ICON

CHAPTER 1

Mathematical Physics

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Quadratic Equations
(b) Logarithmic Functions
(c) Linear Equations
(d) Determinants
(e) Arithmetic Progression (AP)
(f) Geometric Progression (GP)
(g) Coordinate Geometry
(h) Trigonometry
(i) Factorial
(j) Series Expansions
(k) Functions
(l) Limit of a Function
(m) Differentiation
(n) Integration

All this is followed by an Exercise Set which contains questions for your practice. Please be advised that you must first thoroughly study this chapter for your command on the Mathematical tools used in Physics hereafter.

GENERAL MATHEMATICS: A REVIEW

This part in mathematics is introduced here to give you a fundamental review of operations and methods. To have an extra understanding in Physics you should be totally familiar with basic algebraic techniques, analytic geometry, and trigonometry. Differential and integral calculus are discussed in detail and are intended for those students who have difficulties in applying calculus concepts to physical situations.

Table 1.1 Mathematical symbols used in the text and their meaning

Symbol	Meaning
$=$	is equal to
$\neq$	is not equal to
$\propto$	is proportional to
$>$	is greater than
$<$	is less than

(Continued)

Table 1.1 *(Continued)*

Symbol	Meaning
$\gg (\ll)$	much greater (less) than
$\approx$	is approximately equal to
$\sim$	positive difference between two numbers
Δx	the change in x
$\sum_{i=1}^{N} x_i$	the sum of all quantities x_i from $i=1$ to $i=N$
$\lvert x \rvert$	the magnitude of x (always a positive quantity)
$\Delta x \to 0$	Δx approaches zero
$\frac{dx}{dt}$	the derivative of x with respect to t
$\frac{\partial x}{\partial t}$	the partial derivative of x with respect to t
$\int$	Integral

ICON

ALGEBRA

When algebraic operations are performed, the laws of arithmetic apply. Symbols such as x, y and z are usually used to represent quantities that are not specified, and symbols such as a, b and c are used to represent numbers.

You should be familiar with the following operations:

Fractions

(i) $a\left(\frac{b}{c}\right)=\frac{ab}{c}$ **(ii)** $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)=\frac{ac}{bd}$

(iii) $\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}=\frac{ad}{bc}$ **(iv)** $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm cb}{bd}$

Factoring and Combinations

(i) $ax+bx=x(a+b)$

(ii) $x^2-y^2=(x+y)(x-y)$

(iii) $x^2-2x-15=(x+3)(x-5)$ {Quadratic Equation}

Roots of a Quadratic Equation

If $ax^2+bx+c=0\,(a\neq 0)$, has two roots given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

If α and β are two roots of equation, then

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

ILLUSTRATION 1

Solve the equation $10x^2-27x+5=0$.

SOLUTION

By comparing the given equation with standard equation $a=10$, $b=-27$, and $c=5$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=\frac{-(-27)\pm\sqrt{(-27)^2-4\times10\times5}}{2\times10}=\frac{27\pm23}{20}$$

$$\Rightarrow \quad x_1=\frac{27+23}{20}=\frac{5}{2} \text{ and } x_2=\frac{27-23}{20}=\frac{1}{5}$$

Roots of the equation are $\frac{5}{2}$ and $\frac{1}{5}$.

MULTIPLYING POWERS OF A GIVEN QUANTITY

(i) $x^n x^m=x^{n+m}$ **(ii)** $\frac{x^n}{x^m}=x^{n-m}$

(iii) $(x^n)^m=x^{nm}$

Logarithmic Functions

(i) ln = logarithm to base e, also called as natural log.

(ii) log = logarithm to base 10

(iii) $\ln(e)=1$

(iv) $\ln(e^x)=x$

(v) $\ln(xy)=\ln x+\ln y$

(vi) $\ln\left(\frac{x}{y}\right)=\ln x-\ln y$

(vii) $\ln\left(\frac{1}{x}\right)=-\ln x$

(viii) $\ln\left(x^n\right)=n\left(\ln x\right)$

(ix) $\ln(a)=2.3026\ \log(a)$

(x) $\ln a=\log_e a=2.303\log_{10} a$

(xi) $\log_a x=\frac{\log_e x}{\log_e a}$

(xii) If $\log_e x=\alpha$, then $x=e^\alpha$

Simultaneous Linear Equations

In order to solve two simultaneous equations involving two unknowns, x and y, we solve one of the equations for x in terms of y and substitute this expression into the other equation.

ILLUSTRATION 2

Solve the following given equations

$$5x+y=-8 \quad \ldots(1)$$

$$2x-2y=4 \quad \ldots(2)$$

SOLUTION

From (2), $x = y + 2$, substituting this in (1) gives

$$5(y+2)+y=-8$$
$$\Rightarrow\quad 6y=-18$$
$$\Rightarrow\quad y=-3$$
$$\Rightarrow\quad x=y+2=-1$$

Alternate Solution:

Multiplying (1) by 2 and add the result to (2)

$$\begin{aligned}10x+2y&=-16\\ 2x-2y&=4\\ \hline 12x&=-12\end{aligned}$$
$$\Rightarrow\quad x=-1$$
$$\Rightarrow\quad y=x-2=-3$$

Determinants

2^{nd} order (four elements) i.e., a 2×2 determinant is evaluated as

$$\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}=a_1b_2-a_2b_1$$

3^{rd} order (nine elements) i.e., a 3×3 determinant is evaluated as

$$\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}$$
$$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$

POWERS OF TEN

You should be familiar with the usage of powers of ten. It is a compact form of writing very large or very small numbers. For example, instead of 10000, we write 10^4, where the exponent represents the number of zeros; that is $10^4 = 10\times10\times10\times10 = 10000$. Likewise, a small number like 0.0001 can be expressed as 10^{-4}, where the negative exponent indicates that we are dealing with a number less than one. Some other examples of the use of powers of ten are

$$1000 = 10^3 \qquad 0.003 = 3\times10^{-3}$$
$$85000 = 8.5\times10^4 \qquad 0.00085 = 8.5\times10^{-4}$$
$$3\,200\,000 = 3.2\times10^6 \qquad 0.00002 = 2\times10^{-5}$$

If numbers written as powers of ten are multiplied, we simply add the exponents, maintaining their signs. For example,

$$(3\times10^3)\times(5\times10^4)=15\times10^7=1.5\times10^8$$
$$(2\times10^5)\times(4\times10^{-2})=8\times10^3$$
$$(5.6\times10^4)\times(4.3\times10^8)=24\times10^{12}$$

When numbers written as powers of ten are divided, we can bring the power of ten from the denominator to the numerator by changing its sign. For example,

$$\frac{8\times10^5}{2\times10^2}=4\times10^5\times10^{-2}=4\times10^3$$
$$\frac{12\times10^{-4}}{4\times10^{-9}}=3\times10^{-4}\times10^9=3\times10^5$$

In general,

$$10^n10^m=10^{n+m}$$
$$\frac{10^n}{10^m}=10^{n-m}$$
$$(10^n)^m=10^{nm}$$

ARITHMETIC PROGRESSION (AP)

It is a sequence of numbers which are arranged in increasing order and having a constant difference between them.

EXAMPLE: 1, 3, 5, 7, 9, 11, 13, ... or 2, 4, 6, 8, 10, 12, ...

In general arithmetic progression can be written as $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5\ \ldots$

(a) n^{th} term of an arithmetic progression is given by

$$a_n=a_0+(n-1)d$$

where a_0 is the first term, n is the number of terms, d is the common difference given by

$$d=(a_1-a_0)=(a_2-a_1)=(a_3-a_2)$$

(b) Sum of arithmetic progression

$$S_n=\frac{n}{2}\left(2a_0+(n-1)d\right)=\frac{n}{2}(a_0+a_n)$$

ILLUSTRATION 3

Find the sum of series $7+10+13+16+19+22+25$.

SOLUTION

Since $n=7$, $a_0=7$ and $a_n=a_7=25$, so

$$S_n=\frac{n}{2}(a_0+a_n)=\frac{7}{2}(7+25)=112$$

GEOMETRIC PROGRESSION (GP)

It is a sequence of numbers in which every successive term is obtained by multiplying the previous term by a constant quantity. This constant quantity is called the common ratio (r).

EXAMPLE: 4, 8, 16, 32, 64, 128 ... where $a = 4$ and $r = 2$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots \text{ where } a = 1 \text{ and } r = \frac{1}{2}$$

In general geometric progression can be written as a, ar, ar^2, ar^3, ar^4, ... where a is the first term and r is the common ratio

(a) Sum of n terms of G.P. is

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{if} \quad r < 1$$

$$S_n = \frac{a(r^n-1)}{r-1} \quad \text{if} \quad r > 1$$

(b) Sum of infinite terms of G.P.

$$S_\infty = \frac{a}{1-r} \quad \text{if} \quad r < 1$$

ILLUSTRATION 4

Find the sum of series $Q = 2q + \frac{q}{3} + \frac{q}{9} + \frac{q}{27} + \ldots\ldots$

SOLUTION

Above equation can be re-written as

$$Q = q + \left[q + \frac{q}{3} + \frac{q}{9} + \frac{q}{27} + \ldots\ldots\right]$$

By using the formula of sum of infinite terms of G.P.

$$Q = q + \left[\frac{q}{1-\frac{1}{3}}\right] = q + \frac{3}{2}q = \frac{5}{2}q$$

COORDINATE GEOMETRY

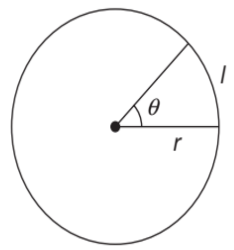
(a) The distance d between two points whose coordinates are (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

(b) **Radian:** The arc length l is proportional to the radius r for a fixed value of θ (in radians)

$$l = r\theta$$

$$\Rightarrow \quad \theta = \frac{l}{r}$$



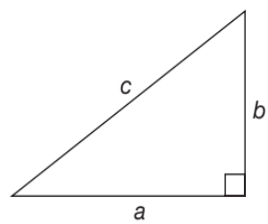
(c) Circumference of a circle is $C = 2\pi r$

(d) Area of a circle is $A = \pi r^2$

(e) Sector area $= \frac{1}{2}r^2\theta$, where θ is in radian

(f) The pythagorean theorem, which relates the three sides of a right triangle

$$c^2 = a^2 + b^2$$



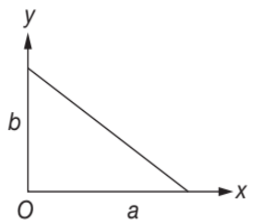
(g) Area of a triangle is $A = \frac{1}{2}(\text{Base})(\text{Altitude})$

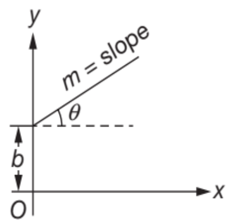
(h) Surface area of a sphere is $A = 4\pi r^2$

(i) Volume of a sphere is $V = \frac{4}{3}\pi r^3$. If a spherical shell (hollow sphere) of radius x and thickness dx is cut out from the centre then the surface area and volume of the solid part is given by $4\pi r^2$ and $4\pi x^2 dx$ respectively

(j) Volume of a cylinder is $V = \pi r^2 l$

(k) Equation of a straight line is $y = mx + b$ ($b = y$ intercept, $m =$ slope $= \tan\theta$). The equation of a straight line in intercept form is $\frac{x}{a} + \frac{y}{b} = 1$.



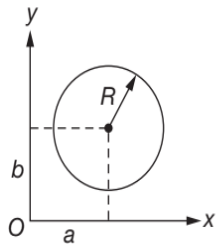


(l) Equation of a circle of radius R centred at the origin is $x^2 + y^2 = R^2$. Equation of a circle of radius R centred at $(a,\ b)$ is the general form of equation of a circle is

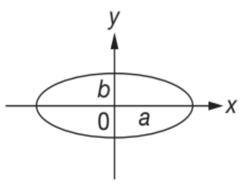
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

This circle has centre at $(-g,\ -f)$ with radius $r = \sqrt{g^2 + f^2 - c}$.

$$(x-a)^2 + (y-b)^2 = R^2$$

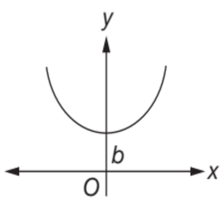


(m) Equation of an ellipse with the origin at its center is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a =$ semi-major axis, $b =$ semi-minor axis).



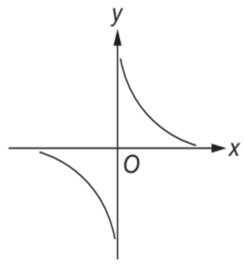
Ellipse

(n) Equation of a parabola whose vertex is at $y = b$ is $y = ax^2 + b$.



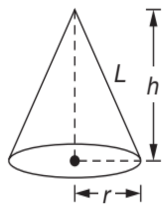
Parabola

(o) Equation of a rectangular hyperbola is $xy = \text{constant}$.



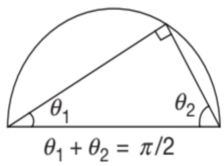
Rectangular hyperbola

(p) For a right circular cone of height h, radius r, the volume is $V = \frac{1}{3}\pi r^2 h$, Lateral surface area $= \pi r L$, where $L = \sqrt{r^2 + h^2}$.



Right circular cone

(q) Every triangle inscribed within a semicircle is a right triangle.



Trigonometry

(a) The angle θ is measured in radian.

$$180^\circ = \pi \text{ radian}$$

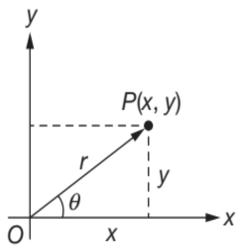
$$\Rightarrow \quad 1^\circ = \frac{\pi}{180} \text{ radian} \approx 0.02 \text{ radian}$$

Similarly 1 radian $= \frac{180}{\pi}$ degree $\approx 57^\circ$

> Degree to Radian: Multiply by $\frac{\pi}{180}$
>
> Radian to Degree: Multiply by $\frac{180}{\pi}$

(b) Basic Trigonometric Quantities

The sine, cosine and tangent functions in trigonometry are defined in terms of the ratio of the sides of a right triangle:



(i) $\sin\theta = \dfrac{\text{Side opposite } \theta}{\text{Hypotenuse}}$

$\Rightarrow \quad \sin\theta = \dfrac{a}{c} = \dfrac{\text{Perpendicular }(P)}{\text{Hypotenuse }(H)}$

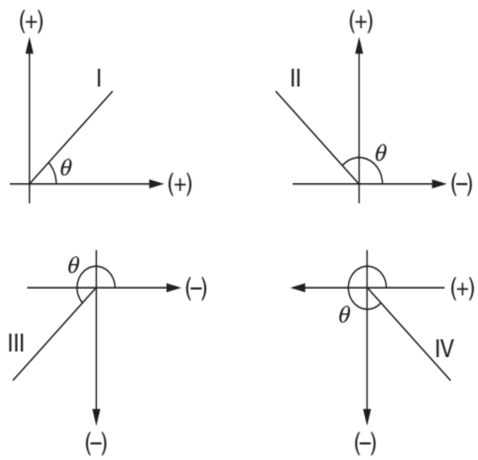
(ii) $\cos\theta = \dfrac{\text{Side adjacent } \theta}{\text{Hypotenuse}}$

$\Rightarrow \quad \cos\theta = \dfrac{b}{c} = \dfrac{\text{Base }(B)}{\text{Hypotenuse }(H)}$

(iii) $\tan\theta = \dfrac{\text{Side opposite } \theta}{\text{Side adjacent } \theta}$

$\Rightarrow \quad \sin\theta = \dfrac{a}{b} = \dfrac{\text{Perpendicular }(P)}{\text{Base }(B)}$

(c) Signs in the four quadrants.



	sinθ	cosθ	tanθ	cosecθ	secθ	cotθ
Quad I (All)	⊕	⊕	⊕	⊕	⊕	⊕
Quad II (sine)	⊕	⊖	⊖	⊕	⊖	⊖
Quad III (tan)	⊖	⊖	⊕	⊖	⊖	⊕
Quad IV (cos)	⊖	⊕	⊖	⊖	⊕	⊖

(d) From the above basic units and using the Pythagorean theorem, it follows that

$$\sin^2\theta + \cos^2\theta = 1; \quad \sec^2\theta = 1 + \tan^2\theta;$$

$$\text{cosec}^2\theta = 1 + \cot^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

(e) The cosecant, secant, and cotangent functions are defined by

(i) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ (ii) $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$

(f) The relations at the right follows directly from the right triangle above:

In First Quadrant

$\sin\theta = \cos(90° - \theta)$

$\cos\theta = \sin(90° - \theta)$

$\cot\theta = \tan(90° - \theta)$

In Second Quadrant:

$\sin(90° + \theta) = \cos\theta$	$\sin(180° - \theta) = \sin\theta$
$\cos(90° + \theta) = -\sin\theta$	$\cos(180° - \theta) = -\cos\theta$
$\tan(90° + \theta) = -\cot\theta$	$\tan(180° - \theta) = -\tan\theta$

In Third Quadrant:

$\sin(180° + \theta) = -\sin\theta$	$\sin(270° - \theta) = -\cos\theta$
$\cos(180° + \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$
$\tan(180° + \theta) = \tan\theta$	$\tan(270° - \theta) = \cot\theta$

In Fourth Quadrant:

$\sin(270° + \theta) = -\cos\theta$	$\sin(360° - \theta) = -\sin\theta$
$\cos(270° + \theta) = \sin\theta$	$\cos(360° - \theta) = \cos\theta$
$\tan(270° + \theta) = -\cot\theta$	$\tan(360° - \theta) = -\tan\theta$

(g) Some properties of trigonometric functions:

(i) $\sin(-\theta) = -\sin\theta$

(ii) $\cos(-\theta) = \cos\theta$

(iii) $\tan(-\theta) = -\tan\theta$

(h) Values for certain angles

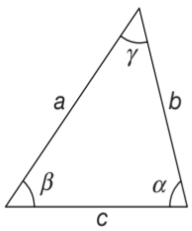
θ (in degree)	θ (in radian)	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	Not Defined
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	–1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
180°	π	0	–1	0

Also take a note that $\sin(37°) = \cos(53°) = \frac{3}{5}$

Sides and Angles of Triangle

(i) Sum of all the angles of Triangle is 180°.

$$\Rightarrow \alpha + \beta + \gamma = 180°$$



(j) Law of cosine

(i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

(ii) $b^2 = a^2 + c^2 - 2ac\cos\beta$

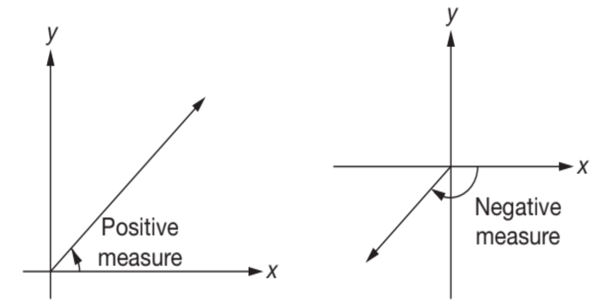
(iii) $c^2 = a^2 + b^2 - 2ab\cos\gamma$

(k) Law of sines (Lami's Theorem), for any triangle

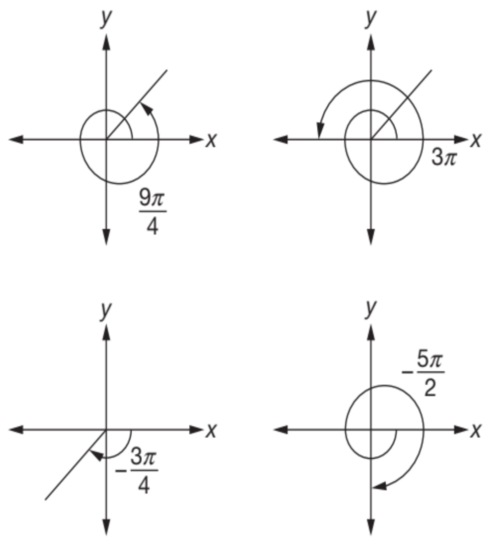
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

MEASUREMENT OF POSITIVE AND NEGATIVE ANGLES

Angles measured counterclockwise (CCW) from the positive x-axis are assigned positive measures whereas angles measured clockwise (CW) are assigned negative measures.



(Counter Clockwise Sense) (Clockwise Sense)



Some Trigonometric Identities

(a) $\sin^2\theta + \cos^2\theta = 1$

(b) $\text{cosec}^2\theta = 1 + \cot^2\theta$

(c) $\sec^2\theta = 1 + \tan^2\theta$

(d) $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta)$

(e) $\sin(2\theta) = 2\sin\theta\cos\theta$

(f) $\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 + \cos\theta)$

(g) $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

(h) $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

(i) $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$

(j) $\tan\left(\dfrac{\theta}{2}\right)=\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

(k) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

(l) $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

(m) $\sin C+\sin D=2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(n) $\cos A+\cos B=2\cos\left[\dfrac{1}{2}(A+B)\right]\cos\left[\dfrac{1}{2}(A-B)\right]$

(o) $\sin(3\theta)=3\sin\theta-4\sin^3\theta$

(p) $\cos(3\theta)=4\cos^3\theta-3\cos\theta$

(q) $\tan(3\theta)=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

(r) $\cos(2\theta)=\cos^2\theta-\sin^2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(s) $\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$

(t) $\cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)$

(u) $\cos C+\cos D=2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(v) $\sin C-\sin D=2\sin\left(\dfrac{C-D}{2}\right)\cos\left(\dfrac{C+D}{2}\right)$

(w) $\cos C-\cos D=-2\sin\left(\dfrac{C-D}{2}\right)\cos\left(\dfrac{C+D}{2}\right)$

ILLUSTRATION 5

Convert the following angles to radian.

(a) 45°	**(b)** 60°	**(c)** 120°
(d) 135°	**(e)** 210°	**(f)** 225°
(g) 270°	**(h)** 300°	**(i)** 330°

SOLUTION

(a) $45^\circ=45\times(1^\circ)=45\times\left(\dfrac{\pi}{180}\right)\text{ radian}=\dfrac{\pi}{4}\text{ radian}$

(b) $60^\circ=60\times(1^\circ)=60\times\left(\dfrac{\pi}{180}\right)\text{ radian}=\dfrac{\pi}{3}\text{ radian}$

(c) $120^\circ=120\times(1^\circ)=120\times\left(\dfrac{\pi}{180}\right)\text{ radian}$

$\Rightarrow\ 120^\circ=\dfrac{2\pi}{3}\text{ radian}$

(d) $135^\circ=135\times(1^\circ)=135\times\left(\dfrac{\pi}{180}\right)\text{ radian}$

$\Rightarrow\ 135^\circ=\dfrac{3\pi}{4}\text{ radian}$

(e) $210^\circ=210\times(1^\circ)=210\times\left(\dfrac{\pi}{180}\right)\text{ radian}$

$\Rightarrow\ 210^\circ=\dfrac{7\pi}{6}\text{ radian}$

(f) $225^\circ=225\times(1^\circ)=225\times\left(\dfrac{\pi}{180}\right)\text{ radian}$

$\Rightarrow\ 225^\circ=\dfrac{9\pi}{4}\text{ radian}$

(g) $270^\circ=270\times(1^\circ)=270\times\left(\dfrac{\pi}{180}\right)\text{ radian}$

$\Rightarrow\ 270^\circ=\dfrac{3\pi}{2}\text{ radian}$

(h) $300^\circ=300\times(1^\circ)=300\times\left(\dfrac{\pi}{180}\right)\text{ radian}$

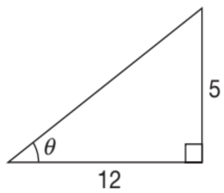
$\Rightarrow\ 300^\circ=\dfrac{5\pi}{3}\text{ radian}$

(i) $330^\circ=330\times(1^\circ)=330\times\left(\dfrac{\pi}{180}\right)\text{ radian}$

$\Rightarrow\ 330^\circ=\dfrac{11\pi}{6}\text{ radian}$

ILLUSTRATION 6

Find the six trigonometric ratios from the given figure.



SOLUTION

By Pythagoras Theorem, we have

$H^2=P^2+B^2$

$\Rightarrow\ H^2=5^2+12^2=169$

$\Rightarrow\ H=13$

$\Rightarrow\ \sin\theta=\dfrac{P}{H}=\dfrac{5}{13},\ \cos\theta=\dfrac{B}{H}=\dfrac{12}{13},\ \tan\theta=\dfrac{P}{B}=\dfrac{5}{12}$

$\Rightarrow\ \text{cosec}\,\theta=\dfrac{13}{5},\ \sec\theta=\dfrac{13}{12},\ \cot\theta=\dfrac{12}{5}$

ILLUSTRATION 7

Find the values of

(a) $\cos(120°)$ **(b)** $\sin(-1485°)$ **(c)** $\sin(300°)$

(d) $\cos(-60°)$ **(e)** $\tan(210°)$

SOLUTION

(a) $\cos(120°) = \cos(90° + 30°) = -\sin(30°) = -\frac{1}{2}$

(b) $\sin(-1485°) = -\sin(3 \times 360° + 45°)$

$\Rightarrow \sin(-1485°) = -\sin(45°) = -\frac{1}{\sqrt{2}}$

(c) $\sin(300°) = \sin(360° - 60°) = -\sin(60°) = \frac{-\sqrt{3}}{2}$

(d) $\cos(-60°) = \cos(60°) = \frac{1}{2}$

(e) $\tan(210°) = \tan(180° + 30°) = \tan(30°) = \frac{1}{\sqrt{3}}$

ILLUSTRATION 8

If $A = 60°$, then find the value of $\sin(2A)$.

SOLUTION

Since $\sin(2A) = 2\sin A\cos A$

So for $A = 60°$, we have

$$\sin(2A) = 2\sin A\cos A = 2\sin(60°)\cos(60°)$$

$$\Rightarrow \sin(2A) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

FACTORIAL

Factorial is always defined for a positive integral number, say n. Then

$$\underline{|n} = n! = n(n-1)(n-2)(n-3)..........\times 3 \times 2 \times 1$$

(read as n factorial)

Also, we observe that $\underline{|n} = n\underline{|n-1} = n(n-1)!$

Further more, we have $\underline{|1} = 1$ and $\underline{|0} = 1$.

SERIES EXPANSIONS

(a) $(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + ...$ {Binomial Expansion}

(b) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$ {Binomial Series}

(c) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ {Exponential Series}

(d) $\ln(1 \pm x) = \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \cdots$ {Logarithmic Series}

(e) $\sin x = x - \frac{x^3}{3!} + \frac{x^3}{5!} - \cdots$

(f) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ {x in radian}

(g) $\tan x = x + \frac{x^3}{3} + \frac{2x^3}{15} + \cdots |x| < \frac{\pi}{2}$ {Trigonometric Expansion}

(h) For $x \ll 1$, the following approximations can be used:

(i) $(1+x)^n \approx 1 + nx$ {Binomial Approximation}

(ii) $\ln(1 \pm x) \approx \pm x$ {Logarithmic Approximation}

(iii) $e^x \approx 1 + x$ {Exponential Approximation}

(iv) $\sin x \approx x$ {Trigonometric Approximation}

(v) $\cos x \approx 1$ {Trigonometric Approximation}

(vi) $\tan x \approx x$ {Trigonometric Approximation}

Conceptual Note(s)

In Binomial Series, if $|x| \ll 1$, then only the first two terms are significant. It is so because the values of second and the higher order terms become very small and hence can be neglected. So the following expressions can be re-written as

$$(1+x)^n = 1 + nx$$

$$\Rightarrow (1+x)^{-n} = 1 - nx$$

$$\Rightarrow (1-x)^n = 1 - nx$$

$$\Rightarrow (1-x)^{-n} = 1 + nx$$

ILLUSTRATION 9

Evaluate $(1001)^{1/3}$ upto six places of decimal.

SOLUTION

$$(1001)^{1/3}=(1000+1)^{1/3}=10(1+0.001)^{1/3}$$

By comparing the given equation with standard equation

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots$$

$$\Rightarrow\quad x=0.001 \text{ and } n=\frac{1}{3}$$

$$\Rightarrow\quad 10(1+0.001)^{1/3}=10\left[1+\frac{1}{3}(0.001)+\begin{pmatrix}\text{Neglected}\\ \text{Terms}\end{pmatrix}\right]$$

$$\Rightarrow\quad (1001)^{1/3}=10\left[1+0.00033-\frac{1}{9}(0.000001)+\ldots\ldots\right]$$

$$\Rightarrow\quad (1001)^{1/3}=10(1.0003301)=10.003301 \text{ (Approx.)}$$

ILLUSTRATION 10

The value of acceleration due to gravity (g_h) at a height h above the surface of earth is given by $g_h=\frac{gR^2}{(R+h)^2}$. Find the approximate value of g_h when $h \ll R$.

SOLUTION

$$g_h=g\left(\frac{R}{R+h}\right)^2$$

$$\Rightarrow\quad g_h=g\left(\frac{1}{1+\frac{h}{R}}\right)^2$$

$$\Rightarrow\quad g_h=\left(1+\frac{h}{R}\right)^{-2}$$

$$\Rightarrow\quad g_h=g\left[1+(-2)\frac{h}{R}+\frac{(-2)(-3)}{2!}\left(\frac{h}{R}\right)^2+\ldots\ldots\right]$$

$$\Rightarrow\quad g_h=g\left(1-\frac{2h}{R}\right)$$

FUNCTION: AN INTRODUCTION

A key idea in mathematical analysis and in Physics is the idea of dependence of one quantity on the other. A quantity depends on another if the variation of one of them is accompanied by a variation of other. We must have seen Mathematicians speaking about an **independent variable** and the **dependent variable**. Similarly in Physics, it is better to think in terms of **cause** and **effect** or **interdependent quantities**.

We are aware of the fact that the area of a circle depends upon its radius. Mathematically speaking, the area of a circle is a function of its radius.

Similarly in Physics we observe that the volume of a given mass of a gas at a fixed temperature is a function of the pressure of the gas. That is we can say that the **cause** of the change in temperature will produce an **effect** that produces the change in pressure of the gas.

REPRESENTATION OF A FUNCTION

A function is denoted by symbols like of $f(x)$, $F(x)$, $\phi(x)\ldots$ and is read as function of x. Thus if y is a function of x, we may write $y=f(x)$.

Conceptual Note(s)

It must be clearly understood that $f(x)$ does not mean f into x, but is only a symbolic way of representing some function of x.

The dependence of one quantity on another can be quantitatively expressed in three different ways:

(a) Tabular Presentation
(b) Graphical Presentation
(c) Mathematical Presentation/Equations

Tabular Presentation

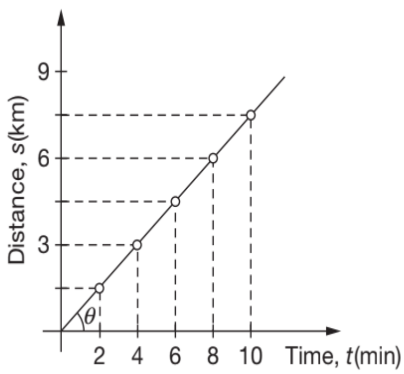
Let us consider the distance covered by an automobile, moving at constant speed, as a function of time. Data for a particular example of such motion may be presented numerically, as in the following **Table.** The exact mathematical relationship between the time and the distance in this example is not immediately obvious while examining the table. This is one of the disadvantages of tabular presentation. Although the numerical values can be precisely specified, they do not at once convey the clear picture of the dependence of variables on one another. This can be better visualised by drawing the graph for the varying quantities.

Table 1.2 Time and Distance for a moving Automobile

Elapsed Time (min)	Distance (km)
0	0
2	1.5
4	3.0
6	4.5
8	6.0
10	7.5

Graphical Presentation

Let us plot the same data on a graph as shown in the figure.



Variation of distance with time

Here we plot the Time (an independent variable) horizontally and the Distance (a dependent variable) vertically. Each pair of numbers in the Table 10.2 gives a single point on the graph. It is immediately seen that the points when joined give us a straight line.

Mathematical Presentation/Equations

The equation that fits the above tabular and graphical data is

$$s = 0.75t$$

Where s represents the distance in kilometre and t represents the time in minutes.

This equation can also be expressed as

$$s = v_0 t$$

where v_0 is a constant whose value in this example is

$$v_0 = 0.75 \text{ kmmin}^{-1}$$

The equation provides the most concise expression of a functional relationship.

SLOPE OF A LINE

The **slope** of a line in a graph is defined as the **tangent of the angle** (measured in anticlockwise direction) that the line makes with the positive direction of the horizontal axis. This angle is designated by θ. From the distance vs time plot shown in "Graphical Presentation", we have

$$\tan\theta = \frac{s}{t} = v_0$$

is the slope of the straight line that has been plotted for the variation of the distance with time.

Here we observe that the quantity $\tan\theta$ is a dimensional quantity. We always measure slope as a vertical increment divided by a horizontal increment on a graph, each increment being measured in the appropriate unit for the quantity in the problem.

ILLUSTRATION 11

If $y = f(x) = x^2 - 3x + 5$, find $f(0)$ and $f(1)$.

SOLUTION

To find the value of a function at a particular value of the independent variable, let us put the given value in the expression of the function and simplify. The result followings.

Here, $f(x) = x^2 - 3x + 5$

$$f(0) = (0)^2 - 3(0) + 5 = 5$$

$$f(1) = (1)^2 - 3(1) + 5 = 3$$

CONCEPT OF LIMIT OF FUNCTIONS: MEANING OF THE SYMBOL $x \to a$

When the independent variable is gradually taken to a definite value, say a, the dependent variable, i.e., the function will lead to another definite value, say l. This value is defined as the limiting value of the function as the independent variable approaches the given value a.

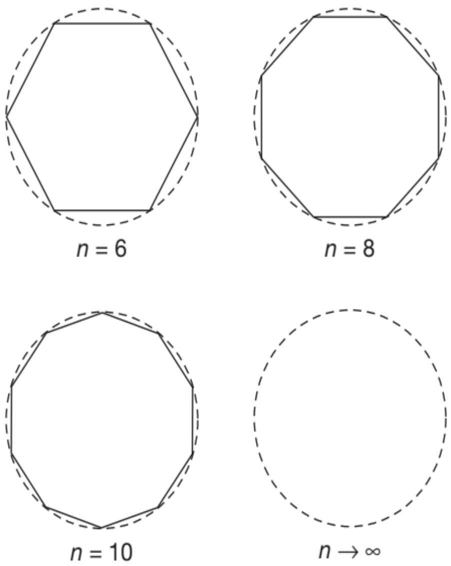
The arrow in the above symbol stands for gradual approach of x to a and the symbol is read as x tending to a. If $y = f(x)$ approaches a value l, as x approaches a, we say that the limiting value of $f(x)$ is l when x approaches a and this is symbolically written as

$$\lim_{x \to a} f(x) = l$$

This symbol is read as the limit of the function is l when x tends to a.

EXAMPLES

(a) Let us inscribe a polygon of n sides in a circle of radius a. The area A of the polygon will depend on the number of sides n. Hence $A = f(n)$.



As we increase the number of sides, the sides will be shorter and shorter in size, the area of the polygon will increase and ultimately when n is made infinitely large, the area of the polygon will become equal to the area of circle. Thus we may say that the limiting value of the area of a polygon of n sides inscribed in a circle is the area of the circle itself, as n tends to infinity. (Just think that the circle is a polygon having infinite number of sides)

Mathematically, $\lim_{n \to \infty} \text{Area} = \pi a^2$

(b) Let us consider the function $y = f(x) = x^2 - 1$. Let us find $\lim_{x \to 2} f(x)$.

When x approaches 2 from left, we observe $f(x)$ approaches 3 from the left.

x	$y = f(x) = x^2 - 1$
1.9	2.6100
1.99	2.96010
1.999	2.996001
1.9999	2.99960

When x approaches 2 from right, we observe $f(x)$ approaches 3 from the right.

x	$y = f(x) = x^2 - 1$
2.2	3.84000
2.1	3.41000
2.01	3.04010
2.001	3.004
2.0001	3.00040

From the above two tables, it is clear that as x tends to 2, $y = f(x) = x^2 - 1$ approaches or tends to 3

$$\Rightarrow \lim_{x \to 2} x^2 - 1 = 3$$

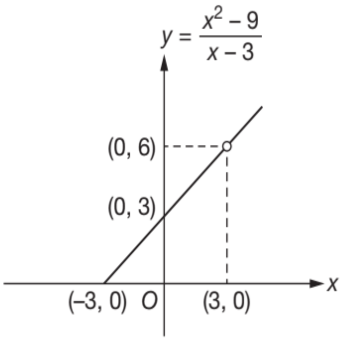
The process of finding the limiting value of a function in the above way is actually the fundamental way of finding the limiting value of a function called the **first principle**. But this is considered to be a lengthy process. Every time it is not possible to find the limiting value by this process and that too for all kinds of functions.

Simply by putting $x = 2$ in the above expression will give us the limiting value. Hence the shortcut method of finding the limiting value of a function is to make direct substitution of the limiting the value of the independent variable in the expression of the function. But this technique may not work for all kinds of functions. Actually we have learnt the concept of limit in an easy manner, but this concept is useful whenever we have to find the value of the functions at a point where they do not have indeterminate values. Consider another example to have a clear concept.

(c) Let us take the function

$$y = f(x) = \frac{x^2 - 9}{x - 3}$$

Here the student generally concludes that the above expression could also have been easily written as $y = f(x) = x + 3$. But this is only true when $x \neq 3$. Actually we shall not be able to find the value of this function at $x = 3$, because at $x = 3$ the function has a value $\frac{0}{0}$ which is an indeterminate value. For this let us again consider the values of x approaching from the left as well as the right towards 3.



When x approaches 3 from left, we observe $f(x)$ approaches 6 from the left.

x	$y = f(x) = \frac{x^2 - 9}{x - 3}$
2.9	5.9
2.99	5.99
2.999	2.9999
5.999	5.9999

When x approaches 3 from right, we observe $f(x)$ approaches 6 from the right.

x	$y = f(x) = \frac{x^2 - 9}{x - 3}$
3.1	6.1
3.01	6.01
3.001	3.0001
6.001	6.0001

From the above two tables, it is clear that as x tends to 3, $y = f(x) = \frac{x^2 - 9}{x - 3}$ approaches (or tends to) 6

$$\Rightarrow \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

Conceptual Note(s)

(a) Please note here that the function $y = \frac{x^2 - 9}{x - 3}$ is discontinuous at $x = 3$. A function is said to be continuous when you can draw it in one go with your pen without its tip losing contact with the paper. However, if we consider the function $y = x + 3$, we observe that it is continuous.

(b) Remember that you must not cancel the common factor from the numerator and the denominator till you are sure that the common factor is non zero.

So, the expression $\frac{x^3 y^4}{x y^2} = x^2 y^2$ only if $x \neq 0$ and $y \neq 0$.

(c) A very important formula of limit extensively used in Physics is

$$\lim_{\phi \to 0} \frac{\sin\phi}{\phi} = 1$$

DERIVATIVE OF A FUNCTION

Meaning of Δx

For a finite but small increment in x, we use symbol Δx. Please note that this is also not to be read as Δ multiplied by x. It stands for a small but finite increment in x and is treated as a single quantity. This should be read as 'delta x'.

Meaning of dx

Whenever a variable is changed by an infinitesimal (extremely small) amount, then that change is called the differential of the variable. This is denoted by dx and read as 'dee x'. Again please do not misinterpret this as d into x. dx is merely a representation standing for a very, very small increment in x. This should be treated as a single quantity just as $\sin\theta$ which again is not the product of sine and θ.

Slope has a simple physical meaning. It is the rate of change of the quantity being plotted vertically with respect to the quantity being plotted horizontally or we can say that it is the rate of change of dependent variable with respect to an independent variable. Mathematically, **derivative of the function** gives the instantaneous slope of that function.

Slope and Derivative of a Function

Slope has a simple physical meaning. It is the rate of change of the quantity being plotted vertically with respect to the quantity being plotted horizontally or we can say that it is the rate of change of dependent variable with respect to an independent variable. Mathematically,

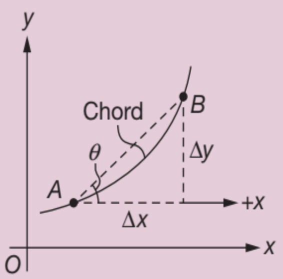
(a) $\frac{dy}{dx}$ (derivative of the function y w.r.t. x) gives the instantaneous slope of the function at a point. $\frac{dy}{dx}$ is also called as the Rate Measurer.

(b) $\frac{\Delta y}{\Delta x}$ gives the average slope of a curve y between two points.

Conceptual Note(s)

Please note that:

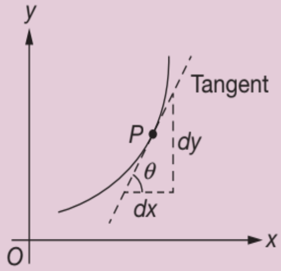
(a) Average slope is always in an interval i.e. between two points and it is the slope of the chord that joins the two points.



So, average slope of the curve between the points A and B is

$$\frac{\Delta y}{\Delta x} = \tan\theta = \tan\begin{pmatrix}\text{of angle which chord} \\ \text{joining points } A \text{ and } B \\ \text{makes with} + x \text{ direction}\end{pmatrix}$$

(b) Instantaneous slope is at a point or an instant and it is the slope of the tangent drawn to the curve at that point.



So, instantaneous slope of the curve at the point P is

$$\frac{dy}{dx} = \tan\theta = \tan\begin{pmatrix}\text{of angle which tangent to} \\ \text{point } P \text{ makes with} \\ +x \text{ direction}\end{pmatrix}$$

DEFINITION OF DIFFERENTIAL COEFFICIENT

Consider a function $y = f(x)$. Let the value of x changes to $x + \Delta x$. Correspondingly the value of y changes from y to $y + \Delta y$. The limiting value of the ratio $\frac{\Delta y}{\Delta x}$ when Δx tends to zero is called the differential coefficient of y with respect to x. It is denoted by the symbol $\frac{dy}{dx}$.

Thus, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

The process of finding the differential coefficient of a function is called differentiation or the derivative of the function and this signifies the instantaneous slope of that function.

MATHEMATICAL DEFINITION

Consider a function $y = f(x)$. Let the value of x changes to $x + \Delta x$. Correspondingly the value of y changes from y to $y + \Delta y$. The limiting value of the ratio $\frac{\Delta y}{\Delta x}$ when Δx tends to zero is called the differential coefficient of y with respect to x. It is denoted by the symbol $\frac{dy}{dx}$.

Thus, $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

So, $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(x)] = f'(x)$

If $y = f(x)$

then, $y + \Delta y = f(x + \Delta x)$

$\Rightarrow \quad y + \Delta y - y = f(x + \Delta x) - f(x)$

$\Rightarrow \quad \Delta y = f(x + \Delta x) - f(x)$

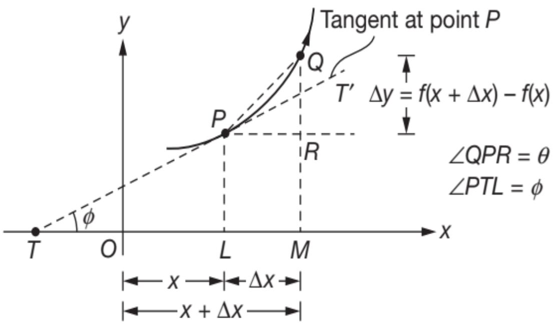
$\Rightarrow \quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$\Rightarrow \quad \dfrac{dy}{dx} = f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

The derivative of $f(x)$ at $x = a$ is denoted by $f'(a)$. In other words the slope of the function $y = f(x)$ at $x = a$ is given by $f'(a)$.

GEOMETRICAL INTERPRETATION OF DERIVATIVE

Let us consider the graph of $y = f(x)$ as shown in figure.



Let P and Q be the two points on it. Then

$PR = LM = \Delta x$

$QR = \Delta y$

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The average slope of the curve in the internal PQ.
As $Q \to P$ along the curve, $\Delta x \to 0$, $\theta \to \phi$ and PQ becomes tangent TPT' at P.

$\therefore \quad \tan\phi = \lim\limits_{\theta \to \phi} \tan\theta = \dfrac{d}{dx}(x^n) = \dfrac{dy}{dx}$ at P

$\tan\phi\left(\text{i.e., } \dfrac{dy}{dx} \text{ at } P\right)$ is the slope of tangent at P.

RULES OF DIFFERENTIATION

The process of finding the derivative of a function is called differentiating the function.
Differentiation obeys several simple rules that are worth **Committing To Memory (CTM)**.

RULE 1:

Derivative of a constant is zero.

$$\frac{d}{dx}(\text{constant}) = 0$$

RULE 2:

The derivative of a constant times a function is the constant times the derivative of the function.

$$\frac{d}{dx}(au) = a\frac{du}{dx}$$

RULE 3:

The derivative of the sum of the functions is the sum of their derivatives.

$$\frac{d}{dx}(u \pm v \pm w \pm ....) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm .....$$

RULE 4: PRODUCT RULE

Derivative of product of two functions is given as

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(①\times②) = ①\frac{d}{dx}② + ②\frac{d}{dx}①$$

RULE 5: QUOTIENT RULE

Derivative of a quotient is given as

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\left(\frac{\text{Num}^r}{\text{Den}^r}\right)=\frac{(\text{Den}^r)\frac{d}{dx}(\text{Num}^r)-(\text{Num}^r)\frac{d}{dx}(\text{Den}^r)}{\left(\text{Den}^r\right)^2}$$

RULE 6: CHAIN RULE

Suppose f is a function of u, which in turn is a function of x. The derivative $\frac{df}{dx}$ can be written as the product of two derivatives.

$$\frac{df}{dx}=\frac{df}{du}\times\frac{du}{dx}$$

IMPORTANT DIFFERENTIAL FORMULAE

$\frac{d}{dx}(\text{constant})=\text{Zero}$	$\frac{d}{dx}[\cos(ax+b)]=-a\sin(ax+b)$
$\frac{d}{dx}(u\pm v\pm w)=\frac{du}{dx}\pm\frac{dv}{dx}\pm\frac{dw}{dx}$	$\frac{d}{dx}[\tan(ax+b)]=a\sec^2(ax+b)$
$\frac{d}{dx}(x^n)=nx^{n-1}$	$\frac{d}{dx}[\sec(ax+b)]=a\sec(ax+b)\tan(ax+b)$
$\frac{d}{dx}(kx^n)=k(nx^{n-1})$	$\frac{d}{dx}[\cot(ax+b)]=-a\,\text{cosec}^2(ax+b)$
$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	$\frac{d}{dx}[\text{cosec}(ax+b)]=-a\,\text{cosec}(ax+b)\cot(ax+b)$
$\frac{d}{dx}(uvw)=uv\frac{dw}{dx}+uw\frac{dv}{dx}+vw\frac{du}{dx}$	$\frac{d}{dx}(\log_e x)=\frac{1}{x}$
$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$	$\frac{d}{dx}[\log_e(ax+b)]=\frac{a}{ax+b}$
$\frac{d}{dx}\left(\frac{\text{Num}^r}{\text{Den}^r}\right)=\frac{(\text{Den}^r)\frac{d}{dx}(\text{Num}^r)-(\text{Num}^r)\frac{d}{dx}(\text{Den}^r)}{(\text{Den}^r)^2}$	$\frac{d}{dx}(e^x)=e^x$
If y is a function of u and u is a function of x, then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$	$\frac{d}{dx}(a^x)=a^x\log_e a$
If y is a function of u, u a function of u, v a function of w and w a function of x, then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dw}\times\frac{dw}{dx}$	$\frac{d}{dx}(e^{kx})=ke^{kx}$
$\frac{d}{dx}(\sin x)=\cos x$	$\frac{d}{dx}(a^{kx})=ka^{kx}\log_e a$
$\frac{d}{dx}(\cos x)=-\sin x$	$\frac{d}{dx}[\log_e(\sec x+\tan x)]=\sec x$

$\frac{d}{dx}(\tan x)=\sec^2 x$	$\frac{d}{dx}\left[\log_e(\cot x+\operatorname{cosec}x)\right]=-\operatorname{cosec}x$
$\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$	$\frac{d}{dx}(x\log_e x-x)=\log_e x$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\frac{d}{dx}\left[\log_e(\cos x)\right]=-\tan x$
$\frac{d}{dx}(\operatorname{cosec}x)=-\operatorname{cosec}x\cot x$	$\frac{d}{dx}\left[\log_e(\sin x)\right]=\cot x$
$\frac{d}{dx}[\sin(ax+b)]=a\cos(ax+b)$	$\frac{d}{dx}(ax^2+bx+c)^n=(2ax+b)\left[n(ax^2+bx+c)^{n-1}\right]$

ILLUSTRATION 12

Find the derivative of $y=3x^2$.

SOLUTION

$$\frac{dy}{dx}=3\frac{d}{dx}(x^2)=3(2x)=6x$$

$$\left\{\because \frac{d}{dx}(x^n)=nx^{n-1}\right\}$$

ILLUSTRATION 13

Find the derivative of $y=x^3+3x^2$.

SOLUTION

Since the derivative of the sum is the sum of the derivatives, so

$$\frac{dy}{dx}=\frac{d}{dx}(x^3+3x^2)$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x^3)+\frac{d}{dx}(3x^2) \qquad \text{(using rule 3)}$$

$$\Rightarrow \frac{dy}{dx}=3x^2+6x$$

ILLUSTRATION 14

Find the derivative of $y=\sin(x^2)$.

SOLUTION

Let us assume $u=x^2$, then $y=\sin u$.

Then $\frac{dy}{du}=\cos u$ and $\frac{du}{dx}=2x$

Since, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$ ($\because$ of chain rule)

$$\Rightarrow \frac{dy}{dx}=(\cos u)(2x)=2x\cos u$$

$$\Rightarrow \frac{dy}{dx}=2x\cos(x^2)$$

ILLUSTRATION 15

If $y=\sin x+\cos x$, then find $\frac{dy}{dx}$.

SOLUTION

Since $y=\sin x+\cos x$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(\sin x+\cos x)$$

Using RULE 3, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sin x)+\frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{dy}{dx}=\cos x-\sin x$$

ILLUSTRATION 16

Find the derivative of $y=x\sin x$.

SOLUTION

$$\frac{dy}{dx}=x\frac{d}{dx}(\sin x)+(\sin x)\left(\frac{dx}{dx}\right)=x\cos x+\sin x$$

(using product rule)

ILLUSTRATION 17

Differentiate the following w.r.t. x.

(a) $\sin x-\cos x$

(b) $\sin x+e^x$

SOLUTION

(a) $\frac{d}{dx}(\sin x - \cos x) = \frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x)$

$\Rightarrow \frac{d}{dx}(\sin x - \cos x) = \cos x + \sin x$

(b) $\frac{d}{dx}(\sin x + e^x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) = \cos x + e^x$

ILLUSTRATION 18

Find the derivative of $y = \frac{x^2+1}{x}$.

SOLUTION

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(x^2+1) - \left[\frac{d}{dx}(x)\right](x^2+1)}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2-(x^2+1)}{x^2} = \frac{x^2-1}{x^2} \quad \text{(using quotient rule)}$$

ILLUSTRATION 19

If $y = \sqrt{x}$, then find $\frac{dy}{dx}$.

SOLUTION

Given $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}} \quad \left\{\because \frac{d}{dx}(x^n) = nx^{n-1}\right\}$$

ILLUSTRATION 20

Differentiate the following w.r.t. x.

(a) x^3 **(b)** $\sqrt{x}$ **(c)** ax^2+bx+c

(d) $2x^3 - e^x$ **(e)** $6\log e^x - \sqrt{x} - 7$

SOLUTION

(a) $\frac{d}{dx}(x^3) = 3x^2$

(b) $\frac{d}{dx}(x)^{1/2} = \frac{1}{2}(x)^{\frac{1}{2}-1} = \frac{1}{2}(x)^{-1/2} = \frac{1}{2\sqrt{x}}$

(c) $\frac{d}{dx}(ax^2+bx+c) = a\frac{d}{dx}(x^2) + b\frac{d}{dx}(x) + \frac{d}{dx}(c)$

$\Rightarrow \frac{d}{dx}(ax^2+bx+c) = 2ax+b$

(d) $\frac{d}{dx}(2x^3-e^x) = 2\frac{d}{dx}(x^3) - \frac{d}{dx}(e^x) = 6x^2 - e^x$

(e) Let $y = 6\log e^x - \sqrt{x} - 7$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(6\log_e x - \sqrt{x} - 7\right)$

$\Rightarrow \frac{dy}{dx} = 6\frac{d}{dx}(\log_e x) - \frac{d}{dx}(x^{1/2}) - \frac{d}{dx}(7)$

$\Rightarrow \frac{dy}{dx} = \frac{6}{x} - \frac{1}{2\sqrt{x}}$

ILLUSTRATION 21

Differentiate the following w.r.t. t.

(a) $\sin(t^2)$ **(b)** $e^{\sin t}$

(c) $\sin(\omega t + \theta)$

SOLUTION

(a) $\frac{d}{dt}(\sin t^2) = \cos t^2 \frac{d}{dt}(t^2) = 2t\cos t^2$

(b) $\frac{d}{dt}(e^{\sin t}) = e^{\sin t}\frac{d}{dt}(\sin t) = e^{\sin t}.\cos t$

(c) $\frac{d}{dt}[\sin(\omega t+\theta)] = \cos(\omega t+\theta)\frac{d}{dt}(\omega t+\theta)$

$\Rightarrow \frac{d}{dt}[\sin(\omega t+\theta)] = \omega\cos(\omega t+\theta)$

ILLUSTRATION 22

Differentiate $\frac{x^2+e^x}{\log x+20}$ w.r.t. x.

SOLUTION

Let $y = \frac{x^2+e^x}{\log x+20}$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+e^x}{\log x+20}\right)$

$$\frac{dy}{dx} = \frac{(\log x+20)\frac{d}{dx}(x^2+e^x) - (x^2+e^x)\frac{d}{dx}(\log x+20)}{(\log x+20)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\log x+20)(2x+e^x) - (x^2+e^x)\left(\frac{1}{x}+0\right)}{(\log x+20)^2}$$

ILLUSTRATION 23

If $y = a\sin(\omega t)$, where a and ω are constants, find $\frac{dy}{dt}$.

SOLUTION

Given $y = a\sin\omega t$

$$\Rightarrow \frac{dy}{dt} = \frac{d}{dt}(a\sin\omega t) = a\frac{d}{dt}(\sin\omega t)$$

$$\Rightarrow \frac{dy}{dt} = a\cos(\omega t)\frac{d}{dt}(\omega t) \qquad (\because \text{ of chain rule})$$

$$\Rightarrow \frac{dy}{dt} = a\omega\cos(\omega t)$$

ILLUSTRATION 24

Find the slope of the function

$$y = \sin x + \frac{1}{x^2} - 3\log_e x \text{ at } x = \frac{\pi}{2}.$$

SOLUTION

We have $y = \sin x + \frac{1}{x^2} - 3\log_e x$

Slope of the function is the derivative of the function. So,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) + \frac{d}{dx}\left(\frac{1}{x^2}\right) - 3\frac{d}{dx}(\log_e x)$$

$$\Rightarrow \frac{dy}{dx} = \cos x - \frac{2}{x^3} - \frac{3}{x}$$

Slope at $x = \frac{\pi}{2}$ is given by

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - \frac{2}{(\pi/3)^3} - \frac{3}{(\pi/2)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = 0 - \frac{16}{\pi^3} - \frac{6}{\pi} = -\frac{16}{\pi^3} - \frac{6}{\pi}$$

ILLUSTRATION 25

If $y = 3e^x - 5x^3 + 3$, then find $\frac{dy}{dx}$.

SOLUTION

Given $y = 3e^x - 5x^3 + 3$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(3e^x) - \frac{d}{dx}(5x^3) + \frac{d}{dx}(3) \qquad (\because \text{ of rule 3})$$

$$\Rightarrow \frac{dy}{dx} = 3e^x - 5(3x^2) + 0 = 3e^x - 15x^2.$$

ILLUSTRATION 26

If $y = x^2\sin x$, find $\frac{dy}{dx}$.

SOLUTION

Given $y = x^2\sin x$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2\sin x)$$

$$\Rightarrow \frac{dy}{dx} = x^2 \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(x^2)$$

(using product rule)

$$\Rightarrow \frac{dy}{dx} = x^2\cos x + 2x\sin x.$$

ILLUSTRATION 27

Find the slope of the tangent to the curve $y = 3x^2 - 5$ at the point $(2, 7)$.

SOLUTION

We have

$$y = 3x^2 - 5$$

$$\Rightarrow \frac{dy}{dx} = 6x$$

At point $(2,\ 7)$, we have $\frac{dy}{dx} = \tan\theta = 6(2) = 12$

$$\Rightarrow \tan\theta = 12$$

ILLUSTRATION 28

Find the inclination with the x-axis of the tangent to the curve $y^2 = 4x$ at $(1,\ 2)$.

SOLUTION

Given $y^2 = 4x$

$$\Rightarrow 2y\frac{dy}{dx} = 4 \qquad \text{(taking derivative of both sides)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

At the point $(1, 2)$

$$\frac{dy}{dx}=\frac{2}{2}=1$$

$\Rightarrow$ $\tan\theta=1$

$\Rightarrow$ $\theta=45^\circ$

ILLUSTRATION 29

Find $\frac{dy}{dx}$ for $y=\frac{2}{\sqrt{x}}+x\sqrt{x}$.

SOLUTION

$$\frac{dy}{dx}=\frac{d}{dx}\left(2x^{-\frac{1}{2}}+x^{\frac{3}{2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=2\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}+\frac{3}{2}x^{\frac{3}{2}-1}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{x^{\frac{3}{2}}}+\frac{3}{2\sqrt{x}}$$

ILLUSTRATION 30

If we have $2y-y^2-x+x^3+9=0$, then calculate $\frac{dy}{dx}$ at $x=2$.

SOLUTION

Let us firstly calculate value of y, when x equals 2. For this, we have

$$2y-y^2-2+8+9=0$$

$$\Rightarrow 2y-y^2+15=0$$

$$\Rightarrow y^2-2y-15=0$$

$$\Rightarrow y^2-5y+3y-15=0$$

$$\Rightarrow (y-5)(y+3)=0$$

$$\Rightarrow y=5,\ y=-3$$

So, we are to calculate $\frac{dy}{dx}$ at $(2,\ 5)$ and $(2,\ -3)$.

Now, $2y-y^2-x+x^3+9=0$

$$\Rightarrow 2\frac{dy}{dx}-2y\frac{dy}{dx}-1+3x^2=0$$

$$\Rightarrow 2\frac{dy}{dx}-2y\frac{dy}{dx}=1-3x^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-2x^2}{2-2y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2,\ y=-5}=\frac{1-2(2)^2}{2-2(5)}=\frac{1-8}{2-10}=\frac{7}{8}$$

and $$\left.\frac{dy}{dx}\right|_{x=2,\ y=-3}=\frac{1-2(-3)^2}{1-2(-3)}=\frac{1-18}{1+6}=-\frac{17}{7}$$

APPLICATIONS OF DERIVATIVE

With the help of differentiation, we can

(a) check whether the function is increasing or decreasing.

(b) find the maximum and minimum value(s) of a function.

(c) calculate the rate of change of quantity (say y) w.r.t. another quantity (say x).

INCREASING AND DECREASING FUNCTION

A function $f(x)$ is said to be increasing if $f(x)$ increases as x increases, and decreasing if $f(x)$ decreases as x increases.

In other words, if $x_1<x_2$

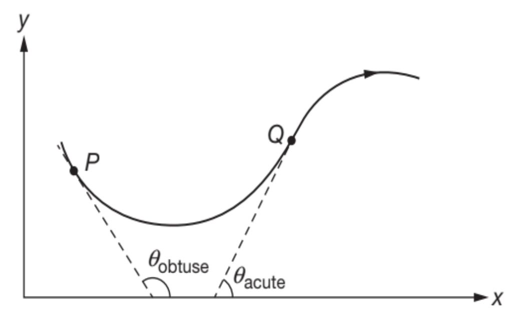
$\Rightarrow$ $f(x_1)<f(x_2)$ then $f(x)$ is increasing if $x_1<x_2$

$\Rightarrow$ $f(x_1)>f(x_2)$ then $f(x)$ is decreasing

As shown in the figure, when $f(x)$ is increasing, the tangent to the curve at any point, say P, makes an acute angle with positive x-axis. The slope of the tangent is positive.

Thus, $\tan\theta=\frac{dy}{dx}>0$

As shown in the figure, when $f(x)$ is decreasing, the tangent to the curve at any point, say P, makes an obtuse angle with positive x-axis. The slope of the tangent is negative.



Thus, $\tan\theta=\frac{dy}{dx}<0$

MAXIMUM AND MINIMUM VALUES OF A FUNCTION

Let us suppose a quantity y depends upon another quantity x in a manner shown in the figure. It becomes maximum at x_1 and minimum at x_2. At those points, the tangent to the curve is parallel to the x-axis and hence its slope is $\tan\theta = 0$.

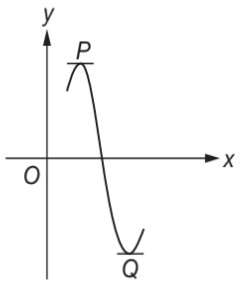
But the slope of curve $= \frac{dy}{dx}$

So, for the maximum or minimum value of y, we have $\frac{dy}{dx} = 0$

Just before the maximum point, the slope is positive. At the maximum point, it is zero And just after the maximum point, it is negative. Thus, $\frac{dy}{dx}$ decreases at the maximum point, i.e., the rate of change of $\frac{dy}{dx}$ is negative at the maximum point,

So $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$

$\Rightarrow \frac{d^2y}{dx^2} < 0$



The maximum and minimum of a function

Hence, condition for the maximum value of y is

$\frac{dy}{dx} = 0$ and

$\frac{d^2y}{dx^2} < 0$

Similarly, at a minimum point, the slope changes from negative to positive. The slope increases at such a point. Hence, $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$

Hence, condition for the minimum value of y is

$\frac{dy}{dx} = 0$ and

$\frac{d^2y}{dx^2} > 0$

As shown in the figure, at the point of maximum and minimum of a function the slope of the tangent at the point is zero.

Thus, $\tan\theta = \frac{dy}{dx} = 0$

Conceptual Note(s)

(a) $f(x)$ is maximum at a point $x = a$, if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes in sign from positive to negative when x passes through the point $x = a$. In other words, the second derivative of the function at $x = a$ is negative

i.e., $f''(a) < 0$.

(b) $f(x)$ is minimum at a point $x = a$, if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes in sign from negative to positive when x passes through the point $x = a$. In other words, the second derivative of the function at $x = a$ is positive.

i.e., $f''(a) > 0$.

ILLUSTRATION 31

Consider a function $y = \sin x + \cos x$. Find the maximum value of the function.

SOLUTION

$y = \sin x + \cos x$

$\Rightarrow \frac{dy}{dx} = \cos x - \sin x$

For a function to be a MAXIMUM

$\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow \cos x - \sin x = 0$

$\Rightarrow \tan x = 1$

$\Rightarrow \alpha = \frac{\pi}{4}$

Verification

$$\frac{d^2y}{dx^2} = -\sin x - \cos x$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{\text{at } x=\frac{\pi}{4}} = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\sqrt{2}$$

Since, at $x = \frac{\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So y is MAXIMUM at $x = \frac{\pi}{4}$

Maximum Value

$$y = \sin x + \cos x$$

$$\Rightarrow \left. y\right|_{\text{at } x=\frac{\pi}{4}} = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \sqrt{2}$$

ILLUSTRATION 32

The height reached in time t by a particle thrown upward with a speed u is given by $h = ut - \frac{1}{2}gt^2$. Find the time taken in reaching the maximum height.

SOLUTION

For maximum height $\frac{dh}{dt} = 0$

$$\frac{d}{dt}\left[ut - \frac{1}{2}gt^2\right] = u - \frac{2gt}{2} = 0$$

$$\Rightarrow t = \frac{u}{g}$$

ILLUSTRATION 33

The distance travelled by a body as a function of time is given by $x = t^3 - 3t^2 + 6$, where x is m and t is in S. Find the maximum and minimum displacement of the body from the origin. Also find the time at which it occurs.

SOLUTION

Differentiating x with respect to t, we get

$$\frac{dx}{dt} = 3t^2 - 6t$$

For maximum and minimum displacements $\frac{dx}{dt} = 0$

$$\therefore \quad 3t^2 - 6t = 0$$

$$\Rightarrow \quad 3t(t-2) = 0$$

$$\Rightarrow \quad t = 0 \text{ and } t = 2 \text{ s}$$

Once again differentiating w.r.t. time t, we get

$$\frac{d^2x}{dt^2} = 6t - 6$$

At $t = 0$, $\frac{d^2x}{dt^2} = -6 < 0$ (maximum)

At $t = 2$ s, $\frac{d^2x}{dt^2} = 6 > 0$ (minimum)

The maximum displacement occurs at $t = 0$ and is equal to

$$(x)_{max} = (0)^3 - 3(0)^2 + 6 = 6 \text{ m}$$

The minimum displacement occurs at $t = 2$ s and is equal to

$$(x)_{min} = (2)^3 - 3(2)^2 + 6 = 2 \text{ m}$$

$\frac{dy}{dx}$ AS RATE MEASURE

The rate of change of a quantity (y) with respect to another quantity (x) is defined as the ratio of the change of y to change is x, however small the change in x may be

If Δy be the change in y corresponding to a change Δx is x, then according to the definition, the rate of change of y with respect to x is the limiting value of the ratio $\frac{\Delta y}{\Delta x}$ when Δx tends to zero.

So, rate of change of y with respect to x is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Conceptual Note(s)

When we simply say rate of change y, we mean change of y with respect to time. So, the rate of change of y is $\frac{dy}{dt}$.

ILLUSTRATION 34

The area of a blot of ink is growing such that after t second, its area is given by $A = (3t^2 + 7)$ cm^2. Calculate the rate of increase of area at $t = 5$ seconds.

SOLUTION

Given $A = 3t^2 + 7$

$$\therefore \quad \frac{dA}{dt} = 6t$$

$$\Rightarrow \left(\frac{dA}{dt}\right)_{t=5} = 6 \times 5 = 30 \text{ cms}^{-2}$$

ILLUSTRATION 35

A metal ring is being heated so that at any instant of time t in second, its area is given by $A = 3t^2 + \frac{t}{3} + 2 \text{ m}^2$. What will be the rate of increase of area at $t = 10$ sec.

SOLUTION

Rate of increase of area

$$\frac{dA}{dt} = \frac{d}{dt}\left(3t^2 + \frac{t}{3} + 2\right) = 6t + \frac{1}{3}$$

$$\left(\frac{dA}{dt}\right)_{t=10\text{sec}} = 6 \times 10 + \frac{1}{3} = \frac{181}{3} \text{ m}^2\text{s}^{-1}$$

ILLUSTRATION 36

Find the rate of change in area of a square of side 4 cm when its side is increasing at the rate of 0.01 cm per second.

SOLUTION

Let A be the area of the square; a be the length of the side

We have

$$A = a^2$$

$$\Rightarrow \frac{dA}{dt} = \frac{d}{dt}\left(a^2\right) = \frac{d}{da}\left(a^2\right) \times \frac{da}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2a \times \frac{da}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2 \times 4 \times .01 = 0.8 \text{ cms}^{-2}$$

ILLUSTRATION 37

The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cms^{-1}. Determine the rate of increase in its volume when the radius is 1 cm.

SOLUTION

Volume of the spherical bubble $V = \frac{4}{3}\pi R^3$

Differentiating both sides w.r.t. time

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi R^3\right)$$

$$\Rightarrow \frac{dV}{dt} = \left(\frac{4}{3}\pi\right)\left(3R^2 \frac{dR}{dt}\right)$$

$$\Rightarrow \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$$

at $R = 1$ cm and when $\frac{dR}{dt} = \frac{1}{2}$ cms^{-1}, we have

$$\frac{dV}{dt} = 4\pi \times (1)^2 \times \frac{1}{2} = 2\pi \text{ cm}^3\text{s}^{-1}$$

INTEGRATION: AN INTRODUCTION

The word "integral" simply means "the whole", and the process of adding or summing up a large number of infinitesimal elements of a quantity is called integration.

If x be supposed to be made up of a large number of infinitesimal elements each equal to dx, it is obvious that if we add up all these dx together, we shall get the total x. Mathematically we put it as $\int dx = x$ (read it as integral of dx equals x). **Integration is the inverse operation of differentiation**.

Integration of $f(x)$ simply means finding the function $I(x)$ whose derivative is equal to $f(x)$.

Mathematically, $f(x) = \frac{dI}{dx}$

$$\Rightarrow I(x) = \int f(x)\, dx$$

In the above expression, $f(x)$ is called the integrand. The symbol, $\int$ is the symbol of integration and dx indicates the variable of integration.

The function $I(x)$ is also known sometimes as the anti derivative of $f(x)$.

"The result of an indefinite integral, when differentiated, will give you the function that has been integrated."

Table 1.3 Some Indefinite Integrals

$\int x^n dx = \frac{x^{n+1}}{n+1}$ (provided $n \neq -1$)	$\int xe^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$
$\int \frac{dx}{x} = \int x^{-1} dx = \ln x$	$\int \frac{dx}{a+be^{cx}} = \frac{x}{a} - \frac{1}{ac}\ln(a+be^{cx})$
$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$	$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$
$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$	$\int \cos(ax)dx = \frac{1}{a}\sin(ax)$
$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	$\int \tan(ax)dx = -\frac{1}{a}\ln(\cos(ax)) = \frac{1}{a}(\sec(ax))$
$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ $(a^2-x^2>0)$	$\int \cot(ax)dx = \frac{1}{a}\ln(\sin(ax))$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$ $(x^2-a^2>0)$	$\int \sec(ax)dx = \frac{1}{a}\ln(\sec(ax)+\tan(ax)) = \frac{1}{a}\ln\left[\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right]$
$\int \frac{xdx}{a^2 \pm x^2} = \pm\frac{1}{2}\ln(a^2 \pm x^2)$	$\int \text{cosec}(ax)dx = \frac{1}{a}\ln(\text{cosec}(ax) - \cot(ax)) = \frac{1}{a}\left[\ln\left(\tan\frac{ax}{2}\right)\right]$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right)(a^2-x^2>0)$	$\int \sin^2(ax)dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$
$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$	$\int \cos^2(ax)dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)$	$\int \frac{dx}{\sin^2(ax)} = \int \text{cosec}^2(ax)dx = -\frac{1}{a}\cot(ax)$
$\int \frac{xdx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$	$\int \frac{dx}{\cos^2(ax)} = \int \sec^2(ax)dx = \frac{1}{a}\tan(ax)$
$\int \sqrt{a^2-x^2}dx = -\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right)$	$\int \tan^2(ax)dx = \frac{1}{a}(\tan(ax)) - x$
$\int x\sqrt{a^2-x^2}dx = -\frac{1}{3}\left(a^2-x^2\right)^{\frac{3}{2}}$	$\int \cot^2(ax)dx = -\frac{1}{a}(\cot(ax)) - x$
$\int \sqrt{x^2 \pm a^2}dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln(x+\sqrt{x^2 \pm a^2})\right]$	$\int \sin^{-1}(ax)dx = x(\sin^{-1}(ax)) + \frac{\sqrt{1-a^2x^2}}{a}$
$\int x\sqrt{x^2 \pm a^2}dx = \frac{1}{3}\left(x^2 \pm a^2\right)^{3/2}$	$\int \cos^{-1}(ax)dx = x(\cos^{-1}(ax)) - \frac{\sqrt{1-a^2x^2}}{a}$
$\int e^{ax}dx = \frac{1}{a}e^{ax}$	$\int \tan^{-1}(ax)dx = x(\tan^{-1}(ax)) - \frac{1}{2a}\ln(1+a^2x^2)$
$\int e^{-ax}dx = -\frac{1}{a}e^{-ax}$	$\int \cot^{-1}(ax)dx = x(\cot^{-1}(ax)) + \frac{1}{2a}\ln(1+a^2x^2)$

(an arbitrary constant should be added to each of these integrals)

After each integral one must add a constant. The reason for adding a constant is given as follows:

The differential of x^{n+1} is $(n+1)x^n dx$. The differential of $\left(x^{n+1}+c\right)$ is $(n+1)x^n dx$ because the differential co-efficient of a constant is zero. Hence in general one has to add a constant after performing an integration. This constant is called the **constant of integration**.

RULES FOR INTEGRATION

(a) $\int c\,dx = c\int dx$, where c is a constant.

(b) $\int (u \pm v)dx = \int u\,dx \pm \int v\,dx$, where u and v are the function of x.

(c) $\int u\,dv = uv - \int v\,du$

$$\left\{\because \int u\,dv + \int v\,du = \int d(uv) = uv\right\}$$

ILLUSTRATION 38

Evaluate $\int dx$.

SOLUTION

Let $I = \int dx = \int (1)dx = \int x^0 dx = \dfrac{x^0+1}{0+1}$

$\Rightarrow \quad I = x + c$, where c is a constant of integration.

ILLUSTRATION 39

Evaluate $\int x^2 dx$

SOLUTION

$I = \int x^2 dx = \dfrac{x^{2+1}}{2+1} = \dfrac{x^3}{3} + c$, where c is a constant of integration.

ILLUSTRATION 40

Evaluate $\int \dfrac{1}{x^2}dx$

SOLUTION

$$I = \frac{dx}{x^2} = \int x^{-2}dx = \frac{x^{-2}+1}{-2+1} = -\frac{1}{x} + c$$

ILLUSTRATION 41

Find $\int \sin^2 x\,dx$.

SOLUTION

Let $I = \int \sin^2 x\,dx$

Since, $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

$$\Rightarrow \quad I = \int\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\int(1-\cos 2x)dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left(\int dx - \int \cos(2x)dx\right)$$

$$\Rightarrow \quad I = \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + c$$

where c is a constant of integration.

ILLUSTRATION 42

Integrate the following w.r.t. x.

(a) $x^{1/2}$ **(b)** $\cot^2 x$ **(c)** $\dfrac{1}{1-\sin x}$

SOLUTION

(a) $\int x^{\frac{1}{2}}dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \dfrac{2}{3}\left(x^{\frac{3}{2}}\right)$

(b) $I = \int \cot^2 x\,dx$

$$\Rightarrow \quad I = \int(\operatorname{cosec}^2 x - 1)dx$$

$$\Rightarrow \quad I = \int \operatorname{cosec}^2 x\,dx - \int dx$$

$$\Rightarrow \quad I = -\cot x - x$$

(c) $I = \int \dfrac{1}{1-\sin x}dx$

$$\Rightarrow \quad I = \int\left(\frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\right)dx$$

$$\Rightarrow \quad I = \int \frac{1+\sin x}{1-\sin^2 x}dx$$

$$\Rightarrow \quad I = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}dx$$

$$\Rightarrow \quad I = \int(\sec^2 x + \tan x \sec x)dx = \tan x + \sec x$$

ILLUSTRATION 43

Evaluate $\int \cos^2 x\, dx$

SOLUTION

Let $I = \int \cos^2 x\, dx$

$$\Rightarrow \quad I = \int\left(\frac{1+\cos(2x)}{2}\right)dx \quad \left\{\because \cos^2 x = \frac{1+\cos(2x)}{2}\right\}$$

$$\Rightarrow \quad I = \frac{1}{2}\int\left[dx + \int \cos 2x\right]dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\int dx + \int \cos(2x)dx\right] = \frac{1}{2}\left(x + \frac{\sin(2x)}{2}\right) + c$$

ILLUSTRATION 44

Evaluate $\int e^{5x}\, dx.$

SOLUTION

Let $I = \int e^{5x} dx = \frac{e^{5x}}{5} + c$, where c is a constant of integration.

DEFINITE INTEGRALS

When an integral is defined between two limits, it is called a **definite integral**. The lower value of the limit is called the **lower limit** and the higher value of the limit is called **the upper limit**.

For example, if the integral of the function $f(x)$ is to be determined between the limits, $x = a$ and $x = b$, we represent it symbolically as

$$\int_a^b f(x)dx$$

where b is the upper limit and a is the lower limit. We read it as "integral from a to b of the function $f(x)$ with respect to x".

If $\int f(x)dx = \phi(x) + c$, then

$$\int_a^b f(x)dx = \left(\phi(x) + c\right)\Big|_a^b$$

$$\Rightarrow \quad \int_a^b f(x)dx = \left(\phi(b) + c\right) - \left(\phi(a) + c\right) = \phi(b) - \phi(a)$$

The constant of integration has disappeared during the process of integrating a function within limits.

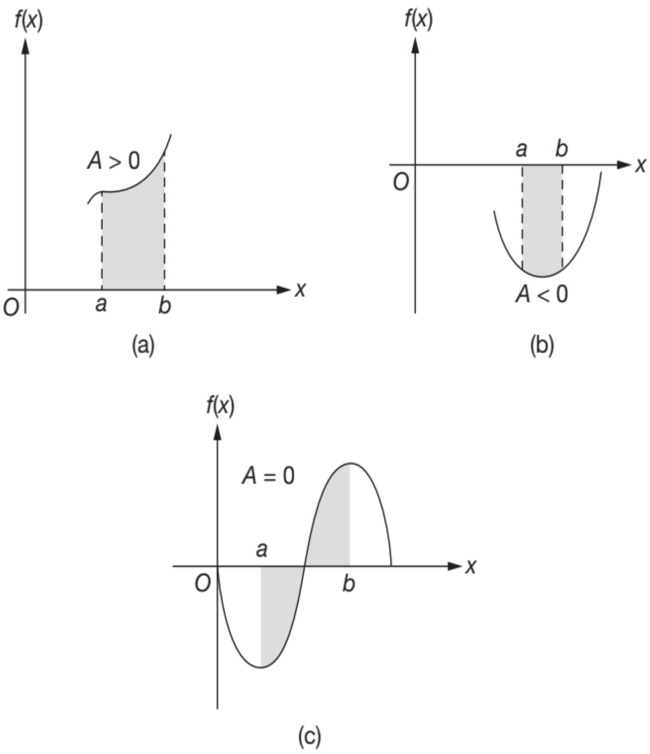
GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRATION

As we have learnt, the graphical interpretation of differentiation is finding the slope of a curve. Integration also has a simple graphical meaning. It is related to finding the area under a curve.

If a function $f(x)$ is expressed graphically in the form $f(x)$ vs x, the **area under the curve between the limits** a **and** b means the area bounded by the curve of $f(x)$, the x-axis and two lines $x = a$ and $x = b$.

The area under the graph of a positive function is defined to be positive. The area under (actually, above) the graph of a negative function is defined to be negative. As shown in the figure (c), positive and negative area add algebraically and may even cancel. **The total area between definite limits of** x **is called a definite integral.** The notation for the definite integral is

$$\text{Area} = A = \int_a^b f(x)dx$$



Definition of the definite integral as area under the curve. Area above the x-axis is defined to be positive, area below the x-axis is negative.

Conceptual Note(s)

A definite integral between fixed limits is a fixed quantity, not a function. It has a specific numerical value and generally has a unit, which need not be a unit of area. Just as the idea of slope is generalized from its purely geometric meaning and acquires a unit determined by the quotient of the vertically plotted quantity and the horizontally plotted quantity, the idea of areas is also generalized and acquires a **unit determined by the product of the vertically and horizontally plotted quantities**.

ILLUSTRATION 45

Evaluate $\int_2^3 x^2 dx$

SOLUTION

Let $I = \int_2^3 x^2 dx = \left(\frac{x^3}{3}\right)\Big|_2^3 = \left[\frac{3^3}{3} - \frac{2^3}{2}\right]$

$$\Rightarrow \quad I = \frac{27}{3} - \frac{8}{2} = \frac{54-24}{6} = \frac{30}{6} = 5$$

ILLUSTRATION 46

Evaluate $\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta.$

SOLUTION

Let $I = \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = (-\cos\theta)\Big|_0^{\frac{\pi}{2}}$

$$\Rightarrow \quad I = -\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -[0-1] = 1$$

ILLUSTRATION 47

Find $\int_a^b \frac{dr}{r}.$

SOLUTION

Let $I = \int_a^b \frac{dr}{r}$

$$\Rightarrow \quad I = \int_a^b \frac{dr}{r} = \left(\log_e r\right)\Big|_a^b$$

$$\Rightarrow \quad I = \log_e b - \log_e a = \log_e\left(\frac{b}{a}\right)$$

ILLUSTRATION 48

The momentum p of a particle changes with time t according to the relation $\frac{dp}{dt} = (10+2t)$. If the momentum is zero at $t = 0$, what will be the momentum at $t = 10$ s?

SOLUTION

Given

$$\frac{dp}{dt} = (10+2t)$$

$$\Rightarrow \quad dp = (10+2t)\,dt$$

$$\Rightarrow \quad \int_0^p dp = \int_0^{10} (10+2t)\,dt$$

$$\Rightarrow \quad p = \left(10t + t^2\right)\Big|_0^{10}$$

$$\Rightarrow \quad p = \left[10(10)+(10)^2\right] - \left[10(0)+(0)^2\right]$$

$$\Rightarrow \quad p = 200 \text{ kgms}^{-1}$$

ILLUSTRATION 49

Evaluate the integral $\int_0^t A\sin(\omega t)\,dt$, where A and ω are constants.

SOLUTION

Let $I=\int_0^t A\sin(\omega t)\,dt$

$$\Rightarrow \quad I=A\left[-\frac{\cos(\omega t)}{\omega}\bigg|_0^t\right]$$

$$\Rightarrow \quad I=-\frac{A}{\omega}\left[\cos(\omega t)-\cos 0°\right]$$

$$\Rightarrow \quad I=\frac{A}{\omega}\left[1-\cos(\omega t)\right]$$

ILLUSTRATION 50

The velocity v and displacement x of a particle executing simple harmonic motion are related as

$$v\frac{dv}{dx}=-\omega^2 x$$

At $x=0$, $v=v_0$. Find the velocity v when the displacement becomes x.

SOLUTION

The given equation is

$$v\frac{dv}{dx}=-\omega^2 x$$

$$\Rightarrow \quad vdv=-\omega^2 x\,dx$$

$$\Rightarrow \quad \int_{v_0}^{v} vdv=-\omega^2\int_0^x x\,dx$$

$$\Rightarrow \quad \left|\frac{v^2}{2}\right|_{v_0}^{v}=-\omega^2\left|\frac{x^2}{2}\right|_0^x$$

$$\Rightarrow \quad v^2-v_0^2=-\omega^2x^2$$

$$\Rightarrow \quad v=\sqrt{v_0^2-\omega^2x^2}$$