04 Three Dimensional Geometry

Introduction:

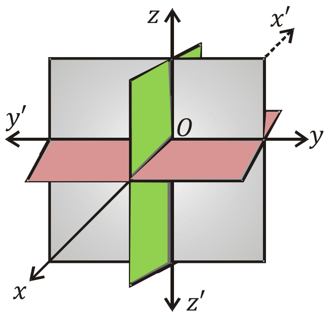
In earlier classes we have learnt about points, lines, circles and conic section in two dimensional geometry. In two dimensions a point represented by an ordered pair (x, y) (where x & y are both real numbers). In space, each body has length, breadth and height i.e. each body exist in three-dimensional space. Therefore, three independent quantities are essential to represent any point in space. Three axes are required to represent these three quantities.

Rectangular co-ordinate System:

In cartesian system of three lines which are mutually perpendicular, such a system is called rectangular cartesian co-ordinate system.

Co-ordinate axes and co-ordinate planes:

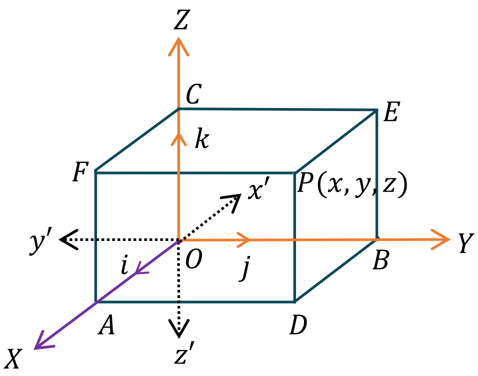
When three mutually perpendicular planes intersect at a point, then mutually perpendicular lines are obtained, and these lines also pass through that point. If we assume the point of intersection as origin, then the three planes are known as co-ordinate planes and the three lines are known as co-ordinate axes.



Octants:

Every plane bisects the space. Hence three co-ordinate plane divide the space in eight parts. These parts are known as octants.

Co-ordinates of a Point in Space:



Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x −axis, y −axis and z-axis respectively, in such a way that they form a right-handed system.

The planes XOY, YOZ and ZOX are known as xy −plane, yz −plane and zx-plane respectively.

Let P be a point in space and distances of P from yz, zx and xy planes be x, y, z respectively (with proper signs) then we say that Co-ordinates of P are (x, y, z). Also $OA = |x|$, $OB = |y|$, $OC = |z|$

Distance Formula:

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

(a) Distance from Origin:

Let O be the origin and $P(x, y, z)$ be any point, then $OP = \sqrt{(x^2 + y^2 + z^2)}$

(b) Distance of a point from Co-ordinate axes:

Let $P(x, y, z)$ be any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX, OY and OZ respectively. Then

$$PA = \sqrt{(y^2 + z^2)}\ ;\ PB = \sqrt{(z^2 + x^2)}\ ;\ PC = \sqrt{x^2 + y^2}$$

Illustration 1:

Prove by using distance formula that the points $P(1,2,3)$, $Q(-1,-1,-1)$ and $R(3,5,7)$ are collinear.

Solution:

We have $PQ = \sqrt{(-1-1)^2 + (-1-2)^2 + (-1-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

$QR = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$

and $PR = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

Since $QR = PQ + PR$. Therefore, the given points are collinear.

Illustration 2:

Find the locus of a point the sum of whose distances from $(1, 0, 0)$ and $(-1, 0, 0)$ is equal to 10.

Solution:

Let the points $A(1,0,0)$, $B\ (-1,0,0)$ and $P(x, y, z)$
Given: $PA + PB = 10$

$$\sqrt{(x-1)^2 + (y-0)^2 + (z-0)^2} + \sqrt{(x+1)^2 + (y-0)^2 + (z-0)^2} = 10$$

$$\Rightarrow \sqrt{(x-1)^2 + y^2 + z^2} = 10 - \sqrt{(x+1)^2 + y^2 + z^2}$$

Squaring both sides, we get;

$$\Rightarrow (x-1)^2 + y^2 + z^2 = 100 + (x+1)^2 + y^2 + z^2 - 20\sqrt{(x+1)^2 + y^2 + z^2}$$

$$\Rightarrow -4x - 100 = -20 \Rightarrow \sqrt{(x+1)^2 + y^2 + z^2} \quad x + 25 = 5\sqrt{(x+1)^2 + y^2 + z^2}$$

Again squaring both sides we get $x^2 + 50x + 625 = 25\left\{\left(x^2 + 2x + 1\right) + y^2 + z^2\right\}$

$$\Rightarrow 24x^2 + 25y^2 + 25z^2 - 600 = 0$$

i.e., required equation of locus

Section Formulae:

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let $R\ (x, y, z)$ divide PQ in the ratio $m_1 : m_2$.

Then co-ordinates of $R(x, y, z) = \dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}, \dfrac{m_1z_2 + m_2z_1}{m_1 + m_2}$

If (m_1/m_2) is positive, R divides PQ internally and if (m_1/m_2) is negative, then externally.

Mid-Point: Midpoint of PQ is given by $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

Illustration 3:

Find the ratio in which the plane $x - 2y + 3z = 17$ divides the line joining the points $(-2, 4, 7)$ and $(3, -5, 8)$.

Solution:

Let the required ratio be $k : 1$

The co-ordinates of the point which divides the join of $(-2, 4, 7)$ and $(3, -5, 8)$ in the ratio $k : 1$ are

$$\left(\frac{3k-2}{k+1}, \frac{-5k+4}{k+1}, \frac{8k+7}{k+1}\right)$$

Since this point lies on the plane $x - 2y + 3z - 17 = 0$

$$\therefore \left(\frac{3k-2}{k+1}\right) - 2\left(\frac{-5k+4}{k+1}\right) + 3\left(\frac{8k+7}{k+1}\right) - 17 = 0$$

$$\Rightarrow (3k-2) - 2(-5k+4) + 3(8k+7) = 17k + 17$$

$$\Rightarrow 3k + 10k + 24k - 17k = 17 + 2 + 8 - 21$$

$$\Rightarrow 37k - 17k = 6 \Rightarrow 20k = 6; = \frac{6}{20} = \frac{3}{10}$$

Hence the required ratio = $k : 1 = \frac{3}{10} : 1 = 3 : 10$

Centroid of a Triangle:

Let $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Then its centroid G is given by

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

Illustration 4:

If the centre of a tetrahedron $OABC$ where A, B, C are given by 3 and $(2,1,c)$ respectively is $(1,2,-1)$, then distance of $P(a, b, c)$ from origin is -

(A) $\sqrt{107}$ (B) $\sqrt{14}$ (C) $\sqrt{107}/14$ (D) none of these

Ans. (A)

Solution:

Centre is $\left(\frac{1}{4}\Sigma x, \frac{1}{4}\Sigma y, \frac{1}{4}\Sigma z\right) = (1, 2, -1)$

$$\Rightarrow \frac{a+1+2+0}{4} = 1, \frac{2+b+1+0}{4} = 2, \frac{3+2+c+0}{4} = -1$$

$\Rightarrow a = 1, b = 5, c = -9$

$\therefore$ OP $= \sqrt{a^2 + b^2 + c^2} = \sqrt{107}$

Illustration 5:

If the points $(0,1,2)$, $(2,-1,3)$ and $(1,-3,1)$ are the vertices of a triangle, then the triangle is

(A) Right angled (B) Isosceles right angled

(C) Equilateral (D) None of these

Ans. (B)

Solution:

Here $a = \sqrt{4+4+1} = \sqrt{9}, b = \sqrt{1+4+4} = \sqrt{9}$ and $c = \sqrt{1+16+1} = \sqrt{18}$.

Obviously, it is a right angled and isosceles triangle.

Illustration 6:

If the points $(-1,3,2),(-4,2,-2)$ and $(5,5,\lambda)$ are collinear, then $\lambda =$

(A) –10 (B) 5 (C) –5 (D) 10

Ans. (D)

Solution:

$\frac{-4+1}{5+4}=\frac{2-3}{5-2}=\frac{-2-2}{\lambda+2}$ or $\lambda+2=12$ or $\lambda=10$.

Illustration 7:

If the co-ordinates of the points A,B,C,D be $(2,3,-1),(3,5,-3),(1,2,3)$ and $(3,5,7)$ respectively, then the projection of AB on CD is

(A) 0 (B) 1 (C) 2 (D) $\sqrt{3}$

Ans. (A)

Solution:

Since AB and CD are perpendicular, therefore projection of AB on CD is 0.

Direction Cosines of Line:

If α,β,γ be the angles made by a line with x –axis, y –axis & z –axis respectively then $\cos\alpha,\cos\beta$ & $\cos\gamma$ are called direction cosines of a line, denoted by ℓ,m & n respectively.

Note:

(i) If line makes angles α,β,γ with x,y & z axis respectively then $\pi-\alpha,\ \pi-\beta$ & $\pi-\gamma$ is another set of angles that line makes with principle axes. Hence if ℓ,m & n are direction cosines of line then $-\ell,-m$ & $-n$ are also direction cosines of the Same line.

(ii) Since parallel lines have same direction. So, in case of lines, which do not pass through the origin. We can draw a parallel line passing through the origin and direction cosines of that line can be found.

Important points :

(i) Direction cosines of a line :

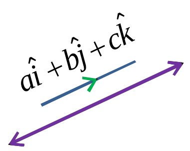
Take a vector $\vec{A}=a\hat{i}+b\hat{j}+c\hat{k}$ parallel to a line whose $D.C's$ are to be found out.

$\vec{A}.\hat{i}=a$

$|\vec{A}|\cos\alpha=a$

$\cos\alpha=\frac{a}{|\vec{A}|}$ similarly, $\cos\beta=\frac{b}{|\vec{A}|}$; $\cos\gamma=\frac{c}{|\vec{A}|}$

$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \Rightarrow \ell^2+m^2+n^2=1$



(ii) Direction cosines of axes :

Since the positive x –axes makes angle 0°, 90°, 90° with axes of x,y and z respectively,

$\therefore$ $D.C.'s$ of x – axis are 1, 0, 0.

$D.C.'s$ of y – axis are 0, 1, 0

$D.C.'s$ of z – axis are 0, 0, 1

Illustration 8:

A line OP makes with the x −axis an angle of measure 120° and with y −axis an angle of measure 60°. Find the angle made by the line with the z −axis.

Solution:

$\alpha = 120°$ and $\beta = 60°$

$\therefore \cos\alpha = \cos 120° = -\frac{1}{2}$ and $\cos\beta = \cos 60° = \frac{1}{2}$ but $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$

$\Rightarrow \cos\gamma = \pm \frac{1}{\sqrt{2}}$

$\therefore\ \gamma = 45°$ or $135°$

Illustration 9:

If α, β, γ be the direction angles of a vector and $\cos\alpha = \frac{14}{15}$, $\cos\beta = \frac{1}{3}$ then $\cos\gamma =$

(A) $\pm\frac{2}{15}$ (B) $\frac{1}{5}$ (C) $\pm\frac{1}{15}$ (D) None of these

Ans.(A)

Solution:

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos\gamma = \sqrt{1 - \left(\frac{14}{15}\right)^2 - \left(\frac{1}{3}\right)^2} = \sqrt{\frac{8}{9} - \left(\frac{196}{225}\right)} = \pm\frac{2}{15}$.

Illustration 10:

If α, β, γ be the angles which a line makes with the positive direction of co-ordinate axes, then

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$

(A) 2 (B) 1 (C) 3 (D) 0

Ans.(A)

Solution:

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \Sigma \sin^2\alpha = 3 - 1 = 2$.

Illustration 11:

If a line makes angles of 30^o and 45^o with x −axis and y −axis, then the angle made by it with z −axis is

(A) 45^o (B) 60^o (C) 120^o (D) None of these

Ans.(D)

Solution:

$\cos\gamma = \sqrt{1 - \frac{3}{4} - \frac{1}{2}} = \sqrt{\frac{-1}{4}}$, which is not possible.

Direction Ratios:

Any three numbers a, b, c proportional to direction cosines ℓ, m, n are called direction ratios of the line. i.e. $\frac{\ell}{a}=\frac{m}{b}=\frac{n}{c}$.

There can be infinitely many sets of direction ratios for a given line.

Direction Ratios and Direction Cosines of The Line Joining Two Points:

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points, then $d.r.$'s of AB are $x_2-x_1, y_2-y_1, z_2-z_1$ and the $d.c.$'s of AB are $\frac{1}{r}(x_2-x_1)$, $\frac{1}{r}(y_2-y_1)$, $\frac{1}{r}(z_2-z_1)$ where $r=\sqrt{[\Sigma(x_2-x_1)^2]}$

Illustration 12:

Find the length of projection of the line segment joining the points (– 1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Solution:

The direction cosines ℓ, m, n of the line are given by $\frac{\ell}{6}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{6^2+2^2+3^2}}=\frac{1}{\sqrt{49}}=\frac{1}{7}$

$\therefore\ \ell=\frac{6}{7},\ m=\frac{2}{7}, n=\frac{3}{7}$

The required length of projection is given by

$= |\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)| = \left|\frac{6}{7}[2-(-1)]+\frac{2}{7}(5-0)+\frac{3}{7}(1-3)\right|$

$= \left|\frac{6}{7}\times 3+\frac{2}{7}\times 5+\frac{3}{7}\times -2\right| = \left|\frac{18}{7}+\frac{10}{7}-\frac{6}{7}\right| = \left|\frac{18+10-6}{7}\right| = \frac{22}{7}$

Illustration 13:

xy –plane divides the line joining the points (2, 4, 5) and (– 4,3, – 2) in the ratio

(A) 3 : 5 (B) 5 : 2 (C) 1 : 3 (D) 3 : 4

Ans.(B)

Solution:

Required ratio $=-\left(\frac{5}{-2}\right)=\frac{5}{2}$ *i.e.*, 5: 2.

Illustration 14:

If the projections of a line on the co-ordinate axes be 2, – 1,2, then the length of the lines is

(A) 3 (B) 4 (C) 2 (D) $\frac{1}{2}$

Ans.(A)

Solution:

$r=\sqrt{4+1+4}=3$.

Illustration 15:

The direction ratios of the line joining the points (4, 3, –5) and (–2, 1, –8) are

(A) $\frac{6}{7},\frac{-2}{7},\frac{3}{7}$ (B) 6,2,3 (C) 2, 4, -13 (D) None of these

Ans.(B)

Solution:

Direction ratios are, $a=4-(-2)=6$, $b=3-1=2$ and $c=-5+8=3$.

Relation Between D.C's & D.R's :

$$\frac{\ell}{a}=\frac{m}{b}=\frac{n}{c}$$

$$\therefore\ \frac{\ell^2}{a^2}=\frac{m^2}{b^2}=\frac{n^2}{c^2}=\frac{\ell^2+m^2+n^2}{a^2+b^2+c^2}$$

$$\therefore \ell=\frac{\pm a}{\sqrt{a^2+b^2+c^2}}\ ;m=\frac{\pm b}{\sqrt{a^2+b^2+c^2}}\ ;n=\frac{\pm c}{\sqrt{a^2+b^2+c^2}}$$

Important point:

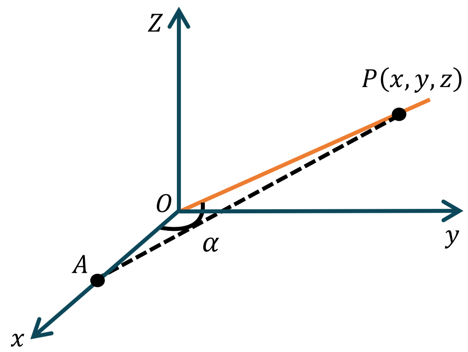
Direction cosines of a line have two sets, but direction ratios of a line have infinite possible sets.

Projections:

(a) Projection of line segment OP on co-ordinate axes:

Let line segment make angle α with x-axis

Thus, the projections of line segment OP on axes are the absolute values of the co-ordinates of P. i.e.



Projection of OP on $x-$ axis $= |x|$

Projection of OP on $y-$ axis $= |y|$

Projection of OP on $z-$ axis $= |z|$

Now, in ΔOAP, angle A is a right angle and $OA = x$

$$OP=\sqrt{x^2+y^2+z^2}$$

$$\therefore\ \cos\alpha=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{|OP|}$$

if $|OP| = r$, then $x = |OP|\cos\alpha = \ell r$

Similarly, y = $|OP|\cos\beta = mr, z = nr$, where ℓ, m, n are $DC's$ of line

(B) Projection of a line segment AB on coordinate axes:

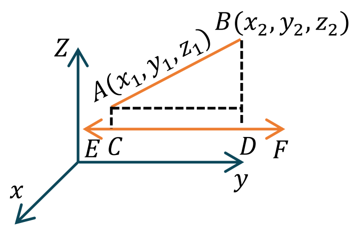
Projection of the point $A(x_1, y_1, z_1)$ on x –axis is $E(x_1, 0,0)$. Projection of point $B(x_2, y_2, z_2)$ on x –axis is $F(x_2, 0,0)$.Hence projection of AB on x –axis is $EF = |x_2 - x_1|$.

Similarly, projection of AB on y and z –axis are $|y_2 - y_1|, |z_2 - z_1|$ respectively.

(c) Projection of line segment AB on a line having direction cosines ℓ, m, n :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Now projection of AB on $EF = CD = AB\cos\theta$



$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\times\frac{(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

$$= |(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n|$$

Illustration 16:

The co-ordinates of the point P are (x, y, z) and the direction cosines of the line OP when O is the origin, are l, m, n. If $OP = r$, then

(A) $l = x, m = y, n = z$ (B) $l = xr, m = yr, n = zr$

(C) $x = lr, y = mr, z = nr$ (D) None of these

Ans.(C)

Solution:

$Op = r = \sqrt{x^2 + y^2 + z^2}$

$\frac{x-0}{r} = \ell,\ \frac{y-0}{r} = m,\ \frac{z-0}{r} = n$

$x = \ell r,\ y = mr,\ z = nr$

Illustration 17:

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two perpendicular lines, then the direction cosine of the line which is perpendicular to both the lines, will be

(A) $(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$ (B) $(l_1l_2 - m_1m_2), (m_1m_2 - n_1n_2), (n_1n_2 - l_1l_2)$

(C) $\frac{1}{\sqrt{l_1^2 + m_1^2 + n_1^2}}, \frac{1}{\sqrt{l_2^2 + m_2^2 + n_2^2}}, \frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Ans(A)

Solution:

Let lines are $l_1x + m_1y + n_1z + d = 0$ ...(i)

and $l_2x + m_2y + n_2z + d = 0$ ...(ii)

If $lx + my + nz + d = 0$ is perpendicular to (i) and (ii), then, $ll_1 + mm_1 + nn_1 = 0,\ ll_2 + mm_2 + nn_2 = 0$

$\Rightarrow \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - l_1n_2} = \frac{n}{l_1m_2 - l_2m_1} = d$

Therefore, direction cosines are

$(m_1n_2 - m_2n_1), (n_1l_2 - l_1n_2), (l_1m_2 - l_2m_1)$.

Definition:

A plane is a surface such that a line segment joining any two points on the surface lies wholly on it.

Equations of a Plane:

The equation of every plane is of the first degree i.e., of the form $ax + by + cz + d = 0$, in which a, b, c are constants, not all zero simultaneously.

Vector form:

If is the position vector of a point on the plane and is a vector normal to the plane then its vectoral equation is given by $(\vec{r} - \vec{a}).\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = d$, where constant.

Cartesian form:

If $\vec{a}(x_1, y_1, z_1)$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$, then cartesian equation of plane will be

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Illustration 18:

The equation of the plane which is parallel to $y-$axis and cuts off intercepts of length 2 and 3 from $x-$axis and $z-$axis is

(A) $3x + 2z = 1$ (B) $3x + 2z = 6$ (C) $2x + 3z = 6$ (D) $3x + 2z = 0$

Ans. (B)

Solution:

Equation of plane parallel to y −axis is, $ax+bz+1=0$

Also $2a+1=0 \Rightarrow a=-\frac{1}{2}$ and $3b+1=0 \Rightarrow b=-\frac{1}{3}$

$\therefore\ 3x+2z=6$.

Aliter : Equation of plane $\frac{x}{2}+\frac{z}{3}=1 \Rightarrow 3x+2z=6$.

Different Form of The Equation of Planes.

(A) Equation of Plane Passing Through a Fixed Point:

Vector form :

If $\vec{a}$ is the position vector of a point on the plane and $\vec{n}$ is a vector normal to the plane then its vectoral equation is given by $(\vec{r}-\vec{a}).\vec{n}=0 \Rightarrow \vec{r}.\vec{n}=d$, where $d=\vec{a}.\vec{n}=$ constant.

Cartesian Form:

If $\vec{a}(x_1, y_1, z_1)$ and $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$, then cartesian equation of plane will be

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

(B) Plane Parallel to the Coordinate Planes:

(i) Equation of yz plane is $x=0$.

(ii) Equation of zx plane is $y=0$.

(iii) Equation of xy plane is $z=0$.

(iv) Equation of the plane parallel to xy plane at a distance c is $z=c$ or $z=-c$.

(v) Equation of the plane parallel to yz plane at a distance c is $x=c$ or $x=-c$

(vi) Equation of the plane parallel to zx plane at a distance c is $y=c$ or $y=-c$.

(C) Equations of Planes Parallel to the Axes:

If $a=0$, the plane is parallel to x −axis i.e. equation of the plane parallel to x −axis is $by+cz+d=0$. Similarly, equations of planes parallel to y −axis and parallel to z −axis are $ax+cz+d=0$ and $ax+by+d=0$, respectively.

(D) Equation of a Plane in Intercept Form:

Equation of the plane which cuts off intercepts a, b, c from the axes x, y, z respectively is.

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

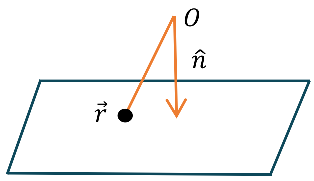
(E) Equation of a Plane in Normal Form:

Vector form:

If $\hat{n}$ is a unit vector normal to the plane from the origin and d be the perpendicular distance of plane from origin, then its vector equation is $\vec{r}.\hat{n}=d$.

Cartesian Form:

If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (ℓ, m, n), then the equation of the plane is $\ell x+my+nz=p$.



(F) Equation of a Plane Through Three Points:

Vector form:

If A, B, C are three points having $P.V.'s\ \vec{a}, \vec{b}, \vec{c}$ respectively, then vector equation of the plane is

$[\vec{r}\ \vec{a}\ \vec{b}] + [\vec{r}\ \vec{b}\ \vec{c}] + [\vec{r}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$.

Cartesian Form:

The equation of the plane through three non-collinear points (x_1, y_1, z_1),

(x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

Illustration 19:

Find the equation of the plane through the points $A(2,2,-1), B(3,4,2)$ and $C(7,0,6)$.

Solution:

The general equation of a plane passing through $(2,2,-1)$ is

$a(x-2) + b(y-2) + c(z+1) = 0$ ...(i)

It will pass through $B(3,4,2)$ and $C(7,0,6)$

if $a(3-2) + b(4-2) + c(2+1) = 0$ or $a + 2b + 3c = 0$ ...(ii)

and $a(7-2) + b(0-2) + c(6+1) = 0$ or $5a - 2b + 7c = 0$ ...(iii)

Solving (ii) and (iii) by cross-multiplication, we have

$\frac{a}{14+6} = \frac{b}{15-7} = \frac{c}{-2-10}$ or $\frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda$ (say)

$\Rightarrow a = 5\lambda, b = 2\lambda$ and $c = -3\lambda$

Substituting the values of a, b and c in (i), we get

$5\lambda(x-2) + 2\lambda(y-2) - 3\lambda(z+1) = 0$ or $5(x-2) + 2(y-2) - 3(z+1) = 0$

$\Rightarrow 5x + 2y - 3z = 17$, which is the required equation of the plane

Illustration 20:

A plane meets the co-ordinates axes in A, B, C such that the centroid of the ΔABC is the point (p, q, r) show that the equation of the plane is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$

Solution:

Let the required equation of plane be :

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ...(i)

Then, the co-ordinates of A, B and C are $A(a,0,0), B(0,b,0), C(0,0,c)$ respectively

So, the centroid of the triangle ABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

But the co-ordinate of the centroid are (p, q, r)

$\frac{a}{3} = p,\ \frac{b}{3} = q,\ \frac{c}{3} = r$

Putting the values of a, b and c in (i), we get the required plane as $\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$

$\Rightarrow \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$

Illustration 21:

The equation of the plane which is parallel to y –axis and cuts off intercepts of length 2 and 3 from x –axis and z –axis is

(A) $3x+2z=1$ (B) $3x+2z=6$ (C) $2x+3z=6$ (D) $3x+2z=0$

Ans(B)

Solution:

Equation of plane parallel to y-axis is, $ax+bz+1=0$

Also $2a+1=0 \Rightarrow a=-\frac{1}{2}$ and $3b+1=0 \Rightarrow b=-\frac{1}{3}$

$\therefore$ $3x+2z=6$.

Aliter : Equation of plane $\frac{x}{2}+\frac{z}{3}=1 \Rightarrow 3x+2z=6$.

Illustration 22:

If a plane cuts off intercepts $-6, 3, 4$ from the co-ordinate axes, then the length of the perpendicular from the origin to the plane is

(A) $\frac{1}{\sqrt{61}}$ (B) $\frac{13}{\sqrt{61}}$ (C) $\frac{12}{\sqrt{29}}$ (D) $\frac{5}{\sqrt{41}}$

Ans(C)

Solution:

Equation is $\frac{x}{-6}+\frac{y}{3}+\frac{z}{4}=1$ or $-2x+4y+3z=12$

$\therefore$ Length of perpendicular from origin

$=\frac{12}{\sqrt{4+16+9}}=\frac{12}{\sqrt{29}}$.

Illustration 23:

The equation of the plane which is parallel to xy –plane and cuts intercept of length 3 from the z –axis is

(A) $x=3$ (B) $y=3$ (C) $z=3$ (D) $x+y+z=3$

Ans(C)

Solution:

Equation of a plane parallel to xy –plane is $z = k$

$\therefore$ $z=3$.

Illustration 24:

The equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x+2y+3z=5$ and $3x+3y+z=0$, is

(A) $7x-8y+3z-25=0$ (B) $7x-8y+3z+25=0$

(C) $-7x+8y-3z+5=0$ (D) $7x-8y-3z+5=0$

Ans. (B)

Solution:

Given, equation of plane is passing through the point $(-1, 3, 2)$

$\therefore$ $A(x+1)+B(y-3)+C(z-2)=0$ ...(i)

Since plane (i) is perpendicular to each of the planes $x+2y+3z=5$ and $3x+3y+z=0$

So, $A+2B+3C=0$ and $3A+3B+C=0$

$\therefore \frac{A}{2-9}=\frac{B}{9-1}=\frac{C}{3-6}=K$

$\Rightarrow A=-7K, B=8K, C=-3K$

Put the values of A, B and C in (i)

we get, $7x-8y+3z+25=0$, which is the required equation of the plane.

Perpendicular Distance of A Point From The Plane:

Vector form:

If $\vec{r}.\vec{n}=d$ be the plane, then perpendicular distance p, of the point $A(\vec{a})$

$$p=\frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$$

Distance between two parallel planes $\vec{r}.\vec{n}=d_1$ & $\vec{r}.\vec{n}=d_2$ is $\left|\frac{d_1-d_2}{|\vec{n}|}\right|$.

Cartesian form:

Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane $ax+by+cz+d=0$ is given by

$$p=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{(a^2+b^2+c^2)}}$$

Distance between two parallel planes $ax+by+cz+d_1=0$ & $ax+by+cz+d_2=0$ is $\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$

Illustration 25:

Find the perpendicular distance of the point $(2, 1, 0)$ from the plane $2x+y+2z+5=0$

Solution:

We know that the perpendicular distance of the point (x_1, y_1, z_1) from the plane

$ax+by+cz+d=0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

so required distance = $\frac{|2\times2+1\times1+2\times0+5|}{\sqrt{2^2+1^2+2^2}}=\frac{10}{3}$

Illustration 26:

Find the distance between the parallel planes $2x-y+2z+3=0$ and $4x-2y+4z+5=0$.

Solution:

Let $P(x_1, y_1, z_1)$ be any point on $2x-y+2z+3=0$, then $2x_1-y_1+2z_1+3=0$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $4x-2y+4z+5=0$ is

$$\left|\frac{4x_1-2y_1+4z_1+5}{\sqrt{4^2+(-2)^2+4^2}}\right| = \left|\frac{2(2x_1-y_1+2z_1)+5}{\sqrt{36}}\right| = \frac{|2(-3)+5|}{6} = \frac{1}{6} \qquad \text{[using (i)]}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{6}$

Illustration 27:

The distance between the planes $x+2y+3z+7=0$ and $2x+4y+6z+7=0$ is

(A) $\frac{\sqrt{7}}{2\sqrt{2}}$ (B) $\frac{7}{2}$ (C) $\frac{\sqrt{7}}{2}$ (D) $\frac{7}{2\sqrt{2}}$

Ans.(A)

Solution:

Required distance = $\dfrac{7-\frac{7}{2}}{\sqrt{1+4+9}}=\dfrac{\sqrt{7}}{2\sqrt{2}}$.

Illustration 28:

Distance of the point (2,3,4) from the plane $3x-6y+2z+11=0$ is

(A) 1 (B) 2 (C) 3 (D) 0

Ans.(A)

Solution:

Required distance = $\left|\dfrac{6-18+8+11}{7}\right|=1$.

Angle Between Two Planes:

Vector form:

If $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$ be two planes, then angle between these planes is the angle between their normal $\cos\theta=\dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

$\therefore$ Planes are perpendicular if $\vec{n}_1.\vec{n}_2=0$ and they are parallel if $\vec{n}_1=\lambda\vec{n}_2$.

Cartesian form:

Consider two planes $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$. Angle between these planes is the angle between their normal.

$$\cos\theta=\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}$$

$\therefore$ Planes are perpendicular if $aa'+bb'+cc'=0$ and they are parallel if $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$.

Planes Parallel to a Given Plane:

Equation of a plane parallel to the plane $ax+by+cz+d=0$ is $ax+by+cz+d'=0$. d' is to be found by other given condition.

Illustration 29:

Find the angle between the planes $x+y+2z=9$ and $2x-y+z=15$

Solution:

We know that the angle between the planes $a_1x+b_1y+c_1z+d=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Therefore, angle between $x+y+2z=9$ and $2x-y+z=15$ is given by

$$\cos\theta=\frac{(1)(2)+(1)(-1)+(2)(1)}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+(-1)^2+1^2}}=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$$

Illustration 30:

Find the equation of the plane through the point $(1,4,-2)$ and parallel to the plane $-2x+y-3z=7$.

Solution:

Let the equation of a plane parallel to the plane $-2x+y-3z=7$ be $-2x+y-3z+k=0$

This passes through $(1,4,-2)$, therefore $(-2)(1)+4-3(-2)+k=0$

$$-2+4+6+k=0, k=-8$$

Putting $k=-8$ in (i), we obtain $-2x+y-3z-8=0$ or $-2x+y-3z=8$

This is the equation of the required plane.

Illustration 31:

The angle between the planes $3x-4y+5z=0$ and $2x-y-2z=5$ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{6}$ (D) None of these

Ans.(B)

Solution:

$$\theta=\cos^{-1}\left[\frac{6+4-10}{\sqrt{50}\sqrt{9}}\right]=\cos^{-1}(0)=\frac{\pi}{2}.$$

Illustration 32:

The value of k for which the planes $3x-6y-2z=7$ and $2x+y-kz=5$ are perpendicular to each other, is

(A) 0 (B) 1 (C) 2 (D) 3

Ans.(A)

Solution:

Planes are perpendicular, if $6-6+2k=0 \Rightarrow k=0$.

Bisectors of Angles Between Two Planes:

Let the equations of the two planes be $ax+by+cz+d=0$ and $a_1x+b_1y+c_1z+d_1=0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=\pm\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

(A) Equation of bisector of the angle containing origin:

First make both constant terms positive. Then positive sign give the bisector of the angle which contains the origin.

(B) Bisector of acute/obtuse angle:

First making both constant terms positive,

$aa_1+bb_1+cc_1>0 \Rightarrow$ origin lies in obtuse angle

$aa_1+bb_1+cc_1<0 \Rightarrow$ origin lies in acute angle

Illustration 33:

Find the equation of the bisector planes of the angles between the planes $2x-y+2z+3=0$ and $3x-2y+6z+8=0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.

Solution:

The two given planes are $2x-y+2z+3=0$ and $3x-2y+6z+8=0$

where $d_1, d_2>0$

and $a_1a_2 + b_1b_2 + c_1c_2 = 6 + 2 + 12 > 0$

$\therefore \dfrac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\dfrac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ (acute angle bisector)

And $\dfrac{a_1x+b_1y+c_1z+d}{\sqrt{a_1^2+b_1^2+c_1^2}} = + \dfrac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ (obtuse angle bisector)

i.e., $\dfrac{2x-y+2z+3}{\sqrt{4+1+4}} = \pm \dfrac{3x-2y+6z+8}{\sqrt{9+4+36}}$

$\Rightarrow (14x - 7y + 14z + 21) = \pm(9x - 6y + 18z + 24)$

Taking positive sign on the right-hand side,

we get $5x - y - 4z - 3 = 0$ (obtuse angle bisector)

and taking negative sign on the right-hand side,

we get $23x - 13y + 32z + 45 = 0$ (acute angle bisector)

Position of Two Points w.r.t. a Plane:

Two points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ are on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ & $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P & Q externally or internally according to P and Q lying on same or opposite sides of the plane.

Family of Planes:

A Plane Through The Line of Intersection of Two Given Planes:

Consider two planes $u \equiv ax + by + cz + d = 0$ and $v \equiv a'x + b'y + c'z + d' = 0$.

The equation $u + \lambda v = 0, \lambda$ a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

Illustration 34:

Find the equation of plane containing the line of intersection of the plane $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through (1,1,1).

Solution:

The equation of the plane through the line of intersection of the given planes is,

$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$ ...(i)

If it is passes through (1,1,1)

$\Rightarrow (1 + 1 + 1 - 6) + \lambda(2 + 3 + 4 + 5) = 0 \Rightarrow \lambda = \dfrac{3}{14}$

Putting $\lambda = 3/14$ in (i); we get $(x + y + z - 6) + \dfrac{3}{14}(2x + 3y + 4z + 5) = 0$

$\Rightarrow 20x + 23y + 26z - 69 = 0$

Illustration 35:

A point moves so that its distances from the points $(3, 4, -2)$ and $(2, 3, -3)$ remains equal. The locus of the point is

(A) a line
(B) a plane whose normal is equally inclined to axes
(C) a plane which passes through the origin
(D) a sphere

Ans. (B)

Solution:

According to question,

$(x-3)^2+(y-4)^2+(z+2)^2=(x-2)^2+(y-3)^2+(z+3)^2$

$\therefore$ The equation reduces to a plane as 2^{nd} degree terms cancel out.

The equation is $2x+2y+2z=7$

Hence equally inclined to axes.

Illustration 36:

The equation of the plane containing the line of intersection of the planes $2x-y=0$ and $y-3z=0$ and perpendicular to the plane $4x+5y-3z-8=0$ is

(A) $28x-17y+9z=0$

(B) $28x+17y+9z=0$

(C) $28x-17y+9x=0$

(D) $7x-3y+z=0$

Ans. (A)

Solution:

Equation of plane containing the line of intersection of planes is, $(2x-y)+\lambda(y-3z)=0$, ...(i)

Also, plane (i) is perpendicular to $4x+5y-3z-8=0$

$\therefore 4(2)+5(\lambda-1)-3(-3\lambda)=0$

$\Rightarrow 14\lambda=-3 \Rightarrow \lambda=-\dfrac{3}{14}$

Put the value of λ in (i), we get $28x-17y+9z=0$, which is the required plane.

Definition:

A straight line in space is characterised by the intersection of two planes which are not parallel and, therefore, the equation of a straight line is present as a solution of the system constituted by the equations of the two planes :

$$a_1x+b_1y+c_1z+d_1=0;\ \ a_2x+b_2y+c_2z+d_2=0$$

This form is also known as **unsymmetrical form**.

Some particular straight lines:

	Straight lines	Equation
(i)	through the origin	$y=mx,\ z=nx$
(ii)	$x-$ axis	$y=0,\ z=0$ or $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$
(iii)	$y-$ axis	$x=0,\ z=0$ or $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$
(iv)	$z-$ axis	$x=0,\ y=0$ or $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$
(v)	parallel to $x-$ axis	$y=p, z=q$
(vi)	parallel to $y-$ axis	$x=h, z=q$
(vii)	parallel to $z-$ axis	$x=h, y=p$

Equation of A Straight Line in Symmetrical Form :

(A) One Point Form:

Let $A(x_1, y_1, z_1)$ be a given point on the straight line and ℓ, m, n be the $d.c's$ of the line, then its equation is

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ (say)}$$

It should be noted that $P(x_1+\ell r, y_1+mr, z_1+nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e., $AP=r$. One should note that for $AP=r$; ℓ,mn must be $d.c.'s$ not $d.r.'s$. If a, b, c are direction ratios of the line, then equation of the line is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=r \text{ but here } AP \neq r$$

(B) Equation of the Line Through Two Points:

$A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Illustration 37:

Find the co-ordinates of those points on the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{6}$ which is at a distance of 3 units from point $(1,-2,3)$.

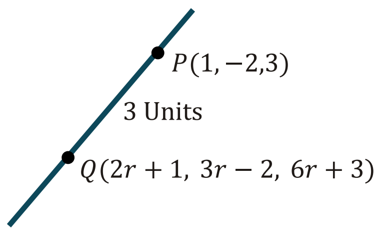
Solution:

Here, $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{6}$ ...(i)

is the given straight line

Let, $P=(1,-2,3)$ on the straight line

Here direction ratios of line (i) are (2,3,6)



$\therefore$ Direction cosines of line (i) are : $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

$\Rightarrow$ Equations of line (i) can be written as

$$\frac{x-1}{2/7}=\frac{y+2}{3/7}=\frac{z-3}{6/7} \quad \text{...(ii)}$$

Co-ordinates of any point on the line (ii) can be taken as $\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

Let, $Q\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$

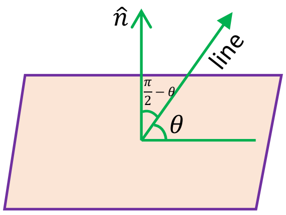
Given $|\vec{r}| = 3, \therefore r = \pm 3$

Putting the value of r, we $Q\left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right)$ have or $Q = \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right)$

Angle Between a Line and a Plane:

Let equations of the line and plane be $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and $ax+by+cz+d=0$ respectively and θ be the angle which line makes with the plane. Then $(\pi/2-\theta)$ is the angle between the line and the normal to the plane.

So, $\sin\theta=\frac{a\ell+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(\ell^2+m^2+n^2)}}$



Line is parallel to plane
if $\theta=0$ i.e. if $a\ell+bm+cn=0$.
Line is perpendicular to the plane
if line is parallel to the normal of the plane i.e., if $\frac{a}{\ell}=\frac{b}{m}=\frac{c}{n}$.

Illustration 38:

Find the angle between the line $\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-3}{-2}$ and the plane $3x+4y+z+5=0$.

Solution:

The given line is $\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-3}{-2}$ ...(i)

and the given plane is $3x+4y+z+5=0$ ...(ii)

If the line (i) makes angle θ with the plane (ii), then the line (i) will make angle $(90°-\theta)$ with the normal to the plane(i). Now direction-ratios of line (i) are $3,-1,-2$ and direction-ratios of normal to plane (ii) are 3,4,1

$\therefore\ \cos(90°-\theta)=\frac{(3)(3)+(-1)(4)+(-2)(1)}{\sqrt{9+1+4}\sqrt{9+16+1}}$

$\Rightarrow\ \sin\theta=\frac{9-4-2}{\sqrt{14}\sqrt{26}}=\frac{3}{\sqrt{14}\sqrt{26}}$

Hence

Angle Between Two Lines:

Let θ be the angle between the lines with $d.c.'s$ ℓ_1,m_1,n_1 and ℓ_2,m_2,n_2 then
$\cos\theta=\ell_1\ell_2+m_1m_2+n_1n_2$. If a_1,b_1,c_1 and a_2,b_2,c_2 be $D.R.'s$ of two lines then angle θ between them is given by

$$\cos\theta=\frac{(a_1a_2+b_1b_2+c_1c_2)}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

Illustration 39:

If a line makes angles α, β, γ, δ with four diagonals of a cube, then $\cos^2\alpha+cos^2\beta+\cos^2\gamma+\cos^2\delta$ equals -

(A) 3 (B) 4 (C) 4/3 (D) 3/4.

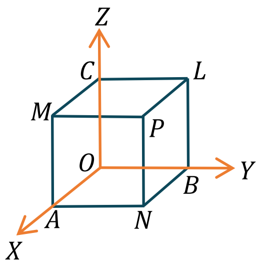
Ans. (C)

Solution:

Let OA, OB, OC be coterminous edges of a cube and $OA=OB=OC=a$, then co-ordinates of its vertices are $O(0,0,0), A(a,0,0), B(0,a,0), C(0,0,a), L(0,a,a), M(a,0,a), N(a,a,0)$ and $P(a,a,a)$

Direction ratio of diagonal AL, BM, CN and OP are

$$\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$



Let ℓ, m, n be the direction cosines of the given line, then

$$\cos\alpha = \ell\left(-\frac{1}{\sqrt{3}}\right)+m\left(\frac{1}{\sqrt{3}}\right)+n\left(\frac{1}{\sqrt{3}}\right)=\frac{-\ell+m+n}{\sqrt{3}}$$

Similarly, $\cos\beta = \frac{\ell-m+n}{\sqrt{3}}$, $\cos\gamma=\frac{\ell+m-n}{\sqrt{3}}$ and $\cos\delta=\frac{\ell+m+n}{\sqrt{3}}$

$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

Perpendicular and Parallel Lines:

Let the two lines have their $d.c.'s$ given by ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90°$

i.e. $\cos\theta = 0$, i.e. $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\frac{\ell_1}{\ell_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$

Note:

If instead of $d.c.'s$, $d.r.'s$ a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if . $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Illustration 40:

The equation of the straight line passing through (1, 2, 3) and perpendicular to the plane $x+2y-5z+9=0$ is

(A) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{-5}$

(B) $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+5}{3}$

(C) $\frac{x+1}{1}=\frac{y+2}{2}=\frac{z+3}{-5}$

(D) $\frac{x+1}{1}=\frac{y+2}{2}=\frac{z-5}{3}$

Ans.(A)

Solution:

The line passes through point (1,2,3) is $\frac{x-1}{a}=\frac{y-2}{b}=\frac{z-3}{c}$ and it is perpendicular to the plane $x+2y-5z+9=0$, therefore the line must be $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{-5}$ because

$$\sin\theta=\frac{1\,.\,1+2\,.\,2+(-5)\,(-5)}{\sqrt{1^2+2^2+5^2}\,.\,\sqrt{1^2+2^2+5^2}}=1$$

$\Rightarrow \theta=90^o$.

Illustration 41:

The distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$, is

(A) 10 (B) 11 (C) 12 (D) 13

Ans.(D)

Solution:

Point on line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ is $P(3r+2, 4r-1, 12r+2)$

$\therefore (3r+2)-(4r-1)+(12r+2)=5$

$\Rightarrow 11r+5=5$

$\Rightarrow r=0$

So, $P(2,-1,2)$ and $Q(-1,-5,-10)$

Hence, $PQ=\sqrt{9+16+144}=13$

Condition That a Line Lies on The Given Plane:

The line $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ will lie on the plane $Ax+By+Cz+D=0$ if

(A) $A\ell+Bm+Cn=0$ and (B) $Ax_1+By_1+Cz_1+D=0$

Illustration 42:

The equation of plane through the line of intersection of planes $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ and parallel to the line $y=0, z=0$ is

(A) $(ab'-a'b)x+(bc'-b'c)y+(ad'-a'd)=0$

(B) $(ab'-a'b)x+(bc'-b'c)y+(ad'-a'd)z=0$

(C) $(ab'-a'b)y+(ac'-a'c)z+(ad'-a'd)=0$

(D) None of these

Ans(C)

Solution:

The equation of a plane through the line of intersection of the planes $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$ is

$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$

or $x(a+\lambda a')+y(b+\lambda b')+z(c+\lambda c')+d+\lambda d'=0$ ...(i)

This is parallel to x-axis *i.e.*, $y=0$, $z=0$

$\therefore 1(a+\lambda a')+0(b+\lambda b')+0(c+\lambda c')=0 \Rightarrow \lambda=-\frac{a}{a'}$

Putting the value of λ in (i), the required plane is

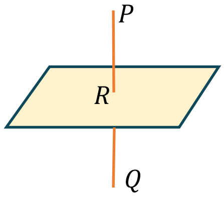
$y(a'b-ab')+z(a'c-ac')+a'd-ad'=0$

i.e., $(ab'-a'b)y+(ac'-a'c)z+ad'-a'd=0$.

Foot and Image of a Point w.r.t a Plane and a Line

Image of a Point in The Plane:

In order to find the image of a point $P(x_1, y_1, z_1)$ in a plane $ax + by + cz + d = 0$, assume It as a mirror. Let $Q(x_2, y_2, z_2)$ be the image of the point $P(x_1, y_1, z_1)$ in the plane, then



(A) Line PQ is perpendicular to the plane. Hence equation of PQ is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=r$$

(B) Hence, Q satisfies the equation of line then $\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=\frac{z_2-z_1}{c}=r$. The plane passes through the middle point of line PQ, therefore the middle point satisfies the equation of the plane i.e.

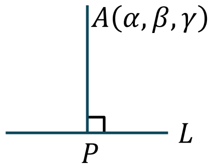
$$a\left(\frac{x_2+x_1}{2}\right)+b\left(\frac{y_2+y_1}{2}\right)+c\left(\frac{z_2+z_1}{2}\right)+d=0$$

The co-ordinates of Q can be obtained by solving these equations.

Foot, Length and Equation of Perpendicular From a Point to A Line:

Let equation of the line be $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$ ...(i)

and $A(\alpha, \beta, \gamma)$ be the point. Any point on the line (i) is $P(\ell r + x_1, mr + y_1, nr + z_1)$ ...(ii)



If it is the foot of the perpendicular, from A on the line, then AP is $\perp$ to the line, so

$\ell(\ell r + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0$

i.e. $r = (\alpha - x_1)\ell + (\beta - y_1)\, m + (\gamma - z_1)\, n$

since $\ell^2 + m^2 + n^2 = 1$

Putting this value of r in (ii), we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP=\sqrt{(\ell r+x_1-\alpha)^2+(mr+y_1-\beta)^2+(nr+z_1-\gamma)^2}$$

Equation of perpendicular is given by

$$\frac{x-\alpha}{\ell r+x_1-\alpha}=\frac{y-\beta}{mr+y_1-\beta}=\frac{z-\gamma}{nr+z_1-\gamma}$$

Illustration 43:

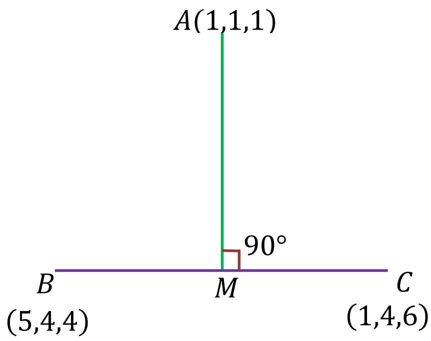
Find the co-ordinates of the foot of the perpendicular from (1, 1, 1) on the line joining (5, 4, 4) and (1,4,6).

Solution:

Let $A(1,1,1)$, $B(5,4,4)$ and $C(1,4,6)$ be the given points. Let M be the foot of the perpendicular from A on BC.

If M divides BC in the ratio $\lambda : 1$, then

co-ordinates of M are $\left(\frac{\lambda+5}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right)$



Direction ratios of BC are $1-5, 4-4, 6-4$

i.e. $-4, 0, 2$

$D.R.'s$ of AM are $\frac{\lambda+5}{\lambda+1}-1, \frac{4\lambda+4}{\lambda+1}-1, \frac{6\lambda+4}{\lambda+1}-1$

$\Rightarrow \frac{4}{\lambda+1}, \frac{3\lambda+3}{\lambda+1}, \frac{5\lambda+3}{\lambda+1} \Rightarrow 4, 3\lambda+3, 5\lambda+3$

Since $AM \perp BC$

$\therefore -4(4) + 0(3\lambda+3) + 2(5\lambda+3) = 0 \Rightarrow -16 + 10\lambda + 6 = 0 \Rightarrow \lambda = 1$

Hence the co-ordinates of M are $(3, 4, 5)$

Illustration 44:

Find the length of perpendicular from $P(2,-3,1)$ to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$

Solution:

$\frac{x+1}{2}=\frac{y-3}{3}=\frac{z+2}{-1}$ Given line is ...(i)

and $P(2,-3,1)$

Co-ordinates of any point on (i) may be taken as

$(2r-1, 3r+3, -r-2)$

Let $Q=(2r-1, 3r+3, -r-2)$

Direction ratios of PQ are : $(2r-3, 3r+6, -r-3)$

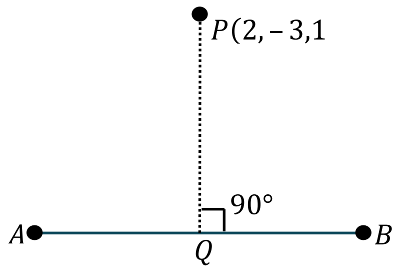
Direction ratios of AB are : $(2, 3, -1)$

Since, $PQ \perp AB$

$2(2r-3)+3(3r+6)-1(-r-3)=0$

$\Rightarrow r = -\frac{15}{14}$

$\therefore\ Q=\left(-\frac{22}{7},-\frac{3}{14},-\frac{13}{14}\right)$



Equation of Plane Containing Two Intersecting Lines:

Let the two lines be

$$\frac{x-\alpha_1}{\ell_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad \text{...(i)}$$

And $\frac{x-\alpha_2}{\ell_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$ ...(ii)

These lines will coplanar if $\begin{vmatrix}\alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1\\ \ell_1 & m_1 & n_1\\ \ell_2 & m_2 & n_2\end{vmatrix}=0$ (It is condition for intersection of two lines)

the plane containing the two lines is $\begin{vmatrix}x-\alpha_1 & y-\beta_1 & z-\gamma_1\\ \ell_1 & m_1 & n_1\\ \ell_2 & m_2 & n_2\end{vmatrix}=0$

Illustration 45:

Find the equation of the plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2}$ and parallel to the line

$\frac{x-4}{2}=\frac{y-1}{-3}=\frac{z+3}{5}$.

Solution:

Any plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2}$ is

$a(x-1)+b(y+6)+c(z+1)=0$ ...(i)

where, $3a+4b+2c=0$ ...(ii)

Also, it is parallel to the second line and hence, its normal is perpendicular to this line

$\therefore\ 2a-3b+5c=0$ ...(iii)

Solving (ii) & (iii) by cross multiplication, we get $\frac{a}{26}=\frac{b}{-11}=\frac{c}{-17}=k$

$\Rightarrow a=26k, b=-11k\ \&\ c=-17k$

Putting these values in (i), we get $26k(x-1)-11k(y+6)-17k(z+1)=0$

$\Rightarrow\ 26x-11y-17z=109$, which is the required equation of the plane

$PQ^2\& = \left(2+\frac{22}{7}\right)^2+\left(-3+\frac{3}{14}\right)^2+\left(1+\frac{13}{14}\right)^2=\frac{531}{14}$

$PQ=\sqrt{\frac{531}{14}}$ units

Illustration 46:

Image point of $(1,3,4)$ in the plane $2x-y+z+3=0$ is

(A) (– 3, 5, 2) (B) (3, 5, – 2) (C) (3, – 5, 3) (D) None of these

Ans.(A)

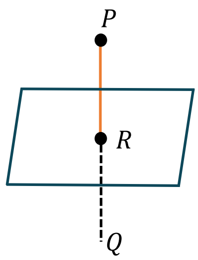
Solution:

Let Q be image of the point $P(1,3,4)$ in the given plane, then PQ is normal to the plane.

The $d.r.'s$ of PQ are 2, – 1,1

Since PQ passes through (1, 3, 4) and has $d.r's$ 2, 1, – 1;

therefore, equation of plane is $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r$, (say)



$\therefore\ x=2r+1, y=-r+3, z=r+4$

So, co-ordinates of Q be

$(2r+1,-r+3,r+4)$

Let R be the mid point of PQ, then co-ordinates of R are $\left(\frac{2r+1+1}{2},\frac{-r+3+3}{2},\frac{r+4+4}{2}\right)$

i.e., $\left(r+1,\frac{-r+6}{2},\frac{r+8}{2}\right)$

Since R lies on the plane

$\therefore\ 2(r+1)-\left(\frac{-r+6}{2}\right)+\left(\frac{r+8}{2}\right)+3=0\ \Rightarrow r=-2$

So, co-ordinates of Q are $(-3,5,2)$.

Trick : From option (A), mid point of (–3, 5, 2) and (1,3,4) satisfies the equation of plane $2x-y+z+3=0$.

Illustration 47:

If P be the point (2, 6, 3), then the equation of the plane through P at right angle to OP, O being the origin, is

(A) $2x+6y+3z=7$
(B) $2x-6y+3z=7$
(C) $2x+6y-3z=49$
(D) $2x+6y+3z=49$

Ans.(D)

Solution:

Distance of point P from origin $OP=\sqrt{4+36+9}=7$

Now $d.r$'s of $OP = 2-0,\ 6-0,\ 3-0 = 2,6,3$

$\therefore$ $d.c$'s of OP = $\frac{2}{7},\frac{6}{7},\frac{3}{7}$

$\therefore$ Equation of plane in normal form is $lx+my+nz=p$

$\Rightarrow \frac{2}{7}x+\frac{6}{7}y+\frac{3}{7}z=7$

$\Rightarrow 2x+6y+3z=49$.

Unsymmetrical Form of Straight Line:

The equations $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ together represents a line in unsymmetrical form.

Note:

Vector along the line of intersection of planes $\vec{r}.\vec{n}_1=q_1$ and $\vec{r}.\vec{n}_2=q_2$ is $\vec{n}_1\times\vec{n}_2$.

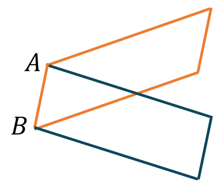
Procedure to Convert Unsymmetrical Form of Straight Line to Symmetrical Form

Let the direction ratios of the line of intersection (AB) of two planes

$a_1x+b_1y+c_1z+d_1=0$ ...(i)

And $a_2x+b_2y+c_2z+d_2=0$ ...(ii)

are a,b,c.



Direction ratios of normal to plane (i) are a_1,b_1,c_1 and

Direction ratios of normal to plane (ii) are a_2,b_2,c_2

Line AB lies in both the plane (i) and (ii)

hence normal to (i) and (ii) are perpendicular to AB.

Hence $aa_1+bb_1+cc_1=0$ and $aa_2+bb_2+cc_2=0$

these two will give the proportional values of a,b,c.

Let the line AB cuts the xy plane at $(x_1,y_1,0)$

Hence $a_1x_1+b_1y_1=-d_1$ and $a_2x_1+b_2y_1=-d_2$. This will give a point on the line AB

$\therefore$ equation of AB is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-0}{c}$.

Note: 3 planes $a_rx+b_ry+c_rz=d_r$ $r = 1,2,3$

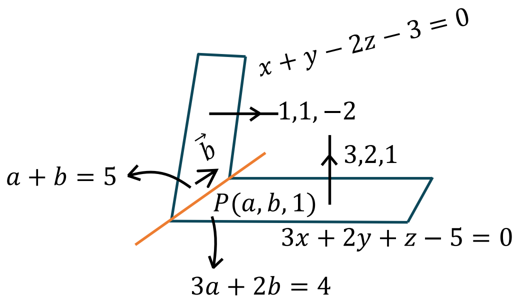
(i) Can intersect at a point $\equiv$ system of equations in 3 variables having unique solution.

(ii) Can intersect coaxially $\equiv$ system of equations in 3 variables having infinite solutions.

(iii) May not have a common point $\equiv$ system of equations in 3 variables having no solution.

Illustration 48:

Obtain the equation of the line of intersection of the planes $3x + 2y + z - 5 = 0$ & $x + y - 2z - 3 = 0$ in symmetrical Form or Convert line: $3x + 2y + z - 5 = 0$ & $x + y - 2z - 3 = 0$ in symmetrical form



Solution:

Let $P = (a, b, 1)$

$3a + 2b = 4$

$a + b = 5$

On solving

$a = -6$

$b = 11$

Point P: $(-6,11,1)$

$\vec{b}$ is perpendicular to $\left(3\hat{i}+2\hat{j}+\hat{k}\right)$ & $\left(\hat{i}+\hat{j}-2\hat{k}\right)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} \Rightarrow -5\hat{i}+7\hat{j}+\hat{k}$$

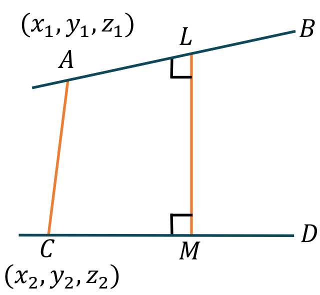
direction of line: $\left(-5\hat{i}+7\hat{j}+\hat{k}\right)$

required line:- $\dfrac{x-(-6)}{-5}=\dfrac{y-11}{7}=\dfrac{z-1}{1}$

Shortest Distance Between the Two Lines:

The length of the line intercepted between two lines which is perpendicular to both the lines is the shortest distance between them.

Let the equations of the lines AB and CD be



$$\frac{x-x_1}{\ell_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

And $\dfrac{x-x_2}{\ell_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$

Let ℓ, m, n be the direction cosines of shortest distance, which is perpendicular to both the lines, then

$\ell\ell_1+mm_1+nn_1=0$

And $\ell\ell_2+mm_2+nn_2=0$

$$\therefore \frac{\ell}{m_1n_2-m_2n_1}=\frac{m}{n_1\ell_2-n_2\ell_1}=\frac{n}{\ell_1m_2-\ell_2m_1}$$

$$=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1\ell_2-n_2\ell_1)^2+(\ell_1m_2-\ell_2m_1)^2}}$$

$$\therefore \ell=\frac{m_1n_2-m_2n_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}},\ m=\frac{n_1\ell_2-n_2\ell_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}},\ n=\frac{\ell_1m_2-\ell_2m_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}$$

Now, shortest distance LM =projection of AC on LM

$$=(x_1-x_2)\ell+(y_1-y_2)m+(z_1-z_2)n$$

$$=\frac{(x_1-x_2)(m_1n_2-m_2n_1)+(y_1-y_2)(n_1\ell_2-n_2\ell_1)+(z_1-z_2)(\ell_1m_2-\ell_2m_1)}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}$$

$$=\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2-m_2n_1)^2}$$

Illustration 49:

Find the shortest distance between the lines

$$\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z-6}{1} \text{ and } \frac{x+3}{7}=\frac{y+3}{-6}=\frac{z+3}{1}$$

Solution:

Let the shortest distance meet the lines in P and Q respectively. Then the coordinates of

$P(r+2, 1-2r, 6+r)$

And $Q(7r_1-3, -6r_1-3, r_1-3)$

Thus, direction cosines of PQ are proportional to

$r-7r_1+5, 4-2r+6r_1, 9+r-r_1$

Since PQ is at right angles to both the lines, then

$1\cdot(r-7r_1+5)-2(4-2r+6r_1)+1\cdot(9+r-r_1)=0$

or, $3r-10r_1+3=0$ ...(i)

and, $7(r-7r_1+5)-6(4-2r+6r_1)+1\cdot(9+r-r_1)=0$

or, $10r-43r_1+10=0$ ...(ii)

Solving (i) and (ii), we get

$r=-1$ and $r_1=0$

Thus $P(1,3,5)$ and $Q(-3,-3,-3)$

$PQ^2=4^2+6^2+8^2=116$

$PQ=2\sqrt{29}$

Illustration 50:

The equations of the line passing through the point $(1,2,-4)$ and perpendicular to the two lines

$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$, will be

(A) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$

(B) $\frac{x-1}{-2}=\frac{y-2}{3}=\frac{z+4}{8}$

(C) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$

(D) None of these

Ans.(A)

Solution:

Line passing through the point $(1,\ 2,\ -4)$ is $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z+4}{n}$

Now, according to question, $3l-16m+7n=0$ and $3l+8m-5n=0$

Hence required line is, $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$.

Coplanarity of Two Lines

The Given Lines Are

$L_1 : \dfrac{x-x_1}{\ell_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$

$L_2 : \dfrac{x-x_2}{\ell_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$

Condition for Coplanarity of (L_1) and (L_2) is

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$$

The Given Lines Are

$L_1 : \dfrac{x-x_1}{\ell_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$

$L_2 : \dfrac{x-x_2}{\ell_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$

Equation of Plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$$

Illustration 51:

The lines $\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$ are coplanar and then equation to the plane in which they lie, is

(A) $x+y+z=0$ (B) $x-y+z=0$

(C) $x-2y+z=0$ (D) $x+y-2z=0$

Ans.(D)

Solution:

The lines will be coplanar

$$\begin{vmatrix} a-d-b+c & a-b & a+d-b-c \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0$$

Add 3^{rd} column to first and it becomes twice the second and hence the determinant is zero as the two columns are identical. Again, the equation of the plane in which they lie is $\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0$

Adding 1^{st} and 3^{rd} columns and subtracting twice the 2^{nd}, we get $\begin{vmatrix} x+z-2y & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix}=0$

$\Rightarrow \{\alpha(\beta+\gamma)-\beta(\alpha+\delta)\}\,(x+z-2y)=0$

$\Rightarrow x+z-2y=0$.

Illustration 52:

The line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ lies in the plane $4x+4y-kz-d=0$. The values of k and d are

(A) 4, 8 (B) –5, –3 (C) 5, 3 (D) – 4, – 8

Ans.(D)

Solution:

Since the line lies on the given plane, therefore any point on the line will satisfy the plane *i.e.*, the points (3, 4, 5) and (5, 7, 9) will lie on the plane. Hence $k=5,\ d=3$.

Line of Greatest Slope:

Consider two planes G-plane and H-plane. H-plane is treated as a horizontal plane or reference plane. G-plane is a given plane. Let AB be the line of intersection of G-plane & H–plane. Line of greatest slope is a line which is contained by G-plane & perpendicular to line of intersection of G-plane & H-plane. Obviously, infinitely many such lines of greatest slopes are contained by G-plane. Generally, an additional information is given in problem so that a unique line of greatest slope can be found out.

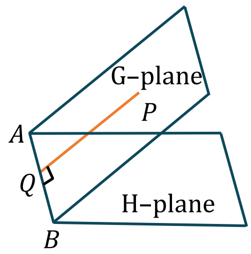


Illustration 53:

Assuming the plane $4x-3y+7z=0$ to be horizontal, find the equation of the line of greatest slope on the plane $2x+y-5z=0$, passing through the point (2,1,1).

Solution:

The required line passing through the point $P(2,1,1)$ in the plane $2x+y-5z=0$ and is having greatest slope, so it must be perpendicular to the line of intersection of the planes

$2x+y-5z=0$ ...(i)

and

$4x-3y+7z=0$ ...(ii)

Let the $d.r'.s$ of the line of intersection of (i) and (ii) be a, b, c

$\Rightarrow 2a+b-5c=0$ and $4a-3b+7c=0$

{as $dr'.s$ of straight line (a, b, c) is perpendicular to $d.r'.s$ of normal to both the planes}

$\Rightarrow \frac{a}{4}=\frac{b}{17}=\frac{c}{5}$

Now let the direction ratio of required line be proportional to ℓ, m, n then its equation be

$\frac{x-2}{\ell}=\frac{y-1}{m}=\frac{z-1}{n}$

where $2\ell+m-5n=0$ and $4\ell+17m+5n=0$ so, $\frac{\ell}{3}=\frac{m}{-1}=\frac{n}{1}$

Thus, the required line is $\frac{x-2}{3}=\frac{y-1}{-1}=\frac{z-1}{1}$ **Ans.**

Area of Triangle:

To find the area of a triangle in terms of its projections on the co-ordinates planes. Let $\Delta_x, \Delta_y, \Delta_z$ be the projections of the plane area of the triangle on the planes yOz, zOx, xOy respectively. Let ℓ, m, n be the direction cosines of the normal to the plane of the triangle.

Then the angle between the plane of the triangle and yOz plane is the angle between the normal to the plane of the triangle and the x-axis.

$\Delta_x = \Delta\ell$

Similarly, $\Delta_y = \Delta_m$; $\Delta_z = \Delta_n \Rightarrow \Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

If $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ be the three vertices of the triangle then

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Do yourself

(i) Prove that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ are coplanar. Find their point of intersection.

(ii) Find the area of the triangle whose vertices are the points $(1, 2, 3), (-2, 1, -4), (3, 4, -2)$.

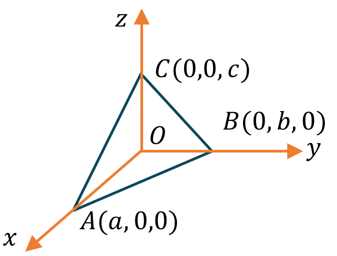
Miscellaneous Illustrations:

Illustration 54:

If a variable plane cuts the coordinate axes in A, B and C and is at a constant distance p from the origin, find the locus of the centre of the tetrahedron $OABC$.

Solution:

Let $A \equiv (a, 0, 0), B \equiv (0, b, 0)$ and $C \equiv (0,0,c)$



Equation of plane ABC is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Now p = length of perpendicular from O to this plane

$$= \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \text{ or } p^2 = \frac{1}{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2} \quad \text{...(i)}$$

Let $G(\alpha, \beta, \gamma)$ be the centre of the tetrahedron $OABC$, then

$$\alpha = \frac{a}{4}, \beta = \frac{b}{4}, \gamma = \frac{c}{4} \qquad \left[\because \ \alpha = \frac{a+0+0+0}{4} = \frac{a}{4}\right]$$

or, $a = 4\alpha, b = 4\beta, c = 4\gamma$

Putting these values of a, b, c in equation (i), we get

$$p^2 = \frac{16}{\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)} \text{ or } \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{16}{p^2}$$

locus of (α, β, γ) is $x^{-2} + y^{-2} + z^{-2} = 16^{-2}$ **Ans.**

Illustration 55:

Through a point $P(h, k, \ell)$ a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If $OP = p$, show that the area of ΔABC is $\frac{p^5}{2|hk\ell|}$.

Solution:

$OP = \sqrt{h^2+k^2+\ell^2} = p$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

Since OP is normal to the plane, therefore, equation of the plane will be,

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x+\frac{k}{\sqrt{h^2+k^2+\ell^2}}y+\frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z=\sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$

$A \equiv \left(\frac{p^2}{h}, 0, 0\right)$, $B \equiv \left(0, \frac{p^2}{k}, 0\right)$, $C \equiv \left(0, 0, \frac{p^2}{\ell}\right)$

Now area of $\Delta ABC, \Delta^2 = A_{xy}^2 + A_{yz}^2 + A_{zx}^2$

Now A_{xy} =area of projection of ΔABC on xy − plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \frac{p^2}{h} & 0 & 1 \\ 0 & \frac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|}$$

Similarly, $A_{yz} = \frac{1}{2}\frac{p^4}{|k\ell|}$ and $A_{zx} = \frac{1}{2}\frac{p^4}{|\ell h|}$

$$\Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2}+\frac{1}{4}\frac{p^8}{k^2\ell^2}+\frac{1}{4}\frac{p^8}{h^2\ell^2}=\frac{p^{10}}{4h^2k^2\ell^2}$$

Or $\Delta = \frac{p^5}{2|hk\ell|}$ Ans.

Illustration 56:

Consider the lines $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ then the equation of the line which

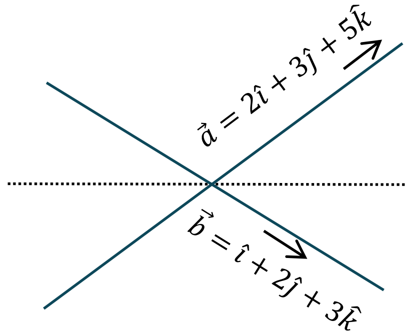
(A) bisects the angle between the lines is $\frac{x}{3}=\frac{y}{3}=\frac{z}{8}$

(B) bisects the angle between the lines is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

(C) passes through origin and is perpendicular to the given lines is $x=y=-z$

(D) none of these

Solution:



$\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$ ...(i)

$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ ...(ii)

$\hat{a}+\hat{b}=\frac{2\hat{i}+3\hat{j}+5\hat{k}}{\sqrt{38}}+\frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}} \Rightarrow$ (A) and (B) will be incorrect

Let the $dr's$ of line $\perp$ to (1) and (2) be a, b, c

$\Rightarrow 2a + 3b + 5c = 0$ ...(iii)

and

$a + 2b + 3c = 0$ ...(iv)

$\therefore \frac{a}{9-10}=\frac{b}{5-6}=\frac{c}{4-3} \Rightarrow \frac{a}{-1}=\frac{b}{-1}=\frac{c}{1} \Rightarrow \frac{a}{1}=\frac{b}{1}=\frac{c}{-1}$

$\therefore$ equation of line passing through (0, 0, 0) and is $\perp r$ to the lines (1) and (2) is

$\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$

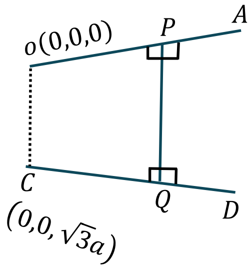
Illustration 57:

Find the shortest distance between any two opposite edges of a tetrahedron formed by the planes $y+z=0$, $x+z=0$, $x+y=0$, $x+y+z=\sqrt{3}a$.

(A) a (B) $2a$ (C) $a/\sqrt{2}$ (D) $\sqrt{2}a$

Ans.(D)

Solution:



Let the equation to one of the pair of opposite edges OA and BC be

$y+z=0$, $x+z=0$ ...(1)

And $x+y=0$, $x+y=0$, $x+y+z=\sqrt{3}\,a$ ...(2)

equation (1) and (2) can be expressed in symmetrical form as

$\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-0}{-1}$ ...(3)

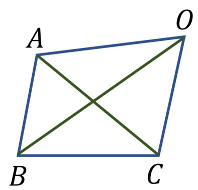
and, $\frac{x-0}{1}=\frac{y-0}{-1}=\frac{z-\sqrt{3}\ a}{0}$ ...(4)

$d.r.$ of OA and BC are respectively $(1,1,-1)$ and $(1,-1,0)$.

Let PQ be the shortest distance between OA and BC having direction cosines (ℓ, m, n)

$\therefore$ PQ is perpendicular to both OA and BC.

$\therefore \ell + m - n = 0$ and $\ell - m = 0$

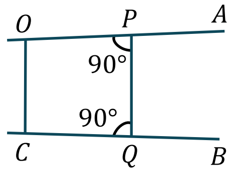


Solving (5) and (6), we get, $\frac{\ell}{1}=\frac{m}{1}=\frac{n}{2}=k$ (say)

also, $\ell^2 + m^2 + n^2 = 1$

$\therefore k^2 + k^2 + 4k^2 = 1 \Rightarrow k = \pm\frac{1}{\sqrt{6}}$

$\therefore\ \ell = \pm\frac{1}{\sqrt{6}},\ m = \pm\frac{1}{\sqrt{6}},\ n = \pm\frac{2}{\sqrt{6}}$



Shortest distance between OA and BC

i.e. PQ =The length of projection of OC on PQ

$= \left|(x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n\right|$

$= \left|0.\frac{1}{\sqrt{6}} + 0.\frac{1}{\sqrt{6}} + \sqrt{3}a.\frac{2}{\sqrt{6}}\right| = \sqrt{2}\,a.$

Illustration 58:

Equation of plane which passes through the point of intersection of lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$ and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and at greatest distance from the point $(0, 0, 0)$ is:

(A) $4x + 3y + 5z = 25$ (B) $4x + 3y + 5z = 50$

(C) $3x + 4y + 5z = 49$ (D) $x + 7y - 5z = 2$

Ans.(B)

Solution:

$\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2} = r$ ...(1)

$\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ ...(2)

$\therefore$ coordinates of any point P on line (1)

$\therefore\ P(3r + 1, r + 2,\ 2r + 3)$ for point of intersection of (1) and (2)

$\frac{3r+1-3}{1} = \frac{r+2-1}{2} = \frac{2r+3-2}{3} \Rightarrow \frac{3r-2}{1} = \frac{r+1}{2} = \frac{2r+1}{3}$ $\therefore\ r = 1$

$\therefore$ point of intersection is $(4, 3, 5)$

$\therefore$ the equation of required plane

$\Rightarrow 4(x-4) + 3(y-3) + 5(z-5) = 0 \Rightarrow 4x + 3y + 52 = 50$

Illustration 59:

The non-zero value of 'a' for which the lines $2x - y + 3z + 4 = 0 = ax + y - z + 2$ and $x - 3y + z = 0 = x + 2y + z + 1$ are co-planar is :

(A) – 2 (B) 4 (C) 6 (D) 0

Ans.(A)

Solution:

$2x - y + 3z + 4 = 0 = ax + y - z + 2$ ...(1)

$\because$ equation of plane through (1) is $(2x - y + 3z + 4) + \lambda(ax + y - z + 2) = 0$

$x(2 + a\lambda) + y(\lambda - 1) + z(3 - \lambda) + (4 + 2\lambda) = 0$ ...(2)

$x - 3y + z = 0 = x + 2y + 2 + 1$ ...(3)

$\because$ equation of plane passing through (3) is

$(x - 3y + z) + \mu(x + 2y + z + 1) = 0$

$\Rightarrow\ x(1 + \mu) + y(2\mu - 3) + z(\mu + 1) + \mu = 0$ ...(4)

if lines (1) and (3) are coplanar, then $\frac{2+a\lambda}{\mu+1} = \frac{\lambda-1}{2\mu-3} = \frac{3-\lambda}{\mu+1} = \frac{4+2\lambda}{\mu}$

Solving this we get $\lambda = -1,\ \mu = 1 \therefore a = -2$

Illustration 60:

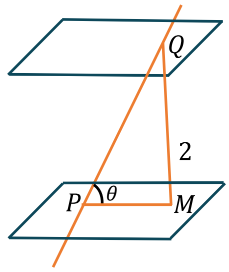
A line having direction ratios 3, 4, 5 cuts 2 planes $2x - 3y + 6z - 12 = 0$ and $2x - 3y + 6z + 2 = 0$ at point P & Q, then find length of PQ

(A) $\frac{35\sqrt{2}}{12}$ (B) $\frac{35\sqrt{2}}{24}$ (C) $\frac{35\sqrt{2}}{6}$ (D) $\frac{35\sqrt{2}}{8}$

Ans.(A)

Solution:

Distance between planes = $\left|\frac{2+12}{7}\right| = 2$



Angle between line and plane = θ

$\sin\theta = \frac{6-12+30}{5\sqrt{2}.7} = \frac{24}{35\sqrt{2}}$

$\sin\theta = \frac{2}{PQ} \Rightarrow PQ = 2\times\frac{35\sqrt{2}}{24} = \frac{35\sqrt{2}}{12}$

Illustration 61:

A line L_1 having direction ratios 1, 0,1 lies on xz plane. Now this xz plane is rotated about z-axis by an angle of 90°. Now the new position of L_1 is L_2. The angle between L_1 & L_2 is:

Solution:

$L_1 : 1,0,1 \Rightarrow \alpha = 45°, \alpha = 90°, \gamma = 45° \Rightarrow \ell_1 = \frac{1}{\sqrt{2}}, m_1 = 0, n_1 - \frac{1}{\sqrt{2}}$

$L_2 : \alpha = 90°, \beta = 45°, \gamma = 45°$

So $\ell_1 = 0, m = \frac{1}{\sqrt{2}}, n_1 = \frac{1}{\sqrt{2}}$

Let angle between L_1 & L_2 be θ.

So $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2 \Rightarrow \cos\theta = \frac{1}{2}, \theta = 60°$

Illustration 62:

A line $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z-k}{3}$ cuts the yz –plane and the xy –plane at A and B respectively. If $\angle AOB = \frac{\pi}{2}$, then $2k$, where O is the origin, is

Ans. (9)

Solution:

$\frac{x+2}{1} = \frac{y-3}{2} = \frac{z-k}{3} = \lambda$

$\Rightarrow (\lambda-2, 2\lambda+3, 3\lambda+k)$ for $A, \lambda = 2$

$A(0, 7, 6+k) \Rightarrow$ for B $\lambda = -\frac{k}{3} \Rightarrow B\left(-2-\frac{k}{3}, 3-\frac{2k}{3}, 0\right)$

$\angle AOB = 90° \Rightarrow \overrightarrow{AO}.\overrightarrow{OB} = 0$

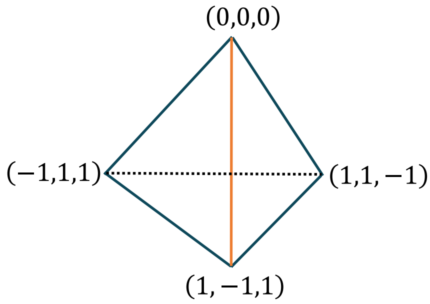
$\Rightarrow 7\left(-3+\frac{2k}{3}\right) = 0$ or $k = \frac{9}{2} \Rightarrow 2k = 9$

Illustration 63:

If the volume of tetrahedron formed by planes whose equations are $y+z=0, z+x=0, x+y=0$ and $x+y+z=1$ Is t cubic unit, then the value of $3t$ is

Ans. (2)

Solution:



The planes are

$y+z=0$ ...(1)

$z+x=0$ ...(2)

$x+y=0$ ...(3)

$x+y+z=1$ ...(4)

Solving above equations we get vertices of the tetrahedron as $(0,0,0), (-1,1,1), (1,-1,1)$ and $(1,1,-1)$

$$\text{Required volume} = \left| \frac{1}{6}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \right| = \left| \frac{1}{6}\begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix} \right| = \frac{4}{6} = \frac{2}{3} \Rightarrow t = \frac{2}{3} \Rightarrow 3t = 2$$

Illustration 64:

L is the equation of the straight line which passes through the point $(2,-1,-1)$; is parallel to the plane $4x+y+z+2=0$ and is perpendicular to the line of intersection of the planes $2x+y=0=x-y+z$. If the point $(3, \alpha, \beta)$ lies on line L, then $|\alpha+\beta|$ is

Ans. (6)

Solution:

$$\frac{x-2}{a} = \frac{y+1}{b} = \frac{z+1}{c} \Rightarrow 4a+b+c=0 \quad \text{...(i)}$$

$$2x+y=0=x-y+z \Rightarrow \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(1-0)-\hat{j}(2-0)+\hat{k}(-2-1) = \hat{i}-2\hat{j}-3\hat{k}$$

$a-2b-3c=0$ ...(ii)

From (i) & (ii) (i) (ii)

$$4a+b+c=0 \Rightarrow a-2b-3c=0 \Rightarrow \frac{a}{-3+2} = \frac{b}{1+12} = \frac{c}{-8-1} \Rightarrow \frac{a}{-1} = \frac{b}{13} = \frac{c}{-9}$$

$\therefore$ equation of the line

$$\frac{x-2}{-1} = \frac{y+1}{13} = \frac{z+1}{-9} \Rightarrow \frac{3-2}{-1} = \frac{\alpha+1}{13} = \frac{\beta+1}{-9} \Rightarrow \alpha = -14 \text{ and } \beta = 8 \Rightarrow |\alpha+\beta| = 6.$$

Illustration 65:

The lines $\dfrac{x+4}{3} = \dfrac{y+6}{5} = \dfrac{z-1}{-2}$ and $3x-2y+z+5=0=2x+3y+4z-k$ are coplanar, then the value of k is

Ans. (4)

Solution:

$L_1 : \frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2} = r$; $L_2 : 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - k$

Any point on the first line is $(3r - 4, 5r - 6, -2r + 1)$

As lines are coplanar therefore this point must lie on both the planes representing the second line

$3(3r - 4) - 2(5r - 6) + (-2r + 1) + 5 = 0 \Rightarrow r = 2$

And $2(3r - 4) + 3(5r - 6) + 4(-2r + 1) - k = 0 \Rightarrow k = 4$

Illustration 66:

About the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ the plane $3x + 4y + 6z + 7 = 0$ is rotated till the plane passes through the origin. Now $4x + \alpha y + \beta z = 0$ is the equation of plane in new position. The value of $\alpha^2 + \beta^2$ is

Ans. (32)

Solution:

Since $3(2) + 4(-3) + 6(1) = 0$ and $3(1) + 4(2) + 6(-3) + 7 = 0$.

$\therefore$ the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ lies in the plane $3x + 4y + 6z + 7 = 0$.

In the new position again, the line lies in the plane. Let the equation of the new position of the plane be $ax + by + cz = 0$, then $2a - 3b + c = 0$ and $a + 2b - 3c = 0$

$\therefore \frac{a}{9-2} = \frac{b}{1+6} = \frac{c}{4+3}$ i.e. $a = b = c$

$\therefore$ equation of the required plane is $x + y + z = 0$.