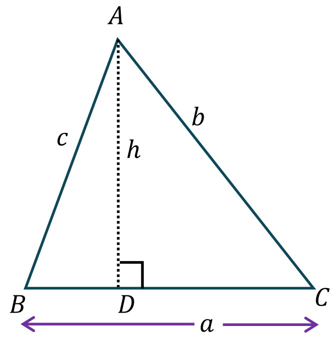
03 Solution of Triangles

The process of calculating the sides and angles of triangle using given information is called solution of triangle.

In a ΔABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. Sine Formulae:



In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\lambda=\frac{abc}{2\Delta}=2R$$

where R is circumradius and Δ is area of triangle.

Illustration 1:

Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Ans. (B)

Solution:

Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side. Now from sine formula $\frac{a}{\sin 120°}=\frac{b}{\sin 30°}=\frac{c}{\sin 30°}$

$$\Rightarrow \frac{a}{\sqrt{3}/2}=\frac{b}{1/2}=\frac{c}{1/2}$$

$$\Rightarrow \frac{a}{\sqrt{3}}=\frac{b}{1}=\frac{c}{1}=k \text{ (say)}$$

then $a=\sqrt{3}k$, perimeter = $(2+\sqrt{3})k$

$\therefore$ required ratio = $\frac{\sqrt{3}k}{(2+\sqrt{3})k}=\frac{\sqrt{3}}{2+\sqrt{3}}$

Illustration 2:

In triangle ABC, if $b = 3, c = 4$ and $\angle B = \pi/3$, then number of such triangles is -

(A) 1 (B) 2 (C) 0 (D) infinite

Ans. (C)

Solution:

Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

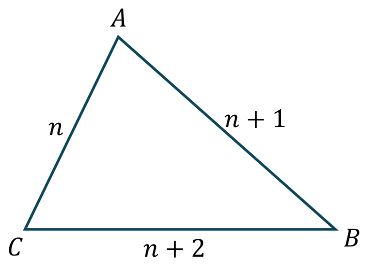
$\Rightarrow \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6} = \frac{\sin C}{4} \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1$ which is not possible.

Hence there exist no triangle with given elements.

Illustration 3:

The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution:



Let the sides be $n, n + 1, n + 2$ cms.

i.e. $AC = n, AB = n + 1, BC = n + 2$

Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow 3\angle B + \angle C = 180° \Rightarrow \angle C = 180° - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$

$$\Rightarrow \underset{(i)}{\frac{\sin 2B}{n+2}} = \underset{(ii)}{\frac{\sin B}{n}} = \underset{(iii)}{\frac{\sin 3B}{n+1}}$$

from (i) and (ii);

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \Rightarrow \cos B = \frac{n+2}{2n} \quad \text{...(iv)}$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \Rightarrow \frac{\sin B}{n} = \frac{\sin B(3-4\sin^2 B)}{n+1}$$

$$\Rightarrow \frac{n+1}{n} = 3 - 4(1-\cos^2 B) \quad \text{...(v)}$$

from (iv) and (v), we get

$$\frac{n+1}{n}=-1+4\left(\frac{n+2}{2n}\right)^2 \Rightarrow \frac{n+1}{n}+1=\left(\frac{n^2+4n+4}{n^2}\right)$$

$$\Rightarrow \frac{2n+1}{n}=\frac{n^2+4n+4}{n^2} \Rightarrow 2n^2+n=n^2+4n+4$$

$\Rightarrow n^2-3n-4=0 \Rightarrow (n-4)(n+1)=0$

$n=4$ or -1

where $n \neq -1$

$\therefore\ n=4$. Hence the sides are 4, 5, 6

2. Cosine Formulae:

(a) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ or $a^2=b^2+c^2-2bc\cos A$

(b) $\cos B=\dfrac{c^2+a^2-b^2}{2ca}$

(c) $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

Illustration 4:

In a triangle ABC, if $B=30°$ and $c=\sqrt{3}\,b$, then A can be equal to -

(A) 45° (B) 60° (C) 90° (D) 120°

Ans. (C)

Solution:

We have $\cos B=\dfrac{c^2+a^2-b^2}{2ca}$

$$\Rightarrow \frac{\sqrt{3}}{2}=\frac{3b^2+a^2-b^2}{2\times\sqrt{3}b\times a}$$

$\Rightarrow a^2-3ab+2b^2=0$

$\Rightarrow (a-2b)\,(a-b)=0$

$\Rightarrow$ Either $a=b \Rightarrow A=30°$

Or $a=2b \Rightarrow a^2=4b^2=b^2+c^2 \Rightarrow A=90°$.

Illustration 5:

In a triangle ABC, $(a^2-b^2-c^2)\tan A+(a^2-b^2+c^2)\tan B$ is equal to -

(A) $(a^2+b^2-c^2)\tan C$ (B) $(a^2+b^2+c^2)\tan C$

(C) $(b^2+c^2-a^2)\tan C$ (D) none of these

Ans. (D)

Solution:

Using cosine law :

The given expression is equal to $-2\,bc\cos A\tan A+2\,ac\cos B\tan B$

$$=2abc\left(-\frac{\sin A}{a}+\frac{\sin B}{b}\right)=0$$

3. Projection Formulae:

(a) $b\cos C + c\cos B = a$

(b) $c\cos A + a\cos C = b$

(c) $a\cos B + b\cos A = c$

Illustration 6:

In a ΔABC, $c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=\frac{3b}{2}$, then show a, b, c are in A.P.

Solution:

Here, $\frac{c}{2}(1+\cos A)+\frac{a}{2}(1+\cos C)=\frac{3b}{2}$

$\Rightarrow a + c + (c\cos A + a\cos C) = 3b$

$\Rightarrow a + c + b = 3b$ {using projection formula}

$\Rightarrow a + c = 2b$

which shows a, b, c are in A.P.

4. Napier's Analogy (Tangent Rule):

(a) $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$

(b) $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$

(c) $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

Illustration 7:

In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution:

Here, $\tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ ...(i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ ...(ii)

from (i) & (ii) ;

$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right) \Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$

{as $A + B + C = \pi$ $\therefore \tan\left(\frac{B+C}{2}\right) = \tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$}

$\Rightarrow \frac{a-b}{a+b}=\frac{1}{3}$ or $3a - 3b = a + b$

$2a = 4b$ or $\frac{a}{b}=\frac{2}{1}\Rightarrow\frac{b}{a}=\frac{1}{2}$

Thus the ratio of the sides opposite to the angles is b : a = 1 : 2. **Ans.**

5. Half Angle Formulae:

$s=\frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}$ (ii) $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}=\frac{\Delta}{s(s-b)}$ (iii) $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\frac{\Delta}{s(s-c)}$

(d) Area of Triangle

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C=\frac{1}{2}ap_1=\frac{1}{2}bp_2=\frac{1}{2}cp_3,$$

where p_1, p_2, p_3 are altitudes from vertices A, B, C respectively.

Illustration 8:

If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to-

(A) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (B) $\frac{2ab}{a+b}\sin\frac{C}{2}$ (C) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (D) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Ans. (C,D)

Solution:

$\Delta CAB=\Delta CAD+\Delta CDB$

$\Rightarrow \frac{1}{2}ab\sin C=\frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right)+\frac{1}{2}a.CD\sin\left(\frac{C}{2}\right)$

$\Rightarrow CD(a+b)\sin\left(\frac{C}{2}\right)=ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$

So $CD=\frac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\frac{CD}{\sin\angle DAC}=\frac{b}{\sin\angle CDA}$ (by sine rule)

$\Rightarrow CD=\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Illustration 9:

If Δ is the area and $2s$ the sum of the sides of a triangle, then show $\Delta\le\frac{s^2}{3\sqrt{3}}$.

Solution:

We have, $2s=a+b+c, \Delta^2=s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

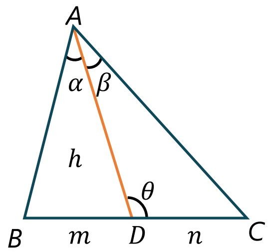
$\frac{(s-a)+(s-b)+(s-c)}{3}\ge\{(s-a)(s-b)(s-c)\}^{1/3}$

or $\dfrac{3s-2s}{3} \geq \left(\dfrac{\Delta^2}{s}\right)^{1/3}$

or $\dfrac{s}{3} \geq \left(\dfrac{\Delta^2}{s}\right)^{1/3}$

or $\dfrac{\Delta^2}{s} \leq \dfrac{s^3}{27} \Rightarrow \Delta \leq \dfrac{s^2}{3\sqrt{3}}$ **Ans**

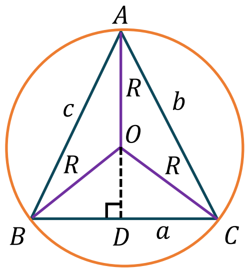
6. *m-n* Theorem:



$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

$(m+n)\cot\theta = n\cot B - m\cot C.$

7. Radius of The Circumcircle 'R':

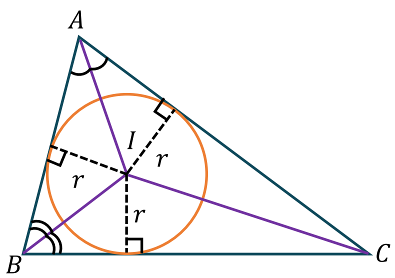


Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

8. Radius of The Incircle 'r':

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.



$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$$

$$= (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

Illustration 10:
In a triangle ABC, if $a : b : c = 4 : 5 : 6$, then ratio between its circumradius and inradius is-

(A) $\frac{16}{7}$ (B) $\frac{16}{9}$ (C) $\frac{7}{16}$ (D) $\frac{11}{7}$

Ans. (A)

Solution:

$$\frac{R}{r}=\frac{abc}{4\Delta}\Big/\frac{\Delta}{s}=\frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r}=\frac{abc}{4(s-a)(s-b)(s-c)} \quad \text{...(i)}$$

$\because a : b : c = 4 : 5 : 6 \Rightarrow \frac{a}{4}=\frac{b}{5}=\frac{c}{6}=k$ (say)

$\Rightarrow a = 4k, b = 5k, c = 6k$

$\therefore\ s=\frac{a+b+c}{2}=\frac{15k}{2}, s-a=\frac{7k}{2}, s-b=\frac{5k}{2}, s-c=\frac{3k}{2}$

using (i) in these values $\frac{R}{r}=\frac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)}=\frac{16}{7}$

Illustration 11:

If A, B, C are the angles of a triangle, prove that : $\cos A + \cos B + \cos C = 1+\frac{r}{R}$.

Solution:

$\cos A + \cos B + \cos C = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)+\cos C$

$=2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2} = 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]$

$= 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right] \qquad \left\{\because \frac{C}{2}=90°-\left(\frac{A+B}{2}\right)\right\}$

$= 1+2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2}=1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

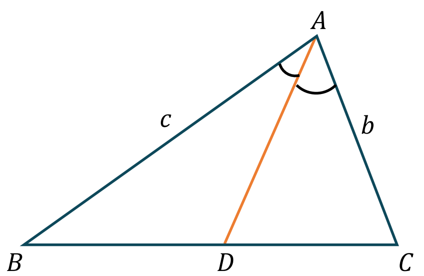
$= 1+\frac{r}{R} \qquad$ {as, $r = 4R \sin A/2 . \sin B/2 . \sin C/2$}

$\Rightarrow \cos A + \cos B + \cos C = 1+\frac{r}{R}$. Hence proved.

9. Angle Bisectors & Medians:

An angle bisector divides the base in the ratio of corresponding sides.

$\frac{BD}{CD}=\frac{c}{b} \Rightarrow BD = \frac{ac}{b+c}$ & $CD=\frac{ab}{b+c}$

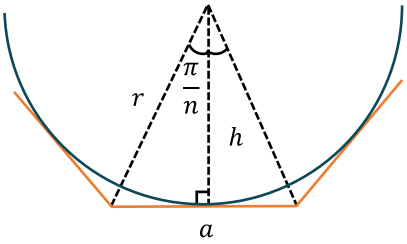


If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$$m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2} \text{ and } \beta_a=\frac{2bc\cos\frac{A}{2}}{b+c}$$

Note that $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$

10. Regular Polygon:



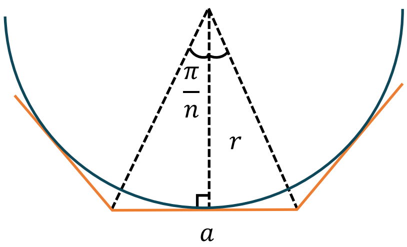
A regular polygon has all its sides equal. It may be inscribed or circumscribed.

(a) Inscribed in Circle of Radius r:

(i) $a = 2h \tan\frac{\pi}{n} = 2r\sin\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

(b) Circumscribed About a Circle of Radius r:



(i) $a = 2r\tan\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r is given by $p = 2nr\tan\frac{\pi}{n}$ and $A = nr^2\tan\frac{\pi}{n}$