

02

Quadratic Equation

1. Polynomial:

A function f defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

where $a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$ is called a polynomial of degree n with real coefficients ($a_n \neq 0, n \in \mathbb{W}$).

If $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$, it is called a polynomial with complex coefficients.

2. Quadratic polynomial & Quadratic equation:

A polynomial of degree 2 is known as quadratic polynomial. Any equation $f(x) = 0$, where f is a quadratic polynomial, is called a quadratic equation. The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \quad \dots(i)$$

Where a, b, c are real numbers, $a \neq 0$.

If $a = 0$, then equation (i) becomes linear equation.

3. Difference between equation & identity:

If a statement is true for all the values of the variable, such statements are called as identities. If the statement is true for some or no values of the variable, such statements are called as equations.

Example:

(i) $(x + 3)^2 = x^2 + 6x + 9$ is an identity

(ii) $(x + 3)^2 = x^2 + 6x + 8$, is an equation having no root.

(iii) $(x + 3)^2 = x^2 + 5x + 8$, is an equation having -1 as its root.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

$ax^2 + bx + c = 0$ is:

★ a quadratic equation if	$a \neq 0$	Two Roots
★ a linear equation if	$a = 0, b \neq 0$	One Root
★ a contradiction if	$a = b = 0, c \neq 0$	No Root
★ an identity if	$a = b = c = 0$	Infinite Roots

If $ax^2 + bx + c = 0$ is satisfied by at least three distinct values of ' x ', then it is an identity.

Illustration 1:

Check these statements are equation or identity

(i) $3x^2 + 2x - 1 = 0$

(ii) $(x + 1)^2 = x^2 + 2x + 1$

Solution:

(i) Given statement is a quadratic equation because here $a = 3$.

(ii) Here highest power of x in the given relation is 2 and this relation is satisfied by three different values $x = 0, x = 1$ and $x = -1$ and hence it is an identity in x because a polynomial equation of n^{th} degree cannot have more than n distinct roots

Illustration 2:

If the equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ has more than two roots, then find the value of λ ?

Solution:

As the equation has more than two roots so it becomes an identity. Hence

$$\begin{aligned} \lambda^2 - 5\lambda + 6 = 0 &\Rightarrow \lambda = 2, 3 \\ \text{and } \lambda^2 - 3\lambda + 2 = 0 &\Rightarrow \lambda = 1, 2 \\ \text{and } \lambda^2 - 4 = 0 &\Rightarrow \lambda = 2, -2 \\ \text{So } \lambda = 2 \end{aligned}$$

4. Relation Between Roots & Co-efficient:

(i) The solutions of quadratic equation, $ax^2 + bx + c = 0$, ($a \neq 0$) is given by α, β

The expression, $b^2 - 4ac \equiv D$ is called discriminant of quadratic equation.

(ii) If α, β are the roots of quadratic equation,

$$\Rightarrow ax^2 + bx + c = 0 \quad \dots(i)$$

then equation (i) can be written as

$$\begin{aligned} \Rightarrow a(x - \alpha)(x - \beta) &= 0 \\ \Rightarrow ax^2 - a(\alpha + \beta)x + a\alpha\beta &= 0 \quad \dots(ii) \end{aligned}$$

equations (i) and (ii) are identical,

\therefore by comparing the coefficients

$$\text{sum of the roots, } \alpha + \beta = -\frac{b}{a}$$

$$\text{and product of the roots, } \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

(iii) Dividing the equation (i) by a ,

$$\begin{aligned} \Rightarrow x^2 - \left(\frac{-b}{a}\right)x + \frac{c}{a} &= 0 \\ \Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta &= 0 \\ \Rightarrow x^2 - (\text{sum of the roots})x + (\text{product of the roots}) &= 0 \end{aligned}$$

Hence, we conclude that the quadratic equation whose roots are α & β is

$$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Illustration 3:

If α and β are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $\alpha + 2$ and $\beta + 2$.

Solution:

Replacing x by $x - 2$ in the given equation, the required equation is

$$\Rightarrow a(x - 2)^2 + b(x - 2) + c = 0 \text{ i.e., } ax^2 - (4a - b)x + (4a - 2b + c) = 0.$$

Illustration 4:

The coefficient of x in the quadratic equation $x^2 + px + q = 0$ was taken as 17 in place of 13, its roots were found to be -2 and -15 . Find the roots of the original equation.

Solution:

Here $q = (-2) \times (-15) = 30$, correct value of $p = 13$. Hence original equation is

$$\Rightarrow x^2 + 13x + 30 = 0 \text{ as } (x + 10)(x + 3) = 0$$

\therefore roots are $-10, -3$

Illustration 5:

If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

- (A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$
 (C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Ans. (B)

Solution:

Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\Rightarrow \beta^2 - 3\beta + 5 = 0$

$\therefore \alpha^2 - 3\alpha = -5$

$\Rightarrow \beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$... (i)

$\Rightarrow -5 + 7, -5 + 7$

$\therefore 2$ and 2 are the roots.

\therefore The required equation is $x^2 - 4x + 4 = 0$.

Illustration 6:

If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution:

If $\alpha + \beta$ are roots of $ax^2 + bx + c = 0$

then, $a\alpha^2 + b\alpha + c = 0 \Rightarrow a\alpha + b = -\frac{c}{\alpha}$

same as $a\beta + b = -\frac{c}{\beta}$

$\Rightarrow (a\alpha + b)^{-2} = (a\beta + b)^{-2} = \frac{\alpha^2}{c^2} + \frac{\beta^2}{c^2}$

$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{c^2} = \frac{(-b/a)^2 - 2(c/a)}{c^2} = \frac{b^2 - 2ac}{a^2c^2}$

Illustration 7:

If the roots of the equation $x^2 + 2ax + b = 0$ are real and distinct and they differ by atmost $2m$, then b lies in the interval

- (A) $(a^2 - m^2, a^2)$ (B) $[a^2 - m^2, a^2)$ (C) $(a^2, a^2 + m^2)$ (D) none of these

Ans. (B)

Solution:

$x^2 + 2ax + b = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$\Rightarrow 0 < |\alpha - \beta| \leq 2m$

$\Rightarrow 0 < \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} \leq 2m$

$\Rightarrow 0 < 4a^2 - 4b \leq 4m^2$

$\Rightarrow a^2 - m^2 \leq b < a^2$

$\Rightarrow b \in [a^2 - m^2, a^2)$

Illustration 8:

The set of possible values of λ for which $x^2 - (\lambda^2 - 5\lambda + 5)x + (2\lambda^2 - 3\lambda - 4) = 0$ has roots, whose sum and product are both less than 1, is

- (A) $\left(-1, \frac{5}{2}\right)$ (B) $(1, 4)$ (C) $\left[1, \frac{5}{2}\right]$ (D) $\left(1, \frac{5}{2}\right)$

Ans. (D)

Solution:

Sum of roots < 1

$$\Rightarrow \lambda^2 - 5\lambda + 5 < 1$$

$$\Rightarrow (\lambda - 1)(\lambda - 4) < 0$$

$$\Rightarrow 1 < \lambda < 4 \quad \dots(1)$$

\Rightarrow Product of roots < 1

$$\Rightarrow 2\lambda^2 - 3\lambda - 4 < 1$$

$$\Rightarrow 2\lambda^2 - 3\lambda - 5 < 0$$

$$\Rightarrow (2\lambda - 5)(\lambda + 1) < 0$$

$$\Rightarrow -1 < \lambda < \frac{5}{2} \quad \dots(2)$$

(1) & (2)

$$\Rightarrow 1 < \lambda < \frac{5}{2}.$$

5. Nature of Roots:

- (a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{R}$ & $a \neq 0$ then; $x = \frac{-b \pm \sqrt{D}}{2a}$
- (i) $D > 0$ roots are real & distinct (unequal).
 - (ii) $D = 0$ roots are real & coincident (equal)
 - (iii) $D < 0$ roots are imaginary.
 - (iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. ($p, q \in \mathbb{R}$)
- (b) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{Q}$ & $a \neq 0$ then;
- (i) If D is a perfect square, then roots are rational.
 - (ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & \sqrt{q} is an irrational) then other root will be $p - \sqrt{q}$

Illustration 9:

For what values of 'm' the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots.

Solution:

Given equation is $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0 \quad \dots(i)$

Let D be the discriminant of equation (i).

Roots of equation (i) will be equal if $D = 0$.

$$\Rightarrow 4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0$$

$$\Rightarrow 4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$$

$$\Rightarrow m^2 - 3m = 0 \text{ or } m(m - 3) = 0$$

$$\therefore m = 0, 3.$$

Illustration 10:

Find all the integral values of 'a' for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution:

Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

$$\text{From (i) } D = a^2 - 20a + 96.$$

$$\Rightarrow D = (a - 10)^2 - 4$$

$$\Rightarrow 4 = (a - 10)^2 - D$$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when

$$\Rightarrow (a - 10)^2 = 4 \text{ and } D = 0.$$

$$\Rightarrow (a - 10) = \pm 2$$

$$\Rightarrow a = 12, 8$$

Illustration 11:

Show that the expression $x^2 + 2(a + b + c)x + 3(bc + ca + ab)$ will be a perfect square if $a = b = c$.

Solution:

Given quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero.

$$\text{i.e. } 4(a + b + c)^2 - 4 \cdot 3(bc + ca + ab) = 0$$

$$\Rightarrow (a + b + c)^2 - 3(bc + ca + ab) = 0$$

$$\Rightarrow \frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2) = 0$$

which is possible only when $a = b = c$.

Illustration 12:

If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1, then find the equation one of whose roots is $\sqrt{2} - 1$.

Solution:

Irrational roots always occur in conjugate pairs.

Hence if one root is $(-1 + \sqrt{2})$, the other root will be $(-1 - \sqrt{2})$. Equation is

$$(x - (-1 + \sqrt{2}))(x - (-1 - \sqrt{2})) = 0$$

$$\Rightarrow x^2 + 2x - 1 = 0$$

Illustration 13:

Find the set of all real values of λ such that the root of the equation

$$x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$$

are always real for any choice of a, b, c (where a, b, c represents sides of scalene triangle).

- (A) $(-\infty, \frac{4}{3})$ (B) $(\frac{4}{3}, \infty)$ (C) $(\frac{1}{3}, \frac{5}{3})$ (D) $(\frac{4}{3}, \frac{5}{3})$

Ans. (A)

Solution:

We, know that $a + b > c, b + c > a$ and $c + a > b$

$$\Rightarrow c - a < b, a - b < c, b - c < a$$

squaring on both sides and adding $(c - a)^2 + (a - b)^2 + (b - c)^2 < a^2 + b^2 + c^2$

$$a^2 + b^2 + c^2 - 2(ab + bc + ca) < 0$$

$$\Rightarrow (a + b + c)^2 - 4(ab + bc + ca) < 0$$

$$\Rightarrow \frac{(a+b+c)^2}{ab+bc+ca} < 4 \quad \dots(i)$$

Now roots of equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then $D \geq 0$

$$\Rightarrow 4(a + b + c)^2 - 4 \cdot 3\lambda(ab + bc + ca) \geq 0$$

$$\Rightarrow \frac{(a+b+c)^2}{ab+bc+ca} \geq 3\lambda$$

$$\text{So } 3\lambda \leq \frac{(a+b+c)^2}{ab+bc+ca} < 4 \Rightarrow \lambda < \frac{4}{3}$$

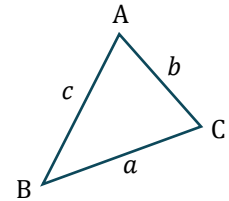


Illustration 14:

If coefficients of biquadratic equation are all distinct and belong to the set $\{-9, -5, 3, 4, 7\}$, then equation has

- (A) atleast two real roots
 (B) four real roots, two are conjugate surds and other two are also conjugate surds
 (C) four imaginary roots
 (D) None of these

Ans. (A)

Solution:

Let biquadratic is $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\Rightarrow a + b + c + d + e = 0 \text{ as } a, b, c, d, e \in \{-9, -5, 3, 4, 7\}$$

Hence $x = 1$ is a root. So real root will be atleast two.

Illustration 15:

Let $p, q, r, s \in R, x^2 + px + q = 0, x^2 + rx + s = 0$ such that $2(q + s) = pr$ then

- (A) atleast one of the equation have real roots.
 (B) either both equations have imaginary roots or both equations have real roots.
 (C) one of equations have real roots and other equation have imaginary roots
 (D) atleast one of the equations have imaginary roots.

Ans. (A)

Quadratic Equation

Solution:

$$\Rightarrow x^2 + px + q = 0 \Rightarrow D_1 = p^2 - 4q \quad \dots(1)$$

$$\Rightarrow x^2 + rx + s = 0 \Rightarrow D_2 = r^2 - 4s \quad \dots(2)$$

$$\Rightarrow D_1 + D_2 = p^2 + r^2 - 4(q + s) \quad [\because pr = 2(q + s)]$$

$$\Rightarrow (p - r)^2 > 0$$

Since $D_1 + D_2$ is +ve, so atleast one of the equations has real roots.

6. Transformation of the equation:

Let $ax^2 + bx + c = 0$ be a quadratic equation with two roots α and β . If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α and β , then this equation can be found by finding α in terms of y . Now as α satisfies given equation, put this α in terms of y directly in the equation.

$$y = f(\alpha)$$

By transformation, $\alpha = g(y)$

$$a(g(y))^2 + b(g(y)) + c = 0$$

This is the required equation in y .

Illustration 16:

If the roots of $ax^2 + bx + c = 0$ are α and β , then find the equation whose roots are:

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (c) α^2, β^2

Solution:

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$

$$\text{put, } y = \frac{-2}{\alpha} \Rightarrow \alpha = \frac{-2}{y}$$

$$\Rightarrow a\left(\frac{-2}{y}\right)^2 + b\left(\frac{-2}{y}\right) + c = 0 \Rightarrow cy^2 - 2by + 4a = 0$$

$$\text{Required equation is } cx^2 - 2bx + 4a = 0$$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

$$\text{put, } y = \frac{\alpha}{\alpha+1} \Rightarrow \alpha = \frac{y}{1-y}$$

$$\Rightarrow a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0$$

$$\Rightarrow (a + c - b)y^2 + (-2c + b)y + c = 0$$

$$\text{Required equation is } (a + c - b)x^2 + (b - 2c)x + c = 0$$

(c) α^2, β^2 put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$$\Rightarrow ay + b\sqrt{y} + c = 0$$

$$\Rightarrow b^2y = a^2y^2 + c^2 + 2acy$$

$$\Rightarrow a^2y^2 + (2ac - b^2)y + c^2 = 0$$

$$\text{Required equation is } a^2x^2 + (2ac - b^2)x + c^2 = 0$$

Illustration 17:

If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$.

Solution:

$$\text{Put } y = \frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = -\frac{a\gamma}{d} \quad (\alpha\beta\gamma = -\frac{d}{a})$$

$$\text{Put } x = -\frac{dy}{a}$$

$$\Rightarrow a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$$

$$\text{Required equation is } d^2x^3 - bdx^2 + acx - a^2 = 0$$

7. Common Roots:

Consider two quadratic equations, $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$.

(i) If two quadratic equations have both roots common, then the equations are identical and their co-efficient are in proportion.

$$\text{i.e. } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(ii) If one root is common, then the common root ' α ' will be:

$$\Rightarrow \alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

Hence the condition for common root is:

$$\Rightarrow (c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$$

Note:

If $f(x) = 0$ & $g(x) = 0$ are two polynomial equation having some common root(s) then those common root(s) is/are also the root(s) of $h(x) \equiv a f(x) + b g(x) = 0$.

Illustration 18:

Determine ' a ' such that $x^2 - 11x + a$ and $x^2 - 14x + 2a$ may have a common factor.

Solution:

Let $x - \alpha$ be a common factor of $x^2 - 11x + a$ and $x^2 - 14x + 2a$.

Then $x = \alpha$ will satisfy the equations $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$.

$$\therefore \alpha^2 - 11\alpha + a = 0 \text{ and } \alpha^2 - 14\alpha + 2a = 0$$

Solving (i) and (ii) by cross multiplication method, we get $a = 0, 24$.

Illustration 19:

If $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ have a root in common and the second equation has equal roots, show that $b + q = \frac{ap}{2}$.

Quadratic Equation

Solution:

$$\text{Given equations are : } x^2 - ax + b = 0 \quad \dots(i)$$

$$\text{and } x^2 - px + q = 0. \quad \dots(ii)$$

Let α be the common root. Then roots of equation (ii) will be α and α . Let β be the other root of equation (i).

Thus, roots of equation (i) are α, β and those of equation (ii) are α, α .

Now

$$\Rightarrow \alpha + \beta = a \quad \dots(iii)$$

$$\Rightarrow \alpha\beta = b \quad \dots(iv)$$

$$\Rightarrow 2\alpha = p \quad \dots(v)$$

$$\Rightarrow \alpha^2 = q \quad \dots(vi)$$

$$\text{L.H.S.} = b + q = \alpha\beta + \alpha^2 = \alpha(\alpha + \beta) \quad \dots(vii)$$

$$\text{and R.H.S.} = \frac{ap}{2} = \frac{(\alpha + \beta)2\alpha}{2} = \alpha(\alpha + \beta) \quad \dots(viii)$$

from (vii) and (viii), L.H.S. = R.H.S.

Illustration 20:

If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, show that $a : b : c = 1 : 2 : 9$.

Solution:

$$\text{Given equations are: } x^2 + 2x + 9 = 0 \quad \dots(i)$$

$$\text{and } ax^2 + bx + c = 0 \quad \dots(ii)$$

Clearly roots of equation (i) are imaginary since equation (i) and (ii) have a common root, therefore common root must be imaginary and hence both roots will be common.

Therefore equations (i) and (ii) are identical

$$\Rightarrow \frac{a}{1} = \frac{b}{2} = \frac{c}{9}$$

$$\Rightarrow a : b : c = 1 : 2 : 9$$

Illustration 21:

Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Solution:

$$a_1 = p, b_1 = 5, c_1 = 2 \text{ and } a_2 = 3, b_2 = 10, c_2 = q$$

We know that:

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{p}{3} = \frac{5}{10} = \frac{2}{q}$$

$$\Rightarrow p = \frac{3}{2}; q = 4$$

Illustration 22:

Find the conditions on a, b, c and d such that equations

$2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.

Solution:

Let ' α ' be a common root of the given equations,

$$\text{then } 2a\alpha^3 + b\alpha^2 + c\alpha + d = 0 \quad \dots(1)$$

$$\text{and } 2a\alpha^2 + 3b\alpha + 4c = 0 \quad \dots(2)$$

Multiply (2) by α and then subtract (1) from it, to get

$$2b\alpha^2 + 3c\alpha - d = 0 \quad \dots(3)$$

Now (2) and (3) are quadratic having a common root α , so

$$\Rightarrow \frac{\alpha^2}{-3bd-12c^2} = \frac{\alpha}{8bc+2ad} = \frac{1}{6ac-6b^2}, \alpha^2 = \frac{bd+4c^2}{2b^2-2ac}, \alpha = \frac{4bc+ad}{3ac-3b^2}$$

Eliminating α from these two equations, we get

$$(4bc + ad)^2 = \frac{9}{2} (bd + 4c^2)(b^2 - ac), \text{ which is the required condition.}$$

Illustrations 23:

If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$,

evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

in terms of p, q, r and s . Deduce the condition that the equations have a common root.

Solution:

α, β are the roots of $x^2 + px + q = 0$

$$\Rightarrow \alpha + \beta = -p, \alpha\beta = q \quad \dots(1)$$

and γ, δ are the roots of $x^2 + rx + s = 0$

$$\Rightarrow \gamma + \delta = -r, \gamma\delta = s \quad \dots(2)$$

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$$\Rightarrow [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$$

$$\Rightarrow (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$$

$$\Rightarrow \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$$

$$\Rightarrow \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s((\alpha + \beta)^2 - 2\alpha\beta) + sr(\alpha + \beta) + s^2$$

$$\Rightarrow q^2 - pqr + r^2q + s(p^2 - 2q) + sr(-p) + s^2$$

$$\Rightarrow (q - s)^2 - r pq + r^2q + sp^2 - p rs$$

$$\Rightarrow (q - s)^2 - r q(p - r) + sp(p - r)$$

$$\Rightarrow (q - s)^2 + (p - r)(sp - r q)$$

$$\text{For a common root (Let } \alpha = \gamma \text{ or } \beta = \delta) \quad \dots(3)$$

$$\text{Then } (\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0 \quad \dots(4)$$

from (3) and (4), we get

$$(q - s)^2 + (p - r)(sp - r q) = 0$$

$$\Rightarrow (q - s)^2 = (p - r)(r q - s p), \text{ which is the required condition.}$$

8. General Method to Solve Inequalities:

(Method of intervals (Wavy curve method))

$$\text{Let } g(x) = \frac{(x-b_1)^{k_1}(x-b_2)^{k_2}\dots(x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2}\dots(x-a_n)^{r_n}} \quad \dots(i)$$

Where k_1, k_2, \dots, k_n and $r_1, r_2, \dots, r_n \in \mathbb{N}$ and b_1, b_2, \dots, b_n and a_1, a_2, \dots, a_n are real numbers.

Then to solve the inequality following steps are taken.

Steps:

Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

- (i) First we find the zeros and poles of the function.
- (ii) Then we mark all the zeros and poles on the real line and put a vertical bar there dividing the real line in many intervals.
- (iii) Determine sign of the function in any of the interval and then alternates the sign in the neighbouring interval if the poles or zeros dividing the two intervals has appeared odd number of times otherwise retain the sign.
- (iv) Thus, we consider all the intervals. The solution of the $g(x) > 0$ is the union of the intervals in which we have put the plus sign and the solution of $g(x) < 0$ is the union of all intervals in which we have put the minus sign.

Illustration 24:

Find x such that $3x^2 - 7x + 6 < 0$

Solution:

$$f(x) = 3x^2 - 7x + 6$$

$$\Rightarrow D = 49 - 72 < 0 \text{ and } a > 0$$

$$\Rightarrow 3x^2 - 7x + 6 > 0 \forall x \in \mathbb{R} \Rightarrow x \in \phi$$

Illustration 25:

Find x such that $2x^2 + 4x + 9 > 0$

Solution:

$$f(x) = 2x^2 + 4x + 9$$

$$\Rightarrow a > 0 \text{ and } D < 0$$

$$\Rightarrow 2x^2 + 4x + 9 > 0 \forall x \in \mathbb{R}$$

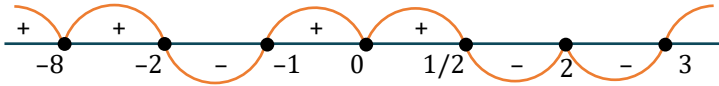
$$\Rightarrow x \in (-\infty, \infty)$$

Illustration 26:

Solve the inequality if $f(x) = \frac{(x-2)^{10}(x+1)^3(x-\frac{1}{2})^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ is > 0 or < 0 .

Solution:

Let $f(x) = \frac{(x-2)^{10}(x+1)^3(x-\frac{1}{2})^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ the poles and zeros are $0, 3, -2, -1, \frac{1}{2}, -8, 2$



If $f(x) > 0$, then $x \in (-\infty, -8) \cup (-8, -2) \cup (-1, 0) \cup (0, \frac{1}{2}) \cup (3, \infty)$

and if $f(x) < 0$, then $x \in (-2, -1) \cup (\frac{1}{2}, 2) \cup (2, 3)$.

Illustration 27:

Number of positive integral values of x satisfying the inequality

$$\frac{(x-4)^{2013} \cdot (x+8)^{2014} (x+1)}{x^{2016} (x-2)^3 \cdot (x+3)^5 \cdot (x-6)(x+9)^{2012}} \leq 0$$

- (A) 0 (B) 1 (C) 2 (D) 3

Ans. (D)

Solution:

Poles and zeros are $4, -8, -1, 0, 2, -3, 6, -9$



$x \in (-\infty, -9) \cup (-9, -3) \cup [-1, 0] \cup (0, 2) \cup [4, 6)$

so +ve integral solutions are $x = 1, 4, 5$

Illustration 28:

Number of non-negative integral values of x satisfying the inequality

$$\frac{2}{x^2 - x + 1} - \frac{1}{x + 1} - \frac{2x - 1}{x^3 + 1} \geq 0$$

- (A) 0 (B) 1 (C) 2 (D) 3

Ans. (D)

Solution:

$$\frac{2}{x^2 - x + 1} - \frac{1}{x + 1} - \frac{2x - 1}{x^3 + 1} \geq 0$$

$$\frac{2x + 2 - x^2 + x - 1}{x^3 + 1} - \frac{(2x - 1)}{x^3 + 1} \geq 0 \Rightarrow \frac{3x + 1 - x^2 - 2x + 1}{x^3 + 1} \geq 0$$

$$\Rightarrow \frac{-x^2 + x + 2}{x^3 + 1} \geq 0 \Rightarrow \frac{x^2 - x - 2}{x^3 + 1} \leq 0$$

$$\Rightarrow \frac{(x + 1)(x - 2)}{(x + 1)(x^2 - x + 1)} \leq 0, x \neq -1 \Rightarrow \frac{(x - 2)}{(x^2 - x + 1)} \leq 0$$

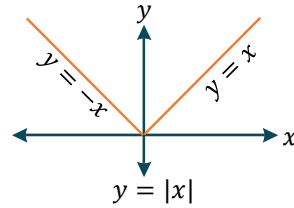


$x \in (-\infty, -1) \cup (-1, 2]$

required value of $x = \{0, 1, 2\}$.

9. Absolute value function/modulus function:

The symbol of modulus function is $f(x) = |x|$ and is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



Properties of modulus:

For any $a, b \in \mathbb{R}$

- (i) $|a| \geq 0$
- (ii) $|a| = |-a|$
- (iii) $|a| \geq a, |a| \geq -a$
- (iv) $|ab| = |a| |b|$
- (v) $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$
- (vi) $|a + b| \leq |a| + |b|$; Equality holds when $ab \geq 0$
- (vii) $|a - b| \geq ||a| - |b||$; Equality holds when $ab \geq 0$

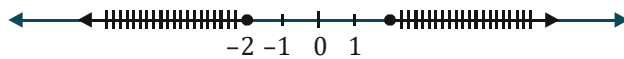
Illustration 29:

Represent x on the number line when

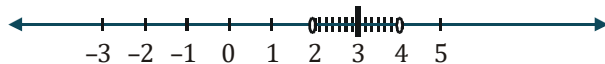
- (i) $|x| \geq 2$
- (ii) $|x - 3| \leq 1$
- (iii) $|5 - x| < 2$
- (iv) $1 \leq |x + 2| < 5$
- (v) $|2x - 5| < 1$

Solution:

(i) $|x| \geq 2 \Rightarrow$ distance of x from O on the number line is greater than or equal to 2.



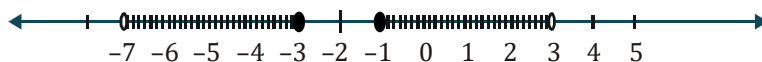
(ii) $|x - 3| \leq 1 \Rightarrow$ distance of x from 3 is less than or equal to 1.



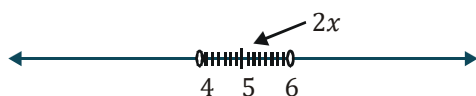
(iii) $|5 - x| < 2 \Rightarrow$ distance of x from 2 is less than 2.



(iv) $1 \leq |x + 2| < 5 \Rightarrow 1 \leq |x - (-2)| < 5$
 \Rightarrow distance of x from -2 is greater than or equal to 1 and less than 5.



(v) $|2x - 5| < 1 \Rightarrow$ distance of $2x$ from 5 on the number line is less than 1.



$\Rightarrow 4 < 2x < 6 \Rightarrow 2 < x < 3$

Illustration 30:

Find the values of m so that the inequality: $\left| \frac{x^2+mx+1}{x^2+x+1} \right| < 3$ holds for all $x \in \mathbb{R}$.

Solution:

We know that

$$|a| < b \Rightarrow -b < a < b \text{ (for } b > 0)$$

$$\text{Hence, } \left| \frac{x^2+mx+1}{x^2+x+1} \right| < 3 \Rightarrow -3 < \frac{x^2+mx+1}{x^2+x+1} < 3$$

Case I :

$$\frac{x^2+mx+1}{x^2+x+1} < 3 \Rightarrow \frac{(x^2+mx+1)-3(x^2+x+1)}{x^2+x+1} < 0 \Rightarrow \frac{-2x^2+(m-3)x-2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} < 0$$

Multiplying both sides by denominator, we get :

$$\Rightarrow -2x^2 + (m-3)x - 2 < 0 \text{ (because denominator is always positive)}$$

$$\Rightarrow 2x^2 - (m-3)x + 2 > 0$$

A quadratic expression in x is always positive if coefficient of $x^2 > 0$ and $D < 0$.

$$\Rightarrow (m-3)^2 - 4(2)(2) < 0 \Rightarrow (m-3)^2 - 4^2 < 0$$

$$\Rightarrow (m-7)(m+1) < 0 \Rightarrow m \in (-1, 7) \quad \dots(i)$$

Case II:

$$-3 < \frac{x^2+mx+1}{x^2+x+1}$$

$$\Rightarrow \frac{(x^2+mx+1)+3(x^2+x+1)}{x^2+x+1} > 0 \Rightarrow 4x^2 + (m+3)x + 4 > 0$$

For this to be true for all $x \in \mathbb{R}$, coefficient of $x^2 > 0$ and $D < 0$

$$\Rightarrow (m+3)^2 - 4(4)(4) < 0$$

$$\Rightarrow (m+3-8)(m+3+8) < 0$$

$$\Rightarrow (m-5)(m+11) < 0 \Rightarrow m \in (-11, 5) \quad \dots(ii)$$

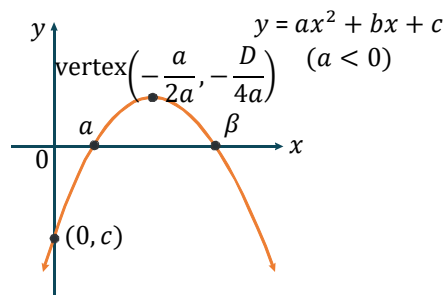
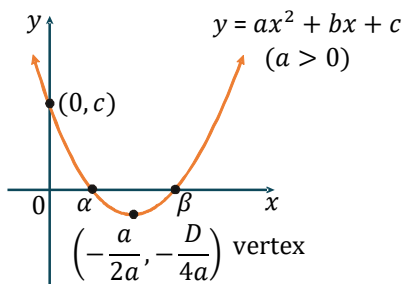
We will combine (i) and (ii) because both must be satisfied.

\Rightarrow The common solution is $m \in (-1, 5)$.

10. Graph of Quadratic Expression:

$$y = f(x) = ax^2 + bx + c \text{ or } \left(y + \frac{D}{4a}\right) = a \left(x + \frac{b}{2a}\right)^2$$

- ★ the graph between x, y is always a parabola.
- ★ the co-ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$
- ★ If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

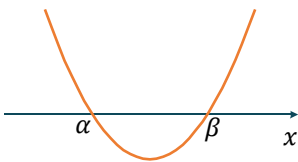
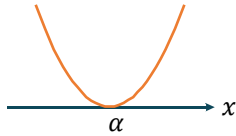
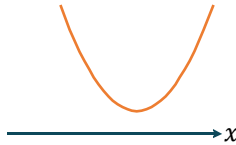
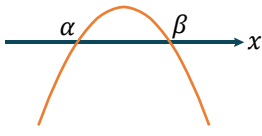
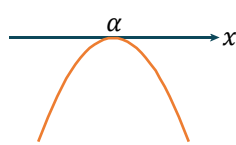
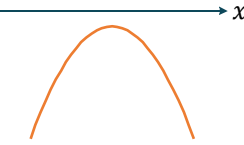


- ★ the parabola intersects the y -axis at point $(0, c)$.
- ★ the x -co-ordinate of point of intersection of parabola with x -axis are the real roots of the quadratic equation $f(x) = 0$. Hence the parabola may or may not intersect the x -axis.

Sign of Quadratic Expressions:

The value of expression $f(x) = ax^2 + bx + c$ at $x = x_0$ is equal to y -co-ordinate of the point on parabola $y = ax^2 + bx + c$ whose x -co-ordinate is x_0 . Hence if the point lies above the x -axis for some $x = x_0$, then $f(x_0) > 0$ and vice-versa.

We get six different positions of the graph with respect to x -axis as shown.

	Graph	Conclusions
(i)		(a) $a > 0$ (b) $D > 0$ (c) Roots are real & distinct. (d) $f(x) > 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$ (e) $f(x) < 0$ in $x \in (\alpha, \beta)$
(ii)		(a) $a > 0$ (b) $D = 0$ (c) Roots are real & equal. (d) $f(x) > 0$ in $x \in \mathbb{R} - \{\alpha\}$
(iii)		(a) $a > 0$ (b) $D < 0$ (c) Roots are imaginary. (d) $f(x) > 0 \forall x \in \mathbb{R}$.
(iv)		(a) $a < 0$ (b) $D > 0$ (c) Roots are real & distinct. (d) $f(x) < 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$ (e) $f(x) > 0$ in $x \in (\alpha, \beta)$
(v)		(a) $a < 0$ (b) $D = 0$ (c) Roots are real & equal. (d) $f(x) < 0$ in $x \in \mathbb{R} - \{\alpha\}$
(vi)		(a) $a < 0$ (b) $D < 0$ (c) Roots are imaginary. (d) $f(x) < 0 \forall x \in \mathbb{R}$.

11. Range of Quadratic Expression $f(x) = ax^2 + bx + c$.

(a) Range:

$$\text{If } a > 0 \Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty \right)$$

$$\text{If } a < 0 \Rightarrow f(x) \in \left(-\infty, -\frac{D}{4a} \right]$$

Hence maximum and minimum values of the expression $f(x)$ is $-\frac{D}{4a}$ in respective cases and it occurs at

$$x = -\frac{b}{2a} \text{ (at vertex).}$$

(b) Range in restricted domain:

Given $x \in [x_1, x_2]$

(A) If $-\frac{b}{2a} \notin [x_1, x_2]$ then, $f(x) \in [\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}]$

(B) If $-\frac{b}{2a} \in [x_1, x_2]$ then,

$$\Rightarrow f(x) \in \min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}$$

Illustration 31:

If $a < 0$ and $ax^2 + bx + c = 0$ does not have any real roots then prove that

(i) $a - b + c < 0$

(ii) $9a + 3b + c < 0$.

Solution:

$a < 0$ and $D < 0$

$\Rightarrow f(x) = ax^2 + bx + c < 0$ for all $x \in R$

$\Rightarrow f(-1) = a - b + c < 0$ and $f(3) = 9a + 3b + c < 0$

Illustration 32:

Find the maximum and minimum values of $f(x) = x^2 - 5x + 6$.

Solution:

minimum of $f(x) = -\left[\frac{D}{4a}\right]$ at $x = -\frac{b}{2a} = -\left(\frac{25-24}{4}\right)$ at $x = \frac{5}{2} = -\frac{1}{4}$

maximum of $f(x) \rightarrow \infty$

Hence range is $\left[-\frac{1}{4}, \infty\right)$.

Illustration 33:

Find the range of rational expression $y = \frac{x^2+x+1}{x^2+1}$ if x is real.

Solution:

$$y = \frac{x^2+x+1}{x^2+1} \Rightarrow (y-1)x^2 - x + (y-1) = 0 \quad \dots(i)$$

Case I:

if $y \neq 1$, then equation (i) is quadratic in x

$$\therefore D \geq 0 \Rightarrow 1 - 4(y-1)^2 \geq 0$$

$$\Rightarrow -\frac{1}{2} \leq (y-1) \leq \frac{1}{2} \Rightarrow \frac{1}{2} \leq y \leq \frac{3}{2}$$

$$y \in \left[\frac{1}{2}, \frac{3}{2}\right] - \{1\}$$

Case II:

if $y = 1$, then equation becomes

$$-x = 0$$

$\Rightarrow x = 0$ which is possible as x is real.

$$\therefore \text{Range } \left[\frac{1}{2}, \frac{3}{2}\right]$$

Illustration 34:

Find the least value of $y = \frac{(6x^2-22x+21)}{5x^2-18x+17}$ for real x ?

Solution:

$$y = \frac{(6x^2-22x+21)}{5x^2-18x+17}$$

$$\text{or } (6-5y)x^2 - 2x(11-9y) + 21-17y = 0$$

$$\Rightarrow D \geq 0 \Rightarrow 4(11-9y)^2 - 4(6-5y)(21-17y) \geq 0$$

$$\Rightarrow -4y^2 + 9y - 5 \geq 0$$

$$\Rightarrow 4y^2 - 9y + 5 \leq 0$$

$$\Rightarrow 4(y-1)\left(y-\frac{5}{4}\right) \leq 0$$

$$\Rightarrow y \in \left[1, \frac{5}{4}\right]$$

least value is 1.

Illustration 35:

Find the values of a for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x .

Solution:

$$\text{Let } y = \frac{ax^2+3x-4}{3x-4x^2+a}$$

$$\Rightarrow x^2(a+4y) + 3(1-y)x - (4+ay) = 0$$

If $x \in R, D \geq 0$

$$\Rightarrow 9(1-y)^2 + 4(a+4y)(4+ay) \geq 0 \Rightarrow (9+16a)y^2 + (4a^2+46)y + (9+16a) \geq 0$$

for all $y \in R, (9+16a) > 0$ & $D \leq 0$

$$\Rightarrow (4a^2+46)^2 - 4(9+16a)(9+16a) \leq 0 \Rightarrow 4(a^2-8a+7)(a^2+8a+16) \leq 0$$

$$\Rightarrow a^2 - 8a + 7 \leq 0 \Rightarrow 1 \leq a \leq 7$$

$$\Rightarrow 9 + 16a > 0 \text{ \& } 1 \leq a \leq 7$$

Taking intersection, $a \in [1, 7]$

Now, checking the boundary values of a

For $a = 1$

$$\Rightarrow y = \frac{x^2 + 3x - 4}{3x - 4x^2 + 1} = -\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$$\because x \neq 1 \Rightarrow y \neq -1$$

$\Rightarrow a = 1$ is not possible.

If $a = 7$

$$\Rightarrow y = \frac{7x^2 + 3x - 4}{3x - 4x^2 + 7} = \frac{(7x-4)(x+1)}{(7-4x)(x+1)} \quad \because x \neq -1 \Rightarrow y \neq -1$$

So y will assume all real values for some real values of x .

So $a \in (1, 7)$

12. Location of Roots:

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in \mathbb{R}$. Let α, β are roots of the equations $f(x) = 0$ and $\alpha < \beta$

Restrictions on roots	Required conditions	Graphical Representation
$\alpha > x_0$ and $\beta > x_0$	$b^2 - 4ac \geq 0$ & $f(x_0) > 0$ & $-\frac{b}{2a} > x_0$	
$\alpha < x_0$ and $\beta < x_0$	$b^2 - 4ac \geq 0$ & $f(x_0) > 0$ & $-\frac{b}{2a} < x_0$	
$\alpha < x_0 < \beta$	$f(x_0) < 0$	
$x_1 < \alpha < \beta < x_2$	$b^2 - 4ac > 0$ & $f(x_1) > 0$ & $f(x_2) > 0$ & $x_1 < -\frac{b}{2a} < x_2$	
exactly one root of $f(x) = 0$ lies in the interval (x_1, x_2) i.e. $x_1 < \alpha < x_2 < \beta$ or $\alpha < x_1 < \beta < x_2$	$f(x_1) \cdot f(x_2) < 0$ (check end points)	

Illustration 36:

Let $x^2 - (m - 3)x + m = 0 (m \in \mathbb{R})$ be a quadratic equation, then find the values of 'm' for which

- (a) both the roots are greater than 2.
- (b) both roots are positive.
- (c) one root is positive and other is negative.
- (d) one root is greater than 2 and other smaller than 1.
- (e) roots are equal in magnitude and opposite in sign.
- (f) both roots lie in the interval (1, 2).

Solution:

(a) **Condition -I:**

$$D \geq 0 \Rightarrow (m - 3)^2 - 4m \geq 0$$

$$\Rightarrow m^2 - 10m + 9 \geq 0$$

$$\Rightarrow (m - 1)(m - 9) \geq 0$$

$$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$$

...(i)

Condition -II:

$$f(2) > 0 \Rightarrow 4 - (m - 3)2 + m > 0$$

$$\Rightarrow m < 10$$

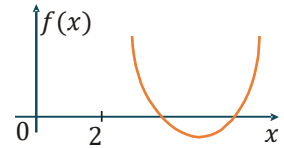
...(ii)

Condition -III:

$$\Rightarrow -\frac{b}{2a} > 2 \Rightarrow \frac{m-3}{2} > 2 \Rightarrow m > 7$$

...(iii)

Intersection of (i), (ii) and (iii) gives $m \in [9, 10)$

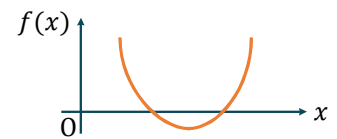


(b) **Condition -I:** $D \geq 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

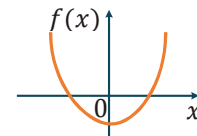
Condition -II: $f(0) > 0 \Rightarrow m > 0$

Condition -III: $-\frac{b}{2a} > 0 \Rightarrow \frac{m-3}{2} > 0 \Rightarrow m > 3$

intersection gives $m \in [9, \infty)$



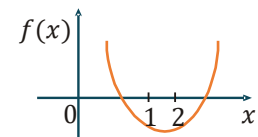
(c) **Condition -I:** $f(0) < 0 \Rightarrow m < 0$



(d) **Condition -I:** $f(1) < 0 \Rightarrow 4 < 0 \Rightarrow m \in \phi$

Condition -II: $f(2) < 0 \Rightarrow m > 10$

Intersection gives $m \in \phi$ **Ans.**



(e) sum of roots = 0 $\Rightarrow m = 3$

and $f(0) < 0 \Rightarrow m < 0$

$\therefore m \in \phi$ **Ans.**

(f) **Condition -I:** $D \geq 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

Condition -II: $f(1) > 0 \Rightarrow 1 - (m - 3) + m > 0 \Rightarrow 4 > 0$ which is true $\forall m \in \mathbb{R}$

Condition -III: $f(2) > 0 \Rightarrow m < 10$

Condition -IV: $1 < -\frac{b}{2a} < 2 \Rightarrow 1 < \frac{m-3}{2} < 2 \Rightarrow 5 < m < 7$

intersection gives $m \in \phi$

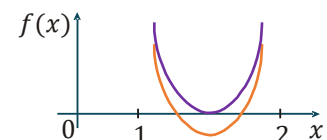
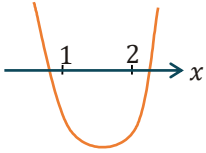


Illustration 37:

if α, β are roots of quadratic equation $x^2 - 2p(x - 4) - 15 = 0$ then set of value of p for which one root is less than 1 and other root is greater than 2?

Solution:



This is clear from graph that

$$f(1) < 0 \text{ and } f(2) < 0$$

Condition -I:

$$f(1) < 0 \Rightarrow 1 - 2p(-3) - 15 < 0$$

$$6p < 14 \Rightarrow p < \frac{7}{3} \quad \dots(i)$$

Condition -II:

$$f(2) < 0 \Rightarrow 4 + 4p - 15 < 0$$

$$4p < 11 \Rightarrow p < \frac{11}{4} \quad \dots(ii)$$

by taking intersection of (i) and (ii)

$$p \in \left(-\infty, \frac{7}{3}\right)$$

Illustration 38:

Find value of k for which one root of equation $x^2 - (k + 1)x + k^2 + k - 8 = 0$ exceeds 2 & other is less than 2.

Solution:

$$4 - 2(k + 1) + k^2 + k - 8 < 0 \Rightarrow k^2 - k - 6 < 0$$

$$(k - 3)(k + 2) < 0 \Rightarrow -2 < k < 3$$

Taking intersection, $k \in (-2, 3)$.

Illustration 39:

Find all possible values of a for which exactly one root of $x^2 - (a + 1)x + 2a = 0$ lies in interval $(0, 3)$.

Solution:

$$f(0).f(3) < 0$$

$$\Rightarrow 2a(9 - 3(a + 1) + 2a) < 0 \Rightarrow 2a(-a + 6) < 0$$

$$\Rightarrow a(a - 6) > 0 \Rightarrow a < 0 \text{ or } a > 6$$

Checking the extremes.

$$\text{If } a = 0, x^2 - x = 0$$

$$x = 0, 1$$

$$1 \in (0, 3)$$

$$\text{If } a = 6, x^2 - 7x + 12 = 0$$

$$x = 3, 4 \text{ But both roots } 3, 4 \notin (0, 3)$$

Hence solution set is

$$a \in (-\infty, 0] \cup (6, \infty)$$

13. Theory of Equations:

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of the equation;

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are all real & $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}, \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \dots, \alpha_1\alpha_2\alpha_3 \dots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note:

- (i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.
- (ii) Every equation of n^{th} degree ($n \geq 1$) has exactly n roots & if the equation has more than n roots, it is an identity.
- (iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.
- (iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in \mathbb{Q}$ & β is not square of a rational number.
- (v) If there be any two real numbers ' a ' & ' b ' such that $f(a)$ & $f(b)$ are of opposite signs, then $f(x) = 0$ must have odd number of real roots (also at least one real root) between ' a ' and ' b '.

Illustration 40:

If $2x^3 + 3x^2 + 5x + 6 = 0$ has roots α, β, γ then find $\alpha + \beta + \gamma, \alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

Solution:

Using relation between roots and coefficients, we get

$$\Rightarrow \alpha + \beta + \gamma = -\frac{3}{2}, \alpha\beta + \beta\gamma + \gamma\alpha = \frac{5}{2}, \alpha\beta\gamma = -\frac{6}{2} = -3.$$

Illustration 41:

If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ then find the equation whose roots are

- (A) $\alpha + \beta, \beta + \gamma, \gamma + \alpha$
- (B) $\alpha\beta, \beta\gamma, \gamma\alpha$

Solution:

(A) Given equation $x^3 + qx + r = 0$ having roots α, β and γ

Sum of roots = $\alpha + \beta + \gamma = 0$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = +q$

$\Rightarrow \alpha\beta\gamma = -r$

For equation having roots $\alpha + \beta, \beta + \gamma, \gamma + \alpha$

Sum of roots = $\alpha + \beta + \beta + \gamma + \gamma + \alpha$

$\Rightarrow 2(\alpha + \beta + \gamma) = 0$

$\Rightarrow (\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\alpha + \beta)(\gamma + \alpha) = (\alpha + \beta + \gamma)^2 + \alpha\beta + \beta\gamma + \gamma\alpha$

$\Rightarrow 0 + q$

$\Rightarrow q$

Product of roots

$\Rightarrow (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = (\alpha\beta + \gamma\alpha + \beta^2 + \gamma\beta)(\gamma + \alpha)$

$\Rightarrow \alpha\beta\gamma + \alpha^2\beta + \gamma^2\alpha + \gamma\alpha^2 + \beta^2\gamma + \alpha\beta^2 + \gamma^2\beta + \alpha\beta\gamma$

$\Rightarrow -2r + \alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta$

$$\begin{aligned} &\Rightarrow -2r + \alpha^2(-\alpha) + \beta^2(-\beta) + \gamma^2(-\gamma) \\ &\Rightarrow -2r - (\alpha^3 + \beta^3 + \gamma^3) \\ &\Rightarrow -2r - (3\alpha\beta\gamma) \quad [\alpha + \beta + \gamma = 0; \alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma] \\ &\Rightarrow -2r - (3 \times (-r)) \\ &\Rightarrow r \end{aligned}$$

Equation $\Rightarrow x^3 + qx - r = 0$

(B) For equation having roots $\alpha\beta, \beta\gamma, \gamma\alpha$

$$\begin{aligned} \text{Sum of roots} &= \alpha\beta + \beta\gamma + \gamma\alpha = q \\ &\Rightarrow (\alpha\beta)(\beta\gamma) + (\beta\gamma)(\gamma\alpha) + (\gamma\alpha)(\alpha\beta) = \alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\alpha^2\beta \\ &\Rightarrow \alpha\beta\gamma(\alpha + \beta + \gamma) \\ &\Rightarrow -r(0) \\ &\Rightarrow 0 \end{aligned}$$

Product $\Rightarrow (\alpha\beta)(\beta\gamma)(\gamma\alpha) = \alpha^2\beta^2\gamma^2 = r^2$

Equation $\Rightarrow x^3 - qx^2 - r^2 = 0$

Illustration 42:

If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find:

- (i) $\sum \alpha^3$
- (ii) $\alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta)$

Solution:

We know that $\alpha + \beta + \gamma = p$

$$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\Rightarrow \alpha\beta\gamma = r$$

$$\begin{aligned} \text{(i)} \quad \alpha^3 + \beta^3 + \gamma^3 &= 3\alpha\beta\gamma + (\alpha + \beta + \gamma)\{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\} \\ &= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) &= \alpha^2(p - \alpha) + \beta^2(p - \beta) + \gamma^2(p - \gamma) \\ &= p(\alpha^2 + \beta^2 + \gamma^2) - 3r - p^3 + 3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r \end{aligned}$$

14. Newton’s formula in Quadratic Equation

It α & β are roots of quadratic equation

$$ax^2 + bx + c = 0 \text{ and } S_n = p\alpha^n + q\beta^n ; p, q \in \text{complex number}$$

$$a\alpha^2 + b\alpha + c = 0 \text{ \& } a\beta^2 + b\beta + c = 0$$

Multiply both side by $p\alpha^n$ & $q\beta^n$ in both equation respectively.

$$ap\alpha^{n+2} + bp\alpha^{n+1} + cp\alpha^n = 0 \quad \dots\text{(i)}$$

$$aq\beta^{n+2} + bq\beta^{n+1} + cq\beta^n = 0 \quad \dots\text{(ii)}$$

(i) + (ii)

$$a(p\alpha^{n+2} + q\beta^{n+2}) + b(p\alpha^{n+1} + q\beta^{n+1}) + c(p\alpha^n + q\beta^n) = 0$$

$$aS_{n+2} + bS_{n+1} + cS_n = 0$$

Quadratic Equation

Idea for Cubic Equation

If α, β, γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ and

$$S_n = p\alpha^n + q\beta^n + t\gamma^n; p, q, t \in \text{complex number}$$

We can write

$$a\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

$$a\beta^3 + b\beta^2 + c\beta + d = 0$$

$$a\gamma^3 + b\gamma^2 + c\gamma + d = 0$$

Multiply both side by $p\alpha^n, q\beta^n$ and $t\gamma^n$ in above three equations respectively

$$ap\alpha^{n+3} + bp\alpha^{n+2} + cp\alpha^{n+1} + dp\alpha^n = 0 \quad \dots(i)$$

$$aq\beta^{n+3} + bq\beta^{n+2} + cq\beta^{n+1} + dq\beta^n = 0 \quad \dots(ii)$$

$$at\gamma^{n+3} + bt\gamma^{n+2} + ct\gamma^{n+1} + dt\gamma^n = 0 \quad \dots(iii)$$

Adding equation (i), (ii) & (iii)

$$\alpha(p\alpha^{n+3} + q\beta^{n+3} + t\gamma^{n+3}) + b(p\alpha^{n+2} + q\beta^{n+2} + t\gamma^{n+2}) + c(p\alpha^{n+1} + q\beta^{n+1} + t\gamma^{n+1}) +$$

$$d(p\alpha^n + q\beta^n + t\gamma^n) = 0$$

$$aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$$

Note

We can extend this idea for higher order polynomial in same way also

Illustration 43:

If a and b are the roots of equation $x^2 - 7x - 1 = 0$, then the value of $\frac{a^{21} + b^{21} + a^{17} + b^{17}}{a^{19} + b^{19}}$ is equal to _____.

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Ans. (51)

Solution:

$$x^2 - 7x - 1 = 0 \quad \left\{ \begin{matrix} a \\ b \end{matrix} \right.$$

By newton's theorem

$$S_{n+2} - 7S_{n+1} - S_n = 0$$

$$S_{21} - 7S_{20} - S_{19} = 0$$

$$S_{20} - 7S_{19} - S_{18} = 0$$

$$S_{19} - 7S_{18} - S_{17} = 0$$

$$\frac{S_{21} + S_{17}}{S_{19}} = \frac{S_{21} + (S_{19} - 7S_{18})}{S_{19}}$$

$$= \frac{S_{21} + S_{19} - 7(S_{20} - 7S_{19})}{S_{19}}$$

$$= \frac{50S_{19} + (S_{21} - 7S_{20})}{S_{19}}$$

$$= 51 \cdot \frac{S_{19}}{S_{19}} = 51$$