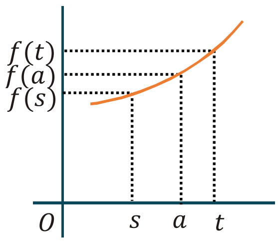
03 Monotonicity

Monotonicity (Introduction):

Increasing / Decreasing / Strictly increasing / Strictly Decreasing Nature of a Function at a Point:

I. Increasing at $x = a$:



If $f(s) \leq f(a) \leq f(t)$ whenever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be increasing at $x = a$.

(1) When $'a'$ be left end of the interval

$f(a) \leq f(x)\ \forall x \in (a, a + h) \cap D_f$ for some $h > 0$

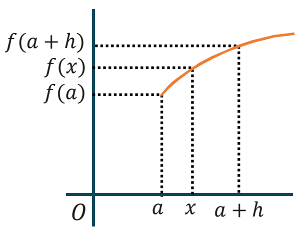
$\Rightarrow f$ is increasing at $x = a$.

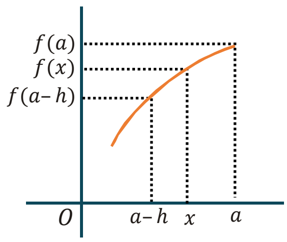
(2) When $'a'$ be right end of the interval

$f(x) \leq f(a) \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow f$ is increasing at $x = a$.

II. Strictly increasing at $x = a$:





If $f(s) < f(a) < f(t)$ when ever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be strictly increasing at $x = a$.

(1) When $'a'$ be left end of the interval

$f(a) < f(x) \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow f$ is strictly increasing at $x = a$.

(2) When $'a'$ be right end of the interval

$f(x) < f(a) \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow f$ is strictly increasing at $x = a$.

III. Decreasing at $x = a$:

If $f(s) \geq f(a) \geq f(t)$ when ever $s < a < t$, where $s, t \in (a-h, a+h) \cap D_f$ for some $h > 0$, then f is said to be decreasing at $x = a$.

(1) When $'a'$ be left end of the interval

$f(a) \geq f(x)\ \forall\ x \in (a, a+h) \cap D_f$ for some $h > 0$

$\Rightarrow f$ is decreasing at $x = a$.

(2) When $'a'$ be right end of the interval

$f(x) \geq f(a)\ \forall\ x \in (a-h, a) \cap D_f$ for some $h > 0$

$\Rightarrow f$ is decreasing at $x = a$.

IV. Strictly decreasing at $x = a$:

If $f(s) > f(a) > f(t)$ when ever $s < a < t$, where $s, t \in (a-h, a+h) \cap D_f$ for some $h > 0$, then f is said to be strictly decreasing at $x = a$.

(1) When $'a'$ be left end of the interval

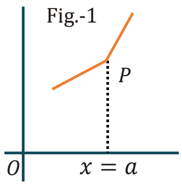
$f(a) > f(x) \forall\ x \in (a, a+h) \cap D_f$ for some $h > 0$

$\Rightarrow f$ is strictly decreasing at $x = a$.

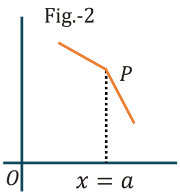
(2) When $'a'$ be right end of the interval

$f(x) > f(a) \forall\ x \in (a-h, a) \cap D_f$ for some $h > 0$

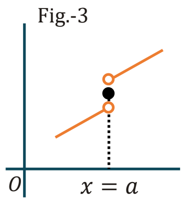
$\Rightarrow f$ is strictly decreasing at $x = a$.



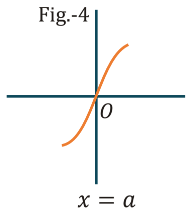
Strictly increasing at $x = a$



Strictly decreasing at $x = a$



Strictly increasing at $x = a$



Strictly increasing at $x = a$

Illustration 1:

Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points $x = 0, 1$ & 2.

Solution:

at $x = 0$,

$f(0) = 2, f(0+h) = h(h-3) + 2 < 2$

$f(0-h) = h(3-h) + 2 > 2$

$\Rightarrow f(0-h) > f(0) > f(0+h)$

f is strictly decreasing at $x = 0$.

at $x = 1, f(1-h) > f(1) < f(1+h)$ neither increasing nor decreasing

Similarly, at $x = 2, f(2-h) < f(2) < f(2+h)$ increasing at $x = 2$.

Increasing Decreasing Nature of a Function in an Interval:

Consider an interval $I \subseteq D_f$

I. Increasing Over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$, then f is increasing over interval I.

II. Decreasing Over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$, then f is decreasing over interval I.

III. Strictly increasing over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$, then f is strictly increasing over interval I.

IV. Strictly decreasing over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Leftrightarrow f(x_1) > f(x_2)$, then f is strictly decreasing over interval I.

Monotonic Function:

If a function is either increasing or decreasing over an interval then it is said to be monotonic function over the interval.

If a function is either strictly increasing or strictly decreasing over an interval then it is said to be strictly monotonic function over the interval.

Monotonicity of Differentiable Functions:

For Differentiable Functions:

Consider an interval I ($\subseteq D_f$) that can be $[a,b]$ or (a,b) or $[a,b)$ or $(a,b]$.

(1) $f'(x) > 0 \ \forall \ x \in \ I \ \Rightarrow \ f$ is strictly increasing function over the interval I.

(2) $f'(x) \geq 0 \ \forall \ x \in \ I \Rightarrow f$ is increasing function over the interval I.

(3) $f'(x) > 0 \ \forall \ x \in \ I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points)
$\Rightarrow \ f$ is strictly increasing function over the interval I.

(4) $f'(x) < 0 \ \forall \ x \in \ I \Rightarrow f$ is strictly decreasing function over the interval I.

(5) $f'(x) < 0 \ \forall \ x \in \ I \ \Rightarrow \ f$ is decreasing function over the interval I.

(6) $f'(x) \leq \ 0 \ \forall \ x \in \ I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points)
$\Rightarrow \ f$ is strictly decreasing function over the interval I.

Illustration 2:

Prove that the function $f(x)=\log\left(x^3+\sqrt{x^6+1}\right)$ is strictly increasing.

Solution:

Now, $f(x)=\log\left(x^3+\sqrt{x^6+1}\right)$

$$f'(x)=\frac{1}{x^3+\sqrt{x^6+1}}\left(3x^2+\frac{6x^5}{2\sqrt{x^6+1}}\right)=\frac{3x^2}{\sqrt{x^6+1}}\geq 0$$

$\Rightarrow f(x)$ is strictly increasing.

Examples using Monotonicity of Differentiable Functions:

Illustration 3:

The function $f(x)=\tan x-x$

(A) always increases
(B) always decreases
(C) never decreases
(D) sometimes increases and sometimes decreases

Ans. (A)

Solution:

$f(x)=\tan x-x$

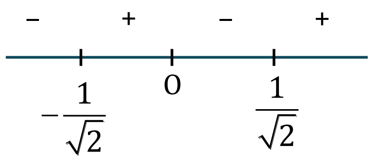
$f'(x) = \sec^2 x - 1 > 0 \Rightarrow$ Always increases.

Illustration 4:

Find the intervals of monotonicity of the function $y=x^2-\log_e|x|$, $(x\neq 0)$.

Solution:

Let $y = f(x) = x^2 - \log_e|x|$



$f'(x) = 2x - \frac{1}{x}$; for all $x(x \neq 0)$

$f'(x) = \frac{2x^2 - 1}{x} \Rightarrow f'(x) = \frac{(\sqrt{2}x - 1)(\sqrt{2}x + 1)}{x}$

So $f'(x) \geq 0$ when $x \in \left[-\frac{1}{\sqrt{2}}, 0\right) \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and $f'(x) \leq 0$

when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left(0, \frac{1}{\sqrt{2}}\right]$

$\therefore f(x)$ is strictly increasing when $x \in \left[-\frac{1}{\sqrt{2}}, 0\right); \left[\frac{1}{\sqrt{2}}, \infty\right)$

and strictly decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right]; \left(0, \frac{1}{\sqrt{2}}\right]$

Illustration 5:

If $g(x) = f(x) + f(1 - x)$ and $f''(x) < 0$; $0 \leq x \leq 1$, show that $g(x)$ strictly increasing in $x \in (0, 1/2)$ and strictly decreasing in $x \in (1/2, 1)$

Solution:

$\therefore f''(x) < 0 \Rightarrow f'(x)$ is strictly decreasing function.

Now, $g(x) = f(x) + f(1 - x)$

$\therefore g'(x) = f'(x) - f'(1 - x)$ ...(i)

Case I : If $x > (1 - x) \Rightarrow x > 1/2$

$\therefore f'(x) < f'(1 - x) \Rightarrow f'(x) - f'(1 - x) < 0$

$\Rightarrow g'(x) < 0$

$\therefore g(x)$ strictly decreases in $x \in \left(\frac{1}{2}, 1\right)$

Case II : If $x < (1 - x) \Rightarrow x < 1/2$

$\therefore f'(x) > f'(1 - x) \Rightarrow f'(x) - f'(1 - x) > 0 \Rightarrow g'(x) > 0$

$\therefore g(x)$ strictly increases in $x \in (0, 1/2)$

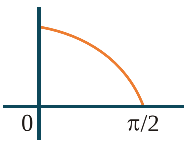
Illustration 6:

Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

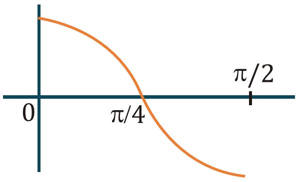
(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

Ans. (A,B)

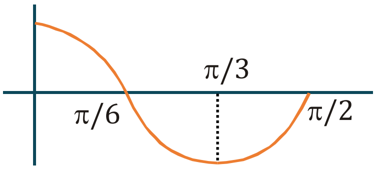
Solution:



$f(x) = \cos x$



$f(x) = \cos 2x$



$f(x) = \cos 3x$

Non-monotonic

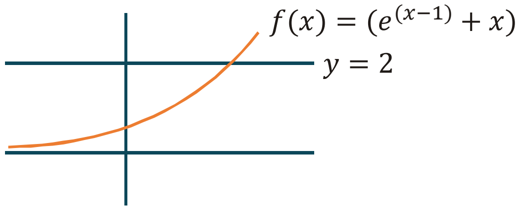
$f(x) = \tan x$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

Option A & B are correct.

Illustration 7:

Prove that the equation $e^{(x-1)} + x = 2$ has one solution.

Solution:



Let $f(x) = e^{(x-1)} + x$

$f'(x) = e^{(x-1)} + 1$

$f(x)$ is strictly increasing funtion

$\lim\limits_{x\to\infty} f(x) = \infty$ & $\lim\limits_{x\to-\infty} f(x) = 0$

$f(x) = 2$ has exactly one solution.

Illustration 8:

The function f, defined by $f(x) = (x+2)e^{-x}$ is -

(A) decreasing for all x

(B) decreasing in $(-\infty, -1)$ and increasing $(-1, \infty)$ in

(C) increasing for all x

(D) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

Solution:

$f(x) = (x+2)e^{-x}$

$f'(x) = (x+2)e^{-x}(-1) + e^{-x} = -e^{-x}(x+1)$

Increasing it $f'(x) > 0 \Rightarrow -e^{-x}(x+1) > 0$

$\Rightarrow (x+1) < 0$ [$\because e^{-x}$ always positive]

$x \in (-\infty, -1)$ & decreasing in $(-1, \infty)$ as $f'(x) < 0$.

Illustration 9:

Function $f(x) = x^3 + 6x^2 + (9+2k)x + 1$ is increasing function if

(A) $k \geq \frac{3}{2}$ (B) $k > \frac{3}{2}$ (C) $k < \frac{3}{2}$ (D) $k \leq \frac{3}{2}$

Ans. (A)

Solution:

$f(x) = x^3 + 6x^2 + (9+2k)x + 1$

$f'(x) = 3x^2 + 12x + (9+2k)$

$\Rightarrow 3x^2 + 12x + (9+2k) \geq 0 \ \forall \ x \in R \Rightarrow D \leq 0$

$\Rightarrow 12.12 - 12(9+2k) \leq 0$

$3 - 2k \leq 0$

$\Rightarrow k \geq \frac{3}{2}$.

Illustration 10:

The function $f(x) = \cos x - 2px$ is monotonically decreasing for

(A) $p < \frac{1}{2}$ (B) $p > \frac{1}{2}$ (C) $p < 2$ (D) $p > 2$

Ans. (A)

Solution:

$f(x) = \cos x - 2px$

$f'(x) = -\sin x - 2p$

For monotonic decreasing $f'(x) < 0$

$\Rightarrow 2p > -\sin x$

$\Rightarrow p < \frac{1}{2}$ $[\because -1 \le \sin x \le 1]$

Illustration 11:

The value of $'a'$ for which the function $f(x) = \sin x - \cos x - ax + b$ decreases for all real values of x, is

(A) $a \ge -\sqrt{2}$ (B) $a \le -\sqrt{2}$ (C) $a \le \sqrt{2}$ (D) $a \ge \sqrt{2}$

Ans. (D)

Solution:

$f(x) = \sin x - \cos x - ax + b$

$f'(x) = \cos x + \sin x - a \le 0 \; \forall \; x \in R$

$\Rightarrow a \ge \cos x + \sin x \;\; \forall \; x \in R$

$\Rightarrow a \ge \sqrt{2}$

Greatest and Least Value of Functions:

(a) If a continuous function $y = f(x)$ is increasing in the closed interval $[a, b]$, then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in $[a, b]$ (Figure-1)

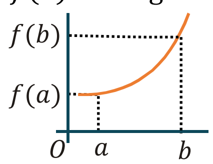


Figure-1

(b) If a continuous function $y = f(x)$ is decreasing in $[a, b]$, then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in $[a, b]$. (Figure-2)

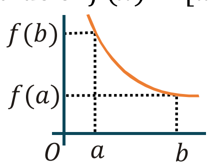


Figure-2

(c) If a continuous function $y = f(x)$ is increasing/decreasing in the (a, b), then no greatest and least value exist.

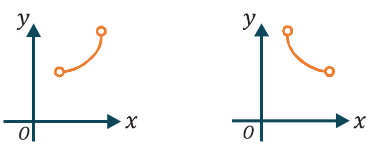


Illustration 12:

Show that $f(x) = \sin^{-1}\frac{x}{\sqrt{1+x^2}} - \ell nx$ is strictly decreasing in $x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$. Also find its range.

Solution:

$f(x) = \sin^{-1}\frac{x}{\sqrt{1+x^2}} - \ell nx = \tan^{-1}x - \ell nx \Rightarrow f'(x) = \frac{1}{1+x^2} - \frac{1}{x} = \frac{-(1+x^2-x)}{x(1+x^2)}$

$\therefore f'(x) \le 0\ \forall\ x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$

$\Rightarrow f(x)$ is strictly decreasing.

$f(x)|_{max} = f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{2}\ell n3$ & $f(x)|_{min} = f(\sqrt{3}) = \frac{\pi}{3} - \frac{1}{2}\ell n3$

$\therefore$ Range of $f(x) = \left[\frac{\pi}{3} - \frac{1}{2}\ell n3, \frac{\pi}{6} + \frac{1}{2}\ell n3\right]$

Illustration 13:

Find the minimum and maximum values of y in $4x^2 + 12xy + 10y^2 - 4y + 3 = 0$.

Solution:

$(2x)^2 + 2.2x\,.3y + (3y)^2 + (y-1)(y-3) = 0$

$\Rightarrow (2x+3y)^2 + (y-1)(y-3) = 0$

$x \in R$

So $D \ge 0$

$144y^2 - 16(10y^2 - 4y + 3) \ge 0$

$16[-y^2 + 4y - 3] \ge 0$

$y^2 - 4y + 3 \le 0$

$(y-1)(y-3) \le 0$

$\therefore 1 \le y \le 3$

So $y_{max} = 3$ and $y_{min} = 1$

Establishing Inequality:

Proving Inequalities using Monotonicity:

Comparison of two functions $f(x)$ and $g(x)$ can be done by analysing their monotonic behaviour.

Illustration 14:

Let $f(x)$ and $g(x)$ are two function which are defined and differentiable for all $x \ge x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$ then

(A) $f(x) < g(x)$ for some $x > x_0$
(B) $f(x) = g(x)$ for some $x > x_0$
(C) $f(x) > g(x)$ only for some $x > x_0$
(D) $f(x) > g(x)$ for all $x > x_0$

Ans. (D)

Solution:

Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) > 0\ \forall\ x > x_0$

$h(x_0) = 0$

$\Rightarrow h(x) > 0\ \forall\ x > x_0$

Illustration 15:

For $x \in \left(0, \frac{\pi}{2}\right)$ prove that $\sin x < x < \tan x$

Solution:

Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f(x)$ is M.I. $\Rightarrow f(x) > f(0)$

$\Rightarrow x - \sin x > 0 \Rightarrow x > \sin x$

Similarly consider another function $g(x) = x - \tan x \Rightarrow g'(x) = 1 - \sec^2 x$

$g'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow g(x)$ is M.D.

Hence $g(x) < g(0)$

$x - \tan x < 0 \Rightarrow x < \tan x$

$\sin x < x < \tan x$ Hence proved

Illustration 16:

For $x \in (0,1)$ prove that $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$.

Solution:

Let $f(x) = x - \frac{x^3}{3} - \tan^{-1} x$

$\Rightarrow f'(x) = 1 - x^2 - \frac{1}{1+x^2} = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1)$

$\Rightarrow f(x)$ is M.D. $\Rightarrow f(x) < f(0)$

$\Rightarrow x - \frac{x^3}{3} - \tan^{-1} x < 0 \Rightarrow x - \frac{x^3}{3} < \tan^{-1} x$ ...(i)

Similarly, $g(x) = x - \frac{x^3}{6} - \tan^{-1} x$

$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2} = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0$ for $x \in (0, 1) \Rightarrow g(x)$ is M.I.

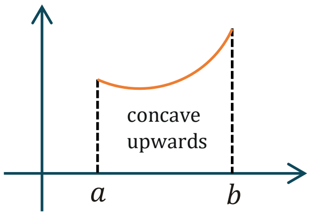
$x - \frac{x^3}{6} - \tan^{-1} x > 0 \Rightarrow x - \frac{x^3}{6} > \tan^{-1} x$ ...(ii)

from (i) and (ii), we get $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$ Hence Proved.

Concavity – Convexity of Function:

Significance of the Sign of IInd Order Derivative:

The sign of the 2^{nd} order derivative determines the concavity of the curve. If $f''(x) > 0\ \forall\ x \in (a, b)$ then graph of $f(x)$ is concave upward in (a, b). Similarly, if $f''(x) < 0\ \forall\ x \in (a, b)$ then graph of $f(x)$ is concave downward in (a, b).



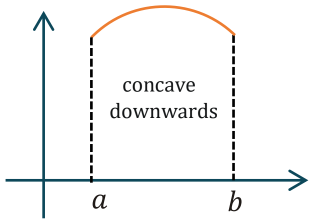


Illustration 17:

Separate the intervals of concavity and convexity of $f(x) = x + \sin x$

Solution:

$f'(x) = 1 + \cos x$

$f''(x) = -\sin x$

$f''(x) > 0\ \forall\ x \in ((2n-1)\,\pi, 2n\pi)\ n \in Z$

where graph is concave upward

$f''(x) < 0\ \forall\ x \in ((2n\pi, (2n+1)\pi), n \in Z$ Graph is concave downward.

Illustration 18:

If graph of $f(x) = 3x^4 + 2x^3 + ax^2 - x + 2$ is concave up for all real x, find values of a

Solution:

$f''(x) = 36x^2 + 12x + 2a$

$f''(x) > 0 \Rightarrow 36x^2 + 12x + 2a > 0 \forall\ x \in R$

$D < 0$

$\Rightarrow 12^2 - 4(36)(2a) < 0$

$1 - 2a < 0$

$a > \dfrac{1}{2}$

Rolle's Theorem:

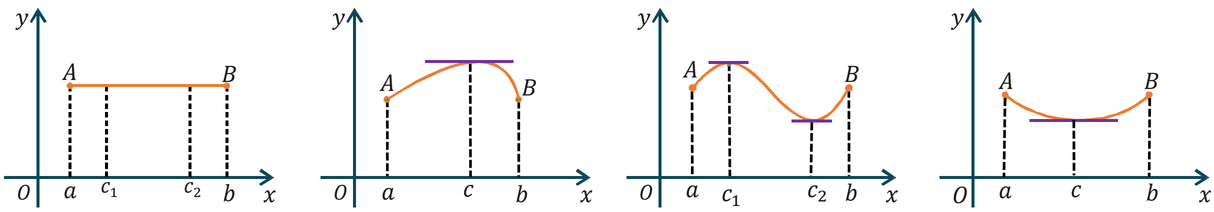
Let f be a function that satisfies the following three hypotheses:

(a) f is continuous on the closed interval $[a, b]$.

(b) f is differentiable on the open interval (a, b)

(c) $f(a) = f(b)$

Then there exist at least one number c in (a, b) such that $f'(c) = 0$.



(a) (b) (c) (d)

Note :

If f is differentiable function then between any two consecutive roots of $f(x) = 0$, there is atleast one root of the equation $f'(x) = 0$.

(d) Geometrical Interpretation:

Geometrically, the Rolle's theorem says that somewhere between A and B the curve has at least one tangent parallel to x-axis.

Illustration 19:

Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval $[0, 2]$.

Solution:

Here we observe that

(a) $f(x)$ is polynomial and since polynomial are always continuous, as well as differentiable. Hence $f(x)$ is continuous in the $[0,2]$ and differentiable in the $(0, 2)$.

&

(b) $f(0) = 0, f(2) = 2^3 - 3.(2)^2 + 2(2) = 0$

$\therefore f(0) = f(2)$

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$$\Rightarrow f'(c) = 3c^2 - 6c + 2 = 0 \Rightarrow \ c = 1 \pm \frac{1}{\sqrt{3}}$$

where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0,2)$ thus Rolle's theorem is verified.

Illustration 20:

Consider the function for $x = [-2,3]$, $f(x) = \begin{cases} \dfrac{x^3 - 2x^2 - 5x + 6}{x - 1} & \text{if} \quad x \neq 1 \\ -6 & \text{if} \quad x = 1 \end{cases}$, then

(A) f is discontinuous at $x = 1 \Rightarrow$ Rolle's theorem is not applicable in $[-2, 3]$

(B) $f(-2) \neq f(3) \Rightarrow$ Rolle's theorem is not applicable in $[-2, 3]$

(C) f is not derivable in $(-2, 3) \Rightarrow$ Rolle's theorem is not applicable

(D) Rolle's theorem is applicable as f satisfies all the conditions and c of Rolle's theorem is $\frac{1}{2}$

Ans. (D)

Solution:

$f(-2) = f(3) = 0$

$f(x)$ is continuous in $[-2, 3]$ & derivable in $(-2, 3)$ so Rolle's theorem is applicable.

So, in $c \in (-2, 3)$ such that $f'(c) = 0$

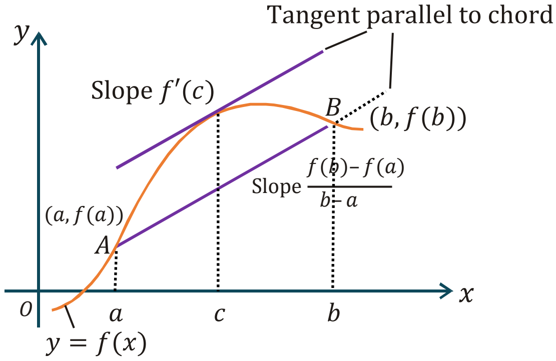
$$\Rightarrow \frac{2c^3 - 5c^2 + 4c - 1}{(c-1)^2} = 0 \Rightarrow c = \frac{1}{2}$$

Lagrange's Mean Value Theorem (LMVT):

Let f be a function that satisfies the following hypotheses:

(i) f is continuous in $[a, b]$

(ii) f is differentiable in (a, b).



Then there is a number c in (a, b) such that

(a) Geometrical Interpretation:

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

(b) Physical Interpretations:

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Illustration 21:

Find c of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval $[1, 3]$.

Solution:

Given $f(x) = 3x^2 + 5x + 7$ ...(i)

$\Rightarrow f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$

Again $f'(x) = 6x + 5$

Here $a = 1, b = 3$

Now from Lagrange's mean value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 6c + 5 = \frac{f(3)-f(1)}{3-1} = \frac{49-15}{2} = 17 \text{ or } c = 2.$$

Illustration 22:

If $f(x)$ is continuous and differentiable over $[-2, 5]$ and $-4 \le f'(x) \le 3$ for all x in $(-2, 5)$, then the greatest possible value of $f(5) - f(-2)$ is

(A) 7 (B) 9 (C) 15 (D) 21

Ans. (D)

Solution:

Apply LMVT

$$f'(x) = \frac{f(5)-f(-2)}{5-(-2)} \text{ for some } x \text{ in } (-2, 5)$$

$$\text{Now, } -4 \le \frac{f(5)-f(-2)}{7} \le 3$$

$-28 \le f(5) - f(-2) \le 21$

$\therefore$ Greatest possible value of $f(5) - f(-2)$ is 21.

Graph of Miscellaneous Function:

General Tips for Plotting the Graph

- Compute points where the curve crosses the x −axis and also where it cuts the y −axis by putting $y = 0$ and $x = 0$ respectively.
- Compute $\frac{dy}{dx}$ and find the intervals where $f(x)$ is increasing or decreasing and also where it has horizontal tangent $\left(\frac{dy}{dx}=0\right)$ or vertical tangent $\left(\frac{dy}{dx}\text{ not defined}\right)$.
- In regions where curve is monotonic compute y if $x \to \infty$ or $x \to -\infty$ to find whether y is asymptotic or not.
- If denominator vanishes say at $x = a$ then examine $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ to find whether f approaches ∞ or $-\infty$.

Illustration 23:

$f(x) = x^4 - 4x^2$

Solution:

$f(x) = x^4 - 4x^2 = 0$

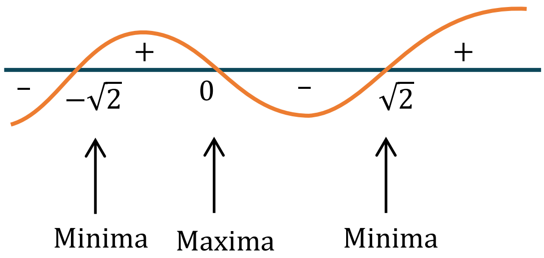
$x^2(x^2 - 4) = 0$

$x = \pm 2, x = 0$

$f'(x) = 4x^3 - 8x = 0$

$= 4x(x^2 - 2)$

$\Rightarrow x = 0,\ x = \pm\sqrt{2}$



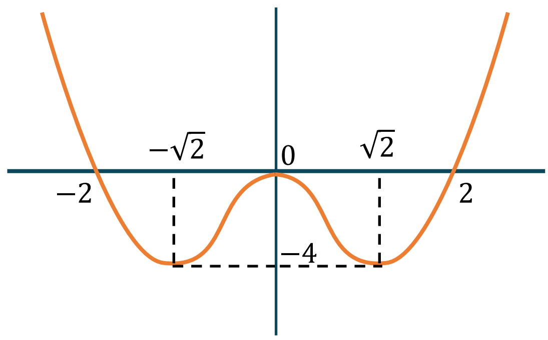


Illustration 24:

$f(x) = \frac{x}{\ell n\, x} - 2$

Solution:

$f(x) = \frac{x}{\ell nx} - 2$ Domain of function $\Rightarrow x > 0, x \neq 1$ $f'(x) = \frac{1}{\ell nx} - \frac{x}{(\ell nx)^2}.\frac{1}{x} \Rightarrow \frac{\ell nx - 1}{(\ell nx)^2}$

$f'(x) = 0 \Rightarrow \ell nx = 1$

$\Rightarrow x = e$