01 Permutations and Combinations

Introduction

Counting is the most fundamental application of mathematics. There are many natural methods used for counting.

In this chapter we will be going to deal with various known techniques those are much faster than the usual counting methods.

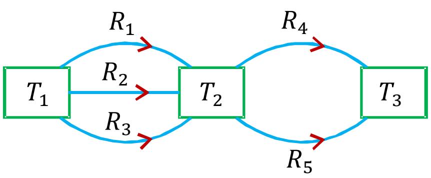
Important Point

FPC (Fundamental Principal of Counting) is used to count some event without actually counting them.

Let us take help of some model.

Model- I :

Find number of ways of in which one can travel from T_1 (town1) to T_3 (town3) via T_2 (town2).



Total ways :-

T_1 — R_1 — T_2 — R_4 — T_3

T_1 — R_1 — T_2 — R_5 — T_3

T_1 — R_2 — T_2 — R_4 — T_3

T_1 — R_2 — T_2 — R_5 — T_3

T_1 — R_3 — T_2 — R_4 — T_3

T_1 — R_3 — T_2 — R_5 — T_3 $=$ 6 ways

It is easy to proceed by FPC T_1 to $T_2 \longrightarrow 3$

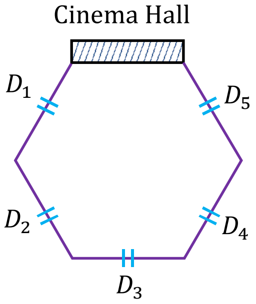
T_2 to $T_3 \longrightarrow 2$

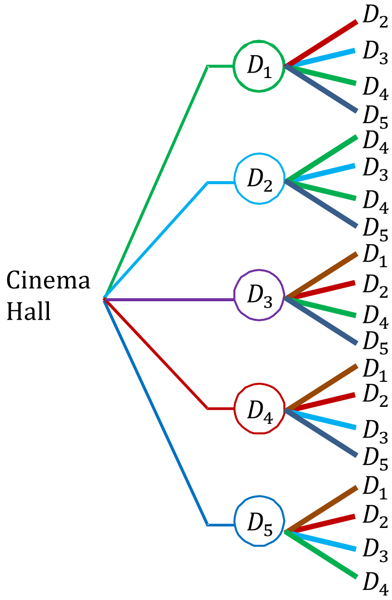
Total ways = $3 \times 2 = 6$

Model- II:

To find the number of ways by which a person can enter and leave cinema hall by a different door.

$4+4+4+4+4=20$





By $F.P.C.$

(i) A person can enter in cinema hall by 5 ways & leave by 4 ways $= 5 \times 4 = 20$.

(ii) If he can enter and leave by any door then number of ways = $5 \times 5 = 25$.

(iii) He can enter by D_1 , D_2 and leaves by D_3 , D_4 , $D_5 = 2 \times 3 = 6$

(iv) He enters with odd number gate and leaves the even number gate = $3 \times 2 = 6$

Basic Steps to Remember :

Step-I : Identify the independent events involved in a given problem.

Step-II : Find the number of ways performing/occurring each event

Step-III : Multiply these numbers to get the total number of ways of performing/occurring all the events

Example :

There are 15 IITs in India and let each IIT has 10 branches, then the IITJEE topper can select the IIT and branch in $15 \times 10 = 150$ number of ways.

Example :

There are 15 IITs & 20 NITs in India, then a student who cleared both IITJEE & AIEEE exams can select an institute in $(15 + 20) = 35$ number of ways.

Illustration 1:

Number of ways in which IITJEE topper can select the IIT and its branch, if there are 23 IITs in India and each IIT has 10 branches.

Solution:

Number of ways $= 23 \times 10 = 230$

Illustration 2:

Number of ways in which a student who cleared both IITJEE & AIEEE exams can select an institute if there are 23 IITs & 31 NITs in India is :

Solution:

Number of ways $= 23 + 31 = 54$

Fundamental Counting Problems

Illustration 3:

Shubham has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

Solution:

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways.

Hence, there are $6 \times 2 = 12$ different ways in which, Shubham can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2 these possibilities can be illustrated in the Figure.

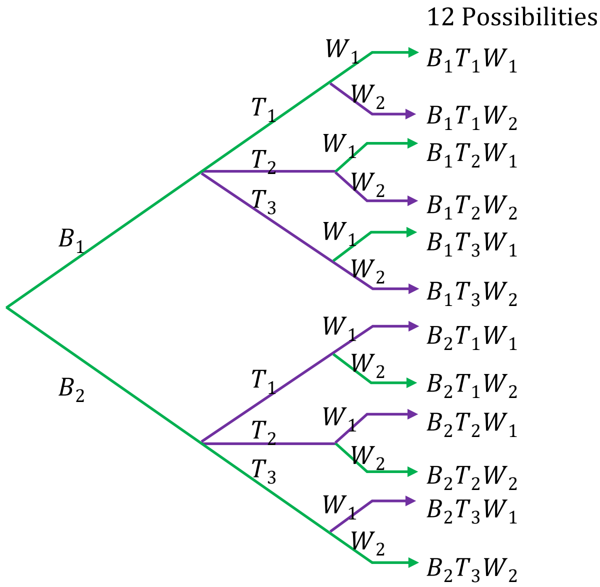


Illustration 4:

A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-

(A) 24 (B) 2 (C) 12 (D) 10

Ans. (A)

Solution:

The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 \times 4 = 24$.

Illustration 5:

A college offers 6 courses in the morning and 4 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening-

(A) 6 (B) 4 (C) 10 (D) 24

Ans. (C)

Solution:

The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 + 4 = 10$.

Illustration 6:

Tossing of a coin & Tree diagram

Solution:

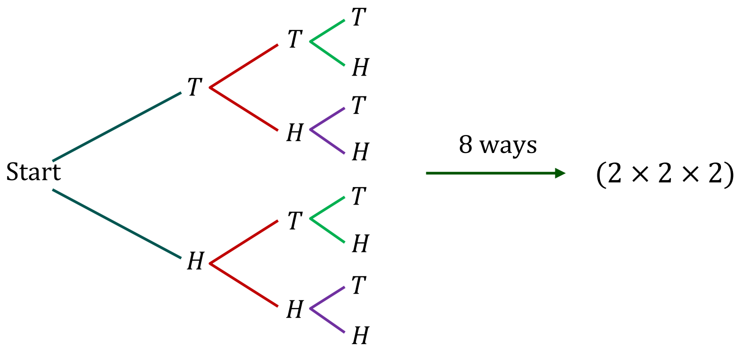


Illustration 7:

In an examination of 10 T/F question, How many sequence of answers are possible.

Solution:

Any question can be answered in two ways , i.e. true or false.

So total task of answering tan question can be done in

$2 \times 2 \times 2 \times \ldots\ldots\ldots.10$ times $= 2^{10}$ ways

Illustration 8:

10 students complete in a swimming race. In how many ways can they occupy the first 3 positions.

Solution:

1^{st} place can be occupied in 10 ways

2^{nd} place can be occupied in 9 ways

3^{rd} place can be occupied in 8 ways.

So total number of ways = $10 \times 9 \times 8 = 720$

Illustration 9:

There are 7 flags of different colour. Find the number of different signals that can be transmitted by the use of 2 flags one above the other.

Solution:

1^{st} place can be occupied in 7 ways

2^{nd} place can be occupied in 6 ways

So total number of ways = $7 \cdot 6 = 42$

Fundamental Principle of Counting

Multiplication Principle (Fundamental Principle of Counting)

Suppose an event E can occur in m different ways and associated with each way of occurring of E, another event F can occur in n different ways, then the total number of occurrences of the two events in the given order is $m \times n$.

Addition Principle

If an event E can occur in m ways and another event F can occur in n ways, and suppose that both cannot occur together, then E or F can occur in $m + n$ ways.

Notation of Factorial

1. Notation of Factorial & its Algebra

The continued product of first n, natural number is called as "n factorial" and denoted by $n!$

$n! = n.(n-1).(n-2).......3.2.1$

$4! = 4.3.2.1 = 24$

$3! = 3.2.1 = 6$

$5! = 120$; $6! = 720$; $7! = 5040$

Special Results :

(I) $0! = 1$ i.e. factorial of zero is 1

Proof : $n! = n.(n-1)!$

Putting $n = 1$

$1! = 1.0! \Rightarrow 0! = 1$

(II) Factorial of negative number is undefined

$(n-1)! = \frac{n!}{n}$ if $n = 0$ then $(-1)! = \frac{0!}{0} = \frac{1}{0}$ Not defined

Illustration 10:

Find n if $(n+1)! = 12 \times (n-1)!$

Solution:

$(n+1)\,n(n-1)! = 12 \times (n-1)!$

$n^2 + n - 12 = 0; (n+4)(n-3) = 0 \quad \therefore \quad n = 3$

Illustration 11:

$(n+2)! = 2550\,n!$

Solution:

$(n+2)(n+1) = 2550; (n+52)(n-49) = 0 \therefore n = 49$

2. Exponent of Prime Number (P) in n!

Let p be a prime number and n be a positive integer. Then, the last integer amongst 1, 2, 3,........ $(n-1), n$ which is divisible by p is $\left[\frac{n}{p}\right] p$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to $\frac{n}{p}$

For example, $\left[\frac{10}{3}\right] = 3, \left[\frac{12}{3}\right] = 2, \left[\frac{15}{3}\right] = 5$ etc.

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^s}\right]$$

Where s is the largest positive integer such that $p^s \le n < p^{s+1}$

$$E_2(100!) = E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$= 50 + 25 + 12 + 6 + 3 + 2$

Illustration 12:

Exponent of 3 in 100! Is equal to.

Solution:

$$E_3 = E_3 = \left[\frac{100}{3}\right] + \left[\frac{100}{9}\right] + \left[\frac{100}{27}\right] + \left[\frac{100}{81}\right]$$

$= [33.3] + [11.1] + [3.7] + [1.2] = 33 + 11 + 3 + 1 = 48$

Illustration 13:

Find number of zeros at the end of (1000)!

Solution:

In any usual factorial of a natural number of 2s are more than number of 5s. Hence number of 10s are same as number of 5s.

Objective approach:

$$E_5(1000!) = \left[\frac{1000}{5}\right] + \left[\frac{1000}{5^2}\right] + \left[\frac{1000}{5^3}\right] + \left[\frac{1000}{5^4}\right]$$

= 200 + 40 + 8 + 1 = 249

Forming Numbers and Arranging Digits

Divisibility of Numbers:

The following chart shows the conditions of divisibility of numbers.

Divisible by Condition

2Whose last digit is even (0, 2, 4, 6, 8)

4Whose last two digits number is divisible by 4

8Whose last three digits number is divisible by 8

3Sum of whose digits is divisible by 3

9Sum of whose digits is divisible by 9

6Which is divisible by both 2 and 3

5Whose last digit is either 0 or 5

25Whose last two digits are divisible by 25

11 Difference of sum of digits at odd place with even place should be divisible by 11.

10Divisible by 2 and 5

Illustration 14:

How many 3 digits numbers can be formed by the digit 1, 2, 3, 4, 5 without repetition.

Solution:

Hundred's place digit can be selected in 5 ways.

Ten's place digit can be selected in 4 ways.

Unit's place digit can be selected in 3 ways.

So, $5 \times 4 \times 3 = 60$

Illustration 15:

How many four digits numbers can be made by using 0, 1, 3, 4, 7, 9

(i) If repetition allowed

(ii) If repetition not allowed

(iii) Even Numbers (Repetition allowed)

(iv) Odd (Repetition allowed)

Solution:

Given digits: 0, 1, 3, 4, 7, 9 $\rightarrow$ 6 digits

(i)

□ □ □ □

↓ ↓ ↓ ↓

$5 \times 6 \times 6 \times 6 = 1080$

(zero not include)

(ii) $5 \times 5 \times 4 \times 3 = 300$

(zero not include)

(iii) [ ] [ ] [ ] [0 or 4]

$5 \times 6 \times 6 \times 2 = 360$

(zero not include)

(iv) [ ] [ ] [ ] [1 or 3 or 7 or 9]

$5 \times 6 \times 6 \times 4 = 720$

Illustration 16:

How many 6 digits odd number greater than 6,00,000 can be formed from the digits 5,6,7, 8, 9, 0 if repetition of digit is allowed?

Solution:

Numbers greater than 6,00,000 and formed with the digit 5, 6, 7, 8, 9, 0 are of 6 digit but begin with 6, 7, 8 or 9.

Also, the numbers which end with 5, 7, 9 are odd.

Hence, first place can be filled by 4 ways (out of 6, 7, 8 or 9). Last place can be filled by 3 ways.

Hence, first and last place can be filled by 4×3 ways.

Also 2^{nd} place can be filled by 6 ways.

3^{rd} place can be filled by 6 ways

4^{th} place can be filled by 6 ways.

5^{th} place can be filled by 6 ways

Hence, all the 6 places can be filled by

$4 \times 3 \times 6 \times 6 \times 6 \times 6 = 15552$ ways.

Definition of Permutation & Combination:

Permutation:

Permutation means arrangement in a definite order of things which may be alike or different taken some or all at a time. Hence permutation refers to the situation where order of occurrence of the events is important.

Example:

(i) Out of A, B, C, D take 3 letters & form number plate of car.

(ii) Selection of cricket team of 11 players from 16 players is combination but deciding with batting order is permutation.

Theorm-1 :

Number of permutations of n distinct things taken all at a time symbolised as :

$${}^{n}P_{n} = P(n, n) = A_{n}^{n} = n!$$

Proof :

Let these are n things arranged at n places

$n.(n-1).(n-2)........3.2.1 = n!$

We also say that number of ways in which n distinct objects can be arranged amongst themselves in ${}^nP_n = n!$ i.e. Find total number of words that of 10 letters that can be formed from all the letters of word GANESHPURI.

$A = {}^nP_n = 10! = n!$

Theorm-2 :

Number of permutations of n distinct things taken r at a time

$0 \le r \le n$

$${}^nP_r = P(n, r) = A_r^n = \frac{n!}{(n-r)!}$$

Things $T_1, T_2 \ldots\ldots T_n$

Places $1, 2, 3 \ldots\ldots\ldots\ldots r$

Choice $n.(n-1).(n-2)\ldots\ldots\ldots[n-(r-1)]$

Total way = $\dfrac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{(n-r)!} = \dfrac{n!}{(n-r)!}$

Hence, we can say that

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

$$= (n)(n-1)(n-2)\ldots\ldots(n-r+1)$$

$\downarrow$

r factors

Note :

1. ${}^{100}P_2 = 100 \times 99$

2. ${}^nP_1 = n$

3. ${}^nP_0 = \dfrac{\lfloor n}{\lfloor n} = 1$

Illustration 17:

Simplify:

i. ${}^3P_2$ **ii.** ${}^{10}P_5$ **iii.** ${}^{100}P_2$

Solution:

i. ${}^3P_2 = \dfrac{3!}{(3-2)!} = 6$

ii. ${}^{10}P_5 = \dfrac{10!}{(10-5)!} = \dfrac{10.9.8.7.6.5!}{5!} = 10.9.8.7.6$

iii. ${}^{100}P_2 = 100 \times 99$

Illustration 18:

In how many ways can 5 persons be made to occupy five different chairs.

Solution:

$^5P_5 = 5! = 120$

Illustration 19:

In how many ways can 5 persons be made to occupy three different chairs.

Solution:

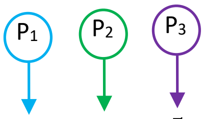
$^5P_3 = 5 \times 4 \times 3 = 60$

Illustration 20:

In how many ways first three prizes are distributed in seven athletes?

Solution:

M-1 :



$7 \times 6 \times 5 \Rightarrow 210$

M-2 : $^7P_3$

$= 7.6.5 = 210$

Combination:

Combination/selection/collection/committee refers to the situation where order of occurrence of the event is not important. Combination is selection of one or more things out of n things which may be alike or different.

Note:

Things which are alike and which are different. All god made things in general are treated to be different and all man made things are to be spelled whether like or different.

Hence, we say that permutation is arrangement of things in definite order.

Example:

(i) Out of four letters A, B, C, D take any 3 letters & form triangle (possible).

Theorm-1 :

Number of combination/selections of n distinct things taken r at a time

$$^nC_r = c(n,r) = \binom{n}{r} = \frac{n!}{(r)!(n-r)!}$$

Proof:

Let 10 different objects are given as $A, B, C, D, E, F, G, H, I, J$

Let combinations taking 3 at a time = x

Arrangement $= (x) \times (3!)$

$x.3! = {}^{10}P_3$

$$x = \frac{^{10}P_3}{3!} = \frac{10!}{(10-3)!\cdot 3!}$$

Illustration 21:

Find the value of

i. $^{5}C_2$ **ii.** $^{10}C_2$ **iii.** $^{10}C_3$

Solution:

i. $^{5}C_2 = \frac{\lfloor 5}{\lfloor 5-2 \cdot \lfloor 2} = \frac{5.4}{2} = 10$

ii. $^{10}C_2 = \frac{\lfloor 10}{\lfloor 8 \lfloor 2} = \frac{10\times 9\times \lfloor 8}{\lfloor 8 \lfloor 2} = \frac{10\times 9}{2} = 45$

iii. $^{10}C_3 = \frac{10\times 9\times 8}{\lfloor 3}$

Keep in Mind :

- $^{n}C_0 = 1$
- $^{n}C_1 = n$
- $^{n}C_n = 1$

Illustration 22:

Find the value of n such that $^{n}P_5 = 42\ ^{n}P_3, n > 4$

Solution:

Given that

$^{n}P_5 = 42\ ^{n}P_3$

or $n(n-1)(n-2)(n-3)(n-4) = 42n(n-1)(n-2)$

Since $n > 4$ so $n(n-1)(n-2) \neq 0$

Therefore, by dividing both sides by $n(n-1)(n-2)$, we get

$(n-3(n-4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30 = 0$

or $(n-10)(n+3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$.

Illustration 23:

Find the value of n such that $\frac{^{n}P_4}{^{n-1}P_4} = \frac{5}{3}, n > 4$

Solution:

Given that $\frac{^{n}P_4}{^{n-1}P_4} = \frac{5}{3}$

Therefore $3n(n-1)(n-2)(n-3) = 5(n-1)(n-2)(n-3)(n-4)$

or $3n = 5(n-4)[as(n-1)(n-2)(n-3) \neq 0, n > 4]$ or $n = 10$.

Illustration 24:

Find r, if $5\ ^{4}P_r = 6\ ^{5}P_{r-1}$.

Solution:

We have $5\,{}^4P_r = 6\,{}^5P_{r-1}$

Or $5\times\dfrac{4!}{(4-r)!}=6\times\dfrac{5!}{(5-r+1)!}$

Or $\dfrac{5!}{(4-r)!}=\dfrac{6\times5!}{(5-r+1)(5-r)(5-r-1)!}$

or $(6-r)(5-r)=6$

or $r^2-11r+24=0$

or $r^2-8r-3r+24=0$

or $(r-8)(r-3)=0$

or $r=8$ or $r=3$.

Hence $r=8,3$.

Rank Problem or Dictionary Problems

Lexicography : The theory and practising of writing and editing dictionary is known as LEXICOGRAPHY

Note :

RANK of the word means, the no. from the starting at which the required word occurs in a special dictionary which is formed by using all the letters of the given word.

Illustration 25:

Find rank of the word $'CAB'$.

Solution:

A, B, C

A __ __ = $\lfloor 2$

B __ __ = $\lfloor 2$

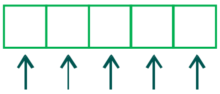
C A B = 1

Rank = 5

Illustration 26:

Find total number of 5 letter word that can be formed from letters of word "TOUGH".

Solution:



5 × 4 × 3 × 2 × 1 = 120

Illustration 27:

Find the rank of "TOUGH" if all the letters of the word are arranged in all possible orders & written out as in a dictionary.

Solution:

The number of letters in the word "TOUGH" is 5 & all the five letters are different.

Alphabetical order of all the letters is G,H,O,T,U

Number of words beginning with G $= 4\times3\times2\times1$

Number of words beginning with H $= 4\times3\times2\times1$

Number of words beginning with O $= 4\times3\times2\times1$

Number of words beginning with TG $= 3\times2\times1$

Number of words beginning with TH $= 3\times2\times1$

Number of words beginning with TOG $= 2\times1$

Number of words beginning with TOH $= 2\times1$

Next words beginning with "TOU" and it is "TOUGH" = 1.

Rank $= 24+24+24+6+6+2+2+1=89$

Illustration 28:

Find rank of the word 'BIHAR'.

Solution:

A, B, H, I, R

A __ __ __ __ $= \lfloor 4 = 24$

B A __ __ __ $= \lfloor 3 = 6$

B H __ __ __ $= \lfloor 3 = 6$

B I A __ __ $= \lfloor 2 = 2$

B I H A R $= 1$

Rank = $24 + 6 + 6 + 2 + 1 = 39$

Problems based on words

Illustration 29:

Number of words which can be made by using all the letters of the word $JODHPUR$.

(i) If no condition.

(ii) If word starts with $'J'$ and ends with $'R'$.

(iii) If word starts with $'J'$ or ends with $'R'$.

(iv) If word neither starts with $'J'$ nor ends with $'R'$.

(v) If all vowels occupy odd places.

(vi) If position of vowels or consonants remain same.

Solution:

(i) $J, O, D, H, P, U, R \rightarrow 7$

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 17$

Number of words are $7! = 5040$

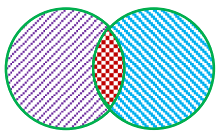
(ii) Starts with $'J'$ and ends with $'R'$ then the number are comes in middle part is 5!.

$1 \times 5 \times 4 \times 3 \quad 2 \times 1 \times 1 = 120$

(iii) (Starts with $'J'$) + (End with $'R'$)(Start J and End R)

OR

(J..........) + (............R) – (J............R)



A or $B = A \cup B$

OR

$= A + B - (A \cap B)$

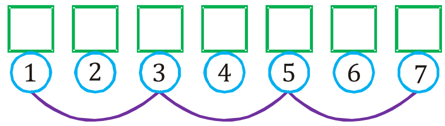
$(1 \times 6!) + (6! \times 1) - (5!)$

$6! + 6! - 5! = 1320$

(iv) Total words – [Case-iii]

= 5040– 1320 = 3720

(v)



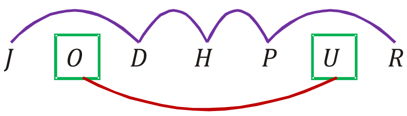
$^{4}C_{2}$ × 2! × 5!

↓ ↓ ↓

(2 odd places for vowels) (O,U) (consonants)

(consonants)

(vi)



vowels

= 2! × 5!

↓ ↓

vowels consonant

Geometric Based Selections:

If there are n points in a plane of which $m(< n)$ are collinear, then

(a) Total number of different straight lines obtained by joining these n points is

$$^{n}C_{2} - {}^{m}C_{2} + 1$$

(b) Total number of different triangles formed by joining these n points is

$$^{n}C_{3} - {}^{m}C_{3}$$

(c) Number of diagonals in polygon of n sides is

$$^{n}C_{2} - n \text{ i.e. } \frac{n(n-3)}{2}$$

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is

$$^{m}C_{2} \times {}^{n}C_{2} \text{ i.e. } \frac{mn(m-1)(n-1)}{4}$$

(e) Number of triangles formed by joining vertices of convex polygon of n sides is $^{n}C_{3}$ of which

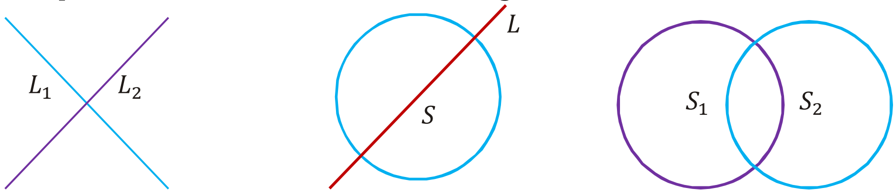
(i) Number of triangles having exactly two sides common to the polygon = n

(ii) Number of triangles having exactly one side common to the polygon = $n(n-4)$

(iii) Number of triangles having no side common to the polygon

$$^{n}C_{3} - n(n-4) - n = \frac{n(n-4)(n-5)}{6}$$

(f) In a plane there are 'm' circle & 'n' straight lines then maximum number of intersection point is



$\left({}^nC_2 \times 1\right) \quad + \quad \left({}^mC_1 . {}^nC_1 \times 2\right) \quad + \quad \left({}^mC_2 \times 2\right)$

(g) In a chase board :

(i) Total number of rectangle = ${}^9C_2 . {}^9C_2 = \frac{9 \cdot 8}{2} \cdot \frac{9 \cdot 8}{2} = (36)^2 = 1296$

(ii) Number of square =

Square size	**Number of squares**
1×1	$8 \times 8 = (8)^2$
2×2	$7 \times 7 = (7)^2$
3×3	$(6)^2$
4×4	$(5)^2$
5×5	$(4)^2$
6×6	$(3)^2$
7×7	$(2)^2$
8×8	$(1)^2$

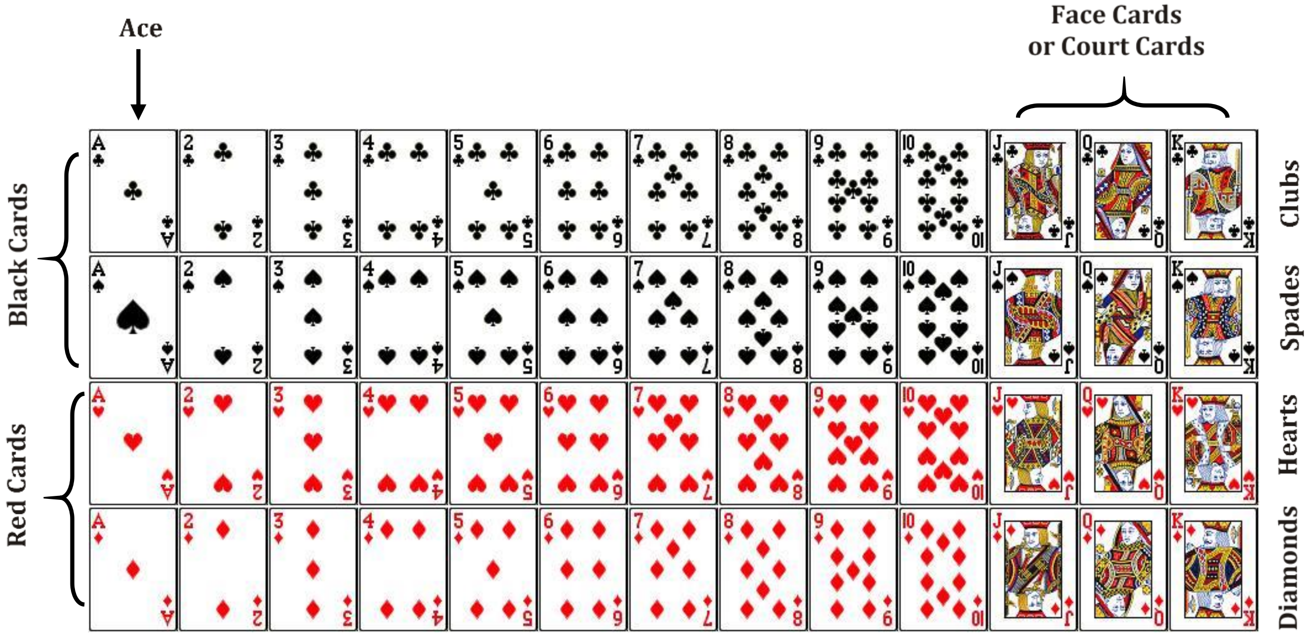
Total = $1^2 + 2^2 + \ldots\ldots\ldots + 8^2$

$$= \frac{n(n+1)(2n+1)}{6} = \frac{8(9)(17)}{6} = 204$$

(iii) Number of rectangle which are not square = $1296 - 204 = 1092$

Playing Cards:

A pack of playing cards consists of 52 cards of 4 suits, 13 in each, as shown in figure.



Recognition of Cards:

	K King	Q Queen	J Jack	A Ace
♥	1	1	1	1
♦	1	1	1	1
♠	1	1	1	1
♣	1	1	1	1
	4	4	4	4

Face Cards:

Face cards contain 12 cards all of K, Q and J having designed a figure of a person.

i.e., Face cards = $4 + 4 + 4 = 12$,

String Method and Gap Method

Illustration 30:

Number of words which can be made by using all the letters of the word JODHPUR.

(i) If all vowels are together.

(ii) If word $'PUR'$ always comes.

(iii) If letters $'PUR'$ always together.

(iv) If letters $'PUR'$ occurs together but $'J'$ never comes with $'PUR'$.

(v) If no vowels comes together.

Solution:

(i) Vowels Consonants TIE Method

O, U J, D, H, P, R

(O, U) (J) (D) (H) (P) (R)

$= 6! \times 2!$

↓ ↓

Blocks (O,U)

(ii) (J) (O) (D) (H) (PUR)

5! blocks

(iii) (J), (O), (D), (H), (P, U, R)

$= 5! \times 3!$

↓

P, U, R

(iv) — (O) —, (D) —, (H) —, (P, U, R)

= 4! × 3! × $^3C_1$ × 1!

↓ Blocks ↓ (P,U,R) ↓ Place for ↓ J

(v) — (J) —, (D) —, (H) —, (P) —, (R) GAP Method

= 5! × $^6C_2$ × 2!

↓ Blocks (consonants) ↓ place for ↓ (O, U) (vowels)

Combinatorial Arguments for Identities on ^nC_r

(1) $^nC_0 = 1$

(2) $^nC_1 = n$

(3) $^nC_n = 1$

(4) $^nC_{n-r} = {^nC_r}$

Number of selections = Number of rejections $\Rightarrow$ $\boxed{^nC_r = {^nC_{n-r}}}$

Proof : $^nC_r = \dfrac{\underline{|n}}{\underline{|r}\ \underline{|n-r}} = \dfrac{\underline{|n}}{\underline{|n-(n-r)}\ \underline{|n-r}} = {^nC_{n-r}}$

(5) If $^nC_x = {^nC_y} \Rightarrow x + y = n$ or $x = y$

(6) $^nP_r = {^nC_r}\, r!$

i.e. permutation is defined total number of combinations of object then arrangements of objects.

(7) $^nC_r = \dfrac{n}{r} . {^{n-1}C_{r-1}}$

Proof : $\dfrac{\underline{|n}}{\underline{|n-r}\ \underline{|r}} = \dfrac{n\underline{|n-1}}{\underline{|(n-1)-(r-1)}\ .\ r\underline{|r-1}} = \dfrac{n}{r} . {^{n-1}C_{r-1}}$

Illustration 31:

$^{10}C_2 = \dfrac{10}{2} . {^9C_1} = \dfrac{10.9}{2}$

(8) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

Solution:

Proof : L.H.S = $^nC_r + {^nC_{r-1}}$ = $\dfrac{\underline{|n}}{\underline{|n-r} \cdot \underline{|r}} + \dfrac{\underline{|n}}{\underline{|n-r+1} \cdot \underline{|r-1}}$

$= \dfrac{\underline{|u}}{\underline{|n-r} \cdot \underline{|r-1}} + \dfrac{\underline{|n}}{(n-r+1)\underline{|n-r} \cdot \underline{|r-1}}$

$= \dfrac{\underline{|n}}{\underline{|n-r}\ \underline{|r-1}}\left[\dfrac{1}{r} + \dfrac{1}{n-r+1}\right]$

$= \dfrac{\underline{|n}}{\underline{|n-r}\ \underline{|r-1}}\left[\dfrac{(n-r+1)+r}{r(n-r+1)}\right] = \dfrac{\underline{|n+1}}{\underline{|n-r+1}\underline{|r}} = {^{n+1}C_r}$ = R.H.S.

Illustration 32:

$^{52}C_{15} + ^{52}C_{36} = ?$

Solution:

$^{52}C_{37} + ^{52}C_{36} = ^{53}C_{37}$

Illustration 33:

$^{r}C_{r} + ^{r+1}C_{r} + ^{r+2}C_{r} + ... + ^{n}C_{r} = ?$

Solution:

$^{r}C_{r} + ^{r+1}C_{r} + ^{r+2}C_{r} + ... + ^{n}C_{r}$

$= ^{r+1}C_{r+1} + ^{r+1}C_{r} + ^{r+2}C_{r} + ... + ^{n}C_{r}$ $\quad \left[^{r}C_{r} = ^{r+1}C_{r+1} = 1 \right]$

$= ^{r+2}C_{r+1} + ^{r+2}C_{r} + ... + ^{n}C_{r}$

$= ^{r+3}C_{r+1} + ^{r+3}C_{r} + ... + ^{n}C_{r}$

$= ^{n}C_{r+1} + ^{n}C_{r}$

$= ^{n+1}C_{r+1} = ^{n+1}C_{n-r}$

Illustration 34:

(9) $\dfrac{^{n}C_{r}}{^{n}C_{r-1}} = \dfrac{n-(r-1)}{r}$

Solution:

Proof :

LHS $= \dfrac{^{n}C_{r}}{^{n}C_{r-1}}$

$$^{n}C_{r} = \left\{ \frac{\lfloor n}{\lfloor n-r \, \lfloor r} \right\} = \left\{ \frac{\lfloor n}{\lfloor n-r \, r \lfloor r-1} \right\} \quad ...(1)$$

$$^{n}C_{r-1} = \left\{ \frac{\lfloor n}{\lfloor n-r+1 \, \lfloor r-1} \right\} = \left\{ \frac{\lfloor n}{(n-r+1)\lfloor n-r . \lfloor r-1} \right\} \quad ...(2)$$

Divide equation (1) by (2)

$$\frac{^{n}C_{r}}{^{n}C_{r-1}} = \frac{n-(r-1)}{r}$$

Arrangement of Alike Objects

Permutations when all the Objects are not Distinct Objects

Suppose we have to find the number of ways of rearranging the letters of the word $BOOT$. In this case, the letters of the word are not all different. There are $2Os$, which are of the same kind.

Let us treat, temporarily, the $2Os$ as different, say, O_1 and O_2. The number of permutations of 4 −different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, $BOOT$. Corresponding to this permutation, we have 2! permutations BO_1O_2T and BO_2O_1T which will be exactly the same permutation if O_1 and O_2 are not treated as different, i.e., if O_1 and O_2 are the same O at both places.

Therefore, the required number of permutations $\frac{4!}{2!}=3\times4=12$

$$\left.\begin{matrix}BO_1O_2T\\BO_2O_1T\end{matrix}\right]\longrightarrow BOOT$$
$$\left.\begin{matrix}TO_1O_2B\\TO_2O_1B\end{matrix}\right]\longrightarrow TOOB$$
$$\left.\begin{matrix}BO_1TO_2\\BO_2TO_1\end{matrix}\right]\longrightarrow BOTO$$
$$\left.\begin{matrix}TO_1BO_2\\TO_2BO_1\end{matrix}\right]\longrightarrow TOBO$$
$$\left.\begin{matrix}BTO_1O_2\\BTO_2O_1\end{matrix}\right]\longrightarrow BTOO$$
$$\left.\begin{matrix}TBO_1O_2\\TBO_2O_1\end{matrix}\right]\longrightarrow TBOO$$
$$\left.\begin{matrix}O_1O_2BT\\O_2O_1BT\end{matrix}\right]\longrightarrow OOBT$$
$$\left.\begin{matrix}O_1BO_2T\\O_2BO_1T\end{matrix}\right]\longrightarrow OBOT$$
$$\left.\begin{matrix}O_1TO_2B\\O_2TO_1B\end{matrix}\right]\longrightarrow OTOB$$
$$\left.\begin{matrix}O_1BTO_2\\O_2BTO_1\end{matrix}\right]\longrightarrow OBTO$$
$$\left.\begin{matrix}O_1TBO_2\\O_2TBO_1\end{matrix}\right]\longrightarrow OTBO$$
$$\left.\begin{matrix}O_1O_2TB\\O_2O_1TB\end{matrix}\right]\longrightarrow OOTB$$

Let us now find the number of ways of rearranging the letters of the word $INSTITUTE$. In this case there are 9 letters, in which I appears 2 times and T appears 3 times. Temporarily, let us treat these letters different and name them as I_1, I_2, T_1, T_2, T_3. The number of permutations of 9 different letters, in this case, taken all at a time is 9!. Consider one such permutation, say, $I_1\ NT_1\ SI_2\ T_2\ U\ E\ T_3$ • Here if I_1, I_2 are not same and T_1, T_2, T_3 are not same, then I_1, I_2 can be arranged in 2! Ways and T_1, T_2, T_3 an be arranged in 3! Ways. Therefore, $2!\times3!$ Permutations will be just the same permutation corresponding to this chosen permutation $I_1NT_1SI_2T_2UET_3$. Hence total number of different permutations will be $\frac{9!}{2!3!}$

Illustration 35:

Find the number of permutations of the letters of the word $ALLAHABAD$.

Solution:

Here, there are 9 objects (letters) of which there are $4A's$, $2L's$ and rest are all different.

Therefore, the required number of arrangements = $\frac{9!}{4!2!}=\frac{5\times6\times7\times8\times9}{2}=7560$

Illustration 36:

In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable?

Solution:

Total number of discs are $4 + 3 + 2 = 9$. Out of 9 discs, 4 are of the first kind (red), 3 are of the second kind (yellow) and 2 are of the third kind (green).

Therefore, the number of arrangements $\frac{9!}{4!3!2!} = 1260$

Illustration 37:

In how many ways the letters of the word "*Aeroplane*" can be arranged without altering the relative positions of vowels & consonants?

(A) 620 (B) 720 (C) 820 (D) 920

Ans. (B)

Solution:

The consonants in their positions can be arranged in $4! = 24$ ways.

The vowels in their positions can be arranged in $\frac{5!}{2!2!} = 30$ ways

$\Rightarrow$ Total number of arrangements = $24 \times 30 = 720$

Total Number of Combinations:

(a) Given n different objects, the number of ways of selecting atleast one of them is, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(b) (i) Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \ldots\ldots$ things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by :

$(p+1)(q+1)(r+1)\ldots\ldots - 1$

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by :

$\{(p+1)(q+1)(r+1)\ldots\ldots 2^n\} - 1$

Illustration 38:

There are 3 books of mathematics, 4 of science and 5 of english. How many different collections can be made such that each collection consists of-

(i) one book of each subject?

(ii) at least one book of each subject?

(iii) at least one book of English?

Solution:

(i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 60$

(ii) $(2^3 - 1)(2^4 - 1)(2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1)(2^3)(2^4) = 31 \times 128 = 3968$ Ans.

Illustration 39:

Find the number of groups that can be made from 5 red balls, 3 green balls and 4 black balls, if at least one ball of all colours is always to be included. Given that all balls are identical except colours.

Solution:

After selecting one ball of each colour, we have to find total number of combinations that can be made from 4 red. 2 green and 3 black balls. These will be $(4 + 1)(2 + 1)(3 + 1) = 60$

Divisors:

Let $N = p^a . q^b . r^c$....... where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(a) The total numbers of positive divisors of N including 1 & N is $= (a + 1)\ (b + 1)\ (c + 1)$.......

(b) The sum of these divisors is

$= (p^0 + p^1 + p^2 + \ldots + p^a)\ (q^0 + q^1 + q^2 + \ldots + q^b)\ (r^0 + r^1 + r^2 + \ldots + r^c)\ldots$

(c) Number of ways in which N can be resolved as a product of two positive factor is =

$\frac{1}{2}(a+1)(b+1)(c+1)$...... if N is not a perfect square

$\frac{1}{2}[(a+1)(b+1)(c+1)......+1]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Note :

(i) Every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.

(ii) A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.

(iii) Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, eg. 4 and 25.

(iv) 1 is neither prime nor composite however it is co-prime with every other natural number.

(v) Two prime numbers are said to be twin prime numbers if their non-negative difference is 2 (e.g. 5 & 7, 19 & 17 etc).

(vi) All positive divisors except the number itself are called proper divisors.

Illustration 40:

Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Solution:

(i) The number $38808 = 2^3 . 3^2 . 7^2 . 1^1$

Hence the total number of divisors (excluding itself i.e. 38808)

$= (3 + 1)(2 + 1)(2 + 1)(1 + 1) - 1 = 71$

(ii) The sum of these divisors

$=(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)(11^0 + 11^1) - 38808$

$= (15)(13)(57)(12) - 38808 = 133380 - 38808 = 94572.$

Permutation & Combination of Things which are NOT all Different

Combination of Things which are not all Different :

(i) Number of ways of selection of $'r'$ identical things out of $'n'$ identical things $=\ 1$

(ii) Number of ways of selecting zero or more things out of $'n'$ identical things $= n+1$

Proof :

Selecting none thing = 1 way

Selecting 1 thing $=$ 1 way

Selecting 2 things $=$ 1 way

:

:

Selecting n things $=$ 1 way

Total number of ways = $1+1+1\ldots\ldots(n+1)$ times $= n+1$

(iii) Number of ways of selection of one or more things out of which,

$'p'$ are alike of one kind,

$'q'$ are alike of second kind,

$'r'$ alike of third kind and

remaining $'s'$ are different is $=(p+1)(q+1)(r+1)2^S - 1$

Proof :

Selecting none thing (out of p alike things) = 1 way

Selecting 1 thing (out of p alike things) = 1 way

Selecting 2 things (out of p alike things) = 1 way

:

:

Selecting p things (out of p alike things) = 1 way

Total number of ways = $1+1+1\ldots\ldots\ldots(p+1)$ times $= p+1$

Similarly for q alike, total ways $= q+1$

Similarly for r alike, total ways $= r+1$

For s different things total ways of selection will be 2^S, i.e. any item is selected or not.

So total number of required ways = $(p+1)(q+1)(r+1)2^S - 1$

(1 is subtracted when no item is selected)

(iv) Number of ways of selection of atleast one thing of each kind in point (iii) is = $p.q.r \ldots = (2^s - 1)$

Illustration 41:

Find the number of ways in which one or more letter be selected from the letters "$AAAABBCCCDEF$"

Solution:

Total number of ways = $(4+1)(2+1)(3+1)2^3 - 1 = 479$

$A \quad B \quad C \quad DEF$

Illustration 42:

How many total no. of ways of selections of letter can be possible from the word $MISSISSIPPI$.

OR

How many ways of selection of atleast one letter of word $MISSISSIPPI$.

Solution:

$MISSISSPPI$

$M^1, \quad I^4, \quad S^4, \quad P^2$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$(1+1)(4+1)(4+1)(2+1) - 1 = 2\cdot 5\cdot 5\cdot 3 - 1 = 149$

Illustration 43:

It is given that 4 Apples, 3 Mangoes, 2 Bananas, 2 Oranges, consider the following cases.

Case-I : Fruits of same species are alike and rests are different ,then

(i) Find the number of ways if atleast one fruit is selected.

(ii) Find the number of ways if atleast one fruit of each kind are selected.

(iii) Find the number of ways if atleast two apples & two mangoes are selected.

Case-II: Fruits of same species are different and rests are also different, then

(i) Find the number of ways, atleast one fruit is selected.

(ii) Find the number of ways, atleast one fruit of each kind is selected.

Solution:

Case-I :

(i) Apples can be selected in $(4+1)$ ways

Total number of ways = $(4+1)(3+1)(2+1)(2+1)-1=179$

(ii) Since we need atleast one fruit of each kind, Apples can be selected in 4 ways.

So total number of ways = $4\times3\times2\times2=48$

(iii) Since we need atleast two apples & two mangoes which can be selected in one way.

So total number of ways = $(2+1)(1+1)(2+1)(2+1)=54$

Case-II :

(i) Since fruits of same kind are different.

Apples can be selected in 2^4 ways

Total number of ways = $2^4\times2^3\times2^2\times2^2-1=2047$

(ii) Since we need atleast one fruit of each kind, Apples can be selected in 2^4-1 ways.

So total number of ways = $(2^4-1)\times(2^3-1)\times(2^2-1)\times(2^2-1)$

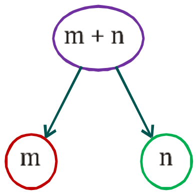
Formation of Groups

Number of ways in which $(m+n)$ different things can be divided into two groups containing m & n things

(i) If $m\neq n$, then number of ways is $\dfrac{(m+n)!}{m!\,n!}$

Explanation:

To find the number of ways in which $(m+n)$ different things can be divided into two unequal groups, it is equivalent to select $'m'$ persons. Since for each selection of $'m'$ persons there will be a corresponding rejection of n persons hence each selection of m and a corresponding rejection of n people will give a group.



$\therefore$ Number of groups = ${}^{m+n}C_m = \dfrac{(m+n)!}{m!\,n!}$

Note : If these groups are to be distributed among two persons or groups are to be named, then number of ways is = $\dfrac{(m+n)!}{m!\,n!}\times2!$

(ii) If $m=n$, then number of ways is $\dfrac{(2n)!}{n!\,n!\,2!}$

Explanation:

Consider 4 different toys $T_1T_2T_3T_4$

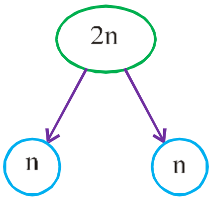
When T_1T_2 is selected and T_3T_4 is rejected $\Rightarrow$ one way of forming the group.

When T_3T_4 is selected and T_1T_2 is rejected is not a different group hence $\frac{4!}{2!\,2!}$ gives double answer.

Therefore, the correct answer is $\frac{4!}{2!\,2!\,2!}$.

Hence the number of ways in which $2n$ different things can be divided into two equal groups

$$= \frac{^{2n}C_n}{2!} = \frac{2n!}{n!\,n!\,2!} \qquad \text{...(i)}$$



Note : If these groups are to be distributed among two persons or groups are to be named, then number of ways is $= \frac{2n!}{n!\,n!\,2!} \times 2!$

Proof : Divide P_1, P_2, P_3, P_4 in two groups

Team - A	Team - B
P_1P_2	P_3P_4
P_1P_3	P_2P_4
P_1P_4	P_2P_3
P_2P_3	P_1P_4
P_2P_4	P_1P_3
P_3P_4	P_1P_2

We see that half of the case are repeated.

Thus $\frac{4!}{2!\,2!}$ gives us wrong answer.

Correct answer = $\frac{4!}{(2!)\,(2!)\,(2!)}$

Actually, counting all such cases we observe that regrouping appears when equal size groups are required. To avoid false counting we divided by factorial of number of equal size groups.

2. Similarly $(m+n+p)$ different things can be divided into 3 unequal groups of m, n and p things is $\frac{(m+n+p)!}{m!\,n!\,p!}$

(i) If the groups are all equal then the number of ways = $\frac{(3n)!}{(n!)^3\,3!}$

(ii) If these groups are to be distributed among 3 persons or group are to be named, then number of ways $= \frac{(3n)!.3!}{(n!)^3 3!}$

Proofs and Explanation of Above:

To understand the article considers 10 children to be divided into three unequal groups of 2, 3 and 5.

First make two groups of 2 and 8 and this can be done in $\frac{10!}{2!\,8!}$ way say. $AB/CDEFGHIJ.$

Consider one such group of 8 which can be divided into two groups of 3 and 5 in $\frac{8!}{3!\,5!}$ ways.

Hence total $= \frac{10!}{2!\,8!}.\frac{8!}{3!5!} = \frac{10!}{2!3!5!}$

Similar explanation will be valid if initial groups in 3 and 7 and then split 7 in 2 and 5. However if 10 is divided into two groups of 5 each initially, which can be done in $\frac{10!}{5!5!2!}$ ways ...(i)

One such grouping is say

$A\ B\ C\ D\ E\ F\ G\ H\ I\ J$

Consider $F\ G\ H\ I\ J$ keeping $A\ B\ C\ D\ E$ as it is. Now the group $F\ G\ H\ I\ J$ can be divided into two groups of 2 and 3 in $\frac{5!}{2!3!}$ ways and similarly when $F\ G\ H\ I\ J$ is kept as it is, $A\ B\ C\ D\ E$ can be divided into two groups of 2 and 3 in $\frac{5!}{2!3!}$ ways. Hence one group (each of 5) given by (i) generated 2. $\frac{5!}{2!3!}$ different groups of 2, 3, 5.

$\therefore$ Total number of groups $= \frac{10!}{5!5!2!}.\frac{2.5!}{2!3!} = \frac{10!}{2!3!5!}$

Similarly if $m = n = p$ situation becomes different.

Consider $T_1T_2T_3T_4T_5T_6$ to be divided into 3 equal groups.

When we say ${}^6C_2\ .\ {}^4C_2 = \frac{6!}{2!2!2!}$ is totally wrong why?

Selected in ${}^6C_2$	Selected in ${}^4C_2$	Rejected in ${}^4C_2$
T_1T_2	T_3T_4	T_5T_6
T_1T_2	T_5T_6	T_3T_4
T_3T_4	T_1T_2	T_5T_6
T_3T_4	T_5T_6	T_1T_2
T_5T_6	T_1T_2	T_3T_4
T_5T_6	T_3T_4	T_1T_2
R	S	G

(A)

Note the all these six groups are counted in (A), however they are identical. Hence the answer in (A) is as many numbers of times more as many numbers of times these equal groups can be arranged i.e. 3! times.

Hence the correct number of groups is equal to $\frac{6!}{2!2!2!3!}$.

In case these 6 toys are to be distributed between $R/S/G$ then our answer will be $= \frac{6!\times 3!}{2!2!2!3!}$.

Illustration 44:

In how many ways 10 children be divided into

(i) 2 groups having 4 & 6 children.**Error! Bookmark not defined.**

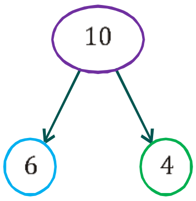
(ii) 2 groups each having equal number of children.

(iii) 3 groups, if groups having 2, 3, & 5 children. **Error! Bookmark not defined.**

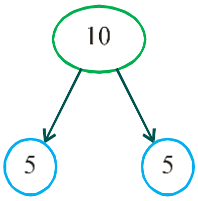
(iv) 3 groups, if groups having 4, 4 & 2 children.

(v) 5 groups A, B, C, D, E having equal number of children.

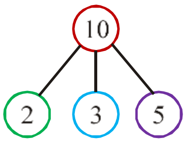
Solution:

(i) 

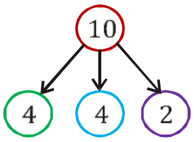
$\Rightarrow \frac{\lfloor 10}{\lfloor 6\cdot\lfloor 4}$

(ii) 

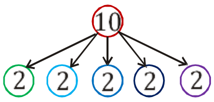
$\Rightarrow \frac{\lfloor 10}{\lfloor 5\cdot\lfloor 5\times\lfloor 2}$

(iii) 

$\Rightarrow \frac{\lfloor 10}{\lfloor 2\cdot\lfloor 3\cdot\lfloor 5}$

(iv) 

$\Rightarrow \frac{\lfloor 10}{\lfloor 4\cdot\lfloor 4\cdot\lfloor 2\times\lfloor 2}$

(v) 

$\Rightarrow \left\{\frac{\lfloor 10}{\lfloor 2\cdot\lfloor 2\cdot\lfloor 2\cdot\lfloor 2\cdot\lfloor 2\times\lfloor 5}\right\}\times\lfloor 5=\frac{\lfloor 10}{(\lfloor 2)^5}$

Illustration 45:

Find number of ways by which 30 Jawan's can be divided into three groups of 12, 10, & 8 and send to three different boarder's.

Solution:

Total ways $=\frac{(30!)\times 3!}{(8!)(10!)(12!)}$

In above case if group are equal size (i.e., group of 10 each)

Send to three boarder's

$=\frac{(30!)\times(3!)}{(10!)^3(3)!}$

Three equal size groups

Illustration 46:

Find number of ways by which five different objects given to three students.

Solution:

Two cases possible {1, 1, 3} {1, 2, 2}

$$\left[\frac{5!}{(1!)^2 3!\times 2!}+\frac{5!}{1!(2!)^2\times 2!}\right]3!$$

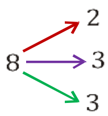
Illustration 47:

Number of ways in which 8 persons can be seated in three diff. taxies each having 3 seats for passengers and duly numbered if

(a) If internal arrangement of persons inside the taxi is immaterial.

(b) If internal arrangement also matters

Solution:

(a) 

$$\left[\frac{8!}{2!3!3!}\times\frac{1}{2!}\right]3!$$

X	X
X	D

(b) Using grouping $\left[\left(\frac{8!}{2!\,3!\,3!}\times\frac{1}{2!}\right)3!\right]3!\,3!\,3! = 9!$

or arrange 8 people in 9 seat $^9C_8 \times 8! = 9!$

Illustration 48:

Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

(A) $\frac{48!}{(12!)^4}\times 4!$ (B) $\frac{47!}{(13!)^4}\times 5!$ (C) $\frac{46!}{(14!)^4}\times 8!$ (D) $\frac{44!}{(15!)^4}\times 5!$

Ans. (A)

Solution:

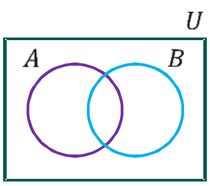
Total number of ways of dividing 48 cards (Excluding 4 Aces) in 4 groups $=\frac{48!}{(12!)^4 4!}$

Now, distribute exactly one Ace to each group of 12 cards. Total number of ways $=\frac{48!}{(12!)^4 4!}\times 4!$

Now, distribute these groups of cards among four players $=\frac{48!}{(12!)^4 4!}\times 4!\,4! = \frac{48!}{(12!)^4}\times 4!$

Principle of Inclusion and Exclusion

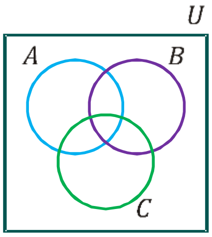
In the Venn's diagram (i), we get



(i)

$n(A\cup B) = n(A) + n(B) - n(A\cap B)$

In the Venn's diagram (ii), we get



(ii)

$n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Illustration 49:

How many words can be formed using all the letters of the word $HONOLULU$ if no two alike letters are together.

Solution:

Let A represents ways when OO together, B when LL together, C when UU together.

Required ways = Total ways – [When all three alike letters together + when 2 alike letters together + when one alike letter together]

= Total ways – $[n(E_3) + n(E_2) + n(E_1)]$ ...(i)

Total ways = $\dfrac{8!}{(2!)\,(2!)(2!)} = 5040$

(a) $n(E_3) = A \cap B \cap C$ (Region 7)

i.e.,

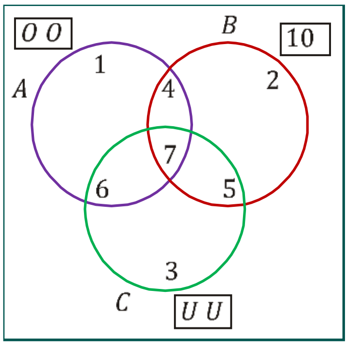
H	N	OO	LL	UU

$n(E_3) = 5! = 120$

(b) $n(E_2) = 3\,[(A \cap B) - (A \cap B \cap C)]$ or [Region 4 + 5 + 6]

→ $n(E_3)$

$= 3\left(\dfrac{6!}{2!} - 5!\right) = 720$



i.e.,

H	OO	N	LL	UU

(c) $n(E_1) = 3\,[A - \{(A \cap B) + (A \cap C)\} + (A \cap B \cap C)]$

$= \left[\dfrac{7!}{(2!)(2!)} - \left(\dfrac{6!}{2!} \times 2\right) + 5!\right] = 1980$

Put in 1^{st}

Required ways = 5040 – [120 + 720 + 1980] = 2220 Ans.

Alternate solution

No. of words = Total words – [At least on pair exist]

= 5040– $n(A \cup B \cup C)$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$$\Rightarrow \left[\frac{\lfloor 7}{2\lfloor 2}+\frac{\lfloor 7}{2\lfloor 2}+\frac{\lfloor 7}{2\lfloor 2}\right]-\frac{\lfloor 6}{\lfloor 2}-\frac{\lfloor 6}{\lfloor 2}-\frac{\lfloor 6}{\lfloor 2}+\lfloor 5 = (5040) - (2820) = 2220$$

Illustration 50:

In How many ways two Americans, two Britishers, two Chinese and one person each of France, Germany, Egypt and Dutch can be sitted if persons of same nationality are to be separated.

Solution:

Number of ways = Total arrangements – [when atleast 1 pair exist together]

= 10! – $n(A \cup B \cup C)$

$$= \lfloor 10 - \begin{bmatrix} n(A)+n(B)+n(C)-n(A\cap B)- \\ n(B\cap C)-n(C\cap A)+n(A\cap B\cap C) \end{bmatrix}$$

$$= \lfloor 10 - \left[\left(\lfloor 9 \times \lfloor 2\right)\times 3 - \left(\left(\lfloor 8 \times \lfloor 2 \times \lfloor 2\right)\times 3 + \left(\lfloor 7 \times \lfloor 2 \times \lfloor 2 \times \lfloor 2\right)\right]$$

= 3628800 – [2177280 – 483840 + 40320]

= 3628800 – [1733760] = 1895040

Circular Permutation

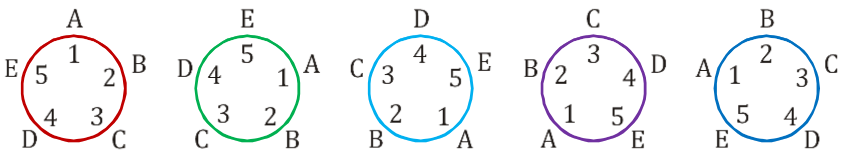
Permutation of objects in a row is called as linear permutation. If we arrange the objects along a closed curve it is called as circular permutation.

Thus in, circular permutation, we consider one object fixed and the remaining objects are arranged as in the case of a linear arrangements.

Case-I : When object are different:

Theorm-1: The number of circular permutation of n distinct objects is $(n-1)!$

Proof : Consider 5 objects A, B, C, D, E to be arranged around a closed curve is called circular permutation.



All are same

Let the total number of circular permutation be x. Above circular permutation is equivalent to 5 linear permutations given by $ABCDEF, EABCD, DEABC, CDEAB, BCDEA$ that is one circular permutation is equivalent to $5x$ linear permutation given by

$x \,.\, 5 = 5\,!$

$$x = \frac{5!}{5} = \frac{5.(5-1)!}{5} = (5-1)!$$

Similarly for n objects $nx = n!$

$$\boxed{x = \frac{n!}{n} = (n-1)!}$$

(i) n distinct things taken all at a time and arranged along circle in $(n-1)!$ ways

(ii) Taken r things out of n distinct things at a time and arranged along circle in ${}^{n}C_{r}.(r-1)!$ ways.

Note : In the above theorem anti-clockwise and clockwise order of arrangements are considered as distinct permutations.

Illustration 51:

Find the number of ways in which 10 children can sit in a mary go round relative to one another.

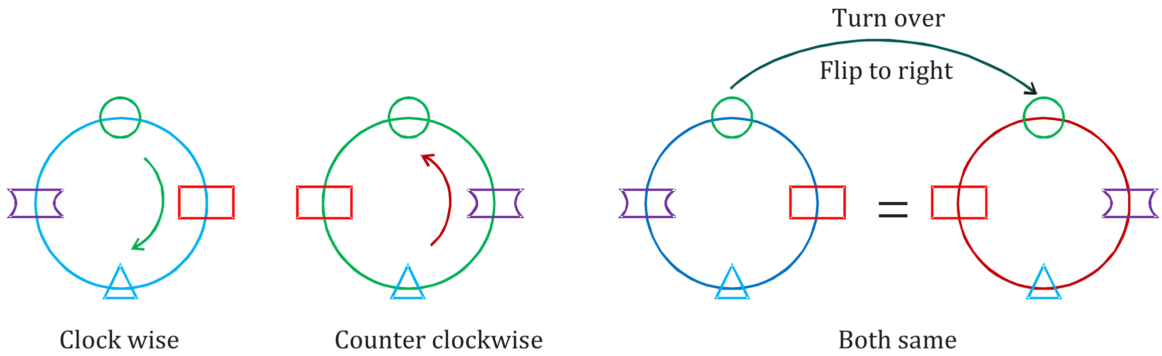
Solution:

Here clockwise and anticlockwise arrangements are different.

Thus required ways = (10 – 1)! = 9!

Theorm-2 : If anticlockwise and clockwise are considered to be same total number of circular permutation given by $\frac{(n-1)!}{2}$.

If we arrange flowers or garland beads in a necklace then there is no distinction between clockwise & anticlockwise direction.



Note : If we have n different things taken r at a time in form of a garland or necklace.

Required number of arrangements = $\frac{{}^nC_r.(r-1)!}{2}$

Illustration 52:

In how many ways garlands can be formed out of 10 different flowers, if each garland consist of :

(i) 10 flowers(ii) 6 flowers

Solution:

(i) Here clockwise and anticlockwise permutations are same

Hence total ways = $\frac{(10-1)!}{2} = \frac{9!}{2}$

(ii) Hence total ways = ${}^{10}C_6 \cdot \frac{(6-1)!}{2} = {}^{10}C_6 \cdot \frac{5!}{2}$

Important note :

The distinction between clockwise and anticlockwise is ignored when a number of people have to be seated around a table so as not to have the same neighbours.

Illustration 53:

Find the number of ways in which 9 people can be seated on a round table so that all shall not have the same neighbours in any 2 arrangements.

Solution:

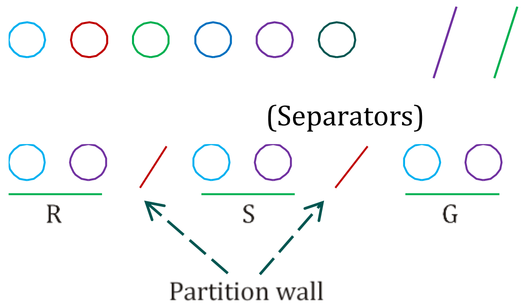
For same neighbour, clockwise and anticlockwise arrangements are same.

So total number of ways will be arrangement of 9 people taken clockwise and anticlockwise same and equal to $\frac{8!}{2}$.

Distribution of Alike Objects

TYPE-1: Total number of ways in which $'n'$ identical coins can be distributed among $'r'$ persons so that each person may get any number of coin is $^{n+r-1}C_{r-1} = \dfrac{(n+r-1)!}{(r-1)!\,(n)!}$

Proof: Let 6 identical coins can be distributed among 3 persons $R|S|G$



Out of eight places if we select any two, then 3 partitions are formed by $^8C_2$ ways

Illustration 54:

In how many ways 10 identical coins can be distributed among four persons (beggars) if each beggar can get?

(i) any number of coins.

(ii) at least one coin.

Solution:

$n = 10$

$r = 4$

(i) Number of ways $= {}^{n+r-1}C_{r-1}$

$= {}^{10+4-1}C_{4-1}$

$= {}^{13}C_3 = 286$

(ii)

B_1	B_2	B_3	B_4
1	1	1	1

Give one coin to each beggars

$\therefore$ Remaining coins = $10 - 4 = 6$

$\therefore n = 6 \;\&\; r = 4$

$\therefore$ Number of ways = $^{6+4-1}C_{4-1} = {}^9C_3$

Illustration 55:

Number of ways in which 5 identical balls can be kept into 3 different boxes so that no box remains empty will be

(A) 1 (B) 3 (C) 6 (D) 15

Ans. (C)

Solution:

Keep one ball to each box

$\therefore$ Remaining balls = $5 - 3 = 2$

The required number of ways = $^{n+r-1}C_{r-1} = {}^{2+3-1}C_{3-1} = {}^4C_2 = \dfrac{4.3}{2} = 6$

Problems Based on Integral Solutions

Type-2 : Number of non-negative integral solutions of an equation $x_1 + x_2 + x_3 \ldots\ldots + x_r = n$ is $= {}^{n+r-1}C_{r-1}$

Illustration 56:

For an equation $x + y + z = 20$, then find

(i) No. of non – ve integral solution.

(ii) No. of integral solution if $x > -1, y \geq 0, z > 3$

(iii) No. of non – ve even integral solution.

(iv) No. of non – ve odd integral solution.

$x + y + z = 20$

Solution:

(i) $x \geq 0 y \geq 0 z \geq 0$

$n = 20 r = 3$

$\therefore$ Number of solutions = ${}^{n+r-1}C_{r-1} = {}^{22}C_2$

(ii) $x + y + z = 20$

$x > -1 \;\; y \geq 0 \quad z > 3$

$x \geq 0 y \geq 0 \quad z \geq 4$

$N = 20 – 4 = 16$

$\therefore$ Number of solutions = ${}^{n+r-1}C_{r-1} = {}^{16+3-1}C_{3-1} = {}^{18}C_2$

(iii) $x + y + z = 20$

$x = 2p \,;\, p = 0,1,2.......;\; p \geq 0$

$y = 2q \,;\, q = 0,1,2.......;\; q \geq 0$

$z = 2r \,;\, r = 0,1,2.......;\; r \geq 0$

$x + y + z = 2p + 2q + 2r$

$20 = 2(p + q + r)$

$p + q + r = 10$

$\therefore$ Number of solutions = ${}^{n+r-1}C_{r-1} = {}^{10+3-1}C_{3-1} = {}^{12}C_2$

(iv) $x + y + z = 20$

Not possible

Illustration 57:

$x + y + z = 21$; Non negative odd.

Solution:

$x + y + z = 21$

$x = 2p + 1 \,;\, p \geq 0$

$y = 2q + 1 \,;\, q \geq 0$

$z = 2r + 1 \,;\, r \geq 0$

$x + y + z = 2p + 2q + 2r + 3$

$21 = 2(p + q + r) + 3$

$p + q + r = 9$

$n = 9 r = 3$

$\therefore$ Number of solutions = ${}^{n+r-1}C_{r-1} = {}^{11}C_2$

Some More Combinatorial Problems

Illustration 58:

Number of ways in which 8 people can be arranged in a line if A and B must be next each other and C must be somewhere behind D, is equal to

(A) 10080 (B) 5040 (C) 5050 (D) 10100

Ans. (B)

Solution:

$ABC \quad DE \quad F \quad G \quad H$

$$\frac{\lfloor 7 \times \lfloor 2}{\lfloor 2} = 5040$$

A & B are tied with string

So there are total 7 units.

Which can be arranged by $\lfloor 7 \times \lfloor 2$ ways

↓

(A & B can beinterchanged)

Now C is somewhere being D

$\therefore$ We divide it by $\lfloor 2$

Hence option (B) is correct.

Illustration 59:

Number of ways in which $'m'$ different toys can be distributed in $'n'$ children if every child may receive any number of toys, is

(A) n^m (B) ${}^mC_n$ (C) ${}^nC_m$ (D) m^n

Ans. (A)

Solution:

Number of ways in which $'m'$ different toys can be distributed in $'n'$ children if every child may receive any number of toys $=\ n^m$.

Illustration 60:

Six married couple are sitting in a room. Find the number of ways in which 4 people can be selected so that they do not form a couple.

(A) 240 (B) 280 (C) 255 (D) 480

Ans. (A)

Solution:

${}^6C_4 \cdot {}^2C_1 \cdot {}^2C_1 \cdot {}^2C_1 \cdot {}^2C_1 = \ 240$

Illustration 61:

Number of 5 digit numbers divisible by 25 that can be formed using only the digits 1, 2, 3, 4, 5 & 0 taken five at a time is

(A) 2 (B) 32 (C) 42 (D) 52

Ans. (C)

Solution:

If the number is divisible by 25 and digit used to form the number are 1, 2, 3, 4, 5, 0 the can last 2 digit can be 25 or 50.

Case-I

			2	5
↑ 3	↑ 3	↑ 2		

(1, 3, 4)

Case-II

			5	0
↑ 4	↑ 3	↑ 2		

(1, 2, 3, 4)

$= 3 \times 3 \times 2 = \ 4 \times 3 \times 2$

$= 18 = \ 24$

Total No. $= 18 + 24 \ = 42$

Station Problems

Illustration 62:

There are n intermediate stations on a railway line from one terminus to another. In how many ways can the train stop at 3 of these intermediate stations if

(a) all the three stations are consecutive

(b) at least two of the stations are consecutive

(c) no two of these stations are consecutive.

Solution:

(a) The number of triples of consecutive stations, viz.

$S_1S_2S_3, S_2S_3S_4, S_3S_4S_5, \ldots\ldots.. S_{n-2}S_{n-1}S_n$

(b) The total number of consecutive pair of stations, viz.

$S_1S_2, S_2S_3 \ldots\ldots\ldots S_{n-1}S_n$

is $(n - 1)$.

Each of the above pair can be associated with a third station in $(n-2)$ ways. Thus, choosing a pair of stations and any third station can be done in $(n-1)\,(n-2)$ ways. The above count also includes the case of three consecutive stations. However, we can see that each such case has counted twice. For example, the pair S_4S_5 combined with S_6 and the pair S_5S_6 combined with S_4 are identical.

Hence, subtracting the excess counting, the number of ways which three stations can be chosen so that at least two of them are consecutive

$$= (n-1)\,(n-2) - (n-2) = \frac{n(n-1)(n-2)}{1.2.3}\,(n-2)^2.$$

(c) Without restriction, the train can stop at any three stations in ${}^{n}C_3$ ways.

Hence, the number of ways the train can stop so that no two stations are consecutive

$$= {}^{n}C_3 - (n-2)^2 = - (n-2)^2$$

$$= (n - 2)\left(\frac{n^2 - n - 6n + 12}{6}\right) = \frac{(n-2)(n-3)(n-4)}{6} = {}^{n-2}C_3$$

Summation of Numbers

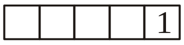
Illustration 63:

Find sum of all the numbers greater then 10000 formed by the digits 1,3,5,7,9 if no digit being repeated.

Solution:

All possible numbers = $5! = 120$

If one occupies the units place then total numbers = 24.

Hence 1 enjoys units place 24 times 

|||ly 1 enjoys each place 24 times

Sum due to 1 $= 1 \times 24\,(1 + 10 + 10^2 + 10^3 + 10^4)$

|||ly Sum due to the

digit 3 $= 3 \times 24\,(1 + 10 + 10^2 + 10^3 + 10^4)$

: : : : : : :

Required total sum = $24\,(1 + 10 + 10^2 + 10^3 + 10^4)\,(1 + 3 + 5 + 7 + 9)$

Illustration 64:

Find sum of all the numbers greater than 10000 formed by the digit 0, 1, 2, 4, 5 no digit being repeated.

Solution:

Using all the given digits we can form a five digit number except when zero is at first place.

So to find the sum of all the possible five digit number

= (Sum of all possible arrangement) – (Sum of all the arrangements when zero is at first place)

$\Rightarrow$ 5 different digits can be arranged in 5! ways so each digit will appear at every place = $\frac{5!}{5}$ times

i.e. 24 times

Sum of all digits at unit place = $24(0 + 1 + 2 + 4 + 5)$

Sum of all digits at ten's place = $24(0 + 1 + 2 + 4 + 5)$

....

Sum of all digits at 10000^{th} place $= 24(0 + 1 + 2 + 4 + 5)$

In this way sum of all possible arrangement = $24(0 + 1 + 2 + 4 + 5)\,[1 + 10 + 10^2 + 10^3 + 10^4]$

when zero is at first place 4 digit number will be formed.

Each number will appear 6 times at every place.

Sum of all 4 digit number at unit place $= 6\,(1 + 2 + 4 + 5)$

Sum of all 4 digit number at ten's place$= 6\,(1 + 2 + 4 + 5)$

....

Hence sum of all four digit numbers = $6\,(1 + 2 + 4 + 5)\,(1 + 10 + 10^2 + 10^3)$

Required sum = $24[0 + 1 + 2 + 3 + 4 + 5]\,[1 + 10 + 10^2 + 10^3 + 10^4]$

$- 6(1 + 2 + 4 + 5)\,(1 + 10 + 10^2 + 10^3)$

Illustration 65:

Find the sum of the five digit numbers that can be formed using the digits 3, 4, 5, 6, 7 not using any digit more than once in any number.

Ans. (6666600)

Solution:

If 3 is placed at units place, the remaining 4 places can be filled in $4! = 24$ ways

Thus, 3 occurs at unit place 24 times.

The other digits similarly, each occurs at the unit places 24 times.

Similarly, each of the digit occurs at the other places tens, hundreds and so on, 24 times.

Hence, the required sum, is

= $24\,(3 + 4 + 5 + 6 + 7)\,(10^0 + 10^1 + 10^2 + 10^3 + 10^4)$

= $24 \times 25 \times 11111$

= 6666600

Grid Problem:

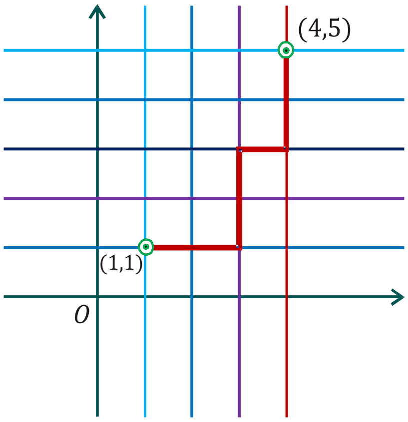
Illustration 66:

Complete cartesian plane is partitioned by drawing line | | to x and y —axis equidistant apart like the lines on a chess board, then the number of ways in which an ant can reach from (1, 1) to (4, 5) via shortest path.

Solution:

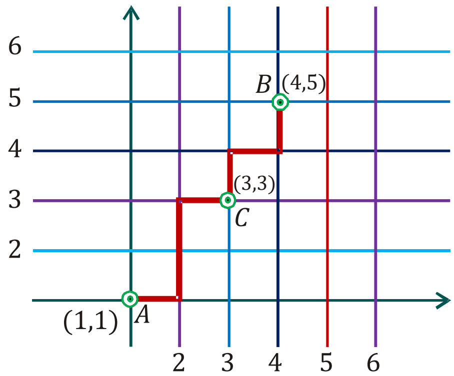
What ever may be the mode of travel of the ant; it has to traverse $3H$ (Horizontal) and $4V$ (Vertical) paths.

Hence required number of ways $= \frac{7!}{4!\,3!} = {}^7C_3$



Note: If there are n vertical and m horizontal lines then there will be $(n-1)$ horizontal and $(m-1)$ vertical paths

Illustration 67:



Number of ways in which an insect can reach from (1, 1) to (4, 5) via C but having shortest path.

Solution:

$(A\ to\ C)$ and $(C\ to\ B)$

(1, 1) – (3, 3) and (3, 3) – (4, 5)

(X^{3-1}, Y^{3-1}) and (X^{4-3}, Y^{5-3})

(X^2, Y^2) and (X^1, Y^2)

Number of ways = ${}^4C_2 \times {}^3C_1 = 6 \times 3 = 18$

Problems Based on Sets and Subsets

Illustration 68:

Set A contains 4 different elements and $B \subseteq A$. In how many ways set B can be formed?

Solution:

Let $a \in A$ then

$a \in B$ ...(i)

$a \notin B$ ...(ii)

for each element of A we have 2 options for that element to be present or absent from B.

So total cases = $2 \times 2 \times 2 \times 2 = 2^4$

B can be formal in 2^4 ways.

Illustration 69:

Set $A = \{1, 2, 3, 4, 5, 6\}$ and $B \subseteq A$ & $C \subseteq A$. In how many ways sets B & C can be formed such that $B \cap C = \phi$?

(A) 3^5 (B) 3^6 (C) 2^6 (D) $3^5 - 1$

Ans. (B)

Solution:

Let $a \in A$ then total 4 cases are possible

$a \in B, a \in C$ ...(i)

$a \in B, a \notin C$ ...(ii)

$a \notin B, a \in C$ ...(iii)

$a \notin B, a \notin C$ ...(iv)

we require $B \cap C = \phi$

thus, for a, equation (ii) (iii) and (iv)

can satisfy $B \cap C = \phi$

therefore $B \cap C = \phi$ *has* 3 options

thus, No. of equation possible for 6 elements $= 3^6$

Illustration 70:

Set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B \subseteq A$ & $C \subseteq A$. In how many ways set B & C can be formed such that $B \cap C = \{2, 5, 7\}$?

Solution:

$\left.\begin{matrix} 1 \in B & 1 \notin C \\ 1 \notin B & 1 \in C \\ 1 \notin B & 1 \notin C \end{matrix}\right\}$ 3 options $\qquad$ $2 \in B \quad 2 \in C\}$ 1 option

$\left.\begin{matrix} 3 \in B & 3 \notin C \\ 3 \notin B & 3 \in C \\ 3 \notin B & 3 \notin C \end{matrix}\right\}$ 3 options $\qquad$ $\left.\begin{matrix} 4 \in B & 4 \notin C \\ 4 \notin B & 4 \in C \\ 4 \notin B & 4 \notin C \end{matrix}\right\}$ 3 options

$5 \in B \quad 5 \in C\}$ 1 option $\qquad$ $\left.\begin{matrix} 6 \in B & 6 \notin C \\ 6 \notin B & 6 \in C \\ 6 \notin B & 6 \notin C \end{matrix}\right\}$ 3 options

$7 \in B \quad 7 \in C\}$ 1 option $\qquad$ $\left.\begin{matrix} 8 \in B & 8 \notin C \\ 8 \notin B & 8 \in C \\ 8 \notin B & 8 \notin C \end{matrix}\right\}$ 3 options

So total possible ways = $3 \times 1 \times 3 \times 3 \times 1 \times 3 \times 1 \times 3 = 3^5$

Illustration 71:

Let $X = \{1, 2, 3, \ldots\ldots, 10\}$. Find the number of pair $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{5, 7, 8\}$.

Solution:

Let $A \cup B = Y, B \backslash a = M, A \backslash B = N$ and $x \backslash y = L$. then X is the disjoint union of M, N, L and $A \cap B$. Now $A \cap B = \{5, 7, 8\}$ is fixed. The remaining seven elements $1, 2, 3, 4, 6, 9, 10$ can be distributed in any of the remaining sets M, N, L. this can be done in 3^7 ways. Of these if all the elements are in the set L, then $A = B = \{5, 7, 8\}$ and this case has to be omitted. Hence the total number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X, A \neq B$ and $A \cap B = \{5, 7, 8\}$is $3^7 - 1$.

Illustration 72:

Let $S = \{1, 2, 3, 4, 5\}$. The total number of unordered pairs of disjoint subsets of S is equal to –

(A) 121 (B) 122 (C) 123 (D) 124

Ans. (B)

Solution:

$S = \{1,2,3,4,5\}$

According to the definition of disjoint sets, if there exist two sets A and B, $A \cap B = \phi$.

Every element of S can be an element of A or B or of neither of the subsets.

There exist 3 possibilities for each element

Since there are for 5 elements, there are 3^5 possibilities are present

Total number of ordered pairs of subsets = $3^5 + 1 = 244$

Total number of unordered pairs = $244/2 = 122$

Illustration 73:

A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is :-

(A) $2^{2n} - {}^{2n}C_n$ (B) 2^n (C) $2^n - 1$ (D) 3^n

Ans. (D)

Solution:

Let $A = \{a_1, a_2, a_3, \ldots\ldots a_n\}$. For $a_i \in A$, we have the following choices:

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, \ldots. a_n$ belong to $P \cap Q$ is 3^n .

Derangement Theorem

Derangement means arrangement in which none can occupy its own place.

(i) If n things are arranged in a row, the number of a ways they can be deranged so that r things occupy wrong places while $(n - r)$ things occupy their original places, is

$= {}^nC_{n-r}\, D_r$

where $D_r = r!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + (-1)^r \frac{1}{r!}\right)$

(ii) If n things are arranged in a row, the number of ways they can be deranged so that none of them occupies its original place, is

$$= {}^{n}C_0 D_n = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+.....+(-1)^n\frac{1}{n!}\right)$$

Alter :

D_n = ways in which n things are arranged so that all n things occupy wrong places

= arranged without restriction

– ways in which 1 thing is in correct position while (n – 1) things are deranged

– ways in which 2 things are in correct position while (n– 2) things are deranged

....

....

– ways in which all n things are in correct position and there is no derangement

$= n! - {}^{n}C_1 D_{n-1} - {}^{n}C_2 D_{n-2} - \ldots\ldots\ldots - {}^{n}C_n D_0$

$= n! - \sum_{r=1}^{n} {}^{n}C_r D_{n-r}$

Thus, we have

$D_0 = 0! = 1$

$D_1 = 1! - {}^{1}C_1 D_0 = 0$

$D_2 = 2! - {}^{2}C_1 D_1 - {}^{2}C_2 D_0 = 1$

$D_3 = 3! - {}^{3}C_1 D_2 - {}^{3}C_2 D_1 - {}^{3}C_3 D_0 = 2$

$D_4 = 4! - {}^{4}C_1 D_3 - {}^{4}C_2 D_2 - {}^{4}C_3 D_1 - {}^{4}C_4 D_0 = 9$

$D_5 = 5! - {}^{5}C_1 D_4 - {}^{5}C_2 D_3 - {}^{5}C_3 D_2 - {}^{5}C_4 D_1 - {}^{5}C_5 D_0 = 44$

and so on.

Short Trick :

Number of derangement of n-different objects =

$$D_n = n!\left[\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+.......+(-1)^n\frac{1}{n!}\right]$$

$$D_2 = 2!\left[\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}\right] = 2\left[\frac{1}{1}-\frac{1}{1}+\frac{1}{2}\right] = 1$$

$$D_3 = 3!\left[\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right] = 6\left[\frac{1}{2}-\frac{1}{6}\right] = 6\left[\frac{3-1}{6}\right] = 2$$

$$D_4 = 4!\left[\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right] = 24\left[\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right] = 24\left[\frac{12-4+1}{24}\right] = 9$$

$$D_5 = 5!\left[\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right] = 120\left[\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right] = 120\left[\frac{60-20+5-1}{120}\right] = 44$$

Illustration 74:

In how many ways three letters be posted in three addressed envelopes if

(i) all are at right place.

(ii) exactly two of them are at right place.

(iii) exactly one is at right place.

(iv) No one letter is at right place.

Solution:

(i) All 3 are at correct place = 1 way

(ii) Zero way

(iii) ${}^{3}C_{1} \times 1 = 3$ ways

(iv) Number of ways in which no one at right place is = K(let)

K = Total ways – (case (i) + (ii) + (iii))

$K = 3! - (1 + 0 + 3)$

$K = 6 - 4$

$K = 2$ ways

$D_2 = 1; D_3 = 2; D_4 = 9; D_5 = 44$

Short Trick :

Number of derangement of n −different objects = $D_3 = 2$

Illustration 75:

A person writes letters to five friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelops so that

(a) all letters are in the wrong envelopes.

(b) at least three of them are in the wrong envelopes.

Solution:

(a) Required number of ways = $5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)=\frac{5!}{2!}-\frac{5!}{3!}+\frac{5!}{4!}-\frac{5!}{5!}$

(b) Required number of ways

$= \sum_{r=1}^{n} {}^{n}C_r D_{n-r}$ where $n = 5$

$= {}^{5}C_2 D_3 + {}^{5}C_1 D_4 + {}^{5}C_0 D_5$

$= 10 \times 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) + 5 \times 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) + 1 \times 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$

$= 10(3 - 1) + 5(12 - 4 + 1) + (60 - 20 + 5 - 1) = 20 + 45 + 44 = 109.$

Illustration 76:

A family consists of a grandfather, m sons and daughters and $2n$ grand children. They are to be seated in a row for dinner. The grand children wish to occupy the n seats at each end and the grandfather refuses to have a grand children on either side of him. In how many ways can the family be made to sit.

(A) $(2n)!\ m!\,(m-1)$ (B) $(2n)!\ m!\ m$

(C) $(2n)!\,(m-1)!\,(m-1)$ (D) $(2n-1)!\,m!\,(m-1)$

Ans. (A)

Solution:

First we select n grand children from $2n$ grand children is ${}^{2n}C_n$

Now arrangement of both group is $n! \times n!$

Now Rest all $(m + 1)$ place where we occupy the grandfather and m sons but grandfather refuse the sit to either side of grand children so the out of $m-1$ seat one seat can be selected

Now required number of sitting in ${}^{2n}C_n \times n! \times n! \times {}^{(m-1)}C_1 . m!$

$= \frac{12n}{n! \times n!} \times n! \times n! \times {}^{(m-1)}C_1 . m! = 2n! . m! . (m-1)$

Illustration 77:

$'n'$ digits positive integers formed such that each digit is 1, 2 or 3. How many of these contain all three of the digits 1, 2 and 3 atleast once?

(A) $3(n-1)$ (B) $3^n - 2.2^n + 3$ (C) $3^n - 3.2^n - 3$ (D) $3^n - 3.2^n + 3$

Ans. (D)

Solution:

Total n –digit numbers using 1, 2 or 3 $= 3^n$

total n –digit numbers using any two digits out of 1, 2 or 3$=\ ^3C_2 \times 2^n - 6 = 3 \times 2^n - 6$

total n –digit numbers using only one digit of 1, 2 or 3 $= 3$

$\therefore$ the numbers containing all three of the digits

1, 2 and 3 at least once = $3^n - (3 \times 2^n - 6) - 3 = 3^n - 3\,.2^n + 3$

Illustration 78:

There are $'n'$ straight line in a plane, no two of which are parallel and no three pass through the same point. Their points of intersection are joined. Then the maximum number of fresh lines thus introduced is

(A) $\frac{1}{12}n(n-1)^2(n-3)$ (B) $\frac{1}{8}n(n-1)(n+2)(n-3)$

(C) $\frac{1}{8}n(n-1)(n-2)(n-3)$ (D) $\frac{1}{8}n(n+1)(n+2)(n-3)$

Ans. (C)

Solution:

If $'n'$ straight line intersect each other then total

number of intersection point is $^nC_2 = \frac{n(n-1)}{2}$

Now, from these $^nC_2$ points we can make $^{\frac{n(n-1)}{2}}C_2$

lines. (total old + new lines) and number of old lines are $^{n-1}C_2 \times n$

So fresh lines are $^{\frac{n(n-1)}{2}}C_2 - {}^{n-1}C_2 \times n = \frac{1}{8}n(n-1)(n-2)(n-3)$

Illustration 79:

In how many ways can a pack of 52 cards be divided

(i) equally in four sets

(ii) equally among four players

Solution:

(i) $\frac{52}{(13!)^4} \times \frac{1}{4!}$ (ii) $\left(\frac{52!}{(13!)^4} \times \frac{1}{4!}\right) \times 4!$

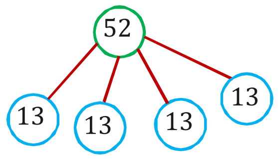
Illustration 80:

In how many ways 52 playing cards can be distributed among 4 players if each get equal no. of cards :

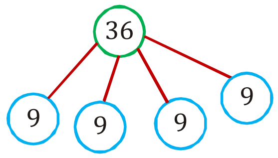
(i) If no condition

(ii) If each player can get A, J, Q, K of same suit.

Solution:

(i) 

$$\left[\frac{\underline{|52}}{\{\underline{|13}\cdot\underline{|13}\cdot\underline{|13}\cdot\underline{|13}\}\cdot\underline{|4}}\right]\times\underline{|4} \Rightarrow \frac{\underline{|52}}{(\underline{|13})^4}$$

(ii) 

$52 - 16 = 36(A, J, Q, K$ of 4 suits)

$$\left[\left\{\frac{\underline{|36}}{(\underline{|9})^4\cdot\underline{|4}}\right\}\times 4\right]\times\underline{|4}$$

Remaining A, J, Q, K of same suits to 4 players.

Illustration 81:

In how many ways can a pack of 52 cards be

(i) distributed among four players having 10, 12, 14 and 16 cards.

(ii) divided into four sets of 7, 15, 15 and 15 cards.

Solution:

(i) $\dfrac{52!}{10!\,12!\,14!\,16!}\times 4!$

(ii) $\dfrac{52!}{7!\,(15!)^3}\times\dfrac{1}{3!}$

Illustration 82:

$X = \{1, 2, 3, 4, \ldots\ldots 2017\}$ and $A \subset X$; $B \subset X$; $A \cup B \subset X$ here $P \subset Q$ denotes that P is subset of $Q(P \neq Q)$. Then number of ways of selecting unordered pair of sets A and B such that $A \cup B \subset X$.

(A) $\dfrac{(4^{2017}-3^{2017})+(2^{2017}-1)}{2}$

(B) $\dfrac{(4^{2017}-3^{2017})}{2}$

(C) $\dfrac{4^{2017}-3^{2017}+2^{2017}}{2}$

(D) None of these

Ans. (A)

Solution:

Ordered pair = total – $(A \cup B = X) = 4^n - 3^n$

Subsets of $X = 2^n$ will not repeat in both but here the whole set X has not been taken

So subsets of x which are not repeated $(2^n - 1)$

Hence unordered pair = $\dfrac{(4^n-3^n)-(2^n-1)}{2} + (2^n - 1)$

Illustration 83:

The number of ways in which 15 identical apples & 10 identical oranges can be distributed among three persons, each receiving none, one or more is:

(A) 5670 (B) 7200 (C) 8976 (D) 7296

Ans. (C)

Solution:

Using multinomial theorem

Total no. of ways = ${}^{15+3-1}C_{15} \times {}^{10+3-1}C_{10} = {}^{17}C_{15} \times {}^{12}C_{10} = \frac{17\times16}{2}\times\frac{12\times11}{2} = 8976$

Illustration 84:

If n identical dice are rolled, then number of possible outcomes are.

(A) 6^n (B) $\frac{6^n}{n!}$ (C) ${}^{(n+5)}C_5$ (D) None of these

Ans. (C)

Solution:

Let i appears on a_i dice $i = 1, 2, 3, 4, 5, 6$

Do number of outcomes is equal to no. of solution of $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = n = {}^{(n+5)}C_5$

Illustration 85:

Find the number of solutions of the equation $xyz = 360$ when $x, y, z \in N$

(A) 150 (B) 180 (C) 210 (D) 240

Ans. (B)

Solution:

$xyz = 360 = 2^3 \times 3^2 \times 5$ $(x, y, z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \le a_1 \le 3, 0 \le a_2 \le 2, 0 \le a_3 \le 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \le b_1 \le 3, 0 \le b_2 \le 2, 0 \le b_3 \le 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \le c_1 \le 3, 0 \le c_2 \le 2, 0 \le c_3 \le 1$)

$\Rightarrow 2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow 2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions = $10 \times 6 \times 3 = 180$.

Illustration 86:

Find the number of solutions of the equation $xyz = 360$ when $x, y, z \in I$

(A) 410 (B) 520 (C) 610 (D) 720

Ans. (D)

Solution:

$xyz = 360 = 2^3 \times 3^2 \times 5$ (if $x, y, z \in N$)

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \le a_1 \le 3, 0 \le a_2 \le 2, 0 \le a_3 \le 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \le b_1 \le 3, 0 \le b_2 \le 2, 0 \le b_3 \le 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \le c_1 \le 3, 0 \le c_2 \le 2, 0 \le c_3 \le 1$)

$\Rightarrow 2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow 2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions = $10 \times 6 \times 3 = 180$.

As in this question $x, y, z \in I$ then, (a) all positive (b) 1 positive and 2 negative.

Total number of ways = $180 + {}^3C_2 \times 180 = 720$.

Illustration 87:

Number of ways in which a pack of 52 playing cards be distributed equally among four players so that each have the Ace, King, Queen and Jack of the same suit, is

(A) $\dfrac{36\,!\,.\,4\,!}{(9\,!)^4}$ (B) $\dfrac{36\,!}{(9\,!)^4}$ (C) $\dfrac{52\,!\,.\,4\,!}{(13\,!)^4}$ (D) $\dfrac{52\,!}{(13\,!)^4}$

Ans. (A)

Solution:

Required number of ways ${}^{36}C_9\,.\,{}^{27}C_9\,.\,{}^{18}C_9\,.\,{}^{9}C_9\,.\,4\,! \;=\dfrac{36!}{(9!)^4}\times 4!$

Illustration 88:

Find total number of positive integral solutions of $15 < x_1 + x_2 + x_3 \leq 20$.

(A) 685 (B) 1140 (C) 455 (D) 1595

Ans. (A)

Solution:

$x_1 + x_2 + x_3 = 20 - t$

$t = 0, 1, 2, 3, 4$

Required value = $\sum_{t=0}^{4} {}^{19-t}C_2 = {}^{20}C_3 - {}^{15}C_3 = 1140 - 455 = 685$

Illustration 89:

Seven persons $P_1, P_2, \ldots\ldots\ldots, P_7$ initially seated at chairs $C_1, C_2 \ldots\ldots\ldots, C_7$ respectively. They all left their chairs simultaneously for hand wash. Now in how many ways they can again take seats such that no one sits on his own seat and P_1 sits on C_2 and P_2 sits on C_3 ?

(A) 52 (B) 53 (C) 54 (D) 55

Ans. (B)

Solution:

If P_3 sits on C_1

$4!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}\right)$

$= 4.3 - 4 + 1 = 9$

If P_3 does not sit on C_1

$= 5!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}-\dfrac{1}{5!}\right) = 44$

total number of ways = 44 + 9 = 53

$$\begin{cases} P_1 & C_1 \\ P_2 & C_2 \;\rightarrow\; P_1 \\ P_3 & C_3 \;\rightarrow\; P_2 \\ P_4 & C_4 \\ P_5 & C_5 \\ P_6 & C_6 \\ P_7 & C_7 \end{cases}$$

Illustration 90:

Given six line segments of length 2, 3, 4, 5, 6, 7 units, the number of triangles that can be formed by these segments is

(A) ${}^6C_3 - 7$ (B) ${}^6C_3 - 6$ (C) ${}^6C_3 - 5$ (D) ${}^6C_3 - 4$

Ans. (A)

Solution:

First we select 3 length from the given 6 length so the no. of ways = ${}^6C_3$

But these some pair i.e. (2, 3, 7), (2, 3, 6), (2, 3, 5) (2, 4, 6), (2, 4, 7), (2, 5, 7), (3, 4, 7) are not form a triangle so that total no. of ways is ${}^6C_3 - 7$ ways

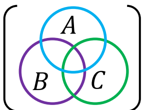
Illustration 91:

In how many ways $A, A, B, B, C, C, D, E, F, G$ can be arranged around a circle. If no two identical letters are together.

Solution:

$A, A, B, B, C, C, D, E, F, G \longrightarrow 10$

$\Rightarrow \left(\frac{\lfloor 10-1}{\lfloor 2\cdot\lfloor 2\cdot\lfloor 2}\right) - \left[\quad\right]$



↓ ↓ ↓
A B C

$n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - (C \cap A) + n(A \cap B \cap C)$

$\Rightarrow \left(\frac{\lfloor 9}{\lfloor 2\cdot\lfloor 2\cdot\lfloor 2}\right) - \left[\left(\frac{\lfloor 9-1}{\lfloor 2\cdot\lfloor 2}\right)3 - \left(\frac{\lfloor 8-1}{\lfloor 2}\right)\times 3 + \left(\lfloor 7-1\right)\right]$

↓ ↓ ↓
B C C
C A A
A B B

Illustration 92:

There are m apples and n oranges to be placed in a line such that the two extreme fruits being both oranges. Let P denotes the number of arrangements if the fruits of the same species are different and Q the corresponding figure when the fruits of the same species are alike, then the ratio P/Q has the value equal to:

(A) ${}^nP_2 . {}^mP_m . (n-2)\,!$ (B) ${}^mP_2 . {}^nP_n . (n-2)\,!$
(C) ${}^nP_2 . {}^nP_n . (m-2)\,!$ (D) none

Ans. (A)

Solution:

For P → If same species are different

Total number of arrangements is ${}^nP_2 . (m+n-2)\,!$

For Q → If same species are alike then number of arrangement is $\frac{(m+n-2)!}{m!\ .(n-2)!}$

Hence $\frac{P}{Q} = {}^nP_2 . m! . (n-2)! = {}^nP_2 . {}^mP_m . (n-2)!$

Illustration 93:

The number of intersection points of diagonals of 2009 sides regular polygon, which lie inside the polygon.

(A) ${}^{2009}C_4$ (B) ${}^{2009}C_2$ (C) ${}^{2008}C_4$ (D) ${}^{2008}C_2$

Ans. (A)

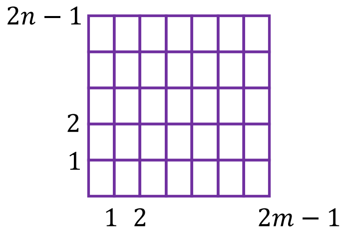
Solution:

We know that in odd sides polygon no two or more then two diagonals are parallel, so if we take any 4 vertices, we get one point of intersection of diagonals.

Hence required no of points will be ${}^{2009}C_4$.

Illustration 94:

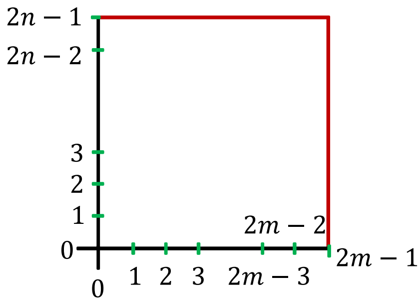
A rectangle with sides $2m - 1$ and $2n - 1$ is divided into squares of unit length by drawing parallel lines as shown in the diagram, then the number of rectangles possible with odd side lengths is



(A) $(m + n - 1)^2$ (B) 4^{m+n-1} (C) $m^2 n^2$ (D) $m(m + 1)n(n + 1)$

Ans. (C)

Solution:



No. of ways of choosing horizontal side of rectangle of one unit length = $2m - 1$

No. of ways of choosing horizontal side of rectangle of 3 unit length = $2m - 3$

$\therefore$ Total no. of ways of choosing horizontal side of rectangle of odd length

$= (2m-1) + (2m-3) + \ldots + 1 = m^2$

Similarly no. of ways of choosing the vertical side of rectangle of odd length = n^2.

$\therefore$ Total no. of ways of choosing the rectangle = $n^2 m^2$