CHAPTER 4

Gravitation and Satellites

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Newton's Laws of Gravitation
(b) Principle of Superposition
(c) Gravitational Field Strength
(d) Acceleration Due to Gravity and its Variation
(e) Gravitational Potential Energy
(f) Gravitational Self Energy
(g) Gravitational Potential
(h) Relation Between Gravitational Field and Gravitational Potential
(i) Conservation Laws for Gravitational Systems
(j) Escape Speed
(k) Satellites and Kepler's Laws
(l) Double Star System or Binary Star System

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

NEWTON'S LAW OF GRAVITATION

The force of gravitation between any two particles having masses m_1 and m_2 separated by a distance r is attractive and acts along the line joining the particles. The magnitude of the force is given by

$$F = \frac{Gm_1m_2}{r^2} \qquad \ldots(1)$$

where G is a universal constant called **Gravitational Constant** having the same value for all pairs of particles. Its numerical value is $6.67 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}$.

The gravitational forces between two particles always form **an action-reaction pair.** The first particle exerts a force on the second particle that is directed toward the first particle along the line joining the two.

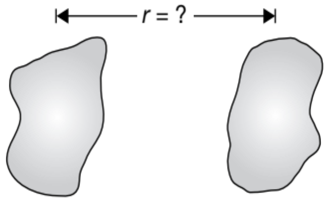
Similarly, the second particle exerts a force on the first particle that is directed towards the second particle along the line joining the two. These forces are equal in magnitude but oppositely directed.

In the vector form, equation (1) is written as

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r} \qquad \ldots(2)$$

where we must remember that $\hat{r}$ **has its origin at the source of the force.**

It is important to realize that Newton's Law of Gravitation is stated only for point particles. For two arbitrary bodies, as shown in figure, there is no unique value for the separation r. So, the equation $F = \frac{Gm_1m_2}{r^2}$ cannot be used for arbitrary bodies.



To compute the force between them requires integral calculus. However, for the special case of a uniform spherical mass distribution, r may be taken as the distance to the center. Also, when the separation between two objects is very much larger than their sizes, they may be approximated as point masses and then, equation may be used.

Problem Solving Technique(s)

(a) The gravitational force between two particles is independent of the presence of other bodies or the properties of the intervening medium.

(b) Gravitational force is conservative force, therefore work done in displacing a body from one place to another is independent of path. It depends only on initial and final positions.

(c) The mass of air bubble in material medium is negative.

(d) The gravitational force obeys Newton's Third Law i.e.

$$\vec{F}_{12}=-\vec{F}_{21}$$

$$\Rightarrow \quad m_1a_1=m_2a_2$$

(e) Gravitational force is a central force.

PROPERTIES OF G

(a) Its value does not depend upon place, time and masses of the bodies and hence it is called a **Universal Constant**.

(b) The value of G is $6.67\times10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}$

(c) Its dimensional formula is $M^{-1}L^3T^{-2}$.

(d) Its value is extremely small. Hence, we do not experience it in our daily life.

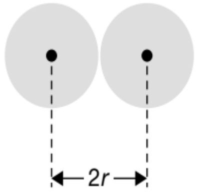
(e) G is numerically equal to gravitational force of potential between two bodies each of mass 1 kg placed at a distance of 1 m.

ILLUSTRATION 1

Spheres of the same material and same radius r are touching each other. Show that gravitational force between them is directly proportional to r^4.

SOLUTION

Since, $m_1=m_2=(\text{volume})(\text{density})$



$$\Rightarrow \quad m_1=m_2=\left(\frac{4}{3}\pi r^3\right)\rho$$

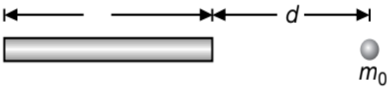
$$\Rightarrow \quad F=\frac{Gm_1m_2}{r^2}$$

$$\Rightarrow \quad F=\frac{G\left(\frac{4}{3}\pi r^3\right)\left(\frac{4}{3}\pi r^3\right)\rho^2}{r^2}$$

$$\Rightarrow \quad F\propto r^4$$

ILLUSTRATION 2

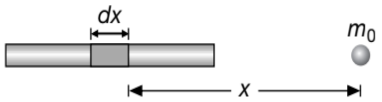
The figure shows a uniform rod of length l whose mass per unit length is λ and the total mass is M. What is the gravitational force of the rod on a particle of mass m_0 located a distance d from one end of the rod, as shown in the figure?



SOLUTION

Consider an infinitesimal element on the rod having length dx at a distance x from the point mass m_0. If dF is the force between the infinitesimal element of mass dm and m_0, then

$$dF=\frac{Gm_0dm}{x^2}\text{, where } dm=\lambda dx$$



To obtain the total force on m_0 due to the rod, we integrate the above expression for force within appropriate limits. So,

$$F=\int_{d}^{d+l}\frac{G(\lambda dx)m_0}{x^2}=G\lambda m_0\left(\frac{1}{d}-\frac{1}{d+l}\right)=\frac{Gm_0(\lambda l)}{d(d+l)}$$

$$\Rightarrow \quad F=\frac{Gm_0M}{d(d+l)}$$

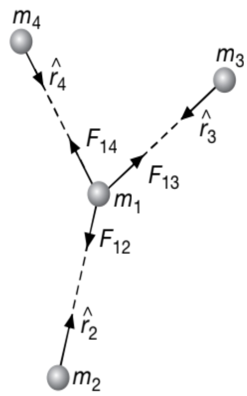
PRINCIPLE OF SUPERPOSITION

Experiments show that when several particles interact, the force between a given pair is independent of the other particles present.

The net force F_1 on the point mass m_1 due to the other particles as shown in the figure is found by first calculating its potential with each of the other particles one at a time.

The **net force** on m_1 is the vector sum of the pair-wise potentials.

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \ldots\ldots + \vec{F}_{1n} \qquad \ldots(3)$$



The net force on m_1 is

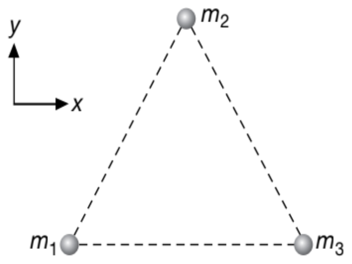
$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}$$

Problem Solving Technique(s)

1. Set up a convenient coordinate system
2. Indicate the directions of the forces acting on the particle under consideration.
3. Calculate the (scalar) magnitudes of the forces.
4. Find the net force by using the component method.

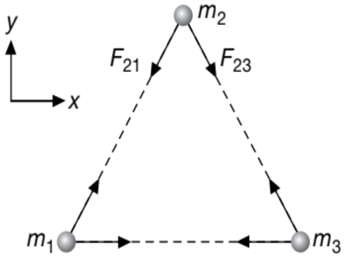
ILLUSTRATION 3

Three point particles with masses $m_1 = 4$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg are at the corners of an equilateral triangle of side $L = 2$ m, as shown in figure. Find the net force on m_2.



SOLUTION

The first two steps have already been done in the figure. The magnitudes of forces are



$$F_{21} = \frac{Gm_2m_1}{L^2} = 1.33 \times 10^{-10}\ \text{N}$$

$$F_{23} = \frac{Gm_2m_3}{L^2} = 1.01 \times 10^{-10}\ \text{N}$$

The net force on m_2 is $\vec{F}_2 = \vec{F}_{21} + \vec{F}_{23}$

Its components are

$$F_{2x} = -F_{21}\cos 60^\circ + F_{23}\cos 60^\circ = -1.6 \times 10^{-11}\ \text{N}$$

$$F_{2y} = -F_{21}\sin 60^\circ - F_{23}\sin 60^\circ = -2.03 \times 10^{-10}\ \text{N}$$

Thus, $F_2 = -\left(1.6\hat{i} + 20.3\hat{j}\right) \times 10^{-11}$ N

ILLUSTRATION 4

Three particles each of mass m are placed at the three corners of an equilateral triangle of side a. Find the force exerted by this system on another particle of mass m placed at

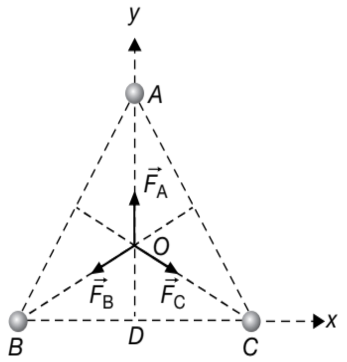
(a) the centre of the triangle and

(b) mid-point of a side.

SOLUTION

To solve the above problem, we apply the gravitational potential which follow the principle of superposition.

(a) When another mass m is placed at O, it experiences three forces $\vec{F}_A$, $\vec{F}_B$ and $\vec{F}_C$. Since AO, BO and CO are equal hence $\left|\vec{F}_A\right| = \left|\vec{F}_B\right| = \left|\vec{F}_C\right|$. Angle between any two forces is same i.e. 120°. Therefore, the resultant force exerted by the system on particle at O is zero.



(b) In this case the particle is placed at point D, which is equidistant from B and C.

$$\left|\vec{F}_B\right| = \left|\vec{F}_C\right|$$

But they are opposite in direction. Therefore, the effective force at D will be due to mass m at A.

By geometry of the figure $AO = a\sin 60 = \dfrac{\sqrt{3}a}{2}$

Therefore, $F_A = \dfrac{4Gm^2}{3a^2}$ along DA

GRAVITATIONAL FIELD STRENGTH (E_g)

Every mass particle is surrounded by a space within which its influence can be felt. This region or space is said to be occupied with **gravitational field**.

Each point in the field is associated with a (vector) force which is experienced by a unit mass placed at that point and is called the **gravitational field strength**.

The region of space around a body in which its gravitational influence is experienced by other bodies is the gravitational field of that body.

OR

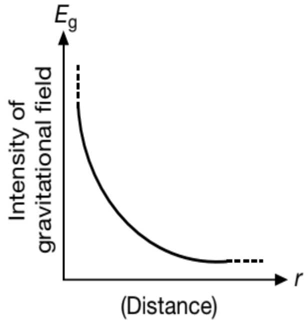
The gravitational force of attraction acting on a body of unit mass at any point in the gravitational field is defined as the intensity of gravitational field $\left(\vec{E}_g\right)$ at that point.

OR

The gravitational field is also defined as the force per unit test mass $\left(\dfrac{F}{m_0}\right)$ placed in the gravitational influence of source mass. Mathematically,

$$E_g = \frac{F}{m_0} = \frac{Gm}{r^2}$$

where m = Source Mass, m_0 = Test Mass.



Its SI unit is newton/kg $\left(\text{Nkg}^{-1}\right)$ and its dimensional formula is LT^{-2}. **It is a vector quantity and its direction is always towards the centre of the body producing the field.** The value of E_g is zero as $r \to \infty$. The graph between E_g and r.

Problem Solving Technique(s)

If a system has got a number of masses, then the total gravitational field at a point P due to all the masses is found in accordance with the Principle of Superposition. So,

$$\vec{E}_g = \vec{E}_{g_1} + \vec{E}_{g_2} + \vec{E}_{g_3} + \ldots + \vec{E}_{g_n}$$

i.e. net field at a point P is the vector sum of the fields due to individual masses (each calculated as if others were absent) at the point of consideration.

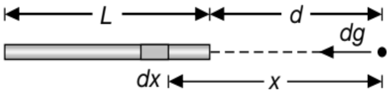
ILLUSTRATION 5

Find the field strength at a point along the axis of a thin rod of length L and mass M, at a distance d from one end.

SOLUTION

First we need to find the field due to an element of length dx. The rod must be thin if we are to assume that all points of the element are at the same distance from the field point. The mass of the element is $dm = \left(\dfrac{M}{L}\right)dx$, so contribution to the field is

$$dg = \frac{G(\lambda dx)}{x^2} = \frac{GM}{L}\frac{dx}{x^2}$$



The total field strength is

$$g = \frac{GM}{L}\int_{d}^{L+d}\frac{dx}{x^2} = \frac{GM}{L}\left(\frac{1}{d} - \frac{1}{L+d}\right) = \frac{GM}{d(L+d)}$$

Notice that when $d \quad L$, we find $g \to \dfrac{GM}{d^2}$, the result for a point particle.

Problem Solving Technique(s)

(a) For calculating the gravitational field at a point *P* at distance *r* due to a source mass *m* we proceed as follows.

STEP-1: Place a test mass of 1 kg at point *P*.

STEP-2: Calculate the force between *m* and 1 kg.

STEP-3: This force is the value of E_g due to *m* at the point *P*.

STEP-4: The direction of force experienced by the 1 kg mass is the direction of E_g due to *m* at the point *P*.

(b) If we are to find the gravitational field due to assembly of masses m_1, m_2, ..., m_n at a point *P*, then we apply Superposition Principle and then

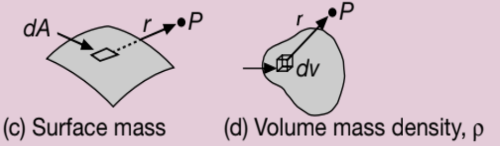
$$\vec{E}_p = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n$$

where we find $\vec{E}_1$, $\vec{E}_2$, ... $\vec{E}_n$ at point *P* by the technique mentioned above.

(c) If we are to find the gravitational field at a point *P* due to a uniform mass distribution, then we calculate the field due to an infinitesimal element of the distribution at the point *P* and then integrate it within appropriate limits.

(a) Continuous distribution

(b) Linear mass density, λ



(c) Surface mass density, σ

(d) Volume mass density, ρ

$$E = \int dE = G\int \frac{dm}{r^2}$$

For mass distributed on a wire with linear mass density λ,

$$E_g = G\int_{\ell} \frac{\lambda d\ell}{r^2}$$

For mass distributed on a surface with surface mass density σ,

$$E_g = G\int_{A} \frac{\sigma dA}{r^2}$$

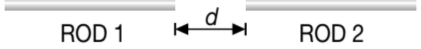
For mass distributed on a volume with volume mass density ρ,

$$E_g = G\int_{V} \frac{\rho dV}{r^2}$$

If the mass distributions are uniform, then mass densities can be taken out of the integral to get desired results.

ILLUSTRATION 6

Two identical uniform rods of length l and mass M are placed along the same line so that their closer ends are a distance d apart, as shown in the figure.



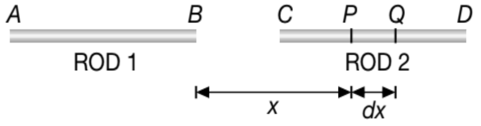
(a) Find the gravitational force of attraction between the rods.

(b) Show that $\lim\limits_{d\to\infty} F = G\dfrac{M^2}{d^2}$.

SOLUTION

(a) The gravitational field due to thin rod at a distance x from one of its end is

$$E = \frac{GM}{x(x+l)}$$



Now consider an infinitesimal element of mass dm at a distance x from end B of rod 1. Then force on this element is

$$dF = (dm)E = \frac{\left(\frac{M}{l}dx\right)GM}{x(x+l)}$$

$$\Rightarrow\ dF = \frac{GM^2}{l}\frac{dx}{x(x+l)} = \frac{GM^2}{l^2}\left[\frac{1}{x} - \frac{1}{x+l}\right]dx$$

$$\Rightarrow\ F = \int_{d}^{d+l} dF = \frac{GM^2}{l^2}\left[\log_e\left(\frac{x}{x+l}\right)\right]_d^{d+l}$$

$$\Rightarrow\ F=\frac{GM^2}{l^2}\log_e\left[\frac{(d+l)^2}{d(d+2l)}\right]$$

(b) The above expression can be written as,

$$F=\frac{GM^2}{l^2}\log_e\left[\frac{\left(1+\frac{l}{d}\right)^2}{\left(1+\frac{2l}{d}\right)}\right]$$

$$\Rightarrow\ F=\frac{GM^2}{l^2}\log_e\left[\left(1+\frac{l^2}{d^2}+\frac{2l}{d}\right)\left(1-\frac{2l}{d}\right)\right]$$

$$\Rightarrow\ F=\frac{GM^2}{l^2}\log_e\left(1+\frac{l^2}{d^2}\right)$$

$$\left\{\text{neglecting the higher terms of } \frac{l}{d}\right\}$$

$$\Rightarrow\ F\approx\frac{GM^2}{l^2}\left(\frac{l^2}{d^2}\right)\qquad\left\{\log_e\left(1+\frac{l^2}{d^2}\right)\approx\frac{l^2}{d^2}\right\}$$

$$\Rightarrow\ F\approx\frac{GM^2}{d^2}$$

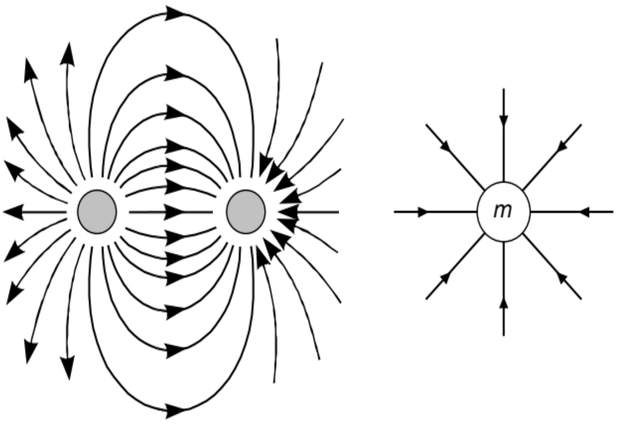
GRAVITATIONAL FIELD LINES

A gravitational field line (or a field line) is a line straight or curved such that a unit mass placed in the field of another mass would always move along this line. Field lines for an isolated mass m are radially inwards.

If two masses are placed close to each other then field lines are shown in figure.

Two field lines never cross each other.

The direction of the field can just be found by drawing a tangent to the field line at the point of consideration.



The Gravitational field lines (for two equal masses).

FIELD DUE TO SPHERE AND SHELL

For an external point $(r>R)$, a sphere (solid or hollow) behaves as if whole of its mass is concentrated at its centre, so the gravitational field outside the sphere/shell is given by

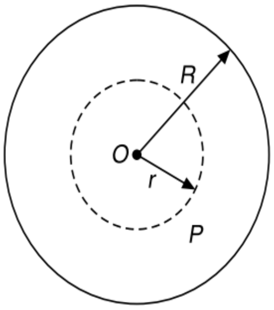
$$E_{\text{outside}}=\frac{GM}{r^2}\qquad\{\text{for } r>R\}$$

In case of a spherical shell, for an internal point, $(r<R)$ the gravitational field inside is zero. So,

$$E_{\text{inside}}=0$$

In case of a spherical volume distribution of mass (i.e., a solid sphere) for an internal point $(r<R)$, the portion of the sphere that lies outside the radius r will not contribute to the field (because the field inside a spherical shell is zero), so

$$E_{\text{inside}}=\frac{GM'}{r^2}$$



Field inside a solid sphere

where $M'=$ mass of sphere of radius r.

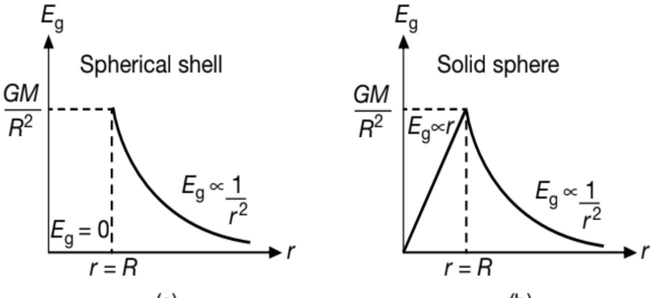
Now if M is the mass of solid sphere, then

$$\rho=\frac{M}{V}=\frac{M}{\left(\frac{4}{3}\right)\pi R^3}$$

and $M'=\frac{4}{3}\pi r^3\rho=\frac{M}{R^3}r^3$

$$\Rightarrow\ E_{\text{inside}}=\frac{G}{r^2}\left(\frac{M}{R^3}r^3\right)=\frac{GM}{R^3}r$$

i.e., intensity inside a solid sphere varies linearly with distance from the centre. So, it is minimum $(E_{\text{centre}}=0)$ at the centre and maximum at the surface $\left(E_{\text{surface}}=\frac{GM}{R^2}\right)$. This is shown graphically in the figure.



(a) (b)

Problem Solving Technique(s)

Gravitational Field For a Shell

$$E_g = \begin{cases} \frac{GM}{r^2} & r \geq R \\ & (\text{outside and at surface}) \\ 0 & r < R \\ & (\text{inside}) \end{cases}$$

Gravitational Field For a Sphere

$$E_g = \begin{cases} \frac{GM}{r^2} & r \geq R \\ & (\text{outside and at surface}) \\ \left(\frac{GM}{R^3}\right)r & r < R \\ & (\text{inside}) \end{cases}$$

ILLUSTRATION 7

There are two concentric shells of masses M_1 and M_2 and radii R_1 and R_2. Find the force on a particle of mass m when the particle is located at

(i) $r_1 > R_2$

(ii) $R_1 < r_2 < R_2$

(iii) $r_3 < R_1$

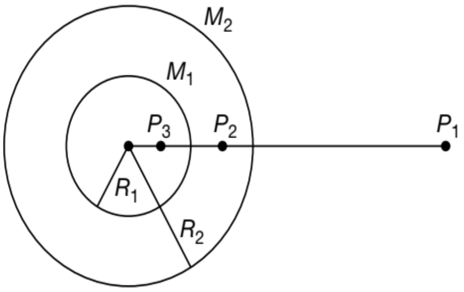
SOLUTION

(i) From the figure, it is clear that the point P_1 lies outside to both the shell. Therefore, gravitational Intensity at the point P_1 is given by

$$E_{P_1} = \frac{G(M_1 + M_2)}{r_1^2}.$$

Therefore, force on the particle of mass m is

$$F = m(E_{P_1}) = \frac{G(M_1 + M_2)m}{r_1^2}.$$



(ii) When $R_1 < r_2 < R_2$, the point P_2 lies outside the smaller shell but inside the larger shell. Therefore, Intensity at P_2 is

$$E_{P_2} = \begin{pmatrix}\text{Intensity} \\ \text{due to} \\ \text{smaller} \\ \text{shell}\end{pmatrix} + \begin{pmatrix}\text{Intensity} \\ \text{due to} \\ \text{larger} \\ \text{shell}\end{pmatrix} = \frac{GM_1}{r_2^2} + 0$$

Therefore, force on mass m is $F = mE_{P_2} = \frac{GM_1 m}{r_2^2}$

(iii) When $r_3 < R_1$, Point P_3 lies inside to both the shells. The intensity of the field at P_3 is

$$E_{P_3} = \begin{pmatrix}\text{Intensity} \\ \text{due to} \\ \text{smaller} \\ \text{shell}\end{pmatrix} + \begin{pmatrix}\text{Intensity} \\ \text{due to} \\ \text{larger} \\ \text{shell}\end{pmatrix} = 0 + 0 = 0$$

ILLUSTRATION 8

A spherically symmetric gravitational system of particles has a mass density $\rho = \begin{cases} \rho_0 \text{ for } r \leq R \\ 0 \text{ for } r > R \end{cases}$, where ρ_0 is a constant. A test mass m_0 can undergo circular motion under the influence of the gravitational field of particles. Calculate its speed v as a function of distance r from the centre of the system. Also draw a plot of speed versus distance r.

SOLUTION

For $r \leq R$, $\frac{m_0 v^2}{r} = \frac{Gm_0 M}{r^2}$ ...(1)

where, $M = \left(\frac{4}{3}\pi r^3\right)\rho_0$

Substituting in Equation (1), we get

$$\frac{v^2}{r} = \frac{G\left(\frac{4}{3}\pi r^3\right)\rho_0}{r^2}$$

$$\Rightarrow \quad v=\sqrt{\frac{4}{3}\pi G\rho_0 r^2}=r\sqrt{\frac{4}{3}\pi G\rho_0}$$

$$\Rightarrow \quad v\propto r$$

i.e. v-r graph is a straight line passing through origin.

For $r>R$, we have

$$M=M_{\text{total}}=\frac{4}{3}\pi R^3\rho_0$$

$$\Rightarrow \quad \frac{m_0v^2}{r}=\frac{Gm_0\left(\frac{4}{3}\pi R^3\right)\rho_0}{r^2}$$

$$v=\sqrt{\frac{4\pi G\rho_0R^3}{3r}}$$

$$\Rightarrow \quad v\propto\frac{1}{\sqrt{r}}$$

The corresponding v-r graph is shown in Figure.

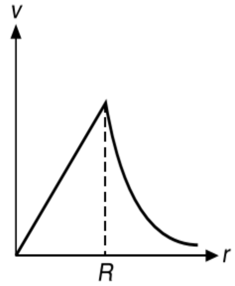
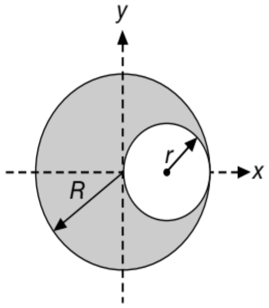


ILLUSTRATION 9

A sphere of radius R has its centre at the origin. It has uniform mass density ρ_0 except that there is a spherical hole of radius $r=\frac{1}{2}R$ whose centre is at $x=\frac{1}{2}R$, as shown in figure. Find the gravitational field at points on the x-axis for $|x|>R$.



SOLUTION

Let, E be the gravitational field at x due to the complete sphere.

If, E_1 be the field due to hole and E_2 be the field due to the remaining portion, then we have

$$E=E_1+E_2$$

$$\Rightarrow \quad E_2=E-E_1$$

$$\Rightarrow \quad E_2=\frac{GM}{x^2}-\frac{Gm}{\left(x-\frac{R}{2}\right)^2} \qquad \ldots(1)$$

where, $M=\frac{4}{3}\pi R^3\rho_0$ and

$$m=\frac{4}{3}\pi\left(\frac{R}{2}\right)^3\rho_0$$

Substituting the values in equation (1), we get

$$E_2=-\left(\frac{\pi G\rho_0R^3}{6}\right)\left[\frac{1}{\left(x-\frac{R}{2}\right)^2}-\frac{8}{x^2}\right]$$

Problem Solving Technique(s)

For **symmetrical mass distributions,** if we are asked to calculate the gravitational field at a point which lies symmetrically with respect to the mass distribution, then we proceed as follows

STEP-1: Consider an infinitesimal element having a mass dm (say) that lies at a distance r from the point P. If dE is the gravitational field due to this element then

$$dE=\frac{Gdm}{r^2}$$

STEP-2: Now consider another identical mirror infinitesimal element having same mass and located at the same distance from P. Then field due to this element will also be dE.

STEP-3: On resolving dE due to both the elements along suitable axis we observe that one set of components cancels and the net gravitational field will then be calculated by taking the integral of the component of the gravitational field due to the contribution of the single element.

$$E_{\text{net}}=\int(\text{contribution due to a single element})$$

However, in the case of **unsymmetrical mass distributions**, if we have to calculate the gravitational field at any point P, then we proceed as follows:

STEP-1:

Consider an infinitesimal element having a mass *dm* (say) that lies at a distance *r* from the point *P*. If *dE* is the gravitational field due to this element then

$$dE = \frac{Gdm}{r^2}$$

STEP-2:

Resolve *dE* along the selected axes (say *x* and *y*) so as to get the components dE_x and dE_y.

STEP-3:

Integrate dE_x and dE_y separately to get E_x and E_y.

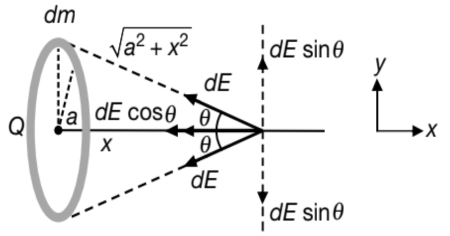
STEP-4:

Then $\vec{E} = E_x\hat{i} + E_y\hat{j}$ such that $|\vec{E}| = E = \sqrt{E_x^2 + E_y^2}$ and if $\vec{E}$ makes an angle β with *x*-axis, then $\tan\beta = \frac{E_y}{E_x}$.

GRAVITATIONAL FIELD AT THE AXIS OF A CIRCULAR UNIFORM RING

Consider a ring of mass M, radius a and uniform linear mass density λ. Then

$$\lambda = \frac{M}{2\pi a} \qquad \ldots(1)$$



Let us calculate the gravitational field due to this ring at a point lying on the axis of the ring at a distance x from the centre of the ring. Since this point is located symmetrically with respect to the ring so let us consider two elements of length dl placed symmetrically on the two diametrically opposite ends. If dE is the field due to each such element, then $dE\sin\theta$ components will cancel out such that the net field is just due to $dE\cos\theta$ components summed up over the entire ring. So,

$$E_{net} = E_x = \int dE\cos\theta = \int \frac{Gdm}{\left(\sqrt{a^2+x^2}\right)^2}\cos\theta$$

$$\Rightarrow \quad E_x = \frac{G\lambda\cos\theta}{\left(a^2+x^2\right)}\int_0^{2\pi a} dl = \frac{G\lambda\cos\theta}{\left(a^2+x^2\right)}(2\pi a)$$

But from equation (1), we have $\lambda(2\pi a) = M$ and also, we have from the figure, $\cos\theta = \frac{x}{\sqrt{a^2+x^2}}$, so we get

$$\Rightarrow \quad E_x = E_{axis} = \frac{GMx}{\left(a^2+x^2\right)^{3/2}} \qquad \text{(along } +x \text{ axis)}$$

Remark(s)

(a) For the point P to lie at far off distance from the centre of the ring i.e., for $x >> a$ we have $\left(x^2+a^2\right)^{3/2} \cong x^3$.

$$\Rightarrow \quad E_x \simeq \frac{GM}{x^2}$$

This result under the specified condition just matches with the field due to a point mass at a distance *x* from it.

(b) For this field to be a MAXIMUM $\frac{dE}{dx} = 0$

$$\Rightarrow \quad \frac{d}{dx}\left[x\left(x^2+a^2\right)^{-3/2}\right] = 0$$

$$\Rightarrow \quad \left(x^2+a^2\right)^{-3/2} + x\left(-\frac{3}{2}\right)\left(x^2+a^2\right)^{-5/2}(2x) = 0$$

$$\Rightarrow \quad 1 = \frac{3x^2}{x^2+a^2}$$

$$\Rightarrow \quad x^2 + a^2 = 3x^2$$

$$\Rightarrow \quad x = \pm\frac{a}{\sqrt{2}}$$

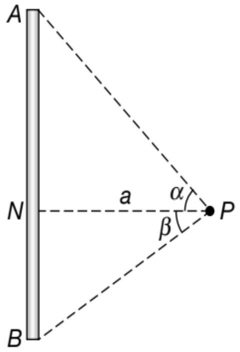
i.e., the field will attain a maximum value at $x = \pm\frac{a}{\sqrt{2}}$ along the axis of the ring and the maximum value equals E_{max} given by

$$E_{max} = \frac{GM\left(a/\sqrt{2}\right)}{\left(a^2+\frac{a^2}{2}\right)^{3/2}}$$

$$\Rightarrow \quad E_{max} = \frac{2GM}{3\sqrt{3}a^2}$$

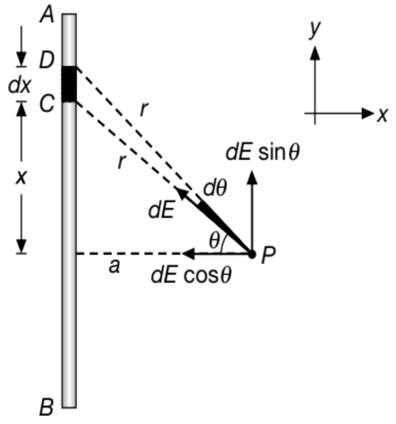
ILLUSTRATION 10

Calculate the gravitational field due to a uniform rod AB at a point P at perpendicular distance a from the rod as shown in figure. Assume that the rod has a linear mass density λ.



SOLUTION

Here it is very important to note the unsymmetrical placement of the point P and hence we must calculate the components of the gravitational fields E_x and E_y separately. For this, let us consider an infinitesimal element of length dx at a distance x as shown.



The net gravitational field at point P due to this infinitesimal element is dE. This dE is resolved into components.

(a) $dE_x = dE\cos\theta$

(b) $dE_y = dE\sin\theta$

The net field E can be calculated by finding E_x and E_y from the above expressions. So

$$E_x = \int dE\cos\theta, \text{ where } dE = \frac{G(\lambda dx)}{r^2}$$

$$\Rightarrow \quad E_x = G\lambda\int\frac{dx\cos\theta}{r^2} = G\lambda\int\frac{dx\cos\theta}{(a^2+x^2)}$$

Also, we observe that

$$\tan\theta = \frac{x}{a}$$

$$\Rightarrow \quad x = a\tan\theta$$

$$\Rightarrow \quad dx = a\sec^2\theta d\theta$$

$$\Rightarrow \quad E_x = G\lambda\int\frac{a\sec^2\theta\cos\theta d\theta}{a^2\sec^2\theta}$$

$$\Rightarrow \quad E_x = \frac{G\lambda}{a}\int_{-\beta}^{\alpha}\cos\theta d\theta$$

$$\Rightarrow \quad E_x = \frac{G\lambda}{a}(\sin\beta+\sin\alpha)$$

Similarly, let us calculate the value of E_y, given by

$$E_y = \int dE_y = \int dE\sin\theta$$

$$\Rightarrow \quad E_y = G\lambda\int\frac{dx\sin\theta}{(a^2+x^2)}$$

Since $x = a\tan\theta$

$$\Rightarrow \quad dx = a\sec^2\theta d\theta$$

$$\Rightarrow \quad E_y = G\lambda\int\frac{a\sec^2\theta\sin\theta d\theta}{a^2\sec^2\theta}$$

$$\Rightarrow \quad E_y = \frac{G\lambda}{a}\int_{-\beta}^{\alpha}\sin\theta d\theta$$

$$\Rightarrow \quad E_y = \frac{G\lambda}{a}\left(-\cos\theta\big|_{-\beta}^{\alpha}\right)$$

$$\Rightarrow \quad E_y = \frac{G\lambda}{a}(\cos\beta-\cos\alpha)$$

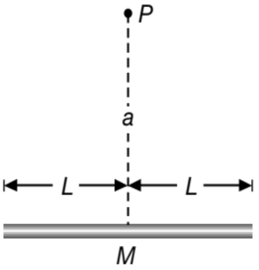
So, the gravitational field due to a rod of length having uniform mass density λ at a point P, that subtends an angle α at one end and β at the other is given by

$$E_x = \frac{G\lambda}{a}(\sin\alpha+\sin\beta) \text{ and}$$

$$E_y = \frac{G\lambda}{a}(\cos\beta-\cos\alpha)$$

ILLUSTRATION 11

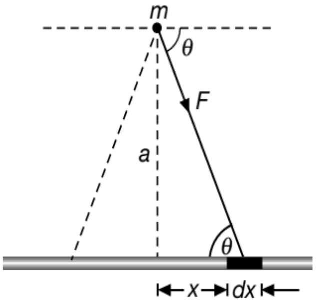
Mass M is distributed uniformly along a line of length $2L$. A particle of mass m_0 is at a point that is a distance a above the centre of the line on its perpendicular bisector (point P in figure).



For the gravitational force that the line exerts on the particle, calculate the components perpendicular and parallel to the line. What happens when $a \quad L$?

SOLUTION

Consider an infinitesimal element of mass dm, as shown in figure.



Then $dm = \left(\frac{M}{2L}\right)dx$

$$dF = \frac{Gm_0 dM}{(a^2+x^2)}$$

Consider another identical mirror element of the already taken infinitesimal element, then we observe that the components $dF\cos\theta$ (i.e., parallel to line) due to these components cancel each other and the net force will be perpendicular to the rod.

$$F_{\text{net}} = \int\left(\begin{array}{c}\text{contribution due to}\\ \text{a single element}\end{array}\right) = \int_{x=-L}^{x=L} dF\sin\theta$$

$$\Rightarrow \quad F_{\text{net}} = \int_{-L}^{L} \frac{Gm_0 M dx}{2L(a^2+x^2)} \frac{a}{\sqrt{a^2+x^2}}$$

$$F_{\text{net}} = \frac{GMm_0 a}{2L}\int_{-L}^{L}\frac{dx}{(a^2+x^2)^{3/2}}$$

$$F_{\text{net}} = \frac{GMm_0}{a\sqrt{L^2+a^2}}$$

ILLUSTRATION 12

Calculate the gravitational field at the centre of a uniform wire in the form of an arc of radius r. The wire subtends an angle ϕ at the centre.

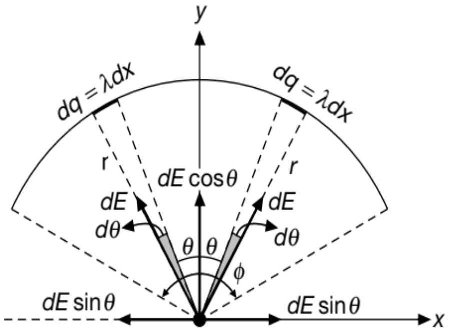
SOLUTION

Consider an element $dm(=\lambda dx)$ of the wire that makes an angle θ with y-axis and subtends an angle $d\theta$ at the centre. Then

$$dm = \lambda(r d\theta)$$

Since, $dE = \frac{Gdm}{r^2} = \frac{\lambda(rd\theta)}{r^2}$

$$\Rightarrow \quad dE = \frac{G\lambda d\theta}{r}$$



Again, take another element which is the mirror image of the element already taken. On resolution, we observe that the x components cancel, while the net field is equal to integral of contribution due to a single element. Hence

$$E = E_y = \int dE\cos\theta$$

$$\Rightarrow \quad E = \int dE\cos\theta$$

$$\Rightarrow \quad E = \frac{G\lambda}{r}\int_{-\frac{\phi}{2}}^{+\frac{\phi}{2}}\cos\theta d\theta$$

$$\Rightarrow \quad E = \frac{G\lambda}{r}\sin\theta\Big|_{-\frac{\phi}{2}}^{\frac{\phi}{2}}$$

$$\Rightarrow \quad E = \frac{G\lambda}{r}\left[\sin\left(\frac{\phi}{2}\right) - \sin\left(-\frac{\phi}{2}\right)\right]$$

$$\Rightarrow \quad E = \frac{G\lambda}{r}2\sin\left(\frac{\phi}{2}\right)$$

$$\Rightarrow \quad E = \frac{2G\lambda}{r}\sin\left(\frac{\phi}{2}\right)$$

Problem Solving Technique(s)

(a) For semi-circle, $\phi = \pi$

So, $E = \dfrac{2G\lambda}{r}$

(b) For Quarter-circle $\phi = \dfrac{\pi}{2}$

$\Rightarrow \quad E = \dfrac{2G\lambda}{r\sqrt{2}}$

(c) For circle, $\phi = 2\pi$

$\Rightarrow \quad E = 0$

ILLUSTRATION 13

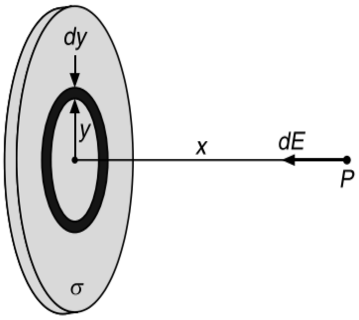
Calculate the gravitational field at point P that lies on the axis of a uniformly disc at a distance x from the centre. Assume the disc to have a radius R and uniform surface mass density σ.

SOLUTION

To find gravitational field at point P due to this disc let us consider an elemental ring of radius y and width dy. If dm and dA be the mass and area of this infinitesimal element, then

$$dm = \sigma dA = \sigma(2\pi y dy)$$

$$\{\because dA = 2\pi y\, dy\}$$



Now we know that gravitational field strength due to a ring of radius R, mass M, at a distance x from its centre on its axis can be given as

$$E_{\text{axis}} = \frac{GMx}{(x^2+R^2)^{3/2}} \qquad \text{(Derived earlier)}$$

So, due to the infinitesimal elemental ring the gravitational field strength dE at point P is

$$dE = \frac{G(dm)x}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \quad dE = \frac{G\sigma(2\pi y dy)x}{(x^2+y^2)^{3/2}}$$

Please note that, here once point P is taken, then value of x remains fixed. So,

$$E = E_{\text{axis}} = \int_0^R dE$$

$$\Rightarrow \quad E = \int dE = 2G\sigma\pi x \int_0^R \frac{y dy}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \quad E = G\sigma\pi x \int_0^R \frac{2y dy}{(x^2+y^2)^{3/2}}$$

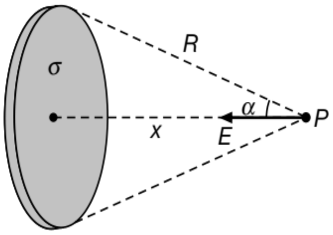
$$\Rightarrow \quad E = G\sigma\pi x \left[-\frac{2}{\sqrt{x^2+y^2}} \Bigg|_{y=0}^{y=R} \right]$$

$$\Rightarrow \quad E = 2G\sigma\pi \left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)$$

Since, $\sigma = \dfrac{M}{\pi R^2}$, so the above expression becomes

$$E = \frac{2GM}{R^2}\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)$$

If we assume that the complete disc subtends an apex angle α at P, as shown in the figure, then



$$\cos\alpha = \frac{x}{\sqrt{R^2+x^2}}$$

So, the gravitational field can also be expressed as

$$E = \frac{2GM}{R^2}\left(1 - \frac{x}{\sqrt{R^2+x^2}}\right) = \frac{2GM}{R^2}(1 - \cos\alpha)$$

ILLUSTRATION 14

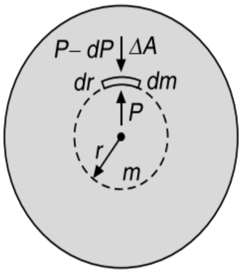
A spherical body of radius R is made of material having constant density ρ. The body is in equilibrium under its own gravity. If $P(r)$ is the pressure at a distance r from the centre of the body inside it, then calculate $P(r)$.

SOLUTION

Gravitational field at a distance r due to mass $m\left(=\frac{4}{3}\pi r^3\rho\right)$ is

$$E=\frac{G\rho\left(\frac{4}{3}\pi r^3\right)}{r^2}=\frac{4G\rho\pi r}{3}$$

Consider a small element of width dr and area ΔA at a distance r from the centre. Pressure force on this element is due to the gravitational force on dm from m inwards towards the centre.



$$\Rightarrow \quad (dP)\Delta A=E(dm)$$

where $dm=(\Delta A)(dr)\rho$ and $m=\left(\frac{4}{3}\pi r^3\right)\rho$

$$\Rightarrow \quad -dP\Delta A=\left(\frac{4}{3}G\pi\rho r\right)(\rho\Delta A dr)$$

$$\Rightarrow \quad -\int_0^P dP=\int_R^r\left(\frac{4G\rho^2\pi}{3}\right)r\,dr$$

$$\Rightarrow \quad -P=\frac{4G\rho^2\pi}{3\times 2}(r^2-R^2)$$

$$\Rightarrow \quad P=\frac{2G\rho^2\pi}{3}(R^2-r^2)$$

ACCELERATION DUE TO GRAVITY (*g*)

Earth attracts all bodies towards its centre. This property of the earth is called 'gravity' and the force with which it attracts a body is called the 'force of gravity' acting on that body. Thus, when a body falls freely towards the earth's surface, the force of gravity $\vec{F}$ produces an acceleration $\vec{g}$ in it given by

$$\vec{g}=\frac{\vec{F}}{m}$$

This acceleration is called acceleration due to gravity. Its magnitude g is independent of the mass, size, shape and composition of the body. It is directed radially inward to the centre of the earth.

If in Newton's law of gravitation one body (say earth) is taken as 'reference body', the force with which the reference body attracts any other body towards its centre is called force due to gravity (of reference body) and the phenomenon 'gravity'. Now if the body is free to move, force of gravity (of reference body) in accordance with Newton's Second law will produce an acceleration in it. The acceleration produced in a body by the force of gravity of reference body (usually earth) is called acceleration due to gravity and is represented by g.

If the reference body has mass M and radius R, the force on a body of mass m at the surface of reference body by Newton's law of gravitation will be

$$F=\left(\frac{GMm}{R^2}\right)$$

and if g is acceleration due to gravity of reference body on its surface, by Newton's Second law,

$$F=mg$$

$$\Rightarrow \quad g=\frac{F}{m}=\frac{GM}{R^2}$$

This is the relation between g and G showing that g depends on reference body. **Also, from this relation we observe that actually the acceleration due to gravity is also the measure of the gravitational field of the earth So,**

$$g=E_g=\frac{F}{m}=\frac{GM}{R^2}$$

It is independent of mass, shape, size, etc., of 'falling body', i.e., a given reference body produces same acceleration in a light and a heavy falling body.

VARIATION IN *g*

With Altitude

As for an external point a spherical distribution of mass behaves as if the whole of its mass were concentrated at the centre, i.e., $g=E_g=\left(\frac{GM}{r^2}\right)$

So, at the surface of earth

$$g = \frac{GM}{R^2}$$

and for a height h above the surface of earth

$$g_h = \left[\frac{GM}{(R+h)^2}\right] \qquad \{\text{as } r = R+h\}$$

$$\Rightarrow \quad \frac{g_h}{g} = \frac{R^2}{(R+h)^2}$$

So, with increase in height, g decreases. However, if $h \quad R$, then

$$g_h = \frac{g}{\left[1+\left(\frac{h}{R}\right)\right]^2}$$

$$\Rightarrow \quad g_h = g\left(1+\frac{h}{R}\right)^{-2}$$

$$\Rightarrow \quad g_h = g\left(1-\frac{2h}{R}\right)$$

With Depth

As in case of spherical distribution of mass, for an internal point

$$g = E_g = \left(\frac{GM}{R^3}\right)r$$

So, at the surface of earth $g = \left(\frac{GM}{R^2}\right)$ and for a point at a depth d below the surface,

$$g_d = \frac{GM}{R^3}(R-d) \qquad \{\because \ r = R-d\}$$

$$\frac{g_d}{g} = \left(\frac{R-d}{R}\right)$$

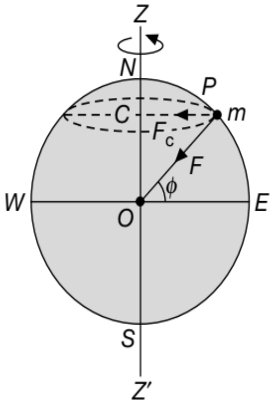
$$\Rightarrow \quad g_d = g\left(1-\frac{d}{R}\right)$$

So, with increase in depth below the surface of earth g decreases and at the centre of earth it becomes zero.

Due to Rotation of Earth

The earth rotates from west to east on its axis, due to which every object on the surface of earth experiences a centrifugal force in the reference frame of the earth. The effective value of acceleration due to gravity at a place of latitude ϕ is given by

$$g' = \sqrt{g^2 + \left(\omega^2 r\right)^2 + 2gr\omega^2\cos(\pi-\phi)}$$



Now $r\omega^2 \quad g$

$$\Rightarrow \quad g' \cong \sqrt{g^2 - 2gR\omega^2\cos^2\phi} \qquad \{\because \ r = R\cos\phi\}$$

$$\Rightarrow \quad g' \approx g - R\omega^2\cos^2\phi$$

At equator $\phi = 0°$ i.e. $g_E \approx g - R\omega^2$

At the poles $\phi = 90°$ i.e. $g_P = g$

Note that the vector g' is not exactly towards the center of earth. Therefore, we note that the decrease in g is maximum at equator and zero at poles. As we go from equator to the pole, value of g increases by

$$\Delta g = \left(g_p - g_e\right) = R\omega^2$$

The fractional change in the value of g is

$$\frac{\Delta g}{g} = \frac{R\omega^2}{g} \quad \frac{1}{291}$$

ILLUSTRATION 15

Find the height at which the gravitational field of the earth becomes one fourth the field at the surface.

SOLUTION

The acceleration due to gravity is the measure of the gravitational field. Let the gravitational field be one fourth the field at the surface at a height h above the earth's surface. So, we have

$$g_h = \frac{g}{4} = \frac{gR^2}{(R+h)^2}$$

$$\Rightarrow \quad \frac{1}{4} = \left(\frac{R}{R+h}\right)^2$$

$$\Rightarrow \quad \frac{R}{R+h} = \pm\frac{1}{2}$$

$$\Rightarrow \quad h = R$$

ILLUSTRATION 16

Assuming earth to be a sphere of uniform mass density, how much would a body weigh half way down the centre of the earth if it weighed 100 N on the surface?

SOLUTION

Given, $mg = 100\text{ N}$

$$g' = g\left(1 - \frac{h}{R}\right)$$

$$\frac{h}{R} = \frac{1}{2}$$

$$\Rightarrow \quad g' = g\left(1 - \frac{1}{2}\right) = \frac{g}{2}$$

$$\Rightarrow \quad mg' = \frac{mg}{2} = \frac{100}{2} = 50\text{ N}$$

ILLUSTRATION 17

What is the acceleration due to gravity of earth at the surface of moon if the distance between earth and moon is 3.8×10^5 km and radius of earth is 6.4×10^3 km ?

SOLUTION

If M and R be the mass and radius of the earth then the acceleration due to gravity due to earth on the surface of earth i.e.,

$$g = \frac{GM}{R^2} \qquad \ldots(1)$$

Similarly, acceleration due to gravity at a distance $r(>R)$ of the earth i.e.,

$$g' = \frac{GM}{r^2} \qquad \ldots(2)$$

If r be the distance between earth and moon then g' will give you the value of acceleration due to gravity on the moon due to earth. Therefore, from equation (1) and (2)

$$g' = \frac{\left(6.4 \times 10^3\right)^2}{\left(3.8 \times 10^5\right)^2} g$$

$$\Rightarrow \quad g = 0.00275\text{ ms}^{-2} \qquad \{\because\ g = 9.8\text{ ms}^{-2}\}$$

ILLUSTRATION 18

A planet of radius R equal to one-tenth the radius of earth has the same mass density as earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density 10^{-3} kgm^{-1} into it. If the wire is not touching anywhere calculate the force applied at the top of the wire by a person holding it in place.
Take the radius of earth to be 6×10^6 m and the acceleration due to gravity of earth to be 10 ms^{-2}.

SOLUTION

Given, $R_{\text{planet}} = R = \dfrac{R_{\text{earth}}}{10}$

Since, density $\rho = \dfrac{M_{\text{earth}}}{\frac{4}{3}\pi R_{\text{earth}}^3}$

Also, $\rho = \dfrac{M_{\text{planet}}}{\frac{4}{3}\pi R_{\text{planet}}^3}$

$$\Rightarrow \quad M_{\text{planet}} = \frac{M_{\text{earth}}}{10^3} = \frac{M_e}{1000}$$

Let the acceleration due to gravity at surface of planet and at the surface of earth be g_p and g_e respectively. Then

$$g_p = \frac{GM_{\text{planet}}}{R_{\text{planet}}^2} = \frac{GM_e(10)^2}{(10)^3 R_e^2} = \frac{GM_e}{10R_e^2}$$

$$\Rightarrow \quad g_p = \frac{g_e}{10}$$

The value of g inside the planet at a distance x from centre of the planet is

$$g_{\text{inside}} = g_{\text{surface of planet}}\left(\frac{x}{R}\right) = g_p\left(\frac{x}{R}\right)$$

So, total force acting on wire is

$$F = \int_{\frac{4R}{5}}^{R} (\lambda dx) g_p\left(\frac{x}{R}\right)$$

$$\Rightarrow \quad F = \frac{\lambda g_p}{R}\left(\frac{x^2}{2}\right)\Bigg|_{4R/5}^{R}$$

Substituting the given values, we get

$$F = 108\text{ N}$$

ILLUSTRATION 19

Two equal masses m and m are hung from a balance whose scale pans differ in vertical height by h. Calculate the error in weighing, if any, in terms of density of earth ρ.

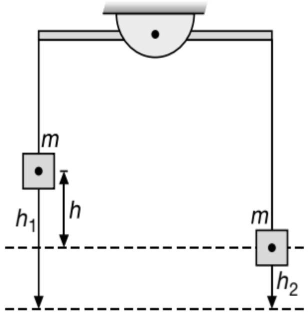
SOLUTION

Since g varies with height as $g' = \dfrac{gR^2}{(R+h)^2}$

For $h \quad R$, we have

$$g' = g\left(1 - \frac{2h}{R}\right)$$

From the figure we see that $h_1 > h_2$, so W_1 will be lesser than W_2 and hence



$$W_2 - W_1 = mg_2 - mg_1 = 2mg\left(\frac{h_1}{R} - \frac{h_2}{R}\right)$$

Since $g = \dfrac{GM}{R^2}$ and $h_1 - h_2 = h$, so we get

$$W_2 - W_1 = 2m\left(\frac{GM}{R^2}\right)\frac{h}{R} = \frac{2GMmh}{R^3}$$

Since $M = \left(\dfrac{4}{3}\pi R^3\right)\rho$

$$\Rightarrow \quad W_2 - W_1 = \frac{2mhG}{R^3}\left(\frac{4}{3}\pi R^3 \rho\right) = \frac{8}{3}\pi\rho G m h$$

ILLUSTRATION 20

Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator becomes zero? Take $g = 10\ \text{ms}^{-2}$ and radius of earth $R = 6400\ \text{km}$.

SOLUTION

The weight will become zero when

$$g' = 0$$

$$\Rightarrow \quad g - R\omega^2 = 0 \qquad \{\text{on the equator } g' = g - R\omega^2\}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{g}{R}}$$

$$\Rightarrow \quad \frac{2\pi}{T} = \sqrt{\frac{g}{R}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{R}{g}}$$

Substituting the values,

$$T = \frac{2\pi\sqrt{\dfrac{6400 \times 10^3}{10}}}{3600}\ \text{hr}$$

$$\Rightarrow \quad T = 1.4\ \text{hr}$$

Thus, the new time period should be 1.4 hr instead of 24 hr for the weight of a body to be zero on the equator.

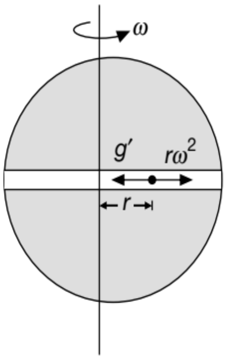
ILLUSTRATION 21

A straight smooth tunnel is dug through a spherical planet whose mass density ρ_0 is constant. The tunnel passes through the centre of the planet and is perpendicular to the planet's axis of rotation, which is fixed in space. The planet rotates with the angular velocity ω so that objects in the tunnel have no acceleration relative to the tunnel. Find ω.

SOLUTION

For no acceleration, we have

$$mg' - mr\omega^2 = 0$$



$$\Rightarrow \quad g' = r\omega^2$$

$$\Rightarrow \quad \left(\frac{GM}{R^3}\right)r = r\omega^2$$

$$\Rightarrow \quad \frac{G\left(\dfrac{4}{3}\pi R^3 \rho_0\right)}{R^3} = \omega^2$$

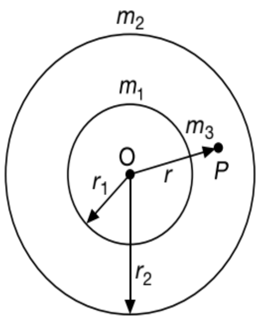
$$\Rightarrow \quad \omega = \sqrt{\frac{4}{3}\pi G \rho_0}$$

ILLUSTRATION 22

Two concentric spherical shells have masses m_1 and m_2 and radii r_1 and $r_2 \left(r_2 > r_1\right)$. What is the force exerted by this system on a particle of mass m_3 if it is placed at a distance $r\left(r_1 < r < r_2\right)$ from the centre?

SOLUTION

The outer shell will have no contribution in the gravitational field at point P, because P lies inside the outer shell. So, at P, the field will only be due to the inner shell of mass m_1.



$$\Rightarrow \quad E_P = \frac{Gm_1}{r^2}$$

Thus, force on mass m_3 placed at P is,

$$F = \left(m_3 E_P\right)$$

$$\Rightarrow \quad F = \frac{Gm_1 m_3}{r^2}$$

The field $\vec{E}_P$ and the force $\vec{F}$ both are towards centre O.

ILLUSTRATION 23

A body is suspended on a spring balance in a ship sailing along the equator with a speed v. Show that the scale reading will be very close to $W_0\left(1 \pm \frac{2v\omega}{g}\right)$ where ω is the angular speed of the earth and W_0 is the scale reading when the ship is at rest. Also explain the significance of plus and minus sign.

SOLUTION

Let R be the radius of earth and ω be its angular speed.

When the ship is at rest, we have

$$W_0 = mg - m\omega^2 R \qquad \ldots(1)$$

When the ship moves with a speed in the sense of rotation of earth, then its effective speed is $v + R\omega$, otherwise it is $v - R\omega$. So, we have

$$W = mg - \frac{m\left(v \pm R\omega\right)^2}{R}$$

$$\Rightarrow \quad W = mg - \left(\frac{mv^2}{R} + m\omega^2 R \pm 2mv\omega\right)$$

$$\Rightarrow \quad W = W_0 - \left(\frac{mv^2}{R} \pm 2mv\omega\right) \qquad \ldots(2)$$

From (1), we get

$$m = \frac{W_0}{g - R\omega^2} = \frac{W_0}{g\left(1 - \frac{R\omega^2}{g}\right)}$$

Since $\frac{R\omega^2}{g} \approx \frac{1}{291} \quad 1$

$$\Rightarrow \quad m = \frac{W_0}{g}$$

$$\Rightarrow \quad W \approx W_0\left(1 \pm \frac{2v\omega}{g}\right)$$

Test Your Concepts-I

Based on Acceleration Due to Gravity, Gravitational Field and Applications

(Solutions on page H.255)

1. Calculate the escape velocity from the surface of moon. The mass of the moon is 7.4×10^{22} kg and radius is 1.74×10^6 m.
2. A planet of mass m_1 revolves round the sun of mass m_2. The distance between the sun and the planet is r. Considering the motion of the sun, find the total energy of the system assuming the orbits to be circular.
3. Consider the ring-shaped body of mass M and radius a. A particle of mass m is placed a distance x from the centre of the ring, along the line through the centre of the ring and perpendicular to its plane.
   (a) Calculate the gravitational potential energy of this system. Take the potential energy to be zero when the two objects are far apart.

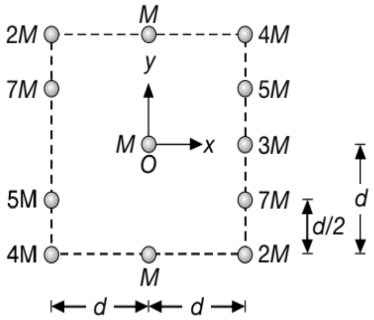
(b) Show that your answer to part (a) reduces to the expected result when x is much larger than the radius a of the ring.

(c) Use $F_x = -\dfrac{dU}{dx}$ to calculate the force between the objects.

(d) Show that your answer to part (c) reduces to the expected result when x is much larger than a.

(e) What is the force when $x = 0$? Explain why this result makes sense.

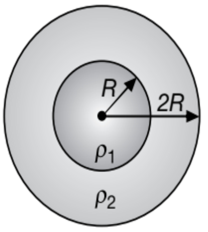
4. In the arrangement shown, find the magnitude and direction of the net gravitational force acting on the central particle at O.



5. The distance between two bodies A and B is r. Taking the gravitational force according to the law of inverse square of r, the acceleration of the body A is a. If the gravitational force follows an inverse fourth power law, then what will be the acceleration of the body A?

6. A particle of mass 20 g experiences a gravitational force of 4 N along positive x-direction. Find the gravitational field at that point.

7. The density of the core of a planet is ρ_1 and that of the outer shell is ρ_2. The radii of the core and that of the planet are R and $2R$ respectively. If the gravitational acceleration at the surface of the planet is same as at a depth R, find the ratio $\dfrac{\rho_1}{\rho_2}$.



8. Calculate the gravitational force of attraction on a particle of mass m placed at the centre of a semicircular wire of length L and mass M.

9. Inside a solid sphere of radius R, the density ρ is given by $\rho = \dfrac{\rho_0 R}{r}$, where ρ_0 is the density at the surface and r is the distance from the centre. Find the gravitational field due to this sphere at a distance $2R$ from its centre.

10. A system consists of a thin ring of radius R and a very long uniform wire oriented along axis of the ring with one of its ends coinciding with the centre of the ring. If mass of ring be M and the linear mass density of the wire be λ, then calculate the interaction force between the ring and the wire.

11. Determine the speed with which the earth would have to rotate on its axis so that a person on the equator would weigh 60% of his weight at the pole. Take $R = 6400\ \text{km}$.

12. A thick spherical shell has an inner radius R_1 and an outer radius R_2. It has mass M and uniform density. Find the gravitational field E_r as a function of r.

13. At what height from the surface of earth will the value of g be reduced by 36% in comparison to the value at the surface? Given $R = 6400\ \text{km}$.

Gravitational Potential Energy (U)

Gravitational potential energy of a system of particles is defined as the external work required to assemble the particles from infinity to a given configuration (as that required in system).

When particles/masses are lying at infinity, their potential energy is taken to be zero, because no potential exists between them. Since the gravitational force between the particles is attractive in nature, so the work will be done by the system and final potential energy of system will be negative.

In giving these arguments please keep this thing in mind that work done by a conservative force equals the decrease in potential energy of the system.

When all particles of a system are separated far apart by infinite distance, there exists no interaction between them. This state, we take as reference of Zero Potential Energy (ZPE).

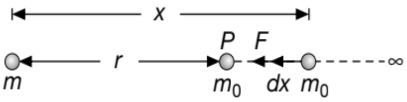
Gravitational potential energy is categorised in two ways.

(a) Gravitational interaction energy (U) of a system of particles.

(b) Gravitational self energy (U_s) of a body.

GRAVITATIONAL POTENTIAL ENERGY OF A SYSTEM OF TWO PARTICLES

Figure shows two masses, a source mass m and a test mass m_0 separated by a distance r. The gravitational potential energy of this system is found by calculating the work done in bringing test mass m_0 from infinity to the given point P at a distance r from the source mass m. Let, at any instant the test mass m_0 be at a distance x from the source mass m. Let m_0 be displaced through dx towards m. The gravitational force of attraction F acts on m_0 towards m.



If dW be the work done, then

$$dW = \vec{F} \cdot d\vec{x}$$

$$\Rightarrow \quad dW = \frac{Gmm_0}{x^2} dx \cos(0^\circ)$$

$$\Rightarrow \quad dW = Gmm_0 x^{-2} dx$$

$$\Rightarrow \quad W = Gmm_0 \int_0^r x^{-2} dx$$

$$\Rightarrow \quad W = Gmm_0 \left(\left. \frac{x^{-2+1}}{-2+1} \right|_{\infty}^{r} \right)$$

$$\Rightarrow \quad W = -Gmm_0 \left(\frac{1}{r} - \frac{1}{\infty} \right)$$

$$\Rightarrow \quad W_{\infty \to P} = -\frac{Gmm_0}{r} = W$$

So, work done by the external force in bringing the test mass m_0 from ∞ to the point P under the influence of source mass m is

$$W = W_{\infty \to P} = -\frac{Gmm_0}{r}$$

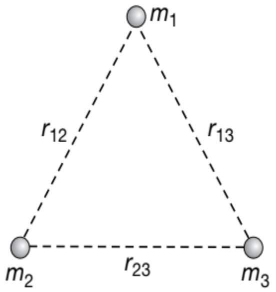
This work done is stored in the form of gravitational potential energy (U) of the system. So,

$$U = -\frac{Gmm_0}{r}$$

GRAVITATIONAL POTENTIAL ENERGY FOR A SYSTEM OF PARTICLES

When more than two particles are there in a system, the potential energy can be given by sum of potential energy of all the pairs of particles with no pair repeated.

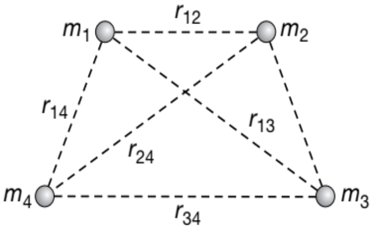
For example, if a system of three particles having masses m_1, m_2 and m_3 is given as shown in figure.



The total potential energy of this system can be given as

$$U = -G\left[\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_1 m_3}{r_{13}} \right] = U_{12} + U_{13} + U_{23}$$

Similarly, for an assembly of four masses m_1, m_2, m_3 and m_4 (having total number of six potential $= {}^4C_2$), we have



$$U = -G\left[\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_4}{r_{34}} + \frac{m_4 m_1}{r_{14}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_4}{r_{24}} \right]$$

$$\Rightarrow \quad U = U_{12} + U_{23} + U_{34} + U_{41} + U_{13} + U_{24}$$

From above we see that the total potential energy is simply the sum of contributions due to distinct pairs. Generalising to N masses, we get

$$U = -G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ (j>i)}}^{N} \frac{m_i m_j}{r_{ij}}$$

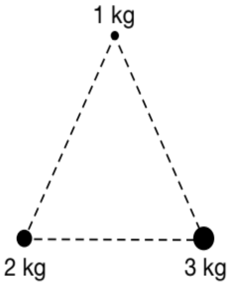
where $j > i$ assures that the pairs are not repeated. Other way round, we can count every pair twice and multiply the result by $\frac{1}{2}$. So,

$$U = -G\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{m_i m_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N} m_i\left(-G\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j}{r_{ij}}\right)$$

ILLUSTRATION 24

Three masses of 1 kg, 2 kg and 3 kg are placed at the vertices of an equilateral triangle of side 1 m. Find the gravitational potential energy of this system. Take $G = 6.67\times10^{-11}\ \text{Nm}^2\text{kg}^{-2}$.

SOLUTION



$$U = -G\left(\frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right)$$

where, $r_{32} = r_{31} = r_{21} = 1.0\ \text{m}$, $m_1 = 1\ \text{kg}$

$m_2 = 2\ \text{kg}$ and $m_3 = 3\ \text{kg}$

Substituting in above, we get

$$U = -\left(6.67\times10^{-11}\right)\left(\frac{3\times2}{1} + \frac{3\times1}{1} + \frac{2\times1}{1}\right)$$

$$\Rightarrow \quad U = -7.337\times10^{-10}\ \text{J}$$

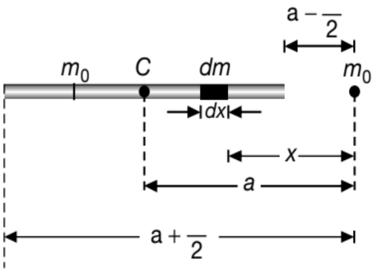
ILLUSTRATION 25

Find the gravitational potential energy of a point mass m_0 and a thin uniform rod of mass m_0 and length l, if they are located along a straight line at a distance a measured from the centre of the rod.

SOLUTION

Let us consider an infinitesimal element of length dx, mass dm at a distance x from the point mass m_0. Then,

$$dm = \left(\frac{m_0}{l}\right)dx$$



If dU be the gravitational potential energy between dm and m_0, then

$$dU = -\frac{Gm_0 dm}{x} = -\frac{Gm_0^2}{l}\frac{dx}{x}$$

$$\Rightarrow \quad U = \int dU = -\frac{Gm_0^2}{l}\int_{\left(a-\frac{l}{2}\right)}^{\left(a+\frac{l}{2}\right)}\frac{dx}{x}$$

$$\Rightarrow \quad U = -\frac{Gm_0^2}{l}\log_e x\Big|_{\left(a-\frac{l}{2}\right)}^{\left(a+\frac{l}{2}\right)}$$

$$\Rightarrow \quad U = -\frac{Gm_0^2}{l}\left[\log_e\left(a+\frac{l}{2}\right) - \log_e\left(a-\frac{l}{2}\right)\right]$$

$$\Rightarrow \quad U = -\frac{Gm_0^2}{l}\log_e\left(\frac{a+\frac{l}{2}}{a-\frac{l}{2}}\right)$$

$$\Rightarrow \quad U = -\frac{Gm_0^2}{l}\log_e\left(\frac{2a+l}{2a-l}\right)$$

GRAVITATIONAL SELF ENERGY FOR A THIN UNIFORM SHELL

To calculate the gravitational self energy of a body, it is supposed that initially the particles of the body are scattered at infinite distance from each other. Therefore, in the formation of a body some external agent has to do some work in assembling the body. This energy is stored in the body as gravitational potential energy and is known as gravitational self energy of the body.

Let us consider a thin uniform shell of mass M and radius R whose gravitational self energy is to be calculated. Let the shell at any instant have a mass m.

Since the shell is uniform and thin, so this mass is lying at its surface. Now let us bring additional mass dm to the surface of this shell, then the gravitational potential energy between m and dm is

$$dU = -\frac{Gmdm}{R}$$

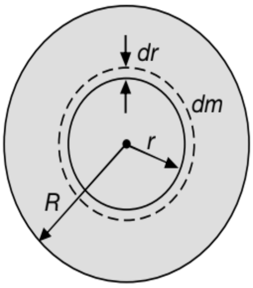
On integrating, the gravitational self energy is obtained as

$$U_s = -\int_0^M \frac{Gmdm}{R} = -\frac{GM^2}{2R}$$

GRAVITATIONAL SELF ENERGY FOR A UNIFORM SPHERE

Let us consider a uniform sphere of mass M and radius R whose gravitational self energy is to be calculated. Let the sphere at any instant have a mass m and radius r. Since the sphere is uniform, so this mass m is distributed uniformly on the sphere. If the sphere has a uniform mass density ρ, then

$$m = \left(\frac{4}{3}\pi r^3\right)\rho$$



Now, let us bring an additional layer of mass dm in the form of a thin spherical shell of inner radius r and outer radius $r+dr$ to be placed on the sphere and repeat the process until it becomes a full fledged solid sphere of radius R. Then

$$dm = \left(4\pi r^2 dr\right)\rho$$

The gravitational potential energy between the sphere and this infinitesimal shell is given by

$$dU = -\frac{Gmdm}{r} = -\frac{G\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2\rho dr\right)}{r}$$

On integrating, the gravitational self energy is obtained as

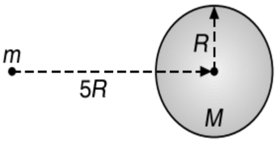
$$U_s = -\frac{16\pi^2\rho^2 G}{3}\int_0^R r^4 dr = -\frac{16\pi^2\rho^2 G}{3}\left(\left.\frac{r^5}{5}\right|_0^R\right)$$

But $\rho = \frac{M}{\frac{4}{3}\pi R^3}$, so on substituting the value of ρ in the above equation, we get

$$U_s = -\frac{3}{5}\frac{GM^2}{R}$$

ILLUSTRATION 26

A solid sphere of mass M and radius R is initially placed at a distance $5R$ from the centre of a point mass m as shown in figure.



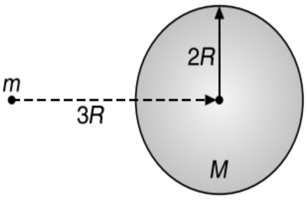
Now it is shifted to a position at a distance $3R$ from the point mass. During displacement, it is also uniformly expanded to a radius $2R$ so that its density decreases uniformly throughout its volume. Find the work required in this process.

SOLUTION

In this process during displacement the size of the sphere is also changing. So, we have to also take into account the self energy of the sphere. Initial total energy of system is the sum of self energy of M and the interaction energy of M and m. If E_m be the self energy of m, then

$$E_i = E_m + \left(-\frac{3}{5}\frac{GM^2}{R}\right) + \left(-\frac{GMm}{5R}\right)$$

Finally the radius of M becomes $2R$ and it is situated at a distance $3R$ from m as shown in figure.



So, final total energy of system is given by

$$E_f = E_m + \left(-\frac{3}{5}\frac{GM^2}{2R}\right) + \left(-\frac{GMm}{3R}\right)$$

External work W required in the process is given by

$$W = E_f - E_i$$

$$\Rightarrow \quad W = \frac{3}{10}\frac{GM^2}{R} - \frac{GMm}{R}\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$\Rightarrow \quad W = \frac{GM}{30R}(9M-4m)$$

POTENTIAL ENERGY OF A PARTICLE ON EARTH'S SURFACE

If M be the mass of earth, R be the radius of earth and m is the mass of particle on earth's surface, then potential energy of particle on earth's surface is

$$U = -\frac{GMm}{R} = -mgR \qquad \left\{\because g = \frac{GM}{R^2}\right\}$$

At height h,

$$U_h = -\frac{GMm}{R+h}$$

If a particle is taken from the surface of the earth to a height h close to the surface then $h \quad R$. So,

$$U_{\text{initial}} = -\frac{GMm}{R} = U_i \text{ and}$$

$$U_{\text{final}} = -\frac{GMm}{R+h} = U_f$$

Hence, work done is equal to change in energy.

$$\Rightarrow \quad W = U_f - U_i = -\frac{GMm}{R+h} + \frac{GMm}{R}$$

$$\Rightarrow \quad W = GMm\left(\frac{1}{R}-\frac{1}{R+h}\right)$$

$$\Rightarrow \quad W = \frac{GMm}{R}\left[1-\left(1+\frac{h}{R}\right)^{-1}\right]$$

$$\Rightarrow \quad W \quad \frac{GMm}{R}\left[1-\left(1-\frac{h}{R}\right)\right]$$

$$\left\{\because \text{ for } x \quad 1;\ (1+x)^n \quad 1+nx\right\}$$

$$\Rightarrow \quad W \quad \frac{GMm}{R}\left(\frac{h}{R}\right)$$

$$\Rightarrow \quad W \quad m\left(\frac{GM}{R^2}\right)h = mgh$$

Conceptual Note(s)

Let us find the difference in potential energy of a mass m in two positions shown in figure. The potential energy of the mass on the surface of earth (at B) is,

$$U_B = -\frac{GMm}{R}$$

and potential energy of mass m at height h above the surface of earth (at A) is,

$$U_A = -\frac{GMm}{R+h} \qquad (U_A > U_B)$$

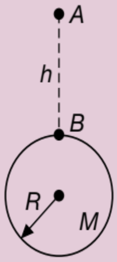
$$\Rightarrow \quad \Delta U = U_A - U_B = -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right)$$

$$\Rightarrow \quad \Delta U = GMm\left(\frac{1}{R}-\frac{1}{R+h}\right) = \frac{GMmh}{R(R+h)}$$

$$\Rightarrow \quad \Delta U = \frac{GMmh}{R^2\left(1+\frac{h}{R}\right)} \qquad \left\{\frac{GM}{R^2} = g\right\}$$

$$\Rightarrow \quad \Delta U = \frac{mgh}{1+\frac{h}{R}}$$

For $h \ll R$, $\Delta U \approx mgh$



Thus, what we read the mgh is actually the difference in potential energy (not the absolute potential energy), that too for $h \ll R$.

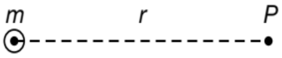
GRAVITATIONAL POTENTIAL (V)

The gravitational potential at a point P due to a source mass is the work done per unit test mass in bringing the test mass (m_0) from infinity to the point P under the gravitational influence of source mass m. So,

$$V_P = \text{Potential at point } P = \frac{W_{\infty \to P}}{m_0}$$

Since $W_{\infty\to P} = -\dfrac{Gmm_0}{r}$

$$\Rightarrow \quad V_P = \frac{W_{\infty\to P}}{m_0} = -\frac{Gm}{r}$$



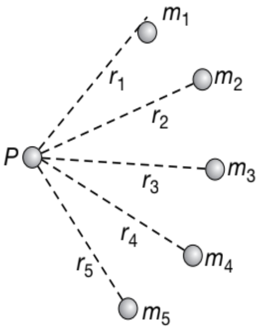
It is a scalar quantity. Its SI unit is joule/kg $\left(\text{Jkg}^{-1}\right)$ and its dimensional formula is $M^0L^2T^{-2}$

The gravitational potential due to a source mass m at a distance r from it is

$$V = -\frac{Gm}{r}$$

GRAVITATIONAL POTENTIAL DUE TO AN ASSEMBLY OF MASSES

Before moving further with this discussion, let us know this and keep in mind that gravitational potential is a scalar quantity. Let us now calculate the gravitational potential due to an assembly of masses at point P shown.



The gravitational potential V at point P due to the assembly of masses shown is the algebraic sum of the potential due to each of the mass in the assembly. So,

$$V = -G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \frac{m_4}{r_4} + \frac{m_5}{r_5}\right)$$

Problem Solving Technique(s)

(a) The gravitational potential at earth's surface is

$$V = \frac{-GM_e}{R_e} = -gR_e$$

(b) For an assembly of masses m_1, m_2, m_3, ..., at distances r_1, r_2, r_3,... from point P, net gravitational potential equals the algebraic sum of gravitational potential due to each of the mass at point P. So,

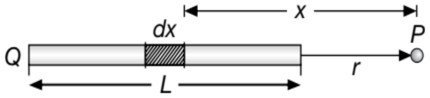
$$V = V_1 + V_2 + V_3 + \ldots$$

$$\Rightarrow \quad V = -G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \ldots\right)$$

$$\Rightarrow \quad V = -G\sum_i^N \frac{m_i}{r_i}$$

GRAVITATIONAL POTENTIAL DUE TO A THIN ROD

Consider a thin rod of length L, having a uniform mass M. Let us find the gravitational potential at a point P due to the rod at a distance r from one end of the rod.



For this we consider an infinitesimal element of length dx at a distance x from the point P. Mass on this element is

$$dm = \frac{M}{L}dx$$

The potential dV due to this element at point P is

$$dV = -\frac{Gdm}{x}$$

$$\Rightarrow \quad V = -\frac{G\left(\dfrac{M}{L}\right)dx}{x}$$

Net gravitational potential at point P is obtained by integrating this expression. So

$$\Rightarrow \quad V = \int dV = -\frac{GM}{L}\int_r^{r+L} \frac{dx}{x}$$

$$\Rightarrow \quad V = -\frac{GM}{L}\log_e x\Big|_r^{r+L}$$

$$\Rightarrow \quad V = -\frac{GM}{L}\log_e\left(\frac{r+L}{r}\right)$$

If mass density of the rod is λ, then $\lambda = \dfrac{M}{L}$. So,

$$V = -G\lambda\log_e\left(\frac{r+L}{r}\right)$$

GRAVITATIONAL POTENTIAL DUE TO A RING AT ITS CENTRE

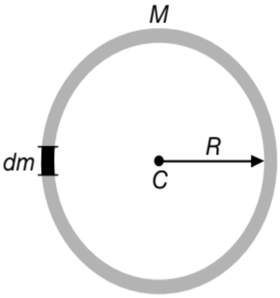
Let us first find potential dV at centre C due to an infinitesimal mass dm on ring which is

$$dV = -\frac{Gdm}{R}$$

Total potential at C is $V = \int dV$

$$\Rightarrow \quad V = -G\int \frac{dm}{R} = -\frac{GM}{R}$$

Since all the infinitesimal dm's of the ring are situated at same distance R from the ring centre C so, we can directly say that the total gravitational potential at centre of ring is $V_C = -\frac{GM}{R}$.



Here we must see that even if mass M is non-uniformly distributed on ring, the gravitational potential at C will remain same (we shall discuss this with an example below) because in that case too

$$V = -\frac{G}{R}\int dm = -\frac{G\int dm}{R}$$

$$\Rightarrow \quad V_{\text{centre}} = -\frac{GM_{\text{total on ring}}}{R}$$

GRAVITATIONAL POTENTIAL DIFFERENCE (V)

Gravitational potential difference (ΔV) is the work done per unit test mass m_0 in bringing it from point A to point B under the gravitational influence of a source mass m.

$$\Rightarrow \quad \Delta V = V_B - V_A = \frac{W_{A\to B}}{m_0} = -Gm\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

$$\Rightarrow \quad W_{A\to B} = m_0\left(V_B - V_A\right) = -Gmm_0\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

THE GRAVITATIONAL POTENTIAL AND FIELD STRENGTH DUE TO A THIN SPHERICAL SHELL

Let M be the mass and R, the radius of thin spherical shell.

$$V = \begin{cases} -\dfrac{GM}{r} & r \ge R \\ & (\text{outside and at surface}) \\ -\dfrac{GM}{R} & r < R \\ & (\text{inside}) \end{cases}$$

Also, as discussed earlier, we know that the gravitational field due to a thin spherical shell is

$$E_g = \begin{cases} \dfrac{GM}{r^2} & r \ge R \\ & (\text{outside and at surface}) \\ 0 & r < R \\ & (\text{inside}) \end{cases}$$

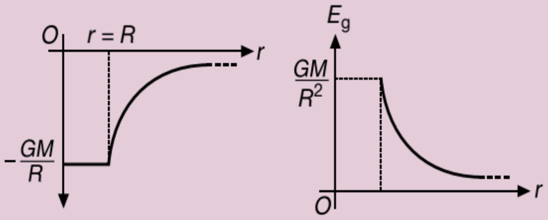
Problem Solving Technique(s)

(a) Gravitational field E_g exhibits discontinuity at the surface of shell $(\text{at } r = R)$. Because inside it is zero whereas outside and at surface it suddenly shoots up to a value $\frac{GM}{R^2}$.

(b) Gravitational potential exhibits continuity at the surface of shell.

(c) Gravitational potential inside the shell is equal to the potential at the surface of shell i.e.

$$V_{\text{inside}} = V_{\text{surface}} = -\frac{GM}{R}$$



THE GRAVITATIONAL POTENTIAL AND FIELD STRENGTH DUE TO A SOLID SPHERE

Let M be the mass and R, the radius of solid sphere

$$V = \begin{cases} -\dfrac{GM}{r} & r \ge R \\ & (\text{outside and at surface}) \\ -\dfrac{GM}{2R^3}\left(3R^2 - r^2\right) & r < R \\ & (\text{inside}) \end{cases}$$

Also, we observe that at the centre of the sphere, where $r = 0$, the gravitational potential is

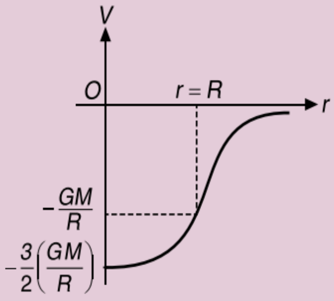
$$V_{\text{centre}} = -\frac{3}{2}\frac{GM}{R}$$

Also, as discussed earlier, we know that the gravitational field due to a sphere is

$$E_g = \begin{cases} \dfrac{GM}{r^2} & r \geq R \\ & (\text{outside and at surface}) \\ \left(\dfrac{GM}{R^3}\right)r & r < R \\ & (\text{inside}) \end{cases}$$

Problem Solving Technique(s)

Both field and potential exhibit continuity at the surface. The graphs also show the continuous nature.



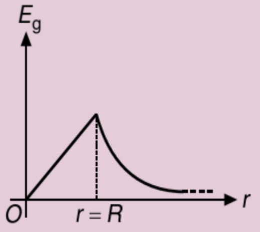


ILLUSTRATION 27

A particle of mass 1 kg is kept on the surface of a uniform sphere of mass 20 kg and radius 1.0 m. Find the work to be done against the gravitational force between them to take the particle away from the sphere.

SOLUTION

Potential at the surface of sphere,

$$V = -\frac{GM}{R} = -\frac{(6.67 \times 10^{-11})(20)}{1}\ \text{Jkg}^{-1}$$

$$V = -1.334 \times 10^{-9}\ \text{Jkg}^{-1}$$

i.e., 1.334×10^{-9} J work is obtained to bring a mass of 1 kg from infinity to the surface of sphere. Hence, the same amount of work will have to be done to take the particle away from the surface of sphere. Thus,

$$W = 1.334 \times 10^{-9}\ \text{J}$$

Problem Solving Technique(s)

To find the gravitational potential due to a shell of radius R, mass m we should keep in mind the following two points:

(a) Gravitational potential on the surface and at any point inside the shell is

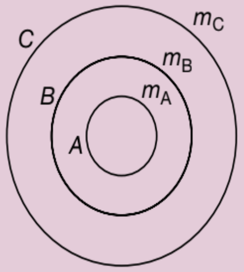
$$V = -\frac{Gm}{R}$$

(b) Gravitational potential at any point outside the sphere/shell at a distance r from the centre is

$$V = -\frac{Gm}{r}$$

For example, in the figure shown, potential at A is

$$V_A = -G\left(\frac{m_A}{r_A} + \frac{m_B}{r_B} + \frac{m_C}{r_C}\right)$$



Similarly, potential at B is

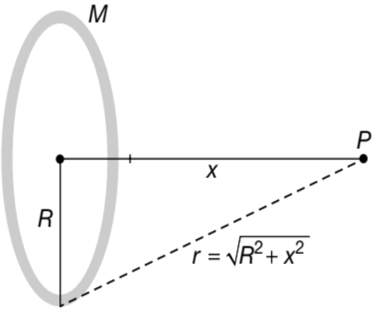
$$V_B = -G\left(\frac{m_A}{r_B} + \frac{m_B}{r_B} + \frac{m_C}{r_C}\right)$$

and potential at C is

$$V_C = -G\left(\frac{m_A}{r_C} + \frac{m_B}{r_C} + \frac{m_C}{r_C}\right)$$

GRAVITATIONAL POTENTIAL AT A POINT *P* ON THE AXIS OF THE RING AT DISTANCE *X* FROM ITS CENTRE

If we wish to find the gravitational potential at a point P lying on the axis of ring, we can directly calculate the result because here too, all points of ring are at same distance $r = \sqrt{R^2 + x^2}$ from the point P.



So, potential at P is

$$V_P = -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 + x^2}}$$

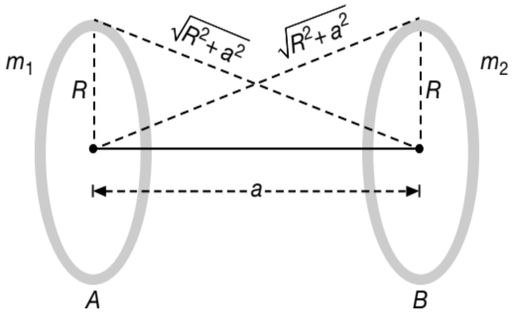
ILLUSTRATION 28

There are two thin wire rings, each of radius R, whose axes coincide. The masses of the rings are m_1 and m_2. Find the gravitational potential difference between the centres of the rings separated by a distance a. Also calculate the work done to move a mass m_0 from centre of first ring to the centre of other ring.

SOLUTION

Net potential at centre of ring A is

$$V_A = \begin{pmatrix} \text{Potential at } A \\ \text{due to itself} \end{pmatrix} + \begin{pmatrix} \text{Potential at } A \\ \text{due to } B \end{pmatrix}$$



$$\Rightarrow \quad V_A = -G\left(\frac{m_1}{R} + \frac{m_2}{\sqrt{R^2 + a^2}}\right)$$

Similarly, net potential at centre of ring B is

$$V_B = \begin{pmatrix} \text{Potential at } B \\ \text{due to itself} \end{pmatrix} + \begin{pmatrix} \text{Potential at } B \\ \text{due to } A \end{pmatrix}$$

$$\Rightarrow \quad V_B = -G\left(\frac{m_2}{R} + \frac{m_1}{\sqrt{R^2 + a^2}}\right)$$

Thus, potential difference,

$$\Delta V = V_B - V_A$$

$$\Rightarrow \quad \Delta V = G\left(-\frac{m_2}{R} - \frac{m_1}{\sqrt{R^2 + a^2}} + \frac{m_1}{R} + \frac{m_2}{\sqrt{R^2 + a^2}}\right)$$

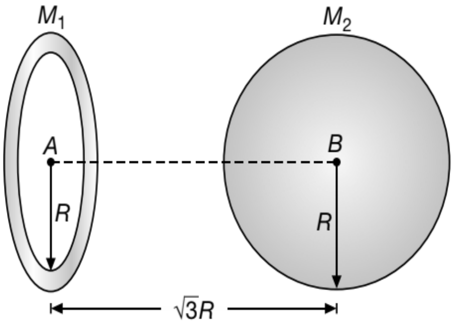
$$\Rightarrow \quad \Delta V = G(m_1 - m_2)\left(\frac{1}{R} - \frac{1}{\sqrt{R^2 + a^2}}\right)$$

Since, $W_{A \to B} = m_0(V_B - V_A)$

$$\Rightarrow \quad W_{A \to B} = Gm_0(m_1 - m_2)\left(\frac{1}{R} - \frac{1}{\sqrt{R^2 + a^2}}\right)$$

ILLUSTRATION 29

Figure shows a ring of mass M_1 and a sphere of mass M_2 separated by a distance $\sqrt{3}R$. A small object of mass m is displaced from A to B. Find the work done by gravitational forces.



SOLUTION

In this case as shifting is from A to B so, work done by the gravitational forces is given by

$$W_g = m(V_A - V_B)$$

where V_A and V_B are gravitational potentials at points A and B respectively, given by

$$V_A = -\frac{GM_1}{R} - \frac{GM_2}{2R}$$

and $$V_B = -\frac{3GM_2}{2R} - \frac{GM_1}{R}$$

So, work done by gravitational force is given by

$$W_g = m\left[\left(-\frac{GM_1}{R} - \frac{GM_2}{2R}\right) - \left(\frac{3}{2}\frac{GM_2}{R} - \frac{GM_1}{2R}\right)\right]$$

$$\Rightarrow \quad W_g = \frac{GM_2}{R} - \frac{GM_1}{2R}$$

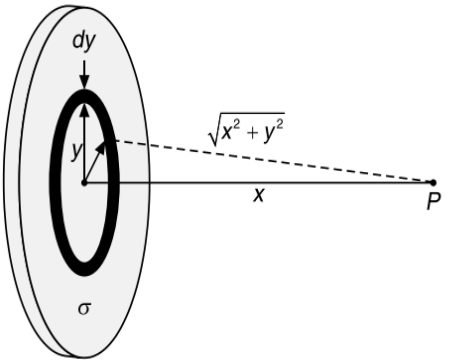
$$\Rightarrow \quad W_g = \frac{GM}{2R}(2M_2 - M_1)$$

GRAVITATIONAL POTENTIAL DUE TO A UNIFORM DISC AT A POINT P ON ITS AXIS

Consider a uniform disc of radius R with surface mass density σ. We wish to find gravitational potential at point P lying on its axis at distance x from the centre. For this we consider an infinitesimal elemental ring of radius y and thickness dy concentric with the disc. Then mass on this infinitesimal element is

$$dm = \sigma \text{ (Area of element)}$$

$$\Rightarrow \quad dm = \sigma(2\pi y dy)$$



The gravitational potential at point P due to this infinitesimal element is

$$dV = -\frac{Gdm}{r} = -\frac{Gdm}{\sqrt{x^2+y^2}}$$

(Please note here that once the point P is taken, then x becomes a fixed value)

$$\Rightarrow \quad dV = -\frac{G\sigma(2\pi y dy)}{\sqrt{x^2+y^2}}, \text{ where } \sigma = \frac{M}{\pi R^2}$$

Net gravitational potential at point P due to entire disc is calculated by integrating this expression.

$$\Rightarrow \quad V = \int dV = -2\pi\sigma G \int_0^R \frac{y dy}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \quad V = -2\pi\sigma G \int_0^R \frac{y dy}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \quad V = -2\pi\sigma G \int_0^R \left(x^2+y^2\right)^{-\frac{1}{2}} 2y dy \qquad \ldots(1)$$

Again using $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1$

$$\Rightarrow \quad V = -2\pi\sigma G \left(\left. \frac{\left(x^2+y^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right|_0^R \right)$$

$$\Rightarrow \quad V = -2\pi\sigma G \left(\left. \sqrt{x^2+y^2} \right|_0^R \right)$$

$$\Rightarrow \quad V_P = -2\pi\sigma G \left(\sqrt{R^2+x^2} - x \right)$$

Since $\sigma = \frac{M}{\pi R^2}$

$$\Rightarrow \quad V_P = -\frac{2GM}{R^2} \left(\sqrt{R^2+x^2} - x \right)$$

Also, **at the centre of the disc**, we have $x = 0$, so

$$V_{\text{centre}} = V_C = -2\pi\sigma G R$$

$$\Rightarrow \quad V_{\text{centre}} = V_C = -2\pi \left(\frac{M}{\pi R^2} \right) G R$$

$$\Rightarrow \quad V_{\text{centre}} = V_C = -\frac{2GM}{R}$$

If we are asked to find the potential due to an annular disc of inner radius R_1 and outer radius R_2, then we shall be integrating equation (1), within the new limits as

$$V = -2\pi\sigma G \int_{R_1}^{R_2} \left(x^2+y^2\right)^{-\frac{1}{2}} 2y dy, \text{ where}$$

$$\sigma = \frac{M}{\pi\left(R_2^2 - R_1^2\right)}$$

$$\Rightarrow \quad V = -2\pi\sigma G \left(\left. \frac{\left(x^2+y^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right|_{R_1}^{R_2} \right)$$

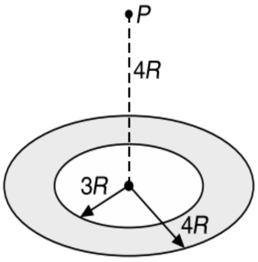
$$\Rightarrow \quad V = -2\pi\sigma G \left(\left. \sqrt{x^2+y^2} \right|_{R_1}^{R_2} \right)$$

$$\Rightarrow \quad V_P = -2\pi\sigma G \left(\sqrt{R_2^2+x^2} - \sqrt{R_1^2+x^2} \right)$$

$$\Rightarrow \quad V_P = -\frac{2GM}{\left(R_2^2 - R_1^2\right)} \left(\sqrt{R_2^2+x^2} - \sqrt{R_1^2+x^2} \right)$$

ILLUSTRATION 30

A thin uniform annular disc (shown in figure) of mass M has outer radius $4R$ and inner radius $3R$. Calculate the work required to take a unit mass from point P lying on its axis to infinity.



SOLUTION

For annular disc, gravitational potential at the point P lying on the axis at a distance x from centre is

$$V_P = -\frac{2GM}{\left(R_2^2 - R_1^2\right)}\left(\sqrt{R_2^2 + x^2} - \sqrt{R_1^2 + x^2}\right)$$

where $R_2 = 4R$, $R_1 = 3R$ and $x = 4R$

$$\Rightarrow \quad V_P = -\frac{2GM}{7R^2}\left(4\sqrt{2}R - 5R\right)$$

Also, $V_\infty = 0$

Since $W_{P\to\infty} = m_0\left(V_\infty - V_P\right)$, where $m_0 = 1$ unit

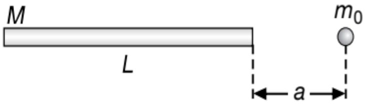
$$\Rightarrow \quad W_{P\to\infty} = \frac{2GM}{7R}\left(4\sqrt{2} - 5\right)$$

Test Your Concepts-II

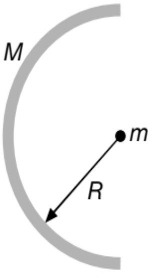
Based on Gravitational Potential, Potential Energy and Applications

(Solutions on page H.257)

1. Find the gravitational potential energy of a system of eight identical masses each of mass m placed at the corners of a cube of side a.
2. Calculate the gravitational potential energy of a point mass m_0 and a uniform thin rod of mass M, length L. Assume that m_0 is at a distance a from one end of the rod on the extended portion of the rod, as shown in figure.



3. A particle of mass m is placed at the centre of a fixed, uniform semi-circular ring of radius R and mass M as shown in figure. Calculate the work required to displace the particle from centre of the ring to infinity.



4. A particle of mass m is located at a distance r from the centre of a sphere of radius $R(<r)$ and mass M. Find the gravitational potential energy of the particle-sphere system. What happens when the sphere is replaced by a thin shell?
5. A particle of mass m is transferred from the centre of the base of a uniform solid hemisphere of mass M and radius R to infinity. Find the work performed in the process by the gravitational force exerted on the particle by the hemisphere.
6. Two bodies of masses m and M are placed a distance d apart. Find the gravitational potential at the position where the gravitational field due to them is zero.
7. A uniform spherical shell of mass M and radius R has a particle of mass M placed at its centre. Find the gravitational potential at a distance $\frac{R}{2}$ from the centre.
8. Find the gravitational self energy of a uniform
   - **(a)** thin spherical shell of mass M and radius R.
   - **(b)** solid sphere of mass M and radius R.
9. A thin uniform rod of length 2ℓ has mass per unit length λ. Calculate the gravitational potential as a function of distance a from centre of the rod along the straight line
   - **(a)** perpendicular to the rod and passing through the centre,
   - **(b)** coinciding with the rod's axis (at points lying outside the rod).

RELATION BETWEEN GRAVITATIONAL FIELD AND GRAVITATIONAL POTENTIAL

Since $dV = -\vec{E}_g \cdot d\vec{l}$

In cartesian coordinates, we have

$$\vec{E}_g = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} \text{ and } d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k},$$

so we get

$$dV = -\left(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$\Rightarrow \quad dV = -\left(E_x dx + E_y dy + E_z dz\right)$$

$$\Rightarrow \quad E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y} \text{ and } E_z = -\frac{\partial V}{\partial z}$$

Now, by introducing a new differential quantity called the "del (gradient) operator $(\vec{\nabla})$", we get Gravitational field as the negative gradient of potential. So,

$$\vec{E}_g = -\vec{\nabla}V \qquad \ldots(1)$$

where $\vec{\nabla} = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$

$$\Rightarrow \quad \vec{E}_g = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

where

$$\frac{\partial V}{\partial x} = \left(\begin{array}{c}\text{Partial Derivative of } V \text{ w.r.t. } x \quad \text{OR} \\ \text{Derivative of } V \text{ w.r.t. } x \text{ keeping} \\ y \text{ and } z \text{ constant}\end{array}\right)$$

$$\frac{\partial V}{\partial y} = \left(\begin{array}{c}\text{Partial Derivative of } V \text{ w.r.t. } y \quad \text{OR} \\ \text{Derivative of } V \text{ w.r.t. } y \text{ keeping} \\ x \text{ and } z \text{ constant}\end{array}\right)$$

$$\frac{\partial V}{\partial z} = \left(\begin{array}{c}\text{Partial Derivative of } V \text{ w.r.t. } z \quad \text{OR} \\ \text{Derivative of } V \text{ w.r.t. } z \text{ keeping} \\ x \text{ and } y \text{ constant}\end{array}\right)$$

Just to make you understand the concept discussed here, I have a small problem discussed here for your comfort.

PROBLEM: **Find the gravitational field $\vec{E}_g$ for the potential $V = kxy$.**

SOLUTION: Since

$$\vec{E}_g = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right),$$

where

$$\frac{\partial V}{\partial x} = ky\left(\frac{\partial x}{\partial x}\right) = ky,$$

$$\frac{\partial V}{\partial y} = kx\left(\frac{\partial y}{\partial y}\right) = kx \text{ and}$$

$$\frac{\partial V}{\partial z} = 0$$

$$\Rightarrow \quad \vec{E} = -k\left(y\hat{i} + x\hat{j}\right)$$

Notice that $\vec{\nabla}$ operates on a scalar quantity (such as gravitational potential) and results in a vector quantity (such as gravitational field). Mathematically, we can think of $\vec{E}_g$ as the negative of the gradient of the gravitational potential V. Physically, the negative sign implies that if $\vec{E}_g$ goes from higher V to lower V.

ILLUSTRATION 31

Suppose that the gravitational potential in some region of space is given by

$$V(x,\ y,\ z) = V_0\exp(-az)\cos ax,$$

where a is a positive constant. Find the gravitational field everywhere.

SOLUTION

$$\vec{E}_g = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$E_x = -\frac{\partial V}{\partial x} = -V_0\exp(-az)\frac{\partial}{\partial x}\left[\cos(ax)\right]$$

$$\Rightarrow \quad E_x = \frac{\partial V}{\partial x} = +aV_0\sin(ax)\exp(-az)$$

Since the given expression has no dependence on y, so

$$E_y = \frac{\partial V}{\partial y} = 0$$

$$E_z = -\frac{\partial V}{\partial z} = -V_0\cos(ax)\frac{\partial}{\partial z}\left[\exp(-az)\right]$$

$$\Rightarrow \quad E_z = -\frac{\partial V}{\partial z} = aV_0\cos(ax)\exp(-az)$$

$$\Rightarrow \quad \vec{E}_g = E_x\hat{i} + E_z\hat{k} \qquad \{\because E_y = 0\}$$

$$\Rightarrow \quad \vec{E}_g = aV_0\exp(-az)\left[\sin(ax)\hat{i} + \cos(ax)\hat{k}\right]$$

Problem Solving Technique(s)

If we are given $\vec{E}_g(x, y, z) = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$, then

$$V = -\int \vec{E}_g \cdot d\vec{\ell}$$

where, $d\vec{\ell} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\Rightarrow \quad V = -\left(\int E_x dx + \int E_y dy + \int E_z dx\right)$$

The integrals are to be calculated within specified limits.

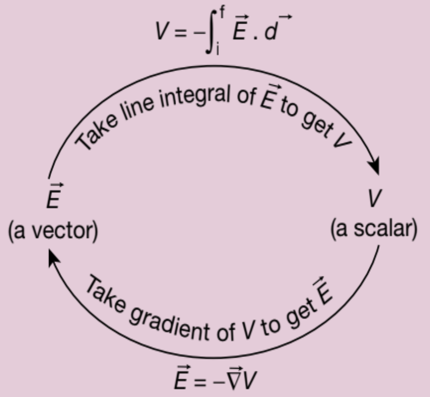


ILLUSTRATION 32

A gravitational field is given by $\vec{E}_g = -k\left(y\hat{i} + x\hat{j}\right)$. Find the gravitational potential generating such a field.

SOLUTION

Since $V = -\int \vec{E}_g \cdot d\vec{l}$

Let $\quad d\vec{l} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\Rightarrow \quad V = \int k\left(y\hat{i} + x\hat{j}\right) \cdot \left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$\Rightarrow \quad V = k\int y\,dx + x\,dy$$

$$\Rightarrow \quad V = k\int d(xy) \qquad \{\because d(xy) = x\,dy + y\,dx\}$$

$$\Rightarrow \quad V = k(xy) + \text{constant}$$

ILLUSTRATION 33

Find the potential function $V(x, y)$ of a gravitational field $\vec{E}_g = 2axy\,\hat{i} + a\left(x^2 - y^2\right)\hat{j}$ where a is a constant.

SOLUTION

Let V_0 be the potential at origin and V be the potential at the point (x, y), then we have

$$dV = -\vec{E}_g \cdot d\vec{\ell}$$

$$\Rightarrow \quad dV = -\left(2axydx + a\left(x^2 - y^2\right)dy\right)$$

$$\Rightarrow \quad \int_{(0,0)}^{(x,y)} dV = -\int_{(0,0)}^{(x,y)} 2axydx + a\left(x^2 - y^2\right)dy$$

$$\Rightarrow \quad V - V_0 = -\int_{(0,0)}^{(x,y)} \left(2axy\,dx + ax^2\,dy\right) - ay^2dy$$

$$\Rightarrow \quad V - V_0 = -\int_{(0,\,0)}^{(x,\,y)} \left\{d\left(ax^2y\right) - ay^2dy\right\}$$

$$\Rightarrow \quad V - V_0 = \left[\left(-ax^2y + \frac{ay^3}{3}\right)\Bigg|_{(0,0)}^{(x,y)}\right]$$

$$\Rightarrow \quad V = V_0 - ax^2y + \frac{ay^3}{3}$$

Remark(s)

(a) For an attractive system U is always **NEGATIVE**.
(b) For a repulsive system U is always **POSITIVE**.
(c) For a stable system U must be **MINIMUM**

i.e., $\dfrac{dU}{dx} = 0$

Since $F = -\dfrac{dU}{dx}$

$\Rightarrow \quad F = -\dfrac{dU}{dx} = 0 \qquad$ (FOR A STABLE SYSTEM)

(d) If two points A and B are at potentials V_A and V_B, then work done in taking a test mass m_0 from A to B is $W_{A\to B} = m_0\left(V_B - V_A\right)$

(e) Consider two points A and B situated in a uniform gravitational field at a distance d such that the line joining A and B is parallel to the field. If V be the potential difference between them, then

$V = Ed$ (in magnitude)

Test Your Concepts-III

Based on Relation Between Gravitational Field and Potential

(Solutions on page H.259)

1. In a region of space the gravitational field is given by $\vec{E}=\left(x\hat{i}-2y\hat{j}+z\hat{k}\right)\,\text{Nkg}^{-1}$. Calculate the potential difference V_{AB} between $A(2,1,0)\,\text{m}$ and $B(0,2,4)\,\text{m}$.
2. Determine the potential $V(x,y,z)$ of a gravitational field $\vec{E}_g=ay\hat{i}+(ax+bz)\hat{j}+by\hat{k}$, where a and b are constants, $\hat{i},\hat{j},\hat{k}$ are the unit vectors of the axis x, y, z.
3. Find potential difference V_{AB} between $A(0,0,0)\,\text{m}$ and $B(1,1,1)\,\text{m}$ in a gravitational field given by
   (a) $\vec{E}=y\hat{i}+x\hat{j}$
   (b) $\vec{E}=3x^2y\hat{i}+x^3\hat{j}$.
   What can you say about the nature of the fields? Explain.
4. Find the gravitational field due to the gravitational potential given by $V(x,y,z)=3x^2y+y^3z$.
5. Determine the gravitational field strength vector if the potential of this field depends on x, y coordinates as
   (a) $V=a\left(x^2-y^2\right)$
   (b) $V=axy$
   where a is a positive constant.
6. Over a certain region of space, the gravitational potential is $V=5x-3x^2y+2yz^2$. Find the expressions for the x, y and z components of the gravitational field over this region. Also calculate the magnitude of the field at the point P that has coordinates $(1,0,-2)\,\text{m}$?
7. In a region of space, the gravitational potential is given by, $V=20(x+y)\,\text{Jkg}^{-1}$. Find the magnitude of the gravitational force on a particle of mass 0.5 kg placed at the origin.
8. The potential of a certain gravitational field has the form $V=a\left(x^2+y^2\right)+bz^2$, where a and b are constants. Find the magnitude and direction of the gravitational field strength vector.
9. In a region of space, the gravitational potential is represented by $V=2x+3y-z$. Obtain an expression for the gravitational field strength.

CONSERVATION LAWS FOR GRAVITATIONAL SYSTEMS

For a variety of problems, we have to understand the Conservation Laws that can be applied to the gravitational system of particles.

LAW I: Law of conservation of Linear Momentum

For a gravitational system of masses, when no external force acts on the system, then we observe that no net internal forces will be acting on the system because gravitational forces always form an action reaction pair. So, by a suitable selection of a system or subsystem we can apply Law of Conservation of Linear Momentum.

$$\begin{pmatrix}\text{Total Initial}\\ \text{Momentum}\end{pmatrix}=\begin{pmatrix}\text{Total Final}\\ \text{Momentum}\end{pmatrix}$$

LAW II: Law of Conservation of Mechanical Energy

Since gravitational forces are conservative in nature, so, in the absence of an external force and a dissipative force (such as forces of friction) we can easily use this law as

$$\begin{pmatrix}\text{Total Initial Energy}\\ \text{of the system}\end{pmatrix}=\begin{pmatrix}\text{Total Final Energy}\\ \text{of the system}\end{pmatrix}$$

Here we need to read the problem carefully for the energies possessed by the system (as selected) initially and finally.

Another point which should be kept in mind is that gravitational field is a conservative field. Work done by gravitational field on a moving mass only depends on the positions of the mass. If a mass is moving freely in the gravitational field, work done by gravitational field on mass is equal to the change in kinetic energy of the mass i.e.,

$$W=\Delta K$$

In the presence of an external force and the dissipative non- conservative force, the law will be suitably modified as below i.e.,

$$W_{\text{ext}}+W_{nc}=\Delta U+\Delta K$$

LAW III: Law of Conservation of Angular Momentum

Since gravitational forces are central in nature, so torque due to gravitational force will be zero. So, we can easily use Law of Conservation of Angular Momentum too.

ILLUSTRATION 34

Two masses m_1 and m_2 at an infinite distance from each other are initially at rest, start interacting gravitationally. Find their velocity of approach when they are at a distance r apart. Assume air drag and other dissipative forces to be absent.

SOLUTION

Since both the particles, initially at rest and lying at infinity, start interacting gravitationally, i.e. the gravitational force between them is an internal force and also no external force is acting on them, so according to Law of Conservation of Linear Momentum we have

$$\begin{pmatrix}\text{Total Initial}\\ \text{Momentum}\end{pmatrix}_{\text{at }\infty}=\begin{pmatrix}\text{Total Final}\\ \text{Momentum}\end{pmatrix}_{\text{at }r}$$

$$\Rightarrow \quad 0+0=m_1v_1+m_2(-v_2)$$

$$\Rightarrow \quad m_1v_1=m_2v_2 \qquad \text{...(1)}$$

Further, applying the Law of Conservation of Energy, we get

$$(U+K)_{at\ \infty}=(U+K)_{at\ r}$$

$$\Rightarrow \quad 0+0=-\frac{Gm_1m_2}{r}+\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2 \qquad \text{...(2)}$$

From equation (1), we have $v_2=\dfrac{m_1v_1}{m_2}$. So, from (2), we get

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2\left(\frac{m_1v_1}{m_2}\right)^2=\frac{Gm_1m_2}{r}$$

$$\Rightarrow \quad \frac{1}{2}m_1v_1^2\left(1+\frac{m_1}{m_2}\right)=\frac{Gm_1m_2}{r}$$

$$\Rightarrow \quad v_1=m_2\sqrt{\frac{2G}{(m_1+m_2)r}}$$

Similarly using the above equations, we get

$$v_2=m_1\sqrt{\frac{2G}{(m_1+m_2)r}}$$

If v_r be their velocity of approach, then we have

$$v_r=v_1+v_2$$

$$\Rightarrow \quad v_r=m_2\sqrt{\frac{2G}{(m_1+m_2)r}}+m_1\sqrt{\frac{2G}{(m_1+m_2)r}}$$

$$\Rightarrow \quad v_r=(m_1+m_2)\sqrt{\frac{2G}{(m_1+m_2)r}}$$

$$\Rightarrow \quad v_r=\sqrt{\frac{2G(m_1+m_2)}{r}}$$

Problem Solving Technique(s)

This problem can also be solved quickly by using the concept of reduced mass $\mu=\dfrac{m_1m_2}{m_1+m_2}$ and the Law of Conservation of Energy according to which, we have

$$(U+K)_{\text{at }\infty}=(U+K)_{\text{at }r}$$

$$0+0=\frac{1}{2}\mu v_r^2-\frac{Gm_1m_2}{r} \qquad \text{...(1)}$$

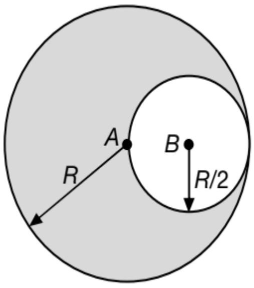
where, $\mu=\text{Reduced Mass}=\dfrac{m_1m_2}{m_1+m_2}$

Substituting in equation (1), we get

$$v_r=\sqrt{\frac{2G(m_1+m_2)}{r}}$$

ILLUSTRATION 35

Inside a fixed sphere of radius R and uniform density ρ, there is spherical cavity of radius $\dfrac{R}{2}$ such that surface of the cavity passes through the centre of the sphere as shown in figure. A particle of mass m is released from rest at centre B of the cavity. Calculate velocity with which particle strikes the centre A of the sphere. Neglect earth's gravity. Initially sphere and particle are at rest.



SOLUTION

Applying conservation of mechanical energy, we have

$$(U+K)_{\text{at }B} = (U+K)_{\text{at }A}$$

$$\Rightarrow \quad U_B + K_B = U_A + K_A$$

$$\Rightarrow \quad U_B + 0 = U_A + \frac{1}{2}mv_A^2$$

where $U_B = mV_B$ and $U_A = mV_A$, so we have

$$\frac{1}{2}mv_A^2 = U_B - U_A = m(V_B - V_A)$$

$$\Rightarrow \quad v_A = \sqrt{2(V_B - V_A)} \qquad \ldots(1)$$

Potential at A

$$V_A = \begin{pmatrix}\text{Potential Due}\\ \text{to the Complete}\\ \text{Sphere at A}\end{pmatrix} - \begin{pmatrix}\text{Potential Due}\\ \text{to the Spherical}\\ \text{Cavity at A}\end{pmatrix}$$

$$V_A = -\frac{3}{2}\frac{GM}{R} - \left(-\frac{GM'}{r}\right) = \frac{GM'}{r} - \frac{3}{2}\frac{GM}{R}$$

where

$$M = \frac{4}{3}\pi R^3\rho,\ r = \frac{R}{2} \text{ and } M' = \frac{4}{3}\pi r^3\rho = \frac{\pi\rho R^3}{6}$$

Substituting the values, we get

$$V_A = \frac{G}{R}\left(\frac{\pi\rho R^3}{3} - 2\pi\rho R^3\right) = -\frac{5}{3}\pi G\rho R^2$$

Potential at B

$$V_B = \begin{pmatrix}\text{Potential Due}\\ \text{to the Complete}\\ \text{Sphere at } B\end{pmatrix} - \begin{pmatrix}\text{Potential Due}\\ \text{to the Spherical}\\ \text{Cavity at } B\end{pmatrix}$$

$$\Rightarrow \quad V_B = -\frac{GM}{2R^3}(3R^2 - r^2) - \left(-\frac{3}{2}\frac{GM'}{r}\right)$$

where

$$M = \frac{4}{3}\pi R^3\rho,\ r = \frac{R}{2} \text{ and } M' = \frac{4}{3}\pi r^3\rho = \frac{\pi\rho R^3}{6}$$

$$\Rightarrow \quad V_B = -\frac{11}{8}\frac{GM}{R} + \frac{3GM'}{R}$$

$$\Rightarrow \quad V_B = \frac{G}{R}\left(\frac{\pi\rho R^3}{2} - \frac{11\pi\rho R^3}{6}\right) = -\frac{4}{3}\pi G\rho R^2$$

$$\Rightarrow \quad V_B - V_A = \frac{1}{3}\pi G\rho R^2$$

So, from equation (1)

$$v = \sqrt{\frac{2}{3}\pi G\rho R^2}$$

LAUNCHING SPEED OF A PROJECTILE

Let us suppose we have to launch a projectile having mass m to reach a height h. The gravitational potential energy of the projectile on the surface of Earth is

$$U(R) = -\frac{GMm}{R} \qquad \ldots(1)$$

Its gravitational potential energy at height h from the surface of earth is

$$U(R+h) = -\frac{GMm}{(R+h)} \qquad \ldots(2)$$

Therefore, the change in potential energy is

$$\Delta U = U(R+h) - U(R) = -GMm\left\{\frac{1}{R+h} - \frac{1}{R}\right\}$$

$$\Rightarrow \quad \Delta U = \frac{GM}{R^2}\frac{mh}{\left(1+\frac{h}{R}\right)} = \frac{mgh}{\left(1+\frac{h}{R}\right)}$$

If $h \quad R$, $\Delta U = mgh$, which we have already used in the **Chapter Work, Energy and Power.**

This difference in the potential energy is fulfilled by providing an initial kinetic energy. If v be the velocity then

$$\frac{1}{2}mv^2 = \frac{mgh}{\left(1+\frac{h}{R}\right)}$$

$$\Rightarrow \quad v = \sqrt{\frac{2gh}{\left(1+\frac{h}{R}\right)}}$$

MAXIMUM HEIGHT ATTAINED BY A PROJECTILE

Let a projectile of mass m be projected vertically upward with velocity v, so that it attains a maximum height h. At maximum height h, the velocity of particle is zero, so kinetic energy is zero. When a body is thrown vertically upwards with a velocity v then by Law of Conservation of Energy,

$$(U+K)_{\text{at surface}} = (U+K)_{\text{at height }h}$$

$$\Rightarrow \quad -\frac{GMm}{R}+\frac{1}{2}mv^2=-\frac{GMm}{R+h}+0$$

$$\Rightarrow \quad \frac{v^2}{2}=GM\left(\frac{1}{R}-\frac{1}{R+h}\right)$$

$$\Rightarrow \quad \frac{v^2}{2}=\frac{GMh}{R(R+h)}$$

$$\Rightarrow \quad \frac{2GM}{v^2R}=\frac{R+h}{h}$$

$$\Rightarrow \quad \frac{2GM}{v^2R}=1+\frac{R}{h}$$

Since, $v_e=\sqrt{2gR}=\sqrt{\frac{2GM}{R}}$

$$\Rightarrow \quad h=\frac{R}{\left(\frac{2GM}{v^2R}-1\right)}=\frac{Rv^2}{2gR-v^2}$$

$$\Rightarrow \quad h=R\left(\frac{v^2}{v_e^2-v^2}\right)$$

From this we can see that

(i) if $v=v_e$ or $v^2=v_e^2=2gR$, $h\to\infty$ and if we have

(ii) v to be small, then we have $h=\frac{v^2}{2g}$

IMPORTANT

If $h \ll R$ then

$$v^2=\frac{2GM}{R}\left[1-\left(1+\frac{h}{R}\right)^{-1}\right]$$

$$\Rightarrow \quad v^2=\frac{2GM}{R}\left[1-\left(1-\frac{h}{R}\right)\right]$$

$$\Rightarrow \quad v^2=\frac{2GM}{R}\left(\frac{h}{R}\right)$$

$$\Rightarrow \quad v^2=2\left(\frac{GM}{R^2}\right)h$$

$$\Rightarrow \quad v^2=2gh$$

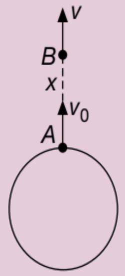
$$\Rightarrow \quad v=\sqrt{2gh}$$

So, the formula $v=\sqrt{2gh}$ only holds good for heights close to surface of the earth.

Problem Solving Technique(s)

Time taken by the particle to reach a height *h*

Suppose a particle of mass m is projected vertically upwards with a speed v_0. We want to find the time taken by the particle to reach a height h. First of all we find the speed of the particle at a height x by applying Law of Conservation of Mechanical Energy, i.e.,



$$\frac{1}{2}m\left(v_0^2-v^2\right)=\frac{mgx}{1+\frac{x}{R}}$$

or $$v=\sqrt{v_0^2-\frac{2gx}{1+\frac{x}{R}}} \quad \text{...(1)}$$

Now, v can be written as $-\frac{dx}{dt}$

Hence, equation (1) reduces to

$$\frac{-dx}{\sqrt{v_0^2-\frac{2gx}{1+\frac{x}{R}}}}=dt$$

By integrating with limits from 0 to h on left hand side and from 0 to t on right hand side we get the desired time.

ILLUSTRATION 36

A particle is projected from the surface of earth with an initial speed of $4\ \text{kms}^{-1}$. Find the maximum height attained by the particle. Radius of earth is $R=6400\ \text{km}$ and $g=9.8\ \text{ms}^{-2}$.

SOLUTION

The maximum height attained by the particle is,

$$h=\frac{v^2}{2g-\frac{v^2}{R}}$$

Substituting the values, we have

$$h = \frac{(4\times10^3)^2}{2\times9.8 - \frac{(4\times10^3)^2}{6.4\times10^6}} = 9.35\times10^5 \text{ m}$$

$$\Rightarrow \quad h \approx 935 \text{ km}$$

ILLUSTRATION 37

A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit to overcome the gravitational pull. Given that, radius of the earth is 6400 km and $g = 9.8 \text{ ms}^{-2}$.

SOLUTION

The speed of the spaceship in a circular orbit close to the earth's surface is given by,

$$v_0 = \sqrt{gR}$$

and escape velocity is given by,

$$v_e = \sqrt{2gR}$$

So, the additional velocity required to escape is given by

$$v_e - v_0 = \sqrt{2gR} - \sqrt{gR}$$

$$\Rightarrow \quad v_e - v_0 = (\sqrt{2}-1)\sqrt{gR}$$

Substituting the values of g and R, we get

$$v_e - v_0 = 3.278\times10^3 \text{ ms}^{-1}$$

ESCAPE SPEED

The minimum velocity to be imparted to a body from the surface of the earth (or planet) such that it just escapes the gravitational pull of the earth **(i.e. reaches infinity and stops there)** is called the escape velocity.

When a body is launched from the surface of the earth with the minimum speed, say **escape speed** v_e then it will reach infinity with zero speed. Then by Law of Conservation of Energy, we have

$$(U+K)_{\text{at surface}} = (U+K)_{\text{at }\infty}$$

$$\Rightarrow \quad -\frac{GmM}{R} + \frac{1}{2}mv_e^2 = 0 + 0$$

$$\Rightarrow \quad v_e = \sqrt{\frac{2GM}{R}}$$

Substituting $M = 6\times10^{24}$ kg,

$G = 6.67\times10^{-11} \text{ Nm}^2kg^{-2}$ and

$R = 6.4\times10^6$ m, we get

$$v_{\text{esc}} = 11.2 \text{ kms}^{-1}$$

Note that the escape speed does not depend on the mass of the body launched from the surface of the earth.

Also, escape velocity is independent of the angle of launch or the angle of projection.

Conceptual Note(s)

(a) If the particle is launched from the surface of the earth with a velocity $u > v_e$, then the body will reach infinity with a non zero velocity v which can be calculated by using the Law of Conservation of Energy, so

$$(U+K)_{\text{at surface}} = (U+K)_{\text{at }\infty}$$

$$\Rightarrow \quad -\frac{GmM}{R} + \frac{1}{2}mu^2 = 0 + \frac{1}{2}mv^2$$

(b) If the particle is launched from the surface of the earth with a velocity $u < v_e$, then the body will attain a maximum height h (say) and then return to the surface of the earth. The maximum height h can be calculated by using the Law of Conservation of Energy, so

$$(U+K)_{\text{at surface}} = (U+K)_{\text{at } h}$$

$$\Rightarrow \quad -\frac{GmM}{R} + \frac{1}{2}mu^2 = -\frac{GmM}{R+h} + 0$$

ILLUSTRATION 38

A rocket is fired vertically with half the escape speed. What is its maximum altitude in terms of the radius of the earth R? Ignore the earth's rotation.

SOLUTION

This Illustration can be solved easily by using the Law of Conservation of Energy, according to which

$$(U+K)_{\text{at surface}} = (U+K)_{\text{at height } h}$$

$$\Rightarrow \quad -\frac{GmM}{R} + \frac{1}{2}m\left(\frac{v_e}{2}\right)^2 = -\frac{GmM}{R+h} + \frac{1}{2}m(0)^2$$

However, we know that $v_e = \sqrt{\dfrac{2GM}{R}}$

$$\Rightarrow \quad -\frac{GmM}{R}+\frac{GmM}{4R}=-\frac{GmM}{R+h}$$

$$\Rightarrow \quad -\frac{3}{4}\frac{GmM}{R}=-\frac{GmM}{R+h}$$

$$\Rightarrow \quad 3R+3h=4R$$

$$\Rightarrow \quad h=\frac{R}{3}$$

Test Your Concepts-IV

Based on Conservation Laws, Escape Velocity and Applications

(Solutions on page H.261)

1. If a body of mass m is projected vertically upward from the surface of the earth (of radius R) so as to reach a height nR above the surface. Then find

(a) the increase in its potential energy.

(b) the velocity with which it must be projected.

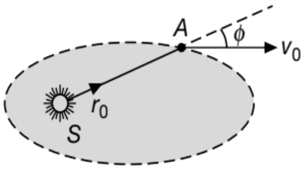
2. If a body is released from a great distance from the centre of the earth, find its velocity when it strikes the surface of the earth. Given, $R = 6400$ km and $g = 9.8\ \text{ms}^{-2}$.

3. Two particles of masses 20 kg and 10 kg are initially at a distance of 1 m. Find the speeds of the particles when the separation between them decreases to 0.5 m, if only gravitational forces are acting.

4. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the mid-point of the line joining their centres, perpendicular to the line joining them. If the universal gravitational constant is denoted by G, then calculate the minimum initial velocity of the mass m so that it escapes the gravitational field of the two bodies.

5. Distance between the centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$ respectively. A body of mass m is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a.

6. Find the maximum and minimum distances of the planet A from the sun S if at a certain moment of time it was at a distance r_0 and travelling with the velocity v_0, with the angle between the radius vector and velocity vector being equal to ϕ.



7. Find the total mechanical energy of a body launched from the surface of earth with a velocity equal to escape velocity.

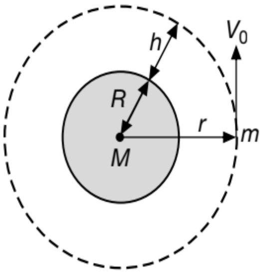
8. A rocket is launched normal to the surface of the Earth, away from the Sun, along the line joining the Sun and the Earth. The sun is 3×10^5 times heavier than the Earth and is at a distance 2.5×10^4 times larger than the radius of Earth. The escape velocity from Earth's gravitational field is $v_e = 11.2\ \text{kms}^{-1}$. Calculate the minimum initial velocity (v_s) required for the rocket to be able to leave the Sun-Earth system. Ignore the rotation and revolution of the Earth and the presence of any other planet.

9. Given a thin homogeneous disc of radius a and mass m_1. A particle of mass m_2 is placed at a distance ℓ from the disk on its axis of symmetry. Initially both are motionless in free space but they ultimately collide because of gravitational attraction. Find the relative velocity at the time of collision. Assume $\ell \gg a$.

10. Two spherical planets 1 and 2 have the same uniform density ρ, masses M_1 and M_2, surface areas A and $4A$ respectively. The escape velocity from the surface of planets 1 and 2 are v_1 and v_2 respectively. The escape velocity from the surface of a spherical planet 3 that has same uniform density and mass equal to the sum of masses of planets 1 and 2 is v_3. Calculate $\dfrac{v_3^3}{v_1^3}$ and $\dfrac{v_2}{v_1}$.

SATELLITES

Consider a satellite of mass m revolving around the earth in circular orbit of radius r.



The gravitational pull between the satellite and earth $\left(=\frac{GMm}{r^2}\right)$ provides the necessary centripetal force to satellite to revolve in the orbit.

$$\Rightarrow \quad \frac{mv_0^2}{r}=\frac{GMm}{r^2}$$

So, orbital speed is given by

$$v_0=\sqrt{\frac{GM}{r}}$$

If h is height of satellite from earth's surface and R is the radius of earth, then

$$r=R+h$$

$$\Rightarrow \quad v_0=\sqrt{\frac{GM}{R+h}}=\sqrt{\frac{R^2g}{R+h}}$$

$$\Rightarrow \quad v_0=R\sqrt{\frac{g}{R+h}}$$

If T is the time period of satellite, then

$$T=\frac{2\pi r}{v_0}=\frac{2\pi r}{\sqrt{\frac{GM}{r}}}=2\pi\sqrt{\frac{r^3}{GM}}$$

$$\Rightarrow \quad T=2\pi\sqrt{\frac{(R+h)^3}{GM}}=2\pi\sqrt{\frac{r^3}{GM}}$$

Problem Solving Technique(s)

(a) The square of the time period is directly proportional to the cube of the distance of separation.

(b) We observe that the time period and the orbital velocity of a satellite of mass m is independent of mass of satellite and depend upon mass of planet (M) and the radius of orbit (r).

(c) If satellite revolves close to the surface of planet, then $r \simeq R$ (= the radius of planet). So

$$T=2\pi\sqrt{\frac{R^3}{G\frac{4}{3}\pi R^3\rho}}$$

$$\Rightarrow \quad T=\sqrt{\frac{3\pi}{G\rho}}$$

i.e., Time period for two satellites revolving about two different planets close to their surface is proportional to $\frac{1}{\sqrt{\rho}}$. Hence

$$\frac{T_1}{T_2}=\sqrt{\frac{\rho_2}{\rho_1}}$$

(d) If a satellite is close to earth's surface such that $h \ll R$, then orbital speed

$$v_0=\sqrt{\frac{GM}{R}}=\sqrt{\frac{R^2g}{R}}=\sqrt{Rg}$$

The periodic time T and the angular velocity ω are given by

$$T=2\pi\sqrt{\frac{R^3}{GM}}=2\pi\sqrt{\frac{R^3}{R^2g}}=2\pi\sqrt{\frac{R}{g}}$$

$$\omega=\sqrt{\frac{GM}{R^3}}$$

For earth $R=6400\text{ km}=6.4\times10^6\text{ m}$ and $g=9.8\text{ ms}^{-2}$

orbital speed, $v_0=\sqrt{6.4\times10^6\times9.8}$

$$v_0=8\times10^3\text{ ms}^{-1}=8\text{ kms}^{-1}$$

and period of revolution i.e.

$$T=2\pi\sqrt{\frac{6.4\times10^6}{9.8}}=5\times10^6\text{ sec}=84.6\text{ min}$$

(e) The bodies are weightless in the satellite because the centrifugal force on body in rotating satellite is balanced by gravitational pull of earth.

ILLUSTRATION 39

A satellite revolving in a circular equatorial orbit of radius $r=2.00\times10^4$ km from west to east appears over a certain point at the equator every 11.6 hr. Using this data, calculate the mass of the earth. The gravitational constant is supposed to be known.

SOLUTION

Here, the absolute angular velocity of satellite is given by

$$\omega = \omega_s + \omega_E$$

where ω_E is the angular velocity of earth, which is also from west to east.

$$\Rightarrow \quad \omega = \frac{2\pi}{t} + \frac{2\pi}{T}$$

where, $t = 11.6$ hr and $T = 24$ hr

Since, $\omega = \sqrt{\frac{GM}{r^3}}$

$$\Rightarrow \quad \sqrt{\frac{GM}{r^3}} = \frac{2\pi}{t} + \frac{2\pi}{T}$$

$$\Rightarrow \quad M = \frac{4\pi^2 r^3}{G}\left(\frac{1}{t} + \frac{1}{T}\right)^2$$

$$\Rightarrow \quad M = \frac{4\pi^2 \left(2\times10^7\right)^3}{\left(6.67\times10^{-11}\right)}\left(\frac{1}{11.6\times3600} + \frac{1}{24\times3600}\right)^2$$

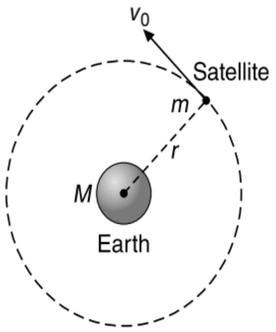
$$\Rightarrow \quad M \approx 6.0\times10^{24}\text{ kg}$$

ILLUSTRATION 40

A satellite of mass m is moving in a circular orbit of radius r. Calculate its angular momentum with respect to the centre of the orbit in terms of the mass of the earth.

SOLUTION

The situation in which a satellite is revolving around the earth is shown in figure.



The angular momentum of the satellite with respect to the centre of orbit is given by

$$L = mv_0 r\sin(90^\circ) = mv_0 r$$

The direction of $\vec{L}$ is perpendicular to the plane of the orbit.

Since the orbital speed v_0 of satellite is given by

$$v_0 = \sqrt{\frac{GM}{r}}$$

$$\Rightarrow \quad L = mv_0 r = m\sqrt{\frac{GM}{r}}r$$

$$\Rightarrow \quad L = \sqrt{GMm^2 r}$$

KINETIC ENERGY OF A SATELLITE

The kinetic energy of a satellite having mass m and orbital speed v_0 is given by

$$K = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} \qquad \left\{\because v_0 = \sqrt{\frac{GM}{r}}\right\}$$

POTENTIAL ENERGY OF A SATELLITE

The potential energy of a satellite of mass m orbiting around a planet of mass M in an orbit of radius r is given by

$$U = -\frac{GMm}{r}$$

TOTAL MECHANICAL ENERGY OF A SATELLITE

Total mechanical energy (E) is

$$E = K + U = -\frac{GMm}{2r}$$

This is negative, indicating that the satellite is bound to the planet.

Problem Solving Technique(s)

$$\text{T.E.} = -\text{K.E.} = \frac{1}{2}(\text{P.E.})$$

$$\Rightarrow \quad \text{P.E.} = -2(\text{K.E.}) = 2(\text{T.E.})$$

BINDING ENERGY OF A SATELLITE

Binding energy is the minimum energy required to make a satellite free from gravitational attraction of the planet so, binding energy (B) is given by

$$B = -E = \frac{GMm}{2r}$$

COMMUNICATION SATELLITE

It is also called Geostationary or Geosynchronous or Parking or Relay or Transmission Satellite. It is an artificial satellite of earth and appears stationary to any observer on the surface of earth. A satellite is said to be Geostationary when it fulfills the following conditions.

(a) The orbit of the satellite must be circular and in the equatorial plane of the earth. (Equatorial plane divides the earth into two equal hemispheres).

(b) The sense of revolution of satellite must be same as the sense of rotation of the earth i.e. from west to east.

(c) The period of revolution of the satellite must be equal to the period of rotation of earth about its axis. Hence,

$$T = 24 \text{ hour} = 86400 \text{ s}$$

Since

$$T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

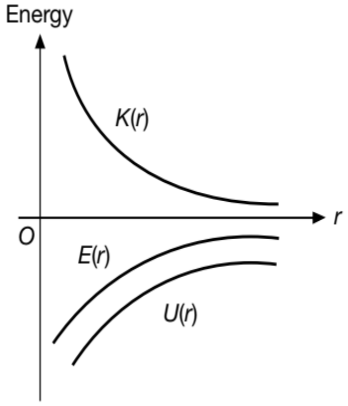
$$\Rightarrow \quad R+h=r \quad 42400 \text{ km}$$

$$\Rightarrow \quad h \quad 36000 \text{ km}$$

(d) A geostationary satellite must revolve at a height of 36000 km above equator or in an orbit of radius 42400 km.

ENERGY GRAPH

The variations of kinetic energy K have been shown by the graph as shown, potential energy U and total energy E with radius for a satellite in a circular orbit. From following graph it is clear that the value of U and E are negative and that of K is positive. As the r increases three curves have the tendency to approach the value of zero.



IMPORTANT

1. Total energy of a closed system is always negative. For example energy of planet-sun, satellite-earth or electron-nucleus system are always negative.
2. Since the Law of Force obeys the inverse square law i.e., $F \propto \frac{1}{r^2}$. Also, $F = \frac{-dU}{dr}$.

$$\Rightarrow \quad K = \frac{|U|}{2} = |E|$$

The same is true for electron-nucleus system because there also, the gravitational force

$$F_e \propto \frac{1}{r^2}$$

ILLUSTRATION 41

A satellite of mass m is moving in a circular orbit of radius r. Due to atmospheric drag if it loses energy at a constant rate W. Find the time in which satellite will fall to the surface of earth. The mass of the earth is M and radius R.

SOLUTION

$$E_i = -\frac{GMm}{2r}, \; E_f = -\frac{GMm}{2R}, \; E_i > E_f$$

$$\Rightarrow \quad \Delta E = E_i - E_f = \frac{GMm}{2}\left(\frac{r-R}{Rr}\right)$$

$$\Rightarrow \quad t = \frac{\Delta E}{W}$$

$$\Rightarrow \quad t = \frac{GMm}{2W}\left(\frac{1}{R} - \frac{1}{r}\right)$$

ILLUSTRATION 42

A rocket with a payload of mass m is at rest at the surface of the earth. Calculate the work needed to raise the payload to the following states:

(a) at rest at an altitude equal to radius of earth (R)

(b) in circular orbit at an altitude R

SOLUTION

In both cases the initial energy of the payload is purely potential energy.

$$E_1 = K_1 + U_1 = 0 - \frac{GmM}{R}$$

(a) At the given altitude the distance to the center of the earth is $r = 2R$.

$$E_2 = K_2 + U_2 = 0 - \frac{GmM}{2R}$$

The work needed is $E_2 - E_1 = +\frac{GmM}{2R}$

(b) Since, the total energy in orbit is

$$E_3 = K_3 + U_3 = -\frac{GmM}{2(2R)} = -\frac{GmM}{4R}$$

The work needed is $E_3 - E_1 = +\frac{3GmM}{4R}$.

Naturally, it takes more work to put the satellite into orbit than merely to raise it to the same altitude.

ILLUSTRATION 43

A sky lab of mass 2×10^3 kg is first launched from the surface of earth in a circular orbit of the radius $2R$ (from the centre of earth) and then it is shifted from this circular orbit to another circular orbit of radius $3R$. Calculate the minimum energy required

(a) to place the lab in the first orbit

(b) to shift the lab from first orbit to the second orbit, Given $R = 6400$ km and $g = 10\ \text{ms}^{-2}$.

SOLUTION

According to the problem sky lab exists in three energy levels, our task is to calculate the total energy of the three level. i.e. on the surface, first orbit and second orbit. Energy difference between first orbit and surface of the earth is the answer of (a) and that between first orbit and second orbit is the answer of (b).

Total mechanical energy of the sky lab on the surface of earth

$$E_1 = KE + PE = 0 + -\frac{GMm}{R} = -\frac{GMm}{R} \qquad \ldots(1)$$

Total mechanical energy of the sky lab in first orbit i.e.

$$E_2 = -\frac{GMm}{4R} \qquad \ldots(2)$$

Total mechanical energy of the skylab in the second orbit i.e.

$$E_3 = -\frac{GMm}{6R}$$

(a) Required energy

$$\Delta E_1 = -\frac{GMm}{4R} + \frac{GMm}{R} = \frac{3}{4}\frac{GMm}{R}$$

$$\Rightarrow \quad \Delta E_1 = \frac{3}{4}mgR = 9.6 \times 10^{10}\ \text{J}$$

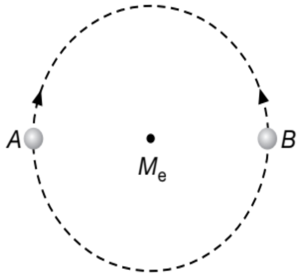
(b) In this case required energy

$$\Delta E_2 = -\frac{GMm}{6R} + \frac{GMm}{4R} = \frac{1}{12}\frac{GMm}{R}$$

$$\Rightarrow \quad \Delta E_2 = \frac{1}{12}mgR = 1.1 \times 10^{10}\ \text{J}$$

ILLUSTRATION 44

Consider two satellite A and B of equal mass moving in the same circular orbit of radius r around the earth but in the opposite sense and therefore, on a collision course as shown in the figure.



(a) Find the total mechanical energy $E_A + E_B$ of the two satellite plus-earth system before collision.

(b) If the collision is completely inelastic find the total mechanical energy immediately after collision. Describe the subsequent motion of the combined satellite.

SOLUTION

(a) $E_A = -\frac{1}{2}\frac{GM_e m}{r}$; $E_B = -\frac{1}{2}\frac{GM_e m}{r}$

$$E_A + E_B = -\frac{GM_e m}{r}$$

(b) Applying Law of Conservation of Momentum along the axis perpendicular to the gravitational force i.e., along the tangent, we get

$$mv + (-mv) = 2mv'$$

$$\Rightarrow \quad v' = 0$$

Energy of the system = Gravitational energy

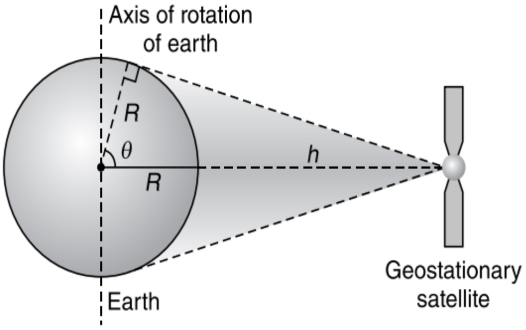
$$E = -\frac{2GM_e m}{r}$$

Since velocity = 0, satellite will fall vertically down.

BROADCASTING REGION OF A SATELLITE

If we know the height of a geostationary satellite, then we can easily find the area of earth exposed to the satellite or area of the region in which the communication can be made using this satellite. Figure shows earth (of radius R) and its area exposed to a geostationary satellite. The angle θ is given by

$$\theta = \cos^{-1}\left(\frac{R}{R+h}\right)$$



The solid angle Ω which the exposed area subtends at the earth's centre is given by

$$\Omega = 2\pi(1-\cos\theta)$$

$$\Rightarrow \quad \Omega = 2\pi\left(1-\frac{R}{R+h}\right) = \frac{2\pi h}{R+h}$$

Since the solid angle Ω subtended at the centre of earth is

$$\Omega = \frac{\text{Area exposed to the satellite}}{R^2} = \frac{A}{R^2}$$

Thus, the area of earth's surface exposed to the geostationary satellite is

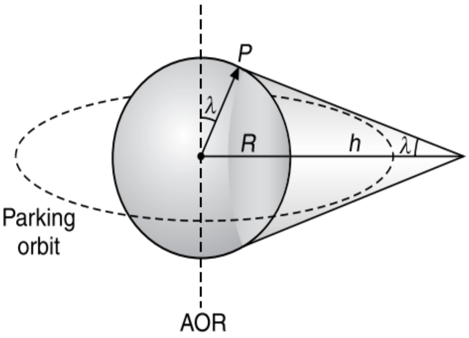
$$A = \Omega R^2 = \frac{2\pi h R^2}{R+h}$$

ILLUSTRATION 45

Find the minimum colatitude angle which can directly receive a signed from a geostationary satellite. Assume that the radius of the geostationary satellite is seven times the radius of earth.

SOLUTION

The farthest point on earth, which can receive signals from the parking orbit is the point where a length is drawn on earth surface from satellite as shown in figure.



The colatitude angle λ of a point P is given by

$$\sin\lambda = \frac{R}{R+h}$$

For a parking orbit, we have $R+h=7R$

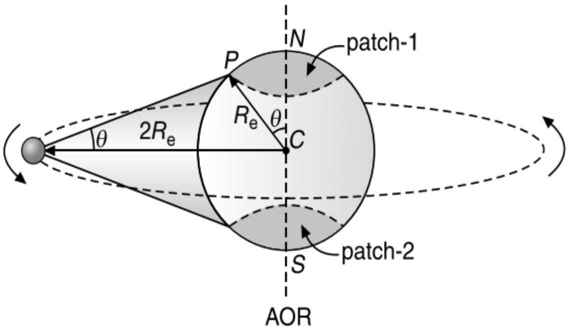
$$\Rightarrow \quad \lambda = \sin^{-1}\left(\frac{1}{7}\right)$$

ILLUSTRATION 46

A satellite is orbiting around the earth in an orbit in equatorial plane of radius $2R_e$ where R_e is the radius of earth. Find the area on earth, this satellite covers for communication purpose in its complete revolution.

SOLUTION

When satellite S revolves around the earth, it covers a complete circular belt on earth's surface for communication.



If the colatitude of the farthest point on surface upto which signals can be received (point P) is θ then we have

$$\sin\theta = \frac{R_e}{2R_e} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

During revolution satellite leaves two spherical patches 1 and 2 on earth's surface at north and south poles where no signals can be transmitted due to the

curvature of earth. The areas of these patches can be obtained by using the concept of solid angle.
The solid angle Ω subtended by a patch on earth's centre is

$$\Omega = 2\pi(1-\cos\theta) = \pi(2-\sqrt{3})$$

So, area of each patch is

$$A_P = \Omega R_e^2 = \pi(2-\sqrt{3})R_e^2$$

Hence total area on earth's surface to which communication can be made is

$$A_C = 4\pi R_e^2 - 2A_P$$

$$\Rightarrow \quad A_C = 4\pi R_e^2 - 2\pi(2-\sqrt{3})R_e^2$$

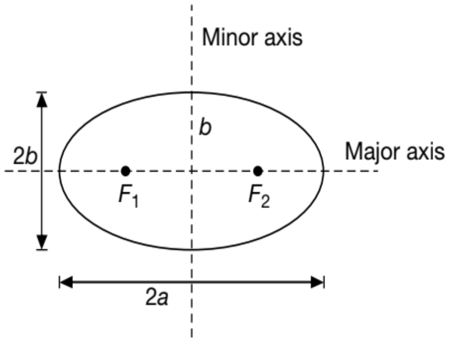
$$\Rightarrow \quad A_C = 2\pi R_e^2(2-2+\sqrt{3})$$

$$\Rightarrow \quad A_C = 2\sqrt{3}R_e^2$$

KEPLER'S LAWS

First Law

The planets move around the sun in elliptical orbits with the sun at one focus.
An ellipse with two foci F_1 and F_2 is shown in the figure.

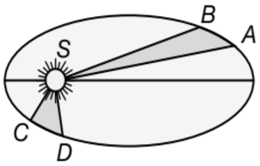


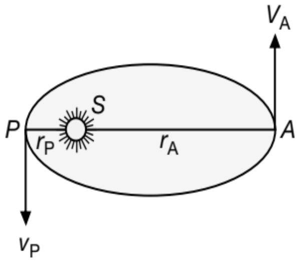
A circle is a special case in which the foci coincide at the center. The short dimension is called the minor axis and has length $2b$; the long dimension is called the major axis and has length $2a$. The closest point P, in the orbit to the sun is called the perihelion; the farthest point A, is called the aphelion.

Second Law

The line joining the sun to a planet sweeps out equal areas in equal time.

Suppose that in a given time interval a planet moves from A to B as shown in the figure, and from C to D during another time interval. According to the second law the areas SAB and SCD are equal. The speed of the planet, therefore, must vary during its orbit. It is greatest at the perihelion, and least at the aphelion. This law is a consequence of the Law of Conservation of Angular Momentum.





Third Law

The square of the period of planet is proportional to the cube of its mean distance from the sun.
The mean distance turns out to be the semi-major axis, a.
Mathematically, $T^2 \propto a^3$ or $T^2 = \kappa a^3$
where κ is a constant that applies to all planets.

ILLUSTRATION 47

According to Kepler's second law of planetary motion, the line joining the sun to a planet sweeps out equal areas in equal time intervals. Show that this is a consequence of the Law of Conservation of Angular Momentum. The path of the planet is an ellipse.

SOLUTION

The gravitational force exerted by the sun on a planet is a central force – it acts along the line joining the two bodies, as shown in figure. The torque on the planet is

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin 180^\circ\ \hat{n} = 0$$

which means that its angular momentum is constant. The constancy in direction implies that the plane of the orbit does not change. The constancy in magnitude leads to the "law of areas."

In a time interval Δt, the planet moves from A to B by a distance $AB = v\Delta t$. The height of the triangle SAB is

$$h = AB\sin(180^\circ - \theta)^\circ = v\Delta t\sin\theta$$

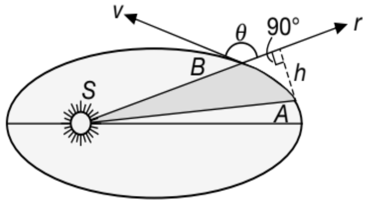
Thus, its area is

$$\Delta A = \frac{1}{2}rh = \frac{1}{2}rv\Delta t\sin\theta$$

so, $$\frac{\Delta A}{\Delta t} = \frac{1}{2}rv\sin\theta \qquad \ldots(1)$$

The angular momentum of the planet is

$$L = rp\sin\theta = mrv\sin\theta \qquad \ldots(2)$$



Combining (1) and (2) we see that

$$\frac{\Delta A}{\Delta t} = \frac{L}{2m} = \text{constant}$$

The rate at which the radial line sweeps out area is constant. Or, equivalently, the radial line sweeps out equal areas in equal time intervals.

Problem Solving Technique(s)

1. The areal velocity of a planet is constant (Kepler's second law) and is given by

$$\frac{dA}{dt} = \frac{L}{2m}$$

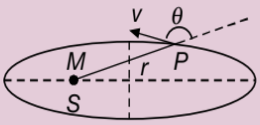
Here, L is the angular momentum of the planet about sun.

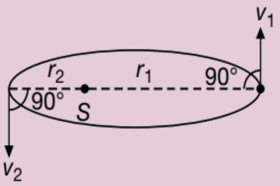
2. Most of the problems of gravitation are solved by two conservation laws:
   - **(i)** Law of Conservation of Angular Momentum about sun and
   - **(ii)** Law of Conservation of Mechanical (potential + kinetic) Energy

   Hence, the following two equations are used in most of the cases,

$$mvr\sin\theta = \text{constant} \qquad \ldots(1)$$

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant} \qquad \ldots(2)$$





At aphelion and perihelion positions $\theta = 90°$

Hence, equation (1) can be written as,

$$mvr\sin 90° = \text{constant}$$

$$\Rightarrow \quad mvr = \text{constant} \qquad \ldots(3)$$

Further, since mass of the planet (m) also remains constant, equation (1) can also be written as

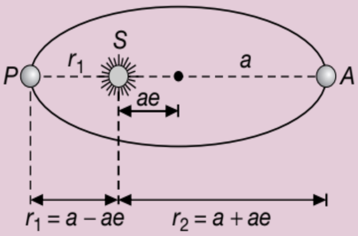
$$vr\sin\theta = \text{constant} \qquad \ldots(4)$$

$$\Rightarrow \quad v_1 r_1 = v_2 r_2 \qquad \{\theta = 90°\}$$

Since, $r_1 > r_2$

$$\Rightarrow \quad v_1 < v_2$$

3. Applying the above mentioned Conservation Laws at the aphelion and perihelion positions along with the knowledge that



$$r_1 = r_{aphelion} = a(1+e) \text{ and } r_2 = r_{perhilion} = a(1-e)$$

where e is the eccentricity of the elliptical orbit. Then, we can show that

$$v_{\min} = v_{aphelion} = v_1 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

$$v_{\max} = v_{perhilion} = v_2 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

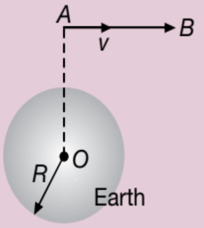
and total energy of the planet is calculated by

$$E = U + K = -\frac{GmM}{r_1} + \frac{1}{2}mv_1^2 = -\frac{GmM}{r_2} + \frac{1}{2}mv_2^2$$

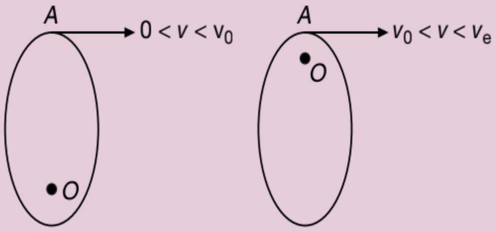
$$\Rightarrow \quad E = -\frac{GMm}{2a}$$

It is interesting to note that the total energy is same for both the circular and elliptical orbit.

4. **Trajectory of a body projected from point *A* in the direction *AB* with different initial velocities:** Let a body be projected from point A with velocity v in the direction AB. For different values of v the paths are different. Here are the possible cases.

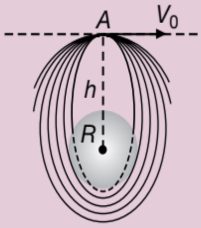


(i) If $v = 0$, path is a straight line from A to O.
(ii) If $0 < v < v_0$, path is an ellipse with centre O of the earth as a focus.
(iii) If $v = v_0$, path is a circle with O as the centre.
(iv) If $v_0 < v < v_e$, path is again an ellipse with O as a focus.
(v) If $v = v_e$, body escapes from the gravitational pull of the earth and path is a parabola.
(vi) If $v > v_e$, body again escapes but now the path is a hyperbola. Here, $v_0 =$ orbital speed $\left(\sqrt{\dfrac{GM}{r}}\right)$ at A and $v_e =$ escape velocity at A.



NOTE:

(a) From case (i) to (iv) total energy of the body is negative. Hence, these are the closed orbits and are also called **Bound Trajectories.** For case (v) total energy is zero and for case (vi) total energy is positive. In these two cases orbits are open and are called **Unbound Trajectories**.



(b) If v is not very large the elliptical orbit will intersect the earth and the body will fall back to earth.

ILLUSTRATION 48

Two satellites S_1 and S_2 revolve around a planet in coplanar circular orbits in the opposite sense. The periods of revolutions are T and ηT respectively. Find the angular speed of S_2 as observed by an astronaut in S_1, when they are closest to each other.

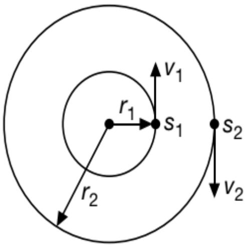
SOLUTION

Since $T^2 \propto r^3$

$$\Rightarrow \quad r \propto T^{\frac{2}{3}}$$

$$\Rightarrow \quad \frac{r_1}{r_2} = \left(\frac{T_1}{T_2}\right)^{2/3} = \left(\frac{T}{\eta T}\right)^{2/3} = \frac{1}{\eta^{2/3}}$$

Now when they are closest to each other, then, the separation between them is minimum i.e., as shown in figure.



So, $\omega_{21} = \dfrac{v_{\text{rel}}}{r_\perp} = \dfrac{v_2 + v_1}{r_2 - r_1}$

$$\Rightarrow \quad \omega_{21} = \frac{\dfrac{2\pi r_2}{T_2} + \dfrac{2\pi r_1}{T_1}}{r_2 - r_1} = \frac{2\pi\left(\dfrac{1}{T_2} + \dfrac{1}{T_1}\dfrac{r_1}{r_2}\right)}{1 - \dfrac{r_1}{r_2}}$$

$$\Rightarrow \quad \omega_{21} = \frac{2\pi\left(\dfrac{1}{\eta T} + \dfrac{1}{T}\dfrac{1}{\eta^{2/3}}\right)}{1 - \dfrac{1}{\eta^{2/3}}}$$

$$\Rightarrow \quad \omega_{21} = \frac{\dfrac{2\pi}{T}\left(\eta^{-1/3} + 1\right)}{\left(\eta^{2/3} - 1\right)}$$

ILLUSTRATION 49

A planet of mass m moves along an ellipse around the sun so that its maximum and minimum distances from the sun are equal to r_1 and r_2 respectively. Find the angular momentum of this planet relative to the centre of the sun. Mass of the sun is M.

SOLUTION

Applying the Law of Conservation of Angular Momentum and Mechanical Energy at P and A, we get

$$mv_1r_1 \sin 90° = mv_2r_2 \sin 90°$$

$$\Rightarrow \quad v_2 = \left(\frac{r_1}{r_2}\right)v_1 \qquad \ldots(1)$$

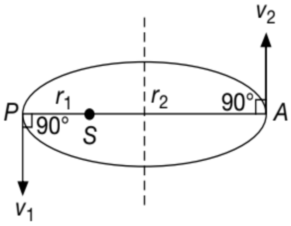
Also, $\dfrac{1}{2}mv_1^2 - \dfrac{GMm}{r_1} = \dfrac{1}{2}mv_2^2 - \dfrac{GMm}{r_2}$ …(2)

Substituting value of v_2 from equation (1) in equation (2), we get

$$\frac{1}{2}m\left(\frac{r_1^2}{r_2^2}-1\right)v_1^2 = GMm\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$

$$\Rightarrow \quad v_1^2\left(\frac{r_1+r_2}{r_2}\right) = \frac{2GM}{r_1}$$

$$\Rightarrow \quad v_1 = \sqrt{\frac{2GMr_2}{r_1\left(r_1+r_2\right)}}$$



Since, angular momentum of the planet about the sun is

$$L = mv_1r_1$$

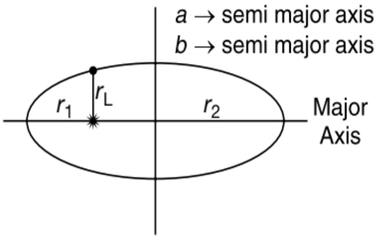
$$\Rightarrow \quad L = m\sqrt{\frac{2GMr_1r_2}{\left(r_1+r_2\right)}}$$

ILLUSTRATION 50

If a planet revolves around the sun in an elliptical orbit such that its minimum distance from sun is r_1 and maximum distance is r_2. Calculate the distance of planet from sun when it is at a position where the line joining the planet and sun is perpendicular to the major axis of the ellipse.

SOLUTION

The situation is shown in Figure.



Since, $r_L = \frac{1}{2}$ (latus rectum)

$$\Rightarrow \quad r_L = \frac{1}{2}\left(\frac{2b^2}{a}\right) = \frac{b^2}{a} \qquad \ldots(1)$$

Here semi major axis is given by

$$a = \frac{r_1+r_2}{2}$$

Also, we know that $b^2 = a^2\left(1-e^2\right)$ and $a - r_1 = ae$

$$\Rightarrow \quad b^2 = a^2 - \left(a-r_1\right)^2 = 2ar_1 - r_1^2$$

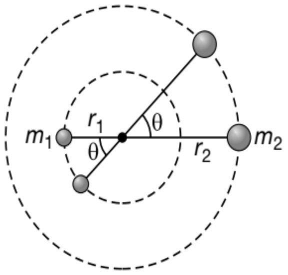
$$\Rightarrow \quad b^2 = 2r_1\left(\frac{r_1+r_2}{2}\right) - r_1^2 = r_2r_1$$

So, from equation (1), we get

$$r_L = \frac{b^2}{a} = \frac{2r_1r_2}{r_1+r_2}$$

DOUBLE STAR SYSTEM OR BINARY STAR SYSTEM

In motion of a planet round the sun we have assumed the mass of the sun to be too large in comparison to the mass of the planet. Under such situation the sun remains stationary and the planet revolves round the sun. If however masses of sun and planet are comparable and motion of sun is also to be considered, then both of them revolve around their centre of mass with same angular velocity but different linear speeds in the circles of different radii. The centre of mass remains stationary.



We use following equations under this condition.

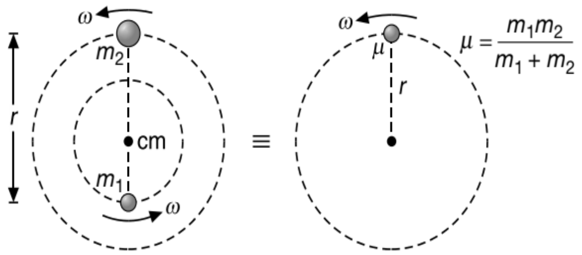
$$m_1r_1 = m_2r_2 \qquad \ldots(1)$$

$$m_1r_1\omega^2 = m_2r_2\omega^2 = \frac{Gm_1m_2}{\left(r_1+r_2\right)^2} \qquad \ldots(2)$$

Solving these two equations, we can find that

$$\omega = \sqrt{\frac{G\left(m_1+m_2\right)}{r^3}} = \sqrt{\frac{GM}{r^3}} \text{ or } T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

Here, $M = m_1 + m_2$ and $r = r_1 + r_2$



Further, angular momentum of the system about centre of mass

$$L = (I_1 + I_2)\omega = \left(\frac{m_1m_2}{m_1+m_2}\right)r^2\omega = \mu r^2\omega$$

Kinetic energy of system,

$$K = \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)r^2\omega^2 = \frac{1}{2}\mu r^2\omega^2$$

and moment of inertia of system,

$$I = \left(\frac{m_1m_2}{m_1+m_2}\right)r^2 = \mu r^2$$

Here, $\mu = \frac{m_1m_2}{m_1+m_2}$ = reduced mass.

Thus, the two bodies can be replaced by a single body whose mass is equal to reduced mass. This single body revolve in a circular orbit whose radius is equal to the distance between two bodies and centripetal force of circular motion is equal to force of potential between two bodies for actual separation.

ILLUSTRATION 51

A double star is a system of two stars moving around the centre of mass of the system due to gravitation. Find the distance between the components of the double star if the total mass equals M and time period is T.

SOLUTION

Let d be the distance between the stars and let d_1 and d_2 be the distances of stars from centre of mass

Therefore, $d_1 = \left(\frac{d}{m_1+m_2}\right)m_2$ and $d_2 = \left(\frac{d}{m_1+m_2}\right)m_1$

Since, we have $\frac{Gm_1m_2}{d^2} = m_1\omega^2 d_1 = m_2\omega^2 d_2$

Also, $m_1 + m_2 = M$ {given}

$$\Rightarrow \quad d_1 = \frac{d}{M}m_2 \text{ and } d_2 = \frac{d}{M}m_1$$

$$\Rightarrow \quad \frac{Gm_1(M-m_1)}{d^2} = m_1\omega^2\left[\frac{d}{M}(M-m_1)\right]$$

$$\Rightarrow \quad GM = \omega^2 d^2$$

$$\Rightarrow \quad d^3 = \frac{GMT^2}{4\pi^2}$$

$$\Rightarrow \quad d = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}$$

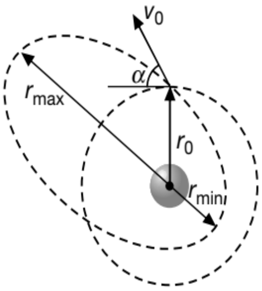
Test Your Concepts-V

Based on Satellites, Kepler's Laws and Applications

(Solutions on page H.263)

1. Two satellites A and B of the same mass are orbiting the earth at altitudes R and $3R$ respectively, where R is the radius of the earth. Taking their orbits to be circular obtain the ratios of their kinetic and potential energies.
2. A satellite of mass m is orbiting a planet of mass M at a radial distance R from the centre of a planet. The satellite explodes by expelling very rapidly a small amount of its mass Δm in an opposite direction to its orbital velocity. The immediate recoil velocity of the satellite is v_r (additional to the velocity already possessed by the satellite).
   (a) Show that the largest possible value of v_r for which the satellite remains within the gravitational field assuming $\Delta m \ll m$ is,
   $$v_r = (\sqrt{2}-1)\sqrt{\frac{GM}{R}}$$
   (b) Deduce the result for v_r in case of Δm not being negligible.
3. If a satellite is revolving close to a planet of density ρ with period T, show that the quantity ρT^2 is a universal constant.

4. A satellite is projected into space with a velocity v_0 at a distance r_0 from the centre of the earth by the last stage of its launching rocket. The velocity v_0 was designed to send the satellite into a circular orbit of radius r_0. However, owing to a malfunction of control, the satellite is not projected horizontally but at an angle α with the horizontal and, as a result, is propelled into an elliptic orbit. Determine the maximum and minimum values of the distance from the center of the earth to the satellite.



5. Two small dense stars rotate about their common centre of mass as a binary system with the period of 1 year for each. One star is of double the mass of the other and the mass of the lighter one is of $\left(\frac{1}{3}\right)$ the mass of the sun. Given the distance between the earth and the sun is R. If the distance between the two stars is r, then obtain the relation between r and R.

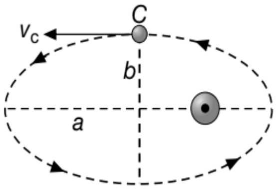
6. In a double star, two stars one of mass m_1 and another of mass m_2, with a separation d, rotate about their common centre of mass. Find:
   (a) an expression for their time period of revolution
   (b) the ratio of their kinetic energies
   (c) the ratio of their angular momenta about the centre of mass and
   (d) the total angular momentum of the system
   (e) the kinetic energy of the system.

7. Two satellites A and B revolve around a planet in two coplanar circular orbits in the same sense with radii 10^4 km and 2×10^4 km respectively. Time period of A is 28 hours. What is time period of another satellite. Find the speed of B with respect to A when A and B are at farthest distance from each other.

8. A satellite is revolving round the earth in a circular orbit of radius r and velocity v_0. A particle is projected from the satellite in forward direction with relative velocity $v=\left(\sqrt{\frac{5}{4}}-1\right)v_0$. Calculate its minimum and maximum distances from earth's centre during subsequent motion of the particle.

9. Two identical stars each of mass M orbit around their centre of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.
   (a) Find the gravitational force of one star on the other.
   (b) Find the orbital speed of each star and the period of the orbit.
   (c) What minimum energy would be required to separate the two stars to infinity?

10. Prove that the velocity of a satellite travelling in an elliptical orbit when it reaches point C on the end of the semi-minor axis is $R\sqrt{\frac{g}{a}}$. Also show that this velocity is the same as that for a circular orbit of radius a.

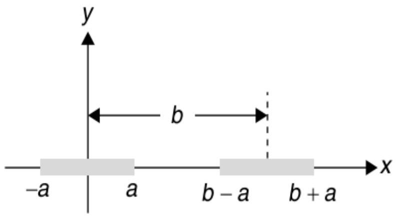


11. Binary stars of comparable masses m_1 and m_2 rotate under the influence of each other's gravity with a time period T. If they are stopped suddenly in their motions, find their relative velocity when they collide with each other. The radii of the stars are R_1 and R_2 respectively. G is the universal constant of gravitation.

SOLVED PROBLEMS

PROBLEM 1

Two identical thin rods of length $2a$ have equal mass M that is distributed uniformly along their lengths. The rods lie along the x-axis with their centres separated by a distance $b > 2a$.



Show that the magnitude of the force exerted by the left rod on the right one is given by

$$F = \frac{GM^2}{4a^2}\log_e\left(\frac{b^2}{b^2 - 4a^2}\right)$$

SOLUTION

This problem is a very special one where we shall learn to calculate the force between two extended bodies using the concept of gravitational field. For this let us divide the problem in two parts.

PART I

Here we shall imagine the rod to the right to be absent and then let us calculate the field due to the rod on the left at a hypothetical point P (say) at a distance d from one end of the rod. Let us consider an element of length dx at a distance x from the point P. If dE_g be the gravitational field due to the rod at P, then

$$dE_g = \frac{Gdm}{x^2} = \left(\frac{GM}{2a}\right)x^{-2}dx$$

$$\Rightarrow \quad E_g = \frac{GM}{2a}\left(\int_d^{d+2a} x^{-2}dx\right) = \frac{GM}{2a}\left(\frac{x^{-2+1}}{-2+1}\right)\Bigg|_d^{d+2a}$$

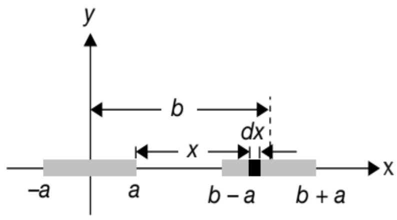
$$\Rightarrow \quad E_g = -\frac{GM}{2a}\left(\frac{1}{x}\right)\Bigg|_d^{d+2a} = -\frac{GM}{2a}\left(\frac{1}{d+2a} - \frac{1}{d}\right)$$

$$\Rightarrow \quad E_g = \frac{GM}{d(d+2a)}$$

PART II

Now after we have calculated the field due to one rod, we actually observe that the second rod will be lying in the field of the first rod. For finding the force on this rod let us again consider an infinitesimal element of length dx at a distance x from the nearest end of the left rod. If dF be the force due to this rod then

$$dF = E_g dm = \left[\frac{GM}{x(x+2a)}\right]\lambda dx$$



Since $\lambda = \dfrac{M}{2a}$

$$\Rightarrow \quad dF = \frac{GM^2}{2a}\left[\frac{dx}{x(x+2a)}\right]$$

$$\Rightarrow \quad F = \int dF = \frac{GM^2}{2a}\int_{b-2a}^{b}\frac{dx}{x(x+2a)}$$

$$\Rightarrow \quad F = \frac{GM^2}{2a}\left[-\frac{1}{2a}\log_e\left(\frac{2a+x}{x}\right)\right]\Bigg|_{b-2a}^{b}$$

$$\Rightarrow \quad F = \frac{GM^2}{4a^2}\left[-\log_e\left(\frac{2a+b}{b}\right) + \log_e\left(\frac{b}{b-2a}\right)\right]$$

$$\Rightarrow \quad F = \frac{GM^2}{4a^2}\log_e\left[\frac{b^2}{(b-2a)(b+2a)}\right]$$

$$\Rightarrow \quad F = \frac{GM^2}{4a^2}\log_e\left(\frac{b^2}{b^2-4a^2}\right)$$

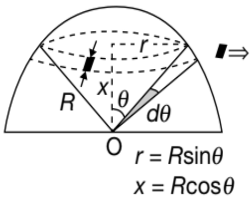
PROBLEM 2

Calculate the gravitational field due to a thin uniform very large hemispherical shell at its centre. Assume the shell to have a radius R and a mass M.

SOLUTION

Let us consider an infinitesimal elemental ring on its surface having angular width $d\theta$ at an angle θ from its axis as shown. The surface area dA of this ring is

$$dA = (2\pi R\sin\theta)(Rd\theta)$$



Mass on this elemental ring is

$$dm = \sigma dA = \sigma\left(2\pi R^2 \sin\theta d\theta\right)$$

where σ is the surface mass density of the shell given by

$$\sigma = \frac{M}{2\pi R^2}$$

Since the gravitational field strength due to a thin ring of mass dm, radius r at a point lying on its axis at a distance x from its centre is

Now due to this ring, gravitational field strength at centre C is

$$dE = \frac{G(dm)x}{\left(r^2+x^2\right)^{3/2}} \qquad \ldots(1)$$

Also, from the figure, we observe that

$$x = R\cos\theta \text{ and } r = R\sin\theta$$

Substituting these values in equation (1), we get

$$dE = \frac{Gdm(R\cos\theta)}{\left(R^2\sin^2\theta + R^2\cos^2\theta\right)^{3/2}}$$

$$\Rightarrow \quad dE = \frac{G\sigma\left(2\pi R^2 \sin\theta d\theta\right)R\cos\theta}{R^3}$$

$$\Rightarrow \quad dE = G\sigma\pi(2\sin\theta\cos\theta d\theta) = G\sigma\pi\sin(2\theta)d\theta$$

Net gravitational field at center is obtained by integrating the above expression for dE between the limits zero to $\frac{\pi}{2}$. So,

$$E = \int dE = G\sigma\pi\int_0^{\pi/2}\sin(2\theta)d\theta$$

$$\Rightarrow \quad E = G\sigma\pi\left(-\frac{\cos(2\theta)}{2}\Big|_0^{\pi/2}\right)$$

$$\Rightarrow \quad E = G\sigma\pi\left(\frac{1}{2}+\frac{1}{2}\right)$$

$$\Rightarrow \quad E = G\sigma\pi = G\left(\frac{M}{2\pi R^2}\right)\pi$$

$$\Rightarrow \quad E = \frac{GM}{2R^2}$$

PROBLEM 3

Calculate the self energy of the sun, taking its mass to be equal to 2×10^{30} kg and its radius to be very nearly 7×10^{8} metre. If its radius contracts by 1 km per year, without affecting its mass, calculate the rate at which it radiates out energy.

SOLUTION

The gravitational self energy of the sun is given by

$$U_s = -\frac{3}{5}\frac{GM^2}{R}$$

This is actually the self energy of the solid sphere. So,

$$U_s = -\frac{3}{5}\frac{\left(6.67\times10^{-11}\right)\left(2\times10^{30}\right)^2}{\left(7\times10^{8}\right)}$$

$$\Rightarrow \quad U_s = -2.29\times10^{41}\text{ J}$$

Now, rate of change of energy of the sun, is

$$\frac{dU_s}{dt} = \frac{3}{5}\frac{M^2G}{R^2}\frac{dR}{dt}$$

where, $\left|\frac{dR}{dt}\right| = \frac{1000}{365\times24\times3600}\text{ ms}^{-1}$

So, the rate of energy radiated out by the sun,

$$\frac{dU_s}{dt} = \frac{3}{5}\frac{M^2G}{R^2}\frac{dR}{dt}$$

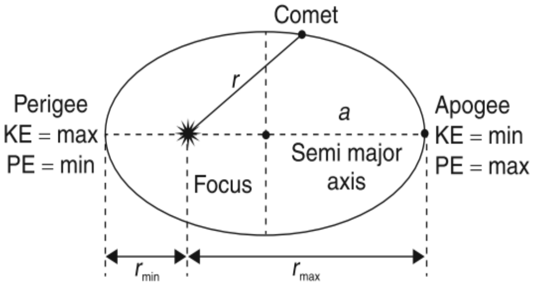
$$\Rightarrow \quad \frac{dU_s}{dt} = \frac{3}{5}\frac{\left(2\times10^{30}\right)^2\left(6.67\times10^{-11}\right)}{\left(7\times10^{8}\right)^2}\left(\frac{1000}{365\times24\times3600}\right)$$

$$\Rightarrow \quad \frac{dU_s}{dt} = 1.03\times10^{28}\text{ Js}^{-1}$$

PROBLEM 4

Halley's comet has a period of 76 years and in the year 1986 the distance of closest approach of the comet to the sun was 8.9×10^{10} m. Calculate the comet's farthest distance from the sun if the mass of sun is 2×10^{30} kg and $G = 6.67\times10^{-11}\text{ Nm}^2\text{kg}^{-2}$ and $1\text{ yr} = 3.15\times10^{7}$ s

SOLUTION



From the problem it is self-evident that the orbit of the comet is elliptical with sun being at one focus. For elliptical orbits, according to Kepler's third law, we have

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right)a^3$$

$$\Rightarrow \quad a = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}$$

where $T = 76 \times 365 \times 86400$

$$\Rightarrow \quad a = \left(\frac{6.67\times10^{-11}\times2\times10^{30}\times(76\times3.15\times10^7)^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad a \approx 2.7\times10^{12}\text{ m}$$

For an ellipse, we know that

$$2a = r_{\text{min}} + r_{\text{max}}$$

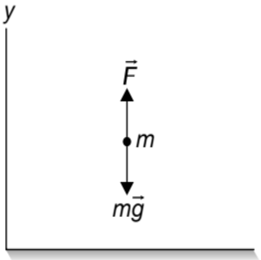
$$\Rightarrow \quad r_{\text{max}} = 2a - r_{\text{min}}$$

$$\Rightarrow \quad r_{\text{max}} = 2(2.7\times10^{12}) - 8.9\times10^{10}$$

$$\Rightarrow \quad r_{\text{max}} \approx 5.3\times10^{12}\text{ m}$$

PROBLEM 5

A body of mass m ascends from the earth's surface with zero initial velocity due to the action of two forces as shown in figure.



The force $\vec{F}$ varying with height y as $\vec{F} = -2m\vec{g}(1-ay)$ where a is a positive constant. Find the work performed by the force $\vec{F}$ over the first half of the ascent and the corresponding increase in the potential energy of the body in the earth's gravity field, which is assumed to be uniform.

SOLUTION

First let us find the total height of ascent. At the beginning and the end of the path, the velocity of the body is happens to be equal to zero and therefore, the change in kinetic energy of the body is also equal to zero. Therefore, from Work-Energy Theorem, we have

$$\text{Work Done by All the Forces} = 0$$

$$\Rightarrow \quad \int_0^h (\vec{F} + m\vec{g})\,dy = 0$$

$$\Rightarrow \quad \int_0^h mg(1-2ay)\,dy = 0$$

$$\Rightarrow \quad mgh(1-ah) = 0$$

$$\Rightarrow \quad h = \frac{1}{a}$$

The work performed by the force $\vec{F}$ over the first half of the ascent is,

$$W = \int_0^{h/2} F\,dy = 2mg\int_0^{1/2a}(1-ay)\,dy = \frac{3mg}{4a}$$

The corresponding increase in the potential energy is

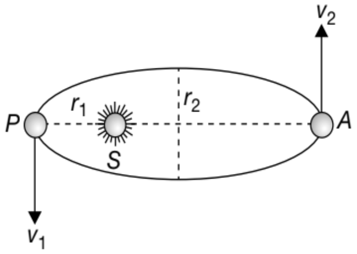
$$\Delta U = \frac{mgh}{2} = \frac{mg}{2a}$$

PROBLEM 6

Find the speeds of a planet of mass m in its perihelion and aphelion positions. The semi-major axis of its orbit is a, eccentricity is e and the mass of the sun is M. Also find the total energy of the planet in terms of the given parameters.

SOLUTION

Let v_1 and v_2 be the speeds of the planet at perihelion



and aphelion positions, then

$$r_1 = a(1-e) \text{ and } r_2 = a(1+e) \qquad \ldots(1)$$

Applying the Law of Conservation of Angular momentum of the planet at P (perihelion) and A (aphelion) about the sun, we get

$$mv_1r_1\sin 90° = mv_2r_2\sin 90°$$

$$\Rightarrow \quad v_1r_1 = v_2r_2 \qquad \ldots(2)$$

Applying the Law of Conservation of Mechanical Energy in these two positions, we get

$$(U+K)_{at\ P} = (U+K)_{at\ A}$$

$$\Rightarrow \quad \frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2} \qquad \ldots(3)$$

Solving equations (1), (2) and (3), we get

$$v_1 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)} \text{ and } v_2 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$$

Further, total energy of the planet is

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2}$$

$$\Rightarrow \quad E = \frac{1}{2}m\left[\frac{GM}{a}\left(\frac{1+e}{1-e}\right)\right] - \frac{GMm}{a(1-e)}$$

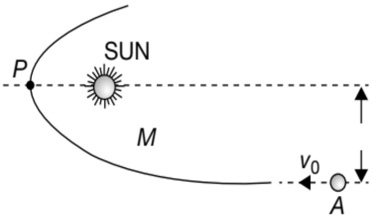
$$\Rightarrow \quad E = \frac{GMm}{a(1-e)}\left[\left(\frac{1+e}{2}\right) - 1\right]$$

$$\Rightarrow \quad E = \frac{GMm}{a(1-e)}\left(\frac{e-1}{2}\right)$$

$$\Rightarrow \quad E = -\frac{GMm}{2a}$$

PROBLEM 7

A cosmic body A coming from infinity with a velocity v_0 is approaching the Sun of mass M, with its line of motion at a distance l from the Sun, as shown in the figure. When it gets closest to the Sun i.e. at P, what will be its distance from the sun?



SOLUTION

By Law of Conservation of Mechanical Energy, we have

$$(U+K)_{at\ \infty} = (U+K)_{at\ P}$$

$$\Rightarrow \quad 0 + \frac{1}{2}mv_0^2 = -\frac{GMm}{r} + \frac{1}{2}mv^2 \qquad \ldots(1)$$

where

M is the mass of the sun
m is the mass of the body
r is the distance of closest approach and
v is the velocity of the body at the point P

Since the torque due to the gravitational force on the body about the sun is zero, so the angular momentum of the body about the sun will remain conserved, therefore by Law of Conservation of Angular Momentum, we have

$$(mv_0)l = (mv)r$$

$$\Rightarrow \quad v_0l = vr \qquad \ldots(2)$$

$$\Rightarrow \quad v = \frac{v_0l}{r}$$

Substituting this value of v in equation (1), we get

$$\Rightarrow \quad \frac{1}{2}mv_0^2 = -\frac{GMm}{r} + \frac{1}{2}m\left(\frac{v_0l}{r}\right)^2$$

$$\Rightarrow \quad v_0^2 = -\frac{2GMr + \left(v_0^2l^2\right)}{r^2}$$

$$\Rightarrow \quad v_0^2r^2 + 2GMr - v_0^2l^2 = 0$$

Solving this quadratic in r, we get

$$r = -\frac{-2GM \pm \sqrt{4G^2M^2 + 4v_0^4l^2}}{2v_0^2}$$

Since we are here calculating the distance of closest approach, i.e. the least distance between the sun and the body, so we will reject the negative sign. Hence, we have

$$r = -\frac{-2GM + \sqrt{4G^2M^2 + 4v_0^4 l^2}}{2v_0^2}$$

$$\Rightarrow \quad r_{\text{min}} = \frac{GM}{v_0^2}\left(-1 + \sqrt{1 + \left(\frac{v_0^2 l^2}{GM}\right)^2}\right)$$

PROBLEM 8

A body is projected vertically upwards from the surface of earth with a velocity sufficient to carry it to infinity. Calculate the time taken by it to reach a height h.

SOLUTION

Let v be the velocity of the body at a distance r from the centre of earth. Applying Law of Conservation of Mechanical Energy, we get

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv_e^2 - \frac{GMm}{R} \qquad \ldots(1)$$

where, $v_e = \sqrt{2gR}$ and $g = \dfrac{GM}{R^2}$

Substituting in equation (1), we get

$$v = \frac{R\sqrt{2g}}{\sqrt{r}}$$

Since $v = \dfrac{dr}{dt}$, so we have

$$\frac{dr}{dt} = \frac{R\sqrt{2g}}{\sqrt{r}}$$

$$\Rightarrow \quad \int_0^t dt = \frac{1}{R\sqrt{2g}} \int_R^{R+h} r^{1/2} dr$$

$$\Rightarrow \quad t = \frac{2}{3}\frac{1}{R\sqrt{2g}}\left[(R+h)^{3/2} - R^{3/2}\right]$$

$$\Rightarrow \quad t = \frac{1}{3}\sqrt{\frac{2R}{g}}\left[\left(1 + \frac{h}{R}\right)^{3/2} - 1\right]$$

PROBLEM 9

An artificial satellite of the earth (radius R and mass M) moves in an orbit whose radius is n times the radius of the earth. Assuming resistance to the motion to be proportional to the square of velocity that is $F = av^2$. Find how long the satellite will take to fall on to the earth.

SOLUTION

Tangential force, $F_t = m\dfrac{dv}{dt}$

$$\Rightarrow \quad -av^2 = m\frac{dv}{dt}$$

Rearranging, we get

$$\int_0^t dt = -\frac{m}{a}\int_{v_i}^{v_f} \frac{dv}{v^2} = -\frac{m}{a}\int_{v_i}^{v_f} v^{-2} dv$$

$$t = -\frac{m}{a}\left(\frac{v^{-2+1}}{-2+1}\right)\Bigg|_{v_i}^{v_f} = -\frac{m}{a}\left(-\frac{1}{v}\right)\Bigg|_{v_i}^{v_f}$$

$$\Rightarrow \quad t = \frac{m}{a}\left(\frac{1}{v_f} - \frac{1}{v_t}\right)$$

Also, $v_i = \sqrt{\dfrac{GM}{nR}}$ and $v_f = \sqrt{\dfrac{GM}{R}}$

$$t = \frac{m}{a}\left(\sqrt{\frac{nR}{GM}} - \sqrt{\frac{R}{GM}}\right)$$

Since $g = \dfrac{GM}{R^2}$, so we get

$$t = \frac{m\sqrt{R}}{a\sqrt{GM}}\left(\sqrt{n} - 1\right)$$

PROBLEM 10

A satellite is to be placed in an orbit just above the earth's atmosphere with a speed $\sqrt{\dfrac{3}{2}}$ times the speed for a circular orbit at that height. The initial velocity imparted is horizontal. Calculate the maximum distance of the satellite from the earth, when it is in the orbit.

SOLUTION

Since the satellite is to be placed in an orbit just above the atmosphere i.e. it is to be placed in an orbit close to surface of earth. Also, it is given that

$$v_i = \sqrt{\frac{3}{2}} v_{\text{orbit}}$$

Close to surface of earth

$$v_0 \approx \sqrt{gR}$$

$$\Rightarrow \quad v_i = \sqrt{\frac{3}{2}}\sqrt{gR}$$

By Law of Conservation of Angular Momentum, we have

$$mv_ir_i = mv_fr_f$$

$$\Rightarrow \quad m\left(\sqrt{\frac{3}{2}}\sqrt{gR}\right)R = mv_fr_{\max} \qquad \ldots(1)$$

By Laws of Conservation of Energy, we have

$$\frac{1}{2}m\left(\frac{3}{2}gR\right) - \frac{GMm}{R} = \frac{1}{2}mv_f^2 - \frac{GMm}{r_{\max}} \qquad \ldots(2)$$

$$\Rightarrow \quad \frac{3}{4}gR - gR = \frac{1}{2}v_f^2 - \frac{gR^2}{r_{\max}}$$

From equation (1), we have

$$v_f = \sqrt{\frac{\frac{3}{2}gR^3}{r_{\max}}}$$

So, equation (2) becomes

$$-\frac{1}{4}gR = \frac{1}{2}\left(\frac{\sqrt{\frac{3}{2}}\sqrt{gR^3}}{r_{\max}}\right)^2 - \frac{gR^2}{r_{\max}}$$

$$\Rightarrow \quad \frac{R^2}{{r_{\max}}^2} - \frac{4}{3}\frac{R}{r_{\max}} + \frac{1}{3} = 0$$

$$\Rightarrow \quad \frac{R}{r_{\max}} = \frac{\frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{4}{3}}}{2} = \frac{\frac{4}{3} \pm \frac{2}{3}}{2} = \frac{2}{3} \pm \frac{1}{3}$$

$$\Rightarrow \quad \frac{R}{r_{\max}} = 1 \text{ or } \frac{1}{3}$$

Since $r_{\max} \neq R$

$$\Rightarrow \quad r_{\max} = 3R$$

So, maximum distance of satellite from the earth surface is

$$h_{\max} = r_{\max} - R$$

$$\Rightarrow \quad h_{\max} = 3R - R = 2R$$

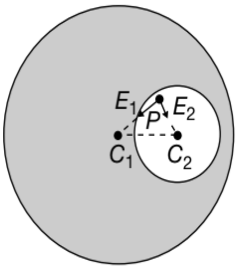
PROBLEM 11

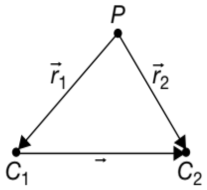
Inside a uniform sphere of density ρ there is a spherical cavity whose centre is at a distance l from the centre of the sphere. Find the strength of the gravitational field inside the cavity.

SOLUTION

Let us consider a point P inside the cavity. Let $\vec{E}_1$ be gravitational field due to sphere at P, then

$$\vec{E}_1 = \left(\frac{4}{3}\pi G\rho\right)\vec{r}_1$$





If E_2 be the gravitational field due to cavity at P, then

$$\vec{E}_2 = \frac{4}{3}\pi G(-\rho)\vec{r}_2$$

$$\Rightarrow \quad \vec{E}_2 = -\frac{4}{3}\pi G\rho\vec{r}_2$$

So, net field at P, due to sphere and the cavity is

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$\Rightarrow \quad \vec{E} = \frac{4}{3}\pi G\rho\left(\vec{r}_1 - \vec{r}_2\right) \qquad \ldots(1)$$

However, in triangle PC_1C_2, we have

$$\vec{r}_1 + \vec{l} = \vec{r}_2$$

$$\Rightarrow \quad \vec{r}_1 - \vec{r}_2 = -\vec{l}$$

Substituting in (1), we get

$$\vec{E} = -\frac{4}{3}\pi G\rho\vec{l}$$

$$\Rightarrow \quad |\vec{E}| = \frac{4}{3}\pi G\rho l$$

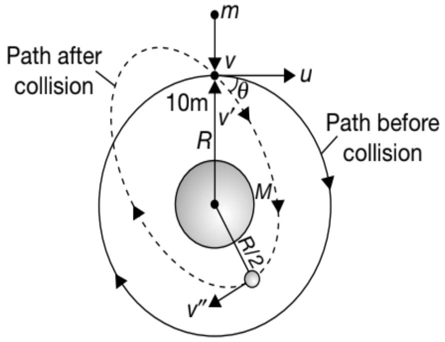
So, we observe that the gravitational field inside the cavity is independent of the location of the point P and just depends upon the distance of the cavity's centre from the centre of sphere.

PROBLEM 12

A meteorite of mass m collides perpendicularly with a satellite which was orbiting around a planet of mass M in a circular path of radius R. Due to collision, the meteorite sticks to the satellite of mass $10m$ and the satellite is seen to have gone into an orbit whose minimum distance from the planet is $\frac{R}{2}$. Determine the velocity v of the meteorite before collision.

SOLUTION

The situation before collision and after collision is shown in the figure.



Before collision, the speed of satellite is the orbital speed given by

$$u_0 = \sqrt{\frac{GM}{R}} \qquad \ldots(1)$$

After collision with meteorite, let the combined mass $11m$ moves at an angle θ with the orbit as shown in figure. If finally, $11m$ moves at speed v' then applying the Law of Conservation of Momentum along horizontal and vertical directions, we get

$$mv = 11mv'\sin\theta$$

$$\Rightarrow \quad v'\sin\theta = \left(\frac{v}{11}\right) \qquad \ldots(2)$$

$$10mu_0 = 11mv'\cos\theta$$

$$\Rightarrow \quad v'\cos\theta = \left(\frac{10v_0}{11}\right) \qquad \ldots(3)$$

After collision applying the Law of Conservation of Angular Momentum, we get

$$11m(v'\cos\theta)R = 11m(v'')\frac{R}{2} \qquad \ldots(4)$$

where v'' is the speed at perigee.

Applying the Law of Conservation of Energy, we have

$$-\frac{GM(11m)}{R} + \frac{1}{2}(11m)v'^2 = -\frac{GM(11m)}{(R/2)} + \frac{1}{2}(11m)v''^2$$

$$\Rightarrow \quad -\frac{GM}{R} + \frac{v'^2}{2} = -\frac{2GM}{R} + \frac{v''^2}{2} \qquad \ldots(5)$$

From equation (2) and (3), we get

$$v'^2 = \left(\frac{v}{11}\right)^2 + \left(\frac{10u_0}{11}\right)^2 = \frac{v^2 + 100u_0^2}{121} \qquad \ldots(6)$$

From equation (4), we get

$$v'' = 2v'\cos\theta = \left(\frac{20}{11}\right)v_0 \qquad \ldots(7)$$

Substituting the values of v'^2 from (6) and v'' from (7) in equation (5), we get

$$-\frac{GM}{R} + \frac{1}{2}\left(\frac{v^2 + 100u_0^2}{121}\right) = -\frac{2GM}{R} + \frac{1}{2}\left(\frac{20}{11}u_0\right)^2$$

$$\Rightarrow \quad \frac{v^2}{242} = -\frac{GM}{R} + \left(\frac{1}{2}\right)\left(\frac{1}{121}\right)\left(400u_0^2 - 100u_0^2\right)$$

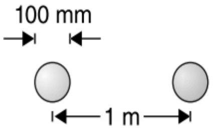
$$\Rightarrow \quad \frac{v^2}{242} = -\frac{GM}{R} + \frac{300}{242}\left(\frac{GM}{R}\right) \qquad \left\{\because\ u_0 = \sqrt{\frac{GM}{R}}\right\}$$

$$\Rightarrow \quad \frac{v^2}{242} = \frac{58GM}{242R}$$

$$\Rightarrow \quad v = \sqrt{\frac{58GM}{R}}$$

PROBLEM 13

Two iron spheres, each of mass 1 kg and 100 mm in diameter, are released from rest with a centre-to-centre separation of $1\ \text{m}$. Assume an environment in space with no forces other than the force of mutual gravitational attraction and calculate the time t required for the spheres to contact each other and the absolute speed v of each sphere upon contact.



SOLUTION

Let v_r be the relative speed at separation r, then by Law of Conservation of Mechanical Energy, we have

$$(U+K)_{\text{at } 1\text{ m}} = (U+K)_r$$

$$-\frac{GmM}{1} + \frac{1}{2}\mu(0)^2 = -\frac{GmM}{r} + \frac{1}{2}\mu v_r^2$$

$$\Rightarrow \quad \frac{1}{2}\mu v_r^2 = Gmm\left(\frac{1}{r} - 1\right) \qquad \ldots(1)$$

where, μ is the reduced mass given by

$$\mu = \frac{m(m)}{m+m} = \frac{m}{2}$$

Substituting in equation (1) with $m = 1\ \text{kg}$, we get

$$v_r = 2\sqrt{G\left(\frac{1}{r}-1\right)} \qquad \ldots(2)$$

$$\Rightarrow \quad -\frac{dr}{dt} = 2\sqrt{G\left(\frac{1}{r}-1\right)}$$

$$\Rightarrow \quad dt = -\frac{dr}{2\sqrt{G\left(\frac{1}{r}-1\right)}}$$

$$\Rightarrow \quad \int_0^t dt = -\frac{1}{2\sqrt{G}}\int_{1.0}^{0.1}\sqrt{\frac{r}{1-r}}\,dr$$

Let the integral be

$$I = \int\sqrt{\frac{r}{1-r}}\,dr$$

Substituting $r = \sin^2\theta$, we get

$$dr = 2\sin\theta\cos\theta d\theta$$

$$\Rightarrow \quad I = \int\sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\,2\sin\theta\cos\theta d\theta$$

$$\Rightarrow \quad I = \int 2\sin^2\theta d\theta = \int(1-\cos 2\theta)\,d\theta$$

$$\Rightarrow \quad I = \theta - \frac{\sin(2\theta)}{2}$$

$$\Rightarrow \quad I = \theta - \sin\theta\cos\theta$$

But $\sin^2\theta = r$

$$\Rightarrow \quad I = \sin^{-1}\sqrt{r} - \sqrt{r(1-r)}$$

$$\Rightarrow \quad t = -\frac{1}{2\sqrt{G}}\left(\sin^{-1}\sqrt{r} - \sqrt{r(1-r)}\right)\Big|_1^{0.1}$$

$$\Rightarrow \quad t = \frac{1}{2\sqrt{G}}\left[\sin^{-1}(1) - 0 - \sin^{-1}\sqrt{0.1} + \sqrt{(0.1)(0.9)}\right]$$

$$\Rightarrow \quad t = \frac{1}{2\sqrt{G}}\left(\frac{\pi}{2} - 0.32 + 0.3\right)$$

$$\Rightarrow \quad t = \frac{1}{2\sqrt{G}}(1.55)$$

$$\Rightarrow \quad t = 95\times10^3\ \text{s}$$

From equation (1), relative speed at the time of collision,

$$v_r = 2\sqrt{\left(6.67\times10^{-11}\right)\left(\frac{1}{0.1}-1\right)} = 4.9\times10^{-5}\ \text{ms}^{-1}$$

So, the speed of each sphere is $\dfrac{v_r}{2} = 2.45\times10^{-5}\ \text{ms}^{-1}$

PROBLEM 14

Two satellites S_1 and S_2 are to be set into the orbits of $\dfrac{R}{4}$ and $\dfrac{R}{6}$ above the earth's surface respectively. They revolve around the earth in coplanar circular orbit in the opposite sense. Determine

(a) the ratio of the speeds of projection from the surface of the earth.

(b) the angular speed of S_2 as observed by an astronaut in S_1 when their radius vectors are making an angle 120° with each other.

SOLUTION

(a) By Law of Conservation of Energy

$$(U+K)_{\text{surface}} = (U+K)_{\text{orbit}}$$

$$\Rightarrow \quad -\frac{GMm_1}{R} + \frac{1}{2}m_1v_1^2 = -\frac{GMm_1}{2\left(R+\frac{R}{4}\right)}$$

$$\Rightarrow \quad \frac{1}{2}m_1v_1^2 = -\frac{GMm_1}{2\left(R+\frac{R}{4}\right)} - \left(-\frac{GMm_1}{R}\right)$$

$$\Rightarrow \quad v_1^2 = \frac{6}{5}\frac{GM}{R} \qquad \ldots(1)$$

Similarly, $-\dfrac{GMm_2}{R} + \dfrac{1}{2}m_2v_2^2 = -\dfrac{GMm_2}{2\left(R+\frac{R}{6}\right)}$

$$\Rightarrow \quad \frac{1}{2}m_2v_2^2 = -\frac{GMm_2}{2\left(R+\frac{R}{6}\right)} - \left(-\frac{GMm_2}{R}\right)$$

$$\Rightarrow \quad v_2^2 = \frac{8}{7}\frac{GM}{R} \qquad \ldots(2)$$

From equations (1) and (2), we get

$$\frac{v_1^2}{v_2^2} = \frac{21}{20}$$

$$\Rightarrow \quad \frac{v_1}{v_2} = \sqrt{\frac{21}{20}}$$

(b) $r_1 = R + \frac{R}{4} = \frac{5R}{4}$ and $r_2 = R + \frac{R}{6} = \frac{7R}{6}$

$$P = (0, r_2)$$

and $Q = (r_2 \cos 30°, -r_1 \sin 30°) = \left(\frac{\sqrt{3}r_1}{2}, -\frac{r_1}{2}\right)$

$$\Rightarrow \quad PQ = \sqrt{\frac{3r_1^2}{4} + \left(r_2 + \frac{r_1}{2}\right)^2} = 2.1R$$

Applying Lami's Theorem, we get

$$\frac{r_2}{\sin(90° - \beta)} = \frac{r_1}{\sin(90° - \alpha)} = \frac{PQ}{\sin(120°)}$$

$$\Rightarrow \quad \cos\alpha = \frac{r_1}{PQ}\sin(120°) = 0.5 \text{ and}$$

$$\cos\beta = \frac{r_2}{PQ}\sin(120°) = 0.48$$

Now, $\omega_r = \dfrac{v_2 \cos\alpha + v_1 \cos\beta}{PQ}$

$$\Rightarrow \quad \omega_r = \frac{\sqrt{\frac{GM}{r_2}}\cos\alpha + \sqrt{\frac{GM}{r_1}}\cos\beta}{PQ}$$

$$\Rightarrow \quad \omega_r = \frac{\sqrt{\frac{gR^2}{r_2}}\cos\alpha + \sqrt{\frac{gR^2}{r_1}}\cos\beta}{2.1R} \qquad \ldots(3)$$

Since, $r_1 = \frac{5R}{4}$ and $r_2 = \frac{7R}{6}$

$$\sqrt{\frac{gR^2}{r_2}}\cos\alpha = \left(\sqrt{\frac{9.81 \times 6400 \times 10^3}{7/6}}\right)0.5$$

$$\Rightarrow \quad \sqrt{\frac{gR^2}{r_2}}\cos\alpha = 3666$$

$$\sqrt{\frac{gR^2}{r_1}}\cos\beta = \left(\sqrt{\frac{9.81 \times 6400 \times 10^3}{5/4}}\right)0.48$$

$$\Rightarrow \quad \sqrt{\frac{gR^2}{r_1}}\cos\beta = 3402$$

Substituting these values in equation (3), we get

$$\omega_r = \frac{3666 + 3402}{2.1 \times 6400 \times 10^3} = 5.25 \times 10^{-4} \text{ rads}^{-1}$$

PROBLEM 15

An earth satellite is revolving in a circular orbit of radius a with velocity v_0. A gun is in the satellite and is aimed directly towards the earth. A bullet is fired from the gun with muzzle velocity $\frac{v_0}{2}$. Neglecting resistance offered by cosmic dust and recoil of gun, calculate maximum and minimum distance of bullet from the centre of earth during its subsequent motion.

SOLUTION

Orbital speed of satellite is

$$v_0 = \sqrt{\frac{GM}{a}} \qquad \ldots(1)$$

Applying the Law of Conservation of Angular Momentum at P and Q, we get

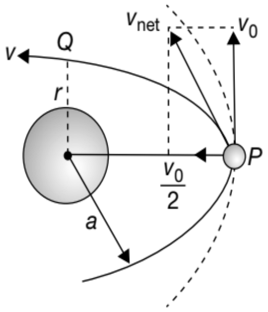
$$mav_0 = mvr$$

$$\Rightarrow \quad v = \frac{av_0}{r} \qquad \ldots(2)$$

Applying the Law of Conservation of Mechanical Energy at P and Q, we get

$$\frac{1}{2}m\left(v_0^2 + \frac{v_0^2}{4}\right) - \frac{GMm}{a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\Rightarrow \quad \frac{5}{8}v_0^2 - \frac{GM}{a} = \frac{v^2}{2} - \frac{GM}{r}$$



Substituting values of v and v_0 from equations (1) and (2), we get

$$\frac{5}{8}\frac{GM}{a} - \frac{GM}{a} = \frac{a^2}{r^2}\cdot\left(\frac{GM}{2a}\right) - \frac{GM}{r}$$

$$\Rightarrow \quad -\frac{3}{8a} = \frac{a}{2r^2} - \frac{1}{r}$$

$$\Rightarrow \quad -3r^2 = 4a^2 - 8ar$$

$$\Rightarrow \quad 3r^2 - 8ar + 4a^2 = 0$$

$$\Rightarrow \quad r = \frac{8a \pm \sqrt{64a^2 - 48a^2}}{6}$$

$$\Rightarrow \quad r = \frac{8a \pm 4a}{6}$$

$$\Rightarrow \quad r = 2a \text{ and } \frac{2a}{3}$$

Hence, the maximum and minimum distances are $2a$ and $\frac{2a}{3}$ respectively.