02 Tangent and Normal

Rate Measure:

Derivative as Rate of Change:

In various fields of applied mathematics, one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

Whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\frac{dy}{dx}$ at $x = a$ (or $f'(a)$) represents the rate of change of y with respect to x at $x = a$.

Illustration 1:

How fast the area of a circle increases when its radius is $5cm$;

(i) with respect to radius

(ii) with respect to diameter

Solution:

(i) $A = \pi r^2, \quad \frac{dA}{dr} = 2\pi r$

$$\therefore \left.\frac{dA}{dr}\right]_{r=5} = 10\pi\ cm^2/cm.$$

(ii) $A = \frac{\pi}{4} D^2, \quad \frac{dA}{dD} = \frac{\pi}{2} D$

$$\therefore \left.\frac{dA}{dD}\right]_{D=10} = \frac{\pi}{2} . 10 = 5\pi\ cm^2/cm.$$

Illustration 2:

If area of circle increases at a rate of $2cm^2/sec$, then find the rate at which area of the inscribed square increases.

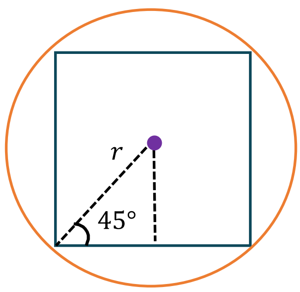
Solution:

Area of circle, $A_1 = \pi r^2$. Area of square, $A_2 = 2r^2$

$$\frac{dA_1}{dt} = 2\pi r \frac{dr}{dt}, \Rightarrow \frac{dA_2}{dt} = 4r . \frac{dr}{dt}$$

$$\because\ 2 = 2\pi r . \frac{dr}{dt} \Rightarrow r \frac{dr}{dt} = \frac{1}{\pi}$$

$$\therefore \frac{dA_2}{dt} = 4 . \frac{1}{\pi} = \frac{4}{\pi}\ cm^2/sec$$



$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm²/sec.

Illustration 3:

The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Solution:

Let at some time t, the length of edge is x cm.

$v = x^3 \Rightarrow \frac{dv}{dt} = 3x^2 \frac{dx}{dt}$ (but $\frac{dv}{dt} = 7$)

$\Rightarrow \frac{dx}{dt} = \frac{7}{3x^2}\ cm/sec.$

Now $S = 6x^2$

$\frac{dS}{dt} = 12x \frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12x.\frac{7}{3x^2} = \frac{28}{x}$

when $x = 4\ cm,\ \frac{dS}{dt} = 7\ cm^2/sec.$

Illustration 4:

Sand is pouring from pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Solution:

$V = \frac{1}{3}\pi r^2 h$ but $h = \frac{r}{6}$

$\Rightarrow V = \frac{1}{3}\pi (6h)^2 h \Rightarrow V = 12\pi h^3 \Rightarrow \frac{dV}{dt} = 36\pi h^2.\frac{dh}{dt}$

when, $\frac{dV}{dt} = 12\ cm^3/s$ and $h = 4\ cm$

$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi}\ cm/sec.$

Error and Approximation:

Let $y = f(x)$ be a function. If these is an error δx in x then corresponding error in y is

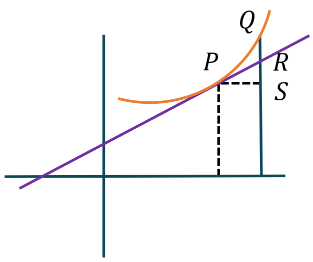
$\delta y = f(x + \delta x) - f(x)$.

We have $\lim\limits_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{dy}{dx} = f'(x)$

We define the differential of y, at point x, corresponding to the increment δx as $f'(x)\delta x$ and denote it by dy.

i.e. $dy = f'(x)\delta x$.

Let $P(x, f(x)), Q((x + \delta x), f(x + \delta x))$ (as shown in figure)



$\delta y = QS$,

$\delta x = PS$,

$dy = RS$

In many practical situations, it is easier to evaluate dy but not δy.

Illustration 5:

Find the approximate value of $26^{\frac{1}{3}}$.

Solution:

Let $y = x^{\frac{1}{3}}$

Let $x = 27$ and $\Delta x = -1$

Now $\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (26)^{\frac{1}{3}} - 3$

$\frac{dy}{dx}\Delta x = 26^{\frac{1}{3}} - 3$

At $x = 27, 26^{\frac{1}{3}} = 3 - 0.037 = 2.963$

Illustration 6:

Find the approximate value of square root of 25.2.

Solution:

Let $f(x) = \sqrt{x}$

Now, $f(x + \Delta x) - f(x) = f'(x) \,.\, \Delta x = \frac{\Delta x}{2\sqrt{x}}$

we may write, $25.2 = 25 + 0.2$

Taking $x = 25$ and $\Delta x = 0.2$, we have

$f(25.2) - f(25) = \frac{0.2}{2\sqrt{25}}$

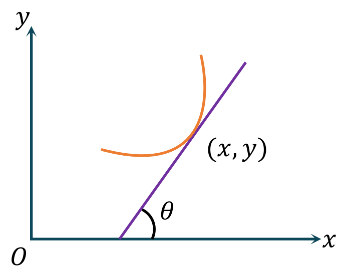
Or $f(25.2) - \sqrt{25} = \frac{0.2}{10} = 0.02 \Rightarrow f(25.2) = 5.02$

Tangent - Normal to Curve at a Point:

Geometrical Interpretation of Derivative:

Let $y = f(x)$ is a function then $\frac{dy}{dx}$ is slope of the tangent i.e. $\left(\frac{dy}{dx}\right)_{(x,y)}$ = $\tan\theta$, where θ is the angle made by the tangent with the positive direction of x-axis.

or, $\theta = \tan^{-1}(dy/dx)$



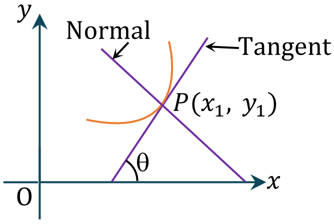
Slopes of the Tangent and the Normal

Slope of the tangent:

Let $y = f(x)$ be a curve, and let $P(x_1, y_1)$ be a point on it. Then, $\left(\frac{dy}{dx}\right)_p$ is the slope of the tangent to the curve $y = f(x)$ at point P. i.e.; $\left(\frac{dy}{dx}\right)_p = \tan\theta$ = slope of the tangent at P, where θ is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of x-axis.

Slope of the Normal:

The normal to a curve at $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P. Slope of the normal at $P = -\frac{1}{\text{slope of tangent at } P} = -\frac{1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$



If normal makes an angle of θ with positive direction of x-axis then $-\frac{dx}{dy} = \tan\theta$ or $\frac{dy}{dx} = -\cot\theta$

Equation of Tangent and Normal to the Curve at a Point

(i) Equation of the tangent at $P\ (x_1, y_1)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$ where P is called point of contact

Equation of normal at $P\ (x_1, y_1)$ is

$$(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1),$$

where P is called foot of normal

Illustration 7:

Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ is-

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$

Solution:

Angle between the tangents $\frac{dy}{dx} = 2x - 5$

$$\left(\frac{dy}{dx}\right)_{(2,0)} = -1,\ \left(\frac{dy}{dx}\right)_{(3,0)} = 1 \Rightarrow \text{Angle} = \frac{\pi}{2}$$

Illustration 8:

The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is :-

(A) $y = 0$ (B) $y = 1$ (C) $y = 2$ (D) $y = 3$

Ans. (D)

Solution:

$$y = x + \frac{4}{x^2}$$

$$\frac{dy}{dx} = 1 - \frac{8}{x^3}$$

Equation of tangent is parallel to x-axis

$$\therefore \frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$$

At, $x = 2, y = 2 + \frac{4}{4} = 3 \Rightarrow y_1 = 3$

$\therefore$ point is $(2, 3)$

equation of tangent is : $y - y_1 = 0(x - x_1)$

$y = 3$

Illustration 9:

The equation of a normal to the curve, sin$y = x$ sin $\left(\frac{\pi}{3}+y\right)$ at $x = 0$, is :

(A) $2y - \sqrt{3}x = 0$ (B) $2y + \sqrt{3}x = 0$ (C) $2x + \sqrt{3}y = 0$ (D) $2x - \sqrt{3}y = 0$

Ans. (C)

Solution:

$$\frac{dx}{dy}=\frac{\sin\frac{\pi}{3}}{\sin^2\left(\frac{\pi}{3}+y\right)}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{(0,0)}=\frac{\sqrt{3}}{2}$$

$\therefore$ Equation of normal $y-0=-\frac{2}{\sqrt{3}}(x-0)$

$\Rightarrow \sqrt{3}y + 2x = 0$

Important Points to Remember:

Cases:

(i) If the tangent at $P(x_1, y_1)$ of the curve $y = f(x)$ is parallel to the x –axis (or perpendicular to y-axis) then $\theta = 0$ i.e. its slope will be zero.

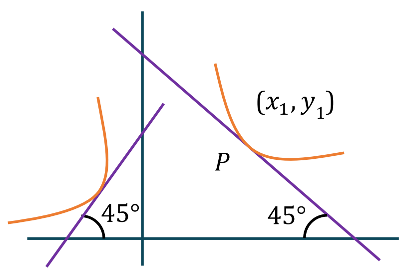
$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

(ii) If the tangent at $P(x_1, y_1)$ of the curve $y = f(x)$ is parallel to y-axis (or perpendicular to x-axis) then $\theta = \pi/2$, and its slope will be not defined $\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = 0$ In this case the tangent is called a vertical tangent.

(iii) If at any point $P(x_1, y_1)$ of the curve $y = f(x)$, the tangent makes equal angles with the axes, then at the point $P, \theta = \pi/4$ or $3\pi/4$. Hence at P, tan $\theta = \frac{dy}{dx} = \pm 1$

(iv) If the tangent at $P(x_1, y_1)$ on the curve is equally inclined to the coordinate axes

$$\Rightarrow \left.\frac{dy}{dx}\right|_{p} = \pm 1$$



(v) If the tangent at $P(x_1, y_1)$ makes equal non-zero intercepts on the coordinate axes then

$$\Rightarrow \left.\frac{dy}{dx}\right|_p = -1$$

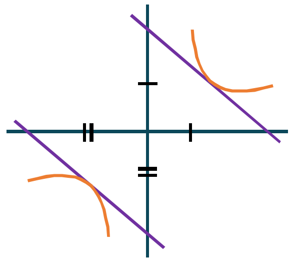


Illustration 10:

The curve $y - e^{xy} + x = 0$ has a horizontal tangent at

(A) (1, 1) (B) (0, 1) (C) (1, 0) (D) no point

Ans. (B)

Solution:

$y - e^{xy} + x = 0$

Differentiating w.r.t. to x

$$y' - e^{xy}\left(\frac{dy}{dx}.\ \ x + y\right) + 1 = 0$$

$$\frac{dy}{dx} = 0$$

$1 - ye^{xy} = 0$

$ye^{xy} = 1 \Rightarrow x = 0, y = 1$

Point is $(0, 1)$

Equation of Tangent at any Point $P(x_1, y_1)$ on the Second-Degree Curve

$S \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ is

$T \equiv axx_1 + byy_1 + h(xy_1 + yx_1) + g(x + x_1) + f(y + y_1) + c = 0$

Illustration 11:

Write down the equation of the tangent to the circle : $x^2 + y^2 - 3x + 10y = 15$ at the point $(4, -11)$.

Solution:

Given circle is $x^2 + y^2 - 3x + 10y - 15 = 0$ and the point is $(4, -11)$

So, the equation of tangent is given by,

$\Rightarrow xx_1 + yy_1 - 3/2(x + x_1) + 5(y + y_1) - 15 = 0$

$\Rightarrow x(4) + y(-11) - 3/2(x + 4) + 5(y - 11) - 15 = 0$

$\Rightarrow 4x - 11y - 3/2\ x - 6 + 5y - 55 - 15 = 0$

$\Rightarrow 4x - 11y - 3/2\,x + 5y - 76 = 0$

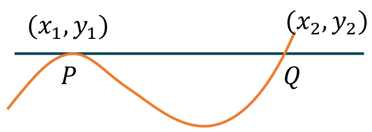
$\Rightarrow 5x - 12y - 152 = 0$

Equation of Tangent / Normal in Some Special Cases:

- If the tangent at P meeting the curve $y = f(x)$ again at Q

  $(x_2, y_2) \equiv (x_2, f(x_2))$

  also $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{y_2 - y_1}{x_2 - x_1}$



- If from an external point Q tangent is drawn to the curve

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{y_2-y_1}{x_2-x_1}$$

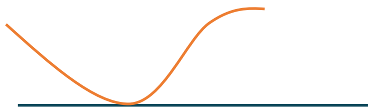


Illustration 12:

If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3, y = 8t^3 - 1, t \in R$, meets the curve again at a point Q, then the coordinates of Q are :

(A) $(t^2 + 3, t^3 - 1)$
(B) $(t^2 + 3, -t^3 - 1)$
(C) $(16t^2 + 3, -64t^3 - 1)$
(D) $(4t^2 + 3, -8t^3 - 1)$

Ans. (B)

Solution:

$x = 4t^2 + 3;\ y = 8t^3 - 1$

Slope of tangent $\frac{dy}{dx}=3t$

Also : $P(x_1\ y_1)=(4t_1^2+3,8t_1^3-1)$

$\therefore$ Slope = $3t_1$

$Q(x_2\ y_2)\ =\ (4t_2^2+3,8t_2^3-1)$

$$\frac{(8t_1^3-1)-(8t_2^3-1)}{(4t_1^2+3)-(4t_2^2+3)}=3t_1$$

$$\frac{2(t_1^2+t_1t_2+t_2^2)}{t_1+t_2}=3t_1$$

$t_1^2 + t_1t_2 - 2t_2^2 = 0$

$t_1 =-2t_2$ OR $t_2\ =\ -\frac{t_1}{2}$

$\therefore$ option (B) is correct.

Illustration 13:

The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at $(1, 1)$:

(A) meets the curve again in the third quadrant
(B) meets the curve again in the fourth quadrant
(C) does not meet the curve again
(D) meets the curve again in the second quadrant

Solution:

$x^2 + 3xy - xy - 3y^2 = 0$

$x(x + 3y) - y(x + 3y) = 0$

$(x - y)\ (x + 3y) = 0$

Normal at $(1, 1)$ will be $x + y = 2$

Now, $x + y = 2$

$x + 3y = 0$

$2y = -2, y = -1\ \&\ x = 3$

$\therefore\ (3, -1)$

which is in fourth quadrant.

Common Parametric Coordinates:

There are many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations**.

Instead of defining y in terms of x or x in terms of y, we define both x and y in terms of a third variable called a parameter as follows,

$x = f(t), y = g(t)$

$(x, y) = (f(t), g(t))$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$

Illustration 14:

The slope of the tangent to the curve $x = 3t^2 + 1, y = t^3 - 1$ at $x = 1$ is

(A) 0 (B) $\frac{1}{2}$ (C) ∞ (D) – 2

Ans. (A)

Solution:

$x = 3t^2 + 1, y = t^3 - 1$

$\therefore \frac{dx}{dt} = 6t, \ \frac{dy}{dt} = 3t^2$

Now $\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{3t^2}{6t} = \frac{t}{2}$

For $x = 1, 3t^2 + 1 = 1 \Rightarrow t = 0 \Rightarrow$ Slope $= \frac{0}{2} = 0$

Illustration 15:

Find the equation of tangent and normal to the curve $x = \frac{2at^2}{1+t^2}$, $y = \frac{2at^3}{1+t^2}$ at the point $t = \frac{1}{2}$.

Solution:

Given that

$x = \frac{2at^2}{1+t^2}, y = \frac{2at^3}{1+t^2}$

at $t = \frac{1}{2}$, $x = \frac{2a}{5}$, $y = \frac{a}{5}$

also $\frac{dx}{dt} = \frac{4at}{(1+t^2)^2}$ and $\frac{dy}{dt} = \frac{2at^2(3+t^2)}{(1+t^2)^2}$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2} t(3 + t^2)$

when $t = \frac{1}{2}, \frac{dy}{dx} = \frac{1}{2}\frac{1}{2}\left(3+\frac{1}{4}\right) = \frac{13}{16}$

$\therefore$ The equation of the tangent when $t = \frac{1}{2}$ is

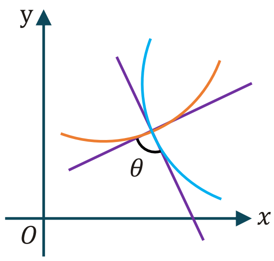
$y - \frac{a}{5} = \left(\frac{13}{16}\right)\left(x - \frac{2a}{5}\right) \Rightarrow 13x - 16y = 2a$

and the equation of the normal is $\left(y - \frac{a}{5}\right)\left(\frac{13}{16}\right) + \left(x - \frac{2a}{5}\right) = 0$

$\Rightarrow 16x + 13y = 9a$

Angle of Intersection of Two Curves:

The angle of intersection of two curves at a point of intersection (say at P) is defined as the angle between the two tangents/normal to the curve at their point of intersection.



Orthogonal Curves:

If the angle of intersection of two curves at **every point of intersection** is 90°, then curves are said to be orthogonal, i.e. $\left(\frac{dy_1}{dx}\right)\left(\frac{dy_2}{dx}\right)=-1$ at every point of intersection.

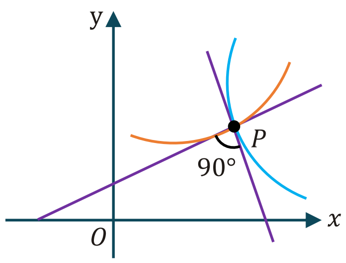


Illustration 16:

The angle of intersection between the curve $x^2 = 32y$ and $y^2 = 4x$ at point $(16, 8)$ is

(A) 60° (B) 90° (C) $\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\left(\frac{4}{3}\right)$

Ans. (C)

Solution:

$$x^2 = 32y \Rightarrow \frac{dy}{dx} = \frac{x}{16} \Rightarrow y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\therefore \; at\ (16, 8), \left(\frac{dy}{dx}\right)_1 = 1, \left(\frac{dy}{dx}\right)_2 = \frac{1}{4}$$

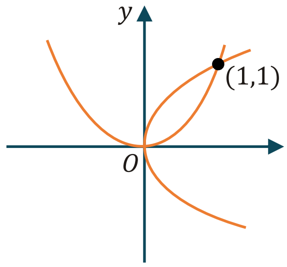
So, required angle $= \tan^{-1}\left(\dfrac{1-\frac{1}{4}}{1+1\left(\frac{1}{4}\right)}\right) = \tan^{-1}\left(\frac{3}{5}\right)$

Illustration 17:

Check the orthogonality of the curves $y^2 = x$ & $x^2 = y$.

Solution:

Solving the curves simultaneously we get points of intersection as $(1,1)$ and $(0,0)$.



At (1,1) for first curve $2y\left(\frac{dy}{dx}\right)_1=1 \Rightarrow m_1=\frac{1}{2}$

& for second curve $2x=\left(\frac{dy}{dx}\right)_2 \Rightarrow m_2=2$

$m_1 m_2 \neq -1$ at (1,1).

But at $(0,0)$ clearly x-axis & y –axis are their respective tangents hence they are orthogonal at (0,0) but not at (1,1). Hence these curves are not said to be orthogonal.

Illustration 18:

If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) 2 (D) 3

Ans. (B)

Solution:

$y=1-ax^2 \Rightarrow \frac{dy}{dx}=-2ax$

$y=x^2 \quad \Rightarrow \frac{dy}{dx}=2x$

Two curves intersect orthogonally if $\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2=-1$

$\Rightarrow(-2ax)(2x)=-1 \;\Rightarrow 4ax^2=1$ ...(i)

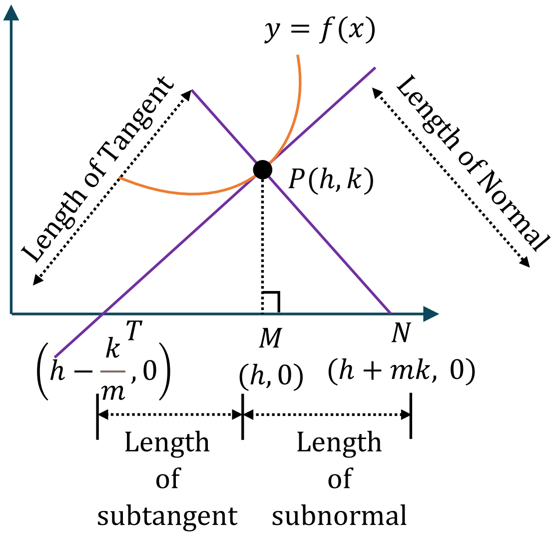
Now eliminating y from the given equations, we have $1-ax^2=x^2$

$\Rightarrow(1+a)x^2=1$ ...(ii)

Eliminating x^2 from (i) and (ii) we get $\frac{4a}{1+a}=1 \Rightarrow a=\frac{1}{3}$

Length of Tangent, Normal, Subtangent and Subnormal:

Let $P\ (h, k)$ be any point on curve $y = f(x)$. Let tangent drawn at point P meets x –axis at T & normal at point P meets x –axis at N. Then the length PT is called the length of tangent and PN is called length of normal. (as shown in figure)



Projection of segment PT on x –axis, TM, is called the subtangent and similarly projection of line segment PN on x axis, MN is called subnormal.

Let $m = \left.\frac{dy}{dx}\right]_{(h,k)}$ = slope of tangent.

Hence equation of tangent is $m\,(x - h) = (y - k)$.

Putting $y = 0$, we get x - intercept of tangent is

$x = h - \frac{k}{m}$

Similarly, the x-intercept of normal is $x = h + km$

Now, length PT, PN, TM, MN can be easily evaluated using distance formula

(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1+m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) $MN = |km|$ = Length of subnormal.

Illustration 19:

At any point of a curve $\sqrt{\frac{\text{Length of subnormal}}{\text{Length of sub tangent}}}$ is equal to –

(A) the abscissa of that point

(B) the ordinate of that point

(C) slope of the tangent at that point

(D) slope of the normal at that point

Solution:

$$\sqrt{\frac{y\frac{dy}{dx}}{\frac{y}{\frac{dy}{dx}}}} = \sqrt{\left(\frac{dy}{dx}\right)^2} = \left|\frac{dy}{dx}\right|$$

Illustration 20:

The length of the tangent to the curve $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ at point with parameter θ, is -

(A) $2a\sin\frac{\theta}{2}$ (B) $a\sin\theta$ (C) $2a\sin\theta$ (D) $a\cos\theta$

Ans. (A)

Solution:

$$\frac{dx}{d\theta} = a + a\cos\theta,\ \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a + a\cos\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$\frac{dy}{dx} = \tan\frac{\theta}{2}$ So length of tangent $= \left|y_1\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right|$

$$= \left|a\left(2\sin^2\frac{\theta}{2}\right)\sqrt{1+\cot^2\frac{\theta}{2}}\right| = 2a\sin\frac{\theta}{2}$$

Illustration 21:

For the curve $y = a\ ln\ (x^2 - a^2)$ show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.

Solution:

Let point of tangency be (x_1, y_1)

$$m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{2ax_1}{x^2{}_1 - a^2}$$

Length of tangent + subtangent $= |y_1|\sqrt{1 + \frac{1}{m^2}} + \left|\frac{y_1}{m}\right|$

$$= |y_1|\sqrt{1 + \frac{(x^2{}_1 - a^2)^2}{4a^2x_1{}^2}} + \left|\frac{y_1(x_1{}^2 - a^2)}{2ax_1}\right| = |y_1|\frac{\sqrt{x_1{}^4 + a^4 + 2a^2x_1{}^2}}{2|ax_1|} + \left|\frac{y_1(x_1{}^2 - a^2)}{2ax_1}\right|$$

$$= \left|\frac{y_1(x_1{}^2 + a^2)}{2ax_1}\right| + \left|\frac{y_1(x_1{}^2 - a^2)}{2ax_1}\right| = \frac{|y_1|(2x_1{}^2)}{2|ax_1|} = \left|\frac{x_1\ y_1}{a}\right|$$

Note :

Initial ordinate: Y intercept of tangent at point $P(x, y) = OA$

Radius vector (Polar radius): Line segment joining origin to point $P(x, y) = OP$

Vectoral angle: Angle made by radius vector with positive direction of x-axis in anticlock wise direction is called vectorial angle.