C}$  and is completely submerged in it at  $50^\circ\text{C}$ . What fraction  $f$  of volume of the body is submerged in the liquid at  $0^\circ\text{C}$ , if  $\gamma_s = 0.3 \times 10^{-5} \text{K}^{-2}$  and of the liquid,  $\gamma_l = 8.0 \times 10^{-5} \text{K}^{-1}$ ?

#### SOLUTION

In both cases, the weight of the body will be balanced by the force of buoyancy on it. At  $t_0 = 0^\circ\text{C}$ , the buoyancy is

$$F_b = f V_0 \rho_0 g \quad \dots(1)$$

where  $V_0$  is the volume of the body and  $\rho_0$  is the density of the liquid at  $t_0 = 0^\circ\text{C}$ .

At  $t = 50^\circ\text{C}$ , the volume of the body becomes  $V = V_0(1 + \gamma_s t)$  and the density of the liquid is

$$\rho_1 = \frac{\rho_0}{1 + \gamma_l t}$$

The buoyancy in this case is

$$F_b = \frac{V_0 \rho_0 g (1 + \gamma_s t)}{1 + \gamma_l t} \quad \dots(2)$$

Equating equations (1) and (2), we get

$$f = \frac{1 + \gamma_s t}{1 + \gamma_l t} = 96\%$$

#### ILLUSTRATION 28

A solid whose volume does not change with temperature floats in a liquid. For two different temperatures  $t_1$  and  $t_2$  of the liquid, fractions  $f_1$  and  $f_2$  of the volume of the solid remain submerged in the liquid. Find the coefficient of volume expansion of the liquid.

#### SOLUTION

With the rise in temperature, the liquid undergoes volume expansion, therefore the fraction of solid submerged in liquid increases. At temperature  $T_1$  if density of liquid is  $\rho_1$  and that at temperature  $T_2$  is  $\rho_2$ , so we have

$$F_{\text{buoyant}} = V_{\text{imm}} \rho_{\text{liq}} g = mg$$

$$\Rightarrow f_1 V_0 \rho_1 = f_2 V_0 \rho_2 = mg$$

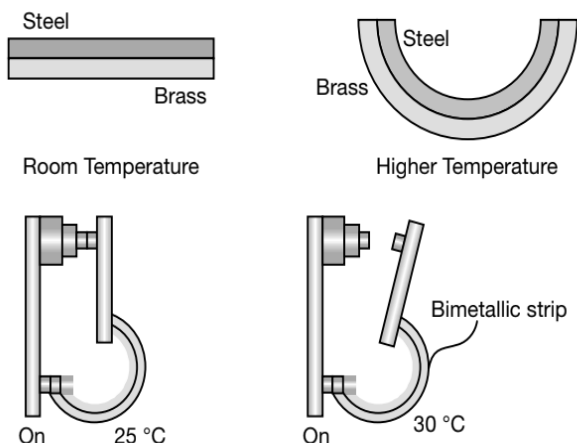
$$\Rightarrow \frac{f_1 \rho_0}{1 + \gamma T_1} = \frac{f_2 \rho_0}{1 + \gamma T_2}$$

$$\Rightarrow f_1 + f_1 T_2 \gamma = f_2 + f_2 T_1 \gamma$$

$$\Rightarrow \gamma = \frac{f_2 - f_1}{f_1 T_2 - f_2 T_1}$$

### BIMETALLIC STRIP

Since each substance has its own characteristic average coefficient of expansion. For example, when the temperatures of a brass rod and a steel rod of equal length are raised by the same amount from some common initial value, the brass rod expands more than the steel rod because brass has a greater average coefficient of expansion than steel. Such type of bimetallic strip is found in practical devices such as **thermostats** to break or make electrical contact.



**ILLUSTRATION 29**

Two metal strips, each of length  $L$  and thickness  $\left(\frac{d}{2}\right)$  at temperature  $T_0$  are riveted together so that their ends coincide. One strip is made of metal A having coefficient of linear expansion  $\alpha_A$  and the other with a coefficient  $\alpha_B$ , where  $\alpha_A > \alpha_B$ . When this bimetallic strip is heated to a temperature  $(T_0 + \Delta T)$ , the bimetallic strip bends into the arc of the circle. Assuming that the thickness  $d$  of the bimetallic strip remains constant at new temperature, calculate the radius of the circle.

**SOLUTION**

Since  $\alpha_A > \alpha_B$ , so metal A will be on outer side and metal B on the inner side. Now,

$$L_A = r_A \theta = L(1 + \alpha_A \Delta T) \quad \dots(1)$$

$$L_B = r_B \theta = L(1 + \alpha_B \Delta T) \quad \dots(2)$$

Adding (1) and (2), we get

$$(r_A + r_B) \theta = L[2 + (\alpha_A + \alpha_B) \Delta T]$$

$$\Rightarrow r = \frac{r_A + r_B}{2} = \frac{L[2 + (\alpha_A + \alpha_B) \Delta T]}{2\theta} \quad \dots(3)$$

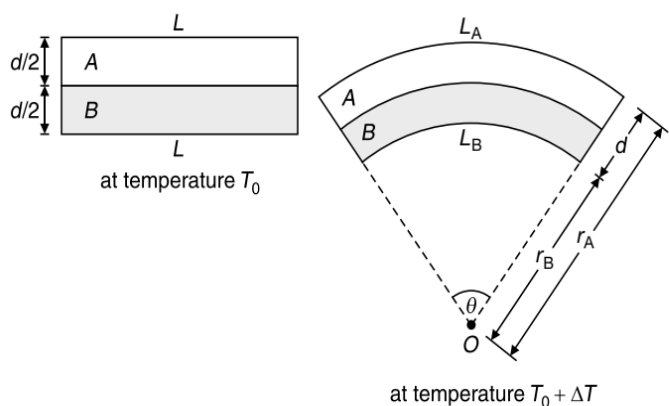
Also,  $r_A - r_B = d \quad \dots(4)$

Subtracting (2) from (1), we get

$$\theta(r_A - r_B) = L(\alpha_A - \alpha_B) \Delta T$$

$$\Rightarrow \theta d = L(\alpha_A - \alpha_B) \Delta T \quad \{\because r_A - r_B = d\}$$

$$\Rightarrow \theta = \frac{L(\alpha_A - \alpha_B) \Delta T}{d} \quad \dots(5)$$



Substituting (5) in (3), we get

$$r = \frac{L[2 + (\alpha_A + \alpha_B) \Delta T]}{\left[ \frac{2L(\alpha_A - \alpha_B) \Delta T}{d} \right]} = \frac{d}{2} \left[ \frac{2 + (\alpha_A + \alpha_B) \Delta T}{(\alpha_A - \alpha_B) \Delta T} \right]$$

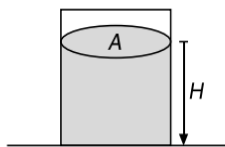
**Test Your Concepts-I**

**Based on Thermometry and Thermal Expansion**

*(Solutions on page H.75)*

- A 20 cm steel ruler is graduated to give correct measurements at 20 °C.

  - Will it give readings that are too long or too short at lower temperatures?
  - What will be the actual length of the ruler be when it is used in the desert at a temperature of 40 °C? Given,  $\alpha_{\text{steel}} = 1.2 \times 10^{-5} (\text{°C})^{-1}$ .
- An aluminium measuring rod which is correct at 5 °C, measures a certain distance as 88.42 cm at 35 °C. Calculate the error in measuring the distance due to expansion of the rod.
  - If the aluminium rod measures the length of a steel rod as 88.42 cm at 35 °C, what is the correct length of the steel rod at 35 °C. Given  $\alpha_{Al} = 2.55 \times 10^{-5} (\text{°C})^{-1}$
- A steel tape measures the length of a copper rod at 90 cm when both are at 10 °C, the calibration temperature for the tape. What would the tape read for the length of the rod when both are at 30 °C? Given  $\alpha_c = 1.7 \times 10^{-5} (\text{°C})^{-1}$  and  $\alpha_s = 1.2 \times 10^{-5} (\text{°C})^{-1}$ .
- A liquid having coefficient of volume expansion  $\gamma$  is filled in a glass vessel. The coefficient of linear expansion of glass is  $\alpha$ . When the arrangement is heated to raise the temperature of the liquid and the glass container by  $\Delta T$ , expansion takes place in both. The expansion may be different or equal. Depending on the values of  $\gamma$  and  $\alpha$  we may find that level of the liquid rises with respect to ground or it may fall with respect to ground. Find the relation between  $\gamma$  and  $\alpha$  for which



- (i) the fraction of volume of container occupied by liquid does not change with rise in temperature.
  - (ii) the level of the liquid in container does not change with respect to the container.
  - (iii) the level of the liquid in container does not change with respect to the ground.
5. The scale on a steel meter stick is calibrated at 15 °C. Find the error in the reading of 60 cm at 27 °C? Given,  $\alpha_{\text{steel}} = 1.2 \times 10^{-5} (\text{°C})^{-1}$ .
  6. In a mercury in glass thermometer the cross-section of the capillary is  $A_0$  and volume of the bulb is  $V_0$  at 0 °C. If mercury just fills the bulb at 0 °C, find that the length of mercury in the capillary at temperature  $t$  °C. Given that coefficient of cubical expansion of mercury is  $\gamma$  and coefficient of linear expansion of glass is  $\alpha$ .
  7. A second's pendulum clock has a steel wire. The clock is calibrated at 20 °C. How much time does the clock lose or gain in one week when the temperature is increased to 30 °C? Given,  $\alpha_{\text{steel}} = 1.2 \times 10^{-5} (\text{°C})^{-1}$ .
  8. A steel wire of cross-sectional area 0.5 mm<sup>2</sup> is held between two rigid clamps so that it is just taut at 20 °C. Calculate the tension in the wire at 0 °C. Given that Young's Modulus of steel is  $Y_s = 2.1 \times 10^{12} \text{ dynecm}^{-2}$  and coefficient of linear expansion of steel is  $\alpha_s = 1.1 \times 10^{-5} (\text{°C})^{-1}$ .
  9. A steel rod 50 cm long has cross-sectional area of 0.8 cm<sup>2</sup>. What force would be required to stretch this rod by the same amount as the expansion produced by heating it through 10 °C. Given, for steel,  $\alpha = 10^{-5} \text{ K}^{-1}$ ,  $Y = 2 \times 10^{11} \text{ Nm}^{-2}$ .
  10. Typical temperatures in the interior of the earth and sun are about  $4 \times 10^3 \text{ °C}$  and  $1.5 \times 10^7 \text{ °C}$  respectively. Calculate these temperatures on absolute scale. Also calculate the percentage error made in each case if the observer forgets to convert °C to K.
  11. At 0 °C, three metal rods form an equilateral triangle. Two rods are of the same material, but the third is made of invar (its expansion is negligible). When the triangle is heated up to 100 °C, the angle formed between the two metal rods of the same material is  $\frac{\pi}{3} - \theta$ . Find the coefficient of linear expansion of the two metal rods.
  12. Calculate the coefficient of volume expansion for an ideal gas at constant pressure.
  13. Two thermometers, one marked in Fahrenheit the other in Celsius, are placed in a bath. At what temperature will both thermometers read the same?

14. A uniform solid brass cylinder of mass 0.50 kg and radius 0.030 m is placed on a friction less bearings and set to rotate about its geometrical axis with an angular velocity of 60 radian s<sup>-1</sup>.
  - (a) Calculate the angular momentum of the cylinder and the work required to reach this state of rotation starting from rest.
  - (b) After the cylinder has reached the specified state of rotation, it is heated without any mechanical contact from room temperature of 20 °C to 100 °C, find the fractional change in the angular velocity of the cylinder. Given  $\alpha_{\text{Brass}} = 2 \times 10^{-5} (\text{°C})^{-1}$

15. In an alcohol-in-glass thermometer, the alcohol column has length 12.45 cm at 0.0 °C and length 21.30 cm at 100.0 °C. Calculate the temperature, if the column has length of
  - (a) 15.10 cm
  - (b) 22.95 cm

16. If the coefficient of linear expansion  $\alpha$  is taken as a variable dependent on temperature  $T$ , show that the length

$$L \text{ at temperature } T \text{ is given by } L = L_0 \left[ 1 + \int_{T_0}^T \alpha(T) dT \right],$$

where  $L_0$  is the length at reference temperature  $T_0$ .

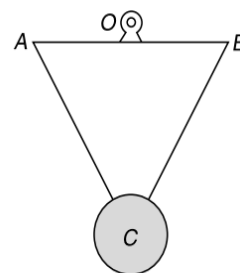
17. What should be the lengths of the steel rod and copper rod so that the length of the steel rod is 5 cm longer than the copper rod at all the temperatures?

Given  $\alpha_{\text{Cu}} = 1.7 \times 10^{-5} (\text{°C})^{-1}$  and

$\alpha_{\text{Steel}} = 1.1 \times 10^{-5} (\text{°C})^{-1}$

18. If the volume of a block of metal changes by 0.12% when it is heated by 20 °C. Calculate the coefficient of linear expansion of the metal.

19. A compensated pendulum is a pendulum made up of two or more metals arranged such that the effective length of centre of mass of the whole body remains unchanged. Such a situation is shown in Figure in which the compensated pendulum is in the form of an isosceles triangle of base length  $l_1 = 5 \text{ cm}$  and coefficient of linear expansion  $\alpha_1 = 18 \times 10^{-6}$  and side length  $l_2$  and coefficient of linear expansion  $\alpha_2 = 12 \times 10^{-6}$ . Find  $l_2$  so that the distance of centre of mass of the bob from suspension centre O may remain the same at all the temperature.

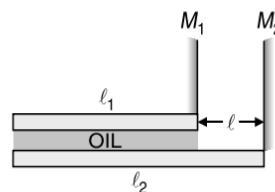


20. A circular hole in an aluminium plate is 2.54 cm in diameter at 0 °C. What is the diameter when the temperature of the plate is raised to 100 °C. Given  $\alpha_{\text{Al}} = 2.3 \times 10^{-5} (\text{°C})^{-1}$ .

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21. If the triple point of neon and carbon dioxide are observed at 24.57 K and 216.55 K respectively, then express these temperatures on the Celsius and Fahrenheit scales.
22. An optical engineering firm needs to ensure that the separation between two mirrors  $M_1$  and  $M_2$  is unaffected by temperature changes. The mirrors are attached to the ends of two bars of different materials that are welded together at one end as shown in figure. The surfaces of the bars in contact are lubricated. Show

that the distance  $\ell$  does not change with temperature if  $\alpha_1 \ell_1 = \alpha_2 \ell_2$ , where  $\alpha_1$  and  $\alpha_2$  are the respective thermal coefficients of temperature. Also determine the required lengths  $\ell_1$  and  $\ell_2$ , in terms of  $\alpha_1$ ,  $\alpha_2$  and  $\ell$ .



### CALORIMETRY: HEAT, WORK AND MECHANICAL EQUIVALENT OF HEAT

Experimentally it has been observed that the work done ( $W$ ) produces a proportional amount of heat ( $Q$ ) i.e.

$$W \propto Q$$

$$W = JQ$$

where  $J$  is called **Joule's Mechanical Equivalent of Heat** whose value is  $4.186 \text{ Jcal}^{-1}$  (or  $4.2 \text{ Jcal}^{-1}$ )

#### Problem Solving Technique(s)

While writing above relation  $W$  must be expressed in joule and  $Q$  must be expressed in calorie i.e., if

$$Q = 1 \text{ calorie}, W = 4.18 \text{ joule}$$

i.e., a work of 4.18 joule has to be done to produce a heat of 1 calorie. Hence  $1 \text{ calorie} \equiv 4.18 \text{ joule}$

### CONCEPT OF GRAM SPECIFIC HEAT

As a student of Physics, you can always think and understand that different masses of bodies made of same material require different amounts of heat to be supplied to them to raise their temperature by the same amount. Similarly, equal masses of bodies of same material require different amounts of heat to be supplied to them to raise their temperature by different amounts and also equal masses of bodies of different material require different amount of heat to be supplied to them to raise their temperature by same amount. (Think of some practical examples!)

From above, we conclude that amount of heat supplied  $Q$  to a body is directly proportional to mass  $m$  of body and rise in temperature  $\Delta T$  of the body i.e.,  $Q \propto m\Delta T$

$$\Rightarrow Q = mc\Delta T$$

where,  $c$  is called the “**gram specific heat**” or “**specific heat**” of the material of the body which gains heat or loses heat.  $c$  is different for different materials (NOTE THIS!) and

is a constant for small temperature variations whereas, for large temperature variations it starts varying as a function of temperature (and hence cannot be taken as a constant). Further if an infinitesimal amount of heat  $dQ$  is supplied to a body of mass  $m$  and rise in temperature is  $dT$  then  $dQ = mcdT$

$$\Rightarrow c = \frac{1}{m} \frac{dQ}{dT} = \frac{1}{m} \lim_{\Delta T \rightarrow 0} \frac{\Delta Q}{\Delta T}$$



### Conceptual Note(s)

- (a) Specific heat of water  $c_{\text{water}} = 1 \text{ calg}^{-1}(\text{°C})^{-1}$
- (b) Specific heat of ice  $c_{\text{ice}} = 0.5 \text{ calg}^{-1}(\text{°C})^{-1}$
- (c)  $1 \text{ calg}^{-1}(\text{°C})^{-1} = 4200 \text{ Jkg}^{-1}(\text{°C})^{-1} = 4200 \text{ Jkg}^{-1}\text{K}^{-1}$

### THERMAL CAPACITY OR HEAT CAPACITY

The amount of heat required to raise the temperature of a body through 1 unit. ( $\Delta T = 1$  unit change) i.e., if  $\Delta T = 1$  unit then Heat capacity =  $mc$

So, heat capacity of a body of mass  $m$ , gram specific heat  $c$  is the product of  $m$  and  $c$ .

### WATER EQUIVALENT ( $w$ )

Consider a body having mass  $m$  and gram specific  $c$ . Let  $dQ$  be the amount of heat supplied to the body such that the rise in temperature is  $dT$ .

The amount (or mass) of water ( $w$ ) to which this same amount of heat  $dQ$  should be supplied such that the rise in temperature of water is also  $dT$  is called **the water equivalent of the body**.

$$\text{For Body} \quad dQ = mcdT \quad \dots(1)$$

$$\text{For Water} \quad dQ = wc_{\text{water}}dT$$

$$\text{Since } c_{\text{water}} = 1 \text{ calg}^{-1}(\text{°C})^{-1}$$

$$\Rightarrow dQ = wdT \quad \dots(2)$$

Using (1) and (2), the water equivalent  $w$  is given by

$$w = mc$$

“water equivalent ( $w$ ) of a body of mass  $m$ , gram specific heat  $c$  is the product of  $m$  and  $c$ .”

SI unit of water equivalent is kg and cgs unit is gram (g).

### A MISCONCEPTION

Two bodies A and B of masses  $m_1$  and  $m_2$  at temperatures  $T_1$  and  $T_2$  with gram specific heats  $c_1$ , and  $c_2$  are placed in contact with each other. As a result of contact heat must flow from the body at higher temperature (say A) to the body at lower temperature and ultimately the equilibrium temperature should be  $\frac{T_1 + T_2}{2}$ .

**Now let's give this thing a thought from an example of everyday life.**

Consider a person of mass 60 kg holding a piece of ice of mass 6 g in his palm. As a result of this heat must flow from the body of person to ice and ultimately the equilibrium temperature should become (in approximation)  $\frac{37+0}{2} = 18.5^\circ\text{C}$

At this body temperature can you think a person to survive? This means that holding a mere 6 g ice piece on the palm may prove to be fatal for the person. CAN THAT BE SO?

### MISSING CONCEPT

Maximum students have this “misconception” that when two bodies at temperatures  $T_1$  and  $T_2$  are placed in contact the equilibrium temperature must be  $\frac{T_1 + T_2}{2}$ , **which is FALSE (in most of the situations)**, but under some specific pre-defined conditions it can be true (see CASE-3 in next article).

### MISCONCEPTION REMOVAL

It is that we must say heat flows from the palm of the person to ice till equilibrium is attained which is always governed by the equation.

$$\left( \begin{array}{l} \text{Heat Lost by} \\ \text{Body at Higher} \\ \text{Temperature} \end{array} \right) = \left( \begin{array}{l} \text{Heat Gained by} \\ \text{Body at Lower} \\ \text{Temperature} \end{array} \right)$$

This is also called as the Law of Calorimetry in which

$$\Sigma Q_{\text{isolated}} = 0 \text{ OR } \Sigma Q_{\text{lost}} = \Sigma Q_{\text{gained}}$$

## BODIES PLACED IN CONTACT

Consider two bodies A and B of masses  $m_1$  and  $m_2$ , at temperatures  $T_1$  and having gram specific heats  $c_1$  and  $c_2$ . When placed in contact, if equilibrium temperature is  $T_{\text{eq}}$  then Heat lost by A = Heat gained by B

$$\Rightarrow m_1 c_1 (T_1 - T_{\text{eq}}) = m_2 c_2 (T_{\text{eq}} - T_2)$$

$$\Rightarrow T_{\text{eq}} = \frac{m_1 c_1 T_1 + m_2 c_2 T_2}{m_1 c_1 + m_2 c_2}$$

### CASE-1: SAME BODIES WITH UNEQUAL MASSES

If bodies are of identical materials, then  $c_1 = c_2$

$$\Rightarrow T_{\text{eq}} = \frac{m_1 T_1 + m_2 T_2}{m_1 + m_2}$$

### CASE-2: DIFFERENT BODIES WITH EQUAL MASSES

If bodies are not of identical materials ( $c_1 \neq c_2$ )

But equal masses, then

$$T_{\text{eq}} = \frac{T_1 c_1 + T_2 c_2}{c_1 + c_2}$$

### CASE-3: SAME BODIES WITH EQUAL MASSES

If bodies are of identical materials,  $c_1 = c_2$  and equal masses i.e.,  $m_1 = m_2$ . Then,

$$T_{\text{eq}} = \frac{T_1 + T_2}{2}$$

### CASE-4: BODIES HAVING EQUAL WATER EQUIVALENT OR HEAT CAPACITIES

If bodies have equal water equivalents or the heat capacities, then we have  $m_1 c_1 = m_2 c_2$ . Hence, we get

$$T_{\text{eq}} = \frac{m_1 c_1 T_1 + m_2 c_2 T_2}{m_1 c_1 + m_2 c_2} = \frac{T_1 + T_2}{2}$$

### Problem Solving Technique(s)

If a number of hot and cold liquids having no chemical affinity for each other are mixed together then equilibrium condition or temperature can be found by using Law of Calorimetry according to which

$$\Sigma Q_{\text{isolated}} = 0 \text{ OR } \Sigma Q_{\text{lost}} = \Sigma Q_{\text{gained}}$$

### ILLUSTRATION 30

The molar heat capacity of a certain substance varies with temperature  $T$  as  $C = a + bT$  where  $a = 27.2 \text{ Jmol}^{-1}\text{K}^{-1}$  and  $b = 4 \times 10^{-3} \text{ Jmol}^{-1}\text{K}^{-2}$ . How much heat is necessary to change the temperature of 2 mol of this substance from  $27^\circ\text{C}$  to  $427^\circ\text{C}$ ?

### SOLUTION

Initial temperature  $T_i = 27 + 273 = 300 \text{ K}$

Final temperature  $T_f = 427 + 273 = 700 \text{ K}$

$$\text{Since, } Q = \int_{T_i}^{T_f} nC dT = \int_{T_i}^{T_f} (a + bT) dT$$

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$$\Rightarrow Q = \int_{300}^{700} (2)(27.2 + 4 \times 10^{-3} T) \cdot dT$$

$$\Rightarrow Q = (54 \cdot 4T + 4 \times 10^{-3} T^2) \Big|_{300}^{700}$$

$$\Rightarrow Q = 23360 \text{ J}$$

### ILLUSTRATION 31

Three liquids  $P$ ,  $Q$  and  $R$  are given. It is observed that 4 kg of  $P$  at  $60^\circ\text{C}$  and 1 kg of  $R$  at  $50^\circ\text{C}$ , when mixed produce a resultant temperature of  $55^\circ\text{C}$ . A mixture of 1 kg of  $P$  at  $60^\circ\text{C}$  and 1 kg of  $Q$  at  $50^\circ\text{C}$  shows a temperature of  $55^\circ\text{C}$ . Find the resulting temperature when 1 kg of  $Q$  at  $60^\circ\text{C}$  is mixed with 1 kg of  $R$  at  $50^\circ\text{C}$ .

### SOLUTION

Let  $c_P$ ,  $c_Q$  and  $c_R$  be the gram specific heats of liquids  $P$ ,  $Q$  and  $R$  respectively.

When  $P$  and  $R$  are mixed, then

$$\Rightarrow 4c_P(60 - 55) = (1)c_R(55 - 50)$$

$$\Rightarrow 4c_P = c_R \quad \dots(1)$$

When  $P$  and  $Q$  are mixed, then

$$(1)c_P(60 - 55) = (1)c_Q(55 - 50)$$

$$\Rightarrow c_P = c_Q \quad \dots(2)$$

When  $Q$  and  $R$  are mixed, then

$$(1)c_Q(60 - T) = (1)c_R(T - 50)$$

$$\Rightarrow c_Q(60 - T) = 4c_Q(T - 50) \quad \{\because \text{of (1) and (2)}\}$$

$$\Rightarrow 60 - T = 4(T - 50)$$

$$\Rightarrow T = 52^\circ\text{C}$$

### ILLUSTRATION 32

A metal container of mass 500 g contains 200 g of water at  $20^\circ\text{C}$ . A block of iron also of mass 200 g at  $100^\circ\text{C}$  is dropped into water. Find the equilibrium temperature of the water. Given that specific heats of metal of container and iron and that of water are  $910 \text{ Jkg}^{-1}\text{K}^{-1}$ ,  $470 \text{ Jkg}^{-1}\text{K}^{-1}$  and  $4200 \text{ Jkg}^{-1}\text{K}^{-1}$  respectively.

### SOLUTION

Here container and water are at  $20^\circ\text{C}$ , thus when iron block is dropped into water it will loose energy to it and its temperature will fall. If equilibrium temperature is  $T_0$ , then we have

heat lost by iron block = heat gained by water plus container

$$m_i s_i (100 - T_0) = m_w s_w (T_0 - 20) + m_c s_c (T_0 - 20)$$

$$\Rightarrow T_0 = \frac{m_i s_i (100) + m_w s_w (20 + m_c s_c (20))}{m_i s_i + m_w s_w + m_c s_c}$$

$$\Rightarrow T_0 = \frac{0.2 \times 470 \times 100 + 0.2 \times 4200 \times 20 + 0.5 \times 910 \times 20}{0.2 \times 470 + 0.2 \times 4200 + 0.5 \times 910}$$

$$\Rightarrow T_0 = \frac{9400 + 16800 + 9100}{94 + 840 + 455}$$

$$\Rightarrow T_0 = \frac{35300}{1329} = 26.56^\circ\text{C}$$

## CONCEPT OF LATENT HEAT OR THE HEAT OF TRANSFORMATION

The word Latent is from the Latin "latere", meaning hidden or concealed.

A substance usually undergoes a change in temperature when heat is transferred between the substance and its surroundings. However, situations arise when the flow of heat does not change the temperature of body. This situation is arising when the physical characteristics of the substance change from one form to another, referred to as **Phase Change**. All such phase changes involve a change in internal energy. The energy required is called the **Heat of Transformation**.

The heat required to change the phase of a given mass  $m$  of a pure substance is given as

$$Q = mL$$

where,  $L$  is the latent heat of substance and depends upon nature of phase change as well as properties of a substance.

The **Latent Heat of Fusion**,  $L_f$ , is used when the phase change is from solid to liquid. The **Latent Heat of Vapourisation** is used when the phase change is from liquid to gas (vapours).

Latent heat of fusion of ice is  $L_{\text{ice}} = 80 \text{ calg}^{-1}$ .

Latent heat of vaporisation of water is also called as latent heat of steam  $L_{\text{steam}} = 540 \text{ calg}^{-1}$ .

When gas (or vapours) cools, it eventually returns to the liquid phase or condenses. The heat per unit mass given up is called **Latent Heat of Condensation** which equals the latent heat of vaporisation. Likewise, when a liquid cools it eventually solidifies and so the **Latent Heat of Solidification** equals the Latent Heat of Fusion.



## Conceptual Note(s)

### (a) ICE-WATER MIXING

Suppose water at temperature  $T_w$   $^\circ\text{C}$  be mixed with ice at  $0^\circ\text{C}$  and let the equilibrium temperature of the mixture be  $T_{\text{mix}} = T$ , then we observe that firstly the ice will melt at  $0^\circ\text{C}$  and then its temperature rises

from  $0^\circ\text{C}$  to  $T$  so as to attain thermal equilibrium. By Law of Calorimetry, we have

Heat lost by water = Heat gained by ice

$$\Rightarrow m_w c_w (T_w - T_{\text{mix}}) = m_i L_i + m_i c_w (T_{\text{mix}} - 0^\circ)$$

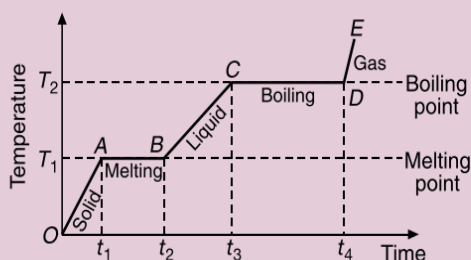
$$\Rightarrow T_{\text{mix}} = T = \frac{m_w T_w - \frac{m_i L_i}{c_w}}{m_w + m_i}$$

$$(i) \text{ If } m_w = m_i \text{ then } T_{\text{mix}} = \frac{\left(T_w - \frac{L_i}{c_w}\right)}{2}$$

(ii) By using this formulae, if  $T_{\text{mix}} < T_i$ , then take  $T_{\text{mix}} = 0^\circ\text{C}$

### (b) HEATING CURVE

Let heat be supplied at constant rate  $P \left( = \frac{dQ}{dt} \right)$  to a given mass  $m$  of a solid and a graph be plotted between temperature and time. The graph is called Heating Curve. From the curve, following conclusions can be made.



(i) In the region OA, temperature of solid is changing with time so

$$Q_1 = mc_s \Delta T$$

$$\Rightarrow P \Delta t = mc_s \Delta T$$

$$\Rightarrow c_s = \frac{P \Delta t}{m \Delta T} = \frac{P}{m \left( \frac{\Delta T}{\Delta t} \right)}$$

Since,  $\frac{\Delta T}{\Delta t}$  is the slope of temperature-time curve, so we have

$$c_s \propto \frac{1}{\text{Slope of line OA}}$$

So, the specific heat (or thermal capacity) is inversely proportional to the slope of temperature-time curve.

(ii) In the region AB temperature is constant, so it represents change of state i.e., melting of solid with melting point  $T_1$ . At A, melting starts and at B all the solid is converted into liquid. So between A and B the substance is partly solid and partly liquid. If  $L_F$  be the latent heat of fusion, then  $Q_2 = mL_F$

$$\Rightarrow L_F = \frac{P(t_2 - t_1)}{m} \quad \left\{ \because Q_2 = P(t_2 - t_1) \right\}$$

$$\Rightarrow L_F \propto \text{length of line AB}$$

So, latent heat of fusion is proportional to the length of line of zero slope.

(iii) In the region BC temperature of liquid increases, so specific heat (or thermal capacity) of liquid will be inversely proportional to the slope of line BC, so

$$c_L \propto \frac{1}{\text{Slope of line BC}}$$

(iv) In the region CD, we observe that temperature is constant, so it represents the change of state, i.e., boiling with boiling point  $T_2$ . At C, all substance is in liquid state while at D it is in vapour state and between C and D it is partly liquid and partly gas. The length of line CD is proportional to latent heat of vapourisation  $L_V$ . So, we have

$$L_V \propto \text{length of line CD}$$

(v) The line DE represents the gaseous state of substance with its temperature increasing linearly with time. The reciprocal of slope of line will be proportional to specific heat or thermal capacity of substance in vapour state.

### ILLUSTRATION 33

How much heat is supplied to convert 40 g of ice at  $-10^\circ\text{C}$  to steam at  $100^\circ\text{C}$ ?

#### SOLUTION

To convert 40 g of ice at  $-10^\circ\text{C}$  to 40 g steam at  $100^\circ\text{C}$ , we have the following steps.

#### STEP-1

Heat supplied,  $Q_1$  to convert 40 g of ice at  $-10^\circ\text{C}$  to 40 g of ice at  $0^\circ\text{C}$  is

$$Q_1 = (40)(0.5)(0 - (-10)) \text{ cal} = 200 \text{ cal}$$

#### STEP-2

Heat supplied,  $Q_2$  to convert 40 g of ice at  $0^\circ\text{C}$  to 40 g of water at  $0^\circ\text{C}$  is

$$Q_2 = (40)(80) \text{ cal} = 3200 \text{ cal}$$

#### STEP-3

Heat supplied,  $Q_3$  to convert 40 g of water at  $0^\circ\text{C}$  to 40 g of water at  $100^\circ\text{C}$  is

$$Q_3 = (40)(1)(100) \text{ cal} = 4000 \text{ cal}$$

#### STEP-4

Heat supplied,  $Q_4$  to convert 40 g of water at  $100^\circ\text{C}$  to 40 g of steam at  $100^\circ\text{C}$  is

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$$Q_4 = (40)(540) \text{ cal} = 21600 \text{ cal}$$

So, total heat supplied is

$$Q = Q_1 + Q_2 + Q_3 + Q_4$$

$$\Rightarrow Q = 200 + 3200 + 4000 + 21600 = 29000 \text{ cal} = 29 \text{ kcal}$$

### ILLUSTRATION 34

In an industrial process 10 kg of water per hour is to be heated from 20 °C to 80 °C. To do this, steam at 150 °C is passed from a boiler into a copper coil immersed in water. The steam condenses in the coil and is returned to the boiler as water at 90 °C. How many kg of steam is required per hour?

Given that specific heat of steam is  $1 \text{ cal g}^{-1} (\text{°C})^{-1}$  and latent heat of steam is  $540 \text{ cal g}^{-1}$

### SOLUTION

Heat required by 10 kg water to increase its temperature from 20 to 80 °C in one hour is given by

$$Q_1 = (mc\Delta T)_{\text{water}} = (10 \times 10^3)(1)(80 - 20)$$

$$Q_1 = 600 \text{ kcal}$$

If  $m$  gram of steam is condensed per hour, the heat released by steam in converting into water at 90 °C

$$Q_2 = mc_s(150 - 100) + mL_v + mc_w(100 - 90)$$

$$\text{i.e., } Q_2 = m(1 \times 50 + 540 + 1 \times 10) = 600m \text{ cal}$$

$$\left\{ \because c_s = c_w = 1 \text{ cal g}^{-1} \text{ °C} \right\}$$

According to given problem  $Q_2 = Q_1$

$$600m = 600 \times 10^3$$

$$\Rightarrow m = 1 \times 10^3 \text{ g} = 1 \text{ kg}$$

### ILLUSTRATION 35

In a pitcher when water is filled some water comes to outer surface slowly through its porous walls and gets evaporated. Most of the latent heat needed for evaporation is taken from water inside and hence this water is cooled down. Assume that 10 kg water is taken in the pitcher and 12 g water comes out and gets evaporated per minute. Neglecting heat transfer by convection and radiation to surroundings, calculate the time in which temperature of water in pitcher decreases by 5 °C.

### SOLUTION

It is given that 12 g water is evaporated per minute, thus heat required per minute for it is

$$Q = mL_v = 12 \times 540 = 6480 \text{ cal min}^{-1}$$

After time  $t$ , mass of water inside the pitcher is

$$m = [10(1000) - 12t] \text{ gram}$$

In a further time  $dt$ , mass of vapourised water is  $dm = 12dt$  and let the temperature of water inside pitcher falls by  $dT$ . So, we get

$$(10000 - 12t)(1)(-dT) = (12dt) 540$$

$$\Rightarrow -dT = 6480 \left( \frac{dt}{10000 - 12t} \right) \quad \dots(1)$$

Integrating equation (1), we get

$$-\int_{T_0}^{T_0-5} dT = 12 \times 540 \int_0^t \frac{dt}{10000 - 12t}$$

$$\Rightarrow 5 = 6480 \left[ \left( -\frac{1}{12} \right) \ln(10000 - 12t) \right]_0^t$$

$$\Rightarrow -\frac{5}{540} = \ln \left( \frac{10000 - 12t}{10000} \right)$$

$$\Rightarrow e^{-5/540} = \frac{10000 - 12t}{10000}$$

$$\Rightarrow 10000e^{-5/540} = 10000 - 12t$$

$$\Rightarrow 12t = 10000(1 - e^{-5/540})$$

$$\Rightarrow t = \frac{10000}{12}(1 - 0.991) \approx 7.5 \text{ minute}$$

Students should note that this method will give the correct answer if the mass of water inside the pitcher nearly remains constant i.e., when the rate of evaporation is very as in this case.

### ILLUSTRATION 36

Ice at 0 °C is added to 200 g of water initially at 70 °C in a vacuum flask. When 50 g of ice has been added and has all melted, the temperature of the flask and contents is 40 °C. When a further 80 g of ice has been added and has all melted, the temperature of the whole becomes 10 °C. Neglecting heat lost to the surroundings, calculate the latent heat of fusion of ice? (Specific heat of water is  $1 \text{ cal g}^{-1} \text{ °C}$ ) and water equivalent of flask.

### SOLUTION

If  $L$  is the latent heat of ice and  $W$  is the water equivalent of flask, According to Principle of Calorimetry

$$\text{Heat gained} = \text{Heat lost}$$

$$\Rightarrow m'L + m'C\Delta T = (m + W)C\Delta T_{\text{water}}$$

$$\Rightarrow 50(L + 1 \times (40 - 0)) = (200 + W) \times 1 \times (70 - 40)$$

$$\Rightarrow 5L = 3W + 400 \quad \dots(1)$$

Now the system contains (200 + 50) g of water at 40 °C, so when further 80 g of ice is added, then

$$80(L+1 \times (10-0)) = (250+W) \times 1 \times (40-10)$$

$$\Rightarrow 8L = 3W + 670 \quad \dots(2)$$

Solving equations (1) and (2), we get

$$L = 90 \text{ calg}^{-1} \text{ and } W = \left(\frac{50}{3}\right) \text{ g}$$

### ILLUSTRATION 37

A thermally insulated piece of metal is heated under atmospheric pressure by an electric current so that it receives electric energy at a constant power  $P$ . This leads to an increase of the absolute temperature  $T$  of the metal with time  $t$  as  $T = T_0 [1 + a(t - t_0)]^{1/4}$ , where  $a$ ,  $t_0$  and  $T_0$  are constants. Determine the heat capacity at constant pressure  $C_p(T)$  of the metal.

### SOLUTION

Let the temperature increase in a small time interval  $dt$  be  $dT$ . During this time interval the metal receives an energy  $Pdt$ . The heat capacity is the ratio between the energy supplied and the temperature increase, so

$$C_p = \frac{Pdt}{dT} = \frac{P}{dT/dt}$$

From the given equation,

$$\frac{dT}{dt} = \left(\frac{T_0}{4}\right) a(1 + a(t - t_0))^{-3/4} = T_0 \frac{a}{4} \left(\frac{T_0}{T}\right)^3$$

$$\Rightarrow C_p = \frac{P}{dT/dt} = \left(\frac{4P}{aT_0^4}\right) T^3$$

We must note that, at low but not extremely low temperatures heat capacities of metals  $C_p \propto T^3$ .



### Test Your Concepts-II

#### Based on Calorimetry

(Solutions on page H.77)

- In a container of negligible mass 30 g of steam at 100 °C is added to 200 g of water that has a temperature of 40 °C. If no heat is lost to the surroundings, what is the final temperature of the system? Also find masses of water and steam in equilibrium. Take  $L_v = 539 \text{ calg}^{-1}$  and  $c_{\text{water}} = 1 \text{ calg}^{-1}(\text{°C})^{-1}$
- A copper cube of mass 200 g slides down a rough inclined plane of inclination 37° at a constant speed. Assuming that the loss in mechanical energy goes into the copper block as thermal energy, find the increase in temperature of the block as it slides down through 60 cms. Specific heat capacity of copper is equal to  $420 \text{ Jkg}^{-1}\text{K}^{-1}$ . Take  $g = 10 \text{ ms}^{-2}$ .
- A pitcher contains 1 kg water at 40 °C. It is given that the rate of evaporation of water from the surface of pitcher is  $0.1 \text{ gs}^{-1}$ . Calculate the time in which water inside the pitcher cools down to 30 °C. Given that latent heat of vaporization of water is  $540 \text{ calg}^{-1}$  and specific heat of water is  $1 \text{ calg}^{-1} \text{ °C}^{-1}$ .
- A vertical close cardboard tube 150 cm long is filled with lead shots to a depth of 30 cm. What is the least number of times the tube has to be inverted end to end to warm the lead shots by 2 °C. Given  $c_{\text{lead}}$  is  $0.031 \text{ calg}^{-1}\text{°C}^{-1}$ .
- A bullet of mass 10 g travelling at a speed of  $100 \text{ ms}^{-1}$  strikes a fixed wooden target. One half of the kinetic energy is transformed into heat to the bullet and the other half is transformed into heat to the wooden block. Find the rise in temperature of the bullet if  $c_{\text{bullet}}$  is  $0.030 \text{ calg}^{-1}\text{°C}^{-1}$ .
- The temperature of a body rises by 44 °C when a certain amount of heat is given to it. The same heat when supplied to 22 g of ice at  $-8 \text{ °C}$ , raises its temperature to 16 °C. Find the water equivalent of the body.  
{Given:  $S_{\text{water}} = 1 \text{ calg}^{-1} \text{ °C}^{-1}$  and  $L_f = 80 \text{ calg}^{-1}$ ,  $S_{\text{ice}} = 0.5 \text{ calg}^{-1} \text{ °C}^{-1}$ }
- When 400 J of heat is added to a 0.1 kg sample of metal, its temperature increases by 20 °C. Calculate the specific heat of the metal.
- 1 kg of ice at 0 °C is mixed with 1 kg of steam at 100 °C. Find the equilibrium temperature and the final composition of the mixture. Given that latent heat of fusion of ice is  $3.36 \times 10^5 \text{ Jkg}^{-1}$  and latent heat of vaporization of water is  $2.27 \times 10^6 \text{ Jkg}^{-1}$  and specific heat of water is  $4200 \text{ Jkg}^{-1}\text{°C}^{-1}$ .
- 10 g of water at 70 °C is mixed with 5 g of water at 30 °C. Find the temperature of the mixture in equilibrium.
- When a block of metal of specific heat  $0.1 \text{ calg}^{-1}\text{°C}^{-1}$  and weighing 110 g is heated to 100 °C and then quickly transferred to a calorimeter containing 200 g of liquid at 10 °C, the resulting temperature is 18 °C. On repeating the experiment with 400 g of same liquid in the same calorimeter at the same initial temperature, the resulting temperature is 14.5 °C. Calculate specific heat of the liquid and water equivalent of the calorimeter.
- The temperatures of equal masses of three different liquids A, B and C are 12 °C, 19 °C and 28 °C respectively. The temperature when liquids A and B are

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- mixed is  $16^\circ\text{C}$  and when liquids B and C are mixed is  $23^\circ\text{C}$ . What should be the temperature when liquids A and C are mixed?
- A lead bullet just melts when stopped by an obstacle. Assuming 25% heat to be absorbed by the obstacle, find the velocity of the bullet if its initial temperature is  $27^\circ\text{C}$ . Given melting point of lead is  $327^\circ\text{C}$ ,  $c_{\text{lead}}$  is  $0.03 \text{ cal g}^{-1}\text{ }^\circ\text{C}^{-1}$ ,  $L_{\text{bullet}} = 6 \text{ cal g}^{-1}$  and  $J = 4.2 \text{ joule cal}^{-1}$ .
  - How should 1 kg of water at  $50^\circ\text{C}$  be divided in two parts so that if one part is turned into ice at  $0^\circ\text{C}$ , then it would release sufficient amount of heat to vaporize the other part. Given that latent heat of fusion of ice is  $3.36 \times 10^5 \text{ J kg}^{-1}$ , latent heat of vapourisation of water is  $22.5 \times 10^5 \text{ J kg}^{-1}$  and specific heat of water is  $4200 \text{ J kg}^{-1}\text{K}^{-1}$ .
  - A steel drill making 180 rpm is used to drill a hole in a block of steel. The mass of the steel block and the drill is 180 g each. If the entire mechanical work is used up in producing heat and the rate of rise of temperature of the block is  $0.5^\circ\text{C s}^{-1}$ . Find
    - the rate of working of drill in watt.
    - the couple required to drive the drill
 (Given: Specific heat of steel is  $0.10 \text{ cal g}^{-1}\text{ }^\circ\text{C}^{-1}$ )
  - A lead bullet strikes against a steel armour plate with a velocity of  $300 \text{ ms}^{-1}$ . If the bullet loses all its velocity after the impact, find the rise in temperature of the bullet assuming that the heat produced is equally shared by bullet and the target. Given  $c_{\text{bullet}} = 0.03 \text{ cal g}^{-1}\text{ }^\circ\text{C}^{-1}$ .
  - An aluminium container of mass 100 g contains 200 g of ice at  $-20^\circ\text{C}$ . Heat is added to the system at the rate 100 calorie per second. What is temperature of the system after four minutes? Specific heat of ice =  $0.5 \text{ cal g}^{-1}\text{ }^\circ\text{C}^{-1}$  and latent heat of fusion of ice =  $80 \text{ cal g}^{-1}$  and specific heat of aluminium is  $0.215 \text{ cal g}^{-1}\text{ }^\circ\text{C}^{-1}$ .
  - In an experiment of measuring specific heat of water, a stream of water flows at a steady rate of  $5 \text{ gs}^{-1}$  over an electrical heater dissipating 135 W and a temperature rise of 5 K is observed. On increasing the rate of flow to  $10 \text{ gs}^{-1}$ , the same temperature rise is produced with a dissipation of 240 W. Find the specific heat of water. Assume heat loss to the surrounding in both cases is the same.
  - How much heat is required to convert 8 g of ice at  $-15^\circ\text{C}$  to steam at  $100^\circ\text{C}$ ? Given that  $c_{\text{ice}} = 0.5 \text{ cal g}^{-1}\text{ }^\circ\text{C}^{-1}$ ,  $L_{\text{ice}} = 80 \text{ cal g}^{-1}$ ,  $L_{\text{steam}} = 540 \text{ cal g}^{-1}$ ,  $c_{\text{water}} = 1 \text{ cal g}^{-1}\text{ }^\circ\text{C}^{-1}$ .
  - 5 g ice at  $0^\circ\text{C}$  is mixed with 5 g of steam at  $100^\circ\text{C}$ . What is the final temperature?
  - A copper calorimeter of mass 100 g contains 200 g of a mixture of ice and water. Steam at  $100^\circ\text{C}$  under normal pressure is passed into the calorimeter and the temperature of the mixture is allowed to rise to  $50^\circ\text{C}$ . If the mass of the calorimeter and its contents is now 330 g, what was the ratio of ice and water in the beginning? Neglect heat losses. Given that
 

Specific heat of copper =  $420 \text{ J kg}^{-1}\text{ K}^{-1}$   
 Specific heat of water =  $4200 \text{ J kg}^{-1}\text{ K}^{-1}$   
 Latent heat of fusion of ice =  $3.36 \times 10^5 \text{ J kg}^{-1}$  and  
 Latent heat of condensation of steam =  $22.5 \times 10^5 \text{ J kg}^{-1}$

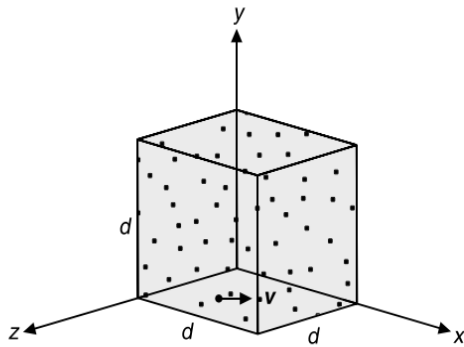
# KINETIC THEORY OF GASES (KTG)

## KINETIC THEORY OF GASES

Since a gas is made up of large number of molecules executing random motion, so the characteristics of a gas depends on the properties of motion of these molecules and this analysis of a gas at the microscopic level is known as Kinetic Theory of Gases (KTG) in which we relate the microscopic properties (like velocity of the molecule) to the macroscopic properties (like pressure, temperature, volume etc).

## MOLECULAR MODEL FOR THE PRESSURE OF AN IDEAL GAS

Let us begin by developing a microscopic model of an ideal gas which shows that the pressure which a gas exerts on the walls of its container is a consequence of the collisions of the gas molecules with the walls of container. The following assumptions will be made



- The number of molecules is large, and the average separation between them is large compared with their dimensions. So, molecules occupy negligible volume in comparison to volume of container.
- The molecules obey Newton's Laws of Motion, but considered collectively move in a random fashion. By random fashion we mean that molecules move in all the directions with equal probability and with various speeds. This distribution in velocities doesn't change with time, despite the collisions between molecules.
- The forces between molecules are negligible except during a collision. The forces between molecules are short range, so that the only time the molecules interact with each other is during a collision.
- The molecules undergo elastic collisions with each other and with the walls of the container. So, the molecules are considered to be structureless (i.e., point masses), and in the collisions both momentum and kinetic energy are conserved.
- The gas under consideration is a pure gas i.e., all molecules of the gas are identical.

- (f) Ideally, all gases obey the ideal gas equation i.e.,

$$PV = nRT$$

where,  $P$  is the pressure of gas,  $V$  is the volume of gas,  $T$  is the temperature of gas and  $R$  is Universal Gas Constant having value

$$R = 8.314 \text{ Jmol}^{-1}\text{K}^{-1}$$

- (g) Typically, a real gas obeys ideal gas equation only at very high temperature and very low pressure. This is because at very high temperatures, the kinetic energy of the molecules is so large that any interaction between the molecules of the gas can be neglected. Also, at very low pressure, the separation between the gas molecules is very large and hence again the interaction between the molecules of the gas can be neglected.

Let's derive an expression for the pressure of an ideal gas consisting of  $N$  molecules in a container of volume  $V (= d^3)$ . The container is assumed to be in the cubic shape with each edge of length  $d$ . Consider the collision of one molecule moving with velocity  $v$  towards the right hand face of the system (box). The molecule has velocity components  $v_x$ ,  $v_y$  and  $v_z$  along  $x$ ,  $y$  and  $z$  axes respectively. When the molecule collides elastically with the wall its  $x$  component of velocity gets reversed whereas  $y$  and  $z$  components remaining unaltered.

The  $x$  component of momentum before collision is  $mv_x$  and after collision is  $-mv_x$ . So, change in momentum of molecule along  $x$  axis is

$$\Delta p_x = -mv_x - mv_x = -2mv_x$$

The momentum delivered to the wall for each collision is  $2mv_x$ , since the momentum of the system (molecule+container) is conserved.

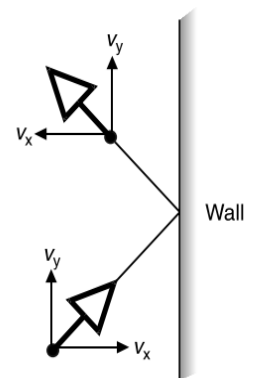
For a molecule to make two successive collisions with the same wall it must travel a distance  $2d$  in the  $x$  direction in time  $\Delta t$ . So

$$\Delta t = \frac{2d}{v_x}$$

If  $F$  is the magnitude of the average force exerted by a molecule on the wall in time  $\Delta t$ , then

$$F\Delta t = \Delta p_x = 2mv_x$$

$$\Rightarrow F = \frac{2mv_x}{\Delta t} = \frac{2mv_x}{2d/v_x} = \frac{mv_x^2}{d} \quad \dots(1)$$



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Total force on the wall is the sum of all such terms for all particles. To get the total pressure  $P$  on the wall we have

$$P = \frac{\Sigma F}{A} = \frac{m}{d^3} (v_{x_1}^2 + v_{x_2}^2 + \dots) \quad \dots(2)$$

where  $v_{x_1}, v_{x_2}, \dots$  refer to the  $x$  components of velocities for particles 1, 2, .....

The average value of  $v_x^2$  is given by

$$\langle v_x^2 \rangle = \frac{v_{x_1}^2 + v_{x_2}^2 + \dots + v_{x_N}^2}{N}$$

and the volume of container is given by  $V = d^3$  using (2), we have

$$P = \frac{Nm}{V} \langle v_x^2 \rangle \quad \dots(3)$$

Further for any particle moving with velocity  $v$

$$v^2 = v_x^2 + v_y^2 + v_z^2 \quad \dots(4)$$

$$\Rightarrow \langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle \quad \dots(5)$$

As there is no preferred direction for the molecules, so average values of  $v_x^2, v_y^2$  and  $v_z^2$  are equal and hence

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{\langle v^2 \rangle}{3}$$

Substituting (5) in (3), we have

$$P = \frac{1}{3} \frac{Nm}{V} \langle v^2 \rangle = \frac{2}{3V} \left[ N \left\{ \frac{1}{2} m \langle v^2 \rangle \right\} \right] \quad \dots(6)$$

Average Kinetic Energy of each molecule is  $\langle K \rangle = \frac{1}{2} m \langle v^2 \rangle$ , so total Kinetic Energy associated with the system of  $N$  molecules is

$$E = N \langle K \rangle = N \left( \frac{1}{2} m \langle v^2 \rangle \right) \quad \dots(7)$$

Using equation (6) and (7), we get

$$P = \frac{2}{3} \frac{E}{V}$$

$$\Rightarrow PV = \frac{2}{3} E$$

The empirical equation of state for an ideal gas is

$$PV = Nk_B T \quad \dots(8)$$

where the Boltzmann constant is

$$k_B = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ JK}^{-1}$$

Substituting (8) in (6), we have

$$\frac{Nk_B T}{V} = \frac{2}{3} \frac{N}{V} \frac{1}{2} m \langle v^2 \rangle$$

$$\Rightarrow \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$$

$$\Rightarrow \langle v^2 \rangle = \frac{3k_B T}{m}$$

Root Mean Square (RMS) velocity of a molecule of mass  $m$  at temperature

$$v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3RT}{M}}$$

where,  $m$  is mass of one molecule of gas and  $M$  is the molar mass of gas or mass of 1 mole of gas.



### Conceptual Note(s)

- To conclude, with this simplified model of ideal gas, we have arrived at an important result that relates the macroscopic quantities of pressure, volume and temperature to a microscopic quantity like rms speed. So, we get a key link between the microscopic world of gas molecules and the macroscopic world.
- In the derivation of this result, note that we have not accounted for collisions between gas molecules. When these collisions are considered, the results **does not alter as collisions do not affect the momenta of the particles, with no net effect on the walls.** Bravo!!! This is consistent with one of our initial assumptions, namely, that the distribution of velocities doesn't change with time.
- In addition, although our result was derived for a cubical container whereas the result is valid for a container of any shape.
- The number density of molecules in gas is  $n_0 = \frac{N}{V}$  and each molecule is assumed to be moving in random direction with rms speed  $v_{\text{rms}}$ . Due to the random character of motion of the gas molecules, it can be assumed that towards "each face" of the cubical container,  $\frac{n_0}{6}$  molecules are moving with this speed. Due to this the number of collisions  $N_c$  of the molecules with "a wall" of the cubical container per square metre of its surface can be written as

$$N_c = \left( \frac{n_0}{6} \right) v_{\text{rms}}$$

### KINETIC INTERPRETATION OF TEMPERATURE

From the point of view of kinetic theory, temperature is a quantity characterizing the average kinetic energy of translatory motion of the molecules of an ideal gas. Since we know that

$$PV = \frac{2}{3} N_A \left( \frac{1}{2} m \langle v^2 \rangle \right)$$

Comparing above equation with  $PV = N_A k_B T$ , we can find the average kinetic energy of a molecule as

$$K_{av} = \frac{1}{2} m v_{rms}^2 = \frac{3}{2} k_B T$$

For an ideal gas, the absolute temperature is a measure of the average translational kinetic energy of the molecules.

Note that the kinetic energy calculated above is associated with the random translational motion of a single molecule. It does not include any orderly motion imposed. The rms speed of molecules varies with temperature as

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3RT}{M}}$$

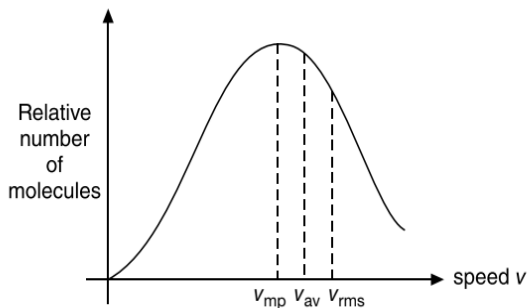
## MAXWELL BOLTZMANN LAW FOR DISTRIBUTION OF MOLECULAR SPEEDS

Since the molecules in a gas are in random motion, so many molecules of gas may have speeds less than the average speed (or rms speed) and others have speeds greater than the average speed (or rms speed).

As an example, the rms speed of an oxygen molecule in a gas sample at 300 K is

$$v_{rms} = \sqrt{\frac{3(8.314)(300)}{32/1000}} \approx 483 \text{ ms}^{-1}$$

This does not mean that the speed of every molecule in the sample is  $483 \text{ ms}^{-1}$ . Many molecules in the sample may have speeds less than  $483 \text{ ms}^{-1}$  and many may molecules in the sample may have speed more than  $483 \text{ ms}^{-1}$ . *This conclusion was made by Maxwell on the basis of kinetic theory and the experiments done in which Maxwell plotted a graph of variation of speeds of molecules with the relative number of molecules as shown in Figure.*



Maxwell derived an equation giving the distribution of molecules in different speeds. If  $dN$  represents the number of molecules with speeds between  $v$  and  $v + dv$ , then based on statistical approach, the number of molecules  $dN$  having speed between  $v$  and  $v + dv$  is given by

$$dN = 4\pi N \left( \frac{m}{2\pi k_B T} \right)^{3/2} v^2 e^{-mv^2/2k_B T} dv$$

The probability that a molecule possesses speed  $v$  is

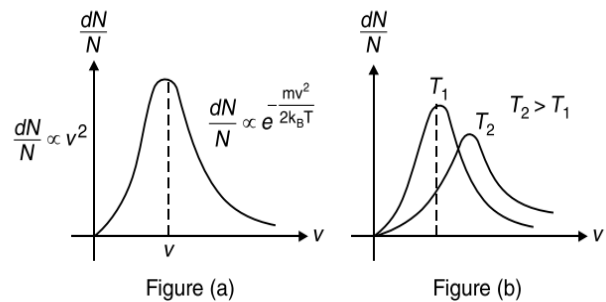
$$\frac{dN}{N} = f(v) dv = 4\pi \left( \frac{m}{2\pi k_B T} \right)^{3/2} v^2 e^{-mv^2/2k_B T} dv$$

where,  $f(v)$  is the number of gas molecules per unit range of speeds.

The expected number of molecules  $N(v_1, v_2)$  in the gas sample lying within speeds between  $v_1$  to  $v_2$  is obtained by integrating  $Nf(v)dv$  from  $v_1$  to  $v_2$  i.e.,

$$N(v_1, v_2) = \int_{v_1}^{v_2} f(v) dv$$

The speed  $v_p$  at which  $\frac{dN}{dv}$  is maximum is called the most probable speed ( $v_{mp}$ ). Maxwell distribution curve is shown in Figure (a). The curves show the  $dN$  increases with increase of speed, becomes maximum and then it decreases.



At low speeds,  $dN \propto v^2$  and hence this part of curve is parabolic. At high speeds  $dN \propto e^{-mv^2/2k_B T}$ , so this part of the curve is decreasing exponentially.

The speed at which  $dN$  is maximum is called the most probable speed and denoted by  $v_{mp}$ . The value of  $v_{mp}$  decreases with increase of temperature. Figure (b) represents  $dN$  versus  $v$  curves at two different temperatures  $T_1$  and  $T_2 (> T_1)$ .

## AVERAGE VELOCITY OF GAS MOLECULES

Since all molecules of the gas in container have random motion due to repeated collisions, so if  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N$  are the instantaneous velocity vectors of all  $N$  molecules of a gas, then average velocity vector of these  $N$  molecules is zero i.e.,

$$\langle \vec{v} \rangle = \frac{\vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_N}{N} = 0$$

Hence, the average velocity of molecules in a gas is always taken as zero. However, average speed of the gas molecules will not be zero.

## AVERAGE SPEED OF GAS MOLECULES

Due to the random motion of the gas molecules, average velocity of all the molecules of a gas is zero. However, it is not same in case of mean or average speed, which is obtained by averaging the speed of the molecules. It can be simply defined as

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$$v_{av} = \langle v \rangle = \frac{|\vec{v}_1| + |\vec{v}_2| + \dots + |\vec{v}_N|}{N}$$

Using Maxwell's distribution function, we can calculate the mean speed as

$$v_{av} = \langle v \rangle = \frac{1}{N} \int_0^{\infty} v f(v) dv = \frac{1}{N} \int_0^{\infty} v dN = \sqrt{\frac{8RT}{\pi M}}$$

### ROOT-MEAN-SQUARE VELOCITY OF GAS MOLECULES

As the name implies this is the square root of mean of squares of velocities of all the molecules of a gas. Mean of squares can be simply written as

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\left( \frac{|\vec{v}_1|^2 + |\vec{v}_2|^2 + \dots + |\vec{v}_N|^2}{N} \right)}$$

and from Maxwell's Law, we can write

$$v_{rms} = \sqrt{\frac{1}{N} \int_0^{\infty} v^2 f(v) dv} = \sqrt{\frac{1}{N} \int_0^{\infty} v^2 dN} = \sqrt{\frac{3RT}{M}}$$

### MOST PROBABLE SPEED OF GAS MOLECULES

The speed  $v_{mp}$  at which  $\frac{dN}{dv}$  is maximum is called the most probable speed ( $v_{mp}$ ). This is the speed which maximum number of molecules have and this speed corresponds to maxima in the Maxwell's distribution curve. So at most probable speed, we have

$$\frac{dN}{dv} = 0$$

$$\Rightarrow v_{mp} = \sqrt{\frac{2RT}{M}}$$

So we conclude that,  $v_{rms} > v_{mean} > v_{mp}$



### Conceptual Note(s)

(a)  $v_{rms} > v_{av} > v_{mp}$  i.e.,  $R > A > M$

One can easily remember above relation by the word RAM, where R stands for rms, A for Average and M for most probable

(b)  $v_{rms} : v_{av} : v_{mp} = \sqrt{3} : \sqrt{\frac{8}{\pi}} : \sqrt{2} = \sqrt{3} : \sqrt{2.5} : \sqrt{2}$

(c) Speed of sound in a gas is given by

$$v_{sound} = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{\gamma k_B T}{m}} = \sqrt{\frac{\gamma P}{\rho}}$$

i.e., the speed of sound in a gas is of the same order as rms speed of its molecules. In nutshell, we can write

$$v = \sqrt{\frac{ART}{M}} = \sqrt{\frac{Ak_B T}{m}} = \sqrt{\frac{AP}{\rho}}$$

where A is a number whose value is given by

$$A = 3 \text{ for } v_{rms}, A = \frac{8}{\pi} \text{ for } v_{av}$$

$$A = 2 \text{ for } v_{mp}, A = \gamma \text{ for } v_{sound}$$

(d) Note that in the above expression the molecular mass M should be taken in kilogram (not in gram). For example, at NTP the rms speed of hydrogen gas is given by

$$(v_{rms})_{H_2} = \sqrt{\frac{3 \times 8.31 \times 273}{2 \times 10^{-3}}} = 1840 \text{ ms}^{-1} \quad (\text{Correct})$$

$$\neq \sqrt{\frac{3 \times 8.31 \times 273}{2}} \quad (\text{Incorrect})$$

### MEAN FREE PATH

If  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$  are N free paths, then mean free path

$$\lambda = \frac{\lambda_1 + \lambda_2 + \dots + \lambda_N}{N} = \frac{\bar{v}t}{N} = \frac{\langle v \rangle t}{N}$$

where,  $\bar{v} = \langle v \rangle$  is average speed.

Also  $\lambda = \frac{1}{\sqrt{2}\pi D^2 n}$  where, n is the number density of molecules i.e.,

$$n = \frac{\text{Number of molecules}}{\text{Volume}} = \frac{N}{V}$$

and D is the molecular diameter. Since for one mole of gas sample, we have

$$n = \frac{N_A}{V} = \frac{N_A}{RT/P} = \frac{P}{k_B T}$$

$$\Rightarrow \lambda = \frac{k_B T}{\sqrt{2}\pi D^2 P}$$

### ILLUSTRATION 38

The speeds (in  $\text{ms}^{-1}$ ) for 8 molecules 2, 4, 5, 5, 8, 9, 12 and 15. Calculate their

(a) average speed and (b) rms speed

### SOLUTION

(a) The average speed is given by

$$v_{av} = \frac{1}{N} (v_1 + v_2 + \dots + v_N)$$

$$\Rightarrow v_{av} = \frac{(2+4+5+5+8+9+12+15) \text{ ms}^{-1}}{8}$$

$$\Rightarrow v_{av} = 7.5 \text{ ms}^{-1}$$

(b) To find the rms speed we first find the average of  $v^2$ :

$$\langle v^2 \rangle = \frac{1}{N}(v_1^2 + v_2^2 + v_3^2 + \dots + v_N^2)$$

$$\Rightarrow \langle v^2 \rangle = \frac{(2^2 + 4^2 + 5^2 + 5^2 + 8^2 + 9^2 + 12^2 + 15^2)}{8}$$

$$\Rightarrow \langle v^2 \rangle = 73 \text{ ms}^{-2}$$

$$\text{Then, } v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = 8.54 \text{ ms}^{-1}$$

In general the average speed is not equal to the rms speed.

### ILLUSTRATION 39

In a certain region of space, on an average, there are only 5 molecules per cubic centimetre. The gas has a temperature of 3 K. Calculate average pressure of this very dilute gas if  $k_B = 1.38 \times 10^{-23} \text{ Jmol}^{-1}\text{K}^{-1}$ .

### SOLUTION

According to ideal gas equation  $PV = nRT$

$$\Rightarrow P = \left( \frac{nN_A k_B T}{V} \right) \quad \{ \because R = N_A k \}$$

Also, the number of molecules  $N = nN_A$

$$\Rightarrow PV = Nk_B T$$

$$\Rightarrow P = \frac{N}{V} k_B T$$

where,  $N/V$  is the number of molecules per unit volume. Since,

$$\frac{N}{V} = 5 \text{ molecules cm}^{-3} = \frac{5 \text{ molecules}}{(10^{-2} \text{ m})^3}$$

$$\Rightarrow \frac{N}{V} = 5 \times 10^6 \text{ molecules m}^{-3}$$

$$\Rightarrow P = (5 \times 10^6)(1.38 \times 10^{-23})(3) = 2.07 \times 10^{-16} \text{ Pa}$$

### ILLUSTRATION 40

Find the average translational kinetic energy of molecules in air at 300 K.

### SOLUTION

Although air consists of a mixture of gases, they all have the same average translational kinetic energy.

$$K_{av} = \frac{3}{2} k_B T = \frac{3}{2} (1.38 \times 10^{-23})(300) = 6.21 \times 10^{-21} \text{ J}$$

### ILLUSTRATION 41

Calculate the root mean square speed of smoke particles of mass  $5 \times 10^{-17} \text{ kg}$  in their Brownian motion in air at NTP. Take  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$ .

### SOLUTION

According to Kinetic Theory of Gases

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3k_B T}{m}}$$

Here  $T = 273 \text{ K}$ ,  $m = 5 \times 10^{-17} \text{ kg}$ ,  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$

$$\Rightarrow v_{\text{rms}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 273}{5 \times 10^{-17}}}$$

$$\Rightarrow v_{\text{rms}} = 1.5 \times 10^{-2} \text{ ms}^{-1} = 1.5 \text{ cms}^{-1}$$

### ILLUSTRATION 42

Two vessels having equal volume contain molecular hydrogen at one atmosphere and helium at two atmospheres respectively. What is the ratio of rms speed of hydrogen molecule to that of helium molecule, assume that both the samples are at the same temperature?

### SOLUTION

According to Kinetic Theory of Gases,  $v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$

$$\Rightarrow \frac{v_H}{v_{He}} = \sqrt{\frac{M_{He}}{M_H}} = \sqrt{\frac{4}{2}} = \sqrt{2}$$



## Conceptual Note(s)

If one uses the relation

$$\frac{v_H}{v_{He}} = \sqrt{\frac{P_H}{P_{He}} \times \frac{M_{He}}{M_H}} = \sqrt{\frac{1}{2} \times \frac{4}{2}} = 1 \text{ i.e., } v_H = v_{He}$$

This solution is **wrong** as in it  $M$  is **not** molecular weight but total mass of gas. Now as pressure of helium is double that of hydrogen at same temperature and volume, in accordance with Avogadro's Law the number of moles of helium will be double that of hydrogen.

$$\Rightarrow \frac{v_H}{v_{He}} = \sqrt{\frac{P_H}{nM_H} \times \frac{(2nM_{He})}{P_{He}}} = \sqrt{\frac{1}{2} \times \frac{2 \times 4}{2}} = \sqrt{2}$$

### ILLUSTRATION 43

At what temperature will the rms speed of oxygen molecule will be sufficient for escaping from the earth? Given that escape velocity from surface of earth is  $v_e = 11.2 \text{ kms}^{-1}$ , mass of oxygen molecule is  $m = 2.76 \times 10^{-26} \text{ kg}$  and Boltzmann's constant is  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$ .

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### SOLUTION

If the temperature is  $T$ , according to kinetic theory of gases

$$\text{translational } KE = \left(\frac{3}{2}\right)k_B T$$

The oxygen molecule will escape from earth if

$$\begin{aligned} \frac{3}{2}k_B T &> \frac{1}{2}mv_e^2 \\ \Rightarrow T &> \frac{mv_e^2}{3k_B} \\ \Rightarrow T &> \frac{2.76 \times 10^{-26} \times (11.2 \times 10^3)^2}{3 \times 1.38 \times 10^{-23}} \\ \Rightarrow T &> 8.36 \times 10^4 \text{ K} \end{aligned}$$

### ILLUSTRATION 44

Calculate the number of molecules per unit volume of air at  $0^\circ\text{C}$ , pressure of  $1.013 \times 10^5 \text{ Pa}$  and the average distance between molecules if Boltzmann constant is  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$ .

### SOLUTION

Pressure of gas is given as  $P = \frac{2}{3}E$ , where  $E$  is the translational kinetic energy per unit volume.

$$\begin{aligned} \Rightarrow P &= \frac{2}{3} \left( \frac{3}{2} k_B T \times n_0 \right) = n_0 k_B T \\ \Rightarrow n_0 &= \frac{P}{k_B T} = \frac{1.013 \times 10^5}{1.38 \times 10^{-23} \times 273} \\ \Rightarrow n_0 &= 2.688 \times 10^{25} \text{ m}^{-3} \end{aligned}$$

Average separation between molecules

$$d = \left( \frac{1}{n_0} \right)^{\frac{1}{3}} = 3.31 \times 10^{-9} \text{ m}$$

### ILLUSTRATION 45

One gram mole  $\text{NO}_2$  at  $57^\circ\text{C}$  and 2 atm pressure is kept in a vessel. Assuming the molecules to be moving with rms velocity, find the number of collisions per second, which the molecules make with one square meter area of the vessel wall.

### SOLUTION

$$\text{Since, } P = \frac{nRT}{V} \quad \dots(1)$$

$$\text{Also, we know that } P = \frac{F}{A}$$

For  $A = 1 \text{ m}^2$ , we have  $P = F$  (numerically)

$$\text{So, } P = F = \frac{dp}{dt} = (2mv_{\text{rms}})f \quad \dots(2)$$

So, from (1) and (2), we get

$$P = \frac{nRT}{V} = 2mv_{\text{rms}}f$$

where,  $f$  is the number of collisions per unit area per unit time and  $m$  is the mass of one gas molecules =  $\frac{M}{N_A}$

$$\Rightarrow P = 2 \frac{M}{N_A} \sqrt{\frac{3RT}{M}} f$$

$$\Rightarrow f = \frac{PN_A}{2} \sqrt{\frac{1}{3MRT}}$$

Substituting the values, we get

$$\begin{aligned} f &= \frac{2 \times 10^5 \times 6.02 \times 10^{23}}{2} \sqrt{\frac{1}{3 \times 46 \times 10^{-3} \times 8.31 \times 330}} \\ \Rightarrow f &= 3.1 \times 10^{27} \text{ s}^{-1} \end{aligned}$$

### ILLUSTRATION 46

A cubical box of side 1 meter contains helium gas (atomic weight 4) at a pressure of  $100 \text{ Nm}^{-2}$ . During an observation time of 1 second, an atom travelling with the root mean square speed parallel to one of the edges of the cube, was found to make 500 hits with a particular wall, without any collision with other atoms. If  $R = \frac{25}{3} \text{ Jmol}^{-1}\text{K}^{-1}$  and  $k = 1.38 \times 10^{-23} \text{ JK}^{-1}$ , then calculate the

- temperature of the gas.
- average kinetic energy per atom.
- total mass of helium gas in the box.

### SOLUTION

Given that, Volume of the box is  $1 \text{ m}^3$ , pressure of the gas is  $100 \text{ Nm}^{-2}$ .

If  $T$  be the temperature of the gas

- Time between two consecutive collisions with one wall is

$$\Delta t = \frac{1}{500} \text{ sec}$$

This time is equal to  $\frac{2l}{v_{\text{rms}}}$ ,

where  $l$  is the side of the cube.

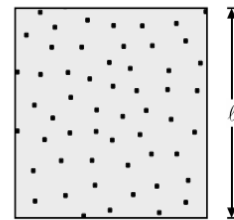
$$\Rightarrow \Delta t = \frac{2l}{v_{\text{rms}}} = \frac{1}{500}$$

$$\Rightarrow v_{\text{rms}} = 1000 \text{ ms}^{-1}$$

$$\{\because l = 1 \text{ m}\}$$

$$\Rightarrow \sqrt{\frac{3RT}{M}} = 1000$$

$$\Rightarrow T = \frac{(1000)^2 M}{3R} = \frac{(10)^6 (4 \times 10^{-3})}{3(25/3)} = 160 \text{ K}$$



(b) Average kinetic energy per atom is  $\frac{3}{2}k_B T$

$$\Rightarrow \langle E \rangle = \frac{3}{2}(1.38 \times 10^{-23})(160) \text{ J}$$

$$\Rightarrow \langle E \rangle = 3.312 \times 10^{-21} \text{ J}$$

(c) Since,  $PV = nRT = \left(\frac{m}{M}\right)RT$

So, mass of helium gas in the box is given by

$$m = \frac{PVM}{RT}$$

Substituting the values, we get

$$m = \frac{(100)(1)(4 \times 10^{-3})}{(25/3)(160)} = 3 \times 10^{-4} \text{ kg}$$

#### ILLUSTRATION 47

Calculate the root mean square speed and mean kinetic energy of one mole of hydrogen at STP (Given that density of hydrogen is  $0.09 \text{ kg m}^{-3}$ ). If mass of a molecule of hydrogen is  $3.33 \times 10^{-27} \text{ kg}$ , then calculate the value of Avogadro's number and Boltzmann's constant. Take  $R = 8.3 \text{ J mol}^{-1} \text{ K}^{-1}$ .

#### SOLUTION

Since, the pressure of a gas is given by  $P = \frac{1}{3} \rho v_{\text{rms}}^2$

$$\Rightarrow v_{\text{rms}} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3 \times 0.76 \times 13.6 \times 10^3 \times 9.8}{0.09}}$$

$$\Rightarrow v_{\text{rms}} = 1837 \text{ ms}^{-1} = 1.837 \text{ kms}^{-1}$$

$$\text{Kinetic energy is } K = N_A \left( \frac{1}{2} m v_{\text{rms}}^2 \right) = \frac{1}{2} M v_{\text{rms}}^2$$

where,  $M = 2 \text{ g} = 2 \times 10^{-3} \text{ kg}$

$$\Rightarrow K = \frac{1}{2} \times 2 \times 10^{-3} \times (1837)^2 = 3374.56 \text{ J}$$

Mass of one molecule of hydrogen is  $m = 3.33 \times 10^{-27} \text{ kg}$  and molar mass of hydrogen is  $M = 2 \times 10^{-3} \text{ kg}$ . So, the Avogadro's number  $N_A$  which is the number of molecules in one-gram molecule of hydrogen is given by

$$N_A = \frac{M}{m} = \frac{2 \times 10^{-3}}{3.34 \times 10^{-27}} = 3 \times 10^{23} \text{ molecules}$$

$$\Rightarrow k_B = \frac{R}{N_A} = \frac{8.3}{6 \times 10^{23}} = 1.38 \times 10^{-23} \text{ J mol}^{-1} \text{ K}^{-1}$$

#### ILLUSTRATION 48

In a container, the molecular density of an enclosed gas is  $10^{26}$  molecules per cubic centimetre, each of mass  $3 \times 10^{-27} \text{ kg}$ . If rms velocity of the gas molecules is  $2000 \text{ ms}^{-1}$ , calculate the number of molecules hitting per

square meter of the container wall every second and the pressure exerted on the walls of container by the molecules.

#### SOLUTION

Number of molecules hitting per square meter of container wall is given as

$$N_c = \left(\frac{n}{6}\right) v_{\text{rms}} = \frac{1}{6} \times 10^{26} \times 2000$$

$$\Rightarrow N_c = 3.33 \times 10^{28} \text{ per second}$$

and pressure is  $P = \frac{1}{3} \rho v_{\text{rms}}^2$

$$\Rightarrow P = \frac{1}{3} (3 \times 10^{-27}) (10^{26}) (2000)^2 = 4 \times 10^5 \text{ Nm}^{-2}$$

#### ILLUSTRATION 49

Calculate the temperature at which rms velocity of a gas molecule is same as that of a molecule of another gas at  $47^\circ \text{C}$ . Molecular weight of first and second gases are 64 and 32 respectively.

#### SOLUTION

$$\text{Since } v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

For first and second gas if temperature is  $T_1$  and  $T_2$  respectively, then

$$(v_{\text{rms}})_1 = (v_{\text{rms}})_2$$

$$\Rightarrow \frac{T_1}{M_1} = \frac{T_2}{M_2}$$

$$\Rightarrow \frac{T_1}{64 \times 10^{-3}} = \frac{320}{32} \quad \{ \because T_2 = 47 + 273 = 320 \text{ K} \}$$

$$\Rightarrow T_1 = 640 \text{ K}$$

#### ILLUSTRATION 50

A vessel of volume  $V = 16.62$  litre contains a mixture of hydrogen and helium at a temperature  $17^\circ \text{C}$  and pressure 6 atmosphere. The mass of the mixture is 10 g. Find the

- ratio of mass of hydrogen to that of helium.
- ratio of rate of collisions of hydrogen molecule per unit area of the container walls to that of helium molecules.

#### SOLUTION

$$(a) PV = (n_1 + n_2)RT = \left(\frac{m_1}{M_1} + \frac{m_2}{M_2}\right)RT$$

$$\Rightarrow \frac{m_1}{2} + \frac{m_2}{4} = \frac{6 \times 10^5 \times 16.62 \times 10^{-3}}{8.31 \times 300}$$

$$\Rightarrow 2m_1 + m_2 = 16 \quad \dots(1)$$

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Also,  $m_1 + m_2 = 10$  ... (2)

Solving equations (1) and (2), we get

$$m_1 = 6 \text{ g and } m_2 = 4 \text{ g}$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{3}{2}$$

(b) Number of moles of  $H_2$ ,  $n_1 = 3$  and number of moles of  $He$ ,  $n_2 = 1$

Since,  $P = \frac{nRT}{V} = 2mv_{rms}f$  ... (3)

where  $f$  is the number of collisions of the molecules per unit area per unit time.

Since,  $m = \frac{M}{N_A}$  and  $v_{rms} = \sqrt{\frac{3RT}{M}}$

Substituting these values in equation (3), we get

$$f \propto \frac{n}{\sqrt{M}}$$

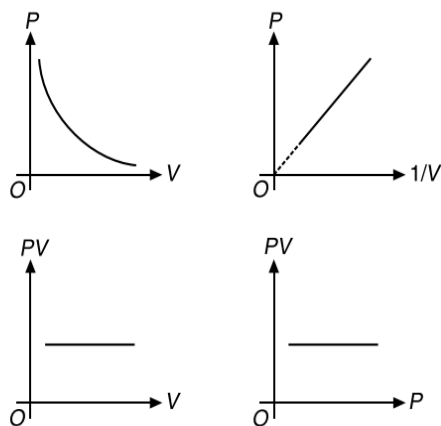
$$\Rightarrow \frac{f_1}{f_2} = \frac{n_1}{n_2} \sqrt{\frac{M_2}{M_1}} = \frac{3}{1} \sqrt{\frac{4}{2}} = 3\sqrt{2}$$

### GAS LAWS

The gases such as hydrogen, oxygen and helium etc. which cannot be liquified easily are called **permanent gases**. The gases whose molecules are point masses (mass without having volume) and do not attract each other are called **ideal or perfect gases**. Assuming permanent gases to be ideal, through experiment it has been observed that, these gases irrespective of their nature obey the following laws.

#### BOYLE'S LAW

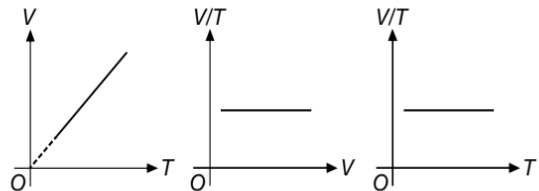
For a given mass of an ideal gas at constant temperature, the volume of a gas is inversely proportional to its pressure. i.e.  $V \propto \frac{1}{P}$



So,  $PV = \text{constant}$ , for given temperature and given mass of gas.

#### CHARLE'S LAW

At constant pressure, the volume of a given quantity of a gas is directly proportional to the absolute temperature i.e.,  $V \propto T$  or  $\frac{V}{T} = \text{constant}$ , for a given mass of gas at constant pressure



#### ILLUSTRATION 51

A glass container encloses a gas at a pressure of  $8 \times 10^5$  Pa and 300 K temperature. The container walls can bear a maximum pressure of  $10^6$  Pa. If the temperature of container is gradually increased, find the temperature at which container will break.

#### SOLUTION

At constant volume, we have  $\frac{P_1}{T_1} = \frac{P_2}{T_2}$

$$\Rightarrow T_2 = \left(\frac{P_2}{P_1}\right) T_1$$

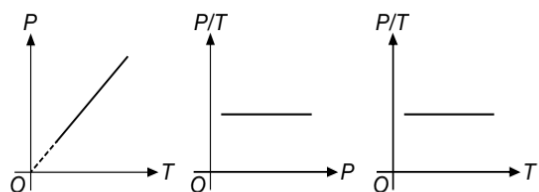
Since,  $P_1 = 8 \times 10^5$  Pa,  $T_1 = 300$  K,  $P_2 = 10^6$  Pa

$$\Rightarrow T_2 = \frac{10^6}{8 \times 10^5} \times 300 = 375 \text{ K}$$

#### GAY-LUSSAC'S LAW

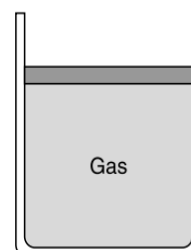
For a given mass of an ideal gas at constant volume, pressure of a gas is directly proportional to its absolute temperature i.e.,  $P \propto T$

$$\Rightarrow \frac{P}{T} = \text{constant, where } m \text{ and } V = \text{constant.}$$



#### ILLUSTRATION 52

A gas is enclosed inside a cylindrical container shown in Figure.



Its initial volume is  $V$  and temperature is  $T$ . No external pressure is applied on the light piston, hence gas pressure must be equal to the atmospheric pressure. If gas temperature is doubled, find its final volume. If in its final state, the piston is clamped and temperature is again doubled, find the final pressure of the gas.

### SOLUTION

Initially  $P = \text{constant}$ , so we have  $V \propto T$   
 So, when the gas temperature is doubled, then gas volume is also doubled i.e.,  $V' = 2V$   
 Now, when piston is clamped after attaining the final state, then  $V' = \text{constant} = 2V$ , so we have  $P \propto T$ .  
 Now, when its temperature is doubled from  $2T$  to  $4T$ , then the pressure also becomes double the value and hence  $P' = 2P$

### AVOGADRO'S LAW

At same pressure and same temperature, equal volumes of all gases contain equal number of molecules. At STP or NTP ( $0^\circ\text{C}$  and  $1\text{ atm}$  i.e.,  $273\text{ K}$  and  $1.01 \times 10^5\text{ Nm}^{-2}$ ), we have 1 mole of an ideal gas  $\equiv N_A (= 6.02 \times 10^{23})$  molecules of gas  $\equiv 22.4$  litre of the gas  $\equiv M$  (molecular weight) gram of the gas.

### DALTON'S LAW OF PARTIAL PRESSURES

If an enclosure contains a number of non-reacting gases, each gas behaves independent of the other and the total pressure exerted by the gases on the walls of enclosure is the sum of pressures exerted by individual gases i.e.,  $P = P_1 + P_2 + P_3 + \dots$

#### ILLUSTRATION 53

A vessel of volume  $2 \times 10^{-3}\text{ m}^3$  contains 0.1 mol of hydrogen gas and 0.2 mol of helium. If the temperature of the mixture is  $300\text{ K}$ , calculate the pressure due to component gases and the mixture. ( $R = 8.31\text{ Jmol}^{-1}\text{K}^{-1}$ )

### SOLUTION

As according to ideal gas equation  $PV = nRT$ ,

$$P_H = n_H \frac{RT}{V} = (0.1) \left( \frac{8.31 \times 300}{2 \times 10^{-3}} \right) = 1.25 \times 10^5\text{ Pa}$$

$$P_{He} = n_{He} \frac{RT}{V} = (0.2) \left( \frac{8.31 \times 300}{2 \times 10^{-3}} \right) = 2.50 \times 10^5\text{ Pa}$$

and by Dalton's Law of partial pressures,

$$P = P_H + P_{He} = (1.25 + 2.5) \times 10^5 = 3.75 \times 10^5\text{ Pa}$$

### GRAHAM'S LAW OF DIFFUSION

At constant temperature and pressure, the rate of diffusion of a gas is inversely proportional to the square root of its density

i.e., Rate of diffusion  $\propto \frac{1}{\sqrt{\rho}}$ , where  $\rho$  is the density of gas.

### IDEAL GAS EQUATION

Combining first four laws (i.e., Boyle's Law, Charles's Law, Gay-Lussac's Law and Avogadro's Law) we get one single equation for an ideal gas, i.e.

$$PV = RT$$

$$PV = nRT \text{ for } n \text{ moles of gas}$$

$$\Rightarrow \frac{PV}{nT} = R = \text{constant} \quad \dots(1)$$

Here,  $R$  is the Universal Gas Constant having value

$$R = 8.31\text{ Jmol}^{-1}\text{K}^{-1} = 2\text{ calmol}^{-1}\text{K}^{-1}$$

For 1 mole of a gas  $n = 1$  i.e.,  $PV = RT$

$$\Rightarrow \frac{PV}{T} = R = \text{constant}$$

#### ILLUSTRATION 54

A vessel of volume 20 litre contains a mixture of hydrogen and helium gas at  $20^\circ\text{C}$  and a pressure of 2 atm. The mass of the mixture is 5 g. Calculate the ratio of mass of hydrogen gas to helium gas.

### SOLUTION

If  $m_1$  be the mass of hydrogen gas and  $m_2$  be the mass of helium gas, then applying the ideal gas equation  $PV = nRT$ , we get

$$(2)(20) = \left( \frac{m_1}{2} + \frac{m_2}{4} \right) (0.082)(293)$$

$$\Rightarrow 2m_1 + m_2 = 6.66 \quad \dots(1)$$

$$\text{Also, } m_1 + m_2 = 5 \quad \dots(2)$$

Solving equations (1) and (2), we get

$$m_1 = 1.66\text{ g and } m_2 = 3.34\text{ g}$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{1}{2}$$

#### ILLUSTRATION 55

In a closed container of volume  $10^{-3}\text{ m}^3$ ,  $\text{O}_2$  gas is filled at temperature  $400\text{ K}$  and pressure  $1.5\text{ atm}$ . A small hole is made in the container from which gas leaks out to open atmosphere. After some time, the temperature and pressure of container become equals to that of surrounding. Find the mass of gas that leaks out from the container. (Atmospheric temperature =  $300\text{ K}$ )

### SOLUTION

If initially, there is mass  $m_1$  of  $\text{O}_2$  gas in the container, then from ideal gas equation, we get

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$$PV = \left(\frac{m_1}{32}\right)RT$$

$$\Rightarrow m_1 = \frac{32PV}{RT} = \frac{32(1.5 \times 10^5)(10^{-3})}{(8.314)(400)} = 1.443 \text{ g}$$

Finally, if  $m_2$  mass of  $O_2$  is left in the container, then from ideal gas equation, we get

$$PV = \left(\frac{m_2}{32}\right)RT$$

$$\Rightarrow m_2 = \frac{32PV}{RT} = \frac{32(10^5)(10^{-3})}{(8.314)(300)} = 1.283 \text{ g}$$

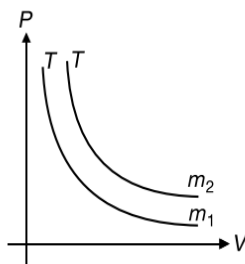
So, mass of gas leaked out is

$$\Delta m = m_1 - m_2$$

$$\Rightarrow \Delta m = 1.443 - 1.283 = 0.16 \text{ g}$$

### ILLUSTRATION 56

The  $P$ - $V$  diagram for two gases having different masses  $m_1$  and  $m_2$  are drawn (as shown) at constant temperature  $T$ . Which of  $m_1$  and  $m_2$  is smaller?



### SOLUTION

$$\text{Since, } PV = nRT = \left(\frac{m}{M}\right)RT$$

$$\Rightarrow m = (PV) \left(\frac{M}{RT}\right)$$

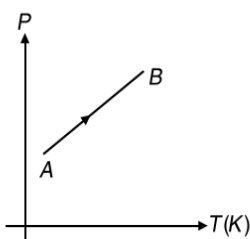
$$\Rightarrow m \propto PV, \text{ when } T = \text{constant}$$

From the graph we observe that  $P_2V_2 > P_1V_1$  (for same  $P$  or  $V$ ).

So,  $m_2 > m_1$ .

### ILLUSTRATION 57

For the given mass of an ideal gas the  $P$ - $T$  diagram is shown in figure. What can you conclude regarding the change in volume (whether it is constant, increasing or decreasing)?



### SOLUTION

When volume is constant,  $P$ - $T$  graph is a straight line passing through origin. The given line does not pass through origin, hence volume is not constant. From Ideal Gas Equation, we get

$$V = (nR) \left(\frac{T}{P}\right)$$

Now, to see the volume of the gas we will have to see whether  $\frac{T}{P}$  is increasing or decreasing.

From the given graph we can write the  $P$ - $T$  equation as,

$$P = aT + b \quad (y = mx + c)$$

where  $a$  and  $b$  are positive constants.

$$\Rightarrow \frac{P}{T} = a + \frac{b}{T}$$

Now since,  $T_B > T_A$

$$\Rightarrow \frac{b}{T_B} < \frac{b}{T_A}$$

$$\Rightarrow \left(\frac{P}{T}\right)_B < \left(\frac{P}{T}\right)_A$$

$$\Rightarrow \left(\frac{T}{P}\right)_B > \left(\frac{T}{P}\right)_A$$

$$\Rightarrow V_B > V_A$$

Thus, as we move from  $A$  to  $B$ , volume of the gas is increasing.

### ILLUSTRATION 58

Equal masses of a gas are sealed in two vessels, one of volume  $V_0$  and other of volume  $2V_0$ . If the first vessel is at temperature 300 K and the other is at 600 K. Find the ratio of pressures in the two vessels.

### SOLUTION

As number of moles in the two vessels are equal, we have

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

It is given that  $V_1 = V_0$ ,  $V_2 = 2V_0$  and  $T_1 = 300 \text{ K}$ ,  $T_2 = 600 \text{ K}$

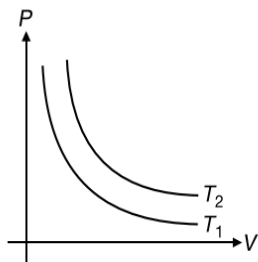
$$\Rightarrow \frac{P_1V_0}{300} = \frac{P_2(2V_0)}{600}$$

$$\Rightarrow P_1 = P_2$$

$$\Rightarrow \frac{P_1}{P_2} = 1$$

### ILLUSTRATION 59

$P$ - $V$  diagrams for two gases of same mass are drawn at two different temperatures  $T_1$  and  $T_2$ . Explain, which of  $T_1$  or  $T_2$  is greater.



### SOLUTION

According to the ideal gas equation, we have  $PV = nRT$

$$\Rightarrow T = \frac{PV}{nR}$$

Here mass of the gas is constant, which implies that number of moles are constant, i.e.,  $T \propto PV$ .

In the given diagram product of  $P$  and  $V$  for  $T_2$  is more than  $T_1$  at all points (keeping either  $P$  or  $V$  same for both graphs). Hence, we conclude that

$$T_2 > T_1$$

### ILLUSTRATION 60

Two glass bulbs of equal volume are connected by a narrow tube and are filled with a gas at  $0^\circ\text{C}$  and a pressure of 76 cm of mercury. One of the bulbs is then placed in melting ice and the other is placed in water bath maintained at  $62^\circ\text{C}$ . What is the new value of the pressure inside the bulbs? The volume of the connecting tube is negligible.

### SOLUTION

Assuming the volume of each bulb to be  $V$ , then the number of moles present in each bulb initially is

$$n = \frac{PV}{RT} = \frac{(76 \text{ cm})V}{R(273)}$$

Total number of moles in the two connected bulbs is

$$2n = \frac{2 \times (76 \text{ cm})V}{R(273)}$$

When one bulb is placed in melting ice ( $273 \text{ K}$ ) and other is placed in water at  $62^\circ\text{C} = 335 \text{ K}$ , still the total number of moles in the two bulbs will be equal to the initial moles i.e.,  $2n$ . If  $P_f$  be the final pressure in the two bulbs, then we have

$$\begin{aligned} \frac{P_i(2V)}{T_i} &= \frac{P_f V}{T_{f1}} + \frac{P_f V}{T_{f2}} \\ \Rightarrow \frac{(76 \text{ cm})(2V)}{273} &= \frac{P_f V}{273} + \frac{P_f V}{335} \end{aligned}$$

$$\Rightarrow P_f = \frac{2 \times 76}{273} \times \frac{273 \times 335}{608}$$

$$\Rightarrow P_f = 83.75 \text{ cm of Hg}$$

### ILLUSTRATION 61

Find the minimum attainable pressure of one mole of an ideal gas if during the expansion, its temperature and volume are related as  $T = T_0 + \alpha V^2$  where  $T_0$  and  $\alpha$  are positive constants.

### SOLUTION

Given that one mole of gas is used, so we have

$$\begin{aligned} PV &= RT \\ \Rightarrow P &= \frac{RT}{V} = \frac{R}{V}(T_0 + \alpha V^2) \quad \left\{ \because T = T_0 + \alpha V^2 \right\} \end{aligned}$$

The pressure  $P$  will be minimum, when

$$\begin{aligned} \frac{dP}{dV} &= 0 \\ \Rightarrow \frac{dP}{dV} &= -\frac{T_0}{V^2} + \alpha = 0 \\ \Rightarrow V &= \sqrt{\frac{T_0}{\alpha}} \end{aligned}$$

Thus, pressure of gas is minimum when its volume is

$$V = \sqrt{\frac{T_0}{\alpha}}$$

At this volume its temperature is given as

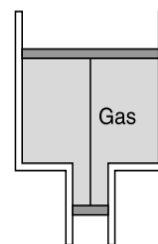
$$\begin{aligned} T &= T_0 + \alpha V^2 \\ \Rightarrow T &= T_0 + \alpha \left( \sqrt{\frac{T_0}{\alpha}} \right)^2 = 2T_0 \end{aligned}$$

Thus, minimum value of pressure is

$$P_{\min} = \frac{RT}{V} = \frac{R(2T_0)}{\sqrt{T_0/\alpha}} = 2R\sqrt{T_0\alpha}$$

### ILLUSTRATION 62

A smooth vertical tube having two different cross sections is open from both the ends but closed by two sliding frictionless pistons tied with an inextensible string as shown in Figure. One mole of an ideal gas is enclosed between the pistons. The difference in cross-sectional areas of the two pistons is given  $\Delta S$ . The masses of pistons are  $m_1$  and  $m_2$  for larger and smaller one respectively. Calculate the temperature through which the tube has to be raised so that the pistons get displaced through a distance  $l$ . Take atmospheric pressure to be  $P_0$ .



### SOLUTION

Let initial pressure of gas be  $P$ ,  $S_1$  and  $S_2$  be the cross-sectional areas of the larger and smaller piston respectively, then for equilibrium of the two pistons we have

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$$P_0 S_1 + m_1 g + T = P S_1 \quad \text{\{For larger piston\}}$$

where,  $T$  is the tension in string

$$P S_2 + m_2 g = T + P_0 S_2 \quad \text{\{For smaller piston\}}$$

Adding these equations and rearranging, we get

$$P_0 (S_1 - S_2) + m_1 g + m_2 g = P (S_1 - S_2)$$

$$\Rightarrow P_0 + \left( \frac{m_1 + m_2}{\Delta S} \right) g = P \quad \dots(1)$$

If gas temperature is increased from  $T_1$  to  $T_2$  the volume of gas increases from  $V$  to  $V + l\Delta S$ , where  $l$  is the displacement of pistons. Then from ideal gas equation, we have

$$\text{for initial state, } PV = RT_1 \quad \dots(2)$$

$$\text{for final state, } P(V + l\Delta S) = RT_2 \quad \dots(3)$$

From equation (1), we see that pressure of gas does not change as it does not depend on temperature

Subtracting equation (2) from (3), we get

$$Pl\Delta S = R(T_2 - T_1)$$

$$\Rightarrow T_2 - T_1 = \frac{Pl\Delta S}{R} = \left[ P_0 + \left( \frac{m_1 + m_2}{\Delta S} \right) g \right] \frac{l\Delta S}{R}$$

$$\Rightarrow T_2 - T_1 = \left[ P_0 \Delta S + (m_1 + m_2) g \right] \frac{l}{R}$$

### ILLUSTRATION 63

A vessel of volume 30 litres contains ideal gas at a temperature  $63^\circ\text{C}$ . After a portion of the gas has been let out, the pressure in the vessel decreased by 0.415 bar. If density of gas at STP is  $1.3 \text{ gltr}^{-1}$ , find the mass of gas released from the vessel.

### SOLUTION

At STP, from ideal gas equation, we have  $P = \frac{\rho RT}{M}$

$$\Rightarrow 10^5 = \frac{1.3 \times 8.314 \times 273}{M}$$

$$\Rightarrow M = 2.95 \times 10^{-3} \text{ kg}$$

Since  $P = \frac{nRT}{V}$ , so  $\Delta P = \frac{(\Delta n)RT}{V}$

$$\Rightarrow 0.415 \times 10^5 = \frac{(\Delta n)(8.314)(336)}{30 \times 10^{-3}}$$

$$\Rightarrow \Delta n = 0.4456 \text{ mole}$$

$$\Rightarrow m = M(\Delta n) = 13.14 \text{ g}$$

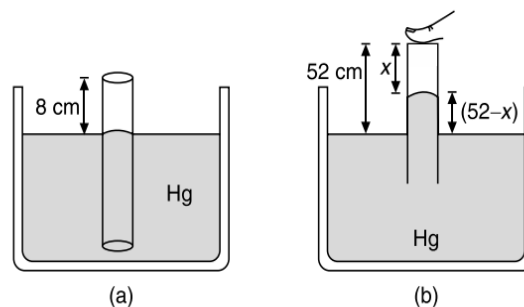
### ILLUSTRATION 64

An open glass tube is immersed in mercury in such a way that the length of 8 cm extends above the mercury level. Now the open end of the tube is closed by a finger and raised further by 44 cm. Calculate the length of air column

above mercury in the tube. Take atmospheric pressure to be 76 cm of mercury. Neglect capillary effect.

### SOLUTION

The initial state of the tube is shown in Figure (a). Now the tube is closed at the top and raised up further by 44 cm so that its length above mercury now becomes  $8 + 44 = 52$  cm as shown in Figure (b).



Initially tube is open to atmosphere, so pressure of air inside the tube is 76 cm of Hg. If  $A$  be the cross-sectional area of the tube then initial volume of air in tube is  $8A$ .

When upper end of tube is closed and raised up, then let the air column be of length  $x$ , so final volume of enclosed air is  $xA$ . Pressure at the mercury level in the container is equal to atmospheric pressure i.e., 76 cm of Hg, hence pressure at the mercury air interface inside the tube is

$$76 - (52 - x) = (24 + x) \text{ cm of Hg}$$

Since during the process, temperature of system remains constant, so we have  $P_1 V_1 = P_2 V_2$

$$\Rightarrow (76)(8A) = (24 + x)(xA)$$

$$\Rightarrow 608 = 24x + x^2$$

$$\Rightarrow x^2 + 24x - 608 = 0$$

$$\Rightarrow x = 15.4 \text{ cm or } x = -39.4 \text{ cm}$$

Hence the acceptable final value of air column is 15.4 cm.

### VANDER WAAL'S EQUATION

Perfect gas equation  $PV = RT$  is most closely obeyed at high temperature and low pressure where the intermolecular forces and the volume of gas molecules become negligible. It fails at high pressure and low temperature, because actual gas molecules possess finite size and intermolecular attraction, Vander Waal introduced these two corrections in perfect gas equation and obtained the following equation for real gases

$$\left( P + \frac{a}{V^2} \right) (V - b) = RT$$

where  $a$  and  $b$  are Vander Waal's constants, depending on the nature of the gas.

Vander Waal's equation for  $n$  moles may be expressed as

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT$$

### CRITICAL TEMPERATURE AND PRESSURE ( $T_C$ )

Critical Temperature is the temperature below which a gas can be liquefied by increasing the pressure alone. The critical temperature is different for different gases, being  $31^\circ\text{C}$  for  $\text{CO}_2$ ,  $-118^\circ\text{C}$  for oxygen,  $-240^\circ\text{C}$  for hydrogen and  $-268^\circ\text{C}$  for Helium.

The minimum pressure required to liquify a gas at critical temperature is called critical pressure ( $P_C$ ). Lesser is the critical temperature, lesser is the critical pressure being  $72.8\text{ atm}$  for  $\text{CO}_2$  and  $2.26\text{ atm}$  for helium. The values of critical temperature, critical pressure and critical volume are

$$T_C = \frac{8a}{27Rb}, P_C = \frac{a}{27b^2}, V_C = 3b$$

### BOYLE TEMPERATURE ( $T_B$ )

The Boyle temperature is that temperature

- (a) below which the product  $PV$  first decreases, becomes minimum and then increases with increase of pressure  $P$ .
- (b) above which the product  $PV$  increases with increase of pressure  $P$ .

So, Boyle temperature is  $T_B = \frac{a}{Rb}$

$$\text{Since } T_C = \frac{8a}{27Rb}, \text{ so } T_B = \frac{27}{8} T_C$$

### VARIATION OF PRESSURE WITH HEIGHT OR ELEVATION ABOVE THE SEA LEVEL

Consider the atmospheric pressure at sea level ( $y = 0$ ) to be  $P_0$ . Let us find the air pressure with elevation in the earth's atmosphere, assuming the temperature to be constant throughout.

We begin with the pressure relation that we have already discussed in the chapter of fluid mechanics i.e.,

$$\frac{dP}{dy} = -\rho g$$

Since  $PV = RT$ , where  $V = \frac{M}{\rho}$

So, the density  $\rho$  is given by,

$$\rho = \frac{PM}{RT} \quad \dots(1)$$

$$\Rightarrow \frac{dP}{dy} = -\left(\frac{PM}{RT}\right)g$$

$$\Rightarrow \int_{P_0}^P \frac{dP}{P} = -\int_0^y \left(\frac{Mg}{RT}\right) dy \quad \dots(2)$$

$$\Rightarrow \int_{P_0}^P \frac{dP}{P} = -\frac{Mg}{RT} \int_0^y dy \quad \{\because T = \text{constant}\}$$

$$\Rightarrow P = P_0 e^{-\left(\frac{Mg}{RT}\right)y}$$

So, the pressure of air decreases exponentially with height ( $y$ ). Also from equation (1), we get

$$\rho = \frac{PM}{RT} = \frac{MP_0}{RT} e^{-\left(\frac{Mg}{RT}\right)y} = \rho_0 e^{-\left(\frac{Mg}{RT}\right)y}$$

i.e., density also varies exponentially with height.

#### Problem Solving Technique(s)

If in some problems  $T \neq \text{constant}$ , but it is given as function of elevation  $y$ , then equation (2), becomes

$$\int_{P_0}^P \frac{dP}{P} = -\frac{Mg}{R} \int_0^y \frac{dy}{T}$$

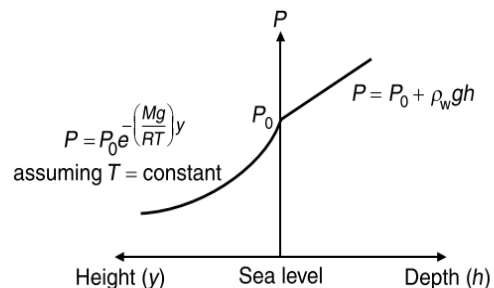
where  $T$  is the given function of  $y$ , so it is inside the integration as done in the following Illustrations.

### VARIATION OF PRESSURE WITH DEPTH BELOW THE SEA LEVEL

Below the sea level, the pressure at a depth  $h$  of water is

$$P = P_0 + \rho_w gh$$

The  $P$  versus  $y$  or  $P$  versus  $h$  graph is shown in figure, so



#### ILLUSTRATION 65

An ideal gas of molar mass  $M$  is contained in a tall vertical cylindrical vessel whose base area is  $A$  and height  $h$ . The temperature of the gas is  $T$ , its pressure on the bottom base is  $P_0$ . Assuming the temperature and the free-fall acceleration  $g$  to be independent of the height, find the mass of gas in the vessel.

**SOLUTION**

At a height  $y$ , let us consider an elemental layer of width  $dy$ . If mass in this layer is  $dm$ , then

$$dm = (A dy) \rho = A dy \left( \frac{PM}{RT} \right) = \frac{MA}{RT} \left( P_0 e^{-\frac{Mgy}{RT}} \right) dy$$

Total mass is given by

$$m = \int dm = \frac{P_0 MA}{RT} \int_0^h e^{-\frac{Mgy}{RT}} dy$$

$$\Rightarrow m = \frac{P_0 MA}{RT} \left( -\frac{RT}{Mg} \right) e^{-\frac{Mgy}{RT}} \Big|_0^h = \frac{P_0 A}{g} \left( 1 - e^{-\frac{Mgh}{RT}} \right)$$

**ILLUSTRATION 66**

The atmospheric pressure at sea level ( $h = 0$ ) is  $P_0$ . If the temperature of atmosphere varies with height  $h$  as  $T = T_0 - ah$  where  $a$  and  $T_0$  are positive constants of proper dimensions. Find the pressure function with height  $h$ . Take molecular mass  $M$  of the air and acceleration due to gravity  $g$  to be constant.

**SOLUTION**

Since,  $\frac{dP}{dh} = -\rho g = -\left( \frac{PM}{RT} \right) g$

$$\Rightarrow \frac{dP}{dh} = -\left\{ \frac{PM}{R(T_0 - ah)} \right\} g$$

$$\Rightarrow \frac{dP}{P} = -\left( \frac{Mg}{R} \right) \frac{dh}{T_0 - ah}$$

$$\Rightarrow \int_{P_0}^P \frac{dP}{P} = -\left( \frac{Mg}{R} \right) \int_0^h \frac{dh}{T_0 - ah}$$

$$\Rightarrow \log_e \left( \frac{P}{P_0} \right) = \frac{Mg}{Ra} \log_e \left( \frac{T_0 - ah}{T_0} \right)$$

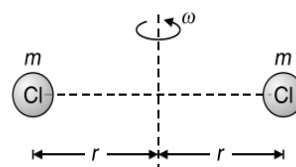
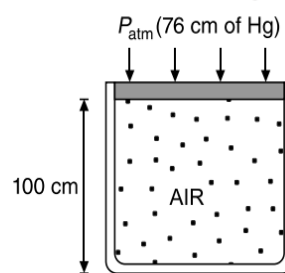
$$\Rightarrow P = P_0 \left( \frac{T_0 - ah}{T_0} \right)^{\frac{Mg}{Ra}}$$

 **Test Your Concepts-III**
**Based on Kinetic Theory of Gases and Ideal Gas Equation**

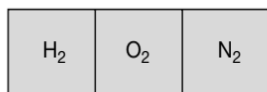
- Two equal bulbs are joined by a very narrow tube and the whole system is initially filled with a gas at NTP and sealed. What will be the pressure of the gas when one of the bulbs is immersed in boiling water and the other in ice?
- The rms velocity of hydrogen molecules at a certain temperature is  $300 \text{ ms}^{-1}$ . If the temperature is doubled and hydrogen gas dissociates into atomic hydrogen. Find the rms speed of the gas molecules now.
- A beam of particles each of mass  $m_0$  and speed  $v$  is directed along the x-axis. The beam strikes an area  $1 \text{ mm}^2$  square with  $10^{15}$  particles striking per second. Find the pressure on the area due to the beam if the particles stick to the area when they hit. Evaluate for an electron beam in a television tube where  $m_0 = 9.1 \times 10^{-31} \text{ kg}$  and  $v = 8 \times 10^7 \text{ ms}^{-1}$ .
- There are two containers, each of volume  $V$  containing ideal gases. The pressure and temperature of the gases in the two vessels are  $p_1, T_1$  and  $p_2, T_2$  respectively. If the vessels are now connected by a thin long tube of negligible volume, the final temperature of the two after mixing is  $T$ . Find the final pressure of the gas.
- A uniform tube closed at one end, contains a pallet of mercury  $10 \text{ cm}$  long. When the tube is kept vertically with the closed end of tube upward, the length of air column trapped by mercury and the closed end of the tube is  $20 \text{ cm}$ . If the tube is inverted so that its open end

(Solutions on page H.80)

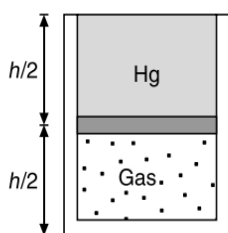
- becomes upward, then calculate the final length of the air column trapped by mercury and closed end of the tube. Take atmospheric pressure to be  $76 \text{ cm}$  of Hg.
- A vertical cylinder of height  $100 \text{ cm}$  contains air at a constant temperature and its top is closed by a frictionless piston at atmospheric pressure ( $76 \text{ cm}$  of Hg) as shown in Figure. If mercury is slowly poured on the piston, then due to weight of mercury, the air inside gets compressed. Calculate the maximum height of the mercury column which can be collected on the piston.
- In a crude model of a rotating diatomic molecule of chlorine ( $\text{Cl}_2$ ), the two  $\text{Cl}$  atoms are  $2 \times 10^{-10} \text{ m}$  apart and rotate about their centre of mass with angular speed  $\omega = 2 \times 10^{12} \text{ rads}^{-1}$ . What is the rotational kinetic energy of one molecules of  $\text{Cl}_2$ , which has a molar mass of  $70 \text{ gmol}^{-1}$ ?



8. Prove that the pressure of an ideal gas is numerically equal to two third of the mean translational kinetic energy per unit volume of the gas.
9. A rubber balloon is filled with air at  $2 \times 10^5 \text{ Nm}^{-2}$  pressure at a temperature  $20^\circ\text{C}$ . When its temperature is increased to  $40^\circ\text{C}$ , the volume of balloon is increased by 2%. Find the final air pressure inside the balloon at  $40^\circ\text{C}$ .
10. A barometer tube 90 cm contains some air above the mercury. The reading is 74.8 cm when the true atmospheric pressure is 76 cm and the temperature is  $30^\circ\text{C}$ . If the reading is observed to be 75.4 cm on a day when the temperature is  $10^\circ\text{C}$ , what is the true pressure?
11. An ideal gas is enclosed between the closed end of a uniform cross-sectional tube and a pellet of mercury of length  $h = 10 \text{ cm}$ . The length of the tube occupied by the gas when held vertical with the closed end upward is  $l_1 = 40 \text{ cm}$ . When it is turned through  $\theta = 60^\circ$  the length occupied by the gas is only  $l_2 = 38 \text{ cm}$ . Calculate the frictional force between the wall of the tube and mercury pellet. The radius of the tube is  $r = 2 \text{ mm}$ . Take  $g = 10 \text{ ms}^{-2}$  and density of mercury  $13600 \text{ kgm}^{-3}$ .
12. A perfectly conducting vessel of volume  $V = 0.4 \text{ m}^3$  contains an ideal gas at a temperature  $T = 273 \text{ K}$ . The density of the gas at STP is  $\rho = 1.2 \text{ kgm}^{-3}$ . A portion of the gas is let out and the pressure of the gas falls by  $\Delta P = 0.24 \text{ atm}$ . Find the mass of the gas which escapes from the vessel.
13. A vessel of volume  $V = 30 \text{ litre}$  is separated into 3 equal parts by stationary semipermeable partitions. The left, middle and right parts are filled with  $m_1 = 30 \text{ g}$  of hydrogen ( $\text{H}_2$ ),  $m_2 = 160 \text{ g}$  of oxygen ( $\text{O}_2$ ), and  $m_3 = 70 \text{ g}$  of nitrogen ( $\text{N}_2$ ) respectively. The left partition lets only hydrogen, while the right partition lets through hydrogen and nitrogen. What will be the pressure in each part of the vessel after the equilibrium has been set in if the vessel is kept at a constant temperature  $T = 300 \text{ K}$ ?



14. A vertical hollow cylinder of height 1.52 m is fitted with a movable piston of negligible mass and thickness. The lower part of cylinder contains an ideal gas and the upper part is filled with mercury as shown in Figure.



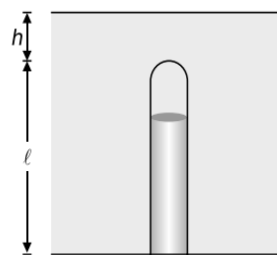
Initially the temperature of system is  $300 \text{ K}$  and the lengths of gas and mercury columns are equal. Calculate the temperature to which system is raised so that half of mercury overflows. Take atmospheric pressure to be  $76 \text{ cm}$  of Hg and ignore thermal expansion of mercury.

15. Assuming the temperature and the molar mass of air, as well as the free fall acceleration, to be independent of height, find the difference in heights at which the air densities at the temperature  $0^\circ\text{C}$  differ ( $M = 28 \text{ gmol}^{-1}$ )
  - (a)  $e$  times;      (b) by  $\eta = 2\%$
16. An ideal gas of molar mass  $M$  is contained in a very tall vertical cylindrical vessel in the uniform gravitational field in which the free-fall acceleration equals  $g$ . Assuming the gas temperature to be the same and equal to  $T$ , calculate the height at which the centre of gravity of the gas is located.
17. The mass of an oxygen molecule is  $5.28 \times 10^{-26} \text{ kg}$ . Its mean or average velocity at NTP is  $4.25 \times 10^2 \text{ ms}^{-1}$ . Calculate the average kinetic energy of an oxygen molecule at  $0^\circ\text{C}$ .
18. A vessel contains a mixture of nitrogen ( $m_1 = 7 \text{ g}$ ) and carbon dioxide ( $m_2 = 11 \text{ g}$ ) at a temperature  $T = 290 \text{ K}$  and pressure  $p_0 = 1 \text{ atm}$ . Find the density of this mixture, assuming the gases to be ideal.
19. A vessel of volume 5 litre contains 1.4 g of  $\text{N}_2$  at 1500 K and 0.4 g of  $\text{He}$ . If 30% of the nitrogen molecules are dissociated into atoms then find the gas pressure.
20. A glass tube, which is closed at one end, is completely submerged with open end downwards in a vessel of mercury, so that air column of length 10 cm is trapped inside the tube. To what height must the upper end of the tube be raised above the level of mercury in the vessel so that the level of mercury in the tube coincides with that in the vessel? Also calculate the mass of the air, if the temperature remains constant at  $27^\circ\text{C}$ . Given that area of cross-section of tube is  $1.0 \text{ cm}^2$ , atmospheric pressure is  $1.013 \times 10^5 \text{ Pa}$  i.e., 76 cm of Hg and molecular weight of air is 29.
21. Show that for any distribution of speeds for an ideal gas  $v_{rms} \geq v_{av}$ .
22. A horizontal cylinder closed from one end is rotated with a constant angular velocity  $\omega$  about a vertical axis passing through the open end of the cylinder. The outside air pressure is  $p_0$ , the temperature is  $T$  and the molar mass of air is  $M$ . Find the air pressure as a function of the distance  $r$  from the rotation axis. The molar mass is assumed to be independent of  $r$ .
23. A very tall cylindrical vessel with an ideal gas of molar mass  $M$  filled in it is placed in a uniform gravitational field. If the temperature of gas varies with height such

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that the gas density remains same throughout the container, then calculate the temperature gradient in the container  $dT/dh$ .

24. A glass tube with a length  $\ell = 50$  cm and a cross section  $a = 0.5$  cm<sup>2</sup> is soldered at one end and is submerged into water ( $\rho = 1000$  kgm<sup>-3</sup>) as shown in figure. What force  $F$  should be applied to hold the tube under the water if the distance from the surface of the water to the soldered end is  $h = 10$  cm and the atmospheric pressure  $P_0 = 1.01 \times 10^5$  Nm<sup>-2</sup>. The weight of the tube is  $W = 1.47 \times 10^{-1}$  N. Take  $g = 9.8$  ms<sup>-2</sup>. Assume that temperature of the gas inside the tube remains constant.



25. An ideal gas of molar mass  $M$  is located in the uniform gravitational field in which the free-fall acceleration is equal to  $g$ . Find the gas pressure as a function of height  $h$ , if  $p = p_0$  at  $h = 0$  and the temperature varies with height as
- (a)  $T = T_0(1 - ah)$   
 (b)  $T = T_0(1 + ah)$

## CONCEPT OF INTERNAL ENERGY (U)

In thermodynamics, the internal energy of a system is the energy contained within the system, excluding the kinetic energy of motion of the system as a whole and the potential energy of the system as a whole due to external force fields.

It keeps account of the gains and losses of energy of the system that are due to changes in its internal state. The internal energy of a system can be changed by transfers of matter or heat or by doing work. So, internal energy is possessed by the system due to molecular configuration and molecular motion. In other words, the internal energy possessed by a system may also be said as the mean energy of 'disordered' motion of molecules. The energy due to molecular configuration is called **internal potential energy** ( $U_p$ ) and the energy due molecular motion is called **internal kinetic energy** ( $U_k$ ). The total internal energy possessed by the system is then given by

$$U = U_p + U_k$$



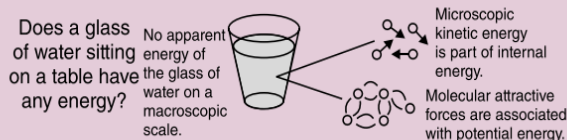
### Conceptual Note(s)

Internal energy is defined as the energy associated with the random, disordered motion of molecules. It is separated in scale from the macroscopic ordered energy associated with moving objects. It refers to the invisible microscopic energy on the atomic and molecular scale.

#### FOR EXAMPLE

At room temperature, a glass of water sitting on a table has no apparent energy, either potential or kinetic. However, on the microscopic scale the liquid is a mass of high speed molecules traveling at hundreds of meters per second. When we superimpose an ordered large scale motion on the water as a whole (may be by tossing/splashing the

water across the room) then this microscopic energy would not necessarily be changed.



## DEGREES OF FREEDOM $f$

Degrees of freedom ( $f$ ) is defined as the total number of possible independent significant ways in which a system can have energy.

The independent motions can be translational, rotational or vibrational or any combination of them. A particle moving in a straight line can have only one degree of freedom. i.e., translational degree of freedom. Similarly, a particle moving in a plane have two translational degrees of freedom and a particle free to move in space have three translational degrees of freedom.

Monatomic gas molecules can be considered as point masses moving in space and they will have only three translational degrees of freedom. However, this does not hold good with polyatomic gases.

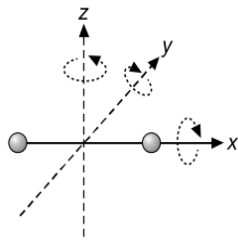
In a polyatomic molecule we have to take rotational motion of atoms about their centre of mass and the vibrational motion of the atoms relative to each other. Thus, we must write the molecular kinetic energy as

$$E = \left( \begin{array}{c} \text{translational} \\ \text{energy} \end{array} \right) + \left( \begin{array}{c} \text{rotational} \\ \text{energy} \end{array} \right) + \left( \begin{array}{c} \text{vibrational} \\ \text{energy} \end{array} \right)$$

As the temperature of a gas increases, the different components of the molecular energy also increase. But each component increases in a different manner. Vibrational energy depends on the strength of the bonding of the atoms in the

molecule. The stronger the ionic bonds, the more difficult it is to increase the vibrational energy. On the other hand, the rotational energy depends on the geometry (moment of inertia) of the molecule. But since vibrational energy comes into picture only at high temperatures, vibrational degrees of freedom are not taken into account unless stated in the question.

- (a) A **monatomic** gas has 3 degrees of freedom (all translational). Although a monatomic molecules can also rotate but due to its small moment of inertia rotational kinetic energy is insignificant.
- (b) A **diatomic** gas such as  $H_2$ ,  $O_2$  etc. are made up of two atoms joined rigidly to one another through a bond. This cannot only move bodily, but also rotate about any one of the three co-ordinate axes. However, its moment of inertia about the axis joining two atoms is negligible. Hence it can have 3 rotational motions out of which only two rotational motions are significant. Thus, a diatomic molecule has five degrees of freedom, three translational and two rotational.
- (c) A nonlinear polyatomic gas molecule such as  $H_2O$ ,  $NH_3$  etc. can rotate about any of three co-ordinate axes. Hence it has six degrees of freedom, three translational and three rotational. However, a linear polyatomic gas molecule like  $CO_2$  has five degrees of freedom three translational and two rotational (like a diatomic molecule).



For a molecule with  $n$  atoms there will be  $3n$  degrees of freedom. These can be attributed as follows:

Type	Translation	Rotation	Vibration
Linear	3	2	$3n - 5$
Non-linear	3	3	$3n - 6$

There is no general way in which one can find the degrees of freedom. However, if a system has  $A$  number of independent particles and  $R$  is the number of independent restrictions between them then the number of degrees of freedom ( $f$ ) is

$$f = 3A - R$$

- (a) For a monatomic gas molecule  
 $A = 1$  and  $R = 0$   
 Hence for monatomic gas molecule  $f = 3$ .
- (b) For di-atomic gas molecule  
 At normal temperature  $A = 2$  and  $R = 1$   
 (as the bond existing between its atoms must be rigid).  
 So, for diatomic gas molecule  $f = 5$ .

## POLYATOMIC GASES

A polyatomic gas molecule should also have three translational degrees of freedom and the number of rotational degrees of freedom depending on the geometry of the molecule. If the molecule is a linear one like  $CO_2$  then it has only two rotational degrees of freedom and if it is non-linear molecule like  $H_2O$ ,  $CH_4$  etc. then it has three rotational degrees of freedom because about all three co-ordinate axes, some significant moment of inertia exists.

Number of vibrational degrees of freedom in complex polyatomic molecules varies in different ways. There is no simple theoretical way to calculate the exact number of active vibrational degrees of freedom in a polyatomic molecule.

Hence, the total internal energy of  $n$  moles of a polyatomic gas at a temperature  $T$  for a linear molecule is given by

$$U = \frac{f}{2} nRT = \left( \frac{5+x}{2} \right) nRT \quad \dots(1)$$

where  $x$  are the number of vibrational degrees of freedom.

For a non-linear molecule which has  $x$  number of vibrational degrees of freedom, the total internal energy of  $n$  moles of a polyatomic gas at a temperature  $T$  is given by

$$U = \frac{f}{2} nRT = \left( \frac{6+x}{2} \right) nRT \quad \dots(2)$$

## THEOREM OF EQUIPARTITION OF ENERGY

According to this theorem, "the energy of a system in thermal equilibrium is equally divided amongst all degrees of freedom."

Since, energy associated with a monatomic gas molecule is

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$$

$$\text{and } \frac{1}{3} \langle v^2 \rangle = \langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$$

$$\Rightarrow \frac{1}{2} m \langle v_x^2 \rangle = \frac{1}{2} m \langle v_y^2 \rangle = \frac{1}{2} m \langle v_z^2 \rangle = \frac{1}{2} k_B T.$$

So, we observe, energy associated with each degree of freedom/molecule is  $\frac{1}{2} k_B T$

Energy associated with each degree of freedom per mole is  $\frac{1}{2} RT$ , where  $T$  is the absolute temperature.

So, for a gas with  $f$  degrees of freedom, energy associated with 1 mole of gas that is its Internal Energy is

$$U = f \left( \frac{1}{2} RT \right)$$

So, according to Theorem of Equipartition of Energy,

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“The energy of a system in thermal equilibrium is equally divided amongst all the degrees of freedom and the energy associated with each degree of freedom per mole is  $\frac{1}{2}RT$  or energy associated with each degree of freedom per molecule is  $\frac{1}{2}k_B T$ ”.

So, the internal energy of 1 mole of monatomic gas is

$$U = \frac{3}{2}RT$$

So, the internal energy of 1 mole of diatomic gas is

$$U = \frac{5}{2}RT$$

In general, we can say that the internal energy of  $n$  mole of a polyatomic gas with  $f$  degrees of freedom is

$$U = n \left( \frac{f}{2} RT \right)$$

### MOLAR SPECIFIC HEAT OF THE GAS (C)

Consider a container containing  $m$  gram of gas of molecular mass  $M$ . If  $n$  is the number of moles of gas in container,  $dQ$  is the heat supplied and rise in temperature is  $dT$  then

$$c = \frac{1}{m} \left( \frac{dQ}{dT} \right)$$

Further  $n = \frac{m}{M}$ , so  $m = nM$

$$\Rightarrow c = \frac{1}{Mn} \frac{dQ}{dT}$$

$$\Rightarrow C = Mc = \frac{1}{n} \left( \frac{dQ}{dT} \right)$$

$$\left( \begin{array}{c} \text{Molar} \\ \text{Specific} \\ \text{Heat (C) of} \\ \text{the Gas} \end{array} \right) = \left( \begin{array}{c} \text{Molar} \\ \text{Mass (M)} \\ \text{of} \\ \text{the Gas} \end{array} \right) \times \left( \begin{array}{c} \text{Gram} \\ \text{Specific} \\ \text{Heat (c) of} \\ \text{the Gas} \end{array} \right)$$



### Conceptual Note(s)

- (a) Please understand that every thermodynamic process has a molar specific heat  $C$  defined as

$$C = Mc = \frac{1}{n} \left( \frac{dQ}{dT} \right)$$

where  $M$  is the molar mass of the gas and  $c$  is its gram specific heat.

- (b) A gas can be supplied heat at constant volume by keeping it inside a rigid closed container OR at constant pressure by keeping it inside a container which has a frictionless piston attached to it.

- (c) When heat is supplied to the gas at constant pressure, then the molar specific heat is called as molar specific heat at constant pressure, and is denoted by  $C_p$ .

- (d) When heat is supplied to the gas at constant volume, then the molar specific heat is called as molar specific heat at constant volume and is denoted by  $C_v$ .

### SPECIFIC HEAT AT CONSTANT VOLUME ( $C_v$ )

When heat is supplied to gas at constant volume the entire heat supplied just increases the internal energy of gas and does nothing else (i.e., no external work is done by the gas), so we have

$$C_v = \frac{1}{n} \left( \frac{dQ}{dT} \right)_v$$

If  $dU$  is the change in internal energy of gas at constant volume then, heat supplied ( $dQ$ ) equals the increase in internal energy ( $dU$ ), so

$$C_v = \frac{1}{n} \left( \frac{dQ}{dT} \right)_v = \frac{1}{n} \left( \frac{dU}{dT} \right)$$

### $C_v$ FOR AN IDEAL GAS

Since the internal energy of  $n$  mole of a polyatomic gas with  $f$  degrees of freedom is  $U = n \left( \frac{f}{2} RT \right)$

By definition we have

$$C_v = \frac{1}{n} \frac{dU}{dT} = \frac{1}{n} \frac{d}{dT} \left( \frac{nfRT}{2} \right)$$

$$\Rightarrow C_v = \frac{fR}{2}$$

As we know that the internal energy of 1 mole of monatomic gas is  $U = \frac{3}{2}RT$

$$\Rightarrow C_v = \frac{dU}{dT} = \frac{3R}{2}$$

### Problem Solving Technique(s)

Please note that for any kind of thermodynamic process, if heat  $dQ$  is supplied, then the rise in internal energy is  $dU = nC_v dT$ . However,  $dQ = dU$ , only when the heat is supplied at constant volume.

### SPECIFIC HEAT AT CONSTANT PRESSURE ( $C_p$ )

$$C_p = \frac{1}{n} \left( \frac{dQ}{dT} \right)_p$$

When heat is supplied to the gas at constant pressure a part of heat increases the internal energy of gas and remaining does an external work (pushing the piston is at the expense of this heat supplied).

So, to increase the internal energy of the gas by the same amount (as in case of heat supplied at constant volume) more amount of heat has to be supplied thus making us to conclude that

$$C_p > C_v$$

### INTERNAL ENERGY (U) FOR AN IDEAL GAS: REVISITED

Since internal energy is possessed by the system due to molecular configuration and molecular motion. As discussed in the earlier text, the energy due to molecular configuration is called **internal potential energy** ( $U_p$ ) and the energy due molecular motion is called **internal kinetic energy** ( $U_k$ ). So, the total internal energy possessed by the system is given by

$$U = U_p + U_k$$

In case of an ideal gas, the intermolecular forces are zero, so the potential energy for an ideal gas is zero and hence the total kinetic energy of the ideal gas is its internal energy  $U$ . Please note that a gas may possess kinetic energy due to

- Translational Motion (due to translatory motion of molecules)
- Rotational Motion and
- Vibrational Motion

For an ideal gas, the internal energy  $U$  depends upon temperature  $T$  only and is directly proportional to it. So, we have  $U \propto T$

Further the change in the internal is given by

$$\Delta U = \Delta U_p + \Delta U_k$$

Since for an ideal gas, no interaction exists between the molecules (meaning  $U_p = \text{constant}$ ) and hence  $\Delta U_p = 0$ .

$$\Delta U = \Delta U_k = nC_v \Delta T = nC_v (T_f - T_i)$$

For infinitesimal temperature increments, we have

$$dU = nC_v dT$$

The Table below shows Molar Heat Capacities of Various Gases.

MOLAR HEAT CAPACITY (cal mole <sup>-1</sup> K <sup>-1</sup> )				
GAS	$C_v$	$C_p$	$C_p - C_v$	$\gamma = \frac{C_p}{C_v}$
<b>MONATOMIC GASES</b>				
He	2.98	4.97	1.99	1.67
Ar	2.98	4.97	1.99	1.67

(Continued)

MOLAR HEAT CAPACITY (cal mole <sup>-1</sup> K <sup>-1</sup> )				
<b>DIATOMIC GASES</b>				
H <sub>2</sub>	4.88	6.87	1.99	1.41
N <sub>2</sub>	4.96	6.95	1.99	1.40
O <sub>2</sub>	5.04	7.03	1.99	1.40
CO	5.02	7.01	1.99	1.40
Cl <sub>2</sub>	6.15	8.29	2.14	1.35
<b>POLYATOMIC GASES</b>				
CO <sub>2</sub>	6.80	8.83	2.03	1.30
SO <sub>2</sub>	7.50	9.65	2.15	1.29
H <sub>2</sub> O	6.46	8.46	2.00	1.30
CH <sub>4</sub>	6.48	8.49	2.01	1.31

### RELATION BETWEEN $C_p$ AND $C_v$

For 1 mole of gas which is supplied heat at constant volume

$$dQ = dU = (1)C_v dT \quad \dots(1)$$

For 1 mole of gas supplied heat at constant pressure

$$dQ = \left( \begin{array}{c} \text{Increase in} \\ \text{Internal Energy} \end{array} \right) + \left( \begin{array}{c} \text{External Work} \\ \text{Done by Piston} \end{array} \right)$$

$$\Rightarrow dQ = dU + PdV \quad \{ \because dW = Fdx = PAdx = PdV \}$$

$$\Rightarrow C_p dT = C_v dT + PdV \quad \dots(2)$$

Since, we know that, for a 1 mole of ideal gas  $PV = RT$

$$\Rightarrow d(PV) = d(RT)$$

$$\Rightarrow PdV + VdP = RdT \quad \dots(3)$$

But as external work is done by piston at constant pressure only i.e.,  $dP = 0$ . Using (3), we have

$$PdV = RdT \quad \dots(4)$$

Put (4) in (2), we have

$$C_p dT = C_v dT + RdT$$

After simplification, we have

$$C_p - C_v = R \quad \text{(Called Mayer's Relation)}$$

where,  $R = 8.314 \text{ Jmole}^{-1} \text{ K}^{-1}$

$$R = 1.99 \text{ calmole}^{-1} \text{ K}^{-1} \approx 2 \text{ calmole}^{-1} \text{ K}^{-1}$$

$C_p$  and  $C_v$  are approximately equal for solids and liquids.

### $C_p$ FOR AN IDEAL GAS

For an ideal gas, according to Mayer's relation, we have

$$C_p - C_v = R \text{ and } C_v = \frac{fR}{2}$$

where  $f$  is the degrees of freedom of the ideal gas.

$$\Rightarrow C_p = C_v + R = \left(\frac{f}{2} + 1\right)R$$

Since  $C_p - C_v = R$ , so we must note that  $C_p > C_v$ .

### ADIABATIC RATIO ( $\gamma$ ) FOR AN IDEAL GAS

The ratio of the molar specific heat at constant pressure to the molar specific heat at constant volume i.e.,  $\frac{C_p}{C_v}$  is denoted by  $\gamma$  and is called as the Adiabatic Ratio. Hence,

$$\gamma = \frac{C_p}{C_v} > 1$$

For an ideal gas having  $f$  degrees of freedom, we have

$$C_v = \frac{fR}{2} \text{ and } C_p = \left(\frac{f}{2} + 1\right)R$$

$$\Rightarrow \gamma = \frac{C_p}{C_v} = 1 + \frac{2}{f}$$

This relation helps us to find the value  $\gamma$  for monatomic, diatomic, polyatomic gases.

### RELATION OF $C_p$ AND $C_v$ WITH $\gamma$

Since  $C_p - C_v = R$

$$\Rightarrow \frac{C_p}{C_v} - \frac{C_v}{C_v} = \frac{R}{C_v}$$

$$\Rightarrow \gamma - 1 = \frac{R}{C_v}$$

$$\Rightarrow C_v = \frac{R}{\gamma - 1} \text{ and } C_p = \frac{\gamma R}{\gamma - 1}$$

#### Problem Solving Technique(s)

##### (a) Mixture of Gases (at same temperatures)

Following results are helpful for a mixture of gases.

(i) The internal energy is

$$U_{mix} = U_1 + U_2 + \dots$$

(ii) The molar mass is

$$M_{mix} = \frac{n_1 M_1 + n_2 M_2 + \dots}{n_1 + n_2 + \dots}$$

(iii) The pressure is

$$P_{mix} = P_1 + P_2 + \dots$$

(iv) The molar specific heat at constant volume is

$$(C_v)_{mix} = \frac{n_1 C_{v1} + n_2 C_{v2} + \dots}{n_1 + n_2 + \dots}$$

(v) The molar specific heat at constant pressure is

$$(C_p)_{mix} = (C_v)_{mix} + R = \frac{n_1 C_{p1} + n_2 C_{p2} + \dots}{n_1 + n_2 + \dots}$$

(vi) The ratio of specific heat at constant pressure to the specific heat at constant volume is

$$\gamma_{mix} = \left(\frac{C_p}{C_v}\right)_{mix}$$

#### (b) Expressions of $U$ , $C_p$ , $C_v$ and $\gamma$ for a monatomic, diatomic and polyatomic gas

For one mole of an ideal gas  $U = \frac{f}{2}RT$

$$\Rightarrow C_v = \frac{dU}{dT} = \frac{fR}{2}$$

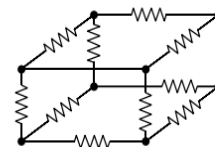
$$\Rightarrow C_p = C_v + R = \left(1 + \frac{f}{2}\right)R$$

$$\Rightarrow \gamma = \frac{C_p}{C_v} = 1 + \frac{2}{f}$$

Gas	$f$	$U$	$C_v = \frac{fR}{2}$	$C_p = \left(1 + \frac{f}{2}\right)R$	$\gamma = 1 + \frac{2}{f}$
Monatomic	3	$\frac{3}{2}RT$	$\frac{3R}{2}$	$\frac{5R}{2}$	1.67
Diatomic	5	$\frac{5}{2}RT$	$\frac{5R}{2}$	$\frac{7R}{2}$	1.4
Polyatomic	6	$3RT$	$3R$	$4R$	1.33

### DULONG AND PETIT'S LAW

In crystalline solids (assumed to be like monatomic gas), the atoms are arranged in a three dimensional array, called a lattice. Each atom in a lattice can vibrate along three mutually perpendicular directions, each of which has two degrees of freedom. One corresponding to vibrational K.E. and the other vibrational P.E.



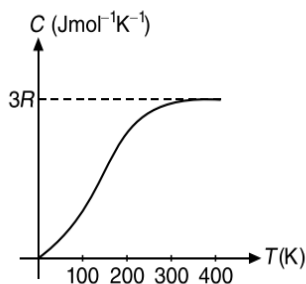
Thus, each atom has a total of six degrees of freedom. The volume of a solid does not change significantly with temperature and so there is little difference between  $C_v$  and  $C_p$  for a solid. The molar heat capacity is expected to be,

$$C = \frac{fR}{2} = \frac{6R}{2} = 3R \quad \text{(ideal crystalline solid)}$$

Its numerical value is

$$C \approx 25 \text{ Jmol}^{-1}\text{K}^{-1} \approx 6 \text{ calmol}^{-1}\text{K}^{-1}$$

This result was first found experimentally by **Dulong and Petit**. Figure shows that the **Dulong and Petit Law** is obeyed quite well at high ( $> 250 \text{ K}$ ) temperatures. At low temperatures, the heat capacities decrease.



## Conceptual Note(s)

### VARIATION OF SPECIFIC HEAT OF SOLIDS "DULONG AND PETIT'S LAW"

Dulong and Petit's Law states that average molar specific heat, of a chemically pure crystalline solid, is constant at room temperature and is, nearly, equal to  $3R$  ( $\approx 6 \text{ cal mole}^{-1} \text{ K}^{-1}$ ) where  $R$  is gas constant for one mole of the substance.

#### ILLUSTRATION 67

A vessel contains a mixture of 7 g of nitrogen and 11 g of carbon dioxide at temperature  $T = 290 \text{ K}$ . If pressure of the mixture  $P = 1 \text{ atm}$ , calculate its density ( $R = 8.31 \text{ Jmol}^{-1}\text{K}^{-1}$ ).

#### SOLUTION

As molecular weights of  $N_2$  and  $CO_2$  are 28 and 44, and

$$n = \left(\frac{m}{M}\right), n_N = \frac{7}{28} = \frac{1}{4} \text{ and } n_C = \frac{11}{44} = \frac{1}{4}$$

$$\Rightarrow n = n_N + n_C = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Now as according to gas equation  $PV = nRT$ ,

$$V = \frac{nRT}{P} = \left(\frac{1}{2}\right) \frac{8.31 \times 290}{1.01 \times 10^5} = 1.19 \times 10^{-2} \text{ m}^3$$

Mass of the gas is given by

$$m = 7 + 11 = 18 \text{ g} = 18 \times 10^{-3} \text{ kg}$$

$$\Rightarrow \rho = \left(\frac{m}{V}\right) = \frac{(18 \times 10^{-3})}{(1.19 \times 10^{-2})} = 1.5 \text{ kgm}^{-3}$$

#### ILLUSTRATION 68

Two perfect gases at absolute temperatures  $T_1$  and  $T_2$  are mixed. There is no loss of energy. Find the temperature of the mixture if the masses of the molecules are  $m_1$  and  $m_2$  and the number of molecules in the gases are  $n_1$  and  $n_2$ .

#### SOLUTION

According to the kinetic theory of gases, the kinetic energy of an ideal gas molecule at temperature  $T$  is given by  $KE = \left(\frac{3}{2}\right)kT$ . And as there is no force of attraction among the molecules of a perfect gas,  $PE$  of the molecule is zero. So, the energy of a molecule of perfect gas,

$$E = KE + PE = \frac{3}{2}kT + 0 = \frac{3}{2}kT$$

Now if  $T$  is the temperature of the mixture, by conservation of energy, i.e.,

$$n_1 E_1 + n_2 E_2 = (n_1 + n_2) E$$

$$\text{we have } n_1 \left(\frac{3}{2}kT_1\right) + n_2 \left(\frac{3}{2}kT_2\right) = (n_1 + n_2) \frac{3}{2}kT$$

$$\text{i.e., } T = \frac{n_1 T_1 + n_2 T_2}{n_1 + n_2}$$

#### ILLUSTRATION 69

Nitrogen gas of mass 15 g is enclosed in a vessel at 300 K. Calculate the amount of heat required to double the root mean square velocity of these molecules.

#### SOLUTION

The rms speed of gas molecules is given by

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

Since,  $v_{\text{rms}}$  is directly proportional to  $\sqrt{T}$ , so to double rms speed, the temperature has to be raised four times i.e.,  $T_f = 1200 \text{ K}$ .

Since the nitrogen gas ( $f = 5$ ) is enclosed in a vessel, so it cannot expand or it cannot lose any energy to external system. Hence, amount of heat required to double the rms speed of nitrogen is

$$Q = \Delta U = nC_V \Delta T = \left(\frac{15}{28}\right) \left(\frac{5}{2}(8.314)\right) (1200 - 300)$$

$$\Rightarrow Q \approx 10021 \text{ J}$$

#### ILLUSTRATION 70

One mole of a monatomic gas is mixed with 3 moles of a diatomic gas. What is the molar specific heat of the mixture at constant volume? ( $R = 8.31 \text{ Jmol}^{-1}\text{K}^{-1}$ )

**SOLUTION**

For monatomic gas  $(C_V)_1 = \frac{3R}{2}$ , while for diatomic gas

$(C_V)_2 = \frac{5R}{2}$ , so by conservation of energy, we get

$$(C_V)_{\text{mix}} = \frac{n_1(C_V)_1 + n_2(C_V)_2}{n_1 + n_2} = \frac{1\left(\frac{3R}{2}\right) + 3\left(\frac{5R}{2}\right)}{1 + 3}$$

$$\Rightarrow (C_V)_{\text{mix}} = \frac{9R}{4} = \frac{9}{4} \times 8.31 = 18.7 \text{ J mol}^{-1} \text{ K}^{-1}$$

**ILLUSTRATION 71**

In a thermally isolated container, 1 g of helium having rms velocity  $1000 \text{ ms}^{-1}$  and 4 g of oxygen having rms velocity  $1000 \text{ ms}^{-1}$  are mixed. Calculate rms velocities of helium and oxygen molecules after equilibrium is attained. Take  $R = 25/3 \text{ J mol}^{-1} \text{ K}^{-1}$ .

**SOLUTION**

The rms velocity of helium molecules is  $1000 \text{ ms}^{-1}$ . If temperature of this gas is  $T_1$ , then

$$v_{\text{rms}} = \sqrt{\frac{3RT_1}{M_1}}$$

$$\Rightarrow 1000 = \sqrt{\frac{3 \times 8.314 \times T_1}{2 \times 10^{-3}}}$$

$$\Rightarrow T_1 = \frac{(2 \times 10^{-3})(10^6)}{3(25/3)} = 80 \text{ K}$$

Also, the rms velocity of oxygen molecules is  $1000 \text{ ms}^{-1}$ . If temperature of this gas is  $T_2$ , then

$$v_{\text{rms}} = \sqrt{\frac{3RT_2}{M_2}}$$

$$\Rightarrow 1000 = \sqrt{\frac{3 \times 8.314 \times T_2}{32 \times 10^{-3}}}$$

$$\Rightarrow T_2 = \frac{(32 \times 10^{-3})(10^6)}{3(25/3)} = 1280 \text{ K}$$

It is given that 1 g of He and 4 g of  $O_2$  is mixed. If their number of moles are  $n_1$  and  $n_2$ , then

$$n_1 = \frac{1}{2} \text{ and } n_2 = \frac{4}{32} = \frac{1}{8}$$

When gases at different temperature are mixed at constant volume (or in a container), the total internal energy of system remains constant before and after mixing, so in this case, if final temperature of mixture is  $T_f$ , then

$$T_f = \frac{f_1 n_1 T_1 + f_2 n_2 T_2}{f_1 n_1 + f_2 n_2}$$

where,  $f_1 = 3$ ,  $f_2 = 5$ ,  $n_1 = 1/2$ ,  $n_2 = 1/8$ ,  $T_1 = 80 \text{ K}$  and  $T_2 = 1280 \text{ K}$

Thus, final temperature of mixture is given as

$$\Rightarrow T_f = \frac{(3)(1/2)(80) + (5)(1/8)(1280)}{3(1/2) + 5(1/8)}$$

$$\Rightarrow T_f = \frac{120 + 800}{2.125} \approx 433 \text{ K}$$

Thus, final rms velocity of He gas molecules is

$$v_{\text{rms}} = \sqrt{\frac{3RT_f}{M_1}} = \sqrt{\frac{3(25/3)(433)}{2 \times 10^{-3}}} = 2326.5 \text{ ms}^{-1}$$

Final rms velocity of  $O_2$  gas molecules is

$$v_{\text{rms}} = \sqrt{\frac{3RT_f}{M_2}} = \sqrt{\frac{3(25/3)(433)}{32 \times 10^{-3}}} = 581.6 \text{ ms}^{-1}$$

**Problem Solving Technique(s)**
 **$C_p$ ,  $C_V$  AND  $\gamma$  OF AN IDEAL GAS**

For an ideal gas, we have to keep the following points in our mind.

- Molar specific heat is the heat required to raise the temperature of 1 mole of a gas by  $1^\circ\text{C}$ .
- The **adiabatic exponent** ( $\gamma$ ) is defined as the ratio of the specific heat of the gas at constant pressure to the specific heat of the gas at constant volume i.e.,  $\gamma = \frac{C_p}{C_V}$ .  $\gamma$  is dimensionless and  $\gamma > 1$ .
- Molar specific heat at constant volume, in terms of the adiabatic exponent is

$$C_V = \frac{dU}{dT} = \frac{R}{\gamma - 1}$$

- Molar specific heat at constant pressure, in terms of the adiabatic exponent is

$$C_p = C_V + R = \frac{R\gamma}{\gamma - 1}$$

- In general, for  $n$  moles of an ideal gas, we have

$$C_V = \frac{1}{n} \left( \frac{dU}{dT} \right).$$

- Please note that the specific heat(s) is/are always expressed per mole of the sample of the gas. So, the answer for the specific heat must remain the same whether the calculations are done by taking one mole or  $n$  moles of the gas sample.

- $U$  is the internal energy of 1 mole of the gas.

For 1 mole of the gas, we have  $dU = (1)C_V dT = C_V dT$  and for  $n$  moles of the gas sample we have  $dU = nC_V dT$ .

**ILLUSTRATION 72**

1 g mole of oxygen at 27 °C and 1 atmospheric pressure is enclosed in a vessel.

- (a) Assuming the molecules to be moving with  $v_{\text{rms}}$ , find the number of collisions per second which the molecules make with one square meter area of the vessel wall.
- (b) The vessel is next thermally insulated and moved with a constant speed  $v_0$ . It is then suddenly stopped. The process results in a rise of temperature of the gas by 1 °C. Calculate the speed  $v_0$ . Given,  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$  and  $N_A = 6.02 \times 10^{23} \text{ mol}^{-1}$ .

**SOLUTION**

- (a) Mass of one oxygen molecule is

$$m = \frac{M}{N_A} = \frac{32}{6.02 \times 10^{23}} \text{ g}$$

$$\Rightarrow m = 5.316 \times 10^{-23} \text{ g} = 5.316 \times 10^{-26} \text{ kg}$$

$$\text{Since, } v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}}$$

$$\Rightarrow v_{\text{rms}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{5.316 \times 10^{-26}}} = 483.35 \text{ ms}^{-1}$$

Change in momentum per collision is

$$\Delta p = mv_{\text{rms}} - (-mv_{\text{rms}}) = 2mv_{\text{rms}}$$

$$\Rightarrow \Delta p = (2)(5.316 \times 10^{-26})(483.35)$$

$$\Rightarrow \Delta p = 5.14 \times 10^{-23} \text{ kgms}^{-1}$$

Now, suppose  $n$  particles strike per second per unit area of vessel wall, then force applied on the vessel wall is

$$F = n\Delta p = (n)(5.14 \times 10^{-23}) \text{ N} \quad \left\{ \because F_{\text{ext}} = \frac{dp}{dt} \right\}$$

Since,  $P = \frac{F}{A}$ , for unit area  $F = P$

$$\Rightarrow (n)(5.14 \times 10^{-23}) = 1.01 \times 10^5$$

$$\Rightarrow n = 1.965 \times 10^{27} \text{ per second}$$

- (b) When the vessel is stopped the ordered motion of the vessel converts into disordered motion and temperature of the gas is increased. So, we have

$$N_A \left( \frac{1}{2} m v_0^2 \right) = \Delta U \quad \dots(1)$$

Since,  $U = \frac{5}{2} RT$  {for  $O_2$ }

$$\Rightarrow \Delta U = \frac{5}{2} R \Delta T$$

where  $m$  is the mass of one gas molecule given by

$$m = 5.316 \times 10^{-26} \text{ kg}$$

Substituting these values in equation (1), we get

$$v_0 = \sqrt{\frac{5R\Delta T}{N_A m}} = \sqrt{\frac{5 \times 8.31 \times 1}{32 \times 10^{-3}}} \left\{ \because N_A m = 32 \times 10^{-3} \text{ kg} \right\}$$

$$\Rightarrow v_0 = 36 \text{ ms}^{-1}$$

**ILLUSTRATION 73**

A cubical vessel of side 1 m contains one mole of nitrogen at a temperature of 300 K. If the molecules are assumed to move with the rms speed, then find the number of collisions per second which the molecules may make with the wall of the vessel.

Further, if the vessel is now thermally insulated, moved with a constant speed  $v$  and then suddenly stopped, then this results in rise of temperature by 2 °C, calculate  $v$ .

**SOLUTION**

Given that volume of container is  $V = 1 \text{ m}^3$ , temperature of gas is  $T = 300 \text{ K}$  and amount of gas is  $n = 1 \text{ mole}$

Let  $n_0$  be the number of molecules per unit volume, then we have  $n = 1 \text{ mole}$

$$\Rightarrow n_0 = 6.023 \times 10^{23} \text{ m}^{-3}$$

The rms velocity of molecules at 300 K temperature is

$$v_{\text{rms}} = \sqrt{\frac{3RT}{M}}$$

$$\Rightarrow v_{\text{rms}} = \sqrt{\frac{3 \times 8.314 \times 300}{28 \times 10^{-3}}} \approx 517 \text{ ms}^{-1}$$

Also, we know from our knowledge of kinetic theory of gases, that the number of collisions taking place per second per square meter of wall is given by

$$N = \frac{n_0 v}{6} = \frac{(6.023 \times 10^{23})(517)}{6} \approx 5.2 \times 10^{25} \text{ s}^{-1} \text{m}^{-2}$$

Now, when the container is moving at speed  $v$ , the kinetic energy associated with one mole of nitrogen gas molecules is

$$K = \frac{1}{2} (N_A m_{N_2}) v^2 = \frac{1}{2} M v^2 = \frac{1}{2} (28 \times 10^{-3}) v^2$$

When the container is suddenly stopped, this kinetic energy is transformed into thermal energy and increases the internal energy of gas (because container is insulated). If temperature increment of gas is  $\Delta T$ , then rise in its internal energy is

$$\Delta U = \left( \frac{nf}{2} \right) R \Delta T = \frac{5}{2} (n R \Delta T)$$

$$\Rightarrow \Delta U = \left( \frac{5}{2} \right) (1)(8.314)(2) = 41.57 \text{ J}$$

Applying law of conservation of energy, we get

$$\frac{1}{2}(28 \times 10^{-3})v^2 = 41.57$$

$$\Rightarrow v = \sqrt{\frac{41.57 \times 2}{28 \times 10^{-3}}} \approx 54.5 \text{ ms}^{-1}$$

### ILLUSTRATION 74

An adiabatic vessel contains 3 mole of a diatomic gas. Moment of inertia of each molecule is  $2.76 \times 10^{-46} \text{ kgm}^2$  and root mean square angular velocity is  $5 \times 10^{12} \text{ rads}^{-1}$ . Another adiabatic vessel contains 5 mole of a monatomic gas at a temperature 470 K. Assume the gases to be ideal, calculate root mean square angular velocity of diatomic molecules when the two vessels are connected by a thin tube of negligible volume. Boltzmann constant  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$ .

### SOLUTION

According to law of equipartition of energy, energy associated with each degree of freedom per molecule is  $\frac{1}{2}k_B T$ . Since a diatomic gas molecule has two rotational degrees of freedom, so its rotational energy must be  $2\left(\frac{1}{2}k_B T\right) = k_B T$ .

If initial temperature of diatomic gas is  $T_1$ , then

$$\frac{1}{2}I\omega_{\text{rms}}^2 = k_B T_1$$

$$\Rightarrow T_1 = \frac{I\omega_{\text{rms}}^2}{2k_B} = \frac{(2.76 \times 10^{-46})(5 \times 10^{12})^2}{2(1.38 \times 10^{-23})}$$

$$\Rightarrow T_1 = 250 \text{ K}$$

When the two vessels containing diatomic and monatomic gases are connected, these gases exchange this thermal energy but no energy is lost to surrounding because the vessels are adiabatic. This mixing of gases takes place at constant volume, so total internal energy of the system remains constant. If  $T_f$  be the final temperature of the system, then

$$T_f = \frac{n_1 f_1 T_1 + n_2 f_2 T_2}{n_1 f_1 + n_2 f_2} \quad \dots(1)$$

For a diatomic gas,  $f_1 = 5$ ,  $n = 3$  and  $T_1 = 250 \text{ K}$

For monatomic gas,  $f_2 = 3$ ,  $n_2 = 5$  and  $T_2 = 470 \text{ K}$

Thus, from equation (1), we get

$$\Rightarrow T_f = \frac{(5)(3)(250) + (3)(5)(470)}{(5)(3) + (3)(5)} = 360 \text{ K}$$

Thus, final mixture of the two gases is at temperature 360 K. If final rms angular velocity of diatomic gas molecules is  $(\omega_{\text{rms}})_f = \omega_f$ , then from law of equipartition of energy, we get

$$\frac{1}{2}I\omega_f^2 = k_B T$$

$$\Rightarrow \omega_f = \sqrt{\frac{2k_B T}{I}} = \sqrt{\frac{2 \times 1.38 \times 10^{-23} \times 360}{2.76 \times 10^{-46}}}$$

$$\Rightarrow \omega_f = 6 \times 10^{12} \text{ rads}^{-1}$$



### Test Your Concepts-IV

#### Based on Internal Energy, Degrees of Freedom and Molar Specific Heats for Ideal Gases

(Solutions on page H.84)

- Two perfect monatomic gases at absolute temperatures  $T_1$  and  $T_2$  are mixed. There is no loss of energy. Find the temperature of the mixture if the number of moles in the gases are  $n_1$  and  $n_2$ .
- The ratio of translational and rotational kinetic energy of a gas at 100 K is 3:2. Calculate the internal energy of one mole gas at this temperature. Take  $R = 8.3 \text{ Jmol}^{-1}\text{K}^{-1}$ .
- A tank used for filling helium balloons has a volume of  $0.3 \text{ m}^3$  and contains 2 mol of helium gas at  $20^\circ\text{C}$ . Assuming that the helium behaves like an ideal gas, calculate the
  - total translational kinetic energy of the molecules of the gas.
  - average kinetic energy per molecule.
- If Avogadro's number is  $N_A = 6.02 \times 10^{23}$  and Boltzmann's constant is  $k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$ , then calculate the
  - average kinetic energy of translation of an oxygen molecule at  $27^\circ\text{C}$
  - total kinetic energy of an oxygen molecule at  $27^\circ\text{C}$  and
  - total kinetic energy in joule of one mole of oxygen at  $27^\circ\text{C}$ .
- Calculate the change in internal energy of 3 mole of helium gas when its temperature is increased by 2 K.
- A cylindrical container of volume  $V$  is divided in two equal parts by a fixed diathermic partition. Identical gases are filled in the two parts at initial pressure and temperature  $p_1, T_1$  and  $p_2, T_2 (> T_1)$  respectively. After a long time when the two gases are in thermal

equilibrium, calculate the final temperature and pressure of the gases in the two parts. Also calculate the amount of heat supplied by gas in one part to that in other part in if gas taken is monatomic.

7. Calculate  $\gamma$  for a gaseous mixture consisting of 2 mole of  $O_2$  and 3 mole of  $CO_2$ . The gases are assumed to be ideal.
8. The temperature of a gas consisting of rigid diatomic molecules is 300 K. Calculate the approximate angular root mean square velocity of a rotating molecule if its moment of inertia is equal to  $2.1 \times 10^{-39} \text{ kgm}^2$ .
9. An insulated gas jar of mass  $M$  which is equal to the molecular mass of the enclosed gas moves with certain kinetic energy. When the jar is stopped, the increment of temperature of the gas is  $\Delta T$ . Find the kinetic energy of the system before stopping. Take the adiabatic exponent of the gas to be  $\gamma$ .

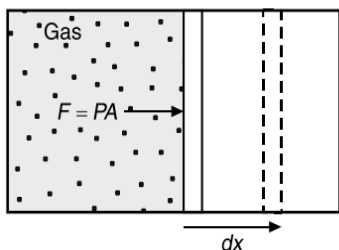
# FIRST LAW OF THERMODYNAMICS AND THERMODYNAMIC PROCESSES

## THERMODYNAMICS

Thermodynamics is the study of the relations between heat, work, temperature, and energy. Thermodynamics is concerned with the work done by a system and the heat it exchanges with its surroundings. Alternatively, it is the study of changes that occur in some part of the universe (we designate as the system) and then everything else (outside the system) is the surrounding. A real or imagined boundary may separate the system from its surroundings. A collection of properties such as pressure, volume, temperature and some other properties to be discussed later characterize the thermodynamical state of a system. The laws of thermodynamics describe how the energy in a system changes and whether the system can perform any useful work on its surroundings.

## CONCEPT OF WORK

Consider a frictionless piston of area  $A$  attached to a container. On heating, the piston is displaced through  $dx$ . If  $dW$  is the work done then



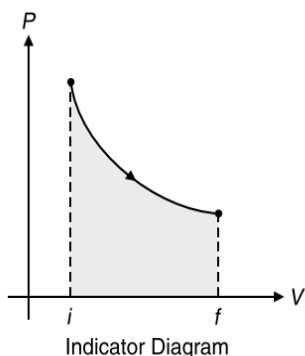
$$dW = Fdx = (PA)dx = P(Adx) \quad \{\because F = PA\}$$

$$\Rightarrow dW = PdV \quad \{\because Adx = dV\}$$

To get total work done from initial state  $i$  to final state  $f$  (along a specified path)

$$W = \int_i^f PdV$$

Also work done is equal to the area under a curve in an **INDICATOR DIAGRAM** called P-V **diagram**.



Now, there are two methods, using which we can calculate the work done by the gas.

**Method 1: To be used when P-V relation (dependence of pressure on volume) is known.**

Suppose  $P = f(V)$  (i.e., pressure is a function of volume) then

$$dW = PdV = f(V)dV$$

$$\Rightarrow W = \int dW = \int_{V_i}^{V_f} f(V)dV$$

### Problem Solving Technique(s)

In the above discussion we have seen that

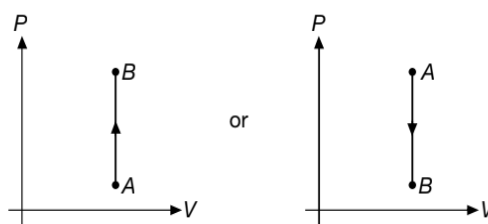
$$W = \int_{V_i}^{V_f} PdV$$

From this equation it seems as if work done can be calculated only when P-V equation is known and the limits  $V_i$  and  $V_f$  are known to us. **But it is not so.** We can calculate work done even if we know the limits of temperature.

**Method 2: To be used when P-V graph is given.**

In this method, the work done by a gas is equal to the area under P-V graph (also called as Indicator Diagram). Following different cases are possible.

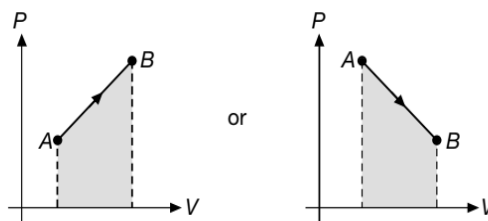
**CASE-1: When volume is constant**



$V = \text{constant}$

$$\Rightarrow W_{AB} = 0$$

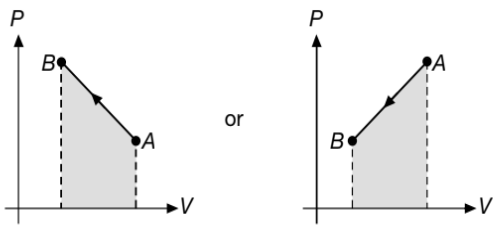
**CASE-2: When volume is increasing (i.e., expansion)**



$V$  is increasing

$$\Rightarrow W_{AB} > 0 \text{ and } W_{AB} = \text{Shaded area}$$

**CASE-3: When volume is decreasing (i.e., compression)**

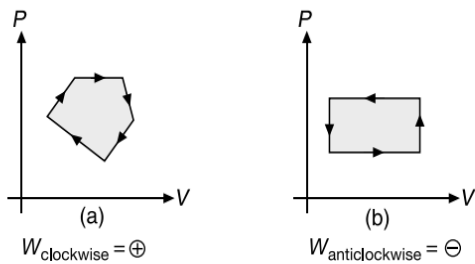


V is decreasing

$\Rightarrow W_{AB} < 0$  and  $W_{AB} = -\text{Shaded area}$

**CASE-4: Cyclic process**

It consists of a series of thermodynamic processes transferring heat and work, while varying pressure, volume, temperature and other state variables, eventually returning the system to its initial state.

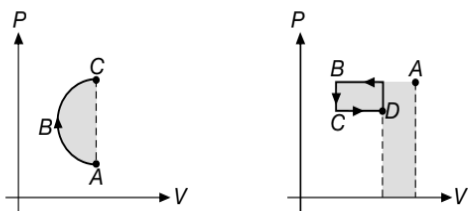


Cyclic process

$W_{\text{clockwise cycle}} = +\text{Shaded area}$  [in figure (a)]

$W_{\text{anticlockwise cycle}} = -\text{Shaded area}$  [in figure (b)]

**CASE-5: Incomplete cycle**

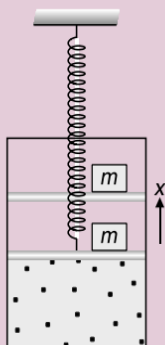


$W_{ABC} = +\text{Shaded area}$

$W_{ABCD} = -\text{Shaded area}$

**Problem Solving Technique(s)**

(a) Sometimes the piston (which is assumed to be light) is attached to a spring of force constant  $k$  and a mass  $m$  is placed over the piston. The area of the piston is  $A$ . The gas expands. To make the calculation easy we assume that initially the spring was in its natural length. We are required to find the work done by the gas. As the piston is assumed to be light, net force on it at every instant is zero.



$PA = kx + mg + P_0A$

$\Rightarrow P = P_0 + \frac{kx}{A} + \frac{mg}{A}$

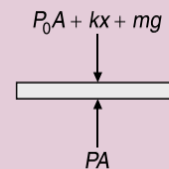
Since,  $dW = PdV = P(Adx) = (AP_0 + kx + mg)dx$

$\Rightarrow W = \int_0^x PdV$

$\Rightarrow W = \int_0^x (AP_0 + kx + mg)dx$

$\Rightarrow W = P_0Ax + \frac{1}{2}kx^2 + mgx$  [as  $Ax = \Delta V$ ]

$\Rightarrow W = P_0\Delta V + \frac{1}{2}kx^2 + mgx$



The result can be stated in a different manner as under. The gas does work

- (i)  $W_1$  against the atmospheric pressure  $P_0$  (which is constant)
- (ii)  $W_2$  against the spring force  $kx$  (which varies linearly with  $x$ ) and
- (iii)  $W_3$  against the gravity force  $mg$  (which is again constant).

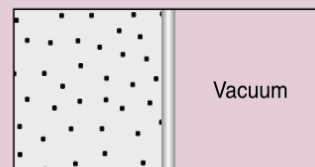
So,  $W_1 = \text{Work done against } P_0 \text{ is } P_0\Delta V$

$W_2 = \text{Work done against } kx \text{ is } \frac{1}{2}kx^2$  and  $W_3 = \text{Work done against } mg \text{ is } mgx$

Hence,

$W_{\text{Total}} = W_1 + W_2 + W_3 = P_0\Delta V + \frac{1}{2}kx^2 + mgx$

- (b) From point number (1). We may conclude that work done by a gas is zero if the other side of the piston is vacuum.



**ILLUSTRATION 75**

The temperature of  $n$  moles of an ideal gas is increased from  $T_0$  to  $2T_0$  through a process  $P = \frac{\alpha}{T}$ . Calculate the work done by the gas.

**SOLUTION**

From ideal gas equation we have,  $PV = nRT$  ... (1)

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Given that  $P = \frac{\alpha}{T}$  ... (2)

Dividing (1) by (2), we get

$$V = \frac{nRT^2}{\alpha}$$

$$\Rightarrow dV = \frac{2nRT}{\alpha} dT$$

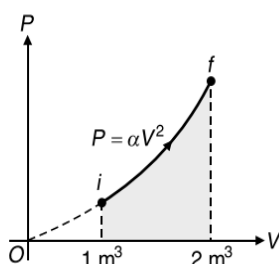
Since,  $W = \int_{V_i}^{V_f} P dV$

$$\Rightarrow W = \int_{T_0}^{2T_0} \left(\frac{\alpha}{T}\right) \left(\frac{2nRT}{\alpha}\right) dT = 2nRT_0$$

So, work done is calculated without substituting the limits of volume.

### ILLUSTRATION 76

A sample of ideal gas is expanded to twice its original volume of  $1 \text{ m}^3$  in a quasi-static process for which  $P = \alpha V^2$ , with  $\alpha = 6 \text{ atm m}^{-6}$  as shown in figure. How much work is done by the expanding gas. Given  $1 \text{ atm} = 10^5 \text{ Nm}^{-2}$ .



### SOLUTION

Since  $dW = PdV$

$$\Rightarrow W = \int_1^2 \alpha V^2 dV = 6 \times 10^5 \left( \frac{V^3}{3} \Big|_1^2 \right)$$

$$\Rightarrow W = \left( \frac{6 \times 10^5}{3} \right) (8 - 1) = 14 \times 10^5 \text{ J}$$

## HEAT VERSUS WORK

**Heat** is the energy transferred between two bodies as a consequence of a temperature difference between them. In contrast, **work** is a mode of energy transfer in which the point of application of a force moves through a displacement and is not associated with a temperature difference. Both heat and work are “energy in transit” from one body to another during the operation of some process, once the process stops, heat and work have no meaning.

## MECHANICAL EQUIVALENT OF HEAT

It has been concluded from Joule’s experiment that the mechanical work required to produce a given change in temperature is in fixed proportion to the heat required for

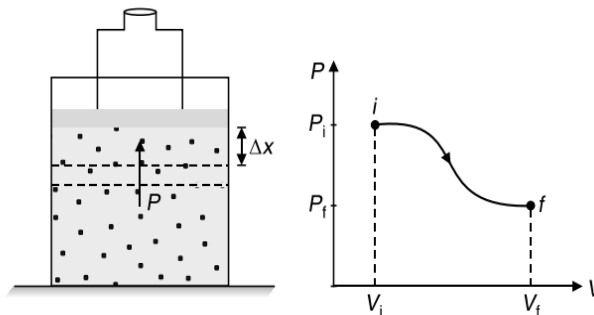
the same change in temperature. This constant factor is called the **mechanical equivalent** of heat.

$$1 \text{ calorie} = 4.186 \text{ J}$$

Thus, a change in the state of a system produced by the addition of 1 calorie of heat may also be produced by the performance of 4.186 J of work on the system.

## THERMODYNAMIC WORK

Figure shows a gas confined to a cylinder by a weight on a movable piston. Our system is the gas, whereas the cylinder and the piston form the environment. If the piston is allowed to move upward, the gas expands and does work on it. To calculate the work done by the gas, we assume that the process is **quasistatic**. In a **quasistatic process** the thermodynamics variables ( $P$ ,  $V$ ,  $T$ ,  $n$ , etc.) of the system and its surroundings change infinitely slowly. Thus, the system is always arbitrarily close to an equilibrium state, in which it has a well-defined volume, and the whole system is characterized by single value of the macroscopic variables. To ensure that the piston moves very slowly, there must be some force, for example, provided by a weight, directed opposite to that due to the pressure. If the piston were to move suddenly, the rapid expansion would involve turbulence and the pressure would not be uniquely defined.



When the piston rises by  $dx$ , the work  $dW$  done by the gas is  $dW = Fdx = (PA)dx$  where  $A$  is the cross-sectional area of the piston. Since the change in volume of the gas is  $dV = Adx$ , the work may be expressed as

$$\text{(Quasistatic)} \quad dW = PdV$$

As a quasistatic process evolves,  $P$  and  $V$  are always uniquely defined. This allows us to depict the process on a  $PV$  diagram such as figure. When the system is taken quasistatically from the equilibrium state  $i$  to another equilibrium state  $f$ , the total work done by the system is

$$W = \int_{V_i}^{V_f} PdV$$

In figure the work is represented by the area under the curve. If  $V_f > V_i$ , the work done by the gas is **positive**. If the

volume decreases, the work done by the gas is **negative**. This may be interpreted as positive work done on the gas by the environment. The work done depends not only on the initial and final states but also on the details of the process, that is, the thermodynamic path between the states. Therefore, we need to know how the pressure varies with the volume.

## FIRST LAW OF THERMODYNAMICS

Consider a system that consists of a gas enclosed by a piston in a cylinder. Suppose the system is taken quasi-statically from an initial state  $P_i, V_i, T_i$  to a final state  $P_f, V_f, T_f$ . At each step the work done and heat exchanged are measured. We know that both the total work done  $W$  and the total heat transfer  $Q$  to or from the system depend on the thermodynamic path. However, the difference  $Q - W$ , is the same for all paths between the given initial and final equilibrium states, and it is equal to the change in internal energy  $\Delta U$  of the system.

$$\Delta U = Q - W$$

$$dQ = dU + dW$$

In the above statement,  $Q$  is **positive** when heat enters the system and  $W$  is **positive** when **work** is done by the system on its surroundings.

The equation, is the mathematical statement of the **First Law of Thermodynamics**. It states that the internal energy of a system changes when work is done on the system (or by it), and when it exchanges heat with the environment.

**Note** that the First Law is valid for all processes, quasistatic or not. However, if friction is present, or the process is not quasistatic, the internal energy  $U$  is uniquely defined only at the initial and final equilibrium states.

The First Law establishes the existence of internal energy  $U$  as a state function – one that depends only on the thermodynamic state of the system.

In the macroscopic approach of thermodynamics, there is no need to specify the physical nature of the internal energy. The experimental results are sufficient to prove that such a function exists. The internal energy is the sum of all possible kinds of energies stored in the system – mechanical, electrical, magnetic, chemical, nuclear, and so on. It does not include the kinetic and potential energies associated with the centre of mass of the system.

## MISCONCEPTION BETWEEN HEAT AND INTERNAL ENERGY

Confusion between heat and internal energy arises from erroneous statements that refer to the “heat content” of a body. Even correct terms like “the heat capacity of a body”

can mislead one to believe that heat is somehow stored in a system. This is not correct.

The physical quantity possessed by a system is **internal energy**, which is the sum of all the kind of energy in the system. As the First Law indicates,  $U$  may be changed either by heat exchange or by work. The internal energy is a state function that depends on the equilibrium state of a system, whereas  $Q$  and  $W$  depend on the thermodynamic path between two equilibrium states. That is,  $Q$  and  $W$  are associated with processes. The heat absorbed by a system will increase its internal energy, only some of which is average translatory kinetic energy. It is therefore incorrect to say that heat is the energy of the random motion.

## FIRST LAW OF THERMODYNAMICS: REVISITED

When the Law of Conservation of Energy was first introduced it was stated that the mechanical energy (kinetic + potential) of a system is conserved in the absence of non-conservative forces such as friction. That is, the changes in the internal energy of the system were not included in this mechanical model.

The First Law of Thermodynamics is a **generalisation of the Law of Conservation of Energy** that includes possible changes in internal energy. It is a universally valid law that can be applied to all kinds of processes. Furthermore it provides us with a connection between the microscopic and macroscopic worlds.

When a system undergoes an infinitesimal change in state, where a small amount of heat,  $dQ$ , is transferred and a small amount of work,  $dW$ , is done, the internal energy also changes by a small amount  $dU$ . Thus, for infinitesimal processes we can write

$$dQ = dU + dW$$

$dQ$  and  $dW$  are not true differential quantities i.e.,  $dQ$  and  $dW$  are path functions and can never be expressed as the difference of final value and initial value whereas,  $dU$  is a true differential i.e. a state function and can always be expressed as the difference of final value and initial value.

### Problem Solving Technique(s)

#### RIGHT SIGN CONVENTIONS SHOULD BE FOLLOWED WHILE USING THE FIRST LAW

- (a) Heat gained by the system is positive
- (b) Heat lost by the system is negative
- (c) Gain in internal energy is positive
- (d) Loss in internal energy is negative
- (e) Work done by the system is positive (i.e., during expansion)
- (f) Work done on the system is negative (i.e., during compression)

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### ILLUSTRATION 77

When a system goes from state  $A$  to state  $B$ , it is supplied with  $400\text{ J}$  of heat and it does  $100\text{ J}$  of work.

- For this transition, what is the system's change in internal energy?
- If the system moves from  $B$  to  $A$ , what is the change in internal energy?
- If in moving from  $A$  to  $B$  along a different path in which  $W'_{AB} = 400\text{ J}$  of work is done on the system, how much heat does it absorb?

### SOLUTION

- (a) According to *FLTD*, we have

$$\Delta U_{AB} = Q_{AB} - W_{AB} = (400 - 100)\text{ J}$$

$$\Rightarrow \Delta U_{AB} = 300\text{ J}$$

- (b) Consider a closed path that passes through the state  $A$  and  $B$ , then internal energy is a state function and so  $\Delta U$  is zero for a closed path.

$$\Rightarrow \Delta U = \Delta U_{AB} + \Delta U_{BA} = 0$$

$$\Rightarrow \Delta U_{BA} = -\Delta U_{AB}$$

$$\Rightarrow \Delta U_{BA} = -300\text{ J}$$

- (c) The change in internal energy is the same for any path, so

$$\Delta U_{AB} = \Delta U'_{AB} = Q'_{AB} - W'_{AB}$$

$$\Rightarrow 300\text{ J} = Q'_{AB} - (-400\text{ J})$$

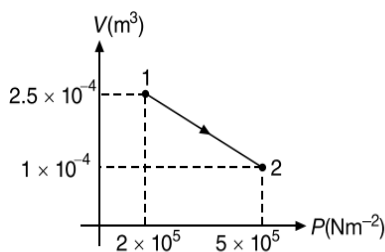
So, the heat exchanged is

$$Q'_{AB} = -100\text{ J}$$

The negative sign indicates that heat is lost by the system in this process.

### ILLUSTRATION 78

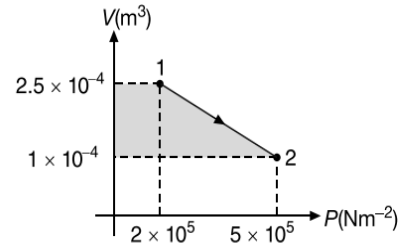
A gas is taken from state-1 to state-2 along the path shown in Figure.



If  $70\text{ cal}$  of heat is extracted from the gas in the process, calculate the change in internal energy of the system.

### SOLUTION

Since work done is equal to the area under the  $P$ - $V$  graph, so in this case work done will be the negative shaded area as shown in Figure. Negative because volume is decreasing.



$$\Rightarrow W = -\frac{1}{2} \times 1.5 \times 10^{-4} \times (2+5) \times 10^5 = -52.5\text{ J}$$

Since,  $70\text{ cal}$  of heat is extracted in the process, so

$$Q = -70\text{ cal} = 70 \times 4.2\text{ J} = -294\text{ J}$$

From First Law of Thermodynamics (*FLTD*), we get

$$Q = W + \Delta U$$

$$\Rightarrow \Delta U = Q - W = (-294) - (-52.5) = -241.5\text{ J}$$

Hence, the internal energy of gas decreases by  $241.5\text{ J}$  in the given process.

### ILLUSTRATION 79

For an ideal gas if the molar heat capacity varies as  $C = C_V + 3aT^2$ . Find the equation of the process in the variables ( $T, V$ ) where  $a$  is a constant.

### SOLUTION

From First Law of Thermodynamics,  $dQ = dU + dW$

$$\Rightarrow CdT = C_V dT + PdV$$

$$\Rightarrow (C_V + 3aT^2)dT = C_V dT + PdV$$

$$\Rightarrow 3aT^2 dT = PdV = \left(\frac{RT}{V}\right)dV \quad \{\because PV = RT\}$$

$$\Rightarrow \left(\frac{3a}{R}\right)TdT = \frac{dV}{V}$$

Integrating, we get

$$\left(\frac{3aT^2}{2R}\right) = \log_e V - \log_e C$$

where  $C$  is a positive constant

$$\Rightarrow \log_e \left(\frac{V}{C}\right) = \frac{3aT^2}{2R}$$

$$\Rightarrow \frac{V}{C} = e^{\frac{3aT^2}{2R}}$$

$$\Rightarrow Ve^{\frac{3aT^2}{2R}} = \text{constant}$$

### ILLUSTRATION 80

A vertical cylinder of cross-sectional area  $A$  contains one mole of an ideal monatomic gas under a piston of mass  $M$ . At a certain instant, a heater that transmits to the gas an

amount of heat  $q$  per unit time is switched on under the piston. Determine the velocity  $v$  of the piston under the condition that the gas pressure under piston is constant and equal to  $P_0$  and the gas under the piston is thermally insulated.

### SOLUTION

Let  $v$  be the speed of piston (upwards). According to First Law of Thermodynamics, we have

$$Q = \Delta U + W$$

where,  $\Delta U = (1)C_V\Delta T = \frac{3}{2}R\Delta T$

$$W = P\Delta V = PA\Delta x$$

where,  $P = P_0 + \frac{Mg}{A}$

Also,  $PV = (1)RT$

Since  $P = \text{constant}$ , so  $P\Delta V = R\Delta T$

$$\Rightarrow Q = P\Delta V + \frac{3}{2}P\Delta V = \frac{5}{2}PA\Delta x$$

Further,  $Q = q\Delta t$

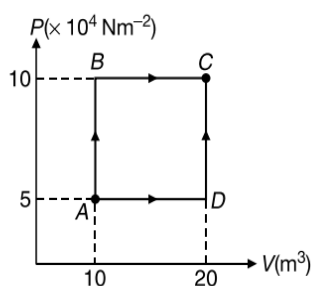
$$\Rightarrow q\Delta t = \frac{5}{2}PA\Delta x$$

$$\Rightarrow q = \frac{5}{2}\left(P_0 + \frac{Mg}{A}\right)A \frac{\Delta x}{\Delta t} = \frac{5}{2}(P_0A + Mg)v$$

$$\Rightarrow v = \frac{2}{5} \frac{q}{(P_0A + Mg)}$$

### ILLUSTRATION 81

A sample of 2 kg monatomic helium (assumed ideal) is taken through the process  $ABC$  and another sample of 2 kg of the same gas is taken through the process  $ADC$  (shown in figure). Given molecular mass of helium is 4.



- What is the temperature of helium in each of the states  $A$ ,  $B$ ,  $C$  and  $D$ ?
- Is there any way of telling afterwards which sample of helium went through the process  $ABC$  and which went through the process  $ADC$ ? Write Yes or No.
- How much is the heat involved the process  $ABC$  and  $ADC$ ?

### SOLUTION

Number of gram moles of  $He$  is given by

$$n = \frac{m}{M} = \frac{2 \times 10^3}{4} = 500$$

$$\begin{aligned} \text{(a)} \quad V_A &= 10 \text{ m}^3, P_A = 5 \times 10^4 \text{ Nm}^{-2} \\ \Rightarrow T_A &= \frac{P_A V_A}{nR} = \frac{(10)(5 \times 10^4)}{(500)(8.31)} = 120.34 \text{ K} \end{aligned}$$

Similarly,  $V_B = 10 \text{ m}^3, P_B = 10 \times 10^4 \text{ Nm}^{-2}$

$$\Rightarrow T_B = \frac{(10)(10 \times 10^4)}{(500)(8.31)} \text{ K} = 240.68 \text{ K}$$

Similarly,  $V_C = 20 \text{ m}^3, P_C = 10 \times 10^4 \text{ Nm}^{-2}$

$$\Rightarrow T_C = \frac{(20)(10 \times 10^4)}{(500)(8.31)} \text{ K} = 481.36 \text{ K}$$

and  $V_D = 20 \text{ m}^3, P_D = 5 \times 10^4 \text{ Nm}^{-2}$

$$\Rightarrow T_D = \frac{(20)(5 \times 10^4)}{(500)(8.31)} \text{ K} = 240.68 \text{ K}$$

- No, it is not possible to tell afterwards which sample went through the process  $ABC$  or  $ADC$ . But during the process if we note down the work done in both the processes, then the process which requires more work goes through process  $ABC$ .

- In the process  $ABC$

$$\Delta U = nC_V\Delta T = n\left(\frac{3}{2}R\right)(T_C - T_A)$$

$$\Rightarrow \Delta U = (500)\left(\frac{3}{2}\right)8.31(481.36 - 120.34) \text{ J}$$

$$\Rightarrow \Delta U = 2.25 \times 10^6 \text{ J}$$

and  $W = \text{area under}$

$$BC = (20 - 10)(10) \times 10^4 \text{ J} = 10^6 \text{ J}$$

So, by  $FLTD$ , we get

$$Q_{ABC} = \Delta U + W = (2.25 \times 10^6 + 10^6) \text{ J}$$

$$\Rightarrow Q_{ABC} = 3.25 \times 10^6 \text{ J}$$

In the process  $ADC$ ,  $\Delta U$  will be same (because it depends on initial and final temperatures only).

Since,  $W = \text{Area under } AD = (20 - 10)(5 \times 10^4) \text{ J}$

$$\Rightarrow W = 0.5 \times 10^6 \text{ J}$$

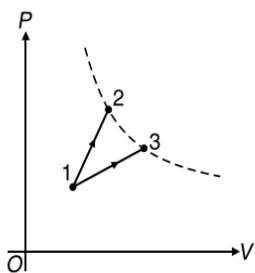
So, by  $FLTD$ , we get

$$Q_{ADC} = \Delta U + W = (2.25 \times 10^6 + 0.5 \times 10^6) \text{ J}$$

$$\Rightarrow Q_{ADC} = 2.75 \times 10^6 \text{ J}$$

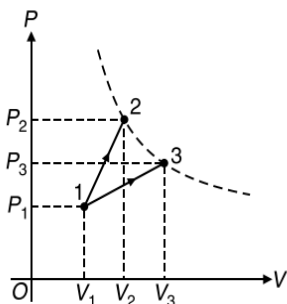
### ILLUSTRATION 82

A gas takes part in two thermal processes in which it is heated from the same initial state to the same final temperature. The  $P$ - $V$  diagram for these two processes are indicated by straight lines 1-3 and 1-2 in figure. Find out in which process the amount of heat supplied to the gas is larger.



**SOLUTION**

Since the temperature remains constant,  $\Delta T = 0$



$$\text{Now, } W_{12} = \frac{1}{2}(P_1 + P_2)(V_2 - V_1)$$

$$\text{and } W_{13} = \frac{1}{2}(P_1 + P_3)(V_3 - V_1)$$

$$\Rightarrow W_{12} - W_{13} = \frac{1}{2}[P_1(V_2 - V_3) + (P_3 - P_1)V_1]$$

Since,  $V_2 - V_3 < 0$  and  $P_3 - P_1 < 0$ , so

$$W_{12} - W_{13} < 0$$

$$\Rightarrow W_{13} > W_{12}$$

$$\Rightarrow Q_{13} > Q_{12}$$

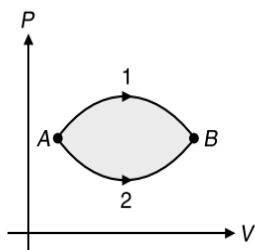


**Test Your Concepts-V**

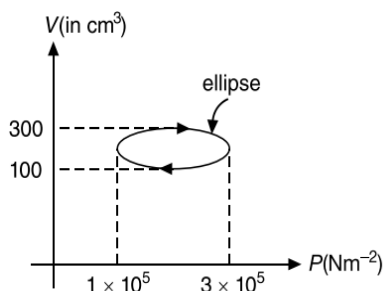
**Based on Work Done and First Law of Thermodynamics**

*(Solutions on page H.85)*

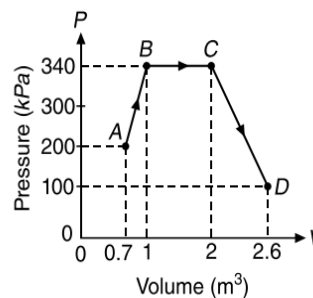
1. A certain amount of an ideal gas passes from state A to B first by means of process 1, then by means of process 2. In which of the process is the amount of heat absorbed by the gas greater.



2. For a thermodynamical system, the pressure, the volume and the temperature are related to each other as new gas law given by  $P = \frac{\alpha T^2}{V}$ , where  $\alpha$  is a constant. Calculate work done by the system in this process when pressure remains constant and its temperature changes from  $T_0$  to  $2T_0$ .
3. Calculate the heat absorbed by a system in going through the cyclic process shown in figure.



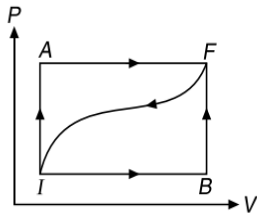
4. The PV diagram for a gas confined to a cylinder with the help of a piston is shown in Figure. Calculate the work done by the gas to expand from A to C and from A to D along the curve.



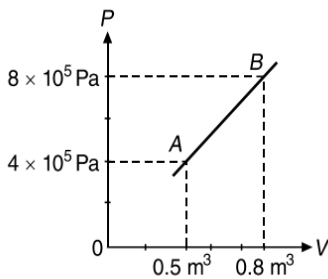
5. The table given below gives some values for different processes. All the data is in joule. Calculate the unknown values using First Law of Thermodynamics.

Process	Q	W	$U_i$	$U_f$	$\Delta U$
1.	35	-15	-	-10	-
2.	-15	-	-	60	-20
3.	-	-20	80	-	40

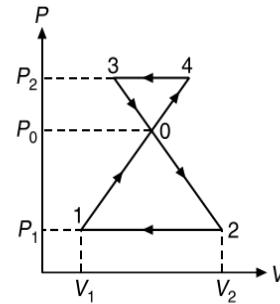
6. When a thermodynamic system is taken from an initial state I to a final state F along the path IAF, as shown in figure, the heat energy absorbed by the system is 55 J and the work done by the system is 25 J. If the same system is taken along the path IBF, the heat energy absorbed by the system is 35 J.



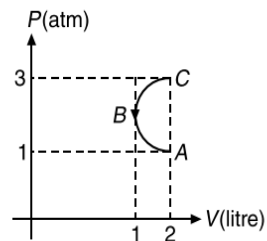
- (a) Calculate the work done along the path  $IBF$ .
- (b) If for the curved path  $FI$ , work done on the system is 15 J, how much heat energy gained or lost by the system along this path.
- (c) If internal energy at  $I$  is 10 J, calculate the internal energy at  $F$ .
- (d) If internal energy is 20 J at the point  $B$ , then calculate the heat gained (or heat lost) for the processes  $BF$  and  $IB$ .
7. The volume of a monatomic ideal gas increases linearly with pressure, as shown in the figure. Calculate the increase in internal energy, work done by the gas, and the heat supplied to the gas.



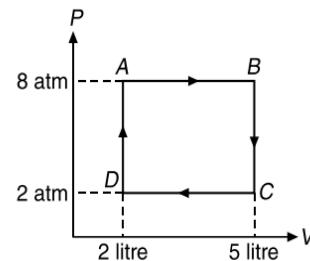
8. Calculate the work done by an ideal gas during a cyclic process 14321 shown in Figure. Take  $P_1 = 10^5$  Pa,  $P_0 = 3 \times 10^5$  Pa,  $P_2 = 4 \times 10^5$  Pa,  $V_2 - V_1 = 100$  litre and segments 43 and 21 of the cycle to be parallel to the  $V$ -axis.



9. The  $PV$  diagram shown in figure for a thermodynamic process is a semicircle. Calculate the work done on the gas in the process  $ABC$ .



10. The  $P$ - $V$  diagram of the thermodynamic process of an ideal gas is shown in Figure. Calculate the work done in the processes  $A \rightarrow B$ ,  $B \rightarrow C$ ,  $C \rightarrow D$ , and  $D \rightarrow A$ . Also calculate the work done in the complete cycle. Take  $1 \text{ atm} = 1.0 \times 10^5 \text{ Nm}^{-2}$ .



## ISOLATED SYSTEM

Consider first an isolated system for which there is no heat exchange and no work is done on the external environment. In this case  $Q = 0$  and  $W = 0$ , so from the First Law we conclude  $\Delta U = 0$

$$\Rightarrow U = \text{constant}$$

The internal energy of an isolated system is constant.

## ISOCORIC PROCESS

A process in which volume remains constant. e.g., a gas heated in a rigid container i.e.,  $V = \text{constant}$

In this case

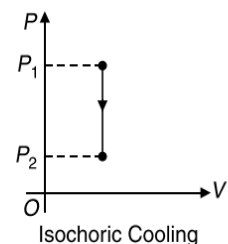
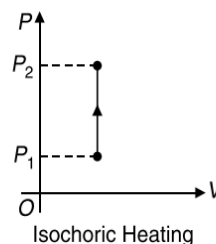
$$dW = PdV = 0$$

So, no external work is done in an Isochoric process.  
{for numerical problems}

According to First Law of Thermodynamics  $dQ = dU + dW$

$$\Rightarrow dQ = dU = nC_V dT$$

i.e., in an isochoric process the entire heat supplied just increases the internal energy of the gas responsible for the increase in temperature of gas.



Also, we must note that, in case of an isochoric process volume of the system remains constant

i.e.,  $V = \text{constant}$

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$$\Rightarrow \frac{P}{T} = \text{constant}$$

Since the boundary of the system does not displace because volume is constant, therefore,  $W = 0$

The change in internal energy is given by

$$\Delta U = nC_V \Delta T = \frac{nR}{\gamma - 1} \Delta T$$

From First Law, we have  $Q = W + \Delta U$

$$\Rightarrow Q = \Delta U = \frac{nR}{\gamma - 1} \Delta T = \frac{P_f V_f - P_i V_i}{\gamma - 1}$$

### ILLUSTRATION 83

Gaseous hydrogen initially at STP in a container of volume  $5 \times 10^{-5} \text{ m}^3$  is cooled by 55 K. Calculate the change in internal energy and amount of heat lost by the gas.

### SOLUTION

Initially at STP, the gas pressure is  $P = P_{\text{atm}} = 10^5 \text{ Pa}$ , gas temperature is  $T = 273 \text{ K}$  and gas volume is  $V = 5 \times 10^{-5} \text{ m}^3$ . If  $n$  mole of gas is there, then

$$PV = nRT$$

$$\Rightarrow n = \frac{PV}{RT} = \frac{10^5 \times 5 \times 10^{-5}}{8.314 \times 273} = 0.22 \text{ mole.}$$

Change in internal energy in the process is

$$\Delta U = \left(\frac{nf}{2}\right) R \Delta T$$

$$\Rightarrow \Delta U = \frac{5}{2} \times 0.22 \times 8.314 \times (-55) = -251.5 \text{ J}$$

The negative sign indicates that there is a loss in internal energy.

Since gas is enclosed in a container, so its volume remains constant during the process and hence work done in this process is zero. Applying FLTD, we get

$$Q = \Delta U + W = -251.5 + 0 = -251.5 \text{ J}$$

The negative sign indicates that heat is lost by the system.

### ILLUSTRATION 84

An ideal gas has specific heat at constant pressure to be  $C_p = \frac{5}{2}R$ . The gas is kept in a closed vessel of volume  $0.0083 \text{ m}^3$  at a temperature of 300 K and a pressure of  $1.6 \times 10^6 \text{ Nm}^{-2}$ . An amount of  $2.49 \times 10^4 \text{ J}$  of heat energy is supplied to the gas. Calculate the final temperature and pressure of gas. Given  $R = 8.3 \text{ Jmol}^{-1}\text{K}^{-1}$ .

### SOLUTION

Here nothing is said about the number of moles of ideal gas. So, let the number of moles of gas be  $n$ . Then from ideal gas equation, we have

$$n = \frac{PV}{RT} = \frac{(1.6 \times 10^6)(0.0083)}{(8.3)(300)} = \frac{16}{3}$$

$$\text{Since } C_p = \frac{5R}{2}, \text{ so } C_V = \frac{3R}{2} \quad \{\because C_p - C_V = R\}$$

Since at constant volume, we have

$$Q = \Delta U = nC_V \Delta T$$

$$\Rightarrow 2.49 \times 10^4 = \left(\frac{16}{3}\right) \left(\frac{3R}{2}\right) \Delta T$$

$$\Rightarrow \Delta T = T_f - T_i = 375 \text{ K}$$

$$\Rightarrow T_f = 300 + 375 = 675 \text{ K}$$

Further at constant volume  $\frac{P_i}{T_i} = \frac{P_f}{T_f}$

$$\Rightarrow P_f = \left(\frac{T_f}{T_i}\right) P_i = \left(\frac{675}{300}\right) (1.6 \times 10^6)$$

$$\Rightarrow P_f = 3.6 \times 10^6 \text{ Nm}^{-2}$$

### ILLUSTRATION 85

Consider two thermally insulated vessels, one with 0.025 mole of helium and other with  $n$  mole of hydrogen. Initially both the gases are at room temperature. Now equal amount of heat is supplied to both the vessels. It is found that in both the gases, temperature rises by same amount. Find the number of moles of hydrogen in second vessel

### SOLUTION

Since the gases are enclosed in closed vessels, so the heating process can be taken as isochoric i.e.  $W = 0$ .

Also, heat supplied to both vessels is same, so applying FLTD, we get

$$Q = n_1 C_{V1} \Delta T = n_2 C_{V2} \Delta T$$

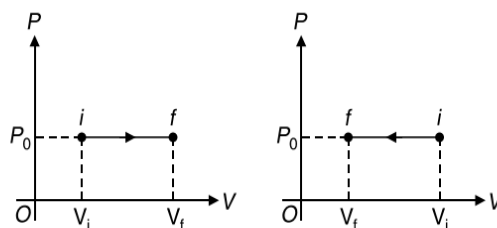
For He,  $C_{V1} = \frac{3}{2}R$  and for  $H_2$ ,  $C_{V2} = \frac{5}{2}R$

$$\Rightarrow 0.025 \times \frac{3}{2} R \Delta T = n \frac{5}{2} R \Delta T$$

$$\Rightarrow n = \frac{0.025 \times 3}{5} = 0.015 \text{ mole}$$

## ISOBARIC PROCESS

In an isobaric process the pressure of the system remains constant i.e.,  $P = \text{constant}$ .



The work done is given by

$$W = \int P dV = P_0 \int_{V_i}^{V_f} dV = P_0 (V_f - V_i)$$

Using gas equation  $PV = nRT$

We get,  $W = P_0 (V_f - V_i) = nR(T_f - T_i)$

$$\Rightarrow W_{\text{isob}} = P\Delta V = nR\Delta T$$

Since the change in internal energy is independent of the path followed, therefore

$$\Delta U = nC_V\Delta T = \frac{nR}{\gamma-1}\Delta T = \frac{nR}{\gamma-1}(T_f - T_i)$$

Using First Law of Thermodynamics,

$$Q = W + \Delta U$$

$$Q = nR(T_f - T_i) + \frac{nR}{\gamma-1}(T_f - T_i)$$

$$\Rightarrow Q = \frac{n\gamma R}{\gamma-1}(T_f - T_i) = nC_p\Delta T$$

So, for isobaric process, we have

$$Q = nC_p\Delta T = nC_p(T_f - T_i), \text{ where } C_p = \frac{\gamma R}{\gamma-1}$$

### Problem Solving Technique(s)

(a)  $C_p - C_v = R$

(b)  $\frac{C_p}{C_v} = \gamma$

(c) A process in which pressure is constant. e.g., boiling of water at atmospheric pressure. So,

$$\Rightarrow W_{\text{isobaric}} = P(V_v - V_\ell) = P\Delta V$$

where,  $V_v$  is volume of vapours,  $V_\ell$  is volume of liquid. Also, the heat that must be transferred to the liquid to vapourise all of it is equal to mass ( $m$ ) of liquid times the latent heat of vapourisation  $L_v$  of liquid, that is,  $dQ = mL_v$ , so, According to First Law of Thermodynamics  $dU = mL_v - P(V_v - V_\ell)$

### ILLUSTRATION 86

Find the ratio of  $\frac{Q}{\Delta U}$  and  $\frac{Q}{W}$  in an isobaric process. Given that the ratio of molar heat capacities  $\frac{C_p}{C_v} = \gamma$ .

#### SOLUTION

In an isobaric process  $P = \text{constant}$ . Therefore,  $C = C_p$

$$\Rightarrow \frac{Q}{\Delta U} = \frac{nC_p\Delta T}{nC_V\Delta T} = \frac{C_p}{C_V} = \gamma$$

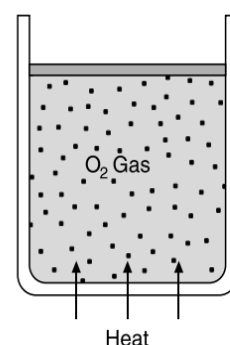
$$\text{Now, } \frac{Q}{W} = \frac{Q}{\Delta Q - \Delta U} \quad \{\because Q = \Delta U + W\}$$

$$\Rightarrow \frac{Q}{W} = \frac{nC_p\Delta T}{nC_p\Delta T - nC_V\Delta T} = \frac{C_p}{C_p - C_V}$$

$$\Rightarrow \frac{Q}{W} = \frac{\frac{C_p}{C_V}}{\frac{C_p}{C_V} - 1} = \frac{\gamma}{\gamma - 1}$$

### ILLUSTRATION 87

A cylindrical container contains oxygen gas and is closed by a movable frictionless piston of mass 50 kg having cross-sectional area  $100 \text{ cm}^2$  as shown in Figure. Some heat is supplied to the cylinder at atmospheric pressure of  $10^5 \text{ Pa}$  so that the piston is slowly displaced up by 20 cm. Calculate the amount of heat supplied to the gas.



#### SOLUTION

Since the piston is open to atmosphere, so the gas is under a constant pressure given by

$$P_{\text{gas}} = P_{\text{atm}} + \frac{Mg}{A} = 10^5 + \frac{50 \times 10}{100 \times 10^{-4}} \text{ Pa}$$

It is given that the piston moves out by 20 cm, so the work done in the process is

$$W = P_{\text{gas}}\Delta V, \text{ where}$$

$$\Delta V = A\Delta x = (100 \times 10^{-4})(20 \times 10^{-2}) = 2 \times 10^{-3} \text{ m}^3$$

$$\Rightarrow W = 1.5 \times 10^5 \times 2 \times 10^{-3} = 300 \text{ J}$$

For a process in which gas pressure is constant, work done can also be given by

$$W = P_{\text{gas}}\Delta V = nR\Delta T$$

$$\Rightarrow nR\Delta T = 300 \text{ J}$$

$$\text{So } Q = nC_p\Delta T, \text{ where } C_p = \frac{7R}{2}$$

$$\Rightarrow Q = n\left(\frac{7R}{2}\right)\Delta T = \frac{7}{2}(nR\Delta T)$$

$$\Rightarrow Q = \frac{7}{2} \times 300 = 1050 \text{ J}$$

### ILLUSTRATION 88

Two moles of an ideal monatomic gas are confined within a cylinder by a massless and frictionless spring loaded piston of cross-sectional area  $4 \times 10^{-3} \text{ m}^2$ . Initially the spring is in its relaxed state. Now the gas is heated by an electric heater, placed inside the cylinder, for some time. During

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this time, the gas expands and does 50 J of work in moving the piston through a distance 0.10 m. The temperature of the gas increases by 50 K. Calculate the spring constant and the heat supplied by the heater. ( $P_0 = 1.0 \times 10^5 \text{ Nm}^{-2}$ )

### SOLUTION

When the piston has displaced by  $x$ , the pressure inside the cylinder is given by

$$P = P_0 + \frac{kx}{A}$$

$$\Rightarrow W = \int P dV = \int \left( P_0 + \frac{kx}{A} \right) A dx$$

$$\Rightarrow W = \left( P_0 Ax + \frac{1}{2} kx^2 \right)$$

$$\Rightarrow 50 = 10^5 \times 4 \times 10^{-3} \times 0.1 + \frac{1}{2} k \times (0.1)^2$$

$$\Rightarrow k = 2000 \text{ Nm}^{-1}$$

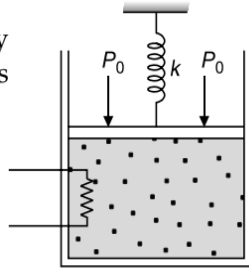
Also, change in internal energy is given by

$$\Delta U = nC_v \Delta T = 2 \times \frac{3R}{2} \times 50 = 150 \times 8.3 = 1245 \text{ J}$$

From First Law of Thermodynamics, we have

$$Q = \Delta U + W$$

$$Q = 1245 + 50 = 1295 \text{ J}$$



### ILLUSTRATION 90

- (a) One mole of oxygen is heated from  $0^\circ \text{C}$  at constant pressure, till its volume becomes 10% more than the initial volume. Find the heat required. The specific heat of  $O_2$  under these conditions is  $0.22 \text{ cal g}^{-1} \text{K}^{-1}$ .
- (b) If the same amount of heat is supplied to the gas at constant volume, calculate the final temperature.

### SOLUTION

- (a) Since, the specific heat of oxygen is given to be  $0.22 \text{ cal g}^{-1} \text{K}^{-1}$ , so molar specific heat of oxygen is

$$C_p = (32)(0.22) \text{ cal mol}^{-1} \text{K}^{-1}$$

At constant pressure, we have  $\frac{V}{T} = \text{constant}$

So, 10% increase in volume increases the temperature by 10% i.e.,  $T_i = 273 \text{ K}$  and  $T_f = 300.3 \text{ K}$

$$\Rightarrow \Delta T = 27.3 \text{ K}$$

Since,  $Q = nC_p \Delta T$

$$\Rightarrow Q = (1)(32)(0.22)(27.3) = 192 \text{ cal}$$

- (b) Since, again  $Q = 192 \text{ cal}$

$$\Rightarrow Q = nC_v \Delta T, \text{ where } C_v = C_p - R$$

Since,  $C_p = (32)(0.22) \text{ cal mol}^{-1} \text{K}^{-1}$

$$\Rightarrow C_p = 7 \text{ cal mol}^{-1} \text{K}^{-1}$$

$$\Rightarrow C_v = C_p - R = 5 \text{ cal mol}^{-1} \text{K}^{-1}$$

$$\left\{ \because R = 2 \text{ cal mol}^{-1} \text{K}^{-1} \right\}$$

$$\Rightarrow 192 = (1)(5)\Delta T$$

$$\Rightarrow \Delta T = T_f - T_i = 38.4 \text{ K}$$

$$\Rightarrow T_f = 311.4 \text{ K}$$

### ILLUSTRATION 89

A cylinder with a piston contains 0.2 kg of water at  $100^\circ \text{C}$ . What is the change in internal energy of the water when it is converted to steam at  $100^\circ \text{C}$  at a constant pressure of 1 atm? The density of water is  $\rho_w = 10^3 \text{ kg m}^{-3}$  and that of steam is  $\rho_s = 0.6 \text{ kg m}^{-3}$ . The latent heat of vaporization of water is  $L_v = 2.26 \times 10^6 \text{ J kg}^{-1}$ .

### SOLUTION

The heat transfer to the water is

$$Q = mL_v = (0.2 \text{ kg})(2.26 \times 10^6 \text{ J kg}^{-1})$$

$$\Rightarrow Q = 4.52 \times 10^5 \text{ J}$$

The work done by the water when it expands against the piston at constant pressure is

$$W = P(V_s - V_w) = P \left( \frac{m}{\rho_s} - \frac{m}{\rho_w} \right)$$

$$\Rightarrow W = (1.01 \times 10^5 \text{ Nm}^2) \left( \frac{0.2 \text{ kg}}{0.6 \text{ kg m}^{-3}} - \frac{0.2 \text{ kg}}{1000 \text{ kg m}^{-3}} \right)$$

$$\Rightarrow W = 3.36 \times 10^4 \text{ J}$$

The change in internal energy is

$$\Delta U = Q - W = 452 \text{ kJ} - 33.6 \text{ kJ} = 418.4 \text{ kJ}$$

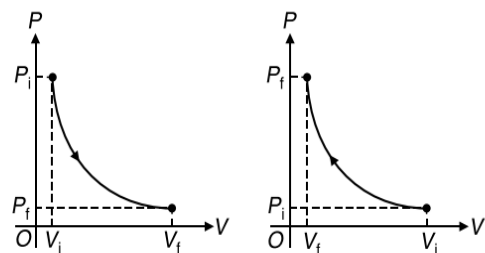
## ISOTHERMAL PROCESS

To a novice we can explain that any process taking place gradually can be taken to be an example of isothermal process e.g., melting process.

Mathematically, a process in which  $T$  is constant or  $dT = 0$  is an isothermal process i.e., for such a process  $dU = 0$  and hence an isothermal process is not accompanied by any change in internal energy. So, the First Law of Thermodynamics takes the form  $dQ = dW$  i.e., the entire heat supplied just does an external work.

The equation of state governing isothermal process is

$$PV = \text{constant.}$$

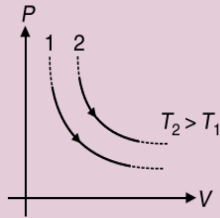


For an isothermal process,  $PV = \text{constant}$ , so

$$\begin{aligned} d(PV) &= 0 \\ \Rightarrow PdV + VdP &= 0 \\ \Rightarrow \frac{dP}{dV} &= -\frac{P}{V} \\ \Rightarrow (\text{Slope})_{\text{isot}} &= \frac{dP}{dV} = -\frac{P}{V} \end{aligned}$$

### Problem Solving Technique(s)

- (a) Two isotherms never intersect each other.  
 (b) Isotherm far from the  $PV$  axis has more temperature compared to the isotherm close to the  $PV$  axis.



### WORK DONE IN AN ISOTHERMAL PROCESS

Since  $W = \int dW = \int PdV$

$$\Rightarrow W_{\text{isot}} = nRT \int_{V_1}^{V_2} \frac{dV}{V} \quad \{\because PV = nRT\}$$

$$\Rightarrow W_{\text{isot}} = nRT \log_e \left( \frac{V_2}{V_1} \right) = 2.303 nRT \log_{10} \left( \frac{V_2}{V_1} \right)$$

where,  $\frac{V_2}{V_1}$  is called the **Expansion Ratio**.

Since  $P_1V_1 = P_2V_2$

$$\Rightarrow W_{\text{isot}} = nRT \log_e \left( \frac{P_1}{P_2} \right) = 2.303 nRT \log_{10} \left( \frac{P_1}{P_2} \right)$$

Since temperature of the system remains constant, therefore, there is no change in internal energy.

$$\Delta U = nC_v \Delta T = 0$$

Using First Law of Thermodynamics,

$$Q = W + \Delta U$$

$$\Rightarrow Q = W = nRT \ln \left| \frac{V_f}{V_i} \right|$$

### ILLUSTRATION 91

Three moles of an ideal gas at 300 K are isothermally expanded to five times its volume and heated at this constant volume so that the pressure is raised to its initial value before expansion. In the whole process 83.14 kJ heat is required. Calculate the ratio  $\left( \frac{C_p}{C_v} \right)$  of the gas. ( $\log_e 5 = 1.61$  and  $R = 8.31 \text{ Jmol}^{-1} \text{ K}^{-1}$ )

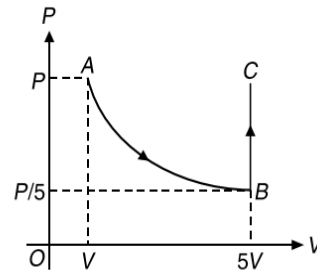
### SOLUTION

According to First Law of Thermodynamics,  $Q = \Delta U + W$

For an isothermal change,

$$T = \text{constant}, U = \text{constant}, \Delta U = 0$$

and  $W = nRT \log_e \left| \frac{V_f}{V_i} \right|$



$$\Rightarrow W = 3 \times 8.31 \times 300 \times \log_e (5) = 12.03 \text{ kJ}$$

$$\Rightarrow Q_{\text{isothermal}} = 0 + 12.03 = 12.03 \text{ kJ} \quad \dots(1)$$

For isochoric change,  $V = \text{constant}$

$$\Rightarrow W = \int PdV = 0$$

So,  $\Delta U = nC_v \Delta T = 3C_v \Delta T \quad \{\because n = 3\}$

Applying gas equation between points A and C

$$\frac{PV}{300} = \frac{P(5V)}{T_C}$$

$$\Rightarrow T_C = 1500 \text{ K}$$

So that  $\Delta T = T_C - T_B = 1500 - 300 = 1200$

$$\Rightarrow \Delta U = 3C_v \times 1200 = 3.6 C_v \text{ kJ}$$

**Note:** To find  $T_C$  you can apply gas equation between points B and C also and hence,

$$Q_{\text{isochoric}} = 3.6 C_v + 0 = 3.6 C_v \text{ kJ} \quad \dots(2)$$

According to given problem,

$$Q_{\text{isothermal}} + Q_{\text{isochoric}} = 83.14 \text{ kJ}$$

Using equation (1) and (2), we get

$$12.03 + 3.6 C_v = 83.14$$

$$\Rightarrow C_v = \left( \frac{71.11}{3.6} \right) = 19.75 \text{ J}$$

Thus  $C_p = C_v + R = 19.75 + 8.3 = 28.05 \text{ Jmol}^{-1} \text{ K}^{-1}$

$$\Rightarrow \gamma = \frac{C_p}{C_v} = \frac{28.05}{19.75} = 1.42$$

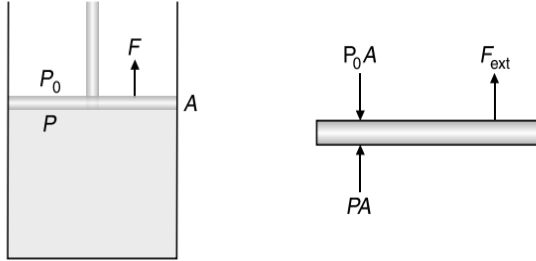
## 2.62 JEE Advanced Physics: Waves and Thermodynamics

### ILLUSTRATION 92

One mole of a gas is put under a weightless piston of a vertical cylinder at temperature  $T$ . The space over the piston opens into atmosphere. How much work should be performed by some external force to increase isothermally the volume under the piston to twice the volume (neglect friction of piston)?

### SOLUTION

Drawing the Free Body Diagram for the piston, we get



Now according to Newton's Second Law, we have for piston  $F_{\text{net}} = m_{\text{piston}} a$

where  $a$  is acceleration of the piston. Since,  $m_{\text{piston}} \rightarrow 0$

$$\Rightarrow F_{\text{net}} = F_{\text{ext}} + PA - P_0 A = 0$$

So, work done by external force is

$$W_{\text{ext}} = \int dW = \int F_{\text{ext}} dx = \int (P_0 - P) A dx$$

$$\Rightarrow W_{\text{ext}} = \int (P_0 - P) dV = P_0 \int_V^{2V} dV - \int_V^{2V} P dV$$

For an ideal gas  $PV = RT$ , so

$$W_{\text{ext}} = P_0 V - RT \int_V^{2V} \frac{dV}{V} = P_0 V - RT (\log_e V|_V^{2V})$$

$$\Rightarrow W_{\text{ext}} = P_0 V - RT \log_e (2) \quad \dots(1)$$

Since temperature is constant,  $T$ , so we have

$$PV = RT \quad (\text{finally})$$

$$P_0 V = RT \quad (\text{initially})$$

So, from (1) we get  $W_{\text{ext}} = RT - RT \log_e (2)$

$$\Rightarrow W_{\text{ext}} = RT(1 - \log_e (2))$$

### ILLUSTRATION 93

A piston can freely move inside a horizontal cylinder closed from both ends. Initially, the piston separates the inside space of the cylinder into two equal parts each of

volume  $V_0$ , in which an ideal gas is contained under the same pressure  $P_0$  and at the same temperature. What work has to be performed in order to increase isothermally the volume of one part of gas is  $\eta$  times compared to that of the other by slowly moving the piston?

### SOLUTION

As the piston is displaced externally some external work is done in the process. If piston is displaced towards right, the gas on left side expands and does some work. Similarly, gas on right is compressed and work is done on it. Work done by external agent is

$$W_{\text{ext}} = W_{\text{by gas in left part}} + W_{\text{on gas in right part}} \quad \dots(1)$$

It is given that initial volume of both the parts is  $V_0$  and in the process final volume of one part is  $\eta$  times that of the other part. If the final volume of right part is  $V$  then that of left part will become  $\eta V$ . Since total volume of container is  $2V_0$ , so we have  $V + \eta V = 2V_0$

$$\Rightarrow V = \frac{2V_0}{\eta + 1}$$

For gas in left part, work done by gas during isothermal expansion is

$$W_{\text{by gas}} = nRT \ln \frac{V_2}{V_1}$$

$$\Rightarrow W_{\text{by gas}} = P_0 V_0 \ln \left( \frac{\eta V}{V_0} \right) = P_0 V_0 \ln \left( \frac{2\eta}{\eta + 1} \right) \quad \dots(2)$$

Similarly, for gas in right part, work done on the gas is isothermal compression is

$$W_{\text{on gas}} = nRT \ln \frac{V_2}{V_1} = P_0 V_0 \ln \frac{V}{V_0}$$

$$\Rightarrow W_{\text{on gas}} = P_0 V_0 \ln \left( \frac{2}{\eta + 1} \right) \quad \dots(3)$$

Now from equations (1), (2) and (3), we get

$$W_{\text{ext}} = W_{\text{by gas in left part}} + W_{\text{on gas in right part}}$$

$$\Rightarrow W_{\text{ext}} = P_0 V_0 \ln \left( \frac{2\eta}{\eta + 1} \right) + P_0 V_0 \ln \left( \frac{2}{\eta + 1} \right)$$

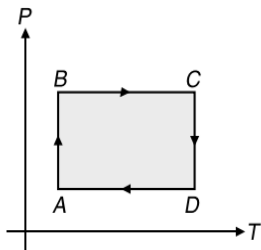
$$\Rightarrow W_{\text{ext}} = P_0 V_0 \ln \left[ \frac{4\eta}{(\eta + 1)^2} \right]$$

## Test Your Concepts-VI

### Based on Isochoric, Isobaric and Isothermal Processes

(Solutions on page H.86)

- What is the heat input needed to raise the temperature of two moles of helium gas from  $0^\circ\text{C}$  to  $100^\circ\text{C}$ .
  - at constant volume
  - at constant pressure
  - what is the work done by the gas in part (a). Give your answer in terms of  $R$ .
- Plot  $P$ - $V$ ,  $V$ - $T$  and  $\rho$ - $T$  graph corresponding to the  $P$ - $T$  graph for an ideal gas shown in figure.

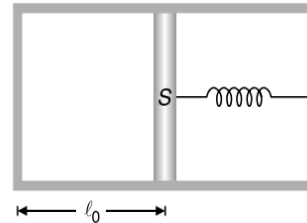


- One mole of an ideal gas is heated from  $0^\circ\text{C}$  to  $100^\circ\text{C}$  at a constant pressure of 1 atm. Calculate the work done in the process. Take  $1\text{ atm} = 10^5\text{ Nm}^{-2}$ .
- Two moles of a certain gas at a temperature  $T_0 = 300\text{ K}$  were cooled isochorically so that the pressure of the gas becomes half the initial pressure. Then as a result of isobaric process, the gas is allowed to expand till its temperature goes back to the initial value. Find the total amount of heat absorbed by gas in this process. Take  $R = 8.3\text{ Jmol}^{-1}\text{K}^{-1}$ .
- One mole of a certain ideal gas is given 500 cal heat. As a result the temperature rises by 72 K at constant pressure. Find the work performed by the gas, the change of internal energy and the value of  $\gamma$ .
- An ideal di-atomic gas is heated at constant pressure such that it performs a work  $W = 2.0\text{ J}$ . Find the amount of heat supplied.
- Three moles of an ideal gas ( $C_p = \frac{7}{2}R$ ) at pressure  $P_0$  and temperature  $T_0$  is isothermally expanded to twice its initial volume, it is then compressed at a constant pressure to its original volume.
  - Sketch  $P$ - $V$  and  $P$ - $T$  diagram for complete process.
  - Calculate net work done by the gas.
  - Calculate net heat supplied to the gas during complete process.

Give all answers in terms of gas constant  $R$ .

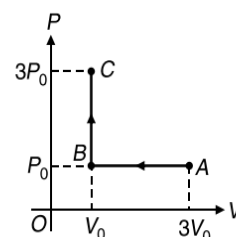
- A piston  $S$  of mass  $M$ , area of cross section  $A$  can slide inside a cylinder without friction. The walls of the cylinder are adiabatic and the piston is diathermic. Length of the cylinder is  $2\ell_0$ . Initial pressure of each chamber is  $P_0$  and volume of both the chambers are equal. Initially spring is unstretched and has spring constant  $k$ . The

spring is fixed with piston and wall of the cylinder as shown in the figure.



Find out its time period when piston is given small displacement.

- A vertical hollow cylinder contains an ideal gas. The gas is enclosed in the cylinder by a 5 kg movable piston having a cross-sectional area of  $5 \times 10^{-3}\text{ m}^2$ . The gas is now heated from 300 K to 350 K and the piston rises by 0.1 m. Now, the piston clamped in this position and gas is cooled back to 300 K. Calculate the difference between the heat energy added during heating and the heat energy lost during cooling. Take  $1\text{ atm} = 10^5\text{ Nm}^{-2}$  and  $g = 10\text{ ms}^{-2}$ .
- A vessel of volume  $V$  is evacuated by means of a piston air pump. One piston stroke removes the volume  $\Delta V$  of air. How many strokes are needed to reduce the pressure in the vessel by  $x$  times the original pressure. The process is assumed to be isothermal and the gas is ideal.
- Four moles of a monatomic ideal gas are at pressure  $3 \times 10^5\text{ Nm}^{-2}$  and temperature 100 K (state A). It is heated isobarically to temperature 400 K (state B). Next it undergoes isothermal expansion to pressure  $1 \times 10^5\text{ Nm}^{-2}$  (state C). It is then cooled isobarically to 100 K (state D). Finally, it is compressed isothermally to return to initial state A. Draw  $P$ - $T$ ,  $P$ - $V$  and  $V$ - $T$  diagrams for the whole process.
- In a gaseous system, a gas expands from  $10^{-4}\text{ m}^3$  to  $2 \times 10^{-4}\text{ m}^3$  while its pressure remains constant at  $10^5\text{ Ntm}^{-2}$ . Calculate the amount of heat absorbed by the gas in the expansion. [ $\gamma = 1.67$ ]
- $P$ - $V$  diagram of an ideal gas for a process ABC is as shown in the figure.



- Find total heat absorbed or released by the gas during the process ABC.

- (b) Change in internal energy of the gas during the process ABC.  
 (c) Plot pressure versus density graph of the gas for the process ABC.
14. One mole of an ideal gas is contained under a weightless piston of a vertical cylinder at a temperature  $T$ . The space over the piston opens into the atmosphere. What work has to be performed to increase isothermally the

gas volume under the piston  $n$  times by slowly raising the piston? Friction is negligible.

15. If 70 calorie of heat is required to raise the temperature of 2 mole of an ideal gas at constant pressure from 30 to 35 °C. If  $R = 2 \text{ cal mol}^{-1} \text{ K}^{-1}$ , calculate  
 (a) the work done by the gas  
 (b) increase in internal energy of the gas

## ADIABATIC PROCESS

Any process carried out suddenly is an example of an Adiabatic process e.g., bursting of a cycle tube. Mathematically a process in which the system and surroundings do not exchange any heat with each other i.e.  $dQ = 0$  and in such a process all  $P$ ,  $V$  and  $T$  must change simultaneously such that  $C = 0$ .

## EQUATION OF STATE FOR AN ADIABATIC PROCESS

According to First Law of Thermodynamics

$$dQ = dU + dW$$

$$\Rightarrow 0 = dU + dW = nC_V dT + PdV \quad \dots(1)$$

For an Ideal Gas, we have  $PV = nRT$

$$\Rightarrow d(PV) = nRdT$$

$$\Rightarrow PdV + VdP = nRdT$$

$$\Rightarrow dT = \frac{PdV + VdP}{nR} \quad \dots(2)$$

Using (2) in (1), we get

$$\Rightarrow \left(\frac{C_V}{R} + 1\right)PdV + \frac{C_V V dP}{R} = 0$$

$$\Rightarrow (C_V + R)PdV + C_V V dP = 0$$

$$\Rightarrow C_p PdV + C_V V dP = 0$$

$$\Rightarrow \gamma PdV + VdP = 0 \quad \left\{ \because \frac{C_p}{C_V} = \gamma \right\}$$

$$\Rightarrow \gamma \int_{V_1}^{V_2} \frac{dV}{V} = - \int_{P_1}^{P_2} \frac{dP}{P}$$

$$\Rightarrow \gamma \log_e \left( \frac{V_2}{V_1} \right) = - \log_e \left( \frac{P_2}{P_1} \right)$$

$$\Rightarrow \log_e \left\{ \left( \frac{V_2}{V_1} \right)^\gamma \right\} = \log_e \left( \frac{P_1}{P_2} \right)$$

$$\Rightarrow P_1 V_1^\gamma = P_2 V_2^\gamma$$

$$\Rightarrow PV^\gamma = \text{constant}$$

For an Adiabatic process the equation of state is/are

$$\left. \begin{aligned} PV^\gamma &= \text{constant} \\ TV^{\gamma-1} &= \text{constant} \\ T^\gamma P^{1-\gamma} &= \text{constant} \end{aligned} \right\} \text{ where } \gamma = \frac{C_p}{C_V}$$

## ILLUSTRATION 94

An ideal monatomic gas at 300 K expands adiabatically to twice its volume. What is the final temperature?

### SOLUTION

For an ideal monatomic gas, we have  $\gamma = \frac{5}{3}$

For an adiabatic process,  $TV^{\gamma-1} = \text{constant}$

$$\Rightarrow T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

$$\Rightarrow T_f = T_i \left( \frac{V_i}{V_f} \right)^{\gamma-1} = (300) \left( \frac{1}{2} \right)^{\frac{5}{3}-1}$$

$$\Rightarrow T_f = 189 \text{ K}$$

## WORK DONE IN AN ADIABATIC PROCESS

Since,  $dQ = 0$

$$\Rightarrow dW = -dU = -nC_V dT$$

$$\Rightarrow W_{\text{adiabatic}} = -nC_V \int_{T_1}^{T_2} dT = -nC_V (T_2 - T_1)$$

Further,  $C_V = \frac{R}{\gamma-1} \quad \left\{ \because C_p - C_V = R \text{ and } \gamma = \frac{C_p}{C_V} \right\}$

$$\Rightarrow W_{\text{adiabatic}} = \frac{nR}{1-\gamma} (T_2 - T_1)$$

$$\Rightarrow W_{\text{adiabatic}} = \frac{1}{1-\gamma} (nRT_2 - nRT_1)$$

Since  $P_2 V_2 = nRT_2$  and  $P_1 V_1 = nRT_1$

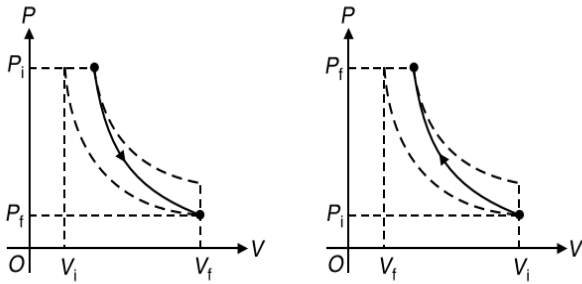
$$\Rightarrow W_{\text{adiabatic}} = \frac{1}{1-\gamma} (P_2 V_2 - P_1 V_1)$$

$$\Rightarrow W = -\Delta U$$

Work done by the system is equal to the decrease in internal energy.

Work done on the system is equal to the increase in internal energy.

## INDICATOR DIAGRAM FOR AN ADIABATIC PROCESS



For an adiabatic process, we have  $PV^\gamma = \text{constant}$

$$\Rightarrow d(PV^\gamma) = 0$$

$$\Rightarrow P(\gamma V^{\gamma-1} dV) + V^\gamma dP = 0$$

$$\Rightarrow \gamma P \left( \frac{V^\gamma}{V} \right) dV + V^\gamma dP = 0$$

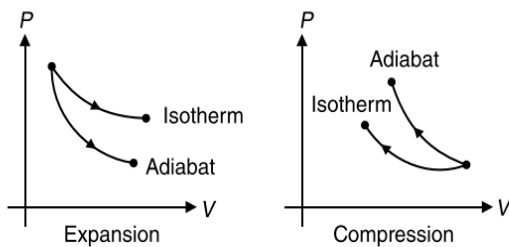
$$\Rightarrow \frac{dP}{dV} = -\gamma \left( \frac{P}{V} \right)$$

Since,  $(\text{Slope})_{\text{isot}} = -\frac{P}{V}$

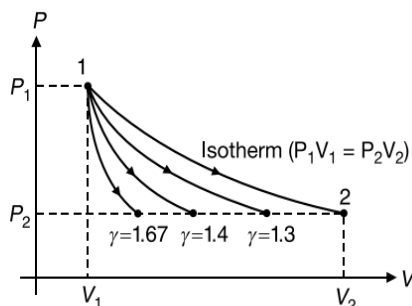
$$\Rightarrow (\text{Slope})_{\text{adia}} = \gamma (\text{Slope})_{\text{isot}}$$

Since  $\gamma > 1$ , so  $(\text{Slope})_{\text{adia}} > (\text{Slope})_{\text{isot}}$

Due to this, during expansion an isotherm lies above the adiabat and during compression an adiabat lies above the isotherm.



Also, we note that the more the value of  $\gamma$ , the more steep will be the curve for adiabatic process as shown in Figure.



### ILLUSTRATION 95

A sample of diatomic gas with  $\gamma = 1.5$  is compressed from a volume of 1600 cc to 400 cc adiabatically. The initial pressure of gas was  $1.5 \times 10^5$  Pa. Find the final pressure and work done by the gas in the process.

#### SOLUTION

We know in an adiabatic process, pressure and volume of gas of its different states are related as

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

$$\Rightarrow P_2 = \left( \frac{V_1}{V_2} \right)^\gamma P_1$$

$$\Rightarrow P_2 = \left( \frac{1600}{400} \right)^{1.5} \times 1.5 \times 10^5$$

$$\Rightarrow P_2 = (4)^{1.5} \times 1.5 \times 10^5 = 1.2 \times 10^6 \text{ Pa}$$

For an adiabatic process work done by a gas is

$$W = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1}$$

$$\Rightarrow W = \frac{1.5 \times 10^5 \times 1600 \times 10^{-6} - 1.2 \times 10^6 \times 400 \times 10^{-6}}{1.5 - 1}$$

$$\Rightarrow W = \frac{240 - 480}{0.5} = -480 \text{ J}$$

Here work done by gas comes out a negative value thus we can state that as gas is being compressed, work is done on the gas and so work done by gas is  $-480 \text{ J}$ .

### ILLUSTRATION 96

Two identical gases whose adiabatic exponent is  $\gamma$  are filled in two identical containers at equal pressure. In both the containers the volume of gas is doubled. In first container it is done by an isothermal process and in second container it is done by an adiabatic process. Find the condition for which the work done by the gas in the two expansion process is same.

#### SOLUTION

In first container, work done is

$$W_1 = W_{\text{isot}} = nRT_1 \ln(2) \quad \dots(1)$$

In second container, work done is

$$W_2 = W_{\text{ad}} = \frac{nR(T_2 - T_1)}{1 - \gamma} \quad \dots(2)$$

where,  $T_2 = T_1(2)^{1-\gamma} \quad \left\{ \because T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \right\}$

According to the problem, we have  $W_1 = W_2$

$$\Rightarrow nRT_1 \ln(2) = \frac{nRT_1(2^{1-\gamma} - 1)}{\gamma - 1}$$

$$\Rightarrow (\gamma - 1) \ln(2) = 1 - 2^{1-\gamma}$$

**ILLUSTRATION 97**

A piston divides a closed gas cylinder into two parts. Initially the piston is kept pressed such that one part has a pressure  $P_0$  and volume  $5V_0$  and the other part has pressure  $8P_0$  and volume  $V_0$ ; the piston is now left free. Find the new pressure and volume for the isothermal and adiabatic process ( $\gamma = 1.5$ ).

**SOLUTION**

Final pressure will be same on both sides. Let it be  $P$ , with volume  $V$ , on the left side and  $(6V_0 - V)$  on the right side.

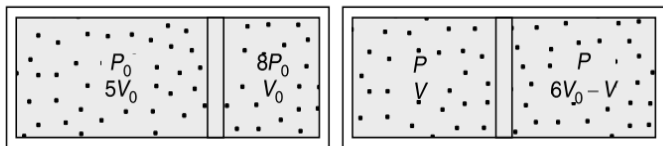
**CASE-1: Isothermal Process**

For the gas enclosed in the left chamber,

$$P_0 \times 5V_0 = PV \quad \dots(1)$$

while for the gas in the right chamber,

$$8P_0 \times V_0 = P(6V_0 - V) \quad \dots(2)$$



After solving, we get

$$V = \frac{30}{13}V_0$$

$$\Rightarrow P = \frac{13}{6}P_0$$

$$\Rightarrow (6V_0 - V) = \frac{48}{13}V_0$$

**CASE-2: Adiabatic Process**

$$P_0 (5V_0)^\gamma = P(V)^\gamma \quad \dots(3)$$

and for the gas in the right chamber,

$$8P_0 (V_0)^\gamma = P(6V_0 - V)^\gamma \quad \dots(4)$$

Dividing (4) by (3), we get

$$\left(\frac{6V_0 - V}{V}\right)^\gamma = \frac{8}{5^\gamma}$$

$$\Rightarrow \frac{6V_0}{V} = 1 + \frac{4}{5} \text{ i.e., } V = \frac{10}{3}V_0$$

Substituting it in equation (3),

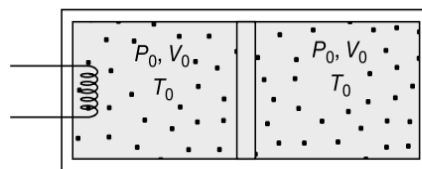
$$P = P_0 \left(\frac{5V_0 \times 3}{10V_0}\right)^{\frac{3}{2}} = \frac{3\sqrt{3}}{2\sqrt{2}}P_0 = 1.84P_0$$

$$\Rightarrow P = 1.84P_0; V = \left(\frac{10}{3}\right)V_0$$

$$\Rightarrow (6V_0 - V) = \left(\frac{8}{3}\right)V_0$$

**ILLUSTRATION 98**

A rectangular box as shown in the figure has a partition which can slide without friction along the length of the box. Initially each of the two chambers of the box has one mole of monatomic ideal gas ( $\gamma = \frac{5}{3}$ ) at a pressure  $P_0$ , volume  $V_0$  and temperature  $T_0$ . The chamber on the left is slowly heated by an electric heater.



The walls of the box and the partition are thermally insulated. The gas in the left chamber expands, pushing the partition until the final pressure in both chambers becomes  $\left(\frac{243}{32}\right)P_0$ . Calculate

- the final temperature of the gas in each chamber
- the work done by the gas in the right chamber

**SOLUTION**

- As no heat is given to the right chamber and it is thermally insulated, so the change in the right chamber is adiabatic. And if  $V_R$  is the final volume of the gas in the right chamber,

$$P_0 V_0^{5/3} = \left(\frac{243}{32}\right)P_0 V_R^{5/3}$$

$$\Rightarrow V_R = \frac{8}{27}V_0$$

Now applying the gas equation to the gas enclosed in the right chamber

(before and after compression),

$$\frac{P_0 V_0}{T_0} = \left(\frac{243}{32}\right)P_0 \times \left(\frac{8}{27}\right)V_0 \times \frac{1}{T_R}$$

$$\Rightarrow T_R = \frac{9}{4}T_0$$

Now for the gas enclosed in the left side, the final volume

$$V_L = 2V_0 - V_R = 2V_0 - \frac{8}{27}V_0 = \frac{46}{27}V_0$$

And from the gas equation

$$\left(\frac{P_0 V_0}{T_0}\right) = \left(\frac{243}{32}\right)P_0 \times \left(\frac{46}{27}\right)V_0 \times \frac{1}{T_L}$$

$$\Rightarrow T_L = \frac{207}{16}T_0 = 13T_0$$

- Work done by the gas in the right chamber is under adiabatic condition.

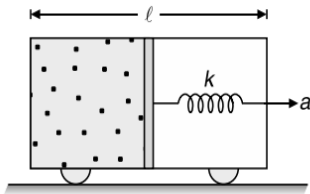
$$W_A = \frac{nR(T_F - T_I)}{1 - \gamma} = \frac{1(8.3)(T_0 - 9T_0/4)}{(5/3) - 1}$$

$$\Rightarrow W_A = -\frac{3}{2} \times 8.3 \times \frac{5}{4} T_0 = -15.5 T_0 \text{ J}$$

Negative sign means that work is done on the gas.

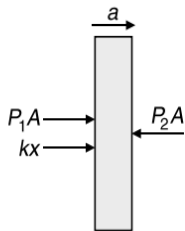
### ILLUSTRATION 99

An adiabatic piston of mass  $m$  equally divides an insulator container of volume  $V_0$  and length  $l$ . A light spring connects the piston to the right wall. Mass of the piston is  $m$ . In equilibrium pressure on both sides of the piston is  $P_0$ . The container starts moving with acceleration  $a$  towards the right. Find the stretch  $x$  of the spring when acceleration of the piston equals acceleration of container. (Assume that  $x \ll l$ ). The gas in the container has the adiabatic exponent (ratio of  $C_p$  and  $C_v$ )  $\gamma$ .



### SOLUTION

Free body diagram of piston is shown in figure below.



From Newton's Second Law, we get

$$P_1 A + kx - P_2 A = ma \quad \dots(1)$$

where  $A$  is the area of cross-section of the piston and  $x$  is the stretch of the spring.

Under adiabatic conditions, we have

$$P_0 \left(\frac{V_0}{2}\right)^\gamma = P_1 \left(\frac{V_0}{2} - Ax\right)^\gamma \quad \dots(2)$$

$$\text{and } P_0 \left(\frac{V_0}{2}\right)^\gamma = P_2 \left(\frac{V_0}{2} + Ax\right)^\gamma \quad \dots(3)$$

From Equation (2), we get

$$P_1 = P_0 \left(\frac{V_0}{V_0 - 2Ax}\right)^\gamma = P_0 \left(1 - \frac{2x}{l}\right)^{-\gamma} \quad \{\because V_0 = Al\}$$

Since,  $x \ll l$ , so we get

$$\left(1 - \frac{2x}{l}\right)^{-\gamma} \cong 1 + \frac{2\gamma x}{l}$$

$$\Rightarrow P_1 = P_0 \left(1 + \frac{2\gamma x}{l}\right)$$

$$\text{Similarly, } P_2 = P_0 \left(1 - \frac{2\gamma x}{l}\right)$$

Substituting the values of  $P_1$  and  $P_2$  in Equation (1), we get

$$P_0 A \left(1 + \frac{2\gamma x}{l}\right) + kx - P_0 A \left(1 - \frac{2\gamma x}{l}\right) = ma \quad \dots(4)$$

$$\text{Since, } V_0 = Al, \text{ so } A = \frac{V_0}{l}$$

Substituting  $A = \frac{V_0}{l}$  in Equation (4), we get

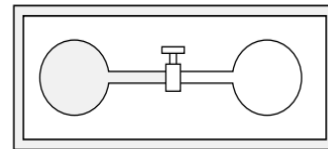
$$\frac{P_0 V_0}{l} \left(1 + \frac{2\gamma x}{l}\right) + kx - \frac{P_0 V_0}{l} \left(1 - \frac{2\gamma x}{l}\right) = ma$$

$$\left(\frac{4\gamma P_0 V_0}{l^2}\right)x + kx = ma$$

$$\Rightarrow x = \frac{ma}{k + \frac{4\gamma P_0 V_0}{l^2}}$$

### ADIABATIC FREE EXPANSION

We now consider what happens when a gas is allowed to expand adiabatically without doing any work. Figure shows two vessels connected by a tube with a stopcock. Initially, one vessel is filled with gas while the other is evacuated. The system is thermally insulated, that is,  $Q = 0$ .



When the stopcock is opened the gas quickly expands to fill the second chamber. The uncontrolled expansion is not quasistatic and cannot be depicted on a  $PV$  diagram. Since the gas does no work,  $W = 0$ . From the First Law we conclude that

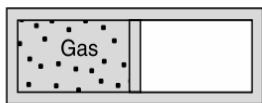
$$\Delta U = 0$$

In an adiabatic free expansion the internal energy of any gas (ideal or real) does not change.

### ILLUSTRATION 100

A cylindrical container of volume  $V$  whose walls are adiabatic is taken. Initially a light adiabatic piston divides the container in two equal parts as shown in Figure. In the left part,  $n$  moles of an ideal gas with adiabatic exponent  $\gamma$  is filled at temperature  $T_0$  and in the right part there is vacuum. If the piston is released, the gas fills the whole container. Calculate the final pressure and temperature of gas. Now if the piston is slowly displaced externally

back to its initial position, then calculate the final pressure and temperature of gas.



**SOLUTION**

When the piston is released the gas expands to fill the complete volume of container. Since there is nothing in the other part of cylinder, so this is the case of free expansion of gas. Hence no work is done by the gas.

Also, the container is thermally insulated from surroundings, so the gas temperature remains constant and hence according to Boyle's Law, as the volume of gas is doubled, then its final pressure is reduced to half. So, we have

$$P_f = \frac{P_i}{2}$$

Initial pressure  $P_i$  of gas is given by applying gas law, so

we get  $P_i = \frac{nRT_0}{V/2}$  and  $P_f = \frac{P_i}{2} = \frac{nRT_0}{V}$ .

Now if piston is displaced back to its initial position, then the process is an adiabatic compression of gas in which gas volume decreases to half i.e.,  $\frac{V}{2}$ .

For an adiabatic process, we have

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

where  $P_1 = \frac{nRT_0}{V}$ ,  $V_1 = V$  and  $V_2 = \frac{V}{2}$

$$\Rightarrow P_2 = P_1 \left( \frac{V_1}{V_2} \right)^\gamma = P_1 (2)^\gamma = (2)^\gamma \left( \frac{nRT_0}{V} \right)$$

Similarly, for an adiabatic process, gas volume and temperature in different states are related as

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

$$\Rightarrow T_2 = T_1 \left( \frac{V_1}{V_2} \right)^{\gamma-1}$$

where  $T_1 = T$ ,  $V_1 = V$  and  $V_2 = \frac{V}{2}$

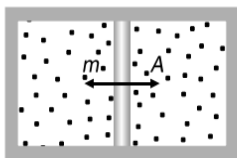
$$\Rightarrow T_2 = T_0 (2)^{\gamma-1}$$

**Test Your Concepts-VII**

**Based on Adiabatic Process**

**(Solutions on page H.89)**

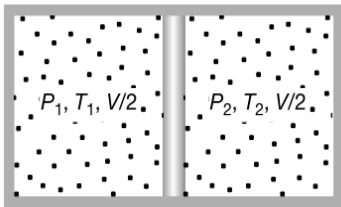
- In a cylinder filled with an ideal gas and closed from both ends there is a piston of mass  $m$  and cross-sectional area  $A$ . In equilibrium position the piston divides the cylinder into two equal parts, each with volume  $V_0$ . The gas pressure is  $P_0$ . The piston is slightly displaced from the equilibrium position and released. Find the time period of the oscillation, assuming the processes in the gas to be adiabatic and friction negligible. (Take  $\frac{C_p}{C_v} = \gamma$ )



- One mole of a gas is isothermally expanded at  $27^\circ\text{C}$  till the volume of doubled. Then it is adiabatically compressed to its original volume. Calculate the total work done if  $\gamma = 1.4$ ,  $R = 25/3 \text{ J mol}^{-1}\text{K}^{-1}$ ,  $\ln(2) \approx 0.7$  and  $(2)^{0.4} \approx 1.3$ .
- Two moles of a gas  $\left(\gamma = \frac{5}{3}\right)$  are initially at temperature  $27^\circ\text{C}$  and occupy a volume of 20 litres. The gas is first expanded at constant pressure until the volume is doubled. Then it is subjected to an adiabatic change until the temperature returns to its initial value.

- Sketch the process on a  $P$ - $V$  diagram.
  - What are final volume and pressure of the gas.
  - What is the work done by the gas.
- One litre of an ideal gas ( $\gamma = 1.5$ ) at 300 K temperature and  $10^5 \text{ Pa}$  pressure, is suddenly compressed to half of its original volume. Calculate the final temperature of the gas. The gas is then cooled isobarically to 300 K and then it is expanded isothermally to achieve its original volume of 1 litre. Calculate the work done by the gas in each process and also calculate the total work done in the cycle. Take  $2 \approx (1.41)^2$  and  $\ln 2 \approx 0.7$
  - There are two vessels, each of them containing one mole of an ideal monoatomic gas. Initial volume of each gas in each vessel is  $8.3 \times 10^{-3} \text{ m}^3$  at  $27^\circ\text{C}$ . Equal amount of heat is supplied to each vessel. In one of the vessels, the volume of gas is doubled without change in its internal energy, whereas the volume of gas is held constant in the other vessel. The vessels are now connected to allow free mixing of the gas. Find the final temperature and pressure of the combined gas system.
  - One mole of oxygen, initially at temperature  $T = 300 \text{ K}$  is compressed adiabatically so that its pressure increases  $\eta = 10$  times. Find the final temperature and work done on it. Take  $(10)^{1/7} \approx 1.4$  and  $R = 8.3 \text{ J mol}^{-1}\text{K}^{-1}$ .

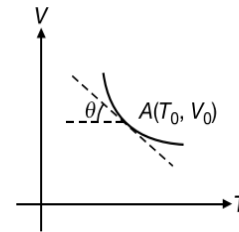
7. On a  $P$ - $V$  diagram starting from an initial state  $(P_0, V_0)$  plot an adiabatic expansion to  $2V_0$ , an isothermal expansion to  $2V_0$  and an isobaric expansion to  $2V_0$ .
- (a) Use this graph to determine in which process the least work is done by the system.
- (b) Plot the same processes on a  $P$ - $T$  diagram starting from  $(P_0, T_0)$ .
- Assume the gas to be a monatomic gas.
8. During the adiabatic expansion of 2 moles of a gas, the change in internal energy was found to be  $-100$  J. Calculate the work done by the gas in the process.
9. The figure shows an insulated cylinder of volume  $V$  containing monatomic gas in both the compartments. The piston is diathermic.



- (a) Initially the piston is kept fixed and the system is allowed to acquire a state of thermal equilibrium.

If the initial pressures and temperatures are as shown, calculate the final temperature and the final pressure.

- (b) Now the pin which was keeping the piston fixed is removed and the piston set free to move. The piston is allowed to slide slowly, such that a state of mechanical equilibrium is also achieved ( $T_1P_2 > P_1T_2$ ). Calculate the final volume of gas in each compartment.
10. The volume of an ideal diatomic gas with  $\gamma = 1.5$  is changed adiabatically from 16 litre to 12 litre. Find the ratio of the final and initial pressure and temperatures.
11. A gas is undergoing an adiabatic process. At a certain stage A, the values of volume and temperature are  $(V_0, T_0)$ . From the details given in the graph, find the value of  $C_p$  and  $C_v$ .



## POLYTROPIC PROCESS

A process in which all  $P$ ,  $V$  and  $T$  change simultaneously such that  $C$  is a non-zero constant is called a polytropic process. The equation of for a polytropic process is given by

$$PV^x = \text{constant} \quad \dots(1)$$

where ( $x \neq 1$  or  $\gamma$ ) is a polytropic process.

For an ideal gas, we have  $P = \frac{RT}{V}$ , so equation (1) can be re-written as

$$\left(\frac{nRT}{V}\right)V^x = \text{constant}$$

$$\Rightarrow TV^{x-1} = \text{constant}$$

Similarly, we have  $v = \frac{nRT}{P}$  and hence equation (1) can again be written as

$$T^x P^{1-x} = \text{constant}$$

## MOLAR SPECIFIC HEAT OF A POLYTROPIC PROCESS

Since, we know that  $C = \frac{dQ}{dT}$  (for 1 mole of gasd)

$$\Rightarrow C = \frac{dU + dW}{dT} \quad \{\because dQ = dU + dW\}$$

$$\Rightarrow C = \frac{dU}{dT} + P \frac{dV}{dT} = C_V + P \frac{dV}{dT} \quad \dots(1)$$

Since,  $TV^{x-1} = \text{constant}$

Taking derivative w.r.t.  $T$ , we get

$$T(x-1)V^{x-2} \frac{dV}{dT} + V^{x-1} = 0$$

$$\Rightarrow (x-1) \frac{T}{V} \frac{dV}{dT} + 1 = 0$$

$$\Rightarrow \frac{dV}{dT} = - \left[ \frac{V}{T(x-1)} \right]$$

Substituting in equation (1), we get

$$C = C_V + P \left[ - \frac{V}{T(x-1)} \right]$$

$$\Rightarrow C = C_V - \frac{PV}{T(x-1)}$$

$$\Rightarrow C = C_V - \frac{RT}{(x-1)T} \quad \{\text{where } T \neq 0\}$$

$$\Rightarrow C = C_V + \frac{R}{1-x}$$

### Problem Solving Technique(s)

A polytropic process is a kind of general process having constant specific heat. It can also be used to denote many special process like

- (i) For Isothermal Process,  $PV = \text{constant}$   
So,  $x = 1$  and hence  $C \rightarrow \infty$
- (ii) For Adiabatic Process,  $PV^\gamma = \text{constant}$   
So,  $x = \gamma$  and hence  $C = 0$
- (iii) For Isochoric Process,  $V = \text{constant}$   
So,  $x \rightarrow \infty$  and hence  $C = C_V$
- (iv) For Isochoric Process,  $P = \text{constant}$   
So,  $x = 0$  and hence  $C = C_V + R = C_p$

### WORK DONE IN A POLYTROPIC PROCESS

The equation for a polytropic process is

$$PV^x = \text{constant}$$

So, work done in a polytropic process is

$$W_{\text{poly}} = \int_{V_1}^{V_2} PdV = \int_{V_1}^{V_2} kV^{-x}dV = k \left( \frac{V^{-x+1}}{-x+1} \right) \Big|_{V_1}^{V_2}$$

$$\Rightarrow W_{\text{poly}} = k \left( \frac{V^{-x+1}}{-x+1} \right) \Big|_{V_1}^{V_2} = \frac{k}{1-x} (V_2^{-x+1} - V_1^{-x+1})$$

$$\Rightarrow W_{\text{poly}} = \frac{1}{1-x} [(kV_2^{-x})V_2 - (kV_1^{-x})V_1] \quad \dots(1)$$

Since  $kV_2^{-x} = P_2$  and  $kV_1^{-x} = P_1$

$$W_{\text{poly}} = \left( \frac{nR}{1-x} \right) \Delta T = \frac{P_2V_2 - P_1V_1}{1-x}, \text{ where } x \neq 1$$

### INDICATOR DIAGRAM FOR A POLYTROPIC PROCESS

For a polytropic process, we have  $PV^x = \text{constant}$

$$\Rightarrow d(PV^x) = 0$$

$$\Rightarrow P(xV^{x-1}dV) + V^x dP = 0$$

$$\Rightarrow Px \left( \frac{V^x}{V} \right) dV + V^x dP = 0$$

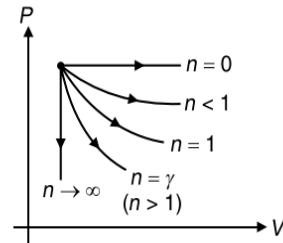
$$\Rightarrow \frac{dP}{dV} = -x \left( \frac{P}{V} \right)$$

$$\text{Since, (Slope)}_{\text{isot}} = -\frac{P}{V}$$

$$\Rightarrow (\text{Slope})_{\text{poly}} = x (\text{Slope})_{\text{isot}}$$

So, for polytropic processes having  $x$  more than 1, the curve will be steeper than the isotherm (i.e., PV curve for isothermal process).

So, for polytropic processes having  $x$  less than 1, the curve will have less slope than the isotherm (i.e., PV curve for isothermal process) as shown in Figure.



### ILLUSTRATION 101

1 mole of an ideal monatomic gas is expanded till the temperature of the gas is doubled under the process  $V^2T = \text{constant}$ . The initial temperature of the gas is 400 K. Calculate the total work done in the process, in terms of  $R$ .

#### SOLUTION

Since,  $T_i = 400$  K and  $T_f = 2T_i = 800$  K

$$\Rightarrow \Delta T = T_f - T_i = 400$$
 K

$$\Rightarrow \Delta U = nC_V \Delta T$$

$$\Rightarrow \Delta U = (1) \left( \frac{3}{2} R \right) (400) = 600R$$

The given process is  $V^2T = \text{constant}$

Substituting  $T = \frac{PV}{R}$ , we get

$$PV^3 = \text{constant}$$

Comparing this equation with equation of a polytropic process i.e.,  $PV^x = \text{constant}$  we observe that  $x = 3$  and so molar heat capacity of polytropic process is

$$C = C_V + \frac{R}{1-x} = \frac{3R}{2} + \frac{R}{1-3} \quad \left\{ \because C_V = \frac{3R}{2} \right\}$$

$$\Rightarrow C = \frac{3}{2}R - \frac{R}{2} = R$$

Since,  $Q = nC\Delta T = (1)(R)(400) = 400R$

Now from First Law, we have

$$W = Q - \Delta U = -200R$$

### Problem Solving Technique(s)

Work done in a polytropic process  $PV^x = \text{constant}$  is

$$W_{\text{poly}} = \left( \frac{nR}{1-x} \right) \Delta T = \frac{P_2V_2 - P_1V_1}{1-x}, \text{ where } x \neq 1$$

Molar specific of the process is

$$C = C_V + \frac{R}{1-x} = \frac{R}{\gamma-1} + \frac{R}{1-x}$$

If  $f$  be the degrees of freedom of the gas, then

$$C_V = \frac{fR}{2}$$

For the above problem, we have  $x = 3$  and the gas is monatomic, so we get

$$W = \left(\frac{1 \times R}{1-3}\right)(400) = -200R \text{ and } C = \frac{3R}{2} + \frac{R}{1-3} = R$$

### ILLUSTRATION 102

An ideal diatomic gas with  $C_V = \frac{5R}{2}$  occupies a volume  $V_i$  at a pressure  $P_i$ . The gas undergoes a process in which the pressure is proportional to the volume. At the end of the process, it is found that the rms speed of the gas molecules has doubled from its initial value. Determine the amount of energy transferred to the gas by heat.

#### SOLUTION

Given that  $P \propto V$ , so  $PV^{-1} = \text{constant}$

As we know, molar heat capacity for the polytropic process  $PV^x = \text{constant}$  is,

$$C = \frac{R}{\gamma-1} + \frac{R}{1-x} = C_V + \frac{R}{1-x}$$

In the given problem, we have

$$C_V = \frac{5R}{2} \text{ and } x = -1, \text{ so we get}$$

$$\Rightarrow C = \frac{5R}{2} + \frac{R}{2} = 3R \quad \dots(1)$$

At the end of the process rms speed is doubled. Since  $v_{\text{rms}} \propto \sqrt{T}$ , so the temperature must have become four times. Since

$$Q = nC\Delta T = nC(T_f - T_i) = nC(4T_i - T_i)$$

$$\Rightarrow \Delta Q = 3nCT_i = 3n(3R)T_i = 9(nRT_i)$$

$$\Rightarrow \Delta Q = 9P_iV_i$$

### ILLUSTRATION 103

A gas undergoes a process such that  $P \propto \frac{1}{T}$ . If the molar heat capacity for this process is  $C = 33.24 \text{ Jmol}^{-1}\text{K}^{-1}$ , find the degrees of freedom of the molecules of the gas.

#### SOLUTION

Since,  $P \propto \frac{1}{T}$

$$\Rightarrow PT = \text{constant} \quad \dots(1)$$

For one mole of an ideal gas, we have

$$PV = RT \quad \dots(2)$$

From Equations (1) and (2), we get

$$P^2V = \text{constant}$$

$$\Rightarrow PV^{\frac{1}{2}} = \text{constant} \quad \dots(3)$$

In the polytropic process  $PV^x = \text{constant}$ , the molar heat capacity is given by

$$C = \frac{R}{\gamma-1} + \frac{R}{1-x}$$

The given process is given by

$$PV^{\frac{1}{2}} = \text{constant}$$

$$\Rightarrow x = \frac{1}{2}$$

$$\Rightarrow C = \frac{R}{\gamma-1} + \frac{R}{1-\frac{1}{2}} = \frac{R}{\gamma-1} + 2R$$

Since  $C = 33.24 \text{ Jmol}^{-1}\text{K}^{-1}$

$$\Rightarrow 33.24 = R\left(\frac{1}{\gamma-1} + 2\right) = 8.31\left(\frac{1}{\gamma-1} + 2\right)$$

Solving this we get

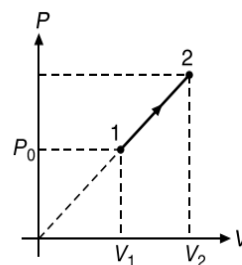
$$\gamma = 1.5$$

Since,  $\gamma = 1 + \frac{2}{f}$

So, degree of freedom is  $f = \frac{2}{\gamma-1} = \frac{2}{1.5-1} = 4$

### ILLUSTRATION 104

$n$  moles of a monatomic ideal gas undergoes a thermodynamic process along a path from 1 to 2 as shown in Figure. The gas pressure at 1 is  $P_0$ . Calculate the amount of heat supplied to gas in this process and work done by the gas in the process if the gas expands from volume  $V_1$  to  $V_2$ .



#### SOLUTION

From the indicator diagram shown in figure, we observe that the  $PV$  curve is a straight line passing through origin, so we have

$$P = kV$$

$$\Rightarrow PV^{-1} = k = \text{constant} \quad \dots(1)$$

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So, we can say that the process is a polytropic with the value of polytropic constant  $x = -1$ . The molar specific heat of the gas is

$$C = C_V + \frac{R}{1-x} = C_V + \frac{R}{1-(-1)}$$

$$\Rightarrow C = \frac{3R}{2} + \frac{R}{2} = 2R \quad \left\{ \because (C_V)_{\text{monatomic}} = \frac{3R}{2} \right\}$$

So, heat supplied is

$$Q = nC\Delta T = nC(T_2 - T_1) = n(2R)(T_2 - T_1) \quad \dots(2)$$

Since  $P = kV$  (from graph) and from ideal gas equation

$$P = \frac{nRT}{V}, \text{ so we get}$$

$$nRT = kV^2$$

$$\text{So, } T_2 = \frac{kV_2^2}{nR} \text{ and } T_1 = \frac{kV_1^2}{nR}$$

Substituting in equation (2), we get

$$Q = 2nR \left( \frac{k}{nR} \right) (V_2^2 - V_1^2) = 2k(V_2^2 - V_1^2) \quad \dots(3)$$

For the given process, from equation (1), we get

$$P_1 = P_0 = kV_1$$

$$\Rightarrow k = \frac{P_0}{V_1}$$

Substituting in equation (3), we get

$$Q = 2P_0 \left( \frac{V_2^2 - V_1^2}{V_1} \right)$$

Work done in a polytropic process is given by

$$W = n(C - C_V)(T_2 - T_1)$$

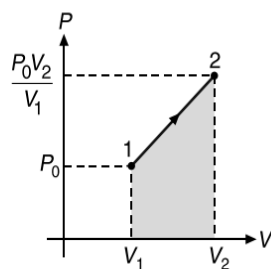
$$\Rightarrow W = n \left( 2R - \frac{3R}{2} \right) \left( \frac{V_2^2 P_0}{nR V_1} - \frac{P_0 V_1}{nR} \right)$$

$$\Rightarrow W = \frac{P_0}{2} \left( \frac{V_2^2 - V_1^2}{V_1} \right)$$

Work done in the process can also be obtained by the calculating the shaded area under the PV graph. So

$$W = \frac{1}{2} \left( P_0 + P_0 \frac{V_2}{V_1} \right) (V_2 - V_1)$$

$$\Rightarrow W = \frac{P_0}{2V_1} (V_2^2 - V_1^2)$$



### ILLUSTRATION 105

One mole of an ideal gas, whose adiabatic exponent equal to  $\gamma$ , is expanded so that the amount of heat transferred to the gas is equal to the decrease in internal energy. Find

- the molar heat capacity of the gas in this process,
- the equation of the process in the variables  $T, V$
- the work performed by one mole of the gas when its volume increases  $\eta$  times if the initial temperature of gas is  $T_0$ .

### SOLUTION

- Since the amount of heat supplied is equal to the decrease in internal energy of gas, so for this process, we have

$$dU = -dQ$$

$$\Rightarrow nC_V dT = -nC dT$$

$$\Rightarrow C = -C_V = -\frac{R}{\gamma - 1} \quad \dots(1)$$

- According to FLTD, we have

$$dQ = dU + dW$$

$$\Rightarrow dQ = -dQ + dW \quad \left\{ \because dU = -dQ \right\}$$

$$\Rightarrow 2dQ = dW$$

$$\text{Since, } dQ = -nC_V dT$$

$$\Rightarrow -\frac{2nR}{\gamma - 1} dT = PdV \quad \dots(2)$$

From ideal gas equation, we have  $PV = nRT$

$$\Rightarrow PdV + VdP = nRdT \quad \dots(3)$$

Substituting value of  $dT$  from equation (3) in (2), we get

$$PdV + VdP = \left( \frac{1-\gamma}{2} \right) PdV$$

$$\Rightarrow \left( \frac{1+\gamma}{2} \right) PdV = -VdP$$

$$\Rightarrow \left( \frac{1+\gamma}{2} \right) \frac{dV}{V} = -\frac{dP}{P}$$

Integrating this equation, we get

$$\left( \frac{1+\gamma}{2} \right) \int \frac{dV}{V} = -\int \frac{dP}{P}$$

$$\Rightarrow \ln V^{\left( \frac{1+\gamma}{2} \right)} = -\ln P + C$$

$$\Rightarrow \ln V^{\left( \frac{1+\gamma}{2} \right)} + \ln P = C$$

$$\Rightarrow PV^{\left( \frac{1+\gamma}{2} \right)} = \text{constant} \quad \dots(4)$$

Since we require process equation in  $T$  and  $V$ , so

substituting  $P = \frac{nRT}{V}$  (from ideal gas equation) in equation (4), we get

$$\left(\frac{nRT}{V}\right)V^{\left(\frac{1+\gamma}{2}\right)} = \text{constant}$$

$$\Rightarrow TV^{\left(\frac{\gamma-1}{2}\right)} = \text{constant} \quad \dots(5)$$

In terms of  $P$  and  $V$ , the process equation (4) can be directly obtained by the molar specific heat of the process by calculating the polytropic constant  $x$ . Since  $C$  is a constant not depending on pressure, volume or temperature of gas thus we can say that this process is a polytropic process whose molar specific heat can be given as

$$C = \frac{R}{\gamma-1} + \frac{R}{1-x} = -\frac{R}{\gamma-1}$$

where  $x$  is the polytropic constant for the process.

$$\Rightarrow \frac{R}{1-x} = -\frac{2R}{\gamma-1}$$

$$\Rightarrow 1-x = -\frac{1}{2}(\gamma-1)$$

$$\Rightarrow 2-2x = 1-\gamma$$

$$\Rightarrow x = \frac{1+\gamma}{2}$$

So, the process equation of this thermodynamic process in  $P$  and  $V$  can simply be written as

$$PV^x = \text{constant}$$

$$\Rightarrow PV^{\left(\frac{1+\gamma}{2}\right)} = \text{constant}$$

which is same as equation (5)

(c) Work done by a gas in a polytropic process is

$$W_{\text{poly}} = \frac{nR}{1-x}(T_2 - T_1), \text{ where } x = \frac{1+\gamma}{2}$$

$$\Rightarrow W_{\text{poly}} = \frac{2nR}{1-\gamma}(T_2 - T_0)$$

Now  $T_2$  is the temperature when volume becomes  $\eta$  times i.e.,  $\eta V_0$ . Since,

$$TV^{\left(\frac{\gamma-1}{2}\right)} = \text{constant}$$

$$\Rightarrow T_2(\eta V_0)^{\left(\frac{\gamma-1}{2}\right)} = T_0 V_0^{\left(\frac{\gamma-1}{2}\right)}$$

$$\Rightarrow T_2 = T_0 \left(\frac{1}{\eta}\right)^{\left(\frac{\gamma-1}{2}\right)} = T_0 \eta^{\left(\frac{1-\gamma}{2}\right)}$$

$$\Rightarrow W = \frac{2R}{1-\gamma} \left[ T_0 (\eta)^{\frac{1-\gamma}{2}} - T_0 \right] \quad \{ \because n = 1 \text{ mole} \}$$

$$\Rightarrow W = \frac{2RT_0}{1-\gamma} \left[ (\eta)^{\frac{1-\gamma}{2}} - 1 \right]$$

### ILLUSTRATION 106

A gas consisting of monatomic molecules (having three degrees of freedom) was expanded in a polytropic process so that the rate of collisions of the molecules against the vessel's wall did not change. Calculate the molar heat capacity of the gas in the process.

### SOLUTION

Since the rate of collisions with vessel wall is

$$N_C = \frac{1}{6} n_0 v_{\text{rms}} = \text{constant}$$

where,  $n_0$  is the number density of molecules.

$$\Rightarrow \frac{1}{6} \left(\frac{N}{V}\right) \sqrt{\frac{3RT}{m}} = \text{constant}$$

$$\Rightarrow \frac{\sqrt{T}}{V} = \text{constant}$$

$$\Rightarrow T = kV^2$$

$$\Rightarrow \frac{dT}{dV} = 2kV$$

Molar specific of the gas is given by

$$C = C_V + \frac{PdV}{ndT}, \text{ where } C_V = \frac{3R}{2}$$

$$\Rightarrow C = \frac{3R}{2} + \left(\frac{RT}{V}\right) \frac{dV}{dT} = \frac{3R}{2} + \frac{R(kV^2)}{V} \frac{1}{2kV}$$

$$\Rightarrow C = \frac{3R}{2} + \frac{R}{2} = 2R$$

## Test Your Concepts-VIII

### Based on Polytropic Process

1. Three moles of a diatomic gas is taken at temperature  $T$ . Its volume is varied according to the law  $V = \alpha T^{-2}$  where  $\alpha$  is a positive constant during the process. If the final temperature of the gas is found to be  $2T$ , find the heat supplied to the gas.

2. An ideal gas has an adiabatic exponent  $\gamma$ . In some process its molar heat capacity varies as  $C = \frac{\alpha}{T}$ , where  $\alpha$  is a constant. Calculate the  
(a) work performed by one mole of the gas during its heating from the temperature  $T_0$  to the temperature  $\eta$  times higher

(Solutions on page H.91)

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- (b) equation of the process in terms of pressure and volume.
- Find the maximum attainable temperature of an ideal gas in a process,  $P = P_0 - \alpha V^2$ , where  $P_0, \alpha$  are constants and  $V$  is the volume of one mole of gas.
  - An ideal gas has a molar heat capacity  $C_V$  at constant volume. Find the molar heat capacity of this gas as a function of its volume  $V$ , if the gas separately undergoes the processes  $T = T_0 e^{\alpha V}$  and  $P = P_0 e^{\alpha V}$ , where  $P_0, T_0$  and  $\alpha$  are positive constants.
  - Consider one mole of an ideal gas whose volume changes with temperature as  $V = \frac{\alpha}{T}$ , where  $\alpha$  is constant. Find the amount of heat required to raise its temperature by  $\Delta T$ , if its adiabatic constant is  $\gamma$ .
  - An ideal gas consisting of rigid diatomic molecules was expanded in a polytropic process so that the rate of collisions of the molecules against the vessel's wall did not change. Calculate molar heat capacity of the gas in this process.
  - Consider one mole of an ideal gas whose pressure changes with volume as  $P = \alpha V$ , where  $\alpha$  is constant. If it is expanded such that its volume increases  $\eta$  times, find the change in its internal energy, work done by the gas and heat capacity of the gas in terms of  $C_V$  and  $R$ .
  - An ideal gas is taken through a process in which the process equation is given as  $P = kV^\alpha$ , where  $k$  and  $\alpha$  are positive constants. Calculate the value of  $\alpha$  for which molar specific heat in this process becomes zero.
  - Two moles of an ideal monatomic gas undergoes the process  $P = \alpha T^{\frac{1}{2}}$ , where  $\alpha$  is a constant. If  $R = 25/3 \text{ J mol}^{-1} \text{ K}^{-1}$ , then calculate
    - the work done by the gas if its temperature increases by 60 K.
    - the molar specific heat of the gas.

### CYCLIC PROCESS

Engines operate in cycles, in which the system – for example, a gas – periodically returns to its initial state. In figure, the system goes from state  $a$  to state  $b$  via path I, for which  $W_I > 0$ , and returns to its initial state via path II, for which  $W_{II} < 0$ . The net work done by the system is the area enclosed by the curve. In a clockwise traversal the network is positive.

Since the system returns to its initial state, the change in internal energy in one complete cycle is zero, that is,  $\Delta U = 0$ .

From the First Law we see that

$$Q = W$$

The net work done by the system in each cycle,  $W = W_I + W_{II}$ , is equal to the net heat input per cycle.

This result is of importance in the discussion of steam engines and diesel engines, for instance, in which the influx of heat is used to perform mechanical work.

The efficiency of a cyclic process is

$$\eta = \frac{W_{\text{total}}}{Q_{\text{input}}} = \frac{W_{\text{total}}}{\sum Q_{\text{positive}}}$$

### EFFICIENCY OF A CYCLIC PROCESS

In a cyclic process, we have

$$\Delta U = 0$$

and hence from First Law, we get  $Q_{\text{net}} = W_{\text{net}}$

Now, let us understand the meaning of efficiency of a cycle. Suppose 100 J of heat is supplied to a system (in our

case it is an ideal gas) and the system does 60 J of work. Then efficiency of the cycle is 60%. Thus, efficiency ( $\eta$ ) of a cycle can be defined as

$$\Rightarrow \eta = \left( \frac{\text{Useful work done}}{\text{Heat supplied}} \right)_{\text{during the cycle}} \times 100\%$$

$$\Rightarrow \eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100\% = \frac{|Q_{+\text{ve}}| - |Q_{-\text{ve}}|}{|Q_{+\text{ve}}|} \times 100\%$$

$$\Rightarrow \eta = \left( 1 - \frac{|Q_{-\text{ve}}|}{|Q_{+\text{ve}}|} \right) \times 100\%$$

$$\text{Thus, } \eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100 = \left( 1 - \frac{|Q_{-\text{ve}}|}{|Q_{+\text{ve}}|} \right) \times 100\%$$

### HEAT ENGINE

A heat engine is a device which converts Thermal energy to other useful forms of energy such as mechanical energy, electrical energy. In other words, any device that transforms heat partly into work or mechanical energy is called a heat engine. Usually, a quantity of matter inside the engine undergoes inflow and outflow of heat, expansion and compression, and sometimes change of phase. We call this matter the working substance of the engine. In internal-combustion engines the working substance is a mixture of air and fuel; in a steam turbine it is water.

The simplest kind of engine to analyse is one in which the working substance undergoes a cyclic process, a sequence of processes that eventually leaves the substance in the same state in which it started. In a steam turbine the water is recycled and used over and over. Internal-combustion engines do not use the same air over and over,

but we can still analyse them in terms of cyclic processes that approximate their actual operation.

All heat engines absorb heat from a source at a relatively high temperature, perform some mechanical work, and discard or reject some heat at a lower temperature. As far as the engine is concerned, the discarded heat is wasted. In internal-combustion engine the waste heat is that discarded in the hot exhaust gases and the cooling system; in a steam turbine it is the heat that must flow out of the used steam to condense and recycle the water.

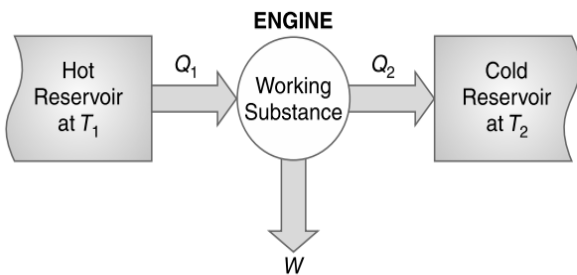
When a system is carried through a cyclic process, its initial and final internal energies are equal. For any cyclic process, the First Law of Thermodynamics requires that

$$U_2 - U_1 = 0 = Q - W$$

$$\Rightarrow Q = W$$

That is, the net heat flowing into the engine in a cyclic process equals the net work done by the engine.

The block diagram of a heat engine is shown in Figure.



When we talk of a heat engine, we happen to think of two bodies with which the working substance of the engine can interact.

One of these, called the hot reservoir, represents the heat source, it can give the working substance large amounts of heat at a constant temperature  $T_1$  without appreciably changing its own temperature. The other body, called the cold reservoir, can absorb large amounts of discarded heat from the engine at a constant lower temperature  $T_2$ .

e.g. In a steam-turbine system the flames and hot gases in the boiler are the hot reservoir, and the cold water and air used to condense and cool the used steam are the cold reservoir.

We denote the quantities of heat transferred from the hot and cold reservoirs to be  $Q_1$  and  $Q_2$ , respectively. Following our sign convention, a quantity of heat  $Q$  is positive when heat is transferred into the working substance and is negative when heat leaves the working substance. Thus, in a heat engine,  $Q_1$  is positive but  $Q_2$  is negative.

When an engine repeats the same cycle over and over,  $Q_1$  and  $Q_2$  represent the quantities of heat absorbed and rejected by the engine during one cycle. The net heat  $Q$  absorbed per cycle is

$$Q = Q_1 - Q_2$$

The useful output of the engine is the net work  $W$  done by the working substance.

From the First Law, we have  $W = Q = Q_1 - Q_2$

Ideally, we would like to convert all the heat  $Q_1$  into work. In that case, we would have  $Q_1 = W$  and  $Q_2 = 0$ . Experience shows that this is impossible; there is always some heat wasted, and  $Q_2$  is never zero. This is also the statement of Second Law of Thermodynamics (discussed afterwards).

By Law of Conservation of Energy, we get

$$Q_1 = Q_2 + W$$

$$\Rightarrow W = Q_1 - Q_2$$

$$\Rightarrow \eta = \frac{W}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

Since  $\frac{Q_2}{Q_1} = \frac{T_2}{T_1}$

$$\Rightarrow \eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}$$

According to this, efficiency is 100% if  $Q_2 = 0$ , that is, no heat is rejected to the cold reservoir or sink that is the entire heat absorbed must be converted to mechanical work, which according to Second Law of Thermodynamics is impossible.

In practice, heat engines convert only a fraction of absorbed heat into mechanical work. For example, a good automobile engine has an efficiency of about 20% and diesel engines have efficiencies ranging from 35% to 40%.

On the basis of this fact, the Kelvin-Planck form of Second Law of Thermodynamics states that "no heat engine, operating in a cycle, can absorb thermal energy from a reservoir and perform an equal amount of work. This is equivalent to say that is impossible to construct a perpetual motion machine."

### ILLUSTRATION 107

A Carnot engine, whose efficiency is 40%, takes in heat from a source maintained at a temperature of 500 K. It is desired to have an engine of efficiency 60%, then calculate the intake temperature for the same exhaust temperature.

### SOLUTION

Efficiency of Carnot engine,  $\eta = 1 - \frac{T_2}{T_1}$

where  $T_1$  is the temperature of the source and  $T_2$  is the temperature of the sink.

For 1<sup>st</sup> case,  $\eta = 40\%$ ,  $T_1 = 500$  K

$$\Rightarrow \frac{40}{100} = 1 - \frac{T_2}{500}$$

$$\Rightarrow T_2 = \frac{3}{5} \times 500 = 300 \text{ K}$$

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For 2<sup>nd</sup> case,  $\eta = 60\%$ ,  $T_2 = 300$  K

$$\Rightarrow \frac{60}{100} = 1 - \frac{300}{T_1}$$

$$\Rightarrow T_1 = \frac{5}{2} \times 300 = 750 \text{ K}$$

### REFRIGERATOR OR HEAT PUMP

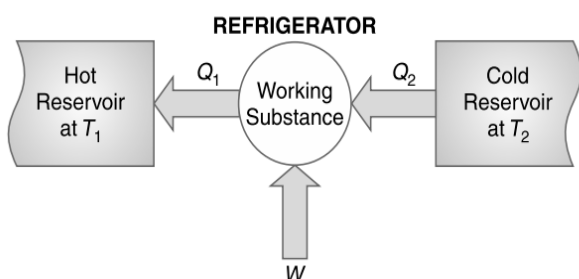
It is a heat Engine running in reverse.

According to Law of Conservation of Energy

$$Q_1 = Q_2 + W$$

Coefficient of performance

$$\beta = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} \quad (\beta > 1)$$



In practice it is desirable to carry out this process with minimum amount of work. If it could be accompanied without doing any work, we would have a “**perfect refrigerator**” which is again in violation with Second Law of Thermodynamics

*On the basis of this fact, the Rudolph-Clausius form of Second Law of Thermodynamics states that “it is impossible to construct a cyclical machine that produces no other effect than to transfer heat continuously from one body at lower temperature to another body at a higher temperature.”*

### RELATION BETWEEN $\eta$ AND $\beta$

$$\text{Since } \eta = 1 - \frac{T_2}{T_1} \quad \dots(1)$$

$$\Rightarrow \eta = \frac{T_1 - T_2}{T_1} \quad \dots(2)$$

$$\text{Also, } \beta = \frac{T_2}{T_1 - T_2} \quad \dots(3)$$

Multiplying (2) and (3), we get

$$\eta\beta = \frac{T_2}{T_1} = 1 - \eta \quad \{\because \text{ of (1)}\}$$

$$\Rightarrow \eta(\beta + 1) = 1$$

$$\Rightarrow \eta = \frac{1}{1 + \beta}$$

For  $\eta$  to be MAXIMUM,  $\beta$  must be MINIMUM i.e., 1

$$\Rightarrow \eta_{\max} = \frac{1}{1+1} = \frac{1}{2} = 50\%$$

### ILLUSTRATION 108

If minimum possible work is done by a refrigerator in converting 100 g of water at 0 °C to ice, how much heat is released to the surrounding at temperature 27 °C. Take latent heat of ice to be 80 calg<sup>-1</sup>.

### SOLUTION

Since  $\frac{Q_2}{W} = \frac{T_2}{T_1 - T_2}$ , where,  $Q_2 = mL_{\text{ice}} = 8000$  cal

Also,  $T_2 = 273$  K and  $T_1 = 300$  K

$$\Rightarrow \frac{8000}{W} = \frac{273}{300 - 273} = \frac{273}{27}$$

$$\Rightarrow W = 791.2 \text{ cal}$$

$$\Rightarrow Q_1 = Q_2 + W = 791.2 + 8000 = 8791.2 \text{ cal}$$

### ILLUSTRATION 109

A Carnot engine having an efficiency of 10% is being used as a refrigerator. If the work done on the refrigerator is 10 J, then calculate the amount of heat absorbed from the reservoir at lower temperature.

### SOLUTION

$$\text{Since, } \eta = \frac{1}{10} = \frac{1}{1 + \beta}$$

Also by definition, we have

$$\beta = \frac{Q_2}{W} = 9$$

$$\Rightarrow Q_2 = 9W = 90 \text{ J}$$

### REVERSIBLE PROCESS

A process which can be retraced back in the opposite direction in such a way that the system passes through the same states as in direct process and finally the system and the surroundings acquire the initial conditions.

### CONDITIONS FOR A PROCESS TO BE REVERSIBLE

- The change must take place at a very slow rate.
- There should be no loss of energy due to conduction, convection or dissipation of energy against any resistance, like friction, viscosity etc.
- No heat should be converted into magnetic or electric energy.
- The system must always be in thermal and chemical equilibrium with the surroundings.

**EXAMPLE:**

- (a) Fusion of ice is a reversible process.
- (b) Vaporisation of water is a reversible process.
- (c) Temperature of two junctions of a thermocouple get reversed on reversing the direction of current in it. Thus Peltier effect a reversible process.
- (d) Gradual compression and extension of a spring is a reversible process.

**IRREVERSIBLE PROCESS**

The process which cannot be retraced back in the opposite direction is defined as irreversible process. The system does not pass through the same intermediate states as in the direction process. Almost all processes of nature are irreversible.

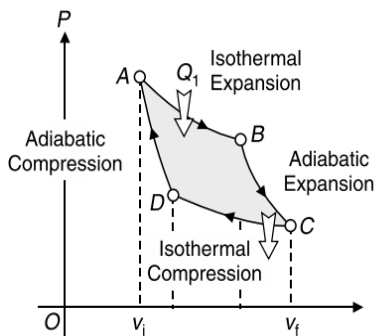
**EXAMPLE:**

- (a) Work done against friction.
- (b) Joule heating effect i.e. heat produced in condition by an electric current.
- (c) Diffusion of gases into one another.
- (d) Magnetic hysteresis.

**CARNOT ENGINE/CYCLE**

Carnot devised an ideal engine which is based on a reversible cycle of four operations in succession.

- (a) Isothermal Expansion
- (b) Adiabatic Expansion
- (c) Isothermal Compression
- (d) Adiabatic Compression



If  $T_1$  and  $T_2$  are absolute temperatures of source and sink, then

Work done in Carnot cycle

$$W = \text{Area enclosed by cycle } ABCDA$$

Also  $W = R(T_1 - T_2) \log_e \left( \frac{V_f}{V_i} \right)$  {for 1 mole of gas}

Heat absorbed from source

$$Q_1 = W_1 = RT_1 \log_e \left( \frac{V_f}{V_i} \right)$$

Since efficiency,  $\eta = \frac{W}{Q_1}$

$$\Rightarrow \eta = \frac{R(T_1 - T_2) \log_e \frac{V_f}{V_i}}{RT_1 \log_e \frac{V_f}{V_i}} = \frac{T_1 - T_2}{T_1} = 1 - \frac{T_2}{T_1}$$

So, Efficiency of Carnot Engine,

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}$$

**Conceptual Note(s)**

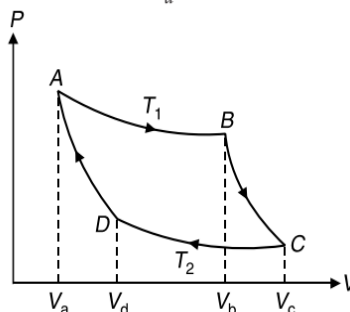
In a Carnot cycle  $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

If an engine consists of a number of sources and sinks, then

$$\left( \sum \frac{Q}{T} \right)_{\text{source}} = \left( \sum \frac{Q}{T} \right)_{\text{sink}}$$

**ILLUSTRATION 110**

Two different adiabatic parts for the same gas intersects two isotherms at  $T_1$  and  $T_2$  as shown in the  $P$ - $V$  diagram in the figure. How does the ratio  $\frac{V_a}{V_d}$  compare with the ratio of  $\frac{V_b}{V_c}$ ?



**SOLUTION**

For adiabatic change  $TV^{\gamma-1} = \text{constant}$

For adiabatic process BC

$$T_1 V_b^{\gamma-1} = T_2 V_c^{\gamma-1} \quad \dots(1)$$

and for adiabatic process DA,

$$T_1 V_a^{\gamma-1} = T_2 V_d^{\gamma-1} \quad \dots(2)$$

Dividing equation (2) by (1),

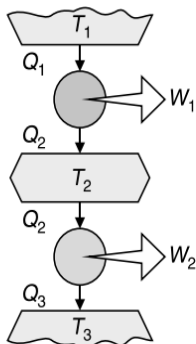
$$\left( \frac{V_a}{V_b} \right)^{\gamma-1} = \left( \frac{V_d}{V_c} \right)^{\gamma-1}$$

$$\Rightarrow \frac{V_a}{V_d} = \frac{V_b}{V_c}$$

Hence, both ratios are same.

### HEAT ENGINES IN SERIES

Let there be three reservoirs at temperatures  $T_1$ ,  $T_2$  and  $T_3$  ( $T_1 > T_2 > T_3$ ) as shown in Figure.



There are two heat engines in this. The

- first engine working between temperatures  $T_1$  and  $T_2$ , and
- second engine working between temperatures  $T_2$  and  $T_3$ .

Let  $Q_1$  be heat taken by first engine from a reservoir ( $T_1$ ). If second engine takes heat  $Q_2$  rejected by first heat engine; then the two engines are said to be in series. Let  $W_1$  and  $W_2$  be work done per cycle by engines. Then the efficiency of combination

$$\eta = \frac{W_1 + W_2}{Q_1} = 1 - \frac{T_3}{T_1}$$

#### ILLUSTRATION 111

Two Carnot engines  $A$  and  $B$  are operated in series. The first one,  $A$ , receives heat at  $T_1 = 600$  K and rejects to a reservoir at temperature  $T_2$ . The second engine  $B$  receives heat rejected by the first engine and, in turn, rejects to a heat reservoir at  $T_3 = 400$  K. Calculate the temperature  $T_2$  if the work outputs of the two engines are equal.

#### SOLUTION

Since,  $\frac{Q_1}{T_1} = \frac{Q_2}{T_2} = \frac{Q_3}{T_3}$

Also,  $W_1 = Q_1 - Q_2$  and  $W_2 = Q_2 - Q_3$

Given that  $W_1 = W_2$ , so we have

$$\Rightarrow Q_1 - Q_2 = Q_2 - Q_3$$

$$\Rightarrow 2Q_2 = Q_1 + Q_3$$

$$\Rightarrow 2T_2 = T_1 + T_3$$

$$\Rightarrow T_2 = \frac{T_1 + T_3}{2} = 500 \text{ K}$$

#### ILLUSTRATION 112

Three Carnot engines operate in series between a heat source at a temperature  $T_1$  and a heat sink at temperature  $T_4$ . There are two other reservoirs at temperature  $T_2$  and  $T_3$

such that  $T_1 > T_2 > T_3 > T_4$ . If the three engines are equally efficient, then calculate  $T_2$  and  $T_3$  in terms of  $T_1$  and  $T_4$ .

#### SOLUTION

Since all engines have equal efficiency, so we have

$$\begin{aligned} \eta &= 1 - \frac{T_2}{T_1} = 1 - \frac{T_3}{T_2} = 1 - \frac{T_4}{T_3} \\ \Rightarrow \frac{T_2}{T_1} &= \frac{T_3}{T_2} = \frac{T_4}{T_3} \end{aligned} \quad \dots(1)$$

$$\Rightarrow T_2 = \sqrt{T_1 T_3} \text{ and } T_3 = \sqrt{T_2 T_4}$$

Substituting value of  $T_2$  in equation (1), we get

$$\Rightarrow \frac{\sqrt{T_1 T_3}}{T_1} = \frac{T_4}{T_3}$$

$$\Rightarrow \frac{T_1 T_3}{T_1^2} = \frac{T_4}{T_3}$$

$$\Rightarrow T_3^3 = T_1 T_4^2$$

$$\Rightarrow T_3 = (T_1 T_4^2)^{1/3}$$

Similarly, we get  $T_2 = (T_4 T_1^2)^{1/3}$

#### ILLUSTRATION 113

An ideal gas is taken through a cyclic thermodynamic process through four steps. The amounts of heat involved in these steps are  $Q_1 = 5960$  J,  $Q_2 = -5585$  J,  $Q_3 = -2980$  J and  $Q_4 = 3645$  J respectively. The corresponding quantities of work involved are  $W_1 = 2200$  J,  $W_2 = -825$  J,  $W_3 = -1100$  and  $W_4$  respectively. Find the value of  $W_4$ . What is the efficiency of the cycle?

#### SOLUTION

In a cyclic process  $dU = 0$

Therefore,  $dQ = dW$

$$\Rightarrow Q_1 + Q_2 + Q_3 + Q_4 = W_1 + W_2 + W_3 + W_4$$

Hence,  $W_4 = (Q_1 + Q_2 + Q_3 + Q_4) - (W_1 + W_2 + W_3)$

$$\Rightarrow W_4 = (5960 - 5585 - 2980 + 3645) - (2200 - 825 - 1100)$$

$$\Rightarrow W_4 = 765 \text{ J}$$

Efficiency,  $\eta = \frac{W_{\text{total}}}{\Sigma Q_{\oplus}} \times 100\%$

$$\Rightarrow \eta = \left\{ \frac{W_1 + W_2 + W_3 + W_4}{Q_1 + Q_4} \right\} \times 100$$

$$\Rightarrow \eta = \left\{ \frac{(2200 - 825 - 1100 + 765)}{5960 + 3645} \right\} \times 100$$

$$\Rightarrow \eta = \frac{1040}{9605} \times 100 = 10.82\%$$

Since, from Conservation of Energy, we have

$$W_{net} = Q_{+ve} - Q_{-ve} \text{ (in a cycle)}$$

$$\Rightarrow \eta = \frac{W_{net}}{Q_{+ve}} \times 100 = \frac{(Q_{+ve} - Q_{-ve})}{Q_{+ve}} \times 100$$

$$\Rightarrow \eta = \left(1 - \frac{Q_{-ve}}{Q_{+ve}}\right) \times 100$$

In the above question

$$Q_{-ve} = |Q_2| + |Q_3| = (5585 + 2980) \text{ J} = 8565 \text{ J}$$

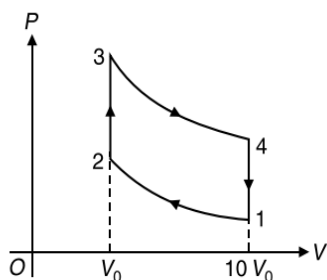
$$\text{and } Q_{+ve} = Q_1 + Q_4 = (5960 + 3645) \text{ J} = 9605 \text{ J}$$

$$\Rightarrow \eta = \left(1 - \frac{8565}{9605}\right) \times 100$$

$$\Rightarrow \eta = 10.82\%$$

### ILLUSTRATION 114

One mole of oxygen undergoes a cyclic process in which volume of the gas changes 10 times within the cycle, as shown in the figure. Processes: 1-2 and 3-4 are adiabatic, 2-3 and 4-1 are isochoric. Find the efficiency of the process.



### SOLUTION

$$W_{12} = \frac{R}{1-\gamma}(T_2 - T_1)$$

$$\text{Since, } T_2 = V_2^{\gamma-1} = T_1 V_1^{\gamma-1}$$

$$\Rightarrow T_2 = T_1 \alpha^{\gamma-1} \text{ where } \alpha = \frac{V_1}{V_2}$$

$$\text{Thus, } W_{12} = \frac{RT_2}{1-\gamma} \left(1 - \frac{1}{\alpha^{\gamma-1}}\right)$$

$$\text{Similarly, } T_3 = T_4 \alpha^{\gamma-1}$$

$$\text{and } W_{34} = \frac{RT_3}{1-\gamma} \left(\frac{1}{\alpha^{\gamma-1}} - 1\right)$$

$$\text{Also } W_{23} = 0, W_{41} = 0$$

$$\Rightarrow W_{net} = W_{12} + W_{23} + W_{34} + W_{41}$$

$$\Rightarrow W_{net} = \frac{R(T_2 - T_3)}{1-\gamma} \left(1 - \frac{1}{\alpha^{\gamma-1}}\right)$$

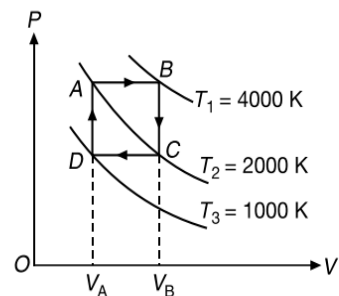
$$Q_{in} = \frac{R}{\gamma-1}(T_3 - T_2) = \frac{R}{1-\gamma}(T_2 - T_3)$$

$$\text{Since } \eta = \frac{W_{net}}{Q_{in}} = 1 - \frac{1}{\alpha^{\gamma-1}}, \text{ where } \alpha = 10, \gamma = 1.4$$

$$\Rightarrow \eta = 60\%$$

### ILLUSTRATION 115

Three isotherms at temperature  $T_1 = 4000 \text{ K}$ ,  $T_2 = 2000 \text{ K}$ ,  $T_3 = 1000 \text{ K}$  are shown in Figure. When one mole of an ideal monatomic gas is taken through the paths AB, BC, CD and DA, calculate the change in internal energy  $\Delta U$ ,



work done by the gas  $W$  and heat  $Q$  absorbed by the gas along each path. Also efficiency for the complete cycle ABCDA if it is given that  $V_A = 1 \text{ m}^3$  and  $V_B = 2 \text{ m}^3$ . Take  $R = 8.3 \text{ Jmol}^{-1}\text{K}^{-1}$

### SOLUTION

Change in its internal energy of one mole of monatomic gas is  $\Delta U = C_V \Delta T$ , where  $C_V$  is the molar specific heat of the gas at constant volume. For monatomic gas,  $C_V = \left(\frac{3}{2}\right)R$ . So, along path BA, we have

$$\Delta U_{AB} = C_V (T_B - T_A)$$

$$\Rightarrow \Delta U_{AB} = \left(\frac{3R}{2}\right)(4000 - 2000) = 3000R$$

$$\Rightarrow \Delta U_{AB} = (3000)(8.3) = 24900 \text{ J}$$

Similarly,  $\Delta U_{BC} = C_V (T_C - T_B)$

$$\Delta U_{BC} = \left(\frac{3R}{2}\right)(2000 - 4000) = -3000R$$

$$\Rightarrow \Delta U_{BC} = -(3000)(8.3) = -24900 \text{ J}$$

$$\Delta U_{CD} = C_V (T_D - T_C)$$

$$\Rightarrow \Delta U_{CD} = \left(\frac{3R}{2}\right)(1000 - 2000) = -1500R$$

$$\Rightarrow \Delta U_{CD} = -(1500)(8.3) = -12450 \text{ J}$$

$$\Delta U_{DA} = C_V (T_A - T_D)$$

$$\Rightarrow \Delta U_{DA} = \left(\frac{3R}{2}\right)(2000 - 1000) = 1500R$$

$$\Rightarrow \Delta U_{DA} = (1500)(8.3) = 12450 \text{ J}$$

So, we observe that for the complete cyclic process

$$\Delta U = 0$$

Work done by gas during path AB at constant pressure is

$$W_{AB} = P(V_2 - V_1) = R(T_1 - T_2)$$

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$$\Rightarrow W_{AB} = (8.3)(2000) = 16600 \text{ J}$$

For both paths  $BC$  and  $DA$ ,  $V = \text{constant}$ , so work done is zero.

Work done by gas during path  $CD$  at constant pressure is

$$W_{CD} = R(T_3 - T_2) = -1000R$$

$$\Rightarrow W_{CD} = -(1000)(8.3) = -8300 \text{ J}$$

Net work done by the gas during the cycle is

$$W = 16600 - 8300 = 8300 \text{ J}$$

For process  $AB$ , we have

$$Q_{AB} = C_P(T_1 - T_2) = \left(\frac{5R}{2}\right)(2000)$$

$$\Rightarrow Q_{AB} = 5000R = 5000(8.3) = 41500 \text{ J}$$

Since,  $Q_{AB}$  is positive, so heat is being absorbed during this process.

For process  $BC$ , we have

$$Q_{BC} = C_V(T_2 - T_1) = \left(\frac{3R}{2}\right)(-2000)$$

$$\Rightarrow Q_{BC} = -3000R = -3000(8.3) = -24900 \text{ J}$$

Since,  $Q_{BC}$  is negative, so heat is being rejected during this process.

For process  $CD$ , we have

$$Q_{CD} = C_P(T_3 - T_2) = \left(\frac{5R}{2}\right)(-1000)$$

$$\Rightarrow Q_{CD} = -2500R = -2500(8.3) = -20750 \text{ J}$$

Since,  $Q_{CD}$  is negative, so heat is being rejected during this process.

For process  $DA$ , we have

$$Q_{DA} = C_V(T_1 - T_3) = \left(\frac{3R}{2}\right)(3000)$$

$$\Rightarrow Q_{DA} = 4500R = 4500(8.3) = 37350 \text{ J}$$

Since,  $Q_{DA}$  is positive, so heat is being absorbed during this process.

Efficiency of the complete cycle is

$$\eta = \frac{W_{\text{total}}}{\Sigma Q_{\oplus}} = \frac{W_{AB} + W_{CD}}{Q_{AB} + Q_{DA}}$$

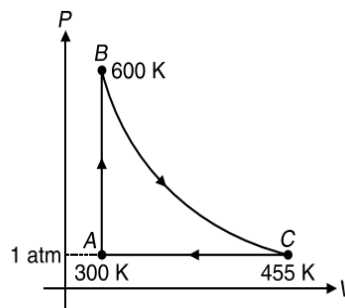
$$\Rightarrow \eta = \left(\frac{8300}{41500 + 37350}\right) \times 100\% = 10.53\%$$

### ILLUSTRATION 116

The  $P$ - $V$  diagram of 0.2 mol of a diatomic ideal gas is shown in figure. Process  $BC$  is adiabatic. The value of  $\gamma$  for this gas is 1.4.

- Find the pressure and volume at points  $A$ ,  $B$  and  $C$ .
- Calculate  $\Delta Q$ ,  $\Delta W$  and  $\Delta U$  for each of the three processes.
- Find the thermal efficiency of the cycle.

Take  $1 \text{ atm} = 1 \times 10^5 \text{ Nm}^{-2}$



### SOLUTION

- $P_A = P_C = 1 \text{ atm} = 1.01 \times 10^5 \text{ Nm}^{-2}$

Process  $AB$  is an isochoric process, so we have

$$P \propto T$$

$$\Rightarrow \frac{P_B}{P_A} = \frac{T_B}{T_A}$$

$$\Rightarrow P_B = \left(\frac{T_B}{T_A}\right)P_A = \left(\frac{600}{300}\right)(1 \text{ atm}) = 2 \text{ atm}$$

$$\Rightarrow P_B = 2.02 \times 10^5 \text{ Nm}^{-2}$$

From ideal gas equation, we get

$$V = \frac{nRT}{P}$$

$$\Rightarrow V_A = V_B = \frac{nRT_A}{P_A}$$

$$\Rightarrow V_A = \frac{(0.2)(8.31)(300)}{(1.01 \times 10^5)} = 5 \times 10^{-3} \text{ m}^3$$

$$\Rightarrow V_A = 5 \text{ litre}$$

$$\text{and } V_C = \frac{nRT_C}{P_C} = \frac{(0.2)(8.31)(455)}{(1.01 \times 10^5)}$$

$$\Rightarrow V_C = 7.6 \times 10^{-3} \text{ m}^3$$

$$\Rightarrow V_C \approx 7.6 \text{ litre}$$

State	$P$	$V$
A	1 atm	5 lt.
B	2 atm	5 lt.
C	1 atm	7.6 lt.

- Process  $AB$  is an isochoric process. Hence,

$$W_{AB} = 0$$

$$\Rightarrow Q_{AB} = \Delta U_{AB} = nC_V\Delta T = n\left(\frac{5}{2}R\right)(T_B - T_A)$$

$$\Rightarrow Q_{AB} = (0.2) \left( \frac{5}{2} \right) (8.31)(600 - 300) \approx 1246 \text{ J}$$

**Process BC** is an adiabatic process. Hence,

$$Q_{BC} = 0$$

$$\Rightarrow W_{BC} = -\Delta U_{BC}$$

Since,  $\Delta U_{BC} = nC_V \Delta T = nC_V (T_C - T_B)$

$$\Rightarrow \Delta U_{BC} = (0.2) \left( \frac{5}{2} R \right) (455 - 600)$$

$$\Rightarrow \Delta U_{BC} = (0.2) \left( \frac{5}{2} \right) (8.31)(-145) \text{ J}$$

$$\Rightarrow \Delta U_{BC} \approx -602 \text{ J}$$

$$\Rightarrow W_{BC} = -\Delta U_{BC} = 602 \text{ J}$$

**Process CA** is an isobaric process. Hence,

$$Q_{CA} = nC_P \Delta T = n \left( \frac{7}{2} R \right) (T_A - T_C)$$

$$\Rightarrow Q_{CA} = (0.2) \left( \frac{7}{2} \right) (8.31)(300 - 455)$$

$$\Rightarrow Q_{CA} \approx -902 \text{ J}$$

Also,  $\Delta U_{CA} = nC_V \Delta T$

$$\Rightarrow \Delta U_{CA} = \frac{Q_{CA}}{\gamma}$$

$$\left\{ \because \gamma = \frac{C_P}{C_V} \right\}$$

$$\Rightarrow \Delta U_{CA} = -\frac{902}{1.4} \approx -644 \text{ J}$$

From FLTD, we get

$$W_{CA} = Q_{CA} - \Delta U_{CA} = -258 \text{ J}$$

(c) Efficiency of the cycle

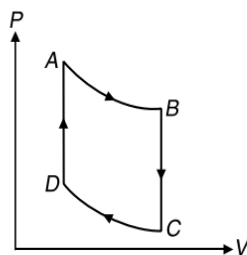
$$\eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|} \times 100\%$$

$$\Rightarrow \eta = \frac{344}{1246} \times 100\%$$

$$\Rightarrow \eta = 27.6\%$$

### ILLUSTRATION 117

One mole of a monatomic ideal gas is taken through the cycle shown in figure:



A → B : adiabatic expansion

B → C : cooling at constant volume

C → D : adiabatic compression

D → A : heating at constant volume

The pressure and temperature at A, B etc., are denoted by  $P_A, T_A, P_B, T_B$  etc., respectively. Given that  $T_A = 1000 \text{ K}$ ,

$P_B = \left( \frac{2}{3} \right) P_A$  and  $P_C = \left( \frac{1}{3} \right) P_A$ , calculate the following quantities:

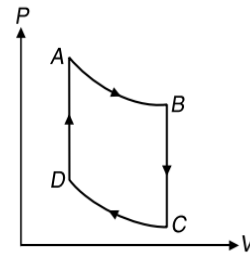
- The work done by the gas in the process A → B
- The heat lost by the gas in the process B → C
- The temperature  $T_D$

Given that  $\left( \frac{2}{3} \right)^{2/5} = 0.85$

### SOLUTION

Given  $T_A = 1000 \text{ K}$ ,  $P_B = \frac{2}{3} P_A$  and  $P_C = \frac{1}{3} P_A$

Number of moles,  $n = 1$  and  $\gamma = \frac{C_P}{C_V} = \frac{5}{3}$  (monatomic)



(a) A → B is an Adiabatic Process, therefore

$$P_A^{1-\gamma} T_A^\gamma = P_B^{1-\gamma} T_B^\gamma$$

$$\Rightarrow T_B = T_A \left( \frac{P_A}{P_B} \right)^{\frac{1-\gamma}{\gamma}} = (1000) \left( \frac{3}{2} \right)^{\frac{1-5/3}{5/3}}$$

$$\Rightarrow T_B = (1000) \left( \frac{3}{2} \right)^{-\frac{2}{5}} = (1000) \left( \frac{2}{3} \right)^{\frac{2}{5}}$$

$$\Rightarrow T_B = (1000)(0.85)$$

$$\Rightarrow T_B = 850 \text{ K}$$

Now, work done in the process A → B is

$$W_{AB} = \frac{R}{1-\gamma} (T_B - T_A) = \frac{831}{1 - \left( \frac{5}{3} \right)} (850 - 1000)$$

$$\Rightarrow W_{AB} = 1869.75 \text{ J}$$

(b) B → C is an Isochoric Process, so  $V = \text{constant}$ , hence

$$\Rightarrow \frac{T_B}{T_C} = \frac{P_B}{P_C}$$

$$\Rightarrow T_C = \left( \frac{P_C}{P_B} \right) T_B = \left( \frac{\left( \frac{1}{3} \right) P_A}{\left( \frac{2}{3} \right) P_A} \right) 850 \text{ K}$$

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$$\Rightarrow T_C = 425 \text{ K}$$

In an Isochoric Process, we have

$$Q_{BC} = \Delta U = nC_V \Delta T = (1) \left( \frac{3}{2} R \right) (T_C - T_B)$$

$$\Rightarrow Q_{BC} = \left( \frac{3}{2} \right) (8.31) (425 - 850)$$

$$\Rightarrow Q_{BC} = -5297.6 \text{ J}$$

Therefore, heat lost in the process BC is 5297.6 J

(c) CD and AB are Adiabatic Process. Therefore, we have

$$P_C^{1-\gamma} T_C^\gamma = P_D^{1-\gamma} T_D^\gamma$$

$$\Rightarrow \frac{P_C}{P_D} = \left( \frac{T_D}{T_C} \right)^{\frac{\gamma}{1-\gamma}} \quad \dots(1)$$

and  $P_A^{1-\gamma} T_A^\gamma = P_B^{1-\gamma} T_B^\gamma$

$$\Rightarrow \frac{P_A}{P_B} = \left( \frac{T_B}{T_A} \right)^{\frac{\gamma}{1-\gamma}} \quad \dots(2)$$

Multiplying equations (1) and (2), we get

$$\frac{P_C P_A}{P_D P_B} = \left( \frac{T_D T_B}{T_C T_A} \right)^{\frac{\gamma}{1-\gamma}} \quad \dots(3)$$

Processes BC and DA are Isochoric i.e.,  $V = \text{constant}$ , so

$$\frac{P_C}{P_B} = \frac{T_C}{T_B} \quad \text{and} \quad \frac{P_A}{P_D} = \frac{T_A}{T_D}$$

Multiplying these two equations, we get

$$\frac{P_C P_A}{P_D P_B} = \frac{T_C T_A}{T_B T_D} \quad \dots(4)$$

From equations (3) and (4), we get

$$\left( \frac{T_D T_B}{T_C T_A} \right)^{\frac{\gamma}{1-\gamma}} = \left( \frac{T_C T_A}{T_B T_D} \right)$$

$$\Rightarrow \left( \frac{T_C T_A}{T_D T_B} \right)^{\frac{1-\gamma}{\gamma}} = \left( \frac{T_C T_A}{T_B T_D} \right)$$

$$\Rightarrow \frac{T_C T_A}{T_B T_D} = 1$$

$$\Rightarrow T_D = \frac{T_C T_A}{T_B} = \frac{(425)(1000)}{850} \text{ K}$$

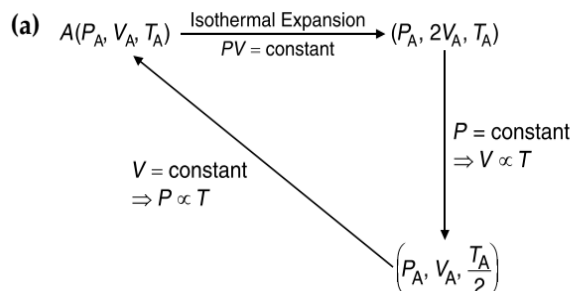
$$\Rightarrow T_D = 500 \text{ K}$$

initial volume. It is then compressed at constant pressure to its original volume. Finally, gas is compressed at constant volume to its original pressure  $P_A$ .

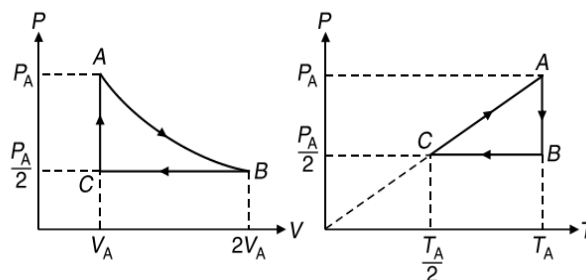
- (a) Sketch  $P$ - $V$  and  $P$ - $T$  diagrams for the complete process.  
 (b) Calculate the net work done by the gas, and net heat supplied to the gas during the complete process.

Take  $\log_e(2) = 0.7$

### SOLUTION



The  $P$ - $V$  and  $P$ - $T$  diagram are shown.



(b) Since, the process is cyclic, so

$$\Delta U = 0$$

From  $FLTD$ , we get

$$Q = W = W_{AB} + W_{BC} + W_{CA}$$

where,  $W_{AB} = (3)RT_A \log_e(2)$

$$W_{BC} = P_A \Delta V = nR \Delta T$$

$$\Rightarrow W_{BC} = (3)R \left( \frac{T_A}{2} - T_A \right)$$

$$\Rightarrow W_{BC} = -\frac{3RT_A}{2}$$

and  $W_{CA} = 0$  {Isochoric Process}

$$\Rightarrow W_{\text{net}} = W = 3RT_A \log_e(2) - \frac{3RT_A}{2}$$

$$\Rightarrow W_{\text{net}} = 0.6RT_A$$

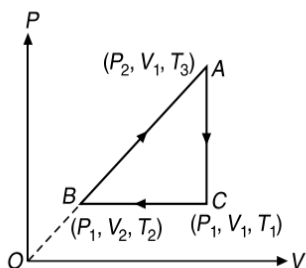
$$\Rightarrow Q_{\text{net}} = W_{\text{net}} = 0.6RT_A$$

### ILLUSTRATION 118

Three moles of an ideal gas  $\left( C_p = \frac{7}{2} R \right)$  at pressure,  $P_A$  and temperature  $T_A$  is isothermally expanded to twice its

### ILLUSTRATION 119

An ideal gas with the adiabatic exponent  $\gamma$  goes through a cycle shown in Figure within which the absolute temperature varies  $n$  fold in each process. Calculate the efficiency of this cycle.



**SOLUTION**

According to the problem, if we take  $T_2 = T$ , then

$$T_1 = nT \text{ and } T_3 = n^2T$$

Equation for process BA is  $P = kV$  i.e.,  $PV^{-1} = k$  i.e.,  $x = -1$ , so specific heat for this process is

$$C = C_V + \frac{R}{1-x} = C_V + \frac{R}{2}$$

$$\Rightarrow C = \frac{R}{\gamma-1} + \frac{R}{2} = \frac{R(\gamma+1)}{2(\gamma-1)}$$

Heat supplied to one mole of gas in process BA is

$$Q_{BA} = C\Delta T = \frac{R(\gamma+1)}{2(\gamma-1)}(n^2 - 1)T$$

Work done by gas equals the area enclosed by the cyclic process, so

$$W_{\text{total}} = \frac{1}{2}(P_2 - P_1)(V_1 - V_2)$$

$$\Rightarrow W_{\text{total}} = \frac{R}{2}(n^2T - nT - nT + T)$$

$$\Rightarrow W_{\text{total}} = \frac{RT}{2}(n-1)^2$$

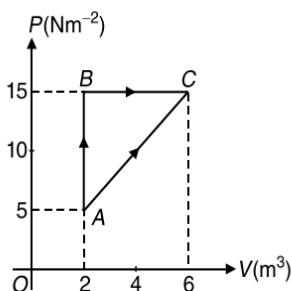
Efficiency of cycle is

$$\eta = \frac{\text{Work done}}{\text{Heat input}} = \frac{W}{Q_{BA}}$$

$$\Rightarrow \eta = \frac{\frac{RT}{2}(n-1)^2}{\frac{RT(\gamma+1)}{2(\gamma-1)}(n^2-1)} = \left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{n-1}{n+1}\right)$$

**ILLUSTRATION 120**

In the given figure, an ideal gas changes its state from A to state C by two paths ABC and AC.



- (a) Find the path along which work done is the least
- (b) The internal energy of gas at A is 10 J and amount of heat supplied to change its state to C through the path AC is 200 J. Calculate the internal energy at C.
- (c) The internal energy of gas at state B is 20 J. Find the amount of heat supplied to the gas to go from A to B.

**SOLUTION**

Since the work done  $W = \int P dV = \text{area under } P-V \text{ curve}$ , so

(a)  $W_{ABC} = W_{AB} + W_{BC}$

i.e.  $W_{ABC} = 0 + 15 \times 4 = 60 \text{ J}$

and  $W_{AC} = \frac{1}{2}(5+15) \times (6-2) = 40 \text{ J}$

thus, the work done along AC is least.

- (b) According to First Law of Thermodynamics,

$$dQ = dU + dW$$

So, for path AC,

$$(U_C - U_A) = dQ - dW = 200 - 40 = 160 \text{ J}$$

$$\Rightarrow U_C = 160 + U_A = 160 + 10 = 170 \text{ J}$$

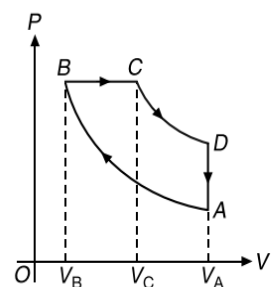
- (c) For path AB, First Law of Thermodynamics yields

$$dQ = (U_B - U_A) + 0 = 20 - 10 = 10 \text{ J}$$

**ILLUSTRATION 121**

One mole of a diatomic ideal gas ( $\gamma = 1.4$ ) is taken through a cyclic process as shown in figure starting from point A.

In this cycle, the process AB is an adiabatic compression, BC is isobaric, CD an adiabatic expansion and DA is isochoric. The



volume ratios are  $\frac{V_A}{V_B} = 16$  and  $\frac{V_C}{V_B} = 2$ . The temperature at A is  $T_A = 300 \text{ }^\circ\text{K}$ . Calculate the

temperature of the gas at the point B and D. Also calculate efficiency of the cycle.

**SOLUTION**

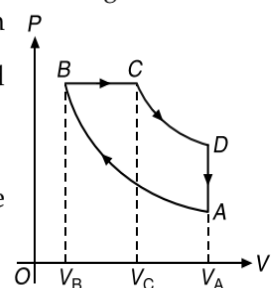
The respective cyclic process is shown in Figure.

The expansion and compression ratio are given to be  $\frac{V_A}{V_B} = 16$  and  $\frac{V_C}{V_B} = 2$

For an adiabatic process AB, we have

$$T_A V_A^{\gamma-1} = T_B V_B^{\gamma-1}$$

$$\Rightarrow T_B = T_A \left(\frac{V_A}{V_B}\right)^{\gamma-1}$$



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$$\Rightarrow T_B = 300 \times (16)^{0.4} \approx 909 \text{ K}$$

Similarly, for isobaric process  $BC$ , we have

$$\frac{T_C}{V_C} = \frac{T_B}{V_B}$$

$$\Rightarrow T_C = T_B \left( \frac{V_C}{V_B} \right) \quad \dots(1)$$

$$\Rightarrow T_C = 909 \times 2 = 1818 \text{ K}$$

Similarly, for adiabatic process  $CD$  we have

$$T_C V_C^{\gamma-1} = T_D V_D^{\gamma-1}$$

$$\Rightarrow T_D = T_C \left( \frac{V_C}{V_D} \right)^{\gamma-1}$$

Since we see that  $V_D = V_A$ , so

$$\frac{V_C}{V_D} = \frac{V_C}{V_A} = \frac{V_C}{V_B} \times \frac{V_B}{V_A} = 2 \times \frac{1}{16} = \frac{1}{8}$$

So, from equation (1), we get

$$T_D = 1818 \times \left( \frac{1}{8} \right)^{0.4} \approx 791 \text{ K}$$

The efficiency of cycle can be given as

$$\eta = 1 - \left| \frac{Q_{\text{out}}}{Q_{\text{in}}} \right|$$

Processes  $AB$  and  $CD$ , both being adiabatic processes, no heat exchange takes place.

In isobaric process  $BC$ , the temperature of gas increases, so heat  $Q_1$  absorbed by the system is given by

$$Q_1 = nC_p(T_C - T_B)$$

Similarly, in isochoric process  $DA$ , the temperature of gas decreases, so heat  $Q_2$  rejected by the system is given by

$$Q_2 = nC_p(T_D - T_A)$$

The efficiency of any cyclic process is

$$\eta = 1 - \left| \frac{Q_{\text{out}}}{Q_{\text{in}}} \right| = 1 - \frac{Q_2}{Q_1}$$

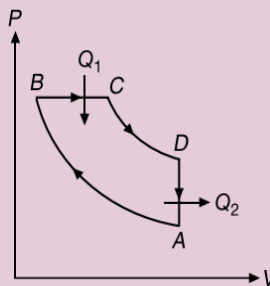
$$\Rightarrow \eta = 1 - \frac{C_V(T_D - T_A)}{C_p(T_C - T_B)} = 1 - \frac{T_D - T_A}{\gamma(T_C - T_B)}$$

$$\Rightarrow \eta = 1 - \frac{791 - 300}{(1.4)(1818 - 909)}$$

$$\Rightarrow \eta = 1 - \frac{491}{1272.6} = 0.614 = 61.4\%$$

### Conceptual Note(s)

While solving problems for calculating efficiency of a cyclic process, students must first identify the paths of heat input i.e. paths in which heat is being absorbed and the paths of heat output i.e. paths in which heat is being rejected. For the discussed problem, heat  $Q_1$  is being absorbed in the path  $BC$  and heat  $Q_2$  is being rejected in the path  $DA$ . So we have shown an incoming arrow on the path  $BC$  and an outgoing arrow on the path  $DA$  as shown in figure.



### LIMITATION OF FIRST LAW AND INTRODUCTION TO SECOND LAW OF THERMODYNAMICS

The First Law of Thermodynamics is just the Law of Conservation of energy generalised to include heat as a form of energy, that is, according to this law an increase in one form of energy must be accompanied by a decrease in some other form of energy and it puts no restrictions on the type of energy conversions that can occur. Furthermore it makes no distinction between heat and work. According to First Law, the internal energy of a body may be increased by either doing work on it or adding heat to it. But a major difference between heat and work exists which is not evident from the first law. For example, it is possible to convert work into heat completely whereas in practice it is impossible to convert the heat completely to work without changing the surroundings.

To overcome this situation the Second Law of Thermodynamics was introduced. This law establishes which processes in nature do or do not occur. Of all the processes permitted by the first law only certain types of energy conversions can take place. The following are some examples of processes that are consistent with the First Law of Thermodynamics but proceed in an order governed by the Second Law of Thermodynamics.

- When two bodies at different temperature are placed in contact with each other heat always flows from hotter body to colder but not vice-versa.
- Salt dissolves spontaneously in water, but extracting it back requires the use of some external agent.
- A rubber ball dropped onto the ground bounces several time and eventually comes to rest. The opposite process does not occur.
- The oscillations of a pendulum will slowly decrease in amplitude due to collisions with air molecules and friction at the point of suspension. Eventually the

pendulum will come to rest. Thus, the initial mechanical energy of the pendulum gets converted to thermal energy. The reverse transformation of energy doesn't occur.

These are all examples of Irreversible Processes, that is a category of processes occurring only in one direction. None of these processes occur in the opposite temporal order; if they did, they would violate the Second Law of Thermodynamics. This one way nature of Thermodynamic processes in fact establishes a direction of time. (which moves forward but never backward).

The Second Law of Thermodynamics, which can be stated in many equivalent way, has a lot of practical applications. For an engineering point of view, the most important application is the limited efficiency of heat engines. That is, the Second Law says that a machine capable of converting the thermal energy continuously into other forms of energy completely, can never be constructed.

## THE SECOND LAW OF THERMODYNAMICS: REVISITED

Experimental evidence suggests strongly that it is impossible to build a heat engine that converts heat completely to work, that is, an engine with 100% thermal efficiency. This impossibility is the basis of one statement of the **Second Law of Thermodynamics**, as follows.

It is impossible for any system to undergo a process in which it absorbs heat from a reservoir at a single temperature and converts the heat completely into mechanical work, with the system ending in the same state in which it began.

The basis of the Second Law of Thermodynamics lies in the difference between the nature of internal energy and that of macroscopic mechanical energy. In a moving body the molecules have random motion, but superimposed on this is a coordinated motion of every molecule in the direction of the body's velocity. The kinetic energy associated with this coordinated macroscopic motion is what we call the kinetic energy of the moving body. The kinetic and potential energies associated with the random motion constitute the internal energy.

If the Second Law were not true, we could power an automobile or run a power plant by cooling the surrounding air. Neither of these impossibilities violates the First Law Thermodynamics. The Second Law, therefore, is not a deduction from the first but stands by itself as a separate law of nature. The First Law denies the possibility of creating or destroying energy; the Second Law limits the availability of energy and the ways in which it can be used and converted.

It is impossible for any process to have as its sole result the transfer of heat from a cooler to a hotter body.

## ENTROPY

Entropy is a measure of disorder of molecular motion of a system. Greater is the disorder, greater is the entropy.

The change in entropy i.e.

$$dS = \frac{\text{Heat absorbed by system}}{\text{Absolute temperature}}$$

$$\text{or } dS = \frac{dQ}{T}$$

This relation is called the mathematical form of Second Law of Thermodynamics.



### Conceptual Note(s)

$dQ$  is a path function but  $\frac{dQ}{T}$  called Entropy is a state function.

$$\Rightarrow \int dS = S_{\text{final}} - S_{\text{initial}} = \Delta S = \frac{dQ}{T}$$

## CALCULATION OF ENTROPY

### For Solids and Liquids

- (a) When heat given to a substance changes its state at constant temperature, then change in entropy

$$dS = \frac{dQ}{T} = \pm \frac{mL}{T}$$

where positive sign refers to heat absorption and negative sign to heat evolution.

- (b) When heat given to a substance raises its temperature from  $T_1$  to  $T_2$ , then change in entropy

$$dS = \int \frac{dQ}{T} = \int_{T_1}^{T_2} mc \frac{dT}{T} = mc \log_e \left( \frac{T_2}{T_1} \right)$$

$$\Rightarrow \Delta S = 2.3mc \log_{10} \left( \frac{T_2}{T_1} \right)$$

### For a Perfect Gas

Perfect gas equation for  $n$  moles is  $PV = nRT$

$$\Delta S = \int \frac{dQ}{T} = \int \frac{nC_V dT + PdV}{T} \quad \{ \because dQ = dU + dW \}$$

$$\Rightarrow \Delta S = \int \frac{nC_V dT + \frac{nRT}{V} dV}{T} \quad \{ \because PV = nRT \}$$

$$\Rightarrow \Delta S = nC_V \int_{T_1}^{T_2} \frac{dT}{T} + nR \int_{V_1}^{V_2} \frac{dV}{V}$$

$$\Rightarrow \Delta S = nC_V \log_e \left( \frac{T_2}{T_1} \right) + nR \log_e \left( \frac{V_2}{V_1} \right)$$

Similarly in terms of  $T$  and  $P$ ,

$$\Delta S = nC_P \log_e \left( \frac{T_2}{T_1} \right) - nR \log_e \left( \frac{P_2}{P_1} \right)$$

and in terms of  $P$  and  $V$

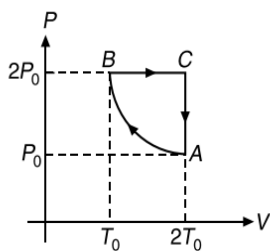
$$\Delta S = nC_V \log_e \left( \frac{P_2}{P_1} \right) + nC_P \log_e \left( \frac{V_2}{V_1} \right)$$

**Test Your Concepts-IX**

**Based on Cyclic Process, Heat Engine and Refrigerator**

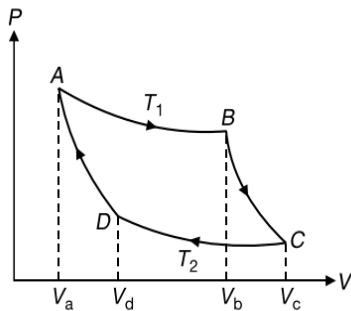
*(Solutions on page H.93)*

1. Two moles of an ideal monoatomic gas is taken through a cycle ABCA whose  $PT$  diagram is shown in Figure.



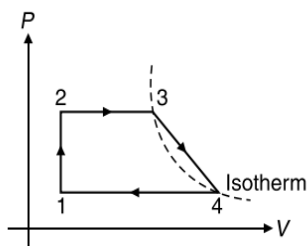
During the process  $AB$ , pressure and temperature of the gas vary such that  $PT = \text{constant}$ . If  $T_0 = 300$  K, calculate work done by the gas in process  $AB$  and the heat absorbed or released by the gas in each of the process.

2. Two different adiabatic paths for the same gas intersect two isotherms at  $T_1$  and  $T_2$  as shown in the given  $P$ - $V$  diagram. Calculate the efficiency of the process.

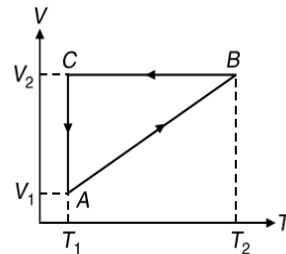


**CYCLIC**

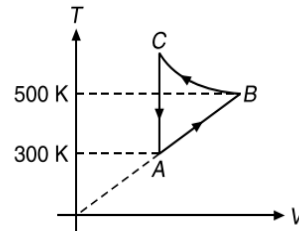
3.  $n$  moles of an ideal gas is made to undergo the cycle  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$  as shown in figure. Process  $3 \rightarrow 4$  is a straight line. The gas temperatures in states 1, 2 and 3 are  $T_1, T_2$  and  $T_3$  respectively. Also points 3 and 4 lie on the same isotherm. Determine the work done by the gas during the cycle.



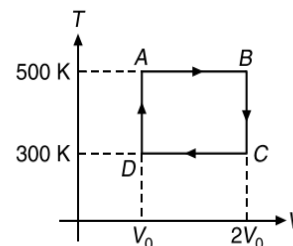
4. A cyclic process ABCA shown in the  $V$ - $T$  diagram is performed with a constant mass of an ideal gas. Show the same process on a  $P$ - $V$  diagram. (In the figure,  $CA$  is parallel to the  $V$ -axis and  $BC$  is parallel to the  $T$ -axis).



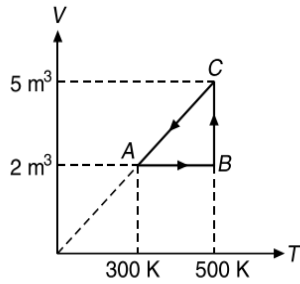
5. Consider the cyclic process ABCA shown in Figure. An ideal gas of 2 moles undergoes this process and a total of 1200 J heat is rejected by the gas in the complete cycle. Calculate work done by the gas during the process  $BC$ .



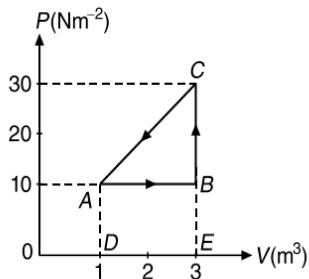
6. The  $VT$  graph for a thermodynamic process in which gas temperature changes from 300 K to 500 K during isochoric part of the cycle and volume of gas is doubled during isothermal part of the cycle is shown in Figure. This graph is plotted for 2 moles of a gas. Calculate total heat supplied by gas in the complete cycle.



7. A process ABCA is performed on one mole of an ideal gas whose  $VT$  diagram is shown in Figure. Calculate the net heat supplied to the gaseous system during the process. Take  $R = 8.3 \text{ J mol}^{-1} \text{ K}^{-1}$  and  $\ln(2.5) \approx 0.9$

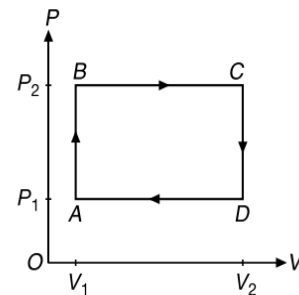


8. An ideal gas whose ratio of specific heats is  $\gamma$  goes through a direct (clockwise) cycle consisting of adiabatic, isobaric and isochoric lines. Find the efficiency of the cycle if in the adiabatic process the volume of ideal gas increase  $n$  fold.
9. An ideal gas is taken round a cyclic thermodynamic process ABCA as shown in Figure.



If the internal energy of the gas at point A is assumed zero while at B it is 50 J and heat absorbed by the gas in process BC is 90 J, then calculate the internal energy of gas at point C, heat energy absorbed by gas in process AB, heat energy rejected or absorbed by gas in the process CA and net work done by the gas in the complete cycle ABCA.

10. An ideal gas whose adiabatic exponent is equal to  $\gamma$  goes through a cycle consisting of two isochoric and two isobaric lines as shown in figure. find the efficiency of such a cycle, if the absolute temperature of the gas rises  $n$  times both in isochoric heating and in isobaric expansion.



# TRANSFER OF HEAT

## MODES OF HEAT TRANSFER

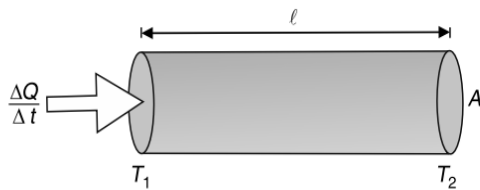
The transfer of heat between a system and its surroundings can occur through these three transfer mechanisms.

- (a) Conduction
- (b) Convection
- (c) Radiation

## HEAT CONDUCTION

It is the process in which heat is transferred from the place of higher temperature to the place of lower temperature due to molecular vibration of particles without their actual motion.

To start with the conduction process every cross section of body absorbs heat handing over excess heat to next section but at a later stage when steady state is reached then temperature of all the parts of the body becomes constant.



Consider a cylindrical metal bar of area  $A$  length  $l$ . Across this area and length is maintained a temperature difference as shown and remaining portion (i.e. curved portion) is kept insulated from surroundings to avoid leakage of heat. At steady state the rate of flow of heat  $\frac{\Delta Q}{\Delta t}$  (or heat current) becomes constant.

Experimentally it has been observed that heat current is directly proportional to

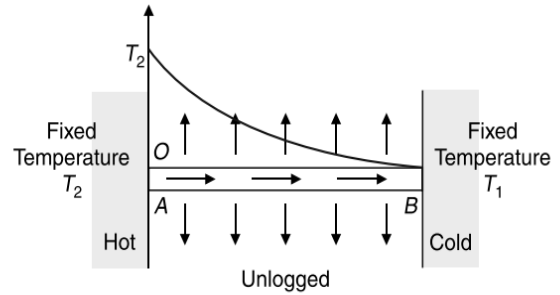
- (a) the temperature gradient  $\left(\frac{T_1 - T_2}{l}\right)$  maintained across the faces
- (b) area  $A$  perpendicular to temperature gradient through which heat enters.

$$\frac{\Delta Q}{\Delta t} = KA \left( \frac{T_1 - T_2}{l} \right)$$

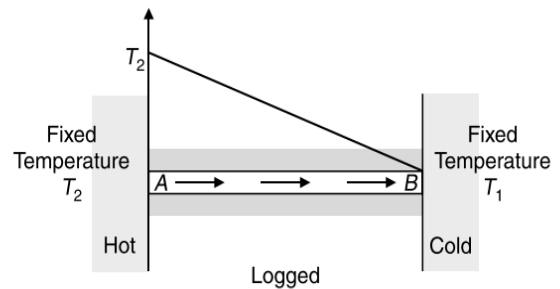
where  $K$  is **coefficient of thermal conductivity** of the bar.

Also, we can write  $\frac{dQ}{dt} = -KA \left( \frac{dT}{dx} \right)$

Please note that the negative sign indicates the decrease in temperature along the direction of heat flow.



Here  $\frac{dQ}{dt}$  is the time rate of heat transfer across the area  $A$ ,  $\frac{dT}{dx}$  is called the **temperature gradient**, and  $k$  is a constant of proportionality called the **thermal conductivity**. We choose the direction of heat flow to be the direction in which  $x$  increases; since heat flows in the direction of decreasing  $T$ , we introduce a minus sign in equation, (i.e., we wish  $\frac{dQ}{dt}$  to be positive when  $\frac{dT}{dx}$  is negative).



The phenomenon of heat conduction also shows that the concepts of heat and temperature are distinctly different. Different rods, having the same temperature difference between their ends, may transfer entirely different quantities of heat in the same time.

### ILLUSTRATION 122

A copper rod 2 m long has a circular cross-section of radius 1 cm. One end is kept at 100 °C and the other at 0 °C and the surface is insulated so that negligible heat is lost through the surface. Calculate the

- (a) thermal resistance of the bar.
- (b) thermal current  $H$ .
- (c) temperature gradient  $\frac{dT}{dx}$  and
- (d) temperature 25 cm from the hot end.

Thermal conductivity of copper is 401 Wm<sup>-1</sup>K<sup>-1</sup>

**SOLUTION**

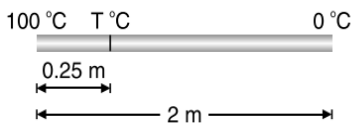
(a) Thermal resistance  $R_{Th} = \frac{l}{kA} = \frac{l}{k(\pi r^2)}$   
 $\Rightarrow R = \frac{(2)}{(401)(\pi)(10^{-2})^2} = 15.9 \text{ KW}^{-1}$

(b) Thermal current,  $H = \frac{\Delta T}{R} = \frac{100}{15.9} = 6.3 \text{ W}$

(c) Temperature gradient

$$\frac{\Delta T}{\Delta x} = \frac{0 - 100}{2} = -50 \text{ Km}^{-1} = -50 \text{ }^\circ\text{Cm}^{-1}$$

(d) Let  $T$  (in  $^\circ\text{C}$ ) be the temperature at 25 cm from the hot end, then we have



$$\frac{100 - T}{0.25} = \frac{100 - 0}{2}$$

$$\Rightarrow T = 87.5 \text{ }^\circ\text{C}$$

**ILLUSTRATION 123**

A copper bar of length 75 cm and a steel bar of length 125 cm, both having circular cross-section with diameter 2 cm are joined together end to end as shown in Figure.



The free ends of the copper and steel bars are maintained at  $100 \text{ }^\circ\text{C}$  and  $0 \text{ }^\circ\text{C}$  respectively. The surfaces of the bars are thermally insulated. Calculate the temperature of the copper steel junction and the heat transmitted per unit time across the junction if coefficient of thermal conductivity of copper is  $9.2 \times 10^{-2} \text{ kcalm}^{-1}\text{ }^\circ\text{C}^{-1}\text{ s}^{-1}$  and that of steel is  $1.1 \times 10^{-2} \text{ kcalm}^{-1}\text{ }^\circ\text{C}^{-1}\text{ s}^{-1}$ .

**SOLUTION**

If  $T$  be the temperature of junction, then in steady state, the rate of flow of heat in the copper bar must be same as that in the steel bar i.e.,

$$\frac{dQ}{dt} = \frac{k_c A (T_1 - T)}{l_c} = \frac{k_s A (T - T_2)}{l_s}$$

$$\Rightarrow \left( \frac{9.2 \times 10^{-2}}{1.1 \times 10^{-2}} \right) \left( \frac{125}{75} \right) = \frac{T - 0}{100 - T}$$

$$\Rightarrow T = 93.3 \text{ }^\circ\text{C}$$

The heat flowing through the junction per second is

$$\frac{dQ}{dt} = \frac{k_c A (T_1 - T)}{l_c} = \frac{(9.2 \times 10^{-2})(3.14 \times 10^{-4})(100 - 93.3)}{0.75}$$

$$\Rightarrow \frac{dQ}{dt} = 2.58 \times 10^{-4} \text{ Js}^{-1}$$

**ILLUSTRATION 124**

A rigid box contains one mole of a monatomic ideal gas. The walls of the box have total surface area  $A$ , thickness  $D$  and thermal conductivity  $k$ . Initially the gas is at a temperature  $T_0$  and pressure  $P_0$ . The temperature of the surroundings is  $\frac{T_0}{2}$ . Find the temperature and pressure of the gas as a function of time.

**SOLUTION**

Since, the box is rigid, so  $V = \text{constant}$

$$\Rightarrow \frac{dQ}{dt} = \frac{dU}{dt}$$

Since,  $dU = (1)C_V dT$  and  $\frac{dQ}{dt} = -kA \frac{\Delta T}{\Delta x}$

$$\Rightarrow \frac{dQ}{dt} = -C_V \left( \frac{dT}{dt} \right) = -\frac{kA \left( T - \frac{T_0}{2} \right)}{D}$$

$$\Rightarrow \int_{T_0}^T - \left( \frac{dT}{T - \frac{T_0}{2}} \right) = \frac{kA}{DC_V} \int_0^t dt$$

$$\Rightarrow T = \frac{T_0}{2} \left( 1 + e^{-\frac{kAt}{DC_V}} \right)$$

For a monatomic gas,  $C_V = \frac{3R}{2}$

$$\Rightarrow T = \frac{T_0}{2} \left( 1 + e^{-\left( \frac{2kA}{3DR} \right)t} \right)$$

Since,  $V = \text{constant}$ , so  $\frac{P_0}{T_0} = \frac{P}{T}$

$$\Rightarrow P = \frac{P_0}{2} \left( 1 + e^{-\left( \frac{2kA}{3DR} \right)t} \right)$$

**ILLUSTRATION 125**

A cylindrical brass boiler of radius 15 cm and thickness 1 cm is filled with water and placed on an electric heater. If the water boils at the rate of  $200 \text{ gs}^{-1}$ , estimate the temperature of the heater filament. Given that the thermal conductivity of brass is  $109 \text{ Js}^{-1}\text{ m}^{-1}\text{ }^\circ\text{C}^{-1}$  and latent heat of vaporisation of water is  $2.256 \times 10^3 \text{ Jg}^{-1}$ .

**SOLUTION**

Since water is boiling at the rate of  $200 \text{ gs}^{-1}$ , the rate at which heat energy is supplied by the heater to water is

$$\frac{Q}{t} = \frac{mL}{t} = \left( \frac{m}{t} \right) L = 4.512 \times 10^5 \text{ Js}^{-1} \quad \dots(1)$$

## 2.90 JEE Advanced Physics: Waves and Thermodynamics

Radius of the boiler is  $r = 15 \text{ cm} = 0.15 \text{ m}$

So, base area of the boiler is

$$A = \pi r^2 = 3.142 \times (0.15)^2 = 0.0707 \text{ m}^2$$

Thickness of brass is  $d = 1 \text{ cm} = 1 \times 10^{-2} \text{ m}$

Thermal conductivity of brass is  $109 \text{ Js}^{-1}\text{m}^{-1}\text{C}^{-1}$

Temperature of boiling water is  $T_w = 100 \text{ }^\circ\text{C}$

If  $T_f$  is the temperature of the filament, then the rate at which heat energy is transmitted through the base is given by

$$\frac{Q}{t} = \frac{kA(T_f - T_w)}{d} \quad \dots(2)$$

Using (1) and substituting the values of  $k$ ,  $A$ ,  $T_w$  and  $d$  in equation (2), we get

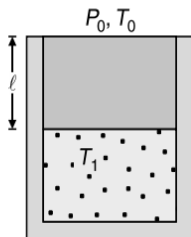
$$\frac{(109)(0.0707)(T_f - 100)}{1 \times 10^{-2}} = 4.512 \times 10^5$$

$$\Rightarrow T_f - 100 = 585.5$$

$$\Rightarrow T_f = 685.5 \text{ }^\circ\text{C}$$

### ILLUSTRATION 126

$n$  moles of a monatomic gas at temperature  $T_1$  are trapped under a piston having area  $A$ , length  $l$ , density  $\rho$  and thermal conductivity  $K$ .  $P_0$  and  $T_0$  are pressure and temperature of the atmosphere. Find length of the gas column as a function of time. Neglect friction and heat loss through the walls of container and sides of piston.



### SOLUTION

If  $P$  be the pressure of the gas, then for equilibrium of piston, we have

$$P_0 A + \rho l g A = \rho A$$

$$\Rightarrow P = P_0 + \rho l g = \text{constant}$$

If  $T$  is the temperature of gas at any time  $t$ , then the rate of heat flow through the piston is

$$\frac{dQ}{dt} = \frac{KA(T - T_\theta)}{l} = -nC_p \left( \frac{dT}{dt} \right) = -\frac{5}{2} nR \frac{dT}{dt}$$

$$\Rightarrow \frac{dT}{T - T_0} = -\frac{2KA}{5nR} dt$$

$$\Rightarrow \int_{T_1}^T \frac{dT}{T - T_0} = -\frac{2KA}{5nR} \int_0^t dt$$

$$\Rightarrow T = T_0 + (T_1 - T_0) e^{-\frac{2KA}{5nR} t}$$

Now if  $x$  is length of gas column at this instant, then

$$P(Ax) = nRT$$

$$\Rightarrow x = \frac{nRT}{PA} = \frac{nR}{A(P_0 + \rho l g)} \left[ T_0 + (T_1 - T_0) e^{-\frac{2KA}{5nR} t} \right]$$

### WIEDEMANN FRANZ LAW

This law deals with the physical ideas about flow of heat through conductors and insulators at microscopic level. According to the modern electronic theory of thermal conduction, the flow of heat in a body from hotter part to colder part is due to the motion of free electrons.

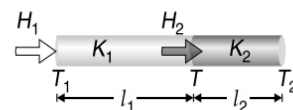
The electrons in the outermost orbit of an atom in a metal are loosely bound to the nucleus. When a metallic rod is heated at one end, the atoms acquire greater kinetic energy and their amplitude of vibration increases. A part of this energy is gained by the electrons in the outermost orbit. These electrons drift away from the atom and move towards the cooler portion of rod. These energetic electrons collide against atoms in cooler portion of the rod and impart their energy to these atoms. These atoms also get agitated and the electrons in their outermost orbits are made free. Thus, the heat energy is transported from the hotter to the colder parts by the motion of free electrons. It may be mentioned here that a good conductor of heat is also a good conductor of electricity.

In the year 1853, Wiedemann and Franz established a relationship between thermal conductivity  $K$  and electrical conductivity  $\sigma$ . The ratio of thermal and electrical conductivities is the same for the metals at a particular temperature and is proportional to the absolute temperature of the metal. If  $T$  is the absolute temperature, then  $\frac{K}{\sigma} \propto T$

$$\Rightarrow \frac{K}{\sigma T} = \text{constant}$$

### TEMPERATURE OF JUNCTION

Let two rods of thermal conductivities  $K_1, K_2$ , lengths  $l_1, l_2$  and equal cross-sectional area  $A$  connected in series. In steady state the temperatures of ends of rod are  $T_1$  and  $T_2$  and the temperature of junction is  $T$ .



In steady state the rate of flow of heat in both rods is the same. Therefore

$$\left( \frac{Q}{t} \right)_1 = \left( \frac{Q}{t} \right)_2$$

$$\Rightarrow \frac{K_1 A (T_1 - T)}{l_1} = \frac{K_2 A (T - T_2)}{l_2}$$

$$\Rightarrow T = \frac{(K_1/l_1)T_1 + (K_2/l_2)T_2}{(K_1/l_1) + (K_2/l_2)} = \frac{K_1 T_1 l_1 + K_2 T_2 l_1}{K_1 l_2 + K_2 l_1}$$

### THERMOMETRIC CONDUCTIVITY (OR DIFFUSIVITY)

The thermometric conductivity or diffusivity is defined as the ratio of the coefficient of thermal conductivity to the thermal capacity per unit volume of the material. We have thermal capacity is  $mc$ . So, thermal capacity per unit volume is  $\frac{mc}{V} = \rho c$ , where  $\rho$  is density of substance. So, diffusivity  $D$  is

$$D = \frac{K}{\rho c}$$

### THERMAL RESISTANCE

The hindrance offered by a body to the flow of heat is called its thermal resistance and is given by

$$R = \frac{\text{Temperature Difference } (\Delta T)}{\text{Heat Current } (H)} = \frac{\Delta T}{H} = \frac{l}{KA}$$

where  $l$  is length of rod area  $A$  having temperature difference  $\Delta T$  across its ends. The unit of thermal resistance is  $^{\circ}\text{Cscal}^{-1}$ .

If different rods are connected in series, then heat flowing per second is same i.e.,  $H_1 = H_2 = H_3$  and net thermal resistance is

$$R_S = R_1 + R_2 + R_3$$

If different rods are connected in parallel, then net thermal resistance  $R$  is given by

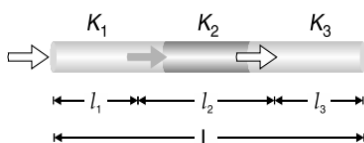
$$\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

and temperature difference  $\Delta T$  is the same.

### SLABS OF DIFFERENT MATERIALS IN SERIES

Let three slabs each of cross-sectional area  $A$ , lengths  $l_1, l_2, l_3$  and conductivities  $K_1, K_2, K_3$  respectively be connected in series. Then in series

$$R_S = R_1 + R_2 + R_3$$



If  $K_S$  is equivalent conductivity, then from relation

$$R = \frac{l}{KA}, \text{ we get}$$

$$\frac{L}{K_S A} = \frac{l_1}{K_1 A} + \frac{l_2}{K_2 A} + \frac{l_3}{K_3 A}$$

$$\Rightarrow \frac{l_1 + l_2 + l_3}{K_S} = \frac{l_1}{K_1} + \frac{l_2}{K_2} + \frac{l_3}{K_3}$$

For 3 slabs of equal length

$$\frac{3l}{K_S} = \frac{l}{K} + \frac{l}{K_2} + \frac{l}{K_3}$$

$$\Rightarrow \frac{1}{K_S} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

Special case: For two slabs of equal length

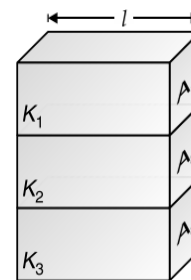
$$\frac{2l}{K_S} = \frac{l}{K_1} + \frac{l}{K_2}$$

$$\Rightarrow K_S = \frac{2K_1 K_2}{K_1 + K_2}$$

### SLABS OF DIFFERENT MATERIALS IN PARALLEL

Let three slabs each of length  $l$ , areas  $A_1, A_2, A_3$  and thermal conductivities  $K_1, K_2, K_3$  respectively be connected in parallel. Then in parallel arrangement

$$\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



If  $K_P$  is equivalent conductivity, then from relation

$$R = \frac{l}{KA}, \text{ we get}$$

$$\frac{K_P A}{l} = \frac{K_1 A_1}{l} + \frac{K_2 A_2}{l} + \frac{K_3 A_3}{l}$$

$$\Rightarrow K_P = \frac{K_1 A_1 + K_2 A_2 + K_3 A_3}{A}$$

$$\Rightarrow K_P = \frac{K_1 A_1 + K_2 A_2 + K_3 A_3}{A_1 + A_2 + A_3}$$

For three slabs of equal areas

$$K_P = \frac{K_1 + K_2 + K_3}{3}$$

Special Case: For two slabs of equal area

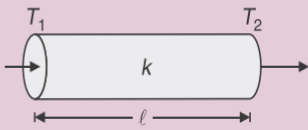
$$K_p = \frac{K_1 + K_2}{2}$$

### Problem Solving Technique(s)

Thermal and Electrical Conductivity. There exists an useful analogue between thermal conductivity and electrical conductivity

### Thermal Conduction

$$\frac{dQ}{dt} = kA \left( \frac{T_1 - T_2}{\ell} \right)$$



- (a) Heat flows from higher temperature to lower temperature. The rate of heat flow is called the heat current.

$$I = \frac{dQ}{dt}$$

- (b) Thermal resistance is defined as

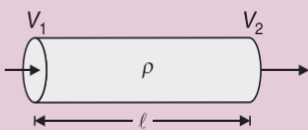
$$R_T = \frac{\ell}{kA}$$

- (c) Ohm's Law for heat conduction may be stated as

$$I = \frac{T_1 - T_2}{R_T}$$

### Electrical Conduction

$$\frac{dq}{dt} = \frac{A}{\rho} \left( \frac{V_1 - V_2}{\ell} \right)$$



- (a) Charges flow from higher potential to lower potential  
 (b) The rate of charge flow is called the electric current

$$I = \frac{dq}{dt}$$

- (c) Electrical resistance is defined as

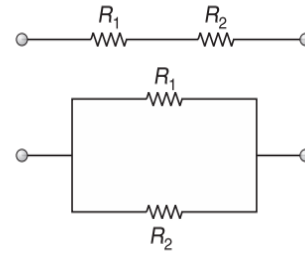
$$R_E = \frac{\rho \ell}{A}$$

- (d) Ohm's Law for electric conduction may be stated as

$$I = \frac{V_1 - V_2}{R_E}$$

Using the above analogy, a problem of heat conduction may be transformed into a problem of electrical conduction and can be easily solved using the formulae of electric circuits.

For series and parallel combination, the equivalent electrical or the equivalent thermal resistance is given by

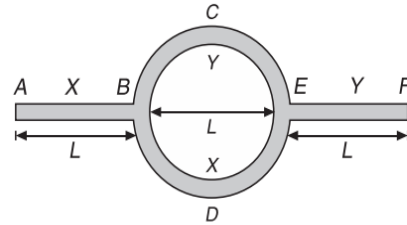


$$R = R_1 + R_2 \quad (\text{series})$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad (\text{parallel})$$

### ILLUSTRATION 127

Rods made of materials X and Y are connected as shown in Figure.



The cross-sectional area of all the rods is same. If the end A is maintained at 80 °C and the end F is maintained at 10 °C. Calculate the temperatures of junctions B and E in steady state. Given that thermal conductivity of material X is double that of Y.

### SOLUTION

The thermal resistances of different rods are

$$R_{AB} = \frac{L}{k_X A}, R_{EF} = \frac{L}{k_Y A}, R_{BDE} = \frac{\pi L/2}{k_X A}, R_{BCE} = \frac{\pi L/2}{k_Y A}$$

If temperature of junction B and E are taken as  $T_B$  and  $T_E$ , then in steady state, at the junction B, we have

$$\begin{aligned} \frac{T_A - T_B}{R_{AB}} &= \frac{T_B - T_E}{R_{BCE}} + \frac{T_B - T_E}{R_{BDE}} \\ \Rightarrow \frac{80 - T_B}{L/k_X A} &= \frac{T_B - T_E}{\pi L/2k_Y A} + \frac{T_B - T_E}{\pi L/2k_X A} \end{aligned}$$

Since,  $k_X = 2k_Y$

$$\Rightarrow 80 - T_B = \left( \frac{T_B - T_E}{\pi} \right) + \left( \frac{T_B - T_E}{2\pi} \right)$$

$$\Rightarrow 80 - T_B = \frac{3}{2\pi} (T_B - T_E) \quad \dots(1)$$

Similarly, at junction E, we have

$$\frac{T_B - T_E}{R_{BCE}} + \frac{T_B - T_E}{R_{BDE}} = \frac{T_E - 10}{R_{EF}}$$

$$\Rightarrow \frac{T_B - T_E}{\pi L / 2k_Y A} + \frac{T_B - T_E}{\pi L / 2k_X A} + \frac{T_E - 10}{L / k_Y A}$$

$$\Rightarrow \frac{2(T_B - T_E)}{\pi} + \frac{4(T_B - T_E)}{\pi} = T_E - 10$$

$$\Rightarrow \frac{3}{4\pi}(T_B - T_E) = T_E - 10 \quad \dots(2)$$

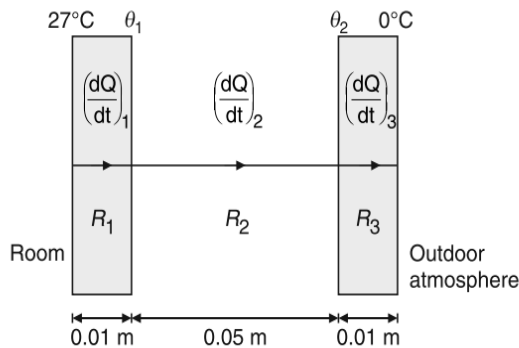
Solving equations (1) and (2), we get  
 $T_E = 19.74^\circ\text{C}$  and  $T_B = 60.52^\circ\text{C}$

**ILLUSTRATION 128**

A double-pane window used for insulating a room thermally from outside consists of two glass sheets each of area  $1\text{ m}^2$  and thickness  $0.01\text{ m}$  separated by a  $0.05\text{ m}$  thick stagnant air space. In the steady state, the room glass inter-face and the glass-outdoor interface are at constant temperatures of  $27^\circ\text{C}$  and  $0^\circ\text{C}$  respectively. Calculate the rate of heat flow through the window pane. Also find the temperatures of other interfaces. Given thermal conductivities of glass and air as  $0.8$  and  $0.08\text{ W m}^{-1}\text{ K}^{-1}$  respectively.

**SOLUTION**

Let  $\theta_1$  and  $\theta_2$  be the temperatures of the two interfaces as shown in figure.



Thermal resistance,  $R = \frac{l}{KA}$

$$\Rightarrow R_1 = R_3 = \frac{(0.01)}{(0.8)(1)} = 0.0125\text{ KW}^{-1}\text{ or }^\circ\text{CW}^{-1}$$

and  $R_2 = \frac{(0.05)}{(0.08)(1)} = 0.625\text{ KW}^{-1}\text{ or }^\circ\text{CW}^{-1}$

Now the rate of heat flow  $\left(\frac{dQ}{dt}\right)$  will be equal from all the three sections and since rate of heat flow is given by

$$\frac{dQ}{dt} = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

and  $\left(\frac{dQ}{dt}\right)_1 = \left(\frac{dQ}{dt}\right)_2 = \left(\frac{dQ}{dt}\right)_3$

Therefore,  $\frac{27 - \theta_1}{0.0125} = \frac{\theta_1 - \theta_2}{0.625} = \frac{\theta_2 - 0}{0.0125}$

Solving this equation, we get  
 $\theta_1 = 26.48^\circ\text{C}$  and  $\theta_2 = 0.52^\circ\text{C}$

and  $\frac{dQ}{dt} = \frac{27 - \theta_1}{0.0125}$

$$\Rightarrow \frac{dQ}{dt} = \frac{27 - 26.48}{0.0125} = 41.6\text{ W}$$

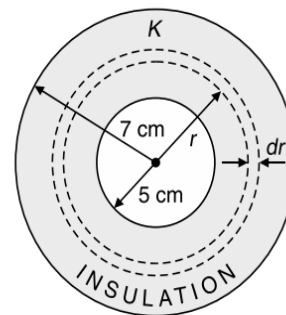
**ILLUSTRATION 129**

A steam pipe of radius  $5\text{ cm}$  carries steam at  $100^\circ\text{C}$ . The pipe is covered by a jacket of insulating material  $2\text{ cm}$  thick having a thermal conductivity  $0.07\text{ W m}^{-1}\text{ K}^{-1}$ . If the temperature at the outer wall of the pipe jacket is  $20^\circ\text{C}$ , how much heat is lost through the jacket per metre length in an hour?

**SOLUTION**

Thermal resistance per meter length of an element at distance  $r$  of thickness  $dr$  is

$$dR = \frac{dr}{k(2\pi r)} \quad \left\{ \because R = \frac{l}{kA} \right\}$$



Total resistance  $R = \int_{r=5\text{ cm}}^{r=7\text{ cm}} dR$

$$\Rightarrow R = \frac{1}{2\pi k} \left[ \int_{5 \times 10^{-2}}^{7 \times 10^{-2}} \frac{dr}{r} \right] = \frac{1}{2\pi k} \log_e \left( \frac{7}{5} \right)$$

$$\Rightarrow R = \frac{1}{(2\pi)(0.07)} \log_e (1.4) = 0.765\text{ KW}^{-1}$$

Since, Heat current  $H = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$

$$\Rightarrow H = \frac{(100 - 20)}{0.765} = 104.6\text{ watt}$$

So, heat lost  $Q$  in one hour is

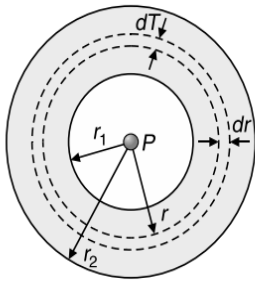
$$Q = Ht = (104.6\text{ Js}^{-1})(3600\text{ s}) = 3.76 \times 10^5\text{ J}$$

**ILLUSTRATION 130**

A point source of heat of power  $P$  is placed at the centre of a spherical shell of mean radius  $R$ . The material of the shell has thermal conductivity  $K$ . Calculate the thickness of the shell if temperature difference between the outer and inner surfaces of the shell in steady state is  $T$ .

**SOLUTION**

Consider a concentric spherical shell of radius  $r$  and thickness  $dr$  as shown in Figure.



In steady state, the rate of heat flow (heat current) through this shell will be

$$H = \frac{\Delta T}{R} = \frac{(-dT)}{dr} \left[ \because R = \frac{l}{KA} \right]$$

$$(K)(4\pi r^2)$$

$$\Rightarrow H = -(4\pi K r^2) \frac{dT}{dr}$$

Here negative sign is used because with increase in  $r$ ,  $T$  decreases.

$$\Rightarrow \int_1^2 \frac{dr}{r^2} = -\frac{4\pi K}{H} \int_{T_1}^{T_2} dT$$

This equation gives,

$$H = \frac{4\pi K r_1 r_2 (T_1 - T_2)}{(r_2 - r_1)}$$

In steady state,  $H = P$ ,  $r_1 r_2 \approx R^2$  and  $T_1 - T_2 = T$

So, thickness of shell is

$$r_2 - r_1 = \frac{4\pi K R^2 T}{P}$$

**INGEN-HAUZ EXPERIMENT**

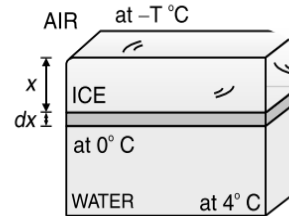
It is used to compare thermal conductivities of different materials. If  $l_1$  and  $l_2$  are the lengths of wax melted on rods, then the ratio of thermal conductivities is

$$\frac{K_1}{K_2} = \frac{l_1^2}{l_2^2}$$

So, in this experiment, we observe that the thermal conductivity  $K$  is proportional to the square of the length i.e.,  $K \propto (\text{length})^2$

**GROWTH OF ICE ON PONDS**

When the atmospheric temperature falls below  $0^\circ\text{C}$  (say  $-T^\circ\text{C}$ ), the cold air above water extracts heat from the water. As a result the water begins to freeze into the ice layers. Consider at any time the thickness of ice is  $x$  and further layer of ice of thickness  $dx$  is formed in time  $dt$ . If  $\rho$  is density of ice and  $L$  be the latent heat of fusion, then



$$\frac{dQ}{dt} = KA \frac{\Delta T}{\Delta x} = KA \left\{ \frac{0 - (-T)}{x} \right\}$$

$$\Rightarrow dQ = KA \left( \frac{T}{x} \right) dt$$

Further,  $dQ = (dm)L = (Adx)\rho L$

$$\Rightarrow Adx\rho L = KA \left( \frac{T}{x} \right) dt$$

$$\Rightarrow dt = \frac{\rho L}{KT} x dx$$

$$\Rightarrow t = \int_0^t dt = \frac{\rho L}{KT} \int_{x_1}^{x_2} x dx = \frac{\rho L}{KT} \left( \frac{x_2^2}{2} - \frac{x_1^2}{2} \right)$$

$$\Rightarrow t = \frac{\rho L}{2KT} (x_2^2 - x_1^2)$$

where,  $L$  is latent heat of fusion of ice,  $K$  is coefficient of thermal conductivity of ice and  $T$  is temperature of air below  $0^\circ\text{C}$ .

Take care and do not apply a negative sign for putting value of temperature in formula and also do not convert it to absolute scale. Since ice is a poor conductor of heat, so the rate of increase of thickness of ice on ponds decreases with time.

**ILLUSTRATION 131**

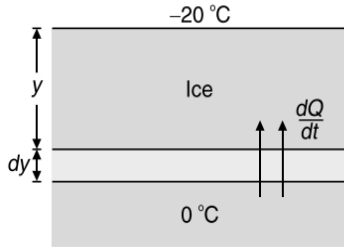
The thickness of ice on a lake is 5 cm and the temperature of air is  $-20^\circ\text{C}$ . Calculate how long will it take for the thickness of ice to be doubled. Thermal conductivity of ice is  $0.005 \text{ cal cm}^{-1} \text{ s}^{-1} \text{ }^\circ\text{C}^{-1}$ , density of ice is  $0.92 \text{ g cc}^{-1}$  and latent heat of ice is  $80 \text{ cal g}^{-1}$ .

**SOLUTION**

Since,  $\frac{dQ}{dt} = L \left( \frac{dm}{dt} \right)$

$$\Rightarrow \frac{KA[0 - (-20)]}{y} = LA\rho \left( \frac{dy}{dt} \right)$$

where,  $A$  is the area of lake.



$$\Rightarrow \int_5^{10} y dy = \frac{20K}{\rho L} \int_0^t dt$$

$$\Rightarrow t = \frac{\rho L}{20K} \left( \frac{10^2}{2} - \frac{5^2}{2} \right)$$

$$\Rightarrow t = \frac{(0.92)(80)}{(20)(0.005)} (50 - 12.5) = 27600 \text{ s}$$

$$\Rightarrow t = 7 \text{ hr } 40 \text{ minute}$$

**ILLUSTRATION 132**

A rod of length  $L$  with thermally insulated lateral surface consists of a material whose heat conductivity coefficient varies with temperature as  $K = \frac{\alpha}{T}$ , where  $\alpha$  is a constant. The ends of the rod are kept at temperatures  $T_1$ , and  $T_2$  ( $T_1 > T_2$ ). Find the function  $T(x)$  where  $x$  is the distance from the end whose temperature is  $T$ , and the heat flow density.

**SOLUTION**

**METHOD-I:**

Since rate of flow of heat is  $q = -KA \frac{dT}{dx}$

So, heat flow density is  $H = \frac{q}{A} = -K \frac{dT}{dx}$

$$\Rightarrow H \int_0^L dx = - \int_{T_1}^{T_2} \frac{\alpha}{T} dT$$

$$\Rightarrow HL = \alpha \log_e \left| \frac{T_1}{T_2} \right|$$

$$\text{Hence } H = \frac{\alpha}{L} \log_e \left| \frac{T_1}{T_2} \right|$$

$$\text{Once again, } H \int_0^x dx = -\alpha \int_{T_1}^T \frac{dT}{T}$$

$$\Rightarrow Hx = \alpha \log_e \left| \frac{T_1}{T} \right|$$

$$\Rightarrow \frac{\alpha x}{L} \log_e \left| \frac{T_1}{T_2} \right| = \alpha \log_e \left| \frac{T_1}{T} \right| \quad \text{[from equation (1)]}$$

$$\Rightarrow T = T_1 \left( \frac{T_2}{T_1} \right)^{\frac{x}{L}}$$

**METHOD-I:**

The rate of heat flow through a conductor is given by

$$q = \frac{dQ}{dt} = \frac{KA\Delta T}{L} \text{ where } K \text{ is the average conductivity.}$$

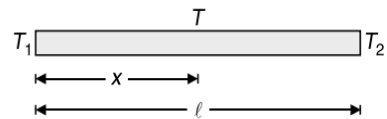
$$K_{av} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} K dT = \frac{\alpha \log_e |T_2/T_1|}{T_2 - T_1}$$

$$\Rightarrow q = \frac{\alpha \log_e |T_2/T_1| \left[ \frac{(T_1 - T_2)A}{l} \right]}{(T_2 - T_1)}$$

So, heat flow density is given by

$$\frac{q}{A} = \frac{\alpha}{l} \log_e \left| \frac{T_1}{T_2} \right| \quad \dots(1)$$

Let  $T$  be the temperature at a distance  $x$  from the left end as shown in the Figure.



Then,  $K'_{av} = \frac{\alpha \log_e |T/T_1|}{T - T_1}$  and heat flow density is

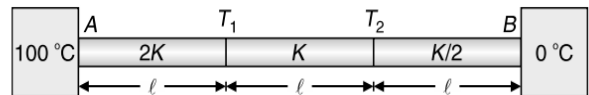
$$\frac{q'}{A} = \frac{\alpha}{x} \ln \left| \frac{T_1}{T} \right| \quad \dots(2)$$

Equating (1) and (2) we get

$$T = T_1 \left( \frac{T_2}{T_1} \right)^{\frac{x}{L}}$$

**ILLUSTRATION 133**

Three rods of same length  $l$  and cross-sectional area  $A$  are joined in series between two heat reservoirs as shown in the figure. Their conductivities are  $2K$ ,  $K$  and  $\frac{K}{2}$  respectively. Assuming that the conductors are lagged from the surroundings find the temperatures  $T_1$  and  $T_2$  at the junction in the steady state condition.



**SOLUTION**

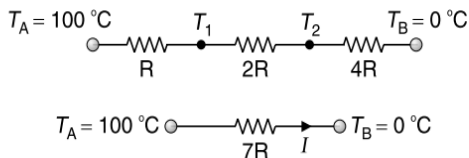
The thermal resistance of the three conductors are

$$R_1 = \frac{l}{2KA}, R_2 = \frac{l}{KA}, R_3 = \frac{2l}{KA}$$

If  $R_1 = R$ , then  $R_2 = 2R$  and  $R_3 = 4R$

Thus, the electric analogy of the heat conduction system is shown in the Figure.

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The equivalent resistance of the system is

$$R_{eq} = R + 2R + 4R = 7R$$

The heat current or thermal current is

$$I = \frac{T_A - T_B}{R_{eq}} = \frac{100 - 0}{7R} = \frac{100}{7R}$$

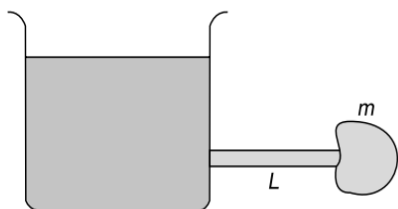
The temperature  $T_1$  and  $T_2$  of the junction are

$$T_1 = T_A - IR = 100 - \left(\frac{100}{7R}\right)R = \frac{600}{7} \text{ } ^\circ\text{C}$$

$$T_2 = T_A - I(R + 2R) = 100 - \left(\frac{100}{7R}\right)(3R) = \frac{400}{7} \text{ } ^\circ\text{C}$$

**ILLUSTRATION 134**

Figure shows a water tank at a constant temperature.  $T_0$  and a small body of mass  $m$  and specific heat  $c$  at a temperature  $T_1$ . Given that  $T_1 < T_0$ . A metal rod of length  $L$ , cross-sectional area  $A$ , having thermal conductivity  $k$  is placed between the tank and the body to connect them. Calculate temperature of body as a function of time. Assume the heat capacity of rod to be negligible.



**SOLUTION**

It is given that the temperature  $T_0$  of water tank is constant and is very large, so heat is conducted through the rod to the small body.

At some instant  $t$ , when the temperature of small body is  $T$ , then rate of heat flow through the rod is

$$\frac{dQ}{dt} = \frac{kA(T_0 - T)}{L} \quad \dots(1)$$

Equation (1) is used for heat conduction in steady state. Since the heat capacity of the rod through which heat is conducting is given to be negligible, so it does not absorb any heat and hence we can assume that it is always in steady state.

Let the temperature of small body rise through  $dT$  on absorbing  $dQ$  amount of heat, so we have

$$dQ = mcdT$$

Now from equation (1), we get

$$\begin{aligned} mc \frac{dT}{dt} &= \frac{kA}{L}(T_0 - T) \\ \Rightarrow \frac{dT}{T_0 - T} &= \frac{kA}{m c L} dt \\ \Rightarrow \int_{T_1}^T \frac{dT}{T_0 - T} &= \frac{kA}{m c L} \int_0^t dt \\ \Rightarrow -\ln\left(\frac{T_0 - T}{T_0 - T_1}\right) &= \frac{kA}{m c L} t \\ \Rightarrow \frac{T_0 - T}{T_0 - T_1} &= e^{-\frac{kAt}{m c L}} \\ \Rightarrow T &= T_0 - (T_0 - T_1)e^{-\frac{kAt}{m c L}} \end{aligned}$$

**ILLUSTRATION 135**

When two bodies of masses  $m_1$  and  $m_2$  with specific heats  $s_1$  and  $s_2$  at absolute temperature  $(T_1)_0$  and  $(T_2)_0$  less than  $(T_1)_0$  are connected by a rod of length  $l$  and cross-sectional area  $A$  with thermal conductivity  $k$ . Calculate the temperature difference of the bodies after time  $t$ . Neglect any heat loss due to radiation at any surface.

**SOLUTION**

At an instant  $t$ , after connecting the two bodies, let their temperatures be  $T_1$  and  $T_2$  respectively ( $T_1 > T_2$ ). At this instant, heat must be flowing from  $m_1$  to  $m_2$  because  $m_1$  is at higher temperature.

If  $dQ$  is the amount of heat flowing through the rod from  $m_1$  to  $m_2$ , then because of flow of heat, if  $dT_1$  is fall in temperature of  $m_1$  and  $dT_2$  is rise in temperature of  $m_2$ , then we have



$$dQ = -m_1 s_1 dT_1 \quad \dots(1)$$

$$\text{and } dQ = m_2 s_2 dT_2 \quad \dots(2)$$

Since this heat  $dQ$  is conducted through the rod, so

$$\begin{aligned} \frac{dQ}{dt} &= \frac{kA(T_1 - T_2)}{l} \\ \Rightarrow dQ &= \frac{kA}{l}(T_1 - T_2) dt \quad \dots(3) \end{aligned}$$

Since this expression for  $dQ$  is given as a function of  $(T_1 - T_2)$ , so from equations (1) and (2), we get

$$\frac{dQ}{m_1 s_1} = -dT_1 \quad \dots(4)$$

and  $\frac{dQ}{m_2 s_2} = dT_2$  ... (5)

So, on adding equations (4) and (5), we get

$$dQ \left( \frac{1}{m_1 s_1} + \frac{1}{m_2 s_2} \right) = -d(T_1 - T_2) \quad \dots (6)$$

Now from equation (3) and (6), we get

$$\frac{kA(T_1 - T_2)}{l} \left( \frac{1}{m_1 s_1} + \frac{1}{m_2 s_2} \right) dt = -d(T_1 - T_2)$$

$$\Rightarrow \frac{d(T_1 - T_2)}{T_1 - T_2} = -\frac{kA}{l} \left( \frac{1}{m_1 s_1} + \frac{1}{m_2 s_2} \right) dt$$

Integrating the expression within proper limits, we get

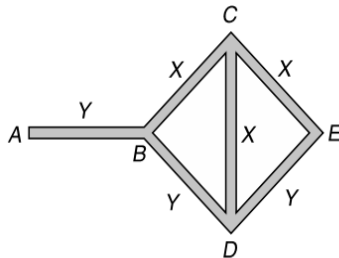
$$\int_{(T_1)_0 - (T_2)_0}^{T_1 - T_2} \frac{d(T_1 - T_2)}{(T_1 - T_2)} = -\int_0^t \frac{kA}{l} \left( \frac{1}{m_1 s_1} + \frac{1}{m_2 s_2} \right) dt$$

$$\Rightarrow \ln \left[ \frac{T_1 - T_2}{(T_1)_0 - (T_2)_0} \right] = -\frac{kA}{l} \left( \frac{1}{m_1 s_1} + \frac{1}{m_2 s_2} \right) t$$

$$\Rightarrow (T_1 - T_2) = [(T_1)_0 - (T_2)_0] e^{-\frac{kA}{l} \left( \frac{1}{m_1 s_1} + \frac{1}{m_2 s_2} \right) t}$$

**ILLUSTRATION 136**

Three rods of material X and three rods of material Y are connected as shown in Figure.



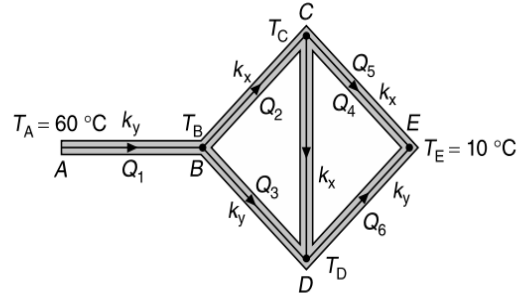
All rods have identical lengths and cross-sectional areas. If the end A is maintained at 60 °C and the junction E at 10 °C, calculate the temperature of junctions B, C and D. The thermal conductivity of X is  $9.2 \times 10^{-2} \text{ kcal m}^{-1} \text{ s}^{-1} \text{ }^\circ\text{C}^{-1}$  and that of Y is  $4.6 \times 10^{-2} \text{ kcal m}^{-1} \text{ s}^{-1} \text{ }^\circ\text{C}^{-1}$ .

**SOLUTION**

Let  $k_x$  and  $k_y$  be the thermal conductivities of X and Y respectively and let  $T_B$ ,  $T_C$  and  $T_D$  be the temperatures of junctions B, C and D respectively. According to the problem, we have

$$T_A = 60 \text{ }^\circ\text{C} \text{ and } T_E = 10 \text{ }^\circ\text{C}$$

In steady state, the rate at which heat enters a junction is equal to the rate at which it leaves that junction. The direction of thermal current also called as rate of flow of heat is denoted by arrows shown in Figure.



At junction B, we have

$$\frac{k_y A (60 - T_B)}{l} = \frac{k_x A (T_B - T_C)}{l} + \frac{k_y A (T_B - T_D)}{l}$$

$$\Rightarrow k_y (T_A - T_B) = k_x (T_B - T_C) + k_y (T_B - T_D)$$

$$\Rightarrow 60 - T_B = \frac{k_x}{k_y} (T_B - T_C) + k_y (T_B - T_D)$$

Given that  $\frac{k_x}{k_y} = \frac{9.2 \times 10^{-2}}{4.6 \times 10^{-2}} = 2$ , so we get

$$(60 - T_B) = 2(T_B - T_C) + (T_B - T_D)$$

$$\Rightarrow 4T_B - 2T_C - T_D = 60 \quad \dots (1)$$

At junction C, we have

$$\frac{k_x A (T_B - T_C)}{l} = \frac{k_x A (T_C - T_D)}{l} + \frac{k_x A (T_C - 10)}{l}$$

$$\Rightarrow -T_B + 3T_C - T_D = 10 \quad \dots (2)$$

At junction D, we have

$$\frac{k_y A (T_B - T_D)}{d} + \frac{k_x A (T_C - T_D)}{d} = \frac{k_y A (T_D - 10)}{d}$$

$$\Rightarrow (T_B - T_D) + \frac{k_x}{k_y} (T_C - T_D) + (T_D - 10)$$

Since  $\frac{k_x}{k_y} = 2$ , so we get

$$(T_B - T_D) + 2(T_C - T_D) = (T_D - 10)$$

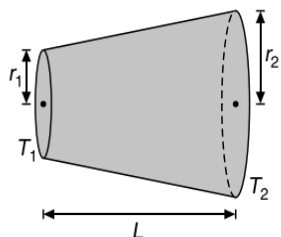
$$\Rightarrow T_B + 2T_C - 4T_D = -10 \quad \dots (3)$$

Solving equation (1), (2) and (3), we get

$$T_B = 30 \text{ }^\circ\text{C} \text{ and } T_C = T_D = 20 \text{ }^\circ\text{C}$$

**ILLUSTRATION 137**

Calculate the heat current through a frustum of a cone made of material having coefficient of thermal conductivity  $k$ , whose two ends are maintained at temperatures  $T_1$  and  $T_2 (> T_1)$  respectively as shown in Figure.

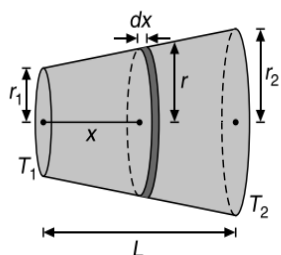


**SOLUTION**

Let us consider an elemental disc at a distance  $x$  from left face. If  $r$  be the radius of this disc, then

$$r = r_1 + \left( \frac{r_2 - r_1}{L} \right) x$$

Thermal resistance of this elemental disc is given by



$$dR_{Th} = \frac{dx}{kA_{disc}} = \frac{dx}{k\pi \left[ r_1 + \left( \frac{r_2 - r_1}{L} \right) x \right]^2}$$

Total thermal resistance of frustum is

$$R_{Th} = \int dR_{Th} = \frac{1}{k\pi} \int_0^L \frac{dx}{\left[ r_1 + \left( \frac{r_2 - r_1}{L} \right) x \right]^2}$$

$$\Rightarrow R_{Th} = \frac{L}{k\pi(r_2 - r_1)} \left[ \frac{1}{r_1 + \left( \frac{r_2 - r_1}{L} \right) x} \right]_0^L$$

$$\Rightarrow R_{Th} = \frac{L}{k\pi(r_2 - r_1)} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{L}{k\pi r_1 r_2}$$

Hence heat current through the frustum is

$$\frac{dQ}{dt} = \frac{T_2 - T_1}{R_{Th}} = \frac{k\pi r_1 r_2 (T_2 - T_1)}{L}$$

**CONVECTION**

Mode of transfer of heat by means of migration of material particles of the medium. It is of two types.

**Natural Convection**

This arises due to difference of densities at two places and is a consequence of gravity because on account of gravity the hot light particles rise up and cold heavy particles try settling down. It mostly occurs on heating a liquid / fluid.

**Forced Convection**

If a fluid is forced to move to take up heat from a hot body then the convection process is called forced convection. In this case Newton's Law of cooling holds good according to which rate of loss of heat from a hot body due to a moving fluid is directly proportional to the surface area of body and excess temperature of body over its surroundings i.e.

$$H \propto A(T - T_0)$$

$$H = -hA(T - T_0)$$

where,  $h$  = constant of proportionality,  $T$  is temperature of body and  $T_0$  is temperature of surrounding. Further

$$H = \frac{dQ}{dt} = -hA(T - T_0)$$

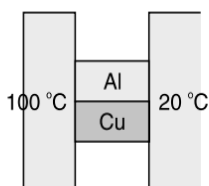
$$\Rightarrow \frac{dT}{dt} = -\frac{hA}{mc}(T - T_0) \quad \{ \because dQ = mcdT \}$$

**Test Your Concepts-X**

**Based on Conduction**

**(Solutions on page H.96)**

- Two metal cubes with 3 cm edges of copper and aluminium are arranged as shown in figure. Calculate the



- total thermal current from one reservoir to the other

- ratio of the thermal current carried by the copper cube to that carried by the aluminium cube. Thermal conductivity of copper is  $401 \text{ Wm}^{-1}\text{K}^{-1}$  and that of aluminium is  $237 \text{ Wm}^{-1}\text{K}^{-1}$ .
- Three rods of copper, brass and steel are welded together to form a Y-shaped structure. The cross-sectional area of each rod is  $4 \text{ cm}^2$ . The end of copper rod is maintained at  $100^\circ\text{C}$  and the ends of the brass and steel rods at  $80^\circ\text{C}$  and  $60^\circ\text{C}$ , respectively. Assume that there is no loss of heat

from the surfaces of the rods. The lengths of rods are: copper 46 cm brass 13 cm and steel 12 cm.

(a) What is the temperature of the junction point?

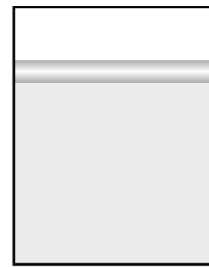
(b) What is the heat current in the copper rod?

$$k(\text{Cu}) = 0.92, k(\text{steel}) = 0.12 \text{ and } k(\text{brass}) = 0.26$$

and all values of  $k$  are in  $\text{calcm}^{-1}\text{s}^{-1}(\text{°C})^{-1}$ .

3. A metallic cylindrical vessel whose inner and outer radii are  $r_1$  and  $r_2$  is filled with ice at  $0\text{ °C}$ . The mass of the ice in the cylinder is  $m$ . Circular portions of the cylinder are sealed with completely adiabatic walls. The vessel is kept in air. Temperature of the air is  $50\text{ °C}$ . How long will it take for the ice to melt completely? Thermal conductivity of the cylinder is  $k$ , its length is  $\ell$  and latent heat of fusion is  $L$ .
4. One end of a copper rod of uniform cross section and of length 1.5 m is kept in contact with ice and the other end with water at  $100\text{ °C}$ . At what point along its length should a temperature of  $200\text{ °C}$  be maintained so that in steady state, the mass of ice melting be equal to that of the steam produced in the same interval of time? Assume that the whole system is insulated from the surroundings. Latent heat of fusion of ice and vaporisation of water are  $80\text{ calg}^{-1}$  and  $540\text{ calg}^{-1}$  respectively.
5. Three rods each of same length and cross-section are joined in series. The thermal conductivity of the materials are  $k$ ,  $2k$  and  $3k$  respectively. If the one end is kept at  $200\text{ °C}$  and the other end is kept at  $100\text{ °C}$ , what would be the temperature of the junctions in the steady state? Assume that no heat is lost due to radiation from the sides of the rods.
6. Two chunks of metal with heat capacities  $C_1$  and  $C_2$  are interconnected by a rod of length  $\ell$  and cross-sectional area  $A$  and fairly low conductivity  $k$ . The whole system is thermally insulated from the environment. At a moment  $t = 0$ , the temperature difference between two chunks of metal equals  $(\Delta T)_0$ . Assuming the heat capacity of the rod to be negligible, find the temperature difference between the chunks as a function of time.
7. A uniform metal bar 100 cm long insulated on the sides, has its ends exposed to ice and steam respectively. If there is a layer of water 0.1 mm thick at each end, calculate the temperature gradient in the bar. Thermal conductivity of the metal is  $1.04\text{ calcm}^{-1}\text{s}^{-1}(\text{°C})^{-1}$  thermal conductivity of the water is  $0.0014\text{ calcm}^{-1}\text{s}^{-1}(\text{°C})^{-1}$ .

8. A cylindrical block of length 0.4 m and area of cross-section  $0.04\text{ m}^2$  is placed coaxially on a thin metal disc of mass 0.4 kg and of the same cross-section. The upper face of the cylinder is maintained at a constant temperature of  $400\text{ K}$  and the initial temperature of the disc is  $300\text{ K}$ . If the thermal conductivity of the material of the cylinder is  $10\text{ Wm}^{-1}\text{K}^{-1}$  and the specific heat of the material of the disk is  $600\text{ Jkg}^{-1}\text{K}^{-1}$ , how long will it take for the temperature of the disc to increase to  $350\text{ K}$ ? Assume for purpose of calculation the thermal conductivity of the disc to be very high and the system to be thermally insulated except for the upper face of the cylinder.
9. The shown container has all insulating surface except bottom through which heat can flow. The bottom has area of cross-section  $A$ , thickness  $a$  and thermal conductivity  $k$ . The movable piston is insulating and massless and no leakage is possible through it. Initially (at time  $t = 0$ ) the gas (monatomic) inside the container is at temperature  $T_0$  and volume  $V_0$ . The surrounding temperature is  $T_s (> T_0)$  and pressure  $P_0$ . Find
  - (a) the temperature of the container as a function of time.
  - (b) the height of piston from the bottom as a function of time if initial height is  $h_0$ .



10. Two identical adiabatic vessels  $A$  and  $B$  each containing  $n$  mole of a monatomic and diatomic gas respectively. Both the vessels are connected by a rod of length  $\ell$  and cross-sectional area  $A$ . Thermal conductivity of rod material is  $k$  and lateral surface of the rod is insulated. At any time  $t = 0$ , the temperature of the gases in the vessel are  $T_1$  and  $T_2$  respectively ( $T_1 > T_2$ ). Neglect heat capacity of the rod and the vessels, find the time when the temperature difference of the vessels becomes half the initial temperature difference.

## RADIATION

Mode of transfer of heat energy in which the heat travels from one point to the other in the form of electromagnetic radiations. *This heat energy is also called radiant energy.*

The process of radiation does not need any material medium for heat transfer. The radiant energy emitted by a body travels in the space just like light.



## Conceptual Note(s)

Thermal radiation, once emitted, is just like an electromagnetic light wave. It, therefore, obeys all the laws of wave theory. The wavelengths are still small compared to the dimensions of usual obstacles encountered, so the rules of geometrical optics are valid, i.e., it travels in a straight line, casts shadow, is reflected and refracted at the change of medium, etc.

### PREVOST'S THEORY OF HEAT EXCHANGE

Prevost's theory of heat exchange states that "each body radiates energy to the surroundings and receives radiant energy from the surroundings". The radiation from each body is emitted regardless of the presence or absence of other bodies. So, all bodies above zero kelvin, absorb and emit energy in the form of radiations. The rise and fall of temperatures, which is observed in a body, is due to the exchange of radiant energy with the surrounding bodies.

The amount of radiant energy emitted per unit time depends on the nature of the emitting surface, its area and its temperature.

The rate of emission of radiant energy is faster at higher temperatures than that at lower temperatures.

*If a body radiates more than what it absorbs, its temperature falls. If a body radiates less than what it absorbs, its temperature rises. In thermal equilibrium with the surroundings, the body radiates and absorbs energy at the same rate.*

So a hotter body placed in a room, will radiate at a faster rate than the rate at which it absorbs, due to which the body suffers a net loss of thermal energy in any given time and hence the temperature of the body falls.

Similarly, when a cold body is kept in warm surroundings, then radiant energy lost to the surroundings is less than the radiant energy absorbed from the surroundings, due to which there is a net increase in the thermal energy of the body and hence the temperature of the body rises.

### PERFECTLY BLACK BODY

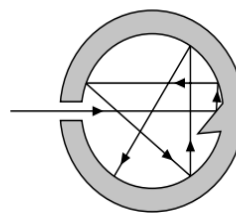
*A body which absorbs all the radiation falling on it is called a blackbody. A black body emits radiation at the fastest rate. The radiation emitted by a blackbody is also called blackbody radiation.*

The radiation inside an enclosure with its inner walls maintained at a constant temperature has the same properties as the blackbody radiation and is also called blackbody radiation. A blackbody is also called an ideal radiator.

A perfect blackbody, absorbing 100% of the radiation falling on it, is only an ideal concept. Among the materials,

lampblack is close to a blackbody. It reflects only about 1% of the radiation falling on it.

The most simple and commonly used black body was designed by Ferry. It consists of an enclosure painted black from inside and a small hole is made in its wall, then the hole acts as a very good blackbody because any radiation that falls on the hole and goes inside the enclosure will have a little or no chance to come out of the hole again. This is because it gets absorbed after multiple reflections inside the enclosure and the wedge shaped obstruction inside the enclosure (directly opposite to the hole) ensures that the incoming radiation does not get reflected directly back to the hole as shown in Figure.



### ABSORPTANCE, REFLECTANCE AND TRANSMITTANCE

Consider an amount of heat radiation  $Q$  incident on a substance. This incident heat gets divided into three parts.

#### Heat Absorbed

A part of the incident heat is kept by the substance and said to be absorbed. Let this amount be  $Q_1$ .

Absorptive power (or absorptance) of the substance is defined as the ratio between amount of heat absorbed by it to the total amount of heat incident upon it.

$$a = \frac{Q_1}{Q} \quad \dots(1)$$

#### Heat Reflected

A part of the incident heat is sent back into the same medium from which it is coming. This portion of heat, say  $Q_2$  is said to be reflected.

Reflective power (or reflectance) of a substance is defined as the ratio between the amount of the heat reflected by the substance to the total amount of heat incident upon it.

$$r = \frac{Q_2}{Q} \quad \dots(2)$$

#### Heat Transmitted

Another part  $Q_3$  of the incident heat radiations pass through body and gets transmitted.

Transmittive power (or transmittance) of a substance is defined as the ratio between amount of heat transmitted by the body to the total amount of heat incident upon it.

$$t = \frac{Q_3}{Q} \quad \dots(3)$$

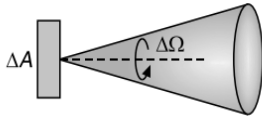
Adding equation (1), (2) and (3), we have

$$a + r + t = \frac{Q_1}{Q} + \frac{Q_2}{Q} + \frac{Q_3}{Q} = \frac{Q}{Q} = 1$$

$$\Rightarrow a + r + t = 1$$

### EMISSIVE POWER

All bodies emit heat radiations at all temperatures. The energy emitted by them comprises of all wavelengths ranging from zero to infinity. Amount of energy emitted by them depends upon the nature and area of the body and also upon the time for which the emission is considered.

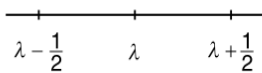


Consider a small area  $\Delta A$  of a body emitting thermal radiation. Consider a small solid angle  $\Delta\Omega$  about the normal to the radiating surface. Let the energy radiated by the area  $\Delta A$  of the surface in the solid angle  $\Delta\Omega$  in time  $\Delta t$  be  $\Delta U$ , then emissive power of the body is defined as the energy radiated per second per unit area per unit solid angle along the normal to the radiating surface. Mathematically, we have

$$E = \frac{\Delta U}{(\Delta t)(\Delta A)(\Delta\Omega)}$$

### Monochromatic Emittance

Monochromatic emittance or monochromatic emissive power or spectral emissive power  $e_\lambda$  of a body at a temperature  $T$  for a wavelength  $\lambda$ , is defined as the energy radiated, in vacuum, per unit time, per unit area per unit range of wavelength i.e. lying between  $\left(\lambda - \frac{1}{2}\right)$  to  $\left(\lambda + \frac{1}{2}\right)$ .



$e_\lambda$  for a body will be different for different values of  $\lambda$  and for different values of  $T$ .

### Radiant Emittance

Radiant emittance of body at a temperature  $T$  is defined as the total amount of energy (for all wavelengths) radiating per unit time, per unit area by the body. If  $dE$  is the amount of energy radiated per second per unit area for wavelength range  $d\lambda$  then  $dE = e_\lambda d\lambda$ . So, radiant emittance  $E$  is given by

$$E = \int_0^E dE = \int_0^\infty e_\lambda d\lambda$$

Units of  $E$  in CGS system are  $\text{ergcm}^{-2}\text{s}^{-1}$  and in SI system are  $\text{Jm}^{-2}\text{s}^{-1}$ .

### EMISSIVITY

The emissivity  $e$  of a body, at a particular temperature, is the ratio of the emissive power of a body to the emissive power of a black body at the same temperature. It is a unitless quantity. Mathematically, we have

$$e = \frac{E_{\text{body}}}{E_{\text{black body}}} = \frac{E}{E_0}$$

Its value lies between 0 and 1.

### ENERGY DENSITY (U)

It deals with the amount of heat energy present, at any point, in the medium surrounding the source.

### Spectral Energy Density

Spectral energy density, at any point, is defined as the radiant energy per unit volume, around that point, for wavelengths lying in a unit range around  $\lambda$  i.e., between  $\left(\lambda - \frac{1}{2}\right)$  and  $\left(\lambda + \frac{1}{2}\right)$ . It depends upon the value of  $\lambda$  as well as temperature of the source of heat.

### Total Energy Density

Total energy density at any point is defined as the radiant energy per unit volume, around the point, for all wavelengths taken together.

If  $du$  is the amount of energy per unit volume for wavelengths lying in between  $\left(\lambda - \frac{d\lambda}{2}\right)$  to  $\left(\lambda + \frac{d\lambda}{2}\right)$ , i.e. in wavelength range  $d\lambda$ , then

$$du = u_\lambda d\lambda$$

$$\Rightarrow u = \int_0^u du = \int_0^\infty u_\lambda d\lambda$$



### Conceptual Note(s)

Emissive power and radiant emittance are constants of the source while the energy density varies from point to point.

### ABSORPTIVE POWER

Absorptive power of a body is defined as the fraction of the incident radiation that is absorbed by the body. If we denote the absorptive power by  $a$ ,

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$$a = \frac{\text{Energy Absorbed}}{\text{Energy Incident}}$$

As all the radiation incident on a blackbody is absorbed, the absorptive power of a blackbody is unity. *Please note that the absorptive power is a dimensionless quantity but the emissive power is not.*

### KIRCHHOFF'S LAW OF HEAT RADIATIONS

Consider two bodies, one polished and the other painted black, having equal surface areas, suspended in a room. After a sufficiently long time, the temperature of both the bodies will be same as the room temperature. Since the bodies have equal surface areas, so equal amount of radiation falls on the two surfaces. The polished body reflects a large part of radiation but absorbs a little, whereas the black painted body absorbs a large part of radiation but reflects a little part. Since the temperature of each body remains constant, so we conclude that the polished surface radiates at a slower rate and the black-painted surface radiates at a faster rate, hence making us conclude that *good absorbers of radiation are also good emitters.*

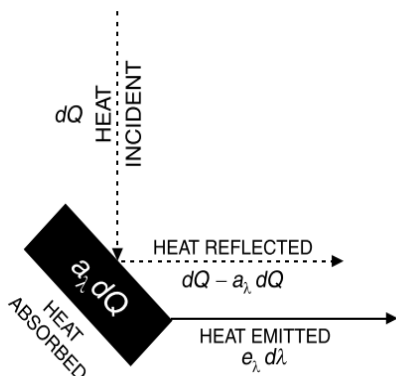
*This can also be concluded by studying Kirchhoff's Law of Heat radiation which states that "at any temperature, the ratio of emissive power  $e_\lambda$  of a body to its absorptive power  $a_\lambda$ , for a particular wavelength, is always constant and is equal to the emissive power of a perfect black body ( $E_\lambda$ ) for that wavelength."*

Hence,  $\frac{e_\lambda}{a_\lambda} = \text{constant}$

If  $E_\lambda$  and  $A_\lambda$  are the emissive power and absorptive power for a perfect black body, then  $\frac{e_\lambda}{a_\lambda} = \frac{E_\lambda}{A_\lambda}$ . For a perfect black body, since  $A_\lambda = 1$  So,

$$\frac{e_\lambda}{a_\lambda} = E_\lambda$$

Let  $dQ$  be a small amount of heat incident per second per unit area on a body. Incident heat radiations have their wavelengths spread in a very narrow range  $d\lambda$ , i.e., between  $\left(\lambda - \frac{d\lambda}{2}\right)$  and  $\left(\lambda + \frac{d\lambda}{2}\right)$ .



Therefore, heat energy absorbed per second per unit area of the body (retained by the body)  $= a_\lambda dQ$  where,  $a_\lambda$  is the absorptive power of the body for that wavelength.

The body emits certain amount of energy by virtue of its own temperature. If  $e_\lambda$  is the emissive power of the body for that wavelength. Then

$$\left( \begin{array}{l} \text{Energy radiated per second} \\ \text{per unit area of the body} \end{array} \right) = e_\lambda d\lambda$$

$$\left( \begin{array}{l} \text{Total amount of heat} \\ \text{going away from body} \end{array} \right) = (1 - a_\lambda) dQ + e_\lambda d\lambda$$

In equilibrium, this heat must be same as that incident on the body. This is in accordance with the Law of Conservation of Energy. So, we have

$$\begin{aligned} dQ &= (1 - a_\lambda) dQ + e_\lambda d\lambda \\ \Rightarrow dQ &= dQ - a_\lambda dQ + e_\lambda d\lambda && \dots(1) \\ \Rightarrow e_\lambda d\lambda &= a_\lambda dQ \end{aligned}$$

In the case of a perfect black body,

$$\begin{aligned} e_\lambda &= E_\lambda \text{ and } a_\lambda = 1 \\ \Rightarrow E_\lambda d\lambda &= 1(dQ) \\ \Rightarrow dQ &= E_\lambda d\lambda \end{aligned}$$

Substituting for  $dQ$ , we get  $e_\lambda d\lambda = a_\lambda (E_\lambda d\lambda)$

$$\Rightarrow \frac{e_\lambda}{a_\lambda} = E_\lambda$$

### ILLUSTRATION 138

Radiant energy of 100 units is incident on a surface. This incident radiant energy has 20 units of wavelength  $\lambda_1$ , 30 units are of wavelength  $\lambda_2$  and the remaining of other wavelengths. The total 60 units of energy is absorbed by the surface. In this 60 units, 5 units are of  $\lambda_1$  and 25 units are of  $\lambda_2$ . Find  $a$ ,  $a_{\lambda_1}$  and  $a_{\lambda_2}$ .

### SOLUTION

$$\text{Total absorptive power } a = \frac{60}{100} = 0.6$$

$$\text{Spectral absorptive power for } \lambda_1 \text{ is } a_{\lambda_1} = \frac{5}{20} = 0.25$$

$$\text{Spectral absorptive power for } \lambda_2 \text{ is } a_{\lambda_2} = \frac{25}{30} = 0.83$$

From this example it is clear that total absorption power of the surface is only 0.6 whereas  $a_{\lambda_2}$  is 0.83 ( $> 0.6$ ) i.e. the surface is good absorber of wavelength  $\lambda_2$ .

### APPLICATIONS OF KIRCHHOFF'S LAW

According to Kirchhoff's Law, the ratio of emissive power to the absorptive power of a body is a constant quantity. It means a body having greater emissive power must have

greater absorptive power also, i.e., good absorbers must be good radiators. A number of application based upon this fact are described below.

- Let a polished metal with a dark spot on it be heated to a high temperature and taken into a dark room. The spot begins to shine brightly indicating that good absorbers are good radiators.
- Let a piece of decorated china be heated to a high temperature and taken into room. The coloured decorations which appeared dark (due to their greater absorptive powers) begin to shine brightly inside a dark room (due to their greater emissive powers).
- Kirchhoff's Law provides an explanation to the Fraunhofer lines (a set of dark lines present in solar spectrum).
- Outer walls of thermos flask are made shining. These shining walls neither absorb more heat from the surrounding nor radiate more heat to the surroundings. Thus, the temperature of bodies inside the flask is preserved.



### Conceptual Note(s)

Kirchhoff's Law of thermal radiations implies that ratio between  $e_\lambda$  and  $a_\lambda$  for any-body is constant

Quantity ( $= E_\lambda$ ). Alternative it can be expressed in the form that the substances

which are good absorbers must be good radiators (or emitters).

### STEFAN'S LAW OR STEFAN'S-BOLTZMANN LAW OR STEFAN'S FOURTH POWER LAW

According to this law, the radiant energy emitted per second i.e.,  $u = -\frac{dQ}{dt}$  by a perfect black body placed in surroundings at 0 K is directly proportional to the fourth power of the absolute temperature ( $T$ ) i.e.,

$$u = -\frac{dQ}{dt} = \sigma AT^4 \quad \dots(1)$$

where,  $\sigma$  a universal constant called the Stefan-Boltzmann constant (or Stefan's constant) whose value is given by

$$\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$$

The negative sign in equation (1) denotes the loss in energy of the body in the form of radiation.

A body which is not a blackbody will emit radiant energy less than that given by equation (1). It will however be proportional to  $T^4$ . The energy emitted per second per unit area by such a body can be written as

$$u' = -\frac{dQ}{dt} = e\sigma AT^4 \quad \dots(2)$$

where  $e$  is a constant for the given surface having a value lying between 0 and 1. This constant is called the emissivity of the surface. It is zero for completely reflecting surface and is unity for a blackbody.

According to Kirchhoff's law, we have

$$\frac{E_{\text{body}}}{E_{\text{black body}}} = a \quad \dots(3)$$

where  $a$  is the absorptive power of the body. The emissive power  $E$  is proportional to the energy radiated per unit time i.e.  $u$ . Substituting equations (1) and (2) in (3), we get

$$\frac{e\sigma AT^4}{\sigma AT^4} = a$$

$$\Rightarrow e = a$$

Hence, emissivity and absorption power have the same value.

If a body of emissivity  $e$  is kept in thermal equilibrium with surroundings at temperature  $T_0$  (also called as Ambient Temperature), then the rate at which radiant energy is absorbed by the body equals the rate at which the radiant energy is emitted by the body. Hence the radiant energy absorbed or emitted per second by the body is

$$u = \left| \frac{dQ}{dt} \right| = e\sigma AT_0^4$$

However, if the temperature of body is changed to  $T (> T_0)$  but the ambient temperature remains  $T_0$ , then the radiant energy emitted per second by the body is

$$u_{\text{emitted}} = e\sigma AT^4$$

The radiant energy absorbed by the body per second is

$$u_{\text{absorbed}} = e\sigma AT_0^4$$

Hence the net loss in thermal energy of the body per unit time is

$$\Delta u = u_{\text{emitted}} - u_{\text{absorbed}} = e\sigma A(T^4 - T_0^4)$$

Similarly, if a black body at temperature  $T$  is placed in surroundings at temperature  $T_0 (< T)$ , then energy radiated per second per unit area is

$$-\frac{dQ}{dt} = \sigma A(T^4 - T_0^4)$$

### ILLUSTRATION 139

A sphere, a cube and a thin circular plate all made of the same material and having the same mass are initially heated to 200 °C. Which of these objects will cool fastest and which one slowest when left in air at room temperature?

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#### SOLUTION

When body cools by radiation, according to Stefan's Law,

$$\frac{dT}{dt} = -\frac{eA\sigma}{mc}(T^4 - T_0^4)$$

Here  $m, c, e, T$  and  $T_0$  are same for all bodies.

$$\Rightarrow \frac{dT}{dt} \propto A \quad \text{where } A \text{ is the surface area}$$

Since for a given mass area of plate is maximum while of sphere is minimum the disc will cool fastest while sphere slowest.

#### ILLUSTRATION 140

A body which has a surface area  $5 \text{ cm}^2$  and a temperature of  $727^\circ\text{C}$  radiates  $300 \text{ J}$  of energy each minute. Calculate the emissivity of the body if Stefan's constant is  $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ .

#### SOLUTION

According to Stefan's Law, we have energy radiated per second by a body is  $P = e\sigma AT^4$ , where

$$P = \left(\frac{300 \text{ J}}{60 \text{ s}}\right) = 5 \text{ Js}^{-1}, T = 273 + 727 = 1000 \text{ K}$$

$$\Rightarrow e = \frac{5}{(5.67 \times 10^{-8})(5 \times 10^{-4})(10^3)^4} = \frac{1}{5.67} = 0.18$$

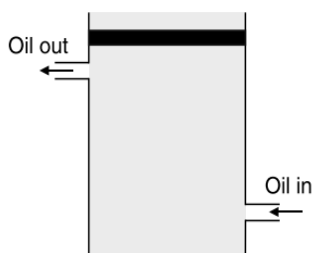
#### ILLUSTRATION 141

The top of an insulated cylindrical container is covered by a disc having emissivity  $0.6$  and conductivity  $0.167 \text{ WKm}^{-1}$  and thickness  $1 \text{ cm}$ . The temperature is maintained by circulating oil as shown.

(a) Find the radiation loss to the surroundings in  $\text{Jm}^{-2}\text{s}$  if temperature of the upper surface of disc is  $127^\circ\text{C}$  and temperature of surroundings is  $27^\circ\text{C}$ .

(b) Also find the temperature of the circulating oil. Neglect the heat loss due to convection.

$$\left(\text{Given: } \sigma = \frac{17}{3} \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}\right)$$



#### SOLUTION

(a) Rate of heat loss per unit area due to radiation is given by

$$I = e\sigma(T^4 - T_0^4)$$

where,  $T = 127 + 273 = 400 \text{ K}$

and  $T_0 = 27 + 273 = 300 \text{ K}$

$$\Rightarrow I = 0.6 \times \frac{17}{3} \times 10^{-8} ((400)^4 - (300)^4)$$

$$\Rightarrow I = 59 \text{ Wm}^{-2}$$

(b) Let  $\theta$  be the temperature of the oil, then

$$\left(\begin{array}{c} \text{Rate of Heat} \\ \text{Through Conduction} \end{array}\right) = \left(\begin{array}{c} \text{Rate of Heat Loss} \\ \text{Due to Radiation} \end{array}\right)$$

$$\Rightarrow \frac{\text{Temperature Difference}}{\text{Thermal Resistance}} = 595A$$

$$\Rightarrow \frac{(\theta - 127)}{\left(\frac{l}{KA}\right)} = 595A$$

where,  $A$  is the area of the disc,  $K$  is the coefficient of thermal conductivity and  $l$  is the thickness (or length) of disc.

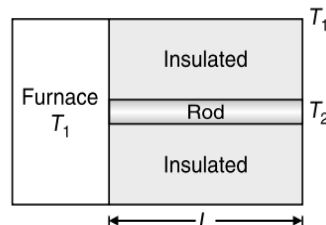
$$\Rightarrow (\theta - 127) \frac{K}{l} = 595$$

$$\Rightarrow \theta = 595 \left(\frac{l}{K}\right) + 127 = \frac{595 \times 10^{-2}}{0.167} + 127$$

$$\Rightarrow \theta = 162.6^\circ\text{C}$$

#### ILLUSTRATION 142

One end of a rod of length  $L$  and cross-sectional area  $A$  is kept in a furnace of temperature  $T_1$ . The other end of the rod is kept at a temperature  $T_2$ . The thermal conductivity of the material of the rod is  $K$  and emissivity of the rod is  $e$ . It is given that  $T_2 = T_S + \Delta T$ , where  $\Delta T \ll T_S$ ,  $T_S$  being the temperature of the surroundings. If  $\Delta T = \alpha(T_1 - T_S)$ , find the proportionality constant  $\alpha$ . Consider that heat is lost only by radiation at the end where the temperature of the rod is  $T_2$ .



#### SOLUTION

$$\text{Since, } \left(\begin{array}{c} \text{Rate of Heat} \\ \text{Conduction} \\ \text{Through Rod} \end{array}\right) = \left(\begin{array}{c} \text{Rate of Heat} \\ \text{Lost from Right} \\ \text{End of the Rod} \end{array}\right)$$

$$\Rightarrow \frac{KA(T_1 - T_2)}{L} = eA\sigma(T_2^4 - T_S^4) \quad \dots(1)$$

Given that  $T_2 = T_S + \Delta T$

$$\Rightarrow T_2^4 = (T_S + \Delta T)^4 = T_S^4 \left(1 + \frac{\Delta T}{T_S}\right)^4$$

Using Binomial Expansion, we get

$$T_2^4 = T_S^4 \left(1 + 4 \frac{\Delta T}{T_S}\right) \quad \{\because \Delta T \ll T_S\}$$

$$\Rightarrow T_2^4 - T_S^4 = 4(\Delta T)(T_S^3)$$

Substituting in Equation (1), we have

$$\frac{K(T_1 - T_S - \Delta T)}{L} = 4e\sigma T_S^3 \Delta T$$

$$\Rightarrow \frac{K(T_1 - T_S)}{L} = \left(4e\sigma T_S^3 + \frac{K}{L}\right) \Delta T$$

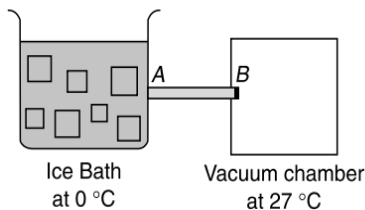
$$\Rightarrow \Delta T = \frac{K(T_1 - T_S)}{4e\sigma L T_S^3 + K} = \alpha(T_1 - T_S)$$

Comparing with the given relation, we get proportionality constant  $\alpha$  to be

$$\alpha = \frac{K}{4e\sigma L T_S^3 + K}$$

### ILLUSTRATION 143

A cylindrical rod of 50 cm length having cross-sectional area 1 cm<sup>2</sup> is used as a conducting material between an ice bath at 0 °C and a vacuum chamber at 27 °C as shown in Figure.



The end of rod which is inside the vacuum chamber behaves like a black body and is at temperature 27 °C in steady state. Find the thermal conductivity of the material of rod and rate at which ice is melting in the ice bath. Given that latent heat of fusion of ice is  $3.35 \times 10^5$  Jkg<sup>-1</sup>.

### SOLUTION

In steady state, heat absorbed per second by the end of the rod which is in vacuum chamber by radiation is fully conducted per second to the ice bath through the rod.

$$\Rightarrow \frac{kA(T_B - T_A)}{L} = \sigma A(T_{vc}^4 - T_B^4)$$

$$\Rightarrow \frac{k(27^\circ\text{C} - 0^\circ\text{C})}{0.5} = 5.67 \times 10^{-8} [(300)^4 - (290)^4]$$

$$\Rightarrow k = \frac{(5.67 \times 10^{-8}) [(300)^4 - (290)^4] (0.5)}{27}$$

$$\Rightarrow k = 1.08 \text{ Wm}^{-1}\text{°C}$$

So, the rate of heat obtained by the ice bath is

$$\frac{dQ}{dt} = \frac{kA(T_A - T_B)}{l} = \frac{1.08 \times 1 \times 10^{-4} \times 27}{0.5}$$

$$\Rightarrow \frac{dQ}{dt} = 5.83 \times 10^{-3} \text{ Js}^{-1}$$

This heat is used to melt the ice in ice bath. If  $m$  is the mass of ice being melted per second, then

$$\frac{dQ}{dt} = mL$$

$$\Rightarrow 5.83 \times 10^{-3} = m \times 3.35 \times 10^5$$

$$\Rightarrow m = 1.74 \times 10^{-8} \text{ kgs}^{-1}$$

### ILLUSTRATION 144

A solid copper sphere, at an initial temperature of 200 K, having density  $\rho$ , emissivity  $e$ , gram specific heat  $c$  and radius  $r$  is suspended inside a chamber whose walls are almost at 0 K. Calculate the time required for the temperature of the sphere to drop to 100 K.

### SOLUTION

According to Stefan's Law the rate of loss of energy due to radiation

$$P = eA\sigma T^4 \quad \dots(1)$$

Now if the rate of change of temperature of a body of mass  $m$  and specific heat  $c$  is  $\left(\frac{dT}{dt}\right)$  the rate of loss of heat will be

$$\frac{dQ}{dt} = mc \frac{dT}{dt} \quad \dots(2)$$

As Equation (1) and (2) represents the same quantity of energy

$$-mc \frac{dT}{dt} = e\sigma AT^4$$

The negative sign shows that temperature decreases with time. Since  $m = \left(\frac{4}{3}\right)\pi r^3 \rho$  and  $A = 4\pi r^2$ ,

$$-\frac{dT}{dt} = \left(\frac{3e\sigma}{\rho cr}\right) T^4$$

$$\Rightarrow -\int_0^t dt = \frac{r\rho c}{3e\sigma} \int \frac{dT}{T^4}$$

$$\Rightarrow t = \frac{r\rho c}{9e\sigma} \left(\frac{1}{T^3}\right)_{200}^{100} = \frac{7r\rho c}{72e\sigma} \times 10^{-6} \text{ s}$$

### NEWTON'S LAW OF COOLING (SPECIAL CASE OF STEFAN'S LAW)

This law is the special case of Stefan's law applicable for comparable temperatures of body  $T$  and surroundings  $T_0$  i.e.,  $T \approx T_0$  but  $T - T_0 \neq 0$ .

According to Newton's Law of Cooling (NLC), "for comparable temperatures of the body and surroundings, the rate of loss of heat radiated by a body (or the rate of cooling) is proportional to average excess temperature".

This relation can be established using Stefan's Law, according to which

$$-\frac{dQ}{dt} = \sigma A(T^4 - T_0^4)$$

Negative sign indicates loss of heat and  $A$  is the surface area of the body.

$$\Rightarrow -\frac{dQ}{dt} = \sigma A(T^2 + T_0^2)(T + T_0)(T - T_0) \quad \dots(1)$$

For comparable temperatures of body and surroundings, we have  $T \approx T_0$  but  $T - T_0 \neq 0$ , so

$$T + T_0 = 2T_0 \text{ and } T^2 + T_0^2 = 2T_0^2$$

$$\Rightarrow -\frac{dQ}{dt} = (4\sigma AT_0^3)(T - T_0) \quad \dots(2)$$

$$\Rightarrow -\frac{dQ}{dt} \propto (T - T_0)$$

where  $T$  is mean temperature of body and  $T_0$  that of surroundings.

If a body cools from temperature  $T_i$  to a temperature  $T_f$  when placed in surroundings at temperature  $T_0$  such that  $T_i > T_f > T_0$ , then  $(T - T_0)$  is called the average excess temperature given by

$$(T - T_0) = (T_{av} - T_0) = \left(\frac{T_i + T_f}{2}\right) - T_0$$

If  $m$  is the mass of body,  $c$  its specific heat and  $dT$  the change in temperature, then  $dQ = mc dT$

So from equation (2), we get

$$-mc \left(\frac{dT}{dt}\right) \propto A(T - T_0)$$

For a spherical body

$$A = 4\pi r^2, m = V\rho = \frac{4}{3}\pi r^3 \rho$$

So, rate of loss of heat by the body

$$\frac{dQ}{dt} = k(T - T_0)4\pi r^2$$

where  $k$  is constant of proportionality and depends on emissive power of body.

Rate of cooling  $-\frac{dT}{dt}$  is given by

$$-mc \frac{dT}{dt} = k(T - T_0)A$$

$$\Rightarrow -\left(\frac{4}{3}\pi r^3 \rho\right)c \frac{dT}{dt} = k(T - T_0)4\pi r^2$$

$$\Rightarrow -\frac{dT}{dt} = \frac{3k(T - T_0)}{r\rho c} \propto \frac{1}{r\rho c}$$

### SOLUTION TO DIFFERENTIAL EQUATION INVOLVED IN NEWTON'S LAW OF COOLING

According to Newton's Law of Cooling

$$-\frac{1}{A} \frac{dQ}{dt} = 4\sigma T_0^3 (T - T_0)$$

$$\Rightarrow -\frac{1}{A} mc \left(\frac{dT}{dt}\right) = 4\sigma T_0^3 (T - T_0)$$

$$\Rightarrow -\frac{dT}{T - T_0} = \frac{4\sigma AT_0^3}{mc} dt$$

$$\Rightarrow -\int_{T_1}^{T_2} \frac{dT}{T - T_0} = \frac{4\sigma AT_0^3}{mc} \int_0^t dt$$

$$\Rightarrow \log_e (T - T_0) \Big|_{T_1}^{T_2} = -\left(\frac{4\sigma AT_0^3}{mc}\right)t$$

$$\Rightarrow \log_e (T_2 - T_0) - \log_e (T_1 - T_0) = -\left(\frac{4\sigma AT_0^3}{mc}\right)t$$

$$\Rightarrow \log_e \left(\frac{T_2 - T_0}{T_1 - T_0}\right) = -\left(\frac{4\sigma AT_0^3}{mc}\right)t \quad \dots(1)$$

$$\Rightarrow t = \frac{mc}{4\sigma AT_0^3} \log_e \left(\frac{T_1 - T_0}{T_2 - T_0}\right)$$

Alternately from (1), we get

$$\frac{T_2 - T_0}{T_1 - T_0} = e^{-kt}, \text{ where } k = \frac{4\sigma AT_0^3}{mc}$$

$$\Rightarrow T_2 = T_0 + (T_1 - T_0)e^{-kt} \quad \dots(2)$$

Equation (2), shows that rate of fall of temperature also falls exponentially with time.

#### ILLUSTRATION 145

A body cools in 7 minutes from  $60^\circ\text{C}$  to  $40^\circ\text{C}$ . What will be its temperature after the next 7 minutes? The temperature of surroundings is  $10^\circ\text{C}$ .

#### SOLUTION

##### METHOD-I:

According to Newton's Law of Cooling,

$$\left(\frac{T_1 - T_2}{t}\right) = K \left(\frac{T_1 + T_2}{2}\right) - T_0$$

$$\text{So that } \left(\frac{60-40}{7}\right) = K\left(\frac{60+40}{2}\right) - 10$$

$$\Rightarrow K = \frac{1}{14} \quad \dots(1)$$

Now if after cooling from 40 °C for 7 minutes the temperature of the body becomes  $T$ , according to Newton's Law of Cooling

$$\left(\frac{40-T}{7}\right) = K\left(\frac{T+40}{2}\right) - 10$$

which in the light of equation (1), i.e., gives

$$\left(\frac{40-T}{7}\right) = \frac{1}{14}\left(\frac{20+T}{2}\right)$$

$$\text{i.e. } 160 - 4T = 20 + T$$

$$\Rightarrow T = 28 \text{ }^\circ\text{C}$$

**METHOD-II (NOT ADVISABLE):**

According to Newton's Law of Cooling, we have

$$-\frac{dT}{dt} = K(T - T_0)$$

$$\Rightarrow \int_0^t dt = \frac{1}{K} \int_{T_1}^{T_2} -\frac{dT}{(T - T_0)}$$

$$\Rightarrow t = \frac{1}{K} \log_e \left( \frac{T_1 - T_0}{T_2 - T_0} \right)$$

$$\text{In first case, } 7 = \frac{1}{K} \log_e \left( \frac{60-10}{40-10} \right) \quad \dots(1)$$

$$\text{In second case, } 7 = \frac{1}{K} \log_e \left( \frac{40-10}{T-10} \right) \quad \dots(2)$$

$$\Rightarrow \log \left( \frac{50}{30} \right) = \log \left( \frac{30}{T-10} \right)$$

$$\Rightarrow \frac{5}{3} = \frac{30}{T-10}$$

$$\Rightarrow 5T - 50 = 90$$

$$\Rightarrow T = 28 \text{ }^\circ\text{C}$$

**ILLUSTRATION 146**

A calorimeter of water equivalent 100 g cools in air in 18 min from 60 °C to 40 °C. When a block of metal of mass 60 g is heated to 60 °C and placed inside the calorimeter. Assume heat loss only by radiation and Newton's Law of cooling to be valid. Find the specific heat of metal if now the system cools from 60 °C to 40 °C in 20 min.

**SOLUTION**

According to Newton's Law of Cooling, we have

$$\frac{T_i - T_f}{t} = \frac{4\sigma AT_0^3}{mc} \left( \frac{T_i + T_f}{2} - T_0 \right)$$

$$\Rightarrow \frac{T_i - T_f}{t} = \frac{k}{w} \left( \frac{T_i + T_f}{2} - T_0 \right)$$

where,  $w = mc$  is the water equivalent (or the heat capacity of the body) and  $k = 4\sigma AT_0^3$  is the constant which depends on surrounding temperature and surface area of body exposed to surrounding.

Since the calorimeter of water equivalent 100 g takes 18 min to cool down from 60° to 40 °C, so we have

$$\frac{60-40}{18} = \frac{k}{100} \left( \frac{60+40}{2} - T_0 \right)$$

$$\Rightarrow \frac{10}{9} = \frac{k}{100} (50 - T_0) \quad \dots(1)$$

Now when metal block of specific heat  $c$  be placed in the calorimeter, then it takes 20 min to cool down from 60 °C to 40 °C, so we have

$$\frac{60-40}{20} = \frac{k}{100+60c} \left( \frac{60+40}{2} - T_0 \right)$$

$$\Rightarrow 1 = \frac{k}{100+60c} (50 - T_0) \quad \dots(2)$$

Dividing equation (1) by (2), we get

$$\frac{10}{9} = \frac{100+60c}{100}$$

$$\Rightarrow 100 = 900 + 540c$$

$$\Rightarrow c = \frac{100}{540} = 0.185 \text{ cal g}^{-1}\text{K}^{-1}$$

**ILLUSTRATION 147**

A metal ball of 1 kg is heated by a 20 W heater in a room at 20 °C. After sometime temperature of ball becomes steady at 50 °C. Calculate the rate of loss of heat to the surrounding when its temperature was 30 °C. Also find the rate at which it loses heat to the surrounding when its temperature was 30 °C.

**SOLUTION**

According to the problem, when ball is at 50 °C, its temperature becomes steady. So, the rate at which heat is being lost by the ball equals the rate at which heat is being supplied to the ball by the heater i.e. 20 W. So, from Newton's Law of cooling, we get

$$20 = k(50 - 20)$$

$$\Rightarrow k = \frac{2}{3} \quad \dots(1)$$

When ball was at 30 °C, then according to Newton's Law of cooling, we have

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$$\left| \frac{dQ}{dt} \right| = k(30 - 20) = 10k \quad \dots(2)$$

$$\Rightarrow \frac{dQ}{dt} = 10 \times \frac{2}{3} = \frac{20}{3} \text{ W}$$

### SOLAR CONSTANT AND TEMPERATURE OF SUN

Solar constant is defined as the amount of energy received from the sun by the earth per minute per of surface placed normally to the sun's rays at mean distance of the earth from the sun in the absence of atmosphere. The value of solar constant is

$$S = 2 \text{ calmin}^{-1}\text{cm}^{-2} = 1.4 \text{ kWm}^{-2}$$

The temperature  $T$  of sun is given by

$$T^4 = \frac{S}{\sigma} \left( \frac{r}{R} \right)^2$$

where,  $S$  is the solar constant,  $\sigma$  is Stefan's constant,  $r$  is the mean distance of earth from sun and  $R$  is the radius of sun. This formula can be derived as follows.

Let  $R$  and  $T$  be the radius and absolute temperature of the sun. Considering the sun as a black body, the heat radiated per unit time is given by

$$u = -\frac{dQ}{dt} = \sigma T^4 (4\pi R^2)$$

The above radiated energy would spread over a surface area of  $4\pi r^2$ . So, energy received per unit time per unit surface area is

$$S = \frac{\sigma T^4 (4\pi R^2)}{4\pi r^2} = \sigma T^4 \left( \frac{r}{R} \right)^2 = \frac{\sigma T^4 R^2}{r^2}$$

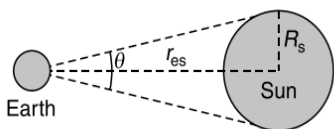
$$\Rightarrow T^4 = \frac{S r^2}{\sigma R^2} = \frac{S}{\sigma} \left( \frac{r}{R} \right)^2$$

#### ILLUSTRATION 148

The earth receives solar energy at the rate of  $2 \text{ calmin}^{-1}\text{cm}^{-2}$ . Assuming the radiation to be black body in nature, estimate the surface temperature of the sun. Given that  $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$  and angular diameter of the sun 32 minute of arc.

#### SOLUTION

Let the surface temperature of sun be  $T_s$ , then total energy radiated by sun per second is given by



$$u = -\frac{dQ}{dt} = \sigma T_s^4 (4\pi R_s^2)$$

Energy received per second by the earth per unit area is given by

$$u_{\text{earth}} = \frac{|u|}{4\pi r_{es}^2} = \sigma T_s^4 \left( \frac{R_s}{r_{es}} \right)^2 \quad \dots(1)$$

Since the angular diameter of sun as observed from earth is 32 minute of arc, so we gave

$$R_s = r_{es} \left( \frac{\theta}{2} \right)$$

$$\frac{R_s}{r_{es}} = \frac{\theta}{2} = \frac{1}{2} \left( \frac{32}{60} \right) = 4.655 \times 10^{-3} \text{ radian}$$

Since, we are given that

$$u_{\text{earth}} = \frac{2(4.2)(10^4)}{60} \text{ Js}^{-1}\text{m}^{-2} \quad \dots(2)$$

From equations (1) and (2), we get

$$\frac{2 \times 4.2 \times 10^4}{60} = (5.67 \times 10^{-8}) T_s^4 (4.655 \times 10^{-3})^2$$

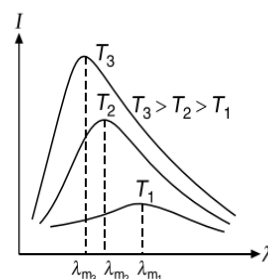
$$\Rightarrow T_s^4 = \frac{2 \times 4.2 \times 10^4}{(60)(5.67 \times 10^{-8})(4.655 \times 10^{-3})^2} = 1.14 \times 10^{15}$$

$$\Rightarrow T_s = 5810.67 \text{ K}$$

### BLACK BODY RADIATION SPECTRUM AND WIEN'S LAW (OR WIEN'S DISPLACEMENT LAW)

It has been observed that the radiation emitted by a body is a mixture of waves of different wavelengths. However, only a small range of wavelength contributes significantly to the total radiation, e.g. the radiation emitted by a body at 300 K i.e., room temperature, has significant contribution from wavelengths around 9550 nm which lies in the long infrared region (visible light has a range of about 380–780 nm). When temperature of the emitter increases, this dominant wavelength is observed to decrease. At around 1100 K, the radiation has a good contribution from red region of wavelengths and the object appears red. At temperatures around 3000 K, the radiation contains enough shorter wavelengths and so the object appears white. Even at such a high temperature most of the significant contribution comes from wavelengths around 950 nm.

The relative importance of different wavelengths in a thermal radiation can be studied qualitatively from the intensity  $I$  vs wavelength  $\lambda$  graph shown in Figure.



In this graph, the intensity of radiation in the neighborhood of a given wavelength is plotted against the wavelength at different temperatures. It is observed that, as the temperature is increased, the wavelength  $\lambda_m$  corresponding to the maximum intensity of radiation decreases.

Actually, this wavelength  $\lambda_m$  is observed to be inversely proportional to the absolute temperature of the body emitting the radiation. So,

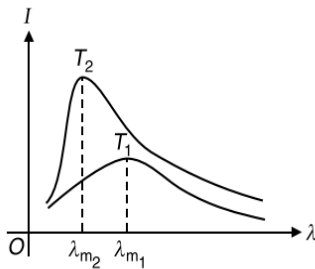
$$\lambda_m T = b$$

where  $b$  is a constant.

This equation is known as the Wien's Displacement Law. For a blackbody, the constant  $b$  appearing in equation is measured to be 0.288 cmK and is known as the Wein's constant. So, we have

$$b = 2.888 \times 10^{-3} \text{ kelvinmetre} = 2.888 \times 10^{-3} \text{ Km}$$

Graphically it can be shown that  $\lambda_{m_1} T_1 = \lambda_{m_2} T_2$



Since  $T_2 > T_1$ , so  $\lambda_{m_2} < \lambda_{m_1}$

Also, it is observed that as the temperature of the body increases, the wavelength at which the radiant intensity is maximum shifts toward left. It is called **Wein's displacement Law**.

In solar radiations  $\lambda_m = 4753 \text{ \AA}$

Hence temperature of sun is approximately

$$T = \frac{b}{\lambda_m} = \frac{2.89 \times 10^{-3}}{4.753 \times 10^{-7}} = 6080 \text{ K}$$

Also, it has been observed that, area under the curve represents the radiant intensity of radiation at a particular temperature. Since total radiant energy is proportional to fourth power of  $T$ , area should also be proportional to fourth power of absolute temperature  $T$ . Hence

$$\text{Area} \propto T^4$$

### ILLUSTRATION 149

A black body at 1500 K emits maximum energy of wavelength 2200 Å. If sun emits maximum energy of wavelength 5500 Å, calculate temperature of sun.

### SOLUTION

According to Wein's displacement law, we have

$$\lambda_{m_1} T_1 = \lambda_{m_2} T_2$$

$$\Rightarrow (2200 \times 10^{-10}) 1500 = (5500 \times 10^{-10}) T_2$$

$$\Rightarrow T_2 = \left( \frac{220}{55} \right) (1500) = 6000 \text{ K}$$

### ILLUSTRATION 150

Two bodies  $A$  and  $B$  have thermal emissivity of 0.01 and 0.81 respectively. The outer surface areas of the two bodies are same. The two bodies emit total radiant power at the same rate. The wavelength  $\lambda_B$  corresponding to maximum spectral radiancy from  $B$  is shifted from the wavelength corresponding to maximum spectral radiancy in the radiation from  $A$  by  $1 \mu\text{m}$ . If the temperature of  $A$  is 5802 K, calculate

- (a) the temperature of Band
- (b) the wavelength  $\lambda_B$ .

### SOLUTION

- (a)  $P_A = P_B$

$$\Rightarrow e_A \sigma A_A T_A^4 = e_B \sigma A_B T_B^4$$

$$\Rightarrow T_B = \left( \frac{e_A}{e_B} \right)^{\frac{1}{4}} T_A \quad (\because A_A = A_B)$$

Substituting the values, we get

$$T_B = \left( \frac{0.01}{0.81} \right)^{\frac{1}{4}} (5802) = 1934 \text{ K}$$

- (b) According to Wein's Displacement Law, we have

$$\lambda_A T_A = \lambda_B T_B$$

$$\Rightarrow \lambda_B = \left( \frac{5802}{1934} \right) \lambda_A = 3 \lambda_A \quad \dots(1)$$

Also, we are given that

$$\lambda_B - \lambda_A = 1 \mu\text{m}$$

$$\Rightarrow \lambda_B - \left( \frac{1}{3} \right) \lambda_B = 1 \mu\text{m}$$

$$\Rightarrow \lambda_B = 1.5 \mu\text{m}$$

### ILLUSTRATION 151

The radiant emittance of a black body is  $R = 250 \text{ kWm}^{-2}$ . At what wavelength will the emissivity of this black body be maximum? Take  $b = 2.9 \times 10^{-3} \text{ Km}$  and  $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ .

### SOLUTION

According to Stefan's law, the energy radiated per unit time per unit area is given by  $R = \sigma T^4$ , so

$$T = \left( \frac{R}{\sigma} \right)^{\frac{1}{4}}$$

$$\Rightarrow T = \left( \frac{250 \times 10^3}{5.67 \times 10^{-8}} \right)^{1/4} = 1.449 \times 10^3 \text{ K} = 1449 \text{ K}$$

According to Wein's displacement law, the maximum spectral radiance will be at wavelength  $\lambda_m$  given by

$$\lambda_m = \frac{b}{T} = \frac{2.89 \times 10^{-3}}{1449} = 19.9447 \times 10^{-7} \text{ m} = 19944.7 \text{ \AA}$$


**Test Your Concepts-XI**
**Based on Radiation**

**(Solutions on page H.98)**

- A liquid takes 5 minutes to cool from 80 °C to 50 °C. How much time will it take to cool from 60 °C to 30 °C ? The temperature of surrounding is 20 °C.
- An electric heater of power 1 kW emits thermal radiations the surface area of heating element of heat is 200 cm<sup>2</sup>. If this heating element is treated like a black body, calculate the temperature at its surface. Assume its temperature is very much higher than its surroundings.
- One end A of a metallic rod of length 10 cm is inserted in a furnace whose temperature is 827 °C. Curved surface of rod is insulated. The room temperature is 27 °C. When the steady state is attained, temperature of other end B of rod is 702 °C. Calculate thermal conductivity of rod if Stefan's constant is  $5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ .
- A metal sphere of radius  $r = 4 \text{ cm}$  is coated black. Density of the metal is  $9 \times 10^3 \text{ kgm}^{-3}$  and its specific heat capacity is  $4.1 \text{ kJkg}^{-1}(\text{°C})^{-1}$ . How much time is required for the sphere to cool down from 800 K to 300 K? Stefan's constant  $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ .
- A copper ball of diameter  $d$  was placed in an evacuated vessel whose walls are kept at the absolute zero temperature. The initial temperature of the ball is  $T_0$ . Assuming the surface of the ball to be absolutely black, find how soon its temperature decreases  $\eta$  times. Take specific heat of copper  $c$ , density of copper  $\rho$  and emissivity  $e$ .
- A solid metallic sphere of diameter 20 cm and mass 10 kg is heated to a temperature of 327 °C and suspended in a box inside which a constant temperature of 27 °C is maintained. Find the rate at which the temperature of the sphere will fall with time. Stefan's constant  $5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$  and specific heat of metal  $420 \text{ Jkg}^{-1}\text{°C}^{-1}$ .
- End A of a rod AB of length  $\ell_0 = 5 \text{ m}$  and cross-sectional area  $A = 1 \text{ m}^2$  is maintained at some constant temperature. The heat conductivity of the rod is varying with the distance  $x$  from, the end A as  $k = k_0(1 + ax)$ , where  $k_0 = 11.4 \text{ Wm}^{-1}\text{K}^{-1}$  and  $a = 0.2 \text{ ms}^{-1}$ . The other end B of this rod is radiating energy at the rate of  $P = 4560 \text{ W}$  and the wavelength with maximum energy density emitted from this end is 14500 Å. Express the variation of temperature as a function of  $x$  and determine the temperature of the end A. Assume that except the ends, the rod is thermally insulated and Wein's constant  $b = 2.9 \times 10^{-3} \text{ Km}$ .
- A cube of mass 1 kg and volume 125 cm<sup>3</sup> is placed in an evacuated chamber at 27 °C. Initially temperature of block is 227 °C. Assume block behaves like a black body, find the rate of cooling of block if specific heat of the material of block is  $400 \text{ Jkg}^{-1}\text{K}^{-1}$ .
- A block having some emissivity is maintained at 500 K temperature in a surrounding of 300 K temperature. It is observed that, to maintain the temperature of the block, 210 W external power is required to be supplied to it. If instead of this block a black body of same geometry and size is used, 700 W external power is required for the same. Calculate the emissivity of the material of the block.
- A black walled metal container of negligible heat capacity is filled with water. The container has sides of length 10 cm. It is placed in an evacuated chamber at 27 °C. How long will it take for the temperature of water to change from 30 °C to 29 °C.
- A calorimeter of negligible heat capacity contains 100 g water at 40 °C. The water cools to 35 °C in 5 minutes. If the water is now replaced by a liquid of same volume as that of water at same initial temperature, it cools to 35 °C in 2 minutes. Given specific heats of water and that liquid are  $4200 \text{ Jkg}^{-1}\text{°C}^{-1}$  and  $2100 \text{ Jkg}^{-1}\text{°C}^{-1}$  respectively. Find the density of the liquid.
- If the filament of a 114.75 W bulb has an area 0.25 cm<sup>2</sup> and behaves as a perfect black body. Find the wavelength corresponding to the maximum in its energy distribution. Given  $\sigma = 5.7 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ ,  $b = 2.89 \times 10^{-3} \text{ Km}$

## SOLVED PROBLEMS

### PROBLEM 1

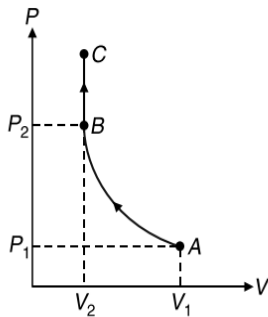
Two moles of an ideal monatomic gas initially at pressure  $P_1$  and volume  $V_1$  undergo an adiabatic compression until its volume is  $V_2$ . Then the gas is given heat  $Q$  at constant volume  $V_2$ .

- Sketch the complete process on a  $P$ - $V$  diagram.
- Find the total work done by the gas, the total change in internal energy and the final temperature of the gas.

[Give your answer in terms of  $P_1$ ,  $V_1$ ,  $V_2$ ,  $Q$  and  $R$ ]

### SOLUTION

- The  $P$ - $V$  diagram for the complete process is shown below.



Process  $A \rightarrow B$  is adiabatic compression and process  $B \rightarrow C$  is isochoric.

- Total work done by the gas**

Process  $AB$

$$\text{Since } W_{ad} = \frac{P_f V_f - P_i V_i}{1 - \gamma}$$

$$\Rightarrow W_{AB} = \frac{P_A V_A - P_B V_B}{\gamma - 1}$$

$$\Rightarrow W_{AB} = \frac{P_1 V_1 - P_2 V_2}{\frac{5}{3} - 1}, \gamma = \frac{5}{3} \text{ for monatomic gas}$$

Since,  $P_1 V_1^\gamma = P_2 V_2^\gamma$

$$\Rightarrow P_2 = P_1 \left( \frac{V_1}{V_2} \right)^\gamma$$

$$\Rightarrow W_{AB} = \frac{P_1 V_1 - P_1 \left( \frac{V_1}{V_2} \right)^\gamma V_2}{2/3}$$

$$\Rightarrow W_{AB} = \frac{3}{2} P_1 V_1 \left[ 1 - \left( \frac{V_1}{V_2} \right)^{\gamma-1} \right]$$

$$\Rightarrow W_{AB} = -\frac{3}{2} P_1 V_1 \left[ \left( \frac{V_1}{V_2} \right)^{\frac{5}{3}-1} - 1 \right]$$

$$\Rightarrow W_{AB} = -\frac{3}{2} P_1 V_1 \left[ \left( \frac{V_1}{V_2} \right)^{2/3} - 1 \right]$$

Since, process  $BC$  is Isochoric, so

$$W_{BC} = 0 \quad \{ \because V = \text{constant} \}$$

$$\Rightarrow W_{\text{Total}} = W_{AB} + W_{BC} = -\frac{3}{2} P_1 V_1 \left[ \left( \frac{V_1}{V_2} \right)^{2/3} - 1 \right]$$

### Total change in internal energy

Since, process  $AB$  is Adiabatic, so

$$Q_{AB} = 0$$

From  $FLTD$ , we have

$$Q = \Delta U + W$$

$$\Rightarrow Q_{AB} = 0 = \Delta U_{AB} + W_{AB}$$

$$\Rightarrow \Delta U_{AB} = -W_{AB} = \frac{3}{2} P_1 V_1 \left[ \left( \frac{V_1}{V_2} \right)^{2/3} - 1 \right]$$

Since, process  $BC$  is Isochoric, so

$$W_{BC} = 0$$

So, from  $FLTD$ , we get

$$\Delta U_{BC} = Q_{BC} = Q \quad \{ \text{given} \}$$

$$\Rightarrow \Delta U_{\text{Total}} = \Delta U_{AB} + \Delta U_{BC} = \frac{3}{2} P_1 V_1 \left[ \left( \frac{V_1}{V_2} \right)^{2/3} - 1 \right] + Q$$

### Final temperature of the gas

$$\text{Since, } \Delta U_{\text{Total}} = n C_V \Delta T = 2 \left( \frac{R}{\gamma - 1} \right) (T_C - T_A)$$

$$\Rightarrow \frac{3}{2} P_1 V_1 \left[ \left( \frac{V_1}{V_2} \right)^{2/3} - 1 \right] + Q = \frac{2R}{\left( \frac{5}{3} - 1 \right)} \left( T_C - \frac{P_A V_A}{2R} \right)$$

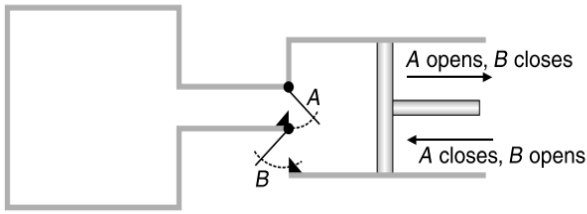
$$\Rightarrow \frac{3}{2} P_1 V_1 \left[ \left( \frac{V_1}{V_2} \right)^{2/3} - 1 \right] + Q = 3R \left( T_C - \frac{P_1 V_1}{2R} \right)$$

$$\Rightarrow T_C = \frac{Q}{3R} + \frac{P_1 V_1}{2R} \left( \frac{V_1}{V_2} \right)^{2/3} = T_{\text{final}}$$

### PROBLEM 2

A vessel of volume  $V$  having pressure  $P_0$  is evacuated by means of a piston air pump as shown in figure. Valve  $A$  opens ( $B$  closed) when the piston moves outward and  $A$  closes ( $B$  opens) when the piston moves inward. Each stroke of the piston captures the volume  $\Delta V$ . How many strokes are needed to reduce the pressure in the vessel  $\eta$  times? The process is assumed to be isothermal and ideal.

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### SOLUTION

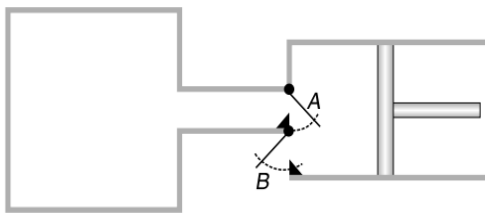
When the piston moves outward, the volume of the gas is  $(V + \Delta V)$  and pressure  $P_1$ , such that

$$P_1(V + \Delta V) = P_0V \quad \dots(1)$$

During return stroke of piston volume  $\Delta V$  is ejected. The moment the return stroke begins, valve A is closed, the remaining gas has pressure  $P_1$  and volume  $V$ . For second stroke, we have

$$P_2(V + \Delta V) = P_1V \quad \dots(2)$$

$$\Rightarrow P_2 = \frac{P_1V}{(V + \Delta V)} = P_0 \left( \frac{V}{V + \Delta V} \right)^2$$



Similarly,  $P_3 = P_0 \left( \frac{V}{V + \Delta V} \right)^2$

After  $N^{\text{th}}$  stroke,  $P_N = P_0 \left( \frac{V}{V + \Delta V} \right)^N \quad \dots(3)$

$$\Rightarrow N = \frac{\log_e \left( \frac{P_N}{P_0} \right)}{\log_e \left( \frac{V}{V + \Delta V} \right)} \quad \dots(4)$$

Since,  $\frac{P_N}{P_0} = \frac{1}{\eta}$

$$\Rightarrow N = \frac{\log_e \left( \frac{1}{\eta} \right)}{\log_e \left( \frac{V}{V + \Delta V} \right)}$$



### Conceptual Note(s)

Note that the mass of the gas ejected in each stroke is not the same. From Ideal Gas equation, we have

$$P_{N-1}V = \frac{m_{N-1}RT}{M} \quad \{\text{mass after } N-1 \text{ strokes}\}$$

$$P_NV = \frac{m_N RT}{M} \quad \{\text{mass after } N \text{ strokes}\}$$

Mass of gas ejected in the  $N^{\text{th}}$  stroke is

$$m_{N-1} - m_N = \frac{MV}{RT} (P_{N-1} - P_N) \quad \dots(5)$$

After  $N^{\text{th}}$  stroke, we have from Boyle's Law,

$$P_{N-1}V = P_N(V + \Delta V)$$

$$\Rightarrow (P_{N-1} - P_N)V = P_N(\Delta V) \quad \dots(6)$$

From equations (5) and (6), we have

$$m_{N-1} - m_N = \frac{M(\Delta V)}{RT} P_N = \frac{\Delta V}{V} m_N$$

$$\Rightarrow \frac{V}{m_N} - \frac{V}{m_{N-1}} = \frac{\Delta V}{V} \frac{V}{m_N - 1} \quad \dots(7)$$

The specific volume of gas  $\frac{V}{m}$  is volume of a unit mass of gas.

$$\Rightarrow \frac{\frac{V}{m_N} - \frac{V}{m_{N-1}}}{\left( \frac{V}{m_{N-1}} \right)} = \frac{\Delta V}{V} = \text{constant}$$

The fractional change in specific volume is constant in this process i.e.,

$$\left( \frac{\Delta V}{V} \right) = \left( \frac{m_{N-1} - m_N}{m_N} \right)$$

### PROBLEM 3

Find the equations of the process for an ideal gas in terms of the variables  $T$  and  $V$  if the molar heat capacity varies as

(a)  $C = C_V + \alpha T$

(b)  $C = C_V + \beta V$  and

(c)  $C = C_V + aP$

where  $\alpha$ ,  $\beta$  and  $a$  are constants.

### SOLUTION

Since,  $C = \frac{dQ}{dT}$  (for 1 mole of an ideal gas)

$$\Rightarrow C = \frac{dU + PdV}{dT}$$

$$\Rightarrow C = C_V + P \left( \frac{dV}{dT} \right)$$

$$\Rightarrow C = C_V + \frac{RT}{V} \left( \frac{dV}{dT} \right) \quad \dots(1)$$

(a) Since, we are given that

$$C = C_V + \alpha T \quad \dots(2)$$

So, comparing (1) and (2), we get

$$\alpha T = \frac{RT}{V} \frac{dV}{dT}$$

$$\Rightarrow \frac{\alpha}{R} dT = \frac{dV}{V}$$

$$\Rightarrow \int \frac{dV}{V} = \frac{\alpha}{R} \int dT$$

$$\Rightarrow \log_e V = \frac{\alpha T}{R} + \text{constant}$$

$$\Rightarrow V e^{-\frac{\alpha T}{R}} = \text{constant}$$

(b) Similarly comparing

$$C = C_V + \frac{RT}{V} \frac{dV}{dT}$$

and  $C = C_V + \beta V$ , we get

$$\beta V = \frac{RT}{V} \frac{dV}{dT}$$

$$\frac{\beta}{R} \frac{dT}{T} = \frac{dV}{V^2}$$

Integrating, we get

$$\int \frac{dV}{V^2} = \frac{\beta}{R} \int \frac{dT}{T}$$

$$\Rightarrow -\frac{1}{V} = \frac{\beta}{R} \log_e T + \text{constant}$$

$$\Rightarrow T e^{\frac{\beta V}{R}} = \text{constant}$$

(c) Similarly comparing

$$C = C_V + P \frac{dV}{dT}$$

and  $C = C_V + aP$ , we get

$$\Rightarrow aP = P \frac{dV}{dT}$$

$$\Rightarrow \frac{dV}{dT} = a$$

$$\Rightarrow V = aT + \text{constant}$$

$V_1 = 64$  in SI units. Calculate the thermal efficiency of the cycle.

**SOLUTION**

The  $P$ - $V$  equation for the process  $AB$  is found by using  $y - y_1 = m(x - x_1)$ , where  $m = \frac{y_2 - y_1}{x_2 - x_1}$

Since,  $AB$  process passes through  $(8, 32)$  so we get

$$P - 32 = \left( \frac{1 - 32}{64 - 8} \right) (V - 8)$$

$$\Rightarrow P = -\frac{31}{56}V + \frac{255}{7}$$

So, the  $P$ - $V$  equation for the process  $AB$  can be written as,

$$P = aV + b, \text{ where } a = -\frac{31}{56} \text{ and } b = \frac{255}{7}$$

The work done by the gas is the area under the curve and is given by

$$W_{AB} = \frac{1}{2}(P_0 + P_1)(V_1 - V_0) = 924 \text{ J}$$

The work done by the gas in the adiabatic compression is given by

$$W_{BA} = \frac{P_B V_B - P_A V_A}{\gamma - 1} = \frac{P_1 V_1 - P_0 V_0}{\gamma - 1}$$

$$\Rightarrow W_{BA} = \frac{64 - 32 \times 8}{\frac{5}{3} - 1}$$

$$\Rightarrow W_{BA} = -288 \text{ J}$$

Thus, the net work done by the gas during this cycle is

$$W_{\text{net}} = W_{AB} + W_{BA} = 924 - 288 = 636 \text{ J}$$

From the First Law net  $Q$  for the whole cyclic process is also 636 J  $\{\because \Delta U = 0\}$

Along the linear path  $AB$ , heat both enters and leaves the system. There is a transition point  $(P_m, V_m, T_m)$  which represents the point on this path at which the heat flow reverses direction changing from heat input to heat output. So, first of all we must find  $Q$  as a function of  $V$  i.e., the heat input as a function of  $V$  and then find the particular volume  $V_m$  at which  $Q$  is a maximum. Using the First Law of Thermodynamics, we have

$$Q = Q(V) = W + \Delta U \quad \dots(1)$$

The  $P$ - $V$  equation along the path  $AB$  is,

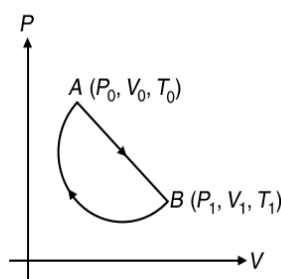
$$P = aV + b$$

Multiplying both sides by  $V$ , we get

$$PV = aV^2 + bV$$

**PROBLEM 4**

An ideal monatomic gas is taken through a cyclic process in which it expands linearly from a state  $(P_0, V_0, T_0)$  to a state  $(P_1, V_1, T_1)$  followed by an adiabatic compression back to the original state. Given that  $P_0 = 32$ ,  $V_0 = 8$ ,  $P_1 = 1$  and



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Since,  $Q = Q(V) = W + \Delta U$

$$\Rightarrow Q(V) = \frac{1}{2}(P_0 + P)(V - V_0) + nC_V \Delta T$$

$$\Rightarrow Q(V) = \frac{1}{2}(P_0 + P)(V - V_0) + \frac{3nR}{2}(T - T_0)$$

$$\Rightarrow Q(V) = \frac{1}{2}(P_0 + aV + b)(V - V_0) + \frac{3}{2}(PV - P_0V_0)$$

$$\Rightarrow Q(V) = \frac{1}{2}(P_0 + aV + b)(V - V_0) + \frac{3}{2}(aV^2 + bV - P_0V_0) \quad \dots(2)$$

Now,  $\frac{dQ(V)}{dV} = 0$

$$\Rightarrow aV - \frac{aV_0}{2} + \frac{1}{2}(P_0 + b) + \frac{3}{2}(2aV + b) = 0$$

$$\Rightarrow 4aV = \frac{aV_0}{2} - \frac{1}{2}(P_0 + b) - \frac{3b}{2}$$

$$\Rightarrow V = \frac{V_0}{8} - \frac{1}{8a}(P_0 + b) - \frac{3b}{8a}$$

Substituting the values, we get

$$V_m = \frac{8}{8} + \frac{56}{8 \times 31} \left( 32 + \frac{255}{7} \right) + \frac{3 \times 255}{7} \times \frac{56}{8 \times 31}$$

$$\Rightarrow V_m = 41.13 \text{ SI units}$$

From equation (2), we get

$$Q_{+ve} = \frac{1}{2} \left( 32 - \frac{31}{56} \times 41.13 + \frac{255}{7} \right) (41.13 - 8) + \frac{3}{2} \left[ \left( -\frac{31}{56} \right) (41.13)^2 + \left( \frac{255}{7} \right) (41.13) - 32 \times 8 \right]$$

$$\Rightarrow Q_{+ve} \approx 1215 \text{ J}$$

Efficiency of the cycle is

$$\eta = \left( \frac{W_{\text{Total}}}{Q_{+ve}} \right) \times 100$$

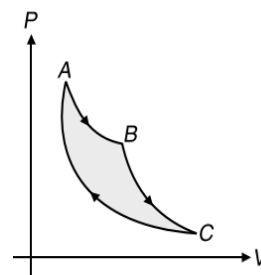
$$\Rightarrow \eta = \frac{636}{1215} \times 100$$

$$\Rightarrow \eta = 52\%$$

### PROBLEM 5

An ideal gas expands isothermally along  $AB$  and does 700 J of work.

- How much heat does the gas exchange along  $AB$ .
- The gas then expands adiabatically along  $BC$  and does 400 J of work. When the gas returns to  $A$  along  $CA$ , it exhausts 100 J of heat to its surroundings. How much work is done on the gas along this path.
- Find the efficiency of the given cycle.



### SOLUTION

- (a)  $AB$  is an isothermal process. Hence,

$$\Delta U_{AB} = 0$$

$$\Rightarrow Q_{AB} = W_{AB} = 700 \text{ J} \quad \{ \because Q_{AB} = \Delta U_{AB} + W_{AB} \}$$

- (b)  $BC$  is an adiabatic process. Hence,

$$Q_{BC} = 0$$

Since,  $W_{BC} = 400 \text{ J}$

$$\Rightarrow \Delta U_{BC} = -W_{BC} = -400 \text{ J}$$

$ABC$  is a cyclic process and internal energy is a state function. Therefore,

$$(\Delta U)_{\text{whole cycle}} = 0 = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA}$$

So, from First Law of Thermodynamics, we have

$$Q_{AB} + Q_{BC} + Q_{CA} = W_{AB} + W_{BC} + W_{CA}$$

Substituting the values, we get

$$700 + 0 - 100 = 700 + 400 + W_{CA}$$

$$\Rightarrow W_{CA} = -500 \text{ J}$$

Negative sign implies that work is done on the gas.

We can summarise the values in different processes in the table below.

Process	Q(J)	W(J)	$\Delta U$ (J)
$AB$	700	700	0
$BC$	0	400	-400
$CA$	-100	-500	400
For complete cycle	600	600	0

Since, total work done is 600 J, so we can also conclude that area of the closed curve is 600 J.

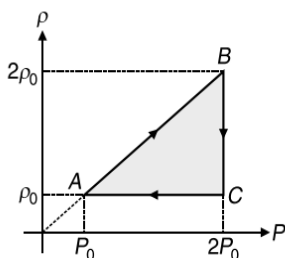
- (c) From table we observe that  $Q_{+ve}$  during the cycle is 700 J, while the total work done in the cycle is 600 J.

$$\text{So, efficiency } \eta = \frac{W_{\text{total}}}{|Q_{+ve}|} \times 100 = \left( \frac{600}{700} \right) \times 100$$

$$\Rightarrow \eta = 85.71\%$$

**PROBLEM 6**

The density versus pressure graph of one mole of an ideal monatomic gas undergoing a cyclic process is shown in figure. The molecular mass of the gas is  $M$ .



- Find the work done in each process.
- Find heat rejected by gas in one complete cycle.
- Find the efficiency of the cycle.

**SOLUTION**

- (a) As  $n = 1$ ,  $m = M$ , also we know that  $\rho = \frac{PM}{RT}$

For process  $AB$ , we observe that  $\rho \propto P$  so it is an isothermal process  $T$  is constant, so  $\rho = \frac{PM}{RT}$

$$\Rightarrow W_{AB} = RT_A \log_e \left( \frac{P_A}{P_B} \right) = RT_A \log_e \left( \frac{1}{2} \right)$$

$$\Rightarrow W_{AB} = -\frac{P_0 M}{\rho_0} \log_e (2)$$

Also,  $\Delta U_{AB} = 0$

$$\Rightarrow Q_{AB} = W_{AB} = -\frac{P_0 M}{\rho_0} \log_e (2)$$

Process  $BC$  is an isobaric process ( $P = \text{constant}$ ), so

$$W_{BC} = P_B (V_C - V_B) = 2P_0 \left( \frac{M}{\rho_C} - \frac{M}{\rho_B} \right) = \frac{2P_0 M}{2\rho_0} = \frac{P_0 M}{\rho_0}$$

So,  $\Delta U_{BC} = C_V \Delta T$

$$\Rightarrow \Delta U_{BC} = \left( \frac{3}{2} R \right) \left( \frac{2P_0 M}{\rho_0 R} - \frac{2P_0 M}{2\rho_0 R} \right) = \frac{3P_0 M}{2\rho_0}$$

So, by FLTD, we get

$$Q_{BC} = W_{BC} + \Delta U_{BC} = \frac{5P_0 M}{2\rho_0}$$

Process  $CA$  has  $\rho = \text{constant}$ , so  $V = \text{constant}$

So, the process  $CA$  is an isochoric process, hence

$$W_{CA} = 0$$

$$\Rightarrow \Delta U_{CA} = C_V \Delta T$$

$$\Rightarrow \Delta U_{CA} = \left( \frac{3}{2} R \right) (T_A - T_C)$$

$$\Rightarrow \Delta U_{CA} = \left( \frac{3}{2} R \right) \left( \frac{P_0 M}{\rho_0 R} - \frac{2P_0 M}{\rho_0 R} \right)$$

$$\Rightarrow \Delta U_{CA} = -\frac{3P_0 M}{2\rho_0}$$

So, by FLTD, we get

$$Q_{CA} = \Delta U_{CA} = -\frac{3P_0 M}{2\rho_0}$$

- (b) Heat rejected by the gas in one complete cycle is

$$Q = Q_{-ve} = -\frac{P_0 M}{\rho_0} \log_e (2) - \frac{3P_0 M}{2\rho_0}$$

$$\Rightarrow Q = -\frac{P_0 M}{\rho_0} \left( \log_e (2) + \frac{3}{2} \right)$$

- (c) Efficiency of the cycle (in fraction)

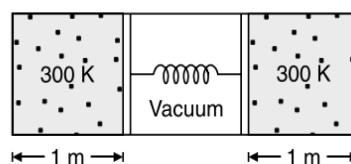
$$\eta = \frac{\text{Total work done}}{\text{Heat supplied}} = \frac{\text{Total work done}}{\text{Heat absorbed}}$$

$$\Rightarrow \eta = \frac{W_{\text{total}}}{Q_{+ve}} = \frac{\frac{P_0 M}{\rho_0} (1 - \log_e (2))}{\frac{5}{2} \left( \frac{P_0 M}{\rho_0} \right)}$$

$$\Rightarrow \eta = \frac{2}{5} (1 - \log_e (2))$$

**PROBLEM 7**

Consider the shown diagram where the two chambers separated by piston-spring arrangement contain equal amounts of certain ideal gas. Initially when the temperatures of the gas in both the chambers are kept at 300 K. The compression in the spring is 1 m. The temperature of the left and the right chambers are now raised to 400 K and 500 K respectively. If the pistons are free to slide, find the final compression in the spring.


**SOLUTION**

Let  $l_1$  and  $l_2$  be the final lengths of the two parts, then from gas equation, we have

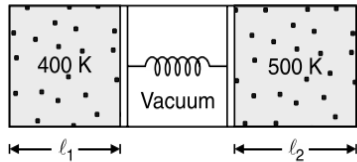
$$\frac{P_0 (Al_0)}{T_0} = \frac{P (Al_1)}{T_1} = \frac{P (Al_2)}{T_2} \quad \dots(1)$$

Considering the equilibrium of piston in initial and final states, we have

$$P_0 A = kx_0 \text{ and } PA = kx$$

$$\Rightarrow \frac{P}{P_0} = \frac{x}{x_0} \quad \dots(2)$$

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Now,  $\left( \begin{matrix} \text{Decrease} \\ \text{in length} \\ \text{of spring} \end{matrix} \right) = \left( \begin{matrix} \text{Total increase} \\ \text{in the lengths of} \\ \text{the two chambers} \end{matrix} \right)$

$$\Rightarrow x - x_0 = (l_1 + l_2) - 2l_0 \quad \dots(3)$$

From equation (1), we get

$$l_1 = \frac{P_0 l_0 T_1}{P T_0} \text{ and } l_2 = \frac{P_0 l_0 T_2}{P T_0}$$

Using equation (2), we get

$$l_1 = \frac{x_0 l_0 T_1}{x T_0} \text{ and } l_2 = \frac{x_0 l_0 T_2}{x T_0}$$

Substituting these in equation (3), we have

$$x - x_0 = \frac{x_0 l_0}{x T_0} (T_1 + T_2) - 2l_0$$

Substituting the values and solving for  $x$ , we get

$$x \approx 1.3 \text{ m}$$

**PROBLEM 8**

A 5 m long cylindrical steel wire with radius  $2 \times 10^{-3}$  m is suspended vertically from a rigid support and carries a bob of mass 100 kg at the other end. If the bob gets snapped, calculate the change in temperature of the wire ignoring losses. (For the steel wire, Young's modulus is  $2.1 \times 10^{11}$  Pa, Density is  $7860 \text{ kg m}^{-3}$ , Specific heat is  $420 \text{ J kg}^{-1} \text{ K}^{-1}$ ).

**SOLUTION**

According to the question

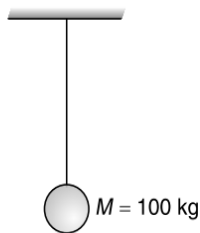
Length of the wire,  $l = 5 \text{ m}$

Radius of the wire,  $r = 2 \times 10^{-3} \text{ m}$

Density of wire,  $\rho = 7860 \text{ kg m}^{-3}$

Young's modulus,  $Y = 2.1 \times 10^{11} \text{ Nm}^{-2}$

and Specific heat,  $c = 420 \text{ J kg}^{-1} \text{ K}^{-1}$



Mass of wire,  $m = (\text{density})(\text{volume}) = (\rho)(\pi r^2 l)$

$$\Rightarrow m = (7860)(\pi)(2 \times 10^{-3})^2 (5) \text{ kg} = 0.494 \text{ kg}$$

Since,  $\frac{\text{Energy}}{\text{Volume}} = \frac{1}{2}(\text{Stress})(\text{Strain})$

So, elastic potential energy stored in the wire is

$$U = \frac{1}{2}(\text{Stress})(\text{Strain})(\text{Volume})$$

$$\Rightarrow U = \frac{1}{2} \left( \frac{Mg}{\pi r^2} \right) \left( \frac{\Delta l}{l} \right) (\pi r^2 l) = \frac{1}{2} (Mg) \Delta l \quad \left\{ \because \Delta l = \frac{Fl}{AY} \right\}$$

$$\Rightarrow U = \frac{1}{2} (Mg) \frac{(Mgl)}{(\pi r^2) Y} = \frac{1}{2} \frac{M^2 g^2 l}{\pi r^2 Y}$$

Substituting the values, we get

$$\Rightarrow U = \frac{1}{2} \frac{(100)^2 (10)^2 (5)}{2 (3.14) (2 \times 10^{-3})^2 (2.1 \times 10^{11})} = 0.9478 \text{ J}$$

When the bob gets snapped, this energy is utilised in raising the temperature of the wire.

So,  $U = mc\Delta T$

$$\Rightarrow \Delta T = \frac{U}{mc} = \frac{0.9478}{0.494(420)} \text{ } ^\circ\text{C or K}$$

$$\Rightarrow \Delta T = 4.568 \times 10^{-3} \text{ } ^\circ\text{C}$$

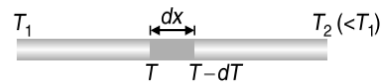
**PROBLEM 9**

A rod of length  $l$  with thermally insulated lateral surface of area  $A$  consists of a material whose heat conduction coefficient varies with temperature as  $k = \frac{k_0}{a + bT}$ . The two ends of the rod are at temperatures  $T_1$  and  $T_2 (< T_1)$ .

- (a) Find the heat transferred across the rod.
- (b) Express the temperature of the rod at a point as a function of,  $x$ , distance from the hot end.

**SOLUTION**

(a)  $H = -kA \frac{dT}{dx}$



$$\Rightarrow H dx = -\frac{k_0}{a + bT} A dT$$

$$\Rightarrow H \int_0^l dx = -k_0 A \int_{T_1}^{T_2} \frac{dT}{a + bT}$$

$$\Rightarrow H = \frac{A k_0}{bl} \log_e \left( \frac{a + bT_1}{a + bT_2} \right)$$

(b) Further  $H \int_0^x dx = -k_0 A \int_{T_1}^T \frac{dT}{a + bT}$

Substituting value of  $H$  from part (a) and integrating the above equation, we get

$$T = \frac{1}{b} \left[ (a + bT_1) \left( \frac{a + bT_2}{a + bT_1} \right)^{x/l} - a \right]$$

**PROBLEM 10**

In the glass tube, half portion is filled by a liquid  $A$  and the other half by liquid  $B$ . The temperature of the whole system is increased by  $\Delta T$ . It is given that initial volume of liquid  $A$  and  $B = \frac{V}{2}$ . Find out  $\Delta V_A$  and  $\Delta V_B$  if the coefficient of volume expansion of liquids  $A$  and  $B$  are  $\gamma_A$  and  $\gamma_B$  respectively and bulk moduli of the two liquids are  $k_A$  and  $k_B$  respectively. Neglect the expansion of glass tube.

**SOLUTION**

Total change in volume due to heating is given by

$$\Delta V_A + \Delta V_B = \frac{V}{2} \gamma_A \Delta T + \frac{V}{2} \gamma_B \Delta T$$

$$\Rightarrow \Delta V_A + \Delta V_B = \frac{V}{2} (\gamma_A + \gamma_B) \Delta T \quad \dots(1)$$

This increase in volume causes stress (pressure) on the liquids (since the glass tube does not expand).

Let  $P$  be this excess pressure then total decrease in volume due to this excess pressure is

$$\Delta V_A + \Delta V_B = \frac{k_A V}{2P} + \frac{k_B V}{2P} \quad \left\{ \because k = \frac{\Delta P}{\left( \frac{\Delta V}{V} \right)} \right\}$$

Since, the tube does not expand, so we have

$$\frac{V}{2} \Delta T (\gamma_A + \gamma_B) = (k_A + k_B) \frac{V}{2P}$$

$$\Rightarrow P = \frac{k_A + k_B}{(\gamma_A + \gamma_B) \Delta T}$$

$$\Rightarrow (\Delta V_A)_{\text{total}} = \frac{V}{2} \gamma_A \Delta T - \frac{k_A V}{2(k_A + k_B)} (\gamma_A + \gamma_B) \Delta T$$

$$\Rightarrow (\Delta V_A)_{\text{total}} = \frac{V}{2} \left( \frac{\gamma_A k_B - \gamma_B k_A}{k_A + k_B} \right) \Delta T$$

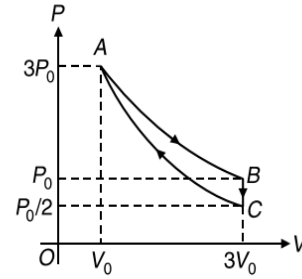
Similarly,  $(\Delta V_B)_{\text{total}} = \frac{V}{2} \left( \frac{\gamma_B k_A - \gamma_A k_B}{k_A + k_B} \right) \Delta T$

**PROBLEM 11**

One mole of a monatomic gas is taken through the cycle  $ABCA$  as shown in the figure. In the process  $C \rightarrow A$  the gas obeys the relation

$$Q_{C \rightarrow A} + W_{C \rightarrow A} = 0$$

where  $Q_{C \rightarrow A}$  is the heat supplied in the process  $C \rightarrow A$  and  $W_{C \rightarrow A}$  is the work done by the gas in that process,  $AB$  is an isothermal process.



- (a) Find the work done and change in internal energy in each process and also find the heat exchanged in each process.
- (b) Find the average molar specific heat for processes  $AB$  and  $CA$ .

**SOLUTION**

- (a) For process  $AB$ ,  $\Delta U_{AB} = 0$

For isothermal process,

$$W_{AB} = nRT_A \log_e \left( \frac{V_B}{V_A} \right)$$

$$\Rightarrow W_{AB} = P_A V_A \log_e (3) = 3P_0 V_0 \log_e (3)$$

So from FLTD, we get

$$Q_{AB} = W_{AB} = 3P_0 V_0 \log_e (3)$$

For process  $BC$ ,  $W_{BC} = 0$

{isochoric}

$$\Rightarrow \Delta U_{BC} = nC_V \Delta T$$

$$\Rightarrow \Delta U_{BC} = (1) \left( \frac{R}{\gamma - 1} \right) (T_C - T_B)$$

$$\Rightarrow \Delta U_{BC} = \left( \frac{1}{5/3 - 1} \right) \left( \frac{3}{2} P_0 V_0 - 3P_0 V_0 \right)$$

$$\Rightarrow \Delta U_{BC} = -\frac{9P_0 V_0}{4}$$

So from FLTD, we get

$$Q_{BC} = \Delta U_{BC} = -\frac{9P_0 V_0}{4}$$

Since for a cyclic process,  $\Delta U_{\text{Total}} = 0$

$$\Rightarrow \Delta U_{CA} = +\frac{9P_0 V_0}{4}$$

Further,  $Q_{AB} + Q_{BC} + Q_{CA} = W_{AB} + W_{BC} + W_{CA}$

{ $\because \Delta U = 0$ }

Since we have,  $Q_{CA} = -W_{CA}$

{given}

$$\Rightarrow 2W_{CA} = Q_{AB} + Q_{BC} - W_{AB} - W_{BC} = -\frac{9P_0 V_0}{4}$$

$$\Rightarrow W_{CA} = -\frac{9P_0 V_0}{8}$$

So from FLTD, we get

$$Q_{CA} = W_{CA} + \Delta U_{CA} = \frac{9P_0 V_0}{8}$$

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The  $Q$ ,  $W$  and  $\Delta U$  values for different processes are summarised in the table below.

Process	$Q$	$W$	$\Delta U$
$AB$	$3P_0V_0 \log_e(3)$	$P_0V_0 \log_e(3)$	0
$BC$	$-\frac{9P_0V_0}{4}$	0	$-\frac{9P_0V_0}{4}$
$CA$	$\frac{9P_0V_0}{8}$	$-\frac{9P_0V_0}{8}$	$\frac{9P_0V_0}{4}$

(b) For isothermal process,  $C_{AB} \rightarrow \infty$

For process  $CA$ ,  $Q_{CA} = \Delta U_{CA} + W_{CA}$

Also,  $W_{CA} = -Q_{CA}$  {given}

$$\Rightarrow 2Q_{CA} = \Delta U_{CA}$$

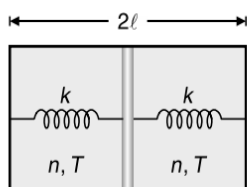
$$\Rightarrow 2C\Delta T = C_V\Delta T$$

$$\Rightarrow C = \frac{C_V}{2} = \left(\frac{1}{2}\right)\left(\frac{3}{2}R\right)$$

$$\Rightarrow C = \frac{3}{4}R$$

### PROBLEM 12

A thin heat insulating piston divides a horizontal cylindrical vessel of length  $2l$  into two equal parts. Each part contains  $n$  moles of ideal monatomic gas at a temperature  $T$ . Two springs of spring constant  $k$  each are connected to the piston on either side as shown in figure. When heat  $Q$  is supplied to the gas in the right part, the piston is displaced to the left by a distance  $x = \frac{l}{2}$ . The left part is in contact with a thermostat at temperature  $T$  all the time. Determine the heat  $Q'$  given away to the thermostat.



### SOLUTION

Net heat added to the system is  $Q - Q'$ . From First Law of Thermodynamics, we have

$$Q = \Delta U + W \Rightarrow Q - Q' = \Delta U + U_s \quad \dots(1)$$

where  $\Delta U$  is the change in internal energy of the system and  $U_s$  is the elastic potential energy stored in the spring. The work done by the gas changes the potential energy of the spring.

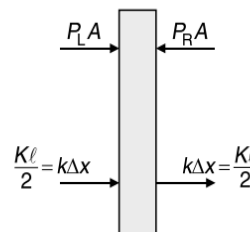
The temperature in the left part has not changed as it is in contact with a thermostat and piston wall is non-conducting. Hence the change in the internal energy of the

gas is due to the heating of the gas in the right part by  $\Delta T$ . So, we have

$$\Delta U = nC_V\Delta T = \frac{3}{2}nR\Delta T \quad \dots(2)$$

Since the gas is monatomic, so  $C_V = \frac{3}{2}R$

Forces acting on the piston are shown in Figure.



In equilibrium, we have

$$P_L A + 2k\Delta X = P_R A \quad \dots(3)$$

When the piston has moved by a distance  $x = \frac{l}{2}$ , the volume of the left part becomes  $V_L = A\left(l - \frac{l}{2}\right)$  and the volume of the right part becomes  $V_R = A\left(l + \frac{l}{2}\right)$ . From Ideal Gas Equation, we have

$$P_L = \frac{nRT}{A\left(l - \frac{l}{2}\right)} \quad \dots(4)$$

$$P_R = \frac{nR(T + \Delta T)}{A\left(l + \frac{l}{2}\right)} \quad \dots(5)$$

Substituting expression for  $P_L$  and  $P_R$  in equation (3), we get

$$\frac{2nR(T + \Delta T)}{3Al} = \frac{2nRT}{Al} + \frac{kl}{A} \quad \dots(6)$$

Solving for  $\Delta T$ , we now get

$$\Delta T = 2T + \frac{3kl^2}{2nR}$$

From equation (1), we have

$$Q' = Q - \Delta U - U_s$$

$$\Rightarrow Q' = Q - \frac{3}{2}nR\left[2T + \frac{3kl^2}{2nR}\right] - \frac{kl^2}{4}$$

$$\Rightarrow Q' = Q - 3nRT - \frac{5}{2}kl^2$$

### PROBLEM 13

A solid body  $X$  of heat capacity  $C$  is kept in an atmosphere whose temperature is  $T_A = 300$  K. At time  $t = 0$ , the temperature of  $X$  is  $T_0 = 400$  K. It cools according to Newton's Law of Cooling. At time  $t_1$  its temperature is found to be

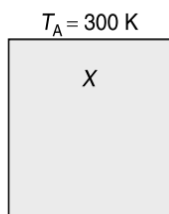
350 K. At this time ( $t_1$ ) the body X is connected to a large body Y at atmospheric temperature  $T_A$  through a conducting rod of length  $L$ , cross-sectional area  $A$  and thermal conductivity  $K$ . The heat capacity of Y is so large that any variation in its temperature may be neglected. The cross-sectional area  $A$  of the connecting rod is small compared to the surface area of X. Find the temperature of X at time  $t = 3t_1$ .

**SOLUTION**

In the first part of the question ( $t \leq t_1$ )

At  $t = 0$ ,  $T_x = T_0 = 400$  K and at  $t = t_1$ ,  $T_x = T_1 = 350$  K

Temperature of atmosphere,  $T_A = 300$  K (constant)



Since, this cools down according to Newton's Law of Cooling, so we have

Rate of Cooling  $\propto$  Temperature Difference

$$\Rightarrow \left(-\frac{dT}{dt}\right) = k(T - T_A)$$

$$\Rightarrow \frac{dT}{T - T_A} = -k dt$$

$$\Rightarrow \int_{T_0}^{T_1} \frac{dT}{T - T_A} = -k \int_0^{t_1} dt$$

$$\Rightarrow \log_e \left( \frac{T_1 - T_A}{T_0 - T_A} \right) = -kt_1$$

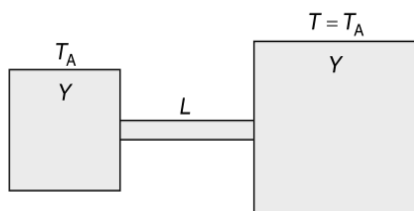
$$\Rightarrow kt_1 = -\log_e \left( \frac{350 - 300}{400 - 300} \right)$$

$$\Rightarrow kt_1 = \log_e (2) \quad \dots(1)$$

In the second part, body X cools by radiations (according to Newton's Law) as well as by conduction ( $t > t_1$ ), so

$$\left( \text{Rate of Cooling} \right) = \left( \text{Cooling by Radiation} \right) + \left( \text{Cooling by Conduction} \right)$$

$$\Rightarrow \left(-\frac{dT}{dt}\right) = k(T - T_A) + \frac{KA}{CL}(T - T_A) \quad \dots(2)$$



$$\text{Since, in conduction, } \frac{dQ}{dt} = \frac{KA(T - T_A)}{L} = C \left(-\frac{dT}{dt}\right)$$

$$\Rightarrow \left(-\frac{dT}{dt}\right) = \frac{KA}{LC}(T - T_A)$$

Where C is the heat capacity of body X,

$$\Rightarrow \left(-\frac{dT}{dt}\right) = \left(k + \frac{KA}{CL}\right)(T - T_A) \quad \dots(3)$$

Let at  $t = 3t_1$ , temperature of X becomes  $T_2$ .

Therefore, from equation (3), we get

$$\int_{T_1}^{T_2} \frac{dT}{T - T_A} = -\left(k + \frac{KA}{LC}\right) \int_{t_1}^{3t_1} dt$$

$$\Rightarrow \log_e \left( \frac{T_2 - T_A}{T_1 - T_A} \right) = -\left(k + \frac{KA}{LC}\right)(2t_1)$$

$$\Rightarrow \log_e \left( \frac{T_2 - T_A}{T_1 - T_A} \right) = -\left(2kt_1 + \frac{2KA}{LC}t_1\right)$$

Since, from equation (1), we have

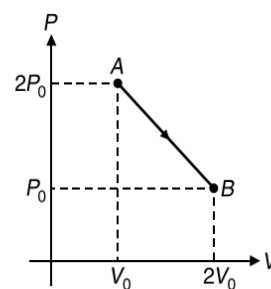
$$kt_1 = \log_e (2)$$

$$\Rightarrow \log_e \left( \frac{T_2 - 300}{350 - 300} \right) = -2\log_e (2) - \frac{2KA t_1}{LC}$$

$$\Rightarrow T_2 = \left( 300 + 12.5e^{-\frac{2KA t_1}{LC}} \right) K$$

**PROBLEM 14**

$P$ - $V$  diagram of  $n$  moles of an ideal gas is as shown in figure. Find the maximum temperature between A and B.



**SOLUTION**

For given number of moles of a gas, we have

$$T \propto PV \quad \{\because PV = nRT\}$$

Although we observe that  $(PV)_A = (PV)_B$

$$\Rightarrow T_A = T_B$$

However still it is not an isothermal process. Because in an isothermal process  $P$ - $V$  graph is a rectangular hyperbola while here it is a straight line. So, to see the behaviour of temperature first we will find either  $T$ - $V$  equation or  $T$ - $P$

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equation and from that equation we can judge how the temperature varies. From the graph we will firstly write  $P$ - $V$  equation and then we will convert it either to  $T$ - $V$  equation or to  $T$ - $P$  equation.

From the graph the  $P$ - $V$  equation can be written as,

$$P = -\left(\frac{P_0}{V_0}\right)V + 3P_0 \quad \left\{ \because y = -mx + c \right\}$$

$$\Rightarrow PV = -\left(\frac{P_0}{V_0}\right)V^2 + 3P_0V$$

Since  $PV = nRT$ , so we get

$$nRT = 3P_0V - \left(\frac{P_0}{V_0}\right)V^2$$

$$\Rightarrow T = \frac{1}{nR} \left[ 3P_0V - \left(\frac{P_0}{V_0}\right)V^2 \right]$$

This is the required  $T$ - $V$  equation. This is quadratic in  $V$ . Hence,  $T$ - $V$  graph is a parabola. Now, to find maximum value of  $T$  we have

$$\frac{dT}{dV} = 0$$

$$\Rightarrow 3P_0 - \left(\frac{2P_0}{V_0}\right)V = 0$$

$$\Rightarrow V = \frac{3}{2}V_0$$

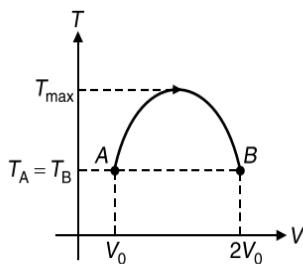
Further  $\frac{d^2T}{dV^2} < 0$ , at  $V = \frac{3}{2}V_0$

Hence,  $T$  is maximum at  $V = \frac{3}{2}V_0$  and this maximum value is

$$T_{\max} = \frac{1}{nR} \left[ \left(3P_0\right)\left(\frac{3V_0}{2}\right) - \left(\frac{P_0}{V_0}\right)\left(\frac{3V_0}{2}\right)^2 \right]$$

$$\Rightarrow T_{\max} = \frac{9P_0V_0}{4nR}$$

Thus,  $T$ - $V$  graph is shown in figure, where



$$T_A = T_B = \frac{2P_0V_0}{nR}$$

and  $T_{\max} = \frac{9P_0V_0}{4nR} = 2.25 \left( \frac{P_0V_0}{nR} \right)$

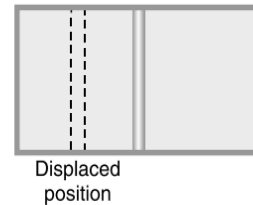
### PROBLEM 15

A heat conducting piston can freely move inside a closed thermally insulated cylinder with an ideal gas. In equilibrium the piston divides the cylinder into two equal parts, the gas temperature being equal to  $T_0$ . The piston is slowly displaced. Find the gas temperature as a function of the ratio  $\eta$  of the volumes of the greater and smaller sections. The adiabatic exponent of the gas is equal to  $\gamma$ .

### SOLUTION

As the piston is heat conducting, the temperature on both the parts is equal at each instant. The system is isolated, so there is no heat transfer to or from the system. From First Law of Thermodynamics (FLTD), we have

$$Q = \Delta U + W = 0$$



From the condition of the problem, we get

$$V_0 + v = \eta(V_0 - v)$$

where  $v$  is the displaced volume

$$\Rightarrow v = \left( \frac{\eta - 1}{\eta + 1} \right) V_0 \quad \dots(1)$$

The work done by an external agent in displacing the piston slowly increases the internal energy of both the compartments is

$$F_{\text{ext}} dx = 2nC_V dT = (P_1 - P_2) dV \quad \dots(2)$$

where  $P_1$  and  $P_2$  are the respective pressures of the two compartments.

From Ideal Gas Equation, we have

$$P_1 = \frac{nRT}{V_0 - v}, P_2 = \frac{nRT}{V_0 + v} \quad \dots(3)$$

On substituting equation (3) in (2), we get

$$\left( \frac{nRT}{V_0 - v} - \frac{nRT}{V_0 + v} \right) dv = 2nC_V dT$$

$$\Rightarrow \frac{R}{C_V} \int_0^v \frac{n}{(V_0^2 - v^2)} dv = \int_{T_0}^T \frac{dT}{T}$$

Since  $C_V = \frac{R}{\gamma - 1}$ , so

$$(\gamma - 1) \int_0^v \frac{v}{V_0^2 - v^2} dv = \log_e \left( \frac{T}{T_0} \right)$$

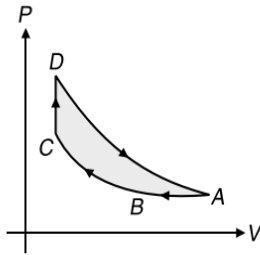
$$\Rightarrow \log_e \left( \frac{T}{T_0} \right) = \left[ \frac{\gamma-1}{2} \right] \left[ -\log_e (V_0^2 - v^2) \right]_0^v$$

$$\Rightarrow \log_e \left( \frac{T}{T_0} \right) = \log_e \left[ \frac{V_0^2}{V_0^2 - v^2} \right]^{\frac{\gamma-1}{2}}$$

$$\Rightarrow T = T_0 \left[ \frac{(\eta+1)^2}{4\eta} \right]^{\frac{\gamma-1}{2}}$$

### PROBLEM 16

Helium is used as working substance in an engine working on the cycle as shown in figure. Processes  $A-B$ ,  $B-C$ ,  $C-D$  and  $D-A$  are isobaric, adiabatic, isochoric and isothermal respectively. The ratio of maximum to minimum volume of helium during the cycle is  $8\sqrt{2}$  and that of maximum to minimum absolute temperature is 4. Calculate efficiency of the cycle.



### SOLUTION

Let  $V_C = V_D = V_0$  (minimum), so  $V_A = 8\sqrt{2}V_0$  (maximum)

Since, process  $AB$  is isobaric, so we have

$$\frac{V_A}{T_A} = \frac{V_B}{T_B} \quad \dots(1)$$

Since  $V_A > V_B$ , so  $T_A > T_B$

Further, process  $BC$  is adiabatic compression. Hence,  $T_C > T_B$

Since, process  $CD$  is isochoric, so we have

$$\frac{P_C}{T_C} = \frac{P_D}{T_D}$$

Since  $P_D > P_C$ , so  $T_D > T_C$

The process  $DA$  is isothermal. Hence,  $T_A = T_D$

Hence, during complete cycle the temperature at  $B$  is minimum while the temperature at  $A$  and  $D$  has the maximum value. So let,  $T_B = T_0$  (minimum),

Then,  $T_A = T_D = 4T_0$  (maximum)

From equation (1), we have

$$V_B = \left( \frac{T_B}{T_A} \right) V_A = \left( \frac{1}{4} \right) (8\sqrt{2}V_0) = 2\sqrt{2}V_0$$

Since,  $BC$  is an adiabatic process, so

$$T_C V_C^{\gamma-1} = T_B V_B^{\gamma-1}$$

$$\Rightarrow T_C = T_B \left( \frac{V_B}{V_C} \right)^{\gamma-1} = T_0 \left( \frac{2\sqrt{2}V_0}{V_0} \right)^{\frac{5}{3}-1}$$

$$\Rightarrow T_C = 2T_0$$

Now let us make a table for  $W$  and  $Q$  for different processes.

Process	$W$	$Q$
$AB$ (isobaric)	$P(V_B - V_A)$ $= nR(T_B - T_A)$ $= -3nRT_0$	$Q$ is negative
$BC$ (adiabatic)	$\frac{nR(T_B - T_C)}{\gamma - 1}$ $= \frac{nR(T_0 - 2T_0)}{\left(\frac{5}{3}\right) - 1}$ $= -\frac{3}{2}nRT_0$	0
$CD$ (isochoric)	0	$nC_V(T_D - T_C)$ $= (n) \left( \frac{3}{2}R \right) (4T_0 - 2T_0)$ $= 3nRT_0$
$DA$ (isothermal)	$nR(4T_0) \log_e \left( \frac{8\sqrt{2}V_0}{V_0} \right)$ $= 14nRT_0 \log_e (2)$	$14nRT_0 \log_e (2)$

$$\text{So, } W_{\text{total}} = 14nRT_0 \log_e (2) - 3nRT_0 - \frac{3}{2}nRT_0$$

$$\Rightarrow W_{\text{total}} = 5.2nRT_0$$

$$\text{and } Q_{+\text{ve}} = 14nRT_0 \log_e (2) + 3nRT_0$$

$$\Rightarrow Q_{+\text{ve}} = 12.7nRT_0$$

So, efficiency of cycle is

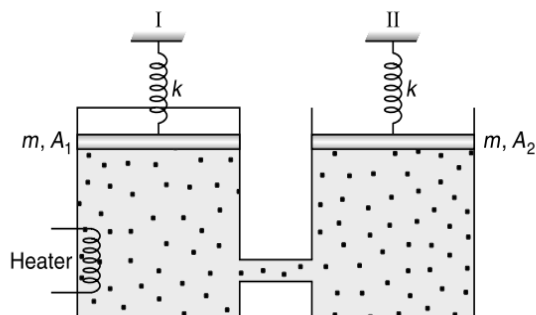
$$\eta = \frac{W_{\text{total}}}{Q_{+\text{ve}}} \times 100 = \left( \frac{5.2}{12.7} \right) \times 100$$

$$\Rightarrow \eta = 43\%$$

### PROBLEM 17

Two vertical cylinders with thermally insulated walls and pistons of same mass fitted on the top as shown in the figure. The cylinders enclose  $n$  moles of a monatomic gas. The initial pressure is  $P$  and it is given that Spring (I) is relaxed. An electric heater slowly supplies heat to the system. Finally, Spring (II) becomes relaxed.

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- (a) What is the initial state of Spring (II)? Given that  $A_2 < A_1$ .  
 (b) What is the final pressure?  
 (c) If the initial volume is  $V$ , what is the final volume?

### SOLUTION

- (a) Let  $P_0$  be the atmospheric pressure, then considering the equilibrium of Piston I, we get

$$PA_1 = mg + P_0A_1$$

$$\Rightarrow P = P_0 + \frac{mg}{A_1} \quad \dots(1)$$

Now consider the forces acting on Piston II. According to given condition, Spring II is relaxed, finally. Before that we have to determine the state of spring by comparing the downward and upward forces. Net upward force acting on Piston II is

$$F_{\text{up}} = PA_2 = P_0A_2 + mg \frac{A_2}{A_1} \quad \left\{ \because \text{of equation (1)} \right\}$$

Net downward force acting on Piston II is

$$F_{\text{down}} = mg + P_0A_2$$

Since,  $A_2 < A_1$ , so net upward force is less than net downward force. Hence to keep the piston in equilibrium, the spring must apply an upward force on it, therefore the spring is stretched initially and the spring force is

$$kx = mg \left( 1 - \frac{A_2}{A_1} \right)$$

where  $x$  is the extension in the Spring II.

- (b) When Spring II is relaxed, let  $P'$  be the pressure. Then from the equilibrium condition of piston of area  $A_2$  from equation (1), we have

$$P' = \frac{mg}{A_2} + P_0 = P - \frac{mg}{A_1} + \frac{mg}{A_2}$$

$$\Rightarrow P' = P + mg \left( \frac{A_1 - A_2}{A_1A_2} \right)$$

- (c) When the pressure increases, Spring I is compressed. Let  $x'$  be the compression, then from the condition of equilibrium of piston of area  $A_1$ , we have

$$P'A_1 = kx' + mg + P_0A_1$$

$$\Rightarrow kx' = P'A_1 - mg - P_0A_1$$

$$\Rightarrow kx' = \left[ P + mg \left( \frac{A_1 - A_2}{A_1A_2} \right) \right] A_1 - mg - \left( P - \frac{mg}{A_1} \right) A_1$$

$$\Rightarrow kx' = mg \left( \frac{A_1}{A_2} - 1 \right)$$

Change in volume of the gas is  $\Delta V = x'A_1 + xA_2$

$$\Rightarrow \Delta V = \frac{mg}{k} \left( \frac{A_1}{A_2} - 1 \right) A_1 + \frac{mg}{k} \left( 1 - \frac{A_1}{A_2} \right) A_2$$

$$\Rightarrow \Delta V = \frac{mg}{k} \left( \frac{A_1^2}{A_2} - A_1 + A_2 - \frac{A_2^2}{A_1} \right)$$

Final volume is

$$V_f = V + \Delta V = V + \frac{mg}{k} \left( \frac{A_1^2}{A_2} - A_1 + A_2 - \frac{A_2^2}{A_1} \right)$$

$$\Rightarrow V_f = V + \frac{mg}{k} \frac{(A_1 - A_2)(A_1^2 + A_2^2)}{A_1A_2}$$

### PROBLEM 18

A cubical rigid container of edge  $l = 10$  cm, thickness of wall  $d = 4$  mm and thermal conductivity  $k = 8.31 \times 10^{-3} \text{ Wm}^{-1}\text{K}^{-1}$  has one mole of a monatomic gas at temperature  $T_0 = 400$  K. If the temperature of the surrounding is  $T_s = 300$  K, find the

- (a) time when pressure becomes  $\frac{5}{6}$  of initial pressure.  
 (b) temperature at that instant.

### SOLUTION

- (a) From the First Law of Thermodynamics, we have

$$dQ = dU + dW$$

Since, volume is constant, therefore,  $dW = 0$ . So, for 1 mole of gas, we have

$$\left( \frac{dQ}{dt} \right) = C_V \left( - \frac{dT}{dt} \right) \quad \dots(1)$$

Further from the equation of heat conduction, we have

$$\left( \frac{dQ}{dt} \right) = \frac{kA(T - T_s)}{d} \quad \dots(2)$$

From equations (1) and (2), we get

$$\frac{dT}{dt} = \frac{kA(T_s - T)}{C_V d} = \frac{2kA(T_s - T)}{3Rd}$$

$$\Rightarrow \int_{T_0}^T \frac{dT}{T_s - T} = \frac{2kA}{3Rd} \int_0^t dt$$

$$\Rightarrow T = T_s - (T_s - T_0) e^{-\frac{2kAt}{3Rd}} \quad \dots(3)$$

As the volume remains constant, so

$$\frac{P}{T} = \frac{P_0}{T_0}$$

$$\Rightarrow P = \left(\frac{P_0}{T_0}\right)T$$

$$\Rightarrow P = \left(\frac{P_0}{T_0}\right)\left(T_s - (T_s - T_0)e^{-\frac{2kAt}{3Rd}}\right) \quad \dots(4)$$

Given,  $\frac{P}{P_0} = \frac{5}{6}$ ,  $A = 6l^2$

$$\Rightarrow \frac{2kA}{3Rd} = \frac{4kl^2}{Rd} = \frac{4 \times 8.31 \times 10^{-3} \times (0.1)^2}{(8.31)(4 \times 10^{-3})} = 10^{-2} \text{ s}^{-1}$$

Substituting the values in equation (4)

$$\frac{5}{6} = \left(\frac{1}{400}\right)(300 + (100)e^{-10^{-2}t})$$

Solving this equation, we get

$$t \approx 110 \text{ sec.}$$

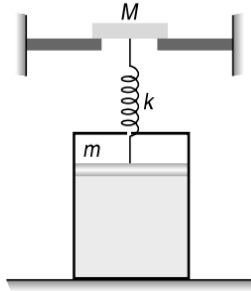
(b) As  $P \propto T$  {  $V = \text{constant}$  }

So, temperature will also remain  $\frac{5}{6}$ th of its initial value, hence

$$T = \frac{5}{6}T_0 = \left(\frac{5}{6}\right)(400) = 333.33 \text{ K}$$

### PROBLEM 19

0.01 moles of an ideal diatomic gas is enclosed in an adiabatic cylinder of cross-sectional area  $A = 10^{-4} \text{ m}^2$ . In the arrangement shown, a block of mass  $M = 0.8 \text{ kg}$  is placed on a horizontal support and another block of mass  $m = 1 \text{ kg}$  is suspended from a spring of stiffness constant  $k = 16 \text{ Nm}^{-1}$ . Initially, the spring is relaxed and the volume of the gas is  $V = 1.4 \times 10^{-4} \text{ m}^3$ .



- Find the initial pressure of the gas.
- If block  $m$  is gently pushed down and released it oscillates harmonically, find its angular frequency of oscillation.
- When the gas in the cylinder is heated up the piston starts moving up and the spring gets compressed so that the block  $M$  is just lifted up. Determine the heat supplied. Take atmospheric pressure  $p_0 = 10^5 \text{ Nm}^{-2}$ ,  $g = 10 \text{ ms}^{-2}$ .

### SOLUTION

(a)  $P = P_0 + \frac{mg}{A} = 10^5 + \frac{(1)(10)}{10^{-4}} = 2 \times 10^5 \text{ Nm}^{-2}$

(b) For adiabatic process,  $\left(\frac{dP}{dV}\right) = -\gamma\left(\frac{P}{V}\right)$

$$\Rightarrow |(dP)| = \left(\frac{7}{5}\right)\left(\frac{2 \times 10^5}{1.4 \times 10^{-4}}\right)(Ax)$$

Net restoring force is  $F = -[|dP|A + kx]$

$$\Rightarrow m \frac{d^2x}{dt^2} = -\left[\left(\frac{7}{5}\right)\left(\frac{2 \times 10^5}{1.4 \times 10^{-4}}\right)(10^{-4})^2 + 16\right]x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -36x = -\omega^2x$$

$$\Rightarrow \omega = 6 \text{ rads}^{-1}$$

(c) Block  $M$  will start moving up when

$$kx = Mg$$

$$\Rightarrow x = \frac{Mg}{k} = \frac{0.8 \times 10}{16} = 0.5 \text{ m}$$

Now,  $\Delta V = Ax = 5 \times 10^{-5} \text{ m}^3$

and  $W_{\text{gas}} = P_0\Delta V + \frac{1}{2}kx^2 + mgx$

$$\Rightarrow W_{\text{gas}} = (10^5)(5 \times 10^{-5}) + \frac{1}{2} \times 16 \times (0.5)^2 +$$

$$(1)(10)(0.5)$$

$$\Rightarrow W_{\text{gas}} = 12 \text{ J}$$

Since,  $\Delta U = nC_V\Delta T$

where,  $T_i = \frac{P_i V_i}{nR} = \frac{(2 \times 10^5)(1.4 \times 10^{-4})}{(0.01)(8.31)} = 337 \text{ K}$

and  $T_f = \frac{P_f V_f}{nR} = \frac{\left(P_0 + \frac{kx}{A} + \frac{mg}{A}\right)(1.9 \times 10^{-4})}{(0.01)(8.31)}$

$$\Rightarrow T_f = \frac{\left(2 \times 10^5 + \frac{16 \times 0.5}{10^{-4}}\right)(1.9 \times 10^{-4})}{(0.01)(8.31)}$$

$$\Rightarrow T_f = 640.3 \text{ K}$$

$$\Rightarrow \Delta U = (0.01)\left(\frac{5}{2}\right)(8.31)(640.3 - 337)$$

$$\Rightarrow \Delta U = 63 \text{ J}$$

So, from  $FLTD$ , we get

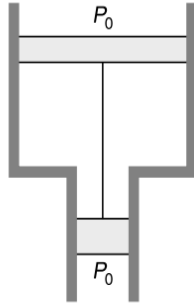
$$Q = \Delta U + W = 63 + 12 = 75 \text{ J}$$

### PROBLEM 20

A smooth vertical cylinder has two different cross-sections open from both ends and equipped with two pistons of different cross-section areas. Each piston slides within a respective section as shown in figure.

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One mole of an ideal gas is enclosed between the pistons tied with a non-stretchable thread. The cross-section area of the upper piston is  $\Delta A$  greater than that of the lower one. The combined mass of the two pistons is equal to  $m$ . The atmospheric pressure outside is  $P_0$ . By how much kelvin must the gas between the pistons be heated to shift the piston through  $l$  unit.



$$\Rightarrow \Delta T = \frac{(P_0 \Delta A + mg)l}{R}$$

**SOLUTION**

Let  $A_1, A_2$  denote the cross-section area of the lower and upper piston and  $l_1, l_2$  the lengths of string in respective sections. Initial volume of the gas enclosed is

$$V_i = l_1 A_1 + l_2 A_2$$

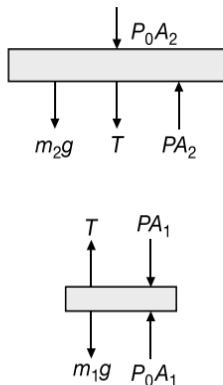
If the piston shifts upward by  $l$ , then final volume of the gas enclosed is

$$V_f = (l_1 - l)A_1 + (l_2 + l)A_2$$

The change in volume is

$$\Delta V = lA_1 - lA_2 = l\Delta A \quad \dots(1)$$

Let us draw the free body diagrams for the two pistons.



Now we consider the equilibrium of the upper and lower pistons, then

$$\text{For Lower Piston, } PA_1 + m_1 g = T + P_0 A_1 \quad \dots(2)$$

$$\text{For Upper Piston, } PA_2 = P_0 A_2 + T + m_2 g \quad \dots(3)$$

From equations (2) and (3), we get

$$P(A_1 - A_2) = P_0(A_1 - A_2) + (m_1 + m_2)g$$

$$\Rightarrow P\Delta V = (P_0 \Delta A + mg)l$$

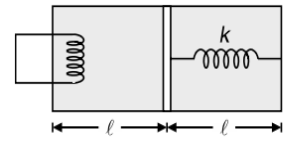
when  $m = m_1 + m_2$  is the combined mass of the piston, as given in the problem.

After temperature rises by  $\Delta T$ , from Ideal Gas Equation applied for one mole of gas, we have

$$\Delta T = \frac{P\Delta V}{R}$$

**PROBLEM 21**

An adiabatic cylinder of length 1 m and area of cross section  $10^{-2} \text{ m}^2$  is closed at both ends.



A freely moving nonconducting thin piston divides the cylinder into two equal parts. Each part contains 28 g of  $N_2$ . The natural length of the spring connected to the piston and right wall of the cylinder is  $l = 50 \text{ cm}$  and  $k = \sqrt{2} \times 10^3 \frac{\text{N}}{\text{m}}$ . Initially one-third molecules of the nitrogen in the right part are dissociated into atoms. If initial pressure in each part is  $P_0 = \sqrt{2} \times 10^5 \text{ Nm}^{-2}$ , then to compress the spring by

$\frac{3l}{4}$ , calculate the

- (a) work done by the gas in right part.
- (b) heat supplied by the heater.

**SOLUTION**

(a) Initial volume of gas in the right part is  $V_0 = Al$ . To compress the spring by  $\frac{3l}{4}$ , the final volume becomes

$$V = \frac{V_0}{4} = \frac{Al}{4}, \text{ so for the gas in the right part, we have}$$

$$P_0 (Al)^{\gamma} = P' \left( \frac{Al}{4} \right)^{\gamma} \quad \dots(1)$$

The right part contains 28 g of  $N_2$ , i.e., 1 mol of  $N_2$  and  $\frac{1}{3}$  rd of molecules are dissociated into atoms.

So, in the right part  $n_1 = \frac{2}{3}$  moles are of diatomic gas

$\left( C_{V_1} = \frac{5}{2} R \right)$  and  $n_2 = 2 \times \frac{1}{3} = \frac{2}{3}$  moles are of mono-

atomic gas  $\left( C_{V_2} = \frac{3}{2} R \right)$ . The  $C_V$  of the mixture is given

by

$$C_V = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2} = \frac{\left( \frac{2}{3} \right) \left( \frac{5}{2} R \right) + \left( \frac{2}{3} \right) \left( \frac{3}{2} R \right)}{\left( \frac{2}{3} + \frac{2}{3} \right)}$$

$$\Rightarrow C_V = 2R$$

$$\text{Since, } C_P = C_V + R = 3R$$

$$\Rightarrow \gamma = \frac{C_P}{C_V} = 1.5$$

Hence, from equation (1), we get

$$P' = 8P_0$$

Work done during compression by the gas in right part is given by

$$W_R = \frac{P_0 V_0 - P' V'}{\gamma - 1}$$

$$\Rightarrow W_R = \frac{P_0 (Al) - (8P_0) \left( \frac{Al}{4} \right)}{1.5 - 1} = -2P_0 Al$$

$$\Rightarrow W_R = -(2)(\sqrt{2} \times 10^5)(10^{-2})(0.5) = -1414 \text{ J}$$

Here negative sign implies that work is done on the gas

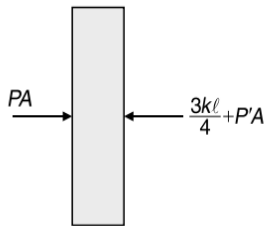
(b) Considering equilibrium of piston, we get

$$PA = \frac{3kl}{4} + P'A$$

So, pressure on the left part is

$$P = 8P_0 + \frac{3kl}{4A}$$

Initial temperature  $T_0 = \frac{P_0 Al}{R}$



and final temperature

$$T = \frac{(P) \left( \frac{5Al}{4} \right)}{R} = \frac{5Al}{4R} \left( 8P_0 + \frac{3kl}{4A} \right)$$

Increase in internal energy of the gas in left part is

$$\Delta U_L = C_v \Delta T = \left( \frac{3}{2} R \right) (T - T_0)$$

$$\Rightarrow \Delta U_L = \left( \frac{3}{2} R \right) \left[ \frac{5Al}{4R} \left( 8P_0 + \frac{3kl}{4A} \right) - \frac{P_0 Al}{R} \right]$$

$$\Rightarrow \Delta U_L = \frac{27}{2} P_0 Al + \frac{45}{32} kl^2$$

Work done by gas in the left part to compress the spring and the gas in right part is given by

$$W_L = \frac{1}{2} k \left( \frac{3l}{4} \right)^2 + |W_R|$$

$$\Rightarrow W_L = \frac{9}{32} kl^2 + 2P_0 Al$$

Hence, heat supplied by heating coil is

$$Q = W_L + \Delta U_L$$

$$\Rightarrow Q = \frac{9}{32} kl^2 + 2P_0 Al + \frac{27}{2} P_0 Al + \frac{45}{32} kl^2$$

Substituting the values, we get

$$Q = 11555 \text{ J}$$

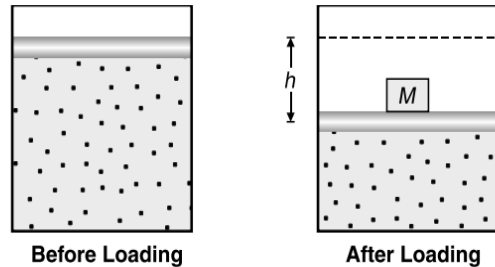
### PROBLEM 22

A vertical cylinder of volume  $V$  has  $n$  moles of an ideal monatomic gas. The walls of the cylinder are thermally insulated, the piston is weightless. When a mass  $M$  is placed on the piston, the piston is displaced by a distance  $h$ . What is the final temperature of the gas after the piston has been displaced, the area of the piston is  $A$  and the atmospheric pressure is  $P_0$ ?

### SOLUTION

Since the gas is thermally insulated, the entire work done on the gas is equal to change in internal energy. When the piston is displaced by  $h$ , the volume becomes  $(V - Ah)$ .

The net pressure on the piston is  $P_0 + \left( \frac{Mg}{A} \right)$ . From Ideal Gas Equation, we have



Before loading,

$$P_0 V = nRT_1 \quad \dots(1)$$

After loading,

$$\left[ P_0 + \left( \frac{Mg}{A} \right) \right] (V - Ah) = nRT_2 \quad \dots(2)$$

The change in internal energy is given by

$$\Delta U = nC_v \Delta T = \frac{3}{2} nR(T_2 - T_1)$$

Work done on the system is  $Mgh$

$$\Rightarrow W = -Mgh$$

Therefore, from First Law of Thermodynamics, we have

$$Q = W + \Delta U$$

Since, the walls of the cylinder are thermally insulated, so

$$Q = 0$$

$$\Rightarrow 0 = -Mgh + \Delta U$$

$$\Rightarrow Mgh = \frac{3}{2} nR(T_2 - T_1) \quad \dots(3)$$

Subtracting equation (1) from equation (2), we get

$$\left[ P_0 V + \left( \frac{Mg}{A} \right) V - P_0 Ah - Mg h - P_0 V \right] = nR(T_2 - T_1)$$

$$\Rightarrow nR(T_2 - T_1) = \left[ \left( \frac{Mg}{A} \right) V - P_0 Ah - Mg h \right] \quad \dots(4)$$

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Substituting  $nR(T_2 - T_1)$  from equation (4) in equation (3), we get

$$\frac{MgV}{A} - Mgh - P_0Ah = \frac{2}{3}Mgh$$

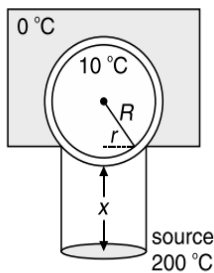
$$\Rightarrow h = \frac{MgV}{A\left(P_0A + \frac{5}{3}Mg\right)}$$

Substituting  $h$  in equation (2) and solving for  $T_2$ , we get

$$T_2 = \frac{(P_0A + mg)(3P_0A - 2Mg)V}{(3P_0A + 5Mg)nAR}$$

### PROBLEM 23

A spherical shell made from iron is placed at a distance  $x$  above a circular heat source of radius  $r$ . Radius of shell is  $R$ . The shell is surrounded by cold bath ( $0^\circ\text{C}$ ) except for the part which is just above the source. The temperature of the source is  $200^\circ\text{C}$ . In equilibrium position the temperature of the inner part of the shell is  $10^\circ\text{C}$ . If the temperature at a distance  $x$  from the source is  $\frac{1}{x}$  times the temperature of the source, find  $x$  at equilibrium. The thickness of the shell is  $t$ .



### SOLUTION

The heat is flowing into the shell through the part just above the source and the remaining part is conducting heat into the bath.

At equilibrium these two rates are equal. The surface area through which the heat is going into, i.e., the area  $ABC$  is,

$$A_1 = 2\pi R^2(1 - \cos \alpha)$$

where,  $\cos \alpha = \frac{\sqrt{R^2 - r^2}}{R}$

$$\Rightarrow A_1 = 2\pi R^2 \left[ 1 - \sqrt{1 - \left(\frac{r}{R}\right)^2} \right]$$

$$\text{So, } A_2 = 4\pi R^2 - A_1 = 2\pi R^2 \left[ 1 + \sqrt{1 - \left(\frac{r}{R}\right)^2} \right]$$

Temperature at the outer surface of part  $ABC$  is

$$T_1 = \frac{200}{x} \text{ } ^\circ\text{C}$$

Temperature of inner surface of shell is  $T_2 = 10^\circ\text{C}$  and temperature of the bath is  $T_3 = 0^\circ\text{C}$

Since,  $\left(\frac{dQ}{dt}\right)_{\text{in}} = \left(\frac{dQ}{dt}\right)_{\text{out}}$

$$\Rightarrow \frac{kA_1(T_1 - T_2)}{t} = \frac{kA_2(T_2 - T_3)}{t}$$

Substituting the values and the solving, we get

$$x = 10 \left[ 1 - \sqrt{1 - \left(\frac{r}{R}\right)^2} \right]$$

### PROBLEM 24

A gaseous mixture enclosed in a vessel of volume  $V$  consists of one gram mole of gas  $A$  with  $\gamma = \left(\frac{C_p}{C_v} = \frac{5}{3}\right)$  and another gas  $B$  with  $\gamma = \frac{7}{5}$  at a certain temperature  $T$ . The gram molecular weights of the gases  $A$  and  $B$  are 4 and 32 respectively. The gases  $A$  and  $B$  do not react with each other and are assumed to be ideal. The gaseous mixture follows the equation  $PV^{19/13} = \text{constant}$ , in adiabatic process

- Find the number of gram moles of the gas  $B$  in the gaseous mixture.
- Compute the speed of sound in the gaseous mixture at 300 K
- If  $T$  is raised by 1 K from 300 K, find the percentage change in the speed of sound in the gaseous mixture.
- The mixture is compressed adiabatically to  $\frac{1}{5}$  of its initial volume  $V$ . Find the change in its adiabatic compressibility in terms of the given quantities.

### SOLUTION

- (a) Number of moles of gas  $A$  are  $n_A = 1$  (given)

Let the number of moles of gas  $B$  be  $n_B = n$

$$\text{Then, } \left( \begin{array}{c} \text{Internal} \\ \text{Energy of} \\ \text{the Mixture} \end{array} \right) = \left( \begin{array}{c} \text{Internal} \\ \text{Energy of} \\ \text{Gas A} \end{array} \right) + \left( \begin{array}{c} \text{Internal} \\ \text{Energy of} \\ \text{Gas B} \end{array} \right)$$

Since,  $U = \frac{nRT}{\gamma - 1}$

$$\Rightarrow \frac{(n_A + n_B)RT}{\gamma_{\text{mixture}} - 1} = \frac{n_A RT}{\gamma_A - 1} + \frac{n_B RT}{\gamma_B - 1} \quad \dots(1)$$

Since, the mixture obeys the law  $PV^{19/13} = \text{constant}$  (in adiabatic process).

Therefore,  $\gamma_{\text{mixture}} = \frac{19}{13} \quad \left\{ \because PV^\gamma = \text{constant} \right\}$

Substituting the values in equation (1), we get

$$\frac{1+n}{\left(\frac{19}{13}\right) - 1} = \frac{1}{\left(\frac{5}{3}\right) - 1} + \frac{n}{\left(\frac{7}{5}\right) - 1}$$

Solving this, we get  $n = 2$

## Conceptual Note(s)

For  $\gamma_{\text{mixture}}$  we can directly use the formula

$$\frac{n}{\gamma_{\text{mixture}} - 1} = \frac{n_1}{\gamma_1 - 1} + \frac{n_2}{\gamma_2 - 1}$$

(b) Molecular weight of the mixture will be given by

$$M = \frac{n_A M_A + n_B M_B}{n_A + n_B} = \frac{(1)(4) + 2(32)}{1 + 2}$$

$$\Rightarrow M = 22.67 \text{ g}$$

Speed of sound in a gas is given by

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Therefore, in the mixture of the gas, we have

$$v = \sqrt{\frac{(19/13)(8.31)(300)}{22.67 \times 10^{-3}}} \text{ ms}^{-1} = 401 \text{ ms}^{-1}$$

(c)  $v \propto \sqrt{T}$

$$\Rightarrow v = KT^{1/2} \quad \dots(2)$$

$$\Rightarrow \frac{dv}{dT} = \frac{1}{2}KT^{-1/2}$$

$$\Rightarrow dv = K \left( \frac{dT}{2\sqrt{T}} \right)$$

$$\Rightarrow \frac{dv}{v} = \frac{K}{v} \left( \frac{dT}{2\sqrt{T}} \right) \frac{K}{v} = \frac{1}{\sqrt{T}} \quad \text{\{from equation (2)\}}$$

$$\Rightarrow \frac{dv}{v} = \frac{1}{\sqrt{T}} \left( \frac{dT}{2\sqrt{T}} \right) = \frac{1}{2} \left( \frac{dT}{T} \right)$$

$$\Rightarrow \frac{dv}{v} \times 100 = \frac{1}{2} \left( \frac{1}{300} \right) \times 100 = 0.167$$

Therefore, percentage change in speed is 0.167%

(d) Compressibility =  $\frac{1}{\text{Bulk modulus}} = \beta$  (say)

Since, adiabatic Bulk modulus is given by

$$B_{ad} = \gamma P \quad \left\{ \because B = - \frac{dP}{dV/V} \right\}$$

So, adiabatic compressibility will be given by

$$\beta_{ad} = \frac{1}{\gamma P} \text{ and } \beta'_{ad} = \frac{1}{\gamma P'} = \frac{1}{\gamma P(5)^\gamma}$$

$$\text{Since, } PV^\gamma = P' \left( \frac{V}{5} \right)^\gamma$$

$$\Rightarrow P' = P(5)^\gamma, \text{ where } \gamma = \gamma_{\text{mixture}} = \frac{19}{13}$$

$$\Rightarrow \Delta\beta = \beta'_{ad} - \beta_{ad} = - \frac{1}{\gamma P} \left[ 1 - \left( \frac{1}{5} \right)^\gamma \right]$$

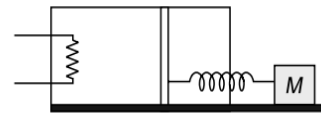
$$\Rightarrow \Delta\beta = \frac{-V}{\gamma(n_A + n_B)RT} \left[ 1 - \left( \frac{1}{5} \right)^\gamma \right] \quad \left\{ \because P = \frac{nRT}{V} \right\}$$

$$\Rightarrow \Delta\beta = \frac{-V}{\left( \frac{19}{13} \right) (1+2) (8.31) (300)} \left[ 1 - \left( \frac{1}{5} \right)^{\frac{19}{13}} \right]$$

$$\Rightarrow \Delta\beta = -8.27 \times 10^{-5} \text{ V}$$

### PROBLEM 25

An adiabatic cylinder has 8 g of helium. A light smooth adiabatic piston is connected to a light spring of force constant  $300 \text{ Nm}^{-1}$ . The other end of the spring is connected with a block of mass  $10 \text{ kg}$  kept on a rough horizontal surface of coefficient of friction  $\mu = 0.3$ . Area of cross-section of cylinder is  $A = 25 \text{ cm}^2$ . Initially the spring is in a relaxed position and the temperature of the gas is  $400 \text{ K}$ , the gas is heated slowly for some time by means of an electric heater so as the block  $M$  just starts moving. Find



- (a) the work done by the gas
- (b) the final temperature
- (c) heat supplied by the heater

Take  $1 \text{ atm} = 10^5 \text{ Nm}^{-2}$  and  $g = 10 \text{ ms}^{-2}$

### SOLUTION

(a) The block  $M$  just starts moving when,



$$kx = \mu(Mg)$$

$$\Rightarrow x = \frac{\mu Mg}{k} = \frac{(0.3)(10)(10)}{300} = 0.1 \text{ m}$$

The force exerted by the gas on the piston is

$$F = (P_0 A + kx)$$

$$\text{Work done by the gas is } W = \int_0^x F dx$$

$$\Rightarrow W = \int_0^x (P_0 A + kx) dx = P_0 A x + \frac{1}{2} kx^2$$

$$\Rightarrow W = (10^5)(25 \times 10^{-4})(0.1) + \left( \frac{1}{2} \right) (300)(0.1)^2$$

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$$\Rightarrow W = 25 + 1.5 = 26.5 \text{ J}$$

(b) The initial temperature is  $T_i = 400 \text{ K}$ . Let the final temperature be  $T_f$ . Then,

$$P_0 V_0 = nRT_i \text{ and } PV' = nRT_f$$

$$\text{where } P = P_0 + \frac{kx}{A} \text{ and } V' = V_0 + Ax$$

$$\text{Hence, } nRT_f = \left( P_0 + \frac{kx}{A} \right) (V_0 + Ax)$$

$$\Rightarrow nRT_f = nRT_i + P_0 Ax + kx^2 + \frac{kx nRT_i}{P_0 A}$$

$$\Rightarrow T_f = T_i + \left( \frac{P_0 Ax + kx^2}{nR} \right) + \left( \frac{kx T_i}{P_0 A} \right)$$

$$\Rightarrow T_f = 400 + \frac{(25 + 3)}{2 \times 8.31} + \frac{(300)(0.1)(400)}{(10^5)(25 \times 10^{-4})}$$

$$\Rightarrow T_f = 449.68 \text{ K}$$

(c) Change in internal energy is given by

$$\Delta U = nC_V \Delta T = \frac{3}{2} nR (T_f - T_i)$$

$$\Rightarrow \Delta U = \left( \frac{3}{2} \right) (2)(8.31)(449.68 - 400) \text{ J}$$

$$\Rightarrow \Delta U = 1238.52 \text{ J}$$

From First Law of Thermodynamics, we have

$$Q = \Delta U + W = 1238.52 + 26.5 \cong 1265 \text{ J}$$

### PROBLEM 26

A wire of length 1 m and radius  $10^{-3} \text{ m}$  is carrying a current. At equilibrium its temperature is 900 K while that of surrounding is 300 K. The resistivity of the material of wire at 300 K is  $10^{-7} \Omega \text{ m}$  and its temperature coefficient of resistance is  $7.8 \times 10^{-3} \text{ per } ^\circ \text{C}$ . Find the current in the wire. Give  $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2} \text{K}^{-4}$ . Take emissivity for wire surface  $e = 1$ .

### SOLUTION

In steady state,

$$\left( \begin{array}{c} \text{Power generated by} \\ \text{the resistance} \end{array} \right) = \left( \begin{array}{c} \text{Power Radiated in} \\ \text{the Atmosphere} \end{array} \right)$$

$$\Rightarrow I^2 R = \sigma A_1 (T^4 - T_0^4)$$

If  $\rho$  be the electrical resistivity of the wire, then

$$R = \frac{\rho_{900 \text{ K}} l}{A_2}$$

$$\Rightarrow I^2 \frac{\rho_{900 \text{ K}} l}{A_2} = \sigma A_1 (T^4 - T_0^4) \quad \dots(1)$$

$$\text{where, } A_1 = 2\pi r l, A_2 = \pi r^2$$

Since,  $\rho_{900 \text{ K}} = \rho_{300 \text{ K}} (1 + \alpha \Delta T)$

$$\Rightarrow \rho_{900 \text{ K}} = 10^{-7} [1 + 7.8 \times 10^{-3} \times (900 - 300)]$$

$$\Rightarrow \rho_{900 \text{ K}} = 5.68 \times 10^{-7} \Omega \text{ m}$$

Substituting in equation (1), we have

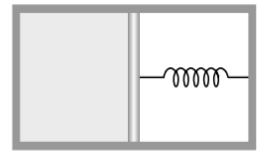
$$I^2 \left( \frac{5.68 \times 10^{-7}}{\pi \times 10^{-6}} \right) = 5.67 \times 10^{-8} \times 2 \times \pi \times$$

$$10^{-3} \times 1 [(900)^4 - (300)^4]$$

$$\Rightarrow I = 35.73 \text{ A}$$

### PROBLEM 27

A non-conducting piston separates a thermally insulated vessel into two parts and it moves in the vessel without friction. The left part of the vessel contains one mole of an ideal monatomic gas and the right part is empty. The piston is connected to the right wall of vessel, through a spring whose length in free state is equal to the length of the vessel. Calculate the heat capacity of the system neglecting the heat capacities of vessel, piston and spring.



### SOLUTION

If  $Q$  be the heat supplied to the system, then this supplied heat will only increase the internal energy of the system because there is no friction and the vessel is thermally insulated, so  $Q = \Delta U$

$$\Rightarrow Q = \Delta U_{\text{spring}} + \Delta U_{\text{gas}} \quad \dots(1)$$

$$\text{where } \Delta U_{\text{spring}} = \frac{1}{2} k (x_2^2 - x_1^2) \quad \dots(2)$$

$$\text{and } \Delta U_{\text{gas}} = (1) C_V \Delta T = \frac{3R}{2} (T_2 - T_1) \quad \dots(3)$$

where  $x_1$  and  $x_2$  are compression in the spring at temperatures  $T_1$  and  $T_2$  respectively.

Now in equilibrium, we have  $PA = kx$

$$\Rightarrow x = \frac{PA}{k}$$

For one mole of an ideal gas, we have  $PV = RT$

$$\Rightarrow P(Ax) = RT$$

But  $PA = kx$

$$\Rightarrow kx^2 = RT$$

$$\Rightarrow \frac{1}{2} k (x_2^2 - x_1^2) = \frac{R}{2} (T_2 - T_1)$$

$$\Rightarrow \Delta U_{\text{spring}} = \frac{R}{2} (T_2 - T_1) \quad \dots(4)$$

$$\Rightarrow Q = \Delta U = \frac{1}{2}k(x_2^2 - x_1^2) + \frac{3R}{2}(T_2 - T_1)$$

$$\Rightarrow Q = \Delta U = \frac{R}{2}(T_2 - T_1) + \frac{3R}{2}(T_2 - T_1)$$

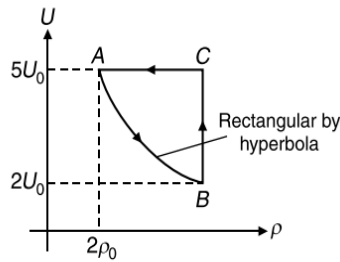
$$\Rightarrow Q = \Delta U = 2R(T_2 - T_1)$$

But  $C = \frac{Q}{\Delta T} = \frac{\Delta U}{\Delta T}$  {by definition}

$$\Rightarrow C = 2R$$

**PROBLEM 28**

Figure shows the variation of the internal energy  $U$  with the density  $\rho$  of one mole of an ideal monatomic gas for a thermodynamic cycle  $A \rightarrow B \rightarrow C \rightarrow A$ . Assume the process  $A \rightarrow B$  to be a part of a rectangular hyperbola.



- (a) Draw the  $P$ - $V$  diagram of the above process.
- (b) Find the total amount of heat absorbed by the system for the cyclic process.
- (c) Find the work done in the process  $AB$ .

**SOLUTION**

(a) Process  $A \rightarrow B$  is part of a rectangular hyperbola, so

$$U\rho = \text{constant}$$

From Ideal Gas Equation, we have

$$PV = RT$$

$$\Rightarrow P\left(\frac{M}{\rho}\right) = RT$$

$$\Rightarrow \frac{P}{\rho} = \frac{RT}{M}$$

Since, internal energy is function of temperature, so  $U = aT$ , where  $a$  is a constant

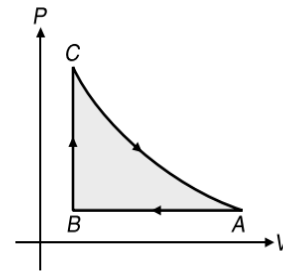
$$\Rightarrow U\rho = aT \frac{PM}{RT} = \text{constant}$$

$$\Rightarrow P = \text{constant}$$

Therefore, the process  $A \rightarrow B$  is isobaric.

Process  $B \rightarrow C$  is isochoric because density is constant.

Process  $C \rightarrow A$  is isothermal because internal energy is constant. Now  $P$ - $V$  diagram is shown below.



(b) Total heat absorbed by the system, is

$$Q = Q_{A \rightarrow B} + Q_{B \rightarrow C} + Q_{C \rightarrow A}$$

Since process  $A \rightarrow B$  is isobaric, so

$$Q_{A \rightarrow B} = C_p \Delta T = C_p \frac{\Delta U}{C_v} = -5U_0$$

As process  $B \rightarrow C$  is isochoric, so

$$Q_{B \rightarrow C} = \Delta U = 3U_0$$

As process  $C \rightarrow A$  is isothermal, so

$$Q_{C \rightarrow A} = nRT \log_e \left( \frac{V_A}{V_C} \right)$$

$$\Rightarrow Q_{C \rightarrow A} = R \left( \frac{10U_0}{3R} \right) \log_e \left( \frac{\rho_C}{\rho_A} \right)$$

Since,  $\rho_A = 2\rho_0$ ,  $\rho_C = \rho_B = 5\rho_0$

$$\Rightarrow Q_{C \rightarrow A} = \frac{10U_0}{3} \log_e \left( \frac{5}{2} \right)$$

$$\Rightarrow Q = -2U_0 + \frac{10U_0}{3} \log_e (2.5)$$

$$\Rightarrow Q = \left( \frac{10}{3} \log_e (2.5) - 2 \right) U_0$$

(c) From First Law of Thermodynamics, we get

$$W_{A \rightarrow B} = \Delta Q_{A \rightarrow B} - \Delta U_{A \rightarrow B}$$

$$\Rightarrow W_{A \rightarrow B} = -5U_0 - (-3U_0)$$

$$\Rightarrow W_{A \rightarrow B} = -2U_0$$