Monotonicity

SOLUTIONS

EXERCISE - O

1. **Ans. (A)**

$f'(x) = 4x - k$

$f''(x) = 4 \Rightarrow f''(x) > 0 \ \forall \ x \in [1, 2\}$

$\Rightarrow f'(x)$ is increasing in [1, 2]

and $f'(1)$ is its least value in [1, 2]

$\therefore f'(1) > 0$

$\Rightarrow 4 - k > 0 \Rightarrow k < 4$

2. **Ans. (C)**

$f(x) = x^3 - 10x^2 + 200x - 10$

$\Rightarrow f'(x) = 3x^2 - 20x + 200$

$a > 0, D < 0 \Rightarrow f'(x) > 0 \ \forall \ x \in R$

$\Rightarrow f(x)$ is strictly increasing $\forall \ x$.

3. **Ans. (C)**

(A) $f(x) = x^3 - 3x^2 + 3x + 3$

$\Rightarrow \quad f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 \geq 0 \ \forall x \in R$

(B) $f(x) = 2x^3 - 3x^2 - 12x + 6$

$\Rightarrow \quad f'(x) = 6x^2 - 6x - 12x = 6(x^2 - x - 2)$

$\Rightarrow \quad f'(x) = 6(x-2)(x+1)$

$+ \quad - \quad +$

$-1 \quad -2$

$\therefore \quad f(x) \uparrow$ in $(-\infty, -1]; [2, \infty)$

(C) $f'(x) = 3x^2 - 2x + 1 \Rightarrow f'(x) = 6x - 2$

$f'(x)$: $- \quad +$; $1/3$ $\quad \therefore f(x) \uparrow$ in $[1/3, \infty]$

(D) $f(x) = x^3 + 6x^2 + 6 \Rightarrow f'(x) = 3x^2 + 12x$

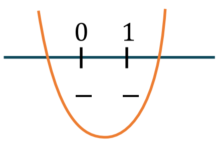
$+ \quad - \quad +$

$\Rightarrow \quad f'(x) = 3x(x+4)$ $\quad -4 \quad 0$

$\therefore \quad f(x) \uparrow$ in $(-\infty, -4]$; $[0, \infty)$

4. **Ans. (A)**

$f(x) = 2x^2 + 3x - m\,\ell nx, \ \ x \in (0, 1)$



$\Rightarrow f'(x) = 4x + 3 - \frac{m}{x}$

$\Rightarrow f'(x) = \frac{4x^2 + 3x - m}{x} \leq 0$ for decreasing

$\because x > 0 \Rightarrow 4x^2 + 3x - m \le 0 \ \forall x \in (0, 1)$

$f(1) \le 0 \Rightarrow 4 + 3 - m \le 0$

$\Rightarrow m \ge 7$

5. **Ans. (C)**

$x > 1 \Rightarrow f(x) \ge f(1)$

$x > 1 \Rightarrow g(x) \le g(1)$

$\Rightarrow f(g(x)) \le f(g(2))$

$\Rightarrow h(x) \le 1$ ...(i)

Range of $h(x)$ is subset of $[1, 10]$

$\Rightarrow h(x) \ge 1$ ...(ii)

By (i), (ii) we have $h(x) = 1 \Rightarrow h(2) = 1$

6. **Ans. (A)**

Consider $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

Continuous and derivable function

$f(0) = 0$; $f(1) = \frac{a}{3} + \frac{b}{2} + C$

$= \frac{2a + 3b + 6c}{6} = 0$

$\therefore$ for at least one, $x \in (0, 1)$, $f'(x) = 0$

$\Rightarrow ax^2 + bx + c = 0$

7. **Ans. (B)**

Let $f(x) = \sin x + x\cos x$

consider $g(x) = \int_0^x (\sin t + t\cos t)\,dt = \sin t]_0^x = x\sin x$

$g(x) = x \sin x$ which is differentiable.

now, $g(0) = 0$ and $g(\pi) = 0$, using Rolles Theorem hence for at least one $x \in (0, \pi)$,

where $g'(x) = 0$

i.e. $x \cos x + \sin x = 0$ for atleast one $x \in (0, \pi)$

8. **Ans. (A)**

$$\left.\begin{array}{l} f(x) \to \text{not differentiable at } x = 1/2 \text{ in } (0,1) \\ g(x) \to \text{Discontinuous at } x = 0 \\ h(x) \to \text{Discontinuous at } x = 1 \end{array}\right\} \begin{array}{c} \text{LMVT} \\ \text{not} \\ \text{applicable} \end{array}$$

$K(x)$ is cont. in $[0,1]$ & derivable in $(0,1)$

$\therefore$ LMVT can be applied

9. **Ans. (B)**

$f'(1) = 1$; $\frac{d}{dx}\left(f(2x)\right) = f'(x), x > 0$

$\Rightarrow 2f'(2x) = f'(x)$

$x = 1: 2f'(2) = f'(1) \Rightarrow f'(2) = \frac{1}{2}$

$x = 2: 2f'(4) = f'(2) \Rightarrow f'(4) = \frac{1}{4}$

$\therefore$ LMVT in [2,4] on $f'(x)$:

$f''(c) = \frac{f'(4) - f'(2)}{4-2} \Rightarrow f''(c) = \frac{1/4 - 1/2}{4-2} = \frac{-1}{8}$

10. **Ans. (A)**

$f(x) = 2x^2 + 3x + 5$

$f'(2) = \frac{f(a) - f(1)}{a-1} \Rightarrow 4x+3|_{x=2} = \frac{f(a)-10}{a-1}$

$\Rightarrow 11(a-1) = f(a) - 10$

$\Rightarrow 2a^2 + 3a + 5 - 10 = 11a - 11$

$\Rightarrow 2a^2 - 8a + 6 = 0$

$\Rightarrow a^2 - 4a + 3 = 0, a = 1; a = 3$

11. **Ans. (A,D)**

$f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}}$

$= 1 - \frac{1}{1+x^2} + 1 - \frac{1}{\sqrt{x^2+1}}$

$= \frac{x^2}{1+x^2} + \left(1 - \frac{1}{\sqrt{x^2+1}}\right) \geq 0$

12. **Ans. (A,C,D)**

LMVT on $f(x)$ in $[A, B]$

$\frac{f(B) - f(A)}{B-A} = f'(C)$

$\Rightarrow \frac{\sin^2 B - \sin^2 A}{B-A} = 2\sin C \cos C = \sin 2C$

$\because A < C < B \Rightarrow 2A < 2C < 2B$

$\Rightarrow \sin 2A < \sin 2C < \sin 2B$

$\left(\because \sin x \text{ is increasing in } x \in \left[0, \frac{\pi}{2}\right]\right)$

$\Rightarrow \quad (B-A)\sin 2A < \sin^2 B - \sin^2 A < (B-A)\sin 2B$

Also $2A < 2B \Rightarrow \cos 2A > \cos 2B$

($\because$ $\cos x$ is derceasing f^n)

Now $\left.\begin{array}{c} B > A \\ \cos 2A > \cos 2B \end{array}\right\} \Rightarrow B\cos 2A > A\cos 2B$

13. **Ans. (A,D)**

$\phi'(x) = (3(f(x))^2 - 6(f(x)) + 4)f'(x) + 5 + 3\cos x - 4\sin x$

$5 - \sqrt{9+16} \leq 5 + 3\cos x - 4\sin x \leq 5 + \sqrt{9+16}$

adding $(3(f(x))^2 - 6(f(x)) + 4)f'(x)$

$(3(f(x))^2 - 6(f(x)) + 4)f'(x) \leq \phi'(x) \leq (3(f(x))^2 - 6(f(x)) + 4)f'(x) + 10$

$\because\ 3(f(x))^2 - 6f(x) + 4 = 3(f(x) - 1)^2 + 1 > 0$

$(3(f(x))^2 - 6(f(x)) + 4)f'(x) \geq 0$ when ever $f(x)$ is increasing.

$\Rightarrow\ \phi'(x) \geq 0 \Rightarrow \phi(x)$ is increasing, when ever $f(x)$ is increasing.

If $f'(x) = -11$ then

$(3(f(x))^2 - 6f(x) + 4)\, f'(x) + 10 = -33\,(f(x) - 1)^2 - 1 < 0$

$\Rightarrow\ \phi'(x) < 0$

$\Rightarrow\ \phi(x)$ is decreasing.

14. **Ans. (B,C)**

$g(x) = 2f\left(\frac{x}{2}\right) + f(1-x)$ and

$g'(x) = f'(x/2) - f'(1 - x)$

Now, $g(x)$ is increasing if $g'(x) \geq 0$

$f'\left(\frac{x}{2}\right) \geq f'(1-x)$

$[\because\ f''(x) < 0$ i.e. $f'(x)$ is decreasing$]$

$\Rightarrow \frac{x}{2} \leq 1 - x \Rightarrow x \leq 2 - 2x$

$\Rightarrow 3x \leq 2 \Rightarrow x \leq 2/3 \Rightarrow 0 \leq x \leq \frac{2}{3}$

$\Rightarrow g(x)$ increases in $0 \leq x \leq 2/3$

and $g'(x) \leq 0$ for decreasing

$\Rightarrow f'\left(\frac{x}{2}\right) \leq f'(1-x) \Rightarrow \frac{x}{2} \geq 1 - x$

$\Rightarrow x \geq 2/3$

$\Rightarrow 2/3 \leq x \leq 1$

15. **Ans. (C)**

$\sin x$ is concave downward in $(0, \pi)$

and $\sin x$ is concave upward in $(\pi, 2\pi)$

16. **Ans. (D)**

$e^x, 2^x, \tan^{-1}x$ (If $x \in R^-$) is concave upward and ℓnx is concave downward

17. **Ans. (A)**

$f(x)$ concave downward $(f''(x) < 0)$

$f(x)$ increasing $(\therefore f'(x) > 0)$

Let $g(x) = f^{-1}(x) = x$

$f(g(x)) = x$

$f'(g(x)).\, g(x) = 1$

$g'(x) = \dfrac{1}{f'(g(x))} > 0$

$g"(x) = -\dfrac{1}{(f'(g(x))^2} \times f"(g(x)).g'(x)$

$g"(x) > 0$

$g(x) = f^{-1}(x)$ concave upward

18. Ans. (B)

(P) $f(x)=2e^{x^2-4x}$

$f'(x)=4(x-2)e^{x^2-4x}$

$\therefore f(x)$ strictly increases $\forall x\in[2,\infty)$ & $f(x)$ strictly decreases $\forall x\in(-\infty,2]$

(P)→(1)

(Q) $f(x)=\dfrac{e^x}{x}$

$f'(x)=e^x\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)=\dfrac{e^x}{x^2}(x-1)$

$\therefore f(x)$ strictly increases $\forall x\in[1,\infty)$ & $f(x)$ strictly decreases $\forall x\in(-\infty,0);(0,1]$

(Q) → (3)

(R) $f(x)=x^2e^{-x}$

$f'(x)=e^{-x}(2x-x^2)$

$\therefore f(x)$ strictly increases $\forall x\in[0,2]$ & $f(x)$ strictly decreases $\forall x\in(-\infty,0];[2,\infty)$

(R) → (2)

(S) $f(x)=2x^2-\ell n(x)$

$f'(x)=4x-\dfrac{1}{x}=\dfrac{(2x-1)(2x+1)}{x}$

$f(x)$ strictly increases $\forall x\in\left[\dfrac{-1}{2},0\right);\left[\dfrac{1}{2},\infty\right)$ & $f(x)$ strictly decreases $\forall x\in\left(-\infty,\dfrac{-1}{2}\right];\left(0,\dfrac{1}{2}\right]$

(S) → (4)

EXERCISE - S

1. Ans. (0)

$f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$

$f(x) = 2e^x + ae^{-x} + (2a+1) > 0 \ \forall\ x \in R$

$= \dfrac{2(e^x)^2+(2a+1)e^x+a}{e^x} > 0$

$\Rightarrow 2(e^x)^2 + (2a+1)e^x + a > 0\ \forall x \in R$

$\Rightarrow 2(e^x)^2 + 2a\cdot e^x + e^x + a > 0$

$\Rightarrow \underbrace{(2e^x+1)}_{+ve}\ (e^x + a) > 0 \Rightarrow e^x + a > 0$

$\Rightarrow a > -e^x \Rightarrow a \geq 0$

2. **Ans. (8)**

For mean value theorem in [0, 2] $f(x)$ must be continuous in [0, 2] and derivable in (0, 2)

$\therefore$ cont. at $x = 0$: $f(0^+) = f(0)$

$\Rightarrow 0 + 0 + a = 3 \Rightarrow \boxed{a=3}$

cont. at $x = 1$: $f(1^+) = f(1^-) = f(1)$

$\Rightarrow m(1) + b = -(1)^2 + 3(1) + a$

$\Rightarrow m + b = a + 2 \Rightarrow m + b = 5$ ...(1)

Derivable at $x = 1$: $f'(1^-) = f'(1^+)$

$\Rightarrow -2x + 3 \,|_{x=1} = m \Rightarrow \boxed{m=1}$

$\therefore$ by (1) : $\boxed{b=4}$

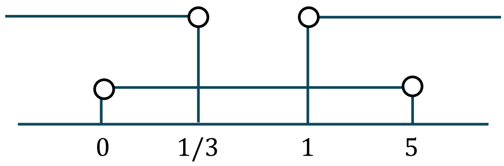
3. **Ans. (9)**

$f(x)$ is st. increasing in $(0, \infty)$

$\therefore f(x_1) < f(x_2) \Rightarrow 0 < x_1 < x_2$

$\therefore f(3a^2 - 4a + 1) < f(2a^2 + a + 1)$

$\Rightarrow 0 < 3a^2 - 4a + 1 < 2a^2 + a + 1$



$3a^2 - 4a + 1 > 0$ & $a^2 - 5a < 0$

$(3a - 1)(a - 1) > 0$ & $a(a - 5) < 0$

$\therefore a \in (0, 1/3) \cup (1, 5)$

4. **Ans. (8)**

$$f(x) = \left(1 - \frac{\sqrt{21-4a-a^2}}{a+1}\right)x^3 + 5x + \sqrt{7}$$

$f'(x) \geq 0$

$$\left(1 - \frac{\sqrt{21-4a-a^2}}{a+1}\right)3x^2 + 5 \geq 0 \;\forall x \in R$$

$(ax^2 + bx + c > 0 \Rightarrow D \leq 0$ & $a > 0)$

$$0^2 - 4 \times 3\left(1 - \frac{\sqrt{21-4a-a^2}}{a+1}\right)5 \leq 0$$

$$1 - \frac{\sqrt{21-4a-a^2}}{a+1} \geq 0 \qquad ...(1)$$

Case-I: $a + 1 < 0$ and $21 - 4a - a^2 \geq 0$

$a < -1$ and $(a + 7)(a - 3) \leq 0$

$a < -1$ and $-7 \leq a \leq 3$

$a \in [-7, -1)$

Case-II: if $a + 1 > 0$

$21 - 4a - a^2 \geq 0$ and $\dfrac{\sqrt{21-4a-a^2}}{a+1} \leq 1$

$(a + 7)(a - 3) \leq 0$ and $(a + 1)^2 \geq (21 - 4a - a^2)$

$-7 \leq a \leq 3$ and $a^2 + 3a - 10 \geq 0$

$a \leq -5$ or $a \geq 2$

$a \in [2, 3]$

so, $a \in [-7, -1) \cup [2, 3]$

5. Ans. (–10)

$$f(x) = x^{\frac{3}{2}} + x^{\frac{-3}{2}} - 4\left(x + \frac{1}{x}\right)$$

$$= x\sqrt{x} + \frac{1}{x\sqrt{x}} - 4\left(x + \frac{1}{x}\right)$$

$\boxed{\text{Put } t = \sqrt{x}}$

$$f(t) = t^2 \cdot t + \frac{1}{t^2 \cdot t} - 4\left(t^2 + \frac{1}{t^2}\right)$$

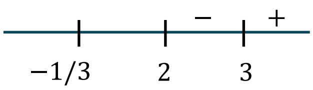
$$= t^3 + \frac{1}{t^3} - 4\left(t^2 + \frac{1}{t^2}\right)$$

$$f(t) = \left(t + \frac{1}{t}\right)^3 - 3.t.\frac{1}{t}\left(t + \frac{1}{t}\right) - 4\left[\left(t + \frac{1}{t}\right)^2 - 2.t.\frac{1}{t}\right]$$

$$= \left(t + \frac{1}{t}\right)^3 - 3\left(t + \frac{1}{t}\right) - 4\left[\left(t + \frac{1}{t}\right)^2 - 2\right]$$

$$= \left(t + \frac{1}{t}\right)^3 - 3\left(t + \frac{1}{t}\right) - 4\left(t + \frac{1}{t}\right)^2 + 8$$

$$= \left(t + \frac{1}{t}\right)^3 - 4\left(t + \frac{1}{t}\right)^2 - 3\left(t + \frac{1}{t}\right) + 8$$



$$\boxed{\begin{array}{l} \text{put } t + \dfrac{1}{t} = a \\ \because t > 0 \Rightarrow a \geq 2 \end{array}}$$

$f(a) = a^3 - 4a^2 - 3a + 8$

$f'(a) = 3a^2 - 8a - 3$

$= 3a^2 - 9a + a - 3$

$= 3a(a - 3) + (a - 3)$

$= (3a + 1)(a - 3)$

Hence $f(a)$ has minimum values at $a = 3$

$\therefore f(a)_{\min} = f(3)$

$= 3^3 - 4.3^2 - 3.3 + 8$

$= 27 - 36 - 9 + 8$

$= 35 - 45$

6. **Ans. (1)**

$f(x) = |(x-a)^3| + |(x-b)^3|$

$$f(x) = \begin{cases} (x-a)^3 + (x-b)^3 & x \geq b \\ (x-a)^3 + (b-x)^3 & a < x < b \\ (a-x)^3 + (b-x)^3 & x \leq a \end{cases}$$

$$f'(x) = \begin{cases} 3(x-a)^2 + 3(x-b)^2 & x > b \\ 3(x-a)^2 - 3(b-x)^2 & a < x < b \\ -(3(a-x)^2 + 3(b-x)^2) & x < a \end{cases}$$

$f'(x) = 0 \Rightarrow (x-a)^2 = (b-x)^2$

$\Rightarrow x - a = b - x \Rightarrow x = \dfrac{a+b}{2}$

$\therefore f(x)_{\min} = \left|\left(\dfrac{a+b}{2} - a\right)^3\right| + \left|\left(\dfrac{a+b}{2} - b\right)^3\right|$

$= \dfrac{|(a-b)^3|}{4} = \dfrac{(b-a)^3}{4}$

7. **Ans. (–2)**

$\ell n(1+x) >, x > \dfrac{x}{1+x}$ – 1 (domain)

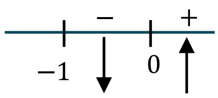
$\Rightarrow \ell n(1+x) - \dfrac{x}{1+x} > 0$

Let $h(x) = \ell n(1+x) - \dfrac{x}{1+x}$

$\Rightarrow h'(x) = \dfrac{1}{1+x} - \left[\dfrac{1+x-x}{(1+x)^2}\right] = \dfrac{1+x-1}{(1+x)^2}$

$\Rightarrow h'(x) = \dfrac{x}{(1+x)^2}$

$h'(x)$ s



$\therefore h(x)$ has a least value at $x = 0$

$h(0) = 0 \Rightarrow h(x) \geq h(0) \ \forall\, x \in (-1, \infty)$

$\Rightarrow \ell n(1+x) - \dfrac{x}{1+x} \geq 0 \ \ \forall\, x \in (-1, \infty)$

equality to zero exists at $x = 0$

$\therefore \ \ell n(1+x) > \dfrac{x}{1+x} \ \ \forall\, x \in (-1, \infty) - \{0\}$

$\Rightarrow \ell n(1+x) > \dfrac{x}{1+x} \ \ \forall\, x \in (-1, 0) \cup (0, \infty)$

EXERCISE - JEE (Main) PYQ

1. **Ans. (3)**

$\frac{f(x)-14}{(x-1)^2} = 0 \Rightarrow f(x) = 14$

Also, $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7, \qquad a \in R$

$f(x)$ is $\uparrow$ in $(0, 1]$ and $\downarrow$ in $[1, 5)$

$\Rightarrow f'(x) = 3(x^2 - 2(a-2)x + a) \Rightarrow f'(x) \geq 0$ in $(0, 1]$ and $f'(x) \leq 0$ in $[1, 5)$

$\Rightarrow f'(x) = 0 \ at \ x = 1 \Rightarrow x^2 - 2(a-2)x + a = 0$ at $x = 1$

$\Rightarrow 1 - 2(a - 2) + a = 0 \Rightarrow 1 - 2a + 4 + a = 0 \Rightarrow a = 5$

$\therefore f(x) = x^3 - 3 \times 3x^2 + 15x + 7$

$f(x) = x^3 - 9x^2 + 15x + 7$

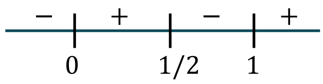
$f(x) = 14 \Rightarrow x^3 - 9x^2 + 15x + 7 = 14$

$\Rightarrow x^3 - 9x^2 + 15x - 7 = 0 \Rightarrow (x-1)^2 (x-7) = 0 \Rightarrow x = 1$ or 7

2. **Ans. (2)**

$h(x) = f(g(x))$

$\Rightarrow h(x) = e^{g(x)} - g(x) \Rightarrow h(x) = e^{x^2-x} - (x^2 - x)$



$\Rightarrow h'(x) = e^{x^2-x}(2x-1) - (2x-1) = (2x-1)(e^{x^2-x} - 1)$

$= (2x-1)\,(e^{x(x-1)} - 1)$

$\Rightarrow h(x) \uparrow : x \in \left[0, \frac{1}{2}\right]; [1, \infty)$

3. **Ans. (2)**

$f(x) = x\sqrt{kx - x^2}$

$kx - x^2 \geq 0$

$\Rightarrow x^2 - kx \leq 0 \Rightarrow x(x-k) \leq 0$

$\Rightarrow x \in [0, 3] \Rightarrow \boxed{k \geq 3} \Rightarrow f'(x) = \sqrt{kx - x^2} + \frac{x(k-2x)}{2\sqrt{kx-x^2}}$

$\Rightarrow f'(x) = \frac{2kx - 2x^2 + xk - 2x^2}{2\sqrt{kx - x^2}} \Rightarrow f'(x) = \frac{3kx - 4x^2}{2\sqrt{kx - x^2}}$

$f'(x) \geq 0 \Rightarrow 3kx - 4x^2 \geq 0$

$\Rightarrow 4x^2 - 3kx \leq 0 \Rightarrow x(4x - 3k) \leq 0 \Rightarrow 4x - 3k \leq 0 \{\because x \in [0, 3]$

$\Rightarrow x \leq \frac{3k}{4} \Rightarrow \frac{3k}{4} \geq 3 \Rightarrow \boxed{k \geq 4}$

$\therefore$ minimum value of $k = 4$

$\therefore f(x) = x\sqrt{4x - x^2}$

$f(x)_{max}$ at $x = 3 \Rightarrow f(x)_{max} = 3\sqrt{3}$

4. Ans. (4)

$f'(c) = \frac{f(1)-f(0)}{1} \Rightarrow 3c^2 - 8c + 8 = \frac{1-4+8+11-11}{1}$

$\Rightarrow 3c^2 - 8c + 8 = 5 \Rightarrow 3c^2 - 8c + 3 = 0$

$\Rightarrow c = \frac{8 \pm \sqrt{64-36}}{6} \Rightarrow c = \frac{8 \pm \sqrt{28}}{6} \Rightarrow c = \frac{4 \pm \sqrt{7}}{3}$

$\Rightarrow \quad c \in (0, 1) \Rightarrow c = \frac{4-\sqrt{7}}{3}$

5. Ans. (2)

LMVT in $[-7, -1] \Rightarrow \frac{f(-1)-f(-7)}{-1+7} = f'(x) \leq 2$

$\Rightarrow f(-1) \leq 9$ ...(i)

LMVT in $[-7, 0] \Rightarrow f(0) - f(-7) = f'(x) \leq 2$

$\Rightarrow f(0) \leq 11$ ...(ii)

(i) + (ii) $\Rightarrow f(-1) + f(0) \leq 20$

6. Ans. (1)

$f(x) = x\cos^{-1}(-\sin|x|), x \in [-\pi/2, \pi/2]$

$f(x) = x(\pi - \cos^{-1} \sin |x|)$

$\Rightarrow f(x) = x\left(\pi - \frac{\pi}{2} + \sin^{-1}\sin|x|\right), |x| \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow f(x) = x\left(\frac{\pi}{2} + \sin^{-1}\sin|x|\right) \Rightarrow f(x) = x\left(\frac{\pi}{2} + |x|\right) \Rightarrow f(x) = \begin{cases} x\left(\frac{\pi}{2}+x\right) & x \geq 0 \\ x\left(\frac{\pi}{2}-x\right) & x < 0 \end{cases}$

Also, $f'(x) = \begin{cases} \frac{\pi}{2} + 2x & x > 0 \\ \frac{\pi}{2} - 2x & x < 0 \end{cases}$

$f'(0^+) = f'(0^-) = \frac{\pi}{2}$

$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases} \quad \therefore \quad f'(x)$ is $\downarrow$ in $\left(-\frac{\pi}{2}, 0\right)$ and $f'(x)$ is $\uparrow$ in $\left(0, \frac{\pi}{2}\right)$

7. Ans. (3)

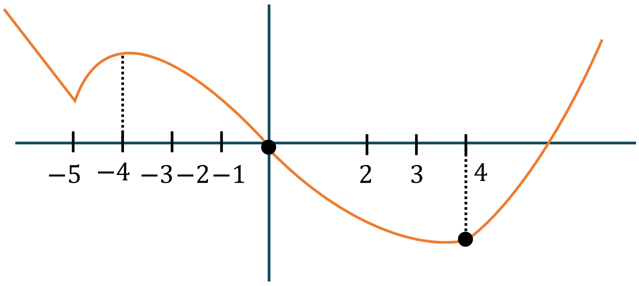
LMVT for $x \in [a, c] \Rightarrow \frac{f(c)-f(a)}{c-a} = f'(\alpha), \alpha \in (a, c)$

LMVT for $x \in [c, b] \Rightarrow \frac{f(b)-f(c)}{b-c} = f'(\beta), \beta \in (c, b)$

$\Rightarrow f''(x) < 0 \Rightarrow f'(x)$ is decreasing & $\alpha < \beta \Rightarrow f'(\alpha) > f'(\beta) \Rightarrow \frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(a)}{b-c}$

$\Rightarrow \frac{f(c)-f(a)}{f(b)-f(c)} > \frac{c-a}{b-c}$ $(\Rightarrow f(x)$ is $\uparrow)$

8. **Ans. (4)**



$$f'(x)=\begin{cases}-55; & x<-5\\ 6(x-5)(x+4); & -5<x<4\\ 6(x-3)(x+2); & x>4\end{cases}$$

$f(x)$ is increasing in

$x \in (-5,-4) \cup (4,\infty)$

9. **Ans. (1)**

$f(1) = f(2)$

$\Rightarrow 1 - a + b - 4 = 8 - 4a + 2b - 4$

$\Rightarrow 3a - b = 7$ ...(1)

Also $f^{1}\left(\frac{4}{3}\right)=0$ given

$$\Rightarrow \left(3x^2-2ax+b\right)_{x=\frac{4}{3}}=0 \Rightarrow \frac{16}{3}-\frac{8a}{3}+b=0$$

$\Rightarrow 8a - 3b - 16 = 0$ ...(2)

Solving (1) and (2)

$a = 5, b = 8$

10. **Ans. (2)**

$$f(x)=\begin{cases}-x\left(2-\sin\left(\frac{1}{x}\right)\right) & x<0\\ 0 & x=0\\ x\left(2-\sin\left(\frac{1}{x}\right)\right) & x>0\end{cases}$$

$$f'(x)=\begin{cases}-\left(2-\sin\frac{1}{x}\right)-x\left(-\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)\right) & x<0\\ \left(2-\sin\frac{1}{x}\right)+x\left(-\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)\right) & x>0\end{cases}$$

$$f'(x)=\begin{cases}-2+\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x} & x<0\\ 2-\sin\frac{1}{x}+\frac{1}{x}\cos\frac{1}{x} & x>0\end{cases}$$

$f'(x)$ is an oscillating function which is non-monotonic in $(-\infty, 0) \cup (0, \infty)$.

11. Ans. (1)

$f(x) = \log_e(x^2+1) - e^{-x} + 1$

$\Rightarrow f'(x) = \dfrac{2x}{x^2+1} + e^{-x} > 0 \ \forall\, x \in R$

$\Rightarrow f$ is strictly increasing

$g(x) = \dfrac{1-2e^{2x}}{e^x} = e^{-x} - 2e^x \Rightarrow g'(x) = -\left(2e^x + e^{-x}\right) < 0 \ \forall\, x \in R$

$\Rightarrow g$ is decreasing

Now $f\left(g\left(\dfrac{(\alpha-1)^2}{3}\right)\right) > f\left(g\left(\alpha - \dfrac{5}{3}\right)\right)$

$\Rightarrow g\left(\dfrac{(\alpha-1)^2}{3}\right) > g\left(\alpha-\dfrac{5}{3}\right) \Rightarrow \dfrac{(\alpha-1)^2}{3} < \alpha - \dfrac{5}{3}$

$\Rightarrow \alpha^2 - 5\alpha + 6 < 0 \Rightarrow (\alpha-2)(\alpha-3) < 0$

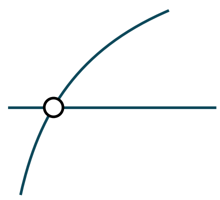
$\Rightarrow \alpha \in (2, 3)$

12. Ans. (2)

$f(x) = x^7 + 5x^3 + 3x + 1$

$f'(x) = 7x^6 + 15x^2 + 3 > 0$

$\therefore f(x)$ is strictly increasing function



$x \to -\infty, \ y \to -\infty$

$x \to \infty, \ y \to \infty$

$\therefore$ no. of real solution = 1

13. Ans. (1)

$f(x) = x\,e^{x(1-x)}$

$f'(x) = -e^{x(1-x)}(2x+1)(x-1)$

$f(x)$ is increasing in $\left(-\dfrac{1}{2}, 1\right)$

14. Ans. (2)

$f(x) = 2x + \tan^{-1}x$ and $g(x) = \ell n\left(\sqrt{1+x^2} + x\right)$

and $x \in [0, 3]$

$g'(x) = \dfrac{1}{\sqrt{1+x^2}}$

Now, $0 \le x \le 3$

$0 \le x^2 \le 9$

$1 \le 1 + x^2 \le 10$

So, $2 + \dfrac{1}{10} \le f'(x) \le 3$

$\dfrac{21}{10} \le f'(x) \le 3$ and $\dfrac{1}{\sqrt{10}} \le g'(x) \le 1$

option (4) is incorrect

From above, $g'(x) < f'(x)\ \forall\ x \in [0, 3]$

Option (1) is incorrect.

$f'(x)$ & $g'(x)$ both positive so $f(x)$ & $g(x)$ both are increasing

So, max $(f(x)$ at $x = 3$ is $6 + \tan^{-1} 3$

Max $(g(x)$ at $x = 3$ is $\ell n\,(3 + \sqrt{10})$

And $6 + \tan^{-1} 3 > \ell n\,(3 + \sqrt{10})$

Option (2) is correct

15. Ans. (4)

$$g(x) = f(-x) - f(x) = \frac{1+e^x}{1-e^x} \qquad \Rightarrow g'(x) = \frac{2e^x}{(1-e^x)^2} > 0$$

$\Rightarrow g$ is increasing in (0, 1) $\quad \Rightarrow g$ is one-one in (0, 1)

16. Ans. (2)

$g(x) = f(x) + f(1-x)$ & $f''(x) > 0, x \in (0, 1)$

$g'(x) = f'(x) - f'(1-x) = 0$

$\Rightarrow f'(x) = f'(1-x)$

$x = 1 - x$

$x = \frac{1}{2}$

$g'(x) = 0$

at $x = \frac{1}{2}$

$g''(x) = f''(x) + f''(1-x) > 0$

g is concave up

hence $\alpha = \frac{1}{2}$

$$\tan^{-1} 2\alpha + \tan^{-1}\frac{1}{\alpha} + \tan^{-1}\frac{\alpha+1}{\alpha}$$

$$\Rightarrow \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$$

EXERCISE - JEE (Advanced) PYQ

1. Ans. (A)

$x \in (0, 1);\ b \in (0, 1);\ f(x) = \frac{b-x}{1-bx}$

$$\Rightarrow \qquad f'(x) = \frac{(1-bx)\cdot(-1) - (b-x)(-b)}{(1-bx)^2} = -\frac{1+bx+b^2-bx}{(1-bx)^2}$$

$$\Rightarrow \qquad f'(x) = \frac{b^2-1}{(1-bx)^2} \Rightarrow -ve \qquad\qquad \because b \in (0,1) \Rightarrow f(x) \text{ is} \downarrow$$

$\therefore$ Range of $f \in (f(1), f(0)) \Rightarrow$ Range $\in (-1, b) \neq$ codomain $\Rightarrow$ Non invertible

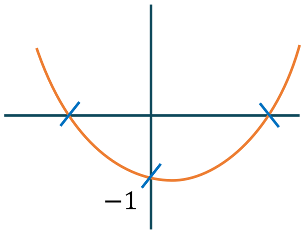
2. Ans. (2)

Consider $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$f'(x) = 4x^3 - 12x^2 + 24x + 1$

$f''(x) = 12x^2 - 24x + 24$

$= \underbrace{12(x^2 - 2x + 2)}_{+ve\ \forall x \in R}$



(Approx. graph)

$\therefore$ $f(x)$ is concave upwards $\forall\ x \in R$

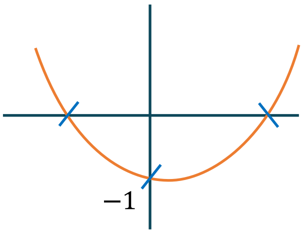
$f(0) = -1$

Also $\lim\limits_{x\to\infty} f(x) \to \infty$

$\lim\limits_{x\to-\infty} f(x) \to \infty$

$\therefore$ $f(x)$ has only 2 real roots.

3. Ans. (C)

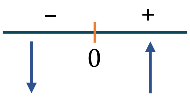


(Approx. graph)

$x^2 - x \sin x - \cos x = 0$

$\Rightarrow$ Let $h(x) = x^2 - x \sin x - \cos x$

$\Rightarrow$ $h'(x) = 2x - x\cos x - \sin x + \sin x$



$h'(x) = x(2 - \cos x)$

$h'(x)$:

$h(0) = -1$

Also $h(x)$ is a continuous function and $\lim\limits_{x\to-\infty} h(x) \to \infty;\ \lim\limits_{x\to\infty} h(x) \to \infty$

$\therefore$ we can estimate a graph of $h(x)$ as :

Decreasing for $x < 0$

Increasing for $x > 0$

-1 at $x = 0$

As $x \to \infty$, $h(x) \to \infty$ and

As $x \to -\infty$, $h(x) \to \infty$

$\therefore$ Exactly 2 distinct roots.

4. **Ans. (B,C)**

$f(x) = x \sin\pi x, x > 0$

$\Rightarrow f'(x) = \pi x \cos\pi x + \sin\pi x = 0 \Rightarrow \tan\pi x = -\pi x$

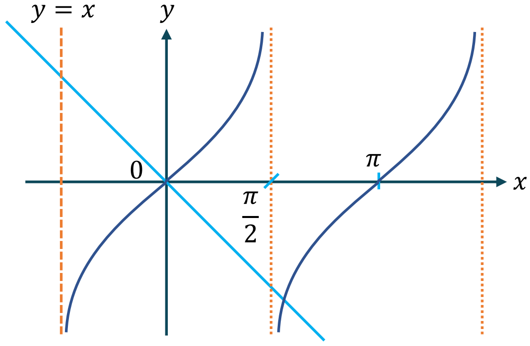
Let $\pi x = t \Rightarrow \tan t = -t$

$\therefore$ unique solution in:

(i) $0 < t < \pi \Rightarrow n\pi < t < n\pi + \pi$

(tan x is periodic)

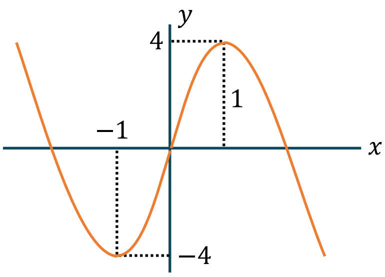
$\Rightarrow n < \pi x < (n+1)\pi \Rightarrow n < x < n+1$



5. **Ans. (B,D)**

Consider $a = 5x - x^5 = g(x)$

graph of $g(x)$:



for only one root: $a > 4$ or $a < -4$

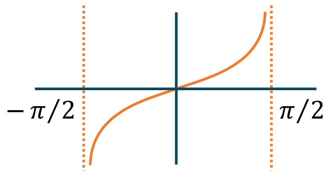
for exactly three roots : $a \in (-4, 4)$

for exactly two real roots : $a \in \{-4, 4\}$

6. **Ans. (A,B,C)**

$$f(x) = (\ell n(\sec x + \tan x))^3$$

$$f'(x) = \frac{3(\ell n(\sec x + \tan x))^2 (\sec x \tan x + \sec^2 x)}{(\sec x + \tan x)} > 0$$



$f(x)$ is an increasing function

$\lim\limits_{x \to -\frac{\pi}{2}} f(x) \to -\infty$ & $\lim\limits_{x \to \frac{\pi}{2}} f(x) \to \infty$

Range of $f(x)$ is R and onto function

$$f(-x) = (\ell n(\sec x - \tan x))^3 = \left(\ell n\left(\frac{1}{\sec x + \tan x}\right)\right)^3$$

$f(-x) = -(\ell n(\sec x + \tan x))^3$

$f(x) + f(-x) = 0 \Rightarrow f(x)$ is an odd function.

7. **Ans. (B,C)**

Consider $h(x) = f(x) - 3g(x)$

$h(-1) = f(-1) - 3g(-1) = 3$

$h(0) = 6 - 3 = 3, h(2) = 3$

$\to h'(x) = 0$ has atleast one root in $(-1, 0)$ & $(0, 2)$

but since $h''(x) = 0$ has no root in $(-1, 0)$ as $(0, 2)$

$\to h'(x) = 0$ has exactly one root each in

$(-1, 0)$ & $(0, 2)$.

8. Ans. (D)

$f(x)=x+\ell nx-x\ell nx$

$f'(x)=\frac{1}{x}-\ell nx$

$f''(x)=-\frac{1}{x^2}-\frac{1}{x}=-\frac{(1+x)}{x^2}$

$f'(1).f'(e)=\left(\frac{1}{e}-1\right)<0\Rightarrow f'(x)=0$ for some $x\in(1,e)$

$\lim\limits_{x\to\infty}\left(\frac{1}{x}-\ell nx\right)=-\infty$

$f'(x)$ is decreasing for $x\in(0,\infty)$

9. Ans. (D)

$\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}1-(x-1)(\ell nx-1)=-\infty$

$f'(e)=\frac{1}{e}-1<0, f''(x)<0$

$\Rightarrow f(x)$ is decreasing in (e, e^2)

10. Ans. (D)

$f'(x)$ is decreases and $f'(1)=1$

$\Rightarrow f'(x)=0$ has no root in $(0,1)$

11. Ans. (C)

$f(1/2)=1/2$; $f(1)=1, f''(x)>0$

Let $h(x)=f(x)-x$

$h(1/2)=h(1)=0\Rightarrow$ by RT ; $f'(\alpha)=1$

for some $\alpha\in(1/2, 1)$

Also, $f''(x)>0\Rightarrow f'(x)$ is $\uparrow$

$\Rightarrow f'(1)>f'(\alpha)$ ($\alpha\in(1/2, 1)$)

$\Rightarrow f'(1)>1$ Ans. (C).

12. Ans. (5)

$f(x) = (x^2-1)^2 h(x) : h(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

Now, $f(1) = f(-1)=0$

$\Rightarrow f'(\alpha)=0, \alpha\in(-1, 1)$ [Rolle's Theorem]

Also, $f'(1)=f'(-1)=0\Rightarrow f'(x)=0$ has atleast 3 root, $-1, \alpha, 1$ with $-1<\alpha<1$

$\Rightarrow f''(x)=0$ will have at least 2 root, say β, γ such that

$-1<\beta<\alpha<\gamma<1$ [Rolle's Theorem]

So, $\min\left(m_{f''}\right)=2$

and we find $\left(m_{f'}+m_{f''}\right)=5$

Thus, Ans. 5.

13. Ans. (C)

$f(x)$ is a non-periodic, continuous and odd function

$$f(x)=\begin{cases}-x^2+x\sin x, & x<0\\ x^2-x\sin x, & x\ge 0\end{cases}$$

$$f(-\infty) = \underset{x\to-\infty}{\text{Lt}}(-x^2)\left(1-\frac{\sin x}{x}\right)=-\infty$$

$$f(\infty) = \underset{x\to\infty}{\text{Lt}}\, x^2\left(1-\frac{\sin x}{x}\right)=\infty$$

$\Rightarrow$ Range of $f(x) = R$

$\Rightarrow$ $f(x)$ is an onto function ...(1)

$$f'(x)=\begin{cases}-2x+\sin x+x\cos x, & x<0\\ 2x-\sin x-x\cos x, & x\ge 0\end{cases}$$

For $(0, \infty)$

$f'(x) = (x - \sin x) + x(1 - \cos x)$

always +ve always +ve

or 0 or 0

$\Rightarrow$ $f'(x) > 0$

$\Rightarrow$ $f'(x) \ge 0,\ \forall\, x\in(-\infty,\infty)$

equality at $x = 0$

$\Rightarrow$ $f(x)$ is one-one function ...(2)

From (1) & (2), $f(x)$ is both one-one & onto.

Ans. (C).

14. Ans. (A,B)

$$f(x)=\frac{x^2-3x-6}{x^2+2x+4}$$

$$f'(x)=\frac{(x^2+2x+4)(2x-3)-(x^2-3x-6)(2x+2)}{(x^2+2x+4)^2}$$

$$f'(x)=\frac{5x(x+4)}{(x^2+2x+4)^2}$$

$f'(x)$: + (−4) − (0) +

$$f(-4)=\frac{11}{6},\ f(0)=-\frac{3}{2},\ \lim_{x\to\pm\infty} f(x)=1$$

Range : $\left[-\frac{3}{2},\frac{11}{6}\right]$, clearly $f(x)$ is into

JEE (Main) Practice Paper

SECTION–A

1. Ans. (3)

$f(x_1) > f(x_2) \Rightarrow x_1 > x_2$; $g(x_1) < g(x_2) \Rightarrow x_1 > x_2$

$\therefore\ f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4)) \Rightarrow g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

$\Rightarrow \alpha^2 - 2\alpha < 3\alpha - 4 \Rightarrow \alpha^2 - 5\alpha + 4 < 0 \Rightarrow (\alpha - 1)(\alpha - 4) < 0$

$\Rightarrow \alpha \in (1,\ 4)$

2. Ans. (3)

$f(x) = |x| + |x-1|,\ x \in [0,1]$

$\Rightarrow f(x) = x + 1 - x \Rightarrow f(x) = 1$ (Constant)

3. Ans. (1)

$f(x) = x^2(x-2)^2$

$\Rightarrow f'(x) = 2x(x-2)^2 + x^2.2(x-2) \Rightarrow f'(x) = 2x(x-2)(x-2+x)$

$\Rightarrow f'(x) = 4x(x-2)(x-1) \Rightarrow f'(x) \uparrow$: [0, 1] ; [2, ∞)

4. Ans. (1)

$f(x), g(x)$ are strictly $\downarrow\ \Rightarrow f'(x) \le 0$; $g'(x) \le 0$

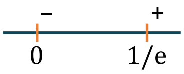
Consider $h(x) = f(g(x)) \Rightarrow h'(x) = \underbrace{f'(g(x))}_{-ve} \cdot \underbrace{g'(x)}_{-ve}$

or Zero

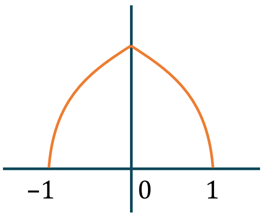
$\Rightarrow h'(x) \ge 0\ \Rightarrow h(x)$ is increasing.

5. Ans. (3)

$f(x) = x^x \Rightarrow f(x) = x^x(1 + \ell nx), x > 0$ (domain)

$f'x$  $\Rightarrow f(x) \downarrow$ in $x \in \left(0, \frac{1}{e}\right)$

6. Ans. (2)



$$\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^1 x & x \in [0,1] \\ 2\pi - 2\cos^{-1} x & x \in [-1,0] \end{cases}$$

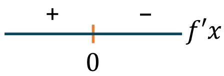
Not differentiable at $x = 0$, non monotoric even function extermum at $x = 0$

7. Ans. (4)

$f(x) = tan^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) \Rightarrow f(x) = tan^{-1}\left(\dfrac{1-x^2}{1+1\cdot x^2}\right)$

$\Rightarrow f(x) = tan^{-1}1 - tan^{-1}x^2 \Rightarrow f(x) = \dfrac{\pi}{4} - \tan^{-1} x^2$

$\Rightarrow f'(x) = 0 - \dfrac{2x}{1+x^4}$



$\Rightarrow f(x)$: strictlty $\uparrow$ in $(-\infty, 0)$; strictlty $\downarrow$ in $(0, \infty)$

8. Ans. (1)

function will be invertible if it is strictly increasing or strictly decreasing in its domain

$f : [a, \infty) \to R$. (R is corresponding range)

$f(x) = 2x^3 - 3x^2 + 6$

$\Rightarrow f'(x) = 6x^2 - 6x \Rightarrow f'(x) = 6x(x-1)$

$\therefore f(x)$ can be invertible if $x \in (-\infty, 0]$ or $x \in [0, 1]$

or $x \in [1, \infty)$

$\because x \in [a, \infty)$ is given $\Rightarrow a \geq 1$

(For any value of $a \geq 1$, the function will be increasing in $[a, \infty) \Rightarrow f(x)$ is invertible.)

9. Ans. (1)

$a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots + a_1 x = f(x)$ (let) α is a root and $x = 0$ is a root and by Rolle's theorem between any two roots of $f(x) = 0$ atleast one root of $(x) = 0$ exists

$\Rightarrow f'(x) = n\, a_n x^{n-1} + (n-1)\, a_{n-1}x^{n-2} + \ldots\ldots + a_1 = 0$

atleast once in $(0, \alpha)$.

10. Ans. (4)

For LMVT to be applicable, $f(x)$ must be continuous in $[-1,1]$ and derivable in $(-1,1)$

CHECK : at $x = 0$: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

$\therefore f(x)$ cont. at $x = 0$, and in $[-1,1]$

Also, at $x = 0$:

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} \Rightarrow f'(0) = \lim_{h\to 0}\frac{h\cos(1/h) - 0}{h} = \lim_{h\to 0}\cos(1/h) = DNE$$

$\therefore f(x)$ is ND at $x = 0$ in $(-1,1)$

$\therefore$ LMVT is not applicable.

11. Ans. (1)

$$g'(x) = \underbrace{2e^{2x} f(x)}_{+ve}\underbrace{(2(f(x))^2 - 3f(x) + 2)}_{\substack{a>0 \\ D<0 \\ \therefore \text{ Always } +ve}} + \underbrace{e^{2x} f'(x)(6(f(x))^2 - 6f(x) + 2)}_{\substack{a>0 \\ D<0 \\ \therefore \text{ Always } +ve}}$$

If $f'(x)$ is $+ve$ then $g'(x)$ is $+ve$

$\therefore g(x)$ is strictly increasing

12. Ans. (2)

Consider $h(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$

$h(x)$ is continuous and derivable

$h(0) = 0$

$h(2) = \frac{a.16}{4} + \frac{b.8}{3} + \frac{c.4}{2} + d2 = \frac{2}{3}(6a + 4d + 2c + 2c)$

$\Rightarrow h(2) = 0$

$\therefore$ Atleast one root of $h'(x) = 0$ in $(0,2)$

13. Ans. (2)

$y = x^2 e^{-2x}$

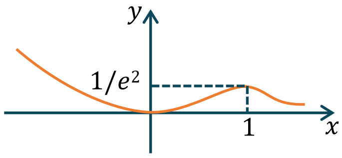
$y' = \left(2x - 2x^2\right) e^{-2x}$

$e^{-2x}\left(2x^2 - 2x\right) = 0$

$x = 2, x = 0$

at $x = 1$, $y = e^{-2}$

at $x = 0$, $y = 0$



14. Ans. (2)

$f'(x) = e^{x(1-x)} + x.\{1-2x\}.e^{x(1-x)}$

$= e^{x(1-x)}\{1 + x - 2x^2\} = -e^{x(1-x)}(2x+1)(x-1)$

15. Ans. (4)

$f'(x) > 0 \Rightarrow f(x)$ is increasing function

$g'(x) < 0 \Rightarrow g(x)$ is decreasing function

Now,

(i) $x > x - 1$

$f(x) > f(x-1)$

$g(f(x)) < g(f(x-1))$

and

(ii) $x + 1 > x$

$f(x+1) > f(x)$

$g(f(x+1)) < g(f(x))$

(iii) $x > x - 1$

$g(x) < g(x-1)$

$f(g(x)) < f(g(x-1))$

(iv) $x + 1 > x$

$g(x+1) < g(x)$

$f(g(x+1)) < f(g(x))$

16. Ans. (3)

$\because\ f(x) = x^3 - 3x^2 + 5x + 7$

$f'(x) = 3x^2 - 6x + 5$

$\because\ D < 0 \Rightarrow f'(x) > 0\ \forall\ x \in R$

$\therefore\ f(x)$ = is increasing $\forall\ x \in R$

17. Ans. (4)

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{d - x}{\sqrt{b^2 + (d-x)^2}}$$

$$f'(x) = \frac{a^2}{\left(a^2 + x^2\right)^{3/2}} + \frac{b^2}{\left(b^2 + (d-x)^2\right)^{3/2}} > 0 \forall x \in R$$

$f(x)$ is an increasing function.

18. **Ans. (2)**

$\phi(x) = f(x) + f(2-x)$

$\phi'(x) = f'(x) - f'(2-x)$ ......(1)

Since $f''(x) > 0$

$\Rightarrow f'(x)$ is increasing $\forall\, x \in (0, 2)$

Case-I : When $x > 2 - x \Rightarrow x > 1$

$\Rightarrow \phi'(x) > 0\ \forall x \in (1, 2)$

$\therefore\ \phi(x)$ is increasing on $(1, 2)$

Case-II : When $x < 2 - x \Rightarrow x < 1$

$\Rightarrow \phi'(x) < 0\ \forall x \in (0, 1)$

$\therefore\ \phi(x)$ is decreasing on $(0, 1)$

19. **Ans. (4)**

$f(0) = 11$

$f(1) = 16$

$\dfrac{f(1) - f(0)}{1 - 0} = 3c^2 - 8c + 8$

$\Rightarrow\ 3c^2 - 8c + 8 = 5\ \Rightarrow\ 3c^2 - 8c + 3 = 0$

$c \in [0, 1]\ \Rightarrow c = \dfrac{4 - \sqrt{7}}{3}$

20. **Ans. (1)**

$|f''(c)| \le 2$

$\Rightarrow |f'(x) - f'(a)| \le 2|x - a|\ \Rightarrow\ |f'(x) - f'(a)| < 2$

a is that point on the interval when

$f'(a) = \left|\dfrac{f(2) - f(1)}{2 - 1}\right| = 0$ as $f(1) = f(2)$

SECTION-B

1. **Ans (2)**

$\dfrac{x^3 - 2x^2 - 5x + 6}{x - 1} = \dfrac{(x-1)(x^2 - x - 6)}{(x-1)} = x^2 - x - 6$

$\Rightarrow\ f(x) = \begin{cases} x^2 - x - 6\ ; & x \ne 1 \\ -6\ ; & x = 1 \end{cases}$

$\lim\limits_{x \to 1^+} f(x) = -6\ ;\ \lim\limits_{x \to 1^-} f(x) = -6$

$\Rightarrow$ LHL = RHL = $f(1)$ $\quad\therefore\ f(x)$ is continuous

LHD at $x = 1$ is 1

RHD at $x = 1$ is 1

$\therefore\ f(x)$ is differentiable at $x = 1$

$f(-2) = 0\ ;\ f(3) = 0$

all the conditions of Rolle's are holding

$f'(x) = 2x - 1 = 0$

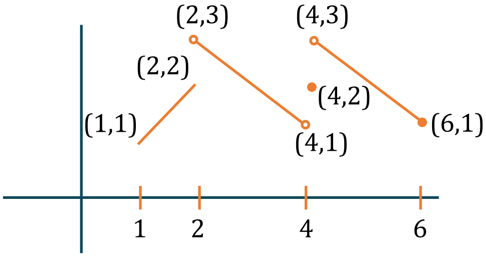
$\Rightarrow x = \frac{1}{2}$ $\qquad\qquad$ $\therefore \frac{1}{2} \in [-2, 3]$

2. **Ans (4)**

$f'(x) = 0 \Rightarrow x = -2, 3$

$x = -2 \in (-3, 0)$ $\qquad$ $\therefore c = -2$

3. **Ans (2)**



$x = 2$ **and 4**

4. **Ans (0)**

$$f(x) = \begin{cases} \dfrac{1-x}{x^2} & x<1, \quad x \neq 0 \\ \dfrac{x-1}{x^2}, & x \geq 1, \end{cases}$$

The given function is not differentiable at $x = 1$

$$f'(x) = \begin{cases} \dfrac{1}{x^2} - \dfrac{2}{x^3}, & x<1, \quad x \neq 0 \\ \dfrac{2}{x^3} - \dfrac{1}{x^2}, & x>1 \end{cases}$$

$$\text{Now } f'(x) < 0 \Rightarrow \begin{cases} \dfrac{x-2}{x^3} < 0 & given \quad x<1 \\ \dfrac{2-x}{x^3} < 0 & when \quad x>1 \end{cases}$$

$f(x)$ decreasing $\forall\, x \in (0, 1) \cup (2, \infty)$ and $f(x)$ increases $\forall\, x \in (-\infty, 0) \cup (1, 2)$

here $f(x)$ is decreasing at all points in $x \in (0, 1) \cup (2, \infty)$ so will also be decreasing at $x = 3$ at $x = 1$ minima and at $x = 2$ maxima

5. **Ans. (1)**

$f(-1) = -2 + a - b, f(1) = 2 + a + b$

$f(-1) = f(1) \Rightarrow -2 + a - b = 2 + a + b$

$-2 = b$

$f'(x) = 6x^2 + 2ax + b$

$f'\left(\frac{1}{2}\right) = 6 \cdot \frac{1}{4} + 2 \cdot a \cdot \frac{1}{2} + b = 0$

$\Rightarrow \frac{3}{2} + a + b = 0$ $(\because b = -2)$

$\Rightarrow a = \frac{1}{2}$ $\qquad\qquad$ $\because 2a + b = -1$

6. **Ans. (4)**

Using LMVT in [–7, –1]

$$\frac{f(-1)-f(-7)}{-1-(-7)} \le 2$$

$f(-1) - f(-7) \le 12$

$\Rightarrow f(-1) \le 9$ ....(1)

Using LMVT in [–7, 0]

$$\frac{f(0)-f(-7)}{0-(-7)} \le 2$$

$f(0) - f(-7) \le 14$

$f(0) \le 11$ ....(2)

from (1) and (2)

$f(0) + f(-1) \le 20$

7. **Ans. (4)**

$$\frac{9+\alpha}{21} = \frac{16+\alpha}{28} \Rightarrow \alpha = 12$$

Also, $f'(x) = \dfrac{7x}{x^2+12} \times \dfrac{x^2-12}{7x^2} = \dfrac{x^2-12}{x\left(x^2+12\right)}$

Hence, $c = 2\sqrt{3}$

Now, $f''(c) = \dfrac{1}{12}$

8. **Ans. (3)**

$f(4) = f(5) = f(6) = f(7) = 0$

By Rolle's theorem on interval

[4, 5], [5, 6], [6, 7] we have

$f'(x) = 0$ for at least once in each intervals (4, 5), (5, 6), (6, 7).

9. **Ans. (1)**

$f(x) \downarrow, g(x) \uparrow$

let $h(x) = f(x) - g(x)$

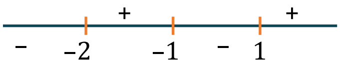
$h(1) = 2, h(2) = -6$

$\Rightarrow$ $h(x) = 0$ has at least one root in (1, 2).

but $h'(x) = f'(x) - g'(x) < 0$ $\Rightarrow h(x)$ is always decreasing so only one root

10. **Ans. (2)**

$f'(x) = 12(x+2)(x+1)(x-1)$



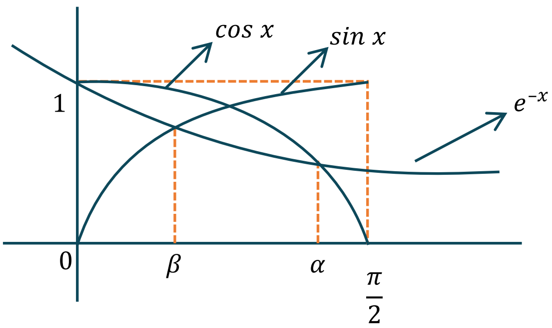
sings of $f'x$

$\Rightarrow a = -2, b = -1$

JEE (Advanced) Practice Paper

1. **Ans. (A,B,C,D)**

$$\begin{cases} \text{According to Rolle's theorem} \\ f(x)=e^{x}\cos x-1 \\ f'(x)=e^{x}(-\sin x)+e^{x}\cos x=0 \\ \tan x=1 \end{cases}$$



Similarly we can get option (B)

for option (c) :

$$e^{x}\cos x = 1 \Rightarrow \cos x = e^{-x} \begin{cases} 0 \\ \alpha \end{cases}$$

b/w 0 and α, we have one value β for which $\sin\beta = e^{-\beta}$

$\therefore$ (c) is correct

similarly for D

2. **Ans. (A,C,D)**

LMVT on $f(x)$ in $[A, B]$

$$\frac{f(B)-f(A)}{B-A}=f'(C)$$

$$\Rightarrow \frac{\sin^2 B-\sin^2 A}{B-A} = 2\sin C\cos C = \sin 2C$$

$A < C < B \Rightarrow 2A < 2C < 2B$

$\Rightarrow \sin 2A < \sin 2C < \sin 2B$ $\left(\because \sin x \text{ is increasing in } x\in\left[0,\frac{\pi}{2}\right]\right)$

$\Rightarrow (B-A)\sin 2A<\sin^2 B-\sin^2 A<(B-A)\sin 2B$

Also $2A < 2B \Rightarrow \cos 2A > \cos 2B$

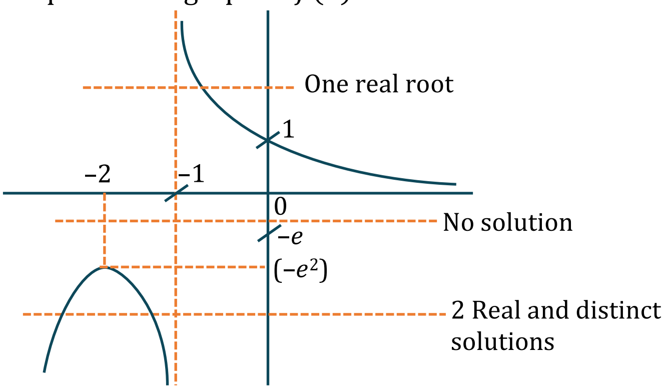
($\cos x$ is derceasing f^n)

Now $\left.\begin{matrix} B>A \\ \cos 2A>\cos 2B \end{matrix}\right\} \Rightarrow B\cos 2A> A\cos 2B$

3. **Ans. (B,C,D)**

$$y=\frac{e^{-x}}{x+1}=f(x)$$

we part of the graph of $f(x)$

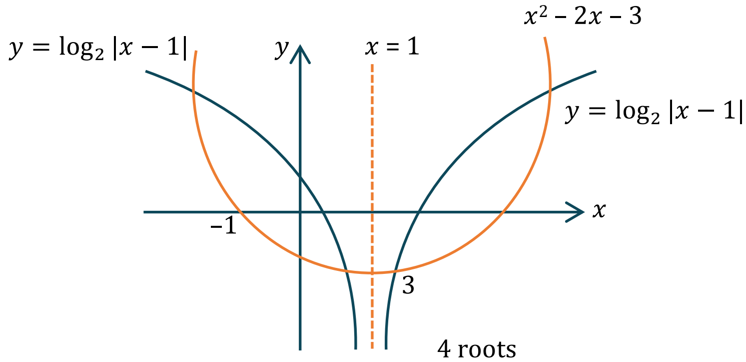


4. **Ans. (A)**

$x^2-2x-\log_2|x-1|=3$

$\Rightarrow x^2-2x-3=\log_2|x-1|=y$

we plot graphs



5. **Ans. (A,B)**

$f'(x)=\frac{m}{n}x^{\left(\frac{m-n}{n}\right)}$

$m-n$ is odd.

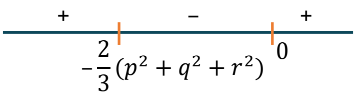
$f'(x)<0\ \forall x\in(-\infty,\ 0)$

$f'(x)>0\ \forall x\in(0,\infty)$

6. **Ans. (A,B)**

$$f(x)=\begin{vmatrix} x+p^2 & pq & pr \\ pq & x+q^2 & qr \\ pr & qr & x+r^2 \end{vmatrix}=x^3+\left(p^2+r^2+q^2\right)x^2$$

$f'(x)=3x^2+2x\left(p^2+q^2+r^2\right)\ =x\left\{3x+2\left(p^2+q^2+r^2\right)\right\}$



Here $f(x)$ is increasing if

$x<-\frac{2}{3}\left(p^2+q^2+r^2\right)$ and $x>0$

decreasing is if $-\left(p^2+q^2+r^2\right)<x<0$

7. **Ans. (A,B,C)**

$f(x)=\frac{x^2}{4}\operatorname{cosec}^2\frac{x}{2}$

$f'(x)=\frac{x}{2}\operatorname{cosec}^2\frac{x}{2}\cot\frac{x}{2}\left(\tan\frac{x}{2}-\frac{x}{2}\right)$

For $0<x<1,\ \tan\frac{x}{2}>\frac{x}{2}$

$\Rightarrow f'(x)>0\ \Rightarrow\ f(x)$ is increasing.

8. **Ans. (B,D)**

$f(0) = 0 \neq f(1)$

there will be no $x \in (0, \infty)$ ($\therefore$ Rolle's theorem is not applicable)

for which $f'(x) = 0$ i.e, $\cot^{-1} x = \frac{x}{1+x^2}$

$$f''(x) = \frac{-1}{1+x^2} - \frac{(1+x^2)-2x^2}{(1+x^2)^2} = \frac{-1}{1+x^2} + \frac{x^2-1}{(x^2+1)^2}$$

$$f''(x) = \frac{-2}{(x^2+1)^2} < 0$$

$f'(x)$ is strictly decreasing

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\frac{-x}{1+x^2} + \cot^{-1} x\right) = 0$$

$$f(0+) = \lim_{x\to 0^+}\left(\cot^{-1} x - \frac{x}{1+x^2}\right) = \frac{\pi}{2}$$

$$\frac{f\left(x+\frac{2}{\pi}\right) - f(x)}{2/\pi} = f'(c)\ c \in \left(0, \frac{\pi}{2}\right)$$ ($\therefore$ LMVT is applicable)

$$\therefore\ f'(c) < \frac{\pi}{2}$$

$$f\left(x+\frac{2}{\pi}\right) - f(x) < \frac{2}{\pi}\times\frac{\pi}{2}$$

$$f\left(x+\frac{2}{\pi}\right) - f(x) < 1$$

$f'(x) \leq 0$; $f(x)$ is increasing

$f(x) \in [f(0), f(\infty))$

$f(0) = 0$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\cot^{-1} x = \lim_{x\to 0}\frac{\cot^{-1} x}{1/x}$$

$$= \lim_{x\to\infty}\frac{-1}{1+x^2}\times(-x^2) = 1$$

$f(x) \in [0, 1)$

$f(x) = \sec x$ will have no solution

9. **Ans. (C)**

10. **Ans. (B)**

Sol. Q.9 to Q.10

Let $g(x) = \frac{x+\sin x}{2}$, $x \in [0, \pi]$. $g(x)$ is increasing function of x.

$\therefore$ range of $g(x)$ is $\left[0, \frac{\pi}{2}\right]$

$\therefore f(x) = \frac{x+\sin x}{2}$, $x \in [0, \pi]$

Now let $\pi \le t \le 2\pi$, then

$f(t)+f(2\pi-t)=\pi$

i.e $f(t)+\dfrac{2\pi-t+\sin(2\pi-t)}{2}=\pi$

i.e $f(t)+\pi-\dfrac{t}{2}-\dfrac{\sin t}{2}=\pi$

i.e $f(t)=\dfrac{t+\sin t}{2}$

$\therefore f(x)=\dfrac{x+\sin x}{2}$ for $\pi \le x \le 2\pi$

Thus $f(x)=\dfrac{x+\sin x}{2}$ for $0 \le x \le 2\pi$

Also $f(x)=f(4\pi-x)$ for all $x \in [2\pi, 4\pi]$

$\Rightarrow f(x)$ is symmetric about $x=2\pi$

$\therefore$ from graph of $f(x)$

$\therefore \alpha = 2\pi - 0 = 2\pi$

$\therefore \beta = \alpha$

11. Ans. (C)

$y=(x-1)^3(x-2)^2$

$\dfrac{dy}{dx}=3(x-1)^2(x-2)^2+2(x-2)(x-1)^3$

$=(x-1)^2(x-2)[3(x-2)+2(x-1)]=(x-1)^2(x-2)(5x-8)$

$(x^2-2x+1)(5x^2-18x+16)$

$\dfrac{d^2y}{dx^2}=(2x-2)(5x^2-18x+16)+(10x-18)(x^2-2x+1)=0$

$=20x^3-42x^2+11x-50=0 \quad =10x^3-42x^2+57x-25=0$

$(x-1)(10x^2-32x+25)=0$

$x=1$ or $x=\dfrac{32\pm\sqrt{24}}{20}$

no. of points of inflections = 3

12. Ans. (B)

$f(x)=x^4+ax^3+\dfrac{3x^2}{2}+1$

$f'(x)=4x^3+3ax^2+3x$

$f''(x)=12x^2+6ax+3$

Now $f(x)$ will be concave upward along the entire real line iff $f''(x) \ge 0 \ \forall\ x \in R$

$12x^2+6ax+3>0 \Rightarrow D \le 0$

$36a^2-144 \le 0$

$a^2-4 \le 0 \Rightarrow a \in [-2, 2]$

13. Ans. (7)

$f'(x)=4\left[\cos^2 x-2(a+1)\cos x+a^2+2a-4\right]$

Let $\phi(t)=t^2-2(a+1)t+a^2+2a-4$, $-1\le t\le 1$

It is sufficient to find values of t when $\phi(t)=0$ has no root in $[-1, 1]$

$D = 20$

Case-I :

$\phi(-1) > 0$, $a + 1 < -1$, $D = 20 > 0$

$\Rightarrow a \in \left(-\infty,-2-\sqrt{5}\right) \cup \left(-2,+\sqrt{5},\infty\right)$ and $a < -2$

$\Rightarrow a \in \left(-\infty,\ -2-\sqrt{5}\right)$

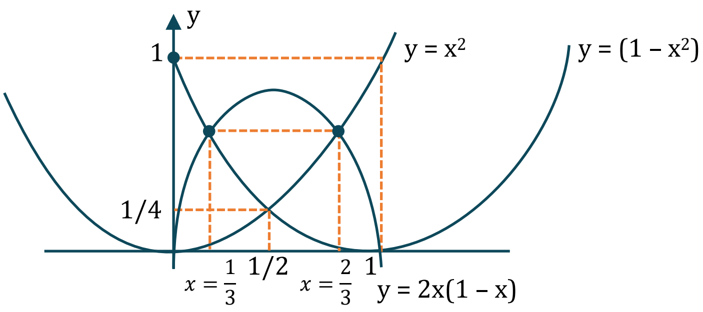
Case - II :

$\phi(1) > 0$, $a + 1 > 1$, $D = 20 > 0$

$\Rightarrow a \in \left(-\infty,-\sqrt{5}\right) \cup \left(\sqrt{5},\infty\right)$ and $a > 0$

$\Rightarrow a \in \left(\sqrt{5},\infty\right)$

14. Ans. (3)



From figure we can see Rolle's theorem is applicable for $x\in\left[\frac{1}{3},\ \frac{2}{3}\right]$ and $f'(c) = 0 = 2 - 4c$

$\Rightarrow c = \frac{1}{2}$

$a + b + c = \frac{1}{3}+\frac{2}{3}+\frac{1}{2}=\frac{3}{2}$

15. Ans. (6)

$f(x) = \frac{\log x}{x}$ is differentiable and continuous for every $x > 0$.

Now for RMV to be applicable $f(a) = f(b)$

$\Rightarrow \frac{\ell na}{a}=\frac{\ell nb}{b} \Rightarrow a^b = b^a \Rightarrow a = 2, b = 4$

hence $a + b = 6$.

16. Ans. (2)

$f(x) = 3x^2 + 5x + 7$

$f(1) = 15 f(3) = 49$

$f'(x) = 6x + 5$

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$6c+5=\frac{49-15}{3-1}=17$$

$c=2$

17. Ans. (6)

$f(x)=\frac{49-7}{3}x^3+(a-3)x^2+x+5$

$f'(x)=(49-7)x^2+2(a-3)x+1$

$f'(x)>0$ or $f'(x)<0$

So, $D<0$ & $49-7\neq 0$

$a\in(2,8)$ & $a\neq\frac{7}{4}$

$a\in(2,8)$

$m=2, M=8$

$M-m=6$

18. Ans. (3)

(i) Let assume $f(x)=\tan^{-1}x\frac{3x}{x^2+3}$

Investigating the behaviour of $f(x)$ i.e.

$$f'(x)=\frac{1}{1+x^2}-\left[\frac{3(x^2+3)-3x(2x)}{(x^2+3)^2}\right]=\frac{4x^4}{(1+x^2)(x^2+3)^2}$$

Hence $f(x)$ is a strictly increasing function.

Moreover $f(0)=0\Rightarrow f(x)>0\forall x>0\Rightarrow \tan^{-1}x>\frac{3x}{3+x^2}$

(ii) Let $f(x)=1+x\ell n\left(x+\sqrt{x^2+1}\right)-\sqrt{1+x^2}$

Then $f'(x)=\ell n\left(x+\sqrt{x^2+1}\right)+\frac{x}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)-\frac{x}{\sqrt{1+x^2}}=\ell n\left(x+\sqrt{x^2+1}\right)$

Now for all $x\geq 0, x+\sqrt{x^2+1}$ is greater than or equal to 1

$\therefore f'(x)\geq \ell n1=0$

Hence $f(x)$ is increasing in $(0,\infty)$. Hence $x>0\Rightarrow f(x)>f(0)>0$

$\therefore 1+x\ell n\left(x+\sqrt{x^2+1}\right)-\sqrt{1+x^2}>0$ or $1+x\ell n\left(x+\sqrt{x^2+1}\right)>\sqrt{1+x^2}$

(iii) Let $f(x)=sinx-x\Rightarrow f'(x)=cosx-1=2\sin^2\left(\frac{x}{2}\right)<0$

$\Rightarrow f(x)$ is a decreasing function.

$\therefore f(x)<f(0)$ in $\left(0,\frac{\pi}{2}\right)\Rightarrow sinx<x$ (i)

Let $g(x)=\frac{x-x^3}{6}-\sin x\Rightarrow g'(x)=\frac{1-x^2}{2}-\cos x=\phi(x)(\text{say})$

This is done to find sign of $g'(x)$.

$\phi'(x)=-x+\sin x\Rightarrow \phi'(x)<0$ from equation (i) $\Rightarrow \phi(x)$ is decreasing function.

$\therefore \phi(x)<\phi(x)<0\Rightarrow g'(x)<0$

$\Rightarrow g(x)$ is a decreasing function $\Rightarrow g(x)<g(0)$

$\Rightarrow g(x)<0$ or $\frac{x-x^3}{6}<\sin x$