ICON

CHAPTER 1

Mechanical Properties of Matter

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Stress, Strain, Hooke's Law, Normal Stress
(b) Longitudinal Strain and Young's Modulus
(c) Elastic Potential Energy and Energy Density
(d) Tangential Stress, Shear Strain and Shear Modulus
(e) Volumetric Stress, Volumetric Strain and Bulk's Modulus
(f) Fluid Properties, Pressure and Pascal's Law
(g) Pressure in Accelerating Fluids
(h) Archimedes' Principle and Buoyancy
(i) Viscosity, Stoke's Law and Terminal Speed
(j) Ideal Fluids, Equation of Continuity, Bernoulli's Theorem and Applications
(k) Surface Tension, Surface Energy, Excess Pressure and Capillarity.

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

ELASTICITY

INTRODUCTION

So far, we have dealt with all the solids that have been modelled as rigid bodies, that is, objects do not change their shape. Real objects, however, deform to some extent when an external force is applied to them. In this chapter we formulate some systematic ways to describe qualitatively the deformation of solids that are subjected to applied forces.

THE STATES OF MATTER

Matter is usually classified into one of three **states** or **phases**: solid, liquid, or gas.
Because they can flow easily, both liquids and gases are called **fluids**.

- A solid has a fixed shape which it tends to retain, whereas fluids have no fixed shape.
- A liquid sinks to the bottom of its container, and a gas expands to fill the available volume.
- The atoms in a solid vibrate about fixed equilibrium positions, whereas the atoms or molecules in a liquid move about relatively freely and collide frequently with each other.
- The atoms in a solid or liquid are quite closely packed, which makes it difficult to reduce their volume, they are almost incompressible.
- On the average, the atoms or molecules in a gas are far apart, typically about ten atomic diameters at room temperature and pressure. They collide much less frequently than those in a liquid. Gases in general are compressible.

ELASTICITY

When external forces are applied on a body, which is not free to move, there is a change in its dimensions. When these forces are removed, the body tends to regain its original shape and size.

The property by virtue of which a body tends to regain its original shape and size when the external deforming forces are removed, is called ***elasticity.***

The property of the material body by virtue of which it does not regain its original configuration when the external force is removed is called plasticity.

DEFORMING FORCE

An external force applied to a body which changes its size or shape or both is called deforming force.

PERFECTLY ELASTIC BODY

A body is said to be perfectly elastic when it completely regains its original configuration on the removal of deforming forces. Since no material can regain completely its original form so the concept of perfectly elastic body is only an ideal concept. A quartz fibre is the nearest approach to the perfectly elastic body.

PERFECTLY PLASTIC BODY

A body is said to be perfectly plastic if it does not regain its original form even slightly when the deforming force is removed. Since every material partially regains its original form on the removal of deforming forces, so the concept of perfectly plastic body is also an ideal concept. Paraffin wax, wet clay are the nearest approach to a perfectly plastic bodies.

CAUSE OF ELASTICITY

In a solid, atoms and molecules are arranged in such a way that each molecule is acted upon by the forces due to the neighbouring molecules. These forces are known as intermolecular forces. When no deforming force is applied on the body, each molecule of the solid (i.e. body) is in its equilibrium position and the inter molecular forces between the molecules of the solid are maximum.

On applying the deforming force on the body, the molecules either come closer or go far apart from each other. As a result of this, the molecules are displaced from their equilibrium position. In other words, intermolecular forces get changed and restoring forces are developed on the molecules. When the deforming force is removed, these restoring forces bring the molecules of the solid to their respective equilibrium positions and hence the solid (or the body) regains its original form.

STRESS

When external deforming forces are applied on a body, the relative positions of the molecules of the body change. This calls into play the internal restoring forces. *The restoring force per unit area is called* ***stress.*** It can also be denoted by

$$\text{Stress} = \frac{\text{Restoring Force}}{\text{Area}}$$

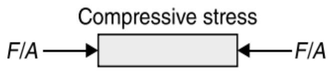
In equilibrium, the restoring force is equal in magnitude to the deforming force. SI unit of stress is Nm^{-2}. Stress is of three types

(a) Longitudinal Stress or Tensile Stress
(b) Tangential Stress or Shearing Stress
(c) Volume Stress or Normal Stress

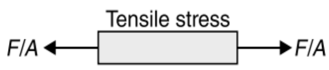
Longitudinal Stress

If the deforming force is applied along some linear dimension of a body, the corresponding stress is called **Longitudinal Stress or Tensile Stress. So, when an object is one dimensional then we use the concept of longitudinal stress.**
It is of two types:

(a) Compressive stress

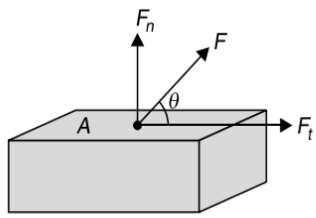


(b) Tensile stress



TANGENTIAL (SHEAR) STRESS AND NORMAL (TENSILE) STRESS

If the force is applied tangentially to one face of a rectangular body, keeping the other face fixed, the stress is called **Tangential or Shearing stress.** So, tangential stress is defined as the restoring force per unit area tangential to the surface of the body.
Consider a block of solid as shown in Figure.



Let a force F be applied to the face which has area A. Resolve $\vec{F}$ into two components $F_n = F\sin\theta$ called normal force and $F_t = F\cos\theta$ called tangential force.
Tangential (shear) stress is given by

$$(\text{Stress})_t = \frac{F_t}{A} = \frac{F\cos\theta}{A}$$

Normal (tensile) stress is given by

$$(\text{Stress})_n = \frac{F_n}{A} = \frac{F\sin\theta}{A}$$

The effect of stress is to produce distortion or a change in size, volume and shape (i.e., configuration of the body).

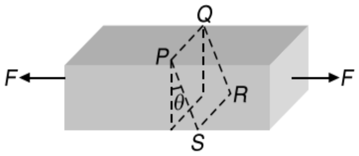
VOLUMETRIC STRESS OR BULK STRESS OR PRESSURE OR NORMAL STRESS

If the force acts all along the surface of the body and normally to its surface, then the stress is called pressure (P) or volumetric Stress because the effect of pressure is to cause a volume change.
The volumetric stress or normal stress or pressure is given by

$$(\text{Stress})_n = \frac{F_n}{A} = P$$

ILLUSTRATION 1

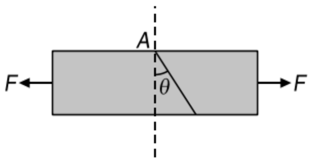
A bar is subjected to equal and opposite forces as shown in the figure. $PQRS$ is a plane making angle θ with the cross-section of the bar? If the area of cross-section of the bar be A, find the tensile stress on $PQRS$, shearing stress on $PQRS$. Also find the condition when tensile stress is maximum and when shearing stress is maximum.



SOLUTION

Since, $\text{Tensile Stress} = \frac{\text{Normal force}}{\text{Area}} = \frac{F_N}{A_N}$

where, $A_N = \frac{A}{\cos\theta}$ and $F_N = F\cos\theta$



$\Rightarrow \quad \text{Tensile Stress} = \frac{F\cos\theta}{\left(\frac{A}{\cos\theta}\right)} = \frac{F\cos^2\theta}{A}$

$\text{Shear Stress} = \frac{\text{Tangential force}}{\text{Area}} = \frac{F_T}{A_T}$

$\Rightarrow \quad \text{Shear Stress} = \frac{F\sin\theta}{A/\cos\theta} = \frac{F\sin\theta\cos\theta}{A}$

$\Rightarrow \quad \text{Shear Stress} = \frac{F\sin\theta\cos\theta}{A} = \frac{F\sin(2\theta)}{2A}$

Since, $\text{Tensile Stress} = \frac{F\cos^2\theta}{A}$

So, for Tensile stress to be maximum,

$\cos^2\theta = \text{Maximum} = 1$

$\Rightarrow \quad \theta = 0°$

Since, $\text{Shear Stress} = \frac{F\sin(2\theta)}{2A}$

So, for Shear stress to be maximum,

$\sin(2\theta) = \text{Maximum} = 1$

$\Rightarrow \quad 2\theta = 90°$

$\Rightarrow \quad \theta = 45°$

STRAIN

When deforming forces are applied on a body, it undergoes a change in shape or size. The fractional (or relative) change in shape or size is called **Strain**. That is

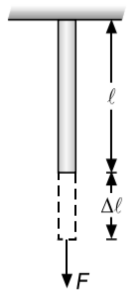
$$\text{Strain} = \frac{\text{Change in Dimension}}{\text{Original Dimension}}$$

Since it is the ratio of two like quantities, so it has no dimensions and units. Strain is of following types

(a) Longitudinal (Linear) Strain
(b) Volume Strain
(c) Shearing Strain

LONGITUDINAL STRAIN

If the deforming force produces a change in length alone, the strain produced in the body is called Longitudinal strain or linear strain or tensile strain. It is the ratio of the change in length (ΔL) to the original length (L).

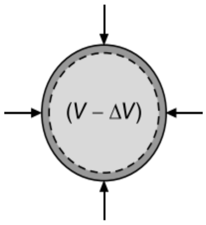


$$\text{Longitudinal Strain} = \frac{\Delta L}{L}$$

Linear strain in the direction of deforming force is called Longitudinal strain and, in a direction, perpendicular to force is called Lateral strain.

VOLUMETRIC STRAIN

When the deforming force produces a change in the volume of the body alone, then the strain produced in the body is called volumetric strain. It is the ratio of the change in volume (ΔV) to the original volume (V).



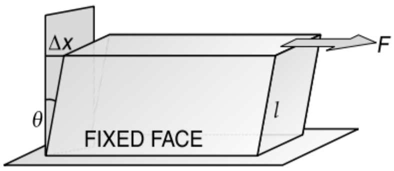
$$\text{Volumetric Strain} = \frac{\Delta V}{V}$$

SHEAR STRAIN

When the deforming force produces a change in the shape of the body without changing its volume, then the strain produced is called shearing strain. It is defined as the angle in radian, through which the face originally perpendicular to the fixed surface of the cubical body gets turned under the effect of tangential force.

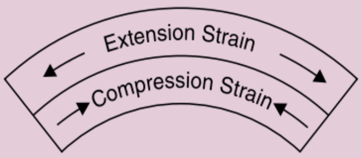
Hence, the angular deformation (θ) in radian is called **shearing strain.** Since θ is generally small, we may write

$$\text{Shearing Strain} = \theta \approx \tan\theta = \frac{\Delta x}{l}$$



Conceptual Note(s)

When a beam is bent both compression strain as well as an extension strain is produced.



RELATION OF STRESS TO STRAIN AND ELASTIC MODULUS

A force applied to an object can change its dimensions and shape. In general, the response of a material to a given type of deforming force is characterized by an **elastic modulus**, which is defined as ratio of stress to strain. So, we have

$$\text{Elastic modulus} = \frac{\text{Stress}}{\text{Strain}}$$

HOOKE'S LAW

This law states that for small deformations (or within elastic limit), stress is directly proportional to strain.

$$\text{Stress} \propto \text{Strain}$$

$$\Rightarrow \quad \text{Stress} = E(\text{Strain})$$

$$\Rightarrow \quad \frac{\text{Stress}}{\text{Strain}} = E$$

where E is a constant called as the modulus of elasticity.

Thus, modulus of elasticity is defined as the ratio of stress to strain. Modulus of elasticity depends on the nature of the material of the body i.e. E is the property of the material of the body and is independent of its shape and dimensions. i.e. length, volume etc. The SI unit of modulus of elasticity is Nm^{-2} or Pascal (Pa), where $1\ \text{Nm}^{-2} = 1\ \text{Pa}$.

Conceptual Note(s)

Modulus of elasticity E (whether it is Y, B or η) is given by

$$E = \frac{\text{stress}}{\text{strain}}$$

Following conclusions can be made from the above expression:

(a) $E \propto$ stress (for same strain), i.e., if we want the equal amount of strain in two different materials, the one which needs more stress is having more E.

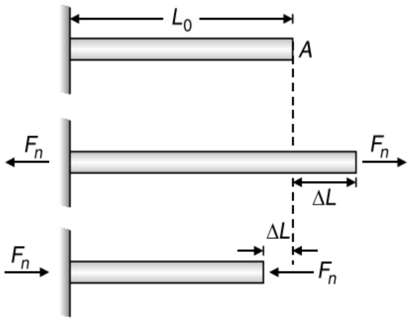
(b) $E \propto \dfrac{1}{\text{strain}}$ (for same stress), i.e., if the same amount of stress is applied on two different materials, the one having the less strain is having more E. Rather we can say that, the one which offers more resistance to the external forces is having greater value of E. So, we can see that modulus of elasticity of steel is more than that of rubber or

$$E_{\text{steel}} > E_{\text{rubber}}$$

(c) E equals stress for unit strain i.e. when $\dfrac{\Delta x}{x} = 1$ or $\Delta x = x$. If the length of a wire is 2 metre, then the Young's modulus of elasticity (Y) is the stress applied on the wire to stretch the wire by the same amount of 2 metre.

YOUNG'S MODULUS

Young's modulus is a measure of the resistance of a solid to a change in its length when a force is applied perpendicular to a face. Consider a rod with an unstressed length L_0 and cross-sectional area A, as shown in the Figure.



When it is subjected to equal and opposite forces F_n along it axis and perpendicular to the end faces its length changes by ΔL. These forces tend to stretch the rod. The tensile stress, sometimes denoted by σ, on the rod is defined as

$$\sigma = \frac{F_n}{A}$$

Forces acting in the opposite direction (as shown) would produce a compressive stress. The resulting strain (ε), a dimensionless ratio is given by

$$\varepsilon = \frac{\Delta L}{L_0}$$

Young's modulus Y for the material of the rod is defined as the ratio of tensile stress to tensile strain. So

$$\text{Young's Modulus} = \frac{\text{Tensile stress}}{\text{Tensile strain}}$$

$$\Rightarrow \quad Y = \frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{\varepsilon} = \frac{\frac{F_n}{A}}{\frac{\Delta L}{L_0}} = \frac{F_n L_0}{A \Delta L}$$

ILLUSTRATION 2

Determine the elongation of the steel bar 1 m long and $1.5\ \text{cm}^2$ cross sectional area when subjected to a pull of 1.5×10^4 N. Take $Y = 2 \times 10^{11}\ \text{Nm}^{-2}$.

SOLUTION

$$Y = \frac{F/A}{\Delta l / l}$$

$$\Rightarrow \quad \Delta l = \frac{Fl}{AY}$$

Substituting the values,

$$\Delta l = \frac{(1.5 \times 10^4)(1.0)}{(1.5 \times 10^{-4})(2 \times 10^{11})} = 0.5 \times 10^{-3}\ \text{m}$$

$$\Rightarrow \quad \Delta l = 0.5\ \text{mm}$$

ILLUSTRATION 3

A metal wire 75 cm long and 0.139 cm in diameter stretches 0.035 cm when a load of 8 kg is hung on its end. Find the stress, the strain, and the Young's modulus of the material of the wire.

SOLUTION

$$\text{Stress} = \frac{F}{A} = \frac{(8\ \text{kg})(9.8\ \text{ms}^{-2})}{\pi (6.5 \times 10^{-4}\ \text{m})^2} = 5.91 \times 10^7\ \text{Nm}^{-2}$$

$$\text{Strain} = \frac{\Delta L}{L} = \frac{0.035\ \text{cm}}{75\ \text{cm}} = 4.67 \times 10^{-4}$$

$$\Rightarrow \quad Y = \frac{\text{Stress}}{\text{Strain}} = \frac{5.91 \times 10^7\ \text{Nm}^{-2}}{4.67 \times 10^{-4}} = 1.27 \times 10^{11}\ \text{Nm}^{-2}$$

ILLUSTRATION 4

A circular steel wire 3 m long is to stretch not more than 0.20 cm when a tensile force of 400 N is applied to each end of the wire. What minimum diameter is required for the wire? $Y = 2 \times 10^{11}\ \text{Pa}$.

SOLUTION

$$A = \frac{Fl}{Y \Delta l}$$

$$\Rightarrow \quad \frac{\pi}{4} d^2 = \frac{Fl}{Y \Delta l}$$

$$\Rightarrow \quad d = \sqrt{\frac{4Fl}{\pi Y \Delta l}} = \sqrt{\frac{4 \times 400 \times 3}{\pi \times 2 \times 10^{11} \times 0.2 \times 10^{-2}}}$$

$$\Rightarrow \quad d = 61.8 \times 10^{-3}\ \text{m} = 61.8\ \text{mm}$$

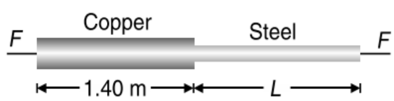
ILLUSTRATION 5

A copper rod with a length of 1.40 m and a cross-section area of $2\ \text{cm}^2$ is fastened to a steel rod with length L and cross-section area $1\ \text{cm}^2$. The compound rod is subjected to equal and opposite pulls of magnitude 6×10^4 N at its ends. Calculate the length L of the steel rod if the elongations of the two rods are equal. Also find the stress and strain in each rod. $Y_{\text{steel}} = 2 \times 10^{11}$ Pa, $Y_{Cu} = 1.1 \times 10^{11}$ Pa.

SOLUTION

Given that $\Delta l_{Cu} = \Delta l_{\text{Steel}}$

$$\Rightarrow \quad \frac{F l_C}{A_C Y_C} = \frac{F l_S}{A_S Y_S}$$



$$\Rightarrow \quad l_S = \left(\frac{A_S}{A_C}\right)\left(\frac{Y_S}{Y_C}\right) l_C$$

$$\Rightarrow \quad L = \left(\frac{1}{2}\right)\left(\frac{2 \times 10^{11}}{1.1 \times 10^{11}}\right)(1.40)$$

$$\Rightarrow \quad L = 1.27\ \text{m}$$

$$\sigma_C = \frac{F}{A_C} = \frac{6 \times 10^4}{2 \times 10^{-4}} = 3 \times 10^8\ \text{Nm}^{-2}$$

and $\sigma_S = \dfrac{F}{A_S} = \dfrac{6\times10^4}{1\times10^{-4}} = 6\times10^8\ \text{Nm}^{-2}$

$$\varepsilon_C = \frac{\sigma_C}{Y_C} = \frac{3\times10^8}{1.1\times10^{11}} = 2.7\times10^{-3}$$

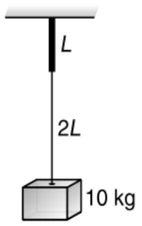
and $\varepsilon_S = \dfrac{\sigma_S}{Y_S} = \dfrac{6\times10^8}{2\times10^{11}} = 3\times10^{-3}$

ILLUSTRATION 6

A length L of copper wire of diameter 1.2 mm is joined to a length $2L$ of steel wire 0.8 mm in diameter, and is hung vertically. When a 10 kg load is suspended from the lower end, the total elongation is 0.65 mm. Find L. $Y_S = 2\times10^{11}\ \text{Nm}^{-2}$, $Y_{Cu} = 1.1\times10^{11}\ \text{Nm}^{-2}$.

SOLUTION

$$\Delta l = \frac{Fl}{AY} \text{ and } \Delta l = \Delta l_c + \Delta l_s$$



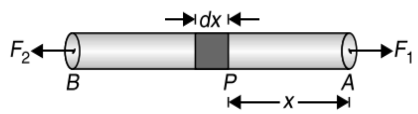
$$\Rightarrow (0.65\times10^{-3}) = \frac{(10\times9.8)L}{\frac{\pi}{4}(1.2\times10^{-3})^2\times1.1\times10^{11}} + \frac{(10\times9.8)(2L)}{\frac{\pi}{4}(0.8\times10^{-3})^2\times2\times10^{11}}$$

Solving this equation, we get

$$L = 23.75\ \text{cm}$$

ILLUSTRATION 7

Consider a metal bar of mass M is pulled by forces F_1 and F_2 applied at its two ends. Consider a small element of length dx at a distance x from end A as shown in Figure.



Calculate the elongation (dL) in small element dx and the total elongation in the bar.

SOLUTION

Acceleration of the bar $= \dfrac{F_1 - F_2}{M}$

Let the tension in bar at distance x from end A is T, then

$$F_1 - T = \frac{Mx}{L}\left(\frac{F_1 - F_2}{M}\right)$$

$$\Rightarrow T = F_1 - \left(\frac{F_1 - F_2}{L}\right)x$$

Stress at this point

$$\frac{T}{A} = \frac{F_1}{A} - \frac{(F_1 - F_2)x}{AL}$$

And Strain $= \dfrac{\text{Stress}}{Y}$

So, $\dfrac{dL}{dx} = \dfrac{F_1}{AY} - \dfrac{(F_1 - F_2)x}{ALY}$

$$\Rightarrow dL = \left(\frac{F_1}{AY} - \frac{(F_1 - F_2)}{ALY}x\right)dx$$

Total elongation in the bar is

$$\Delta L = \int_0^L dL$$

$$\Rightarrow \Delta L = \int_0^L \left(\frac{F_1}{AY} - \frac{(F_1 - F_2)x}{ALY}\right)dx$$

$$\Rightarrow \Delta L = \frac{F_1}{AY}\int_0^L dx - \left(\frac{F_1 - F_2}{ALY}\right)\int_0^L x\,dx$$

$$\Rightarrow \Delta L = \frac{F_1 L}{AY} - \frac{(F_1 - F_2)L}{2AY}$$

$$\Rightarrow \Delta L = \frac{L}{AY}\left(F_1 - \frac{F_1 - F_2}{2}\right)$$

$$\Rightarrow \Delta L = \frac{(F_1 + F_2)L}{2AY}$$

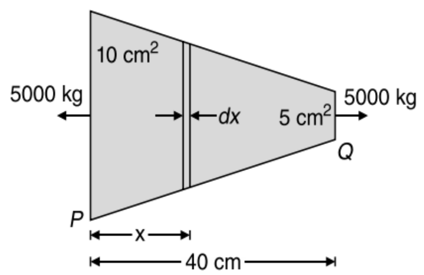
$$\Rightarrow \Delta L = \left(\frac{|F_1| + |F_2|}{2AY}\right)L$$

ILLUSTRATION 8

A steel flat plate PQ tapers uniformly from area $10\ \text{cm}^2$ to $5\ \text{cm}^2$ in length of $40\ \text{cm}$. Calculate the elongation in the plate if an axial tensile force of $5000\ \text{kg}$ is acting on it. Take $Y = 2\times10^6\ \text{kgcm}^{-2}$.

SOLUTION

Consider a small element of length dx of the bar at a distance from P as shown in Figure.



Area of cross section at this section is

$$A(x) = 10 - \left(\frac{10-5}{40}\right)x = \left(10 - \frac{x}{8}\right)\text{ cm}^2$$

Elongation in this elementary length is

$$dl = \frac{F}{A(x)Y}dx = \frac{5000}{\left(10-\frac{x}{8}\right)(2\times 10^6)}dx$$

$$\Rightarrow \quad dl = \frac{dx}{400\left(10-\frac{x}{8}\right)}$$

The total extension in the bar is

$$\Delta l = \frac{1}{400}\int_0^{40}\frac{dx}{\left(10-\frac{x}{8}\right)} = \frac{1}{400}(-8)\ln\left(10-\frac{x}{8}\right)\Big|_0^{40}$$

$$\Rightarrow \quad \Delta l = -\frac{8}{400}(\ln 5 - \ln 10) = \frac{1}{50}\ln\left(\frac{10}{5}\right)$$

$$\Rightarrow \quad \Delta l = 0.14\text{ mm}$$

ILLUSTRATION 9

A light rod of length 2 m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross section 10^{-3} m^2 and the other is of brass of cross section $2\times 10^{-3}\text{ m}^2$. Locate the position along the rod at which a weight may be hung to produce,

(a) equal stresses in both wires
(b) equal strains on both wires.

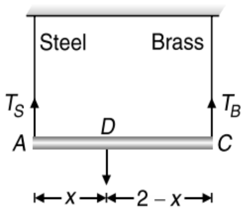
Young's modulus for steel is $2\times 10^{11}\text{ Nm}^{-2}$ and for brass is 10^{11} Nm^{-2}.

SOLUTION

(a) Given, Stress in steel = Stress in brass

$$\Rightarrow \quad \frac{T_S}{A_S} = \frac{T_B}{A_B}$$

$$\Rightarrow \quad \frac{T_S}{T_B} = \frac{A_S}{A_B} = \frac{10^{-3}}{2\times 10^{-3}} = \frac{1}{2} \qquad \ldots(1)$$



As the system is in equilibrium, taking moments about D, we have

$$T_S x = T_B(2-x)$$

$$\Rightarrow \quad \frac{T_S}{T_B} = \frac{2-x}{x} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$x = 1.33\text{ m}$$

(b) $\text{Strain} = \dfrac{\text{Stress}}{Y}$

Given, Strain in steel = Strain in brass

$$\Rightarrow \quad \frac{\frac{T_S}{A_S}}{Y_S} = \frac{\frac{T_B}{A_B}}{Y_B}$$

$$\Rightarrow \quad \frac{T_S}{T_B} = \frac{A_S Y_S}{A_B Y_B} = \frac{(1\times 10^{-3})(2\times 10^{11})}{(2\times 10^{-3})(10^{11})} = 1 \qquad \ldots(3)$$

From equations (2) and (3), we have

$$x = 1\text{ m}$$

ILLUSTRATION 10

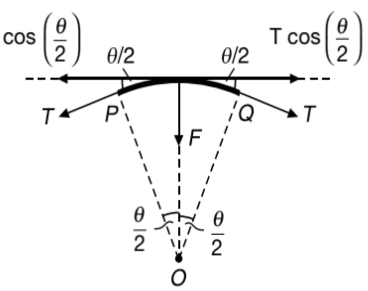
A thin ring of radius R is made of a material of density ρ and Young's modulus Y. If the ring is rotated about its centre in its own plane with angular velocity ω, find the small increase in its radius.

SOLUTION

Consider an element PQ of length dl. Let T be the tension and A the area of cross section of the wire.
Mass of element $dm = \text{volume}\times\text{density} = A(dl)P$
The component of T, towards the centre provides the necessary centripetal force is

$$2T\sin\left(\frac{\theta}{2}\right) = (dm)R\omega^2 \qquad \ldots(1)$$

For small angles $\sin\left(\dfrac{\theta}{2}\right) \approx \dfrac{\theta}{2} = \dfrac{(dl/R)}{2}$



Substituting in equation (1), we get

$$T\frac{dl}{R} = A(dl)\rho R\omega^2$$

$$\Rightarrow \quad T = A\rho\omega^2 R^2$$

Let ΔR be the increase in radius,

$$\text{Longitudinal strain} = \frac{\Delta l}{l} = \frac{\Delta(2\pi R)}{2\pi R} = \frac{\Delta R}{R}$$

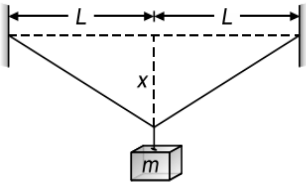
$$\text{Now, } Y = \frac{T/A}{\Delta R/R}$$

$$\Rightarrow \quad \Delta R = \frac{TR}{AY} = \frac{(A\rho\omega^2 R^2)R}{AY}$$

$$\Rightarrow \quad \Delta R = \frac{\rho\omega^2 R^3}{Y}$$

ILLUSTRATION 11

A steel wire of diameter d, area of cross-section A and length $2L$ is clamped firmly at two points A and B which are $2L$ metre apart and in the same plane.



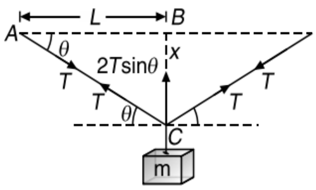
A body of mass m is hung from the middle point of wire such that the middle point sags by x lower from original position. If Young's modulus is Y then find m.

SOLUTION

Let the tension in the string be T, then for the equilibrium of mass m, we have

$$2T\sin\theta = mg$$

$$\Rightarrow \quad T = \frac{mg}{2\sin\theta}$$



Since θ is small, so we have $\sin\theta \approx \frac{x}{L}$ and hence

$$T = \frac{mg}{2\sin\theta} \approx \frac{mgL}{2x}$$

Increment in length of the wire is $\Delta l = AC - AB$

$$\Delta l = \sqrt{L^2 + x^2} - L = \left(L^2 + x^2\right)^{1/2} - L$$

$$\Rightarrow \quad \Delta l = L\left[\left(1 + \frac{x^2}{L^2}\right)^{\frac{1}{2}} - 1\right]$$

Since $(1+x)^n \cong 1 + nx$ for $x \ll 1$, so we have

$$\Delta l = L\left(1 + \frac{1}{2}\frac{x^2}{L^2} - 1\right) = \frac{x^2}{2L}$$

Now, by definition, the Young's modulus is given by

$$Y = \frac{T}{A}\frac{L}{\Delta l}$$

$$\Rightarrow \quad T = \frac{YA\Delta l}{L}$$

Substituting the value of T and Δl in the above equation we get

$$\frac{mgL}{2x} = \left(\frac{YA}{L}\right)\left(\frac{x^2}{2L}\right)$$

$$\Rightarrow \quad m = \frac{YAx^3}{gL^3}$$

ILLUSTRATION 12

The length of an elastic string is a metres when the longitudinal tension is 4 N and b metres when the longitudinal tension is 5 N. Find the length of the string in metres when the longitudinal tension is 9 N.

SOLUTION

Let the original length of elastic string be L and its force constant be k.
When longitudinal tension 4 N is applied on it

$$L + \frac{4}{k} = a \quad \ldots(1)$$

and when longitudinal tension 5 N is applied on it

$$L + \frac{5}{k} = b \quad \ldots(2)$$

Solving (1) and (2) we get

$$k = \frac{1}{b-a} \text{ and } L = 5a - 4b$$

When the longitudinal tension of 9 N is applied on elastic string then let its length be c, which is given by

$$c = L + \frac{9}{k} = 5a - 4b + 9(b-a) = 5b - 4a$$

Conceptual Note(s)

To find the actual length of the wire

If a wire has a length L_1 under the tension T_1 and a length L_2 under a tension T_2, then the original length of the wire is $L = \frac{L_1T_2 - L_2T_1}{T_2 - T_1}$

Since, $Y = \frac{F/A}{\Delta L/L} = \frac{T_1/A}{\frac{(L_1 - L_0)}{L_0}} = \frac{T_1L_0}{A(L_1 - L_0)}$

$$\Rightarrow \quad \frac{L_1 - L_0}{T_1} = \frac{L_0}{AY} \quad \ldots(1)$$

Similarly, $\frac{L_2 - L_0}{T_2} = \frac{L_0}{AY}$ ...(2)

Equating (1) and (2), we get

$$L = \frac{L_1T_2 - L_2T_1}{T_2 - T_1}$$

THERMAL STRESS

If the ends of a rod are rigidly fixed so as to prevent expansion or contraction and the temperature of the rod is changed then, tensile or compressive stresses, called thermal stresses, will be set up in the rod. Hence in the design of many structures which is subject to change in temperature, some provision must be made for expansion to avoid failure of such structures.

Suppose that a rod at a temperature T has its ends rigidly fastened and that while they are thus held, the temperature is reduced to a lower value T_0. The fractional change in length if the rod were free to contract would be

$$\frac{\Delta L}{L} = \alpha(T - T_o) = \alpha\Delta T$$

Since the rod is not free to contract, the tension must increase to produce some fractional change in length. Let F be the tension produced, then

$$F = AY\frac{\Delta L}{L}$$

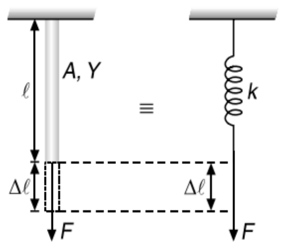
Substituting for $\frac{\Delta L}{L}$, we get,

$$F = AY\alpha\Delta T$$

Hence, the stress in rod is $\frac{F}{A} = Y\alpha\Delta T$

FORCE CONSTANT OF A WIRE

A thin wire or a rod can be imagined to be a series combination of arrays of springs. When we pull the wire by applying a force, then we are actually pulling the springs. Let us take an elastic rod or wire of length l, area of cross-section A, having modulus of elasticity Y and apply a force F to it as shown in Figure.



The force required to produce unit elongation in a wire is called force constant of material of wire, denoted by k. Mathematically

$$k = \frac{F}{\Delta\ell} \quad \ldots(1)$$

By definition, the Young's modulus of the wire is given by

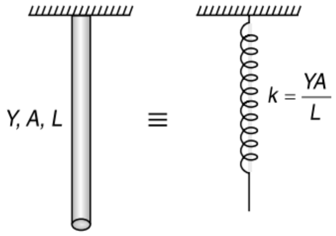
$$Y = \frac{FL}{A\Delta\ell}$$

$$\Rightarrow \frac{F}{\Delta\ell} = \frac{YA}{L} \quad \ldots(2)$$

From (1) and (2), we get

$$k = \frac{YA}{L}$$

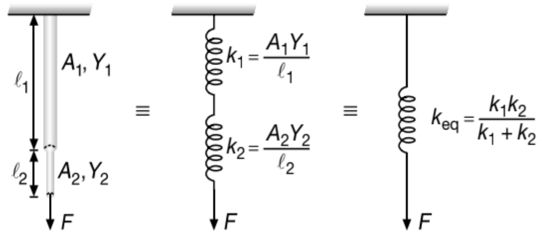
It is clear that the value of force constant depends upon the dimension (length and area of cross section) and material of a substance. So, a wire of length L, Young's Modulus Y and having area of cross section A behaves like a spring of force constant $k = \frac{YA}{L}$. So, all formulae which we use in case of a spring can be applied to a wire also.



For a system of rods joined in series as shown in Figure (a), the replaced spring system is shown in Figure (b) having two springs in series. Figure (c) represents the equivalent spring system having spring constant

$$k_{eq} = \frac{k_1k_2}{k_1 + k_2}$$

where, $k_1 = \frac{Y_1A_1}{l_1}$, $k_2 = \frac{Y_2A_2}{l_2}$ and $k_3 = \frac{Y_3A_3}{l_3}$

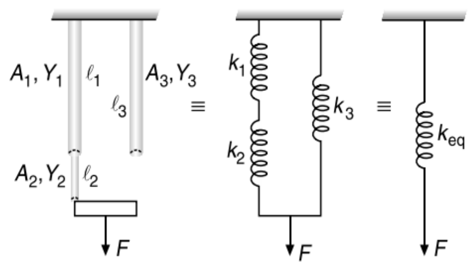


(a) (b) (c)

Another combination of rods and its replaced spring system is shown in Figure such that the equivalent spring constant is given by

$$k_{eq} = \frac{k_1k_2}{k_1 + k_2} + k_3$$

where, $k_1 = \frac{Y_1A_1}{l_1}$, $k_2 = \frac{Y_2A_2}{l_2}$ and $k_3 = \frac{Y_3A_3}{l_3}$



Conceptual Note(s)

Since, we have $k_{eq} = \dfrac{YA}{l}$

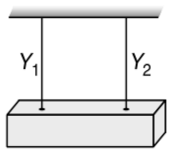
$\Rightarrow \quad k_{eq} l = YA$

This simply means that the product of stiffness constant of an elastic wire and its length is always constant.

So, if we cut a wire in two equal parts, then the stiffness constant of each part will be double that of the original wire. On the same basis, we can say that if a spring is cut into N equal parts, each part will have stiffness N times the stiffness of the original spring.

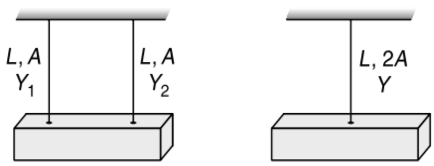
ILLUSTRATION 13

Two wires of equal length and cross-section are suspended as shown. Their Young's moduli are Y_1 and Y_2 respectively. Calculate the equivalent Young's modulus of the arrangement.



SOLUTION

Let the equivalent young's modulus of given combination is Y and the area of cross section is $2A$.



For parallel combination, we have

$$k_1 + k_2 = k_{eq.}$$

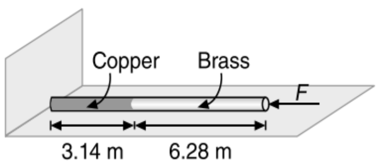
$$\Rightarrow \quad \frac{Y_1 A}{L} + \frac{Y_2 A}{L} = \frac{Y(2A)}{L}$$

$$\Rightarrow \quad Y_1 + Y_2 = 2Y$$

$$\Rightarrow \quad Y = \frac{Y_1 + Y_2}{2}$$

ILLUSTRATION 14

A copper cylinder and a brass cylinder are stacked end to end, as shown in Figure.



Each cylinder has a radius of $0.25\ \text{cm}$. A compressive force of $F = 9000\ \text{N}$ is applied to the right end of the brass cylinder. Find the amount by which the length of the stack decreases if $Y_{\text{copper}} = 1.0 \times 10^{11}\ \text{Nm}^{-2}$ and $Y_{\text{brass}} = 9.0 \times 10^{10}\ \text{Nm}^{-2}$.

SOLUTION

METHOD-I

Since both the cylinders experience the same amount of force F. Since we know that

$$Y = \frac{F/A}{\Delta L/L}$$

$$\Rightarrow \quad \Delta L = \frac{FL}{AY}$$

Due to the compressive force, we observe that each cylinder decreases in length and hence the total decrease in length is equal to the sum of the decrease in length for each cylinder.

The length of the copper cylinder decreases by

$$\Delta L_{\text{Copper}} = \frac{FL_C}{Y_{\text{Copper}} A} = \frac{FL_C}{Y_{\text{Copper}}(\pi r^2)}$$

Similarly, the length of the brass decreases by

$$\Delta L_{\text{Brass}} = \frac{FL_B}{Y_{\text{Brass}} A} = \frac{FL_B}{Y_{\text{Brass}}(\pi r^2)}$$

Total decrease in length is

$$\Delta L = \Delta L_{\text{Copper}} + \Delta L_{\text{Brass}}$$

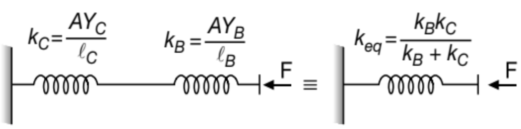
$$\Rightarrow \quad \Delta L = \frac{FL_C}{Y_{\text{Copper}} A} + \frac{FL_B}{Y_{\text{Brass}} A} = \frac{F}{A}\left(\frac{L_C}{Y_C} + \frac{L_B}{Y_B}\right)$$

$$\Rightarrow \quad \Delta L = \frac{9000}{\pi(0.25 \times 10^{-2})^2}\left(\frac{3.14}{1.0 \times 10^{11}} + \frac{6.28}{9.0 \times 10^{10}}\right)$$

$$\Rightarrow \quad \Delta L = 4.64 \times 10^{-2}\ \text{m}$$

METHOD-II

The composite bar can be replaced with a series combination of two springs having spring constants k_C and k_B as shown in Figure.



The equivalent spring constant k_{eq} of the arrangement is

$$\frac{1}{k_{eq}} = \frac{1}{k_C} + \frac{1}{k_B}$$

$$\Rightarrow \quad k_{eq} = \frac{k_B k_C}{k_B + k_C}$$

Deformation in length of the equivalent spring is

$$\Delta L = \frac{F}{k_{eq}} = F\left(\frac{k_B + k_C}{k_B k_C}\right) = F\left(\frac{1}{k_B} + \frac{1}{k_C}\right)$$

where, $k_C = \frac{Y_C A}{L_C}$ and $k_B = \frac{Y_B A}{L_B}$

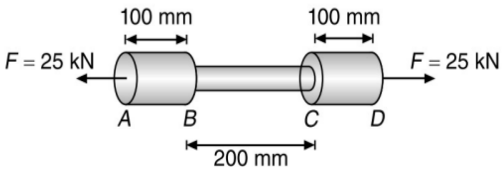
$$\Rightarrow \quad \Delta L = \frac{F}{A}\left(\frac{L_C}{Y_C} + \frac{L_B}{Y_B}\right)$$

$$\Rightarrow \quad \Delta L = \frac{9000}{\pi\left(0.25\times10^{-2}\right)^2}\left(\frac{3.14}{1.0\times10^{11}} + \frac{6.28}{9.0\times10^{10}}\right)$$

$$\Rightarrow \quad \Delta L = 4.64\times10^{-2}\ \text{m}$$

ILLUSTRATION 15

A steel bar $ABCD$ of length 40 cm is made up of three parts AB of diameter $50\ \text{mm}$, BC of diameter $25\ \text{mm}$ and CD of diameter $50\ \text{mm}$ as shown in Figure.

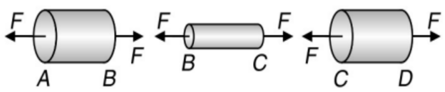


The rod is subjected to a pull of 25 kN. If Young's modulus for steel is $2\times10^{11}\ \text{Nm}^{-2}$, then calculate stress in each part of the bar and the total extension in the bar.

SOLUTION

METHOD-I: Conventional Method

If F is the stretching force applied on the given steel bar, then the same axial force 25 kN is transmitted longitudinally to each of the three bars. However, stresses induced will be different and hence elongations of each bar will also be different. Figure shows the free-body diagram of the different part of the bar.



Stress in part AB and CD will be equal

$$\sigma_{AB} = \sigma_{CD} = \frac{F}{A_{AB}} = \frac{25000\ \text{N}}{\frac{\pi}{4}(50)^2\ \text{mm}^2}$$

$$\Rightarrow \quad \sigma_{AB} = \frac{40}{\pi} = 12.73\ \text{Nmm}^{-2}$$

Stress in part BC is

$$\sigma_{BC} = \frac{F}{A_{BC}} = \frac{25000}{\frac{\pi}{4}(25)^2} = 50.93\ \text{Nmm}^{-2}$$

Total extension in the rod is

$$\Delta l = \Delta l_{AB} + \Delta l_{BC} + \Delta l_{CD} = 2\Delta l_{AB} + \Delta l_{BC}$$

$$\Rightarrow \quad \Delta l = 2\left(\frac{\sigma_{AB} l_{AB}}{Y}\right) + \left(\frac{\sigma_{BC} l_{BC}}{Y}\right)$$

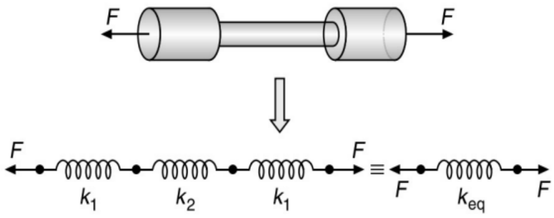
Given that $Y = 2\times10^{11}\ \text{Nm}^{-2} = 2\times10^{5}\ \text{Nmm}^{-2}$

$$\Rightarrow \quad \Delta l = 2\left(\frac{12.73}{2\times10^5}\times100\right) + \left(\frac{50.93}{2\times10^5}\times200\right)$$

$$\Rightarrow \quad \Delta l = 0.0637\ \text{mm}$$

METHOD-II: Equivalent Spring Method

The steel bar $ABCD$ can be replaced with series combination of three springs.



In series combination of springs, the force in each spring should be equal. Hence, each rod will experience same force F. Hence, stress in rods AB and CD is

$$\sigma_{AB} = \sigma_{CD} = \frac{F}{A_1} = \frac{25\times10^3}{\left(\frac{\pi}{4}\times50^2\right)} = \frac{40}{\pi}\ \text{Nmm}^{-2}$$

Stress in rod BC is $\sigma_{BC} = \frac{25\times10^3}{\left(\frac{\pi}{4}\times25^2\right)} = \frac{160}{\pi}\ \text{Nmm}^{-2}$

Equivalent spring contact can be written as

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_1} = \frac{2}{k_1} + \frac{1}{k_2}$$

where, $k_1 = \frac{YA_1}{l_1}$ and $k_2 = \frac{YA_2}{l_2}$

$$\Rightarrow \quad k_{eq} = \frac{YA_1A_2}{2A_2l_1 + A_1l_2}$$

So, net extension produced in the composite bar is

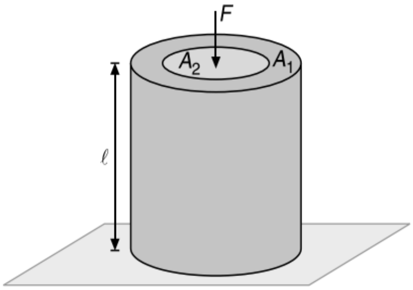
$$x = \frac{F}{k_{eq}}$$

$$\Rightarrow \quad x = F\left(\frac{2A_2l_1 + A_1l_2}{YA_1A_2}\right)$$

$$\Rightarrow \quad x = \frac{1}{5\pi}\ \text{mm} = 0.0637\ \text{mm}$$

ILLUSTRATION 16

Consider a cylindrical column (or strut) made up of two different materials as shown in Figure.



The inner section of the column has area A_1, Young's modulus Y_1 and the outer section of the column has area A_2, Young's modulus Y_2. A compressive force F acts on this column. Calculate the load supported by inner and outer section of the column.

SOLUTION

METHOD-I: Conventional Method

Let σ_1 and σ_2 be the stresses induced in the two materials, then sum of restoring forces in the two materials should balance the externally applied compressive force F. Hence, we have

$$\sigma_1 A_1 + \sigma_2 A_2 = F \quad \ldots(1)$$

Also, the compression in the two portions should be the same. But since their original lengths are same, so strain (ε) will also be the same.

$$\Rightarrow \quad \varepsilon_1 = \varepsilon_2$$

$$\Rightarrow \quad \frac{\sigma_1}{Y_1} = \frac{\sigma_2}{Y_2} \quad \ldots(2)$$

Solving Equations (1) and (2), we get

$$\sigma_1 = \frac{FY_1}{A_1Y_1 + A_2Y_2}$$

Hence load supported by inner section is

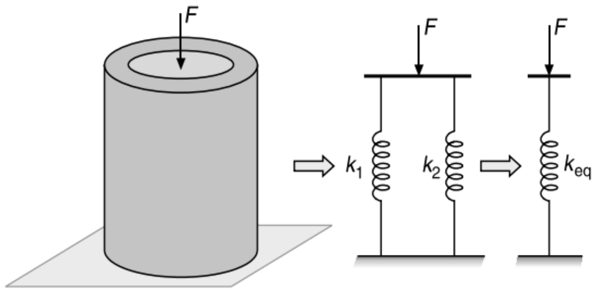
$$F_1 = \sigma_1 A_1 = \frac{FY_1A_1}{A_1Y_1 + A_2Y_2}$$

and load supported by outer section is

$$F_2 = \sigma_2 A_2 = \frac{FY_2A_2}{A_1Y_1 + A_2Y_2}$$

METHOD-II: Equivalent Spring Method

The cylindrical column (or strut) made up of two different materials can be replaced by a combination of two springs in parallel as shown in Figure.



Since the springs are connected in parallel, so the compressions in each spring will be same. Hence

$$x_1 = x_2$$

$$\Rightarrow \quad \frac{F_1}{k_1} = \frac{F_2}{k_2}$$

where, $k_1 = \dfrac{A_1Y_1}{l_1}$ and $k_2 = \dfrac{A_2Y_2}{l_2}$

The equivalent force constant of the arrangement is

$$k_{eq} = k_1 + k_2 = \frac{1}{l}(A_1Y_1 + A_2Y_2)$$

Compression in each spring is

$$x_1 = x_2 = x = \frac{F}{k_{eq}} = \frac{Fl}{A_1Y_1 + A_2Y_2}$$

So, load supported by the inner section is

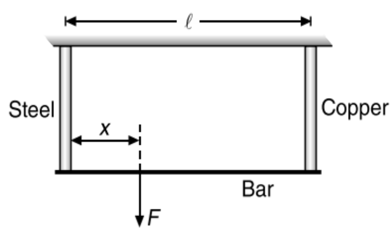
$$F_1 = k_1x_1 = \frac{FY_1A_1}{(A_1Y_1 + A_2Y_2)}$$

and load supported by the outer section is

$$F_2 = k_2x_2 = \frac{FY_2A_2}{A_1Y_1 + A_2Y_2}$$

ILLUSTRATION 17

Two vertical rods of equal lengths, one of steel and the other of copper, are suspended from the ceiling, at a distance l apart and are connected rigidly to a rigid horizontal bar at their lower ends as shown in Figure.



The respective area of cross section and Young's Modulus of steel and copper wires are A_S, Y_S and A_C, Y_C. Calculate the tension in steel and copper wires. Where should a vertical force F be applied to the horizontal bar in order to keep the bar horizontal?

SOLUTION

METHOD-I: Conventional Method

Let the force F be applied at a distance x from the steel bar. Let T_S and T_C be the loads on steel and copper rods, respectively. Then,

$$T_S + T_C = F \quad \ldots(1)$$

Since the rigid horizontal bar remains horizontal, the extensions produced in the two rods and strains remain the same. That is

$$\frac{T_S}{A_S Y_S} = \frac{T_C}{A_C Y_C} \quad \ldots(2)$$

Solving equations (1) and (2), we get

$$T_S = \frac{F A_S Y_S}{A_S Y_S + A_C Y_C} \text{ and } T_C = \frac{F A_C Y_C}{A_S Y_S + A_C Y_C}$$

Since the steel bar is horizontal and in equilibrium, so, we have

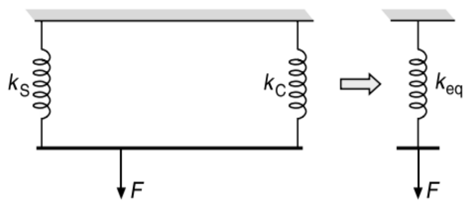
$$\Sigma\tau = 0$$

$$\Rightarrow \quad T_C l = Fx$$

$$\Rightarrow \quad x = \frac{T_C}{F} l = \frac{A_C Y_C l}{A_S Y_S + A_C Y_C}$$

METHOD-II: Equivalent Spring Method

We can treat two rods as two springs, connected in parallel combination



Equivalent spring constant of steel and copper rod

$$k_S = \frac{A_S Y_S}{l} \text{ and } k_C = \frac{A_C Y_C}{l}$$

As the bar always remains horizontal, it means the elongation of the rods should be equal.

$$\Rightarrow \quad \frac{F_S}{k_S} = \frac{F_C}{k_C} = \frac{F}{(k_S + k_C)}$$

$$\Rightarrow \quad F_S = \frac{F k_s}{k_S + k_C} = \frac{F A_S Y_S}{A_S Y_S + A_C Y_C}$$

$$\text{and } F_C = \frac{F k_C}{k_S + k_C} = \frac{F A_C Y_C}{A_S Y_S + A_C Y_C}$$

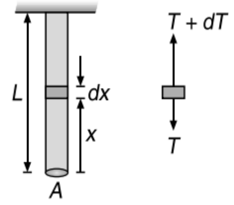
Taking moments about the steel bar, we get

$$T_C l = Fx$$

$$\Rightarrow \quad x = \frac{T_C}{F} l = \frac{A_C Y_C l}{A_S Y_S + A_C Y_C}$$

ELONGATION OF ROD UNDER IT'S SELF WEIGHT

Consider a rope having weight W, area of cross-section A, length L and Young's Modulus Y. Let us consider an element at a distance x from the free end of the rope as shown in Figure.



Consider an element of length dx at distance x from the free end of the rope. The tension in the rope at a distance x from its free end is

$$T = \frac{W}{L} x$$

Elongation dl in this infinitesimal element of length dx of the rope is

$$dl = \frac{T dx}{AY} = \left(\frac{W}{AY}\right) x dx \quad \ldots(1)$$

Total elongation ΔL in the rope is obtained by integrating equation (1). So, we get

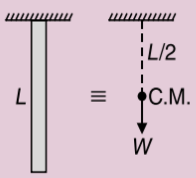
$$\Delta L = \int dl = \int_0^L \frac{T dx}{AY} = \frac{W}{AY} \int_0^L x dx = \frac{WL}{2AY}$$

If m is mass of the rope, then $W = mg$, so we have

$$\Delta L = \frac{WL}{2AY} = \frac{(mg)L}{2AY}$$

Conceptual Note(s)

One can do this directly by considering that the total weight of wire is acting at CM and using effective original length of the wire as $\frac{L}{2}$.



Since $Y = \frac{F/A}{x/L_{eff}} = \frac{W/A}{\frac{x}{(L/2)}}$

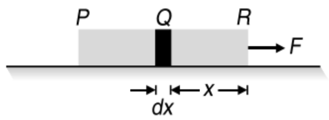
$$\Rightarrow \quad x = \frac{WL}{2AY}$$

ILLUSTRATION 18

A uniform elastic plank moves over a smooth horizontal plane due to a constant force F distributed uniformly over the end face. The surface of the end face is equal to A, and Young's modulus of the material is Y. Find the tensile strain of the plank in the direction of the acting force.

SOLUTION

Let m be the mass of the plank and l its length.



Tension at distance x is

$$T = m_{QR}a$$

$$\Rightarrow \quad T = \left(\frac{m}{l}\right)(l-x)\frac{F}{m}$$

$$\Rightarrow \quad T = F\left(1-\frac{x}{l}\right)$$

Small change in element dx is,

$$dl = \frac{Tdx}{AY} = \frac{F}{AY}\left(1-\frac{x}{l}\right)dx$$

$$\Rightarrow \quad \text{Strain} = \frac{\Delta l}{l} = \frac{F}{2AY}$$

Problem Solving Technique(s)

In this problem, if friction between block and surface is present and coefficient of friction is μ, then following two cases arise.

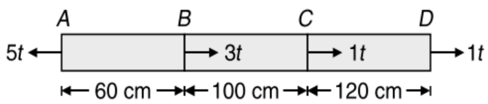
Case (i): $F < \mu mg$

Case (ii): $F > \mu mg$,

For both these cases the answer will be same for elongation i.e., $\Delta \ell = \frac{FL}{2AY}$ for both cases.

ILLUSTRATION 19

A brass bar, having cross sectional area 10 cm^2 is subjected to axial forces as shown in figure. Find the total elongation of the bar. Take $Y = 8\times10^2 \text{ tcm}^{-2}$.



SOLUTION

Given, $A = 10 \text{ cm}^2$, $Y = 8\times10^2 \text{ tcm}^{-2}$

Let Δl be the total elongation of the bar. For sake of simplicity, the force of $3t$ acting at B may be split into two forces of $5t$ and $2t$ as shown in Figure.

Similarly, the force of $1t$ acting at C may be split into two forces of $2t$ and $1t$.

Since the extension in the bar on which both forces of equal magnitude act in opposite direction is

$$\Delta l = \frac{FL}{AY}$$

So, net extension is given by

$$\Delta l = \frac{1}{AY}(F_1l_1 + F_2l_2 + F_3l_3)$$

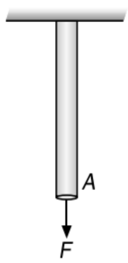
$$\Rightarrow \quad \Delta l = \frac{1}{10\times8\times10^2}[(5)(60)+(2)(100)+(1)(120)]$$

$$\Rightarrow \quad \Delta l = 0.0775 \text{ cm}$$

BREAKING STRESS

If one end of rod or wire is rigidly fixed and a force is applied at the other end, it will stretch. If the force is small, the extension will also be small and upon the withdrawal of the force, the wire will regain its original state. But if the applied force is gradually increased, after some intermediate states, a state is reached when the wire or rod breaks. The stress corresponding to this breaking point is termed as breaking stress.

When the wire is loaded beyond the elastic limit, then strain increases much more rapidly. The maximum stress after which the wire begins to flow and breaks, is called breaking stress or tensile strength and the force by application of which the wire breaks is called the Breaking Force.



The breaking force depends upon the area of cross-section of the wire i.e.,

$$\text{Breaking Force} \propto A$$

$$\text{Breaking Force} = PA$$

where P is a constant of proportionality known as breaking stress (B.S.) of the wire.

Breaking stress P is a constant for a given material and it does not depend upon the dimension (length or thickness) of wire.

BREAKING OF A WIRE UNDER ITS OWN WEIGHT

Consider a wire of length l, area A, density ρ. Let the breaking stress for the material of the wire be P. Since we know that

$$\begin{pmatrix}\text{Breaking}\\ \text{Force}\end{pmatrix}=\begin{pmatrix}\text{Breaking}\\ \text{Stress}\end{pmatrix}\begin{pmatrix}\text{Area of}\\ \text{cross-section}\end{pmatrix}$$

The weight of wire is given by

$$W = Mg = Al\rho g$$

So, breaking stress is given by

$$P = \frac{W}{A} = l\rho g$$

$$\Rightarrow \quad l = \frac{P}{\rho g}$$

This length of the wire can also be said as the maximum length of the wire which can withstand its own weight before breaking.

Conceptual Note(s)

(a) If a wire can bear maximum force F, then wire of same material but double thickness can bear a maximum force $4F$ because breaking force is proportional to area of cross section and hence r^2.

(b) If a wire of length L is cut into two or more parts, then again, it's each part can hold the same weight, because breaking force is independent of the length of wire.

(c) The working stress is always kept lower than that of a breaking stress.

(d) Safety factor which is the ratio of breaking stress to working stress has a large value as

$$\text{Breaking Stress} \gg \text{Working Stress}.$$

ILLUSTRATION 20

Find the greatest length of steel wire that can hang vertically without breaking. Breaking stress of steel is $8\times10^8\ \text{Nm}^{-2}$, density of steel is $8\times10^3\ \text{kgm}^{-3}$ and $g = 10\ \text{ms}^{-2}$.

SOLUTION

Let l be the length of the wire that can hang vertically without breaking. Then the stretching force on it is equal to its own weight. If therefore, A is the area of cross section and ρ the density, then breaking stress P is

$$P = \frac{\text{Weight}}{A}$$

$$\Rightarrow \quad P = \frac{(Al\rho)g}{A}$$

$$\Rightarrow \quad l = \frac{P}{\rho g}$$

Substituting the values, we get

$$l = \frac{8\times10^8}{(8\times10^3)(10)} = 10^4\ \text{m}$$

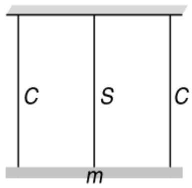
Test Your Concepts-I

Based on Young's Modulus, Longitudinal Stress and Strain

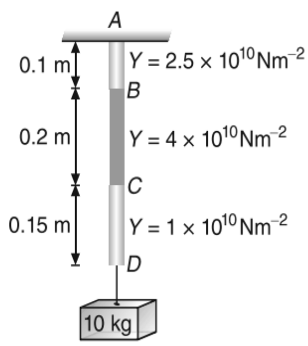
(Solutions on page H.3)

1. One end of a uniform wire of length L and of weight W is attached rigidly to a point in the roof and a weight W_1 is suspended from its lower end. If S is the area of cross-section of the wire, then find the stress in the wire at a height $\frac{3L}{4}$ from its lower end.

2. A homogeneous block with a mass m hangs on three vertical wires of equal length arranged symmetrically. Find the tension of the wires if the middle wire is of steel and the other two are of copper. All the wires have the same cross section. Consider the modulus of elasticity of steel to be double than that of copper.

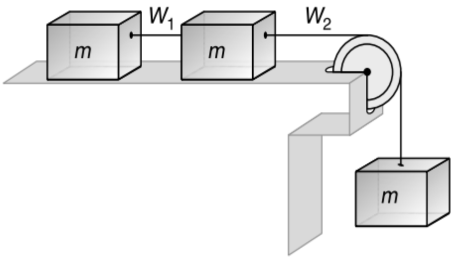


3. A light rod with uniform cross-section of $10^{-7}\ \text{m}^2$ is shown in the figure. The rod consists of three different materials whose lengths are $0.1\ \text{m}$, $0.2\ \text{m}$ and $0.15\ \text{m}$ respectively and whose Young's moduli are $2.5\times10^{10}\ \text{Nm}^{-2}$, $4\times10^{10}\ \text{Nm}^{-2}$ and $1\times10^{10}\ \text{Nm}^{-2}$ respectively. Calculate the displacement of points B, C and D.

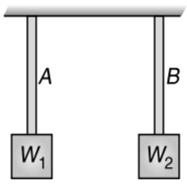


4. Two wires A and B of same dimensions are stretched by same amount of force. Young's modulus of A is twice that of B. Which wire will get more elongation?

5. Three blocks, each of same mass m, are connected with wires W_1 and W_2 of same cross-sectional area a and Young's modulus Y. Neglecting friction, find the strain developed in wire W_2.



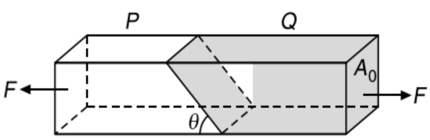
6. Two rods A and B, each of equal length but different materials are suspended from a common support as shown in the figure.



The rods A and B can support a maximum load of $W_1 = 600\text{ N}$ and $W_2 = 6000\text{ N}$, respectively. If their cross-sectional areas are $A_1 = 10\text{ mm}^2$ and $A_2 = 1000\text{ mm}^2$, respectively, then identify the stronger material.

7. A wire elongates by 1 mm when a load W is hanged from it. If this wire goes over a pulley and two weights W each are hung at the two ends, find the elongation in the wire.

8. Two bars P and Q are glued together using a strong adhesive. The contact surface of the bars makes an angle θ with its length and the area of cross-section of each bar is A_0. It is known that the adhesive yields if the normal stress at the contact surface exceeds σ_0. Calculate the maximum pulling force F that can be applied to the arrangement without detaching the bars.



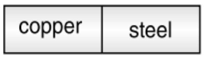
9. The Young's modulus of three materials are in the ratio $2:2:1$. Three wires made of these materials have their cross-sectional areas in the ratio $1:2:3$. For a given stretching force calculate ratio of the elongation's in the three wires.

10. A copper wire has a breaking stress of about $3\times10^8\text{ Nm}^{-2}$. Calculate the maximum load that can be hung from a copper wire of diameter 0.42 mm. If half this maximum load is hung from the copper wire of same diameter, then calculate the percentage of its length that will be stretched.
Given that $Y_{Cu} = 1.1\times10^{11}\text{ Nm}^{-2}$.

11. A solid cylindrical steel column is 4 cm long and 9 cm in diameter. What will be its decrease in length when carrying a load of 80000 kg? $Y = 1.9\times10^{11}\text{ Nm}^{-2}$.

12. A metal rod that is 4.00 m long and 0.50 cm^2 in cross-section area is found to stretch 0.20 cm under a tension of 5000 N. What is the Young's modulus for this metal?

13. A steel wire and a copper wire of equal length and equal cross-sectional area are joined end to end and the combination is subjected to a tension as shown in Figure.



Calculate the ratio of the stresses and the ratio of the strains developed in the two wires.
Given that $Y_{steel} = 2\times10^{11}\text{ Nm}^{-2}$ and $Y_{copper} = 1.3\times10^{11}\text{ Nm}^{-2}$.

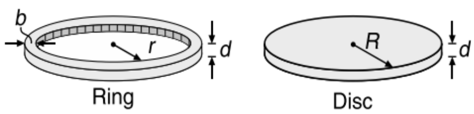
14. A uniform copper bar of density ρ, length L, cross-sectional area S and Young's modulus Y is moving horizontally on a frictionless surface with constant acceleration a_0. Calculate stress at centre of the wire and the total elongation in the wire.

15. A rod PQ of mass m, area of cross section A, length l and Young's modulus of elasticity Y is lying on a smooth table as shown in figure. A force F is applied at P. Find

(a) tension at a distance x from end P,
(b) longitudinal stress at this point,
(c) total change in length and
(d) total strain in the rod.

16. A vertical solid steel post 15 cm in diameter and 3 m long is required to support a load of 8000 kg. The weight of the post can be neglected. What is (a) the stress in the post? (b) the strain in the post? (c) the change in post's length when the load is applied? $Y = 2\times10^{11}\text{ Pa}$.

17. A steel ring is to be fitted on a wooden disc of radius R and thickness d. The inner radius of the ring is r which is slightly smaller than R. The outer radius of the ring is $r+b$ and its thickness is d (same as the disc) as shown in Figure.



There is no change in value of b and d after the ring is fitted over the disc, only the inner radius becomes R. If the Young's modulus of steel is Y, calculate the longitudinal stress developed in it. Also calculate the tension force developed in the ring.

ELASTIC POTENTIAL ENERGY

If a force F is applied to a wire of length l and cross-sectional area A, made of material of Young's Modulus Y, and if the wire suffers an extension x then

$$F=\frac{YAx}{l} \qquad \left\{\because Y=\frac{Fl}{Ax}\right\}$$

Work done in extending the wire through Δl is given by

$$W=\int_0^{\Delta l} F\,dx=\frac{YA}{l}\int_0^{\Delta l} x dx$$

$$\Rightarrow \quad W=\frac{YA}{l}\frac{(\Delta l)^2}{2}=\frac{1}{2}(Al)\left(\frac{Y\Delta l}{l}\right)\left(\frac{\Delta l}{l}\right)$$

$$\Rightarrow \quad W=\frac{1}{2}\times \text{Volume}\times\text{Stress}\times\text{Strain}$$

This work done is stored in the wire as **elastic potential energy** (U_e) in the wire. Work done per unit volume is called the elastic energy density denoted by (u_e). Hence, we have

$$u_e=\frac{W}{Al}=\frac{U_e}{Al}=\frac{1}{2}(\text{Stress})(\text{Strain})$$

The Work can also be written as

$$W=\frac{1}{2}\left(\frac{YA\Delta l}{l}\right)\Delta l$$

$$\Rightarrow \quad \text{Work }=\frac{1}{2}\times\text{Load}\times\text{Extension}$$

This work done is stored in the wire as **elastic potential energy** (U_e) in the wire. **Energy density** (u_e) is the elastic potential energy per unit volume of the wire.

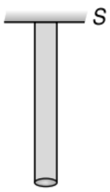
$$\Rightarrow \quad u_e=\frac{W}{Al}=\frac{1}{2}\left(\frac{\text{Load}}{A}\right)\left(\frac{\text{Extension}}{l}\right)$$

$$\Rightarrow \quad u_e=\frac{1}{2}(\text{Stress})(\text{Strain})$$

$$\Rightarrow \quad u_e=\frac{1}{2}Y(\text{Strain})^2=\frac{(\text{Stress})^2}{2Y}$$

ILLUSTRATION 21

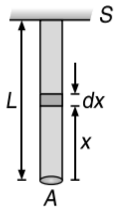
A bar of mass M and length L is hanging from point S as shown in figure.



The Young's modulus of elasticity of the wire is Y and the area of cross-section of the wire is A.

(i) Find the stress at x distance from bottom end.
(ii) Consider a small section dx of the bar at a distance x from lowest point of bar. Find elongation (dL) in that section dx.
(iii) Find total elongation in bar.
(iv) Find energy density at distance x from bottom end.
(v) Find total elastic potential energy stored in bar.

SOLUTION



(i) The weight of x length of the bar is

$$W=\left(\frac{Mg}{L}\right)x$$

So, stress at x distance from bottom

$$\frac{W}{A}=\frac{Mgx}{AL}$$

(ii) Stress $=Y(\text{Strain})$

$$\frac{Mgx}{AL}=Y\frac{dL}{dx}$$

$$\Rightarrow \quad dL=\frac{Mgxdx}{ALY}$$

(iii) Total elongation in wire

$$\Delta L=\int_0^L dL=\frac{Mg}{ALY}\int_0^L x\,dx$$

$$\Rightarrow \quad \Delta L=\frac{MgL}{2AY}$$

(iv) Elastic energy density at x distance is

$$u_e=\frac{1}{2}(\text{stress})(\text{strain})$$

$$\Rightarrow \quad u_e=\frac{1}{2}\frac{(\text{stress})^2}{Y}$$

$$\Rightarrow \quad u_e=\frac{1}{2Y}\left(\frac{Mgx}{AL}\right)^2=\frac{M^2g^2x^2}{2YA^2L^2}$$

(v) Energy stored in Adx volume is

$$dU_e=(u_e)Adx$$

$$\Rightarrow \quad dU_e=\frac{M^2g^2x^2}{2YA^2L^2}(Adx)$$

Total energy is given by

$$U_e=\int_0^L\frac{M^2g^2x^2}{2YAL^2}dx=\frac{M^2g^2L}{6AY}$$

ILLUSTRATION 22

A rubber cord has a cross sectional area 1 mm^2 and total unstretched length 10 cm. It is stretched to 12 cm and then released to project a missile of mass 5 g. Taking Young's modulus Y for rubber as $5\times10^8\text{ Nm}^{-2}$. Calculate the velocity of projection.

SOLUTION

The equivalent force constant of rubber cord is given by

$$k=\frac{YA}{l}=\frac{(5\times10^8)(1\times10^{-6})}{(0.1)}=5\times10^3\text{ Nm}^{-1}$$

Applying Law of Conservation of Mechanical Energy, we get

$$\begin{pmatrix}\text{Elastic Potential}\\ \text{Energy of Cord}\end{pmatrix}=\begin{pmatrix}\text{Kinetic Energy}\\ \text{of Missile}\end{pmatrix}$$

$\Rightarrow\quad \frac{1}{2}k(\Delta l)^2=\frac{1}{2}mv^2$

$\Rightarrow\quad v=\left(\sqrt{\frac{k}{m}}\right)\Delta l=\left(\sqrt{\frac{5\times10^3}{5\times10^{-3}}}\right)(12-10)\times10^{-2}$

$\Rightarrow\quad v=20\text{ ms}^{-1}$

Please note that while solving this example following assumptions have been made.

(i) k has been assumed constant, even though it depends on the length (l).

(ii) The entire elastic potential energy is converted into kinetic energy of missile.

ILLUSTRATION 23

The demolition of a building is to be accomplished by swinging a 400 kg steel ball on the end of a 30 m steel wire of diameter 5 mm hanging from a tall crane. The ball is swung through an arc from side to side, the wire making an angle of 60° with the vertical at the top of the swing. Calculate the amount by which the wire is stretched at the bottom of the swing and estimate the energy stored in the stretched wire at the bottom of the swing if $Y_S=2\times10^{11}\text{ Nm}^{-2}$.

SOLUTION

The height to which the ball is raised when it makes an angle of 60° with the vertical is given by

$$h=l(1-\cos60^\circ)=\frac{l}{2}=15\text{ m}$$

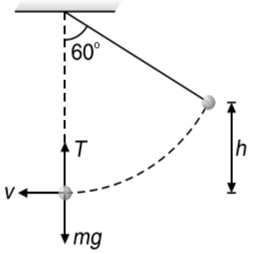
When released from this position, the speed acquired by the ball at the lowest point is obtained by applying the Law of Conservation of Energy, according to which *"loss in gravitational potential energy of the ball equals the gain in kinetic energy of the ball"*, so we get

$$\frac{1}{2}mv^2=mgh$$

$\Rightarrow\quad v^2=2gh=2\times9.8\times15=294\text{ m}^2\text{s}^{-2}$

Also, applying Newton's Second Law to the ball at the lowest point, we get

$$T-mg=\frac{mv^2}{l}$$



$\Rightarrow\quad T=m\left(g+\frac{v^2}{l}\right)=(400)\left(9.8+\frac{294}{30}\right)$

$\Rightarrow\quad T=7840\text{ N}$

Since, $\Delta l=\frac{Tl}{AY}=\frac{(7840)(30)}{\frac{\pi}{4}(5\times10^{-3})^2\times2\times10^{11}}$

$\Rightarrow\quad \Delta l=0.06\text{ m}=6\text{ cm}$

The elastic energy stored in the wire is

$$U=\frac{1}{2}(Y)(\text{Strain})^2(\text{Volume})\text{, where}$$

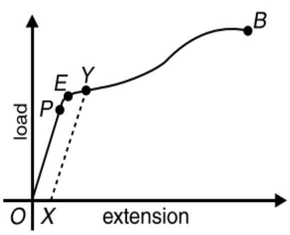
$$\text{Volume}=\left(\frac{\pi}{4}\right)(5\times10^{-3})^2(30)=5.9\times10^{-4}\text{ m}^3$$

$\Rightarrow\quad U=\frac{1}{2}(2\times10^{11})\left(\frac{6\times10^{-2}}{30}\right)^2(5.9\times10^{-4})$

$\Rightarrow\quad U=236\text{ J}$

BEHAVIOUR OF A WIRE UNDER STRESS

A typical load-extension (or stress-strain) curve for a metal wire is shown here.



P	limit of proportionality
E	elastic limit, which may coincide with the limit of proportionality
Y	yield point
B	break point

Up to the point P the curve is straight. This is called **limit of proportionality.** Beyond this point P Hooke's Law is not obeyed.

The point E is called the **elastic limit**. Upto this point the wire returns to its original form when the load is removed. P and E are very close and may not coincide.

Beyond this elastic limit the wire no longer returns to its original size on the removal of load, i.e., a permanent deformation (OX) occurs. At the **yield point** Y, the material begins to flow, i.e., the length increases with little increase in stress. This continues till the **break point** B is achieved and the wire breaks at some weak point.

Materials which undergo a large increase in length beyond the elastic limit before snapping are called **ductile**. Such materials can be drawn into long wires. Substances which break just after the elastic limit is reached are said to be **brittle.**

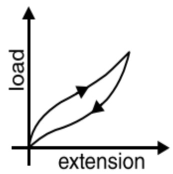
ELASTIC FATIGUE

Consider a wire loaded, within elastic limits, and kept strained for a sufficiently long time. The extension is liable to complete recovery on removal of deforming forces but it is observed that the wire recovers its original configuration slowly on removal of deforming forces, i.e. the recovery lags behind the process of removing one deforming forces.

This phenomenon, by virtue of which a substance exhibits a delay in recovering its original configuration, if it had been subjected to a stress for a longer time, is called **elastic fatigue.**

BEHAVIOUR OF RUBBER UNDER STRESS

For rubber, stress is not proportional to strain at any portion of the curve. However, when the load is removed the specimen recovers its original length – thus it is elastic (but does not obey Hooke's Law). On decreasing the load, the curve is not retraced (as expected) but follows a different path. This property is called **elastic hysteresis.**



ELASTIC HYSTERESIS

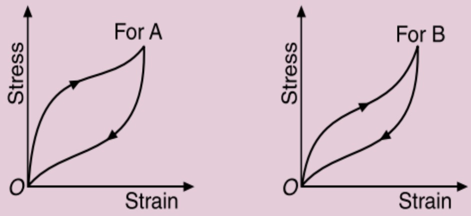
The word Hysteresis actually means "Lagging Behind". When a deforming force is applied on a body then the strain does not change simultaneously with stress rather it lags behind the stress. The lagging of strain behind the stress is called as Elastic Hysteresis. This is the reason due to which the values of strain for same stress are different while increasing the load and while decreasing the load.

Conceptual Note(s)

Significance of Hysteresis Loop

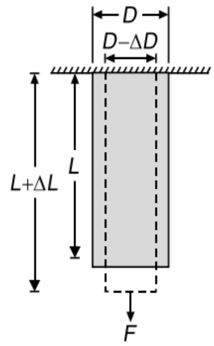
(a) The area of the stress-strain curve is called the hysteresis loop and it is numerically equal to the work done in loading the material and then unloading it.

(b) If we have two tyres of rubber having different hysteresis loop then rubber B should be used for making the car tyres. It is because of the reason that area under the curve i.e. work done in case of rubber B is lesser and hence the car tyre will not get excessively heated and rubber A should be used to absorb vibration of the machinery because of the large area of the curve, a large amount of vibrational energy can be dissipated.



POISSON'S RATIO (σ)

When a long bar is stretched by a force along its length then its length increases and the radius decreases as shown in the Figure.



In that case, we have two types of strain in the wire

(a) **Lateral strain:** The ratio of change in diameter (ΔD) to the original diameter (D) is called lateral strain i.e. $\frac{\Delta D}{D}$.

(b) **Longitudinal strain:** The ratio of change in length (ΔL) to the original length (L) is called longitudinal strain i.e. $\frac{\Delta L}{L}$.

Poisson's Ratio (σ) is defined as the ratio of lateral strain to longitudinal strain.

$$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$\Rightarrow \quad \sigma = -\frac{\Delta D/D}{\Delta L/L}$$

Negative sign indicates that the radius of the bar decreases when it is stretched. Poisson's ratio is a dimensionless and a unit less quantity.

Conceptual Note(s)

(a) The theoretical value of σ lies between -1 and 0.5
(b) The actual value i.e. practical value of σ lies between 0 and 0.5
(c) Poisson's ratio σ is not the modulus of elasticity as it is the ratio of two strains and not of stress to strain.

RELATION BETWEEN VOLUMETRIC STRAIN, LONGITUDINAL STRAIN AND POISSON'S RATIO

Consider a long bar having length L and radius R, then volume of the bar is given by

$$V=\pi R^2 L=\frac{\pi}{4}D^2 L$$

$$\Rightarrow \quad \Delta V=\frac{\pi}{4}\Delta\left(D^2 L\right)$$

$$\Rightarrow \quad \Delta V=\frac{\pi}{4}\left[D^2\Delta L+L\Delta\left(D^2\right)\right]$$

Since, we know that $\Delta\left(D^2\right)=2D\Delta D$

$$\Rightarrow \quad \Delta V=\frac{\pi}{4}\left[D^2\Delta L+L(2D\Delta D)\right]$$

Dividing both the sides of the above equation by volume of the bar i.e. $V=\frac{\pi}{4}D^2 L$, we get

$$\frac{\Delta V}{V}=\frac{D^2\Delta L}{D^2 L}+\frac{2DL\Delta D}{D^2 L}$$

$$\Rightarrow \quad \frac{\Delta V}{V}=\frac{\Delta L}{L}+2\frac{\Delta D}{D}$$

Since $\frac{\Delta D}{D}=-\sigma\left(\frac{\Delta L}{L}\right)$ $\quad \left\{\because\ \sigma=-\frac{\Delta D/D}{\Delta L/L}\right\}$

$$\Rightarrow \quad \frac{\Delta V}{V}=(1-2\sigma)\frac{\Delta L}{L}$$

Conceptual Note(s)

(a) $\frac{\Delta V}{V}=(1-2\sigma)\frac{\Delta L}{L}$

(b) For $\sigma=0.5$, $\Delta V=0$
i.e. if value of Poisson's ratio is 0.5, then change in volume for this value is zero.

DEPRESSION OF A BEAM

If a beam of length l is supported horizontally at the ends and is loaded at the middle by a load W, the depression at the centre is given by

$$\delta=\frac{Wl^3}{48YI}$$

where, Y is the Young's Modulus.

For a beam of circular cross-section $I=\frac{1}{4}\pi r^2$, and

For a beam of rectangular cross-section $I=\frac{bd^3}{12}$

where, b being the breadth and d the depth of the beam.

Test Your Concepts-II

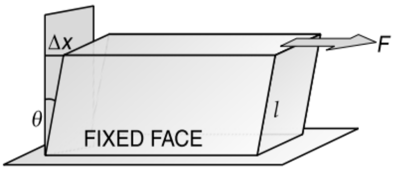
Based on Elastic Energy, Energy Density and Poisson's Ratio

(Solutions on page H.5)

1. Find the elastic deformation energy of a steel rod of mass $m=3.1$ kg stretched to a tensile strain $\varepsilon=1\times10^{-3}$. Y for steel is 2×10^{11} Nm^{-2} and density of steel is 7800 kgm^{-3}.
2. A smooth uniform string of natural length l, cross-sectional area A and Young's modulus Y is pulled along its length by a force F on a horizontal surface. Find the elastic potential energy stored in the string.
3. A wire 4 m long and 0.3 mm in diameter is stretched by a force of 100 N. If extension in the wire is 0.3 mm, calculate the potential energy stored in the wire.
4. A steel wire of length 2 m and 1.2×10^{-7} m^2 in cross sectional area is stretched by a force of 36 N. Calculate stress, strain, increase in length and work done in stretching the wire $\left(Y=1.8\times10^{11}\ \text{Nm}^{-2}\right)$
5. Find the work done in stretching a wire of cross section 1 mm^2 and length 2 m through 0.1 mm. Young's modulus for the material of wire is 2×10^{11} Nm^{-2}.
6. A 45 kg boy whose leg bones are 5 cm^2 in area and 50 cm long falls through a height of 2 m without breaking his leg bones. If the bones can stand a stress of 0.9×10^8 Nm^{-2}, calculate the Young's modulus for the material of the bone. Use $g=10$ ms^{-2}.
7. A catapult consists of two parallel rubber strings, each of lengths 10 cm and cross-sectional area 10 mm^{-2}. When stretched by 5 cm, it can throw a stone of mass 100 g to a vertical height of 25 m. Determine Young's modulus of elasticity of rubber.
8. A steel wire 4.0 m in length is stretched through 2.0 mm. The cross-sectional area of the wire is 2.0 mm^2. If Young's modulus of steel is 2.0×10^{11} Nm^{-2}. Calculate the energy density of wire and the elastic potential energy stored in the wire.

SHEAR MODULUS

The shear modulus of a solid measures its resistance to a shearing force, which is a force applied tangentially to a surface, as shown in the figure.



Since the bottom of the solid is assumed to be at rest, there is an equal and opposite force on the lower surface. The top surface is displaced by Δx relative to the bottom surface. The shear stress is defined as

$$\text{Shear Stress} = \frac{\text{Tangential force}}{\text{Area}} = \frac{F_t}{A} = \frac{F}{A}$$

where A is the area of the surface. The **shear strain** is denoted by θ and is given by

$$\theta = \frac{\Delta x}{l}$$

where l is the separation between the top and the bottom surfaces.

$$\eta = \frac{\text{Shearing Stress}}{\text{Shearing Strain}}$$

Shear modulus is also called **modulus of rigidity.**

Consider a solid rectangular block whose lower face is held fixed and a tangential force F is applied to the upper face (see figure). This deforms the block into a parallelopiped, turning the vertical end faces through an angle θ. If A is the area of the upper face then

$$\eta = \frac{F}{A\theta} = \frac{F}{A\tan\theta} \qquad \{\text{Using Shearing Strain}\}$$

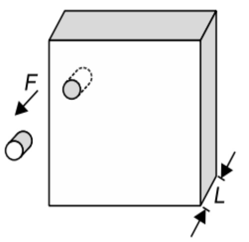
$$\eta = \frac{Fl}{A\Delta x}$$

Conceptual Note(s)

Solids possess all the three moduli of elasticity BUT Liquids and Gases possess only Bulk Modulus.

ILLUSTRATION 24

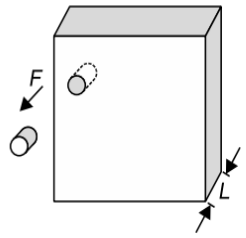
Calculate the force needed to punch a 1.46 cm diameter hole in a steel plate of 1.27 cm thick shown in figure. The ultimate shear stress of steel is $345 \times 10^6\ \text{Nm}^{-2}$.



SOLUTION

According to figure. As in punching, shear elasticity is involved, the hole will be punched if

$$\frac{F}{A} > (\text{Ultimate Shear Stress})$$



$\Rightarrow \quad F > (\text{Shear Stress})(\text{Area})$

Since the area $A = 2\pi rL$

$\Rightarrow \quad A = 2(3.14)(0.73 \times 10^{-2})(1.27 \times 10^{-2})$

$\Rightarrow \quad A = 5.8 \times 10^{-4}\ \text{m}^2$

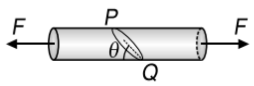
$\Rightarrow \quad F > (3.45 \times 10^8)(5.8 \times 10^{-4}) \approx 200\ \text{kN}$

$\Rightarrow \quad F > 200\ \text{kN}$

Minimum force is required is $F_{\min} = 200\ \text{kN}$

ILLUSTRATION 25

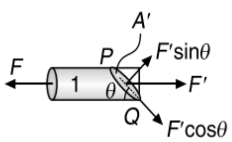
Two equal and opposite forces each of magnitude F act on a rod of uniform cross-sectional area A, as shown in the Figure.



Calculate the shear stress and longitudinal stress on the section PQ.

SOLUTION

Since the net force acting on the rod is zero, so the rod is in equilibrium. Let the tension in the segment PQ be F. Applying Newton's Second Law for the segment 1, we get



$$F_{\text{net}} = F' - F = ma = 0 \qquad \{\because a = 0\}$$

$\Rightarrow \quad F' = F$

Resolving the force F' parallel and perpendicular to the given surface PQ having area A', we get the tangential component of the force F_t to be $F_t = F'\cos\theta$ and the normal component of the force to be $F_n = F'\sin\theta$, so tangential stress i.e. shear stress is given by

$$\sigma_t = \frac{F_{\tan}}{A'} = \frac{F'\cos\theta}{A'}$$

Since, $A' = \dfrac{A}{\sin\theta}$ and $F' = F$

$$\Rightarrow \quad \sigma_t = \frac{F \sin\theta \cos\theta}{A}$$

Similarly, longitudinal stress or normal stress is

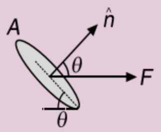
$$\sigma_n = \frac{F_n}{A'} = \frac{F' \sin\theta}{A'}$$

Since, $A' = \frac{A}{\sin\theta}$ and $F' = F$

$$\Rightarrow \quad \sigma_n = \frac{F}{A}\sin^2\theta$$

Conceptual Note(s)

To find the normal (or longitudinal) stress, resolve the force perpendicular to the plane of the given surface and then divide it by the area of the surface, i.e.



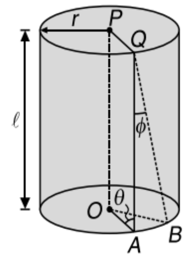
$$\sigma_n = \frac{F_n}{A_{surface}} = \frac{F_\perp}{A_{surface}} = \frac{F\cos\theta}{A_{surface}}$$

To find shearing stress or tangential stress, resolve the force parallel to plane of the given surface and then divide it by the area of the surface, i.e.

$$\sigma_t = \frac{F_t}{A_{surface}} = \frac{F_\parallel}{A_{surface}} = \frac{F\sin\theta}{A_{surface}}$$

TORSION OF A CYLINDER

When one end of a cylinder is clamped and a torque is applied at the other end then the cylinder gets twisted by angle θ due to which a shearing strain ϕ is also produced in the cylinder.



(a) From OAB, we observe that $AB = r\theta$ and from QAB we observe that $AB = l\phi$. So, we have

$$AB = r\theta = l\phi$$

$$\Rightarrow \quad \phi = \frac{r\theta}{l}$$

(b) The restoring torque in the cylinder is given by

$$\tau = \frac{\pi\eta r^4 \phi}{2l}$$

where, η is the modulus of rigidity of the material of the cylinder.

(c) The torque per unit twist i.e. the torque required to produce a unit twist in the cylinder (also called as the torsional constant C of the cylinder) is given by

$$C = \frac{\tau}{\phi} = \frac{\pi\eta r^4}{2l}$$

(d) Work done in twisting the cylinder through an angle θ is given by

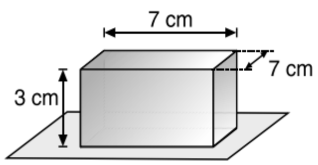
$$W = \frac{1}{2}C\theta^2 = \frac{\pi\eta r^4\theta^2}{4l}$$

Test Your Concepts-III

Based on Shear Modulus, Tangential Stress, Shear Strain

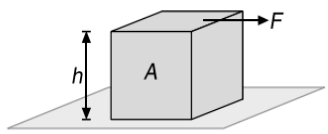
(Solutions on page H.6)

1. A box shaped piece of gelatin dessert has a top area of 15 cm^2 and a height of 3 cm. When a shearing force of 0.50 N is applied to the upper surface, the upper surface displaces 4 mm relative to the bottom surface. Calculate the shearing stress, the shearing strain, and the shear modulus for the gelatin.
2. A $3\text{ cm} \times 7\text{ cm} \times 7\text{ cm}$ block of soft butter block is resting on a surface as shown in Figure.



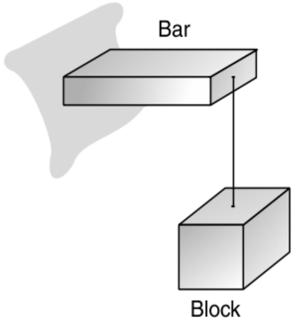
A tangential force of 0.49 N is applied at the top face of the butter block. The top face is observed to move 6 mm relative to the bottom face. Calculate the shear modulus of butter block.

3. A cube of side h is fixed to a horizontal surface as shown in Figure.



If a shearing force distorts the cube such that σ is the shear stress, then calculate the work done by the shearing force, assuming deformation to be small.

4. A block of mass 160 kg is hanging from the end of a steel bar as shown in Figure.

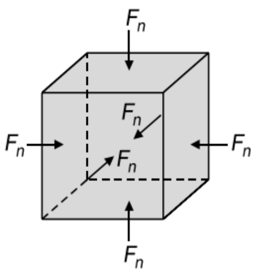


The length of the bar is 0.10 m and its cross-sectional area is $3.2\times10^{-4}\ \text{m}^2$. Ignoring weight of the bar, calculate shear stress on the bar and vertical deflection of the right end of the bar if $\eta_{\text{steel}} = 8.0\times10^{10}\ \text{Nm}^{-2}$ and $g = 10\ \text{ms}^{-2}$.

BULK MODULUS

The bulk modulus of a solid or a fluid indicates its resistance to a change in volume. Consider a cube of some material, solid or fluid, as shown in the figure. We assume that all faces experience the same force F_n normal to each face. (One way to accomplish this is to immerse the body in a fluid as long as the change in pressure over the vertical height of the cube is negligible). The **pressure** on the cube is defined as the normal force per unit area

$$p = \frac{F_n}{A}$$



The *SI* unit of pressure is Nm^{-2} and is given the name **pascal (Pa).**

Pressure is a scalar because on any infinitesimal volume, it acts in all directions; it has no unique direction.

When the pressure on a body is increased, its volume decreases. The change in pressure ΔP is called the volume stress and the fractional change in volume $\frac{\Delta V}{V}$ is called the volume strain. The **bulk modulus** B of the material is defined as

$$\text{Bulk modulus} = \frac{\text{Volume stress}}{\text{Volume strain}}$$

$$\Rightarrow \quad B = \frac{-\Delta p}{\frac{\Delta V}{V}}$$

The negative sign is included to make B a positive number since an increase in pressure $(\Delta p > 0)$ leads to decrease in volume $(\Delta V < 0)$

The inverse of B is called the **compressibility**,

$$k = \frac{1}{B}$$

State	Shear Modulus	Bulk Modulus
Solid	Large	Large
Liquid	Zero	Large
Gas	Zero	Small

RELATIONS BETWEEN ELASTIC CONSTANTS

The elastic constants i.e. Y, K, η and σ are found to depend on each other through the relations given by

(a) $Y = 3B(1-2\sigma)$ (b) $Y = 2\eta(1+\sigma)$

(c) $\sigma = \frac{3B-2\eta}{6B+2\eta}$ (d) $Y = \frac{9\eta B}{3B+\eta}$

DENSITY OF COMPRESSED LIQUIDS

When a pressure ΔP is applied on a substance its density is changed. If a liquid of density ρ, volume V and bulk modulus K is compressed, then its mass remains same, volume decreases and hence density increases. Since density is

$$\rho = \frac{M}{V} \qquad \ldots(1)$$

When compressed, we have the density ρ' given by

$$\rho' = \frac{M}{V-\Delta V} = \frac{M}{V\left(1-\frac{\Delta V}{V}\right)} \qquad \ldots(2)$$

From (1) and (2), we get

$$\rho' = \frac{\rho}{\left(1-\frac{\Delta V}{V}\right)}$$

Since, $B = -\dfrac{\Delta P}{\left(\dfrac{\Delta V}{V}\right)}$

$\Rightarrow \quad -\dfrac{\Delta V}{V} = \dfrac{\Delta P}{B}$

$\Rightarrow \quad \rho' = \dfrac{\rho}{1 - \dfrac{\Delta P}{B}}$

From this expression we can see that ρ' increases as pressure is increased (ΔP is positive) and vice-versa.

ILLUSTRATION 26

Sea water has a Bulk modulus of $23 \times 10^{10}\ \text{Nm}^{-2}$. Calculate the density of sea water at a depth where the pressure is 800 atm if the density at the surface is $1024\ \text{kgm}^{-3}$.

SOLUTION

The changed density is given by

$$\rho' = \frac{\rho}{1 - \dfrac{\Delta P}{B}}$$

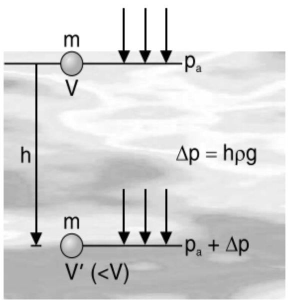
Substituting the value, we get

$$\rho' = \frac{1024}{1 - \dfrac{799 \times 1.01 \times 10^5}{23 \times 10^{10}}} = 1024.4\ \text{kgm}^{-3}$$

ILLUSTRATION 27

Find the depth of lake at which density of water is 1% greater than at the surface. Given compressibility $50 \times 10^{-6}\ \text{atm}^{-1}$.

SOLUTION



$$B = \frac{\Delta p}{-\dfrac{\Delta V}{V}}$$

$\Rightarrow \quad \dfrac{\Delta V}{V} = -\dfrac{\Delta p}{B} \quad \ldots(1)$

Since, $p = p_{\text{atm}} + h\rho g$

$m = \rho V = \text{constant}$

$\Rightarrow \quad V d\rho + \rho dV = 0$

$\Rightarrow \quad \dfrac{d\rho}{\rho} = -\dfrac{dV}{V}$

$\Rightarrow \quad \dfrac{\Delta\rho}{\rho} = \dfrac{\Delta p}{B}$

Since, $\dfrac{\Delta\rho}{\rho} = \dfrac{1}{100}$

$\Rightarrow \quad \dfrac{1}{100} = \dfrac{h\rho g}{B}$

Assuming ρ to be constant, we get

$$h\rho g = \frac{B}{100} = \frac{1}{100k}$$

$\Rightarrow \quad h\rho g = \dfrac{1 \times 1 \times 10^5}{100 \times 50 \times 10^{-6}}$

$\Rightarrow \quad h = \dfrac{10^5}{5000 \times 10^{-6} \times 1000 \times 10}$

$\Rightarrow \quad h = \dfrac{100 \times 10^3}{50} = 2\ \text{km}$

ILLUSTRATION 28

The pressure P of the gas varies with volume V according to the relation $P = P_0 e^{\alpha V}$. Find the bulk modulus of the gas.

SOLUTION

Since $P = P_o e^{\alpha V}$

$\Rightarrow \quad \dfrac{dP}{dV} = \dfrac{d}{dV}\left(P_o e^{\alpha V}\right) = P_o e^{\alpha V}\alpha = P\alpha \qquad \{\because P = P_o e^{\alpha V}\}$

Since by definition of Bulk's Modulus, we have

$$B = -\frac{dP}{dV/V} = -V\left(\frac{dP}{dV}\right)$$

$$B = \left|\frac{dP}{dV}\right| V = \alpha P V$$

Conceptual Note(s)

(a) A solid will have all the three modulus of elasticity Y, B and η. But in case of a liquid or a gas only B can be defined as a liquid or a gas cannot be framed into a wire or no shear force can be applied on them.

(b) For a liquid or a gas,

$$B = \left(\frac{-dP}{dV/V}\right)$$

So instead of P we are more interested in change in pressure dP.

(c) In case of a gases, the Bulk's modulus is given by

$$B = XP$$

In the process $PV^x = \text{constant}$

For example, for $x = 1$, or $PV = \text{constant}$ (isothermal process) $B = P$.

i.e., isothermal bulk modulus of a gas (denoted by $B_{\text{isothermal}}$) is equal to the pressure of the gas at that instant of time or

$$B_{\text{isothermal}} = P$$

Similarly, for $x = \gamma = \dfrac{C_P}{C_V}$ or $PV^\gamma = \text{constant}$ (adiabatic process) $B = \gamma P$.

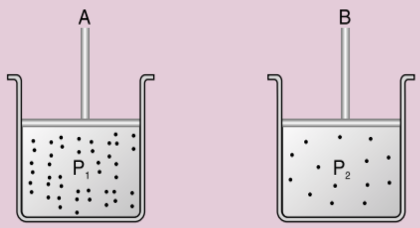
i.e., adiabatic bulk modulus of a gas (denoted by $B_{\text{adiabatic}}$) is equal to γ times the pressure of the gas at that instant of time or

$$B_{\text{adiabatic}} = \gamma P$$

(d) For a gas $B \propto P$

whether it is an isothermal process or an adiabatic process. Physically this can be understood as under. Suppose we have two containers A and B. Some gas is filled in both the containers. But the pressure in A is more than the pressure in B, i.e.,

$$P_1 > P_2$$



So, bulk modulus of A should be more than the bulk modulus of B, or

$$B_1 > B_2$$

and this is quiet obvious, because it is more difficult to compress the gas in chamber A, i.e., it provides more resistance to the external forces. And as we have said in point number 1 (ii) the modulus of elasticity is greater for a substance which offers more resistance to external forces.

(e) For liquids, the energy density is given by

$$U = \frac{(\text{Stress})^2}{2B} = \frac{(h\rho g)^2}{2B}$$

ILLUSTRATION 29

In a vertical cylindrical vessel of base area $A = 80\ \text{cm}^2$ water is filled to a height $h = 30$ cm. If density and the Bulk modulus of water be $\rho = 1000\ \text{kgm}^{-3}$ and $B = 2\times10^9\ \text{Nm}^{-2}$. Calculate elastic deformation energy of water in the vessel. $(g = 10\ \text{ms}^{-2})$.

SOLUTION

At a depth h below the free surface of water volume density of elastic deformation energy,

$$U = \frac{(\rho g h)^2}{2B}$$

So, total deformation energy E of water in the vessel is given by

$$E = \int_0^h u\,dV$$

$$\Rightarrow \quad E = \int_0^h \frac{(\rho g h)^2}{2B} A\,dh = \frac{A(\rho g)^2 h^3}{6B}$$

Substituting the values, we have

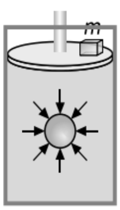
$$E = \frac{(80\times10^{-4})(10^3\times10)^2(0.3)^3}{6\times2\times10^9} = 1.8\times10^{-6}\ \text{J}$$

Test Your Concepts-IV

Based on Bulk's Modulus, Normal Stress and Volumetric Strain

(Solutions on page H.6)

1. A solid sphere of radius R made of a material of bulk modulus B is surrounded by a liquid in a cylindrical container. A heavy piston of mass m and area A floats on the surface of the liquid as shown. Calculate the fractional change in the radius of the sphere.



2. A specimen of oil having an initial volume of $800\ \text{cm}^3$ subjected to a pressure increase of 1.8×10^6 Pa, and the volume is found to decrease by $0.30\ \text{cm}^3$. Calculate the bulk modulus and compressibility of the material.
3. In taking a solid ball of rubber from the surface to the bottom of a lake of 180 m depth, reduction in the volume of the ball is 0.1%. The density of water of the lake is $1\times10^3\ \text{kgm}^{-3}$. Determine the value of the bulk modulus of elasticity of rubber. $(g = 9.8\ \text{ms}^{-2})$.

4. What is the percentage change in the volume of a glass ball if it is placed in vacuum than in air. Bulk modulus of glass $B = 4\times10^{10}\ \text{Nm}^{-2}$.
5. Bulk modulus of water is $2.3\times10^{9}\ \text{Nm}^{-2}$. Taking average density of water $\rho = 10^{3}\ \text{kgm}^{-3}$, find increase in density at a depth of $1\,\text{km}$. Take $g = 10\ \text{ms}^{-2}$.
6. Calculate the density of lead under a pressure of $2\times10^{8}\ \text{Nm}^{-2}$, if the bulk modulus of lead is $8\times10^{9}\ \text{Nm}^{-2}$ and initially the density of lead is $11.4\ \text{gcm}^{-3}$.
7. The bulk modulus of water is $2.3\times10^{9}\ \text{Nm}^{-2}$.
   (a) Find its compressibility in the units atm^{-1}.
   (b) How much pressure in atmospheres is needed to compress a sample of water by 0.1%?
8. Calculate the volume density of the elastic deformation energy in fresh water at the depth of $h = 1000\ \text{m}$. Compressibility of water $k = 4.9\times10^{-10}\ \text{m}^2\text{N}^{-1}$.
9. The pressure of a medium is changed from $1.01\times10^{5}\ \text{Pa}$ to $1.165\times10^{5}\ \text{Pa}$ and change in volume is 10% keeping temperature constant. Calculate the Bulk's modulus of the medium.

FLUID STATICS

FLUIDS: INTRODUCTION AND ASSUMPTIONS

A fluid is a substance that can flow and does not have a shape of its own. All liquids and gases are fluids. However, in this chapter our main study is applicable for

(a) Incompressible Liquids, which have their density to be independent of the variations in pressure and is assumed to be constant.

(b) Non-viscous Liquids, in which the parts of the liquid in contact do not exert any tangential force on each other. The force acted by one part of the liquid on the other part is perpendicular to the surface of contact due to which no friction between the adjacent layers of liquid in contact.

DENSITY

The **density** ρ of a substance is defined as the mass per unit volume of a sample of the substance.
If a small mass element Δm occupies a volume ΔV, the density is given by

$$\rho = \frac{\Delta m}{\Delta V}$$

In general, the density of an object depends on position, so that

$$\rho = f(x, y, z)$$

If the object is homogeneous, its physical parameters do not change with position throughout its volume. Thus, for a homogeneous object of mass M and volume V, the density is defined as

$$\rho = \frac{M}{V}$$

The SI units of density are kgm^{-3}.
In a fluid, at a point, density ρ is defined as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

Conceptual Note(s)

(a) In case of homogenous isotropic substance, density has no directional properties, so is a scalar. It has dimensional formula ML^{-3}, SI unit kgm^{-3} and cgs unit gcc^{-1} where

$$1\ \text{gcc}^{-1} = 10^3\ \text{kgm}^{-3}$$

(b) When immiscible liquids of different densities are poured in a beaker, the liquid having the highest density will be at the bottom of the beaker while the liquid having the lowest density will be at the top of the beaker and interfaces separating the liquids will be plane surfaces.

(c) With the rise in temperature (ΔT) due to thermal expansion of a given body, the volume of the body increases whereas the mass of the body remains unchanged. So, the density of the body decreases according to the relation

$$\frac{\rho}{\rho_0} = \frac{m/V}{m/V_0} = \frac{V_0}{V}$$

Since, we know that $V = V_0(1+\gamma\Delta T)$

$$\Rightarrow \rho = \frac{\rho_0}{1+\gamma\Delta T}$$

where, γ is the coefficient of volume expansion of the liquid. (Detailed discussion done in Thermal Expansion)

(d) With increase in pressure (ΔP) the volume of the liquid decreases (ΔV), so the density will increase according to the relation

$$\rho = \frac{\rho_0}{1-\frac{\Delta V}{V}} = \frac{\rho_0}{1-\frac{\Delta P}{B}}$$

where, B is the Bulk's Modulus of the liquid. (Detailed discussion done in Elasticity)

RELATIVE DENSITY OR SPECIFIC GRAVITY

Sometimes instead of density we use the term relative density (RD) or specific gravity (SG) which is defined as:

$$\text{RD} = \frac{\text{Density of body}}{\text{Density of water at } 4\ ^\circ\text{C}}$$

Relative density is a pure ratio and has no units. Density of water at $4\ ^\circ\text{C}$ is $1\ \text{gcc}^{-1}$. So, in cgs system of units, the relative density is equal to the density.
For example, the relative density of mercury is 13.6, so its density will be $13.6\times10^3\ \text{gcc}^{-1}$.

ILLUSTRATION 30

Find the density and specific gravity of gasoline if $51\ \text{g}$ occupies $75\ \text{cm}^3$?

SOLUTION

$$\text{Density} = \frac{\text{mass}}{\text{volume}} = \frac{0.051\ \text{kg}}{75\times10^{-6}\ \text{m}^3} = 680\ \text{kgm}^{-3}$$

$$\Rightarrow \quad \text{Sp. gravity} = \frac{\text{density of gasoline}}{\text{density of water}}$$

$$\Rightarrow \quad \text{Sp. gravity} = \frac{680\ \text{kgm}^{-3}}{1000\ \text{kgm}^{-3}} = 0.68$$

$$\Rightarrow \quad \text{Sp. gravity} = \frac{\text{mass of } 75\ \text{cm}^3 \text{ gasoline}}{\text{mass of } 75\ \text{cm}^3 \text{ water}}$$

$$\Rightarrow \quad \text{Sp. gravity} = \frac{51\ \text{g}}{75\ \text{g}} = 0.68$$

DENSITY OF A MIXTURE OF TWO OR MORE LIQUIDS

If a mass M_1 of liquid of density ρ_1, volume V_1 is mixed with a liquid of mass M_2, density ρ_2, volume V_2 and so on, then

$$\rho_{\text{mixture}} = \frac{M_{\text{mixture}}}{V_{\text{mixture}}} = \frac{M_1 + M_2 + \ldots}{V_1 + V_2 + \ldots}$$

CASE-1

If equal masses of the liquids are mixed, then

$$M_{\text{mixture}} = M + M = 2M \text{ and } V_{\text{mixture}} = V_1 + V_2$$

$$\Rightarrow \quad \rho = \frac{2M}{(M/\rho_1) + (M/\rho_2)}$$

$$\Rightarrow \quad \rho_{\text{mixture}} = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2} = \text{Harmonic Mean}$$

CASE-2

If equal volumes of the liquids are mixed, then

$$M_{\text{mixture}} = M_1 + M_2 \text{ and } V_{\text{mixture}} = 2V$$

$$\Rightarrow \quad \rho = \frac{\rho_1 V + \rho_2 V}{2V} = \frac{\rho_1 + \rho_2}{2}$$

$$\Rightarrow \quad \rho_{\text{mixture}} = \frac{\rho_1 + \rho_2}{2} = \text{Arithmetic Mean}$$

Problem Solving Technique(s)

(a) If n liquids of equal masses are mixed, then

$$\frac{1}{\rho_{\text{mixture}}} = \frac{\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \ldots}{n}$$

(b) If n liquids of equal volumes are mixed, then

$$\rho_{\text{mixture}} = \frac{\rho_1 + \rho_2 + \rho_3}{n}$$

ILLUSTRATION 31

Two substances of densities ρ_1 and ρ_2 are mixed in equal volume and the relative density of mixture is 4. When they are mixed in equal masses, the relative density of the mixture is 3. Find ρ_1 and ρ_2.

SOLUTION

When substances are mixed in equal volume then density of the mixture is

$$\rho_{\text{mix}} = \frac{\rho_1 + \rho_2}{2} = 4$$

$$\Rightarrow \quad \rho_1 + \rho_2 = 8 \qquad \ldots(1)$$

When substances are mixed in equal masses then density is

$$\frac{2\rho_1\rho_2}{\rho_1 + \rho_2} = 3$$

$$\Rightarrow \quad 2\rho_1\rho_2 = 3(\rho_1 + \rho_2)$$

$$\Rightarrow \quad 2\rho_1\rho_2 = 3 \times 8 = 24$$

$$\Rightarrow \quad \rho_1\rho_2 = 12 \qquad \ldots(2)$$

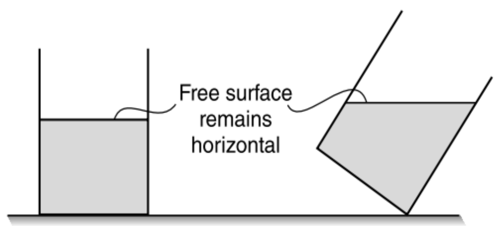
Solving equations (1) and (2), we get

$$\rho_1 = 6 \text{ and } \rho_2 = 2.$$

FLUID AT REST

It has been observed that a fluid at rest cannot sustain a tangential force. If such a force is applied to a fluid, the different layers simply slide over one another. Hence, for a fluid at rest, the forces acting on it have to be normal to the surface.

An important consequence of this property is that the free surface of a liquid at rest, under gravity, in a container, is always horizontal.



Since, the force of gravity acts vertically downwards so, in equilibrium, the liquid surface has to be perpendicular to it and hence is horizontal.

Problem Solving Technique(s)

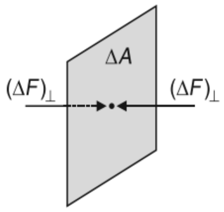
(a) A fluid exerts force on any surface in contact with it.
(b) This force always acts at right angles (normally) to the surface and is also called **Thrust.**

PRESSURE

The normal force or the thrust per unit area is called **Fluid Pressure.** Its S.I. unit is Nm^{-2} or pascal (Pa). Mathematically, we have

$$P = \frac{(\Delta F)_\perp}{\Delta A}$$

where $(\Delta F)_\perp$ is the normal force acting on surface of area A. If we consider an imaginary surface within the liquid, then the fluid on both the sides of the surface exerts equal and opposite force $(\Delta F_\perp)$ on each surface as shown. If this is not the case then the liquid surface will accelerate and the fluid will not remain at rest.



If the pressure is same at all points of a finite plane surface of area A, then

$$P = \frac{F_\perp}{A}$$

where $F_\perp$ is the normal force acting on one side of the imaginary surface. The **average pressure** in the fluid at the position of the element is given by

$$P_{av} = \frac{\Delta F}{\Delta A}$$

When $\Delta A \to 0$, the element reduces to a point, and thus, pressure at a point is defined as

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

When the force is constant over the surface of area A, then the above equation reduces to

$$P = \frac{F}{A}$$

The SI unit of pressure is Nm^{-2} and is also called **pascal (Pa)**. where, $1 \text{ Pa} = 1 \text{ Nm}^{-2}$

The other common units of pressure are the **atmosphere** and **bar**.

$$1 \text{ atm} = 1.01325 \times 10^5 \text{ Pa}$$

$$1 \text{ bar} = 1.00000 \times 10^5 \text{ Pa}$$

ATMOSPHERIC PRESSURE (P_0)

It is the pressure due to the earth's atmosphere. It changes with weather conditions and elevation. The normal average atmospheric pressure at the sea level is 1.01×10^5 Pa also called as 1 atmosphere i.e.

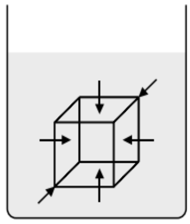
$$1 \text{ atm} = 1.01 \times 10^5 \text{ Pa}$$

and $1 \text{ atm} = 1.01 \times 10^5 \text{ Pa} = 1.01 \text{ bar} = 760 \text{ torr}$

The atmospheric pressure is maximum at the surface of earth and goes on decreasing as we move up into the earth's atmosphere.

PRESSURE IS ISOTROPIC

Imagine a static fluid and consider a small cubic element of it deep within the fluid as shown in Figure. Since this fluid element is in equilibrium, therefore, forces acting on each lateral face of this element must also be equal in magnitude. Because the areas of each face are equal, therefore, the pressure on each of the lateral faces must also be the same. In the limit as the cube element reduces to a point, the forces on the top and bottom surfaces also become equal. Thus, the pressure exerted by a fluid at a point is the same in all directions – **the pressure is isotropic**.



Since the fluid cannot support a shear stress, the force exerted by a fluid pressure must also be perpendicular to the surface of the container that holds it.

VARIATION OF PRESSURE WITH DEPTH

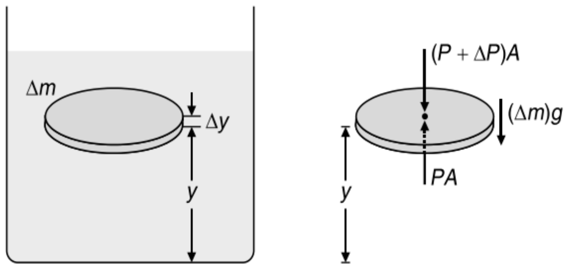
Consider a small cylindrical element of mass Δm at a height y from the base of the container as shown in Figure. Let A be the area of the top and bottom surface of this cylinder.

We know that the fluid forces on the opposite vertical faces of the cylinder are equal in magnitude and opposite in direction, and therefore, cancel. The other forces acting on the fluid contained within the imaginary boundary is the gravitational force

$$\Delta W = (\Delta m) g$$

and the forces due to fluid pressure.

The pressure force acting on the lower face of the element is pA and that on the upper face is $(p + \Delta p)A$. The free body diagram of the element is shown in Figure.



Applying the condition of equilibrium, we get

$$PA-(P+\Delta P)A-(\Delta m)g=0 \qquad \ldots(1)$$

If ρ be the density of the fluid at the position of the element, then

$$\Delta m=\rho A(\Delta y)$$

$$\Rightarrow \quad PA-(P+\Delta P)A-\rho g A(\Delta y)=0 \qquad \{\text{from (1)}\}$$

$$\Rightarrow \quad \frac{\Delta P}{\Delta y}=-\rho g$$

In the limit Δy approaching to zero, $\frac{\Delta P}{\Delta y}$ becomes

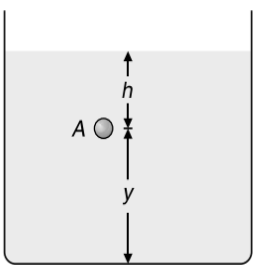
$$\frac{dP}{dy}=-\rho g$$

The above equation indicates that the slope of P versus y is negative. That is, the pressure P decreases with height y from the bottom of the fluid.
In other words, the pressure p increases with depth h, i.e.

$$\frac{dP}{dh}=\rho g$$

THE INCOMPRESSIBLE FLUID MODEL

For an incompressible fluid, the density ρ of the fluid remains constant throughout its volume. It is a good assumption for liquids. To find pressure at the point A in a fluid column as shown in Figure is obtained by integrating.



$$dP=\rho g dh$$

$$\Rightarrow \quad \int_{P_0}^{P} dP=\rho g\int_{0}^{h} dh$$

$$\Rightarrow \quad P-P_0=\rho g h$$

$$\Rightarrow \quad P=P_0+\rho g h$$

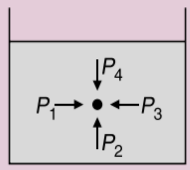
where ρ is the density of the fluid, and ρ_0 is the atmospheric pressure at the free surface of the liquid.

Conceptual Note(s)

(a) Thus, pressure is same at all points at the same depth. The force due to pressure acts normally on any area, irrespective of the orientation of the area.

(b) At same point inside a fluid, the pressure is same in all directions. In the Figure,

$$P_1=P_2=P_3=P_4$$



ABSOLUTE PRESSURE AND GAUGE PRESSURE

Absolute pressure is the total pressure at a point while gauge pressure is relative to the local atmospheric pressure. Gauge pressure may be positive or negative depending upon the fact whether the pressure is more or less than the atmospheric pressure.

$$P_{\text{gauge}}=P_{\text{absolute}}-P_{\text{atm}}$$

ILLUSTRATION 32

If pressure at half the depth of a lake is two third the pressure at the bottom of the lake then find the depth of the lake.

SOLUTION

Let P_0 be the atmospheric, then pressure at bottom of the lake is

$$P=P_0+h\rho g$$

Pressure at half the depth of a lake is

$$P'=P_0+\frac{h}{2}\rho g$$

According to given condition, we have

$$P'=\frac{2}{3}P$$

$$\Rightarrow \quad P_0+\frac{1}{2}h\rho g=\frac{2}{3}(P_0+h\rho g)$$

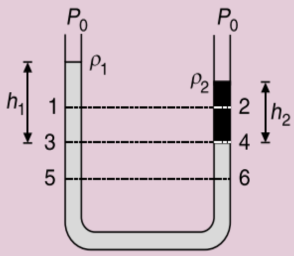
$$\Rightarrow \quad \frac{1}{3}P_0=\frac{1}{6}h\rho g$$

$$\Rightarrow \quad h=\frac{2P_0}{\rho g}=\frac{2\times 10^5}{10^3\times 10}=20 \text{ m}$$

Conceptual Note(s)

(a) Forces acting on a fluid in equilibrium have to be perpendicular to its surface. Because it cannot sustain the shear stress.

(b) In the same liquid pressure will be same at all points at the same level.



For example, in the Figure, we observe that

$$P_1 \neq P_2,\ P_3 = P_4,\ P_5 = P_6$$

For $P_3 = P_4$, we get

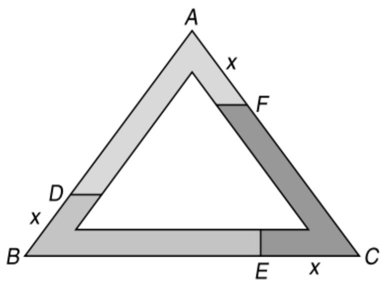
$$P_0 + \rho_1 g h_1 = P_0 + \rho_2 g h_2$$

$$\Rightarrow \rho_1 h_1 = \rho_2 h_2$$

$$\Rightarrow h \propto \frac{1}{\rho}$$

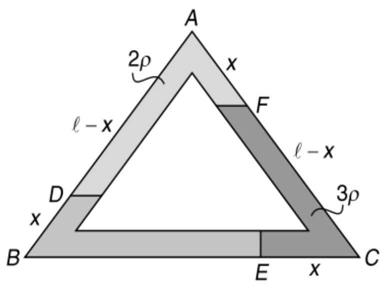
ILLUSTRATION 33

A closed tube in the form of an equilateral triangle of side l contains equal volumes of three liquids which do not mix and is placed vertically with its lowest side horizontal. Find the value of x in the Figure, if the densities of liquids are in arithmetic progression.



SOLUTION

Pressure at E due to the left limb AB must be equal to the pressure at E due to the right limb AC, so using Pascal's equation, we get



$$P_{AB} = (l-x)\sin 60°(2\rho)g + x\sin 60°(3\rho)g \text{ and}$$

$$P_{AC} = (l-x)\sin 60°(3\rho)g + (x\sin 60°)\rho g$$

Equating, these two, we get

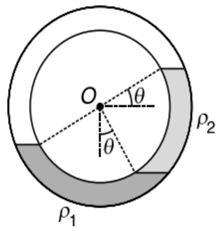
$$2(l-x) + 3x - 3(l-x) - x = 0$$

$$\Rightarrow 3x = l$$

$$\Rightarrow x = \frac{l}{3}$$

ILLUSTRATION 34

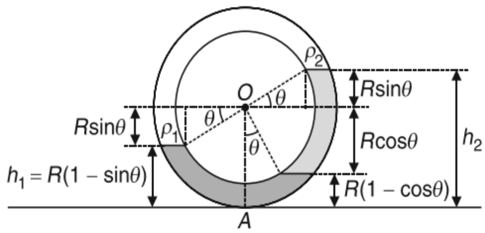
A circular tube of uniform cross section is filled with two liquids of densities ρ_1 and ρ_2 such that half of each liquid occupies a quarter of volume of the tube as shown in Figure. If the line joining the free surfaces of the liquid makes an angle θ with horizontal, then calculate θ.



SOLUTION

Since pressure at A due to the left column and the right column of the liquid should be the same, so we have

$$(P_A)_{\text{due to left column}} = (P_A)_{\text{due to right column}} \quad \ldots(1)$$



Due to left column

Pressure at A is given by

$$(P_A)_{\text{left}} = R(1-\sin\theta)\rho_1 g$$

Due to right column

Pressure at A is given by

$$(P_A)_{\text{right}} = R(\sin\theta + \cos\theta)\rho_2 g + (1-\cos\theta)\rho_1 g$$

Equating both these equations, we get

$$\rho_1(1-\cos\theta) + \rho_2(\sin\theta + \cos\theta) = \rho_1(1-\sin\theta)$$

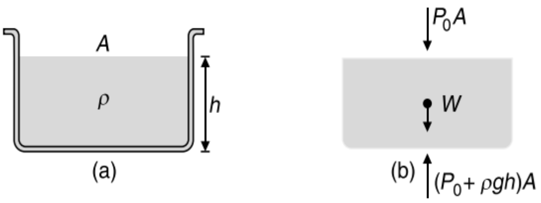
$$\Rightarrow \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{\rho_1}{\rho_2}$$

$$\Rightarrow \tan\theta = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right)$$

SIMPLE FREE BODY DIAGRAM OF A LIQUID IN A CONTAINER

The free body diagram of the liquid (showing the vertical forces only) is shown in Figure (b). For the equilibrium of liquid, the Net downward force must equal the net upward force.

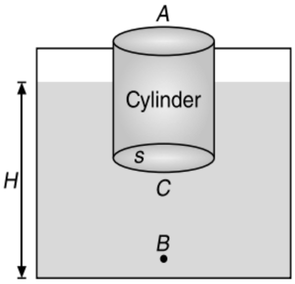


$\Rightarrow \quad P_0A + W = (P_0 + \rho gh)A$

$\Rightarrow \quad W = \rho ghA$ {which is true}

ILLUSTRATION 35

A liquid of density ρ is filled in a beaker of cross-section S to a height H and then a cylinder of mass m and cross-section s is made to float in it as shown in Figure.



If the atmospheric pressure is p_0, find the pressure at the top face A of the cylinder, at the bottom face C of the cylinder and at the base B of the beaker. Can you think of a condition for which these three pressures are equal?

SOLUTION

Above the cross-section A there is external pressure due to atmosphere only. So

$P_A = \text{Atmospheric pressure} = P_0$

At the point C the pressure will be due to atmosphere and also due to the weight of the cylinder, so, we have

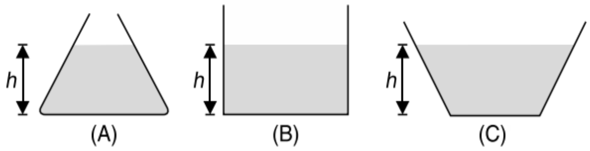
$$P_C = P_0 + \frac{mg}{s}$$

If the system is in free fall (as in a satellite), then $g_{eff} = 0$

$\Rightarrow \quad P_A = P_B = P_C = P_0$

HYDROSTATIC PARADOX

Hydrostatic pressure depends on the depth of the point below the surface (h), nature of liquid (ρ) and acceleration due to gravity (g) while it is independent of the amount of liquid, shape of the container or cross-sectional area considered. So, if a given liquid is filled in vessels of different shapes to same height, the pressure (P) at the base of each vessel's will be the same, though the volume or weight of the liquid in different vessels will be different. This situation is called **Hydrostatic Paradox**.



$$P_A = P_B = P_C = P_0 + h\rho g \text{ but } W_A < W_B < W_C$$

In a liquid at same level, the pressure will be same at all points, if not, due to pressure difference the liquid cannot be at rest. This is why the height of liquid is same in vessels of different shapes containing different amounts of the same liquid at rest when they are in communication with each other.

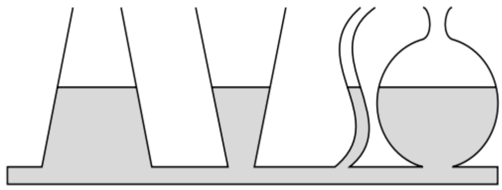
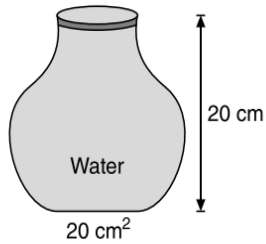


ILLUSTRATION 36

A half-litre vessel of height 20 cm is full of water. The vessel base has an area of 20 cm^2 and its mouth area is also 20 cm^2 as shown in Figure.



Calculate the force exerted by the water on the base of the vessel. Assuming the water to be in equilibrium calculate the resultant force exerted by the curved surface of the vessel on water if the atmospheric pressure is $P_0 = 1.0\times10^5\text{ Nm}^{-2}$, density of water is $\rho = 1000\text{ kgm}^{-3}$ and $g = 10\text{ ms}^{-2}$.

SOLUTION

Pressure of water at the base of vessel is

$P = P_0 + h\rho g$

$\Rightarrow \quad P = 1\times10^5 + (0.2)(1000)(10)$

$\Rightarrow \quad P = 1.02\times10^5\text{ Nm}^{-2}$

Force on the base of vessel is

$F = PA_{\text{base}} = (1.02\times10^5)(20\times10^{-4})$

$\Rightarrow \quad F = 204\text{ N}$

So, upward force on water by the base of vessel is

$$F_{\text{upward}} = 204 \text{ N}$$

Weight of water in the vessel is

$$W = (0.5\times10^{-3})(1000)(10) = 5 \text{ N}$$

Downward force on water by atmosphere is

$$F_{\text{downward}} = P_0 A = (1\times10^5)(20\times10^{-4}) = 200 \text{ N}$$

Assuming the force applied by the curved surface of vessel on water to be F_s, then for equilibrium, we have upwards, for equilibrium of water we have

$$\Sigma F = 0$$

$$\Rightarrow \quad F_{\text{up}} + F_{\text{down}} + W + F_s = 0$$

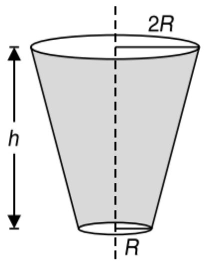
Taking upward direction as positive, we get

$$204 - 200 - 5 + F_s = 0$$

$$\Rightarrow \quad F_s = 1 \text{ N (upwards)}$$

ILLUSTRATION 37

Consider a frustum of a cone h having radius $2R$ and R. A liquid of density ρ is filled in it. Calculate the force exerted by the walls of the container on the liquid.

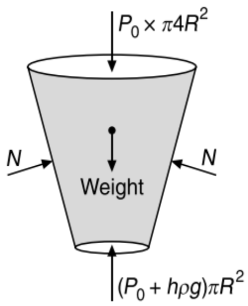


SOLUTION

For vertical equilibrium of the fluid

$$P_0\left[\pi(2R)^2\right] + \text{Weight} = (P_0 + h\rho g)\pi R^2 + N_y$$

Here N_y is the net vertical component of the normal reaction by the wall



Weight of liquid is

$$\rho\left(\frac{7}{3}\pi r^2 h\right)g = \frac{1}{3}\pi h\left[R^2 + (2R)^2 + 2R^2\right]\rho g$$

$$\Rightarrow \quad 4P_0\pi R^2 + \frac{7}{3}\pi R^2 h\rho g = P_0\pi R^2 + h\rho g\pi R^2 + N_y$$

$$\Rightarrow \quad 3P_0\pi R^2 + \frac{4}{3}\pi R^2 h\rho g = N_y$$

For horizontal equilibrium net force by the wall on the liquid in horizontal direction should be zero.

Thus, total force $F = \left(3P_0\pi R^2 + \frac{4}{3}\pi R^2 h\rho g\right)$

$$\Rightarrow \quad F = \pi R^2\left(3P_0 + \frac{4}{3}h\rho g\right)$$

Conceptual Note(s)

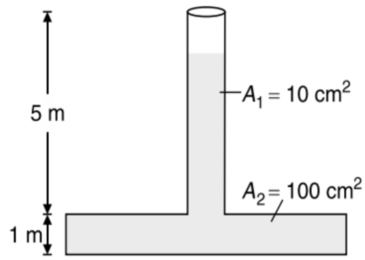
$$V_{\text{frustum}} = \frac{1}{3}\pi h\left(R^2 + r^2 + Rr\right)$$

$$\Rightarrow \quad V_{\text{frustum}} = \frac{1}{3}\pi R^2 h\left(1 + \frac{r^2}{R^2} + \frac{r}{R}\right)$$

$$\Rightarrow \quad V = \frac{1}{3}\pi R^2 h\left(\left(\frac{r}{R}\right)^0 + \left(\frac{r}{R}\right)^1 + \left(\frac{r}{R}\right)^2\right)$$

ILLUSTRATION 38

In the Figure shown, calculate the total force acting at the bottom of the tank due to the water pressure, the total weight of water. Can you explain the difference between the two readings?



SOLUTION

Pressure at the base due to water is

$$P = \rho_w g(5+1) = (10)^3(10)(6) = 6\times10^4 \text{ Nm}^{-2}$$

So, force F due to pressure is given by

$$\Rightarrow \quad F = PA_2 = (6\times10^4)(100\times10^{-4}) = 600 \text{ N}$$

Weight of water W is given by

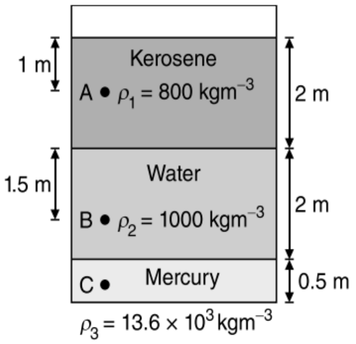
$$W = \rho_w g(5A_1 + A_2)$$

$$\Rightarrow \quad W = 10^4\left[5(10\times10^{-4}) + 100\times10^{-4}\right] = 150 \text{ N}$$

This problem exhibits that a liquid may exert thrust which is more than the weight of the liquid.

ILLUSTRATION 39

Find the absolute pressure and gauge pressure at points A, B and C as shown in Figure.
Take $1\text{ atm} = 10^5\text{ Pa}$.



SOLUTION

Given that

$$P_{\text{atm}} = P_0 = 10^5\text{ Pa} = 100\text{ kPa}$$

At *A*

Gauge Pressure is measuring the pressure without taking the atmospheric pressure into account, so we have

$$\Delta P_A = \rho_1 g h_A = (800)(10)1$$

$$\Rightarrow \quad \Delta P_A = 8\text{ kPa}$$

Absolute pressure is given by

$$P_A = P_0 + \Delta P_A$$

$$\Rightarrow \quad P_A = 10^5\text{ Pa} + 8000\text{ Pa} = 108\text{ kPa}$$

At *B*

Gauge pressure is given by

$$\Delta P_B = \rho_1 g(2) + \rho_2 g(1.5)$$

$$\Rightarrow \quad \Delta P_B = (800)(10)(2) + (10)^3(10)(1.5)$$

$$\Rightarrow \quad \Delta P_B = 31\text{ kPa}$$

Absolute pressure is given by

$$P_B = P_0 + \Delta P_B$$

$$\Rightarrow \quad P_A = 100\text{ kPa} + 31\text{ kPa} = 131\text{ kPa}$$

At *C*

Gauge pressure is given by

$$\Delta P_C = \rho_1 g(2) + \rho_2 g(2) + \rho_3 g(0.5)$$

$$\Rightarrow \quad \Delta P_C = g\left[2\rho_1 + 2\rho_2 + \frac{\rho_3}{2}\right]$$

$$\Rightarrow \quad \Delta P_C = 10\left(1600 + 2000 + \frac{13.6\times 10^3}{2}\right)$$

$$\Rightarrow \quad \Delta P_C = 104\text{ kPa}$$

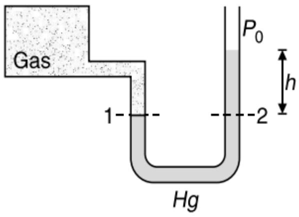
Absolute pressure is given by

$$P_C = P_0 + \Delta P_C$$

$$\Rightarrow \quad P_A = 100\text{ kPa} + 104\text{ kPa} = 204\text{ kPa}$$

PRESSURE MEASURING DEVICE: MANOMETER

It is a device used to measure the pressure of a gas inside a container. A **manometer** is a tube open at both the ends and bent into the shape of a U and partially filled with mercury. When one end of the tube is subjected to an unknown pressure p, the mercury level drops on that side of the tube and rises on the other so that the difference in mercury level is h as shown in Figure.



According to Pascal's Law, when we move down in a fluid pressure increases with depth and when we move up the pressure decreases with depth. When we move horizontally in the same fluid, the fluid pressure remains constant. At equilibrium, we have

$$P_1 = P_2$$

where P_1 is the pressure of the gas in the container and P_2 is given by

$$P_2 = P_{\text{gas}} = P_0 + h\rho g$$

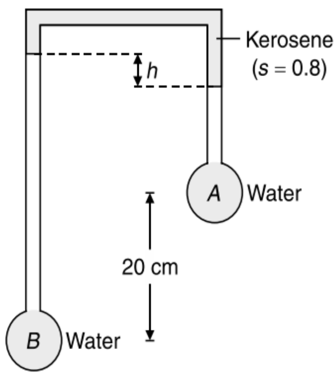
So, gauge pressure is given by

$$P - P_0 = h\rho g$$

where, ρ is the density of the liquid used in U-tube and P_0 is the atmospheric pressure. So, by measuring h we can find absolute (or gauge) pressure in the vessel.

ILLUSTRATION 40

For the arrangement shown in Figure, calculate h if the pressure difference between the vessels A and B is 3 kNm^{-2}.

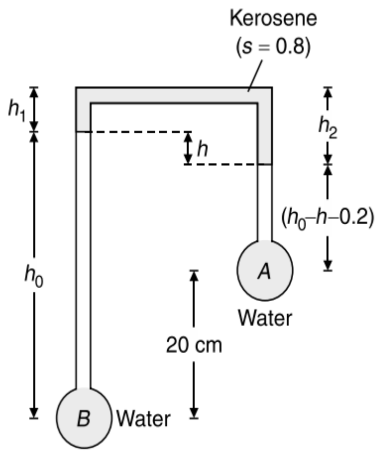


SOLUTION

Let pressure in the horizontal tube be P, then in the left vertical tube, we have

$$P + \rho_k g h_1 + \rho_w g h_0 = P_B$$

$$\Rightarrow \quad P + \rho_k g h_2 + \rho_w g \left(h_0 - h - 0.2\right) = P_A$$



Given that, $P_B - P_A = 3\times10^3\ \text{Nm}^{-2}$, $\rho_w = 10^3\ \text{kgm}^{-3}$

$$\Rightarrow \quad \rho_k = 800\ \text{kgm}^{-3}$$

$$\Rightarrow \quad h = 0.5\ \text{m} = 50\ \text{cm}$$

ILLUSTRATION 41

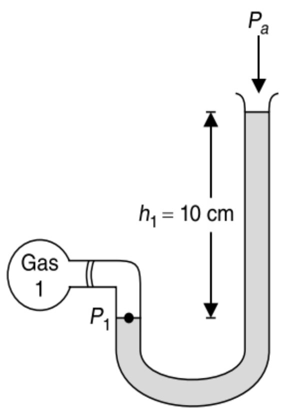
A manometer tube contains a liquid of density $3\times10^3\ \text{kg m}^{-3}$. When connected to a vessel containing a gas, the liquid level in the other arm of the tube is higher by $10\ \text{cm}$. When connected to another sample of enclosed gas, the liquid level in the other arm of the manometer tube falls $7\ \text{cm}$ below the liquid level in the first arm. Which of the two samples exerts more pressure and by what amount?

SOLUTION

For Sample Gas 1:

$$h_1 = 10\ \text{cm} = 0.1\ \text{m}$$

$$P_1 = P_a + \rho g h_1 \quad \ldots(1)$$



For Sample Gas 2:

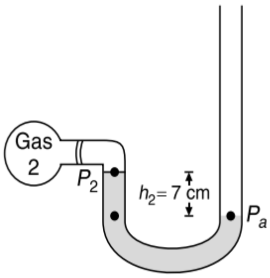
In this case, level of the liquid in the left arm is higher than that in the right arm by $7\ \text{cm}$.

So, atmospheric pressure P_a is greater than the pressure exerted by the sample

$$\Rightarrow \quad P_a = P_2 + \rho g h_2$$

$$\Rightarrow \quad P_2 = P_a - \rho g h_2 \quad \ldots(2)$$

Comparing equations (1) and (2), we observe that $P_1 > P_2$. Therefore the gas in sample 1 exerts greater pressure than that in sample 2.



The difference in the two pressures is given by

$$\Delta P = P_1 - P_2 = \left(P_a + \rho g h_1\right) - \left(P_a - \rho g h_2\right)$$

$$\Rightarrow \quad \Delta P = \rho g \left(h_1 + h_2\right)$$

where $h_1 + h_2 = 17\ \text{cm} = 0.17\ \text{m}$

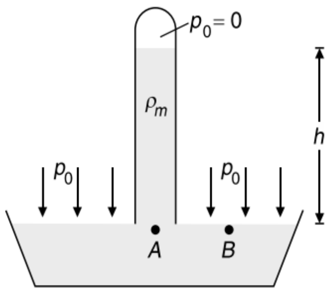
$$\Rightarrow \quad \Delta P = \left(3\times10^3\ \text{kg m}^{-3}\right)\times\left(9.8\ \text{ms}^{-2}\right)\left(0.17\ \text{m}\right)$$

$$\Rightarrow \quad \Delta P = 4.99\times10^3\ \text{Pa}$$

$$\Rightarrow \quad \Delta P \approx 5\ \text{kPa}$$

THE MERCURY BAROMETER

It is a straight glass tube (closed at one end) completely filled with mercury and inserted into a dish which is also filled with mercury as shown in Figure. Atmospheric pressure supports the column of mercury in the tube to a height h. The pressure between the closed end of the tube and the column of mercury is zero, $P = 0$.



Therefore, pressure at points A and B are equal, so

$$P_{\text{atm}} = P_0 = 0 + h\rho_m g$$

At the sea level, p_0 can support a column of mercury about $76\ \text{cm}$ in height. Hence

$$P_0 = \left(13.6\times10^3\right)\left(9.81\right)\left(0.76\right) = 1.01\times10^5\ \text{Nm}^{-2}$$

ILLUSTRATION 42

What must be the length of a barometer tube used to measure atmospheric pressure if we are to use water instead of mercury.

SOLUTION

We know that

$$P_0 = \rho_m g h_m = \rho_w g h_w$$

where ρ_w and h_w are the density and height of the water column supporting the atmospheric pressure P_0.

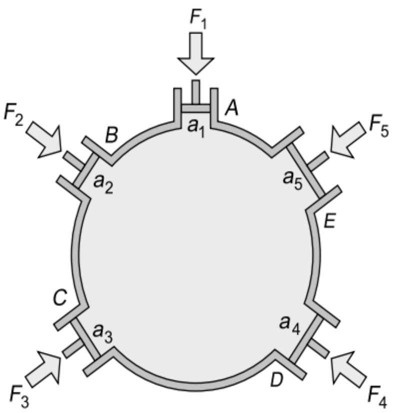
$$\Rightarrow \quad h_w = \left(\frac{\rho_m}{\rho_w}\right) h_m$$

Since, $\dfrac{\rho_m}{\rho_w} = 13.6$ and $h_m = 0.76$ m

$$\Rightarrow \quad h_w = (13.6)(0.76) = 10.33 \text{ m}$$

PASCAL'S LAW

Liquids, being incompressible are capable of transmitting the pressure applied at one point to any other point. A principle regarding this was formulated by Blaise Pascal (1623-1662), a French mathematician, physicist and philosopher. The formulation is called **Pascal's Law.**



The excess pressure, applied anywhere in a mass of a confined incompressible fluid (or liquid), is transmitted by the fluid in all directions and acts undiminished at every point of the fluid and at right angles to the surfaces exposed to the fluid.

Consider a vessel full of water and filled with air tight pistons in different position as shown. Let the piston at A be pushed down with a force F_1 (shown in Figure). Pressure P on the piston is

$$P = \frac{F_1}{a_1}$$

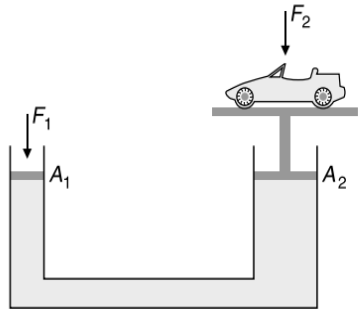
where a_1 is the area of cross-section of piston at A. It will be observed that to hold pistons at B, C, D and E we have to apply forces F_2, F_3, F_4 and F_5 on them such that

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3} = \frac{F_4}{a_4} = \frac{F_5}{a_5}$$

where a_2, a_3, a_4 and a_5 are the area of cross-section of piston at B, C, D and E respectively. This indicates that pressure is transmitted equally in all directions as stated by Pascal's law.

HYDRAULIC LIFT

Hydraulic lift is used to support or lift heavy objects and works on the principle of Pascal's Law. This principle is used in a hydraulic jack or lift, as shown in Figure.



The pressure due to a small force F_1 applied to a piston of area A_1 is transmitted to the larger piston of area A_2. The pressure at the two pistons is the same because they are at the same level.

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

$$\Rightarrow \quad F_2 = \left(\frac{A_2}{A_1}\right) F_1$$

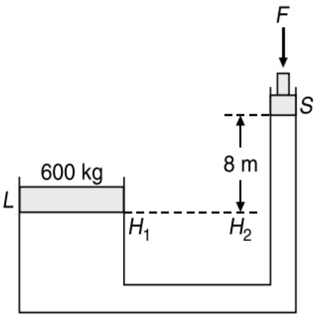
Consequently, the force on the larger piston is large.

Thus, a small force F_1 acting on small area A_1 results in a larger force F_2 acting on larger area A_2. This ratio of the larger force to the smaller force is called the Mechanical Advantage (MA). It is always greater than 1. So,

$$\text{MA} = \frac{F_{\text{larger}}}{F_{\text{smaller}}} = \frac{A_{\text{larger}}}{A_{\text{smaller}}}$$

ILLUSTRATION 43

For the system shown in Figure, the cylinder on the left, at L, has a mass of 600 kg and a cross-sectional area of 800 cm^2. The piston on the right, at S, has cross-sectional area 25 cm^2 and negligible weight. If the apparatus is filled with oil $\left(\rho = 0.78 \text{ gcm}^3\right)$, find the force F required to hold the system in equilibrium as shown in Figure.



SOLUTION

The pressures at point H_1 and H_2 are equal because they are at the same level in the single connected fluid. Therefore,

Pressure at H_1 = Pressure at H_2

$$\Rightarrow \begin{pmatrix}\text{Pressure}\\ \text{due to}\\ \text{left piston}\end{pmatrix} = \begin{pmatrix}\text{Pressure due}\\ \text{to } F \text{ and}\\ \text{right piston}\end{pmatrix} + \begin{pmatrix}\text{Pressure}\\ \text{due to}\\ \text{8 m of oil}\end{pmatrix}$$

$$\Rightarrow \frac{(600)(9.8)}{0.08} = \frac{F}{25\times10^{-4}} + (8)(780)(9.8)$$

$$\Rightarrow F = 31 \text{ N}$$

ILLUSTRATION 44

Two pistons of a hydraulic machine have diameters of 30 cm and 2.5 cm. What is the force exerted on the larger piston when 40 kg wt is placed on the smaller piston? If the smaller piston moves in through 6 cm, how much does the other piston move out?

SOLUTION

For smaller piston, Area $a_1 = \pi\times(1.25)^2\ \text{cm}^2$

For larger piston, Area $a_2 = \pi\times(15)^2\ \text{cm}^2$

Mechanical advantage at the larger piston is $\frac{a_2}{a_1}$

$$\Rightarrow F_2 = \frac{a_2}{a_1}\times F_1 = \frac{\pi(15)^2}{\pi(1.25)^2}\times 40 \text{ kg wt}$$

$$\Rightarrow F_2 = \frac{225}{1.25\times1.25}\times40\times9.8 \text{ N}$$

$$\Rightarrow F_2 = 56,448 \text{ N}$$

This is the force exerted on the larger piston.

The liquids are considered incompressible. Therefore, volume covered by the movement of smaller piston inwards is equal to that moved outwards by larger piston.

$$\Rightarrow l_1 a_1 = l_2 a_2$$

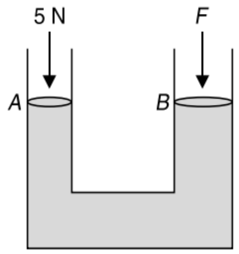
$$\Rightarrow l_2 = \frac{a_1}{a_2} l_1 = \frac{(1.25)^2}{(15)^2}\times 6 \text{ cm}$$

$$\Rightarrow l_2 = 0.042 \text{ cm}$$

So, the distance moved out by the larger piston is 0.042 cm.

ILLUSTRATION 45

The area of cross section of the two arms of a hydraulic pressure are $5\ \text{cm}^2$ and $15\ \text{cm}^2$ respectively (shown in Figure). A force of 5 N is applied on water in the thicker arm so that the water may remain in equilibrium?



SOLUTION

In equilibrium, the pressures at the two surfaces should be equal as they lie in the same horizontal level. If atmospheric pressure is P_0 and force F is applied to maintain the equilibrium then

$$P_A = P_B \qquad \ldots(1)$$

The pressure at point A, $P_A = P_0 + \frac{5 \text{ N}}{5\times10^{-4}\ \text{m}^2}$

Pressure at point B, $P_B = P_0 + \frac{F}{15\times10^{-4}\ \text{m}^2}$

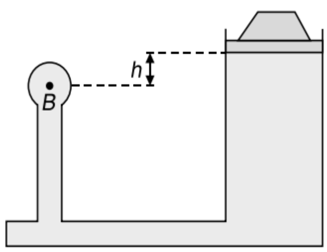
From equation (1), i.e. $P_A = P_B$, we get

$$\Rightarrow P_0 + \frac{5 \text{ N}}{5\times10^{-4}\ \text{m}^2} = P_0 + \frac{F}{15\times10^{-4}\ \text{m}^2}$$

$$\Rightarrow F = \frac{5}{5\times10^{-4}}\times15\times10^{-4} = 15 \text{ N}$$

ILLUSTRATION 46

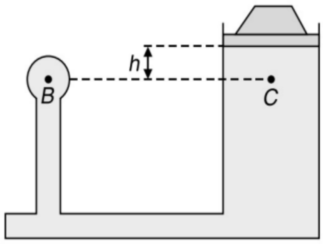
A weighted piston confines a fluid of density ρ in a closed container, as shown in Figure. The combined weight of piston and weight is $W = 200$ N, and the cross-sectional area of the piston is $A = 8\ \text{cm}^2$. Find the total pressure at point B if the fluid is mercury and $h = 25$ cm $(\rho_m = 13600\ \text{kgm}^{-3})$. What would an ordinary pressure gauge read at B?



SOLUTION

Since we know that the pressure at points which are at same level is the same, so we have

$$P_B = P_C$$



where, $P_C = P_0 + \frac{W}{A} + h\rho g$

$$\Rightarrow \quad P_B = P_C = P_0 + \frac{W}{A} + h\rho g \text{, where}$$

Atmospheric pressure is $P_0 = 1 \times 10^5$ Pa
Pressure due to piston and weight is

$$\frac{W}{A} = \frac{200 \text{ N}}{8 \times 10^{-4} \text{ m}^2} = 2.5 \times 10^5 \text{ Pa}$$

Pressure due to height h of fluid is

$$h\rho g = 0.33 \times 10^5 \text{ Pa}$$

$$\Rightarrow \quad P_B = P_0 + \frac{W}{A} + h\rho g$$

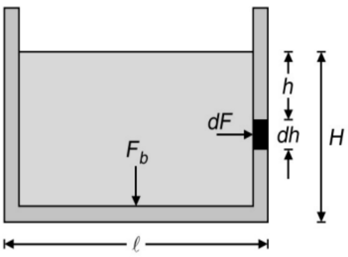
$$\Rightarrow \quad P_B = 3.8 \times 10^5 \text{ Pa} = 380 \text{ kPa}$$

Since the gauge pressure does not include atmospheric pressure, so we have gauge pressure at

$$\Delta P_B = P_B - P_0 = 280 \text{ kPa}$$

FORCE AND TORQUE DUE TO HYDROSTATIC PRESSURE

Whenever a fluid comes in contact with solid boundaries it exerts a force on it. Consider a rectangular vessel of base size $l \times b$ filled with water to a height H as shown in Figure.



The force F_b acting at the base of the container is given by

$$F_b = P_{\text{base}} (\text{Area of the base})$$

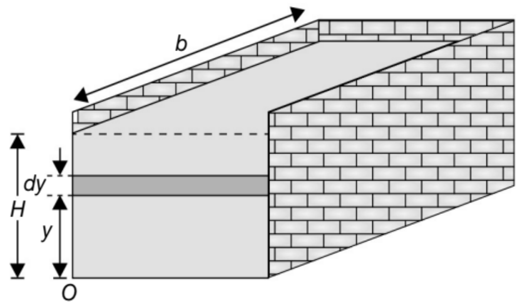
because pressure is same everywhere at the base and is equal to $P_{\text{base}} = \rho g H$.
Therefore, $F_b = \rho g H (lb) = \rho g l b H$
In this particular case, we see that

$$F_b = \rho g V = \begin{pmatrix} \text{weight of the liquid} \\ \text{inside the vessel} \end{pmatrix}$$

where $V = lbH$ is the volume of the liquid inside the beaker.

Unlike the base, the pressure on the vertical wall of the vessel is not uniform but increases linearly with depth from the free surface. Therefore, we have to perform the integration to calculate the total force on the wall.

Suppose water stands at a depth H behind the vertical face of a dam. It exerts a certain resultant force on the dam tending to slide it and a certain torque tending to overturn the dam's wall. Let's assume width of dam is b as shown in Figure.



The pressure at height y is

$$P = \rho g (H - y)$$

Atmospheric pressure can be omitted since it acts against the other face of the dam also. The force against shaded strip is

$$dF = P dA = \rho g (H - y) b\, dy$$

and the total force is

$$F = \int_0^H \rho g b (H - y) dy = \frac{\rho g b H^2}{2}$$

The **total force** acting per unit width of the vertical wall is

$$\frac{F}{b} = \frac{1}{2} \rho g H^2$$

The moment of the force dF about an axis through O is

$$d\tau = y\, dF$$

$$\Rightarrow \quad d\tau = \rho g b y (H - y) dy$$

The total torque about O is

$$\tau = \int d\tau = \int_0^H \rho g b y (H - y) dy = \frac{1}{6} \rho g b H^3$$

The torque due to hydrostatic forces per unit width of the wall is

$$\frac{\tau}{b} = \frac{\rho g H^3}{6}$$

If H' is the height above O at which the total force F would have to act to produce this torque

$$H' = \frac{1}{3} H$$

Hence, the line of action of the resultant force is at the depth of $\frac{2H}{3}$.

Alternatively, we can keep in mind that the point of application (the centre of force) of the total force from the free surface is given by

$$h_c = \frac{1}{F}\int_0^H h\,dF$$

where $\int_0^H hdF$ is the **moment of force** about the free surface. Here,

$$\int_0^H hdF = \int_0^H h(\rho gbhdh) = \rho gb\int_0^H h^2 dh = \frac{1}{3}\rho gbH^3$$

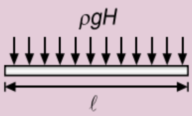
Since, $F = \frac{1}{2}\rho gbH^2$, therefore,

$$h_c = \frac{2}{3}H$$

The **net resultant force** acts at a depth $2H/3$ from the free surface.

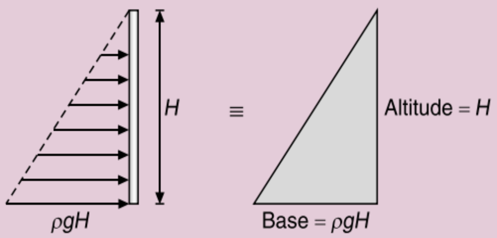
Conceptual Note(s)

(a) The force acting at the base per unit width $\left(\frac{F}{b}\right)$ is equal to the area of the pressure diagram as shown in Figure. That is,



$$\frac{F}{b} = P\ell = \rho gH\ell$$

The force acting per unit width is equal to the area of the pressure diagram shown in Figure.



$$\frac{F}{b} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(\rho gH)(H) = \frac{1}{2}\rho gH^2$$

Hence, force per unit width of an immersed surface is equal to the area of the pressure diagram on the surface.

(b) Please note that, for the Figures shown, torque due to hydrostatic force about point O, the centre of a semi-cylindrical (or hemispherical) gate is zero as the hydrostatic force at all points passes through point O.

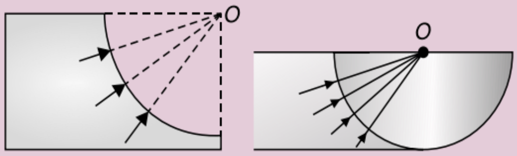
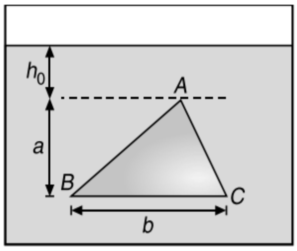


ILLUSTRATION 47

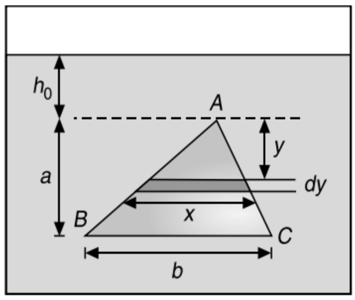
A triangular plate is submerged completely in a liquid of density ρ as shown in Figure. Calculate the hydrostatic force acting on one face of the plate.



SOLUTION

METHOD I:

Consider an infinitesimal strip of width dy, length x at a distance y from the edge A of the triangular plate as shown in Figure.



Pressure at the position of this infinitesimal strip is

$$P = (h_0 + y)\rho g$$

Force on this elemental strip is

$$dF = PdA = (h_0 + y)\rho g(xdy)$$

$$F = \int dF = \int_0^a \rho g(h_0 + y)xdy \quad \ldots(1)$$

Using similarity of triangles, we get

$$\frac{x}{b} = \frac{y}{a}$$

$$\Rightarrow \quad x=\left(\frac{b}{a}\right)y \qquad \ldots(2)$$

Substituting (2) in (1) and integrating, we get

$$F=\int dF=\int_0^a \rho g\left(h_0+y\right)\left(\frac{b}{a}y\right)dy$$

$$\Rightarrow \quad F=\frac{\rho g b}{a}\left[\int_0^a\left(h_0 y dy+y^2 dy\right)\right]$$

$$\Rightarrow \quad F=\frac{\rho g b}{a}\left(\frac{h_0 a^2}{2}+\frac{a^3}{3}\right)$$

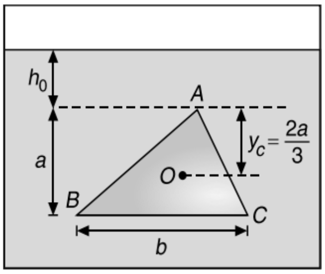
$$\Rightarrow \quad F=\rho g a b\left(\frac{a}{3}+\frac{h_0}{2}\right)$$

METHOD II:

The horizontal force F_H on the plate due to water is

$$F_H=\left(\begin{array}{c}\text{Pressure at}\\ \text{Centroid of Plate}\end{array}\right)\left(\begin{array}{c}\text{Area}\\ \text{of Plate}\end{array}\right)$$

$$\Rightarrow \quad F_H=P_O A$$



The centroid of the plate is at a distance $\frac{2a}{3}$ from the edge A of the plate. So, pressure at the centroid of the plate is given by

$$P_O=\rho g\left(h_0+\frac{2a}{3}\right)$$

Area A of the plate is given by

$$A=\frac{1}{2}ab$$

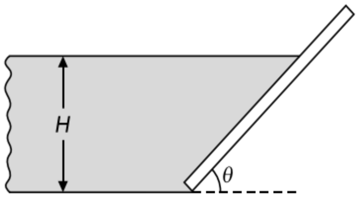
Hence the horizontal force on the plate is

$$\Rightarrow \quad F_H=P_O A=\rho g\left(\frac{2a}{3}+h_0\right)\left(\frac{ab}{2}\right)$$

$$\Rightarrow \quad F_H=\rho g a b\left(\frac{a}{3}+\frac{h_0}{2}\right)$$

ILLUSTRATION 48

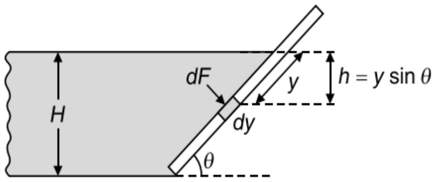
Find the force acting per unit width on a plane wall inclined at an angle θ with the horizontal as shown in Figure.



SOLUTION

METHOD-I:

Consider a small element of thickness dy at a distance y measured along the wall from the free surface as shown in Figure.



The pressure at the position of the element is

$$P=\rho g h=\rho g\left(y\sin\theta\right)$$

The force is given by

$$dF=PdA=p\left(bdy\right)=\rho g b\left(ydy\right)\sin\theta$$

The **total force** per unit width b is given by

$$\frac{F}{b}=\rho g\sin\theta\int_0^{\frac{H}{\sin\theta}} y dy=\rho g\sin\theta\left(\frac{y^2}{2}\right)\Bigg|_0^{\frac{H}{\sin\theta}}$$

$$\Rightarrow \quad \frac{F}{b}=\frac{1}{2}\left(\frac{\rho g H^2}{\sin\theta}\right)$$

Note that the above formula reduces to $\frac{1}{2}\rho g H^2$ for a vertical wall $\left(\theta=90^\circ\right)$ as discussed earlier.

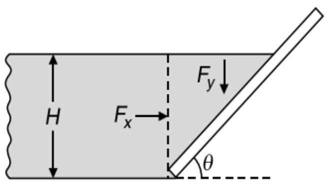
METHOD-II:

Alternatively, the force on the inclined wall may be obtained in two parts viz. horizontal and vertical.

The **horizontal force** F_x acts on the vertical projection of the incline wall, i.e., $F_x=\frac{1}{2}\rho g b H^2$

The **vertical force** F_y acts due to weight of the liquid supported by the wall, i.e.,

$$F_y=\frac{1}{2}\rho g b(H)(H\cot\theta)=\frac{1}{2}\rho g b H^2\cot\theta$$



The **magnitude** of the resultant force is given by

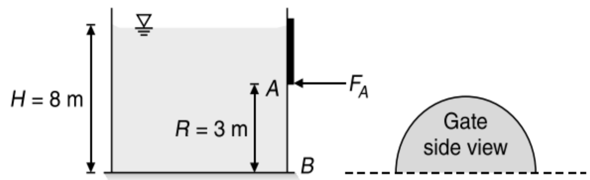
$$F = \sqrt{F_x^2 + F_y^2} = \frac{1}{2}\rho g b H^2 \operatorname{cosec}\theta$$

$$\Rightarrow \quad F = \frac{1}{2}\rho g \frac{bH^2}{\sin\theta}$$

$$\Rightarrow \quad \frac{F}{b} = \frac{1}{2}\left(\frac{\rho g H^2}{\sin\theta}\right)$$

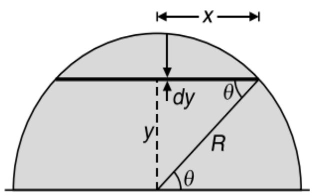
ILLUSTRATION 49

Semi-circular plane gate AB is hinged along B and held by horizontal force F_A applied at A. The liquid to the left of the gate is water, calculate the force F_A required for equilibrium. Take $g = 10\ \text{ms}^{-2}$ and density of water is $10^3\ \text{kgm}^3$.



SOLUTION

For equilibrium of the gate, net clockwise torque due to hydrostatic force is balanced by the anticlockwise torque due to force F_A.



Torque due to hydrostatic force

$$y = R\sin\theta$$

$$\Rightarrow \quad dy = R\cos\theta d\theta$$

$$x = R\cos\theta$$

$$\Rightarrow \quad dA = 2xdy = (2R\cos\theta)(R\cos\theta)d\theta$$

$$\Rightarrow \quad dA = (2R^2\cos^2\theta)d\theta$$

$$\Rightarrow \quad dF = \rho g(8-y)dA$$

$$\Rightarrow \quad dF = \rho g(8 - R\sin\theta)(2R^2\cos^2\theta)d\theta$$

$$\Rightarrow \quad dF = 2\rho g R^2(8 - R\sin\theta)(\cos^2\theta)d\theta$$

$$\Rightarrow \quad d\tau = ydF = 2\rho g R^3(8 - R\sin\theta)(\sin\theta\cos^2\theta)d\theta$$

$$\Rightarrow \quad \tau_C = \int_0^{\pi/2} d\tau$$

$$\Rightarrow \quad \tau_C = 2\rho g R^3 \int_0^{\pi/2} (8 - R\sin\theta)(\sin\theta\cos^2\theta)d\theta$$

Substituting the values, $\rho = 10^3\ \text{kgm}^{-3}$, $g = 10\ \text{ms}^{-2}$ and $R = 3\ \text{m}$, we get

$$\tau_C = 2\times10^3\times10\times(3)^3 \int_0^{\pi/2} (8 - 3\sin\theta)(\sin\theta\cos^2\theta)d\theta$$

$$\Rightarrow \quad \tau_C = 5.4\times10^5\left(\frac{8}{3} - \frac{3\pi}{16}\right)\text{Nm}$$

$$\Rightarrow \quad \tau_C = 1.12\times10^6\ \text{Nm} \qquad \ldots(1)$$

Anticlockwise torque due to F_A

$$\tau_A = F_A r_\perp = 3F_A \qquad \ldots(2)$$

Equating equations (1) and (2), we get

$$F_A = 3.73\times10^5\ \text{N} \approx 373\ \text{kN}$$

THE COMPRESSIBLE FLUID MODEL

For gases, the constant density assumed in the compressible model is often not adequate. However, an alternative simplifying assumption can be made that the density is proportional to the pressure, i.e.,

$$\rho = kP$$

Let ρ_0 be the density of air at the earth's surface where the pressure is atmospheric p_0, then

$$\rho_0 = kP_0$$

After eliminating k, we get

$$\rho = \left(\frac{P}{P_0}\right)\rho_0$$

Substituting the value of ρ in equation $dP = -\rho g dy$, we get

$$dP = -\left(\frac{P}{P_0}\right)\rho_0 g dy$$

$$\Rightarrow \quad \int_{p_o}^{p}\frac{dP}{P} = -\frac{\rho_0 g}{P_0}\int_0^h dy$$

where P is the pressure at a height $y = h$ above the earth's surface. After integrating, we get

$$\ln\left|\frac{P}{P_0}\right| = -\left(\frac{\rho_0 g}{P_0}\right)h$$

$$\Rightarrow \quad P = P_0 e^{-\left(\frac{\rho_0 g}{P_o}\right)h}$$

Note that instead of a linear decrease in pressure with increasing height as in the case of an incompressible fluid, in this case pressure decreases exponentially.

ILLUSTRATION 50

The density of air in atmosphere decreases with height and can be expressed by the relation

$$\rho = \rho_0 e^{-Ah}$$

where ρ_0 is the density at sea-level, A is a constant and h is the height. Calculate the atmospheric pressure at sea-level. Assume g to be constant. $g = 9.8\ \text{ms}^{-2}$, $\rho_0 = 1.3\ \text{kgm}^{-3}$ and $A = 1.2\times10^{-4}\ \text{m}^{-1}$.

SOLUTION

Considering an atmospheric layer at a height y of width dy, the pressure due to this layer is

$$dP = \rho g dy$$

$$\Rightarrow \quad dP = \rho_0 e^{-Ay} g dy$$

$$\Rightarrow \quad P = \int dP = \rho_0 g \int_0^\infty e^{-Ay} dy$$

$$\Rightarrow \quad P = \rho_0 g \left(-\frac{1}{A} e^{-Ay} \right)\Bigg|_0^\infty = -\frac{\rho_0 g}{A}\left(e^{-\infty} - e^0\right)$$

$$\Rightarrow \quad P = \frac{\rho_0 g}{A} = \frac{1.3\times9.8}{1.2\times10^{-4}}$$

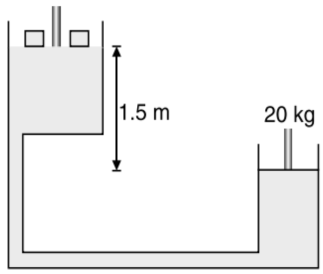
$$\Rightarrow \quad P = 1.0616\times10^5\ \text{Nm}^{-2}$$

Test Your Concepts-V

Based on Fluid Properties, Pressure and Pascal's Law

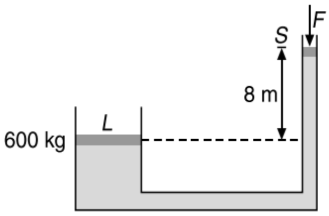
(Solutions on page H.7)

1. Relative density of an oil is 0.8. Find the absolute density of oil in CGS and SI unit.
2. The mass of a litre of milk is $1.032\ \text{kg}$. The butterfat that it contains has a density of $865\ \text{kgm}^{-3}$ when pure, and it constitutes 4 percent of the milk by volume. Calculate the density of the fat-free skimmed milk?
3. Atmospheric pressure is about $1.01\times10^5\ \text{Pa}$. How large a force does the atmosphere exert on a $2\ \text{cm}^2$ area patch on the top of your head assuming it to be flat?
4. Two vessels have the same base area but different shapes. The first vessel takes twice the volume of water that the second vessel requires to fill up to a particular common height. Is the force exerted by water on the base of the vessel the same in the two cases? If so, why do the vessels filled with water to that same height give different readings on a weighing scale?
5. A beaker of circular cross-section of radius $4\ \text{cm}$ is filled with mercury up to a height of $10\ \text{cm}$. Find the force exerted by the mercury at the bottom of beaker. The atmospheric pressure is $10^5\ \text{Nm}^{-2}$, specific gravity of mercury is 13.6 and $g = 10\ \text{ms}^{-2}$.
6. At a depth of $500\ \text{m}$ in an ocean, what is the absolute pressure? Given that the density of sea water is $1.03\times10^3\ \text{kgm}^{-3}$ and $g = 10\ \text{ms}^{-2}$.
7. An inverted bell lying at the bottom of a lake $47.6\ \text{m}$ deep has $50\ \text{cm}^3$ of air trapped in it. The bell is brought gradually to the surface of the lake. Find the volume of the trapped air if atmospheric pressure is taken to be $70\ \text{cm of Hg}$ and density of $Hg = 13.6\ \text{gcm}^{-3}$.
8. To what height should a cylindrical vessel of radius R be filled with a homogeneous liquid such that the force exerted by the liquid on the curved surface of the vessel is equal to the force exerted by the liquid at the bottom of the vessel?
9. The height of a mercury barometer is $75\ \text{cm}$ at sea level and $50\ \text{cm}$ at the top of a hill. If the ratio of density of mercury to that of air is 10^4, calculate the height of this hill.
10. A hydraulic press has a larger piston of diameter 35 cm at a height of 1.5 m relative to the smaller piston of diameter 10 cm as shown in Figure. The mass on the smaller piston is 20 kg. Calculate the force exerted on the load by the larger piston if the density of oil in the press is $750\ \text{kgm}^{-3}$ and $g = 9.8\ \text{ms}^{-2}$.



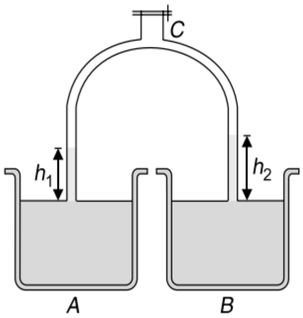
11. Find the pressure in the air column at which the piston remains in equilibrium. Assume the piston to be massless and frictionless.
12. An open U-tube of uniform cross-section contains mercury. When $27.2\ \text{cm}$ of water is poured into one limb of the tube, how high does the mercury rise in the other limb from its initial level? Also calculate the difference in levels of liquids of the two sides if specific gravity of water is 1 and that of mercury is 13.6.
13. The upper edge of a gate in a dam runs along water surface. The gate is $2\ \text{m}$ high and $3\ \text{m}$ wide and is hinged along a horizontal line through its center. Calculate the torque about hinge.

14. For the system shown in Figure, the cylinder on the left at L has a mass of 600 kg and a cross sectional area of 800 cm^2.



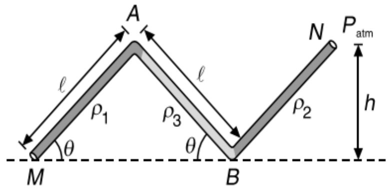
The piston on the right, at S, has cross sectional area 25 cm^2 and negligible weight. If the apparatus is filled with oil $(\rho = 0.75 \text{ gcm}^{-3})$. Find the force F required to hold the system in equilibrium. Take $g = 10 \text{ ms}^{-2}$.

15. The limbs of a glass U-tube are lowered into vessels A and B (Figure).

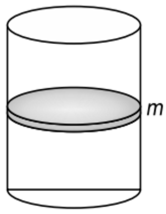


Some air is pumped out through the top of the tube C. The liquid in the left-hand limb then rises to a height h_1 and in the right-hand one to a height h_2. Determine the density of the liquid in limb B if water is present in limb A, $h_1 = 10$ cm and $h_2 = 12$ cm.

16. A zig-zag tube open at N, having liquids of densities ρ_1, ρ_2 and ρ_3, is placed in a vertical plane as shown in Figure. (The pressure at M is equal to atmospheric pressure). Calculate the pressures at points A and B. Also find the angle θ.

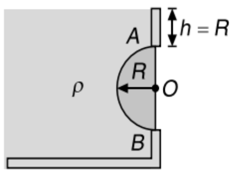


17. A cylindrical vessel containing a liquid is closed by a smooth piston of mass m as shown in Figure.



The area of cross-section of the piston is A. If the atmospheric pressure is P_0, find the pressure of the liquid just below the piston.

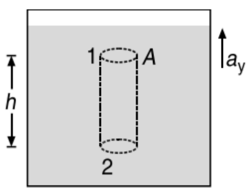
18. A semi-cylindrical massless gate of length ℓ, radius R pivoted at the point O is holding a stationary liquid of density ρ as shown in Figure. Calculate, the horizontal force exerted by the liquid on the gate.



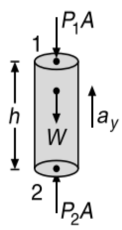
LIQUIDS IN ACCELERATED CONTAINERS

Container Having Vertical Acceleration

Consider a container shown in Figure. The container is accelerating upwards with an acceleration a_y.



Now to write pressure at point P at depth h. Lets consider a hypothetical cylinder of liquid with area of cross section A and length h. If m be the mass of the cylinder, then $m = Ah\rho$.



This hypothetical cylinder of liquid is moving up with an acceleration a_y, so on applying Newton's Second Law, we get

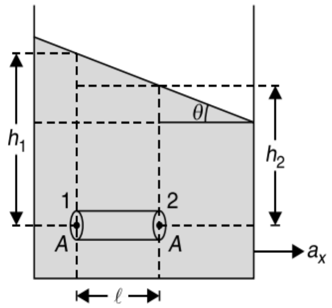
$$P_2A - (P_1A + W) = ma_y$$

$$\Rightarrow \quad P_2A - Ah\rho g - P_1A = (Ah\rho)a_y$$

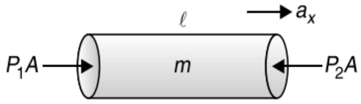
$$\Rightarrow \quad P_2 - P_1 = h\rho(g + a_y)$$

Container Having Horizontal Acceleration

If container has acceleration a_x in the horizontal direction, then its free surface gets tilted as shown in Figure.



Let us apply Newton's Second Law on an imaginary cylinder of liquid of length l and area of cross-section A as shown in Figure.



$$\Rightarrow \quad (P_1 - P_2)A = \rho(Al)a_x$$

$$\Rightarrow \quad P_1 - P_2 = l\rho a_x$$

Now it can be seen very clearly that pressure on a horizontal level is not same throughout. So, we have

$$P_1 = h_1\rho g \text{ and } P_2 = h_2\rho g$$

$$\Rightarrow \quad P_1 - P_2 = (h_1 - h_2)\rho g = l\rho a_x$$

Also, from the Figure, we see that

$$\tan\theta = \frac{h_1 - h_2}{l}$$

$$\Rightarrow \quad \tan\theta = \frac{a_x}{g}$$

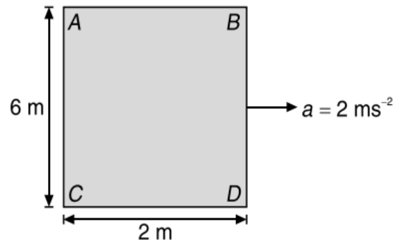
Problem Solving Technique(s)

If the beaker is accelerating both along x and y directions with respective accelerations a_x and a_y, then the liquid surface makes an angle θ with the horizontal given by

$$\tan\theta = \frac{a_x}{g_{\text{eff}}} = \frac{a_x}{g + a_y}$$

ILLUSTRATION 51

A closed container is filled with water as shown in figure. This container is accelerated in horizontal direction with an acceleration, $a = 2 \text{ ms}^{-2}$. Taking the density of water to be 1000 kgm^{-3}, calculate $P_C - P_D$ and $P_A - P_D$.



SOLUTION

Since the container is closed. So, we do not need to take the atmospheric pressure inside the container.
Along the horizontal direction, pressure decreases in the direction of acceleration, so $P_C > P_D$.

$$\Rightarrow \quad P_C - P_D = +\rho ax$$

Substituting the values, we get

$$P_C - P_D = (10^3)(2)(2)$$

$$\Rightarrow \quad P_C - P_D = 4.0 \times 10^3 \text{ Nm}^{-2}$$

In vertical direction, pressure increases with depth, so $P_C > P_A$.

$$\Rightarrow \quad P_A - P_C = -\rho gh = -(10^3)(10)(6)$$

$$\Rightarrow \quad P_A - P_C = -60 \times 10^3 \text{ Nm}^{-2}$$

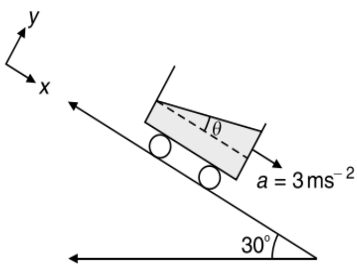
Now, $P_A - P_D = (P_A - P_C) + (P_C - P_D)$

$$\Rightarrow \quad P_A - P_D = (-60 \times 10^3) + (4.0 \times 10^3)$$

$$\Rightarrow \quad P_A - P_D = -56 \times 10^3 \text{ Nm}^{-2}$$

ILLUSTRATION 52

A rectangular container of water undergoes constant acceleration down an incline as shown. Determine the slope $\tan\theta$ of the free surface using the coordinate system shown. Take $g = 10 \text{ ms}^{-2}$.



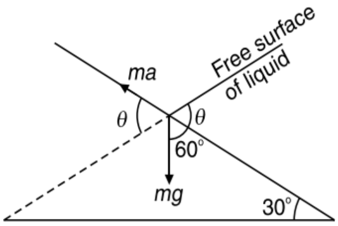
SOLUTION

No force acts along the fluid surface, so net force on a fluid particle of mass m at the surface of the liquid should be perpendicular to its surface when seen from accelerating frame of reference. Two forces are acting on the fluid particle are

(i) weight (mg), acting vertically downwards.

(ii) pseudo force (ma) along negative x-direction.

Since we know that the resultant of these two should be perpendicular to the free surface i.e. along the free surface the components of these two forces should cancel each other.



$$\Rightarrow \quad ma\cos\theta = mg\cos(60^\circ + \theta)$$

$$\Rightarrow \quad 3\cos\theta = 10(\cos 60^\circ \cos\theta - \sin 60^\circ \sin\theta)$$

$$\Rightarrow \quad 3\cos\theta = 5\cos\theta - 5\sqrt{3}\sin\theta$$

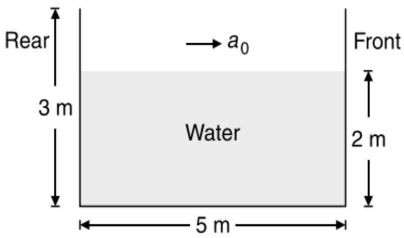
$$\Rightarrow \quad 5\sqrt{3}\sin\theta = 2\cos\theta$$

$$\Rightarrow \quad \tan\theta = \frac{2}{5\sqrt{3}} = 0.23$$

Therefore, the desired slope is 0.23.

ILLUSTRATION 53

An open rectangular tank $5\text{ m} \times 4\text{ m} \times 3\text{ m}$ high containing water up to a height of 2 m is accelerated horizontally along the longer side as shown in Figure.



Calculate the maximum acceleration that can be given without spilling the water. If this acceleration is increased by 20%, calculate the percentage of water spilt over.

If initially, the tank is closed at the top and is accelerated horizontally by 9 ms^{-2}, find the gauge pressure at the bottom of the front and rear walls of the tank. (Take $g = 10\text{ ms}^{-2}$)

SOLUTION

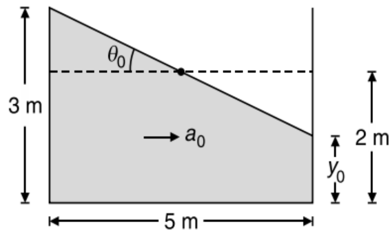
Volume of water inside the tank remains constant

$$\left(\frac{3 + y_0}{2}\right)5 \times 4 = 5 \times 2 \times 4$$

$$\Rightarrow \quad y_0 = 1\text{ m}$$

$$\Rightarrow \quad \tan\theta_0 = \frac{3-1}{5} = 0.4$$

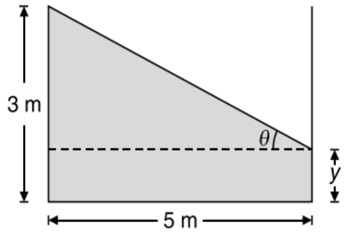
Since, $\tan\theta_0 = \frac{a_0}{g}$, therefore $a_0 = 0.4g = 4\text{ ms}^{-2}$



When acceleration is increased by 20%, then

$$a = 1.2a_0 = 0.48g$$

$$\Rightarrow \quad \tan\theta = \frac{a}{g} = 0.48$$



Now, $y = 2 - 5\tan\theta = 3 - 5(0.48) = 0.6\text{ m}$

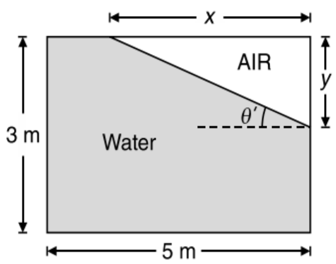
Fraction (f) of water spilt over is

$$f = \frac{(4)(2)(5) - \left(\frac{3+0.6}{2}\right)(5)(4)}{(2)(5)(4)} = 0.1$$

So, percentage of water spilt over is 10%

If $a' = 0.9g$, then we have

$$\tan\theta' = \frac{a'}{g} = 0.9 \qquad \ldots(1)$$



Volume of air remains constant

$$4 \times \frac{1}{2}yx = (5)(1) \times 4$$

$$\Rightarrow \quad xy = 10 \qquad \ldots(2)$$

Since $y = x\tan\theta'$

$$\Rightarrow \quad xy = x^2\tan\theta'$$

So, from equation (2), we get

$$\Rightarrow \quad x^2\tan\theta' = 10$$

Using equations (1) and (2), we get

$$x = 3.33\text{ m} \text{ and } y = 3.0\text{ m}$$

Hence, gauge pressure at the bottom of the

(i) Front wall is

$$p_f = \text{zero}$$

(ii) Rear wall is

$$p_r = (5\tan\theta')\rho_w g = 5(0.9)(10^3)(10)$$

$$\Rightarrow p_r = 4.5\times10^4\text{ Pa}$$

HORIZONTALLY ACCELERATING *U*-TUBE

When the U tube accelerates horizontally, difference of levels of liquid satisfies the relation,

$$\tan\theta = \frac{a}{g} = \frac{h}{L}$$

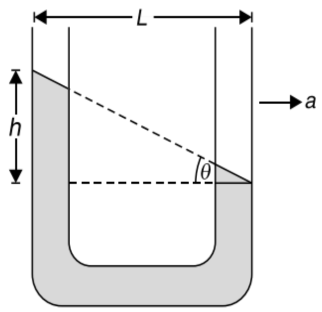
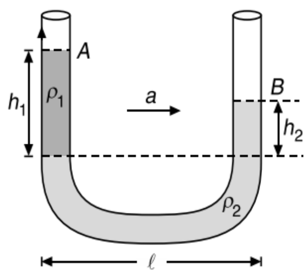


ILLUSTRATION 54

A U-tube containing two liquids of densities ρ_1 and ρ_2. The tube is given an acceleration a in the horizontal direction as shown in Figure.



Calculate the ratio of the densities of liquids.

SOLUTION

Let A and B be the two free liquid levels in the two limbs. Before we solve the problem, we must keep in mind that

1. As we go away from the free surface inside the liquid, pressure increases.
2. As we go towards the free surface inside the liquid, pressure decreases.
3. As we move in the direction of acceleration, the pressure decreases inside the liquid.

Let us start from point A and go to the point B through the tube (i.e. inside the liquid). If P_0 be the atmosphere pressure, then we have

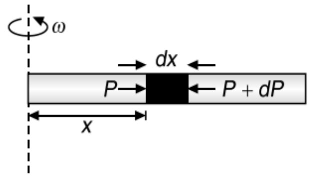
$$P_0 + h_1\rho_1 g - l\rho_2 a - h_2\rho_2 g = P_0$$

$$\Rightarrow h_1\rho_1 g = l\rho_2 a + h_2\rho_2 g = \rho_2(la + h_2 g)$$

$$\Rightarrow \frac{\rho_1}{\rho_2} = \frac{la + h_2 g}{h_1 g}$$

PRESSURE DIFFERENCE IN ROTATING FLUIDS

Consider a liquid of density ρ to be kept inside a tube of area of cross-section A. Let the tube be rotating with an angular velocity ω as shown in Figure.



Consider a small element of length dx at a distance x from the axis of rotation. Mass of this element is,

$$dm = \rho dV = \rho(Adx)$$

This element is rotating in a circle of radius x. So, this element must be accelerated towards centre with centripetal acceleration given by

$$a_c = x\omega^2 \qquad \{\because a_c = R\omega^2\}$$

For providing this acceleration, pressure on right hand side of the element should be more than that on the left-hand side, so we have

$$(P + dP)A - PA = (dm)a = (\rho Adx)(x\omega^2)$$

$$\Rightarrow dP = (\rho x\omega^2)dx$$

$$\Rightarrow \Delta P = \int_0^x (\rho\omega^2)xdx$$

$$\Rightarrow \Delta P = \frac{\rho\omega^2 x^2}{2}$$

The pressure difference between two points at distances x_1 and $x_2 (> x_1)$ will be

$$\Delta P = \frac{1}{2}\rho\omega^2(x_2^2 - x_1^2)$$

Conceptual Note(s)

For a rotating fluid (also accelerating) pressure increases in moving away from the rotational axis. At a distance x from the rotational axis, pressure difference is given by

$$\Delta P = \pm\frac{\rho\omega^2 x^2}{2}$$

When moving away from the axis of rotation, the pressure increases, so we take

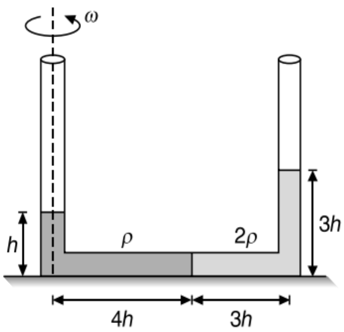
$$\Delta P = \frac{\rho\omega^2 x^2}{2}$$

When moving towards the axis of rotation, the pressure decreases, so we take

$$\Delta P = -\frac{\rho\omega^2 x^2}{2}$$

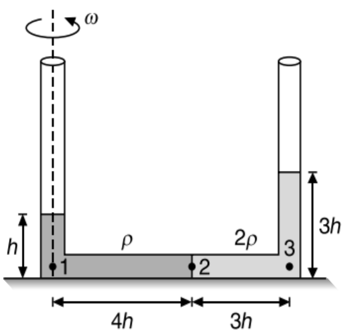
ILLUSTRATION 55

A vertical U-tube having two liquids of densities ρ and 2ρ is rotating with an angular speed ω about a vertical axis passing through its left limb such that the liquid columns in the limbs have heights h and $3h$ as shown in Figure. Calculate ω.



SOLUTION

Let P_1, P_2 and P_3 be the pressures at points 1, 2 and 3. The point 2 lies on the interface separating the two liquids as shown in Figure.



Since $\Delta P = \frac{\rho\omega^2}{2}\left(x_1^2 - x_2^2\right)$, so we have

$$P_2 - P_1 = \frac{1}{2}\rho\omega^2(4h)^2 = 8\rho\omega^2h^2 \quad \ldots(1)$$

Similarly,

$$P_3 - P_2 = \frac{1}{2}(2\rho)\omega^2\left[(7h)^2 - (4h)^2\right]$$

$$\Rightarrow \quad P_3 - P_2 = 33\rho\omega^2h^2 \quad \ldots(2)$$

Also $P_1 = h\rho g \quad \ldots(3)$

and $P_3 = (3h)(2\rho)g = 6h\rho g \quad \ldots(4)$

Adding equations (1) and (2), we get

$$P_3 - P_1 = 41\rho\omega^2h^2 \quad \ldots(5)$$

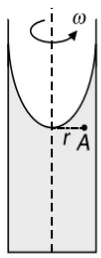
Substituting in equation (5) the value of P_1 from equation (3) and P_3 from equation (4), we get

$$5\rho gh = 41\rho\omega^2h^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{5g}{41h}}$$

ILLUSTRATION 56

A liquid of density ρ is in a bucket that spins with angular velocity ω. Calculate the pressure at a point A at radial distance r from the axis as shown in Figure, if p_0 is the atmospheric pressure.

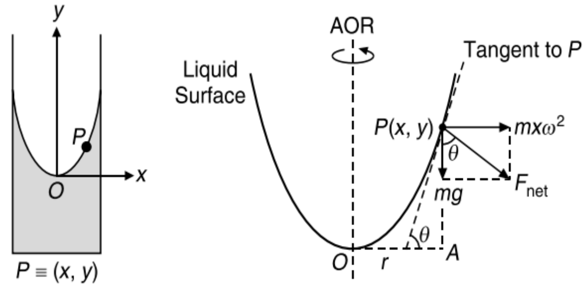


SOLUTION

Consider a fluid particle of mass m at the point $P(x, y)$. From a non-inertial rotating frame of reference two forces are acting on it,

(i) pseudo force $(mx\omega^2)$, radially outwards

(ii) weight (mg), vertically downwards



Net force F_{net} on the particle should be perpendicular to the free surface (in equilibrium). Hence,

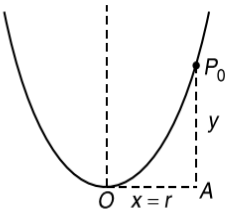
$$\tan\theta = \frac{mx\omega^2}{mg} = \frac{x\omega^2}{g}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x\omega^2}{g}$$

$$\Rightarrow \quad \int_0^y dy = \int_0^x \frac{x\omega^2}{g}dx$$

$\Rightarrow \quad y = \dfrac{x^2\omega^2}{2g}$

This is the equation of the free surface of the liquid, which is a parabola



At radial distance r, we have $x = r$

$\Rightarrow \quad y = \dfrac{r^2\omega^2}{2g}$

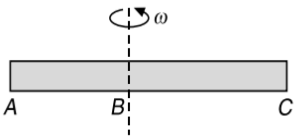
If P_A be the pressure in the liquid at the point A, then we have

$\Rightarrow \quad P_A = P_0 + \rho g y$

$\Rightarrow \quad P_A = P_0 + \dfrac{\rho\omega^2 r^2}{2}$

ILLUSTRATION 57

A closed tube of length 6 m filled with water $\left(\rho = 10^3\text{ kgm}^{-3}\right)$ is rotating with an angular velocity $\omega = 2\text{ rads}^{-1}$ about an axis perpendicular to the tube at 2 m from its one end as shown in Figure.



Calculate the pressure difference between the points A and C i.e. $P_A - P_C$.

SOLUTION

Since pressure decreases in moving towards the axis of rotation and increases in moving away from the axis i.e. $\Delta P = \pm\dfrac{\rho\omega^2 x^2}{2}$, so we must have $P_A > P_B$ and $P_B < P_C$.

Now, $P_A - P_C = \left(P_A - P_B\right) + \left(P_B - P_C\right)$

$$\Rightarrow \quad P_A = \left(+\frac{\rho\omega^2 x_1^2}{2}\right) + \left(\frac{-\rho\omega^2 x_2^2}{2}\right)$$

$$\Rightarrow \quad P_A = \frac{\rho\omega^2}{2}\left(x_1^2 - x_2^2\right)$$

where, $x_1 = AB = 2\text{ m}$ and $x_2 = BC = 4\text{ m}$

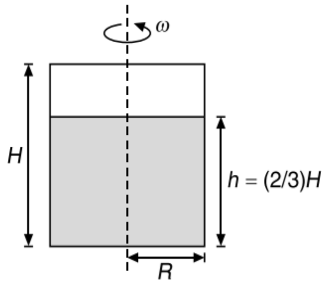
Substituting the values, we get

$$P_A - P_C = \frac{(10^3)(2)^2}{2}\left[(2)^2 - (4)^2\right]$$

$$\Rightarrow \quad P_A - P_C = -2.4 \times 10^3\text{ Nm}^{-2}$$

ILLUSTRATION 58

A cylinder of radius $R = 1\text{ m}$ and height $H = 3\text{ m}$, is filled with water to two thirds its height. It is rotated about its vertical axis, as shown in Figure.



Calculate the angular speed of rotation for which the water just starts spilling over the rim. Also calculate the angular speed of rotation for which the centre of base of the cylinder is just exposed.

SOLUTION

According to the problem, we have

$y_{max} = H = 3\text{ m}$

Also, $h = \dfrac{2}{3}H = 2\text{ m}$, $R = 1\text{ m}$ and $g = 10\text{ ms}^{-2}$

Since, the maximum height available to the liquid just before spilling is

$$y = H - h = \frac{r^2\omega^2}{2g}$$

$$3 - 2 = \frac{(1)^2\omega^2}{2(10)}$$

$$\Rightarrow \quad \omega = \sqrt{20}\text{ rads}^{-1}$$

For the centre of the base of the cylinder to be just exposed, we have

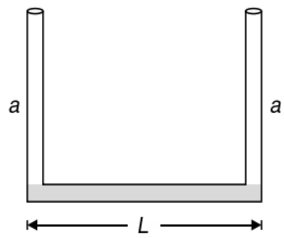
$y = H = 3\text{ m}$

$$\Rightarrow \quad y = H = \frac{r^2\omega^2}{2g}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{2gH}{R^2}} = \sqrt{\frac{2(10)(3)}{1^2}} = \sqrt{60}\text{ rads}^{-1}$$

ILLUSTRATION 59

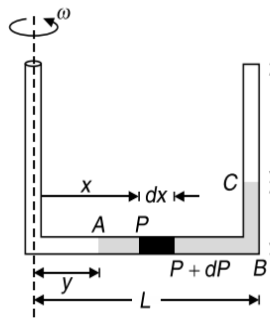
The length of horizontal arm of a U-tube is L and its each vertical arm has a length a. Both the vertical arm ends are open to atmospheric pressure P_0. A liquid of density ρ is poured in the tube such that liquid just fills the horizontal part of the tube as shown in Figure.



Now one of the open ends is sealed and the tube is rotated about the vertical axis passing through the other vertical arm with an angular speed ω_0. If liquid rises to a height y in the sealed end, then calculate pressure in the sealed tube during rotation.

SOLUTION

When tube is rotated, liquid starts to flow radially outward and air in sealed arm is compressed. Let the shift of liquid be y as shown in Figure.



Let the cross-sectional area of tube be S. The pressure difference between points A and B is obtained by integrating the pressure difference dP across an infinitesimal element of width dx, which is given as

$$dP = \rho\omega^2 x dx$$

Integrating from A to B, we get

$$P_B - P_A = \int_y^L \rho\omega^2 x dx$$

$$\Rightarrow \quad P_B - P_A = \frac{\rho\omega^2}{2}\left(L^2 - y^2\right) \quad \ldots(1)$$

Since at point A, the pressure is atmospheric, so we have $P_A = P_{atm} = P_0$. So, from equation (1), we get

$$P_B = P_A + \frac{\rho\omega^2}{2}\left(L^2 - y^2\right) \quad \ldots(2)$$

Also, pressure at point C is given by

$$P_C = P_B - y\rho g \quad \ldots(3)$$

Substituting the value of P_B from equation (2) in equation (3), we get

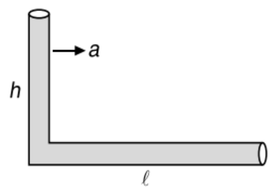
$$P_C = \frac{\rho\omega^2}{2}\left(L^2 - y^2\right) + \rho_0 - y\rho g$$

Test Your Concepts-VI

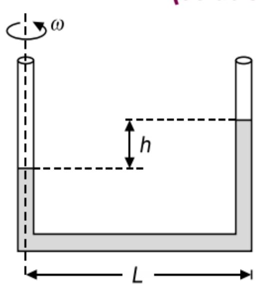
Based on Pressure in Accelerating Fluids

(Solutions on page H.10)

1. An L-shaped tube is filled with a liquid of density ρ as shown in Figure. Calculate the horizontal acceleration a to which the tube must be subjected towards right so that no liquid falls out of the tube.

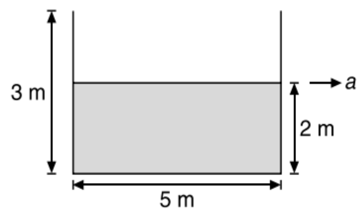


2. A cylindrical tank of radius 20 cm and height 50 cm has water up to 30 cm of height. Calculate the rise in level of liquid at the periphery if the cylinder if it is rotating with an angular velocity of 10 rad s^{-1} about the axis of the cylinder. Also find the frequency of rotation for which the water just starts spilling over sides of the vessel.
3. A U-tube of length L contains a liquid. It is rotated with angular velocity ω about an axis which is co-axial to one of its vertical limbs as shown in Figure.

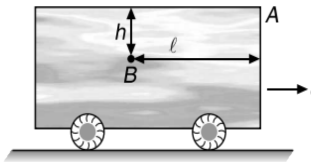


Calculate the difference in heights between the liquid columns in two vertical arms.

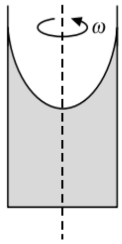
4. A container is accelerated horizontally with acceleration a as shown in Figure. Calculate the minimum value of a for which liquid just starts spilling out. Take $g = 10\ ms^{-2}$



5. A closed tank filled with water is mounted on a cart. The cart moves with an acceleration a on a plane road. Calculate the difference in pressure between points B and A shown in Figure.



6. A liquid of density ρ is in a bucket that spins with angular velocity ω as shown in figure. Prove that the free surface of the liquid has a parabolic shape and also find its equation.



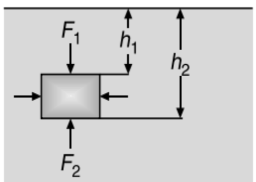
ARCHIMEDES' PRINCIPLE

Archimedes discovered that when a body is immersed partly or completely in a fluid, then the body experiences an upward force that equals the weight of fluid displaced by the immersed part of body. This principle is called Archimedes principle and is a necessary consequence of the Laws of Fluid Statics.

So, according to Archimedes', "*whenever a body is immersed partially or completely in a liquid (or fluid), then it experiences an upward force (called upthrust or buoyant force), which is equal to the weight of the liquid (or fluid) displaced by the immersed part of the body and this upward force or upthrust or buoyant force is equal to the loss in weight of the body*".

Upthrust acts at a point called as the **Centre of Buoyancy,** which is **actually the centre of gravity of the immersed part of the body.**

To determine the magnitude and direction of this force consider a body having volume V, density ρ to be immersed in a fluid of density σ as shown in Figure.



The forces on the vertical sides of the body will cancel each other. The top surface of the body which is at a depth h_1 below the free surface of the liquid will experience a downward force F_1 given by

$$F_1 = P_1 A = (h_1 \sigma g + P_0) A \qquad \{\because P_1 = h_1 \sigma g + P_0\}$$

The lower face of the body will experience an upward force F_2 given by

$$F_2 = P_2 A = (h_2 \sigma g + P_0) A \qquad \{\because P_2 = h_2 \sigma g + P_0\}$$

Since $h_2 > h_1$, so $F_2 > F_1$ and hence the body will experience a net upward force $(F_2 - F_1)$ vertically upwards and this force is called upthrust (U) or the buoyant force (F_B) acting on the body vertically upwards (opposite to the weight of the body).

$$U = F_B = F_2 - F_1 = A\sigma g (h_2 - h_1)$$

If h be the vertical height of the body, then

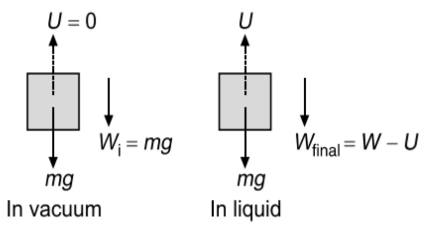
$$h = h_2 - h_1$$

Hence, upthrust is given by

$$U = F_B = (Ah)\sigma g = V\sigma g$$

where, $V = Ah$ is the volume of the liquid displaced by the immersed part of the body, $V\sigma$ is the mass of the liquid displaced and $V\sigma g$ is the weight of the liquid displaced by the immersed part of the body.

Also, we observe that if W_i be the initial weight of body and W_f be the final weight of the body, then



$$W_f = W_i - U$$

$$\Rightarrow \quad U = W_i - W_f = (\text{Loss in Weight of the Body})$$

$$\Rightarrow \quad U = F_B = (V_{\text{immersed}})(\rho_{\text{liquid}})g = W_{\text{initial}} - W_{\text{final}}$$

Please keep in mind that U (or F_B) acts through the centre of gravity of displaced fluid also called as Centre of Buoyancy.

Conceptual Note(s)

(a) Though we have derived this result for a body fully immersed in a fluid, however, this also holds good for bodies partially immersed in a liquid or a body immersed in more than one liquid.

(b) Upthrust is independent of all factors of the body such as its mass, size, density etc. except the volume of the body inside the fluid.

(c) Upthrust depends upon the nature of displaced fluid. This is why upthrust on a fully submerged body is more in sea water than in fresh water because density of sea water is more than fresh water.

(d) **Finding the Volume (V) of the Body:** If a body of volume V is immersed in a liquid of density σ then its weight reduces. If W_1 be the weight of the body in air and W_2 be the weight of the body in liquid, then loss in weight of the body is

$$W_1 - W_2 = V\sigma g$$

$$\Rightarrow V = \frac{W_1 - W_2}{\sigma g}$$

ARCHIMEDES PRINCIPLE: A GENERAL PROOF FOR AN ARBITRARY SHAPED BODY

A body immersed in a fluid experiences an upward buoyant force equivalent to the weight of the fluid displaced by it. To prove this, let us consider a body of any arbitrary shape completely immersed in a liquid of density ρ as shown in Figure (a). A body is being acted upon by the forces from all directions. Let us consider a vertical element of height h and cross-sectional area dA as shown in Figure (b).

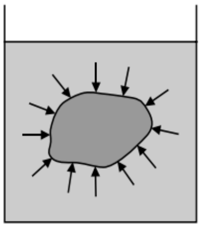


Figure (a)

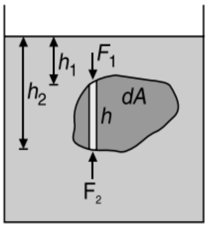


Figure (b)

The force acting on the upper surface of the element is F_1 (downward) and that on the lower surface is F_2 (upward). Since $F_2 > F_1$, therefore, the net upward force acting on the element is

$$dF = F_2 - F_1$$

It can be easily seen from Figure (b), that

$$F_1 = (\rho g h)dA \text{ and } F_2 = (\rho g h_2)dA$$

so, $dF = \rho g(h)dA$

Also, $h_2 - h_1 = h$ and $h(dA) = dV$

The net upward force is

$$F = \int \rho g dV = V\rho g$$

Hence, for the entire body, **the buoyant force is the weight of the volume of the fluid displaced.**

The buoyant force acts through the centroid of the displaced fluid.

Conceptual Note(s)

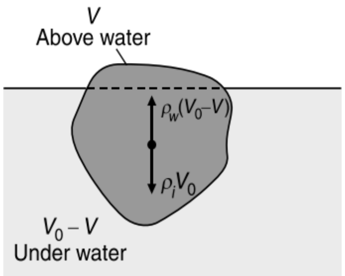
The buoyant force arises because the pressure in the fluid is not uniform because it increases with depth.

An object floats on water if it can displace a volume of water whose weight is greater than that of the object. If the density of the material is less than that of the liquid, it will float even if the material is a uniform solid, such as a block of wood floats on water surface.

If the density of the material is greater than that of water, such as iron, the object can be made to float provided it is not a uniform solid. An iron hulled ship is an example to this case.

ILLUSTRATION 60

An iceberg with a density of $920\ \text{kgm}^{-3}$ floats on an ocean of density $1025\ \text{kgm}^{-3}$. What fraction of the iceberg is visible?



SOLUTION

Let V be the volume of the iceberg above the water surface, then volume under water will be $V_0 - V$.

Under floating conditions, the weight $(\rho_i V_0 g)$ of the iceberg is balanced by the buoyant force $\rho_w(V_0 - V)g$. Thus,

$$\rho_i V_0 g = \rho_w (V_0 - V) g$$

$$\Rightarrow \rho_w V = (\rho_w - \rho_i) V_0$$

$$\Rightarrow \rho_w V = (\rho_w - \rho_i) V_0$$

$$\Rightarrow \quad \frac{V}{V_o}=\left(\frac{\rho_w-\rho_i}{\rho_w}\right)$$

Since, $\rho_w = 1025 \text{ kgm}^{-3}$ and $\rho_i = 920 \text{ kgm}^{-3}$

Therefore, $\dfrac{V}{V_0}=\dfrac{1025-920}{1025}=0.10$

Hence 10% of the total volume is visible.

FINDING THE RELATIVE DENSITY (R.D.) OR SPECIFIC GRAVITY OF A BODY

Let a body of volume V and density $\rho(=\rho_b)$ be completely immersed in water of density $\sigma(=\rho_\ell)$, then upthrust acting on the body is equal to the loss in weight $(W_i - W_f)$ of the body. So,

$$U = W_i - W_f = V\sigma g = V\rho_w g \quad \ldots(1)$$

Now, the weight of the body (W) is given by

$$W = V\rho g = V\rho_b g \quad \ldots(2)$$

Dividing (2) by (1), we get

$$\frac{W}{U}=\frac{V\rho_b g}{V\rho_w g}=\frac{\rho_b}{\rho_w}=R.D.$$

So, relative density $(R.D.)$ is the ratio of the weight of the body in air (or vacuum) to the loss in weight of the body completely immersed in water.

$$(R.D.)_{\text{body}}=\frac{\text{Weight of Body in Air}}{\left(\begin{array}{c}\text{Loss in Weight of Body}\\ \text{Fully Immersed in Water}\end{array}\right)}$$

So, if W_1 the weight of body in air and W_2 be the weight of body completely immersed in water, then

$$R.D.=\frac{W_1}{W_1-W_2}$$

FINDING THE RELATIVE DENSITY (R.D.) OR SPECIFIC GRAVITY OF A LIQUID

If the loss in weight of a body completely immersed in water is "a" while the loss in weight of body completely immersed in a liquid is "b". If W is the weight of body in air, W_ℓ be the weight of body in liquid and W_w be the weight of body in water, then

$$W - W_\ell = a = V\rho_\ell g \quad \ldots(1)$$

$$W - W_w = b = V\rho_w g \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{W-W_\ell}{W-W_w}=\frac{a}{b}=\frac{\rho_\ell}{\rho_w}=(R.D.)_{\text{liquid}}$$

$$(R.D.)_{\text{liquid}}=\frac{\left(\begin{array}{c}\text{Loss in Weight of Body}\\ \text{Fully Immersed in Liquid}\end{array}\right)}{\left(\begin{array}{c}\text{Loss in Weight of Body}\\ \text{Fully Immersed in Water}\end{array}\right)}$$

ILLUSTRATION 61

When a 2.5 kg crown is immersed in water, it has an apparent weight of 22 N. What is the density of the crown?

SOLUTION

Let $W =$ actual weight of the crown
$W' =$ apparent weight of the crown
$\rho =$ density of crown
$\rho_0 =$ density of water

The buoyant force is given by

$$U = W - W'$$

$$\Rightarrow \quad \rho_0 V_g = W - W'$$

Since, $W = \rho V g$, therefore, $V = \dfrac{W}{\rho g}$

Eliminating V from the above two equations, we get

$$\rho = \frac{\rho_0 W}{W-W'}$$

Here $W = 25 \text{ N}$; $W' = 22 \text{ N}$; $\rho_0 = 10^3 \text{ kgm}^{-3}$

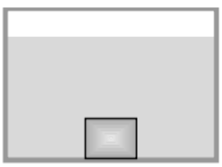
$$\Rightarrow \quad \rho=\frac{(10^3)(25)}{25-22}=8.3\times10^3 \text{ kgm}^{-3}$$

LAWS OF FLOATATION

For the different situations that may arise we have summarised the Laws of Floatation in the following Cases.

CASE-1

When $\rho_{\text{body}} > \rho_{\text{liquid}}$ i.e. $\rho > \sigma$, then $W_{\text{app}} > 0$ (or $W > U$) and hence the body will sink to the bottom of the beaker as shown in Figure.



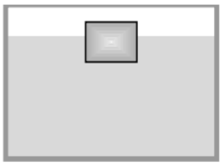
CASE-2

If, $\rho_{\text{body}} = \rho_{\text{liquid}}$ i.e. $\rho = \sigma$, then $W_{\text{app}} = 0$ OR $W = U$

So, the weight of the body will be equal to the upthrust. Hence the body will float fully submerged in neutral equilibrium anywhere in the liquid i.e. the body will just float or remain hanging at whatever height it is left inside the liquid. So, a small downward push to the body in this case will make it sink to the bottom of the beaker.

CASE-3

If, $\rho_{\text{body}} < \rho_{\text{liquid}}$, i.e. $\rho < \sigma$, then $W < U$ and so the Weight will be less than upthrust so the body will move upwards till the state of equilibrium is attained i.e. The body will rise partly out of the free surface of the liquid until the upward thrust acting on the body due to its immersed part in the liquid becomes equal to the total weight of the body W **(Condition for Floating)**. The body will then float with a fraction of its volume inside (or outside).



APPARENT WEIGHT OF A BODY IMMERSED IN LIQUID

For a body of true weight W, immersed in a liquid, the apparent weight (W' or W_{app}) or the effective weight of the is given by

$$W_{\text{app}} = W' = W - U$$

CASE-1: For a Floating Body

Since a body floats when the upthrust acting on the body balances the weight of the body. So, for a floating body, $W = U$ and hence its apparent weight is zero.

$$\begin{pmatrix}\text{Apparent Weight} \\ \text{of a Floating Body}\end{pmatrix} = \text{ZERO}$$

CASE-2: For a Body That Sinks In The Liquid

The body appears to have a lesser weight when immersed in a fluid. So apparent weight of the body is given by

$$W_{\text{app}} = W' = W - U$$

Suppose a body of volume V and density ρ is fully immersed in a liquid of density σ. Then

True weight of the body is $W = V\rho g$

Weight of the liquid displaced is $U = V\sigma g$

So, net downward force or apparent weight is

$$W_{\text{app}} = W' = (\rho - \sigma) gV$$

$$\Rightarrow \quad W_{\text{app}} = W' = \rho g V\left(1 - \frac{\sigma}{\rho}\right)$$

$$\Rightarrow \quad W_{\text{app}} = W' = W\left(1 - \frac{\sigma}{\rho}\right)$$

PRINCIPLE OF FLOATATION

For a body to float in a liquid, the weight of the liquid displaced by the immersed portion of the body (i.e. the upthrust) must be equal to its own weight.

FRACTIONAL VOLUME OF A FLOATING BODY INSIDE THE LIQUID

Suppose a body of volume V and density ρ_b floats in a liquid of density ρ_l and let V_{imm} be the volume of the body immersed in the liquid. Then, for the body to be in equilibrium,

$$\begin{pmatrix}\text{Upthrust acting due to} \\ \text{immersed part of the body}\end{pmatrix} = \begin{pmatrix}\text{Weight of} \\ \text{the body}\end{pmatrix}$$

$$\Rightarrow \quad V_{\text{imm}}\rho_l g = V\rho_b g$$

$$\Rightarrow \quad \frac{V_{\text{imm}}}{V} = \frac{\rho_b}{\rho_l}$$

If the liquid is water, then

$$\frac{V_{\text{imm}}}{V} = \text{Relative Density of the Body}$$

So, fraction of volume of body immersed in a liquid is

$$f_{\text{immersed}} = \frac{V_{\text{imm}}}{V} = \frac{\rho_b}{\rho_l}$$

So, fraction of volume of body immersed in a liquid is independent in the variation of g.

Further, fraction of volume of body outside liquid surface is

$$f_{\text{outside}} = 1 - f_{\text{immersed}} = 1 - \frac{\rho_b}{\rho_l}$$

Conceptual Note(s)

(a) Translatory Equilibrium of a Floating Body: When a body of density ρ and volume V is immersed in a liquid of density σ, the forces acting on the body are

- **(i)** Weight of body $W = mg = V\rho g$, acting vertically downwards through centre of gravity of the body.
- **(ii)** Upthrust force $= V\sigma g$ acting vertically upwards through the centre of gravity of the displaced liquid i.e., centre of buoyancy.

(b) The body will float in liquid only if

$$U \geq W \text{ OR } \sigma \geq \rho$$

(c) For a floating body

$$V\rho_{\text{body}} g = V_{\text{imm}}\rho_{\text{liquid}} g$$

So, the equilibrium of a floating body is simply not affected by the variation(s) in the acceleration due to gravity g, though both thrust and weight depend on g.

(d) If a platform of mass M and cross-section A is floating in a liquid of density σ with its height h inside the liquid

$$Mg = hA\sigma g \quad ...(1)$$

Now if a body of mass m is placed on it and the platform sinks by y then

$$(M+m)g = (y+h)A\sigma g \quad ...(2)$$

Subtracting equation (1) and (2), we get

$$mg = A\sigma yg$$

$$\Rightarrow W \propto y \quad ...(3)$$

So, we can determine the weight of a body by placing it on a floating platform and noting the depression by which the platform further sinks in the liquid.

ILLUSTRATION 62

A block of wood floats in water with one third of its volume submerged. In oil, the block floats with two third of its volume submerged. Find the density of wood and oil if the density of water is 10^3 kgm^{-3}.

SOLUTION

For the case of a floating bod, we have

$$W = U$$

$$\Rightarrow V\rho = V_{\text{in}}\sigma$$

$$\Rightarrow V\rho = \frac{1}{3}V\sigma_w \qquad \left\{\because V_{\text{in}} = \frac{V}{3}\right\}$$

$$\Rightarrow \rho = \frac{1}{3}\sigma_w = \frac{1}{3}\times 10^3 = 333 \text{ kgm}^{-3}$$

So, density of wood is $\rho_{\text{wood}} = 333$ kgm^{-3}

For oil, we have

$$V\rho = \frac{2}{3}V\sigma_{\text{oil}}$$

$$\Rightarrow \frac{2}{3}V\sigma_{\text{oil}} = \frac{1}{3}V\sigma_w$$

$$\Rightarrow \sigma_{\text{oil}} = \frac{1}{2}\sigma_w = \frac{10^3}{2} = 500 \text{ kgm}^{-3}$$

$$\Rightarrow \sigma_{\text{oil}} = 500 \text{ kgm}^{-3}$$

ILLUSTRATION 63

A cubical block of iron 5 cm on each side is floating on mercury in a vessel. Calculate the height of the block above mercury level. If water is poured in the vessel until it just covers the iron block, then calculate the height of water column assuming that $\rho_{Hg} = 13.6$ gcm^{-3} and $\rho_{Fe} = 7.2$ kgm^{-3}.

SOLUTION

CASE-1:

Suppose h be the height of cubical block of iron above mercury.

Volume of iron block is $V_{Fe} = 5\times5\times5 = 125$ cm^3

Mass of iron block is $m_{Fe} = 125\times7.2 = 900$ g

Volume of mercury displaced by the block is

$$V_{Hg} = 5\times5\times(5-h) \text{ cm}^3$$

Mass of mercury displaced by the block is

$$m_{Hg} = 5\times5(5-h)\times13.6 \text{ gram}$$

Since upthrust U equals the weight of liquid displaced by the immersed part of the body, so we have

$$U = (m_{Hg})g$$

According to the Laws of Floatation, this upthrust balance the weight of the iron block, so we have

$$U = W_{Fe}$$

$$\Rightarrow 5\times5(5-h)\times13.6 = 900$$

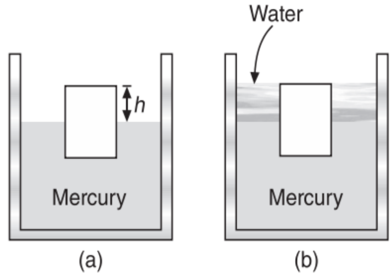
$$\Rightarrow (5-h) = \frac{900}{25\times13.6} = 2.65$$

$$\Rightarrow h = 5-2.65 = 2.35 \text{ cm}$$

CASE-2:

Suppose in this case the height of iron block in water be x, so the height of iron block in mercury will be $(5-x)$ cm.

Mass of the water displaced is



$$m_{\text{water}} = 5\times5\times(x)\times1$$

Mass of mercury displaced is

$$m_{Hg} = 5\times5\times(5-x)\times13.6$$

For the iron block to float, we must have

$$U_{\text{water}} + U_{Hg} = W_{Fe}$$

$$\Rightarrow (m_{\text{water}})g + (m_{Hg})g = (m_{Fe})g$$

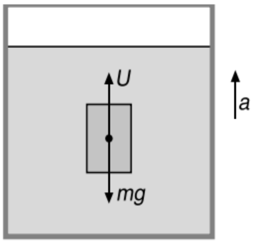
$$\Rightarrow (1)(5)^2x + (13.6)(5)^2(5-x) = 900$$

$$\Rightarrow x + (5-x)\times13.6 = 36$$

$$\Rightarrow x = 2.54 \text{ cm}$$

BUOYANT FORCE IN ACCELERATING FLUIDS

Consider a body to be submerged in a liquid of density ρ. Let the liquid be accelerating up with an acceleration a as shown in Figure.



For the block, we have

$$U - mg = ma$$

$$\Rightarrow \quad U = m(g+a)$$

where m is the mass of the displaced liquid, i.e.

$$m = V\rho$$

$$\Rightarrow \quad U = \rho V(g+a)$$

Conceptual Note(s)

(a) While writing the expression for the buoyant force for an accelerated fluid, instead of taking gravity g we take the effective value of gravity i.e. $g_{eff} = g \pm a$. So, we write

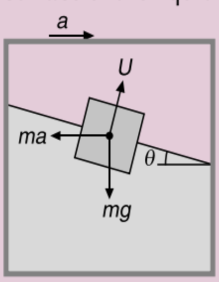
$$F_{buoyant} = U = V_{immersed}\rho_{liquid}g_{eff}$$

where, $V_{immersed} = V_{displaced}$

(b) When the floating object is accelerating in vertical direction, then $g_{eff} = g + a$, if the arrangement is accelerating up or retarding down.

(c) When the floating object is accelerating in vertical direction, then $g_{eff} = g - a$, if the arrangement is accelerating down or retarding up.

(d) When an object is floating in liquid filled in a container and the arrangement is accelerating in horizontal direction, then the liquid surface in the container will be inclined at angle θ with horizontal where, $\tan\theta = \frac{a}{g}$. Since upthrust or the buoyant force acts perpendicular to equi-pressure surface hence, in this case buoyant force will not be vertical but it will act normal to free surface of the liquid as shown in Figure.



The upthrust or the buoyant force is given by

$$U = V_{imm}\rho_{liq}g_{eff} = V_{imm}\rho_{liq}\sqrt{g^2+a^2}$$

ILLUSTRATION 64

A cube of length L, density σ is floating in a beaker filled with water having density ρ_w. Calculate the change in submerged length of cube in water if the arrangement is accelerating in the horizontal direction with an acceleration a.

SOLUTION

Let l be the submerged length of the cube in water when the beaker is at rest, then by applying the Laws of Floatation, we get

$$Mg = V_{immmersed}\rho_w g \quad \text{...(1)}$$

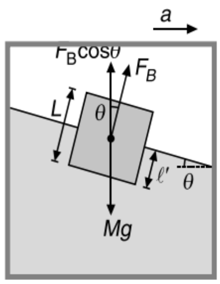
where, $M = (AL)\sigma$ and $V_{immersed} = Al$

So, from equation (1), we get

$$AL\sigma = Al\rho_w$$

$$\Rightarrow \quad l = L\left(\frac{\sigma}{\rho_w}\right)$$

When the beaker is accelerating, then let the new length of the cube immersed in the liquid be l'.



When the arrangement accelerates, then the free surface of the liquid in the beaker makes an angle θ

$$\tan\theta = \frac{a}{g}$$

$$\Rightarrow \quad \cos\theta = \frac{g}{\sqrt{g^2+a^2}}$$

Since the upthrust $F_B = U$ acts normally to the free surface of the liquid, so we have

$$F_B\cos\theta = Mg$$

$$(Al')\rho_w g_{eff}\cos\theta = (AL\sigma)g$$

$$\Rightarrow \quad Al'\rho_w\sqrt{g^2+a^2}\left(\frac{g}{\sqrt{g^2+a^2}}\right) = AL\sigma g$$

$$\Rightarrow \quad l'\rho_w = L\sigma$$

$$\Rightarrow \quad l' = L\left(\frac{\sigma}{\rho_w}\right) = l$$

So, there is no change in the submerged length of the cube in water when the arrangement is accelerated horizontally.

STABILITY OF A FLOATING BODY

The stability of a floating body depends on the effective point of application of the buoyant force. The weight of the body acts at its centre of gravity. *The buoyant force acts at the centre of gravity of the displaced liquid. This is called the* ***centre of buoyancy***. Under equilibrium condition, the centre of gravity G and the centre of buoyancy B lie along the vertical axis of the body as shown in Figure (a).

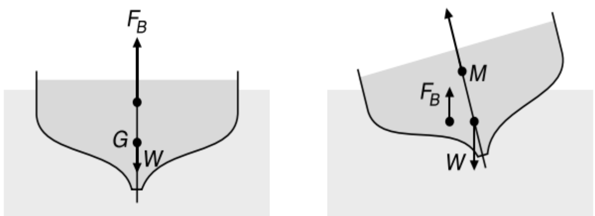


Figure (a) Figure (b)

When the body tilts to one side, the centre of buoyancy shifts relative to the centre of gravity as shown in Figure (b). The two forces act along different vertical lines. As a result, the buoyant force exerts a torque about the centre of gravity. The line of action of the buoyant force crosses the axis of the body at the point M, called the **metacentre**. If G is below M, the torque will tend to restore the body to its equilibrium position.

If G is above M, the torque will tend to rotate the body away from its equilibrium position and the body will be unstable.

ILLUSTRATION 65

A rubber ball of mass m and radius r is submerged in water to a depth h and released. Calculate the height to which the ball will jump above the surface of the water. Neglect resistance of water and air and assume density of water to be ρ.

SOLUTION

Let the ball rise to a height H above water. According to Modified Work Energy Theorem, we have

$$W_{ext} = \Delta U + \Delta K$$

For both the initial and the final positions of the ball, kinetic energy is zero, so $\Delta K = 0$. Also,

$$W_{ext} = Uh\cos 0° = Uh = \frac{4}{3}\pi r^3 \rho g h$$

Assuming Zero Potential Energy Level (ZPEL) at the free surface of water, then

$$\Delta U = mgH - (-mgh) = mg(H+h)$$

$$\Rightarrow \quad \left(\frac{4}{3}\pi r^3 \rho g\right)h = mg(H+h)$$

$$\Rightarrow \quad H = \left(\frac{\frac{4}{3}\pi r^3 \rho - m}{m}\right)h$$

ILLUSTRATION 66

A cube of wood supporting a 200 g mass just floats in water. When the mass is removed, the cube rises by 2 cm. Calculate the side length of the cube.

SOLUTION

Initially, we have

$$(m_{cube} + 200)g = U_{initial} \quad \text{...(1)}$$

Since the cube of wood supporting a 200 g mass just floats in water, so we have

$$U_{initial} = l^3 \rho_{water} g \quad \text{...(2)}$$

From equations (1) and (2), we get

$$m_{cube} + 200 = l^3 \rho_{water}$$

$$\Rightarrow \quad l^3 \rho_{wood} + 200 = l^3 \rho_{water} \quad \text{...(3)}$$

On removing the block, cube moves up by 2 cm, so we have

$$W = U_{final}$$

$$\Rightarrow \quad (l^3 \rho_{wood})g = l^2(l-2)\rho_{water} g$$

$$\Rightarrow \quad l^3 \rho_{wood} = l^2(l-2)\rho_{water} \quad \text{...(4)}$$

Substituting (4) in (3), we get

$$l^2(l-2)\rho_{water} + 200 = l^3 \rho_{water}$$

Since, $\rho_{water} = 1\text{ gcc}^{-1}$

$$\Rightarrow \quad l^2(l-2) + 200 = l^3$$

$$\Rightarrow \quad l^3 - 2l^2 + 200 = l^3$$

$$\Rightarrow \quad -2l^2 + 200 = 0$$

$$\Rightarrow \quad 2l^2 = 200$$

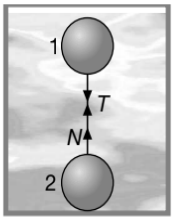
$$\Rightarrow \quad l = 10\text{ cm}$$

ILLUSTRATION 67

Two solid uniform spheres each of radius 5 cm are connected by a light string and totally immersed in a tank of water. If the specific gravities of the spheres are 0.5 and 2, calculate the tension in the string and the contact force between the bottom of the tank and the heavier sphere.

SOLUTION

Let the volume of each sphere be $V\ \text{m}^3$ and density of water be $\rho_w\ \text{kgm}^{-3}$. The situation given in the problem is shown in Figure.



Upward thrust acting on heavier sphere 2 is

$$U_2 = V\rho g$$

Weight of the heavier sphere 2 is

$$W_2 = V(2\rho_w)g$$

For the heavier sphere 2, we have

$$T + N + V\rho_w g = V(2\rho_w)g \quad \ldots(1)$$

where, N is the contact force between the bottom of the tank and the heavier sphere.
Similarly, for lighter sphere 1, we have

$$T + W_1 = U_1$$

$$\Rightarrow \quad T + V(0.5\rho_w)\rho g = V\rho_w g \quad \ldots(2)$$

$$\Rightarrow \quad T = 0.5V\rho_w g$$

$$\Rightarrow \quad T = (0.5)\left(\frac{4}{3}\times 3.14\times 5^3\times 10^{-6}\right)(1000)(9.8)$$

$$\Rightarrow \quad T = 2.56 \text{ newton}$$

Also, from equation (1), we get

$$0.5V\rho_w g + N = V\rho_w g$$

$$\Rightarrow \quad N = 0.5V\rho_w g$$

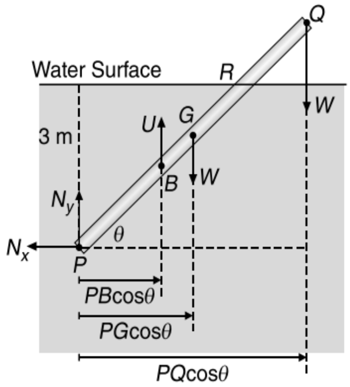
$$\Rightarrow \quad N = 2.56 \text{ newton}$$

ILLUSTRATION 68

A rod of length 6 m has a mass of 12 kg. If it is hinged at one end at a distance of 3 m below a water surface, calculate the weight which must be attached to other end of the rod so that 5 m of the rod is submerged. Also calculate the magnitude and direction of the force exerted by the hinge on the rod. The specific gravity of the material of the rod is 0.5.

SOLUTION

Let PR be the submerged part of the rod PQ hinged at P as shown in Figure.



If G is the centre of gravity of the rod and B is the centre of buoyancy through which force of buoyancy $F_B = U$ acts vertically upwards. Since the rod is uniform, so the mass of the immersed part PR of the rod will be

$$m_{PR} = \frac{5}{6}\times 12 = 10\text{ kg}$$

The buoyant force on rod at B is

$$F_B = U = V_{\text{imm}}\rho_{\text{liq}}g = \left(\frac{10}{0.5\rho_w}\right)\rho_w g = 20\text{ kgwt}$$

Let w be the weight attached at the end Q of the rod, then for the rotational equilibrium of the rod, the torques about the point P due to various forces are balanced. So, we have

$$\Sigma\tau_{\text{about } P} = 0$$

$$\Rightarrow \quad W(PG\cos\theta) + w(PQ\cos\theta) = U(PB\cos\theta)$$

$$\Rightarrow \quad W(PG) + w(PQ) = U(PB), \text{ where}$$

$$PG = 3\text{ m},\ PQ = 6\text{ m and } PB = \frac{PR}{2} = 2.5\text{ m}$$

$$\Rightarrow \quad 12\times 3 + w\times 6 = 20\times(2.5)$$

$$\Rightarrow \quad w = 2.33\text{ kgwt}$$

Let N_x and N_y be the x and y components of the force exerted by hinge on the rod. Since the rod is in equilibrium, so force acting on it is zero. Hence, we have $\Sigma F_x = 0$ and $\Sigma F_y = 0$. As no force acts along the x direction, so we get

$$N_x = 0$$

Also, we have

$$W + w = U + N_y$$

$$\Rightarrow \quad N_y = W + w - U$$

$$\Rightarrow \quad N_y = 12 + 2.33 - 20$$

$$\Rightarrow \quad N_y = -5.67\text{ kgwt}$$

The negative indicates that the reaction at the hinge is acting in the downward direction and has a magnitude of 5.67 kgwt.

ILLUSTRATION 69

A piece of ice is floating in a glass vessel filled with water. How will the level of water in the vessel change when the ice melts?

SOLUTION

Let m be the mass of ice piece floating in water. In equilibrium (i.e. initially)

$$(\text{Weight of Ice} + \text{Stone}) = \text{Upthrust}$$

$$\Rightarrow \quad mg = V_i \rho_w g$$

$$\Rightarrow \quad V_i = \frac{m}{\rho_w} \quad \ldots(1)$$

where, V_i is the volume of ice piece immersed in water or the volume of water displaced by the immersed part of ice in water.

When the ice melts (i.e. finally), let V be the volume of water formed by m mass of ice. Then

$$V = \frac{m}{\rho_w} \quad \ldots(2)$$

From equations (1) and (2), we get

$$V_i = V$$

Hence, the level of water in the vessel will not change.

ILLUSTRATION 70

A piece of ice having a stone frozen in it floats in a glass vessel filled with water. How will the level of water in the vessel change when the ice melts?

SOLUTION

Let, m_1 be the mass of ice, m_2 be the mass of stone, ρ_S be the density of stone and ρ_w be the density of water.

In equilibrium (i.e. initially), when the piece of ice having stone in it floats in water, then we have

$$(\text{Weight of Ice} + \text{Stone}) = \text{Upthrust}$$

$$\Rightarrow \quad (m_1 + m_2)g = V_i \rho_w g$$

$$\Rightarrow \quad V_i = \frac{m_1}{\rho_w} + \frac{m_2}{\rho_w} \quad \ldots(1)$$

where, V_i is the volume of ice immersed in water or the volume of water initially displaced by ice stone combination while floating.

When the ice melts (i.e. finally), the mass m_1 of ice converts into water and stone of mass m_2 get sunk completely in water.

The volume of water formed by m_1 mass of ice is

$$V_1 = \frac{m_1}{\rho_w}$$

The volume of stone is also equal to the volume of water displaced by the stone when it sinks in water. So, we have

$$V_2 = \frac{m_2}{\rho_S}$$

Since, $\rho_S > \rho_w$

$$\Rightarrow \quad V_1 + V_2 < V_i$$

Hence the level of water will decrease.

ILLUSTRATION 71

A solid frozen body is floating in a liquid of different material contained in a beaker. Carry out an analysis to find whether the level of liquid in the beaker will rise or fall when the solid melts.

SOLUTION

Let M be the mass of the solid frozen body floating in the liquid, ρ_1 be the density of liquid formed by the melting of the solid, ρ_2 be the density of the liquid in which the solid is floating.

The mass of liquid displaced by the solid is M and hence the volume of liquid displaced by the body is $\frac{M}{\rho_2}$. When the solid melts, the volume occupied by it is $\frac{M}{\rho_1}$.

So, level of liquid in container will rise, when

$$\frac{M}{\rho_1} > \frac{M}{\rho_2} \text{ i.e., } \rho_1 < \rho_2$$

Level of liquid in container will fall, when

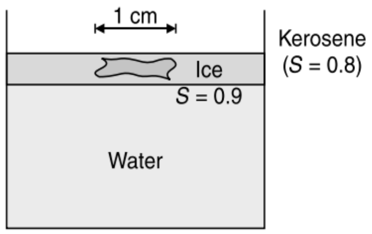
$$\frac{M}{\rho_2} > \frac{M}{\rho_1} \text{ i.e., } \rho_1 > \rho_2$$

Level of liquid in container will remain same, when

$$\frac{M}{\rho_2} = \frac{M}{\rho_1} \text{ i.e., } \rho_1 = \rho_2$$

ILLUSTRATION 72

An ice cube of side 1 cm is floating at the interface of kerosene and water in a beaker of base area 10 cm^2. The level of kerosene is just covering the top surface of the ice cube. Calculate the depth of submergence in the kerosene and that in the water. Also calculate change in total level of the liquid when the whole ice melts into water.



SOLUTION

Condition of floating

$$0.8\ \rho_w g h_k + \rho_w g h_w = 0.9\ \rho_w g h$$

$$\Rightarrow \quad 0.8 h_k + h_w = (0.9) h \qquad \ldots(1)$$

where h_k and h_w be the submerged depth of the ice in the kerosene and water, respectively.

Also, $h_k + h_w = h$ ...(2)

Solving equations (1) and (2), we get

$$h_k = 0.5\ \text{cm}\ ,\ h_w = 0.5\ \text{cm}$$

Also, when $1\ \text{cm}^3$ of ice melts, then we get $0.9\ \text{cm}^3$ of water i.e.

$$\underset{(\text{Ice})}{1\ \text{cm}^3} \xrightarrow{\text{melts}} \underset{(\text{Water})}{0.9\ \text{cm}^3}$$

Fall in the level of kerosene is

$$\Delta h_k = \frac{0.5}{A}$$

Rise in the level of water is

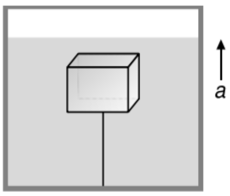
$$\Delta h_w = \frac{0.9 - 0.5}{A} = \frac{0.4}{A}$$

So, net fall in the overall level is

$$\Delta h = \frac{0.1}{A} = \frac{0.1}{10} = 0.01\ \text{cm} = 0.1\ \text{mm}$$

ILLUSTRATION 73

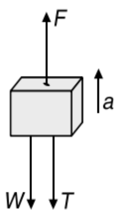
A block of mass 1 kg and density $0.8\ \text{gcm}^{-3}$ is held stationary with the help of a string as shown in Figure.



The tank is accelerating vertically upwards with an acceleration $a = 1.0\ \text{ms}^{-2}$. Calculate the tension in the string. Now, if the string is cut, then calculate acceleration of the block if $g = 10\ \text{ms}^{-2}$ and density of water is $10^3\ \text{kgm}^{-3}$.

SOLUTION

The free body diagram of the block is shown in Figure.



If F_B is the upthrust force, then we have

$$F_B = V\rho_w (g + a)$$

$$\Rightarrow \quad F_B = \left(\frac{\text{mass of block}}{\text{density of block}}\right)\rho_\omega (g + a)$$

$$\Rightarrow \quad F_B = \left(\frac{1}{800}\right)(1000)(10 + 1) = 13.75\ \text{N}$$

Also, the weight of the block is

$$W = mg = 10\ \text{N}$$

So, equation of motion of the block is,

$$F_B - T - W = ma$$

$$\Rightarrow \quad 13.75 - T - 10 = 1 \times 1$$

$$\Rightarrow \quad T = 2.75\ \text{N}$$

When the string is cut, then we have

$$T = 0$$

$$\Rightarrow \quad a = \frac{F_B - W}{m}$$

$$\Rightarrow \quad a = \frac{13.75 - 10}{1}$$

$$\Rightarrow \quad a = 3.75\ \text{ms}^{-2}$$

ILLUSTRATION 74

An iron casting containing a number of cavities weighs 6000 N in air and 4000 N in water. Calculate the volume of the cavities in the casting if the density of iron is $7.87\ \text{gcm}^{-3}$, $g = 9.8\ \text{ms}^{-2}$ and the density of water is $10^3\ \text{kgm}^{-3}$.

SOLUTION

If v be the volume of cavities and V the volume of solid iron, then we have

$$V = \frac{\text{Mass}}{\text{Density}} = \left(\frac{6000/9.8}{7.87 \times 10^3}\right) = 0.078\ \text{m}^3$$

Further, we know that the loss in weight of the body equals the upthrust acting on the body, so we have

$$(6000 - 4000) = (V + v)\rho_w g$$

$$\Rightarrow \quad 2000 = (0.078 + v) \times 10^3 \times 9.8$$

$$\Rightarrow \quad 0.078 + v \approx 0.2$$

$$\Rightarrow \quad v = 0.12\ \text{m}^3$$

ILLUSTRATION 75

An ornament weighing 50 g in air weighs only 46 g is water. Assuming that some copper is mixed with gold to prepare the ornament. Find the amount of copper in it. Specific gravity of gold is 20 and that of copper is 10.

SOLUTION

Let m be the mass of the copper in ornament. Then mass of gold in it is $(50 - m)$. Since,

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$$

So, volume of copper is $V_1 = \dfrac{m}{10}$

and volume of gold is $V_2 = \dfrac{50-m}{20}$

When immersed in water $\left(\rho_w = 1\ \text{gcm}^{-3}\right)$, we know that the loss in weight of the body equals the upthrust acting on it, so we have

$$(50-46)g = (V_1+V_2)\rho_w g$$

$$\Rightarrow \quad 4 = \frac{m}{10} + \frac{50-m}{20}$$

$$\Rightarrow \quad 80 = 2m + 50 - m$$

$$\Rightarrow \quad m = 30\ \text{g}$$

ILLUSTRATION 76

A hollow sphere of inner radius 9 cm and outer radius 10 cm floats half-submerged in a liquid of specific gravity 0.8. Calculate the density of the material of which the sphere is made. What would be the density of a liquid in which the hollow sphere would just float completely submerged?

SOLUTION

For equilibrium of sphere, the upthrust acting on the body due to the immersed part must balance the weight of the body, so we have

$$\frac{1}{2}\left[\frac{4}{3}\pi(10)^3\right](0.8) = \frac{4}{3}\pi\left((10)^3-(9)^3\right)\rho$$

$$\Rightarrow \quad \rho = \frac{(0.4)(1000)}{1000-729} = \frac{400}{271} = 1.476\ \text{gcm}^{-3}$$

If ρ_L be the density of liquid in which sphere will just float then

$$\frac{4}{3}\pi(10)^3\rho_L = \frac{4}{3}\pi(10)^3 \times \frac{1}{2} \times 0.8$$

$$\Rightarrow \quad \rho_L = 0.4\ \text{gcm}^{-3}$$

ILLUSTRATION 77

A piece of gold sample weighs 36 g in air and 34 g in water. If this sample has some copper mixed in it, then calculate the amount of copper in the sample if specific gravity of gold is 19.3 and that of copper is 8.9. Now assume that the same mass of sample is made of pure gold but with air cavities inside it and it weighs the same in water, then calculate the volume of the air cavities in the sample.

SOLUTION

Let the sample contain x g of copper, then the amount of gold in this sample is $(36-x)$ g. Since we know that the loss in weight of the sample equals the upthrust acting on it, so we have

$$36-34 = \left(\frac{x}{\rho_{Cu}} + \frac{36-x}{\rho_{\text{gold}}}\right)\rho_\omega$$

$$\Rightarrow \quad \frac{x}{8.9} + \frac{36-x}{19.3} = 2$$

$$\Rightarrow \quad 19.3x + 8.9\times 36 - 8.9x = 2\times 8.9\times 19.3$$

$$\Rightarrow \quad 10.4x = 343.54 - 320.4 = 23.14$$

$$\Rightarrow \quad x = 2.225\ \text{g}$$

If V_c is volume of air cavities inside 36 g of pure gold sample, then

$$2 = \left(V_c + \frac{36}{\rho_{\text{gold}}}\right)\rho_w$$

$$\Rightarrow \quad V_c = \frac{2}{\rho_w} - \frac{36}{\rho_{\text{gold}}}$$

$$\Rightarrow \quad V_c = 2 - \frac{36}{19.3} = 0.135\ \text{cm}^3$$

ILLUSTRATION 78

A wooden stick of length L, radius R and density ρ has a small metal piece of mass m (of negligible volume) attached to its one end. Find the minimum value for the mass m (in terms of given parameters) that would make the stick float vertically in equilibrium in a liquid of density $\sigma(>\rho)$.

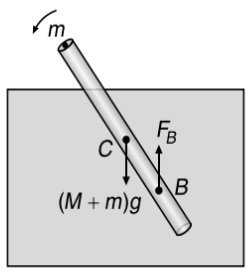
SOLUTION

Let M be the mass of the stick, then we have

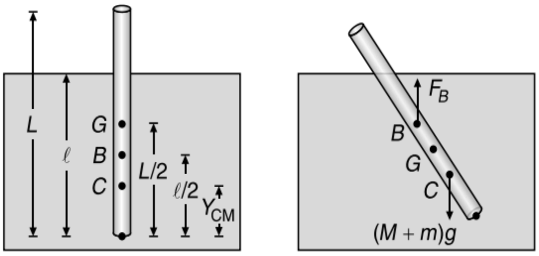
$$M = \pi R^2 \rho L$$

If ℓ be the immersed length of the rod, G be the centre of mass (CM) of rod, B be the centre of buoyant force (F_B), C be the centre of mass of rod plus metal piece system and Y_{CM} be the distance of C from bottom of the rod.

If the metal piece is attached to the top end of the rod, then the centre of buoyancy B will be below the centre of mass C of the rod and combined centre of mass C of rod plus metal piece will be above the centre of mass G of the rod as shown in Figure.



If this is the case, then the torque of the couple of two equal and opposite forces F and $(M+m)g$ will be counter clockwise on displacing the rod leftwards and hence the rod cannot be in rotational equilibrium when m is attached to the upper end of the rod.



When the mass m is attached to the lower end of the rod, then for translational equilibrium of the system, we have

$$F_B = Mg + mg$$

$$\Rightarrow \quad (\pi R^2 l)\sigma g = (\pi R^2 L)\rho g + mg$$

$$\Rightarrow \quad l = \left(\frac{\pi R^2 L\rho + m}{\pi R^2 \sigma}\right) \quad \ldots(1)$$

The position of CM of rod plus metal piece from the bottom of the rod is given by

$$Y_{CM} = \frac{M\left(\frac{L}{2}\right)}{M+m} = \frac{(\pi r^2 L\rho)\left(\frac{L}{2}\right)}{(\pi R^2 L\rho)+m} \quad \ldots(2)$$

Since the centre of buoyancy (B) is at a height $\frac{l}{2}$ from the bottom of the rod, so for rotational equilibrium of the rod, B should either lie above C or at the same level of C. Therefore, we have

$$\frac{l}{2} \geq Y_{COM}$$

$$\Rightarrow \quad \frac{\pi R^2 L\rho + m}{2\pi R^2 \sigma} \geq \frac{(\pi R^2 L\rho)\frac{L}{2}}{(\pi R^2 L\rho)+m}$$

$$\Rightarrow \quad m + \pi R^2 L\rho \geq \pi R^2 L\sqrt{\rho\sigma}$$

$$\Rightarrow \quad m \geq \pi R^2 L\left(\sqrt{\rho\sigma} - \rho\right)$$

So, minimum value of m is $\pi R^2 L\left(\sqrt{\rho\sigma} - \rho\right)$.

ILLUSTRATION 79

A flat bottomed thin-walled glass tube has a diameter of 4 cm and it weighs 30 g. The centre of gravity of the empty tube is 10 cm above the bottom. Calculate the amount of water that must be poured into the tube so that when it is floating vertically in a tank of water, the centre of mass of the tube and its contents lies at the midpoint of the immersed length of the tube.

SOLUTION

Assuming that water is filled to a height h and l is length of tube submerged in water, then upthrust acting on the system balances the weight of the system. So, we have

$$(l\pi r^2)\rho_w g = 30g + (\pi r^2 h)\rho_w g$$

$$\Rightarrow \quad (l\pi r^2)\rho_w = 30 + (\pi r^2 h)\rho_w$$

$$\Rightarrow \quad l = \frac{30 + (\pi r^2 h)\rho_w}{(\pi r^2)\rho_w} \quad \ldots(1)$$

Also, it is given that the centre of mass of the tube and its contents lies at the midpoint of the immersed length of the tube. So, we have

$$\frac{l}{2} = \frac{(30)(10) + (\pi r^2 h\rho_w)\left(\frac{h}{2}\right)}{30 + \pi r^2 h\rho_w}$$

$$\Rightarrow \quad (30 + \pi r^2 h\rho_w)\left(\frac{30 + \pi r^2 h\rho_w}{2\pi r^2 \rho_w}\right) = 300 + \frac{\pi r^2 h^2 \rho_w}{2}$$

$$\Rightarrow \quad \frac{(30 + \pi r^2 h\rho_w)^2}{2\pi r^2 \rho_w} = 300 + \frac{\pi r^2 h^2 \rho_w}{2}$$

$$\Rightarrow \quad \frac{900 + (\pi r^2 h\rho_w)^2 + 60(\pi r^2 h\rho_w)}{2\pi r^2 \rho_w} = 300 + \frac{(\pi r^2 \rho_w h)^2}{2}$$

$$\Rightarrow \quad 450 + 30(\pi r^2 h\rho_w) = 300\pi r^2 \rho_w$$

$$\Rightarrow \quad h = \frac{300\pi r^2 \rho_w - 450}{30\pi r^2 \rho_w}$$

$$\Rightarrow \quad h = 10 - \frac{450}{30 \times 3.14 \times (2)^2 \times 1}$$

$$\Rightarrow \quad h = 8.8\text{ cm}$$

Mass of water is given by

$$m = \pi r^2 h\rho_w = 3.14 \times (2)^2 \times 8.8 \times 1$$

$$\Rightarrow \quad m = 110.53\text{ g}$$

ILLUSTRATION 80

A cylinder of area 300 cm^2 and length 10 cm made of material of specific gravity 0.8 is floated in water with its axis vertical. It is then pushed slowly downwards, so as to be just immersed. Calculate the work done by the external force to push the cylinder completely in water. Take $g = 10\text{ ms}^{-2}$.

SOLUTION

Weight of the cylinder is

$$mg = (300 \times 10^{-4})(10 \times 10^{-2})(800)\text{ kgf}$$

$$\Rightarrow \quad mg = 2.4\text{ kgf} = 24\text{ N}$$

If l be the length of the cylinder inside the water, then by Law of Floatation, we have

$$2.4g = (300 \times 10^{-4})l(1000)g$$

$$\Rightarrow \quad l = 0.08\text{ m} = 8\text{ cm}$$

When completely immersed in water, the buoyant force is given by

$$U = F_B = (300\times10^{-4})(0.1)(1000)g$$

$$\Rightarrow \quad U = 30\ \text{N}$$

So, to immerse the cylinder inside water, the external agent has to push it against the upthrust force by

$$h = 0.1 - 0.08 = 0.02\ \text{m} = 2\ \text{cm}$$

Increase in upthrust is

$$\Delta U = 3g - 2.4g = 0.6g = 6\ \text{N}$$

Since this increase in upthrust takes place linearly from zero to $6\ \text{N}$, so the average upthrust against which work has to be done is

$$\langle U\rangle = \frac{0+6}{2} = 3\ \text{N}$$

Hence the work done by the external force is

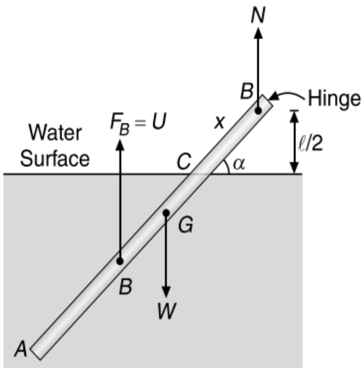
$$W = F_{\text{ext}}h = |\langle U\rangle|h = (3)(0.02) = 0.06\ \text{J}$$

ILLUSTRATION 81

A thin uniform rod of length $2l$ and specific gravity $\frac{3}{4}$ is hinged at one end to a point height $\frac{l}{2}$ above the surface of water, with the other end immersed. If the rod is in equilibrium, then calculate the angle of inclination of rod with the water surface.

SOLUTION

Let the length of rod outside water be x and its cross-sectional area be A. The situation is shown in Figure.



If W be the weight of rod and d be the density of water, then weight of the rod is

$$W = A(2l)\left(\frac{3}{4}d\right)g \qquad \ldots(1)$$

The buoyant force $F_B = U$ acting on the rod at the centre of buoyancy B is

$$U = F_B = A(2l - x)(d)g \qquad \ldots(2)$$

Let N be the upward force on rod by the hinge, then for equilibrium of rod we have

$$N + U = W$$

$$\Rightarrow \quad N = W - U$$

$$\Rightarrow \quad N = (Al)\left(\frac{3d}{2}\right)g - A(2l - x)(d)g$$

$$\Rightarrow \quad N = A\left(\frac{3}{2}l - 2l + x\right)(d)g$$

$$\Rightarrow \quad N = A\left(x - \frac{l}{2}\right)(d)g \qquad \ldots(3)$$

Also, for the rotational equilibrium of the rod, we take the net torque about point A to be zero i.e.

$$U\left(\frac{2l - x}{2}\right)\cos\alpha - Wl\cos\alpha + N(2l\cos\alpha) = 0$$

$$\Rightarrow \quad \left[F_B\left(\frac{2l - x}{2}\right) - Wl + 2Nl\right]\cos\alpha = 0$$

$$\Rightarrow \quad F_B\left(\frac{2l - x}{2}\right) - Wl + 2Nl = 0 \qquad \ldots(4)$$

Substituting the values of W, F_B and N from equations (1), (2) and (3) respectively in equation (4), we get

$$\left[\frac{(2l - x)^2}{2} - \frac{3}{2}l^2 + 2l\left(x - \frac{l}{2}\right)\right]A(d)g = 0$$

$$\Rightarrow \quad (2l - x)^2 - 3l^2 + 4l\left(x - \frac{l}{2}\right) = 0$$

$$\Rightarrow \quad 4l^2 + x^2 - 4lx - 3l^2 + 4lx - 2l^2 = 0$$

$$\Rightarrow \quad x^2 - l^2 = 0$$

$$\Rightarrow \quad x^2 = l^2$$

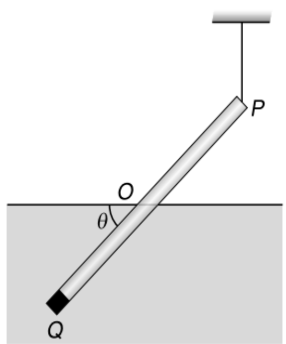
$$\Rightarrow \quad x = l$$

$$\Rightarrow \quad \sin\alpha = \frac{l/2}{l} = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = 30^\circ$$

ILLUSTRATION 82

A uniform rod PQ, $4\ \text{m}$ long and weighing 12 kg, is supported at end P, with a $6\ \text{kg}$ lead weight attached at Q. The rod floats with one half of its length submerged as shown in Figure.

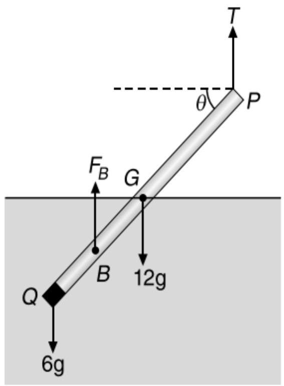


Assuming the lead mass to be of negligible volume, calculate tension in the cord and the total volume of the rod. Take $g = 10\ \text{ms}^{-2}$.

SOLUTION

If A be the area of cross section of the rod, then the buoyant force F_B on submerged part of rod is

$$U = F_B = \left(\frac{4}{2}\right)A\rho_w g = 2A\rho_w g \qquad \ldots(1)$$



For translational equilibrium of rod, we have

$$F_B + T = 18g \qquad \ldots(2)$$

Please note that, since the lead weight has negligible volume, so the buoyant force on it is negligible and hence is not to be taken into account.

For rotational equilibrium of the rod, taking the torque due to all forces about point P to be zero, we get

$$(12g)(2\cos\theta) + (6g)(4\cos\theta) = F_B(3\cos\theta)$$

$$\Rightarrow \quad F_B = 16g = 160\ \text{N} \qquad \ldots(3)$$

$$\Rightarrow \quad T = 18g - F_B = 2g = 20\ \text{N}$$

From equation (1), we get

$$2A(10^3)(10) = 160$$

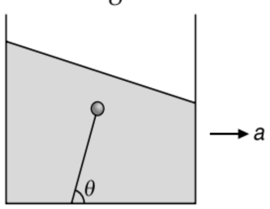
$$\Rightarrow \quad A = 80 \times 10^{-4}\ \text{m}^2$$

So, volume of the rod is

$$V = Al = (80 \times 10^{-4})(4) = 3.2 \times 10^{-2}\ \text{m}^3$$

ILLUSTRATION 83

A container partially filled with water is moved horizontally with acceleration $a = \frac{g}{\sqrt{3}}$. A small wooden ball of mass m is tied to the bottom of the container using a string. The ball remains inside water with the string inclined at an angle θ to the horizontal as shown in Figure. Assuming that the density of ball is half the density of water, calculate the tension in the string.



SOLUTION

As the container is accelerating the liquid surface will be inclined with horizontal such that

$$\tan\theta = \frac{a}{g} = \frac{g/\sqrt{3}}{g} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2} \text{ and } \cos\theta = \frac{\sqrt{3}}{2}$$

In this case the buoyant force on the ball will be normal to free surface of the liquid. The magnitude of buoyant force F_B is

$$F_B = V_{\text{imm}}\rho_w g_{\text{eff}}$$

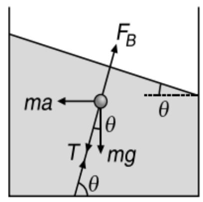
Please note that the immersed volume is equal to the volume of liquid displaced by the immersed part of the body i.e.

$$V_{\text{imm}} = V_{\text{displaced}} = \frac{m}{\rho_w}$$

$$\Rightarrow \quad F_B = \left(\frac{m}{\rho_w}\right)\rho_w\sqrt{g^2 + a^2} = 2m\sqrt{g^2 + \frac{g^2}{3}}$$

$$\Rightarrow \quad F_B = \frac{4mg}{\sqrt{3}}$$

The force acting on the ball are shown in FBD (w.r.t container)



Since the ball is in equilibrium with respect to container, so we have

$$F_B = T + mg\cos\theta + ma\sin\theta$$

$$\Rightarrow \quad \frac{4mg}{\sqrt{3}} = T + mg\frac{\sqrt{3}}{2} + m\left(\frac{g}{\sqrt{3}}\right)\frac{1}{2}$$

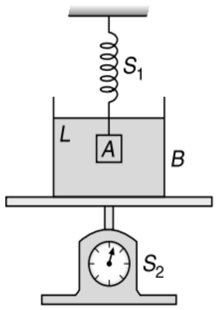
$$\Rightarrow \quad \frac{4mg}{\sqrt{3}} = T + \frac{mg}{2}\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad \frac{4mg}{\sqrt{3}} = T + \frac{2mg}{\sqrt{3}}$$

$$\Rightarrow \quad T = \frac{2mg}{\sqrt{3}}$$

ILLUSTRATION 84

A block A is hanging from spring balance S_1 and immersed in liquid L which is contained in beaker B as shown in Figure. The mass of beaker B is $1\ \text{kg}$ and mass of liquid L is $1.5\ \text{kg}$. The spring balances S_1 and S_2 read $2.5\ \text{kg}$ and $7.5\ \text{kg}$, respectively. Calculate the readings of spring balances S_1 and S_2 when block A is pulled up out of the liquid.



SOLUTION

When body is immersed in liquid, then it experiences an upthrust U, which makes spring balance S_1 to measure the apparent weight of the body. So, the reading R_{S_1} of spring balance S_1 is equal to the apparent weight W_{app} and is given by

$$R_{S_1} = W_{app} = W - U$$

$$\Rightarrow \quad R_{S_1} = W_{app} = W_A - U = 2.5 \text{ kgwt}$$

If W_L be the weight f the liquid, then the reading R_{S_2} of spring balance S_2 is

$$R_{S_2} = W_B + W_L + U$$

$$\Rightarrow \quad W_B + W_L + U = 7.5 \text{ kgwt}$$

Since, $W_B = 1 \text{ kgwt}$, $W_L = 1.5 \text{ kgwt}$

$$\Rightarrow \quad 1 + 1.5 + U = 7.5 \text{ kgwt}$$

$$\Rightarrow \quad U = 5 \text{ kgwt}$$

So, $W_A = U + 2.5 = 7.5 \text{ kgwt}$

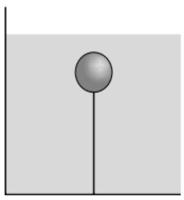
Hence the spring balance S_1 reads 7.5 kg and S_2 reads 2.5 kg.

Test Your Concepts-VII

Based on Archimedes' Principle and Buoyancy

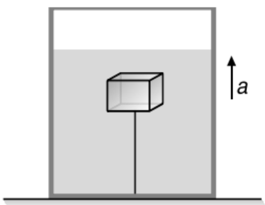
(Solutions on page H.10)

1. A piece of ice is floating in water. Calculate the fraction of volume of the piece of ice outside the water. Density of ice is 900 kgm^{-3} and density of water is 1000 kgm^{-3}.
2. A block is found to weigh W in air when suspended from a spring scale. When completely immersed in water while being attached to the spring scale, it weighs W'. Calculate density ρ of the block in terms of the scale reading and the density of water i.e. ρ_w.
3. A piece of an alloy of mass 96 g is composed of two metals whose specific gravities are 11.4 and 7.4. If the alloy weighs 86 g in water, calculate mass of each metal in the alloy.
4. A solid sphere of mass 2 kg and specific gravity 0.5 is held stationary relative to a tank filled with water as shown. The tank is accelerating vertically upward with an acceleration of 2 ms^{-2}.



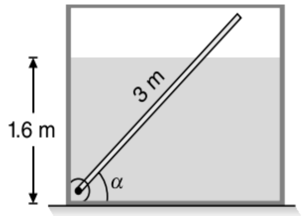
Calculate tension in the thread connecting sphere to the bottom of the tank. If the thread snaps, then calculate acceleration of the sphere with respect to the tank. Assume density of water to be is 1000 kgm^{-3} and $g = 10 \text{ ms}^{-2}$.

5. A rubber ball of mass 10 g and volume 15 cm^3 is dipped in water to a depth of 10 m. If the ball is released from rest, calculate the acceleration of the ball and the time taken by it to reach the surface. Assume density of water to be uniform throughout the depth and $g = 10 \text{ ms}^{-2}$.
6. A solid block is held below the surface of a liquid (of density greater than that of solid block) with the help of a light string as shown in Figure.



The tension in the string is T_0 when the system is at rest. If the system accelerates upwards with an acceleration a, then calculate the tension in the string.

7. A rough surfaced metal cube of size 4 cm and mass 100 g is placed in an empty vessel. Now water is filled in the vessel so that the cube is just immersed in the water. Calculate the average pressure at the bottom surface of vessel which is in contact with the cube. Take $g = 10 \text{ ms}^{-2}$.
8. A wooden rod weighing 25 N having length 3 m, uniform cross sectional area of $9.5 \times 10^{-4} \text{ m}^2$ is mounted on a hinge 1.6 m below the free surface of water as shown.

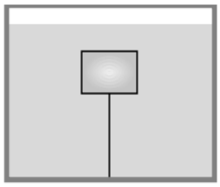


Initially the rod is in the vertical position and when pushed gently it comes to rest making an angle α with the base of the beaker, then calculate α. Also calculate the reaction on the hinge in this position if $\rho_{\text{water}} = 1000\ \text{kgm}^{-3}$ and $g = 9.8\ \text{ms}^{-2}$.

9. A piece of copper having an internal cavity weighs $264\ \text{g}$ in air and $221\ \text{g}$ in water. Calculate volume of the cavity if density of copper is $8.8\ \text{gcm}^{-3}$.

10. A cubical block of length L is floating vertically in a beaker filled with water. Calculate the change in submerged length of block in water if the beaker starts accelerating upwards with an acceleration a.

11. A solid block of volume $1000\ \text{cm}^3$ and density $0.8\ \text{gcm}^{-3}$ dipped in liquid and tied to the bottom of a container (with the help of a light string) filled with liquid of density $1.2\ \text{gcm}^{-3}$ as shown in Figure.

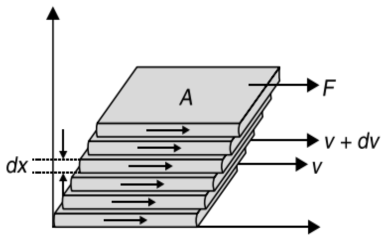


Calculate the tension in the string when the container is stationary and when it is moving up with an acceleration of $5.2\ \text{ms}^{-2}$. Take $g = 9.8\ \text{ms}^{-2}$.

VISCOSITY

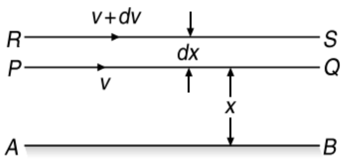
VISCOSITY

For a liquid flowing steadily, it is observed that a layer of liquid slips or tends to slip on adjacent layers in contact. Due to this relative motion between the adjacent layers of liquid, the two layers exert a tangential force on each other which tries to destroy the relative motion between them. The property of a fluid due to which it opposes the relative motion between its different layers is called viscosity (or fluid friction) and the force between the layers opposing the relative motion is called viscous force or viscous drag.



The internal friction of the fluid, which tends to oppose relative motion between different layers of the fluid is called **viscosity.**

Suppose a fluid flows over a fixed surface AB. The layer of the fluid in contact with AB remains at rest and the uppermost layer moves with maximum speed. This shows that every layer opposes the flow of the adjacent upper layer by a tangential viscous force along the surface of the layer.



Consider two layers PQ and RS at distances x and $x+dx$, respectively from AB. Also v and $v+dv$ be the speeds of fluid at PQ and RS respectively. It is found that the viscous force per unit area is proportional to the speed gradient, that is

$$\frac{F}{A} \propto \frac{dv}{dx}$$

$$\Rightarrow \quad F = -\eta A \frac{dv}{dx}$$

where the constant η is called the coefficient of viscosity or simply viscosity of the fluid. The negative sign indicates the opposing nature of the viscous force.

Examples of Viscosity

(i) A stirred liquid, when left, comes to rest on account of viscosity. Thicker liquids like honey, coal tar, glycerine, etc. have a larger viscosity than thinner ones like water. If we pour coal tar and water on a table, the coal tar will stop soon while the water will flow upto quite a large distance.

(ii) If we pour water and honey in separate funnels, water comes out readily from the hole in the funnel while honey takes enough time to do so. This is because honey is much more viscous than water. As honey tends to flow down under gravity, the relative motion between its layers is opposed strongly.

(iii) We can walk fast in air, but not in water. The reason is again viscosity which is very small for air but comparatively much larger for water.

(iv) The cloud particles fall down very slowly because of the viscosity of air and hence appear floating in the sky.

(v) Viscosity comes into play only when there is relative motion between the layers of the same material. This is why it does not act in solids.

COEFFICIENT OF VISCOSITY (η)

It is defined as the tangential force per unit area offered by a fluid layer to create a unit speed gradient. It is measured in $kgm^{-1}s^{-1}$ called decapoise in SI system of units. The cgs unit is $gcm^{-1}s^{-1}$ and is called **poise**.

$$1 \text{ decapoise} = 10 \text{ poise}$$

OR $\quad 1 \text{ poise} = \frac{1}{10} \text{ decapoise}$

The dimensional formula for coefficient of viscosity is $ML^{-1}T^{-1}$.

Conceptual Note(s)

(a) Viscosity comes into play only when there is a relative motion between the layers of the same material. This is why it does not act in solids.

(b) Viscosity of liquid is much greater (about 100 times more) than that of gases i.e. $\eta_L > \eta_G$

EXAMPLE: Viscosity of water is 0.01 poise while that of air is 200 μpoise.

(c) The viscosity of thick liquids like honey, glycerine, coal tar etc. is more than that of thin liquids like water.

SIMILARITIES BETWEEN VISCOSITY AND SOLID FRICTION

Viscosity and solid friction are similar in the following manner.

(a) Both oppose relative motion. Viscosity opposes the relative motion between two adjacent liquid layers, solid friction opposes the relative motion between two solid layers.

(b) Both come into play, whenever there is relative motion between layers of liquid or solid surfaces as the case may be.

(c) Both are due to molecular attractions.

DIFFERENCES BETWEEN VISCOSITY AND SOLID FRICTION

Viscosity and solid friction are different in the following manner.

(a) Viscosity (or viscous drag) between layers of liquid is directly proportional to the area of the liquid layers, whereas friction between two solids is independent of the area of solid surfaces in contact.

(b) Viscous drag is proportional to the relative velocity between two layers of liquid, whereas friction is independent of the relative velocity between two surfaces.

(c) Viscous drag is independent of normal reaction between two layers of liquid, whereas friction is directly proportional to the normal reaction between two surfaces in contact.

SOME APPLICATIONS OF VISCOSITY

Knowledge of viscosity of various liquids and gases have been put to use in daily life. Some applications of its knowledge are discussed as under:

(a) As the viscosity of liquids vary with temperature, proper choice of lubricant is made depending upon season.

(b) Liquids of high viscosity are used in shock absorbers and buffers at railway stations.

(c) The phenomenon of viscosity of air and liquid is used to damp the motion of some instruments.

(d) The knowledge of the coefficient of viscosity of organic liquids is used in determining the molecular weight and shape of the organic molecules.

(e) It finds an important use in the circulation of blood through arteries and veins of human body.

ILLUSTRATION 85

A square plate of area 4 m^2 is made to move horizontally with a speed of 4 ms^{-1}, by applying a horizontal tangential force over the free surface of liquid. The depth of liquid is 2 m and the liquid in contact with the bed is stationary. If coefficient of viscosity of liquid is 0.02 poise, then calculate the tangential force required to move the plate.

SOLUTION

Velocity gradient is $\dfrac{\Delta v}{\Delta x} = \dfrac{4-0}{2-0} = 2 \text{ s}^{-1}$

From Newton's Law, viscous force is given by

$$|F| = \eta A \frac{\Delta v}{\Delta x}$$

$$\Rightarrow \quad F = (0.02 \times 10^{-1}) \times 4 \times (2)$$

$$\Rightarrow \quad F = 0.16 \times 10^{-1}$$

$$\Rightarrow \quad F = 16 \times 10^{-3} \text{ N}$$

So, to keep the plate moving, a force of 16×10^{-3} N must be applied.

ILLUSTRATION 86

A man is rowing a boat with a constant velocity v_0 in a river the contact area of boat is A and coefficient of viscosity is η. If depth of river is D, then calculate the force required to row the boat.

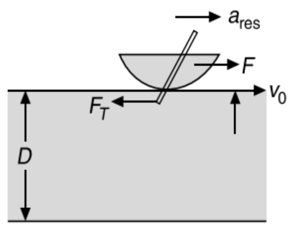
SOLUTION

According to Newton's Second Law, we have

$$F - F_T = ma$$

Since boat moves with constant velocity, so $a = 0$.

$$\Rightarrow \quad F = F_T$$



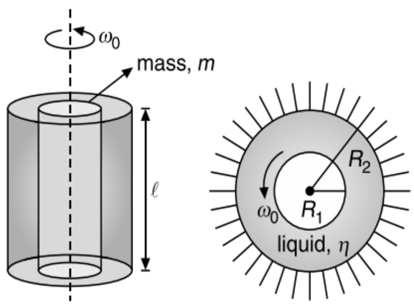
Also, we know that $F_T = \eta A \dfrac{dv}{dx}$

where, $\dfrac{dv}{dx} = \dfrac{v_0 - 0}{D} = \dfrac{v_0}{D}$

$$\Rightarrow \quad F = F_T = \frac{\eta A v_0}{D}$$

ILLUSTRATION 87

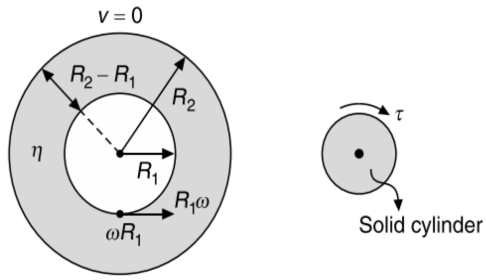
A liquid of viscosity η is filled between an outer fixed cylinder and an inner cylinder as shown in Figure.



The central solid cylinder starts rotating with an initial angular velocity ω_0. Calculate the time after which the angular velocity of the central cylinder becomes half its initial velocity.

SOLUTION

At any instant, the speed of the liquid layer in contact with outer fixed cylinder is zero, whereas the speed of the liquid layer attached with the inner rotating cylinder will be $R_1\omega$, where $\omega < \omega_0$.



The viscous force developed is given by

$$|F| = \eta A\frac{dv}{dx}, \text{ where } \frac{dv}{dx} = \frac{\omega R_1 - 0}{R_2 - R_1}$$

$$\Rightarrow \quad F = \eta(2\pi R_1 l)\left(\frac{\omega R_1}{R_2 - R_1}\right)$$

The torque due to this viscous force about axis is

$$\tau = FR_1 = \frac{2\pi\eta R_1^3 l\omega}{R_2 - R_1}$$

Since $\tau = I\alpha$, where $\alpha = -\frac{d\omega}{dt}$ and $I = \frac{1}{2}mR_1^2$

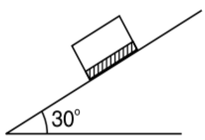
$$\Rightarrow \quad I\alpha = \frac{mR_1^2}{2}\left(-\frac{d\omega}{dt}\right) = \frac{2\pi\eta R_1^3 l\omega}{R_2 - R_1}$$

$$\Rightarrow \quad -\int_{\omega_0}^{\frac{\omega_0}{2}} \frac{d\omega}{\omega} = \frac{4\pi\eta R_1 l}{m(R_2 - R_1)}\int_0^t dt$$

$$\Rightarrow \quad t = \frac{m(R_2 - R_1)\ln 2}{4\pi\eta R_1 l}$$

ILLUSTRATION 88

A cubical block of side 2 m having mass 20 kg slides with constant velocity of 10 ms^{-1} on an inclined plane lubricated with the oil of viscosity $\eta = 10^{-1}$ poise. Taking $g = 10\text{ ms}^{-2}$, calculate the thickness of layer of liquid.

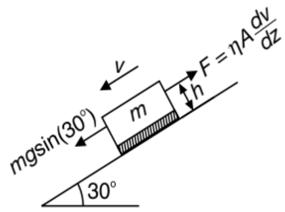


SOLUTION

Since, we have $F = \eta A\frac{dv}{dx}$

$$\Rightarrow \quad F = \eta A\frac{dv}{dx} = mg\sin\theta, \text{ where } \frac{dv}{dx} = \frac{v}{h}$$

$$\Rightarrow \quad (20)(10)(\sin 30°) = \eta(4)\left(\frac{10}{h}\right)$$



Since $1\text{ poise} = \frac{1}{10}$ decapoise

$$\Rightarrow \quad \eta = 10^{-1}\text{ poise} = 10^{-2}\text{ decapoise}$$

$$\Rightarrow \quad h = \frac{40 \times 10^{-2}}{100}$$

$$\Rightarrow \quad h = 4 \times 10^{-3}\text{ m} = 4\text{ mm}$$

ILLUSTRATION 89

A plate of area 100 cm^2 is placed on the upper surface of castor oil, 2 mm thick. Taking the coefficient of viscosity to be 15.5 poise, calculate the horizontal force necessary to move the plate with a velocity 3 cms^{-1}.

SOLUTION

The (horizontal tangential) viscous force is given by

$$F = -\eta A\frac{dv}{dx}$$

Given that, $\eta = 15.5$ poise, $A = 100\text{ cm}^2$ and

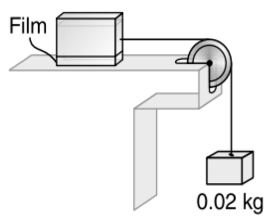
$$\frac{dv}{dx} = \frac{v_2 - v_1}{x_2 - x_1} = \frac{(0-3)\text{ cms}^{-1}}{(2-0)\text{ mm}} = -\frac{3}{0.2}\text{ s}^{-1} = -15\text{ s}^{-1}$$

$$\Rightarrow \quad F = -15.5 \times 100 \times (-15) = 2.235 \times 10^4\text{ dyne}$$

$$\Rightarrow \quad F = 0.2325\text{ N}$$

ILLUSTRATION 90

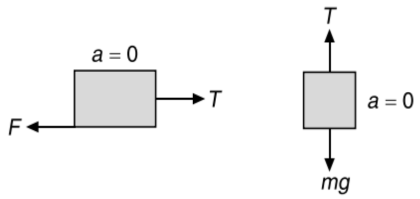
A metal block of area 0.10 m^2 is connected to a 0.02 kg mass through a light. The string passes over an ideal pulley (considered massless and frictionless) as shown in Figure.



A liquid film of thickness 0.15 mm is placed between the metal block and the table. When released the block moves to the right with a constant speed of 0.075 ms^{-1}. Calculate the coefficient of viscosity of the liquid if $g = 10\text{ ms}^{-2}$.

SOLUTION

Since the block is moving with a constant speed, so tension developed in the string is equal to the viscous force F. The free body diagram of the block and the mass is shown in Figure.



From the FBD of ck and the mass, we have

$$T = F = mg$$

where, $F = \eta A\left(\frac{\Delta v}{\Delta x}\right)$

$$\Rightarrow \quad mg = \eta A\left(\frac{\Delta v}{\Delta x}\right)$$

Given that, area of the film is $A = 0.10\ \text{m}^2$, thickness of the film is $\Delta x = 0.15\ \text{mm} = 0.15\times10^{-3}\ \text{m}$ and the relative velocity of plate $\Delta v = 0.075\ \text{ms}^{-1}$.

$$\Rightarrow \quad (0.02)(10) = \eta(0.10)\left(\frac{0.075}{0.15\times10^{-3}}\right)$$

$$\Rightarrow \quad \eta = \frac{0.2\times0.15\times10^{-3}}{0.10\times0.075}$$

$$\Rightarrow \quad \eta = 4\times10^{-3}\ \text{Pas}$$

$$\Rightarrow \quad \eta = 0.004\ \text{Pas}$$

VARIATION OF VISCOSITY

With Temperature

The viscosity of liquids is mainly due to the cohesive forces between the molecules of the neighbouring layers. When the temperature increases, the kinetic energy of the molecules increases, resulting in decrease of cohesive forces. Thus, the viscosity of liquids decreases (or fluidity increases) with rise in temperature.

The viscosity of gases is mainly due to the diffusion of molecules from one moving layer to the neighbouring layers. With increase of temperature, the rate of diffusion increases. Hence the viscosity of gases increases with increase of temperature.

Problem Solving Technique(s)

(a) For liquids, the viscosity decreases with rise in temperature.
(b) For gases, the viscosity increases with rise in temperature.

With Pressure

As pressure increases, the molecules come marginally closer, resulting in an increase in viscosity of liquids. However, in gases, viscosity is found to be independent of pressure, provided the pressure is not too small.

Conceptual Note(s)

(a) For liquids, as pressure increases viscosity increases.
(b) For gases, as pressure increases or decreases, viscosity remains same provided pressure is not too small.
(c) With increase in pressure, the viscosity of liquids (except water) increases while that of gases is practically independent of pressure. The viscosity of water decreases with increase in pressure.
(d) For temperatures above $+32\ °\text{C}$, water behaves like other liquids. Its viscosity increases with increasing pressure.
For temperatures below $+32\ °\text{C}$ and under pressures of up to 20 MPa, the water's viscosity decreases with increasing pressure.
The reason is that the structure of the three-dimensional network of hydrogen bridges is destroyed. This network is rather stronger than the structures of other low-molecular liquids.

STOKE'S LAW

When a body moves through a liquid, then the liquid layer in immediate contact with the body is dragged with it. This establishes relative motion between consecutive liquid layers near the body, due to which a viscous force starts acting on the body. The liquid exerts a viscous force on the body to oppose its motion. The magnitude of the viscous force depends on the shape and size of the body, its speed and the viscosity of the liquid.

Stoke's law governs the motion of a small sphere through a liquid column of infinite length. When a body moves through a fluid, it experience a viscous drag. This viscous drag experienced by a small sphere of radius r, moving with a speed v through a fluid of viscosity η is given by

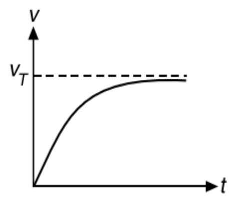
$$F = 6\pi\eta rv$$

This is called **Stoke's Law.**

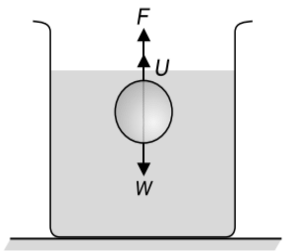
TERMINAL VELOCITY

It has been observed that when a spherical body starts falling in a liquid column of infinite length, then initially the viscous drag on the body is zero as initial velocity of the body is zero. However, as the velocity of the body increases, then the viscous drag on the body also increases in accordance with the Stoke's Law. At a certain maximum velocity called as Terminal velocity (v_T) it is observed that the weight (W) of the body is balanced by the combined effect of Upthrust (U) and the viscous drag (F) acting on the body.

For a body falling through a liquid column of infinite length, the variation in the velocity of a body with time is shown in Figure.



When a sphere is falling under gravity in a liquid of infinite column, the forces acting on it are its weight W (downwards), the upthrust U (upwards) and the viscous drag F (also upwards), as shown in Figure.



The net downward force when $v < v_T$ is

$$F_{net} = W - (U + F)$$

However, when $v = v_T$, the net force on the sphere becomes zero, so

$$F + U = W, \text{ where}$$

$$F = 6\pi\eta r v_T,\ U = \left(\frac{4}{3}\pi r^3\right)\sigma g \text{ and } W = \left(\frac{4}{3}\pi r^3\right)\rho g$$

So, we have

$$6\pi\eta r v_T = W - U$$

$$\Rightarrow \quad 6\pi\eta r v_T = \frac{4}{3}\pi r^3(\rho - \sigma)g$$

where, ρ is the density of the sphere and
σ is the density of the liquid.

$$\Rightarrow \quad v_T = \frac{2}{9}\frac{(\rho - \sigma)r^2 g}{\eta} \qquad \text{\{Terminal Velocity\}}$$

If $\sigma > \rho$, the sphere will move upwards with a constant speed. So, we observe that,

(a) Terminal velocity depends on the radius of the sphere, so if radius of the spherical body is made n times, then the terminal velocity will become n^2 times.

(b) Increasing the density of body (ρ), will increase the terminal velocity.

(c) Increasing the density of the liquid (σ) and its viscosity (η) will decrease the terminal velocity.

(d) If $\rho > \sigma$, then terminal velocity will be positive and hence the spherical body will attain constant velocity in downward direction.

(e) If $\rho < \sigma$, then terminal velocity will be negative and hence the spherical body will attain constant velocity in upward direction.

EXAMPLE: Air bubble in a liquid and clouds in sky.

ILLUSTRATION 91

Eight spherical raindrops of equal size are falling vertically through air with a terminal velocity of $0.5\ \text{ms}^{-1}$. What would be the terminal speed of the bigger spherical drop formed if these drops coalesce to form the bigger drop?

SOLUTION

If r be the radius of small rain drop, then the terminal speed of the small drop is

$$v_T \propto r^2 \qquad \ldots(1)$$

If R be the radius of large drop, then equating the initial and the final volumes, we get

$$\frac{4}{3}\pi R^3 = 8\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad R = 2r$$

$$\Rightarrow \quad \frac{v_T'}{v_T} = \left(\frac{R}{r}\right)^2 = 4$$

$$\Rightarrow \quad v_T' = 4v_T = (4)(0.5) = 2\ \text{ms}^{-1}$$

ILLUSTRATION 92

Calculate the terminal speed with which an air bubble 0.8 mm in diameter will rise in a liquid of coefficient of viscosity $0.15\ \text{Nsm}^{-2}$ and specific gravity 0.9, if the density of air is $1.293\ \text{kgm}^{-3}$.

SOLUTION

The terminal speed of bubble is

$$v_T = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}$$

where, $r = 0.4 \times 10^{-3}\ \text{m}$, $\sigma = 0.9 \times 10^3\ \text{kgm}^{-3}$

$\rho = 1.293\ \text{kgm}^{-3}$, $\eta = 0.15\ \text{Nsm}^{-2}$

and $g = 9.8\ \text{ms}^{-2}$

$$\Rightarrow \quad v_T = \frac{2}{9} \times \frac{(0.4 \times 10^{-3})^2(1.293 - 0.9 \times 10^3) \times 9.8}{0.15}$$

$$\Rightarrow \quad v_T = -0.0021\ \text{ms}^{-1}$$

$$\Rightarrow \quad v_T = -0.21\ \text{cms}^{-1}$$

Please note that the negative sign means that the bubble will rise up.

ILLUSTRATION 93

A spherical ball of radius 3×10^{-4} m and density 10^4 kgm^{-3} falls freely under gravity through a distance h before entering a tank of water. If after entering the water the velocity of the ball does not change, calculate h, if the coefficient of viscosity of water is 9.8×10^{-6} Nsm^{-2}.

SOLUTION

Before entering the water, the velocity of ball is $\sqrt{2gh}$. If after entering the water this velocity does not change then this value should be equal to the terminal velocity. Therefore,

$$\sqrt{2gh}=\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}$$

$$\Rightarrow \quad h=\frac{1}{2g}\left[\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}\right]^2$$

$$\Rightarrow \quad h=\frac{2}{81}\times\frac{r^4(\rho-\sigma)^2g}{\eta^2}$$

$$\Rightarrow \quad h=\frac{2}{81}\times\frac{(3\times10^{-4})^4(10^4-10^3)^2\times9.8}{(9.8\times10^{-6})^2}$$

$$\Rightarrow \quad h=1.65\times10^3 \text{ m}$$

ILLUSTRATION 94

A spherical ball is moving with terminal velocity inside a liquid. Determine the relationship of rate of heat loss with the radius of ball.

SOLUTION

Rate of heat loss is power dissipated (P)

$$\Rightarrow \quad P=F\times v=(6\pi\eta rv)v=6\pi\eta rv^2$$

$$\Rightarrow \quad P=6p\eta r\left[\frac{2}{9}\frac{r^2}{\eta}(\rho_0-\rho_l)g\right]^2$$

Rate of heat loss $\propto r^5$

ILLUSTRATION 95

A sphere is dropped under influence of gravity through a fluid of coefficient of viscosity η. If the average acceleration of the sphere is half of the initial acceleration, then show that time taken by the sphere to attain terminal speed is independent of density of fluid.

SOLUTION

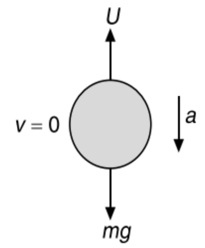
Let r be the radius of the sphere and ρ be its density, then mass (m) of the sphere is

$$m=\frac{4}{3}\pi r^3\rho$$

If σ be the density of the fluid, then mass of fluid displaced by the sphere is

$$m'=\frac{4}{3}\pi r^3\sigma$$

Initially, since $v=0$, so the viscous force acting on the sphere is also zero and hence the free body diagram of the sphere when it is just dropped is shown in Figure.



If a be the initial acceleration of the sphere, then we have

$$mg-U=ma$$

$$\Rightarrow \quad \frac{4}{3}\pi r^3\rho g-\frac{4}{3}\pi r^3\sigma g=\left(\frac{4}{3}\pi r^3\rho\right)a$$

$$\Rightarrow \quad a=\left(\frac{\rho-\sigma}{\rho}\right)g$$

The terminal velocity of the ball is

$$v=\frac{2r^2(\rho-\sigma)g}{9\eta}$$

If the average acceleration of the sphere during the fall (till the drop acquires terminal velocity) is a', then according to the problem, we have

$$a'=\frac{a}{2}=\frac{(\rho-\sigma)g}{2\rho}$$

Also, the average acceleration is given by

$$a_{av}=a'=\frac{\Delta v}{\Delta t}$$

$$\Rightarrow \quad \Delta v=a'\Delta t$$

$$\Rightarrow \quad \frac{2r^2(\rho-\sigma)g}{9\eta}-0=a'(t-0)=a't$$

$$\Rightarrow \quad \frac{2r^2(\rho-\sigma)g}{9\eta}=\left(\frac{\rho-\sigma}{2\rho}\right)gt$$

$$\Rightarrow \quad t=\frac{4r^2\rho}{9\eta}$$

Hence, the time taken to acquire the terminal velocity is independent of the fluid density.

FLOW OF LIQUID IN TUBE: CRITICAL VELOCITY

When a liquid flow in a tube, the viscous forces oppose the flow of the liquid. Hence a pressure difference is applied between the ends of the tube which maintains the flow of the liquid. If all particles of the liquid passing through a particular point in the tube move along the same path, the flow of the liquid is called 'stream-line flow'. This occurs only when the velocity of flow of the liquid is below a certain limiting value called 'critical velocity'.

When the velocity of flow exceeds the critical velocity, the flow is no longer stream-line but becomes turbulent. In this type of flow, the motion of the liquid becomes zig-zag.

It is found that the flow is streamlined only when the rate of flow is small. Above a certain speed, called the critical speed, the flow becomes turbulent.

Reynold showed that there is a particular combination of four factors, namely density (ρ), velocity (v), tube diameter (d) and viscosity (η), which determines the nature of flow of a viscous fluid through the tube. This combination is called Reynolds Number and is given by

$$R = \frac{\rho v d}{\eta}$$

The Reynolds Number R is a **dimensionless variable**. It is found empirically that when, approximately

(a) $R < 2000$: The flow is Laminar.

(b) $R > 3000$: The flow is turbulent.

(c) $2000 < R < 3000$: The flow is unstable and may be changing from one type to another.

ILLUSTRATION 96

Water is flowing in a pipe of radius 1.5 cm with an average velocity $15\ \text{cms}^{-1}$. What is the nature of flow? Given coefficient of viscosity of water is $10^{-3}\ \text{kgm}^{-1}\text{s}^{-1}$ and its density is $10^3\ \text{kgm}^{-3}$.

SOLUTION

Reynolds number for the given situation is given as

$$R = \frac{\rho v d}{\eta}$$

where, $\rho = 10^3\ \text{kg m}^{-3}$, coefficient of viscosity of the liquid is $\eta = 10^{-3}\ \text{kgm}^{-1}\text{s}^{-1}$, average velocity of water is $v = 15\ \text{cms}^{-1} = 0.15\ \text{ms}^{-1}$ and diameter of the pipe, is $d = 2 \times 1.5\ \text{cm} = 3\ \text{cm} = 0.03\ \text{m}$.

$$\Rightarrow \quad R = \frac{10^3 \times 0.15 \times 0.03}{10^{-3}}$$

$$\Rightarrow \quad R = 10^6 \times 0.0045$$

$$\Rightarrow \quad R = 4500$$

Since, $R > 2000$, hence the flow is turbulent.

ILLUSTRATION 97

Calculate the maximum average velocity of water in a tube of diameter 2 cm, so that the flow is laminar. Assume coefficient of viscosity of water to be $10^{-3}\ \text{Nsm}^{-2}$.

SOLUTION

For laminar flow through the tube, the maximum value for Reynolds number R is 2000. If v is the maximum average velocity of water, then

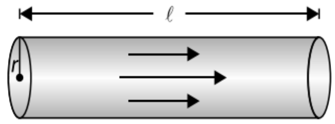
$$R_e = \frac{\rho v d}{\eta}$$

where, $d = 2\ \text{cm} = 0.02\ \text{m}$, $\eta = 10^{-3}\ \text{Nsm}^{-2}$ and $\rho = 10^3\ \text{kgm}^{-3}$.

$$\Rightarrow \quad v = \frac{\eta R}{\rho d} = \frac{(10^{-3})(2000)}{(10^3)(0.02\ \text{m})} = 0.1\ \text{ms}^{-1}$$

STEADY FLOW OF LIQUID THROUGH A CAPILLARY TUBE: POISEUILLE'S FORMULA

Poiseuille formula is designed for a streamlined flow of a liquid through a horizontal tube. If a viscous liquid flow in a tube, the velocity is greatest at the centre of the tube and decreases to zero at the wall. If the velocity is small, the flow is streamlined (also called the steady or laminar flow).



Poiseuille studied the streamline flow of liquid through horizontal capillary or horizontal tube. He found that if a pressure difference (P) is maintained across the two ends of a capillary tube of length l having radius r, then the volume of liquid flowing per second through the tube i.e. the rate of flow of volume of liquid (V) through the capillary is

(a) directly proportional to the pressure difference $(\Delta P = p)$.

(b) directly proportional to the fourth power of radius (r) of the capillary tube.

(c) inversely proportional to the coefficient of viscosity (η) of the liquid.

(d) inversely proportional to the length (l) of the capillary tube.

Let l be the length of the tube and r be its radius. If a fluid of viscosity η flows steadily under a pressure difference p, then the rate of flow of liquid Q i.e. volume V of fluid flowing per unit time is given by

$$Q = \frac{V}{t} = \frac{\pi r^4 p}{8 \eta l} \qquad \text{\{Poiseuille's Formula\}}$$

The quantity $\frac{\Delta P}{l} = \frac{p}{l}$ is also called the pressure gradient.

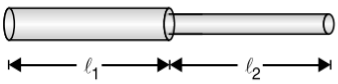
The rate of flow of liquid is due to the pressure difference just like rate of flow of charge (i.e. current) which flows due to the potential difference.

So liquid resistance R is simply defined as the pressure difference (P) per unit rate of flow of liquid (Q).

$$R=\frac{\text{Pressure Difference}}{\text{Rate of Flow of Liquid}}=\frac{\Delta P}{Q}=\frac{8\eta l}{\pi r^4}$$

COMBINATION OF TUBES IN SERIES

When two tubes of length l_1 and l_2 having radii r_1 and r_2 are connected in series across a pressure difference P, then



$$P=P_1+P_2 \qquad \ldots(1)$$

where P_1 and P_2 are the pressure difference across the first and second tube respectively.
Since the volume of liquid flowing through both the tubes *i.e.* rate of flow of liquid is same.

Therefore $V=V_1=V_2$

$$\Rightarrow \quad V=\frac{\pi P_1 r_1^4}{8\eta l_1}=\frac{\pi P_2 r_2^4}{8\eta l_2} \qquad \ldots(2)$$

Substituting the value of P_1 and P_2 from equations (1) and (2), we get

$$P=P_1+P_2=V\left(\frac{8\eta l_1}{\pi r_1^4}+\frac{8\eta l_2}{\pi r_2^4}\right)$$

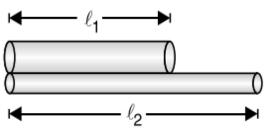
$$\Rightarrow \quad V=\frac{P}{\left(\frac{8\eta l_1}{\pi r_1^4}+\frac{8\eta l_2}{\pi r_2^4}\right)}=\frac{P}{R_1+R_2}=\frac{P}{R_{\text{eff}}}$$

where R_1 and R_2 are the liquid resistances offered to the flow of liquid by the tubes.
So, effective liquid resistance for series combination of the tubes is

$$R_{\text{eff}}=R_1+R_2$$

COMBINATION OF TUBES IN PARALLEL

When two tubes of length l_1 and l_2 having radii r_1 and r_2 are connected in series across same pressure difference P, then



$$P=P_1=P_2$$

The total rate of flow of volume (V) of the liquid will be the sum of volume of liquid flowing per second through tube 1 and volume of liquid flowing per second through tube 2, i.e.

$$V=V_1+V_2$$

$$V=\frac{P\pi r_1^4}{8\eta l_1}+\frac{P\pi r_2^4}{8\eta l_2}=P\left(\frac{\pi r_1^4}{8\eta l_1}+\frac{\pi r_2^4}{8\eta l_2}\right)$$

$$\Rightarrow \quad V=P\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{P}{R_{\text{eff}}}$$

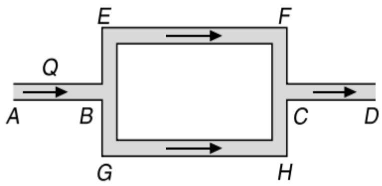
So, the effective liquid resistance in parallel combination

$$\frac{1}{R_{eff}}=\frac{1}{R_1}+\frac{1}{R_2}$$

$$\Rightarrow \quad R_{eff}=\frac{R_1R_2}{R_1+R_2}$$

ILLUSTRATION 98

A liquid is flowing through horizontal pipes as shown in Figure.



Length of different pipes have the ratio

$$L_{AB}=L_{CD}=\frac{L_{EF}}{2}=\frac{L_{GH}}{2}$$

Similarly, radii of different pipes have the ratio,

$$r_{AB}=r_{EF}=r_{CD}=\frac{r_{GH}}{2}$$

If pressure at A is $2P_0$ and pressure at D is P_0 and the rate of flow of volume of the liquid through the pipe AB is Q, then calculate the rate of flow of volume of liquid through the pipes EF and GH. Also calculate the pressure at E and F.

SOLUTION

The liquid resistance R is given by

$$R=\frac{\text{Pressure Difference}}{\text{Rate of Flow of Liquid}}=\frac{\Delta P}{Q}=\frac{8\eta l}{\pi r^4}$$

$$\Rightarrow \quad R\propto\frac{l}{r^4}$$

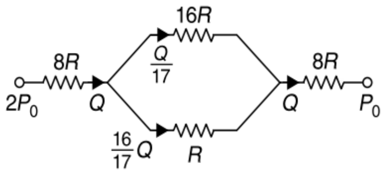
$$\Rightarrow \quad R_{AB}:R_{CD}:R_{EF}:R_{GH}=\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^4}:\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^4}:\frac{(1)}{\left(\frac{1}{2}\right)^4}:\frac{(1)}{(1)^4}$$

$$\Rightarrow \quad R_{AB}:R_{CD}:R_{EF}:R_{GH}=8:8:16:1$$

If $R_{GH}=R$, then

$$R_{AB}=8R,\; R_{CD}=8R \text{ and } R_{EF}=16R$$

The equivalent liquid resistance circuit for the arrangement is shown in Figure.



As the current is distributed in the inverse ratio of the resistance (in parallel). The Q will be distributed in the inverse ratio of R, so the volume flow rate through EF will be $\frac{Q}{17}$ and that from GH will be $\frac{16}{17}Q$. The equivalent liquid resistance offered by the arrangement is

$$R_{net}=8R+\frac{(16R)(R)}{(16R)+(R)}+8R=\frac{288}{17}R$$

Since rate of flow of volume is

$$Q=\frac{\Delta P}{R_{net}}$$

$$\Rightarrow \quad Q=\frac{(2P_0-P_0)}{\frac{288}{17}R}=\frac{17P_0}{288R}$$

Now, let P_1 be the pressure at E, then

$$2P_0-P_1=8QR=8\left(\frac{17P_0}{288R}\right)R=\left(\frac{8\times17}{288}\right)P_0$$

$$\Rightarrow \quad P_1=\left(2-\frac{8\times17}{288}\right)P_0=1.53P_0$$

Similarly, if P_2 be the pressure at F, then

$$P_2-P_0=8QR$$

$$\Rightarrow \quad P_2=P_0+\frac{8\times17}{288}P_0$$

$$\Rightarrow \quad P_2=1.47P_0$$

ILLUSTRATION 99

Water flows through a capillary tube of radius r and length l at a rate of 40 mls^{-1}, when connected to a pressure difference of h cm of water. Another tube of the same length but radius $\frac{r}{2}$ is connected in series with this tube and the combination is connected to the same pressure head. Calculate the pressure difference across each tube and the rate of flow of water through the combination.

SOLUTION

The volume (quantity) of liquid flowing per second through a capillary tube maintained across a pressure difference $P=h\rho g$ is

$$Q=\frac{\pi r^4P}{8\eta\ell}=\frac{\pi(h\rho g)r^4}{8\eta\ell}=40\text{ mls}^{-1} \qquad \ldots(1)$$

where l is length of tube, r is radius of tube and η is coefficient of viscosity of liquid.

When tubes are connected in series, the amount of liquid flowing through them is equal, i.e.,

$$Q=\frac{\pi P_1r^4}{8\eta l}=\frac{\pi P_2\left(\frac{r}{2}\right)^4}{8\eta l} \qquad \ldots(2)$$

and $\quad P_1+P_2=P=\rho gh \qquad \ldots(3)$

From equation (2), we get

$$P_1=\frac{P_2}{16}$$

$$\Rightarrow \quad P_2=16P_1$$

Substituting for P_2 in equation (3), we get

$$P_1+16P_1=\rho gh$$

$$\Rightarrow \quad P_1=\frac{\rho gh}{17} \text{ and } P_2=\frac{16\rho gh}{17}$$

Substituting P_1 or P_2 in equation (2) and using equation (1), we get

$$Q=\frac{\pi(\rho gh)r^4}{17\times8\eta\ell}=\frac{1}{17}\left[\frac{\pi(\rho gh)r^4}{8\eta\ell}\right]$$

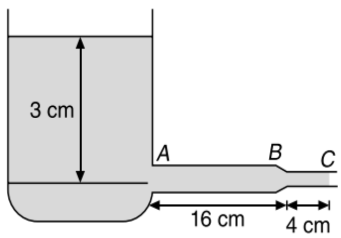
$$\Rightarrow \quad Q=\frac{40}{17}\text{ mLs}^{-1}$$

ILLUSTRATION 100

Two capillary tubes AB and BC are joined end to end at B. Tube AB is 16 cm long, has a diameter 4 mm and tube BC is 4 cm long, has a diameter 2 mm. The composite tube is held horizontally with A connected to a vessel of water giving a constant pressure head of 3 cm and C is open to the air. Calculate the pressure difference between B and C.

SOLUTION

According to the problem, AB and BC are two capillary tubes which are joined end to end at B as shown in Figure.



Let the atmospheric pressure be equal to h cm of water column. So, pressure at A is $P_A=(h+3)$ and pressure at C is $P_C=h$. Further, if h' be the pressure at B, then pressure difference across AB is

$$P_{AB}=(h+3)-h'$$

and pressure difference across BC is

$$P_{BC}=h'-h$$

Since both tubes are in series, so rate of flow of liquid through both the tubes is same, i.e.

$$Q = \text{constant}$$

$$\Rightarrow \quad \frac{\pi P_1 r_1^4}{8\eta l_1} = \frac{\pi P_2 r_2^4}{8\eta l_2}$$

$$\Rightarrow \quad \frac{P_1 r_1^4}{l_1} = \frac{P_2 r_2^4}{l_2}$$

$$\Rightarrow \quad \frac{P_1}{P_2} = \frac{l_1}{l_2}\left(\frac{r_2}{r_1}\right)^4$$

Since, $l_1 = 16$ cm, $l_2 = 4$ cm

$$r_1 = 0.2 \text{ cm and } r_2 = 0.1 \text{ cm}$$

$$\Rightarrow \quad \frac{(h+3-h')}{(h'-h)} = \frac{16}{4} \times \left(\frac{0.1}{0.2}\right)^4$$

$$\Rightarrow \quad \frac{h+3-h'}{h'-h} = \frac{16}{4} \times \frac{1}{16} = \frac{1}{4}$$

$$\Rightarrow \quad h'-h = 4h+12-4h'$$

$$\Rightarrow \quad 5h' = 5h+12$$

$$\Rightarrow \quad h' = h+24$$

So, pressure difference across BC is

$$P_{BC} = h'-h = 2.4 \text{ cm of water column.}$$

ILLUSTRATION 101

A cylindrical vessel of area of cross-section A and filled with liquid to a height of h_1 has a capillary tube of length l and radius r protruding horizontally at its bottom. If the viscosity of liquid is η, density ρ and $g = 9.8 \text{ ms}^{-2}$, find the time in which the level of water in vessel falls to h_2.

SOLUTION

Let h be the height of water level in the vessel at instant t which decrease by dh in time dt.
So, rate of flow of water through capillary tube,

$$Q = -A\left(\frac{dh}{dt}\right) \qquad \ldots(1)$$

The negative sign indicates that as t increases, h decreases. The rate of flow of liquid through a horizontal tube is given by Poiseuille formula, i.e.

$$Q = \frac{\pi p r^4}{8\eta l} \qquad \ldots(2)$$

From (1) and (2), we get

$$-A\frac{dh}{dt} = \frac{\pi \rho g h r^4}{8\eta l} \qquad \{\because\ p = \rho g h\}$$

$$\Rightarrow \quad dt = -\frac{8\eta l A}{\pi \rho g r^4}\frac{dh}{h}$$

So, the required time is obtained by integrating this expression within suitable limits.

$$\Rightarrow \quad t = -\frac{8\eta l A}{\pi \rho g r^4}\int_{h_1}^{h_2}\frac{dh}{h} = -\frac{8\eta l A}{\pi \rho g r^4}\log_e h\Big|_{h_1}^{h_2}$$

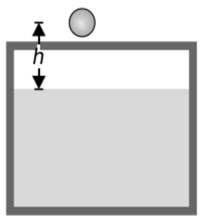
$$\Rightarrow \quad t = -\frac{8\eta l A}{\pi \rho g r^4}\log_e\frac{h_2}{h_1} = \frac{8\eta l A}{\pi \rho g r^4}\log_e\frac{h_1}{h_2}$$

Test Your Concepts-VIII

Based on Viscosity and Terminal Speed

(Solutions on page H.12)

1. The velocity of water in a river is 18 kmhr^{-1} near the surface. If the river is 5 m deep, find the shear stress between the horizontal layers of water. The coefficient of viscosity of water is 10^{-2} poise.
2. Twenty seven identical spherical raindrops are falling vertically through air with a terminal velocity of 1 ms^{-1}. Calculate the terminal speed of the bigger spherical drop formed if these drops coalesce.
3. A solid rubber ball of density d and radius R falls vertically through an air column of very large length. Assume that the air resistance acting on the ball is $F = KRv$, where K is constant and v is its velocity. Calculate the terminal velocity of the ball.
4. A plate of area 2 m^2 is made to move horizontally with a speed of 2 ms^{-1} by applying a horizontal tangential force over the free surface of a liquid. If the depth of the liquid is 1 m and the liquid in contact with the bed is stationary. Coefficient of viscosity of liquid is 0.01 poise. Find the tangential force needed to move the plate.
5. A powder comprising particles of various sizes is stirred up in a vessel filled to a height of 10 cm with water. If the particles are assumed to be spherical, then calculate the size of the largest particle that will remain in suspension after 1 hr if density of powder is 4 gcm^{-3} and coefficient of viscosity of water is 0.01 poise.
6. A ball of radius r and density ρ falls freely under gravity through a distance h before entering water.



Velocity of ball does not change even on entering water. If viscosity of water is η, Find h.

7. Two square metal plates, each of side 10 cm are immersed in water. One plate moves parallel to the other with a speed of 5 cms^{-1}. If the viscous force is 150 dyne, calculate their distance of separation if the coefficient of viscosity of water is $\eta_{\text{water}} = 0.001\text{ Pl}$.

8. Calculate the Reynolds number for blood flowing at 30 cms^{-1} through an aorta of radius 1.0 cm. Assume that the blood has a viscosity of 4 mPa s and a density of 1060 kgm^{-3}. What is the nature of flow?

9. Capillaries of length l, $2l$ and $\frac{l}{2}$ are connected in series. Their radii are r, $\frac{r}{2}$ and $\frac{r}{3}$ respectively. If stream line flow is maintained and pressure across the first capillary is P_1 and across second and third being P_2 and P_3 respectively. Calculate the ratio of P_2 and P_3.

10. An engineer wants to have the same flow rate of water and light machine oil from the pipes of the same length and with the same pressure gradient. Calculate the ratio of the radii of the two pipes, if the coefficient of viscosity of water is 0.01 poise and that of light machine oil is 12.96 poise.

FLUID DYNAMICS

MOTION OF A FLUID

In order to describe the motion of a fluid, in principle one might apply Newton's laws to a particle (a small volume element of fluid) and follow its progress in time. This is a difficult approach. Instead, we consider the properties of the fluid, such as velocity and pressure, at fixed points in space.

In order to simplify the discussion, we make several assumptions:

(i) **The flow is steady:** The velocity and pressure at each point are constant in time

(ii) **The fluid is incompressible:** The density of fluid is constant throughout.

(iii) **The flow is irrotational:** A tiny paddle wheel placed in the liquid will not rotate.
In rotational flow, for example, in eddies, the fluid has net angular momentum about a given point.

(iv) **The fluid is non viscous:** There is no dissipation of energy due to internal friction between adjacent layer in the fluid.

Such type of fluid is called an **Ideal fluid** also called **SIIN fluid**.

FLOW OF IDEAL FLUID: BASIC DEFINITIONS

Ideal Fluid

An incompressible, streamline, irrotational, non-viscous fluid is called an **ideal fluid (SIIN fluid)**. So, an ideal fluid (i.e. a **S**teady, **I**rrotational, **I**ncompressible and **N**on-viscous liquid is an SIIN fluid.

Steady Flow

If the fluid velocity at any point is constant in time, then the flow is said to be **steady**.

Non-Steady Flow

If the fluid velocity at any point varies with time, then the flow is said to be **non-steady**.

Streamline

A **streamline** is a curve, the tangent to which at a point gives the direction of fluid velocity at that point.

✓ It is analogous to a line of force in an electric or magnetic field.

✓ In steady flow, the pattern of streamlines is stationary with time and therefore, also called a **streamline** flow.

✓ No two streamlines can ever cross one another, for if they did, a fluid particle arriving at that point would have possessed two directions (or two velocities) and hence the flow would never be steady.

Tube of Flow

A tubular region of fluid enclosed in by a boundary consisting of streamlines is called a **tube of flow**.

No fluid can cross the boundaries of a tube of flow and, therefore, a tube of flow behaves like a pipe of the same shape.

Rotational Flow

The flow of liquid is said to be **rotational** if the angular velocity is non zero.

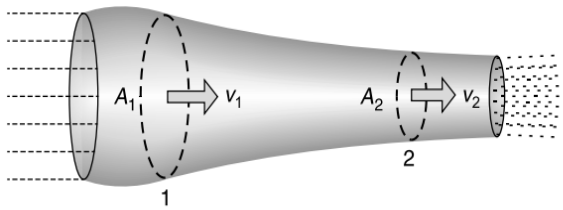
Irrotational Flow

The flow of the fluid is said to be **irrotational** if the element of the fluid at each point has no net angular velocity.

EQUATION OF CONTINUITY

Consider a steady, irrotational flow of an ideal fluid through a tube of varying cross-section having no source or sink between the entry and the exit.

If, A_1 and A_2 are the cross-sectional areas at points 1 and 2 respectively, v_1 and v_2 are the respective velocities of the liquid entering at 1 and leaving at 2 as shown in Figure.



If $\frac{dm}{dt}$ is the mass of liquid entering per second, then we have

$$\frac{dm}{dt} = \frac{A(dx)\rho}{dt} = A\left(\frac{dx}{dt}\right)\rho = Av\rho$$

As there is no source and sink between the entry and the exit, so we have

$$\begin{pmatrix}\text{Mass entering}\\ \text{per second at 1}\end{pmatrix} = \begin{pmatrix}\text{Mass leaving}\\ \text{per second at 2}\end{pmatrix}$$

$\Rightarrow \quad A_1v_1\rho = A_2v_2\rho$

$\Rightarrow \quad A_1v_1 = A_2v_2$

$\Rightarrow \quad Av = \text{constant}$

This is called the **Equation of Continuity** and follows from the Law of Conservation of Mass to the flow of the ideal fluids.

The quantity Av is called the flow rate or volume flux or Volume of liquid flowing per second and is also denoted by Q.

The above equation of continuity states that "for steady incompressible flow the speed of fluid varies inversely with the cross-sectional area".

Conceptual Note(s)

(a) Mathematically, the physical quantity Av is also called **Velocity Flux.** Vectorially,

$$\text{Velocity Flux} = \vec{v} \cdot \vec{A}$$

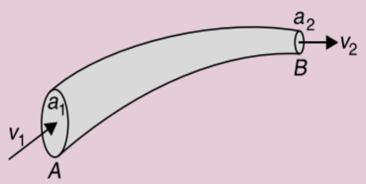
Velocity flux is the measure of the volume of liquid flowing in or out per second for a surface, the surface held normally to the liquid velocity (or liquid flow).

(b) The more the value of velocity flux, the more the number of streamlines cross the surface.

(c) We may also interpret the streamline picture as follows. In a narrow part of the tube the streamlines get closer together than in a wide part. Thus, as the distance between the streamlines decreases, the speed of the fluid increases.

(d) Widely spaced streamlines indicate regions of low speed, whereas closely spaced streamlines indicate regions of high speed.

(e) If a non-viscous compressible liquid in streamline flow passes through a tube AB of varying cross section having cross sectional areas a_1 and a_2 at points A and B. Let the liquid enter at A with normal velocity v_1 and leave at B with velocity v_2 and let ρ_1 and ρ_2 be the densities of the liquid at points A and B respectively, then Equation of Continuity is modified as $a_1 v_1 \rho_1 = a_2 v_2 \rho_2$

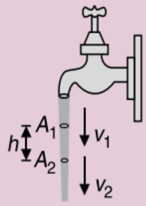


(f) Mathematically av is actually the rate of flow of volume of the liquid (or volume of liquid flowing per second) through a cross section of a tube.

$$av = \frac{dV}{dt} = Q = \begin{pmatrix} \text{Rate of Flow of} \\ \text{Liquid per second} \end{pmatrix}$$

(g) In hilly region, where the river is narrow and shallow (i.e., small cross-section) the water current will be faster, while in plains where the river is wide and deep (i.e., large cross-section) the current will be slower, and so deep water will appear to be still.

(h) When water falls from a tap, the velocity of falling water under the action of gravity will increase with distance from the tap, so $v_2 > v_1$. So, in accordance with continuity equation the cross section of the water stream will decrease and so, $A_2 < A_1$ due to which the falling stream of water becomes narrower.



Consider two such points on the water stream having a separation h, areas A_1, A_2 and respective velocities v_1 and v_2, then from Equation of Continuity, we have

$$A_1 v_1 = A_2 v_2$$

Now, since we have

$$v_2^2 = v_1^2 + 2gh$$

$$\Rightarrow A_2 = \frac{A_1 v_1}{\sqrt{v_1^2 + 2gh}} \qquad (A_2 < A_1)$$

(i) If liquid entering a tube leaves the tube at two other points, and assuming that the tube has no source and sink, then we have

$$a_1 v_1 = a_2 v_2 + a_3 v_3$$

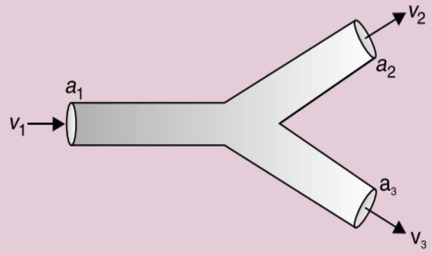


ILLUSTRATION 102

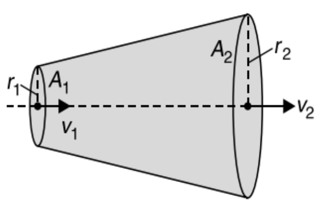
Water is flowing through a horizontal tube of non-uniform cross section. At a place the radius of the tube is 1 cm and the velocity of water is $2\ \text{ms}^{-1}$. What will be the velocity of water where the radius of the pipe is 2 cm?

SOLUTION

Using equation of continuity, $A_1 v_1 = A_2 v_2$

$$v_2 = \left(\frac{A_1}{A_2}\right) v_1$$

$$\Rightarrow \quad v_2 = \left(\frac{\pi r_1^2}{\pi r_2^2}\right) v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1$$



Substituting the values, we get

$$v_2 = \left(\frac{1.0 \times 10^{-2}}{2.0 \times 10^{-2}}\right)^2 (2)$$

$$\Rightarrow \quad v_2 = 0.5\ \text{ms}^{-1}$$

ILLUSTRATION 103

Water is flowing in a circular pipe of varying cross-sectional area, and at all points the water completely fills the pipe.

(a) At one point in the pipe the radius is 0.2 m. What is the magnitude of the water velocity at this point if the volume flow rate in the pipe is $1.20\ \text{m}^3\text{s}^{-1}$?

(b) At a second point in the pipe the water velocity has a magnitude of $3.80\ \text{ms}^{-1}$. What is the radius of the pipe at this point?

SOLUTION

(a) Q = volume flow rate $= Av$

$$v = \frac{Q}{A} = \frac{Q}{\pi r^2} = \frac{1.2}{\pi(0.2)^2} = 9.55\ \text{ms}^{-1}$$

(b) From continuity equation

$$A_1 v_1 = A_2 v_2$$

$$\Rightarrow r_1^2 v_1 = r_2^2 v_2$$

$$\Rightarrow r_2 = \left(\sqrt{\frac{v_1}{v_2}}\right) r_1 = \left(\sqrt{\frac{9.55}{3.8}}\right)(0.2)$$

$$\Rightarrow r_2 = 0.317\ \text{m}$$

ENERGIES POSSESSED BY A LIQUID

A liquid possesses three kinds of energies.

Kinetic Energy (K.E.)

It is the energy possessed by a liquid due to its velocity. So, kinetic energy possessed by a liquid of mass m moving with a speed v is

$$\text{K.E.} = \frac{1}{2}mv^2$$

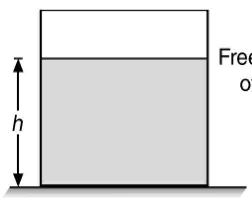
For liquids it is customary to talk about energy per unit mass or energy per unit volume, so we have

$$\frac{\text{K.E.}}{\text{Mass}} = \frac{1}{2}v^2$$

$$\Rightarrow \frac{\text{K.E.}}{\text{Volume}} = \frac{1}{2}\rho v^2$$

Potential Energy (P.E.)

It is the energy possessed by a liquid due to its position. Consider a beaker filled to a height h with liquid of density ρ. The beaker is placed on a horizontal surface as shown in Figure.



Assuming zero potential energy level (ZPEL) to be assigned to the horizontal surface, the potential energy of the free surface of the liquid is

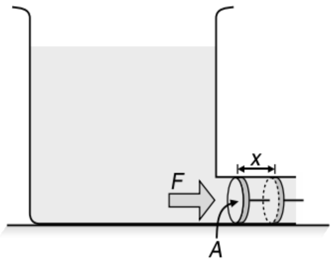
$$\text{P.E.} = mgh$$

$$\Rightarrow \frac{\text{Potential Energy}}{\text{Mass}} = gh$$

$$\Rightarrow \frac{\text{Potential Energy}}{\text{Volume}} = \rho gh$$

Pressure Energy (Pr.E.)

It is the energy possessed by a liquid due to its pressure. Consider a beaker having a frictionless piston attached to its base and filled with liquid of density ρ as shown in Figure.



The piston will move backwards because of the force due to pressure given by $F = PA$. The area of the piston is assumed to be small, so that there is no variation of pressure on the cross-section of the piston. Work done (W) by this force to displace the piston backwards by x is

$$W = Fx = (PA)x = P(Ax) = PV$$

This work done is store as pressure energy in the liquid, so we have

$$\text{Pressure Energy} = PV$$

$$\Rightarrow \frac{\text{Pressure Energy}}{\text{Mass}} = \frac{P}{\rho}$$

$$\Rightarrow \frac{\text{Pressure Energy}}{\text{Volume}} = P$$

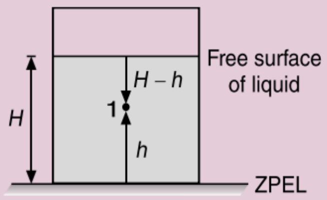
So, from above we see that pressure energy per unit volume is just equal to the pressure of the liquid.

Conceptual Note(s)

To us it may appear that potential energy per unit volume and the pressure energy per unit volume are the same, but let me explain it properly to you, that both are different.

To measure the potential energy per unit volume at a point inside the liquid, we shall be taking the height of the point above the zero potential energy level (ZPEL) and to measure the pressure energy per unit volume (i.e. pressure) at a point inside the liquid, we shall be taking the depth of the point below the free surface of the liquid.

Consider a beaker filled to a height H with a liquid of density ρ. Let us find the potential energy per unit volume and the pressure energy per unit volume at a point 1 inside the liquid as shown in Figure.



Assuming ZPEL to be at the surface on which the beaker is kept, we see that the potential energy per unit volume is

$$\frac{\text{P.E.}}{\text{Volume}} = h\rho g$$

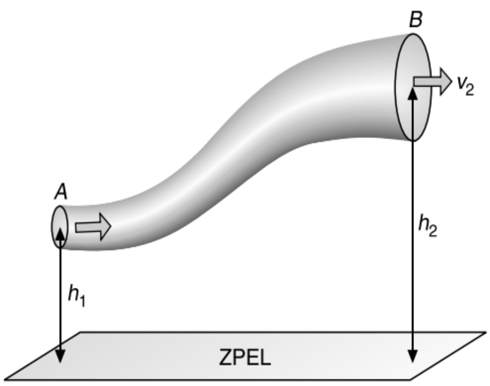
and pressure energy per unit volume is

$$\frac{\text{Pr.E.}}{\text{Volume}} = (H-h)\rho g$$

BERNOULLI'S EQUATION

This theorem is Law of Conservation of Energy for an ideal fluid flowing through a tube of variable cross-section having no source or sink in between.

Consider two cross-sections A and B of area A_1 and A_2 of a tube of flow at heights h_1 and h_2 respectively. Let v_1 and v_2 be the respective fluid velocities at these cross-sections, and let P_1 and P_2 be the respective fluid pressure. Let ρ be the density of the liquid.



Work done at the input A in time t is

$$W_1 = F_1 x_1 \cos 0$$

$$\Rightarrow \quad W_1 = P_1 A_1 v_1 t$$

Work done at the exit point B in time t is

$$W_2 = F_2 x_2 \cos 180$$

$$\Rightarrow \quad W_2 = -P_2 A_2 v_2 t$$

Hence total work done is

$$W = W_1 + W_2 = P_1 A_1 v_1 t - P_2 A_2 v_2 t \quad \ldots(1)$$

Further by Equation of Continuity we have

$$A_1 v_1 = A_2 v_2 = \frac{\left(\begin{array}{c}\text{Volume of liquid}\\ \text{flowing in or out}\end{array}\right)}{\text{Time}}$$

$$\Rightarrow \quad A_1 v_1 = A_2 v_2 = \frac{m}{\rho t} \qquad \left\{\because \text{Volume} = \frac{m}{\rho}\right\}$$

$$\Rightarrow \quad A_1 v_1 t = A_2 v_2 t = \frac{m}{\rho} \quad \ldots(2)$$

Substituting (2) in (1), we get

$$W = (P_1 - P_2)\frac{m}{\rho} \quad \ldots(3)$$

Further by Work Energy Theorem, we have

$$W = \Delta(KE) + \Delta(PE)$$

$$\Rightarrow \quad (P_1 - P_2)\frac{m}{\rho} = \frac{1}{2}m\left(v_2^2 - v_1^2\right) + mg(h_2 - h_1)$$

$$\Rightarrow \quad P_1 + \frac{1}{2}\rho v_1^2 + h_1\rho g = P_2 + \frac{1}{2}\rho v_2^2 + h_2\rho g$$

$$\Rightarrow \quad P + \frac{1}{2}\rho v^2 + h\rho g = \text{constant}$$

This is called **Bernoulli's Equation** for the steady flow of an ideal fluid according to which the sum total of pressure energy per unit volume, kinetic energy per unit volume and potential energy per unit volume is a constant. It is a consequence of the Law of Conservation of Energy.

Further, we have

$$\frac{P}{\rho} + \frac{1}{2}v^2 + gh = \text{constant}$$

i.e. sum total of pressure energy per unit mass, kinetic energy per unit mass and potential energy per unit mass for an SIIN fluid flowing through a tube of variable cross-section is a constant.

Also, we can write

$$\frac{P}{\rho g} + \frac{v^2}{2g} + h = \text{constant}$$

i.e. sum total of pressure head $\left(\frac{P}{\rho g}\right)$, velocity head $\left(\frac{v^2}{2g}\right)$ and gravitational head (h) for an SIIN fluid flowing through a tube of variable cross-section is a constant.

Problem Solving Technique(s)

If the height of the ends of the tube is same throughout, then the equation reduces to

$$P + \frac{1}{2}\rho v^2 = \text{constant}$$

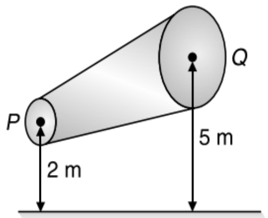
This equation tells us that when pressure is less, then velocity is more and vice-versa.

This pressure arises due to the motion of the fluid and hence is also called as Dynamic Pressure.

There are lot of applications of Bernoulli's theorem in common life. Some of those applications are discussed in detail in the upcoming section but before those examples, it is important for us to discuss the importance of Static, Dynamic and Total Pressure.

ILLUSTRATION 104

A non-viscous liquid of constant density $1000\ \text{kgm}^{-3}$ flows in a streamline motion along a tube of variable cross-section. The tube is kept inclined in the vertical plane as shown in Figure. The area of cross-section of the tube at the points P and Q at heights of 2 metre and 5 metre are respectively $4\times10^{-3}\ \text{m}^2$ and $8\times10^{-3}\ \text{m}^2$. The velocity of the liquid at point P is $1\ \text{ms}^{-1}$. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point P to Q. Take $g = 9.8\ \text{ms}^{-2}$.

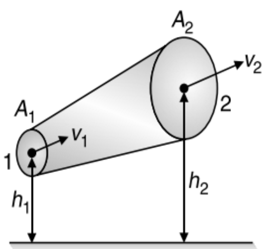


SOLUTION

According to the problem, we have $A_1 = 4\times10^{-3}\ \text{m}^2$, $A_2 = 8\times10^{-3}\ \text{m}^2$, $h_1 = 2\ \text{m}$, $h_2 = 5\ \text{m}$, $v_1 = 1\ \text{ms}^{-1}$ and $\rho = 10^3\ \text{kgm}^{-3}$

From continuity equation, we have

$$A_1v_1 = A_2v_2$$



$$\Rightarrow \quad v_2 = \left(\frac{A_1}{A_2}\right)v_1$$

$$\Rightarrow \quad v_2 = \left(\frac{4\times10^{-3}}{8\times10^{-3}}\right)(1\ \text{ms}^{-1})$$

$$\Rightarrow \quad v_2 = \frac{1}{2}\ \text{ms}^{-1}$$

Applying Bernoulli's equation at section 1 and 2,

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

$$\Rightarrow \quad P_1 - P_2 = \rho g(h_2 - h_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) \qquad \ldots(1)$$

Work done for unit volume by the pressure as the fluid flows from P to Q is

$$W_1 = P_1 - P_2$$

$$\Rightarrow \quad W_1 = \rho g(h_2 - h_1) + \frac{1}{2}\rho(v_2^2 - v_1^2) \qquad \{\text{from (1)}\}$$

$$\Rightarrow \quad W_1 = \left[(10^3)(9.8)(5-2) + \frac{1}{2}(10^3)\left(\frac{1}{4} - 1\right)\right]\ \text{Jm}^{-3}$$

$$\Rightarrow \quad W_1 = (29400 - 375)\ \text{Jm}^{-3}$$

$$\Rightarrow \quad W_1 = 29025\ \text{Jm}^{-3}$$

Work done per unit volume by the gravity as fluid flows from P to Q is

$$W_2 = -\rho g(h_2 - h_1) = -(10^3)(9.8)(5-2)\ \text{Jm}^{-3}$$

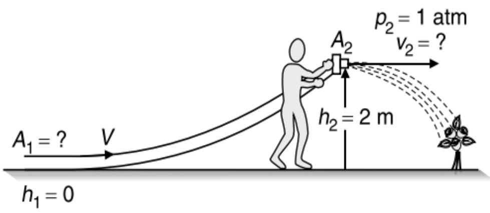
$$\Rightarrow \quad W_2 = -29400\ \text{Jm}^{-3}$$

ILLUSTRATION 105

A garden hose has an inside cross-sectional area of $3\ \text{cm}^2$, and the opening in the nozzle is $0.5\ \text{cm}^2$. If the water velocity is $100\ \text{cms}^{-1}$ in a segment of the hose that lies on the ground, calculate the speed with which water comes out from the nozzle when it is held $2\ \text{m}$ above the ground. Also calculate the water pressure in the hose lying on the ground. Take $g = 10\ \text{ms}^{-2}$ and $P_{\text{atm}} = 10^5\ \text{Nm}^{-2}$.

SOLUTION

The situation given in the problem is shown in Figure.



On applying the Equation of Continuity to find the speed of the fluid at the nozzle, we get

$$v_2 = \frac{A_1}{A_2}v_1 = \left(\frac{3\ \text{cm}^2}{0.5\ \text{cm}^2}\right)(100\ \text{cms}^{-1})$$

$$\Rightarrow \quad v_2 = 600\ \text{cms}^{-1} = 6\ \text{ms}^{-1}$$

According to Bernoulli's Theorem, we have

$$P_1 = P_2 + \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1)$$

where, $h_1 = 0$, $h_2 = 2\ \text{m}$, the pressure at the nozzle is atmospheric pressure, i.e. $P_2 = 10^5\ \text{Pa}$ and the density of water is $\rho = 10^3\ \text{kgm}^{-3}$.

$$\Rightarrow \quad P_1 = (10^5) + \frac{1}{2}(10^3)(36-1) + (10^3)(10)(2)$$

$$\Rightarrow \quad P_1 = 137500\ \text{Nm}^{-2}$$

STATIC, DYNAMIC AND TOTAL PRESSURE

Since we know that Bernoulli's Theorem is fundamental to the dynamics of incompressible (ideal or SIIN) liquids. In many fluid flow situations where the tube ends are at the same level or changes in elevation can be ignored, we can simply re-write Bernoulli's equation as

$$P+\frac{1}{2}\rho v^2 = \text{constant} \quad \ldots(1)$$

where, P is called the **Static Pressure** (sometimes denoted by P_s), $\frac{1}{2}\rho v^2$ is called the **Dynamic Pressure** (denoted by P_d) and $P+\frac{1}{2}\rho v^2$ collectively is called the **Stagnation Pressure or Total Pressure** (denoted by P_0). Interestingly, we see from equation (1) that the Stagnation Pressure is constant. So, we have

$$P+\frac{1}{2}\rho v^2 = P_0 = \text{constant}$$

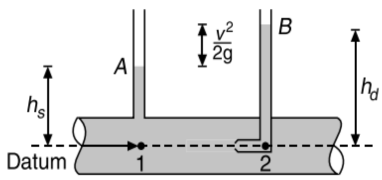
Hence, we conclude that total pressure is constant along any streamline, though static pressure and dynamic pressure may vary.

In simplified form, Bernoulli's Theorem can be summarised as

$$\begin{pmatrix}\text{Static}\\ \text{Pressure}\end{pmatrix}+\begin{pmatrix}\text{Dynamic}\\ \text{Pressure}\end{pmatrix}=\begin{pmatrix}\text{Stagnation}\\ \text{Pressure}\end{pmatrix}$$

The height h_s to which the liquid rises in it with respect to the reference level called Datum or Zero Potential Energy Level (ZPEL) is a measure of the static pressure in the fluid at a point on the axis.

The static pressure at a point in a fluid flow is calculated by connecting a single-tube manometer perpendicular to the direction of the fluid flow as shown by tube A in Figure.



So, we have

$$P_s = \rho g h_s$$

$$\Rightarrow \quad \frac{P_s}{\rho g} = h_s$$

The height h_s is also called the static pressure head.

Now, let a tube B (which is bent at right angle) be inserted in the fluid flow such that its open-end faces against the direction of fluid flow as shown in Figure. Just inside the open end of tube, the velocity of the fluid suddenly reduces to zero and the kinetic energy gets converted into pressure energy. Also, as discussed, the Stagnation Pressure should remain the same on a streamline, so we have

$$P_1+\frac{1}{2}\rho v^2 = P_2+\frac{1}{2}\rho (0)^2$$

$$\Rightarrow \quad P_2 - P_1 = \frac{1}{2}\rho v^2$$

$$\Rightarrow \quad (P_0 + h_d\rho g)-(P_0+h_s\rho g) = \frac{1}{2}\rho v^2$$

$$\Rightarrow \quad h_d - h_s = \frac{v^2}{2g}$$

So, the liquid in tube B rises to a higher level than that in tube A.

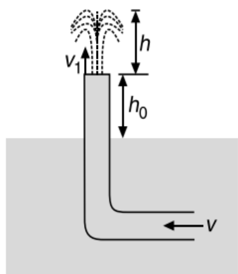
Conceptual Note(s)

The difference in the level of liquid is a measure of velocity of the fluid. Tube B measures the dynamic pressure and the height h_d of the liquid in the tube is called the dynamic head, which is equal to the sum total of the static pressure head and velocity head, i.e.

$$h_d = h_s + \frac{v^2}{2g}$$

ILLUSTRATION 106

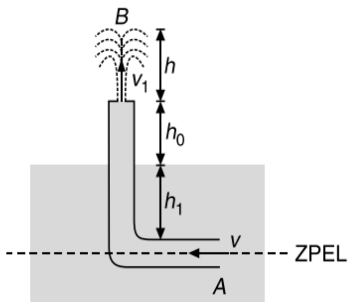
A bent tube is lowered into a water stream as shown in Figure.



The velocity of the stream relative to the tube is equal to $v = 2.5\ \text{ms}^{-1}$. The closed upper end of the tube located at the height $h_0 = 12$ cm has a small orifice. Calculate the height h to which the water jet will spurt.

SOLUTION

Let us consider the zero potential energy level (ZPEL) at the lowest portion of the tube as shown in Figure.



Applying Bernoulli's Theorem at A and B, we get

$$P_A + \rho g h_A + \frac{1}{2}\rho v_A^2 = P_B + \rho g h_B + \frac{1}{2}\rho v_B^2 \quad \ldots(1)$$

where,

$$P_A = P_0 + h_1 \rho g,\ h_A = 0,\ v_A = v \text{ and}$$

$$P_B = P_0,\ h_B = h_1 + h_0 + h,\ v_B = 0$$

Substituting these values in equation (1) we get

$$(P_0 + h_1 \rho g) + \frac{1}{2}\rho v^2 = P_0 + \rho g(h_1 + h_0 + h)$$

$$\Rightarrow \quad \frac{1}{2}\rho v^2 = \rho g(h_0 + h)$$

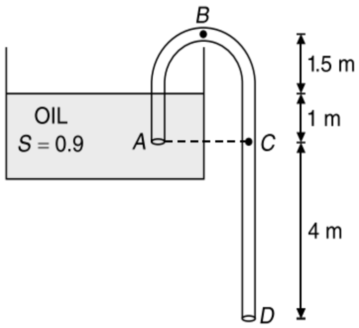
$$\Rightarrow \quad \frac{1}{2}\rho v^2 = \rho g(h_0 + h)$$

$$\Rightarrow \quad h = \frac{v^2}{2g} - h_0$$

$$\Rightarrow \quad h = \frac{(2.5)^2}{2 \times 9.8} - 0.12 = 0.20 \text{ m}$$

ILLUSTRATION 107

A siphon tube is discharging a liquid of specific gravity 0.9 from a reservoir as shown in Figure.



Calculate the velocity of the liquid through the siphon and the pressure at highest point B, at A (out-side the tube) and at C. Also, give correct answer with explanation for each of the following statements.

Statement-1

Would the rate of flow be more, less or the same if the liquid were water?

Statement-2

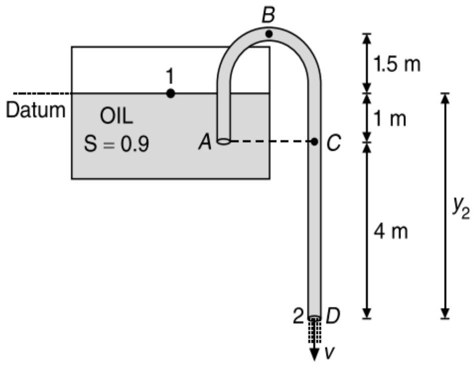
Is there a limit on the maximum height of B above the liquid level in the reservoir?

Statement-3

Is there a limit on the vertical depth of the right limb of the siphon?

SOLUTION

Assume datum at the free surface of the liquid. Applying Bernoulli's equation on point 1 and 2, as shown in Figure.



$$\frac{P_1}{\rho g} + \frac{v_1^2}{2g} + y_1 = \frac{P_2}{\rho g} + \frac{v_2^2}{2g} + y_2$$

where, $P_1 = P_2 = P_0 = 10^5 \text{ Nm}^2$; $y_1 = 0$, $y_2 = -5$ m

Since area of the tube is very small as compared to that of the reservoir, so $v_1 \ll v_2$, hence $\frac{v_1^2}{2g} \approx 0$.

$$\Rightarrow \quad v_2 = \sqrt{2g(y_1 - y_2)} = \sqrt{2(10)(5)} = 10 \text{ ms}^{-1}$$

Applying Bernoulli's equation at 1 and B.

$$\frac{P_B}{\rho g} + \frac{v_B^2}{2g} + y_B = \frac{P_1}{\rho g} + \frac{v_1^2}{2g} + y_1$$

where, $P_1 = 10^5 \text{ Nm}^{-2}$, $\frac{v_1^2}{2g} \approx 0$,

$$y_1 = 0,\ v_B = v_2 = 10 \text{ ms}^{-1},\ y_B = 1.5 \text{ m}$$

$$\Rightarrow \quad P_B = P_1 - \frac{1}{2}\rho v_2^2 - \rho g y_B$$

$$\Rightarrow \quad P_B = 10^5 - \frac{1}{2}(900)(10)^2 = (900)(10)(1.5)$$

$$\Rightarrow \quad P_B = 41.5 \text{ kNm}^{-2}$$

Applying Bernoulli's equation at 1 and A

$$P_A = P_1 + \rho g(y_1 - y_A)$$

$$\Rightarrow \quad P_A = 10^5 + (900)(10)(1) = 109 \text{ kNm}^{-2}.$$

Applying Bernoulli's equation at 1 and C,

$$P_C = P_1 - \frac{1}{2}\rho v_C^2 - \rho g y_C$$

$$\Rightarrow \quad P_C = 10^5 - (900) - (10)^2(900)(10)(-1)$$

$$\Rightarrow \quad P_C = 10^5 - 45000 + 9000 = 64 \text{ kNm}^{-2}$$

Statement-1 (Answer and Explanation)

The velocity of flow is independent of the density of the liquid, therefore, the discharge would remain the same.

Statement-2 (Answer and Explanation)

Since the pressure at B is less than atmospheric, the liquid, therefore, has a tendency to get vaporised if the pressure becomes equal to the vapour pressure of it. Thus, $P_B > P_{\text{vapour}}$.

Statement-3 (Answer and Explanation)

The velocity of flow depends on the depth of the point D, below the free surface

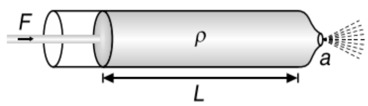
$$\frac{v^2}{2g} = y_1 - y_2 = H$$

$$\Rightarrow \quad P_B = P_1 - \frac{1}{2}\rho v^2 - \rho g y_B = P_1 - \rho g H - \rho g y_B$$

For working of siphon, $H \neq 0$, and H should not be high enough so that P_B may not reduce to vapour pressure.

ILLUSTRATION 108

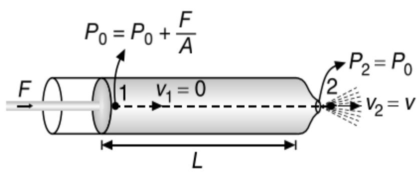
Calculate the work that should be done in order to squeeze all water in time t from a horizontally located cylinder by means of a constant force acting on the piston as shown in Figure. Assume that the volume of water in the cylinder is equal to V, the cross sectional area of the orifice is a, with a being considerably less than the piston area. The friction and viscosity are negligibly small.



SOLUTION

Work done to squeeze all the water from the container in time duration t is

$$W = FL$$



If v be the velocity of the liquid at point 2, then applying Bernoulli's theorem at points 1 and 2, we get

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \qquad \ldots(1)$$

where, $P_1 = P_0 + \frac{F}{A}$ and $P_2 = P_0$

Also, according to the problem, $A \gg a$, so v_1 is very much smaller than v_2 and hence v_1 can be neglected. So, from equation (1), we get

$$P_0 + \frac{F}{A} = P_0 + \frac{1}{2}\rho v^2 \qquad \ldots(2)$$

$$\Rightarrow \quad \frac{F}{A} = \frac{1}{2}\rho v^2$$

$$\Rightarrow \quad F = \left(\frac{1}{2}\rho v^2\right)A \qquad \ldots(3)$$

Volume of liquid coming out per second through orifice is

$$Q_2 = av$$

Volume of liquid coming out through the orifice in time duration t is

$$V = Q_2 t = (av)t$$

So, velocity of liquid coming out through orifice is

$$v = \frac{V}{at}$$

From Equation (3), we get

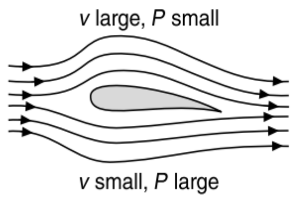
$$F = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 A$$

$$\Rightarrow \quad W = FL = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 AL = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 V$$

$$\Rightarrow \quad W = \frac{1}{2}\rho\left(\frac{V^3}{a^2t^2}\right)$$

WORKING OF AN AIRPLANE OR AERODYNAMIC LIFT

Because of the specific shape of wings of an airplane (as shown), when the airplane runs fast on the runway then air passes at higher speed over the upper surface of the wing as compared to its lower surface as shown in Figure.



This difference of air speeds above and below the wings, in accordance with Bernoulli's principle, creates a pressure difference, due to which an upward force called **dynamic lift** (= pressure difference × area of wing) acts on the plane and when this force becomes greater than the weight of the plane, the plane starts rising up.

ILLUSTRATION 109

An airplane wing is designed so that the speed of the air across the top of the wing is 251 ms^{-1} when the speed of the air below the wing is 225 ms^{-1}. The density of the air is 1.29 kgm^{-3}. What is the lifting force on a wing of area 24.0 m^2?

SOLUTION

According to Bernoulli's theorem, we have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\Rightarrow \quad P_2 - P_1 = \Delta P = \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

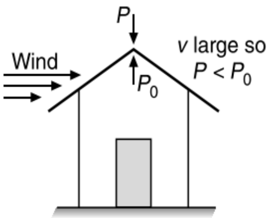
$$\Rightarrow \quad F = A\Delta P = \frac{1}{2}\rho A\left(v_1^2 - v_2^2\right)$$

$$\Rightarrow \quad F = \frac{1}{2}\times 1.29\times 24\left[(251)^2 - (225)^2\right]$$

$$\Rightarrow \quad F = 1.92\times 10^5 \text{ N}$$

ROOF'S BLOWING OFF DURING WIND STORMS

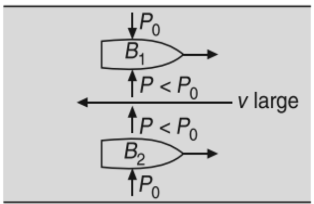
When a high-speed wind blows over a straw or tin roof, it creates a low pressure (P) in accordance with Bernoulli's principle.



However, the pressure below the roof (i.e., inside the room) is still atmospheric $(= P_0)$. So due to this difference of pressure the roof experiences an upward force and hence is then blown off by the wind.

ATTRACTION BETWEEN TWO CLOSELY PARALLEL MOVING BOATS OR TRAINS

It is observed that, when two boats move side by side in the same direction, the water in the region between them moves faster than that on the remote sides.

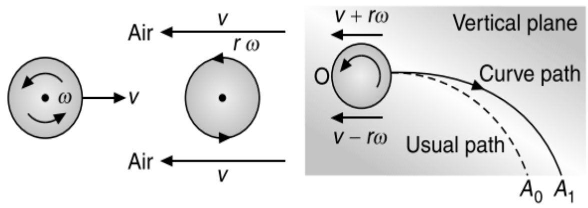


Consequently, in accordance with Bernoulli's principle the pressure between them is reduced and hence due to pressure difference they are pulled towards each other.

MAGNUS EFFECT (SPINNING OF A BALL OR SPINNING OF A BULLET)

When a ball is thrown with a spin, then the ball deviates from its usual path in flight. This effect is called Magnus effect and plays as important role in games like tennis, cricket, soccer, etc. where the player throws the ball by applying appropriate spin so that the ball can be made to follow a curved track in the desired direction.

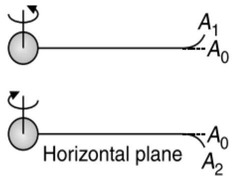
Consider a ball moving from left to right and also spinning about a horizontal axis perpendicular to the direction of motion as shown in Figure.



We observe that air will be moving from right to left relative to the ball. The resultant velocity of air above the ball will be $(v + r\omega)$ while below it $(v - r\omega)$. So, in accordance with Bernoulli's principle pressure above the ball will be less than below it. Due to this difference of pressure an upward force will act on the ball and hence the ball will deviate from its usual path OA_0 and will hit the ground at A_1 following the path OA_1 i.e., if a ball is thrown with back-spin, the pitch will curve less sharply prolonging the flight.

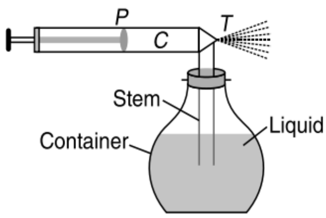
Similarly, if the spin is clockwise i.e., the ball is thrown with top-spin, the force due to pressure difference will act in the direction of gravity and so the pitch will curve more sharply shortening the flight.

Furthermore, if the ball is made to spin about a vertical axis, the curving will be sideways as shown in producing the so called out swing or in swing.



ACTION OF AN ATOMISER OR SPRAYER

The action of gun, scent-spray or insect-sprayer is based on Bernoulli's principle. In all these, by means of motion of a piston P in a cylinder C, high speed air is passed over a tube T dipped in liquid L to be sprayed. High speed air creates low pressure over the tube due to which the liquid (petrol, paint, scent, insecticide) rises in it and is then blown off as tiny droplets along with the expelled air.

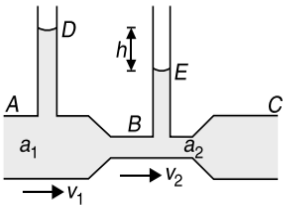


BLOOD FLOW AND HEART ATTACK

The occurrence of Heart Attack can be explained on the basis of Bernoulli's theorem. When the inner walls of a blood artery get constricted due to the deposition of plaque, the heart is required to exert excess pressure so as to derive the blood through this constriction. Blood flow speed increases through it as in accordance with the equation of continuity. The increased blood flow reduces the pressure at the constriction. This leads to the collapse of artery due to external pressure. The heart has to push harder to push the blood through this artery. As the blood rushes through the opening, the pressure inside the artery drops once again. The artery collapses again and this repeated collapse may lead to a heart attack.

VENTURIMETER

Venturimeter is a device used to measure the speed of an incompressible liquid flowing through a pipe. It can also measure the rate of flow of liquid i.e. volume of liquid flowing per second, through a pipe.



It consists of two identical coaxial tubes A and C connected by a narrow co-axial tube B. Two vertical tubes D and E are mounted on the tubes A and B to measure the pressure of the following liquid.

When liquid of density ρ, flows through the tube ABC, the velocity of flow of liquid will be larger in tube B than in the tube A or C (as B has lesser area compared to A and C). So, the pressure in part B will be less than that in tube A or C. By measuring the pressure difference between A and B, the rate of flow of the liquid in the tube can be calculated.

Let a_1, a_2 be the area of cross section of tube A and B respectively and v_1, v_2 be the velocity of flow of liquid through tube A and B respectively. Let P_1, P_2 be the liquid pressure at A and B respectively, then we have

$$P_1 - P_2 = h\rho g \quad \text{...(1)}$$

Applying Bernoulli's theorem for horizontal flow of liquid, we have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) \quad \text{...(2)}$$

From (1) and (2), we get

$$h\rho g = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \quad \text{...(3)}$$

According to Equation of Continuity, we have

$$a_1 v_1 = a_2 v_2$$

$$\Rightarrow \quad v_2 = \frac{a_1 v_1}{a_2}$$

So, from equation (3), we get

$$h\rho g = \frac{1}{2}\rho v_1^2\left(\frac{a_1^2}{a_2^2} - 1\right)$$

$$\Rightarrow \quad v_1 = \sqrt{2gh}\sqrt{\frac{a_2^2}{a_1^2 - a_2^2}}$$

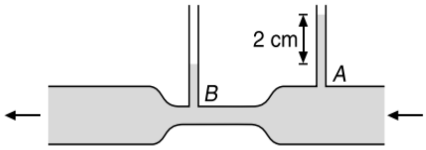
If Q be the volume of liquid flowing per second, then from Equation of continuity, we have

$$Q = a_1 v_1 = a_2 v_2$$

$$\Rightarrow \quad Q = a_1 a_2\sqrt{\frac{2gh}{a_1^2 - a_2^2}}$$

ILLUSTRATION 110

Water flows through a horizontal tube as shown in Figure.



If the difference of heights of water column in the vertical tubes is 2 cm and the areas of cross-section at A and B are 4 cm^2 and 2 cm^2 respectively, calculate the rate of flow of water across any section. Take $g = 10\text{ ms}^{-2}$.

SOLUTION

Using Bernoulli's theorem for venturi tube, flow velocity is

$$v_A = \sqrt{2gh}\sqrt{\frac{a_A^2}{a_A^2 - a_B^2}}$$

$$\Rightarrow \quad v_A = \sqrt{2(10)(0.02)}\sqrt{\frac{(2)^2}{(4)^2 - (2)^2}}$$

$$\Rightarrow \quad v_A = \sqrt{\frac{0.4}{3}} = 0.365\text{ ms}^{-1}$$

Rate of flow of liquid is

$$Q = a_A v_A = \left(4 \times 10^{-4}\right)(0.365)$$

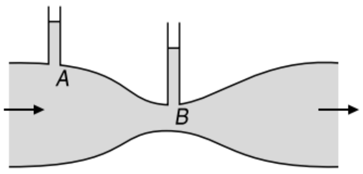
$$\Rightarrow \quad Q = 1.46 \times 10^{-4}\text{ m}^3\text{s}^{-1}$$

Since $1\text{ m}^3 = 10^6\text{ cm}^3$

$$\Rightarrow \quad Q = 146\text{ cm}^3\text{s}^{-1}$$

ILLUSTRATION 111

Water is flowing through a horizontal tube as shown in Figure. If the difference of heights of water column in the vertical tubes is 2 cm and the areas of cross-section at A and B are 4 cm^2 and 2 cm^2 respectively, then calculate the rate of flow of water across any section of the tube.



SOLUTION

Applying Bernoulli's equation at A and B, we get

$$p_A + \frac{1}{2}\rho v_A^2 + \rho g h_A = p_B + \frac{1}{2}\rho v_B^2 + \rho g h_B$$

$$\Rightarrow \quad \frac{1}{2}\rho v_B^2 - \frac{1}{2}\rho v_A^2 = \rho g (h_A - h_B)$$

$$\Rightarrow \quad v_B^2 - v_A^2 = 2g(h_A - h_B) = 2(10)(0.02)$$

$$\Rightarrow \quad v_B^2 - v_A^2 = 0.4 \text{ m}^2\text{s}^{-2} \qquad \ldots(1)$$

Applying Equation of Continuity at A and B, we get

$$v_A a_A = v_B a_B$$

$$\Rightarrow \quad 4v_A = 2v_B$$

$$\Rightarrow \quad v_B = 2v_A \qquad \ldots(2)$$

Solving equations (1) and (2), we get

$$v_A = 0.363 \text{ ms}^{-1}$$

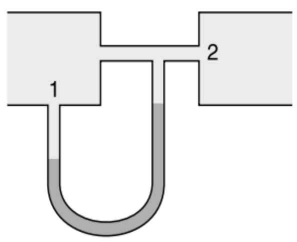
Volume flow rate Q is given by

$$Q = v_A A_A = (0.365)(4\times 10^{-4})$$

$$\Rightarrow \quad Q = 1.46\times 10^{-4} \text{ m}^3\text{s}^{-1} = 146 \text{ cm}^3\text{s}^{-1}$$

ILLUSTRATION 112

Water flows through the tube as shown in Figure. The areas of cross-section of the wide and the narrow portions of the tube are 5 cm^2 and 2 cm^2 respectively. The rate of flow of water through the tube is $500 \text{ cm}^3\text{s}^{-1}$. Find the difference of mercury levels in the U-tube.



SOLUTION

Applying Equation of Continuity at 1 and 2, we get

$$A_1 v_1 = A_2 v_2 = Q \text{ i.e., rate of flow of water}$$

$$\Rightarrow \quad 5v_1 = 2v_2 = 500 \text{ cm}^3\text{s}^{-1}$$

$$\Rightarrow \quad v_1 = 100 \text{ cm}^{-1}\text{s}^{-1} = 1.0 \text{ ms}^{-1}$$

$$\text{and} \quad v_2 = 250 \text{ cm}^{-1}\text{s}^{-1} = 2.5 \text{ ms}^{-1}$$

Applying Bernoulli's equation at 1 and 2, we get

$$p_1 + \frac{1}{2}\rho_w v_1^2 + \rho_w g h_1 = p_2 + \frac{1}{2}\rho_w v_2^2 + \rho_w g h_2$$

Since $h_1 = h_2$, so we get

$$\frac{1}{2}\rho_w (v_2^2 - v_1^2) = p_1 - p_2 = \rho_{Hg} g h_{Hg}$$

$$\Rightarrow \quad \frac{1}{2}\rho_w (v_2^2 - v_1^2) = \rho_{Hg} g h_{Hg}$$

$$\Rightarrow \quad h_{Hg} = \frac{\rho_w (v_2^2 - v_1^2)}{2\rho_{Hg} g}$$

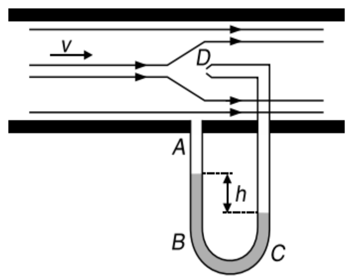
$$\Rightarrow \quad h_{Hg} = \frac{10^3(6.25-1)}{2\times 13.6\times 10^3 \times 9.8} = 0.0196 \text{ m}$$

$$\Rightarrow \quad h_{Hg} = 1.96 \text{ cm}$$

PITOT TUBE

It is based on Bernoulli's principle. It is a device used to measure the velocity of flow of liquid and hence the rate of flow at any depth in a flowing liquid. It is mounted on an aeroplane wing to measure the speed of the plane.

The arrangement consists of bent tube $ABCD$ with small apertures at A and D as shown in Figure. This tube is inserted in a pipe P through which the fluid (air in aeroplanes) is flowing.



Let ρ be the density of the fluid/air. The plane of aperture A is parallel to the direction of the air flow as such the velocity v_1 of air at A is same as its velocity in pipe which is equal to the velocity of the plane relative to air. Let P_1 is the pressure at A. However, plane of aperture at D is normal to the direction of air flow in the pipe. Therefore the velocity v_2 of the air flow rapidly falls to zero at D and pressure increases to P_2.

Applying Bernoulli's theorem to the horizontal flow at A and D, we get

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

where, $v_1 = v$ and $v_2 = 0$

$$\Rightarrow \quad P_1 + \frac{1}{2}\rho v^2 = P_2$$

$$\Rightarrow \quad v = \sqrt{\frac{2(P_2 - P_1)}{\rho}}$$

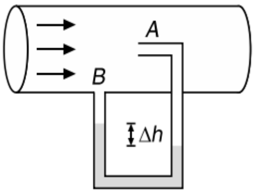
If σ is the density of liquid in tube $ABCD$, then

$$P_2 - P_1 = h\sigma g$$

$$\Rightarrow \quad v = \sqrt{\frac{2h\sigma g}{\rho}}$$

ILLUSTRATION 113

A Pitot tube is mounted along the axis of a gas pipeline whose cross-sectional area is equal to S as shown in Figure.



Assuming the viscosity to be negligible, find the volume of the gas flowing cross the section of the pipe per unit time, if the difference in the liquid columns is equal to Δh and the densities of the liquid and the gas are ρ_0 and ρ respectively.

SOLUTION

Using Bernoulli's theorem at points A and B, we get

$$P_A + \frac{1}{2}\rho v_A^2 = P_B + 0 \qquad \{\because v_B = 0\}$$

$$\Rightarrow \quad \frac{1}{2}\rho v_A^2 = P_B - P_A = \rho_0 g \Delta h$$

$$v_A = \sqrt{\frac{2\rho_0 g \Delta h}{\rho}}$$

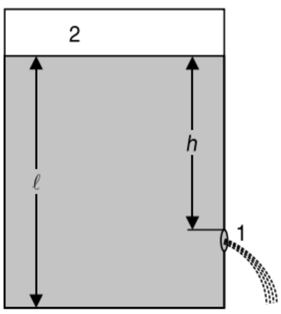
Volume of the gas flowing per unit time is

$$Q = S v_A$$

$$\Rightarrow \quad Q = S\sqrt{\frac{2\rho_0 g \Delta h}{\rho}}$$

SPEED OF EFFLUX

The word 'efflux' means the outflow of the fluid. Let us find an expression for the velocity of efflux for a fluid, from a small hole of its container. Consider a closed vessel filled with a liquid up to height l. A small hole (or orifice) is made in its side at a depth h below the surface of the liquid s shown in Figure.



Taking the liquid to be incompressible and its flow through the hole as streamlined, we can apply the equation of continuity at points 1 and 2

$$A_2 v_2 = A_1 v_1$$

$$\Rightarrow \quad v_2 = \frac{A_1}{A_2} v_1$$

Applying Bernoulli's equation at the two points, we have

$$P_2 + \frac{1}{2}\rho v_2^2 + \rho g l = P_1 + \frac{1}{2}\rho v_1^2 + \rho g (l - h)$$

Conceptual Note(s)

While writing the Bernoulli's equation, we have

(a) considered ZPEL to be at the base of the vessel and hence potential energy per unit volume at 1 and 2 are $\rho g(l-h)$ and $\rho g l$ respectively.

(b) not taken atmospheric pressure into account because the vessel is closed.

If the cross-sectional area of the vessel A_2 is much larger than that of the orifice i.e., A_1 then,

$$v_2 \approx 0$$

$$\Rightarrow \quad P_2 + \rho g l = P_1 + \frac{1}{2}\rho v_1^2 + \rho g (l - h)$$

$$\Rightarrow \quad \frac{1}{2}\rho v_1^2 = (P_2 - P_1) + \rho g h$$

$$\Rightarrow \quad v_1 = \sqrt{\frac{2}{\rho}(P_2 - P_1) + 2gh}$$

Since the hole is open to atmosphere so, the pressure P_1 is the same as the atmospheric pressure P_a.

$$\Rightarrow \quad v_1 = \sqrt{\frac{2}{\rho}(P_2 - P_a) + 2gh} \qquad \ldots(1)$$

This expression gives the velocity of efflux.

CASE-1:

Torricelli's Law: In case the vessel containing the fluid is open i.e., not covered, the pressure P_2 at the top of the liquid surface is same as the atmospheric pressure P_a. Thus, the velocity of efflux given above in equation (1) becomes

$$v_{\text{efflux}} = v = \sqrt{2gh} \qquad \ldots(2)$$

This is same as the speed acquired by a body after falling freely through a height h. The expression (2) is known as Torricelli's law.

CASE-2:

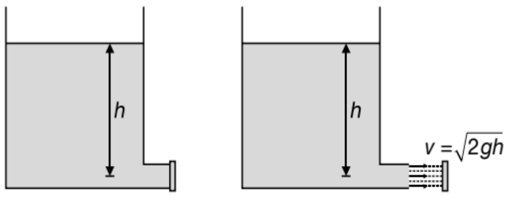
In case pressure $P_2 \gg P_a$, we can ignore the value $2gh$ in equation (1), to get the speed of efflux as

$$v_{\text{efflux}} = \sqrt{\frac{2}{\rho}(P_2 - P_a)}$$

The speed of efflux is determined by the container pressure. This occurs during rocket propulsion.

FORCE OF REACTION DUE TO EJECTION OF THE LIQUID FROM AN ORIFICE

Consider a cylindrical vessel that has an opening of cross-sectional area a near its bottom at a depth h below free surface of liquid. Let a disc be held against the opening to prevent liquid of density ρ from coming out of the opening.



The disc experiences a hydrostatic pressure from the liquid inside the vessel. Pressure at the level of the disc is

$$P_1 = P_{\text{atm}} + \rho gh$$

The air pressure on the outside the disc is $P_2 = P_{\text{atm}}$.
The net outward/rightward force acting on disc is

$$F_{\text{rightwards}} = (P_2 - P_1)a = \rho gha$$

Now, let the disc be moved a short distance away in horizontal direction. Due to this, the liquid comes out and strikes the disc. Assuming the collision of the liquid with the disc to be perfectly inelastic such that the liquid stops after hitting the disc, then the liquid in this case will exert an impulsive (impact) force on the disc.

When the disc is moved away, the liquid moves out with speed $v = \sqrt{2gh}$. The (mass per second) i.e., the rate of mass coming out of the opening is given by

$$\frac{dm}{dt} = \rho av = \rho a\sqrt{2gh}\,.$$

The thrust force acting on the liquid will be leftwards (opposite to v) and is given by

$$F_{\text{leftwards}} = v\frac{dm}{dt} = v(\rho av) = av^2\rho = 2h\rho ga$$

Taking the rightward direction as positive, the net force on the liquid is towards the left and its reaction on the disc is towards the right. Also, we note that the force due to atmospheric pressure is cancelled out from both the sides.

Conceptual Note(s)

Dear Student's, always keep in mind that the thrust force due to the liquid/gas coming out of the orifice is opposite to the velocity of a liquid (or gas) coming out through the orifice. If the liquid having density ρ is coming out of an orifice of cross-sectional area a with a velocity v, then thrust force due to liquid coming out the orifice is given by

$$F = \rho av^2$$

The direction of the thrust force will be just opposite to the direction of velocity as shown in examples taken in the Figure.

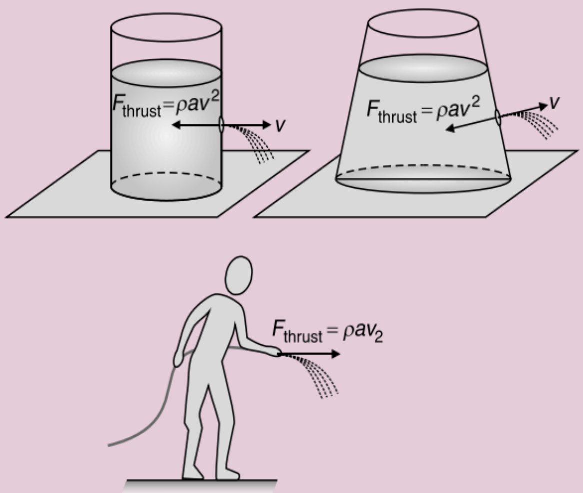


ILLUSTRATION 114

Find the velocity of efflux of water from an orifice near the bottom of a tank in which pressure is 500 gfcm^{-2} above atmosphere.

SOLUTION

Pressure at orifice, $P = 500$ gfcm^{-2}

$$\Rightarrow \quad P = 500 \times 9.8 \times \frac{(100)^2}{1000 \text{ Nm}^{-2}}$$

$$\Rightarrow \quad P = 500 \times 98 \text{ Nm}^{-2}$$

Let h be the depth of orifice below the surface
As $P = h\rho g$

$$\Rightarrow \quad h = \frac{P}{\rho g} = \frac{(500)(98)}{(10^3)(9.8)} = 5 \text{ m}$$

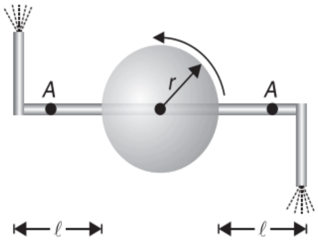
So, the velocity of efflux is

$$v = \sqrt{2gh} = \sqrt{(2 \times 9.8 \times 5)} = 9.893 \text{ ms}^{-1}$$

ILLUSTRATION 115

Figure shows the top view of a cylindrical vessel mounted on a turntable. The vessel is filled with water. At a depth h below the water surface are two horizontal tubes of length

l and cross-sectional area a, with right-angle bends at their ends. Show that, as the water jet just emerges from the tubes, there is a torque τ exerted on the system given by the expression $\tau = 4\rho gh(r+l)a$, where ρ is the density of the water.



SOLUTION

Just when the water jet leaves the tube, the velocity of efflux of water coming out of each tube is

$$v = \sqrt{2gh}$$

Thrust due to the emerging water jet stream on each tube is

$$F = av^2\rho = a(2gh)\rho$$

These forces acting opposite to each other have different line of action and hence they produce a torque given by

$$\tau = 2(Fr_{\perp})$$

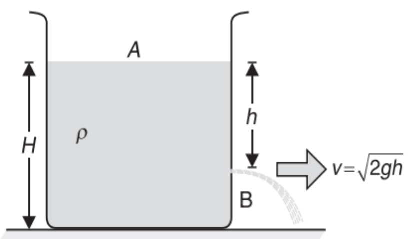
$$\Rightarrow \quad \tau = 2\left[(\rho a v^2)\times(r+l)\right]$$

$$\Rightarrow \quad \tau = 2\left[(\rho a)\left(\sqrt{2gh}\right)^2\times(r+l)\right]$$

$$\Rightarrow \quad \tau = 4\rho gha(r+l)$$

SPEED OF EFFLUX: TORRICELLI'S THEOREM

Consider an ideal liquid of density ρ contained in a tank having a small orifice at a point B in its wall. If the tank is large, the velocity of liquid at any point A on its free surface can be assumed to be zero. Atmospheric pressure P acts at both A and B. If v is the velocity with which the liquid flows out of B, then according to Bernoulli's Equation, we have



$$\begin{pmatrix}\text{Total energy per}\\ \text{unit volume at}\\ \text{the orifice}\end{pmatrix} = \begin{pmatrix}\text{Total energy per}\\ \text{unit volume at}\\ \text{the surface}\end{pmatrix}$$

$$\Rightarrow \quad P_a + (H-h)\rho g + \frac{1}{2}\rho v^2 = P_a + H\rho g$$

$$\Rightarrow \quad \frac{1}{2}\rho v^2 = h\rho g$$

$$\Rightarrow \quad v = \sqrt{2gh}$$

So, according to Torricelli's Theorem , the speed of efflux through an orifice at a depth h below the free surface the liquid equals the speed that is acquired by a body falling freely from a height h.

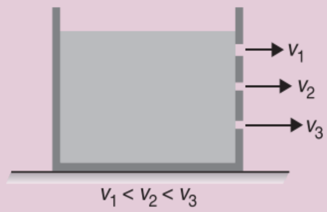
Please note that this result for the speed of efflux is applicable only when the area of the orifice a is very small compared to the area of the beaker A. However, if this is not the case i.e. the area of the orifice a is comparable to the area of the beaker A, then the speed of efflux is given by

$$v = \sqrt{2gh}\sqrt{\frac{A^2}{A^2-a^2}}$$

Problem Solving Technique(s)

(a) The velocity of efflux is independent of the nature of liquid, quantity of liquid in the vessel and the area of orifice.

(b) The greater is the depth of the orifice below the free surface the greater will be the velocity of efflux i.e., $v \propto \sqrt{h}$



RANGE OF LIQUID HITTING THE GROUND

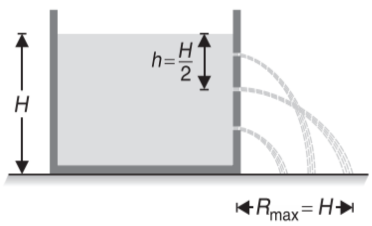
Just when the liquid leaves the orifice, the vertical velocity of liquid at the orifice is zero and since the orifice is at a height $(H-h)$ from the bottom, so the time taken by the liquid to reach the level of the base is

$$t = \sqrt{\frac{2(H-h)}{g}}$$

The range of the liquid is thus given by

$$R = vt = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}}$$

$$\Rightarrow \quad R = 2\sqrt{h(H-h)}$$



For R to be maximum i.e. for R^2 to be maximum, we have

$$\frac{d}{dh}(R^2)=0$$

$$\Rightarrow \frac{d}{dh}[h(H-h)]=0$$

$$\Rightarrow \frac{d}{dh}(Hh-h^2)=0$$

$$\Rightarrow H-2h=0$$

$$\Rightarrow h=\frac{H}{2}$$

So, range will be maximum when

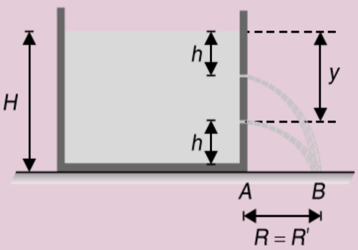
$$h=\frac{H}{2} \text{ and } R_{\max}=H$$

Conceptual Note(s)

INTERESTING OBSERVATION

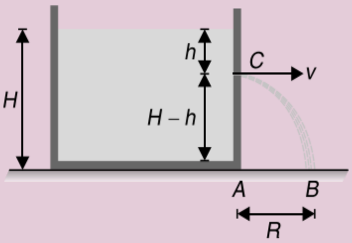
If h is replaced by $H-h$ in the above formula, then range R remains the same.

So, if the level of free surface in a container is at height H from the base and there are two holes one at depth h below the free surface and at the other at height h above the base, then



$$R=2\sqrt{h(H-h)} \text{ and } R'=2\sqrt{(H-h)h}$$

So, the range will be same if the orifice is at a depth h or $(H-h)$ below the free surface. Now as the distance $(H-h)$ from top means $H-(H-h)=h$ from the bottom, so the range is same for liquid coming out of holes at same distance below the top and above the bottom.



TIME TO EMPTY THE BEAKER TO THE ORIFICE

If a be the area of orifice and the depth of the orifice below the free surface of the liquid be y, then speed of efflux (v) of liquid coming out of the orifice is

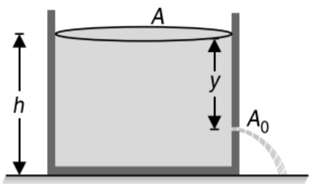
$$v=\sqrt{2gy}$$

If A is the area of the mouth of container, then volume of liquid coming out of the orifice per second will be

$$Q=-\frac{dV}{dt}=av=a\sqrt{2gy} \quad \ldots(1)$$

Due to the liquid coming out of the orifice, the level of liquid in the container will decrease and so if the level of liquid in the container above the hole changes from y to $y-dy$ in time t to $t+dt$, then we have

$$dV=Ady$$



Substituting value of dV in equation (1), we get

$$-A\frac{dy}{dt}=a\sqrt{2gy}$$

$$\Rightarrow \int_0^t dt=-\frac{A}{a}\frac{1}{\sqrt{2g}}\int_{h_1}^{h_2} y^{-\frac{1}{2}}dy$$

So, the time taken for the level to fall from h to h' is

$$t=-\frac{A}{a}\frac{1}{\sqrt{2g}}\left(\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right)\Bigg|_h^{h'}$$

$$\Rightarrow t=\frac{A}{a}\sqrt{\frac{2}{g}}\left(\sqrt{h}-\sqrt{h'}\right)$$

ILLUSTRATION 116

A tank is filled with a liquid up to a height H. A small hole is made at the bottom of this tank. Let t_1 be the time taken to empty first half of the tank and t_2 the time taken to empty rest half of the tank. Then find $\frac{t_1}{t_2}$.

SOLUTION

Substituting the proper limits in equation (1), derived in the theory we have,

$$\int_0^{t_1} dt = -\frac{A}{a\sqrt{2g}}\int_H^{H/2} y^{-1/2}dy$$

$$\Rightarrow \quad t_1 = \frac{2A}{a\sqrt{2g}}\left[\sqrt{y}\right]_{H/2}^{H}$$

$$\Rightarrow \quad t_1 = \frac{2A}{a\sqrt{2g}}\left[\sqrt{H}-\sqrt{\frac{H}{2}}\right]$$

$$\Rightarrow \quad t_1 = \frac{A}{a}\sqrt{\frac{H}{g}}\left(\sqrt{2}-1\right) \qquad \ldots(1)$$

Similarly, $\int_0^{t_2} dt = -\frac{A}{a\sqrt{2g}}\int_{H/2}^{0} y^{-1/2}dy$

$$\Rightarrow \quad t_2 = \frac{A}{a}\sqrt{\frac{H}{g}} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$\frac{t_1}{t_2} = \sqrt{2}-1$$

$$\Rightarrow \quad \frac{t_1}{t_2} = 0.414$$

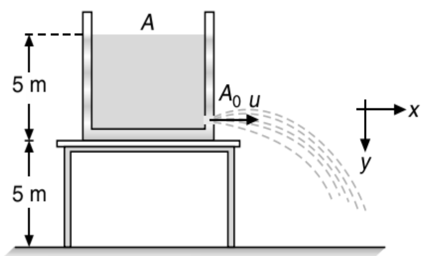
From here we see that $t_1 < t_2$. This is because initially the pressure is high and the liquid comes out with greater speed.

ILLUSTRATION 117

A cylindrical tank 1 m in radius rests on a platform 5 m high. Initially the tank is filled with water up to a height of 5 m. A plug having an area of 10^{-4} m^2 is removed from the orifice at the bottom of the tank. Calculate the speed with which the water leaves the orifice and the speed with which it hits the ground. Also calculate the time taken to empty the tank to half its original volume. Does this time depend on the height of platform above ground?

SOLUTION

The situation given in the problem is shown in Figure.



The speed with which water leaves the orifice is equal to the speed of efflux given by

$$u_x = u = \sqrt{2gh} = \sqrt{2\times10\times5} = 10 \text{ ms}^{-1}$$

Since initial vertical velocity of water is zero, so its vertical velocity when it hits the ground is

$$v_y = \sqrt{2gh} = \sqrt{2\times10\times5} = 10 \text{ ms}^{-1}$$

Also, motion is uniform along the horizontal direction, so

$$v_x = u_x = u$$

Hence the speed (v) with which water strikes the ground is

$$v = \sqrt{v_x^2+v_y^2} = 10\sqrt{2} = 14.1 \text{ ms}^{-1}$$

When height of water level above the orifice is y, then velocity of efflux will be $v = \sqrt{2gy}$ and so rate of flow of water is

$$\frac{dV}{dt} = A_0 v = A_0\sqrt{2gy}$$

$$\Rightarrow \quad -Ady = \left(\sqrt{2gy}\right)A_0 dt \qquad \{\because dV = -Ady\}$$

$$\Rightarrow \quad \int_H^0 \frac{Ady}{\sqrt{2gy}} = \int_0^t A_0 dt$$

$$\Rightarrow \quad t = \frac{A}{A_0}\sqrt{\frac{2}{g}}\left(\sqrt{H}-\sqrt{H'}\right)$$

$$\Rightarrow \quad t = \left(\frac{\pi\times1^2}{10^{-4}}\right)\sqrt{\frac{2}{10}}\left[\sqrt{5}-\sqrt{\left(\frac{5}{2}\right)}\right]$$

$$\Rightarrow \quad t = 9.2\times10^3 \text{ s} \approx 2.5 \text{ h}$$

The expression of t is independent of height of platform on which beaker is kept.

ILLUSTRATION 118

At what velocity does the water emerge from a small orifice in an open tank if the gauge pressure at the orifice is 2×10^5 Nm^{-2} before the flow starts?

SOLUTION

Since the tank is open, velocity of efflux can be directly given by Torricelli's law.

$$v = \sqrt{2gh} \qquad \ldots(1)$$

If the gauge pressure of 2×10^5 Nm^{-2} corresponds to a height h, then we have

$$h\rho_w g = \Delta P = 2\times10^5 \text{ Nm}^{-2}$$

$$\Rightarrow \quad gh = \frac{2\times10^5}{\rho_w}$$

$$\Rightarrow \quad gh = \frac{2\times10^5}{10^3}$$

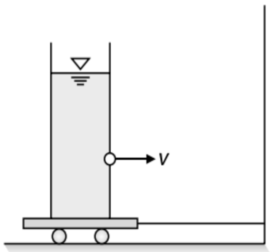
Substituting this value of gh in equation (1), we get

$$v = \sqrt{\frac{2\times2\times10^5}{10^3}}$$

$$\Rightarrow \quad v = 20 \text{ ms}^{-1}$$

ILLUSTRATION 119

A large tank of height $h = 1$ m and diameter $D = 0.6$ m is affixed to a cart as shown. Water issues from a tank through a nozzle of diameter $d = 10$ mm. The speed of the liquid leaving the tank is approximately $v = \sqrt{2gy}$ where y is the height from the nozzle to the free surface. Determine the tension in the wire when $y = 0.8$ m. Plot the tension in the wire as a function of water depth $0 \le y \le 0.8$ m. Density of water is 10^3 kgm^{-2} and $g = 9.8$ ms^{-2}.



SOLUTION

Tension in the thread is equal to the thrust force due to liquid on the beaker, so we have

$$T = \rho a v^2$$

where, $a = \left(\frac{\pi}{4}\right)(d^2)$ and $v^2 = 2gy$

$$\Rightarrow \quad T = \rho\left(\frac{\pi}{4}\right)(d^2)(2gy) \quad \ldots(1)$$

$$\Rightarrow \quad T = (10^3)\left(\frac{\pi}{4}\right)(10^{-2})^2(2)(9.8)(0.8)$$

$$\Rightarrow \quad T = 1.23 \text{ N}$$

From equation (1) $T \propto y$

i.e. T-y graph is a straight line passing through origin as shown in Figure.

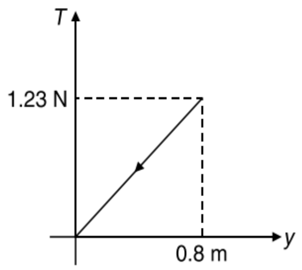
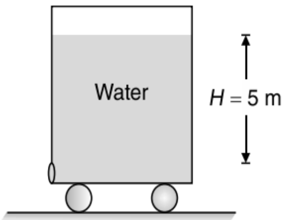


ILLUSTRATION 120

A tank, initially at rest, is filled with water to a height $H = 5$ m. A small orifice is made at the bottom of the wall. Find the velocity attained by the tank when it becomes completely empty. Assume mass of the tank to be negligible. Friction is negligible. Take $g = 10$ ms^{-2}.



SOLUTION

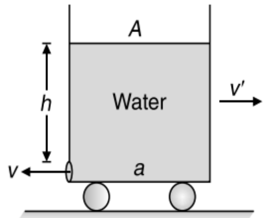
Let v' be the instantaneous velocity of the tank, and v be the instantaneous velocity of efflux with respect to the tank. **Thrust** exerted on the tank is

$$F = \rho a v^2$$

where a is the cross-sectional area of the orifice.

$$v = \sqrt{2gh}$$

where h is the depth of the orifice below the free surface of water in the tank.



If A be the cross-sectional area of the tank, then mass of the tank at any time t is

$$m = \rho A h$$

According to Newton's Second Law, we have

$$F = m\frac{dv'}{dt} = \rho A h \frac{dv'}{dt}$$

$$\Rightarrow \quad \rho A h \frac{dv'}{dt} = \rho a v^2 = 2\rho g a h$$

$$\Rightarrow \quad \frac{dv'}{dt} = 2g\left(\frac{a}{A}\right) \quad \ldots(1)$$

In a time dt, if the water level falls by dh, then by Law of conservation of mass applied to liquids, we have

$$-\rho A dh = \rho a v dt$$

$$\Rightarrow \quad \frac{dh}{dt} = -\frac{av}{A}$$

Equation (1) can be re-written as

$$\left(\frac{dv'}{dh}\right)\left(\frac{dh}{dt}\right) = 2g\left(\frac{a}{A}\right)$$

$$\Rightarrow \quad \frac{dv'}{dh}\left(-\frac{av}{A}\right) = 2g\left(\frac{a}{A}\right)$$

$$\Rightarrow \quad \frac{dv'}{dh} = -\frac{2g}{v} = -\frac{2g}{\sqrt{2gh}} = -\sqrt{\frac{2g}{h}}$$

On integrating, we get

$$\int_0^{v'} dv' = -\sqrt{2g}\int_H^0 \frac{dh}{\sqrt{h}}$$

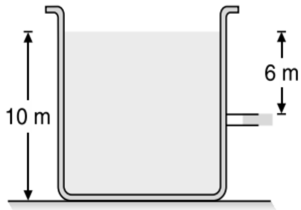
$$\Rightarrow \quad v = 2\sqrt{2gH}$$

Since $H = 5$ m,

$$\Rightarrow \quad v = 2\sqrt{2(10)(5)} = 20 \text{ ms}^{-1}$$

ILLUSTRATION 121

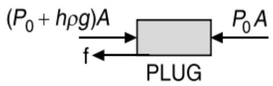
A fresh water on a reservoir is 10 m deep. A horizontal pipe 4 cm in diameter passes through the reservoir 6 m below the water surface as shown in Figure. A plug secures the pipe opening.



Calculate the frictional force between the plug and pipe wall. If the plug is removed, then calculate the volume of water flows out of the pipe in 1 h. Assume area of reservoir to be too large.

SOLUTION

The force of friction between the plug and the inner surface of the pipe is equal to the force due to pressure at the depth of 6 m. The free body diagram for the plug is shown in Figure.



For equilibrium, we have

$$f + P_0A = (P_0 + h\rho g)A$$

$$\Rightarrow \quad f = (h\rho g)A$$

$$\Rightarrow \quad f = (10^3)(9.8)(6.0)(\pi)(2\times10^{-2})^2$$

$$\Rightarrow \quad f = 73.9 \text{ N}$$

Assuming the area of the reservoir to be too large, the velocity of efflux will be

$$v = \sqrt{2gh} \approx \text{constant}$$

$$\Rightarrow \quad v = \sqrt{2\times9.8\times6} = 10.84 \text{ ms}^{-1}$$

Volume of water coming out per second is

$$Q = Av = \pi(2\times10^{-2})^2(10.84)$$

$$\Rightarrow \quad Q = 1.36\times10^{-2} \text{ m}^3\text{s}^{-1}$$

So, the volume of water flowing through the pipe in $t = 1 \text{ hr} = 3600 \text{ s}$ is

$$\Rightarrow \quad V = Qt = \left(\frac{dV}{dt}\right)t = (1.36\times10^{-2})(3600)$$

$$\Rightarrow \quad V = 48.96 \text{ m}^3$$

ILLUSTRATION 122

A cylindrical tank of base area A has a small hole of area a at the bottom. At time $t = 0$, a tap starts to supply water into the tank at a constant rate α m^{3}s^{-1}. Calculate the maximum level of water h_{max} in the tank. Also calculate the time when level of water becomes $h(< h_{max})$.

SOLUTION

Level will be maximum when

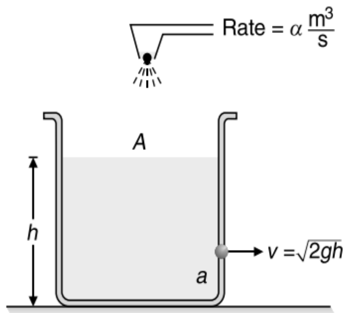
rate of inflow of water = rate of outflow of water

i.e., $\alpha = av$

$$\Rightarrow \quad \alpha = a\sqrt{2gh_{max}}$$

$$\Rightarrow \quad h_{max} = \frac{\alpha^2}{2ga^2}$$

Let at time t, the level of water be h. Then



$$A\left(\frac{dh}{dt}\right) = \alpha - a\sqrt{2gh}$$

$$\Rightarrow \quad \int_0^h \frac{dh}{\alpha - a\sqrt{2gh}} = \int_0^t \frac{dt}{A}$$

On solving this equation, we get

$$t = \frac{A}{ag}\left[\frac{\alpha}{a}\ln\left(\frac{\alpha - a\sqrt{2gh}}{\alpha}\right) - \sqrt{2gh}\right]$$

ILLUSTRATION 123

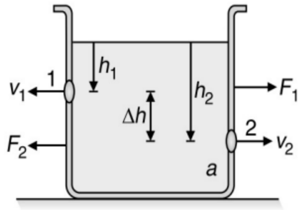
On the opposite sides of a wide vertical vessel filled with water, two identical holes each having cross-sectional area A are opened. The height difference between the holes is equal to Δh. Calculate the force of reaction due to the water flowing out of vessel.

SOLUTION

Force of reaction due to ejection of fluid through a hole is

$$F = Av^2\rho$$

This force is opposite to the velocity of efflux of the liquid coming out from the orifice.



Due to two holes, net force on vessel is

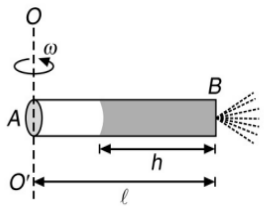
$$F = F_2 - F_1 = A\left(v_1^2 - v_2^2\right)\rho$$

$$\Rightarrow \quad F = A\left[\left(\sqrt{2g(h+\Delta h)}\right)^2 - \left(\sqrt{2gh}\right)^2\right]\rho$$

$$\Rightarrow \quad F = 2\rho g A\Delta h$$

ILLUSTRATION 124

A horizontal oriented tube AB of length l rotates with a constant angular velocity ω about a stationary vertical axis OO' passing through the end A as shown in Figure.



The tube is filled with an ideal fluid. The end A of the tube is open, the closed end B has a very small orifice. Find the velocity of the fluid relative to the tube as a function of the column height h.

SOLUTION

Using Bernoulli's theorem, we get

$$P_B = P_{atm} + \frac{1}{2}\rho v^2 \qquad \ldots(1)$$

Also, we know that

$$P_B - P_{atm} = \int_{l-h}^{l} \rho\omega^2 x\,dx$$

$$\Rightarrow \quad P_B - P_{atm} = \frac{\rho\omega^2}{2}\left[l^2 - (l-h)^2\right]$$

$$\Rightarrow \quad P_B - P_{atm} = \frac{1}{2}\rho\omega^2\left(2lh - h^2\right)$$

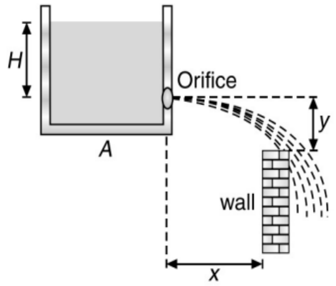
Using equation (1), we get

$$\frac{1}{2}\rho\omega^2\left(2lh - h^2\right) = \frac{1}{2}\rho v^2$$

$$\Rightarrow \quad v = \omega h\sqrt{\frac{2l}{h} - 1}$$

ILLUSTRATION 125

For the arrangement shown in Figure, calculate the time interval after which the water jet ceases to cross the wall. Assume area of the tank to be A and area of orifice to be $a \ll A$.



SOLUTION

At any instant, when liquid level is at a height h, then the speed of ejection of water is

$$v = \sqrt{2gh} \qquad \{\because a \ll A\}$$

The water jet will be crossing the wall when

$$x = vt = v\sqrt{\frac{2y}{g}}$$

$$\Rightarrow \quad x = \sqrt{2gh}\sqrt{\frac{2y}{g}}$$

$$\Rightarrow \quad h = \frac{x^2}{4y}$$

According to Equation of Continuity, we have

$$-A\frac{dh}{dt} = a\sqrt{2gh}$$

$$\Rightarrow \quad \int_H^h \frac{dh}{\sqrt{h}} = -\frac{a}{A}\sqrt{2g}\int_0^t dt$$

$$\Rightarrow \quad 2\left(\sqrt{H} - \sqrt{h}\right) = \frac{a}{A}\sqrt{2g}t$$

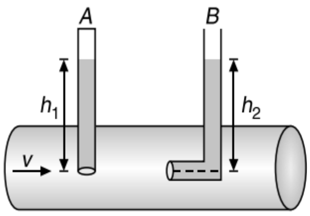
$$\Rightarrow \quad t = \frac{A}{a}\sqrt{\frac{2}{g}}\left(\sqrt{H} - \sqrt{\frac{x^2}{4y}}\right)$$

Test Your Concepts-IX

Based on Equation of Continuity, Bernoulli's Theorem and Applications

(Solutions on page H.14)

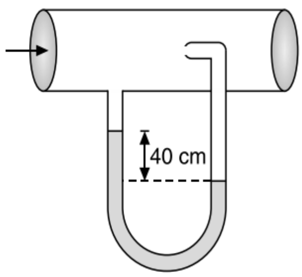
1. A liquid is flowing in a horizontal pipe of uniform cross section with velocity v. Two tubes A and B of small cross-sectional area are inserted into the pipe as shown.



Assuming the flow to remain streamline inside the pipe, calculate the difference in height of the liquid in two tubes

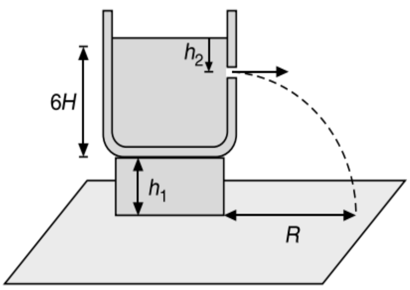
2. A cylindrical bucket, open at the top, is 0.2 m high and 0.1 m in diameter. A circular hole with cross-section area $1\ \text{cm}^2$ is cut in the centre of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of $1.3\times10^{-4}\ \text{m}^3\text{s}^{-1}$. How high will the water in the bucket rise?

3. A pitot tube is mounted on an aeroplane wing to measure the speed of the plane. The tube contains alcohol and shows a level difference of 40 cm as shown in Figure.



Calculate the speed of the plane relative to air if relative density of alcohol is 0.8 and density of air is $1\ \text{kgm}^{-3}$.

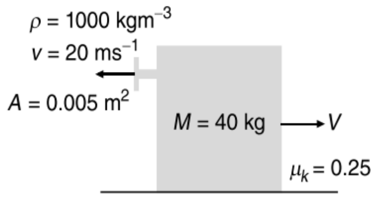
4. A beaker containing liquid to a height $6H$ is having an orifice at a depth h_2 below the free surface of the liquid. The beaker is placed on a horizontal platform of height h_1 as shown in Figure.



Calculate the value of h_2 for range R to be maximum, if $h_1 = 4H$ and $h_1 = 8H$. Also find the value of this maximum range in both cases.

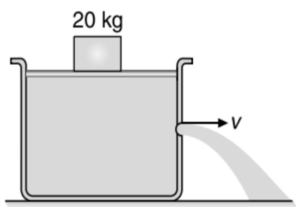
5. On the opposite side of a wide vertical vessel filled with water two identical holes are opened, each having the cross-sectional area $0.50\ \text{cm}^2$. The height difference between them is equal to 51 cm. Calculate the resultant force of reaction of the water flowing out of the vessel.

6. A slider assembly of mass $m = 40$ kg moves under the influence of a liquid jet. At any instant, the slider is moving to the right with a speed $V = 10\ \text{ms}^{-1}$ by emitting a water of density $1000\ \text{kgm}^{-3}$ from an area of $A = 0.005\ \text{m}^2$ with a velocity of $v = 20\ \text{ms}^{-1}$ to the left as shown in Figure.

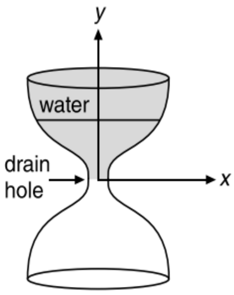


If the coefficient of kinetic friction between the slider and the surface is $\mu_k = 0.25$, then calculate the acceleration of the slider at that instant. Also calculate terminal speed of the slider.

7. A long cylindrical tank of cross-section area $0.5\ \text{m}^2$ is filled with water. It has a hole of cross-section $1\times10^{-4}\ \text{m}^2$ at a height 50 cm from the bottom. A movable piston of cross-sectional area almost equal to $0.5\ \text{m}^2$ is fitted on the top of the tank such that it can slide in the tank freely. A load of 20 kg is applied on the top of the water by piston, as shown in the figure. Calculate the speed of the water jet with which it hits the surface when piston is 1 m above the base. Ignore the mass of the piston and take $g = 10\ \text{ms}^{-2}$.



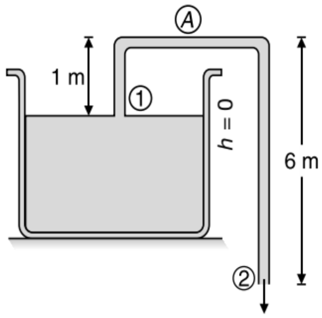
8. The shape of an ancient water clock jug is such that water level descends at a constant rate at all times. If the water level falls by 4 cm every hour, determine the shape of the jar, i.e., specify x as a function of y. Assume that the radius of drain hole is 2 mm which is very small compared to x.



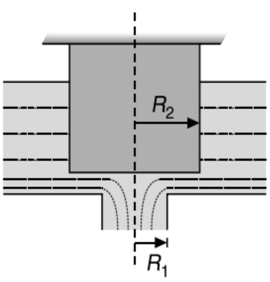
9. Calculate the rate of flow of glycerine of density 1.25×10^3 kgm^{-3} through the conical section of a pipe, if the radii of its ends are 0.1 m and 0.04 m and the pressure drop across its length is 10 Nm^{-2}.

10. A cylindrical vessel is filled with water upto a height of 1 m. The cross-sectional area of the orifice at the bottom is $\left(\frac{1}{400}\right)$ that of the vessel. Calculate the time required to empty the tank through the orifice at the bottom. Also calculate the time required for the same amount of water to flow out, if the water level in tank is maintained always at a level of 1 m above the orifice.

11. The U-tube acts as a water siphon. The bend in the tube is 1 m above the water surface. The tube outlet is 5 m below the water surface. The water issues from the bottom of the siphon as a free jet at atmospheric pressure. Determine the speed of the free jet and the minimum absolute pressure of the water in the bend. Given atmospheric pressure to be 10^5 Nm^{-2}, density of water to be 10^3 kgm^{-3} and $g = 9.8$ ms^{-2}.



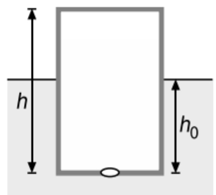
12. The horizontal bottom of a wide vessel with an ideal fluid has a round orifice of radius R_1 over which a round closed cylinder is mounted, whose radius $R_2 > R_1$ as shown in Figure.



The clearance between the cylinder and the bottom of the vessel is very small, the fluid density is ρ. Assuming the atmospheric pressure to be P_0 calculate the static pressure of the fluid in the clearance as a function of the distance r from the axis of the orifice (and the cylinder) if the height of the fluid is equal to h. Assume that $R_1 < r < R_2$.

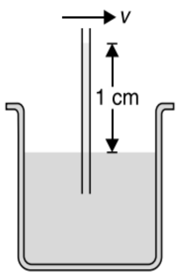
13. A large tank of area A, filled with water to a height H is to be emptied to half the height through a small hole of area a at the bottom. Calculate the ratio of times taken for the level of water to fall from H to $\frac{H}{2}$ and from $\frac{H}{2}$ to zero.

14. A cylindrical can of height h and base area A is immersed in water to a depth h_0 as shown in Figure.

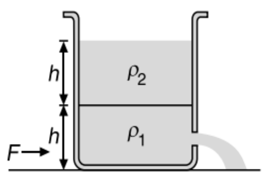


If there is a small hole of area a in the base of the can, then calculate the time taken by the can to sink.

15. When air of density 1.3 kgm^{-3} flows across the top of the tube shown in Figure, water rises in the tube to a height of 1 cm. Calculate the speed of air.

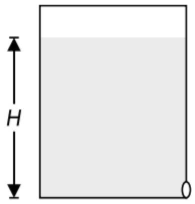


16. A cylindrical tank having cross-sectional area $A = 0.5$ m^2 is filled with two liquids of density $\rho_1 = 900$ kgm^{-3} and $\rho_2 = 600$ kgm^{-3} to a height $h = 60$ cm each as shown in Figure.



A small hole having area $a = 5$ cm^2 is made in right vertical wall at a height $y = 20$ cm from the bottom. Calculate the velocity of efflux and the horizontal force F required to keep the cylinder in static equilibrium, if it is placed on a smooth horizontal plane. If coefficient of friction between the cylinder and the surface on which it is kept is $\mu = 0.01$, then calculate the maximum and minimum values of F to keep the cylinder in equilibrium. Take $g = 10\ \text{ms}^{-2}$.

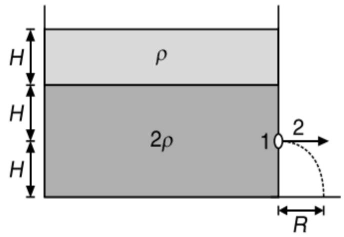
17. Find the time to empty the container filled with liquid up to height H initially. Area of hole at bottom is a and area of cross-section of the base of container is A.



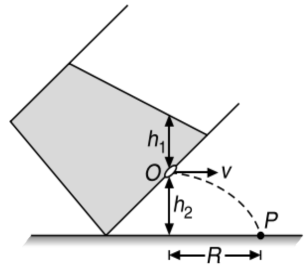
18. The side wall of side vertical cylindrical vessel of height $h = 75\ \text{cm}$ has a narrow vertical slit running all the way down to the bottom of the vessel. The length of the slit is $l = 50\ \text{cm}$ and the width $b = 1.0\ \text{mm}$. With the slit closed, the vessel is filled with water. Calculate the resultant force of reaction of the water flowing out of the vessel immediately after the slit is opened.

19. A water line with an internal radius of $6.5 \times 10^{-3}\ \text{m}$ is connected to a shower head that has 12 holes. The speed of the water in the line is $1.2\ \text{ms}^{-1}$. Calculate the volume flow rate in the line. Also calculate the speed with which the water leaves one of the holes in the head assuming the effective hole radius to be $4.6 \times 10^{-4}\ \text{m}$.

20. A cylindrical tank is filled with two liquids of density ρ and 2ρ as shown in Figure. Calculate the range R of the liquid coming out from the orifice.



21. A beaker is kept such that the orifice O is located as shown in Figure. Calculate the speed v with which liquid comes out of the orifice, the time taken by the liquid to hit the ground and the range R.



22. A wide vessel with a small hole in the bottom is filled with water and kerosene. Neglecting the viscosity, find the velocity of the water flow, if the thickness of the water layer is equal to $h_1 = 30\ \text{cm}$ and that of the kerosene layer to $h_2 = 20\ \text{cm}$. Density of kerosene is $600\ \text{kgm}^{-3}$.

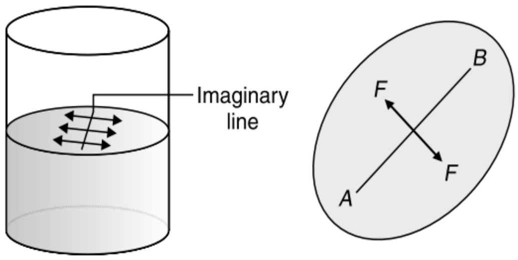
23. Liquid is filled in a container upto a height of H. A small hole is made at the bottom of the tank. Time take to empty from H to $\frac{H}{3}$ is t_0. Find the time taken to empty the tank from $\frac{H}{3}$ to zero.

SURFACE TENSION

SURFACE TENSION: INTRODUCTION

A liquid has the property that its free surface tends to attain minimum possible area and is therefore, in a state of tension, somewhat like a stretched membrane. It is a molecular phenomenon based on electromagnetic interaction between the molecules. The force of contraction at right angles to an imaginary line of unit length, tangential to the surface of liquid, is called its **surface tension.**

Surface tension of a liquid is measured by the force acting per unit length on either side of an imaginary line drawn on the free surface of liquid as shown in Figure.



The direction of this force being perpendicular to the imaginary line and tangential to the free surface of liquid. So if F is the force acting on one side of imaginary line AB of length l, then mathematically surface tension is

$$T = \frac{F}{l}$$

The S.I. unit of surface tension is newton per metre $(\mathrm{Nm^{-1}})$ and dynecm^{-1}. The Dimensional formula of surface tension is MT^{-2} which is same as that of force constant or spring constant.

Surface Tension depends only on the nature of liquid and is independent of the area of surface or length of line considered. It is a scalar as it has a unique direction which is not to be specified.

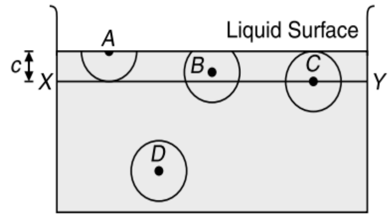
Small liquid drops tend to be spherical due to surface tension, because for a given volume, sphere has the minimum surface area. It follows that in order to increase the surface area of liquid, work has to be done against this force of contraction. This work is stored in the surface as its potential energy.

EXPLANATION TO SURFACE TENSION

Laplace explained the phenomenon of surface-tension on the basis of inter-molecular forces. We know that if the distance between two molecules is less than the molecular range $c(\approx 10^{-9}\ \text{meter})$ then they attract each other, but if the distance is more than this range, then attraction becomes negligible.

Therefore, if we draw a sphere of radius c with a molecule as center, then only those molecules which are enclosed within this sphere can attract, or be attracted by, the molecule at the center of the sphere. This is called 'sphere of molecular activity'.

In order to understand the tension existing in the free surface of a liquid, let us consider four liquid molecules like A, B, C and D along with their spheres of molecular activity as shown in Figure.



The molecule D is well inside the liquid and so it is attracted equally in all directions. Hence the resultant force acting on molecule D is zero.

The sphere of molecule C is just below the liquid surface and the resultant force on it is also zero.

The molecule B which is a little below the liquid surface has its sphere of molecular activity partly outside the liquid. Thus, the number of liquid molecules in upper half (attracting in downward). Hence the molecule B experience a resultant downward force.

The molecule A is at the surface of the liquid, so that its sphere of molecular activity is half outside the liquid and half inside. So it experiences a maximum downward force.

Hence all the molecules situated between the surface and a plane XY, distance c below the surface, experience a resultant downward cohesive force.

When the surface area of liquid is increased, molecules from the interior of the liquid rise to the surface. As these molecules reach near the surface, work is done against the (downward) cohesive force. This work is stored in the molecule in the form of potential energy. Thus, the potential energy of the molecules lying at the surface is greater than that of the molecules in the interior of the liquid.

Since we know that a system is in stable equilibrium when it has minimum potential energy. So, in order to have minimum potential energy, the liquid surface tends to have minimum number of molecules in it (which is possible if the free surface tries to attain the minimum surface area). In other words, the surface tends to contract to a minimum possible area. *This tendency of the free surface of the liquid is exhibited as surface tension.*

Conceptual Note(s)

It must be noted that surface tension is a property of a liquid and it does not depend on the area of the free surface of the liquid. For example, both containers have the same liquid under same conditions. $A_2 > A_1$ but surface tension will be same in both cases.

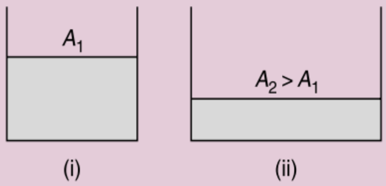


ILLUSTRATION 126

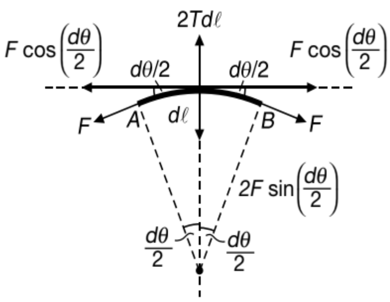
Consider a horizontal film of soap solution, on which a light thread soaked in soap solution is placed in the form of a loop. The film is then pierced inside the loop and the thread becomes a circular loop of radius R. If the surface tension of the soap solution be T, then calculate the tension in the thread.

SOLUTION

Let the tension in the thread be F. Consider a small arc AB of length dl of the circular loop. Let this arc subtend an angle $d\theta$ at the centre of the circular loop of radius R. On this small arc, we observe the following two forces to be acting.

(a) Force due to surface tension $2T\Delta l$, acting outwards.

(b) Component of tension force $2F\sin\left(\dfrac{d\theta}{2}\right)$, acting inwards.



At equilibrium, we have

$$2F\sin\left(\frac{d\theta}{2}\right) = 2Tdl \qquad \ldots(1)$$

Since $d\theta$ is small, so

$$\sin\left(\frac{d\theta}{2}\right) \approx \frac{d\theta}{2}, \text{ where } d\theta = \frac{dl}{R}$$

So, from (1), we get

$$2F\left(\frac{d\theta}{2}\right) = Fd\theta = 2T(Rd\theta)$$

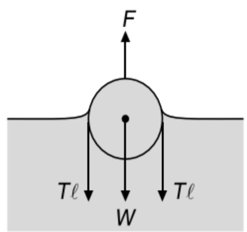
$$\Rightarrow \quad F = 2TR$$

FORCE DUE TO SURFACE TENSION

If a body of weight W is placed on the liquid surface, whose surface tension is T. Let us calculate the value of minimum force F required to pull it away from the water for different body shapes.

CASE-1: Needle on water surface

If a needle of length l is placed on the liquid surface whose cross-sectional view is shown in Figure. as shown in Figure.

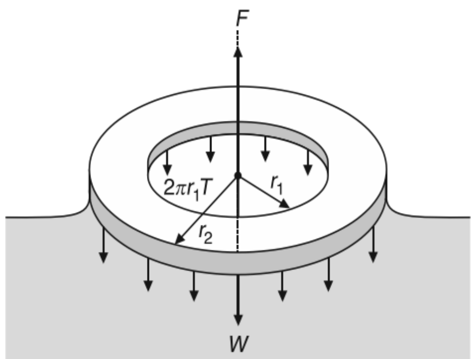


Minimum force required to lift up the needle is

$$F = 2lT + W$$

CASE-2: Annular disc on water surface

If an annular disc of inner radius r_1 and outer radius r_2 is placed on the liquid surface and we try to pull it up with a minimum force F, then forces acting on the disc are shown in Figure.



The weight W acts vertically downwards, the force due to surface tension acting on outer perimeter of the disc is $2\pi r_2 T$ (downwards) and that acting on inner perimeter of the disc is $2\pi r_1 T$ (downwards). The applied minimum force should be equal to the sum of these forces, so we have

$$F = 2\pi(r_1 + r_2)T + W$$

Special Case(s)

(a) **For a disc** of radius r, we can think $r_1 = 0$ and $r_2 = r$, so the minimum force F is

$$F = 2\pi rT$$

(b) **For a ring** of radius r, we can think $r_1 \approx r_2 \approx r$, so the minimum force F is

$$F \approx 2\pi(r + r)T = 4\pi rT$$

(c) **For a square plate** of side l, the minimum force required to pick up the plate is

$$F = 4lT + W$$

(d) **For a square frame** of side l, the minimum force required to pick up the frame is

$$F = (4lT + 4lT) + W = 8lT + W$$

ILLUSTRATION 127

A ring is cut form a platinum tube of 8.5 cm internal and 8.7 cm external diameter. It is supported horizontally from a pan of a balance so that it comes in contact with the water in a glass vessel. Calculate the surface tension of water if an extra 3.97 gwt is required to pull it away from water. Take $g = 980\text{ cms}^{-2}$.

SOLUTION

The ring is in contact with water along its inner and outer circumference. When pulled out, the total force on it due to surface tension will be

$$F = T(2\pi r_1 + 2\pi r_2) + mg$$

$$\Rightarrow \quad T = \frac{F - mg}{2\pi(r_1 + r_2)}$$

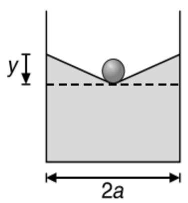
Given that an extra 3.97 gwt is required to pull it away from water so we have

$$F - mg = 3.97\text{ gwt} = (3.97)(980)\text{ dyne}$$

$$\Rightarrow \quad T = \frac{3.97 \times 980}{3.14 \times (8.5 + 8.7)} = 72.13\text{ dynecm}^{-1}$$

ILLUSTRATION 128

A container of width $2a$ is filled with a liquid. A thin wire of linear mass density λ is gently placed over the liquid surface in the middle of surface as shown. As a result, the liquid surface is depressed by a distance $y(y \ll a)$. Determine the surface tension of the liquid.

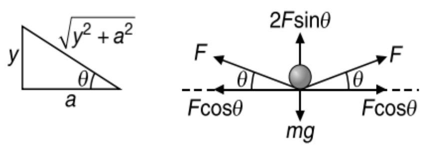


SOLUTION

If l be the length of wire and λ is mass per unit length of wire, then weight of wire is

$$mg = (\lambda x)g\text{, acting vertically downwards}$$

The force due to surface tension acting on each side of the wire is $F = Tl$. The free body diagram showing forces acting on the wire is shown in Figure.



Vertical force acting upwards due to surface tension balances the weight of the wire, so we have

$$2Tl\sin\theta = mg = (\lambda l)g$$

$$\Rightarrow \quad T = \frac{\lambda g}{2\sin\theta} \qquad \ldots(1)$$

Since $y \ll a$, so θ is small and hence $\sin\theta \approx \theta \approx \frac{y}{a}$

So, from equation (1), we get

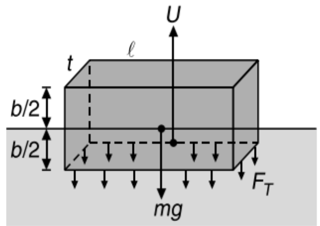
$$T = \frac{\lambda g}{2(y/a)} = \frac{\lambda a g}{2y}$$

ILLUSTRATION 129

A glass plate of length 40 cm, height 20 cm and thickness 10 cm weighs 5 kg in air. If the plate is held vertically with long side horizontal and half its height immersed in a liquid, then calculate the apparent weight of the plate assuming the surface tension of the liquid to be $50 \times 10^{-2}\text{ Nm}^{-1}$.

SOLUTION

The glass plate is held inside liquid as shown in Figure.



According to the problem, we have $l = 40\text{ cm}, b = 20\text{ cm}$, $t = 10\text{ cm}$ and $T = 50 \times 10^{-3}\text{ Nm}^{-1}$. The glass plate is under the action of following forces

(a) weight mg, acting vertically downwards

(b) force (F_T) due to surface tension, acting vertically downwards such that

$$F_T = 2(l + t)T = 2\left(\frac{40}{100} + \frac{10}{100}\right)(0.5) = 0.5\text{ N}$$

(c) upthrust (U), acting vertically upwards, experienced due to the immersed part of the plate such that

$$U = V_{\text{immersed}}\rho_{\text{liquid}}g = \left(\frac{ltb}{2}\right)\rho_w g$$

$$\Rightarrow \quad U = \left[\left(\frac{40 \times 10 \times 20}{2}\right) \times 10^{-6}\right](10^3)(10)$$

$$\Rightarrow \quad U = 40\text{ N}$$

So, apparent weight of the glass plate is

$$W' = mg + F_T - U$$

$$\Rightarrow \quad W' = 50 + 0.5 - 40 = 10.5 \text{ N}$$

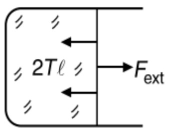
SURFACE ENERGY

A liquid molecule completely inside the liquid is surrounded by similar molecules on all sides and hence experiences no resultant force on it, whereas a molecule at the free surface of liquid is surrounded by similar liquid molecules on one side of the free surface (while on the other side it may be surrounded by air molecules or the molecules of the vapour of the liquid etc). Since air or liquid vapours have negligible density compared to liquid, so they exert only a small force on the molecules at the free surface. Hence, a resultant inward force acts on molecule lying at the surface. This inward force tries to pull the molecule into the liquid due to which the free surface layer remains in microscopic turbulence in which the molecules are pulled back from the free surface layer to the liquid bulk and hence new molecules from the liquid bulk come to the surface in an attempt to fill the empty space.

When a molecule is taken from inside the liquid to the free surface, then work has to be done against the inward resultant force for moving the molecule up to the free surface. The potential energy is increased due to this work. A molecule at the free surface has greater potential energy than a molecule completely inside the liquid. *This extra energy possessed by the free surface layer is called the surface energy*.

RELATION BETWEEN SURFACE ENERGY AND SURFACE TENSION

To find the relation between surface energy and surface tension, let us consider a U-shaped frame that guides a sliding wire on its arm. Let the arrangement be dipped in a soap solution, then taken out and placed in a horizontal position as shown in Figure.



We may think that the soap film formed is extremely thin, however at the molecular scale its thickness is not ignorable as it may possess several hundred thousand molecular layers. So, we say that the soap film has two free surfaces both of which are in contact with the sliding wire and hence exert forces of surface tension on the wire.

If T be the surface tension of the soap solution and l be the length of the sliding wire, then each surface will pull the wire parallel to itself with a force Tl and hence the net force of pull F on the wire due to both the surfaces is

$$F = Tl + Tl = 2Tl$$

In order to keep the wire in equilibrium, we have to apply a constant external force F_{ext} equal and opposite to F.

Now, let the wire be slowly pulled out (so that change in kinetic energy is zero) with the help of external force through a distance x so that the area of the frame is increased by $\Delta A = lx$. Since the film has two free surfaces of the soap solution, hence total change in surface area of the film is $2\Delta A = 2lx$.

The work done by the external force in moving the wire through x is

$$W_{\text{ext}} = F_{\text{ext}} x = (2Tl)x = T(2lx) = 2T\Delta A$$

Since there is no change in kinetic energy, so the work done by external force is stored as the change in potential energy ΔU of the surface, so we have

$$\Delta U = W_{\text{ext}} = 2T\Delta A$$

$$\Rightarrow \quad T = \frac{\Delta U}{2\Delta A} = \frac{W_{\text{ext}}}{\Delta A_{\text{total}}} \qquad \ldots(1)$$

where, $\Delta A = lx$ is the change in surface area of each free surface.

So, we observe that the surface tension of a liquid is equal to the surface energy per unit change in surface area.

So, from equation (1), we observe that the work done by an external force to increase the surface area of the film by ΔA_{total} is

$$W_{\text{ext}} = T\Delta A_{\text{total}} = 2T\Delta A_{\text{each surface}}$$

However, for the case where we may have only one free surface, then $\Delta A_{\text{total}} = \Delta A$ and hence

$$W_{\text{ext}} = T\Delta A_{\text{total}} = T\Delta A$$

The SI unit of surface tension is also written as Jm^{-2} and it can also be verified that $1 \text{ Nm}^{-1} = 1 \text{ Jm}^{-2}$.

Conceptual Note(s)

Please note that the work done to change the surface area of

(a) a soap film or a liquid film will be $W = 2T\Delta A$, because a soap film or a liquid film has two free surfaces.

(b) a soap bubble will be $W = 2T\Delta A$, because a soap bubble has two free surfaces.

(c) an air bubble or a liquid drop will be $W = T\Delta A$, because an air bubble or a liquid drop has only one free surface.

(d) In case of the liquid drop, if the initial radius of liquid drop (having surface tension T) is r_1 and final radius is r_2, then work done in changing the radius from r_1 to r_2 is

$$W = T\Delta A = 4\pi T\left(r_2^2 - r_1^2\right)$$

(e) In case of soap bubble, which has two free surfaces, if the initial radius of soap bubble (having surface tension T) is r_1 and final radius is r_2, then then work done in changing the radius from r_1 to r_2 is

$$W = 8\pi T\left(r_2^2 - r_1^2\right)$$

ILLUSTRATION 130

A U-shaped wire loop having a slider of negligible mass having a length of 0.1 m is dipped in a soap solution and then removed. Due to this, a thin soap film is formed between the wire and the light slider. It is observed that the film can support a weight of 0.006 N, before it can break. Calculate the surface tension of the film.

SOLUTION

Because of the phenomenon of surface tension, the soap film tries to occupy minimum surface area. Since a soap film has two free surfaces, so the total force acting on the slider due to surface tension is

$$F = 2(Tl)$$

Since the film can support a weight of 0.006 N, before it can break, so we have

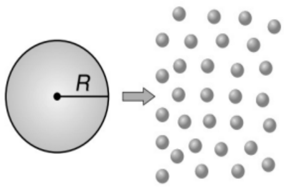
$$2(Tl) = W = mg$$

where, $W = mg = 0.006\text{ N}$ and $l = 0.1\text{ m}$

$$\Rightarrow \quad T = \frac{W}{2l} = \frac{0.006}{2\times 0.1} = 0.03\text{ Nm}^{-1}$$

SPLITTING OF BIGGER DROP INTO SMALL DROPLETS

When a drop of radius R splits into n smaller drops, (each of radius r) then surface area of liquid increases.



Hence the work is to be done against surface tension. Since the volume of liquid remains constant therefore

$$\frac{4}{3}\pi R^3 = n\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad R^3 = nr^3$$

Work done is given by

$$W = T\Delta A = T\left[n(4\pi r^2) - 4\pi R^2\right]$$

$$\Rightarrow \quad W = 4\pi T(nr^2 - R^2)$$

$$\Rightarrow \quad W = 4\pi R^2 T(n^{1/3} - 1)$$

$$\Rightarrow \quad W = 4\pi T r^2 n^{2/3}(n^{1/3} - 1)$$

$$\Rightarrow \quad W = 4\pi T R^3\left(\frac{1}{r} - \frac{1}{R}\right)$$

Conceptual Note(s)

If the work is not done by an external source then internal energy of liquid decreases, subsequently temperature decreases. This is the reason why spraying causes cooling. By Law of Conservation of energy, loss in thermal energy equals the work done against surface tension. So

$$W = JQ$$

where, $Q = mc\Delta\theta$, c is the gram specific heat of the liquid, $\Delta\theta$ is the decrease in temperature an J is the Joule's Mechanical Equivalent of heat.

$$J(mc\Delta\theta) = 4\pi T R^3\left(\frac{1}{r} - \frac{1}{R}\right)$$

Since $m = \left(\frac{4}{3}\pi r^3\right)\rho$

$$\Rightarrow \quad \left(\frac{4}{3}\pi R^3\right)\rho c\Delta\theta = \frac{4\pi R^3 T}{J}\left(\frac{1}{r} - \frac{1}{R}\right)$$

So, decrease in temperature is

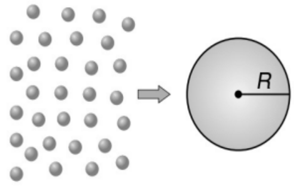
$$\Delta\theta = \frac{3T}{Jc\rho}\left(\frac{1}{r} - \frac{1}{R}\right)$$

In cgs system and for water, we have $s = 1\text{ cal g}^{-1}(^\circ\text{C})^{-1}$ and $\rho = 1\text{ gcc}^{-1}$, so we get

$$\Delta\theta = \frac{3T}{J}\left(\frac{1}{r} - \frac{1}{R}\right)$$

FORMATION OF BIGGER DROP FROM SMALL DROPLETS

If n small drops of radius r coalesce to form a big drop of radius R then surface area of the liquid decreases.



Energy released is

$$\Delta E = T(A_{\text{initial}} - A_{\text{final}})$$

$$\Rightarrow \quad \Delta E = n(4\pi r^2)T - 4\pi R^2 T$$

$$\Rightarrow \quad \Delta E = 4\pi T(nr^2 - R^2)$$

$$\Rightarrow \quad \Delta E = 4\pi R^2 T(n^{1/3} - 1)$$

$$\Rightarrow \quad \Delta E = 4\pi T r^2 n^{2/3}(n^{1/3} - 1)$$

$$\Rightarrow \quad \Delta E = 4\pi T R^3\left(\frac{1}{r} - \frac{1}{R}\right)$$

Conceptual Note(s)

(a) If this released energy is absorbed by a big drop, its temperature increases and rise in temperature can be given by

$$\Delta\theta = \frac{3T}{J\rho c}\left(\frac{1}{r}-\frac{1}{R}\right)$$

In cgs system and for water, we have $c = 1\,\text{cal}\,\text{g}^{-1}(^\circ\text{C})^{-1}$ and $\rho = 1\,\text{g}\,\text{cc}^{-1}$, so we have

$$\Delta\theta = \frac{3T}{J}\left(\frac{1}{r}-\frac{1}{R}\right)$$

(b) If this released energy is converted into kinetic energy of a big drop without dissipation, then by the Law of Conservation of Energy, we get

$$\frac{1}{2}mv^2 = 4\pi R^3 T\left(\frac{1}{r}-\frac{1}{R}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)\rho v^2 = 4\pi R^3 T\left[\frac{1}{r}-\frac{1}{R}\right]$$

$$\Rightarrow v^2 = \frac{6T}{\rho}\left(\frac{1}{r}-\frac{1}{R}\right)$$

$$\Rightarrow v = \sqrt{\frac{6T}{d}\left(\frac{1}{r}-\frac{1}{R}\right)}$$

ILLUSTRATION 131

How much work will be done in increasing the diameter of a soap bubble from 2 cm to 5 cm. Surface tension of soap solution is $3\times10^{-2}\ \text{Nm}^{-1}$.

SOLUTION

Soap bubble has two surfaces. Hence,

$$W = T\Delta A$$

where, $\Delta A = 2\left[4\pi\left\{(2.5\times10^{-2})^2-(1\times10^{-2})^2\right\}\right]$

$$\Rightarrow \Delta A = 1.32\times10^{-2}\ \text{m}^2$$

$$\Rightarrow W = (3\times10^{-2})(1.32\times10^{-2})\ \text{J}$$

$$\Rightarrow W = 3.96\times10^{-4}\ \text{J}$$

ILLUSTRATION 132

Calculate the energy released when 1000 small water drops each of same radius 10^{-7} m coalesce to form one large drop. The surface tension of water is $7\times10^{-2}\ \text{Nm}^{-1}$

SOLUTION

Let r be the radius of smaller drops and R of bigger one. Equating the initial and final volumes, we have,

$$\frac{4}{3}\pi R^3 = (1000)\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow R = 10r = (10)(10^{-7})\ \text{m}$$

$$\Rightarrow R = 10^{-6}\ \text{m}$$

Further, the water drops have only one free surface. Therefore,

$$\Delta A = 4\pi R^2 - (1000)(4\pi r^2)$$

$$\Rightarrow \Delta A = 4\pi\left[(10^{-6})^2-(10^3)(10^{-7})^2\right]$$

$$\Rightarrow \Delta A = -36\pi(10^{-12})\ \text{m}^2$$

Here, negative sign implies that surface area is decreasing. Hence, energy released in the process.

$$\Delta U = T|\Delta A| = (7\times10^{-2})(36\pi\times10^{-12})\ \text{J}$$

$$\Rightarrow \Delta U = 7.9\times10^{-12}\ \text{J}$$

COHESIVE FORCES

The force of attraction between the molecules of the same substance is called force of cohesion or cohesive force. In case of solids, the force of cohesion is very large and due to this, solids have definite shape and size. On the other hand, the force of cohesion in case of liquids is weaker than that of solids. Hence liquids do not have definite shape but have definite volume. The force of cohesion is negligible in case of gases. Because of this fact, gases have neither fixed shape nor volume.

Examples

(a) Two drops of a liquid coalesce into one when brought in mutual contact because of the cohesive force.

(b) It is difficult to separate two sticky plates of glass wetted with water because a large force has to be applied against the cohesive force between the molecules of water.

(c) It is very difficult to break a drop of mercury into small droplets because of large cohesive force between mercury molecules.

ADHESIVE FORCES

The force of attraction between molecules of different substances is called adhesion.

Examples

(a) Adhesive force enables us to write on the black board with a chalk.

(b) Adhesive force helps us to write on the paper with ink.

(c) Large force of adhesion between cement and bricks helps us in construction work.

(d) Due to force of adhesive, water wets the glass plate.

(e) Fevicol and gum are used in gluing two surfaces together because of adhesive force.

WATER WETS THE GLASS SURFACE, BUT MERCURY DOES NOT: EXPLANATION

The adhesive force between water molecules and glass molecules is greater than the cohesive force between the molecules of water. Hence when water is poured on glass, the water molecules cling with the glass molecules and the glass surface is wetted.

However, the adhesive force between mercury-glass molecules is less than the cohesive force between mercury-mercury molecules. So, mercury molecules do not cling with glass molecules i.e. mercury does not wet the glass.

If, however, the glass surface is greasy, then water also does not wet the glass because the adhesive force between water-grease molecules is less than the cohesive force between water-water molecules.

The adhesive force between oil and water is less than the cohesive force of water, but greater than the cohesive force of oil. Therefore, a water drop poured on the surface of oil contracts to take the form of a globule, while a drop of oil poured on the surface of water spreads to a large area in the form of a thin film.

The adhesive force between ink and paper is greater than the cohesive force of ink. That is why ink sticks on paper. Writing on blackboard by chalk is also possible due to adhesive force.

ANGLE OF CONTACT

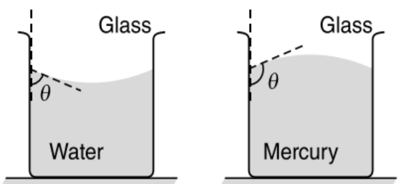
When the free surface of a liquid comes in contact of a solid, it becomes curved near the place of contact.

The angle of contact between a liquid and a solid is defined as the angle inside the liquid between the tangent to the solid surface and the tangent to the liquid surface at the point of contact of the liquid with the solid.

Liquids which wet the surface have acute angle of contact. Angle of contact is zero for pure water and clean glass. For ordinary water and glass angle of contact is about 8°.

Liquids which do not wet the surface have obtuse angle of contact. For mercury and glass the angle of contact is 135°.

The angles of contact θ for water-glass and mercury-glass are shown in Figure.



The angle of contact for water and silver is 90°. Hence in a silver vessel the surface of water at the edges also remains horizontal.

Conceptual Note(s)

(a) Angle of contact θ lies between 0° and 180°.

(b) It does not depend upon the inclination of the solid in the liquid.

(c) If $\theta < 90^\circ$, the liquid wets the surface of solid and if $\theta > 90^\circ$, the liquid does not wet the surface of solid.

(d) The angle of contact depends on
- **(i)** the nature of the liquid.
- **(ii)** the nature of the solid.
- **(iii)** the cleanliness of the surfaces.
- **(iv)** the medium above the liquid surface.
- **(v)** temperature. (on increasing the temperature, angle of contact decreases).

(e) Soluble impurities increase the angle of contact.

(f) Partially soluble impurities decrease the angle of contact.

SHAPE OF LIQUID MENISCUS IN A GLASS TUBE

When a liquid is brought in contact with a solid surface, the surface of the liquid becomes curved near the place of contact. The nature of the curvature (concave or convex) depends upon the relative magnitudes of the cohesive force between the liquid molecules and the adhesive force between the molecules of the liquid and those of the solid. A liquid takes the shape of the vessel in which it is contained, so it cannot permanently oppose any force that tries to change its shape.

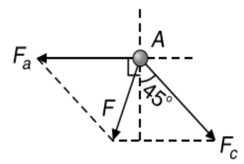
The free surface of liquid at rest always adjusts itself at right angles to the resultant force such that the component of force tangential to the liquid is zero.

When a capillary tube is dipped in a liquid, the liquid surface becomes curved at the point of contact. This surface is curved due to forces of cohesion and forces of adhesion. This curved surface of the liquid is called the Meniscus of the Liquid or Liquid Meniscus. Consider a liquid molecule A in contact with solid (i.e. wall of capillary tube), then forces acting on molecule A are

(a) Weight of the molecule A which acts vertically downwards along the wall of the tube and is negligible compared to forces of adhesion and cohesion.

(b) Force of adhesion F_a (acts outwards at right angle to the wall of the tube).

(c) Force of cohesion F_c (acts at an angle 45° to the vertical) i.e., F_c makes an angle of 135° with F_a as shown in Figure (in which weight of the particle is not shown as it is negligible in comparison to F_a and F_c).



The resultant force F depends on the values of F_a and F_c. If the resultant force F makes an angle α with F_a, then we have

$$\tan\alpha = \frac{F_c \sin(135^\circ)}{F_a + F_c \cos(135^\circ)} = \frac{F_c}{\sqrt{2}\,F_a - F_c} \quad \ldots(1)$$

By knowing the direction of resultant force, we can find out the shape of meniscus because the free surface of the liquid adjusts itself at right angle to this resultant force.

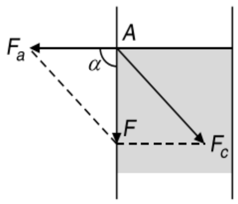
CASE-1: For Horizontal Meniscus

From equation (1) we see that, when $F_c = \sqrt{2}F_a$, then

$\tan\alpha \to \infty$

$\Rightarrow \quad \alpha \to 90^\circ$

So, the resultant force F acts vertically downwards and hence the liquid meniscus must be horizontal.



EXAMPLE: Pure water in silver coated capillary tube.

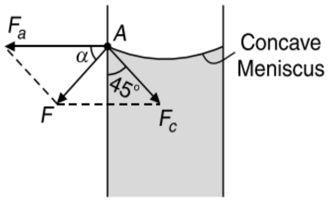
CASE-2: For Concave Meniscus

From equation (1), we see that, if $F_c < \sqrt{2}F_a$, then

$\tan\alpha = \text{positive}$

$\Rightarrow \quad \alpha$ is acute

So, the resultant force is directed outside the liquid and hence the liquid meniscus must be concave upward.



EXAMPLE: Water in glass capillary tube.

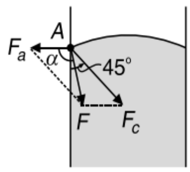
CASE-3: For Convex Meniscus

From equation (1), we see that, if $F_c > \sqrt{2}F_a$, then

$\tan\alpha = \text{negative}$

$\Rightarrow \quad \alpha$ is obtuse

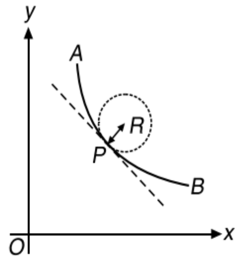
So, the resultant force is directed inside the liquid and hence the liquid meniscus must be convex upward.



EXAMPLE: Mercury in glass capillary tube.

RADIUS OF CURVATURE OF A CURVE

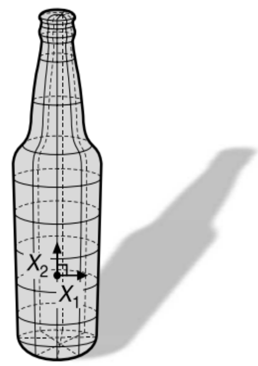
Shape of a curved surface or interface can be described if we know the radius of curvature of the curved surface at some point. Consider a point P on the curve AB as shown in Figure.



The radius of curvature R of the curve at the point P is actually the radius of the circle which is tangent to the curve at the point P.

PRINCIPAL RADII OF CURVATURE OF A SURFACE

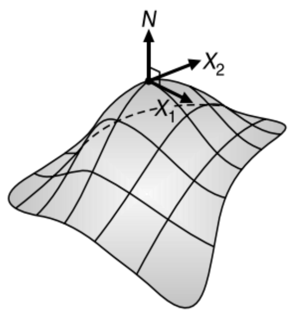
We must understand the fact that all known shapes are made of surfaces which have some curvature. The spherical and cylindrical surfaces are rather simple cases for mathematical treatment. Consider a bottle shown in Figure.



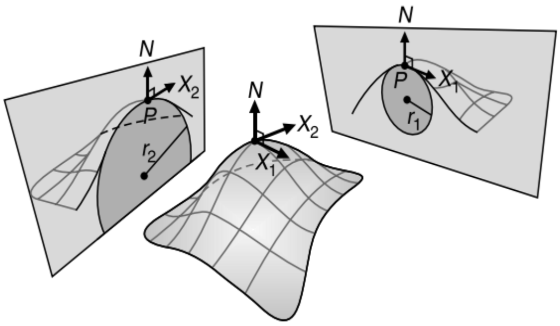
Along one direction X_1, the bottle quickly curves around in a circle but along another direction X_2 it is completely flat and travels along a straight line.

This way of looking at curvature in terms of curves traveling along the surface is often how we treat curvature in general. Also note that the curvature of a curve κ is inversely proportional to the radius of curvature.

In many other cases however, the shapes are more complicated. Let us now consider a curved surface shown in Figure.



At each point on a given surface, two radii of curvature (which are denoted by r_1 and r_2) are required to describe the shape. If we want to determine these radii at any point (say P), a normal to the surface at this point is drawn and a plane containing the normal is constructed through the surface. This plane will intersect the surface in a curve. The radius of curvature of this curve at point P is denoted by r_1 as shown in Figure.



An infinite number of such planes can be constructed each of which intersects the surface at P. For each of these planes, a radius of curvature can be obtained. If we construct a second plane through the surface, containing the normal and perpendicular to the first plane, the second curve of intersection and hence the second radius of curvature at point P (i.e., r_2) is obtained. These two radii define the curvature at P completely.

It can be shown that the mean curvature i.e. $\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$ of the surface is constant, which is independent of the choice of the planes. An infinite set of such pairs of radii is possible.

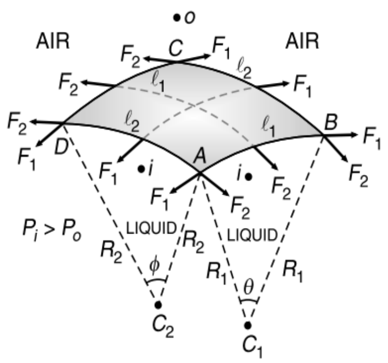
For standardization, the first plane is rotated around the normal until the radius of curvature in that plane reaches maximum and hence the other radius of curvature becomes minimum. These are called the principal radii of curvature and are denoted by R_1 and R_2.

EXCESS PRESSURE INSIDE LIQUID SURFACE WITH TWO CURVATURES: YOUNG-LAPLACE EQUATION

There exists a difference in pressure across a curved liquid surface which is a consequence of surface tension. The pressure is greater on the concave side. The Laplace equation relates the pressure difference to the shape of the Young surface. This difference in pressure is given by

$$\Delta P = T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

where R_1 and R_2 re the principal radii of curvature of the surface. To derive this equation, consider a small section of liquid surface having two curvatures R_1 and R_2 as shown in Figure 1.



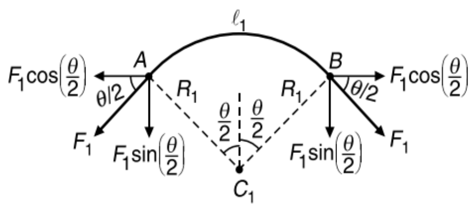
Since the surface is small, the angle subtended at the respective centres are also small. Net outward force on the liquid surface is

$$F_{\text{out}} = (P_i - P_o)A_{\text{surface}} = (P_i - P_o)(R_1\theta)(R_2\phi) \qquad \ldots(1)$$

Let us now calculate the force due to surface tension on each surface. Let us draw the force acting on the portion AB due to surface tension. Interestingly, we see that the force F_1 is acting along the length $l_2 = R_2\phi$ and the force F_2 is acting on the length $l_1 = R_1\theta$ as shown in Figure 1. So, we have

$$F_1 = Tl_2 = TR_2\phi \text{ and } F_2 = Tl_1 = TR_1\theta$$

The force $F_1 = Tl_2 = T(R_2\phi)$, acting along the length AD has components $F_1\cos\left(\frac{\theta}{2}\right)$ and $F_1\sin\left(\frac{\theta}{2}\right)$. Similarly, this same value of force is acting along the length BC has components $F_1\cos\left(\frac{\theta}{2}\right)$ and $F_1\sin\left(\frac{\theta}{2}\right)$ with cross sectional view for the curve AB shown in Figure 2.



The components $F_1\cos\left(\frac{\theta}{2}\right)$ and $F_1\cos\left(\frac{\theta}{2}\right)$ cancel, so the inward force acting due to surface tension on the curves AD and BC is $2F_1\sin\left(\frac{\theta}{2}\right)$.

Similarly, the inward force acting due to surface tension on the curves AB and CD is $2F_2\sin\left(\frac{\phi}{2}\right)$.

If F_{in} is the net inward force, then

$$F_{\text{in}} = 2F_1\sin\left(\frac{\theta}{2}\right) + 2F_2\sin\left(\frac{\phi}{2}\right)$$

Since, θ and ϕ are small, so we have

$$\sin\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2} \text{ and } \sin\left(\frac{\phi}{2}\right) \approx \frac{\phi}{2}$$

Also, $F_1 = Tl_2 = TR_2\phi$ and $F_2 = Tl_1 = TR_1\theta$

$$\Rightarrow \quad F_{in} = F_1\theta + F_2\phi = (TR_2\phi)\theta + (TR_1\theta)\phi$$

$$\Rightarrow \quad F_{in} = T\theta\phi(R_1 + R_2) \quad \ldots(2)$$

For equilibrium of the surface, the net outward force must be balanced by the net inward force, so we have

$$F_{out} = F_{in}$$

$$\Rightarrow \quad (P_i - P_o)(R_1\theta)(R_2\phi) = T\theta\phi(R_1 + R_2)$$

$$\Rightarrow \quad P_i - P_o = \Delta P = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

However, for the case of a film, we have

$$\Delta P = 2T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

because a film has two free surfaces.

Conceptual Note(s)

The simplified forms of spherical, cylindrical and planar surfaces are given below

For a spherical surface, we have

$R_1 = R_2 = R$, therefore $\Delta P = \dfrac{2T}{R}$

For a cylindrical surface, we have

$R_1 \to \infty$ and $R_2 = R$, therefore $\Delta P = \dfrac{T}{R}$

For a planar surface, we have

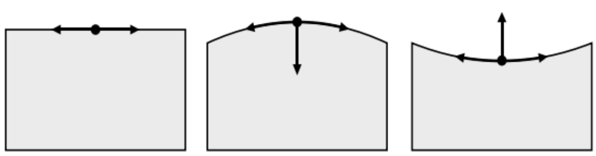
$R_1 \to \infty$ and $R_2 \to \infty$, therefore $\Delta P = 0$

PRESSURE DIFFERENCE BETWEEN THE TWO SIDES FORCE A CURVED LIQUID SURFACE

A molecule lying in the surface of a liquid is attracted by other molecules on the surface in all directions. If the surface is plane, then the molecule is attracted equally in all directions. Hence the resultant force on the molecule due to surface tension is zero.

If the surface is convex, then a resultant component of all the forces of attraction acting on every molecule acts normal to the surface is directed inward.

Similarly, if the surface is concave, then every molecule experiences a resultant force due to surface tension acting normally outward.



(a) (b) (c)

Obviously, for the equilibrium of a curved surface, there must be a difference of pressure between its two sides so that the excess pressure force may balance the resultant force due to surface tension. Hence the pressure on the concave side must be greater than the pressure on the convex side. This difference of pressure is equal to $\dfrac{2T}{R}$, where T is the surface tension and R is radius of curvature of the surface.

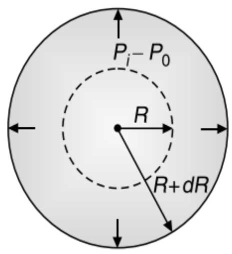
EXCESS PRESSURE

Due to the property of surface tension a drop or bubble tries to contract and so compresses the matter enclosed. This in turn increases the internal pressure which prevents further contraction and equilibrium is achieved. So, in equilibrium the pressure inside a bubble or drop is greater than outside and the difference of pressure between two sides of the liquid surface is called excess pressure.

In case of a drop, excess pressure is provided by hydrostatic pressure of the liquid within the drop while in case of bubble the gauge pressure of the gas confined in the bubble provides it.

EXCESS PRESSURE INSIDE A LIQUID DROP AND AIR BUBBLE

Let P_i be the pressure inside the liquid drop and P_0 be the pressure outside it. Then $(P_i - P_0)$ is the excess pressure inside the liquid drop.



Let T be the surface tension of drop. Let the radius of bubble increase from R to $(R + dR)$ under the influence of excess pressure $(P_i - P_0)$ acting only on the inner surface of area $4\pi R^2$.

So, work done by the force due to pressure is

$$W = (P_i - P_0)4\pi R^2 dR \quad \ldots(1)$$

Also, work done to change surface area of the drop is

$$W = T\left[4\pi(R + dR)^2 - 4\pi R^2\right]$$

$$\Rightarrow \quad W = 4\pi TR^2\left[\left(1 + \frac{dR}{R}\right)^2 - 1\right]$$

$$\Rightarrow \quad W \approx 4\pi TR^2\left[\cancel{1} + 2\frac{dR}{R} - \cancel{1}\right]$$

$$\Rightarrow \quad W \approx 8\pi TR^2 dR \qquad \ldots(2)$$

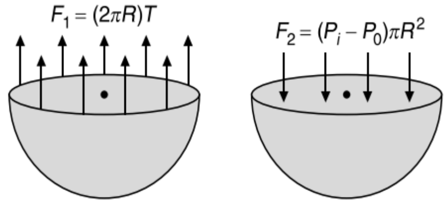
Equating (1) and (2), we get

$$P_i - P_0 = \frac{2T}{R}$$

For an air bubble also, the excess pressure is

$$P_i - P_0 = \frac{2T}{R} \qquad \{\because \text{ Both have a single free surface}\}$$

We can also do this quickly, by equating the force due to surface tension with the force due to excess pressure (as done earlier too) as shown in Figure.



The force due to surface tension is

$$F_1 = (2\pi R)T \qquad \ldots(3)$$

The force due to excess pressure is

$$F_2 = (P_i - P_0)\pi R^2 \qquad \ldots(4)$$

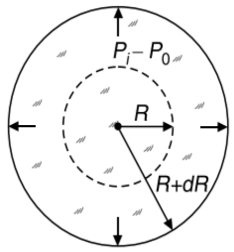
Equating (3) and (4), we get

$$P_i - P_0 = \frac{2T}{R}$$

EXCESS PRESSURE INSIDE A SOAP BUBBLE

Let P_i be the pressure inside the bubble and P_0 be the pressure outside it. Then $(P_i - P_0)$ is the excess pressure inside the bubble.

Let T be the surface tension of the soap bubble. Let the radius of bubble increase from R to $(R + dR)$ under the influence of excess pressure $(P_i - P_0)$ acting only on the inner surface of area $4\pi R^2$.



Work done by the force due to pressure is

$$W = (P_i - P_0)4\pi R^2 dR \qquad \ldots(1)$$

Since a soap bubble has two free surfaces, so work done to change surface area of the soap bubble is

$$W = T\left[4\pi(R + dR)^2 - 4\pi R^2\right] \times 2$$

$$\Rightarrow \quad W = 8\pi TR^2\left[\left(1 + \frac{dR}{R}\right)^2 - 1\right]$$

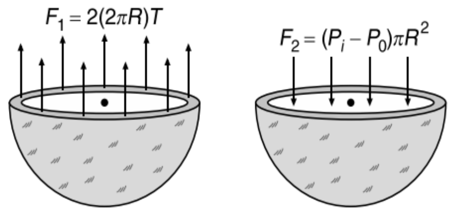
$$\Rightarrow \quad W \approx 8\pi TR^2\left[1 + \frac{2dR}{R} - 1\right]$$

$$\Rightarrow \quad W \approx 16\pi TR^2 dR \qquad \ldots(2)$$

Equating (1) and (2), we get

$$P_i - P_0 = \frac{4T}{R}$$

We can also do this quickly, by equating the force due to surface tension with the force due to excess pressure (as done earlier too) as shown in Figure.



The force due to surface tension is

$$F_1 = 2[(2\pi R)T] \qquad \ldots(3)$$

The force due to excess pressure is

$$F_2 = (P_i - P_0)\pi R^2 \qquad \ldots(4)$$

Equating (3) and (4), we get

$$P_i - P_o = \frac{4T}{R}$$

Conceptual Note(s)

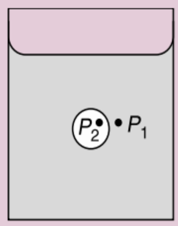
(a) Thus, the pressure difference is greater for smaller drops and bubbles than for larger ones.

(b) Pressure difference is inversely proportional to the radius.

(c) The pressure inside the surface is greater than the pressure outside the surface.

(d) The pressure on the concave side is greater than the pressure on the convex side.

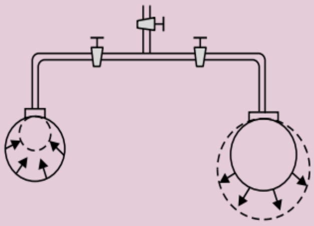
(e) Consider an air bubble inside the liquid as shown in the figure.



We observe that a single surface is formed with air on the concave side and liquid on the convex side. The pressure at the concave side is greater than the pressure at the convex side, by an amount $\frac{2T}{R}$. So, we have

$$P_2 - P_1 = \frac{2T}{R}$$

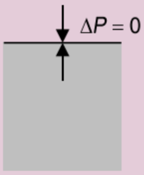
(f) Excess pressure is inversely proportional to the radius of bubble (or drop), i.e., pressure inside a smaller bubble (or drop) is higher than inside a larger bubble (or drop).



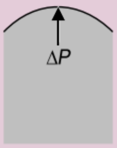
This is why when two bubbles of different sizes are connected to each other, the air will rush from smaller bubble to larger bubble. Due to this the smaller will shrink while the larger will expand till the smaller bubble reduces to droplet.

(g) Excess Pressure in Different Cases

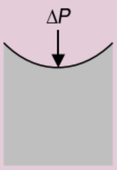
(i) For a plane surface, $\Delta P = 0$



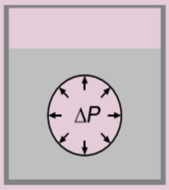
(ii) For a convex surface, $\Delta P = \frac{2T}{R}$



(iii) For a concave surface, $\Delta P = \frac{2T}{R}$



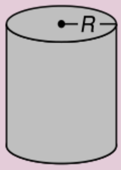
(iv) For an air bubble in liquid, $\Delta P = \frac{2T}{R}$



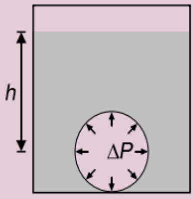
(v) For a liquid Drop, $\Delta P = \frac{2T}{R}$

(vi) For a soap bubble $\Delta P = \frac{4T}{R}$

(vii) For a cylindrical liquid surface, $\Delta P = \frac{T}{R}$

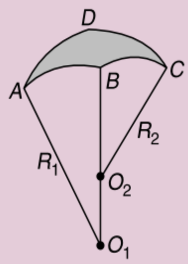


(viii) For an air bubble at depth h below the free surface of liquid of density d, $\Delta P = \frac{2T}{R} + hdg$



(ix) For a liquid surface of unequal radii,

$$\Delta P = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



(x) For a liquid film of unequal radii,

$$\Delta P = 2T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

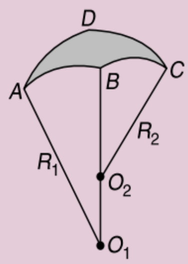


ILLUSTRATION 133

Calculate the pressure inside a small air bubble of 0.1 mm radius situated just below the water surface. Assume the surface tension of water to be $7.2 \times 10^{-2}\ \text{Nm}^{-1}$ and atmospheric pressure to be $1.013 \times 10^{5}\ \text{Nm}^{-2}$.

SOLUTION

Surface tension of water $T = 7.2 \times 10^{-2}\ \text{Nm}^{-1}$
Radius of air bubble $R = 0.1\ \text{mm} = 10^{-4}\ \text{m}$
The excess pressure inside the air bubble is given by,

$$P_2 - P_1 = \frac{2T}{R}$$

Pressure inside the air bubble is

$$P_2 = P_1 + \frac{2T}{R}$$

Substituting the values, we have

$$P_2 = (1.013 \times 10^5) + \frac{(2 \times 7.2 \times 10^{-2})}{10^{-4}}$$

$$\Rightarrow \quad P_2 = 1.027 \times 10^5\ \text{Nm}^{-2}$$

ILLUSTRATION 134

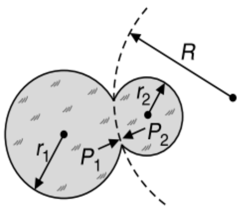
Two separate air bubbles (radii 0.006 m and 0.003 m) formed of the same liquid (surface tension $0.07\ \text{Nm}^{-1}$) come together to form a double bubble. Calculate the radius and the sense of curvature of the internal film surface common to both the bubbles.

SOLUTION

The pressure inside the bubbles is

$$P_1 = P_0 + \frac{4T}{r_1} \text{ and } P_2 = P_0 + \frac{4T}{r_2}$$

Since $r_2 < r_1$, so we get $P_2 > P_1$
The pressure inside the smaller bubble will be more as shown in Figure.



The excess pressure at the interface is

$$P = P_2 - P_1 = 4T\left(\frac{r_1 - r_2}{r_1 r_2}\right) \qquad \ldots(1)$$

This excess pressure acts from concave to convex side, the interface will be concave towards smaller bubble and convex towards larger bubble. Let R be the radius of interface then,

$$P = \frac{4T}{R} \qquad \ldots(2)$$

From equations (1) and (2)

$$R = \frac{r_1 r_2}{r_1 - r_2} = \frac{(0.006)(0.003)}{0.006 - 0.003}$$

$$\Rightarrow \quad R = 0.006\ \text{m}$$

ILLUSTRATION 135

Under isothermal condition two soap bubbles of radii r_1 and r_2 coalesce to form a single bubble of radius r. The external pressure is P_0. Find the surface tension of the soap in terms of the given parameters.

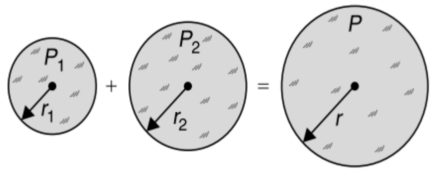
SOLUTION

As mass of the air is conserved,

$$\Rightarrow \quad n_1 + n_2 = n$$

Since, $PV = nRT$

$$\Rightarrow \quad \frac{P_1 V_1}{RT_1} + \frac{P_2 V_2}{RT_2} = \frac{PV}{RT}$$



As temperature is constant,

$$T_1 = T_2 = T$$

$$\Rightarrow \quad P_1 V_1 + P_2 V_2 = PV \qquad \ldots(1)$$

where, $V_1 = \frac{4}{3}\pi r_1^3$, $V_2 = \frac{4}{3}\pi r_2^3$ and $V = \frac{4}{3}\pi r^3$

To avoid confusion with temperature, let us denote surface tension by σ. From equation (1), we get

$$\left(P_0 + \frac{4\sigma}{r_1}\right) r_1^3 + \left(P_0 + \frac{4\sigma}{r_2}\right) r_2^3 = \left(P_0 + \frac{4\sigma}{r}\right) r^3$$

$$\Rightarrow \quad \sigma = \frac{P_0\left(r^3 - r_1^3 - r_2^3\right)}{4\left(r_1^2 + r_2^2 - r^2\right)}$$

Special Case

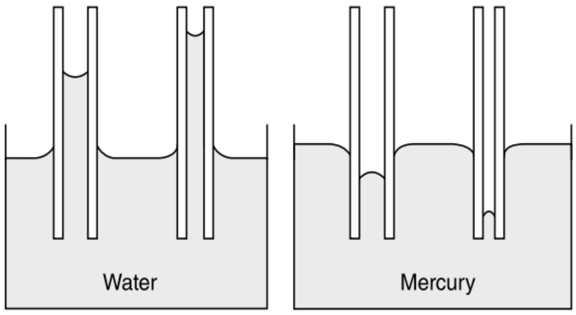
If the bubbles coalesce in vacuum, then $P_0 = 0$ and hence from equation (2), we get

$$r^2 = r_1^2 + r_2^2$$

CAPILLARITY

When a glass capillary tube open at both ends is dipped vertically in water, the water rises up in the tube to a certain height above the water level outside the tube. The narrower the tube, the higher is the rise of water. On the other hand, if the tube is dipped in mercury, the mercury is depressed below the outside level.

The phenomenon of rise or depression of liquids in a capillary tube is called capillarity. The liquids which wet glass (for which the angle of contact is acute) rise up in the capillary tube, while those which do not wet glass (for which the angle of contact is obtuse) are depressed down in the capillary.



(a) (b)

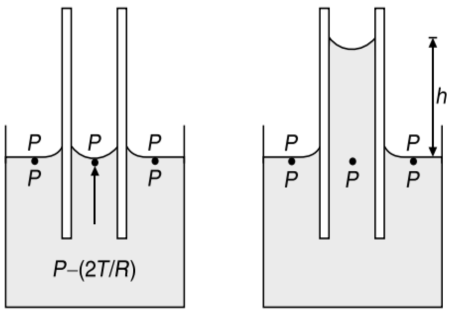
EXPLANATION TO CAPILLARITY

The phenomenon of capillarity arises due to the surface tension of liquids. When a capillary tube is dipped in water, the water meniscus inside the tube is concave. The pressure just below the meniscus is less than the pressure just above it by $\frac{2T}{R}$, where T is the surface tension of water and R is the radius of curvature of meniscus.

The pressure of the surface of water is atmospheric pressure P. The pressure just below plane surface of water outside the tube is also P, but that just below the meniscus inside the tube is $P-\left(\frac{2T}{R}\right)$. We know that pressure at all points in the same level of water must be the same.

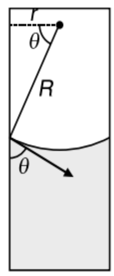
Therefore, to make up the deficiency of pressure, $\frac{2T}{R}$, below the meniscus, water begins to flow from outside into the tube. The rising force water in the capillary stops at a certain height h. In this position the pressure of the water-column of height h becomes equal to $\frac{2T}{R}$, that is,

$$h\rho g=\frac{2T}{R}$$



(a) (b)

where, ρ is the density of water and g is the acceleration due to gravity. If r be the radius of the capillary tube and θ the angle of contact of water-glass, then the radius of curvature R of the meniscus is given by $R=\frac{r}{\cos\theta}$.



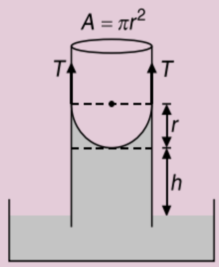
$$\Rightarrow \quad h\rho g=\frac{2T}{\left(\frac{r}{\cos\theta}\right)}$$

$$\Rightarrow \quad h=\frac{2T\cos\theta}{r\rho g}$$

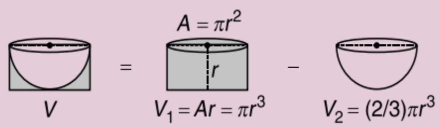
This shows that as r decreases, h increases, that is, narrower the tube, greater is the height to which the liquid rises in the tube.

Conceptual Note(s)

If weight of the liquid in the meniscus is to be considered, then the correction due to the weight of the liquid contained in the meniscus can be made for contact angle of 0° for which the meniscus will be hemispherical in shape as shown in Figure.



The volume of the shaded part of liquid inside the meniscus is $V=V_1-V_2$ as shown in Figure.



$$\Rightarrow \quad V=(\pi r^2)r-\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)=\frac{1}{3}\pi r^3$$

Since the force due to surface tension will balance weight of the liquid W in the capillary, so we have

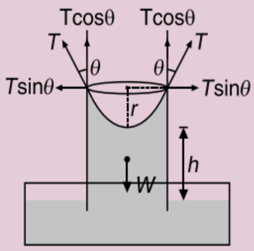
$$T(2\pi r)=W$$

where, $W=\left(\pi r^2h+\frac{1}{3}\pi r^3\right)\rho g$

$$\Rightarrow \quad T(2\pi r)=\left(\pi r^2h+\frac{1}{3}\pi r^3\right)\rho g$$

$$\Rightarrow \left(h+\frac{r}{3}\right)=\frac{2T}{r\rho g}$$

However, if the contact angle is non-zero and acute, then too the meniscus is nearly hemispherical as shown in Figure, then we have



$$(T\cos\theta)(2\pi r)=W$$

where, $W=\left(\pi r^2 h+\frac{1}{3}\pi r^3\right)\rho g$

$$\Rightarrow (T\cos\theta)(2\pi r)=\left(\pi r^2 h+\frac{1}{3}\pi r^3\right)\rho g$$

$$\Rightarrow \left(h+\frac{r}{3}\right)=\frac{2T\cos\theta}{r\rho g}$$

PRACTICAL APPLICATIONS OF CAPILLARITY

(a) The oil in a lamp rises in the wick by capillary action.
(b) The tip of nib of a pen is split up, to make a narrow capillary so that the ink rises upto the tin or nib continuously.
(c) Sap and water rise upto the top of the leaves of the tree by capillary action.
(d) If one end of the towel dips into a bucket of water and the other end hangs over the bucket the towel soon becomes wet throughout due to capillary action.
(e) Ink is absorbed by the blotter due to capillary action.
(f) Sandy soil is more dry than clay. It is because the capillaries between sand particles are not so fine as to draw the water up by capillaries.
(g) The moisture rises in the capillaries of soil to the surface, where it evaporates. To preserve the moisture in the soil, capillaries must be broken up. This is done by ploughing and levelling the fields.
(h) Bricks are porous and behave like capillaries.

Conceptual Note(s)

(a) For liquids which wet the glass tube or capillary tube, angle of contact $\theta<90°$. Hence,

$\cos\theta=$ positive

$\Rightarrow h=$ positive.

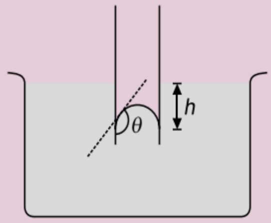
It means that these liquids rise in the capillary tube. So, the liquids which wet capillary tube rise in the capillary tube. For example, water, milk, kerosene oil, petrol etc.

(b) For liquids which do not wet the glass tube or capillary tube, angle of contact $\theta>90°$. Hence,

$\cos\theta=$ negative

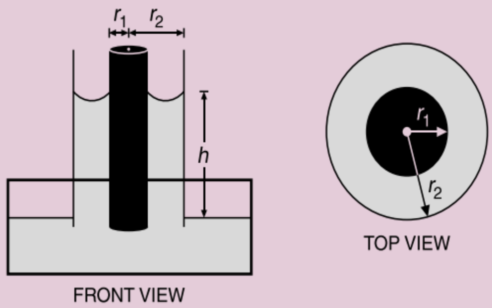
$\Rightarrow h=$ negative.

So, the liquids which do not wet capillary tube are depressed in the capillary tube. For example, mercury.



(c) Since, T, θ, ρ and g are constant and hence $h\propto\frac{1}{r}$. Thus, the liquid rises more in a narrow tube and less in a wider tube. **This is called Jurin's Law.**

(d) If two concentric tubes of radii r_1 and r_2 (inner one is solid) are placed in water reservoir, then height to which the liquid rises is obtained by equating the force due to surface tension with the weight of liquid in the tube. So, we get



$$T(2\pi r_1+2\pi r_2)=(\pi r_2^2 h-\pi r_1^2 h)\rho g$$

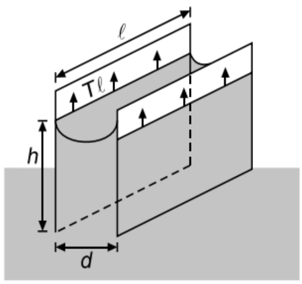
$$\Rightarrow h=\frac{2T}{(r_2-r_1)\rho g}$$

RISE OF LIQUID BETWEEN TWO PARALLEL PLATES

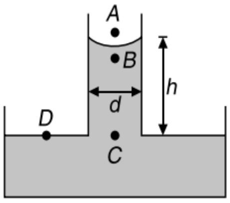
If two parallel plates with the spacing $'d'$ are placed in water reservoir, then liquid will rise till force due to surface tension balances the weight of liquid, so we have

$$2Tl\cos(0°)=(\rho lhd)g$$

$$\Rightarrow h=\frac{2T}{\rho g d}$$



We can also calculate this height by applying a different approach. Since the liquid meniscus between two plates is cylindrical in shape, so the cross-sectional view of the arrangement is shown in Figure.



The pressure at A is equal to the atmospheric pressure $P_0 = P_{\text{atm}}$.

The pressure difference between the point A and the point B (which lies just inside the cylindrical meniscus) is

$$P_A - P_B = \frac{T}{R}, \text{ where } R \approx \frac{d}{2}$$

$$\Rightarrow \quad P_B = P_0 - \frac{T}{R}$$

Also, we see that

$$P_C - P_B = h\rho g$$

$$\Rightarrow \quad P_C = P_B + h\rho g = \left(P_0 - \frac{T}{R}\right) + h\rho g \qquad \ldots(1)$$

Now we must see that the points C and D are at the same level, hence

$$P_C = P_D = P_{\text{atm}} = P_0$$

So, from equation (1), we get

$$P_0 = \left(P_0 - \frac{T}{R}\right) + h\rho g$$

$$\Rightarrow \quad h = \frac{T}{\rho g R} = \frac{2T}{\rho g d}$$

ILLUSTRATION 136

Mercury has an angle of contact of $120°$ with glass. A narrow tube of radius 1.0 mm made of this glass is dipped in a trough containing mercury. By what amount does the mercury dip down in the tube relative to the liquid surface outside. Surface tension of mercury at the temperature of the experiment is 0.5 Nm^{-1} and density of mercury is $13.6 \times 10^3 \text{ kgm}^{-3}$. Take $g = 9.8 \text{ ms}^{-2}$.

SOLUTION

$$h = \frac{2T\cos\theta}{r\rho g}$$

Substituting the values, we get,

$$h = \frac{2 \times 0.5 \times \cos 120°}{10^{-3} \times 13.6 \times 10^3 \times 9.8}$$

$$\Rightarrow \quad h = -3.75 \times 10^{-3} \text{ m}$$

$$\Rightarrow \quad h = -3.75 \text{ mm}$$

Please note that the negative sign indicates that mercury suffers capillary depression.

ILLUSTRATION 137

A glass tube of radius 0.4 mm is dipped vertically in water. Find upto what height the water will rise in the capillary? If the tube in inclined at an angle of $60°$ with the vertical, how much length of the capillary is occupied by water. Surface tension of water is $7 \times 10^{-2} \text{ Nm}^{-1}$, density of water 10^3 kgm^{-3}.

SOLUTION

For glass-water, angle of contact $\theta = 0°$

Now, $h = \dfrac{2T\cos\theta}{r\rho g}$

$$\Rightarrow \quad h = \frac{(2)(7 \times 10^{-2})\cos 0°}{(0.4 \times 10^{-3})(10^3)(9.8)}$$

$$\Rightarrow \quad h = 3.57 \times 10^{-2} \text{ m}$$

$$\Rightarrow \quad h = 3.57 \text{ cm}$$

$$\Rightarrow \quad l = \frac{h}{\cos 60°} = \frac{3.57}{1/2} = 7.14 \text{ cm}$$

ILLUSTRATION 138

Two narrow bores of radius 3.0 mm and 6.0 mm are joined together to form a U-shaped tube open at both ends. If the U-tube contains water, what is the difference in its levels in the two limbs of the tube. Surface tension of water is $7.3 \times 10^{-2} \text{ Nm}^{-1}$. Take the angle of contact to be zero and density of water to be 10^3 kgm^{-3} and $g = 9.8 \text{ ms}^{-2}$.

SOLUTION

$$h\rho g = \Delta P$$

$$\Rightarrow \quad h\rho g = \frac{2T\cos\theta}{r_1} - \frac{2T\cos\theta}{r_2}$$

$$\Rightarrow \quad h = \frac{2T\cos\theta}{\rho g}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

Substituting the values, we have

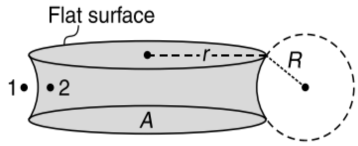
$$h = \frac{2 \times 7.3 \times 10^{-2} \times \cos 0°}{10^3 \times 9.8}\left(\frac{6.0 - 3.0}{6.0 \times 3.0}\right) \times \frac{1}{10^{-3}}$$

$$\Rightarrow \quad h = 2.48 \times 10^{-3} \text{ m}$$

$$\Rightarrow \quad h = 2.48 \text{ mm}$$

LIQUID BETWEEN TWO HORIZONTAL PLATES

When a small drop of water is placed between two glass plates placed face to face, it forms a thin cylindrical film which is concave outward along its boundary.

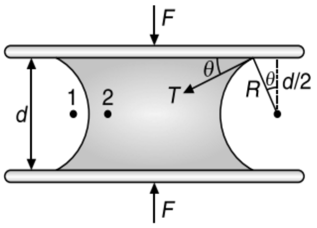


Let R and r be the radii of curvature of the enclosed film in two perpendicular directions.
The pressure difference on two sides of a cylindrical surface is

$$\Delta P = P_1 - P_2 = \frac{T}{R}, \text{ where } P_1 = P_{atm} = P_0$$

$$\Rightarrow \quad P_2 = P_1 - \frac{T}{R} = P_0 - \frac{T}{R} \qquad \ldots(1)$$

If d be the distance between the two plates and θ the angle of contact for water and glass as shown in Figure, then we get



$$\cos\theta = \frac{d/2}{R}$$

$$\Rightarrow \quad \frac{1}{R} = \frac{2\cos\theta}{d}$$

Substituting for $\frac{1}{R}$ in Equation (1), we get

$$P_2 = \left(P_1 - \frac{2T}{d}\cos\theta\right) \approx \left(P_0 - \frac{2T}{d}\cos\theta\right)$$

For water and glass, angle θ can be taken zero, so

$$\cos\theta \approx 1$$

The pressure above the upper pale is equal to the atmospheric pressure, whereas just below the plate inside the liquid, pressure is

$$P_2 = P_0 - \frac{T}{R} \approx P_0 - \frac{2T}{d}$$

So, the upper plate is pressed downwards by a force that corresponds to a pressure of $\frac{2T}{d}$. If $A = \pi r^2$ be the area of plate wetted by the film, then the resultant force F pressing the upper plate downwards is

$$F = A(\Delta P) = \frac{2TA}{d}$$

ILLUSTRATION 139

A drop of water of mass $m = 0.2$ g is placed between two clean glass plates, the distance between which is 0.01 cm. Calculate the force of attraction between the plates. Surface tension of water is $0.07\ \text{Nm}^{-1}$.

SOLUTION

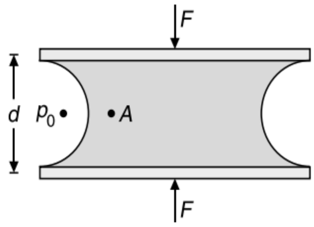
Let R be the radius of the circular layer of water, then

$$m = (\pi R^2 d)\rho \qquad \ldots(1)$$

Since meniscus is cylindrical in shape, so pressure at A is

$$P_A = P_0 - \frac{2T}{d}$$

Thus, pressure between the plates is less than the atmospheric pressure and so the plates are pressed together as though they are attracted towards each other.



If F is the force of attraction, then

$$F = (\Delta P)A = \left(\frac{2T}{d}\right)(\pi R^2) = \left(\frac{2T}{d}\right)\left(\frac{m}{\rho d}\right)$$

$$\Rightarrow \quad F = \frac{2Tm}{d^2\rho} = \frac{2 \times 0.2 \times 10^{-3} \times 0.07}{0.01^2 \times 10^{-4} \times 1000} = 2.8\ \text{N}$$

RISE OF LIQUID IN A CAPILLARY TUBE OF INSUFFICIENT LENGTH

Suppose a liquid of density ρ and surface tension T rises in a capillary tube to a height h. Then

$$h\rho g = \frac{2T}{R}$$

where R is the radius of curvature of the liquid meniscus in the tube. From this we may write

$$hR = \frac{2T}{\rho g} = \text{constant (for a given liquid)}$$

When the length of the tube is greater than h, the liquid rises in the tube to a height so as to satisfy the above relation. But if the length of the tube is less than h, say h', then the liquid rise up to the top of the tube and then spreads out until its radius of curvature R increases to R', such that

$$h'R' = hR = \frac{2T}{\rho g}$$

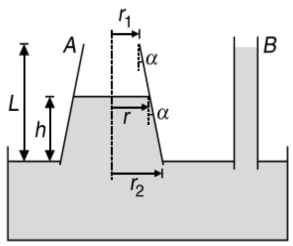
It is clear that liquid cannot emerge in the form of a fountain from the upper end of a short capillary tube.

ILLUSTRATION 140

A conical glass capillary tube A of length 0.1 m has diameters 10^{-3} m and $5\times10^{-4}\text{ m}$ at the ends. When it is just immersed in a liquid at 0 °C with larger diameter in contact with it, the liquid rises to $8\times10^{-2}\text{ m}$ in the tube. If another cylindrical glass capillary tube B is immersed in the same liquid at 0 °C, the liquid rises to $6\times10^{-2}\text{ m}$ height. The rise of liquid in the tube B is only $5.5\times10^{-2}\text{ m}$ when the liquid is at 50 °C. Find the rate at which the surface tension changes with temperature considering the change to be linear. The density of the liquid is $\left(\frac{1}{14}\right)\times10^{4}\text{ kgm}^{-3}$ and angle of contact is zero. Effect of temperature on density of liquid and glass is negligible.

SOLUTION

Let r be radius of meniscus in the conical tube as shown in Figure. Then, we have



$$\tan\alpha=\frac{r-r_1}{L-h}=\frac{r_2-r_1}{L}$$

$$\Rightarrow\quad \frac{r-2.5\times10^{-4}}{0.1-0.08}=\frac{(5.2.5)\times10^{-4}}{0.1}$$

$$\Rightarrow\quad r\times10^{4}-2.5=0.2\times2.5$$

$$\Rightarrow\quad r=3\times10^{-4}\text{ m}$$

Since the phenomenon of capillarity is independent of the shape of tube, so at same temperature of $\theta=0$ °C, we have

$$h_Ar_A=h_Br_B=\frac{2T_0}{\rho g}=\text{constant}$$

$$\Rightarrow\quad r_B=\frac{(0.08)\left(3\times10^{-4}\right)}{0.06}=4\times10^{-4}\text{ m}$$

For a cylindrical tube, we have $h=\frac{2T}{r\rho g}$

$$\Rightarrow\quad T_{0\text{ °C}}=T_0=\frac{h_0\rho gr}{2}$$

$$\Rightarrow\quad T_0=\frac{1}{2}\left[(0.06)\left(\frac{1}{14}\times10^{4}\right)(9.8)\left(4\times10^{-4}\right)\right]$$

$$\Rightarrow\quad T_0=8.4\times10^{-2}\text{ Nm}^{-1}$$

For a given tube and liquid, we have $T=\frac{h\rho gr}{2}$

$$\Rightarrow\quad T\propto h$$

$$\Rightarrow\quad \frac{T_{50}}{T_0}=\frac{h_{50}}{h_0}\text{ so}$$

$$\Rightarrow\quad T_{50\text{ °C}}=T_{50}=\left(\frac{5.5\times10^{-2}}{6\times10^{-2}}\right)\left(8.4\times10^{-2}\right)$$

$$\Rightarrow\quad T_{50}=7.7\times10^{-2}\text{ Nm}^{-1}$$

Hence, rate of change of surface tension (T) with temperature (θ) assuming linearity is given by

$$\frac{\Delta T}{\Delta\theta}=\frac{T_{50}-T_0}{50-0}=\frac{(7.7-8.4)\times10^{-2}}{50}$$

$$\Rightarrow\quad \frac{\Delta T}{\Delta\theta}=-1.4\times10^{-2}\text{ Nm}^{-1}(\text{°C})^{-1}$$

Negative sign shows that with rise in temperature surface tension decreases.

ILLUSTRATION 141

The lower end of a capillary tube of diameter 2 mm is dipped 8.5 cm below the surface of water in a beaker. Calculate the pressure required in the tube in order to blow a hemispherical bubble at its end in the water. Assume that the surface tension of water is $7.5\times10^{-2}\text{ Nm}^{-1}$, density of water is 10^{3} kgm^{-3}, $g=10\text{ ms}^{-2}$ and $1\text{ atm}=10^{5}\text{ Nm}^{-2}$. Also find excess pressure inside the bubble.

SOLUTION

Let P_i be the pressure inside the hemispherical air bubble blown at the lower end of the tube and P_0, just outside it. Then,

$$P_i-P_0=\frac{2T}{r}$$

$$\Rightarrow\quad P_i=P_0+\frac{2T}{r}\qquad\ldots(1)$$

Now, pressure outside hemispherical air bubble is

$$P_0=P_{atm}+h\rho g\qquad\ldots(2)$$

where, h is the length of capillary tube dipped inside water.

The pressure required to blow the bubble is equal to pressure inside the bubble.

$$\Rightarrow\quad P_i=P_0+\frac{2T}{r}=\left(P_{atm}+h\rho g\right)+\frac{2T}{r}$$

According to the problem, we have

$$T=7.5\times10^{-2}\text{ Nm}^{-1},\ \rho=10^{3}\text{ kgm}^{-3},$$

$$r=\frac{2\text{ mm}}{2}=1\text{ mm}=10^{-3}\text{ m},\ P_{atm}=10^{5}\text{ Nm}^{-2},$$

$$g=10\text{ ms}^{-2}\text{ and }h=8.5\text{ cm}=\frac{8.5}{100}\text{ m}$$

$$\Rightarrow \quad P_i = 10^5 + \left(\frac{8.5}{100}\right)(10^3)(10) + \frac{2(7.5\times10^{-2})}{10^{-3}}$$

$$\Rightarrow \quad P_i = 10^5 + 850 + 150 = 101000 \text{ Nm}^{-2}$$

$$\Rightarrow \quad P_i = 1.01\times10^5 \text{ Nm}^{-2}$$

The excess pressure inside the bubble is

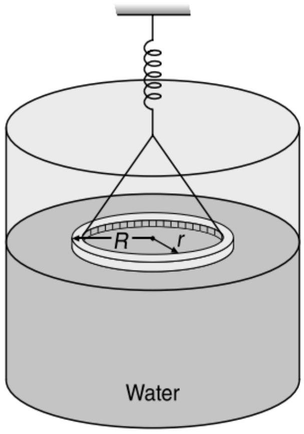
$$P_i - P_0 = \frac{2T}{r} = \frac{2(7.5\times10^{-2})}{10^{-3}} = 150 \text{ Nm}^{-2}$$

Test Your Concepts-X

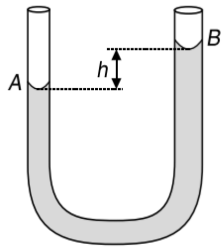
Based on Surface Tension, Surface Energy, Excess Pressure and Capillarity

(Solutions on page H.18)

1. A liquid of specific gravity 1.5 is observed to rise 3 cm in a capillary tube of diameter 0.50 mm and the liquid wets the surface of the tube. Calculate the excess pressure inside a spherical bubble of 1 cm diameter blown from the same liquid if angle of contact is 0°.
2. A circular ring has inner and outer radii equal to r and R respectively. Mass of the ring is m. It gently pulled out vertically from a water surface by a sensitive spring. When the spring is stretched 3.4 cm from its equilibrium position, then ring is on verge of being pulled out from the water surface. If spring constant is k. Calculate the surface tension of water.



3. A capillary tube whose inside radius is 0.5 mm is dipped in water having surface tension 7×10^{-2} Nm^{-1}. To what height is the water raised above the normal water level. Angle of contact of water with glass is 0°. Density of water is 10^3 kgm^{-3} and $g = 9.8$ ms^{-2}.
4. A tube of insufficient length is immersed in water (surface tension $= 0.07$ Nm^{-1}) with 1 cm of it projecting vertically upwards outside the water. What is the radius of the meniscus? Radius of tube is 1 mm.
5. A soap bubble of radius r is placed on another soap bubble of radius R. What is the radius of the film separating the two bubbles?
6. Calculate the maximum possible mass of a greased needle that floats on a water surface. Assume that the length of the needle is l and surface tension of water to be T.
7. Water rises in a capillary tube to a height of 2 cm. In another capillary tube whose radius is one third of it, how much the water will rise?
8. Calculate the amount of energy evolved when 8 droplets of water each of radius 0.5 mm combine to form a single drop. Assume surface tension of water to be 72×10^{-3} Nm^{-1}.
9. Calculate difference (h) in the levels of water in two communicating capillary tubes A and B of radii 1 mm and 1.5 mm assuming that the surface tension of water is 0.07 Nm^{-1}.



10. Calculate the work to be done (in millijoule) against surface tension in blowing a soap bubble from a radius of 10 cm to 20 cm, if the surface tension of soap solution in 25×10^{-3} Nm^{-1}.
11. A film of water is formed between two straight parallel wires each 10 cm long and at separation 0.5 cm. Calculate the work required to increase 1 mm distance between the wires if surface tension of water is 72×10^{-3} Nm^{-1}.
12. Two soap bubbles in vacuum having radii 6 cm and 8 cm respectively coalesce under isothermal conditions to form a single bubble. Calculate the radius of the new bubble formed.
13. A small hollow sphere having a small hole in it is immersed in water to a depth 40 cm before any water enters the sphere. Calculate radius of the hole if surface tension of water is 75×10^{-3} Nm^{-1}, density of water is 10^3 kgm^{-3} and $g = 10$ ms^{-2}.
14. There are 1000 droplets of mercury of 1 mm diameter on a glass plate. Subsequently they merge into one big drop. How will the surface energy of the surface layer change? The process is isothermal. Surface tension of mercury is 0.475 Nm^{-1}.

SOLVED PROBLEMS

PROBLEM 1

A steel wire of diameter 0.40 mm is stretched between rigid supports separated by a horizontal distance of 1.8 m.

(a) What load suspended from the middle of the wire will produce a stress of 10^9 Nm^{-2}?

(b) Find the stored elastic energy in the wire under this load.

(c) Find the drop in potential energy of the object hung from the centre for this stress. $Y_S = 2\times10^{11}$ Nm^{-2}.

SOLUTION

(a) $2T\cos\theta = Mg$

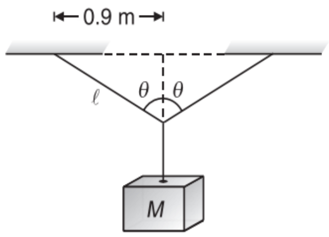
$$\Rightarrow T = \frac{Mg}{2\cos\theta} \quad \ldots(1)$$

Further, $\dfrac{T}{A} = 10^9$ Nm^{-2} ...(2)

$$\Rightarrow T = (10^9)A \quad \ldots(3)$$

From equations (1) and (2)

$$10^9 = \frac{Mg}{2A\cos\theta}$$



$$\Rightarrow Mg = (2A\cos\theta)\times10^9$$

$$\Rightarrow l = \frac{0.9}{\sin\theta}$$

$$\Rightarrow \Delta l = 0.9\left(\frac{1}{\sin\theta}-1\right) = \frac{0.9T}{AY}$$

$$\Rightarrow \sin\theta = \frac{AY}{AY+T}$$

$$A = \frac{\pi}{4}(d^2) = 1.25\times10^{-7}\text{ m}^2$$

$$\Rightarrow T = (10^9)A = 125\text{ N}$$

$$\Rightarrow \sin\theta = \frac{(1.25\times10^{-7})(2\times10^{11})}{(1.25\times10^{-7}\times2\times10^{11})+125} = 0.995$$

$$\Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} = 0.0996$$

$$\Rightarrow Mg = (2A\cos\theta)\times10^9$$

$$\Rightarrow Mg = (2\times1.25\times10^{-7})(0.0996)(10^9) \approx 25\text{ N}$$

(b) $U = \left(\dfrac{(\text{Stress})^2}{2Y}\right)(\text{Volume})$

$$\Rightarrow U = \frac{(10^9)^2}{2(2.0\times10^{11})}(0.125\times10^{-6})(1.8)$$

$$\Rightarrow U = 0.562\text{ J}$$

(c) $10^9 = \dfrac{Mg}{2A\cos\theta}$

$$\Rightarrow \cos\theta = \frac{Mg}{(2A)(10^9)}$$

$$\Rightarrow \cos\theta = \frac{25}{2\times0.125\times10^{-6}\times10^9} = 0.1$$

$$\Rightarrow \theta = 84.26^\circ$$

$$\Rightarrow \Delta U = Mgh = (25)(0.9\cot\theta)$$

$$\Rightarrow \Delta U = 2.25\text{ J}$$

PROBLEM 2

A glass full of water upto a height of 10 cm has a bottom of area 10 cm^2, top of area 30 cm^2 and volume 1 litre.

(a) Calculate the force exerted by the water on the bottom.

(b) Calculate the resultant force exerted by the sides of the glass on the water.

(c) If the glass is covered with a jar and the air inside the jar is completely pumped out, then recalculate the answers to parts (a) and (b).

(d) If a glass of different shape having same height, bottom area and volume is used then again recalculate the answers to parts (a) and (b).

Take $g = 10$ ms^{-2}, density of water to be 10^3 kgm^{-3} and atmospheric pressure to be 1.01×10^5 Nm^{-2}.

SOLUTION

(a) Force exerted by water at the bottom of glass is

$$F_1 = (P_0 + \rho g h)A_1 \quad \ldots(1)$$

where, $P_0 = P_{atm} = 1.01\times10^5$ Nm^{-2}

$\rho = \rho_w = 10^3$ kgm^{-3}, $g = 10$ ms^{-2}

$h = 10$ cm $= 0.1$ m and

Area of the base is

$A_1 = 10\text{ cm}^2 = 10^{-3}\text{ m}^2$

Substituting these values in equation (1), we get

$$F_1 = (1.01\times10^5 + 10^3\times10\times0.1)\times10^{-3}$$

$$\Rightarrow F_1 = 102\text{ N (downwards)}$$

(b) Force exerted by atmosphere on water

$$F_2 = P_0 A_2$$

where, A_2 is area of the top given by

$$A_2 = 30\text{ cm}^2 = 3\times10^{-3}\text{ m}^2$$

$$\Rightarrow F_2 = (1.01\times10^5)(3\times10^{-3})$$

$$\Rightarrow F_2 = 303\text{ N (downwards)}$$

Force exerted by bottom on the water is

$$F_3 = -F_1$$

$$\Rightarrow F_3 = 102\text{ N (upwards)}$$

Weight of water in glass is

$$W = V\rho_w g = (10^{-3})(10^3)(10)$$

$$\Rightarrow W = 10\text{ N (downwards)}$$

Let F be the force exerted by side walls on the water (upwards), then due to the equilibrium of water, we have

$$\Sigma F_{\text{upwards}} = \Sigma F_{\text{downwards}}$$

$$\Rightarrow F + F_3 = F_2 + W$$

$$\Rightarrow F = F_2 + W - F_3 = 303 + 10 - 102$$

$$\Rightarrow F = 211\text{ N (upwards)}$$

(c) When air inside the jar is completely pumped out, then the atmospheric pressure becomes zero and hence we have

$$F_1 = (\rho g h)A_1$$

$$\Rightarrow F_1 = (10^3)(10)(0.1)(10^{-3})$$

$$\Rightarrow F_1 = 1\text{ N (downwards)}$$

In this case, $F_2 = 0$ and $F_3 = 1\text{ N}$ (upwards)

$$\Rightarrow F = F_2 + W - F_3$$

$$\Rightarrow F = 0 + 10 - 1 = 9\text{ N (upwards)}$$

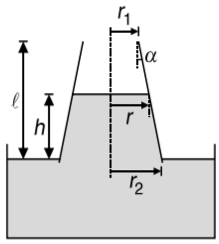
(d) When a glass of different shape having same height, bottom area and volume is used then the answers to (a) and (b) will remain the same, because they depend upon P_0, ρ, g, h, A_1 and A_2.

PROBLEM 3

A tube of conical bore of length 10 cm is just dipped inside the water. The diameters of upper and lower ends are 0.04 cm and 0.06 cm respectively. If the surface tension of water is 70 dyne cm^{-1} and angle of contact $0°$, calculate the height to which the liquid rises in the tube.

SOLUTION

Let r_1, r_2 and r be the respective radii of the tube at the upper end, lower end and the meniscus of the water surface. Suppose that water rises to a height h in the tube. The view of the conical tube of length l dipped inside water is shown in Figure.



Now, $h = \dfrac{2T\cos\theta}{r\rho g}$

where, $T = 70\text{ dynecm}^{-1}$, $\theta = 0°$, $\rho = 1\text{ gcm}^{-3}$ and $g = 980\text{ cms}^{-1}$

$$\Rightarrow h = \frac{2\times70\times\cos0°}{r\times1\times980} = \frac{1}{7r}$$

$$\Rightarrow r = \frac{h}{7} \quad \text{...(1)}$$

If the wall of the conical tube makes an angle α with the vertical, then

$$\tan\alpha = \frac{r_2 - r_1}{l} = \frac{r - r_1}{l - h}$$

where, $r_1 = \dfrac{0.04}{2} = 0.02\text{ cm}$,

$$r_2 = \frac{0.06}{2} = 0.03\text{ cm and } l = 10\text{ cm}$$

$$\Rightarrow \frac{0.03 - 0.02}{10} = \frac{r - 0.02}{10 - h}$$

$$\Rightarrow (10 - h)\times0.001 = r - 0.02$$

$$\Rightarrow 0.001h + r - 0.03 = 0 \quad \text{...(2)}$$

Substituting value of r from equation (1) in equation (2), we get

$$0.001h + \frac{1}{7h} - 0.03 = 0$$

$$\Rightarrow 7h^2 - 210h + 1000 = 0$$

$$\Rightarrow h = \frac{210 \pm \sqrt{(-210)^2 - 4\times7\times1000}}{14}$$

$$\Rightarrow h = \frac{210 \pm 126.9}{14} = 24.06\text{ cm OR } 5.94\text{ cm}$$

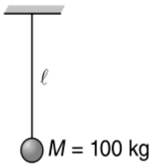
It can be checked that in a tube of uniform bore $r_1 (< r)$, water can rise up to a height of 7.35 cm only. So, the water cannot rise up to 24.06 cm and hence the water will rise up to a height of 5.94 cm in the conical tube.

$$\Rightarrow h = 5.94\text{ cm}$$

PROBLEM 4

A 5 m long cylindrical steel wire with radius $2\times10^{-3}\text{ m}$ is suspended vertically from a rigid support and carries a bob of mass 100 kg at the other end. If the bob gets snapped, calculate the change in temperature of the wire ignoring radiation losses. Take $g = 10\ \text{ms}^{-2}$. (For the steel wire: Young's modulus $= 2.1\times10^{11}\ \text{Nm}^{-2}$. Density $= 7860\ \text{kgm}^{-3}$; specific heat $= 420\ \text{Jkg}^{-1}\text{K}$).

SOLUTION



Given, Length of the wire, $l = 5$ m

Radius of the wire, $r = 2\times10^{-3}$ m

Density of wire, $\rho = 7860\ \text{kgm}^{-3}$

Young's modulus, $Y = 2.1\times10^{11}\ \text{Nm}^{-2}$

and specific heat, $c = 420\ \text{Jkg}^{-1}\text{K}$

Mass of wire, $m = (\text{density})(\text{volume})$

$$m = (\rho)(\pi r^2 l)$$

$$\Rightarrow \quad m = (7860)(\pi)(2\times10^{-3})^2(5)\ \text{kg}$$

$$\Rightarrow \quad m = 0.494\ \text{kg}$$

Elastic potential energy stored in the wire,

$$U = \frac{1}{2}(\text{Stress})(\text{Strain})(\text{Volume})$$

$$\left\{\because \frac{\text{Energy}}{\text{Volume}} = \frac{1}{2}\text{stress}\times\text{strain}\right\}$$

$$\Rightarrow \quad U = \frac{1}{2}\left(\frac{Mg}{\pi r^2}\right)\left(\frac{\Delta l}{l}\right)(\pi r^2 l)$$

$$\Rightarrow \quad U = \frac{1}{2}(Mg)\Delta l \qquad \left(\Delta l = \frac{Fl}{AY}\right)$$

$$\Rightarrow \quad U = \frac{1}{2}(Mg)\frac{Mgl}{(\pi r^2)Y} = \frac{1}{2}\frac{M^2g^2l}{\pi r^2 Y}$$

Substituting the values, we get

$$U = \frac{1}{2}\frac{(100)^2(10)^2(5)}{(3.14)(2\times10^{-3})^2(2.1\times10^{11})}\ \text{J}$$

$$\Rightarrow \quad U = 0.9478\ \text{J}$$

When the bob gets snapped, this energy is utilised in raising the temperature of the wire, through $\Delta\theta$, so

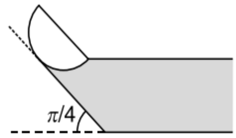
$$U = mc\Delta\theta$$

$$\Rightarrow \quad \Delta\theta = \frac{U}{ms} = \frac{(0.9478)}{(0.494)(420)}\ ^\circ\text{C or K}$$

$$\Rightarrow \quad \Delta\theta = 4.568\times10^{-3}\ ^\circ\text{C}$$

PROBLEM 5

A uniform half cylinder of density ρ_2 is resting in equilibrium on a rough inclined plane (having inclination $\pi/4$) with liquid of density ρ_1 on its right as shown in Figure.



Calculate the minimum coefficient of friction to ensure equilibrium and the ratio $\frac{\rho_1}{\rho_2}$ assuming that the flat surface of the half cylinder is parallel to the inclined plane.

SOLUTION

Let us draw the arrangement in the Figure(s) shown.

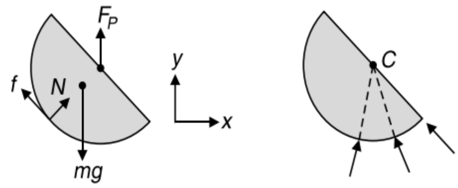


Figure (a) Figure (b)

From Figure (b) following conclusion can be drawn about the pressure force.

(i) Torque of pressure force about C is zero and

(ii) The resultant force acts vertically (by symmetry)

See the force diagram in Figure (a). Cylinder is in equilibrium under the four coplanar forces shown in figure. So, we have

$$\Sigma F_x = 0$$

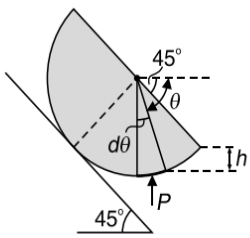
$$\Rightarrow \quad \frac{f}{\sqrt{2}} = \frac{N}{\sqrt{2}}$$

$$\Rightarrow \quad f = N \qquad \ldots(1)$$

Since, we know that $\mu \ge \frac{f}{N}$, so on using equation (1), we get

$$\mu_{\min} = 1$$

Calculation of F_P



Since, $h = R\left(\sin\theta - \sin\frac{\pi}{4}\right)$

$$\Rightarrow \quad F_P = \int_{\pi/4}^{3\pi/4} PdA\sin\theta$$

$$\Rightarrow \quad F_P = \int_{\pi/4}^{3\pi/4} \rho_1 gR\left(\sin\theta - \sin\frac{\pi}{4}\right)(LRd\theta)\sin\theta$$

$$\Rightarrow \quad F_P = \rho_1 g R^2 L\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

Also, $\Sigma F_y = 0$

$$\Rightarrow \quad F_P = mg - \frac{1}{\sqrt{2}}(f+N)$$

$$\Rightarrow \quad \left(\frac{\pi}{4}-\frac{1}{2}\right)LR^2\rho_1 g = mg - \sqrt{2}f \qquad \ldots(2)$$

Since in equilibrium, torque about the point C is also zero, so we have

$$\Sigma\tau_C = 0$$

$$\Rightarrow \quad fR = (mg)\left(\frac{4R}{3\pi}\sin\frac{\pi}{4}\right)$$

$$\Rightarrow \quad f = \frac{4mg}{3\sqrt{2}\pi}$$

Substituting this value of f in equation (2), we get

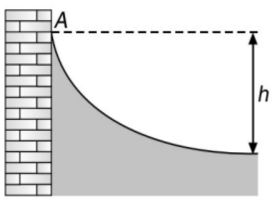
$$\left(\frac{\pi}{4}-\frac{1}{2}\right)LR^2\rho_1 g = mg\left(1-\frac{4}{3\pi}\right)$$

$$\Rightarrow \quad \left(\frac{\pi}{4}-\frac{1}{2}\right)LR^2\rho_1 g = \left(\frac{\pi R^2}{2}\right)(L\rho_2)g\left(\frac{3\pi-4}{3\pi}\right)$$

$$\Rightarrow \quad \frac{\rho_1}{\rho_2} = \frac{2(3\pi-4)}{3(\pi-2)}$$

PROBLEM 6

A liquid having surface tension T and density ρ is in contact with a vertical solid wall due to which the liquid surface gets curved as shown in Figure.



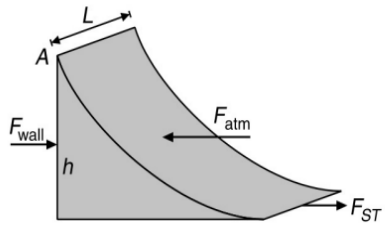
Assuming that at the bottom, the liquid surface is flat and the atmospheric pressure is P_0, Calculate the pressure inside the liquid at the top most point A of the meniscus. Also calculate the difference in height (h) between highest and the lowest points of the meniscus.

SOLUTION

At the lowest level of meniscus liquid surface is flat, so pressure at the lowest level of meniscus is P_0 and hence pressure inside the liquid at A is

$$P_A = P_0 - \rho g h$$

For horizontal equilibrium of liquid in the meniscus, let us consider a depth L of the liquid section perpendicular to the plane as shown in Figure.



Force due to wall on the liquid is

$$F_{\text{wall}} = P_{av}A = P_{av}(hL)$$

$$\Rightarrow \quad F_{\text{wall}} = \left[\frac{P_0 + (P_0 - \rho g h)}{2}\right]hL = \left(P_0 - \frac{\rho g h}{2}\right)hL$$

Force due to the atmospheric pressure is

$$F_{\text{atm}} = P_0 A = P_0(hL)$$

Force due to surface tension is

$$F_{ST} = TL$$

For horizontal equilibrium of liquid in the meniscus, we have

$$F_{\text{wall}} + F_{ST} = F_{\text{atm}}$$

$$\Rightarrow \quad \left(P_0 - \frac{\rho g h}{2}\right)hL + TL = P_0 hL$$

$$\Rightarrow \quad P_0 h - \frac{\rho g h^2}{2} + T = P_0 h$$

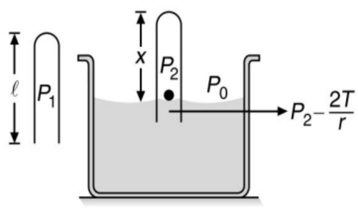
$$\Rightarrow \quad h = \sqrt{\frac{2T}{\rho g}}$$

PROBLEM 7

A glass capillary sealed at the upper end is of length 0.11 m and internal diameter 2×10^{-5} m. The tube is immersed vertically into a liquid of surface tension 5.0×10^{-2} Nm^{-1} and density 10^3 kgm^{-3}. To what length has the capillary to be immersed so that the liquid level inside and outside the capillary becomes the same. What will happen to the level of liquid inside the capillary if the seal is now broken. Assume the temperature to be constant, atmospheric pressure to be $P_0 = 1.01\times10^5$ Nm^{-2} and contact angle to be $\theta = 0°$.

SOLUTION

From the figure shown, we have



$$P_1 = P_0 \qquad \{\text{atmospheric pressure}\}$$

If A is the area of cross section of tube, then we have

$$V_1 = Al$$

Also, we know that

$$P_2 - \frac{2T}{r} = P_0$$

$$\Rightarrow \quad P_2 = \left(P_0 + \frac{2T}{r} \right)$$

and $V_2 = Ax$

Under isothermal ($T =$ constant) conditions, we have

$$P_1V_1 = P_2V_2$$

$$\Rightarrow \quad P_0Al = \left(P_0 + \frac{2T}{r} \right) Ax$$

$$\Rightarrow \quad x = \frac{P_0 l}{P_0 + \frac{2T}{r}}$$

Substituting the values, we get

$$x = \frac{(1.01\times10^5)(0.11)}{(1.01\times10^5) + \frac{2\times5.0\times10^{-2}}{10^{-5}}}$$

$$\Rightarrow \quad x = 0.1 \text{ m}$$

The length of the tube immersed in liquid is

$$l' = l - x = 0.11 - 0.1 = 0.01 \text{ m}$$

If the seal is now broken, then the pressure now becomes P_0 and the rise of liquid will be given by

$$h = \frac{2T}{r\rho g} \qquad (\theta = 0°)$$

$$\Rightarrow \quad h = \frac{2\times5.0\times10^{-2}}{10^{-5}\times10^3\times9.8} = 1.02 \text{ m}$$

However, the length of tube outside the liquid is only 0.1 m, so the liquid will rise only to the top of the tube and will stay there with radius of meniscus

$$r_2 = \frac{h_1 r_1}{h_2}$$

$$\Rightarrow \quad r_2 = \frac{(1.02)(10^{-5})}{0.1} = 1.02\times10^{-4} \text{ m}$$

PROBLEM 8

A thin wire of cross-sectional area a is bent to form a circular ring of radius R. The ring rotates about an axis perpendicular to its plane and through its centre with angular frequency ω. Given density of the wire is ρ, Young's modulus of elasticity is Y. Find the total energy stored in the wire.

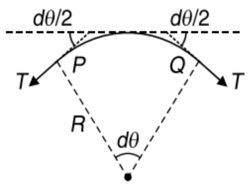
SOLUTION

$dm =$ mass of element PQ of the wire

$$\Rightarrow \quad dm = (Rd\theta)(a\rho)$$

Now, $2T\sin\left(\frac{d\theta}{2}\right) = (dm)R\omega^2$

$$\Rightarrow \quad Td\theta = a\rho R^2\omega^2 d\theta$$



Since, $\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$ for small angles

$$\Rightarrow \quad T = a\rho R^2\omega^2$$

So, stress in the wire is given by $\sigma = \frac{T}{a} = \rho R^2\omega^2$

Potential energy per unit volume,

$$U = \frac{1}{2}\frac{(\text{stress})^2}{Y} = \frac{1}{2}\frac{\rho^2R^4\omega^4}{Y}$$

Total potential energy is the product of energy density and volume. So,

$$E = u(\text{Volume}) \qquad \ldots(1)$$

$$\Rightarrow \quad E = \frac{1}{2}\frac{\rho^2R^4\omega^4}{Y}(2\pi Ra) = \frac{\pi\rho^2R^5a\omega^4}{Y}$$

Kinetic energy

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(mR^2)\omega^2$$

$$\Rightarrow \quad K = \frac{1}{2}(2\pi Ra\rho)(R^2\omega^2)$$

$$\Rightarrow \quad K = \pi R^3 a\rho\omega^2 \qquad \ldots(2)$$

So, total energy

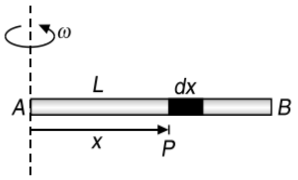
$$E_{\text{total}} = U + K = \pi\omega^2R^3\rho a\left(1 + \frac{\rho R^2\omega^2}{Y}\right)$$

PROBLEM 9

A uniform material rod of length L is rotated in a horizontal plane about a vertical axis through one of its ends. The angular speed of rotation is ω. Calculate the increase in length of the rod if the density and Young's modulus of rod are ρ and Y respectively.

SOLUTION

Consider an element of length dx at a distance x from the axis of rotation of the rod as shown in Figure.



The centripetal force required to move this element in a circle of radius x is

$$dF = (dm)x\omega^2 = (\rho A dx)\omega^2 x$$

Tension at a point P at a distance x from the axis is equal to sum of centripetal forces on all elements lying between P and B. So, at point P, tension is given by

$$T = \rho A\omega^2 \int_x^L x dx = \frac{\rho A\omega^2}{2}(L^2 - x^2)$$

Now assume that dl is extension in an element of length dx located at a distance x from the axis.

$$\text{Strain} = \frac{dl}{dx}$$

$$\text{Stress} = \frac{T}{A} = \frac{1}{2}\rho\omega^2(L^2 - x^2)$$

$$\Rightarrow \quad Y\frac{dl}{dx} = \frac{1}{2}\rho\omega^2(L^2 - x^2)$$

$$\Rightarrow \quad dl = \frac{1}{2}\frac{\rho\omega^2}{Y}(L^2 - x^2)dx$$

So, change in length of the entire rod is given by

$$\Delta l = \int dl = \frac{\rho\omega^2}{2Y}\left(L^2 x - \frac{x^3}{3}\right)\Bigg|_0^L$$

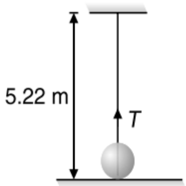
$$\Rightarrow \quad \Delta \ell = \frac{\rho\omega^2}{2Y}\left(L^3 - \frac{L^3}{3}\right) = \frac{1}{3}\frac{\rho\omega^2 L^3}{Y}$$

PROBLEM 10

A sphere of radius 0.1 m and mass 8π kg is attached to the lower end of a steel wire of length 5 m and diameter 10^{-3} m. The wire is suspended from 5.22 m high ceiling of a room. When the sphere is made to swing as a simple pendulum, it just grazes the floor at its lowest point. Calculate the velocity of the sphere at the lowest position. Young's modulus of steel is 1.994×10^{11} Nm^{-2}.

SOLUTION

Let Δl be the extension of wire when the sphere is at mean position. Then, we have



$$l + \Delta l + 2r = 5.22$$

$$\Rightarrow \quad \Delta l = 5.22 - l - 2r$$

$$\Rightarrow \quad \Delta l = 5.22 - 5 - 2 \times 0.1$$

$$\Rightarrow \quad \Delta l = 0.02 \text{ m}$$

Let T be the tension in the wire at mean position during oscillations, then from Hooke's Law

$$Y = \frac{T/A}{\Delta l / l}$$

$$\Rightarrow \quad T = \frac{YA\Delta l}{l} = \frac{Y\pi r^2 \Delta l}{l}$$

Substituting the values, we get

$$T = \frac{(1.994 \times 10^{11}) \times \pi \times (0.5 \times 10^{-3})^2 \times 0.02}{5}$$

$$\Rightarrow \quad T = 626.43 \text{ N}$$

The equation of motion at mean position is, given by

$$T - mg = \frac{mv^2}{R} \qquad \ldots(1)$$

where, $R = 5.22 - r = 5.22 - 0.1 = 5.12$ m

and $m = 8\pi$ kg $= 25.13$ kg

Substituting the values in equation (1), we get

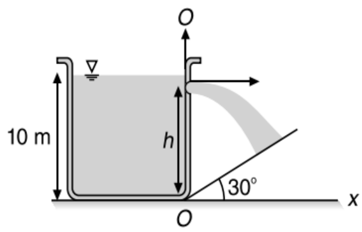
$$(626.43) - (25.13 \times 9.8) = \frac{(25.13)v^2}{5.12}$$

Solving this equation, we get

$$v = 8.8 \text{ ms}^{-1}$$

PROBLEM 11

A rectangular tank of height 10 m filled with water, is placed near the bottom of a plane inclined at an angle 30° with horizontal. At height h from bottom a small hole is made (as shown in figure) such that the stream coming out from hole, strikes the inclined plane normally. Calculate h.



SOLUTION

Speed of the liquid coming out of the orifice is

$$v = \sqrt{2g(10 - h)}$$

Component of its velocity parallel to the plane is $v\cos 30°$. Let stream strike the plane after time t at the point (x, y) i.e. when the velocity of the stream is perpendicular to the plane, so we get

$$0 = v\cos 30° - g\sin 30° t$$

$$\Rightarrow \quad t = \frac{v\cot 30^\circ}{g}$$

Also, we have $\tan(30^\circ) = \frac{y}{x}$

$$\Rightarrow \quad x = vt = \frac{v^2 \cot 30^\circ}{g} = \sqrt{3}y$$

$$\Rightarrow \quad \frac{v^2\cos 30^\circ}{g} = \sqrt{3}\left(h - \frac{1}{2}gt^2\right)$$

$$\Rightarrow \quad \frac{\sqrt{3}v^2}{g} = \sqrt{3}\left(h - \frac{g}{2}\frac{v^2\cot^2 30^\circ}{g^2}\right)$$

$$\Rightarrow \quad \frac{v^2}{g} = h - \frac{3}{2}\frac{v^2}{g}$$

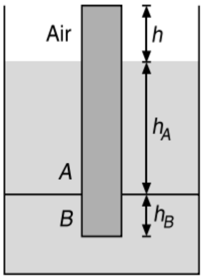
$$\Rightarrow \quad \frac{5}{2}\frac{v^2}{g} = h$$

$$\Rightarrow \quad 5(10 - h) = h$$

$$\Rightarrow \quad h = 8.33 \text{ m}$$

PROBLEM 12

A uniform solid cylinder of density 0.8 gcm^{-3} floats in equilibrium in a combination of two non-mixing liquids A and B with its axis vertical as shown in Figure.



The densities of liquids A and B are 0.7 gcm^{-3} and 1.2 gcm^{-3}, respectively. The height of liquid A is $h_A = 1.2$ cm. The length of the part of the cylinder with liquid B is $h_B = 0.8$ cm. Calculate the net force exerted by liquid A on the cylinder and the length h of part of the cylinder in air. If the cylinder is depressed in such a way that its top surface is just below the upper surface of liquid A and is then released, then calculate the acceleration of cylinder immediately after being released. Take $g = 10 \text{ ms}^{-2}$.

SOLUTION

The liquid A exerts horizontal forces on all surface area elements from all the directions. Due to symmetry, the resultant of these horizontal forces is zero. So, the net force exerted by liquid A on the cylinder is zero.

Let a be the area of cross-section of the cylinder. Then for equilibrium of the cylinder, weight of the cylinder equals the buoyant force, which equals the loss in weight of the cylinder.

$$\Rightarrow \quad W = U$$

$$\Rightarrow \quad a(h + h_A + h_B)(0.8)g = (ah_A)\rho_A g + (ah_B)\rho_B g$$

$$\Rightarrow \quad (h + h_A + h_B)(0.8) = h_A\rho_A + h_B\rho_B$$

$$\Rightarrow \quad (h + 1.2 + 0.8)(0.8) = (1.2)(0.7) + (0.8)(1.2)$$

$$\Rightarrow \quad 0.8h + 1.6 = 1.8$$

$$\Rightarrow \quad 0.8h = 0.2$$

$$\Rightarrow \quad h = 0.25 \text{ cm}$$

Mass of cylinder is

$$M = a(h + h_A + h_B)(0.8)$$

$$\Rightarrow \quad M = a(0.2 + 1.2 + 0.8)(0.8) = 1.8a$$

When the cylinder is depressed just completely, its height h goes inside the liquid B and hence the extra buoyant force due to liquid B acts on it, due to which the cylinder is accelerated upward.

Additional upthrust is given by

$$F = (ah)\rho_B g$$

$$\Rightarrow \quad F = (0.25a)(1.2)g = 0.3ag$$

Acceleration is given by

$$\text{Acceleration} = \frac{F}{M} = \frac{0.3ag}{1.8a} = \frac{g}{6}$$

$$\Rightarrow \quad \text{Acceleration} = \frac{10}{6} = \frac{5}{3} \text{ ms}^{-2}$$

PROBLEM 13

A thin uniform metallic rod of length 0.5 m rotates with an angular velocity 400 rads^{-1} in a horizontal plane about a vertical axis passing through one of its ends. Calculate the elongation of the rod. The density of the material of the rod is 10^4 kgm^{-3} and the Young's modulus is $2\times10^{11} \text{ Nm}^{-2}$.

SOLUTION

Let the mass per unit length of rod be $\lambda\left(=\frac{m}{l}\right)$ (say).

Consider an element of length dx at a distance x from centre. Then centripetal force on this element is

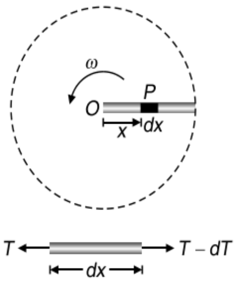
$$dF = (\lambda dx)x\omega^2$$

Tension in the rod is a function of x. The difference in tension at two ends of the section P provides the necessary centripetal force.

$$-dT = \lambda(dx)x\omega^2$$

The negative sign indicates decrease in the value of T and x increases.

$$\Rightarrow \quad -\int_0^T dT = \lambda\omega^2\int_l^x x\,dx$$



Now, we can also write $\lambda = \frac{m}{l} = \rho A = \rho \pi r^2$

$$\Rightarrow \quad T(x) = \frac{\pi r^2 \rho l^2 \omega^2}{2}\left(1 - \frac{x^2}{l^2}\right)$$

Let dl be the extension of the rod in the small element dx at P, then

$$dl = \frac{T(x)}{AY}dx$$

$$\Rightarrow \quad dl = \frac{\pi r^2 \rho l^2 \omega^2}{2AY}\left(1 - \frac{x^2}{l^2}\right)dx$$

So, total extension is given by $\Delta l = \int_0^l dl$

$$\Rightarrow \quad \Delta l = \frac{\pi r^2 \rho l^2 \omega^2}{2AY}\int_0^l \left(1 - \frac{x^2}{l^2}\right)dx$$

$$\Rightarrow \quad \Delta l = \frac{\rho \omega^2 l^3}{3Y} \text{ as } A = \pi r^2$$

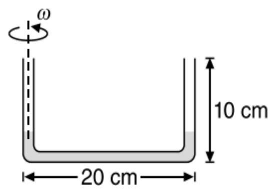
Substituting the values, we get

$$\Delta l = \frac{1}{3}\frac{(10^4)(400)^2(0.5)^3}{2 \times 10^{11}}$$

$$\Rightarrow \quad \Delta l = 3.33 \times 10^{-4} \text{ m}$$

PROBLEM 14

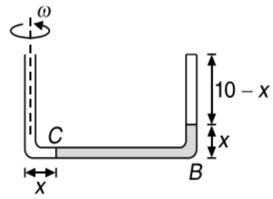
Length of a horizontal arm of a *U*-tube is 20 cm and ends of both the vertical arms are open to a pressure of $1.01 \times 10^3 \text{ Nm}^{-2}$. Water is poured into the tube such that liquid just fills horizontal part of the tube. Now, one of the open ends is sealed and the tube is then rotated about a vertical axis passing through the other vertical arm with angular velocity ω. If length of water in sealed arm is 5 cm, then calculate ω. Assume the density of water to be 10^3 kgm^{-3}, $g = 10 \text{ ms}^{-2}$ and temperature to be constant.



SOLUTION

Let the cross-sectional area of the tube be A. The initial pressure of air in sealed tube is

$$P_i = 1.01 \times 10^3 \text{ Nm}^{-2}$$



Initial volume, $V_i = 0.1A$

Final volume, $V_f = (0.1 - x)A$

If final pressure is P_f, then for constant temperature, we have

$$P_i V_i = P_f V_f$$

$$\Rightarrow \quad P_f = \frac{P_i V_i}{V_f} = (1.01 \times 10^3)\left(\frac{0.1}{0.1 - x}\right) \quad \ldots(1)$$

Also, we have

$$P_B = P_f + \rho g x \quad \ldots(2)$$

and $P_C = 1.01 \times 10^3 \text{ Nm}^{-2}$ ...(3)

So, pressure difference between B and C is

$$\Delta P = P_B - P_C$$

$$\Rightarrow \quad \Delta P = (P_f + \rho g x) - P_C \quad \ldots(4)$$

The centripetal force required for circular motion of vertical column is provided by reaction of the tube while the centripetal force required for circular motion of the horizontal part is provided by the excess pressure at B.

$$\Rightarrow \quad F = (\Delta P)A = (m_{CB})r\omega^2 = [(0.2 - x)\rho A]r\omega^2$$

$$\Rightarrow \quad \Delta P = [(0.2 - x)\rho]r\omega^2 \quad \ldots(5)$$

where, r is the distance of centre of mass of horizontal portion of liquid from the axis of rotation.

$$\Rightarrow \quad r = x + \left(\frac{0.2 - x}{2}\right) = \frac{0.2 + x}{2} \quad \ldots(6)$$

Substituting the value of r from equation (5) and ΔP from equation (4) in equation (5), we get these values we have,

$$(1.01 \times 10^3)\left(\frac{0.1}{0.1 - x}\right) + \rho g x - (1.01 \times 10^3) = (0.2 - x)\rho\left(\frac{0.2 + x}{2}\right)(\omega)^2$$

Substituting the values given in the problem, i.e.

$$x = 0.05 \text{ m}, \ \rho = 10^3 \text{ kgm}^{-3} \text{ and } g = 10 \text{ ms}^{-2}$$

we get

$$2.02 + 0.5 - 1 = 0.01875\omega^2$$

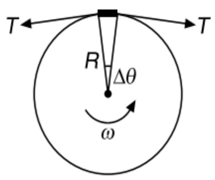
$$\Rightarrow \quad \omega = 9.15 \text{ rads}^{-1}$$

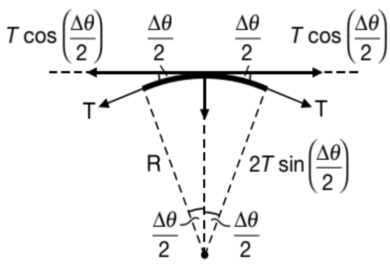
PROBLEM 15

A circular ring of radius R, mass M, made of a uniform wire of cross-sectional area A is rotated about a stationary vertical axis passing through its centre and perpendicular to the plane of the ring. If the breaking stress of the material of the ring is σ_0, then calculate the maximum angular speed ω_{max} at which the ring may be rotated without breaking it. If Young's modulus of the material of ring is Y, then calculate the increment ΔR in the radius of the ring.

SOLUTION

Every element of the ring rotates in a circle of radius R about the axis of rotation. The radial component of tension (T) in the wire provides the centripetal force. The free body diagram of a small element of mass Δm is shown in Figure.





The mass of this element is given by

$$\Delta m = \frac{M\Delta\theta}{2\pi}$$

The net force acting towards the centre is

$$2T\sin\left(\frac{\Delta\theta}{2}\right) \approx 2T\left(\frac{\Delta\theta}{2}\right) \approx T\Delta\theta$$

This force provides the necessary centripetal force to the ring to revolve in a circle of radius R.

$$\Rightarrow \quad T\Delta\theta = \left(\frac{M\Delta\theta}{2\pi}\right)\omega^2 R$$

$$\Rightarrow \quad T = \frac{M\omega^2 R}{2\pi}$$

According to the problem, the breaking stress is σ_0, so the maximum value of tension that the ring can bear is

$$T_{max} = \sigma_0 A$$

$$\Rightarrow \quad \frac{M\omega_{max}^2 R}{2\pi} = \sigma_0 A$$

$$\Rightarrow \quad \omega_{max} = \sqrt{\frac{2\pi\sigma_0 A}{MR}}$$

Stress (σ) in the ring is

$$\sigma = \frac{T}{A} = \frac{MR\omega^2}{2\pi A}$$

If ΔR is the increment in the radius of the ring, then the elongation in the length of the ring is

$$\Delta l = 2\pi(R+\Delta R) - 2\pi R = 2\pi\Delta R$$

The strain (ε) is given by

$$\varepsilon = \frac{\Delta l}{l} = \frac{\Delta l}{2\pi R} = \frac{\Delta R}{R}$$

By definition, we have $Y = \frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{\varepsilon}$

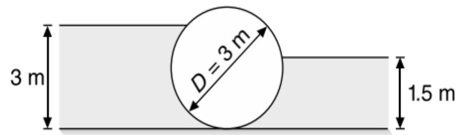
$$\Rightarrow \quad \varepsilon = \frac{\sigma}{Y}$$

$$\Rightarrow \quad \Delta R = \left(\frac{\sigma}{Y}\right)R = \left(\frac{MR\omega^2}{2\pi A}\right)\frac{R}{Y}$$

$$\Rightarrow \quad \Delta R = \frac{MR^2\omega^2}{2\pi AY}$$

PROBLEM 16

A cylindrical weir has a diameter of 3 m and a length of 6 m, whose cross-sectional view is shown in Figure. Calculate the magnitude of the resultant force acting on the weir due to water. Take $g = 10\ \text{ms}^{-2}$ and $\rho_{water} = 1000\ \text{kgm}^{-3}$.

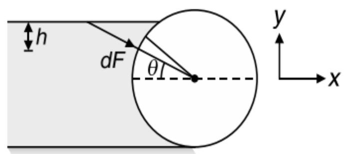


SOLUTION

Let us calculate the force on weir due to water from the left side and the right side.

Hydrostatic force from the left side

Consider an infinitesimal element on the left side of the weir subtending an angle $d\theta$ at the centre of the weir and making an angle θ with the diameter as shown in Figure.



The arc length of the element is $Rd\theta$, so area of the element measured across the length of the weir is

$$dA = (6)Rd\theta$$

Depth of the element below the free surface of water is

$$h = R(1-\sin\theta)$$

Force on this element due to water is

$$dF = (\rho g h)dA = 6\rho g R^2(1-\sin\theta)d\theta$$

The x and y components of this force are

$$dF_x = dF\cos\theta = 6\rho gR^2\cos\theta(1-\sin\theta)d\theta \text{ and}$$

$$dF_y = -dF\sin\theta = -6\rho gR^2(1-\sin\theta)\sin\theta d\theta$$

$$\Rightarrow \quad F_x = \int_{-\pi/2}^{+\pi/2} dF_x = 6\rho gR^2\int_{-\pi/2}^{+\pi/2}\cos\theta(1-\sin\theta)d\theta$$

$$\Rightarrow \quad F_x = 6\rho gR^2\left(\sin\theta+\frac{\cos 2\theta}{4}\right)\Bigg|_{-\pi/2}^{+\pi/2}$$

$$\Rightarrow \quad F_x = 12\rho gR^2 \qquad \ldots(1)$$

Similarly, we have

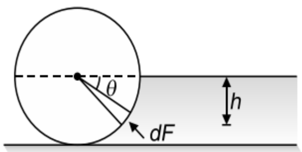
$$F_y = \int_{-\pi/2}^{+\pi/2} dF_y = -6\rho gR^2\int_{-\pi/2}^{+\pi/2}(1-\sin\theta)\sin\theta d\theta$$

$$\Rightarrow \quad F_y = -6\rho gR^2\left(-\cos\theta+\frac{\sin 2\theta}{4}-\frac{\theta}{2}\right)\Bigg|_{-\pi/2}^{+\pi/2}$$

$$\Rightarrow \quad F_y = 3\pi\rho gR^2 \qquad \ldots(2)$$

Hydrostatic force from the right side

Similarly, consider an infinitesimal element on the right side of the weir subtending an angle $d\theta$ at the centre of the weir and making an angle θ with the diameter as shown in Figure.



Area of this element is

$$dA = 6(Rd\theta)$$

Depth of this element below the free surface of the liquid is

$$h = R\sin\theta$$

Force on this element due to water is

$$dF = (\rho gh)dA = 6\rho gR^2\sin\theta d\theta$$

The x and y components of this force are

$$dF_x = -dF\cos\theta = -3\rho gR^2(\sin 2\theta)d\theta \text{ and}$$

$$dF_y = dF\sin\theta = 3\rho gR^2(2\sin^2\theta)d\theta$$

$$\Rightarrow \quad F_x = -3\rho gR^2\int_0^{\pi/2}(\sin 2\theta)d\theta$$

$$\Rightarrow \quad F_x = -3\rho gR^2 \qquad \ldots(3)$$

$$\Rightarrow \quad F_y = 3\rho gR^2\int_0^{\pi/2}(1-\cos 2\theta)d\theta$$

$$\Rightarrow \quad F_y = \frac{3}{2}\pi\rho gR^2 \qquad \ldots(4)$$

From equations (1), (2), (3) and (4), we get

$$(F_x)_{\text{net}} = 12\rho gR^2 - 3\rho gR^2 = 9\rho gR^2 \text{ and}$$

$$(F_y)_{\text{net}} = \frac{9}{2}\pi\rho gR^2$$

$$\Rightarrow \quad F_{\text{net}} = \rho gR^2\sqrt{81+\left(\frac{9\pi}{2}\right)^2} = 16.76\rho gR^2$$

$$\Rightarrow \quad F_{\text{net}} = (16.76)(10^3)(10)(1.5)^2$$

$$\Rightarrow \quad F_{\text{net}} = 377100 \text{ N} \approx 377 \text{ kN}$$

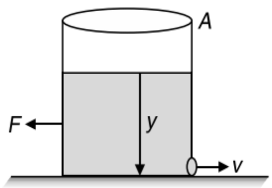
PROBLEM 17

A large open top container of negligible mass and uniform cross-sectional area A has a small hole of cross-sectional area $\frac{A}{100}$ in its side wall near the bottom. The container is kept on a smooth horizontal floor and contains a liquid of density ρ and mass M_0. Assuming that the liquid starts flowing out horizontally through the hole at $t = 0$, calculate

(a) the acceleration of the container and
(b) its velocity when 75% of the liquid has drained out.

SOLUTION

Let y be the depth of the orifice below the free surface of the liquid at any instant t after the start.



At any instant (t)

(a) The velocity of efflux of the liquid coming out of the orifice is

$$v = \sqrt{2gy}$$

The thrust exerted on the beaker due to the ejection of the liquid is

$$F = av^2\rho, \text{ opposite to } v$$

$$\Rightarrow \quad F = a(2gy)\rho$$

So, acceleration of the container plus liquid system at the instant discussed is

$$\text{Acceleration} = \frac{F}{M_{\text{liquid}} + M_{\text{container}}}$$

Since, the container has negligible mass, so

$$\text{Acceleration} \approx \frac{F}{M_{\text{liquid}}} = \frac{a(2gy)\rho}{Ay\rho}$$

$$\Rightarrow \text{Acceleration} = 2g\left(\frac{a}{A}\right) = 2g\left(\frac{1}{100}\right)$$

$$\Rightarrow \text{Acceleration} = 2g\left(\frac{a}{A}\right) = \frac{g}{50}$$

(b) When 75% of the liquid has drained out, then height y' of the free surface of the liquid is given by

$$Ay'\rho = \frac{25}{100}M_0$$

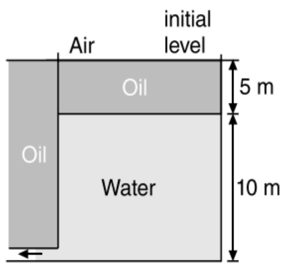
$$\Rightarrow y' = \frac{M_0}{4A\rho}$$

So, the new speed of efflux is

$$v' = \sqrt{2gy'} = \sqrt{2g\left(\frac{M_0}{4A\rho}\right)} = \sqrt{\frac{M_0 g}{2A\rho}}$$

PROBLEM 18

A tank having a small circular hole contains oil on top of water. It is immersed in a large tank of the same oil. Water flows through the hole. What is the velocity of this flow initially? When the flow stops, what would be the position of the oil-water interface in the tank from the bottom. If the ratio of the cross-sectional area of tank to that of hole is 50, determine the time in which the flow stops. The specific gravity of oil is 0.5.



SOLUTION

Since, the difference in pressure is

$$\Delta P = h(\rho_w - \rho_0)g = (10)(1000-500)9.8$$

$$\Rightarrow \Delta P = 49000 \text{ Nm}^{-2}$$

According to Bernoulli's Theorem, we have

$$\Delta P = \frac{1}{2}\rho_w v^2$$

$$\Rightarrow v = \sqrt{\frac{2\Delta P}{\rho_w}} = \sqrt{\frac{2\times 49000}{1000}} = 9.8 \text{ ms}^{-1}$$

When the flow stops, then let h be the height of oil water interface. Also, flow will stop, because the pressure difference will be zero. Hence, we have

$$(10+5)\rho_0 g = 5\rho_0 g + h\rho_w g$$

$$\Rightarrow 10\rho_0 = h\rho_w$$

$$\Rightarrow h = \frac{10\times 500}{1000} = 5 \text{ m}$$

i.e., flow will stop when the water-oil interface is at a height of 5.0 m

When the oil-water interface is at a height of y from ground, then we have

$$\Delta P = 5\rho_0 g + y\rho_w g - 15\rho_0 g$$

$$\Rightarrow \Delta P = y\rho_w g - 10\rho_0 g$$

According to Bernoulli's Theorem, we get

$$\frac{1}{2}\rho_w v^2 = y\rho_w g - 10\rho_0 g$$

$$\Rightarrow v = \sqrt{2g\left(y - \frac{10\rho_0}{\rho_w}\right)} = 4.43\sqrt{y-5}$$

Further, according to the Equation of Continuity, we have

$$av = A\left(-\frac{dy}{dt}\right)$$

$$\Rightarrow dt = \left(\frac{A}{a}\right)\left(-\frac{dy}{v}\right) = 50\left(-\frac{dy}{4.43\sqrt{y-5}}\right)$$

$$\Rightarrow \int_0^t dt = -\frac{50}{4.43}\int_{10}^{5}\frac{dy}{\sqrt{y-5}} = 11.29\int_5^{10}\frac{dy}{\sqrt{y-5}}$$

$$\Rightarrow t = 11.29\left(\frac{\sqrt{y-5}}{1/2}\right)\Bigg|_5^{10} = 22.58\left(\sqrt{5}\right)$$

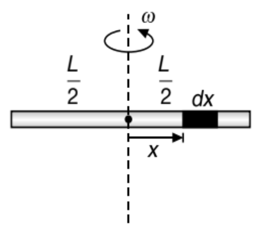
$$\Rightarrow t \approx 50.5 \text{ s}$$

PROBLEM 19

A horizontally oriented copper rod of length l is rotated about a vertical axis passing through its middle. Calculate the angular frequency at which the rod ruptures. Assume that the breaking or rupture strength of copper is σ and density of copper is ρ.

SOLUTION

Since the stress is zero at the free ends and maximum at the axis, hence the rod will rupture at the middle. Let us consider the element of length dx of the rod at a distance x from the axis of rotation of the rod as shown in Figure.



The mass dm of this element is $dm = \rho A dx$

The centripetal force required to move this element in a circle of radius x is

$$dF = (dm)x\omega^2 = (\rho A dx)\omega^2 x$$

Tension at a point P at a distance x from the axis is equal to sum of centripetal forces on all elements lying between P and B. So at point P, tension is given by

$$T = \rho A\omega^2 \int_0^{L/2} x\,dx = \frac{\rho A\omega^2 L^2}{8}$$

So, stress is given by

$$\frac{F}{A} = \frac{\rho\omega^2 l^2}{8} \quad \ldots(1)$$

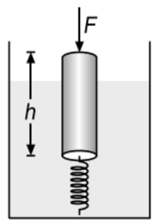
The rod will rupture when the stress calculated in equation (1) equals the breaking stress σ of the rod.

$$\Rightarrow \quad \sigma = \frac{\rho\omega^2 l^2}{8}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{8\sigma}{\rho l^2}}$$

PROBLEM 20

A solid cylinder of radius $r = 10$ cm is floating with its axis vertical in a cylindrical vessel of radius $R = 20$ cm, containing water. A spring of force constant $k = 500\ \text{Nm}^{-1}$ attached at the bottom of the cylinder is initially in its natural position. The cylinder is slowly pushed from top till it gets just immersed in the water and in doing so 25 J of energy was spent. The height of the cylinder is $h = 40$ cm. Calculate the density of cylinder. Neglect viscosity of the liquid and take $g = 10\ \text{ms}^{-2}$.



SOLUTION

For equilibrium, the weight of the cylinder must be balanced by the upthrust acting on the cylinder. If ρ be density of cylinder and h_i be length of cylinder immersed in water, then we have

$$(\pi r^2 h)\rho g = (\pi r^2 h_i)\rho_w g$$

$$\Rightarrow \quad h_i = \left(\frac{\rho}{\rho_w}\right)h = \left(\frac{\rho}{10^3}\right)h \quad \ldots(1)$$

So, length of cylinder above the surface of water is

$$H = h - h_i$$

$$\Rightarrow \quad H = h - \left(\frac{\rho}{10^3}\right)h$$

$$\Rightarrow \quad H = \left(\frac{10^3 - \rho}{10^3}\right)h \quad \ldots(2)$$

When the cylinder is pushed downward by a distance y, let the water level rise up through y'.

$$\Rightarrow \quad \pi r^2 y = (\pi R^2 - \pi r^2)y'$$

$$\Rightarrow \quad y' = \left(\frac{r^2}{R^2 - r^2}\right)y = \left(\frac{100}{400-100}\right)y = \frac{y}{3}$$

Since $y + y' = H$

$$\Rightarrow \quad \frac{4}{3}y = H$$

$$\Rightarrow \quad y = \frac{3h}{4}\left(\frac{10^3 - \rho}{10^3}\right) = 0.3\left(\frac{10^3 - \rho}{10^3}\right) \quad \ldots(3)$$

The net force acting on the cylinder is

$$F = ky + \pi r^2(y + y')\rho_w g = ky + \frac{4}{3}y\pi r^2 \rho_w g$$

The work done by this variable force is

$$W = \int_0^y F\,dy$$

$$\Rightarrow \quad W = \left(k + \frac{4}{3}\pi r^2 \rho_w g\right)\frac{y^2}{2}$$

Substituting the values, we get

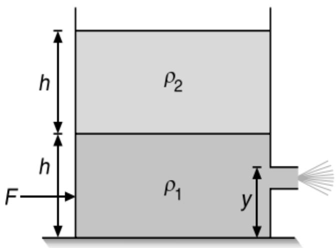
$$25 = \left[\frac{500 + \frac{4\pi}{3}(0.1)^2(10^3)(10)}{2}\right](0.3)^2\left(\frac{10^3 - \rho}{10^3}\right)^2$$

$$\Rightarrow \quad \frac{10^3 - \rho}{10^3} = 0.774$$

$$\Rightarrow \quad \rho = 225.5\ \text{kgm}^{-3}$$

PROBLEM 21

A cylindrical vessel having cross-sectional area $A = 0.5\ \text{m}^2$ is filled with two liquids of density $\rho_1 = 500\ \text{kgm}^{-3}$ and $\rho_2 = 1000\ \text{kgms}^{-3}$, to a height $h = 50$ cm each as shown in Figure.

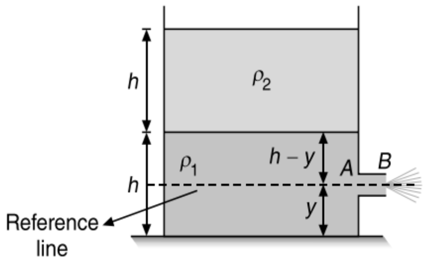


A small hole having area $a = 5\ \text{cm}^2$ is made in right vertical wall at a height $y = 10$ cm from the bottom. Calculate the velocity of efflux of the liquid coming out of the orifice. Calculate the horizontal force F to keep the cylinder in static equilibrium, if it is placed on a smooth horizontal plane. Also find the minimum and maximum values of F to keep the cylinder in static equilibrium, assuming that the coefficient of friction between the cylinder and the plane is $\mu = 0.02$ and $g = 10\ \text{ms}^{-2}$.

SOLUTION

The area a of hole is very small in comparison to base area A of the cylinder, therefore, velocity of liquid inside the cylinder is negligible. Let velocity of efflux be v and atmospheric pressure P_0.

Consider two points A (inside the cylinder) and B (just outside the hole) on the same horizontal line as shown in Figure.



Pressure at A, $P_A = P_0 + h\rho_2 g + (h - y)\rho_1 g$

Pressure at B, $P_B = P_0$

According to Bernoulli's theorem, we have

$$P_A = P_B + \frac{1}{2}\rho_1 v^2$$

$$\Rightarrow \quad v = 3\ \text{ms}^{-1}$$

When cylinder is on smooth horizontal plane, force F required to keep cylinder stationary equals horizontal thrust exerted by the water jet emerging out from the orifice.

$$F_{\text{thrust}} = v_{\text{rel}}\frac{dm}{dt} = (v - 0)(av\rho) = av^2\rho$$

$$\Rightarrow \quad F_{\text{thrust}} = 5 \times 10^{-4} \times 1000 \times (3)^2 = 4.5\ \text{N}$$

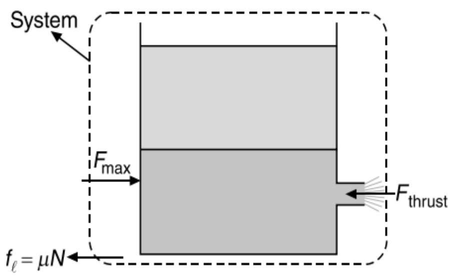
Total mass of the liquid in the cylinder is

$$M = Ah\rho_1 + Ah\rho_2 = 750\ \text{kg}$$

Limiting friction between the cylinder and the surface is

$$f_l = \mu N = \mu M g = 150\ \text{N}$$

Since $F_{\text{thrust}} = 4.5$ N, which is less than the limiting value of the force of friction, so minimum force required to keep the cylinder in static equilibrium is zero.



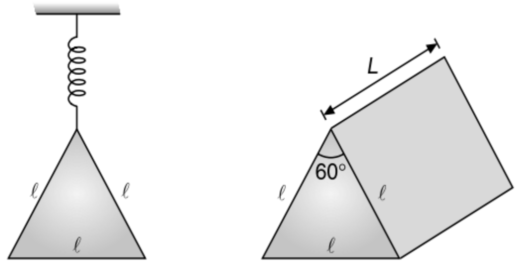
Considering free body diagram of the system for maximum value of force, we get

$$F_{\text{max}} = F_{\text{thrust}} + \mu N$$

$$\Rightarrow \quad F_{\text{max}} = 4.5 + 150 = 154.5\ \text{N}$$

PROBLEM 22

A glass prism of length L has its principal section in the form of an equilateral triangle of side length l whose cross sectional view and full view are shown in the Figure.



Cross-sectional View Full View

The prism, with its base horizontal, is supported by a vertical spring of force constant k such that half the slant surface of the prism is submerged in water. If the surface tension of water is T, contact angle between water and glass is $0°$, density of glass is d and that of water is $\rho (< d)$, then calculate the extension in the spring in this position of equilibrium.

SOLUTION

Volume of prism will be

$$V = AL = \left(\frac{\sqrt{3}}{4}l^2\right)L = \frac{\sqrt{3}}{4}l^2 L$$

Since half the slant surface of prism is submerged in water, so the volume of prism immersed in water is

$$V_{\text{imm}} = V' = \frac{\sqrt{3}}{4}l^2 L - \frac{\sqrt{3}}{4}\left(\frac{l}{2}\right)^2 L = \frac{3\sqrt{3}}{16}l^2 L$$

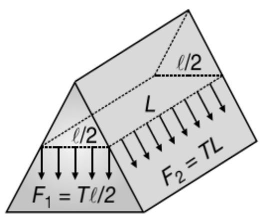
The forces acting on prism are

(i) Weight of the prism, $W = \frac{\sqrt{3}}{4}l^2 L d g$

(ii) Spring force, $F_{\text{sp}} = kx$

(iii) Buoyant force, $F_B = \left(\frac{3\sqrt{3}}{16}l^2 L\right)\rho g$

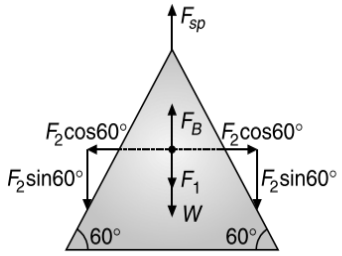
(iv) Surface tension force, F_{ST} made up of two forces $F_1 = T\left(\frac{l}{2}\right)$ and $F_2 = TL$ as shown in Figure.



$$F_{ST}=2(F_1+F_2)=2\left(T\frac{l}{2}+TL\sin 60^\circ\right)$$

$$\Rightarrow \quad F_{ST}=T\left(l+\sqrt{3}L\right)$$

The Free Body Diagram (FBD) of prism showing all the forces acting on it is shown in Figure.



In equilibrium,

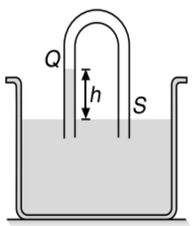
$$F_{sp}+F_B=W+F_{ST}$$

$$\Rightarrow \quad kx+\frac{3\sqrt{3}}{16}l^2L\rho g=\frac{\sqrt{3}}{4}l^2Ldg+T\left(l+\sqrt{3}L\right)$$

$$\Rightarrow \quad x=\frac{1}{k}\left[\frac{\sqrt{3}}{4}l^2Ldg-\frac{3\sqrt{3}}{16}l^2L\rho g+\sqrt{3}TL+Tl\right]$$

PROBLEM 23

A glass U-tube is inverted with its open ends of diameters 0.5 mm and 1.0 mm below the surface of water in a beaker as shown in Figure.



The air pressure in the upper part is increased until the meniscus in one limb is in level with the water outside. If density of water is 10^3 kgm^{-3}, surface tension of water is $7.5\times10^{-2}\text{ Nm}^{-1}$, contact angle is $\theta=0^\circ$, then calculate height of water in the other limb.

SOLUTION

Let P be the pressure of air inside the U-tube and P_0 be the atmospheric pressure, then pressure inside the liquid below Q is $(P_0-h\rho g)$

Since, we know that

$$\Delta P=\frac{2T}{R}$$

$$\Rightarrow \quad P-(P_0-h\rho g)=\frac{2T}{r_1} \qquad \ldots(1)$$

where, r_1 is radius of the left limb.

$$\Rightarrow \quad r_1=0.25\times10^{-3}\text{ m}$$

Pressure just inside the liquid below S is P_0

$$\Rightarrow \quad P-P_0=\frac{2T}{r_2} \qquad \ldots(2)$$

where, r_2 is radius of the left limb.

$$\Rightarrow \quad r_2=0.5\times10^{-3}\text{ m}$$

From equations (1) and (2), we get

$$h=\frac{2T}{\rho g}\left(\frac{r_2-r_1}{r_2r_1}\right)$$

$$\Rightarrow \quad h=\frac{2\times7.5\times10^{-2}}{10^3\times9.8}\left(\frac{0.5-0.25}{0.5\times0.25}\right)\frac{1}{10^{-3}}$$

$$\Rightarrow \quad h\approx3.1\times10^{-2}\text{ m}$$