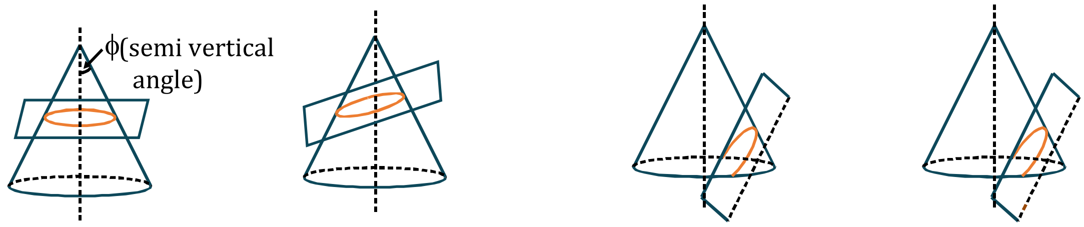
03 Parabola

Introduction to Conic Sections:

(a) Geometrical Interpretation:

The conic sections are the curves generated by intersections of a plane, with one or two cones.



Circle **Ellipse** **Parabola** **Hyperbola**

Case-I: When plane P is not passing through vertex of cone

(i) When $\theta = 90°$; Circle
(ii) When $\phi < \theta < 90°$; Ellipse
(iii) When $\theta = \phi$; Parabola
(iv) When $0 \le \theta < \phi$; Hyperbola

$\left\{ \begin{array}{l} \phi = \text{semi vertical angle} \\ \theta = \text{angle made by plane with vertical axis} \end{array} \right.$

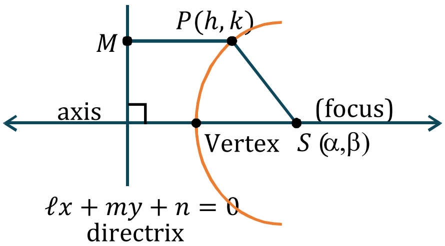
Case-II: When plane P is passing through vertex of cone

(i) When $\theta > \phi$; Two imaginary lines
(ii) When $\theta = \phi$; Two real & coincident lines
(iii) When $\theta < \phi$; Two real & distinct lines

(b) Mathematical Definition:

A conic section or conic is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is a constant i.e.

$\dfrac{PS}{PM}$ = constant = e.



- The fixed point is called the **Focus**.
- The fixed straight line is called the **Directrix**.
- The constant ratio is called the **Eccentricity** denoted by 'e'.
- The line passing through the focus & perpendicular to the directrix is called the **Axis**.
- A point of intersection of a conic with its axis is called a **Vertex**.

General Equation of a Conic:

Focal Directrix Property:

The general equation of a conic with focus (α, β) & directrix $\ell x + my + n = 0$ and eccentricity e is:

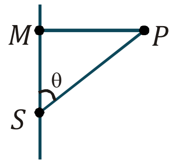
$$[(x-\alpha)^2 + (y-\beta)^2] = \frac{e^2(\ell x + my + n)^2}{\ell^2 + m^2}$$

this simplifies to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Now two cases arise:

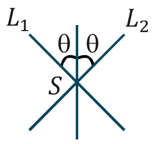
Case (I): When the focus lies on the directrix.

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines:

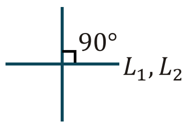


$$e = \frac{PS}{PM} = \operatorname{cosec}\theta$$

(i) If $e > 1$ ($\sin\theta < 1$, hence valid) $\Rightarrow$ the lines L_1 and L_2 will be real & distinct intersecting at S.



(ii) If $e = 1$ ($\sin\theta = 1$, hence $\theta = 90°$) $\Rightarrow$ the lines will coincident and perpendicular to the given line passing through S.



(iii) If $e < 1$ ($\sin\theta > 1$, hence no real lines) $\Rightarrow$ the lines will be imaginary (no such real lines are possible)

Illustration 1:

Consider the example with y-axis as directrix and origin as focus.

Solution:

$x^2 + y^2 = e^2(x)^2 \Rightarrow$ Equation of locus is $y^2 = x^2(e^2 - 1)$

Case(II) : When the Focus does not lie on directrix.

a parabola	an ellipse	a hyperbola	a circle
$e=1$	$0<e<1$	$e>1$	$e = 0$
$\Delta \neq 0$	$\Delta \neq 0$	$\Delta \neq 0$	$\Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h = 0,\ a = b \neq 0$
(if $h = 0$, then $a \neq b$)			

Illustration 2:

Identify the conic $x^2 + 2y^2 - 2xy + 8x + 10y - 36 = 0$

Solution:

$a = 1, b = 1, h = -1, g = 5, f = 9, c = -46$

Now, $\Delta = abc + 2fgh - af^2 - by^2 - ch^2 \neq 0$

and $h^2 < ab$, hence ellipse

Illustration 3:

Locus of a point which moves such that the ratio of its distance from (0, 3) to its perpendicular distance from $x - 3y + 9 = 0$ is 4

Solution:

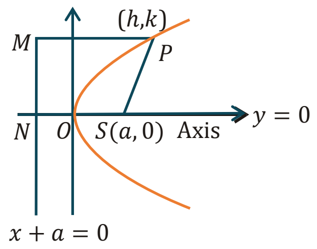
Focus (0, 3) lies on directrix $x - 3y + 9 = 0$ and $e = 4 > 1$

Hence locus is a pair of real and distinct straight lines.

Standard Equation of Parabola:

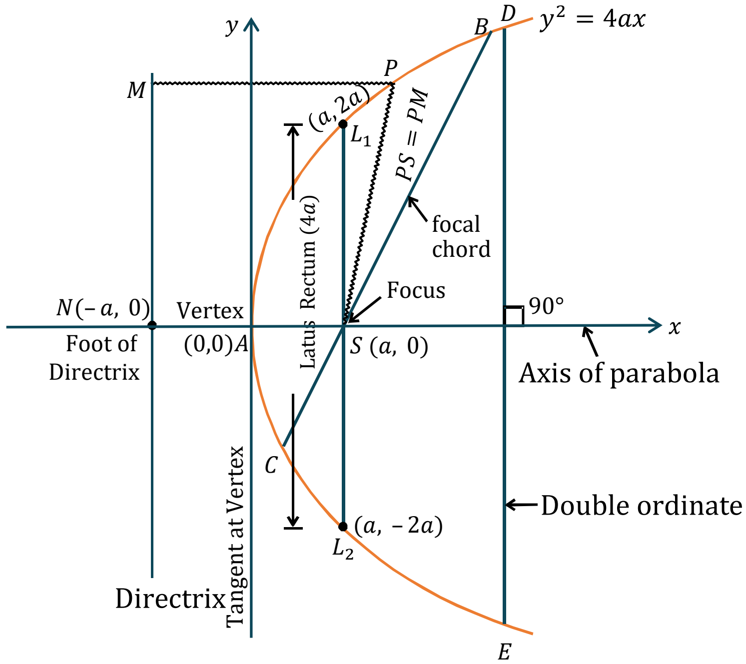
Standard equation of parabola is obtained by placing its focus at $S(a, 0)$ and taking its directrix as the line $x + a = 0$ so that origin lies on the curve

$PS = PM$



$(x - a)^2 + y^2 = (x + a)^2 \Rightarrow y^2 = 4ax \qquad (a > 0)$

(A) Standard parabola $y^2 = 4ax (a > 0)$ at a glance



General Terminology of Parabola:

(a) Axis of parabola : The line passing through the focus & perpendicular to the directrix is called the axis of parabola.

(b) Vertex : A point of intersection of a parabola with axis of parabola is called a Vertex.

(c) Double ordinate : A chord of the parabola perpendicular to the axis of the symmetry is called a double ordinate.

(d) Focal chord : A chord of the parabola, which passes through the focus is called a Focal chord.

(e) Latus rectum : A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus rectum.

(f) Focal distance/Focal radii : The distance of a point P on the parabola from the focus is called the Focal distance of the point and is equal to the distance of point P from the directrix.

(g) Ends of latus rectum : Intersection point of latus rectum and parabola is called ends of latus rectum. For parabola $y^2 = 4ax$, ends of the latus rectum are $L_1(a, 2a)$ & $L_2(a, -2a)$.

(h) Length of Latus rectum : Distance between ends of latus rectum is called length of latus rectum. Length of the latus rectum = $4a$

(i) Semi Latus Rectum : Half of length of latus rectum.

Note:

(i) Perpendicular distance from focus to directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Point of intersection of axis and directrix is called foot of directrix.

(iv) Two parabolas are said to be equal if they have the same length of latus rectum.

(B) Four Standard Parabolas: (a > 0)

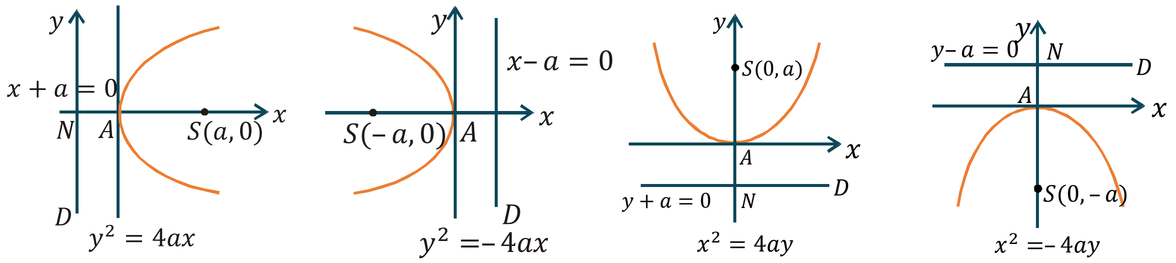


Illustration 4:

Find all the parameters of following parabolas and draw their graphs.

(i) $y^2 = 16x$

Solution:

$a = 4$,

focus $(4, 0)$,

vertex $(0, 0)$,

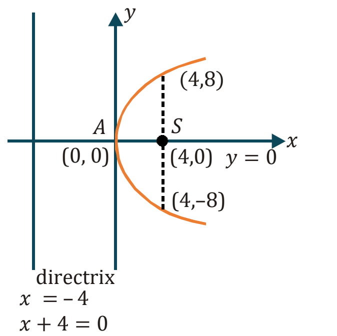
directrix $x + 4 = 0$

axis $y = 0$,

Latus rectum equation $x = 4$

Length of Latus Rectum = 16,

End points of latus rectum (4, 8) & (4, –8)



(ii) $y^2 = -x$

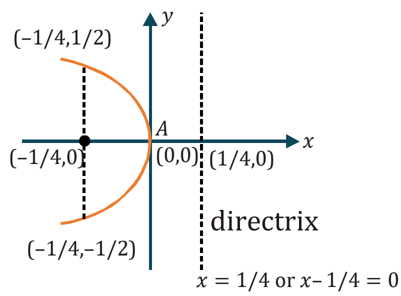
Solution:

$$y^2 = -4\left(\frac{1}{4}\right)x$$

$$a = \frac{1}{4}$$

focus $\left(-\frac{1}{4}, 0\right)$,

vertex (0, 0),



directrix $x - \frac{1}{4} = 0$

axis $y = 0$,

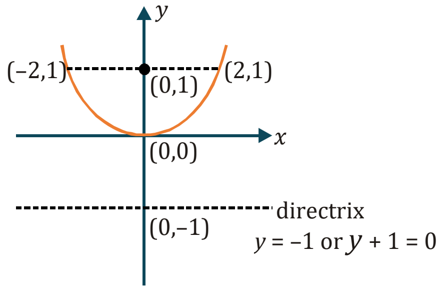
Latus rectum equation $x = -\frac{1}{4}$

Length of Latus Rectum = 1,

End points of latus rectum $\left(-\frac{1}{4}, \frac{1}{2}\right) \& \left(-\frac{1}{4}, -\frac{1}{2}\right)$

(iii) $x^2 = 4y$

Solution:



$x^2 = 4(1)y$

$a = 1$

focus (0, 1),

vertex (0, 0),

directrix $y + 1 = 0$

axis $y = 0$,

Latus rectum equation $y = 1$

Length of Latus Rectum = 4,

End points of latus rectum (–2, 1) & (2, 1)

Illustration 5:

The number of parabolas that can be drawn if two ends of the latus rectum are given

(A) 1 (B) 2 (C) 4 (D) 3

Ans. (B)

Solution:

Fact: Only two parabolas can be drawn with a given latus rectum.

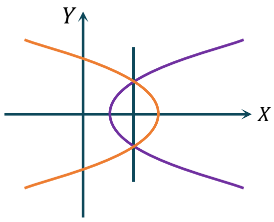


Illustration 6:

Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is:

(A) 4 (B) 2 (C) 1/5 (D) 2/5

Ans. (D)

Solution:

$$(x-2)^2 + (y-3)^2 = \left|\frac{3x-4y+7}{5}\right|^2$$

$\therefore$ focus is (2, 3) & directrix is $3x - 4y + 7 = 0$

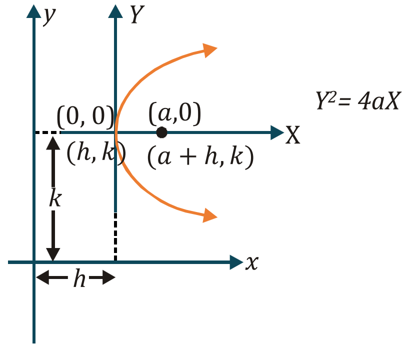
Latus rectum = 2 × perpendicular distance from focus to

Directrix = $2 \times \frac{1}{5} = 2/5$

Shifted Parabola:

In case the vertex of the parabola is not origin, then its equation can be taken as:

(a) $x = ay^2 + by + c$, its axis is parallel to x –axis convertible into the form $(y - k)^2 = 4a(x - h)$



(b) $y = ax^2 + bx + c$, its axis is parallel to y –axis convertible into the form $(x - h)^2 = 4a(y - k)$

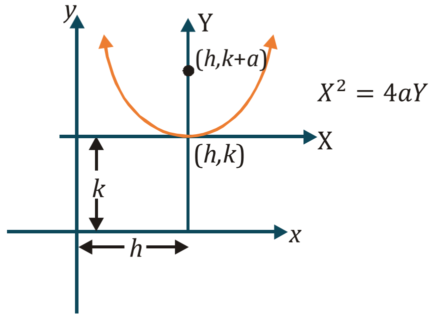
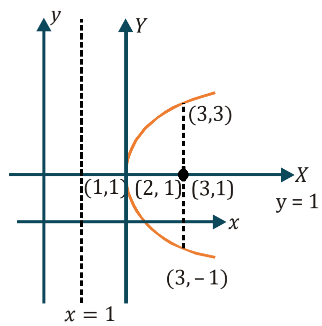


Illustration 7:

Find all parameters of parabola

(i) $y^2 - 2y - 4x + 9 = 0$

Solution:



$(y-1)^2 = 4(x-2)$ $\quad\quad Y^2 = 4.1.X$ $\quad\quad \therefore a = 1$

Vertex : $X = 0 \Rightarrow x = 2$

$Y = 0 \Rightarrow y = 1$

Focus : $X = a \Rightarrow x = 3$

$Y = 0 \Rightarrow y = 1$

Directrix : $X = -a \Rightarrow x - 2 = -1 \Rightarrow x = 1$

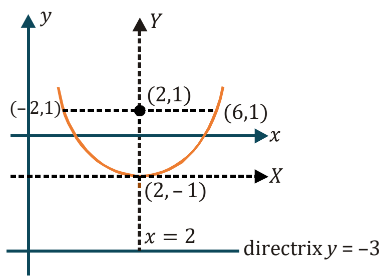
Axis : $Y = 0 \Rightarrow y = 1$

LR : $x = 3$

Length of LR = 4

(ii) $x^2 - 4x - 8y - 4 = 0$

Solution:



$(x-2)^2 = 4.2(y + 1) \Rightarrow a = 2$

$X^2 = 4aY$

Vertex : $(2, -1)$

Focus : $(2, 1)$

Directrix : $y = -3$

Axis : $x = 2$

L.R. : $y = 1$

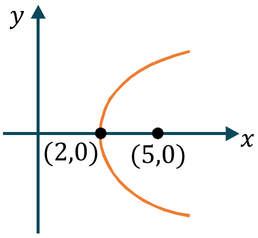
Length of $LR = 8$

Illustration 8:

Find equation of parabola whose

(i) Vertex (2, 0) and Focus (5, 0)

Solution:

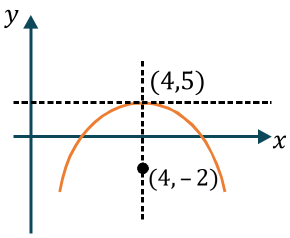


$(y-0)^2 = 4.3(x-2)$

$y^2 = 12(x-2)$

(ii) Vertex (4, 5) and focus (4, – 2)

Solution:

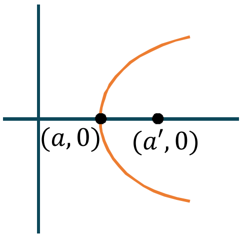


$(x-4)^2 = 4.7(y-5)$

$(x-4)^2 = -28(y-5)$

(iii) Vertex and focus lies on positive $x-$axis at distance a and a' from origin respectively $(a' > a)$

Solution:



$(y-0)^2 = 4(a'-a)(x-a)$

Illustration 9:

Find the equation of the parabola

(a) whose vertex is the point (4, – 3) and whose latus rectum is 4 and whose axis is parallel to the x-axis and focus lies on the right side of vertex.

Solution:

vertex is (4, – 3) and $L_1L_2 = 4$

$\therefore\ (y+3)^2 = 4(x-4)$

$y^2 + 6y + 9 = 4x - 16$

$y^2 - 4x + 6y + 25 = 0$

(b) passing through the point (–4, –7) and whose directrix is parallel to x-axis and whose vertex is the point (4, –3).

Solution:

Since the directrix is parallel to x-axis hence axis of symmetric is parallel to y-axis

As vertex is $(4, -3)$ hence the equation of the parabola can be taken as

$(x-4)^2 = 4a(y+3)$

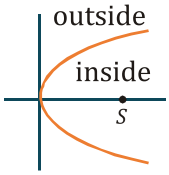
since it passes through $(-4, -7)$

$6y = -16a \Rightarrow a = -4$

$\therefore\ (x-4)^2 = -16(y+3)$

Position of a Point Relative to a Parabola:

Let $P(x, y) = 0$ be the equation of parabola then to find the position of point (x_1, y_1)



Make the coefficient of x^2 and y^2 non-negative

(a) If $P(x_1, y_1) < 0 \Rightarrow$ point lies inside the parabola

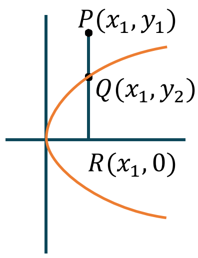
(b) If $P(x_1, y_1) > 0 \Rightarrow$ point lies outside the parabola

(c) If $P(x_1, y_1) = 0 \Rightarrow$ point lies on the parabola

Let $P(x_1, y_1)$ be a given point and

$S \equiv y^2 - 4ax = 0$ be a given parabola. ...(i)

Let the ordinate through P meet the parabola at Q and the X –axis at R. Then the coordinates of Q are (x_1, y_2) where $y_2^2 = 4ax_1$ since Q lies on the parabola. If P lies outside the parabola as shown in the fig., then



$PR^2 > QR^2$

i.e. $y_1^2 > 4ax_1$

i.e. $y_1^2 - 4ax_1 > 0$

i.e. $S_1 > 0$ where $S_1 \equiv y_1^2 - 4ax_1$

Similarly, if $S_1 < 0$, then $P(x_1, y_1)$ is a point lying inside the parabola.

Note:

(i) The above inequalities can be used for establishing the relative position of a point with respect to any parabola.

(ii) If point lies inside the parabola, then no tangent can be drawn.

(iii) If point lies on the parabola, then one tangent can be drawn.

(iv) If point lies outside the parabola, then two tangents can be drawn.

Illustration 10:

For what values of 'a' the point $P(\alpha, \alpha)$ lies inside, on or outside the parabola $(y-2)^2 = 4(x-3)$.

Solution:

Given equation can be written as $y^2 - 4y - 4x + 16 = 0$

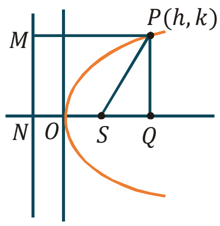
Point $P(\alpha, \alpha)$ lies inside parabola if $\alpha^2 - 8\alpha + 16 < 0$

$\Rightarrow (\alpha - 4)^2 < 0 \Rightarrow$ no such α exist.

Point $P(\alpha, \alpha)$ lies on parabola if $(\alpha - 4)^2 = 0 \Rightarrow \alpha = 4$

Point $P(\alpha, \alpha)$ lies outside parabola if $(\alpha - 4)^2 > 0 \Rightarrow \alpha \in R - \{4\}$

Focal Distance/Focal Radii:



$y^2 = 4ax$

Let a point $P(h, k)$ be on the parabola $y^2 = 4ax$, then its focal distance is PS which is equal to PM.

$PM = ON + OQ = a + h$

- For parabola $y^2 = 4ax$ or $y^2 = -4ax$, focal distance of any point $P(h, k)$ is $(|\boldsymbol{h}| + \boldsymbol{a})$
- For parabola $x^2 = 4ay$ or $x^2 = -4ay$, focal distance of any point $P(h, k)$ is $(|\boldsymbol{k}| + \boldsymbol{a})$
- For shifted parabola $(y - \beta)^2 = \pm 4a(x - \alpha)$, focal distance any point $P(h, k)$ $(|\boldsymbol{h} - \boldsymbol{\alpha}| + \boldsymbol{a})(\boldsymbol{a} > \mathbf{0})$
- For shifted parabola $(x - \alpha)^2 = \pm 4a(y - \beta)$, focal distance any point $P(h, k)$ $(|\boldsymbol{k} - \boldsymbol{\beta}| + \boldsymbol{a})$ $(\boldsymbol{a} > \mathbf{0})$

Note:

(i) Minimum focal distance of any point on the parabola is 'a'.

(ii) If focal distance > $a \Rightarrow$ two such points are possible

If focal distance = $a \Rightarrow$ only one point is possible which is vertex of the parabola

If focal distance < $a \Rightarrow$ no such points are possible

Illustration 11:

Find the points on the parabola $y^2 = 12x$ having focal radii (a) 4 (b) 3 (c) 2

Solution:

$y^2 = 4.3x \Rightarrow a = 3$

Focal distance = $|h| + a$

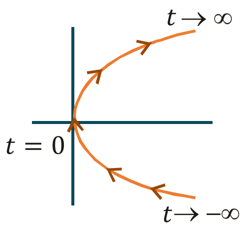
(a) $|h| + 3 = 4 \Rightarrow h = 1$

Now put in parabola $\Rightarrow y^2 = 12 \Rightarrow y = \pm 2\sqrt{3}$

(b) $|h| + 3 \Rightarrow h = 0, k = 0 \Rightarrow$ vertex

(c) $|h| + 3 = 2 \Rightarrow |h| = -1 \Rightarrow$ Not possible

Parametric Co-Ordinates:



Variation of parameter (t)
as point varies on

Parabola	Parametric Coordinates
(a) $y^2 = 4ax$	$(at^2, 2at)$
(b) $y^2 = -4ax$	$(-at^2, 2at)$
(c) $x^2 = 4ay$	$(2at, at^2)$
(d) $x^2 = -4ay$	$(2at, -at^2)$
(e) $(y-\beta)^2 = 4a(x-\alpha)$	$(\alpha + at^2, \beta + 2at)$
(f) $(x-\alpha)^2 = 4a(y-\beta)$	$(\alpha + 2at, \beta + at^2)$

Illustration 12:

The parametric equation of the parabola $y^2 = 8x$ are-

(A) $x = 2t, y = 4t^2$ (B) $x = 2t^2, y = 4t$

(C) $x = t^2, y = 2t$ (D) None of these

Solution:

Here $a = 2;\ y = 2at$

$\Rightarrow\ y = 2.2.t\ =\ 4t$

$x = at^2 \Rightarrow x = 2t^2$

Illustration 13:

The parametric equation of the parabola $(y-1)^2 = 16(x-8)$ are-

Solution:

Here $a = 4$

$y-1\ = 2at \Rightarrow y = 1 + 2.4t = 1 + 8t$

$x-8 = at^2 \Rightarrow x = 8 + 4.t^2 = 8 + 4t^2$

Illustration 14:

Parameter t of a point $(1/2, -2)$ of the parabola $y^2 = 8x$ is-

Solution:

Parametric coordinates of any point on parabola $y^2 = 4ax$ are $(at^2, 2at)$

Here $4a = 8 \Rightarrow a = 2$

$\therefore\ y$ coordinate $2at = -2$

$\therefore\ 2(2)t = -2 \Rightarrow t = -1/2$

Illustration 15:

$x-2 = t^2, y = 2t$ are the parametric equations of the parabola-

(A) $y^2 = -4x$ (B) $y^2 = 4x$

(C) $x^2 = -4y$ (D) $y^2 = 4(x-2)$

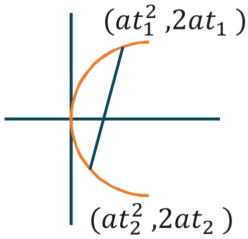
Solution:

Here $\frac{y}{2} = t$ and $x-2 = t^2$

$\Rightarrow (x-2) = \left(\frac{y}{2}\right)^2 \Rightarrow y^2 = 4(x-2)$

Chord Joining Two Points t_1 and t_2:

Parametric equation of a chord joining two points "t_1" and "t_2" of the parabola $y^2 = 4ax$ is



$$(y-2at_1)=\frac{2at_2-2at_1}{at_2^2-at_1^2}(x-at_1^2)$$

$$\Rightarrow\quad y-2at_1=\frac{2}{t_1+t_2}(x-at_1)$$

$$\Rightarrow\quad \boxed{2x-y(t_1+t_2)+2at_1t_2=0}$$

For Memory

$$x-y\frac{(t_1+t_2)}{2}+at_1t_2=0 \Rightarrow x-y(AM)+a(GM)^2=0$$

Remarks

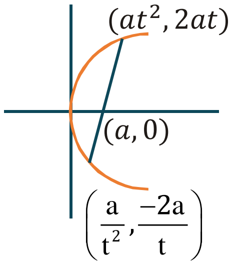
(i) $\boxed{\text{Slope} = \dfrac{2}{t_1+t_2}}$

(ii) Length of the chord : $\sqrt{(at_1^2-at_2^2)^2+(2at_1-2at_2)^2}=\boxed{a|t_1-t_2|\sqrt{(t_1+t_2)^2+4}}$

(iii) If chord is focal chord $\Rightarrow$ Chord passes through $(a, 0)$

$\Rightarrow\quad 2a + 2at_1t_2 = 0 \Rightarrow \boxed{t_1t_2=-1}$

(OR $t_1, t_2, (a, 0)$ are collinear)



$\Rightarrow t_2 = -\dfrac{1}{t_1}$. Hence if one end of the focal chord is t then other end is $-\dfrac{1}{t}$.

$\Rightarrow$ Coordinates of end points of focal chord can be taken as $(at^2, 2at)$ and $\left(\dfrac{a}{t^2}-\dfrac{2a}{t}\right)$

(**Note :** Only one variable)

(iv) Length of the focal chord,

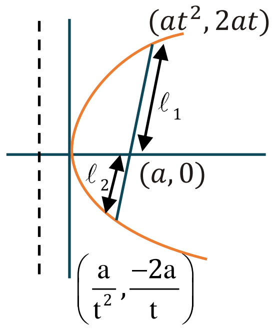
$$\ell_f=a|t_1-t_2|\sqrt{(t_1+t_2)^2+4} = a|t_1-t_2|\sqrt{t_1^2+t_2^2+2}=a|t_1-t_2|\sqrt{t_1^2+t_2^2-2t_1t_2}$$

$$\boxed{\ell_f=a(t_1-t_2)^2=a\left(t+\frac{1}{t}\right)^2}$$

(v) Equation of focal chord

$$\boxed{2x-y\left(t-\frac{1}{t}\right)-2a=0}$$

(vi) If ℓ_1, ℓ_2 are the length segments of a focal chord made by focus then length of its latus rectum is $\dfrac{4\ell_1\ell_2}{\ell_1+\ell_2}$ i.e. $= 4a$.



$\ell_1 = at^2 + a$ (use focal distance property)

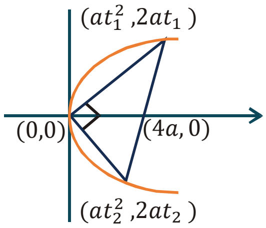
$\ell_2 = \frac{a}{t^2} + a$

Now $\frac{1}{\ell_1} + \frac{1}{\ell_2} = \frac{1}{a+at^2} + \frac{t^2}{a+at^2} = \frac{1}{a}$

$\Rightarrow \frac{\ell_1 + \ell_2}{\ell_1 \ell_2} = \frac{1}{a} \Rightarrow \frac{4\ell_1\ell_2}{\ell_1 + \ell_2} = 4a$

(vii) If the chord subtends 90° angle at the vertex of parabola then $(t_1t_2 = -4)$ and it passes through a fixed point $(4a, 0)$ on the axis

$m_1 = \frac{2}{t_1}$ $\quad m_2 = \frac{2}{t_2}$ $\quad m_1m_2 = -1 \Rightarrow t_1t_2 = -4$



Hence chord $2x - y(t_1 + t_2) + 2a(-4) = 0$

$\Rightarrow 2(x - 4a) - (t_1 + t_2)y = 0$

$\Rightarrow (x - 4a) + \lambda y = 0$

$\Rightarrow$ Passes through $(4a, 0)$

Illustration 16:

Find the length and other end of the focal chord of parabola $y^2 = 8x$ at $(2, -4)$

Solution:

$a = 2, at^2 = 2t^2 = 2 \Rightarrow t = \pm 1$ and $2 \times 2 \times t = -4 \Rightarrow \boxed{t = -1}$

$\Rightarrow$ other end $\equiv \left(\frac{2}{(-1)^2}, -\frac{2 \cdot 2}{(-1)}\right) = (2, 4)$

$\ell_f = 2(1+1)^2 = 8$

Illustration 17:

If (x_1, y_1) and (x_2, y_2) are ends of a focal chord of parabola $y^2 = 4ax$ then find the value of

(i) x_1x_2

(ii) y_1y_2 in terms of a.

Solution:

(i) $x_1 = at^2$ and $x_2 = \frac{a}{t^2} \Rightarrow \boxed{x_1 x_2 = a^2}$

(ii) $y_1 = 2at$ and $y_2 = -\frac{2a}{t} \Rightarrow \boxed{y_1 y_2 = -4a^2}$

Tangents to the Parabola $y^2 = 4ax$:

(i) Slope Form:

Intersection of line and Parabola

$y = mx + c$ & $y^2 = 4ax$

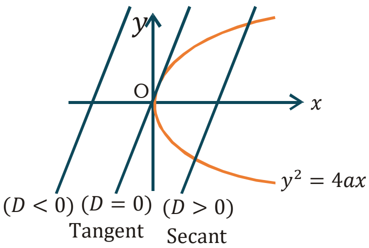
solving them together

$(mx + c)^2 - 4ax = 0$

$m^2x^2 + 2(cm - 2a)x + c^2 = 0$

and $D = 4(mc - 2a)^2 - 4m^2c^2 = 16a(a - mc)$

$D > 0$	$D = 0$	$D < 0$
Secant	Tangent	Neither intersecting nor touching
$(a - mc > 0)$	$(a - mc = 0)$	$(a - mc < 0)$



$\Downarrow$

now, $\boxed{c = \frac{a}{m}}$ gives the condition of tangency

hence $\boxed{y = mx + \frac{a}{m}}$ is always a tangent to $y^2 = 4ax$ for all $m \neq 0$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Note:

(i) Solve equation of tangent and parabola to get point of contact.

$y^2 = 4ax$

$$\Rightarrow \left(mx + \frac{a}{m}\right)^2 = 4ax \Rightarrow m^2x^2 + 2ax + \frac{a^2}{m^2} = 4ax \Rightarrow m^2x^2 - 2ax + \frac{a^2}{m^2} = 0$$

$$\Rightarrow \left(mx - \frac{a}{m}\right)^2 = 0 \Rightarrow x = \frac{a}{m^2} \;\&\; y = \frac{2a}{m}$$

(ii) $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $c = -am^2$ and point of contact is $(2am, am^2)$.

(Tip: Interchange $x \leftrightarrow y$ and $m \leftrightarrow 1/m$)

Illustration 18:

If the line $x + 2y - 1 = 0$ touches the parabola $y^2 = kx$, then the value of k is-

Solution:

We have $y = -\frac{x}{2} + \frac{1}{2}$

If this line touches the parabola $y^2 = kx$, then

$\frac{1}{2} = \frac{k}{4}(-2)$ [Using $c = a/m$]

$\Rightarrow k = -1$

Illustration 19:

At which point the line $x = my + \frac{a}{m}$ touches the parabola $x^2 = 4ay$

(A) $(2am, am^2)$ (B) $(am^2, 2am)$ (C) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ (D) $\left(\frac{2a}{m}, \frac{a}{m^2}\right)$

Solution:

For parabola $x^2 = 4ay$

Points of contact is $(2am, am^2)$, {where m = slope of tangent}

$\because$ equation of tangent is $x = my + \frac{a}{m}$

Then slope of tangent is $\frac{1}{m}$

$\therefore$ Point of contact is $\left(\frac{2a}{m}, \frac{a}{m^2}\right)$

Illustration 20:

The value λ such that line $y = 4x + \lambda$ is tangent to the parabola $y^2 = 8x$

Solution:

$\lambda = c = \frac{a}{m} = \frac{2}{4} = \frac{1}{2}$

Cartesian Form:

$yy_1 = 2a(x + x_1)$ at the point (x_1, y_1)

Proof:

$2y \cdot \frac{dy}{dx} = 4a$

$\Rightarrow m_{(x_1, y_1)} = \frac{2a}{y_1}$

$\therefore$ Equation $\equiv y - y_1 = \frac{2a}{y_1}(x - x_1)$

$yy_1 = 2ax - 2ax_1 + y_1^2$

Put $y_1^2 = 4ax_1$, we get

$yy_1 = 2a(x + x_1)$

Illustration 21:

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (3, 2) is-

(A) $3y + x + 3 = 0$ (B) $3x + y + 3 = 0$ (C) $3x = y + 3$ (D) $3y = x + 3$

Solution:

Since (3, 2) lies on the parabola $y^2 = 4ax$, so

$4 = 12a \Rightarrow a = 1/3$

Now using $T = 0$, the equation of the tangent at (3, 2) is

$y(2) = 2a(x + 3)$

$\Rightarrow y = \frac{1}{3}(x + 3) \Rightarrow 3y = x + 3$

Illustration 22:

The equation of tangent to the parabola $x^2 = y$ at one extremity of latus rectum in the first quadrant is

(A) $y = 4x + 1$ (B) $x = 4y + 1$ (C) $4x + 4y = 1$ (D) $4x - 4y = 1$

Solution:

Given parabola is $x^2 = y$

$4a = 1 \Rightarrow a = \frac{1}{4}$

Ends of Latus rectum $(2a, a)$ {I[st] quadrant}

$\therefore \left(\frac{1}{2}, \frac{1}{4}\right)$

So, equation of tangent $T = 0$

$\frac{1}{2}x = \frac{y + \frac{1}{4}}{2} \Rightarrow 4x - 4y - 1 = 0$

Parametric Form:

The equation of tangent to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ is given by $ty = x + at^2$

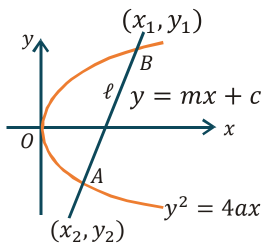
Proof:

Put $x_1 = at^2$ and $y_1 = 2at$ in $yy_1 = 2a(x + x_1)$ $\Rightarrow ty = x + at^2$

$\Rightarrow$ Slope of tangent is $1/t$.

Length of Chord of the Conic Intercepted on Line:

Length of chord of the parabola intercepted on $y = mx + c$



$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_2 - x_1)^2\,[1 + m^2]$

$\ell^2 = [(x_2 + x_1)^2 - 4x_1x_2]\,[1 + m^2]$

$\ell = \sqrt{(1 + m^2)((x_1 + x_2)^2 - 4x_1x_2)}$ or $\left[\ell = |x_1 - x_2|\sqrt{1 + m^2}\right]$ ...(i)

Now to find $(x_1 + x_2)$ & x_1x_2, form quadratic equation in x by solving

$y = mx + c$ & conic

for parabola $y^2 = 4ax \Rightarrow (mx + c)^2 = 4ax$

$\Rightarrow m^2x^2 + 2(mc - 2a)x + c^2 = 0$

$\Rightarrow x_1 + x_2 = -\frac{2(cm - 2a)}{m^2}$; $x_1x_2 = \frac{c^2}{m^2}$ from (i)

Hence $\ell = \frac{4}{m^2}\sqrt{a(1 + m^2)(a - cm)}$

Length of Focal Chord Making Angle α with x-axis.

Chord $y = mx + c$ passes through $(a, 0)$

$\Rightarrow$ $0 = ma + c$ and $m = \tan\alpha$

$$\ell_f = \frac{4}{m^2}\sqrt{a(1 + m^2)\big(a - m(-am)\big)}$$

$$= \frac{4a(1+m^2)}{m^2} = \frac{4a\sec^2\alpha}{\tan^2\alpha} = 4a\,\text{cosec}^2\alpha$$

Common Tangent with Circle:

Common Tangent of Two Curves:

Take the equation of tangent in slope(m) form of one of the curves.

As this is common tangent it also satisfies tangency condition for the second curve, solve for 'm'.

Use this value of m to get the equation of common tangent.

If a line is tangent to a circle then its perpendicular distance from center is equal to the radius.

Common Tangent with Circle:

Consider the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 = r^2$

Let the line $y = mx + c$ be the common tangent

General equation of tangent to parabola

$y^2 = 4ax$ is $y = mx + \frac{a}{m}$

$\Rightarrow\ mx - y + \frac{a}{m} = 0$

$$\Rightarrow \left|\frac{\frac{a}{m}}{\sqrt{1+m^2}}\right| = r$$

$\Rightarrow\ c = \frac{a}{m} = \pm r\sqrt{1+m^2}$

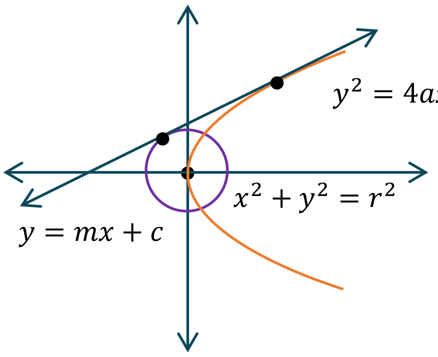


Illustration 23:

Find the equations of the straight lines touching both $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$.

Solution:

The given curves are

$x^2 + y^2 = 2a^2$ ...(i)

and $y^2 = 8ax$ ...(ii)

The parabola (ii) is

$y^2 = 8ax$

or $y^2 = 4(2a)x$

$\therefore$ Equation of tangent of (ii) is

$y = mx + \frac{2a}{m}$

or $m^2x - my + 2a = 0$ ...(iii)

It is also tangent of (i), then the length of perpendicular from centre of (i) i.e. (0, 0) to (iii) must be equal to the radius of (i) i.e., $a\sqrt{2}$.

$$\therefore \left|\frac{0-0+2a}{\sqrt{(m^2)^2+(-m)^2}}\right| = a\sqrt{2}$$

or $\frac{4a^2}{m^4+m^2} = 2a^2$ or $m^4 + m^2 - 2 = 0$

or $(m^2 + 2)(m^2 - 1) = 0$

$\because m^2 + 2 \neq 0$

$\therefore\ m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

Hence from (iii) the required tangents are

$x \pm y + 2a = 0$ (gives the imaginary values)

Illustration 24:

If $y = x + c$ is common tangent to both parabola $y = 4x$ and circle $(x-2)^2 + (y+1)^2 = r^2$, then find the value of r^2

Solution:

For parabola $a = 1$, slope of tangent = 1

$\Rightarrow c = \frac{1}{1} = 1$

Perpendicular distance of tangent from centre is equal to radius

$\Rightarrow \left|\frac{(2)-(-1)+1}{\sqrt{1+1}}\right| = |r| \quad \Rightarrow r^2 = 8$

Rules of Transformation:

In the previous articles all the results have been proved for the particular parabola $y^2 = 4ax (a > 0)$. However, all the results with slight transformations are valid for any shifted parabola.

If any equation derived for the parabola $y^2 = 4ax. (a > 0)$ is $R(x, y, a, m)$ then result of other parabolas are as follows

S. No.	Parabola	Transform
(i)	$y^2 = 4ax$	$R(x, y, a, m)$
(ii)	$y^2 = -4ax$	$R(x, y - a, m)$
(iii)	$x^2 = 4ay$	$R\left(y, x, a, \frac{1}{m}\right)$
(iv)	$x^2 = -4ay$	$R\left(y, x, -a, \frac{1}{m}\right)$
(v)	$(y-\beta)^2 = 4a(x-\alpha)$	$R(x-\alpha, y-\beta, a, m)$
(vi)	$(x-\alpha)^2 = 4a(y-\beta)$	$R(y-\beta, x-\alpha, a, \frac{1}{m})$

Using the above rule, equation of tangent to parabola $x^2 = 4ay$ is $x = \frac{y}{m} + am \Rightarrow y = mx - am^2$

Similarly, equation of normal to the parabola $x^2 = 4ay$ is $x = \frac{y}{m} - \frac{2a}{m} - \frac{a}{m^3} \Rightarrow y = mx + 2a + \frac{a}{m^2}$ and so on.

Illustration 25:

The equation of the tangent to the parabola $x^2 - 4x - 8y + 36 = 0$ with slope -4 is -

(A) $4x + y - 6 = 0$
(B) $4x + y - 44 = 0$
(C) $4x + y - 24 = 0$
(D) None of these

Solution:

Equation of parabola can be written as

$(x-2)^2 = 8(y-4)$ ...(1)

$\Rightarrow 4a = 8 \Rightarrow a = 2$

Slope of tangent = – 4 (given)

$\therefore$ equation of tangent of parabola (1)

$y - 4 = m(x-2) + 2m^2$

$y - 4 = -4(x-2) + 2\cdot 16$

$\Rightarrow y - 4 = -4x + 40 \Rightarrow 4x + y - 44 = 0$

Illustration 26:

For what value of k, the line $4y - x + k = 0$ touches the parabola $x^2 + 8y = 0$

(A) 2 (B) –1/2 (C) –2 (D) 1/2

Solution:

Given parabola $x^2 = -8y$

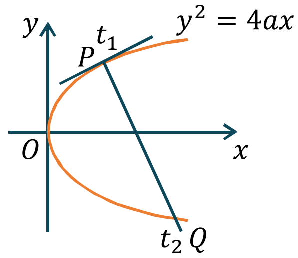
$a = -2$ & tangent $4y - x + k = 0 \Rightarrow y = \frac{x}{4} - \frac{k}{4}$

$\because$ line touches the parabola, then c = $2m^2$

$\Rightarrow -\frac{k}{4} = 2 \cdot \frac{1}{16} \Rightarrow k = -\frac{1}{2}$

Normal to the Parabola (Part-1):

A line perpendicular to the tangent at given point P is called **Normal** at point P.



Slope of Normal = $\frac{-1}{\text{Slope of tangent}} = \frac{-1}{\left(\frac{dy}{dx}\right)}$

(i) Cartesian Form:

$\boxed{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$ at the point (x_1, y_1)

Illustration 27:

Find normal at the point (1,2) to the parabola $y^2 = 4x$.

Solution:

$y - 2 = -\frac{2}{2}(x-1) \Rightarrow x + y - 3 = 0$

Illustration 28:

Find the equation of the normal to the curve $2y = 3 - x^2$ at the point (1, 1)

(A) $x + y + 1 = 0$ (B) $x + y = 0$ (C) $x - y + 1 = 0$ (D) $x - y = 0$

Solution:

Given parabola is $2y = 3 - x^2$ & point $P(1, 1)$

Now $2\left(\frac{dy}{dx}\right) = -2x$

Slope of tangent $\Rightarrow \left(\frac{dy}{dx}\right)_P = -1$

$\therefore$ slope of normal = 1

$\therefore$ equation of normal at (1, 1)

$y - 1 = (x - 1)$

$x - y = 0$

(ii) Parametric form:

Consider the parabola $y^2 = 4ax$.

"At point" $P(t)$ slope of normal" $= -\dfrac{y}{2a} = -t$

The equation of the normal is $(y - 2at) = -t\,(x - at^2)$

$\Rightarrow y + tx = 2at + at^3$ or $at^3 + (2a - x)t - y = 0$

(iii) Slope form:

Replace t by $(-m)$ in parametric form, we get

equation of normal as $y = mx - 2am - am^3$ at $(am^2, -2am)$

Illustration 29:

The equation of the normal to the parabola $y^2 = 16x$ with slope $-1/4$ is-

(A) $x + 4y + 1 = 0$ (B) $4x + 16y = 33$ (C) $4x - 16y = 33$ (D) None of these

Solution:

Parabola is $y^2 = 16x$

$a = 4$

Slope of normal $m = -\dfrac{1}{4}$

equation of normal in slope form $y = mx - 2am - am^3$

$$y = -\frac{1}{4}x + \frac{2.4}{4} + \frac{4}{64}$$

$$\Rightarrow y = -\frac{1}{4}x + 2 + \frac{1}{16}$$

$\Rightarrow 4x + 16y = 33$

Illustration 30:

Identify whether these lines are normal to the $y^2 = 4x$ or not

(A) $x - y - 3 = 0$

(B) $x + y - 3 = 0$

(C) $2x - y - 12 = 0$

Solution:

$a = 1$

Normal : $y = mx - 2am - am^3 \Rightarrow y = mx - 2m - m^3$

(a) $m = -1 \Rightarrow y = x - 2 - 1 \Rightarrow x - y - 3 = 0$

(b) $m = -1 \Rightarrow y = -x + 2 + 1 \Rightarrow x + y - 3 = 0$

(c) $m = 2 \Rightarrow y = 2x - 4 - 8 \Rightarrow 2x - y - 12 = 0$

Illustration 31:

If two normal to a parabola $y^2 = 4ax$ intersect at right angles then the chord joining their feet passes through a fixed point whose co-ordinates are:

(A) $(-2a, 0)$ (B) $(a, 0)$ (C) $(2a, 0)$ (D) $(-a, 0)$

Solution:

$y^2 = 4ax$ ....(i)

$\because$ Slopes of the two normal at the points $P(t_1)$ and $Q(t_2)$are $-t_1$ and $-t_2$ respectively

$\because (-t_1)(-t_2) = -1 \Rightarrow t_1t_2 = -1$

$\because$ equation of chord joining $P(t_1)$ and $Q(t_2)$ is

$2x - y(t_1 + t_2) + 2at_1t_2 = 0$

$\Rightarrow 2x - y\,(t_1 + t_2) - 2a = 0$

$\Rightarrow (2x - 2a) - (t_1 + t_2)(y) = 0$ ...(ii)

(ii) will always pass through $(a, 0)$

Normal to the Parabola (Part-2):

Normal Chord of Parabola:

If the normal to the parabola $y^2 = 4ax$ at the point $P(t_1)$, meets the parabola again at the point $Q(t_2)$, then $\boldsymbol{t_2 = -t_1 - \frac{2}{t_1}}$. Chord PQ is called normal chord.

Proof:

Normal at t_1 is $y = -xt_1 + 2at_1 + at_1^3$

Put $(at_2^2, 2at_2)$ in equation

$\Rightarrow 2at_2 = -at_2^2t_1 + 2at_1 + at_1^3$

$\Rightarrow (t_2 - t_1)(2 + t_1(t_1 + t_2)) = 0$

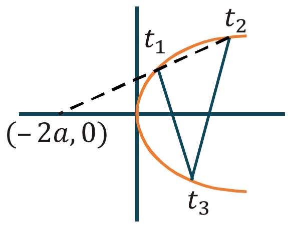
$\Rightarrow \ t_2 = -t_1 - \dfrac{2}{t_1}$

Note:

If the normal to parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point $'t_3'$ then $\boldsymbol{t_1t_2 = 2}$; $\boldsymbol{t_3 = -(t_1 + t_2)}$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

Proof:

$t_3 = -\left(t_1 + \dfrac{2}{t_1}\right) = -\left(t_2 + \dfrac{2}{t_2}\right) \Rightarrow t_1t_2 = 2$



Also $\ t_3 = -\left(t_1 + \dfrac{2}{\frac{2}{t_2}}\right) = -(t_1 + t_2)$

Also equations of chord joining t_1 and t_2 is

$2x - y(t_1 + t_2) + 2at_1t_2 = 0$

$\Rightarrow (2x + 2at_1t_2) - y(t_1 + t_2) = 0$

$\Rightarrow (2x + 4a) - y(t_1 + t_2) = 0$

$\Rightarrow L_1 + \lambda L_2 = 0$

Hence passes through fixed point $(-2a, 0)$

Product of ordinates at t_1 and $t_2 = 2at_1 \times 2at_2 = 8a^2$ and product of abscissa $= at_1 . at_2^2 = 4a^2$

Illustration 32:

If the normal of the parabola $y^2 = 4ax$ drawn at $(4a, -4a)$ meets the parabola again at point $(at^2, 2at)$ then t is equal to-

Solution:

If t' be the parameter of the given point, then $2at' = -4a \Rightarrow t' = -2$

Now $t = -t' - \frac{2}{t'} \Rightarrow t = 2 + \frac{2}{2} = 3$

Illustration 33:

Find the point where the normal at $(1/2, 2)$ meets the parabola $y^2 = 8x$

Solution:

Here $a = 2$ and $2at_1 = 2 \Rightarrow t_1 = 1/2$

$t_2 = -1/2 - 2 \times 2 = -9/2$

Point is $(at_2^2, 2at_2) = \{2(-9/2)^2, 2.2.(-9/2)\} = (81/2, -18)$

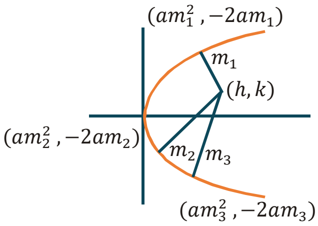
Equation of Normal from any Point and its Analysis:

(i) Take the equation of normal in slope form $y = mx - 2am - am^3$

(ii) Put the given point (h, k)

(iii) Form cubic in m

$am^3 + (2a - h)m + k = 0$ ...(1)



(iv) Solve for m and find the equation of normal.

(v) From equation (1)

$m_1 + m_2 + m_3 = 0$

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$

$m_1m_2m_3 = -\frac{k}{a}$

Where m_1, m_2 & m_3 are the slopes of the three concurrent normals.

- **Conormal points** : Foot of the normals of three concurrent normals are called conormals point.
- From any point at the most three normals can be drawn to a parabola.
- From any point atleast one real normal can be drawn.
- Algebraic sum of the slopes of three concurrent normals is zero.
- Sum of ordinates of the three conormal points on the parabola is zero.

  $\because\ -2am_1 - 2am_2 - 2am_3 = 0$
- Centroid of the triangle formed by three co-normal points lies on the axis of parabola.

  Centroid $\equiv \left(\frac{\Sigma am_1^2}{3}, \frac{-2a\Sigma m_1}{3}\right)$

  Hence y co $-$ ordinate $= 0$

Illustration 34:

Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by-

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

Solution:

Equation of normal is $y = mx - 2m - m^3$

It passes through the point $(9, 6)$ then

$6 = 9m - 2m - m^3 \Rightarrow m^3 - 7m + 6 = 0 \Rightarrow (m-1)(m-2)(m+3) = 0$

$\Rightarrow m = 1, 2, -3$

Equations of normals are $y - x + 3 = 0, y + 3x - 33 = 0$ & $y - 2x + 12 = 0$

Illustration 35:

If two of the normal of the parabola $y^2 = 4x$, that pass through (15, 12)are $4x + y = 72$, and $3x - y = 33$, then the third normal is-

(A) $y = x - 3$ (B) $x + y = 3$ (C) $y = x + 3$ (D) None of these

Ans. (A)

Solution:

Here, If m_1, m_2, m_3 are slopes of normal, then

$m_1 + m_2 + m_3 = 0$ and $m_1m_2m_3 = \frac{y_1}{a}$

$a = 1$ here $m_1 = -4, m_2 = 3$

$\therefore -4 + 3 + m_3 = 0 \Rightarrow m_3 = 1$

Also $(-4)(3)(1) = -\frac{12}{1}$ is satisfied. But $(15, 12)$ satisfies $y = x - 3$

Illustration 36:

If a normal chord of $y^2 = 4ax$ subtends an angle $\pi/2$ at the vertex of the parabola, then its slope is equal to -

(A) ± 1 (B) $\pm\sqrt{2}$ (C) ± 2 (D) None of these

Ans. (B)

Solution:

Let AB be a normal chord, where $A \equiv (at_1^2, 2at_1)$ and $B \equiv (at_2^2, 2at_2)$.

We have $t_2 = -t_1 - \dfrac{2}{t_1}$ and $t_1t_2 = -4$

$\Rightarrow t_1t_2 = -t_1^2 - 2 = -4 \;\; t_1^2 = 2.$

Now slope of chord $AB = \dfrac{2}{t_1+t_2} = -t_1 = \pm\sqrt{2}$

Illustration 37:

If three distinct and real normals can be drawn to $y^2 = 8x$ from the point $(a, 0)$ then -

(A) $a > 2$ (B) $a > 4$ (C) $a \in (2, 4)$ (D) None of these

Solution:

Equation of normal in terms of m is $y = mx - 4m - 2m^3$. If it passes through $(a, 0)$ then

$am - 4m - 2m^3 = 0$

$\Rightarrow m(a - 4 - 2m^2) = 0$

$\Rightarrow m = 0, m^2 = \dfrac{a-4}{2}$.

For three distinct normals, $a - 4 > 0$

$\Rightarrow a > 4$

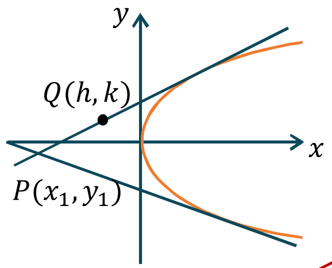
Pair of Tangent:

The equation of the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by: $\boldsymbol{SS_1 = T^2}$ where :

$S \equiv y^2 - 4ax$;

$S_1 \equiv y_1^2 - 4ax_1$;

$T \equiv yy_1 - 2a(x + x_1)$.



Pair of perpendicular tangents:

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **director circle.** It's equation is $\boldsymbol{x + a = 0}$ which is parabola's own directrix.

Let $p(x_1, y_1)$ be the point from which tangents drawn to the parabola $y^2 = 4ax$ are perpendicular

$\Rightarrow (y^2 - 4ax)(y_1^2 - 4ax_1) = \left(yy_1 - 2a(x + x_1)\right)^2$

$\Rightarrow x_1 + a = 0$

$\Rightarrow P(x_1, y_1)$ lies on $x + a = 0$

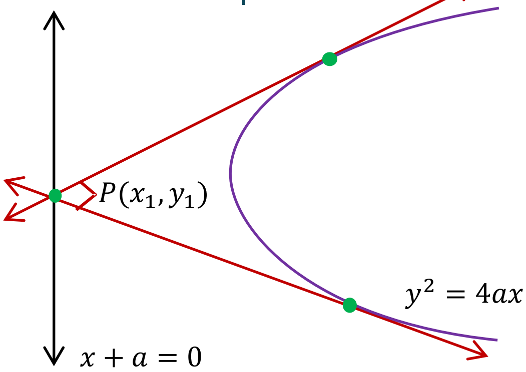


Illustration 38:

Find the equation of pair of tangents to the parabola $y^2 = 8x$ drawn from a point $P(-1, 3)$.

Solution:

We know the equation of pair of tangents are given by $SS_1 = T^2$

$\therefore (y^2 - 8x)(9 + 8) = (2y - 4(x - 1))^2$

$\Rightarrow 17y^2 - 136x = 4y^2 + 16x^2 + 16 - 16xy + 16y - 32x$

$\Rightarrow 16x^2 - 17y^2 - 16xy + 104x + 16y - 16 = 0$

Illustration 39:

The angle between the tangents drawn from a point $(-a, 2a)$ to $y^2 = 4ax$ is -

(A) $\pi/4$ (B) $\pi/2$ (C) $\pi/3$ (D) $\pi/6$

Solution:

The given point $(-a, 2a)$ lies on the directrix $x = -a$ of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle.

Chord of Contact:

Consider the parabola $y^2 = 4ax$ and point $T(x_1, y_1)$.

Tangents are drawn from point $T(x_1, y_1)$ to the parabola.

PQ is called chord of contact.

Equation of the chord of contact of pair of tangents from (x_1, y_1) is

$\boldsymbol{T = 0} \Rightarrow yy_1 = 2a(x + x_1)$.

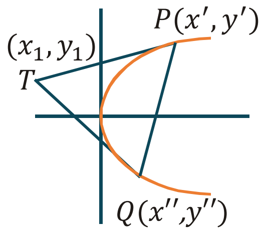


Illustration 40:

If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution:

Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0$ ...(i)

But given line is

$x - y - 1 = 0$ ...(ii)

Comparing (i) and (ii)

$\therefore \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \Rightarrow h = -1, k = 4$

$\therefore$ point $\equiv (-1, 4)$

Illustration 41:

From point $(4, k)$ two tangents are drawn to the parabola $y^2 = 12x$. If slope of chord of contact is 3, then $k = ?$

Solution:

Equation of chord of contact is $T = 0, x_1 = 4, \ y_1 = k$

$yk = 6(x + 4) \Rightarrow$ slope of this line is $\frac{6}{k}$

$\Rightarrow \frac{6}{k} = 3$

$\Rightarrow k = 2$

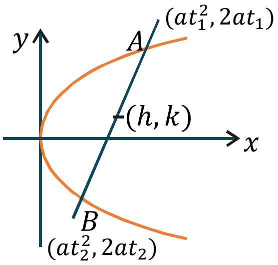
Chord with a Given Middle Point:

General method for finding the equation of a chord of any conic with middle point (h, k) is $\boldsymbol{T = S_1}$.

Consider parabola $y^2 = 4ax$, (h, k) is mid-point of chord AB.

Equation of AB

$y - k = m(x - h)$ ...(i)



But $m_{AB} = \dfrac{2}{t_1 + t_2}$

also $2k = 2a(t_1 + t_2)$

$(t_1 + t_2) = \dfrac{k}{a}$

$\therefore m_{AB} = \dfrac{2a}{k}$

Hence equation of a chord whose mid-point is (h, k)

$y - k = \dfrac{2a}{k}(x - h)$ ...(ii)

or $4ah + ky - k^2 = 2a(x - h) + 4ah = 2a(x + h)$

$\underbrace{ky - 2a(x+h)}_{T} = \underbrace{k^2 - 4ah}_{S_1}$

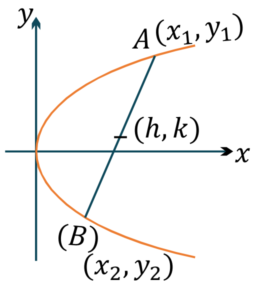
Illustration 42:

For parabola $y^2 = 8x$, find equation of chord whose mid-point is $(h, k) \equiv (2, -3)$

Solution:

The equation of AB is

$y + 3 = m(x - 2)$ ...(i)



now $\quad y_1^2 = 4ax_1$

and $\quad y_2^2 = 4ax_2$

————————

$y_1^2 - y_2^2 = 4a(x_1 - x_2)$

or $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{4a}{y_1 + y_2} = m$

but $y_1 + y_2 = 2k = -6$ and $4a = 8$

$\therefore m = \dfrac{8}{-6} = -\dfrac{4}{3}$

Hence equation of $AB = y + 3 = -\dfrac{4}{3}(x - 2) \Rightarrow 4x + 3y + 1 = 0$

OR

By formula $T = S_1$

$\Rightarrow -3y - 4(x + 2) = (-3)^2 - 8.2$

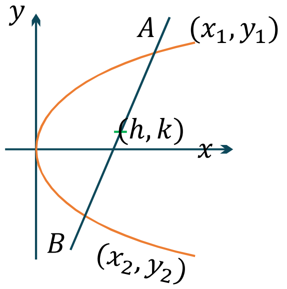
$\Rightarrow 4x + 3y + 1 = 0$

Illustration 43:

Find the mid-point of given chord $4x + 3y + 1 = 0$ of parabola $y^2 = 8x$.

Solution:

Let the mid-point of chord be (h, k)



Here, $m = -\frac{4}{3} = \frac{4a}{y_1 + y_2}$

$\Rightarrow \quad -\frac{4}{3} = \frac{8}{2k} \Rightarrow \quad k = -3$

Since, $4h + 3k + 1 = 0$

$\Rightarrow \quad 4h - 9 + 1 = 0 \Rightarrow h = 2$

hence, mid-point is $(2, -3)$

OR

By formula $T = S_1$

$\Rightarrow ky - 4(x + h) = k^2 - 8h$

$\Rightarrow 4x - ky + k^2 - 4h = 0$

Compare with $4x + 3y + 1 = 0$

$\Rightarrow k = -3$ and $h = 2$

Hence, mid-point is $(2, -3)$

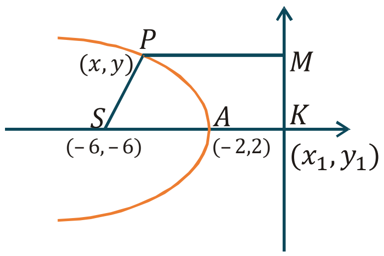
JEE-Advanced (Part-1):

Illustration 44:

Find the equation of the parabola whose focus is $(-6, -6)$ and vertex $(-2, 2)$.

Solution:

Let $S(-6, -6)$ be the focus and $A(-2, 2)$ is vertex of the parabola. On SA take a point $K(x_1, y_1)$ such that $SA = AK$. Draw KM perpendicular on SK. Then KM is the directrix of the parabola. Since A bisects SK,



$\left(\frac{-6 + x_1}{2}, \frac{-6 + y_1}{2}\right) = (-2, 2)$

$\Rightarrow -6 + x_1 = -4$ and $-6 + y_1 = 4$ or $(x_1, y_1) = (2, 10)$

Hence the equation of the directrix KM is

$y - 10 = m(x - 2)$ ...(i)

Also gradient of $SK = \frac{10 - (-6)}{2 - (-6)} = \frac{16}{8} = 2; \Rightarrow m = \frac{-1}{2}$

$y - 10 = \frac{-1}{2}(x - 2)$ (from (i))

$\Rightarrow \quad x + 2y - 22 = 0$ is the directrix

Next, let PM be a perpendicular on the directrix KM from any point $P(x, y)$ on the parabola.

From $SP = PM$, the equation of the parabola is $\sqrt{\left\{(x+6)^2 + (y+6)^2\right\}} = \frac{|x + 2y - 22|}{\sqrt{(1^2 + 2^2)}}$

or $5(x^2 + y^2 + 12x + 12y + 72) = (x + 2y - 22)^2$

or $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$ or $(2x - y)^2 + 104x + 148y - 124 = 0$.

Illustration 45:

The focal chord to $y^2 = 64x$ is tangent to $(x-4)^2 + (y-2)^2 = 4$, then the possible values of the slope of this chord is -

(A) $\left\{0, -\frac{12}{35}\right\}$ (B) $\left\{0, \frac{12}{35}\right\}$ (C) $\left\{0, \frac{35}{12}\right\}$ (D) $\left\{0, -\frac{35}{12}\right\}$

Solution:

Equation of focal chord is $y = m(x-16)$

$\therefore\ 2 = \frac{|4m-16m-2|}{\sqrt{1+m^2}}$

$\Rightarrow 4(1+m^2) = (-12m-2)^2$

$\Rightarrow\ m = 0\ \&\ \frac{-12}{35}$

Illustration 46:

The area of circle touching parabola $y = x^2$ at $(1,1)$ and having directrix of $y = x^2$ as its normal is $125A\pi$, then A is -

(A) $\frac{1}{64}$ (B) $\frac{1}{16}$ (C) $\frac{1}{8}$ (D) $\frac{1}{4}$

Solution:

Equation of normal at (1,1) is

$y - 1 = -1/2(x-1)$

$\therefore\ 2y + x = 3$ ...(i)

Equation of directrix to $y = x^2$ is

$y = -\frac{1}{4}$ ...(ii)

$\therefore$ Centre $\equiv \left(\frac{7}{2}, -\frac{1}{4}\right)$, Radius = $\frac{5\sqrt{5}}{4}$

$\therefore$ Area = $\frac{125\pi}{16}$

Illustration 47:

The curve $(y-a)^2 = bx - bc - 2$, where a, b, c are consecutive integers, passes through $(2c, 2a)$. If tangent to the curve at a point P passes through $(2, 0)$, then the normal at P may pass through

(A) $\left(-\frac{21}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 0\right)$ (C) $\left(\frac{5}{2}, 0\right)$ (D) $\left(\frac{13}{2}, 0\right)$

Solution:

$a^2 = bc - 2$

Case I :

$a = b - 1, c = b + 1 \Rightarrow b = 1, a = 0, c = 2$

$y^2 = x - 4$

we know that focus is mid-point of line segment joining points where tangent and normal at a point cut the axis of parabola.

$(2, 0)$ lies on axis.

Case-II :

$a = b + 1, c = b - 1$

$\Rightarrow a = 0, b = -1, c = -2$

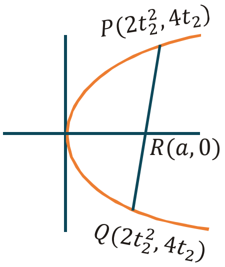
$y^2 = -x - 4$

Illustration 48:

The line $2(x-a) + cy = 0$ cuts the parabola $y^2 = 8x$ at $P(2t_1^2, 4t_1)$ and $Q(2t_2^2, 4t_2)$. If a $\in$ [2,4] and $c \in R$ then $t_1 t_2$ belongs to -

(A) $[-2, -1]$ (B) $[-4, -2]$ (C) $[-4, -3]$ (D) $[-3, -2]$

Solution:



Line $2(x-a) + cy = 0$ always passes through $(a, 0)$ slope of PR = slope of QR

$$\frac{4t_1}{2t_1^2 - a} = \frac{4t_2}{2t_2^2 - a} \Rightarrow t_1 t_2 = -\frac{a}{2}$$

$\because a \in [2,4] \Rightarrow t_1 t_2 \in [-2, -1]$

Illustration 49:

A parabola passes through $(1, 2)$ and $(3, 4)$. The tangents drawn at these points intersect at $(-6, 8)$, then the slope of directrix of the parabola is

(A) $\frac{5}{8}$ (B) $-\frac{5}{8}$ (C) $\frac{8}{5}$ (D) $-\frac{8}{5}$

Solution:

As the line joining the mid-point of the chord joining $(1, 2)$, $(3, 4)$ and the point of intersection of tangents at these points is parallel to axis of parabola

$\therefore$ Slope of axis = $\frac{8-3}{-6-2} = -\frac{5}{8}$

$\therefore$ Slope of directrix = $\frac{8}{5}$

Important Highlights (JEE-Advanced) (Part-2):

Alternate Definition of Parabola

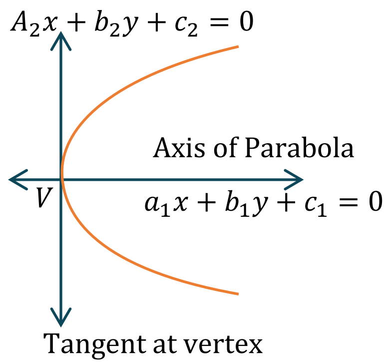
Axis of Parabola $a_1x + b_1y + c_1 = 0$

Tangent at vertex $a_2x + b_2y + c_2 = 0$

Length of latus-rectum ℓ

The equation of parabola is

$$\frac{(a_1x + b_1y + c_1)^2}{a_1^2 + b_1^2} = \ell \frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$



X-Intercepts of Tangent & Normal

Tangent $ty = x + at^2$

Normal $y = -tx + 2a + at^3$

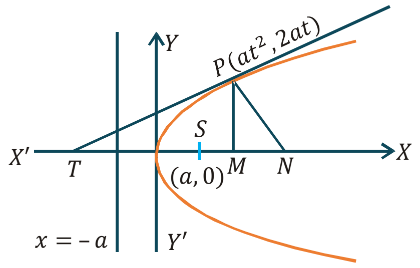
$T(-at^2, 0) N(2a + at^2, 0) M\ (at^2, 0)$

$TM\ =\ 2at^2\ MN = 2a$

Midpoint of NT is $(a, 0)$

$ST = SN = SP$

Tangent at $P(t)$ intersects $y-$ axis at $(0, at)$.

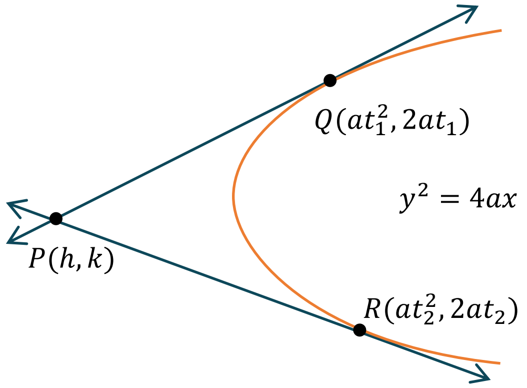


Point of Intersection of Two Tangents

Equation of tangent at Q is $t_1 y = x + at_1^2$

Equation of tangent at R is $t_2 y = x + at_2^2$

$\Rightarrow\ h = at_1t_2\ \&\ k = a(t_1 + t_2)$



Properties of Tangents

In any parabola, foot of perpendicular form focus upon any tangents lies on the tangent at vertex.

In any parabola image of focus in any tangent lies on the directrix.

Length of tangent between the point of contact and the directrix subtends right angle at focus.

Illustration 50:

A tangent is drawn to the parabola $y^2 = 4x$ at the point 'P' whose abscissa lies in the interval [1, 4]. Maximum possible area of the triangle formed by the tangent at 'P', ordinate of the point 'P' and the x –axis is A. Evaluate $\left(\frac{A}{4}\right)$.

Solution:

Equation of tangent to parabola at $P(t)$ is given by

$ty = x + t^2, \tan\theta = \frac{1}{t}$

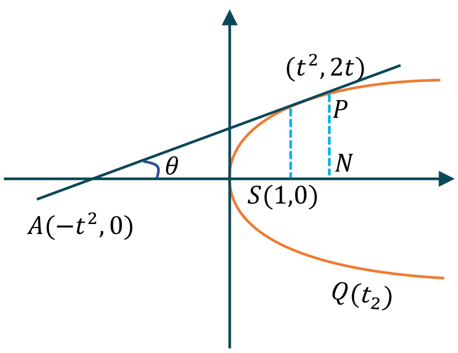
$\therefore$ Area of $\Delta APN = \Delta = \frac{1}{2}\ (AN)(PN)\ =\ \frac{1}{2}\ (2t^2)(2t)$

$$\Delta = 2t^3\ =\ 2(t^2)^{3/2}$$

$\because t^2 \in [1, 4]\ \Rightarrow \Delta_{max}$ when $t^2 = 4$

$\Rightarrow \Delta_{max} = 16$

The maximum area of Δ is 16 square units



$\left(\frac{A}{4}\right) = 4$

Illustration 51:

$y + x = 0$ & $y - x = 0$ are tangents at the vertex & axis of a parabola respectively and $y = 2x + 3$ is a tangent to the same parabola, the coordinate of whose focus is (a, b), then $6(a + b)$ is equal to

Solution:

Intersection point of tangent at vertex & give tangent is $(-1,1)$. perpendicular from focus on given tangent meets it at tangent at vertex.

$\Rightarrow$ focus is the intersection point of $y = x$ & $y-1=-\frac{1}{2}(x+1)$

$\Rightarrow$ coordinate of focus is $\left(\frac{1}{3},\frac{1}{3}\right)$

$\Rightarrow 6(a + b) = 6\times\frac{2}{3}=4$.

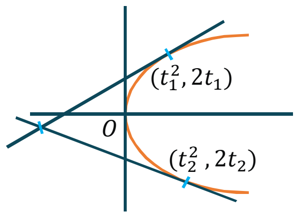
JEE-Advanced (Part-3):

Illustration 52:

If the locus of the point of intersection of two tangents to a parabola $y^2 = 4x$ at the points whose ordinates have the constant ratio 4:1 can be expressed as $y^2 = \frac{p}{q}x$ (where p and q are integers in lowest form), then $(p + q)$ is

Solution:

Given $\frac{2t_1}{2t_2}=\frac{4}{1}$



$\Rightarrow t_1 = 4t_2$

Let point of intersection be (h, k)

$\therefore h = t_1t_2 k = t_1 + t_2$

$\Rightarrow h = 4t_2^2 \Rightarrow k = 5t_2$

$\therefore h=4.\frac{k^2}{25} \Rightarrow y^2=\frac{25}{4}x$

$\therefore p + q = 25 + 4 = 29$

Illustration 53:

Let $y = 3x - 8$ be the equation of tangent at the point (7,13) lying on a parabola whose focus is at (-1,-1). If the latus rectum of the parabola is ℓ, then the value of $[\ell^2]$, (where [.] denote greatest integer function), is

Solution:

Image of focus$(-1,-1)$ upon the tangent $y = 3x - 8$ is the point $(5,-3)$ and it will lie on directrix.

Slope of directrix $=-\frac{7-5}{13+3}=-\frac{1}{8}$. It's equation is $y+3=-\frac{1}{8}(x-5)$.

$\Rightarrow x + 8y + 19 = 0$. $I=2\times\frac{|-1-8+19|}{\sqrt{65}}=\frac{20}{\sqrt{65}}$

Illustration 54:

If the shortest distance of (0,3) from parabola $y = x^2$ is $\frac{\sqrt{p}}{q}$ (where p and q are coprime numbers), then $(p - q)$ is equal to

Solution:

Any point on parabola (x, x^2)

Distance between (x, x^2) and $(0, 3)$ is

$$D=\sqrt{x^2+(x^2-3)^2}$$

$$D^2=\left(x^2-\frac{5}{2}\right)^2+3-\frac{1}{4} \text{ is minimum if}$$

$$x^2-\frac{5}{2}=0$$

$$D_{\min}=\sqrt{\frac{11}{4}}=\frac{\sqrt{11}}{2}$$

$p-q=9$

Illustration 55:

TP and TQ are tangents of the parabola $y^2=8x$ at P and Q respectively. If the chord PQ passes through the point (-2,3) and locus of point T is $y=mx+c$ then $(m+c)$ is equal to -

(A) 0 (B) $\frac{16}{3}$ (C) $-\frac{4}{3}$ (D) $-\frac{8}{3}$

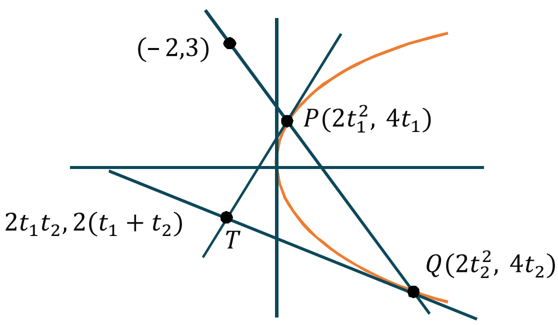
Solution:

$\frac{h}{2}=t_1t_2$ & $\frac{k}{2}=t_1+t_2$ ...(i)

Equation of PQ is $y-4t_1=\frac{4(t_2-t_1)}{2(t_2^2-t_1^2)}(x-2t_1^2)$

Passes through $(-2, 3)$

$\therefore$ Locus of T is $y=\frac{4}{3}x-\frac{8}{3}$ (using (i))



Aliter :

Equation of the locus is the pole of $(-2, 3)$

w.r.t. $y^2=8x$

$\Rightarrow 3y=4(x-2)$

$\Rightarrow 3y+8=4x$

Illustration 56:

Let the locus of centroid of triangle formed by focal chord and tangent drawn at extremities of focal chord of parabola $y^2=4x$ be a conic S then latus rectum of S is -

(A) 8 (B) 3 (C) 4 (D) 16

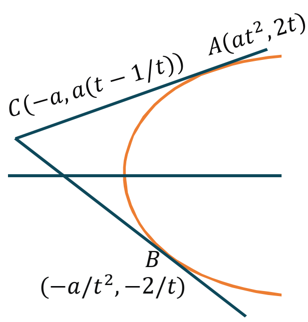
Solution:

For focal chord $t_1t_2=-1$

Coordinates of $G\left(\frac{t^2+\frac{1}{t^2}+2t-\frac{2}{t}-1}{3}, t-\frac{1}{t}\right)$

$\Rightarrow 3x=t^2+\frac{1}{t^2}+2\left(t-\frac{1}{t}\right)-1$

& $y=t-\frac{1}{t}$



$$y^2 = t^2 + \frac{1}{t^2} - 2$$

$\Rightarrow 3x = y^2 + 2 + 2y - 1$

$\Rightarrow (y+1)^2 = 3x$

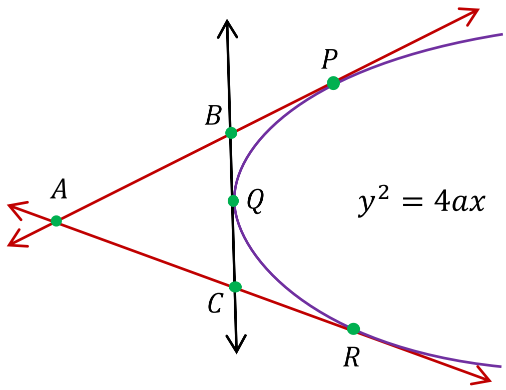
$\Rightarrow$ Length of latus rectum = 3

Important Highlights (JEE-Advanced) (Part-4):

Area of the Triangle formed by Three Tangents

The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

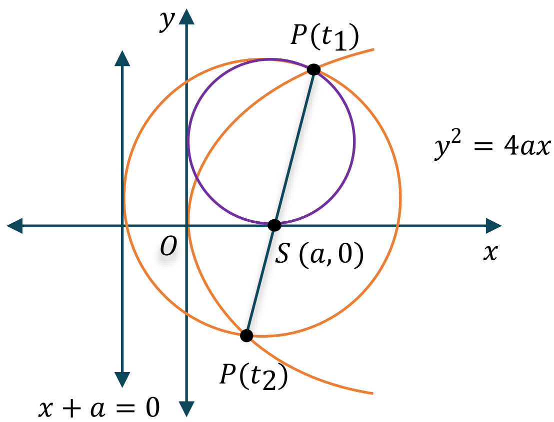
$$\frac{\text{Area of } \Delta(\text{PQR})}{\text{Area of } \Delta(\text{ABC})} = 2$$



Focal Chord and Its Properties

In any parabola circle described on focal length as diameter touches the tangent at vertex.

In any parabola circle described on focal chord as diameter touches the directrix.



Properties of Normal

(1) Normal other than axis parabola never passes through the focus Consider the parabola $y^2 = 4ax$

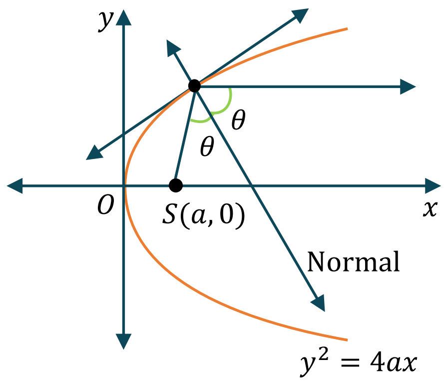
Normal at any point $P(t)$ is $y = -tx + 2at + at^3$

If this normal passes through focus $S(a, 0)$, then

$0 = -at + 2at + at^3$

$\Rightarrow \ t^2 + 1 = 0$

(2) For the parabolic mirror if incident ray passes through focus then the reflected ray is parallel to the axis of the parabola.



(3) Normal at any point bisects the angle between the focal chord and line passing through it perpendicular to the directrix.

Illustration 57:

Circles drawn with diameter as extremities of any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line. Find the equation of line.

Solution:

$y^2 - 4x - y - 4 = 0$

or $y - y + \frac{1}{4} = 4x + \frac{17}{4}$

or $\left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$

Circles drawn with diameter as extremities of any focal chord of the parabola always touches the directrix of the parabola. Thus, the circle will touch the line.

$$x + \frac{17}{16} = -1$$

Or $16x + 33 = 0$

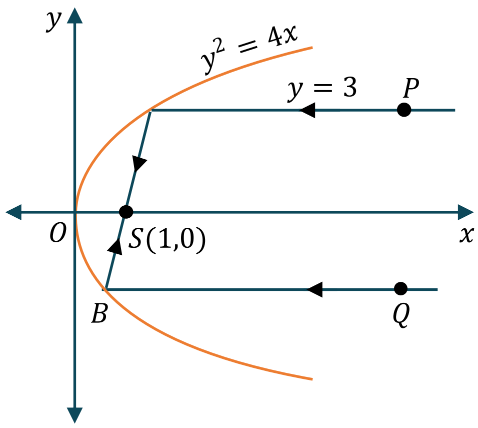
Illustration 58:

A parabolic mirror is kept along $y^2 = 4x$ and two light rays, parallel to its axis, are reflected along one straight line. If one of the incident light rays is at 3 units distance from the axis, then find the distance of the other incident ray from axis.

Solution:

We have parabola $y^2 = 4x$.

Let two incident rays AP and BQ, which are parallel to axis of the parabola, strikes the parabola at points A and B, respectively. After reflection, both the rays pass through the focus $S(1, 0)$



Therefore, AB is focal chord.

Let point A be $(t^2, 2t)$

Distance of ray AP is 3 units from axis.

$\therefore\ 2t = 3$

or $t = \frac{3}{2}$

Coordinates of point B are $\left(\frac{1}{t^2}, \frac{-2}{t}\right)$ (other end of focal chord).

Distance of B from the axis $= \frac{2}{t} = \frac{4}{3}$.