PROBABILITY

INTRODUCTION

There are various phenomena in nature, leading to an outcome, which cannot be predicted apriori e.g. in tossing of a coin, a head or a tail may result. Probability theory aims at measuring the uncertainties of such outcomes. Probability gives us a measure of likelihood that something will happen. However probability can never predict the number of times that an occurrence actually happens. But being able to quantify the likely occurrence of an event is important because most of the decisions that affect our daily lives are based on likelihoods and not on absolute certainties.

IMPORTANT DEFINITION

(A) **Experiment** An action or operation resulting in two or more well defined outcomes. e.g. tossing a coin, throwing a die, drawing a card from a pack of well shuffled playing cards etc.

(B) **Sample space** A set S that consists of all possible outcomes of a random experiment is called a sample space and each outcome is called a sample point. Often, there will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information.

e.g. in an experiment of "throwing a die", following sample spaces are possible :

(i) {even number, odd number}

(ii) {a number less than 3, a number equal to 3, a number greater than 3}

(iii) {1,2,3,4,5,6}

Here 3rd sample space is the one which provides most information.

If a sample space has a finite number of points it is called finite sample space and if it has an infinite number of points, it is called infinite sample space. e.g. **(i)** "in a toss of coin" either a head (H) or tail (T) comes up, therefore sample space of this experiment is S = {H,T} which is a finite sample space. **(ii)** "Selecting a number from the set of natural numbers", sample space of this experiment is S = {1,2,3,4,......} which is an infinite sample space.

(C) **Event** It is subset of sample space. e.g. getting a head in tossing a coin or getting a prime number in throwing a die. In general if a sample space consists 'n' elements, then a maximum of 2^n events can be associated with it.

(i) ϕ is called impossible event and S (sample space) is called sure event.

(ii) Probability of occurrence of an event A is denoted by P(A)

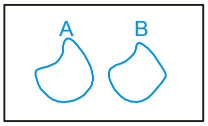
(D) **Compound event** When two or more than two events occur simultaneously, the event is said to be a compound event. Symbolically $A \cap B$ or AB represent the occurrence of both A & B simultaneously.

"$A \cup B$" or $A + B$ represent the occurrence of either A or B.

(E) **Complement of event** The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A. It is usually denoted by A', $\overline{A}$ or A^C.

(F) **Mutually exclusive / disjoint / incompatible events** Two events are said to be mutually exclusive if occurrence of one of them rejects the possibility of occurrence of the other i.e. both cannot occur simultaneously.

In the vein diagram the events A and B are mutually exclusive. Mathematically, we write $A \cap B = \phi$



Events $A_1, A_2, A_3, \ldots\ldots A_n$ are said to be mutually exclusive events iff

$A_i \cap A_j = \phi \ \forall\ i, j \in \{1, 2, \ldots, n\}$ where $i \neq j$

If $A_i \cap A_j = \phi \ \forall\ i, j \in \{1, 2, \ldots, n\}$ where $i \neq j$, then $A_1 \cap A_2 \cap A_3 \ldots \cap A_n = \phi$ but converse need not to be true.

(G) **Equally likely events**

If events have same chance of occurrence, then they are said to be equally likely.

e.g

(i) In a single toss of a fair coin, the events {H} and {T} are equally likely.

(ii) In a single throw of an unbiased die the events {1}, {2}, {3} and {4}, are equally likely.

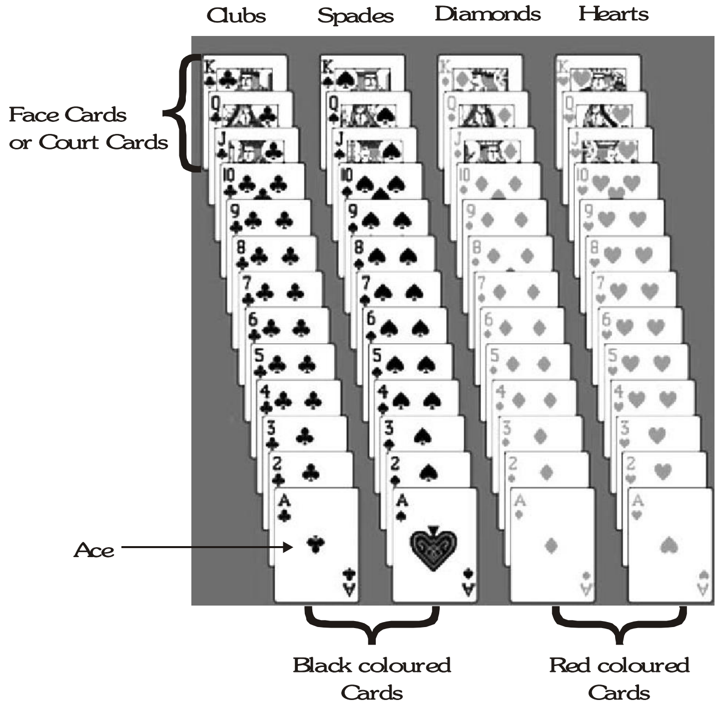
(iii) In tossing a biased coin the events {H} and {T} are not equally likely.

(H) **Exhaustive system of events**

If each outcome of an experiment is associated with at least one of the events $E_1, E_2, E_3, \ldots\ldots E_n$, then collectively the events are said to be exhaustive. Mathematically we write

$E_1 \cup E_2 \cup E_3 \ldots\ldots E_n = S$. (Sample space)

Playing cards : A pack of playing cards consists of 52 cards of 4 suits, 13 in each, as shown in figure.



Comparative study of Equally likely, Mutually Exclusive and Exhaustive events :				
Experiment	**Events**	**E/L**	**M/E**	**Exhaustive**
1. Throwing of a die	A: throwing an odd face {1, 3, 5} B : throwing a composite {4,6}	No	Yes	No
2. A ball is drawn from an urn containing 2 White, 3 Red and 4 Green balls	E1 : getting a White ball E2 : getting a Red ball E3 : getting a Green ball	No	Yes	Yes
3. Throwing a pair of dice	A : throwing a doublet {11, 22, 33, 44, 55, 66} B : throwing a total of 10 or more { 46, 64, 55, 56, 65, 66 }	Yes	No	No
4. From a well shuffled pack of cards a card is drawn	E1 : getting a heart E2 : getting a spade E3 : getting a diamond E4 : getting a club	Yes	Yes	Yes
5. From a well shuffled pack of cards a card is drawn	A = getting a heart B = getting a face card	No	No	No

Ex. Write the sample space of the experiment 'A coin is tossed and a die is thrown'.

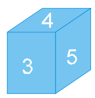
Sol. The sample space S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

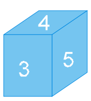
Ex. Find the sample space associated with the experiment of rolling a pair of dice (plural of die) once. Also find the number of elements of the sample space.

Sol. Let one die be blue and the other be green. Suppose '1' appears on blue die and '2' appears on green die. We denote this outcome by an ordered pair (1, 2). Similarly, if '3' appears on blue die and '5' appears on green die, we denote this outcome by (3, 5) and so on. Thus, each outcome can be denoted by an ordered pair (x, y), where x is the number appeared on the first die (blue die) and y appeared on the second die (green die). Thus, the sample space is given by

S = {(x, y) x is the number on blue die and y is the number on green die}

We now list all the possible outcomes (figure)





	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Number of elements (outcomes) of the above sample space is 6×6 i.e., 36

Ex. Write down all the events of the experiment 'tossing of a coin'.

Sol. S = {H, T}

Ex. A die is thrown. Let A be the event ' an odd number turns up' and B be the event 'a number divisible by 3 turns up'. Write the events **(A)** A or B **(B)** A and B

Sol. A = {1, 3, 5}, B = {3, 6}

$\therefore$ A or B = $A \cup B$ = {1, 3, 5, 6}

A and B = $A \cap B$ = {3}

Ex. In a single throw of a die, find whether the events {1, 2}, {2, 3} are mutually exclusive or not.

Sol. Since $\{1, 2\} \cap \{2, 3\} = \{2\} \neq \phi$

$\therefore$ the events are not mutually exclusive.

Ex. In throwing of a die, let A be the event 'even number turns up', B be the event 'an odd prime turns up' and C be the event 'a numbers less than 4 turns up'. Find whether the events A, B and C form an exhaustive system or not.

Sol. A ≡ {2, 4, 6}, B ≡ {3, 5} and C ≡ {1, 2, 3}.

Clearly $A \cup B \cup C$ = {1, 2, 3, 4, 5, 6} = S. Hence the system of events is exhaustive.

Ex. Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Sol. In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on. Hence, the desired sample space is S = {H, TH, TTH, TTTH, TTTTH,...}

CLASSICAL DEFINITION OF PROBABILITY

If n represents the total number of equally likely, mutually exclusive and exhaustive outcomes of an experiment and m of them are favourable to the happening of the event A, then the probability of happening of the event A is given by P(A) = m/n. There are (n–m) outcomes which are favorable to the event that A does not happen. 'The event A does not happen' is denoted by $\bar{A}$ (and is read as 'not A')

Thus $P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n}$

i.e. $P(\bar{A}) = 1 - P(A)$

(i) $0 \le P(A) \le 1$

(ii) $P(A) + P(\bar{A}) = 1$, Where $\bar{A}$ = Not A

This relationship is most useful in the 'at least one' type of problems.

(iii) If x cases are favourable to A & y cases are favourable to $\bar{A}$ then $P(A) = \frac{x}{(x+y)}$ and $P(\bar{A}) = \frac{y}{(x+y)}$.

We say that Odds In Favour Of A are x: y & Odds Against A are y : x

Ex. In throwing of a fair die find the probability of the event ' a number ≤ 4 turns up'.

Sol. Sample space S = {1, 2, 3, 4, 5, 6} ; event A = {1, 2, 3, 4}

∴ n(A) = 4 and n(S) = 6

∴ $P(A) = \frac{n(A)}{n(S)} = \frac{4}{6} = \frac{2}{3}$.

Ex. A coin is tossed successively three times. Find the probability of getting exactly one head or two heads.

Sol. Let S be the sample space and E be the event of getting exactly one head or exactly two heads, then

S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}.

and E = {HHT, HTH, THH, HTT, THT, TTH}

∴ n(E) = 6 and n(S) = 8.

Now required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{6}{8} = \frac{3}{4}$.

Ex. In throwing a pair of fair dice, find the probability of getting a total of 8.

Sol. When a pair of dice is thrown the sample space consists

{(1, 1) (1, 2) .... (1, 6)
(2, 1,) (2, 2,) .... (2, 6)
....
....
(6, 1), (6, 2) .... (6, 6)}

Note that (1, 2) and (2, 1) are considered as separate points to make each outcome as equally likely.
To get a total of ‘8’, favourable outcomes are, (2, 6) (3, 5) (4, 4) (5, 3) and (6, 2).

Hence required probability = $\frac{5}{36}$

Ex. Words are formed with the letters of the word PEACE. Find the probability that 2 E’s come together.

Sol. Total number of words which can be formed with the letters P, E, A, C, E $= \frac{5!}{2!} = 60$

Number of words in which 2 E’s come together $= 4! = 24$ $\therefore$ reqd. prob. $= \frac{24}{60} = \frac{2}{5}$

Ex. A bag contains 5 red and 4 green balls. Four balls are drawn at random, then find the probability that two balls are of red and two balls are of green colour.

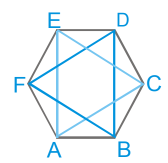
Sol. n(s) = the total number of ways of drawing 4 balls out of total 9 balls : ${}^9C_4$

A : Drawing 2 red and 2 green balls ; n(A) = ${}^5C_2 \times {}^4C_2$

$$\therefore \quad P(A) = \frac{n(A)}{n(s)} = \frac{{}^5C_2 \times {}^4C_2}{{}^9C_4} = \frac{\frac{5\times4\times4\times3}{2\times2}}{\frac{9\times8\times7\times6}{4\times3\times2}} = \frac{10}{21}$$

Ex. Three vertices out of six vertices of a regular hexagon are chosen randomly. The probability of getting a equilateral triangle after joining three vertices is -

Sol. The total no. of cases = ${}^6C_3 = 20$

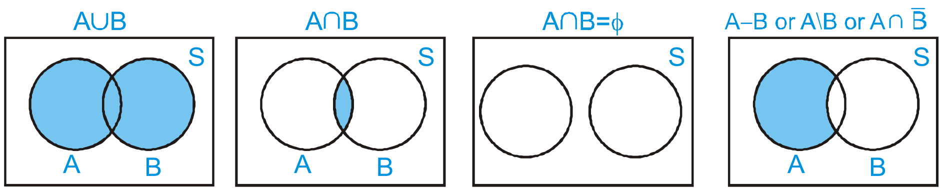


As shown in the figure only two triangles ACE and BDF are equilateral. So number of favourable cases is 2.

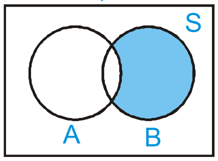
Hence the required probability = $\frac{2}{20} = \frac{1}{10}$

VENN DIAGRAMS

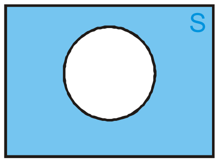
A diagram used to illustrate relationships between sets. Commonly, a rectangle represents the universal set and a circle within it represents a given set (all members of the given set are represented by points within the circle). A subset is represented by a circle within a circle and intersection is indicated by overlapping circles.
Let S is the sample space of an experiment and A, B are two events corresponding to it :



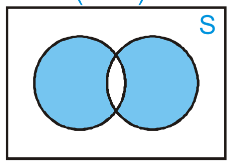
B-A, B\A



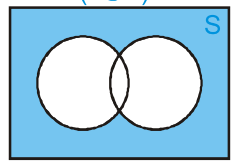
A',A^c or $\bar{A}$



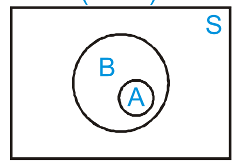
(A∩B)'

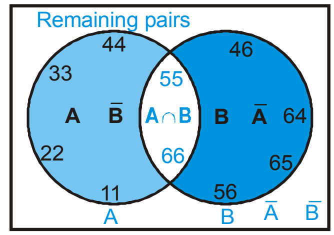


(A∪B)'



(A⊂B)





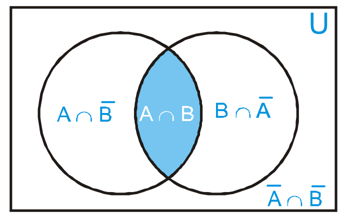
Example : Let us now conduct an experiment of tossing a pair of dice.

Two events defined on the experiment are

A : getting a doublet $\{11, 22, 33, 44, 55, 66\}$

B : getting total score of 10 or more $\{64, 46, 55, 56, 65, 66\}$

5. ADDITION THEOREM OF PROBABILITY



$A \cup B = A + B$ = A or B denotes occurrence of at least A or B.

For 2 events A & B :

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note :

(A)

$$\left.\begin{array}{l} P(A \cup B) \\ P(A+B) \\ P(A \text{ or } B) \\ P(\text{occurence of atleast A or B}) \end{array}\right] = \begin{array}{l} P(A)+P(B)-P(A\cap B) \text{ (This is known as generallised addition theorem)} \\ P(A)+P(B\cap\bar{A}) \\ P(B)+P(A\cap\bar{B}) \\ P(A\cap\bar{B})+P(A\cap B)+P(B\cap\bar{A}) \\ 1-P(A^C\cap B^C) \\ 1-P(A\cup B)^C \end{array}$$

(B) $P(A \backslash B) = P(A - B) = P(A \cap B^C) = P(A) - P(A \cap B)$

(C) Opposite of "atleast A or B" is neither A nor B

i.e. $\overline{A+B} = 1 - (A \text{ or } B) = \bar{A} \cap \bar{B}$

Note that $P(A \cup B) + P(\bar{A} \cap \bar{B}) = 1$.

(D) If A & B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$.

(E) For any two events A & B, P(exactly one of A, B occurs)

$= P(A \cap \bar{B}) + P(B \cap \bar{A}) = P(A) + P(B) - 2P(A \cap B)$

$= P(A \cup B) - P(A \cap B) = P(A^c \cup B^c) - P(A^c \cap B^c)$

(F) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

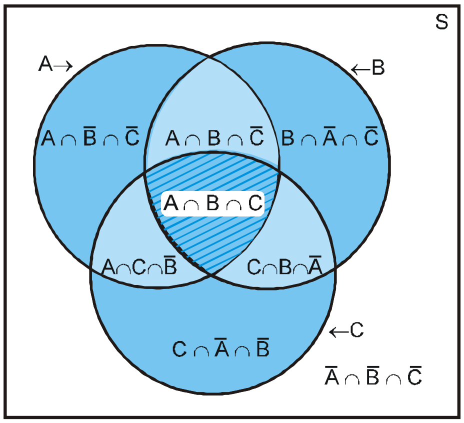
(G) **De Morgan's Law** : If A & B are two subsets of a universal set U, then

(i) $(A \cup B)^c = A^c \cap B^c$ & **(ii)** $(A \cap B)^c = A^c \cup B^c$

(H) $(A \cup B \cup C)^C = A^C \cap B^C \cap C^C$ & $(A \cap B \cap C)^C = A^C \cup B^C \cup C^C$

(I) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

For any three events A, B and C we have the figure



(i) P(A or B or C) = P(A) + P(B) + P(C) – P(A ∩ B) – P(B ∩ C) – P(C ∩ A) + P(A ∩ B ∩ C)

(ii) P (at least two of A, B, C occur) = P(B ∩ C) + P(C ∩ A) + P(A ∩ B) – 2P(A ∩ B ∩ C)

(iii) P(exactly two of A, B, C occur) = P(B ∩ C) + P(C ∩ A) + P(A ∩ B) – 3P(A ∩ B ∩ C)

(iv) P(exactly one of A, B, C occur) =

P(A) + P(B) + P(C) – 2P(B ∩ C) – 2P(C ∩ A) – 2P(A ∩ B) + 3P(A ∩ B ∩ C)

Ex. A bag contains 4 white, 3red and 4 green balls. A ball is drawn at random. Find the probability of the event 'the ball drawn is white or green'.

Sol. Let A be the event 'the ball drawn is white' and B be the event 'the ball drawn is green'.

P(The ball drawn is white or green) = P (A ∪ B) = P(A) + P(B) – P(A ∩ B) = $\frac{8}{11}$

Ex. Given two events A and B. If odds against A are as 2 : 1 and those in favour of A ∪ B are as 3 : 1, then find the range of P(B).

Sol. Clearly P(A) = 1/3, P(A ∪ B) = 3/4.

Now, $P(B) \le P(A \cup B)$

$\Rightarrow$ $P(B) \le 3/4$

Also, $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$\Rightarrow$ $P(B) \ge P(A \cup B) - P(A)$ ($\because P(A \cap B) \ge 0$)

$\Rightarrow$ $P(B) \ge 3/4 - 1/3$

$\Rightarrow$ $P(B) \ge \frac{5}{12}$

$\Rightarrow$ $\frac{5}{12} \le P(B) \le \frac{3}{4}$

Ex. Three numbers are chosen at random without replacement from 1, 2, 3,.......,10. The probability that the minimum of the chosen numbers is 4 or their maximum is 8, is -

Sol. The probability of 4 being the minimum number = $\frac{^6C_2}{^{10}C_3}$

(because, after selecting 4 any two can be selected from 5, 6, 7, 8, 9, 10).

The probability of 8 being the maximum number = $\frac{^7C_2}{^{10}C_3}$.

The probability of 4 being the minimum number and 8 being the maximum number = $\frac{3}{^{10}C_3}$

$\therefore$ the required probability = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{^6C_2}{^{10}C_3} + \frac{^7C_2}{^{10}C_3} - \frac{3}{^{10}C_3} = \frac{11}{40}.$$

CONDITIONAL PROBABILITY AND MULTIPLICATION THEOREM

Let A and B be two events such that P (A) >0. Then P(B|A) denote the conditional probability of B given that A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this we led to the definition.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

which is called conditional probability of B given A

$\Rightarrow$ $P(A \cap B) = P(A)\ P(B|A)$ which is called compound probability or multiplication theorem. It says the probability that both A and B occur is equal to the probability that A occur times the probability that B occurs given that A has occurred.

Note : For any three events A_1, A_2, A_3 we have $P(A_1 \cap A_2 \cap A_3) = P(A_1)\ P(A_2|A_1)\ P(A_3 | (A_1 \cap A_2))$

Ex. If P(A/B) = 0.2 and P(B) = 0.5 and P(A) = 0.2. Find $P(A \cap \overline{B})$.

Sol. $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Also $P(A/B) = \frac{P(A \cap B)}{P(B)}$ $\Rightarrow$ $P(A \cap B) = 0.1$

From given data,

$P(A \cap \overline{B}) = 0.1$

Ex. Two dice are thrown. Find the probability that the numbers appeared have a sum of 8 if it is known that the second die always exhibits 4

Sol. Let A be the event of occurrence of 4 always on the second die

$= \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$; $\therefore$ $n(A) = 6$

and B be the event of occurrence of such numbers on both dice whose sum is 8 = {(6,2), (5,3), (4,4), (3,5), (2,6)}.

Thus, $A \cap B = \{(4,4)\}$

$\therefore$ $n(A \cap B) = 1$

$\therefore$ $P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6}$ or $\frac{P(A \cap B)}{P(A)} = \frac{1/36}{6/36} = \frac{1}{6}$

Ex. A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag?

Sol. Let A be the event of drawing first ball white and B be the event of drawing second ball blue.

Here A and B are dependent events.

$$P(A) = \frac{6}{16},\ P(B \mid A) = \frac{7}{15}$$

$$P(AB) = P(A) \,.\, P(B \mid A) = \frac{6}{16} \times \frac{7}{15} = \frac{7}{40}$$

DEPENDENT AND INDEPENDENT EVENTS

Two events A & B are said to be independent if occurrence or non occurrence of one does not affect the probability of the occurrence or non occurrence of other.

(A) If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be Dependent or Contingent. For two independent events A and B :

$P(A \cap B) = P(A) \,.\, P(B)$. Often this is taken as the definition of independent events.

(B) Three events A, B & C are independent if & only if all the following conditions hold ;

$P(A \cap B) = P(A) \,.\, P(B)$; $P(B \cap C) = P(B) \,.\, P(C)$

$P(C \cap A) = P(C) \,.\, P(A)$ & $P(A \cap B \cap C) = P(A) \,.\, P(B) \,.\, P(C)$

i.e. they must be pairwise as well as mutually independent.

Similarly for n events $A_1, A_2, A_3, \ldots\ldots A_n$ to be independent, the number of these conditions is equal to

$${}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - n - 1.$$

> Independent events are not in general mutually exclusive & vice versa.
>
> Mutually exclusiveness can be used when the events are taken from the same experiment & independence can be used when the events are taken from different experiments.

Ex. The probability that an anti aircraft gun can hit an enemy plane at the first, second and third shot are 0.6, 0.7 and 0.1 respectively. The probability that the gun hits the plane is

Sol. Let the events of hitting the enemy plane at the first, second and third shot are respectively A, B and C. Then as given $P(A) = 0.6,\ P(B) = 0.7,\ P(C) = 0.1$

Since $P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$ and events A,B,C are independent

$\Rightarrow P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A})P(\bar{B})P(\bar{C})$

Required probability $= P(A \cup B \cup C) = 1 - P(\bar{A})P(\bar{B})P(\bar{C})$

$= 1 - (1 - 0.6)(1 - 0.7)(1 - 0.1) = 1 - (0.4)(0.3)(0.9) = 1 - 0.108 = 0.892$

Ex. A pair of fair coins is tossed yielding the equiprobable space S = {HH, HT, TH, TT}. Consider the events:

A = {head on first coin} = {HH, HT}, B = {head on second coin} = {HH, TH}

C = {head on exactly one coin} = {HT, TH}

Then check whether A, B, C are independent or not.

Sol. $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$.

Also $P(A \cap B) = \frac{1}{4} = P(A)\,P(B)$, $P(A \cap C) = \frac{1}{4} = P(A)\,P(C)$, $P(B \cap C) = \frac{1}{4} = P(B)\,P(C)$

but $P(A \cap B \cap C) = 0 \neq P(A)\,P(B)\,P(C)$

$\therefore$ A, B & C are not independent

Ex. If cards are drawn one by one from a well shuffled pack of 52 cards without replacement, until an ace appears, find the probability that the fourth card is the first ace to appear.

Sol. Probability of selecting 3 non-Ace and 1 Ace out of 52 cards is equal to $\frac{{}^{48}C_3 \times {}^{4}C_1}{{}^{52}C_4}$

Since we want 4th card to be first ace, we will also have to consider the arrangement, Now 4 cards in sample space can be arranged in 4! ways and, favorable they can be arranged in 3 ! ways as we want 4th position to be occupied by ace

Hence required probability $= \frac{{}^{48}C_3 \times {}^{4}C_1}{{}^{52}C_4} \times \frac{3!}{4!}$

Ex. In drawing two balls from a box containing 6 red and 4 white balls without replacement, which of the following pairs is independent ?

(A) Red on first draw and red on second draw

(B) Red on first draw and white on second draw

Sol. Let E be the event 'Red on first draw', F be the event 'Red on second draw' and G be the event 'white on second draw'.

$$P(E) = \frac{6}{10},\ P(F) = \frac{6}{10},\ P(G) = \frac{4}{10}$$

(A) $P(E \cap F) = \frac{{}^{6}P_2}{{}^{10}P_2} = \frac{1}{3}$

$P(E)\,.\,P(F) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25} \neq \frac{1}{3}$

$\therefore$ E and F are not independent

(B) $P(E)\,.\,P(G) = \frac{6}{10} \times \frac{4}{10} = \frac{6}{25}$

$P(E \cap G) = \frac{{}^{6}P_1 \times {}^{4}P_1}{{}^{10}P_2} = \frac{4}{15}$

$\therefore$ $P(E)\,.\,P(G) \neq P(E \cap G)$

$\therefore$ E and G are not independent

BERNOULLI TRIAL

The Bernoulli trials process, named after Jacob Bernoulli, is one of the simplest yet most important random processes in probability. Essentially, the process is the mathematical abstraction of coin tossing, but because of its wide applicability, it is usually stated in terms of a sequence of generic trials.

A sequence of Bernoulli trials satisfies the following assumptions:

(A) Each trial has two possible outcomes, in the language of reliability called success and failure.

(B) The trials are independent. Intuitively, the outcome of one trial has no influence over the outcome of another trial.

(C) On each trial, the probability of success is p and the probability of failure is 1-p where p [0,1] is the success parameter of the process.

It happens very often in real life that an event may have only two outcomes that matter. For example, either you pass an exam or you do not pass an exam, either you get the job you applied for or you do not get the job, either your flight is delayed or it departs on time, etc. The probability theory abstraction of all such situations is a Bernoulli trial.

Bernoulli trial is an experiment with only two possible outcomes that have positive probabilities p and q such that p + q = 1. The outcomes are said to be "success" and "failure", and are commonly denoted as "S" and "F" or, say, 1 and 0.

For example, when rolling a die, we may be only interested whether 1 shows up, in which case, naturally, P(S) = 1/6 and P(F) = 5/6. If, when rolling two dice, we are only interested whether the sum on two dice is 11, P(S) = 1/18, P(F) = 17/18.

BINOMIAL PROBABILITY THEOREM

Let p be the probability that an event will happen in any single Bernoulli trial (called the probability of success). Then $q = 1 - p$ is the probability that the event will fail to happen in any single trial (called the probability of failure). The probability that the event will happen exactly x times in n trials (i.e., x successes and $n - x$ failures will occur) is given by the probability function.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x} \quad \text{...... (i)}$$

where the random variable X denotes the number of successes in n trials and x = 0, 1,........., n.

Ex. The probability of getting exactly 2 heads in 6 tosses of a fair coin is

$$P(X = 2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{6!}{2!4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

The discrete probability function (i) is often called the binomial distribution since for x = 0, 1, 2,...,n, it corresponds to successive terms in the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \ldots + p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

The special case of a binomial distribution with n = 1 is also called the Bernoulli distribution.

Ex. A pair of dice is thrown 5 times. Find the probability of getting a doublet twice.

Sol. In a single throw of a pair of dice probability of getting a doublet is $\frac{1}{6}$

considering it to be a success, $p = \frac{1}{6}$

$\therefore \quad q = 1 - \frac{1}{6} = \frac{5}{6}$

number of success $r = 2$

$\therefore \quad P(r = 2) = {}^5C_2\, p^2\, q^3 = 10 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3 = \frac{625}{3888}$

Ex. India and Pakistan play a 5 match test series of hockey, the probability that India wins at least three matches is -

Sol. India win atleast three matches $= {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^5 (16) = \frac{1}{2}$

Ex. In an examination of 10 multiple choice questions (1 or more can be correct out of 4 options). A student decides to mark the answers at random. Find the probability that he gets exactly two questions correct.

Sol. A student can mark 15 different answers to a MCQ with 4 option i.e. ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$

Hence if he marks the answer at random, chance that his answer is correct $= \frac{1}{15}$ and being incorrecting $\frac{14}{15}$.

Thus $\quad p = \frac{1}{15}, q = \frac{14}{15}$.

$$P\ (2\ \text{success}) = {}^{10}C_2 \times \left(\frac{1}{15}\right)^2 \times \left(\frac{14}{15}\right)^8$$

Ex. A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

Sol. Since the man is one step away from starting point mean that either

(i) man has taken 6 steps forward and 5 steps backward.

(ii) man has taken 5 steps forward and 6 steps backward.

Taking, movement 1 step forward as success and 1 step backward as failure.

$\therefore \quad p$ = Probability of success = 0.4

and q = Probability of failure = 0.6

$\therefore \quad$ Required Probability $= P\ \{X = 6 \text{ or } X = 5\} = P\ (X = 6) + P(X = 5) = {}^{11}C_6 p^6 q^5 + {}^{11}C_5 p^5 q^6$

$= {}^{11}C_5(p^6q^5 + p^5q^6) = \frac{11.10.9.8.7}{1.2.3.4.5}\ \{(0.4)^6\,(0.6)^5 + (0.4)^5\,(0.6)^6\}$

$= \frac{11.10.9.8.7}{1.2.3.4.5}\,(0.24)^5$

Hence the required probability = 0.37

TOTAL PROBABILITY THEOREM

If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots, B_n$ and the probabilities $P(A/B_1), P(A/B_2) \ldots P(A/B_n)$ are known, then

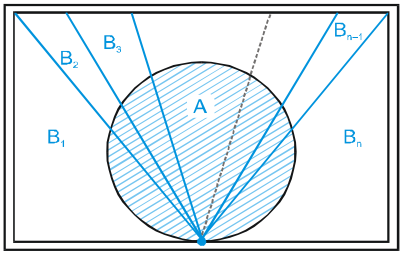
$$P(A) = \sum_{i=1}^{n} P(B_i) \,.\, P(A/B_i)$$

Proof :

The event A occurs with one of the n mutually exclusive and exhaustive events $B_1, B_2, B_3, \ldots, B_n$

$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cup (A \cap B_n)$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$$



Now,

$P(A \cap B_i) = P(A) \,.\, P(B_i/A) = P(B_i) \,.\, P(A/B_i)$

$$\therefore \quad P(A) = \sum_{i=1}^{n} P(B_i) \,.\, P(A/B_i)$$

Ex. Box - I contains 5 red and 4 white balls while box - II contains 4 red and 2 white balls. A fair die is thrown. If it turns up a multiple of 3, a ball is drawn from box - I else a ball is drawn from box - II. Find the probability that the ball drawn is white.

Sol. Let A be the event 'a multiple of 3 turns up on the die' and R be the event 'the ball drawn is white'

then P (ball drawn is white)

$= P(A) \,.\, P(R/A) + P(\bar{A}) \; P(R/\bar{A})$

$$= \frac{2}{6} \times \frac{4}{9} + \left(1 - \frac{2}{6}\right) \frac{2}{6} = \frac{10}{27}$$

Ex. A purse contains 4 copper and 3 silver coins and another purse contains 6 copper and 2 silver coins. One coin is drawn from any one of these two purses. The probability that it is a copper coin is -

Sol. Let $A \equiv$ event of selecting first purse

$B \equiv$ event of selecting second purse

$C \equiv$ event of drawing a copper coin

Then given event has two disjoint cases: AC and BC

$$\therefore \quad P(C) = P(AC + BC) = P(AC) + P(BC) = P(A)P(C\,|\,A) + P(B)P(C\,|\,B)$$

$$= \frac{1}{2} \cdot \frac{4}{7} + \frac{1}{2} \cdot \frac{6}{8} = \frac{37}{56}$$

Ex. Three groups A, B, C are contesting for positions on the Board of Directors of a Company. The probabilities of their winning are 0.5, 0.3, 0.2 respectively. If the group A wins, the probability of introducing a new product is 0.7 and the corresponding probabilities for group B and C are 0.6 and 0.5 respectively. Find the probability that the new product will be introduced.

Sol. Given P(A) = 0.5, P(B) = 0.3 and P(C) = 0.2

$\therefore$ P(A) + P(B) + P(C) = 1

then events A, B, C are exhaustive.

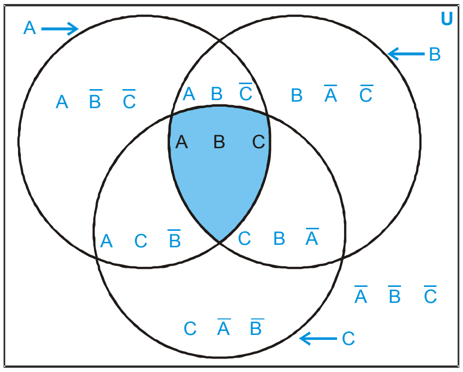
If P(E) = Probability of introducing a new product, then as given

P(E|A) = 0.7, P(E|B) = 0.6 and P(E|C) = 0.5

$\therefore$ P(E) = P(A). P(E|A) + P(B). P(E|B) + P(C). P(E|C)

$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5 = 0.35 + 0.18 + 0.10 = 0.63$

PROBABILITY OF THREE EVENTS



For any three events A,B and C we have

(A) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(B) P(at least two of A,B,C occur)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(C) P(exactly two of A,B,C occur)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(D) P(exactly one of A,B,C occurs)

$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive.

i.e $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0 \Rightarrow P(A \cap B \cap C) = 0$. However the converse of this is not true.

Ex. Let A,B,C be three events. If the probability of occurring exactly one event out of A and B is 1 – a, out of B and C is 1 – 2a, out of C and A is 1 – a and that of occurring three events simultaneously is a^2, then prove that the probability that at least one out of A,B,C will occur is greater than 1/2.

Sol. $P(A) + P(B) - 2P(A \cap B) = 1 - a$ ...... **(i)**

and $P(B) + P(C) - 2P(B \cap C) = 1 - 2a$ ...... **(ii)**

and $P(C) + P(A) - 2P(C \cap A) = 1 - a$ ...... **(iii)**

and $P(A \cap B \cap C) = a^2$ ...... **(iv)**

$\because$ $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= \frac{1}{2}\{P(A) + P(B) - 2P(A \cap B) + P(B) + P(C) - 2P(B \cap C) + P(C) + P(A) - 2P(C \cap A)\} + P(A \cap B \cap C)$

$= \frac{1}{2}\{1 - a + 1 - 2a + 1 - a\} + a^2$

{from (1), (2), (3) & (4)}

$= \frac{3}{2} - 2a + a^2 = (a-1)^2 + \frac{1}{2} > \frac{1}{2}$

BAYE'S THEOREM

If an event A can occur with one of the n mutually exclusive and exhaustive events B_1, B_2, ..., B_n and the probabilities $P(A/B_1)$, $P(A/B_2)$ .... $P(A/B_n)$ are known, then

$$P(B_i / A) = \frac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$$

Proof

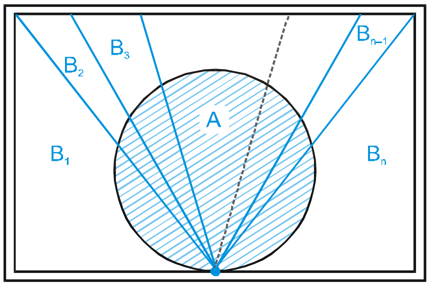
The event A occurs with one of the n mutually exclusive and exhaustive events $B_1, B_2, B_3, ..., B_n$

$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup .... \cup (A \cap B_n)$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + ... + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$$

Now,

$P(A \cap B_i) = P(A) . P(B_i/A) = P(B_i) . P(A/B_i)$



$$P(B_i/A) = \frac{P(B_i).P(A/B_i)}{P(A)} = \frac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(A \cap B_i)}$$

$$P(B_i/A) = \frac{P(B_i).P(A/B_i)}{\sum P(B_i).P(A/B_i)}$$

Ex. Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ?

Sol. Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then $P(A|E_1) = P(\text{a gold coin from box I}) = \frac{2}{2} = 1$

$P(A|E_2) = P(\text{a gold coin from box II}) = 0$

$P(A|E_3) = P(\text{a gold coin from box III}) = \frac{1}{2}$

Now, the probability that the other coin in the box is of gold

= the probability that gold coin is drawn from the box I.

$= P(E_1|A)$

By Baye's theorem, we know that

$$P(E_1 \mid A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)} = \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}$$

Ex. Pal's gardener is not dependable, the probability that he will forget to water the rose bush is $\frac{2}{3}$. The rose bush is in questionable condition any how, if watered the probability of its withering is $\frac{1}{2}$, if not watered, the probability of its withering is $\frac{3}{4}$. Pal went out of station and upon returning, he finds that the rose bush has withered, what is the probability that the gardener did not water the bush.

[Here result is known that the rose bush has withered, therefore. Bayes's theorem should be used]

Sol. Let A = the event that the rose bush has withered

Let A_1 = the event that the gardener did not water.

A_2 = the event that the gardener watered.

By Bayes's theorem required probability,

$$P(A_1/A) = \frac{P(A_1) \,.\, P(A/A_1)}{P(A_1) \,.\, P(A/A_1) + P(A_2) \,.\, P(A/A_2)} \qquad .....(i)$$

Given, $P(A_1) = \frac{2}{3}$ $\therefore$ $P(A_2) = \frac{1}{3}$

$P(A/A_1) = \frac{3}{4}$, $P(A/A_2) = \frac{1}{2}$

$$\text{From (1), } P(A_1/A) = \frac{\frac{2}{3} \cdot \frac{3}{4}}{\frac{2}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{2}} = \frac{6}{6+2} = \frac{3}{4}$$

Ex. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from the bag B.

Sol. Let E_1 = The event of ball being drawn from bag A

E_2 = The event of ball being drawn from bag B.

E = The event of ball being red.

Since, both the bags are equally likely to be selected, therefore

$P(E_1) = P(E_2) = \frac{1}{2}$ and $P(E \mid E_1) = \frac{3}{5}$ and $P(E \mid E_2) = \frac{5}{9}$

$$\therefore \quad \text{Required probability } P(E_2 \mid E) = \frac{P(E_2)P(E \mid E_2)}{P(E_1).P(E \mid E_1) + P(E_2)P(E \mid E_2)} = \frac{\frac{1}{2} \times \frac{5}{9}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{25}{52}$$

Ex. There are 5 brilliant students in class XI and 8 brilliant students in class XII. Each class has 50 students. The odds in favour of choosing the class XI are 2 : 3. If the class XI is not chosen then the class XII is chosen. A student is a chosen and is found to be brilliant, find the probability that the chosen student is from class XI.

Sol. Let E and F be the events 'Class XI is chosen' and 'Class XII is chosen' respectively.

Then $P(E) = \frac{2}{5}$, $P(F) = \frac{3}{5}$

Let A be the event 'Student chosen is brilliant'.

Then $P(A / E) = \frac{5}{50}$ and $P(A / F) = \frac{8}{50}$.

$$\therefore \quad P(A) = P(E) \,.\, P(A / E) + P(F) \,.\, P(A / F) = \frac{2}{5} \cdot \frac{5}{50} + \frac{3}{5} \cdot \frac{8}{50} = \frac{34}{250}.$$

$$\therefore \quad P(E / A) = \frac{P(E)\,.\,P(A/E)}{P(E)\,.\,P(A/E) + P(F)\,.\,P(A/F)} = \frac{5}{17}.$$

EXPECTATION

If there are n possibilities $A_1, A_2, \ldots. A_n$ in an experiment having the probabilities $p_1, p_2, \ldots\ldots\ldots p_n$ respectively. If value $M_1, M_2, \ldots.., M_n$ are associated with the respective possibility. Then the expected value of the experiment is given by $\sum_{i=1}^{n} p_i \,.\, M_i$

Ex. A fair die is tossed. If 2, 3 or 5 occurs, the player wins that number of rupees, but if 1, 4, or 6 occurs, the player loses that number of rupees. Then find the possible payoffs for the player.

Sol.

A_i	2	3	5	1	4	6
M_i	2	3	5	–1	–4	–6
P_i	1/6	1/6	1/6	1/6	1/6	1/6

Then expected value E of the game payoffs for the player

$$= 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) - 1\left(\frac{1}{6}\right) - 4\left(\frac{1}{6}\right) - 6\left(\frac{1}{6}\right) = -\left(\frac{1}{6}\right)$$

COINCIDENCE TESTIMONY

If p_1 and p_2 are the probabilities of speaking the truth of two independent witnesses A and B then

$$P\text{ (their combined statement is true)} = \frac{p_1 p_2}{p_1 p_2 + (1 - p_1)(1 - p_2)}.$$

In this case it has been assumed that we have no knowledge of the event except the statement made by A and B.

However if p is the probability of the happening of the event before their statement then

$$P\text{ (their combined statement is true)} = \frac{p\, p_1 p_2}{p\, p_1 p_2 + (1 - p)(1 - p_1)(1 - p_2)}.$$

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood.

If this is not the case and c is the chance of their coincidence testimony then the

probability that the statement is true = $p\, p_1\, p_2$

probability that the statement is false = $(1-p).c\,(1-p_1)(1-p_2)$

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

Ex. A speaks truth in 75% cases and B in 80% cases. What is the probability that they contradict each other in stating the same fact?

Sol. There are two mutually exclusive cases in which they contradict each other i.e. $\bar{A}B$ and $A\bar{B}$. Hence required probability

$$= P(A\bar{B} + \bar{A}B) = P(A\bar{B}) + P(\bar{A}B)$$

$$= P(A)P(\bar{B}) + P(\bar{A})P(B) = \frac{3}{4}.\frac{1}{5} + \frac{1}{4}.\frac{4}{5} = \frac{7}{20}$$

Ex. A die is thrown, a man C gets a prize of Rs. 5 if the die turns up 1 and gets a prize of Rs. 3 if the die turns up 2, else he gets nothing. A man A whose probability of speaking the truth is $\frac{1}{2}$ tells C that the die has turned up 1 and another man B whose probability of speaking the truth is $\frac{2}{3}$ tells C that the die has turned up 2. Find the expectation value of C.

Sol. Let A and B be the events 'A speaks the truth' and 'B speaks the truth' respectively. Then

$$P(A) = \frac{1}{2} \quad \text{and} \quad P(B) = \frac{2}{3}.$$

The experiment consists of three hypothesis

(i) the die turns up 1 **(ii)** the die turns up 2 **(iii)** the die turns up 3, 4, 5 or 6

Let these be the events E_1, E_2 and E_3 respectively then $P(E_1) = P(E_2) = \frac{1}{6}$ and $P(E_3) = \frac{4}{6}$. Let E be the event that the statements made by A and B agree to the same conclusion.

$$\therefore \quad P(E / E_1) = P(A) \,.\, P(\bar{B}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$P(E / E_2) = P(\bar{A}) \,.\, P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$$

$$P(E / E_3) = P(\bar{A}) \,.\, P(\bar{B}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$\therefore \quad P(E) = P(E_1)\, P(E / E_1) + P(E_2)\, P(E / E_2) + P(E_3)\, P(E / E_3)$$

$$= \frac{1}{6} \,.\, \frac{1}{6} + \frac{1}{6} \,.\, \frac{2}{6} + \frac{4}{6} \,.\, \frac{1}{6} = \frac{7}{36}$$

Thus $$P(E_1 / E) = \frac{P(E_1)P(E/E_1)}{P(E)} = \frac{1}{7}$$

$$P(E_2/E) = \frac{P(E_2)P(E/E_2)}{P(E)} = \frac{2}{7}$$

$$P(E_3/E) = \frac{P(E_3)P(E/E_3)}{P(E)} = \frac{4}{7}$$

$\therefore$ expectation of C $= \frac{1}{7} \times 5 + \frac{2}{7} \times 3 + 0 = \text{Rs. } \frac{11}{7}$

PROBABILITY DISTRIBUTION

(A) A Probability Distribution spells out how a total probability of 1 is distributed over several values of a random variable.

(B) Mean of any probability distribution of a random variable is given by :

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i \quad (\text{Since } \Sigma p_i = 1)$$

(C) Variance of a random variable is given by, $\sigma^2 = \sum (x_i - \mu)^2 . p_i$

$\sigma^2 = \sum p_i x_i^2 - \mu^2$ (Note that Standard Deviation (SD) $= +\sqrt{\sigma^2}$)

(D) The probability distribution for a binomial variate 'X' is given by ; $P(X = r) = {}^nC_r\, p^r q^{n-r}$ where : p = probability of success in a single trial, q = probability of failure in a single trial and p + q = 1. The recurrence formula $\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1} \cdot \frac{p}{q}$, is very helpful for quickly computing P(1), P(2). P(3) etc. if P(0) is known.

(e) Mean of Binomial Probability Distribution (BPD) = np ; variance of BPD = npq.

(f) If p represents a person chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM

$$\boxed{\text{Expectations} = pM}$$

Ex. A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Determine

(i) k **(ii)** $P(X < 3)$ **(iii)** $P(X > 6)$ **(iv)** $P(0 < X < 3)$

Sol. **(i)** $\sum P(X) = 1$

$k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0$

$k = -1$ (not possible) or $k = 1/10$

$k = 1/10$

(ii) $P(X<3)=P(0)+P(1)+P(2)=0+k+2k=3k=3/10$

(iii) $P(X>6)=P(7)=7k^2+k=17/100$

(iv) $P(0<X<3)=P(1)+P(2)=3k=3/10$

Ex. A pair of dice is thrown 5 times. If getting a doublet is considered as a success, then find the mean and variance of successes.

Sol. In a single throw of a pair of dice, probability of getting a doublet = $\frac{1}{6}$

considering it to be a success, p = $\frac{1}{6}$

$\therefore$ $q = 1 - \frac{1}{6} = \frac{5}{6}$

mean = $5 \times \frac{1}{6} = \frac{5}{6}$, variance = $5 \times \frac{1}{6} \cdot \frac{5}{6} = \frac{25}{36}$

GEOMETRICAL PROBABILITY

The following statements are axiomatic :

(i) If a point is taken at random on a given straight line segment AB, the chance that it falls on a particular segment PQ of the line segment is PQ/AB. i.e. probability = $\frac{\text{favourable length}}{\text{total length}}$

(ii) If a point is taken at random on the area S which includes an area σ, the chance that the point falls on σ is σ/S. i.e. $\frac{\text{favourable area}}{\text{total area}}$

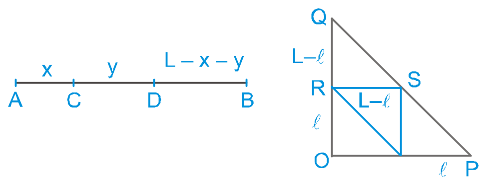
Ex. On a line segment of length L two points are taken at random, find the probability that the distance between them is ℓ, where $\ell < 1$

Sol. Let AB be the line segment

Let C and D be any two points on AB so that AC = x and CD = y. Then $x + y < L$, $y > \ell$

$\therefore$ sample space is represented by the region enclosed by ΔOPQ.

Area of $\Delta OPQ = \frac{1}{2}L^2$



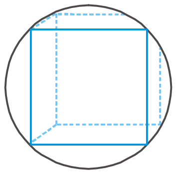
The event is represented by the region, bounded by the ΔRSQ

Area of $\Delta RSQ = \frac{1}{2}(L-\ell)^2$

$\therefore$ probability of the event = $\left(\frac{L-\ell}{L}\right)^2$

Ex. A sphere is circumscribed over a cube. Find the probability that a point lies inside the sphere, lies outside the cube.

Sol. Required probability $= \dfrac{\text{favourable volume}}{\text{total volume}}$



Clearly if edge length of cube is a radius of sphere will be $\dfrac{a\sqrt{3}}{2}$

Thus, volume of sphere $= \dfrac{4}{3}\pi\left(\dfrac{a\sqrt{3}}{2}\right)^3 = \dfrac{\pi a^3\sqrt{3}}{2}$

Hence $P = 1 - \dfrac{1}{\pi\dfrac{\sqrt{3}}{2}} = 1 - \dfrac{2}{\pi\sqrt{3}}$

IMPORTAINT NOTES

(A) If $A_1 \subseteq A_2$ then $P(A_1) \le P(A_2)$ and $P(A_2 - A_1) = P(A_2) - P(A_1)$

(B) If $A = A_1 \cup A_2 \cup .... \cup A_n$ where $A_1, A_2....A_n$ are mutually exclusive events then

$$P(A) = P(A_1) + P(A_2) + ...P(A_n)$$

(C) Let A & B are two events corresponding to sample space S then P(S/A) = P(A/A) =1

(D) Let A and B are two events corresponding to sample space S and F is any other event s.t.

$P(F) \neq 0$ then $P\left((A \cup B)/F\right) = P(A/F) + P(B/F) - P\left((A \cap B)/F\right)$

(E) $P(A'/B) = 1 - P(A/B)$

(F) $P(A \cap B) \le P(A), P(B) \le P(A \cup B) \le P(A) + P(B)$

TIPS & FORMULAS

1. Some Basic Terms and Concepts

(a) **An Experiment :** An action or operation resulting in two or more outcomes is called an experiment.

(b) **Sample space :** The set of all possible outcomes of an experiment is called the sample space, denoted by S. An element S is called a sample point.

(c) **Event :** Any subset of sample space is an event.

(d) **Simple Event :** An event is called a simple event if it is a singleton subset of the sample space S.

(e) **Compound Event :** It is the joint occurence of two or more simple events.

(f) **Equally Likely Events :** If events have same chance of occurrence, then they are said to be equally likely.

(g) **Exhaustive Events :** All the possible outcomes taken together in which an experiment can result are said to be exhaustive or disjoint.

(h) **Mutually Exclusive Events :** If two events cannot occur simultaneously, then they are mutually exclusive. If A and B are mutually exclusive, then $A \cap B = \phi$.

(i) **Complement of an event :** The complement of the event A, denoted by $\overline{A}$, A', A^c is the set of all sample points of the space other then the sample points in A.

2. Mathematical Definition of Probability

Let the outcomes of an experiment consists of n exhaustive mutually exclusive and equally likely cases. Then the sample spaces S has n sample points. If an event A consists of m sample points, $(0 \le m \le n)$, then the probability of event A, denoted by P(A) is defined to be m/n i.e. P(A) = m/n.

Let $S = a_1, a_2, ... , a_n$ be the sample space

(A) $P(S) = \frac{n}{n} = 1$ corresponding to the certain event.

(B) $P(\phi) = \frac{0}{n} = 0$ corresponding to the null event ϕ or impossible event.

(C) If $A_1 = \{a_1\}$, i = 1, ..., n then A_1 is the event corresponding to a single sample point a_1. Then $P(A_1) = \frac{1}{n}$.

(D) $0 \le P(A) \le 1$

3. Odds Against and Odds in Favour of an Event

Let three be m + n equally likely, mutually exclusive and exhaustive cases out of which an event A can occur in m cases and does not occur in n cases. Then by definition of probability of occurences $= \frac{m}{m+n}$

The probability of non-occurence = $= \frac{n}{m+n}$

$\therefore$ P(A) : P(A') = m : n

Thus the odd in favour of occurences of the event A are defined by m : n i.e. P(A) : P(A') ; and the odds against the occurence of event A are defined by n : m i.e. P(A') : P(A).

4. Addition Theorem

(A) If A and B are any events in S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since the probability of an event is a non negative number, it follows that

$P(A \cup B) \le P(A) + P(B)$

For three events A, B and C in S we have

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$

General form of addition theorem

For n events $A_1, A_2, A_3, \ldots\ldots\ldots\ldots A_n$ in S, we have

$P(A_1 \cup A_2 \cup A_3 \cup A_4 \ldots\ldots\ldots \cup A_n)$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<l} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \ldots\ldots\ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \ldots\ldots\ldots \cap A_n)$$

(B) If A and B are mutually exclusive, then $P(A \cap B) = 0$ so that $P(A \cup B) = P(A) + P(B)$.

5. Multiplication Theorem

Independent event :

So if A and B are two independent events then happening of B will have no effect on A.

Difference between independent & mutually exclusive event :

(i) Mutually exclusiveness is used when events are taken from same experiment & independence when events are taken from different experiment.

(ii) Independent events are represented by word "and" but mutually exclusive events are represented by word "OR".

(A) When events are independent

$P(A/B) = P(A)$ and $P(B/A) = P(B)$, then

$P(A \cap B) = P(A).\,P(B)$ or $P(AB) = P(A)\,.\,P(B)$

(B) When events are not independent

The probability of simultaneous happening of two events A and B is equal to the probability of A multipled by the conditional probability of B with respect to A (or probability of B multiplied by the conditional probability of A with respect to B) i.e.

$P(A \cap B) = P(A)\,.\,P(B/A)$ or $P(B)\,.\,P(A/B)$

OR

$P(AB) = P(A)\,.\,P(B/A)$ or $P(B)\,.\,P(A/B)$

(C) Probability of at least one of the n Independent events

If $p_1, p_2, p_3, \ldots\ldots p_n$ are the probabilities of n independent events $A_1, A_2, A_3 \ldots\ldots\ldots A_n$ then the probability of happening of at least one of these event is

$1-[(1-p_1)(1-p_2)\ldots\ldots\ldots\ldots(1-p_n)]$

$P(A_1+A_2+A_3+\ldots\ldots+A_n) = 1\text{-}P(\overline{A}_1)\,P(\overline{A}_2)\,P(\overline{A}_3)\ldots\ldots P(\overline{A}_n)$

6. Conditional Probability

If A and B are any events in S then the conditional probability of B relative to A is given by

$$P(B|A) = \frac{P(A\cap B)}{P(A)}, \qquad \text{If } P(A)\neq 0$$

7. Baye's Theorem or Inverse Probability

Let $A_1, A_2, \ldots\ldots\ldots\ldots A_n$ be n mutually exclusive and exhaustive events of the sample space S and A is event which can occur with any of the events then

$$P\left(\frac{A_i}{A}\right) = \frac{P(A_i).P(A/A_i)}{\sum_{i=1}^{n} P(A_i).P(A/A_i)}$$

8. Binomial Distribution for Repeated Trials

Binomial Experiment : Any experiment which has only two outcomes is known as binomial experiment. Outcomes of such an experiment are known as success and failure. Probability of success is denoted by p and probability of failure by q.

$\therefore \quad p+q=1$

If binomial experiment is repeated n times, then

$$(q+p)^n = {}^nC_0\, q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2q^{n-2} + \ldots\ldots + {}^nC_r\, p^r q^{n-r} + \ldots\ldots\ldots + {}^nC_n\, p^n$$

(A) Probability of exactly r successes in n trials = ${}^nC_r p^r q^{n-r}$

(B) Probability of at most r successes in n trials = $\sum_{\lambda=0}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$

(C) Probability of atleast r successes in n trials = $\sum_{\lambda=r}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$

(D) Probability of having I^{st} success at the r^{th} trials = $p\, q^{r-1}$.

The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

9. SOME IMPORTANT RESULTS

(A) Let A and B be two events, then

(i) $P(A)+P(\bar{A})=1$ **(ii)** $P(A+B)=1-P(\bar{A}\bar{B})$

(iii) $P(A/B)=\dfrac{P(AB)}{P(B)}$ **(iv)** $P(A+B)=P(AB)+P(\bar{A}B)+P(A\bar{B})$

(v) $A\subset B \Rightarrow P(A)\le P(B)$ **(vi)** $P(\bar{A}B)=P(B)-P(AB)$

(vii) $P(AB)\le P(A)\,P(B)\le P(A+B)\le P(A)+P(B)$

(viii) $P(AB)=P(A)+P(B)-P(A+B)$

(ix) $P(\text{Exactly one event})=P(A\bar{B})+P(\bar{A}B)$

$$=P(A)+P(B)-2P(AB)=P(A+B)-P(AB)$$

(x) $P(\text{neither A nor B})=P(\bar{A}\bar{B})=1-P(A+B)$

(xi) $P(\bar{A}+\bar{B})=1-P(AB)$

(B) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) = 2^n

(C) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) = 6^n

(D) Playing Cards :

(i) Total Cards : 52 (26 red, 26 black)

(ii) Four suits : Heart, Diamond, Spade, Club - 13 cards each

(iii) Court Cards : 12 (4 Kings, 4 queens, 4 jacks)

(iv) Honour Cards : 16 (4 aces, 4 kings, 4 queens, 4 jacks)

(E) Probability regarding n letters and their envelopes :

If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) Probability that all letters are in right envelopes = $\dfrac{1}{n!}$

(ii) Probability that all letters are not in right envelopes = $1-\dfrac{1}{n!}$

(iii) Probability that no letters is in right envelopes

$$=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\ldots+(-1)^n\frac{1}{n!}$$

(iv) Probability that exactly r letters are in right envelopes

$$=\frac{1}{r!}\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\ldots+(-1)^{n-r}\frac{1}{(n-r)!}\right]$$